Physical Informed Machine Learning.

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Outline

1. Motivation
   - The success of ML models
   - ML in industry
   - Expert Knowledge & ML Modeling

2. From Physics to Machine Learning based models
   - Numerical methods & Solvers

3. Physics Informed Machine Learning (PIML)
   - Physics Informed Neural Networks (PINNS)
   - Sampling of "collocation" points
   - Fixed-Budget Online Adaptive Learning (FBOAL)

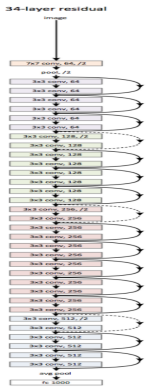
4. Industrial Applications
   - The Michelin Rubber Calendering Process : Inverse Problem
   - The Michelin Rubber Calendering Process. Geometry Aware PINNS
   - Orographic gravity waves Modeling.

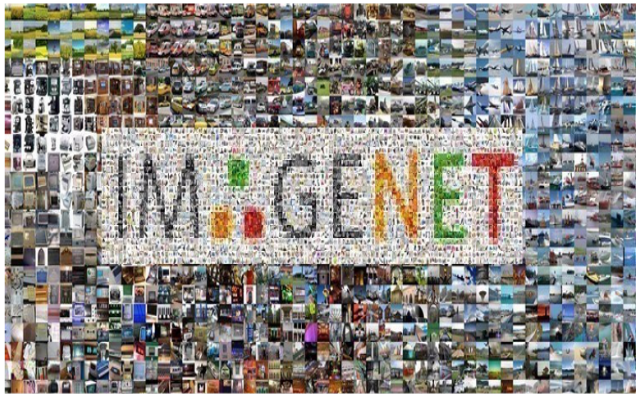
5. PINNS Softwares & Tutorials
   - Pinns softwares & Library

6. To conclude

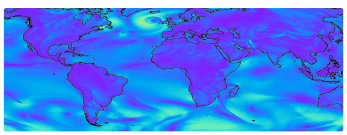
Data Sources & several Successes of "ML/AI" models
AI Foundations models

▶ Imagenet is a huge open source database :
14.10^6 labeled images for 10^3 **categories**,
... "quite expensive" labeling effort....
today pretrained models and DB are open source and may be used fo image classification, object detection...
AlexNet [Krizhevsky et al., 2012], ResNet [Huang et al., 2017]...





▶ GraphCast. This state-of-the-art model delivers 10-day weather predictions at unprecedented accuracy in under one minute based on 39 years (1979–2017) of historical data from ECMWF's ERA5 (21) reanalysis archive.



▶ DeepL. relied on the huge French-English Linguee dictionary.

▶ ChatGPT, LLM. trained on very large corpus of text data.

▶ Foundation models. **open-source and pre-trained models** dedicated to various tasks under various software licences...

Top-performing deep architectures are trained on massive amounts of labeled data.

Learning a model from an algorithmic point of view...

 # The Train Data and your New, Target data
(XTrain,yTrain)= load(MyTrainDataBase);
(XNew,yNew)= load(MyNewData);

 # The NN model
from sklearn.neural_network import MLPClassifier
clf = MLPClassifier(solver='lbfgs', alpha=0.0001,
hidden_layer_sizes=(5, 2), random_state=1)
clf.fit(XTrain, yTrain)

 # The Decision Tree model
from sklearn.tree impot DecisionTreeClassifier
treemod = DecisionTreeClassifier(
min_samples_split=10,min_samples_leaf=5,min_impurity_decrease=0)

 # Decision on new data
(XNew,yNew)=load(MyNewData)
pyNew=clf.predict_proba(Xnew)
treefit= treemod.fit (XTrain,YTrain)

 # Error Evaluation
pyNew=treefit.predict_proba(Xnew)>0.5

1. Input/ output (X, Y)
   (Features, labels set) defined by the operational need.
2. Data set. $S = \{(x_i, y_i)\}_{i=1}^{m} \sim \mathcal{D}^m$ a learning/training sample of m iid pairs.
   with $\mathcal{D}$ an unknown joint probability distribution on the product space $X \otimes Y$
3. Model $\mathcal{H} = \{h_\theta | h_\theta : X \to Y\}$ a hypothesis class, θ parameters, classifiers or regressors depending on the nature of Y.
4. Loss function $\ell(y, h_\theta(x))$ providing a cost of $h_\theta(x)$ deviating form the true output $y \in Y$.

   The best hypothesis is the one that minimizes the true risk, consequently, generalizes well :

$$R_{\mathcal{D}}{}^{\ell}(h_\theta) = \mathop{\mathbb{E}}_{(x,y)\sim\mathcal{D}}[\ell(h_\theta(x), y)]$$

   "Learning" consists of finding a good hypothesis function $h_\theta \in \mathcal{H}$ that captures in the best possible way the relationship between X and Y.

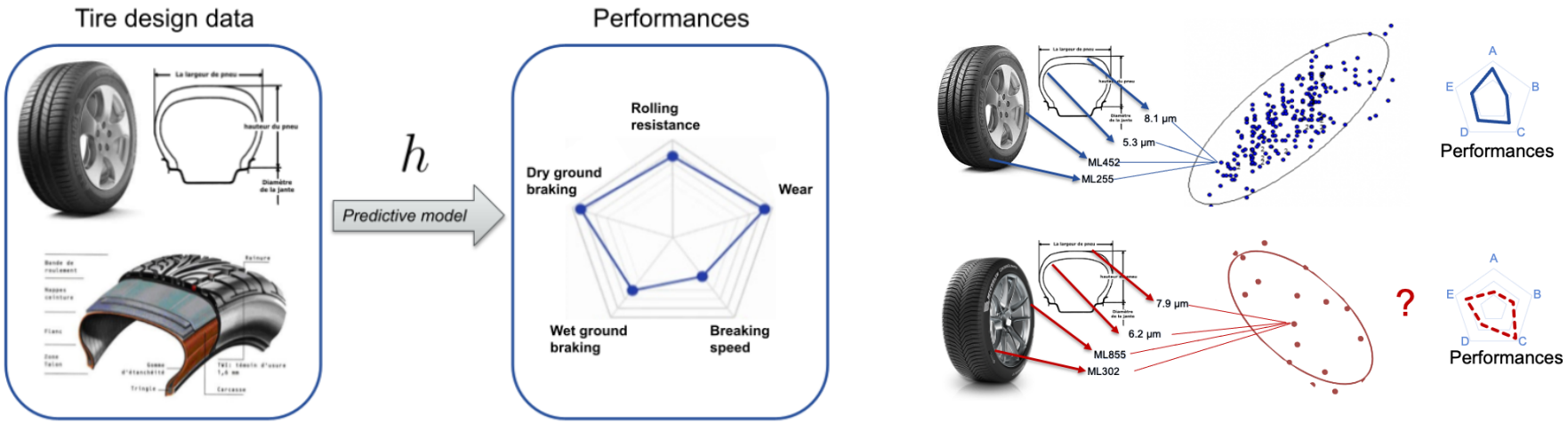
$$h_{\theta_{\text{opt}}} = \arg\min_{h_\theta} R_{\mathcal{D}}{}^{\ell}(h_\theta)$$

5. Optimization Procedure, hyperparameters...

Motivations

Conception : product design.

- New products are regularly manufactured with a long and costly development.
- Relative small data sets are gathered during the development of products such as characteristics (X : color, shape, weight...) and performances (Y).

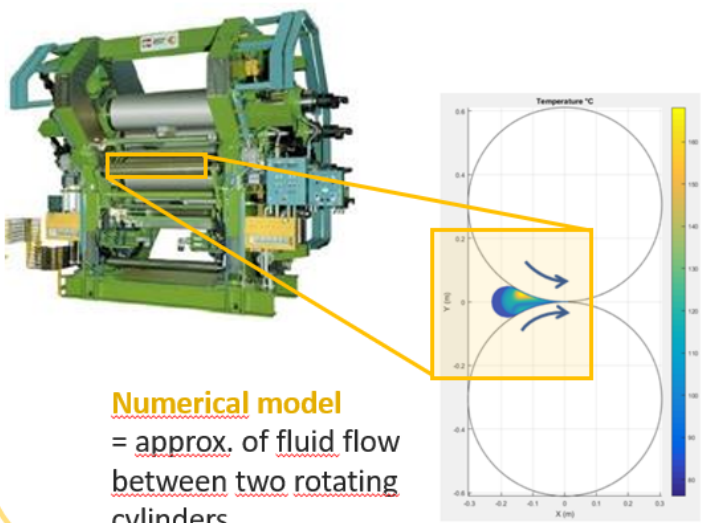


- Is-it possible to predict the performances (Y) of a new tire line given data previously gathered from **other** lines ? [de Mathelin, 2024]

Production : decision making process.

- The final quality of the product strongly depends on raw materials (composition/proportion) and fabrication. Fabrication needs to be fully understood, for automatic decision making Process.
- As an illustration, the Calendering process smoothes out the rubber through contra-rotating cylinders. Physical problem : Assure that the rubber has the desired properties.

Example of Michelin's use-case:



Numerical model
= approx. of fluid flow between two rotating cylinders

• The rubber is assimilated as an incompressible non-Newtonian fluid flow.

$$\begin{cases} (2\eta(\vec{u},T)u_x)_x + \left(\eta(\vec{u},T)(u_y+v_x)\right)_y = p_x \\ \left(2\eta(\vec{u},T)v_y\right)_y + \left(\eta(\vec{u},T)(u_y+v_x)\right)_x = p_y \\ u_xT_x + u_yT_y = \dfrac{\lambda}{\rho C_p}(T_{xx}+T_{yy}) + \dfrac{\eta(\vec{u},T)}{\rho C_p}|\gamma(\vec{u})|^2 \\ u_x + v_y = 0 \end{cases}$$

where $\vec{u}$ is the velocity, p pressure and T temperature, $\bar{\bar{\varepsilon}}(\vec{u})$ is the strain rate tensor, $\gamma(\vec{u}) = \sqrt{2\sum \bar{\bar{\varepsilon}}_{i,j}^2}$, and the dynamic viscosity η:

$$\eta(\vec{u},T) = K|\gamma(\vec{u})|^{n-1}\exp\left(\frac{E_\alpha}{R}\left(\frac{1}{T}-\frac{1}{T_\alpha}\right)\right)$$

• Numerical simulations help to understand the process.

Machine Learning (in the industry)

Main observations :

- Often small, moderate, evolving database. Ex. manufacturing process.
- Few or not labeled data. Ex. Few production defaults.
- labeled-data is often difficult and time-consuming to acquire.
  Ex. Experimental design to help selecting costly observation outputs.
- In many real-world applications, historical (training) data and newly collected (test) data may often exhibit different statistical characteristics.
  However, in many ML scenarios, training and test samples are supposed to be generated by the same (unknown) probability distribution.
- Still Strong Needs for monitoring and diagnosis based on machine learning (ML) .
- Makes sense to re-use knowledge gained from related but distinct datasets, from physics, from conception.....

Knowledge in ML modeling

Extra Prior knowledge can provide rich information not existing or hard to extract in limited training data and helps improve the ability to generalize, and the plausibility of resulting models.

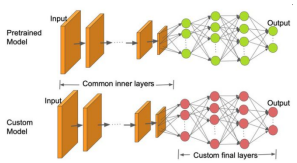
- Data knowledge .
  1. **Augmentation.** Easy for Image classification tasks (symmetry, rotation...).
  2. **Feature engineering.** $x_{raw} \to x \to f_\theta(x) \sim y$
     Ex. Wavelet based scattering transform, Fourier transform.
     Ex : sounds classification for Delphin challenge classification (frequential data).
  3. **Foundation models.** Pre-trained models on alternative large data base, use of TL.

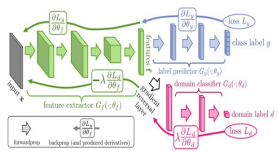
- Model Tailored knowledge [Features, architecture, function properties]
  1. **Design of specialized NN architecture** associated with a given predictive task. Symmetry groups as rotation, homothety, translation may implement an intrinsic geometry of f_θ
     $x \to f_\theta(x) \sim y$
     Ex : Convolutional NN [CNN] by crafty respecting **invariance** along the groups of symmetries.
     Pre-calibrated models (NN warm start).
  2. **Multi-Task Learning**
     Introduction of knowledge/ constraints in the cost function, in the optimisation process. Ex : *Physical Informed Neural Network*

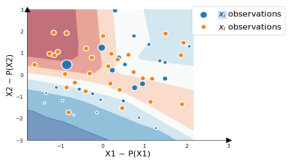
Towards Transfer Learning, Domain adaptation, & Physics Informed Machine Learning..

Transfer Learning & Domain adaptation Methods

- **Several approaches to transfer knowledge from Source to Target domain.**
  - Model-based. Transfer the model parameters learnt on the source data to the target model.
    - Train model available, not necessary the source data- .
    Ex. Image based tire wear estimation based on Deep architecture (Michelin) (Resnet...), Automatic fall detection based on decision trees/ RF (Tarkett).
  - Feature-based. Find a new representation space to bring feature spaces closer.
    -Source and Target Input data available-. *Ex. Domain adversarial neural networks (EDF, Michelin)*
  - Instance-based. Re-weight the source samples to bring the distributions closer.
    -Source and Target Input data available-.
    Ex. Multi-source domain adaptations for Product design (Michelin) or Electricity prediction (EDF)







- **Theoretical guarantees on the Target Risk given the Source Risk and discrepancy**

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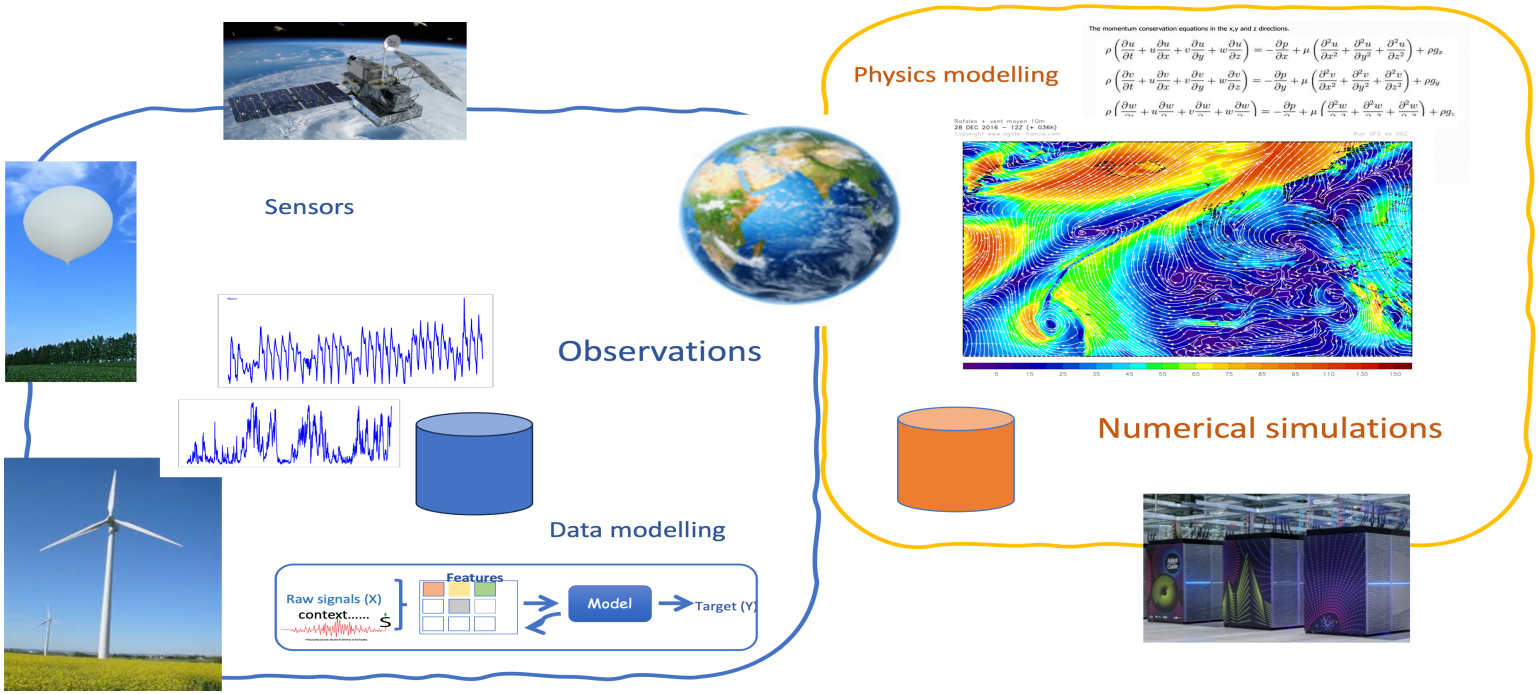
From Physics

to Machine learning

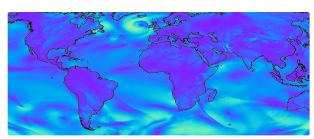
models

Modeling. Observation data and Numerical simulations



▶ Surrogate Models. Meta Model. Machine Learning models and numerical simulations. (speed..)

▶ GraphCast. The state-of-the-art model delivers 10-day weather predictions at unprecedented accuracy in under one minute based on 39 years (1979–2017) of historical data from ECMWF's ERA5 (21) reanalysis archive.



Physics models- Numerical methods-Solvers

The study of a given phenomena may be described thanks to a system of equations, based of Physics knowledge related for example in solid, fluid mechanics....

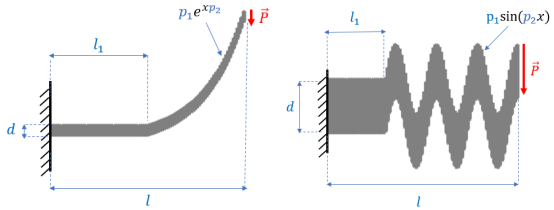
Evolution of a continuous value $v(x, t)$ using a generic form of PDE :

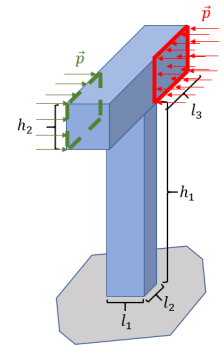
$$\frac{\partial v}{\partial x} = F(t, x, v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j}, \ldots)$$

- Computational methods solve these differential equations using numerical methods with *discretization techniques* like finite volume method mostly used in computational fluid dynamics.
- Depending on the complexity of the problem, these methods tend to be *very computationnaly expensive* and require a high level of methodological understanding in the generation of the discretization (*meshing*).
- Simulation softwares have been developped
  - Opensource : OpenFOAM open source software for computational fluid dynamics (CFD), OpeFEM : A free and open source software to solve partial differential equations (PDE) using the Finite Element Method (FEM)....
  - Proprietary simulation software for conception, design, monitoring...

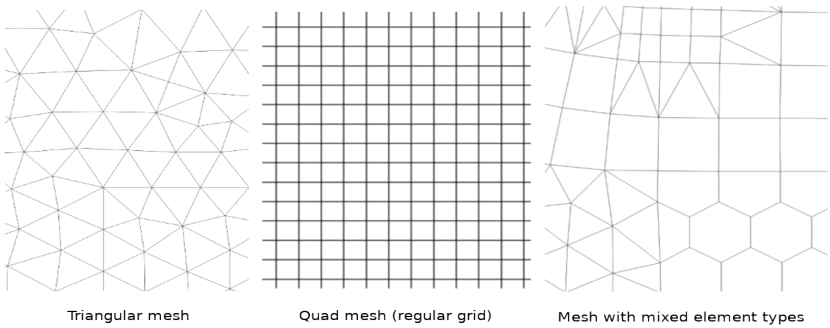
Physics models- Numerical methods-Solvers

Objects, Domain & Boundaries



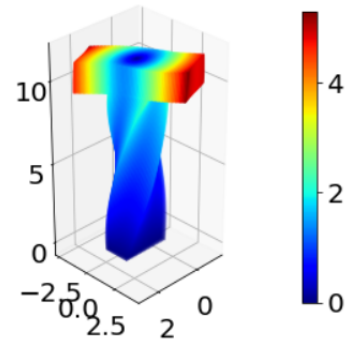


Mesh



credit QGIS

Physical equations
Numerical equations/ solveur



- The complexity arises from the large number of degrees of freedom required to capture multiscale physics, mesh refinement requirements for accurate boundary layer resolution, and iterative solution procedures.
- The computational cost of traditional numerical methods remains a significant bottleneck

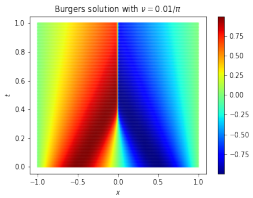
ML surrogate model

Let's consider a toy example : **Burgers equation.**

h_θ (u_θ) : input (x space , t *time*), $\rightarrow$ output : speed of the fluid

given parameter : ν viscosity. Dirichlet boundary condition :

$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0; u(0,x) = -sin(\pi x); u(t,-1) = u(t,1) = 0$



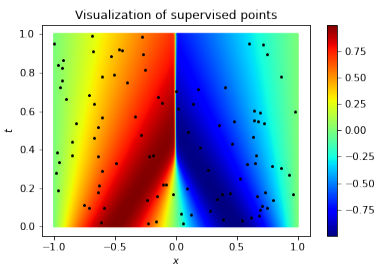
Data Set. $n = 100/1000$ supervised observations.

Neural network - Surrogate model. u_θ : **ML model** : NN architecture : 2-4 (x50)-1.

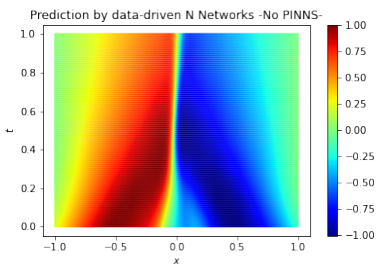
Evaluation on a **Test data set**, regular grid of points (256 (x) , 100 (t)).

$E(u,\hat{u}) = ||u-\hat{u}||_2^2/||u||_2^2$ on a grid (256 (x) , 100 (t))

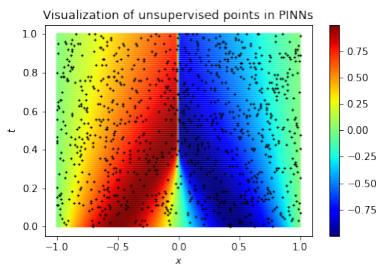
Supervised points



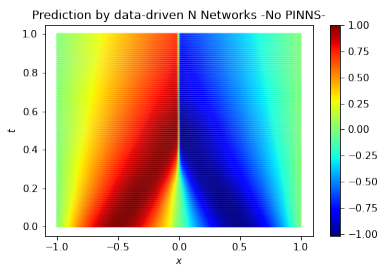
Surrogate model $E(u,\hat{u}) = 0.17$



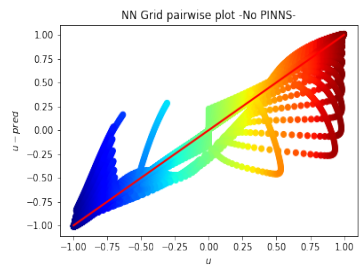
Supervised points



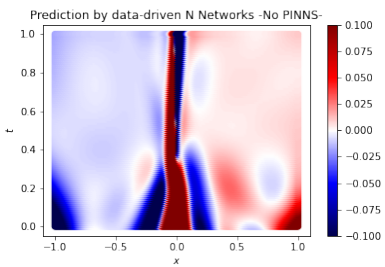
Surrogate model $E(y,\hat{y}) = 0.02$



Pairwise Plot



Error



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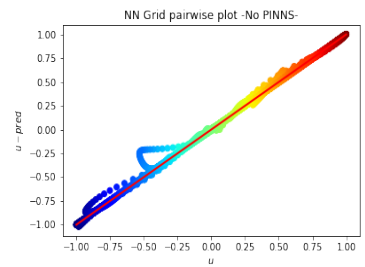

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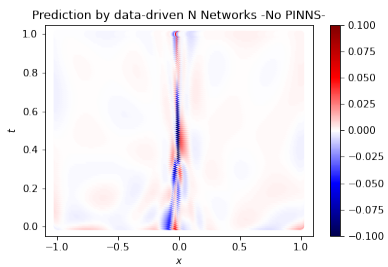
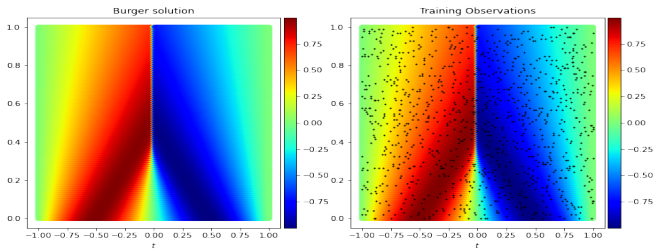


ML Surrogate model

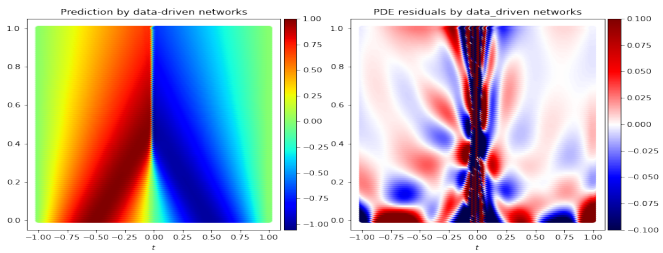
Let's consider a toy example : **Burgers equation.**
Focus on the Pde errors/residuals computed for Burger model (right/bottom figure) :

$$R(u,t) = \frac{\partial \hat{u}(x,t)}{\partial t} + \hat{u}(x,t)\frac{\partial \hat{u}(x,t)}{\partial x} - \nu \frac{\partial^2 \hat{u}(x,t)}{\partial x^2} \ ?= 0\,;$$

$N_{\text{data}} = 1000$, $N_{\text{colloc}=0}$, NN architecture : 2-4 (x50)-1. 50 000 epoch, Adam optimizer.
$E(u,\hat{u}) = ||u-\hat{u}||_2^2/||u||_2^2$ on a grid (256 (x) , 100 (t))



PINNs model $E(u,\hat{u}) = 0.0163$



→The pde constraints are not respected...
→ The NN model mimics the input/output relation but the underlaying physics is not caught.

- Especially, in a "small data regime", the vast majority of state-of-the-art machine learning techniques are lacking robustness.
- The cost of data acquisition may be prohibitive. Experimental design are proposed to chose the observations.
- ML fail to "model" the underlaying physics phenomena (pde constraints not respected).
- We are inevitably faced with the challenge of drawings conclusions and making decision under partial information.

ML Surrogate model

First conclusions :

- Especially, in a "small data regime", the vast majority of state-of-the-art machine learning techniques are lacking robustness and fail to "model" the underlaying physics phenomena (pde constraints not respected).
- The cost of data acquisition may be prohibitive.
  Experimental design are proposed to chose the observations.
- We are inevitably faced with the challenge of drawings conclusions and making decision under partial information.

Physics Informed

Neural Networks

-PINNs-

Physics Informed Neural Networks

PINNs are neural networks trained to solve supervised learning tasks while respecting any given laws of physics described by general nonlinear partial differential equations. [Raissi et al., 2019].

Key ingredients :

1. A deep neural network model : $h_\theta \in \mathcal{H}$

2. A supervised data set. $\mathcal{D}_n = \{(X_i, y_i)\ i = 1..n,\ X \in \mathcal{X}, y \in \mathcal{Y}\}$ the training data (initial and boundary observations).

$$\boxed{\text{MSE}_{\text{data}} = \frac{1}{n}\sum_{i=1}^{n}[h_\theta(X_i) - y_i]^2}$$

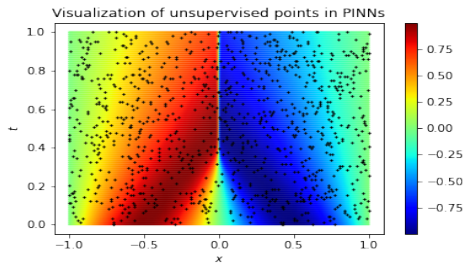
3. **Equations*** describing the physics. ($h_\theta = u_\theta$)
Ex : Burger equation : Input : $X = (x, t)$ output : u :

$$R(u, t, x, \nu) = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0$$

$$\boxed{\text{MSE}_{\text{pde}} = \frac{1}{n}\sum_{i=1}^{\text{Colloc}} R(u_i, t_i, X_i, \nu)}, \textbf{ No mesh !}$$

pde residuals cost on random collocation points.

4. **A appropriate loss function** $\mathcal{L}(h_\theta, \mathcal{D}_n,$ physics equations$)$.



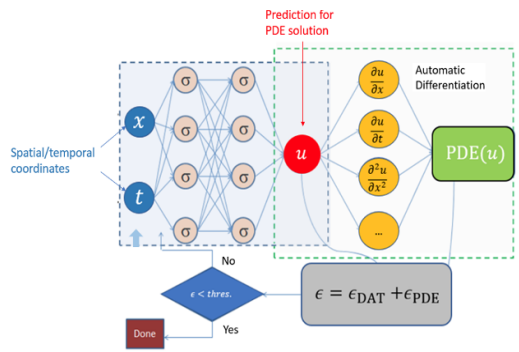


FIGURE – [Cuomo et al., 2022]

Shared parameters θ of the NN model $h_\theta(t, x)$ and $R_\theta^{\text{pde}}(t, x)$ are "learned" by minimizing :

$$\boxed{\text{MSE} = w_{\text{data}}\text{MSE}_{\text{data}} + w_{\text{pde}}\text{MSE}_{\text{pde}}}$$

Physical Informed Neural Netwotks

Open questions with an important impact for PINNs models, September 2020

- Supervised data. Observations (number, localization, boundary or inside the domain).
- Unsupervised data. Collocation points : number, localization, on a mesh grid, at random ...
- Trade-off between the two data and residual pde losses : $\text{MSE} = w_{\text{data}}\text{MSE}_{\text{data}} + w_{\text{pde}}\text{MSE}_{\text{pde}}$
- PDE with given parameters (ν) are easily learned by PINNs.
  What about parameterized PDE ?
- Impact of the geometry of the domain (design) ?

Industrial PINNs Applications

- Surrogate modeling. Data-driven solution of pde. Fluid mechanics [Raissi et al., 2019],... , Solid mechanics,...
  Application to rubber calendering process (non-Newtonian fluid thermo-mechanical problems) [Nguyen et al., 2022]
- Identification of unknown parameters. Data discovery of pde. [Raissi et al., 2019], [Nguyen et al., 2022]
- Inverse problems.

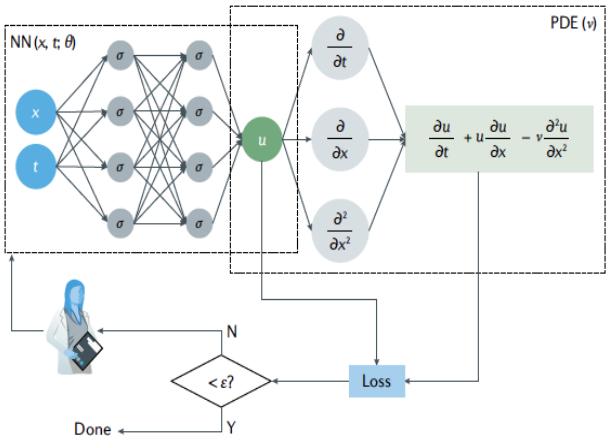
Khoa Nguyen, PhD thesis, September 20^{th} 2024.
Development and assessment of physics-informed deep learning methods : towards mutliphysics simulation in industrial contexts.

Physics Informed Neural Netwotks

[Raissi et al., 2019]

$f_\theta(t,x)$ is defined by : $f_\theta := u_t + \mathcal{N}[u]$

- $u(t,x)$ is approximated by a deep NN.
- Automatic differentiation helps to minimize $R(t,x)$ applying the chain rule for differentiating compositions of functions.
  Pytorch, TensorFlow



The shared parameters θ between the NN $h_\theta(t,x)$ and $R_\theta^{\text{pde}}(t,x)$ can be learned by minimizing the mean squared error.

$$\text{MSE} = w_{\text{data}}\text{MSE}_{\text{data}} + w_{\text{pde}}\text{MSE}_{\text{pde}}$$

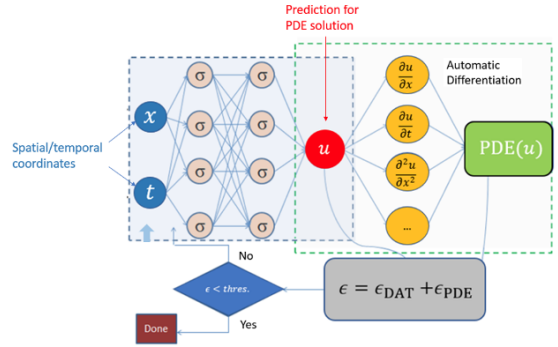
- $\text{MSE}_{\text{data}} = \frac{1}{N_{\text{data}}}\sum_{i=1}^{N_{\text{data}}}[h_\theta(t_u^i, x_u^i) - u^i]^2$ for chosen training data (initial and boundary).
- $\text{MSE}_{\text{pde}} =$ is the **cost of pde residuals on random collocation points**. **No mesh !**

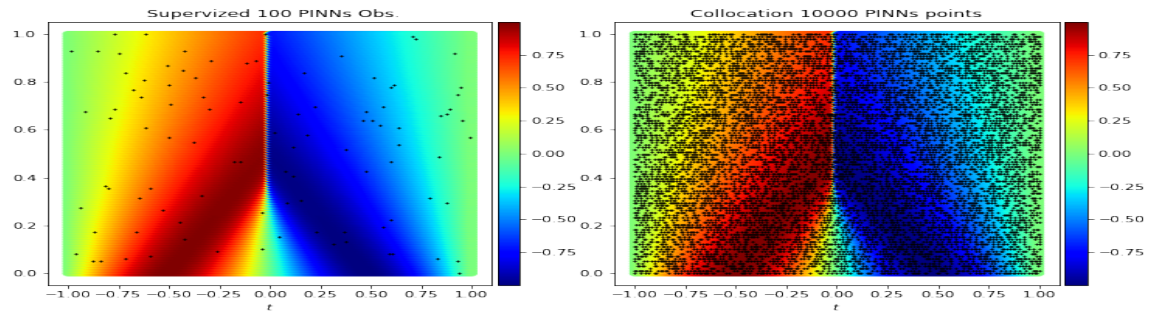
Supervised and Unsupervised Smart Mix

PINNs models

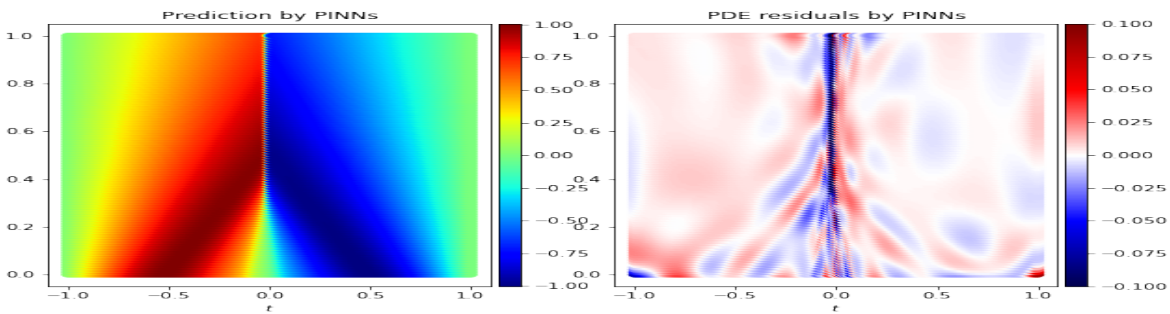
Importance of the Training observations.
Impact on the pde errors computed for Burger's model.

$N_{\text{data}} = 100$, $N_{\text{colloc}=10000}$,
ML model. NN architecture : 2-4 (x50)-1. 50 000 epoch, Adam optimizer.





PINNs model $E(u, \hat{u}) = 0.0157$



Outline

1. Motivation
   The success of ML models
   ML in industry
   Expert Knowledge & ML Modeling

2. From Physics to Machine Learning based models
   Numerical methods & Solvers

3. Physics Informed Machine Learning (PIML)
   Physics Informed Neural Networks (PINNS)
   Sampling of "collocation" points
   Fixed-Budget Online Adaptive Learning (FBOAL)

4. Industrial Applications
   The Michelin Rubber Calendering Process : Inverse Problem
   The Michelin Rubber Calendering Process. Geometry Aware PINNS
   Orographic gravity waves Modeling.

5. PINNS Softwares & Tutorials
   Pinns softwares & Library

6. To conclude

Sampling methods for collocation points in PINNS

In the PINNS framework, PDE residuals are computed and minimized on a set of collocation points.

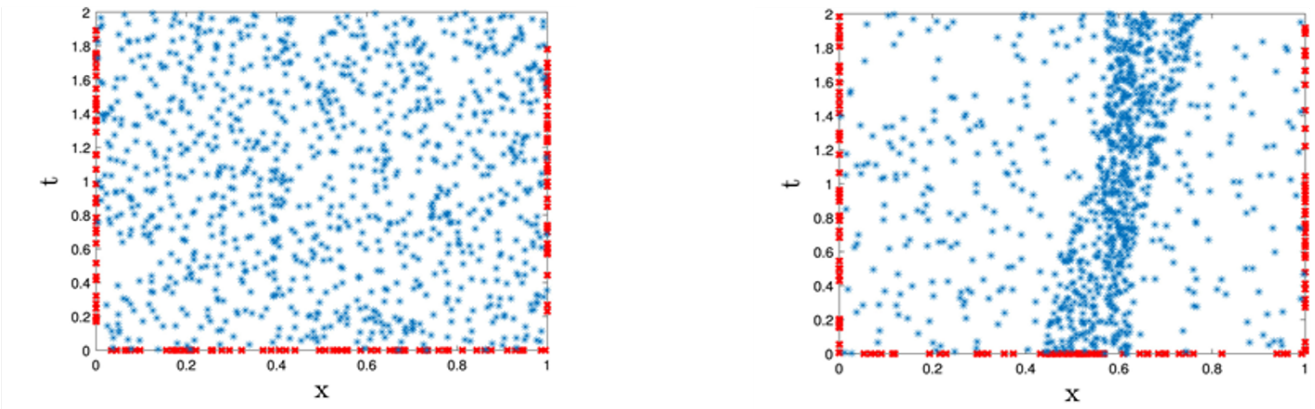
Various sampling strategies have been proposed for these training points, which can be divided into two main sub-classes :

- the non-adaptive sampling approaches.
  - The collocation points are often uniformly and randomly distributed, as in the original work of [Raissi et al., 2019] that employed the Latin Hypercude Sampling (LHS), or
  - manually and non-uniformly based on prior knowledge of the physical phenomena, as in the work of [Mao et al., 2020] and [Nguyen et al., 2022] which is however problem dependent.
- the adaptive sampling approaches.

Non adaptive Sampling methods

Vanilla PINNS with randomly distributed collocation points often fail to predict accurately the PDE solution in complex problems. If one dispose of **prior Physical knowledge** (discontinuity, coupled physics, interest zone, expert guided). Properly chosen locations can increase significantly the accuracy.

- Manual sampling in PINNs [Mao et al., 2020]. Error : left : $O(10^{-2})$; right : $O(10^{-5})$

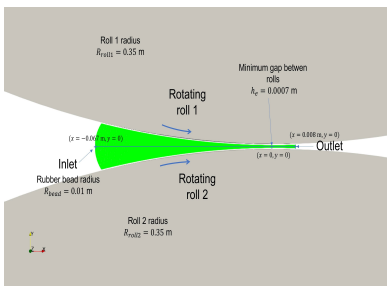


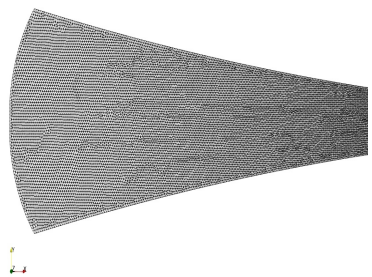
Non adaptive Sampling methods

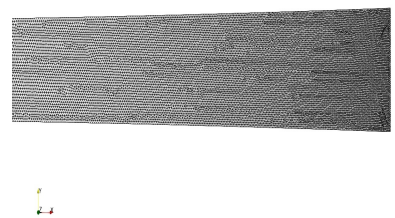
- Prior Mesh knowledge based Information [Nguyen et al., 2022]

The non-adaptive sampling approaches are widely used in PINNS literature : straightforward and efficient when dealing with various PDS problems, including PDEs providing that expert physical knowledge is available.

Illustration : calendar simulation process and pre-defined mesh grid.







→ However, physical knowledge is not always available...

Adaptive Sampling methods

Several adaptive collocation approaches have been developed. The Probability Density Function (PDF) $p(x)$, based on the PDE residuals [Wu et al., 2023], is defined as follows :

$$p(x) \propto \frac{\epsilon^k(x)}{\mathbb{E}[\epsilon^k(x)]} + c$$

where $\epsilon(x)$ denotes the PDE residual for any point x, $k \geq 0$ and $c \geq 0$ are two hyperparameters.

- Residual-based Adaptive Refinement (RAR) approach, first introduced by [Lu et al., 2021], adds new training collocation points to the location where the PDE residual errors are large.
  $\rightarrow$ leads to an uncontrollable amount of collocation points (∞) and computational cost at the end of the training process.
- Residual-based Adaptive Refinement with Distribution (RAR-D) : after a certain number of iterations, m points are randomly sampled according to the PDF previously defined. Then these points are added to the set of training collocation points.
- Residual-based Adaptive Distribution (RAD) : after a certain number of iterations, the collocation points are randomly resampled according to the PDF previously defined. [Wu et al., 2023]

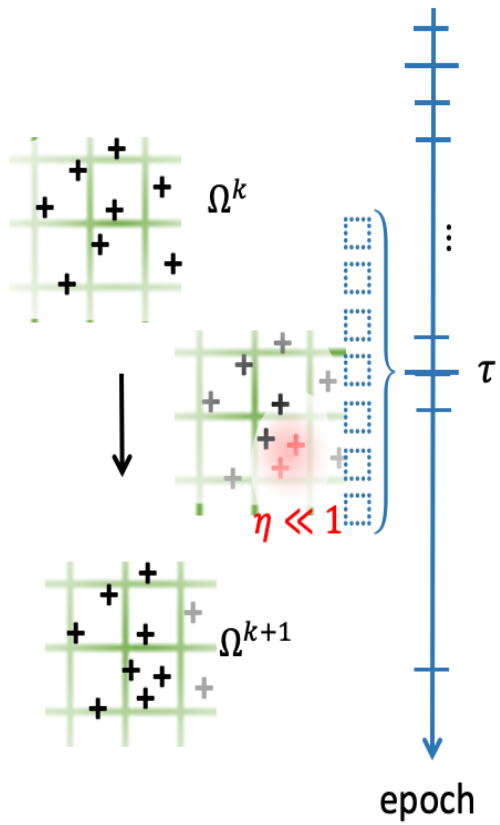
$\rightarrow$ only RAD allows the number of training points to be fixed during the training, while in RAR-D, this number gradually increases and may lead to higher computational costs.
The performance of RAD and RAR-D also depends on the period of resampling method.
hyperparameters k and c of RAD and RAR-D, need to be adjusted.

Adaptive Sampling methods

Fixed-Budget Online Adaptive Learning (FBOAL) [Nguyen et al., 2023]
Principle : The domain is decomposed into different sub-domains.
The collocation set is divided equally for each equation.

Notations : Ω_j : collocation set for eq. j. L^j_{pde} : PDE res. of eq. j.
x_{ji} : collocation points in Ω_j at loc. i. x'_{ji} : new collocation points.

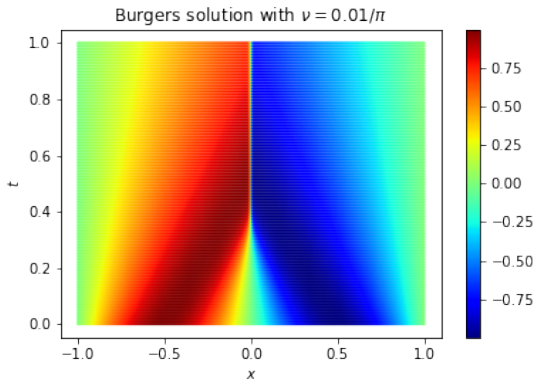
- The x_{ji} are distributed so as to maximize L^{ji}_{pde} i.e. Ω_j is updated by adding n new points x'_{ji}.
  Typically $n = \eta N_j$ $\eta \leq 1$ that yield larger residuals L^{ji}_{pde}.
  $\forall i \leq n \qquad L^j_{pde}(x'_{ji}) \geq \min_{x \in \Omega^k_j} L^j_{pde}(x)$
  Update rule : $\Omega^{k+1}_j = \{x'_{ji}\}_{i \leq n} \cup \Omega^k_j \setminus \{x_{ji}\}_{i \geq n+1}$
- The $N_j - n$ points of Ω^k_j with smallest residuals are discarded
- Updating is applied during the training process every τ epochs, which introduces two new parameters : η and τ.



Impact on collocation points/ Precision.

Burgers equation.

$$\begin{cases} u_t + uu_x - \nu u_{xx} = 0 & \text{for } x \in [-1, 1], t \in [0, 1] \\ u(x, 0) = -\sin(\pi x) \\ u(-1, t) = u(1, t) = 0 \end{cases}$$



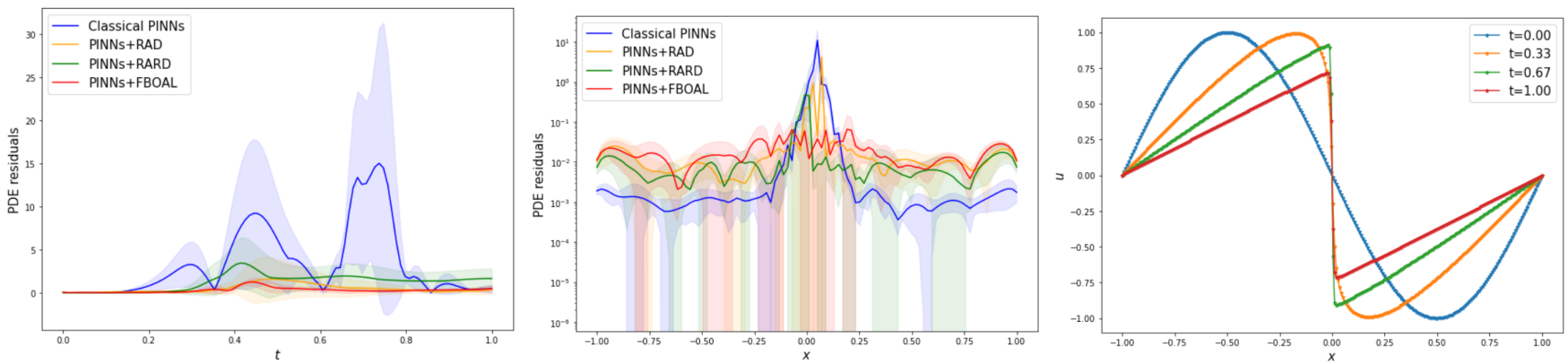
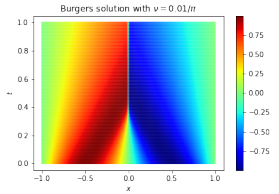


FIGURE – *Burgers equation : Absolute value of PDE residuals for* $\nu = 0.0025$ *after the training process for different approaches.* The curves and shaded regions represent the geometric mean and one standard deviation of five runs. On the line $x = 0$ (left), At instant $t = 1$ (right).

Impact on collocation points/ feed-back on the density

Burgers equation. ML Model $u(x, t, \nu = \nu_0)$

$$\begin{cases} u_t + uu_x - \nu u_{xx} = 0 \quad \text{for } x \in [-1, 1], t \in [0, 1] \\ u(x, 0) = -\sin(\pi x) \\ u(-1, t) = u(1, t) = 0 \end{cases}$$



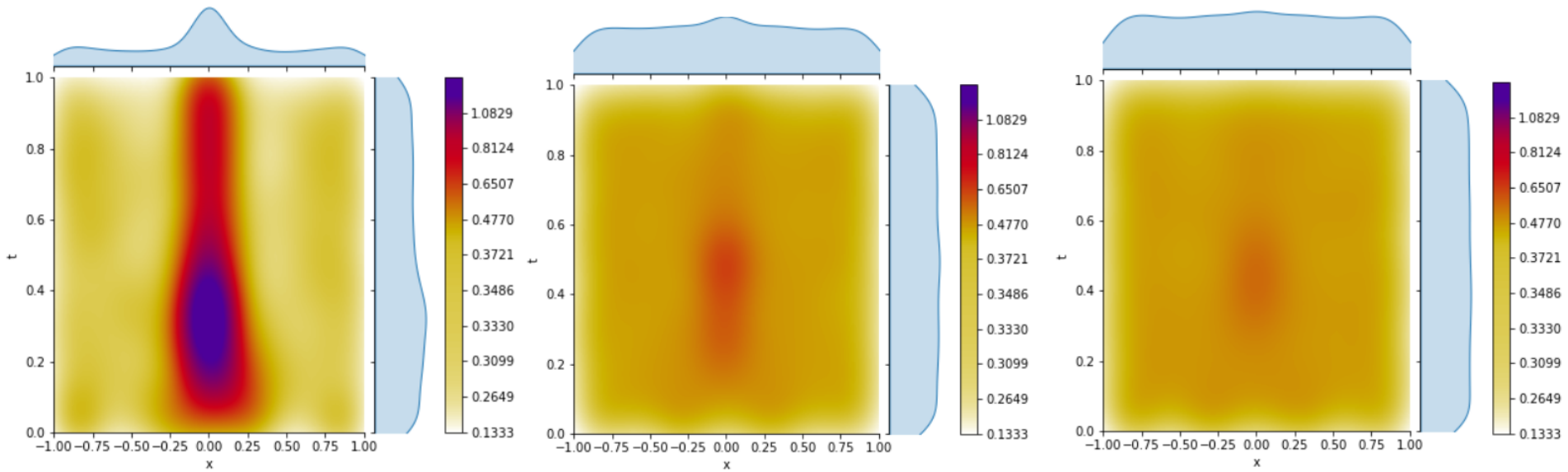
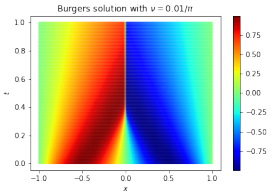


FIGURE – *Burgers equation : Density of collocation points after the training with FBOAL.* $\nu = 0.0025$ (left); $\nu = 0.0076$ (center); $\nu = 0.0116$ (right)

Impact on collocation points/ Training cost benefits

Burgers equation.

$$\begin{cases} u_t + uu_x - \nu u_{xx} = 0 \quad \text{for } x \in [-1, 1], t \in [0, 1] \\ u(x, 0) = -\sin(\pi x) \\ u(-1, t) = u(1, t) = 0 \end{cases}$$



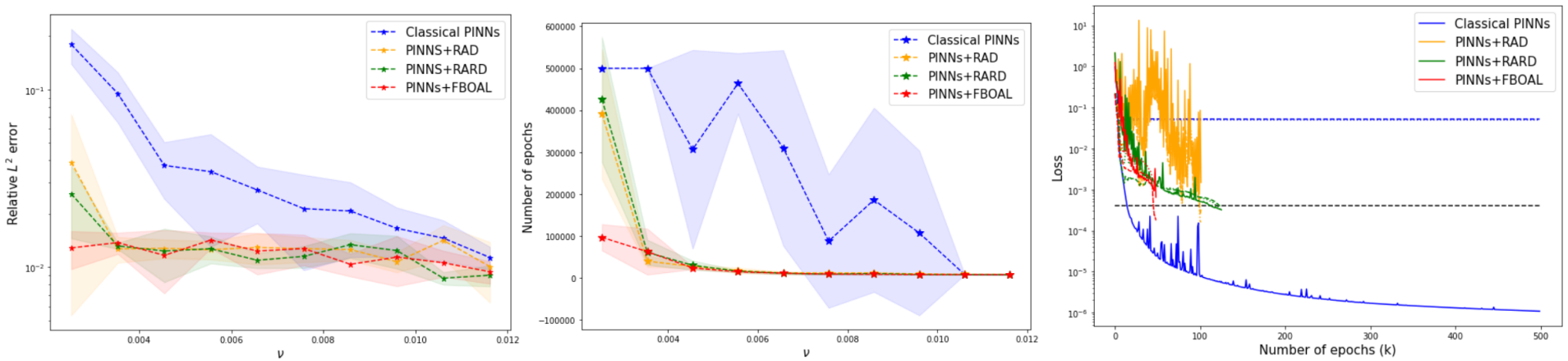


FIGURE – *Burgers equation : Performance of classical PINNs and PINNs with adaptive sampling approaches.* The curves and shaded regions represent the geometric mean and one standard deviation of five runs. In (c) the solid lines show the cost function during the training, the dashed lines show the errors on the testing mesh, and the black line shows the threshold to stop the training. Relative $\mathcal{L}^2$ error (left); Number of training iterations; Loss for $\nu = 0.0025$ (right).

	Classical PINNs	PINNs + RAR-D	PINNs + RAD	PINNs + FBOAL
Training time	33.7 ± 1.5	41.0 ± 5.8	38.8 ± 2.3	$\mathbf{21.5 \pm 2.7}$
Number of resampling	0 ± 0	201 ± 75	210 ± 77	$\mathbf{48 \pm 2}$

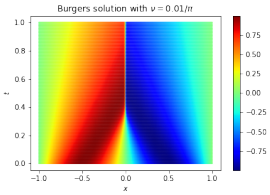
TABLE – *Burgers equation : Training time (in minutes) and the number of resampling for $\nu = 0.0025$.* The training is effectuated on an NVIDIA V100 GPU card.

Impact on collocation points/ ν Adaptation.

Parametrized Burgers equation. ML model $u(x, t, \nu)$

Training, 40 values of $\nu \in [0.0025, 0.0124]$.

$$\begin{cases} u_t + uu_x - \nu u_{xx} = 0 \quad \text{for } x \in [-1, 1], t \in [0, 1] \\ u(x, 0) = -\sin(\pi x) \\ u(-1, t) = u(1, t) = 0 \end{cases}$$



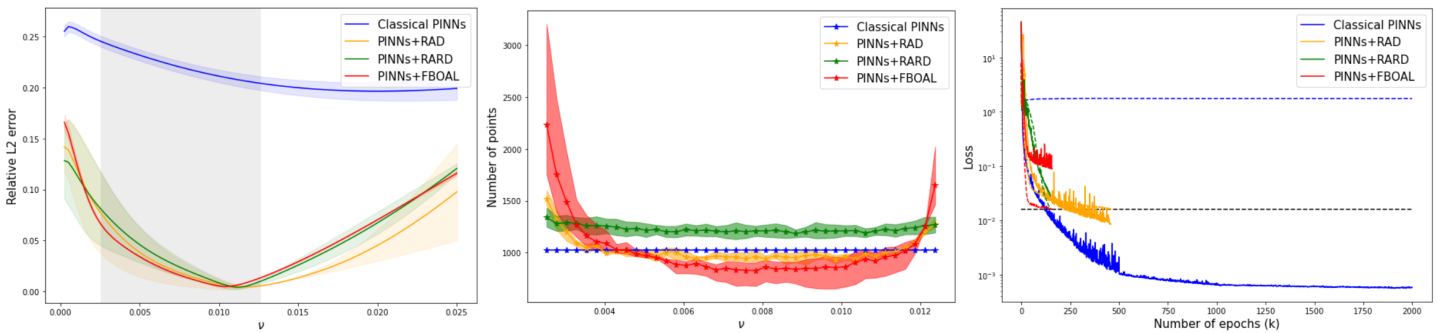


FIGURE – *Parametrized Burgers equation : Comparison of classical PINNs and PINNs with adaptive sampling approaches.* The zone in gray is the learning interval for ν (interpolation zone). The curves and shaded regions represent the geometric mean and one standard deviation of five runs. In (c) the solid lines show the cost function during the training, the dashed lines show the errors on the testing data set, and the black line shows the threshold to stop the training. Relative $\mathcal{L}^2$ error (left); Number of collocation points; Cost function during the training (right).

	Classical PINNs	PINNs + RAR-D	PINNs + RAD	PINNs + FBOAL
Training time	13.2 ± 0.0	$\mathbf{1.4 \pm 0.3}$	9.1 ± 3.5	7.8 ± 1.9
Number of resampling	0 ± 0	$\mathbf{34 \pm 8}$	204 ± 71	173 ± 25

TABLE – *Parametrized Burgers equation : Training time (in hours) and the number of resampling of each methodology. The training is effectuated on an NVIDIA A100 GPU card.*

PINNS Sampling methods

Dont't use PINNs methods without adaptation!

- FBOAL adaptively adds more points to important zones (coupling, discontinuity)
- With appropriate chosen hyperparameters, FBOAL provides better accuracy and faster convergence than vanilla PINNS with random points
- FBOAL can provide prior knowledge of high residuals.
  $\rightarrow$ Help conventional numerical solver in the construction of mesh.

Limitations and Perspectives

- No theoretical results for convergence
- Optimal hyperparameters of FBOAL are problem-dependant
- The decomposition domain step can be further investigated (e.g. high-dimensional)
- The division of set points for multi-physics problems may not be optimal.

Physics Informed

Neural Networks

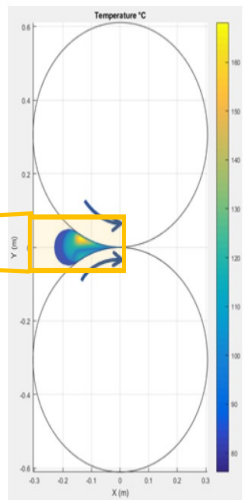
-PINNs- (Inverse problems)

Motivation : Operationnal Use cases & Pinns

Context: Two use-cases...

... And common equations

MICHELIN



2. Decision making

Validate a fabrication decision

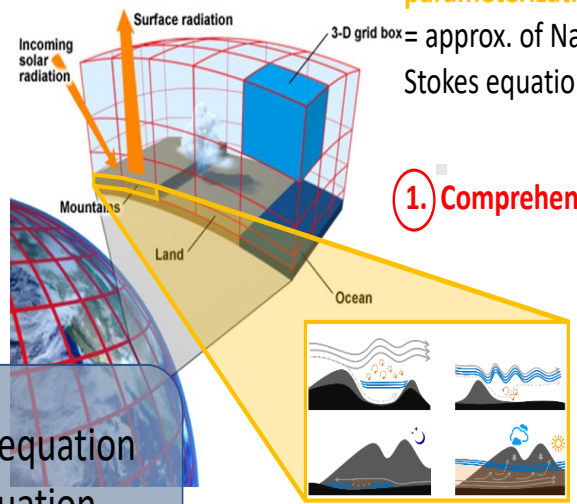
3. Conception

Create new products

cea

Subscale physical parameterizations

= approx. of Navier-Stokes equations



1. Comprehension

Momentum equation
Energy equation
Mass equation

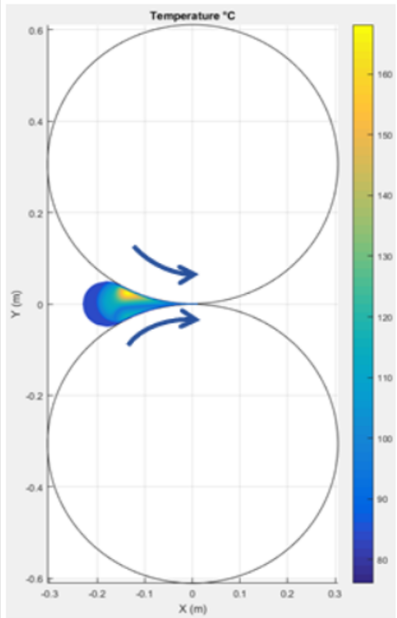
High computational cost

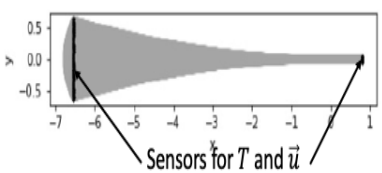
Interest in **Machine learning**

3

Rubber calendering process

- Calendering process : smooth out the rubber through contra-rotating cylinders
- Physical problem : assure that the rubber has the desired properties





Rubber: assimilated as **an incompressible non-Newtonian fluid flow.**

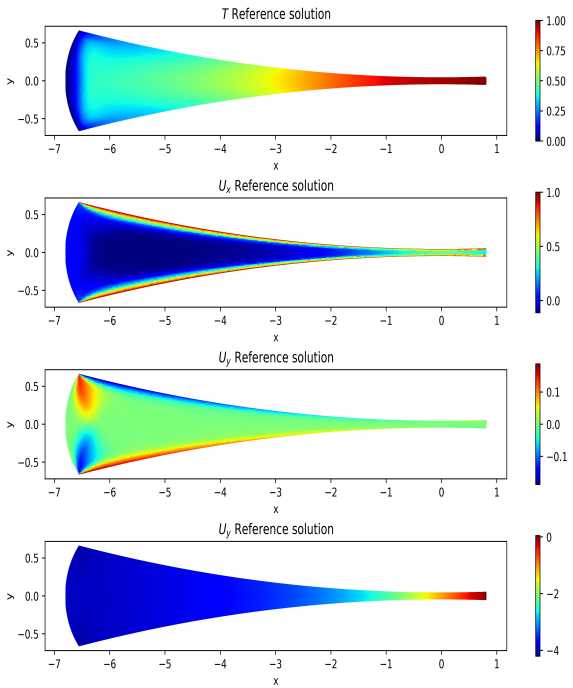
$$\begin{cases} (2\eta(\vec{u},T)u_x)_x + \left(\eta(\vec{u},T)(u_y+v_x)\right)_y = p_x \\ \left(2\eta(\vec{u},T)v_y\right)_y + \left(\eta(\vec{u},T)(u_y+v_x)\right)_x = p_y \\ u_x T_x + u_y T_y = \dfrac{\lambda}{\rho C_p}\left(T_{xx}+T_{yy}\right) + \dfrac{\eta(\vec{u},T)}{\rho C_p}|\gamma(\vec{u})|^2 \\ u_x + v_y = 0 \end{cases}$$

where $\vec{u}$ is the velocity, p pressure and T temperature, $\bar{\bar{\varepsilon}}(\vec{u})$ is the strain rate tensor, $\gamma(\vec{u}) = \sqrt{2\sum \bar{\bar{\varepsilon}}_{i,j}^2}$, and the dynamic viscosity η:

$$\eta(\vec{u},T) = K|\gamma(\vec{u})|^{n-1}\exp\left(\frac{E_\alpha}{R}\left(\frac{1}{T}-\frac{1}{T_\alpha}\right)\right)$$

Industrial Physics models. Rubber calendering modeling

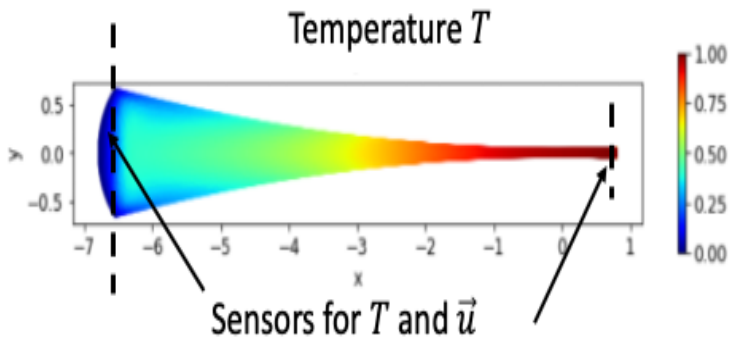
To generate reference High-Fidelity (HF) solutions of velocity, pressure, and temperature fields, the equations are discretized and solved using an in-house generic and multi-purpose finite element solver named MEF++ and co-developed by Laval University and Michelin. [Nguyen et al., 2022].

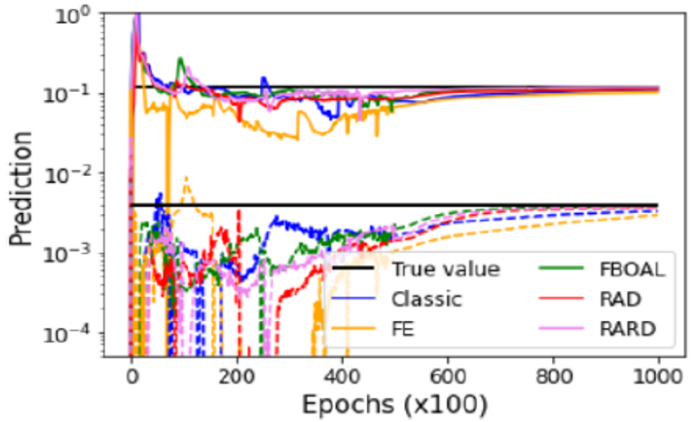


Rubber calendering process - inverse problem

Goal : identify the true value of the thermal conductivity from sensors measurements.
Dimensionless equation :

$$u_x T_x + u_y T_y = \frac{1}{Pe}\left(T_{xx} + T_{yy}\right) + \frac{Br}{Pe}\eta(\vec{u}, T)|\gamma(\vec{u})|^2$$





Strategy	$\epsilon_{Br/Pe}$	$\epsilon_{1/Pe}$
Random points	6.50	12.9
FE mesh	12.2	24.7
FBOAL	**0.35**	8.05
RAD	0.71	7.00
RAR-D	**0.69**	**0.11**

- Using only two lines of sensors, PINNs can identify accurately the true values of unknown parameters.
- Adaptive strategies outperform fixe strategies

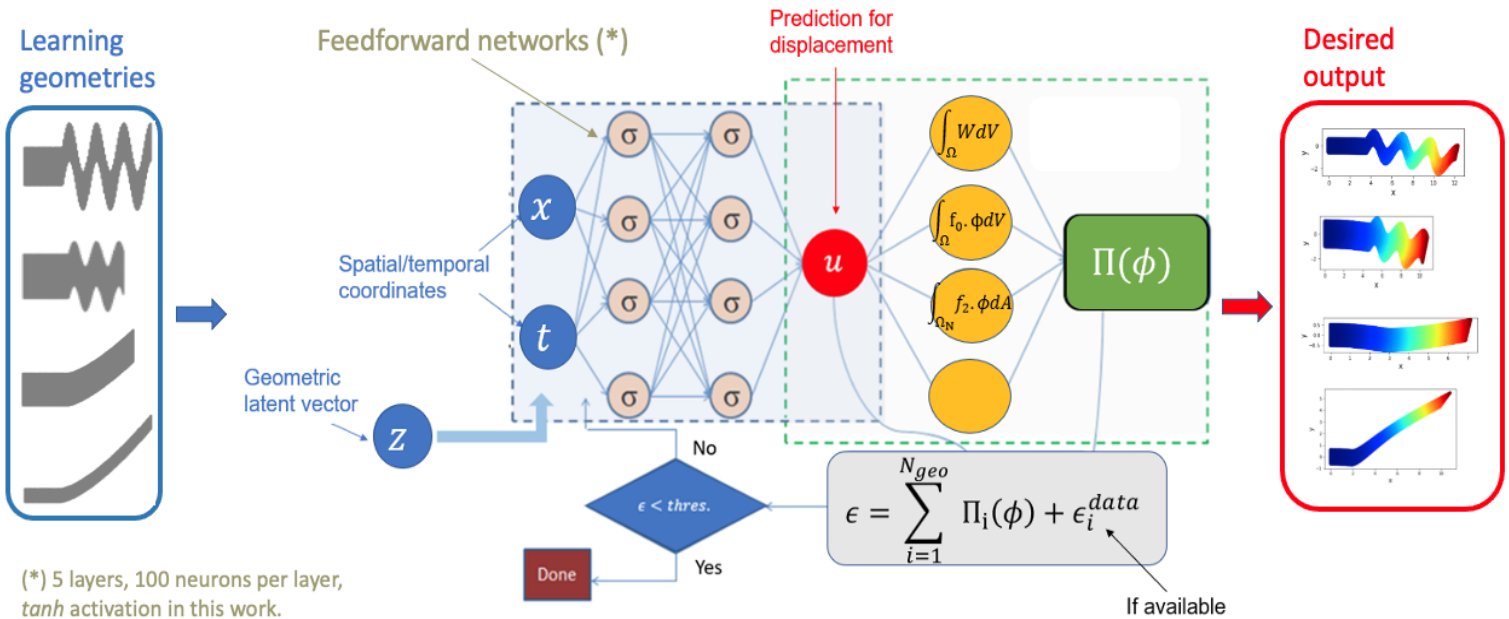
Physics Informed

Neural Networks

-PINNs- (geometry of the domain)

Geometry Aware Deep Energy Method

- [Nguyen et al., 2025] proposed to encodes the geometric knowledge into the PINNS model and to minimize a loss function on the potential energy of the system and not on residual equations.
- The potential energy of the system is minimum at the equilibrium state and computed by the difference between internal and external energy, here approximated by Monte Carlo.
- The potential energy of the systems over all geometries are minimized.



- Focus is made of mechanical problem using weak formulation.

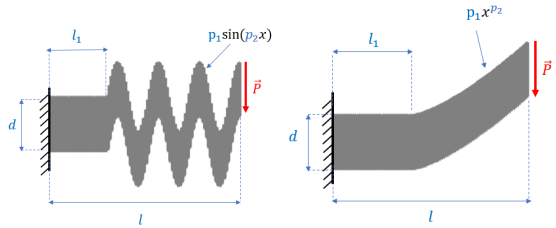
Geometry Aware PIML.

Displacement of a Beam with **Linear Elasticity**.

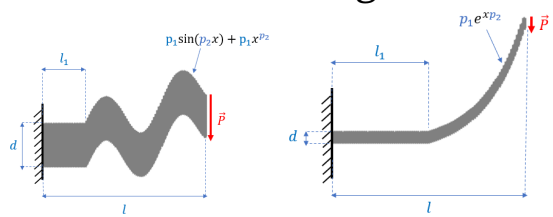
Experimental design.

- 5 parameters encoding the geometry
- LHS to compute 50 geometries
- The left side clamped, right side subjected to a traction $\vec{P} = (0, -1)N$.
- homogeneous, isotropic material with Young's modulus $E = 1000N/m^2$, and Poisson's ratio $\nu = 0.3$.
- Reference solution obtained by Finite Element Method and Fenics software

Ex. Training :

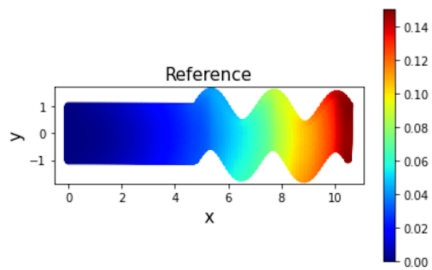


Ex. Testing



Coding the Geometry...

- **Parametric :** explicit parametric encoding.
- **PCA-Coord :** spatial coordinates for the geometric representation and PCA for the encoding (objects' boundaries).
- **VAE-Coord :** spatial coordinates for the geometric representation and VAE for the encoding.
- **PCA-Image :** images for the geometric representation and PCA for the encoding. (objects' images are available.)
- **VAE-Image :** images for the geometric representation and VAE for the encoding.



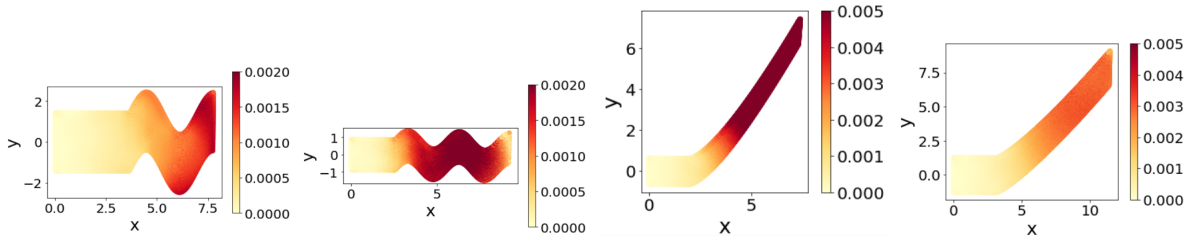
Geometry Aware PIML.

Displacement of a Beam with **Linear Elasticity**.

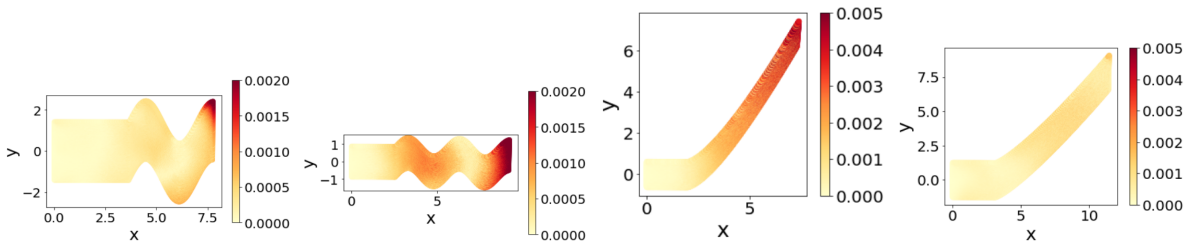
- **Importance of adaptive sampling**

Parametric encoding. Illustration.

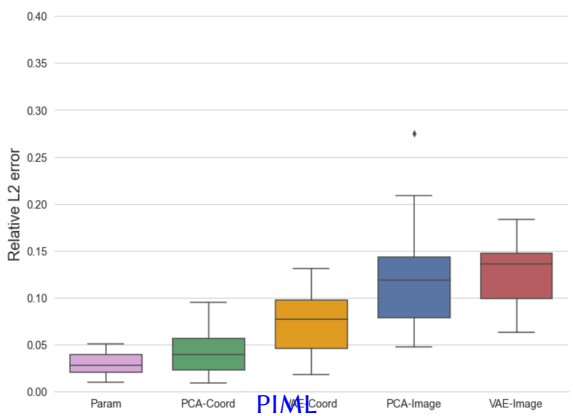
Without FBOAL (absolute error)



With FBOAL(absolute error)



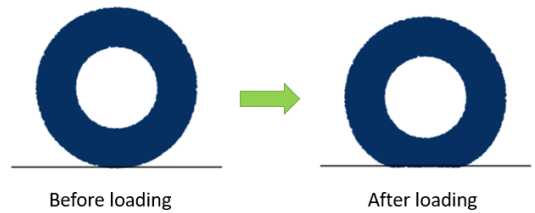
- **Impact of encoding.** Testing data set.



Geometry Aware PIML

Toy tire loading simulation with Hyperelasticity

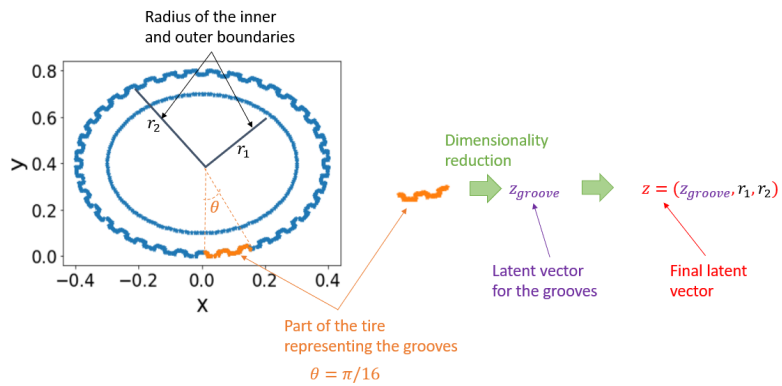
- Model the 2D displacement of toy tires composed of hyperelastic materials (rubber).
- Tires are made of rubber with a Young modulus $E = 21.10^6 Pa$ and a Poisson's ratio $\nu = 0.3$.
- Tires follow the Saint-Venant Kirchhoff hyperelastic model.



Sketch of a tire before and after loading.

About the geometry.

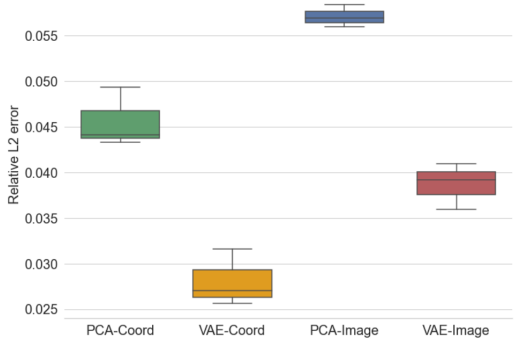
- 55 Tire geometries (5 x 11) given radius of the inner and outer boundaries, and the grooves.
- Different coding schemes : PCA-coord, VAE-coord, PCA-image, VAE-image



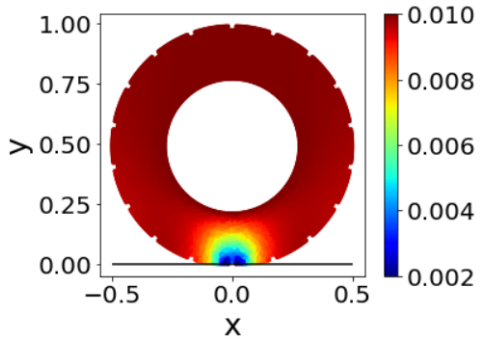
Geometry Aware PIML

Toy tire loading simulation with Hyperelasticity

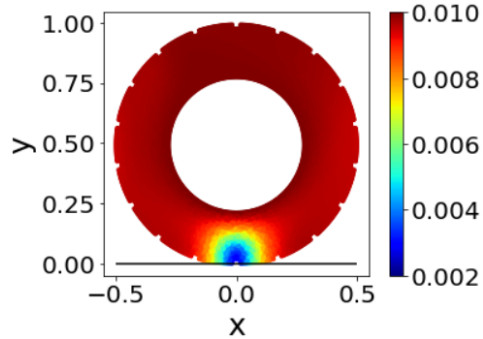
Performance of tire load simulation.
Best VAE-coord encoding (non linear)



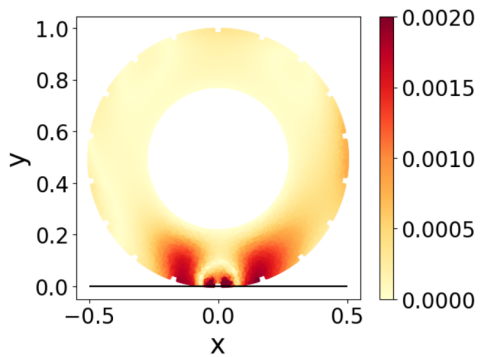
Improvement using adaptive sampling. Illustration



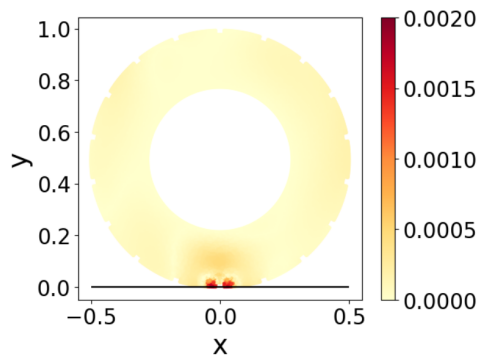
Truth



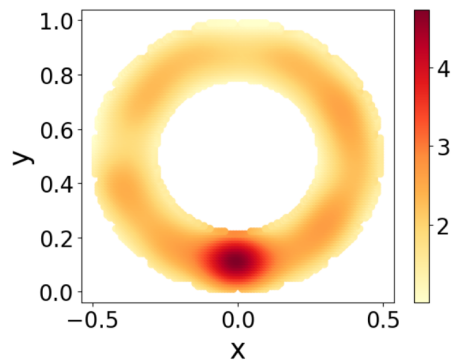
Pred Test



Err w/o FBOAL



Err w FBOAL



Density by FBOAL

Orographic Gravity Waves Modeling

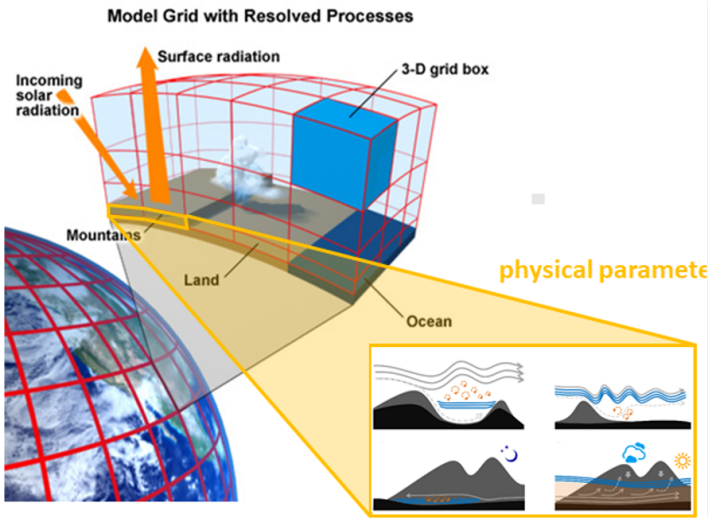
with
Physical Informed Machine Learning...

Weather modeling & Infrasound propagation

Joint work with Khoa Nguyen (PhD), C. Millet (CEA), T. Dairay, R. Meunier (Michelin) [Nguyen, 2024]

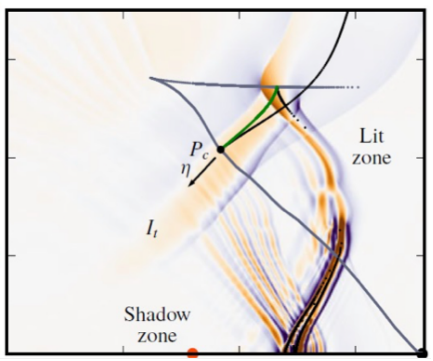
Weather modeling

- Direct Numerical Simulation is prohibitive
- GCM (Global Circulation Models) require parameterizations of many sub-scale physical processes (that are not explicitly resolved!)
- These parameterizations provide forcing at the global scale (i.e. for the dynamical component of the GCM)



Infrasound propagation

- Need to have atmospheric fields at various scales (from local/regional to global) for a comprehensive description of the propagation medium
- Acoustic propagation depends on waveguides that are affected by small-scale processes
- Parameterizations can be used for downscaling

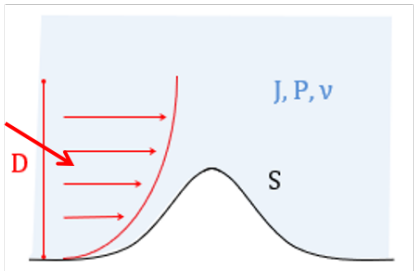


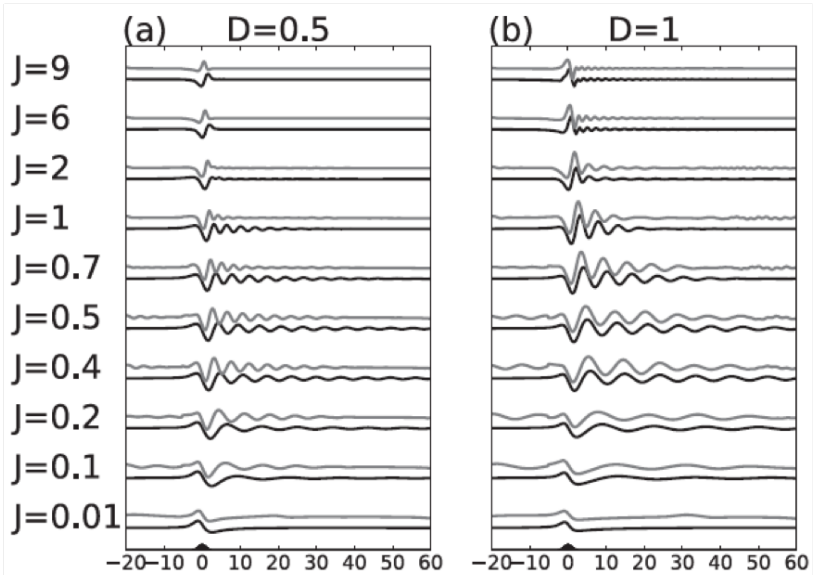
Motivation. Orographic gravity waves Modeling.

Example of parameterization :

The physical problem

$u_0(z) = \mathrm{D}\tanh(z/\mathrm{D})$





Linearized 2D Boussinesq **dimensionless** equations:

$$\left.\begin{aligned} u_0\partial_x u + u_{0z} w &= -\partial_x p + \nu\partial_z^2 u \\ u_0\partial_x w &= -\partial_z p + b + \nu\partial_z^2 w \\ u_0\partial_x b + \mathrm{J} w &= \mathrm{P}^{-1}\nu\partial_z^2 b \\ \partial_x u + \partial_z w &= 0 \end{aligned}\right\} \mathcal{L}(\mathrm{J}, \ldots)\mathbf{X} = \mathbf{0}$$

u, w, p and b: horizontal/vertical velocity, pressure and buoyancy.

Boundary conditions at ground level are given by:

$$h(x) + u(x,h) = w(x,h) = \mathrm{J}h(x) + b(x,h) = 0$$

Dimensionless parameters:
J: Richardson number
P (Prandlt), S (Slope)
D (BL depth), ν^{-1} (Reynolds)

$w(z = D)$
MITgcm

Open questions : models based on data ? simulation models based on physics ? hybrid ? PINNS...
[Nguyen, 2024]

Orographic Gravity Waves. Forward problem

Numerical results

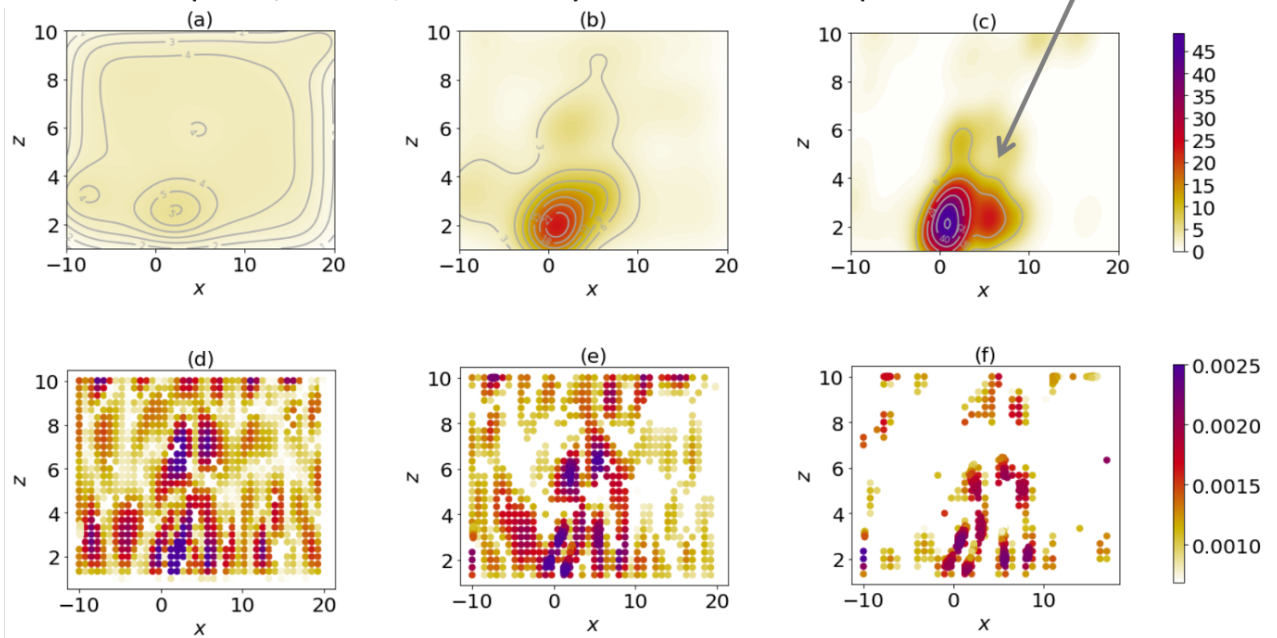
The forward problem with FBOAL

- PDE parameter is fixed, $J = 1$
- With FBOAL, after each 25k epochs, we add/remove only 1% of colocation points.

FBOAL captures couplings that trigger gravity waves aloft

Number of collocation points per area unit

PDE residuals



After 100k epochs

After 500k epochs

After 1000k epochs

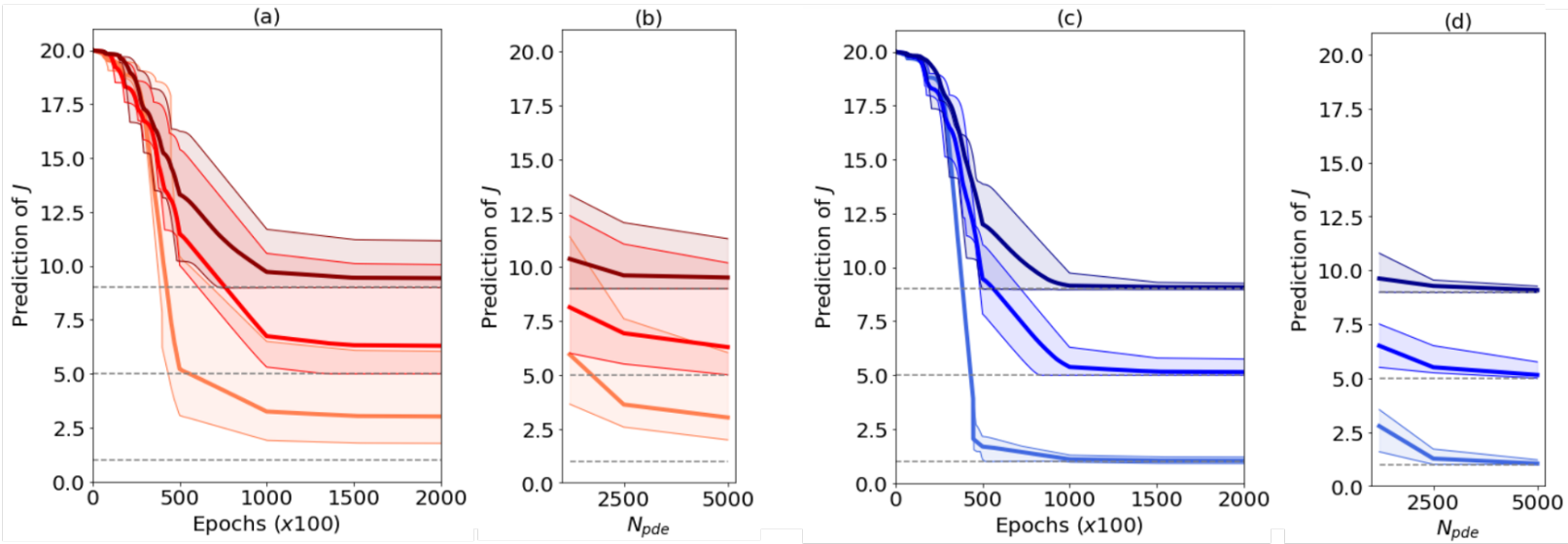
Orographic Gravity Waves.

Numerical results

The inverse problem

- J is supposed to be unknown.
- With FBOAL, after each 25k epochs, we add/remove 1% collocation points.

Prediction of J as a function of the number of epochs



Classical PINNs

PINNs + FBOAL

Geometry Aware PINNS

- The vanilla PINNS [Raissi et al., 2019] are only capable of inferring the solution on one fixed configuration. One needs to retrain PINNs when dealing with new PDE parameters, new Initial Conditions (IC) and boundary conditions (BC). High computational cost..
- Several works proposed to include geometry information/ [Oldenburg et al., 2022] proposed to solve Navier-Stokes equations in a forward scheme on various irregularly shaped vessels (GAPINNs), as [Gao et al., 2021] for PhyGeoNet, [Cameron et al., 2024]...

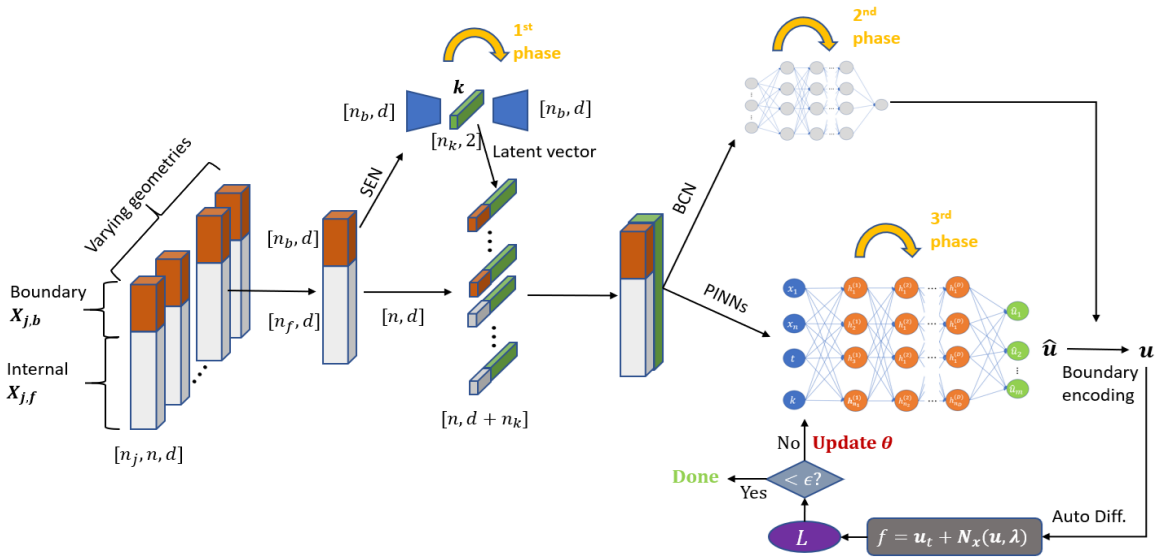


FIGURE – Geometry Aware PINNS schematic diagram,. The Shape Encoding Network (SEN) computes a latent representation using a variationnal Auto-encodeur (VAE). The Boundary Encoding Network (BCN) enforces the BC during training. [Nguyen, 2024]

Beyond PINNS : Geometry Aware PINNSs (GAPINNS).

- Integrate geometric information into the model.
- Use dimensionality reduction methods to extract geometric latent vectors.
- Able to infer the solution for various geometries using only one trained model.

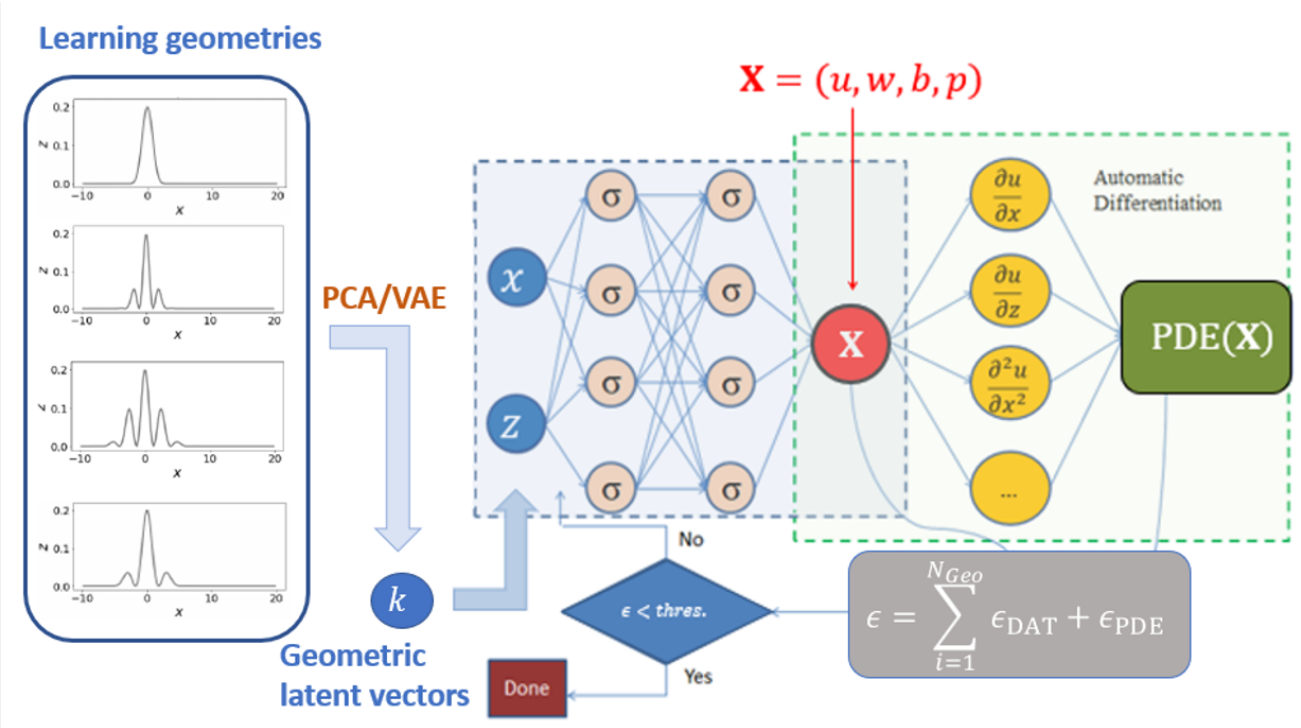


Illustration of GAPPINS : Numerical results

Supposing that the mountain has the following shape:

$$h(x) = Se^{\frac{(\epsilon x)^2}{2}}(1 + \cos(kx))$$

and:

- k, ϵ are two varying geometric parameters.
- Using Latin Hypercube Sampling to sample 20 geometries for $k \in [1,3], \epsilon \in [0.4,1]$.
- Training GAPINNs with 11 geometries, other geometries are used for testing.

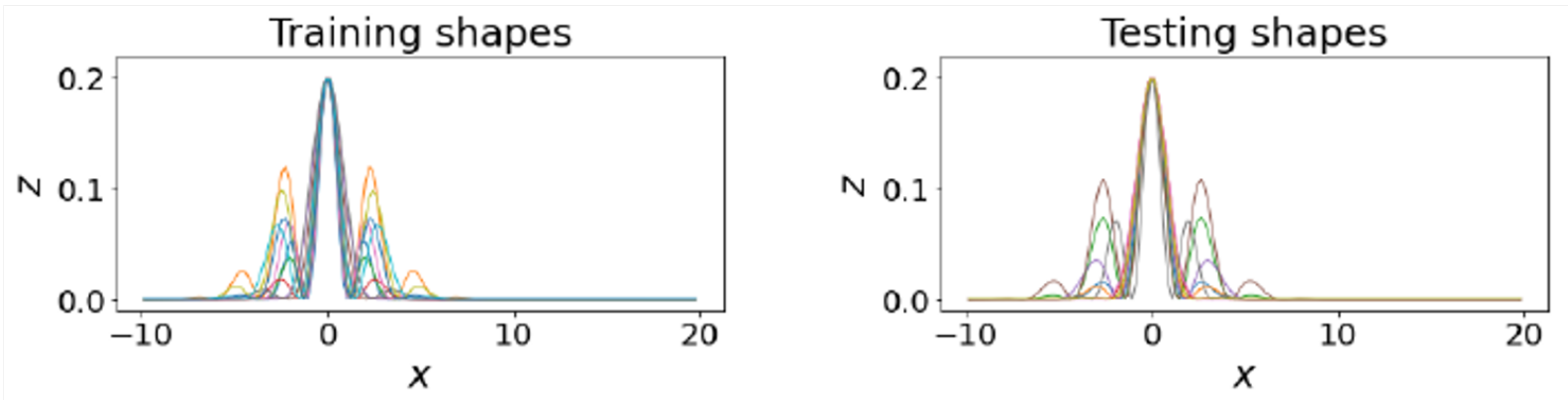


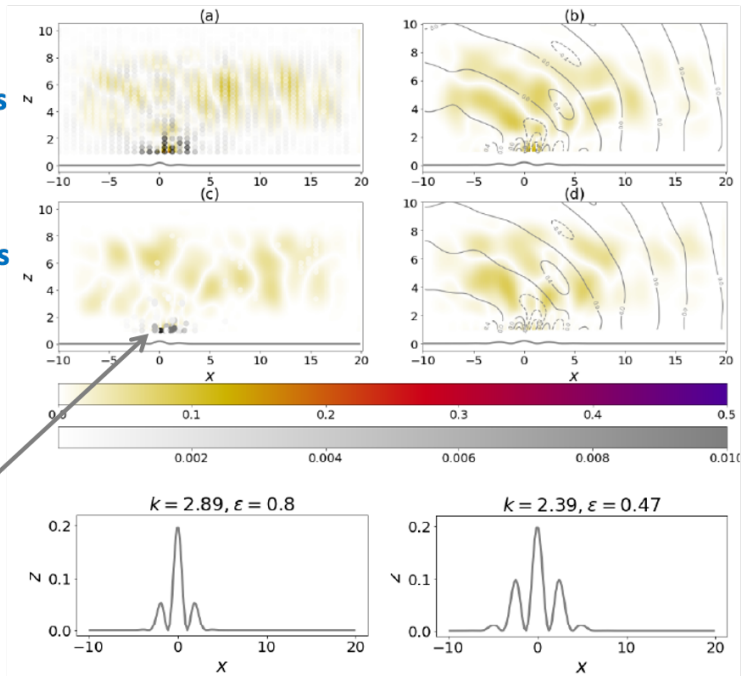
Illustration of GAPPINS : Numerical results

- The Richarson number is fixed, $\mathrm{J} = 1$.
- The supervised is distributed around the boundary with the thickness $1km$.
- For each geometry, the number of supervised points is $N_{sup} = 700$, the number of collocation points is $N_{col} = 4800$.
- With FBOAL, after each 25k epochs, we add/remove only 1% of colocation points.

Prediction errors for w

GAPINNs

GAPINNs +FBOAL



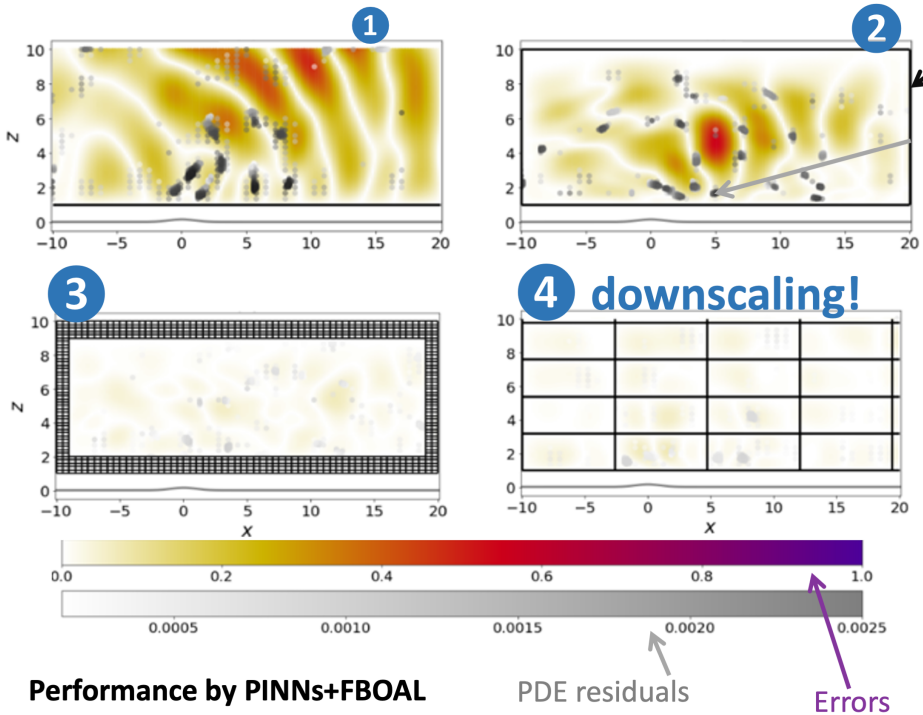
FBOAL again captures couplings above the mountain.

Training geo. Testing geo.

Meteorological problem. Downscaling

2. FBOAL application: Interaction BL-mountain – Mountain waves prediction

- Weather data obtained by ECMWF is **sampled on a coarse mesh** (with resolution around 30km)
- **Goal:** Using **PINNs** to reproduce the flow on **a finer mesh** (around hundreds of meters)



Performance by PINNs+FBOAL

- **FBOAL captures couplings that trigger gravity waves aloft.**
- **Prediction accuracy is improved with FBOAL**

	W/o FBOAL	With FBOAL
u	1.17e-02	**7.34e-03**
w	9.34e-03	**8.36e-03**
b	1.14e-02	**1.06e-02**
p	9.90e-03	**7.76e-03**

Error

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Geometry Aware PINNS

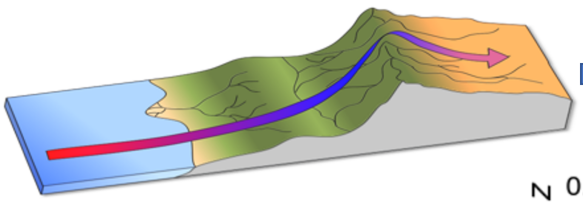
GAPINNs application: Interaction between a BL and a mountain

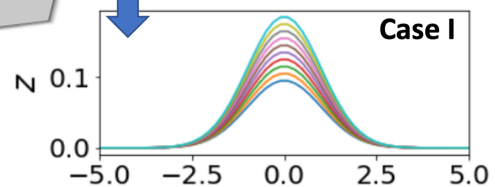
- A **more general form of boundary condition** (or mountain shape):

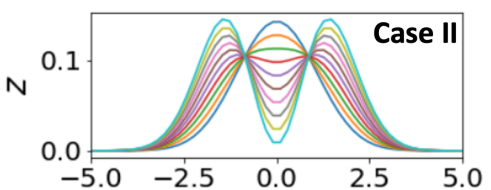
$$h = h(\mathbf{x}, t)$$

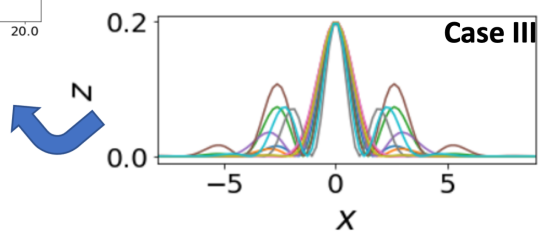
where t is a vector of parameters.

- Different cases are considered:



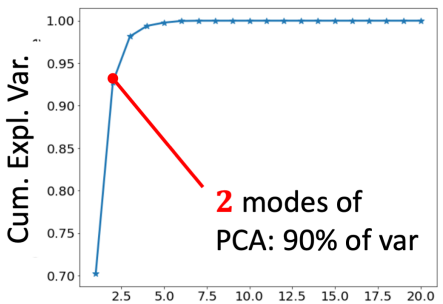






Complexity ↑

- Modeling the orography with simple models:
  - Databases are generated with model of increasing complexity (from case 1 to case 3).
  - PCA is performed to produce the geometric latent vector.



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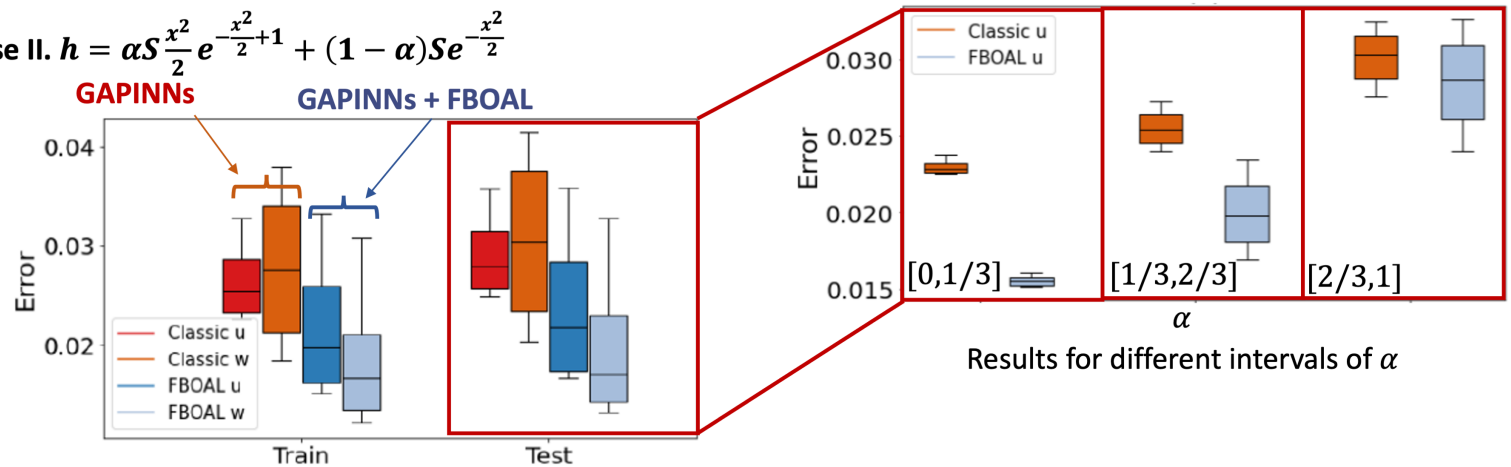
Geometry Aware PINNS

Benefits of adaptive collocation points evolution and Geometrical inputs

GAPINNs application: Interaction between a BL and a mountain

Case II. $h = \alpha S \frac{x^2}{2} e^{-\frac{x^2}{2}+1} + (1-\alpha) S e^{-\frac{x^2}{2}}$



Results for different intervals of α

Boxplots for **GAPINNs** and **GAPINNs + FBOAL** for training

- **GAPINNs successfully infer the solution on various geometries ($\mathbf{errors} < 5\%$)**
- **FBOAL helps to increase the accuracy of GAPINNs.**

Physics Informed

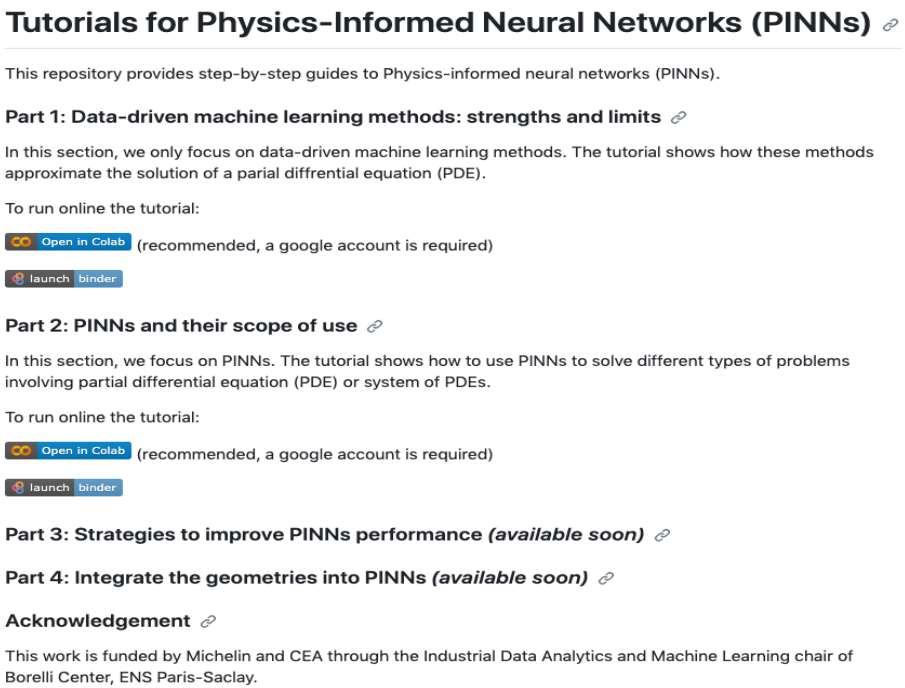
Neural Networks

-PINNs- (software and tutorial)

Softwares & Tutorials

- Several Softwares packages
  DeepXDE, PyDEns, NeuroDiffEq PyTorch-based library, Modulus....
  see [Cuomo et al., 2022]

- PINNS Tutorial provided by Khoa Nguyen. More to come soon....



Tutorials for Physics-Informed Neural Networks (PINNs)

This repository provides step-by-step guides to Physics-informed neural networks (PINNs).

Part 1: Data-driven machine learning methods: strengths and limits

In this section, we only focus on data-driven machine learning methods. The tutorial shows how these methods approximate the solution of a parial diffrential equation (PDE).

To run online the tutorial:

Open in Colab (recommended, a google account is required)

launch binder

Part 2: PINNs and their scope of use

In this section, we focus on PINNs. The tutorial shows how to use PINNs to solve different types of problems involving partial differential equation (PDE) or system of PDEs.

To run online the tutorial:

Open in Colab (recommended, a google account is required)

launch binder

Part 3: Strategies to improve PINNs performance *(available soon)*

Part 4: Integrate the geometries into PINNs *(available soon)*

Acknowledgement

This work is funded by Michelin and CEA through the Industrial Data Analytics and Machine Learning chair of Borelli Center, ENS Paris-Saclay.

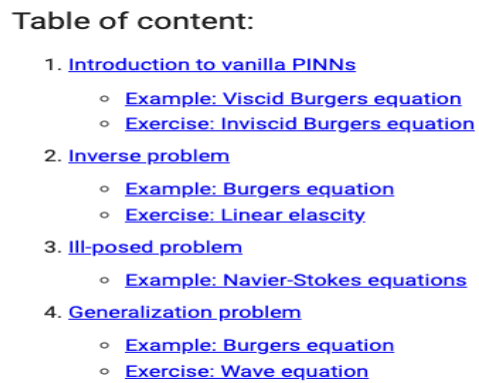


Table of content:

1. Introduction to vanilla PINNs
   - Example: Viscid Burgers equation
   - Exercise: Inviscid Burgers equation
2. Inverse problem
   - Example: Burgers equation
   - Exercise: Linear elascity
3. Ill-posed problem
   - Example: Navier-Stokes equations
4. Generalization problem
   - Example: Burgers equation
   - Exercise: Wave equation

https ://github.com/nguyenkhoa0209/pinns_tutorial

A joint work

thanks to the Industrial Data Analytics and
Machine Learning chair

Industrial Data Analytics
& Machine Learning

with

- Antoine de Mathelin, Towards reliable machine learning under domain shift and costly labeling, with applications to engineering design Michelin & IDAML, Centre Borelli
- Khoa Nguyen, Development and assessment of physically informed learning methods : enhancement of multi-physical simulation in industrial contexts, CEA, Michelin & IDAML, Centre Borelli
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