A 3D variant of Vogel's model to observe prime patterns in a spherical coordinate system

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Abstract

The prime numbers are a mystery as no complete formula exist to enumerate them. Existing visualization methods show prime numbers as patterns in 2D space but these patterns are too complex to formulate. We describe a novel 3D visualization approach that shows strong natural patterns, e.g., sea shells, which should ease formulation.

1 Introduction

Personal and corporate security have become one of the biggest concern in modern society. In order to increase security, to combat more and more powerful hackers who make a living in stealing identities and corporate secrets, it became necessary to create new highly secure encryption algorithms. The cornerstone of this encryption is surprisingly, prime numbers.

The way encryption systems use prime numbers is through the concepts of keys, specifically a public and a private key. The public key is the result of the multiplication of two enormous prime numbers and the private key is the two prime factors of the public key. Even for a computer, the calculation of the prime factors of the public key is really difficult and can take years to process. This is why systems are secure for now [1].

The reason prime numbers are used is that they are very mysterious as no formula exists to generate them all. There exist some formulas to enumerate limited sequences like the Mersenne method which uses this formula:

$$M_n = 2^{\mathrm{n}} - 1$$

It was using this method that the largest prime number was ever found [2].

$$2^{74,207,281} - 1$$

Currently sequences of large prime numbers can only be enumerated using very powerful computer farms and by waiting a very long time.

The holy grail in mathematics is to find the formula that could generate all prime numbers. Many mathematicians have spent their whole life hunting down this formula. Recently, they started looking into non-traditional approaches to crack the mystery. Stan Ulam and Robert Sacks used computers to generate visual representations of prime number distributions in the hope of seeing strong patterns. If one can see a pattern then one could hope to find a formula for the pattern, however complex it is [9]. Using his namesake spiral, Ulam observed patterns as diagonal lines, for most of which he found a parametric formulation, and some of which could generate a very large number of primes.[3]

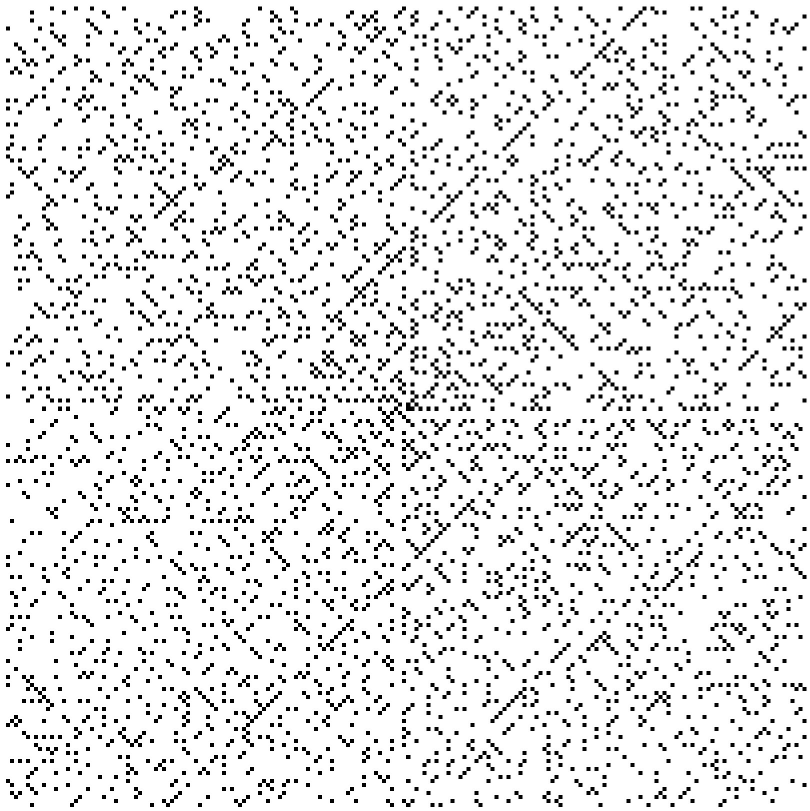


Figure 1: The Ulam Spiral [3].

The Sacks spiral is a variation of Ulam's, using a different formula to create the visualization of the prime numbers [3]. Sacks changed the square spiral formula to an Archimedean spiral so that at each complete rotation the perfect squares were on the positive x axis [4].

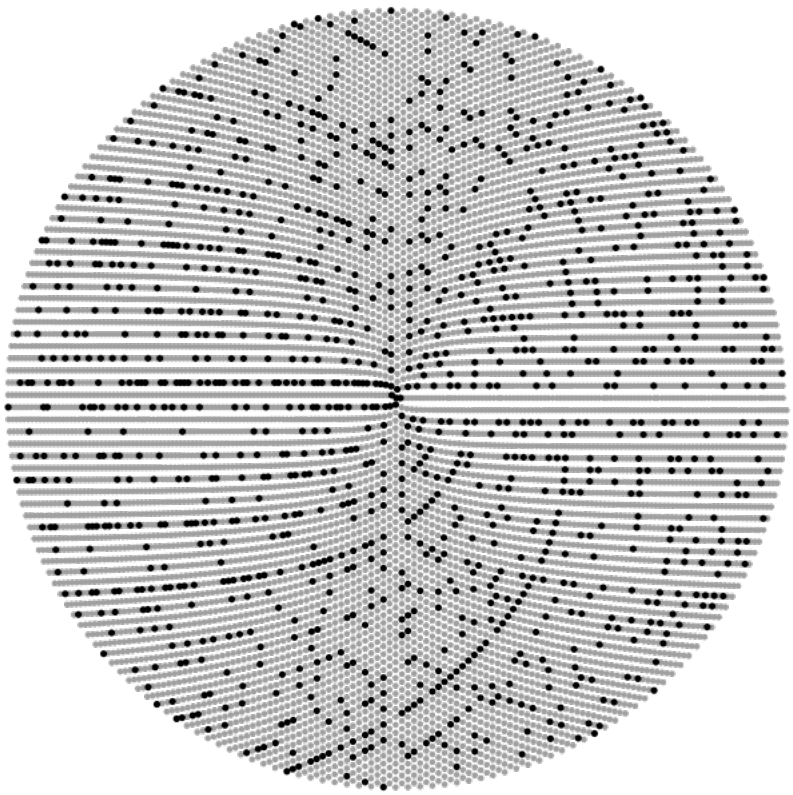


Figure 2: The Sacks spiral [4].

Other mathematicians found natural patterns in their spiral, such as the shell morphology or the organization of sunflower seeds. The study of patterns in nature is known as spiral phyllotaxis. In the study of spiral phyllotaxis, mathematicians found a recurring ratio number known has the golden ratio. This ratio is wildly used in computer graphics to represent plants and generate fractals [6].

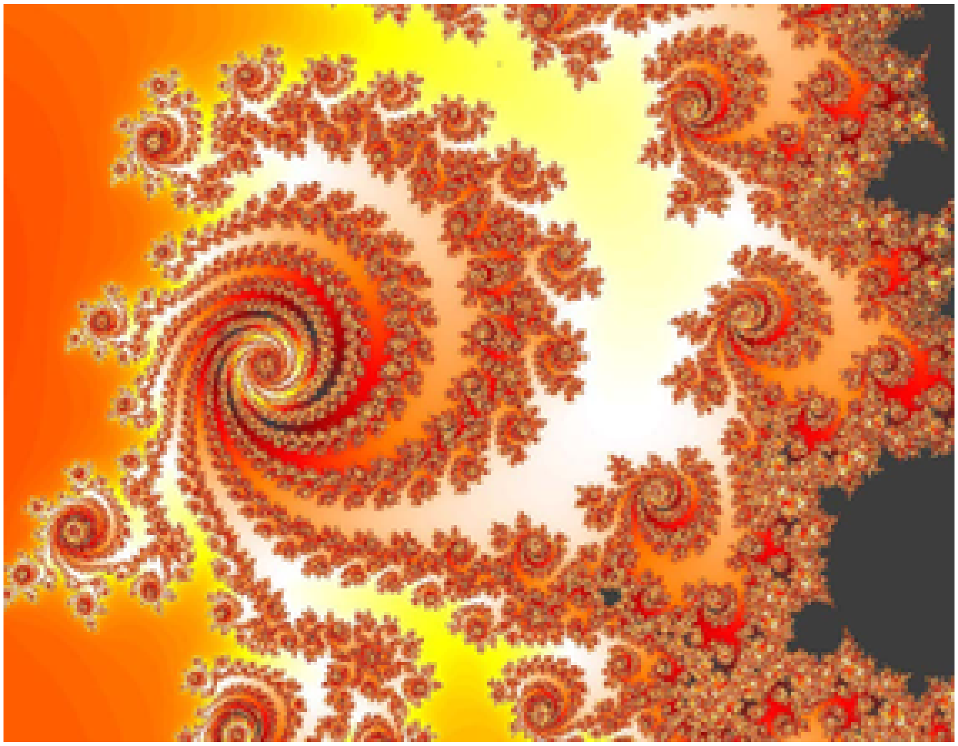


Figure 3: Computer generated fractal using the golden ratio.

My results use these principles to create a new and interesting way to visualize primes in three dimensional space. My model is based on existing formulas used to create 2D natural patterns such as sunflowers. By extending these formulas for 3D visualization, elegant spirals similar to sea shells can be observed. These strong and more discernible patterns, as compared to 2D, provides a possible new path to the discovery of the complete formulation of prime numbers.

2 Prerequisite

The following knowledge is required to understand my prime number visualization model:

- polar coordinate system,
- spherical coordinate system,
- basic understanding of Matlab's 3D plotting functions,
- prime numbers,
- 2D spirals used to visualize prime numbers.

The Cartesian coordinate system is known to all but isn't the only one use in the world of mathematics. Other systems like the polar and spherical coordinate systems use really different definitions of axis and coordinate to visualize functions. For example, the polar coordinate system is a two-dimensional system in which each point in the space is described by the length of the vector originating from the reference point of the system (origin) and the angle of the vector to the x axis of the Cartesian system.The spherical coordinate system is its equivalent in 3D where the third dimension is called elevation. Elevation correspond to the angle of rotation from the reference direction.

Basic knowledge of primes and visualization methods will be required to generate the prime distributions. There are many 2D spirals used for prime number visualization [9]. The Vogel model is well known and has been used to generate the sunflower pattern in computer graphic systems. This model is a variant of the Fermat spiral, using only discrete points in the polar graph. Using this approach Vogel observed that sunflower seeds grow in a very efficient manner which can be formulated using the golden angle [8]. The golden angle is obtained as follows:

$$(360 - \frac{360}{1.618...}) = 137.508...$$

The Vogel spiral is defined as follows:

$$r(n) = \sqrt{n}$$

and

$$\theta(n) = n\psi$$

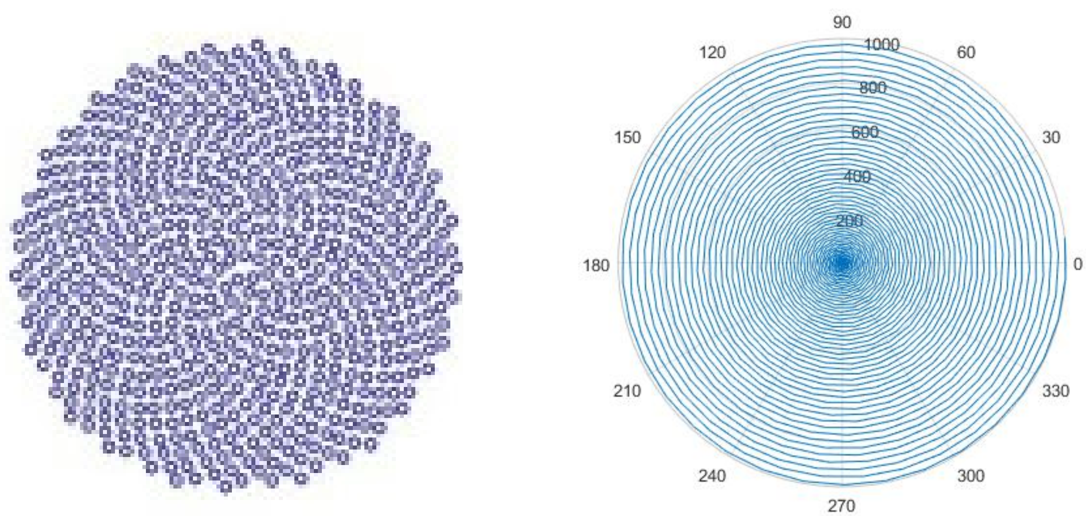
$$\psi \approx 137.5$$

It is as variant of the Fermat spiral:

$$\theta(n) = n\psi$$

$$\psi \approx 137.5$$

$$r = \sqrt{\theta}$$



(a) The Vogel model [6]   (b) The Fermat spiral [6]

Figure 4: The spiral used in the sunflower formula.

Matlab Code

```
tic
clf
allPoint=cell(1,1000);
prime=1;
allPointPrimeExist=cell(1,prime);
square=1;
allPointSquare=cell(1,square);
allPointSphere=cell(1,1000);
for i=1:1000
%azimuth=i*1.61803398875;
%elevation=i*1.61803398875;
%r=sqrt(elevation);
%azimuth=i*137.508;
%elevation=i*137.508;
%r=sqrt(elevation);
azimuth=i;
elevation=i;
r=sqrt(elevation);
allPointSphere{i}=[azimuth,elevation,r];
[x,y,z] = sph2cart(allPointSphere{i}(1),allPointSphere{
    i}(2),allPointSphere{i}(3));
allPoint{i} = [x,y,z];
%fft(gpuArray(allPointSphere{i}(:)));
%fft(gpuArray(allPoint{i}(:)));
view(270, 90);
if(allPoint{i}(3) > 0)
scatter3(x,y,z,[],[1 0 0],'*');
hold on;
end
if isprime(i)
allPointPrimeExist{prime}=[azimuth,elevation,r];
if(allPoint{i}(3) > 0)
view(270, 90);
[x,y,z] = sph2cart(allPointPrimeExist{prime}(1),
    allPointPrimeExist{prime}(2),allPointPrimeExist{
    prime}(3));
```

```
scatter3(x,y,z,[],[0 1 0],'*');
hold on;
prime=prime+1;
end
end

if rem(allPointSphere{i}(3),1) ==0
%if rem(sqrt(allPointSphere{i}(1)/137.508),1) ==0
%if rem((sqrt(allPointSphere{i}(1)/1.61803398875)),1)
    ==0
allPointSquare{square}=[azimuth,elevation,r];
view(270, 90);
if(allPoint{i}(3) > 0)
[x,y,z] = sph2cart(allPointSquare{square}(1),
    allPointSquare{square}(2),allPointSquare{square}(3))
    ;
scatter3(x,y,z,[],[0 0 1],'*');
hold on;
square=square+1;
end
end
end
drawnow
toc
```

Explication

In my Matlab code, we initialize 4 cell arrays: one for all numbers in the spherical coordinate system, one for all numbers in the 3D Cartesian system, one for the primes and one for the squares. Also, we initialize other variables including "i" for the FOR loop with a maximum count that can be varied for each rendering. In my model, I use the following formulas to generate the 3D scatter plots [12].

$$r(n) = \sqrt{n}$$
$$\theta(n) = n * 1.61803398875$$
$$\psi(n) = n * 1.61803398875$$

$$r(n) = \sqrt{n}$$
$$\theta(n) = n * 137.508$$
$$\psi(n) = n * 137.508$$

$$r(n) = \sqrt{n}$$
$$\theta(n) = n$$
$$\psi(n) = n$$

Matlab stores the resulting matrix of calculations in the spherical coordinate system cell array in. Then the built-in Matlab function **sph2cart** converts these into Cartesian coordinate points.

$$x = r. * cos(\theta). * cos(\psi)$$
$$y = r. * cos(\theta). * sin(\psi)$$
$$z = r. * sin(\theta)$$

The program then plots the points using the **scatter3** function at each loop cycle. Each number category is colored differently.

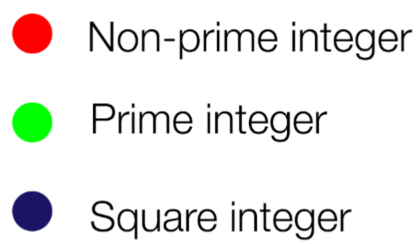


Figure 5: Color coding for each number category.

The built-in function **isprime** automatically identifies primes places the information in a final cell array for rendering. We use a similar process for identifying squares.

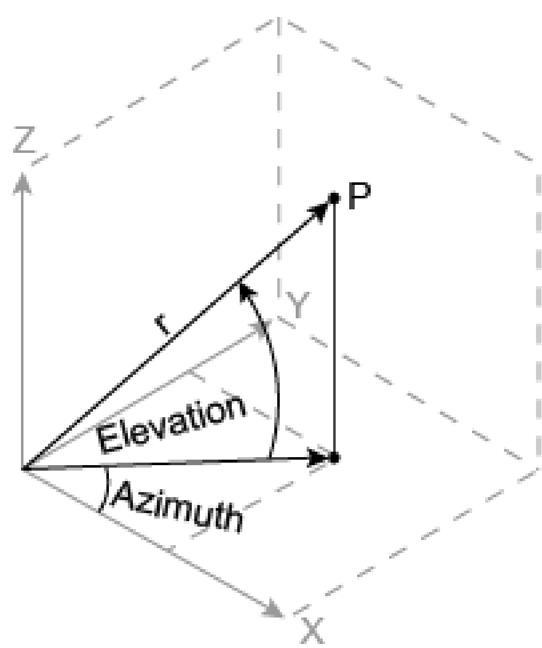


Figure 6: Spherical coordinate system [12]

3 Mathematical Results

We started our search for a better visualization formula for primes by studying existing spirals that mathematicians used. We used an empirical approach of trial and error to find the best way to enhance prime number spiral techniques. We first studied the 2D Ulam spiral, previously shown in Figure 1, to understand how it was generated. We then modified it for 3D rendering by mapping the square spiral approach to a cube spiral by taking every 3rd step into the z axis (as opposed to a sequence of x, then y axis and then returning to the x axis).

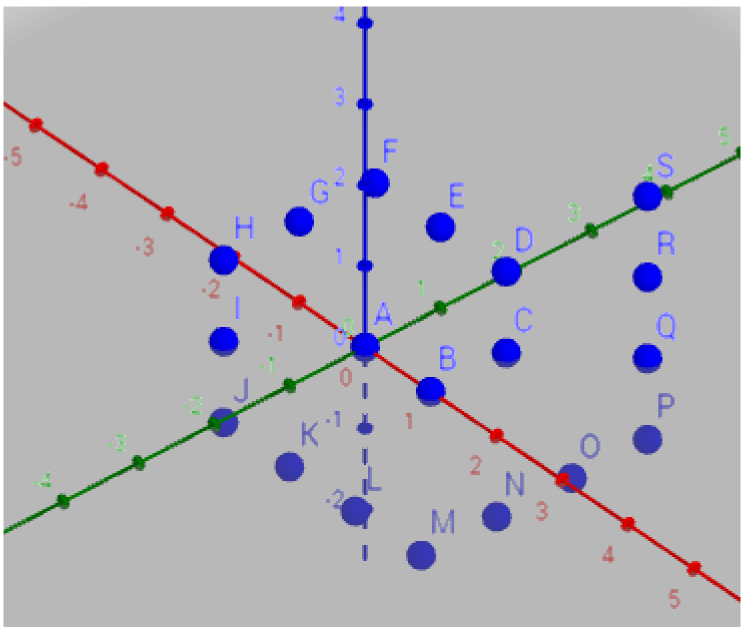


Figure 7: My first variation of the 3D Ulam spiral.

Unfortunately this approach didn't improve the visualization of prime numbers. We then looked into other more promising approaches which used polar coordinates. The most well known approach is Fermat's spiral also known in the discrete space as Vogel's approach, previously shown in Figure 4. It is also called the sunflower approach since the visualization generates a sunflower seed pattern. Interestingly Vogel used the golden angle in mapping to polar coordinates, since this angle occurs naturally in sunflowers and many other natural phenomena. We were inspired by this method as a basis for our 3D approach in the spherical coordinate system. Our first attempt in 3D rendering used the Vogel formula for the first two spherical dimensions and simply the integer for the 3rd dimension:

$$r(n) = \sqrt{n}$$

$$\theta(n) = n * 137.508$$

$$\psi(n) = n$$

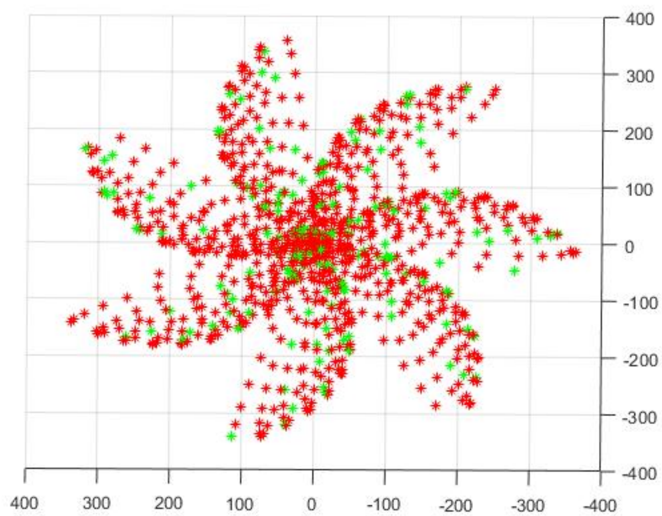
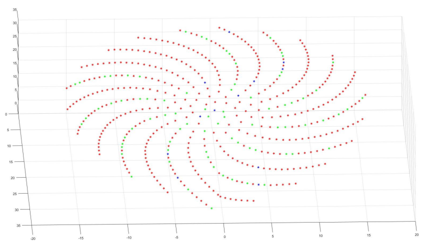


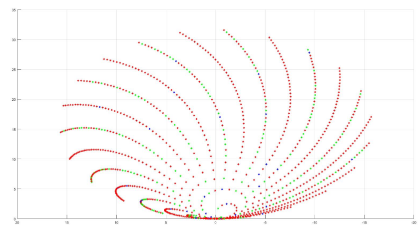
Figure 8: A 2D view of our first 3D rendering based on Vogel's formula.

Boucher Conjecture 1. *By extending the Vogel model to 3D, more systematic prime number patterns should be observable.*

Observe the strong natural patterns resulting from our conjectured method, show below:



(a) Interior view



(b) Top view

Figure 9: Positive z quadrant view of a 1000 points of the Boucher spiral

2D visualization is inherently very limited as compared to 3D, since 3D offers infinitely more view points. Human vision is highly effective in detecting the faintest 3D pattern in nature. 3D visualization should in turn significantly increase the likelihood of discovering strong systematic patterns.

In 2D prime visualization models, patterns are not strong enough to predict succeeding prime numbers. In my model, patterns are more systematic. Specifically, we observe that every other spiral line (arm) doesn't contain primes, which is significantly better than all other investigated spirals, e.g., Sacks, Ulam's and variations of these.

A deeper study of the model should allow us to determine the equation for the lines with prime numbers, allowing us to possibly predict large prime numbers with more ease than with 2D approaches. Furthermore, this study could explain an observed particularity: three successive prime-less lines. This might be a macro-pattern that repeats itself if the spiral was grown much further if we had access to more powerful computers.

Boucher Conjecture 2. *Observed patterns have natural analogues such as sea shell morphologies and structures.*

Interestingly the rendering, when viewed in 2D in the x-y plane, has a striking similarity to the morphology of sea shells, specifically the Virginica bivalve shell and others like Nuttall's Lucine and Ark shells. We need to make more research to see if this similarity is only a coincidence or the actual morphological way that a shell forms in nature.

(a) Top view of the Lucine sea shell. (b) Interior view of Ark sea shells.

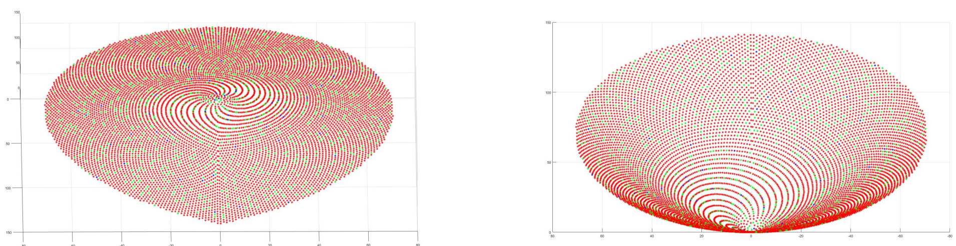
Figure 10: Photos of sea shells. Notice the similarity to the Boucher spiral.

3.1 Equations and Images

We first modified the formula for Vogel's model to put it in the spherical coordinate system:

$$r(n) = \sqrt{n}$$
$$\theta(n) = n$$
$$\psi(n) = n$$

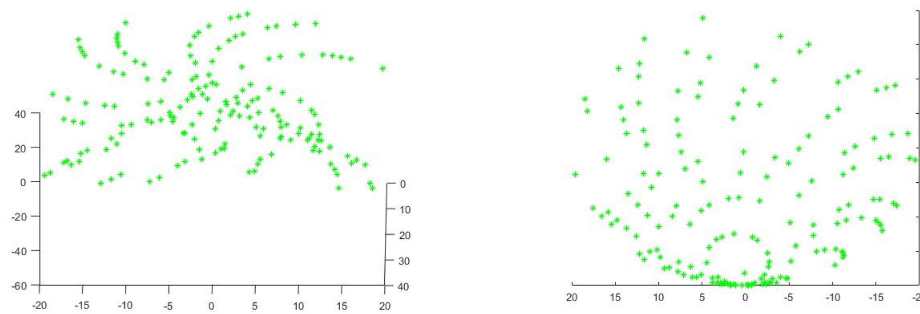
We will call this the "normal" formula. The total number of points generated in the rendering affect the way we see the pattern of the sea shell and the prime patterns. With a small amount of points, we can clearly distinguish the repetition of a spiral line (arm) with primes followed by a line with no primes. As we increase the number of points, the patterns become more easily visible.



(a) Interior of the spiral

(b) Top view

Figure 11: A rendering of 20 000 points of the normal formula.



(a) Interior of the spiral

(b) Top view

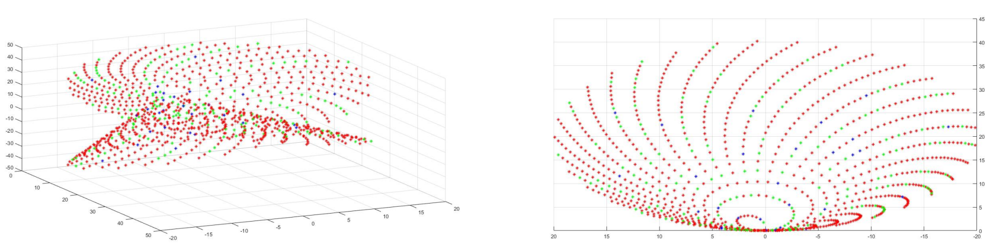
Figure 12: Displaying only the primes in the 1000 point rendering of the normal formula

We then tried a variant of the "normal" formula by using the golden ratio as a multiplicative constant, to try to generate a stronger pattern for primes in the spiral.

$$r(n) = \sqrt{n}$$

$$\theta(n) = n * 1.61803398875$$

$$\psi(n) = n * 1.61803398875$$

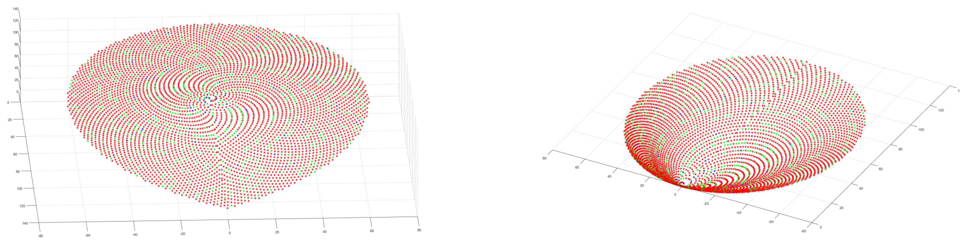


(a) Interior of the spiral

(b) Top view

Figure 13: 1000 points for the golden ratio formula

One can observe in Figure 11, that by plotting only numbers in the positive quadrant of z, only half of the sea shell is generated. Then when plotting in both the positive and negative z quadrants, the two sides of the sea shell appear back to back as can be seen in Figure 13. The two sides of the shell are a perfect reflection of one another.

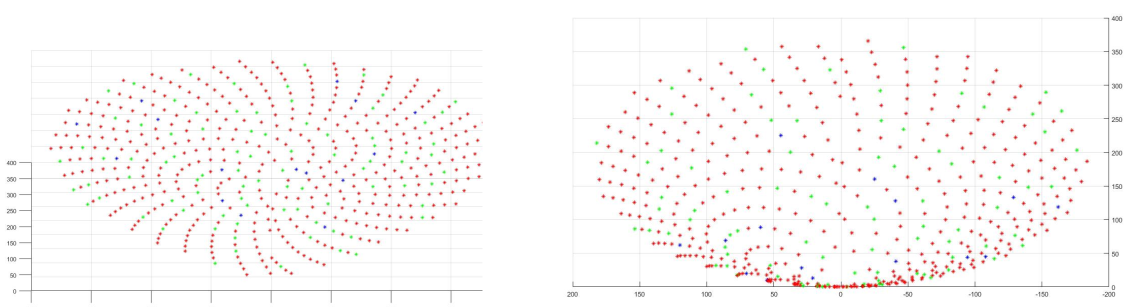


(a) Interior of the spiral (b) Top view

Figure 14: 10000 points for the golden ratio formula

We then used the golden angle as the multiplicative constant:

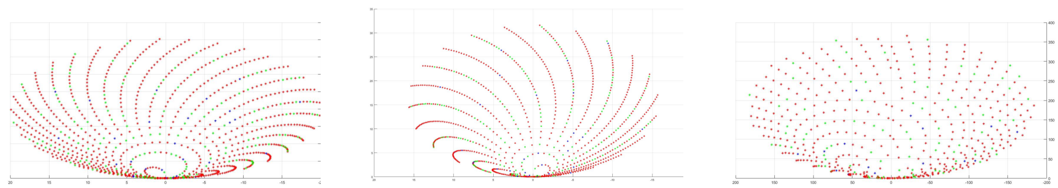
$$r(n) = \sqrt{n}$$
$$\theta(n) = n * 137.508$$
$$\psi(n) = n * 137.508$$



(a) Interior of the spiral (b) Top view

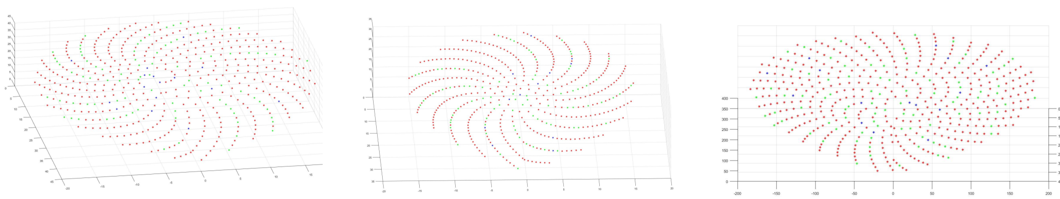
Figure 15: 1000 points for the golden angle formula

When comparing the three approaches, we observe that in the case of the golden ratio, a change in the orientation of the spin of the spiral. Notice also that in the case of the golden angle, the pattern is more similar to a sunflower than a sea shell, and that the prime number patterns are not as visually strong as in the normal case. Hence, surprisingly the simple normal approach produces the strongest prime number patterns.



(a) The golden ratio (b) Normal (c) The golden angle

Figure 16: Top view comparison with three different constants(1000 points).



(a) The golden ratio (b) Normal (c) The golden angle

Figure 17: Side view comparison with three different constants (1000 points)

4 Conclusion

We faced a number of limitations. One important limitation is computing power. I was limited to my home computer which is much much less powerful than supercomputer farms used in encryption. I could not generate long sequences of prime numbers within reasonable time with a home computer. For example, once I reached approximately 10,000 points in my spiral, I observed that the verification for the next prime number took 20 minutes and that this interval was exponentially increasing. I thus had no hope to grow my spiral much further.

In terms of future work, I would like to improve my software to more quickly identify prime patterns in the spiral, for example if I could first identify spiral lines with no prime numbers this could accelerate the identification of lines of interest containing prime numbers.

I would also like to investigate and understand why my visualization model generates such strong similarity to the morphology of animals or plants such as in sea shells.

With the advent of quantum computers, promising a huge jump in computing performance, I hope that my model would be able to shed more light on the mystery of prime numbers.

Given the encouraging results of 3D rendering, we should explore the use of higher dimensional spaces. One could start by adding a 4th dimension: time. One could then detect patterns in the time lapse rendering. However, one wonders what could be discovered with even more dimensions, but only computers could possibly detect patterns in hyperspace. Hyperspace rendering could push Mathematics in a new direction, just like when great mathematicians discovered the beauty of numbers though 2D visualization functions. Hyperspace the final frontier, these are the voyages of Mathematics, to explore strange new worlds, to seek out new conjectures and theorems, to boldly go where no man has gone before. .

5 Other

Acknowledgement

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