Non homothetic Preferences

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Non Homothetic derivation

$$\mathcal{L}(U, C_1, ..., C_I) = U + \Upsilon \left[1 - \sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} \right] + \lambda \left[E - \sum_i^I p_i C_i \right]$$

FOCs of $\mathcal{L}$ wrt $U, (C_1, ..., C_I), \Upsilon, \lambda$:

$$\frac{\partial \mathcal{L}}{\partial U} = 1 - \Upsilon \frac{(1-\sigma)^2}{\sigma} \sum_{i=1}^I \gamma_i^{\frac{1}{\sigma}} \epsilon_i \frac{1}{U} \left[\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0$$

$$\forall i \in \mathcal{I}, \quad \frac{\partial \mathcal{L}}{\partial C_i} = \Upsilon \frac{1-\sigma}{\sigma} \left[\gamma_i^{\frac{1}{\sigma}} \frac{1}{C_i} \left(\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right)^{\frac{\sigma-1}{\sigma}} \right] - \lambda p_i = 0 \tag{1}$$

Then

$$\forall i, j \in \mathcal{I}, \quad \frac{\gamma_i^{\frac{1}{\sigma}} \frac{1}{C_i} \left(\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right)^{\frac{\sigma-1}{\sigma}}}{p_i} = \frac{\gamma_j^{\frac{1}{\sigma}} \frac{1}{C_j} \left(\frac{C_j}{U^{(1-\sigma)\epsilon_j}} \right)^{\frac{\sigma-1}{\sigma}}}{p_j}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = E - \sum_i^I p_i C_i = 0$$

$$\frac{\partial \mathcal{L}}{\partial \Upsilon} = 1 - \sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0$$

This system of equation should then hold at the optimum F(.)= 0.

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial U} = 1 - \Upsilon \frac{(1-\sigma)^2}{\sigma} \sum_{i=1}^I \gamma_i^{\frac{1}{\sigma}} \epsilon_i \frac{1}{U} \left[\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0 \\ \forall i, j \in \mathcal{I}, \quad \frac{\gamma_i^{\frac{1}{\sigma}} \frac{1}{C_i} \left(\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right)^{\frac{\sigma-1}{\sigma}}}{p_i} = \frac{\gamma_j^{\frac{1}{\sigma}} \frac{1}{C_j} \left(\frac{C_j}{U^{(1-\sigma)\epsilon_j}} \right)^{\frac{\sigma-1}{\sigma}}}{p_j} \\ \frac{\partial \mathcal{L}}{\partial \lambda} = E - \sum_i^I p_i C_i = 0 \\ \frac{\partial \mathcal{L}}{\partial \Upsilon} = 1 - \sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0 \end{cases} \tag{2}$$

Optimal demands from resource constraints are equal to:

$$\forall i, j \in \mathcal{I}, \quad C_i = \left(\frac{p_i}{E} \right)^{-\sigma} \gamma_i U^{\epsilon_i (1-\sigma)^2}, \quad C_j = \left(\frac{p_j}{E} \right)^{-\sigma} \gamma_j U^{\epsilon_j (1-\sigma)^2}$$

By substituting this demand into $\frac{\partial \mathcal{L}}{\partial \Upsilon} = 1 - \sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0$
We end up with:

$$\frac{\partial \mathcal{L}}{\partial \Upsilon} = 1 - \sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{\left(\frac{p_i}{E} \right)^{-\sigma} \gamma_i}{U^{(1-\sigma)\sigma\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0$$

We know $\gamma_i, p_i, E, \sigma, \epsilon_i, \forall i \in \mathcal{I}$, then we can solve for U by having the optimal demand given endowment level.

The system of equation will have the following form:

$$
\begin{cases}
\frac{\partial \mathcal{L}}{\partial U} = 1 - \Upsilon \frac{(1-\sigma)^2}{\sigma} \sum_{i=1}^{I} \gamma_i^{\frac{1}{\sigma}} \epsilon_i \frac{1}{U} \left[\frac{C_i}{U^{(1-\sigma)\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0 \\
\forall i, j \in \mathcal{I}, \quad C_i = \left(\frac{p_i}{E} \right)^{-\sigma} \gamma_i U^{\epsilon_i (1-\sigma)^2}, \quad C_j = \left(\frac{p_j}{E} \right)^{-\sigma} \gamma_j U^{\epsilon_j (1-\sigma)^2} \\
\frac{\partial \mathcal{L}}{\partial \Upsilon} = 1 - \sum_i^I \gamma_i^{\frac{1}{\sigma}} \left[\frac{\left(\frac{p_i}{E} \right)^{-\sigma} \gamma_i}{U^{(1-\sigma)\sigma\epsilon_i}} \right]^{\frac{\sigma-1}{\sigma}} = 0 \\
Solve \quad for \quad U
\end{cases}
\tag{3}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial U} &= 1-\Upsilon\frac{(1-\sigma)^2}{\sigma}\sum_{i=1}^{I}\gamma_i^{\frac{1}{\sigma}}\epsilon_i\frac{1}{U(C_1,...,C_I)}\left[\frac{C_i}{U(C_1,...,C_I)^{(1-\sigma)\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}}=0\\
\frac{\partial \mathcal{L}}{\partial C_i} &= U_i' + \Upsilon\frac{1-\sigma}{\sigma}\left[\gamma_i^{\frac{1}{\sigma}}\frac{1}{C_i}\left(\frac{C_i}{U^{(1-\sigma)\epsilon_i}}\right)^{\frac{\sigma-1}{\sigma}}\right]\\
&\quad -\Upsilon\frac{(1-\sigma)^2}{\sigma}\left[\sum_{i=1}^{I}\gamma_j^{\frac{1}{\sigma}}\epsilon_j\left(\frac{U_j'}{U}\right)\left[\frac{C_j}{U^{(1-\sigma)\epsilon_j}}\right]^{\frac{\sigma-1}{\sigma}}\right]-\lambda p_i=0\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= E-\sum_i^I p_iC_i=0\\
\frac{\partial \mathcal{L}}{\partial \Upsilon} &= 1-\sum_i^I\gamma_i^{\frac{1}{\sigma}}\left[\frac{C_i}{U(C_1,...,C_I)^{(1-\sigma)\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}}=0\\
&\Upsilon\left[1-\sum_i^I\gamma_i^{\frac{1}{\sigma}}\left[\frac{C_i}{U(C_1,...,C_I)^{(1-\sigma)\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}}\right]=0\\
&\lambda\left[E-\sum_i^I p_iC_i\right]=0
\end{aligned}
\tag{4}
$$

Υ has to be positive to hold. Then we have $\left[1-\sum_i^I\gamma_i^{\frac{1}{\sigma}}\left[\frac{C_i}{U(C_1,...,C_I)^{(1-\sigma)\epsilon_i}}\right]^{\frac{\sigma-1}{\sigma}}\right]=0$

$$
\frac{U_i'}{U_j'}=\frac{-\Upsilon\frac{1-\sigma}{\sigma}\left[\gamma_i^{\frac{1}{\sigma}}\frac{1}{C_i}\left(\frac{C_i}{U^{(1-\sigma)\epsilon_i}}\right)^{\frac{\sigma-1}{\sigma}}\right]+\Upsilon\frac{(1-\sigma)^2}{\sigma}\left[\sum_{k=1}^{I}\gamma_k^{\frac{1}{\sigma}}\epsilon_k\left(\frac{U_k'}{U}\right)\left[\frac{C_k}{U^{(1-\sigma)\epsilon_k}}\right]^{\frac{\sigma-1}{\sigma}}\right]-\lambda p_i}{-\Upsilon\frac{1-\sigma}{\sigma}\left[\gamma_j^{\frac{1}{\sigma}}\frac{1}{C_j}\left(\frac{C_j}{U^{(1-\sigma)\epsilon_i}}\right)^{\frac{\sigma-1}{\sigma}}\right]+\Upsilon\frac{(1-\sigma)^2}{\sigma}\left[\sum_{k=1}^{I}\gamma_k^{\frac{1}{\sigma}}\epsilon_k\left(\frac{U_k'}{U}\right)\left[\frac{C_k}{U^{(1-\sigma)\epsilon_k}}\right]^{\frac{\sigma-1}{\sigma}}\right]-\lambda p_j}
$$