Non homothetic Preferences

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Homothetic CES preferences

Suppose utility defined as a consumption bundle with CES form. Let good be indexed by $i \in \mathcal{I}$. Homothetic preferences will take the following form such that σ correspond to the elasticity of substitution across goods and is designed to be constant. Analoguous to a CES production function, our household utility function is composed of how household choose their consumption among a set of consumption goods in $\mathcal{I}$

$$U(C_1, C_2, ...C_n) = \left[\sum_i^I \gamma_i C_i^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

Our utility is then monotonic increasing and concave. It enables a well defined utility function over a consumption bundle $\mathcal{C}$. γ_i captures the relative weight in household consumption and $\sum_i^I \gamma_i = 1$ so that it accounts for constant return to scale. Consumption bundle consist of having $C_i \geq 1$ when goods are consumed and 0 when not. Utility increases by diversity. In this case, the relative demand across goods depend on the relative price between goods and thus the marginal rate of substitution.

$$\mathcal{L} = U(C_1, ...C_n) + \lambda[E - \sum_{i=1}^I P_i C_i]$$

FOC with respect to $C_i, \forall i \in \mathcal{I}$

$$U'(C_i) = P_i$$

$$C_i = U^{-1'}(P_i)$$

In case $\sigma = 1$, we get

$$C_i = \frac{\gamma_i}{p_i} U(C_1, ..., C_n)$$

The demand in good i is then linear in U that is highly linked to household expenditure level E.

Namely, we would have for all $i, j \in \mathcal{I}$:

$$\log \frac{C_i}{C_j} = \sigma \log \frac{P_j}{P_i}$$

Equivalently with $\frac{C_i}{C_j} = D_{ij}$ and $\frac{P_i}{P_j} = \mathbf{P}_{ij}$

$$\log D_{ij} = \sigma \log \mathbf{P}_{ij}.$$

When $\sigma = 1$ we have our Cobb Douglas utility function of the following form:

$$U(C_1, C_2, ..., C_n) = \prod_i^I C_i^{\gamma_i}.$$

This form presents constant return to scale and insures our elasticity of substitution across good is equal to 1 ($\sigma = 1$). Consequently, the relative demand between good i and j is independent of the level of utility and don't tackle preferences concerns. Intuitively, someone with a higher revenue will just consume X times more goods than someone with a poor revenue over the same consumption bundle. This relative demand would then only be defined by the relative product prices and thus the marginal rate of substitution between goods.

Non-homothetic CES preferences

We are thus interested in supplying an utility form that takes into account change in consumption behavior across the income ladder. Depending on income/wealth level, people exhibit different preferences over goods. In other words a parameter that captures income elasticity of relative demand would be welcomed. Some goods may not be bought by some individuals due to revenue constraints that cannot satisfy this consumption. Some would be heavily consumed by some income groups, some not at all. Consequently, I permit different consumption bundle to be formed according to their income level. Following Comin et al. [2021], Sato [1977], Hanoch [1975] non homothetic CES utility function adopts an implicit form in which an utility level is reached with minimal cost. Constraints are thus included. Last constraint indicates that consumption is feasible under budget constraint.

household face then an implicit form of non-homothetic utility:

$$\sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{g(U)^{\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}} = 1$$

When $g(U)^{\epsilon_i} = U$, we go back to homothetic CES preference. When it is not we have a non homothetic CES preference utility function that takes into account non homothetic parameter ϵ_i and holds for income elasticity for relative demand. Each good has then its own non homothetic parameter so that we can look for income effect in consumption. Indeed the maximization problem would take the following form

$$\mathcal{L} = U + \rho[1 - \sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{g(U)^{\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}}] + \lambda[E - \sum_i^I p_i c_i].$$

When we derive with respect to C_i it takes

$$\frac{\rho}{\lambda}\frac{1-\sigma}{\sigma}\omega_i = p_i c_i$$

with $\omega_i = \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{g(U)^{\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}}$. Since $\sum_i^I \omega_i = 1$ then we get

$$\sum_i^I p_i c_i = \frac{\rho}{\lambda}\frac{1-\sigma}{\sigma}.$$

Then $E\omega_i = p_i C_i$ so that $E\gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{g(U)^{\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}} = p_i C_i$ and isolating C_i gives the hicksian demand function

$$C_i = \left(\frac{p_i}{E}\right)^{-\sigma} \gamma_i g(U)^{\epsilon_i(1-\sigma)}.$$

Environment

Household.

- Type $h \in \mathcal{H}$ household maximizes utility given budget constraint

$$\max_{\{C_1^h\}_{i=1}^{\mathcal{I}}} U(C_1^h, ..., C_{\mathcal{I}}^h)$$

$$st \quad E^h \geq \sum_{i=1}^{I} p_i C_i^h.$$

- In this model, household differs by their productivity level. We evaluate household productivity level for an individual by looking at their individual production. Suppose y^h correspond to individual h production then $\frac{y^h}{\theta^h} = L^h$ accounts for number of hour worked with θ^h the individual productivity type. higher θ^h accounts for higher productivity and lower hours worked L^h for a same production y^h. Let us further assume that a distribution and thus a ranking is available so that and higher indexed $h \in \mathcal{H}$ constitutes a higher value for household expenditure level (E^h).

- To account for an income effect in the allocation of resources across consumption goods, I adopt the implicit utility function with non-homothetic parameter proposed by Sato [1977]Comin et al. [2021]:

$$\sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i^h}{g(U)^{\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}} = 1.$$

- Let $U = U(C_1^h, ..., C_{\mathcal{I}}^h)$ The maximization problem would take the following form

$$\mathcal{L} = U + \rho[1 - \sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{g(U)^{\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}}] + \lambda[E^h - \sum_i^I p_i C_i^h]. \tag{1}$$

- When we derive (1) with respect to C_i^h it gives

$$\frac{\rho}{\lambda} \frac{1-\sigma}{\sigma} \omega_i^h = p_i C_i^h$$

where $\omega_i^h = \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i^h}{g(U)^{\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}}$ and stands for share of consumption in good i by household type h Since $\sum_i^I \omega_i = 1$ then we get

$$E^h \equiv \sum_i^I p_i c_i = \frac{\rho}{\lambda} \frac{1-\sigma}{\sigma}.$$

Then $E^h \omega_i^h = p_i C_i^h$ so that $E^h \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i^h}{g(U)^{\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}} = p_i C_i^h$ and isolating C_i gives the Hicksian demand function

$$C_i^h = \left(\frac{p_i}{E^h}\right)^{-\sigma} \gamma_i g(U)^{\epsilon_i(1-\sigma)}. \tag{2}$$

- Substituting this demand (2) into the expenditure share, we obtain the true expenditure share

$$\omega_i^h = \gamma_i \left[\frac{p_i}{E^h} g(U)^{\epsilon_i} \right]^{(1-\sigma)}. \tag{3}$$

- Total expenditure by household h is then

$$E^h \equiv \sum_{i=1}^{I} p_i C_i^h = \left[\sum_{i=1}^{I} \gamma_i \left(p_i g(U)^{\epsilon_i} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \tag{4}$$

- Relative demand $\forall i, j \in \mathcal{I}$ for household h is then found by taking the log of $\frac{C_i^h}{C_j^h}$ and delivers us:

$$\log \left(\frac{C_i^h}{C_j^h} \right) = \sigma \log \left(\frac{p_j}{p_i} \right) + (\epsilon_i - \epsilon_j)(1-\sigma) \log g(U) + \log \left(\frac{\gamma_i}{\gamma_j} \right).$$

- We consequently have the elasticity of relative demand with respect to a monotonic transformation of $g(.)$ equals

$$\frac{\partial \log \left(\frac{C_i^h}{C_j^h} \right)}{\partial \log g(U)} = (1-\sigma)(\epsilon_i - \epsilon_j).$$

  We can also see that the elasticity of relative demand with respect to change in relative prices equals

$$\frac{\partial \log \left(\frac{C_i^h}{C_j^h} \right)}{\partial \log \left(\frac{p_j}{p_i} \right)} = \sigma.$$

- Using (3) relative Hicksian demand on expenditure shares yield

$$\log \left(\frac{\omega_i^h}{\omega_j^h} \right) = \log \left(\frac{\gamma_i}{\gamma_j} \right) + (1-\sigma) \log \left(\frac{p_i}{p_j} \right) + (1-\sigma)(\epsilon_i - \epsilon_j) \log g(U)$$

- Let us take a good as reference good indexed as $b \in \mathcal{I}$. We could then evaluate the relative demand of other goods with respect to the reference one as goods $\forall i \in \mathcal{I}_{-b}$

$$\log \omega_i^h = \log \gamma_i + (1-\sigma) \log \left(\frac{p_i}{E^h} \right) + (1-\sigma) \epsilon_i \log g(U), \quad \forall i \in \mathcal{I}. \tag{5}$$

  In the programming part, our share in consumption of good i by household h correspond to

$$\omega_i^h = \gamma_i \left[\frac{p_i}{E^h} g(U)^{\epsilon_i} \right]^{(1-\sigma)}.$$

  Where $\gamma_i g(U)^{\epsilon_i (1-\sigma)} = \mathbf{U}$

  Rearranging the term in (5), we have then

$$\frac{1}{1-\sigma} \log \gamma_i + \epsilon_i \log g(U) = \frac{1}{1-\sigma} \log \omega_i^h + \log \frac{E^h}{p_i}.$$

By considering $\mathbf{U}$ we end up with

$$\log \omega_i^h = \log \mathbf{U} + (1-\sigma) \log \frac{p_i}{E^h}. \tag{6}$$

By taking a reference good b we find:

$$\log \omega_i^h = \frac{\epsilon_i}{\epsilon_b} \log \omega_b^h + (1-\sigma) \left(\frac{\epsilon_i}{\epsilon_b} - 1\right) \log \left(\frac{E^h}{p_b}\right) + \log \left(\frac{\gamma_i}{\gamma_b^{\frac{\epsilon_i}{\epsilon_b}}}\right)$$

Firm.

- Each sector i firm produces sectoral output under competition. We adopt a constant-return-to-scale production function with only labor supply and sector specific productivity,

  $$Y_i = A_i L_i$$

  where L_i denotes the labor supply used in producing output Y_i.

- Competitive goods market firm profit maximization imply that wage are same for all sectors and price equal the marginal cost of hiring an additional hour corrected by sector specific productivity.

  It is good to remark that L_i stands for the sum of labor supplied by different household h. So that $L_i = \sum_{h=1}^{H} \sum_{i=1}^{I} \omega_i^h L^h$ where L^h correspond to the total labor supply regardless of the production sector by household h.

  $$\max_{L_i} \Pi = \max p_i Y_i - w L_i$$

  $$Note: \quad Y_i = \sum_{h=1}^{H} C_i^h$$

  $$L_i = \sum_{h=1}^{H} \sum_{i=1}^{I} \omega_i^h L^h.$$

  When we derive with respect to L_i, we find $w = p_i A_i$. that can be rewritten as

  $$w L_i = p_i Y_i.$$

- Goods market clearing ensures that $\sum_{h=1}^{H} \omega_i^h E^h = p_i Y_i$.

  $$w L_i = \sum_{h=1}^{H} \omega_i^h E^h.$$

  So that at the aggregate level, total expenditure across household and sectors equal total production

  $$\sum_{i=1}^{I} \sum_{h=1}^{H} \omega_i^h E^h = \mathbf{pY}$$

  with $\mathbf{p}$ the price level weighted by production size $\mathbf{Y}$.

- Regarding relative labor supply across sectors, we see that relative sector size across sectors determine its value.

$$\frac{L_i}{L_j} = \frac{\sum_{h=1}^{H} \omega_i^h E^h}{\sum_{h=1}^{H} \omega_j^h E^h}$$

- Considering that $w = p_i A_i \quad \forall i \in \mathcal{I}$, then relative price is defined with relative sector productivity.

$$\frac{p_i}{p_j} = \frac{A_j}{A_i}$$

Programming

To account for non-homothetic preference and its income effect on households consumption choice.

- We first set an endowment bundle achieved by household H composed of goods from $i \in \mathcal{I}$ sector.
- I allow 3 products to exist in this economy so that $\mathcal{I} = \{Primary, Normal, Luxury\}$. These products differs by their income effect on the demand function. There is a higher taste for luxury product as utility/wealth grows. In our setup, it is embodied by sector specific income elasticities ϵ_i for $i \in \mathcal{I}$.
- These products follow different price dynamics. Primary goods price grows by 5%, Normal goods price grows by 2% and Luxury goods price by 1% at each date t. We take a base date t to construct a price index. We can remark that relative price of Luxury good over Primary good will drop gradually as t grows. It translates into a higher demand for Luxury good as it becomes relatively less expensive to consume it.
- With the utility level in mind, I compute the expenditure level by using equation (4). With that I compute consumption share in good h with equation (6) and consumption unit in good h with (2).

What we see at the end is that by allowing for non homothetic preferences, higher utility household consume more from luxury good relatively to primary one. Moreover, as Primary good becomes relatively more expensive, consumption unit in luxury good grows over time at the expense of Primary and Normal good.

References

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