Non homothetic Preferences

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Non Homothetic derivation

$$\mathcal{L} = U(C_1, ..., C_I) + \Upsilon \left[1 - \sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{U(C_1, ..., C_I)^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} \right] + \lambda \left[E - \sum_i^I p_i C_i \right]$$

FOCs of $\mathcal{L}$ wrt $U, (C_1, ..., C_I), \Upsilon, \lambda$:

$$\frac{\partial \mathcal{L}}{\partial U} = 1 - \Upsilon \frac{(1-\sigma)^2}{\sigma} \sum_{i=1}^I \gamma_i^{\frac{1}{\sigma}} \epsilon_i \frac{1}{U(C_1, ..., C_I)} \left[\frac{C_i}{U(C_1, ..., C_I)^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0$$

Note: $i \in \mathcal{I}, U_i' = \frac{\partial U}{\partial C_i} = U'_{C_i}(C_1, ..., C_I)$

$$\begin{aligned} \forall i \in \mathcal{I}, \quad \frac{\partial \mathcal{L}}{\partial C_i} =& U_i' + \Upsilon \frac{1-\sigma}{\sigma} \left[\gamma_i^{\frac{1}{\sigma}} \frac{U^{(1-\sigma)\epsilon_i} - C_i(1-\sigma)\epsilon_i U^{(1-\sigma)\epsilon_i - 1} U_i'}{U^{2(1-\sigma)\epsilon_i}} \left(\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right)^{-\frac{1}{\sigma}} \right] \\ & - \Upsilon \frac{1-\sigma}{\sigma} \left[(1-\sigma) \sum_{j=1}^{I \backslash \{i\}} \gamma_j^{\frac{1}{\sigma}} \epsilon_j \frac{U_j'}{U} \left[\frac{C_j}{U^{(1-\sigma)\epsilon_j}} \right]^{\frac{\sigma-1}{\sigma}} \right] - \lambda p_i = 0 \end{aligned} \tag{1}$$

Rearranging the term, we have:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_i} =& U_i' + \Upsilon \frac{1-\sigma}{\sigma} \left[\gamma_i^{\frac{1}{\sigma}} \frac{1}{C_i} \left(\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right)^{\frac{\sigma-1}{\sigma}} \right] \\ & - \Upsilon \frac{(1-\sigma)^2}{\sigma} \left[\gamma_i^{\frac{1}{\sigma}} \epsilon_i \left(\frac{U_i'}{U} \right) \left[\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} \right] \\ & - \Upsilon \frac{(1-\sigma)^2}{\sigma} \left[\sum_{j=1}^{I \backslash \{i\}} \gamma_j^{\frac{1}{\sigma}} \epsilon_j \left(\frac{U_j'}{U} \right) \left[\frac{C_j}{U^{(1-\sigma)\epsilon_j}} \right]^{\frac{\sigma-1}{\sigma}} \right] - \lambda p_i = 0 \end{aligned} \tag{2}$$

1st line without U_i' is in the Comin et al. paper. With Expenditure equation, it gives optimal demand C_i^*. 2nd and 3rd lines are new terms for income effect term i.e cross effects between U and C.

Rearranging again, 2nd and 3rd lines add ups and form a better looking:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_i} =& U_i' + \Upsilon \frac{1-\sigma}{\sigma} \left[\gamma_i^{\frac{1}{\sigma}} \frac{1}{C_i} \left(\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right)^{\frac{\sigma-1}{\sigma}} \right] \\ & - \Upsilon \frac{(1-\sigma)^2}{\sigma} \left[\sum_{i=1}^{I} \gamma_j^{\frac{1}{\sigma}} \epsilon_j \left(\frac{U_j'}{U} \right) \left[\frac{C_j}{U^{(1-\sigma)\epsilon_j}} \right]^{\frac{\sigma-1}{\sigma}} \right] - \lambda p_i = 0 \end{aligned} \tag{3}$$

I was thinking about using $\frac{\partial \mathcal{L}}{\partial U} = 1 - \Upsilon \frac{(1-\sigma)^2}{\sigma} \sum_{i=1}^I \gamma_i^{\frac{1}{\sigma}} \epsilon_i \frac{1}{U} \left[\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0$ since the second term looks alike but there is no U_j'. Need help. Maybe try to rearrange again blue and red terms so that it equals 0 and get back to the term in the paper?

For this part we have something of this form

$$\frac{\partial \mathcal{L}}{\partial C_i} = F_i(...) - \lambda p_i = 0$$

Our system should satisfy this equation:

$$\frac{F_i(C_1, ..., C_I)}{p_i} = \frac{F_j(C_1, ..., C_I)}{p_j} \quad \forall i, j \in \mathcal{I}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = E - \sum_i^I p_i C_i = 0$$

$$\frac{\partial \mathcal{L}}{\partial \Upsilon} = 1 - \sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{U(C_1, ..., C_I)^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0$$

This system of equation should then hold at the optimum.

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial U} = 1 - \Upsilon \frac{(1-\sigma)^2}{\sigma} \sum_{i=1}^I \gamma_i^{\frac{1}{\sigma}} \epsilon_i \frac{1}{U} \left[\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0 \\ \text{From} \quad \frac{\partial \mathcal{L}}{\partial C_i} = 0 = \frac{\partial \mathcal{L}}{\partial C_j}, \quad \text{obtain} \quad \frac{F_i(C_1, ..., C_I)}{p_i} = \frac{F_j(C_1, ..., C_I)}{p_j} \quad \forall i, j \in \mathcal{I} \\ \frac{\partial \mathcal{L}}{\partial \lambda} = E - \sum_i^I p_i C_i = 0 \\ \frac{\partial \mathcal{L}}{\partial \Upsilon} = 1 - \sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{U(C_1, ..., C_I)^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0 \end{cases} . \tag{4}$$