1. a) We want to reduce the error by going to the next order in the truncation error as that we saw in class. Since even degrees will not contribute to the error, we need to Taylor expands our functions to the 5th degree to get a better approxiamtion than what we had before.

This gives that:

$$f(x \pm \delta x) = f(x) \pm f'(x)\delta x + \frac{f''(x)\delta x^2}{2} \pm \frac{f^{(3)}(x)\delta x^3}{6} + \frac{f^{(4)}(x)\delta x^4}{24} \pm \frac{f^{(5)}(x)\delta x^5}{120} + ...$$

$$f(x \pm 2\delta x) = f(x) \pm f'(x)\delta x + 2f''(x)\delta x^2 \pm \frac{4}{3}f^{(3)}(x)\delta x^3 + \frac{2}{3}f^{(4)}(x)\delta x^4 \pm \frac{2}{15}f^{(5)}(x)\delta x^5 + ...$$

We then have that:

$$\frac{f(x+\delta x) - f(x-\delta x)}{2\delta x} = f'(x) + \frac{1}{6}f^{(3)}(x)\delta x^2 + \frac{1}{120}f^{(5)}\delta x(4) + ...$$

$$\frac{f(x+2\delta x) - f(x-2\delta x)}{4\delta x} = f'(x) + \frac{2}{3}f^{(3)}(x)\delta x^2 + \frac{2}{15}f^{(5)}\delta x(4) + ...$$

We thus want to find $a, b, c \in \mathbb{R}$ such that

$$a\frac{f(x+\delta x) - f(x-\delta x)}{2\delta x} - b\frac{f(x+2\delta x) - f(x-2\delta x)}{4\delta x} = f'(x) + c * f^{(5)}\delta x^4$$

$$\Rightarrow a\Big(f'(x)+\frac{1}{6}f^{(3)}(x)\delta x^2+\frac{1}{120}f^{(5)}\delta x(4)\Big)+b\Big(f'(x)+\frac{2}{3}f^{(3)}(x)\delta x^2+\frac{2}{15}f^{(5)}\delta x(4)\Big) = f'(x)+c*f^{(5)}\delta x^{(4)}$$

Solving this system of equation then gives $a = \frac{4}{3}, b = -\frac{1}{3} and c = \frac{1}{30}$.

Isolating $f'(x)$ we then get that:

$$f'(x) = \frac{8\Big(f(x+\delta x) - f(x-\delta x)\Big) - \Big(f(x+2\delta x) - f(x-2\delta x)\Big)}{12\delta x} + \frac{\delta x^4}{30}f^{(5)}(x)$$

Our error is thus now $\frac{\delta x^4}{30}f^{(5)}(x)$ and we use

$$f'(x) = \frac{8\Big(f(x+\delta x) - f(x-\delta x)\Big) - \Big(f(x+2\delta x) - f(x-2\delta x)\Big)}{12\delta x}$$

to compute the derivative.

b) To compute the optimal δx, we need to consider the error coming from the representation of the number in binary. Let

$$\overline{f}(x) = f(x) + \epsilon f(x)$$

where $\overline{f}(x)$ is the representation of the function in binary, $f(x)$ is the true value of the function and ϵ accounts for the difference between the two. To find the optimal δx we first need to compare the difference between the real value of the derivative and its representation:

$$\begin{aligned} E(\epsilon) &= \left| f'(x) - \frac{8\Big(\overline{f}(x+\delta x) - \overline{f}(x-\delta x)\Big) - \Big(\overline{f}(x+2\delta x) - \overline{f}(x-2\delta x)\Big)}{12\delta x} \right| \\ \Rightarrow E(\epsilon) &\leq \left| f'(x) - \frac{8\Big(f(x+\delta x) - f(x-\delta x)\Big) - \Big(f(x+2\delta x) - f(x-2\delta x)\Big)}{12\delta x} \right| \\ &+ \left| \frac{8\Big(\epsilon_1 f(x+\delta x) - \epsilon_2 f(x-\delta x)\Big) - \Big(\epsilon_3 f(x+2\delta x) - \epsilon_4 f(x-2\delta x)\Big)}{12\delta x} \right| \end{aligned}$$

bn using the form of the represetation and the triangular inequality. Now, let $\epsilon > 0$ such that $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 < \epsilon$. By using the result from a), we then get that:

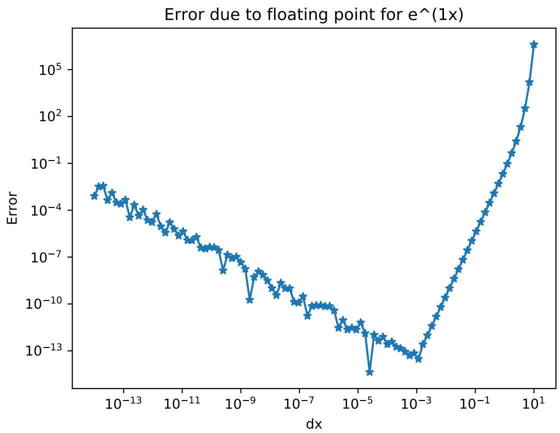
$$\begin{aligned} E(\epsilon) &\leq \left| \frac{\delta x^4}{30} f^{(5)}(x) \right| + \left| \frac{8\Big(\epsilon_1 f(x+\delta x) - \epsilon_2 f(x-\delta x)\Big) - \Big(\epsilon_3 f(x+2\delta x) - \epsilon_4 f(x-2\delta x)\Big)}{12\delta x} \right| \\ &\leq \left| \frac{\delta x^4}{30} f^{(5)}(x) \right| + \epsilon \Big| f(x) \Big| \left| \frac{1}{6\delta x} + \frac{4}{3\delta x} \right| \\ &= \frac{\delta x^4}{30} \Big| f^{(5)}(x) \Big| + \frac{3\epsilon}{2\delta x} \Big| f(x) \Big| \end{aligned}$$

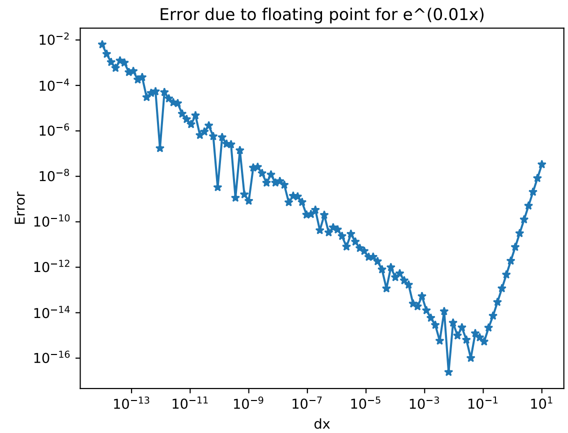
To find the optimal δx, we minimize E with respect to dx:

$$\frac{\partial E}{\partial(\delta x)} = \frac{2 * \delta x^3}{15} \Big| f^{(5)}(x) \Big| - \frac{3\epsilon}{2 * \delta x^2} \Big| f(x) \Big| = 0$$

$$\Rightarrow \frac{2\delta x^3}{15} \Big| f^{(5)}(x) \Big| = \frac{3\epsilon}{2\delta x^2} \Big| f(x) \Big| \Rightarrow \delta x = \sqrt[5]{\frac{45\epsilon}{4} \frac{\Big| f(x) \Big|}{\Big| f^{(5)}(x) \Big|}}$$

Thus since $\frac{45}{4} \approx 10$ and $\epsilon \approx$ 1e-16 for double precision, if $f^{(5)}(x)$ has the same order as $f(x)$ like it is the case for exp(x), we should find approximately 1e-3 for the best δx. For exp(0.001), we should find around 1e-1 since $f^{(5)}(x)$ is of order 1e-10 times and $f(x)$ is of order 1. As we can see in the two plots below:


Error due to floating point for e^(1x)
Error
dx


Error due to floating point for e^(0.01x)
Error
dx