How Large Is Too Large? A Risk-Benefit Framework for Quantitative Easing

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Abstract

This work proposes a framework to study the risk-benefit trade-off of quantitative easing (QE) for the consolidated government, integrating the central bank and treasury department. In a simple model with distortionary taxes, nominal frictions, and a zero lower bound, we characterize the optimal size of a QE program as equalizing the marginal benefit from stimulating output to the marginal cost of induced rollover risk for taxpayers. A conservative quantification of this trade-off suggests that QE programs in the US made a positive net present contribution to welfare.

Keywords: Large Scale Asset Purchase Programs, Central Bank Losses, Rollover Risk, Interest Rate Risk, Optimal Maturity of Government Debt

JEL Classifications: E5, E58, E6, E63, G10, G12

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1 Introduction

Since the Great Financial Crisis, many central banks have engaged in large-scale asset purchase programs, also called quantitative easing (QE). These policies entail purchasing large amounts of long-maturity government debt financed by short-term interest-bearing reserves. As a result of this maturity mismatch, central banks have accumulated exposure to interest rate risk. When interest rates increase, the cash flow of long-term bond assets remains constant while the interest the central bank pays on its reserves liabilities increases, resulting in net operational losses.[1] Following the recent hikes in interest rates over 2021 to 2023, that exposure turned into significant losses for central banks, with the Federal Reserve (Fed) reporting a mark-to-market loss of 4 pp of nominal GDP over 2022.[2] Although economically significant, the interest rate risk exposure is typically absent from academic and policy discussions of the opportunity and design of QE programs.

This paper aims to fill this gap by proposing a framework to study the optimal size of a QE program in the presence of rollover risk. Our model features distortionary taxes, nominal rigidity, and a zero lower bound (ZLB), as well as two types of agents: bondholders and hand-to-mouth households without access to financial markets. Outside the ZLB, the government chooses its consolidated debt maturity to fully hedge against interest risk and perfectly smooth taxes across time and states, as in the baseline model of Greenwood, Hanson, and Stein (2015b). This result is overturned once the economy reaches the ZLB and conventional monetary policy becomes ineffective in preventing the economy from entering a demand recession (Caballero and Simsek, 2020, 2021). In such circumstances, it may be optimal for the government to engage in a QE program that consists of a shortening of the consolidated debt maturity to prop up aggregate demand by redistributing risk to hand-to-mouth households. Such a policy is, however, costly because it requires the government to take interest rate rollover risk and deviate from the full tax-smoothing solution.

We characterize the optimal size of a program at the ZLB as an interior solution that equalizes the marginal cost of rollover risk and the marginal benefit of output stimulation. We then quantify this trade-off using granular holdings data from the Fed. Separately for each US QE program, we compute the increase in expected tax deadweight losses for taxpayers induced by QE using a standard affine term structure model. This exercise yields a cumulative risk-neutral cost estimate of 0.25% of GDP (USD 70 bn.) and a conservative upper bound of 0.69% of GDP (USD 195 bn.) across the five programs. We

[1] The net present value of those operating losses is also directly measured by the market value of long-term government debt declines reflecting changes in the term structure of interest rates.

[2] Authors' calculation.

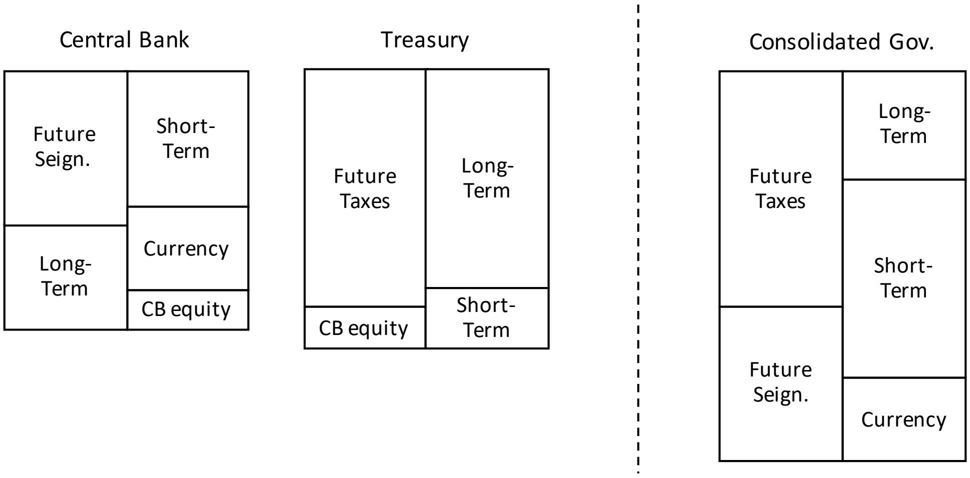


Figure 1: Sketch of Consolidated Government Balance Sheets. The figure sketches the balance sheet of the government in our model, both unconsolidated (left side) and consolidated (right side). The treasury issues both long-term and short-term debt against agents' future tax liabilities and central bank equity. The central bank issues currency and short-term debt, holds long-term government debt, and generates seigniorage revenues from currency.

compare those figures with the gains in output generated by QE policies, as estimated for specific programs by the previous literature of 3.36% of GDP (USD 946.85 bn.) cumulatively across the five programs when considering all studies and 1.19% of GDP (USD 335.34 bn.) when excluding articles written by central bank researchers (Fabo, Jančoková, Kempf, and Pástor, 2021). Our analysis, therefore, suggests that the Fed's QE programs made a positive net present contribution to welfare at origination.

The paper takes a consolidated government perspective, as illustrated in Figure 1. When consolidating the treasury and central bank, any intra-governmental position—such as long-term government bonds purchased by the central bank through its QE program or the central bank equity held by the government—cancels out. In particular, when the central bank engages in QE policy by purchasing long-term government debt financed by issuing short-term central bank debt (i.e., reserves), it results in a net substitution of long-term for short-term debt. This policy, therefore, results in a net shortening of the consolidated government debt and increased exposure to interest rate hikes. This consolidated view highlights the equivalence between the rollover risk governments face when issuing short-term debt and the mark-to-market return of the central bank QE portfolio. Under this equivalence, our model defines QE policy as a shortening of the consolidated government debt maturity for the purpose of aggregate demand stimulation at the ZLB and remains agnostic regarding its implementation.[3]

[3]Although our primary empirical application will be the evaluation of QE programs from the Fed buying long-term government debt by issuing (short-term) reserves, a similar maturity shortening could

In our model, QE operates through a "duration risk extraction" channel, similar to previous work (Ray, 2019; Caballero and Simsek, 2020; Vayanos and Vila, 2021; Caballero and Simsek, 2021). By buying long-term assets, the central bank reduces the interest risk bondholders have to bear in equilibrium, which generates a decline in long-term rates and a boost in aggregate demand. In the aforementioned studies, however, risk is not only transferred to the central bank but also assumed to disappear once on the central bank's balance sheet. Although this assumption is justifiable for the focus of those studies, it makes these models unsuitable for studying the trade-off central banks face when considering a QE program. Our model deviates by explicitly modeling interest rate risk as a systematic risk factor that has to be borne by some agent in equilibrium. In our model, a QE policy that shortens the consolidated government debt maturity results in a redistribution of interest rate risk from Ricardian bondholders to hand-to-mouth households through their tax liabilities. As a consequence of the policy, aggregate demand from bondholders increases because their precautionary savings motives are reduced. Importantly, this increase in demand from bondholders is not compensated by a reduction in demand from non-Ricardian households because, by assumption, they continue to consume the maximum allowed by their constraints.

When the economy is at the ZLB and conventional monetary policy cannot provide further macroeconomic accommodation, the consolidated government can still use QE policy to stimulate aggregate demand and output. These gains, however, come at the cost of additional rollover risk. Because QE policy shortens the government debt maturity below full tax smoothing, the government has to roll over its debt at an uncertain interest rate. The convexity of distortionary taxes implies that the low deadweight losses from low interest rate states do not compensate the high tax deadweight losses from high interest rate states. Therefore, the expected deadweight losses from taxation are increasing in the size of a QE program. From this trade-off, we derive the optimal size of a QE program as the maturity shortening that equalizes the marginal benefits from additional output to the marginal cost of additional expected deadweight losses.

We also show in an extension of our model that this trade-off is not affected by the presence of non-interest-bearing currency. In doing so, our framework provides further insight into recent controversies regarding the economic nature of central bank losses and how those losses could be mitigated. In particular, De Grauwe and Ji (2023) suggest that the central bank could avoid those losses altogether by stopping paying interest rates on infra-marginal reserves. In our model, however, this proposal would render

be implemented by the Treasury Department by increasing the share of T-bills relative to T-bonds, an operation that some economic commentators have referred to as "stealth QE" or "activist Treasury issuance" (Roubini and Miran, 2024).

QE ineffective for boosting aggregate demand because non-interest-bearing reserves are effectively long-term assets. Thus, when QE consists of swapping one long-term asset for another, bondholders' exposure to interest rate risk is unchanged, as are long-term interest rates and aggregate demand. Moreover, our model shows that the presence of countercyclical central bank seigniorage revenues does not affect the calculation of marginal rollover risk once the consolidated government is considered.

We then evaluate this trade-off quantitatively for the five QE programs implemented in the US since 2008 using granular data on the Fed's holdings. Applying term structure forecasts at the time of the Fed's interventions and guided by our model, we measure the change in expected tax deadweight losses brought about by the different programs. We find that, cumulatively, the five programs resulted in an expected deadweight loss of 0.25% of GDP under a risk-neutral assumption and of 0.69% of GDP for a conservative estimate allowing for risk-aversion. Those figures appear to be smaller than the average output effect across those programs as surveyed by Fabo, Jančoková, Kempf, and Pástor (2021) both when all published articles are taken into account (3.36% of GDP) and when excluding articles written by researchers at central banking institutions (1.19% of GDP). Lastly, we conduct a scenario analysis by comparing the output gain from QE with the hypothetical economic loss from QE for a series of adverse scenarios. We find that around 81% of interest rate paths result in a net positive contribution to welfare, including the realized one. Taken together, those estimates suggest that QE programs, as enacted by the Fed, had a positive net present value at their origination. This analysis, however, does not account for additional potential costs of QE that are not directly related to interest rate risk, such as how these programs may affect the central bank's independence or the role of the US dollar as the reserve currency.

Related Literature Our paper relates to a section of the literature that studies the effect of QE policies through risk extraction (Greenwood and Vayanos, 2014; Vayanos and Vila, 2021; Ray, 2019; Caballero and Simsek, 2020, 2021). Our work differs by treating interest rate risk as a systematic risk factor that is redistributed to taxpayers in general equilibrium. In this regard, our work is particularly related to Silva (2016), who studies how QE may optimally affect risk sharing in a model with financial market segmentation but does not feature distortionary taxes. Instead, we focus on the role of QE in propping up aggregate demand at the cost of taxation deadweight losses. In a New Keynesian setting with segmented markets but no tax distortion, Abadi (2023) finds that the optimal policy uses both interest rate cuts and asset purchases to stabilize asset prices during downturns. In a New-Keynesian setting with tax distortion but no market segmentation, Bhattarai, Eggertsson, and Gafarov (2022) study how future state-contingent losses from the central bank from QE policy act as a commitment device to keep interest rates lower for longer. Instead, we study the general equilibrium fiscal consequences when the central bank does not modify its policy rule based on those losses.

Our paper also relates to a long literature on the optimal maturity of government debt (Barro, 1979; Lucas Jr. and Stokey, 1983; Bohn, 1990). Angeletos (2002) and Buera and Nicolini (2004) study how the maturity of government debt can be used in the absence of state-contingent securities to smooth taxes and reduce the negative effect of convex distortionary tax costs. Greenwood, Hanson, and Stein (2015b) further study the optimal maturity of government debt in a setting where convex distortionary tax costs create a tax-smoothing motive interacting with agents' preference for short-term assets. They find that the moneyness of short-term debt and its ability to crowd out risky liquidity transformation implies that the government should deviate from perfect tax smoothing in the direction of shorter duration. Our model builds on theirs by adding risk aversion and a production economy constrained at the ZLB to study the trade-off between output stabilization and rollover risk. Greenwood, Hanson, Rudolph, and Summers (2015a) also describes a similar trade-off without a formal model and quantitative analysis.

Lastly, our paper relates to previous work on how treasury and central bank risk exposure affect their ability to achieve macroeconomic objectives. Corhay, Kind, Kung, and Morales (2023) finds that maturity operations have sizable effects on expected inflation and output through a risk transmission mechanism. Hall and Reis (2015), Del Negro and Sims (2015), Reis (2015b) and Reis (2015a) study how an insolvent central bank may cause the institution to lose its control over inflation. Christensen, Lopez, and Rudebusch (2015) propose a stress-test methodology to assess the risk of central bank insolvency following the first two QE programs. Instead, we assume that monetary dominance is always maintained and estimate the expected deadweight losses from additional tax dispersion across states induced by QE policy.

2 A Stylized 3-period Model

In this section, we present a simple 3-period model to study the key trade-offs that determine the optimal maturity of government debt in the presence of a ZLB and tax distortions. We start from the baseline scenario in which the government determines the optimal maturity of its debt to perfectly smooth taxes. We then introduce a binding ZLB, which pushes the government to optimally reduce its debt maturity below the full-tax-smoothing benchmark, causing it to be exposed to rollover risk. Although our main interpretation and empirical application for this maturity tilt is a QE program from the central bank, the model adopts a consolidated government perspective and remains agnostic about its implementation. We derive the optimal size of a QE policy at the ZLB and show that it is robust to the presence of zero-interest currency. All proofs are relegated to Appendix A.

2.1 Environment

Consider an economy with 3 periods, $t \in \{0, 1, 2\}$, heterogeneous agents, one consumption good, and one factor of production. Let $s \in \mathcal{S}$ be the state of the economy that determines capital's period-1 productivity $a_1(s)$ with probability $\pi(s)$. This uncertainty is resolved in period 1 and is the only source of risk in the economy. Productivity in period 0 and period 2, a_0 and a_2, are assumed to be constant. Potential output per unit of capital equals capital's productivity a_t, but actual output per unit of capital y_t can fall below due to a shortage of aggregate demand when the ZLB is binding. The model is populated by bondholders, households, and a government that funds its initial spending with government debt and taxes. Bondholders are split between domestic and foreign bondholders. Domestic bondholders and households receive consumption goods in each period from their capital holdings, but only the former choose consumption and government bond holdings. Households are hand-to-mouth—that is, they consume all goods net of taxes in each period.

Demography We normalize the total population in the economy to 1, in which θ are domestic bondholders and $1 - \theta$ are households. There are also foreign bondholders that hold a fixed fraction ϕ of the bonds. Each domestic bondholder and household holds 1 unit of capital used to produce intermediate goods by intermediate firms they own. We denote variables related to households, domestic bondholders, and foreign bondholders with superscripts h, b, and f, respectively. When referring to the value in a variable at a specific time period, we write it as a function of the state s of the economy whenever it is dependent on s. For example, we write $a_0, a_1(s), a_2$ when referring to productivity in periods 0, 1, and 2, respectively. However, for ease of notation, we write a_t when referring to productivity in any period t.

Preferences Domestic bondholders have CRRA utility over consumption c_t^b with risk aversion γ and time discount rate β:

$$V_0^b = \mathbb{E}_0 \left[\sum_{t=0}^{2} \beta^t \frac{(c_t^b)^{1-\gamma}}{1-\gamma} \right]. \tag{1}$$

Government The government spends G_0 in period 0 and does not spend in periods 1 and 2. To fund the initial spending in period 0, it can raise taxes τ_t and issue short-term bonds B_t^S and long-term bonds B_t^L. For expositional simplicity, we assume that taxes are raised only from households, whereas bonds are only held by bondholders.[4] Thus, taxes to each household are given by $\tau_t^h = \tau_t/(1-\theta)$.

Tax Distortions Taxes incur deadweight welfare losses of $\alpha(\tau_t)^2/2$, where the parameter α controls the magnitude of these losses. Therefore, the total deadweight loss in the economy in period t is given by $\alpha(\tau_t)^2/2(1-\theta)$.

Government's Objective The government maximizes the total sum of aggregate output net of tax deadweight losses discounted by the stochastic discount factor of domestic bondholders.[5] In each period t, the government maximizes

$$V_t^g = E_t\left[\sum_{u=t}^{2} \frac{\Lambda_u}{\Lambda_t}\left(y_u - \frac{\alpha}{2}\frac{(\tau_u)^2}{(1-\theta)}\right)\right], \tag{2}$$

where Λ_t is the stochastic discount factor of domestic bondholders and is taken as given by the government.

Sticky Prices Homogeneous firms in the final good production sector take prices as given, buy intermediate goods $x_t(i)$ from intermediate firms i, and produce the final good y_t as demanded according to a CES technology:

$$y_t = \left(\int_i x_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\epsilon-1}}, \tag{3}$$

for some elasticity of substitution $\varepsilon > 1$. Thus, given (3), final good producers solve

$$\max_{x_t(i)} P_t y_t - \int_\nu p_t(i)x_t(i)d\nu,$$

where P_t is the price of the aggregate good and $p_t(i)$ the price of the intermediate good i. To allow output to fall below potential, we follow Caballero and Simsek (2020) and assume that intermediate firms face fixed nominal prices. Hence, the problem of each

[4] In Online Appendix OA.1, we show that all our results remain valid as long as domestic bondholders hold a share of bonds larger than their tax incidence.

[5] As discussed in 2.3 below, this assumption is made to abstract from market-completion motives.

individual intermediate firm $i \in [0,1]$ is given by

$$\max_{0 \leq \eta_t(i) \leq 1} p_t(i) a_t \eta_t(i), \quad \text{s.t.} \quad a_t \eta_t(i) \leq x_t(i),$$

where $\eta_t(i)$ is the capital utilization rate. Because of nominal frictions, intermediate firms cannot adjust the price of their product. The solution to their maximization problem characterizes the aggregate capital utilization rate:

$$\eta_t = \min\left\{\frac{y_t}{a_t}, 1\right\}. \tag{4}$$

Following Keynesian logic, a demand-driven recession is possible when demand for the final good y_t in the economy is lower than output capacity a_t per unit of capital.

2.2 Agents' Maximization Problem

Domestic Bondholders At time 0, domestic bondholders maximize their lifetime expected value:

$$\max_{c_0^b, c_1^b(s), c_2^b(s), B_0^{S,b}, B_0^{L,b}, B_1^{S,b}(s)} \mathbb{E}_0 \left[\sum_{t=0}^{2} \beta^t \frac{(c_t^b)^{1-\gamma}}{1-\gamma}\right], \tag{5}$$

subject to the budget constraint in three periods:

$$c_0^b = y_0 - B_0^{S,b} p_0^S - B_0^{L,b} p_0^L, \tag{6}$$

$$c_1^b(s) = y_1(s) - B_1^{S,b}(s) p_1^S(s) + B_0^{S,b}, \tag{7}$$

$$c_2^b(s) = y_2(s) + B_1^{S,b}(s) + B_0^{L,b}. \tag{8}$$

Bondholders have access to the saving technology provided by the government: short-term bonds $B_t^{S,b}$ with price p_t^S in period $t \in \{0,1\}$ and long-term bonds $B_t^{L,b}$ with price p_t^L in period $t = 0$.

Foreign Bondholders Foreign bondholders purchase bonds, with consumption goods produced abroad, in proportion ϕ of the total supply. Thus, consumption by foreign bondholders c_t^f in each period is given by

$$c_0^f = -\phi(p_0^S B_0^S + p_0^L B_0^L), \tag{9}$$

$$c_1^f(s) = \phi(B_0^S - p_1^S(s) B_1^S(s)), \tag{10}$$

$$c_2^f(s) = \phi(B_1^S(s) + B_0^L). \tag{11}$$

Households Hand-to-mouth households pay taxes $\tau_t/(1-\theta)$ and consume their income net of taxes and tax deadweight losses:

$$c_0^h = y_0 - \frac{\tau_0}{1-\theta} - \frac{\alpha}{2}\left(\frac{\tau_0}{1-\theta}\right)^2, \tag{12}$$

$$c_1^h(s) = y_1(s) - \frac{\tau_1(s)}{1-\theta} - \frac{\alpha}{2}\left(\frac{\tau_1(s)}{1-\theta}\right)^2, \tag{13}$$

$$c_2^h(s) = y_2(s) - \frac{\tau_2(s)}{1-\theta} - \frac{\alpha}{2}\left(\frac{\tau_2(s)}{1-\theta}\right)^2. \tag{14}$$

Tax and Debt Policies Given its initial spending G_0, the government chooses taxes $\{\tau_0, \tau_1(s), \tau_2(s)\}$ and bonds $\{B_0^S, B_0^L, B_1^S(s)\}$ to maximize its objective function while satisfying its budget constraint. Importantly, we assume that the government takes bond prices as given, which allows us to abstract from the government's cost reduction motive.[6] Also, the government cannot commit to a prespecified policy rule.[7] Thus, in period 0, the government's problem is given by

$$\max_{\tau_0, B_0^S, B_0^L} \mathbb{E}_0 \left[\sum_{t=0}^{2} \frac{\Lambda_t}{\Lambda_0}\left(y_t - \frac{\alpha}{2}\frac{\tau_t^2}{1-\theta}\right)\right], \tag{15}$$

subject to the budget constraint:

$$G_0 = \tau_0 + B_0^S p_0^S + B_0^L p_0^L. \tag{16}$$

In period 1, the government solves

$$\max_{\tau_1(s), B_1^S(s)} \sum_{t=1}^{2} \frac{\Lambda_t}{\Lambda_1}\left(y_t - \frac{\alpha}{2}\frac{\tau_t^2}{1-\theta}\right), \tag{17}$$

[6]Since the government issues bonds to fund its spending, it has incentives to manipulate bond prices to increase its revenue and reduce the level of taxes. When a larger short-term bond share reduces the term premium, and the consolidated government's asset duration is larger than its liability duration, the government has the additional incentive to issue more short-term bonds.

[7]The absence of commitment, which follows from Greenwood et al. (2015b) and Bhattarai et al. (2022), corresponds to a more realistic assumption and yields a simpler solution.

subject to the budget constraint:

$$0 = \tau_1(s) + B_1^S(s)p_1^S(s) - B_0^S. \tag{18}$$

Finally, in period 2, taxes $\tau_2(s)$ are raised to close the budget: $0 = \tau_2(s) - B_1^S(s) - B_0^L$.

Conventional Monetary Policy The central bank sets short-term interest rates $\{p_0^S, p_1^S(s)\}$ in order to maximize output.[8] Thus, the central bank's problem is given by

$$\max_{p_t^S} \; y_t \qquad \text{for } t = 0, 1. \tag{19}$$

This assumption follows Caballero and Simsek (2020), where the short-term interest rate r_t^S is set such that output in the current period is maximized whenever possible. In particular, this target is not always admissible when the effective lower bound is binding.

2.3 Discussion of Assumptions

Market Incompleteness Some market incompleteness is required to break the Ricardian-Wallace neutrality (Wallace, 1981) so that a QE policy results in real economic effect. To capture the trade-off between tax smoothing and output stabilization in a tractable manner, we introduce this market incompleteness by assuming that households are hand-to-mouth.

Government's Objective Function Although our model features market incompleteness for the above reason, the paper's main focus is on the trade-off between tax smoothing and output stabilization. This market incompleteness, however, implies an additional motive for the government to dynamically adjust its risk exposure to allow households and bondholders to implicitly trade risk through government tax liabilities. To abstract from this additional motive, we assume that the government discounts aggregate consumption using bondholders' discount factor and refer to Silva (2016) for an analysis of optimal QE policy in the presence of such market-completeness motives from which we abstract.

[8] Note that allowing the government to set the interest rate instead of delegating this task to a central bank would yield a different solution since the optimal interest rate does not necessarily maximize output even if the ZLB is not binding. This discrepancy arises because the government is incentivized to reduce the debt burden by increasing the interest rate. In our analysis, we purposely abstract from the government's incentive to exploit monetary policy for pure fiscal cost reduction purposes because those have additional normative implications that are beyond the focus of this study.

Price-taking Government We assume the government acts as a price taker when deciding on its debt policy. We do so to abstract from tax-reduction motives (similarly to Greenwood, Hanson, and Stein (2015b)) that would otherwise incentivize the central bank to manipulate consumption patterns to reduce the government tax burden, which would have additional political economy implications beyond the focus of this article.

Tax Distribution We assume all taxes are collected from households, so bondholders are not exposed to tax risk. Part of the interest rate risk born by bondholders in equilibrium is then transferred to households when the government shortens the maturity of bonds. If some part of taxes is collected from bondholders instead, the risk redistribution effect becomes less effective. We relax this corner assumption in Appendix OA.1 and discuss the impact on the government's optimal policy. All our results remain valid as long as domestic bondholders' bond-holding share is larger than their tax incidence.

2.4 Equilibrium

We provide a definition for the sequential competitive equilibrium and derive first-order conditions.

Equilibrium Definition Given government spending and productivity processes $\{G_t, a_t : t \in \{0,1,2\}\}$, the sequential competitive equilibrium is a set of (i) long-term bond price p_0^L; (ii) decisions for domestic bondholders $\{c_0^b, c_1^b(s), c_2^b(s), B_0^{S,b}, B_0^{L,b}, B_1^{S,b}(s)\}$; (iii) consumption by households $\{c_0^h, c_1^h(s), c_2^h(s)\}$; (iv) consumption by foreign bondholders $\{c_0^f, c_1^f(s), c_2^f(s)\}$; (v) tax policies $\{\tau_0, \tau_1(s), \tau_2(s)\}$; (vi) debt policies $\{B_0^S, B_0^L, B_1^S(s)\}$; and (vii) conventional monetary policies $\{p_0^S, p_1^S(s)\}$ such that

(1) Domestic bondholders' decisions and the government's policies are solutions to their respective problems given long-term bond price (i);

(2) The short-term bond price in period 0: p_0^S is bounded above by 1;

(3) Markets for consumption goods, short-term bonds, and long-term bonds clear:

   (a) *consumption*: $\theta c_t^b + (1-\theta)c_t^h + c_t^f = y_t - G_t - \alpha\tau_t^2/2(1-\theta)$;

   (b) *short-term bonds*: $\theta B_t^{S,b} = (1-\phi)B_t^S$ for $t = 0, 1$;

   (c) *long-term bonds*: $\theta B_0^{L,b} = (1-\phi)B_0^L$.

Domestic Bondholders' First-Order Conditions The first-order conditions for domestic bondholders yield the Euler equations for bondholders:

$$p_0^S = \mathbb{E}_0\left[\frac{\Lambda_1(s)}{\Lambda_0}\right], \quad p_0^L = \mathbb{E}_0\left[\frac{\Lambda_2(s)}{\Lambda_0}\right], \quad p_1^S(s) = \frac{\Lambda_2(s)}{\Lambda_1(s)}, \tag{20}$$

where the stochastic discount factor is defined as $\Lambda_t \equiv \beta^t (c_t^b)^{-\gamma}$.

Since households are hand-to-mouth, their consumption in each period is equal to their endowment net of taxes:

$$c_t^h = y_t - \frac{\tau_t}{1-\theta} - \frac{\alpha}{2}\left(\frac{\tau_t}{1-\theta}\right)^2 \qquad \forall t = 0, 1, 2. \tag{21}$$

We then obtain domestic bondholders' consumption through their budget constraints and market-clearing conditions:

$$c_t^b = y_t - \frac{1-\phi}{\theta} G_t + \frac{1-\phi}{\theta}\tau_t \qquad \forall t = 0, 1, 2. \tag{22}$$

The central bank sets short-term rates such that output is maximized in period 1 when the ZLB is assumed not to be binding. It does so by following the Wicksellian prescription of setting the interest rate equal to the natural rate. Therefore, we have $y_1(s) = a_1(s)$ and we set $y_2 = a_2$ to pin down the equilibrium. Combining equations (20) and (22) and defining the short-term rate in period 0 as $r_0 \equiv 1/p_0^S - 1$, we get

$$r_0 = \max\left\{\left(\beta \mathbb{E}_0\left[\left(\frac{\theta a_1(s) + (1-\phi)\tau_1(s)}{\theta a_0 - (1-\phi)(G_0 - \tau_0)}\right)^{-\gamma}\right]\right)^{-1} - 1, 0\right\}, \tag{23}$$

where the max operator reflects that the ZLB is potentially binding. In period 1, we get

$$r_1(s) = \left(\beta\left(\frac{\theta a_2 + (1-\phi)\tau_2(s)}{\theta a_1(s) + (1-\phi)\tau_1(s)}\right)^{-\gamma}\right)^{-1} - 1, \tag{24}$$

since the ZLB can be binding only in period 0.

2.5 The Government's Problem

Period-1 Problem We solve the problem using backward induction and first characterize the solution in period 1. In period 1, because the ZLB is never binding by assumption, monetary policy can set the short-term rate such that output is always maximized ($y_1(s) = a_1(s)$), and the government minimizes the present value of tax deadweight losses. Given the optimality condition in (20) and the government's budget constraint, the solution to the government's problem (17) is characterized by tax smoothing:

$$\tau_1(s) = \tau_2(s) = \frac{B_0^S + p_1^S(s)B_0^L}{1 + p_1^S(s)} \tag{25}$$

and the corresponding short-term bond issuance:

$$B_1^S(s) = \frac{B_0^S - B_0^L}{1 + p_1^S(s)}. \tag{26}$$

To minimize deadweight losses from taxation, the government rolls over short-term bonds to smooth taxes across time. If the amounts of short- and long-term bonds issued in period 0 are not equal, short-term bond issuance in period 1 is not equal to 0 ($B_1^S(s) \neq 0$), the government is exposed to rollover risk due to its new position in short-term bonds, and taxes become risky across states. Because of the convexity of the tax cost function, this tax dispersion across states results in higher expected deadweight losses in period zero, which the government aims to minimize.

Period-0 Problem In Proposition 1, we characterize the optimal government policy when the ZLB is not binding.

Proposition 1 (Optimal Government Policy without the ZLB). *In period 0, if the ZLB is not binding, the government achieves first-best: Output is maximized, and tax deadweight losses are minimized. The optimal policy matches the duration of bonds with the duration of taxes as follows:*

(a) Tax plan: $\tau_0 = \tau_1(s) = \tau_2(s) = G_0/(1 + p_0^S + p_0^L)$;

(b) Bond issuance: $B_0^S = B_0^L = G_0/(1 + p_0^S + p_0^L)$.

When the ZLB is not binding, as in period 1, the only consideration determining the optimal bond issuance scheme is minimizing quadratic tax deadweight losses. Thus, the government smooths taxes across periods and states by matching bond and tax duration to hedge against interest rate risk so that it never issues short-term bonds in period 1—that is, $B_1^S(s) = 0$. This result is a generalization of the benchmark solution by Greenwood, Hanson, and Stein (2015b) to a setting with risk-averse agents.

Period-0 Reformulation To ease the exposition of our results, we use the solution to the period-1 problem and reformulate the government problem in period 0 in terms of debt level and maturity. To do so, we define the short-term bond share in period 0 as the proportion of short-term bond value, $S = p_0^S B_0^S / D$, where D is the total value of debt outstanding in period 0, $D = p_0^S B_0^S + p_0^L B_0^L$. The short-term bond share S captures the interest rate risk embedded in the government's bond position. We denote $S^\star = p_0^S/(p_0^S + p_0^L)$, the share of short-term debt that implements the full tax smoothing solution from Proposition 1. We also further define $R_1^c(s) = p_1^S(s)/p_0^L - 1/p_0^S$ as the return on a borrow-short-invest-long (carry) position. The following lemma reformulates the government's period-0 problem according to those variable redefinitions:

Lemma 1. *The government's period-0 problem can be rewritten as:*

$$\max_{S,D} \left\{ y_0 - \frac{\alpha}{2(1-\theta)} \left((G_0 - D)^2 + D^2 \left(m\,(S - S^\star)^2 + \frac{1}{p_0^S + p_0^L} \right) \right) \right\}, \tag{27}$$

where $m = \mathbb{E}_0 \left[\frac{\Lambda_1(s)}{\Lambda_0(1+p_1^S(s))} (R_1^c(s))^2 \right]$.

In Lemma 1 above, m corresponds to the discounted variance of the bond market carry trade return, which affects the magnitude of rollover risk assumed by the government.

2.6 Demand Recession at the ZLB

When the ZLB is binding in period 0, conventional monetary policy is constrained. Lemma 2 shows that a demand recession is possible, and debt policies can stimulate the economy. When the ZLB is binding and policy changes, output y_0 must also change to satisfy the short-term rate pricing equation (23) and bondholders' intertemporal consumption-saving trade-off. Therefore, debt policy may affect output during a demand recession.

Lemma 2 (Demand Recession at the ZLB). *If the ZLB is binding, output is given by*

$$\begin{aligned} y_0 = {} & \frac{1-\phi}{\theta} D + \left(\beta \mathbb{E}_0 \left[\left(a_1(s) + \frac{1-\phi}{\theta} D \left[\frac{R_1^c(s)}{1+p_1^S(s)} (S^\star - S) + \frac{1}{p_0^S + p_0^L} \right] \right)^{-\gamma} \right] \right)^{-\frac{1}{\gamma}} \\ & \leq a_0. \end{aligned} \tag{28}$$

Lemma 3 characterizes how a change in the share of short-term debt S affects aggregate output y_0 through agents' consumption demand.

Lemma 3 (Output Effect). *If the ZLB is binding, the impact of changing debt maturity S on output y_0 is given by*

$$\frac{\partial y_0}{\partial S} = \frac{1-\phi}{\theta} D \mathbb{E}_0 \left[\beta \left(\frac{c_1^b(s)}{c_0^b} \right)^{-(1+\gamma)} \frac{-R_1^c(s)}{1+p_1^S(s)} \right]. \tag{29}$$

We can further decompose (29) into

$$\frac{\partial y_0}{\partial S} = \frac{1-\phi}{\theta} D \Bigg(\underbrace{\mathbb{E}_0 \left[\beta \left(\frac{c_1^b(s)}{c_0^b} \right)^{-(1+\gamma)} \right] \mathbb{E}_0 \left[\frac{-R_1^c(s)}{1+p_1^S(s)} \right]}_{\text{intertemporal substitution}} + \underbrace{\mathrm{Cov}_0 \left[\beta \left(\frac{c_1^b(s)}{c_0^b} \right)^{-(1+\gamma)}, \frac{-R_1^c(s)}{1+p_1^S(s)} \right]}_{\text{precautionary saving}} \Bigg). \tag{30}$$

The output effect of increasing short-term bond share S is composed of both intertemporal-substitution and precautionary-saving components.[9] Since the two terms might have opposite effects, the sign of the partial derivative (29) is ambiguous.

To gain intuition about the sign determination, consider a set of parameters such that $\mathbb{E}_0[R_1^c(s)/(1+p_1^S(s))] > 0$; that is, the discounted expected return of investing in short-term bonds is lower than that in long-term bonds and the intertemporal-substitution component is negative. This would correspond, for instance, to a case in which long-term bonds are risky assets (positive-beta), so they require a positive term premium. Under this assumption, reducing debt maturity (i.e., increasing S through QE policy) has two consequences.

First, the shortening of debt maturity lowers the interest rate risk taken by bondholders and transfers it to hand-to-mouth households. Because bondholders are paying a lower proportion of taxes than their holdings of bonds by assumption (respectively 100% vs. 0% in our baseline model), the maturity shift from QE makes their consumption less pro-cyclical (i.e., decreasing in bad states). This effect reduces their precautionary saving motives and boosts consumption demand in period 0. In the meantime, demand for consumption from households remains unaffected because those are assumed to be hand-to-mouth.

Second, due to reduced risk exposure, the expected return on the portfolio of bondholders, and therefore their expected consumption in period 1, is lowered. Consequently, bondholders' intertemporal-substitution motives are increased, which reduces their demand for consumption at time zero.

The overall direction of the output effect, therefore, depends on the relative magnitude of the two components, which depends on bondholders' relative risk-aversion coefficient γ and the interest rate risk exposure of long-term bonds. In Online Appendix OA.2, we provide a sufficient condition for the precautionary savings motive to dominate and the

[9] In this paper, we focus on QE policy, which is related to S in the model, and not on pure fiscal policy which would instead consist in changing total debt outstanding D.

output effect to be positive, in line with empirical studies on the real effects of QE (Fabo, Jančoková, Kempf, and Pástor, 2021). In what follows, we assume that this condition is met.[10]

2.7 Optimal Size of a QE Program at the ZLB

We now consider how being constrained by the ZLB alters the government's optimal maturity decision. By taking the first-order condition with respect to S in our reformulated problem Lemma 1, we get

$$\underbrace{\frac{\alpha m}{1-\theta}\left(S-S^{\star}\right)D}_{\text{rollover risk}} = \underbrace{\frac{\partial y_0}{\partial S}\frac{1}{D}}_{\text{output effect}}. \tag{31}$$

This optimality condition states that the government sets its debt maturity so that the marginal cost of deviating from the full tax-smoothing solution—and bearing rollover risk—equals the marginal benefit of increasing output. Specifically, the marginal cost it pays due to non-smoothing taxes depends on four factors: the parameter that determines the magnitude of the cost of tax distortions, α; the total value of debt outstanding, D; the magnitude of rollover risk, m; and finally the difference between the current short-term bond share and the short-term bond share in the perfect tax-smoothing solution, $S - S^{\star}$. The right-hand side represents the marginal benefit of boosting aggregate demand when the government shifts toward more short-term bonds, which we refer to as the output effect. When the ZLB is not binding, the solution reverts to Proposition 1 with full tax smoothing, $S = S^{\star}$, as the latter term is zero. Proposition 2 builds on this optimality condition and characterizes the optimal size of a QE program at the ZLB.

Proposition 2 (Optimal Size of QE at the ZLB). *If the ZLB is binding and $\partial y_0/\partial S > 0$, the optimal deviation from full tax smoothing (or, equivalently, the optimal size of a QE program) is given by*

$$Q^{\star} \equiv (S-S^{\star})D = \min\left\{\frac{1-\theta}{\alpha m}\frac{1}{D}\frac{\partial y_0}{\partial S}, \overline{Q}\right\}, \tag{32}$$

[10] This model is meant to provide a minimal set of assumptions that can generate and illustrate the trade-off we measure in Section 3 but falls short of capturing all the channels through which QE may affect aggregate output. For instance, in a HANK model (Kaplan, Moll, and Violante, 2018), the output effect would feature richer dynamics through general equilibrium wages and investments. QE can also increase the share of long-term investments in the economy and increase output if these investments are more productive and are underprovisioned at the ZLB (Hubert de Fraisse, 2024). Because our quantitative analysis takes estimates of the output effect of QE from previous empirical studies, none of our results depend on a specific channel.

where $\overline{Q}$ is the minimum QE program size that pushes the economy out of the ZLB.

Proposition 2 defines the optimal outstanding debt maturity swap in deviation from the perfect tax-smoothing solution $S^\star$. This definition follows from Proposition 1's result that the full tax-smoothing solution is optimal without a binding ZLB. The variable $Q^\star$, therefore, isolates the nominal value of short-term debt that is optimally issued for the specific purpose of alleviating demand recession at the ZLB while keeping total debt constant. From a consolidated view of the government's balance sheet, this operation corresponds to QE programs whereby the central bank purchases long-term debt from the treasury and issues central bank reserves paying a floating rate to finance the purchase. Although our model is agnostic regarding the implementation of the maturity shortening, we now refer to $Q^\star$ as the optimal QE program size.[11]

The optimal QE program size has several components. First, it depends negatively on the variable α scaling the tax distortion cost and m capturing the dispersion of interest rates along the yield curve, thereby scaling the marginal rollover risk taken by the government when deviating from $S^\star$. Second, it depends positively on the share of taxpaying households in the economy $(1-\theta)$ since a larger share allows quadratic tax costs to be spread across more agents. Third, it also depends positively on the marginal output effect $\partial y_0/\partial S$, which captures the efficiency of QE in propping up output. Lastly, the minimum operator follows from Proposition 1, in that any deviation from full tax smoothing harms the government's objective when the ZLB is not binding since conventional monetary policy is sufficient to ensure that output is at capacity. This operator, therefore, caps the optimal QE size to the minimum level $\overline{Q}$ that is required for the ZLB not to bind. In Section 3, we apply Proposition 2 to evaluate the five US QE programs quantitatively.

2.8 Zero-interest Liabilities

Until this point, we have referred generically to "short-term" debt without specifying whether this debt was issued as central bank reserves by the Fed or Treasury bills by the Treasury Department because the two assets represent the same risk exposure and, hence, require the same return in equilibrium. This assumption corresponds to the post-2008 US monetary policy implementation regime, according to which the Fed pays a floating interest rate on reserves. Motivated by two conjectures about how zero-interest central bank liabilities could mitigate the effect of central bank losses, this section relaxes this assumption. First, De Grauwe and Ji (2023) propose to stop paying interest on inframarginal reserves so that the central bank would not bear losses in high interest rates

[11] See footnote 3.

states. Second, because currency does not pay interest, seigniorage revenues of central banks increase in times of high interest rates. This implies that central banks have a natural hedge against interest rate risk that is similar to that of banks. For this reason, some central banks record their operational losses from their QE portfolio as "deferred assets" (Faria-e Castro and Jordan-Wood, 2023).

To formally study how zero-interest central bank liabilities affect QE policy, we extend our model in two directions. First, we study how the existence of outstanding currency (i.e., banknotes and coins), which also does not pay interest, affects the optimal size of a QE program. In this case, we find that the expression for the optimal QE size has the same form as the one derived in the previous section, once accounting for the extra seigniorage revenues generated by currency. Second, we study a counterfactual scenario in which the Fed would finance a QE program by issuing reserves that do not pay interest (as pre-2008) that bondholders (banks) are required to hold as advocated by De Grauwe and Ji (2023). Under this assumption, reserves are effectively long-term assets. Therefore, a QE program swapping these against other long-term assets does not generate any positive output effect.

Modified Setup Assume that bondholders must hold a certain amount of zero-interest currency, M.[12] We assume the amount of money outstanding in periods 0 and 1 is fixed at M and redeemed in period 2. The budget constraint of the government becomes

$$G_0 = \tau_0 + B_0^S p_0^S + B_0^L p_0^L + M, \tag{33}$$
$$0 = \tau_1(s) + B_1^S(s) p_1^S(s) - B_0^S, \tag{34}$$
$$0 = \tau_2(s) - B_1^S(s) - B_0^L - M. \tag{35}$$

The solution to the period-1 problem is then given by

$$\tau_1(s) = \tau_2(s) = \frac{B_0^S + p_1^S(s)(B_0^L + M)}{1 + p_1^S(s)}.$$

That is, it implements the full tax-smoothing solution across time, accounting for currency acting as a long-term asset. We can define $\widetilde{B}_0^L \equiv B_0^L + M$ and rewrite the period-0 budget constraint as

$$\widetilde{G}_0 = \tau_0 + B_0^S p_0^S + \widetilde{B}_0^L p_0^L \quad \text{where} \quad \widetilde{G}_0 \equiv G_0 - M(1 - p_0^L).$$

[12]This policy can be implemented through an increase in the reserve requirement for banks as advocated by De Grauwe and Ji (2023) or simply because currency is required for daily purchases and, hence, results in an unmodeled convenience yield.

Because currency does not require the government to pay interest, the government generates a seigniorage of which the present value is given by $M(1-p_0^L)$. Hence, $\widetilde{G}_0$ represents the government spending that remains to be financed after accounting for those seigniorage revenues.

QE not Financed by Zero-interest Currency We first consider how the optimal size of a QE program is altered in the presence of zero-interest currency that generates seigniorage revenues to the government. We define the effective short-term bond share in period 0 as $\widetilde{S} = p_0^S B_0^S / \widetilde{D}$, where $\widetilde{D}$ is the total effective value of debt outstanding in period 0, $\widetilde{D} = p_0^S B_0^S + p_0^L \widetilde{B}_0^L$.

Proposition 3 (Optimal QE Size at the ZLB with Currency). *If the ZLB is binding and $\partial y_0/\partial\widetilde{S} > 0$, the optimal deviation from full tax smoothing (or equivalently, the optimal size of a QE program) is given by*

$$\widetilde{Q}^\star = \min\left\{\frac{1-\theta}{\alpha m}\frac{1}{\widetilde{D}}\frac{\partial y_0}{\partial \widetilde{S}}, \overline{Q}\right\}, \tag{36}$$

where $\overline{Q}$ is the minimum QE program size that pushes the economy out of the ZLB.

Proposition 3 shows that once accounting for the additional seigniorage and the long-term nature of currency paying no interest, the optimal QE size is isomorphic to that of Proposition 2. In other words, seigniorage revenues are helpful in reducing the level of distortionary taxes and alter the implementation of the full tax smoothing solution but do not affect the optimal size of a QE program on the margin.

QE Financed by Zero-interest Reserves Second, we consider a case in which the government finances its QE program by issuing zero-interest reserves M to bondholders.[13]

Proposition 4 (Currency-Financed QE's Output Effect). *If the ZLB is binding and $p_0^L \leq 1$, the impact of funding QE with zero-interest reserves M instead of long-term debt B_0^L on output y_0 is given by*

$$\frac{\partial y_0}{\partial M} = \frac{1-\phi}{\theta}\mathbb{E}_0\left[\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-(1+\gamma)}\frac{-R_1^m(s)}{1+p_1^S(s)}\right] \leq 0, \tag{37}$$

[13]Note that we refer here to M as zero-interest *reserves*. In contrast, in previous sections, we referred to S_t as short-term debt while implicitly thinking about the change in this short-term debt in the case of QE as being driven by an increase in (interest-bearing) reserves. We did so because interest-bearing reserves and short-term government debt are equivalent so long as both are correctly priced. In this section, because bondholders are required to hold these zero-interest reserves, the interest on reserves is nil and differs from the interest rate on short-term assets.

where $R_1^m(s) = \frac{p_1^S(s)}{p_0^L} - p_1^S(s) \geq 0$ *is the return on a borrow-reserves-invest-long position.*

Proposition 4 shows that, unlike a regular QE program financed by interest-bearing reserves, the output effect of a QE program financed by zero-interest reserves is weakly negative. This result is the consequence of the two properties of zero-interest reserves: It is (i) effectively a long-term asset that (ii) generates seigniorage revenues to the government. To understand this result, we can decompose equation (37) into

$$\frac{\partial y_0}{\partial M} = \frac{1-\phi}{\theta} \left(\mathbb{E}_0 \left[\beta \left(\frac{c_1^b(s)}{c_0^b} \right)^{-(1+\gamma)} \frac{\frac{p_1^S(s)}{p_0^L} - \frac{p_1^S(s)}{p_0^L}}{1+p_1^S(s)} \right] + \mathbb{E}_0 \left[\beta \left(\frac{c_1^b(s)}{c_0^b} \right)^{-(1+\gamma)} \frac{-R_1^m(s)}{1+p_1^S(s)} \right] \right).$$

The first term is comparable to equation (29) with $-R_t^c$ being replaced by $p_1^S(s)/p_0^L - p_1^S(s)/p_0^L = 0$. Unlike when swapping short-term debt for long-term debt, the interest rate risk exposure of bondholders remains unchanged by a policy that swaps long-term debt for zero-interest reserves because such reserves are effectively also long-term debt. The second term captures that when bondholders have to hold additional zero-interest reserves, their expected future consumption is lower because part of their interest is taken away as seigniorage and redistributed to households. This additional redistribution effect reduces bondholders' demand for consumption in period 0. If this seigniorage was rebated to bondholders[14] and the transfers they receive do not incur deadweight losses, QE funded by zero-interest reserves would be perfectly neutral because it would simply amount to swapping two assets with the same interest rate risk exposure without any redistributive effect.

3 Quantification

In this section, we present a quantitative assessment of the trade-off behind QE programs from the consolidated government's perspective described in equation (36). We study the five QE programs conducted by the Federal Reserve: the first round of large-scale asset purchases during 2008-2010 (QE1); the second round of large-scale asset purchases between 2010 and 2011 (QE2); the Maturity Extension Program during 2011-2012 (MEP); the third round of large-scale asset purchases between 2012 and 2014 (QE3); and the most recent fourth round of large-scale asset purchases between 2020 and 2022 (QE4).

[14] This policy would be implemented by taxing households $MR_1^m(s)/(1+p_1^S(s))$ in periods 1 and 2 and transferring the proceeds to bondholders.

3.1 Data

We collect data on the Fed's purchases associated with each QE program from the system open market account (SOMA) portfolio holdings available on the Fed's website,[15] which reports the Fed's holdings in Treasury securities, agency debt, and mortgage-backed securities (MBS) at a weekly frequency. The maturity date and coupon information for each security is available for Treasury and agency debt but not for MBS holdings. To compute MBS duration, we make the conservative assumption that MBS holdings have the same maturity structure as Treasury and agency debt holdings in aggregate.[16]

We retrieve term structure data from Liu and Wu (2021), who construct and regularly update the U.S. Treasury yield curve at monthly frequency from 1961 to the present and have smaller pricing errors than the leading alternative data of Gürkaynak, Sack, and Wright (2007). The availability of long-term yield data increases over time. For dates before August 1971, only the 7-year yield or shorter are available. The 10-year yield, 15-year yield, and 20-year yield became available in August 1971, November 1971, and July 1981, respectively. The entire yield curve up to 30 years is only available starting in November 1985.

For historical tax rates, we use the time series of annual Federal Receipts over GDP from 1929 from Federal Reserve Economic Data (FRED) by the St. Louis Fed. We also retrieve the nominal GDP series from FRED.

3.2 History of QE Programs

This section provides a brief introduction to the history of QE programs in the U.S. Figure 2 summarizes the evolution of the Fed's bond portfolio size in terms of its 10-year duration equivalent over the last two decades. The 10-year duration equivalent is computed as the weighted sum of all coupon and principal payments of the portfolio, where each payment is weighted by its maturity relative to 10 years, capturing the overall interest rate risk exposure. The blue area depicts the 10-year duration equivalent values of Treasury and federal agency debt holdings, and the red area the 10-year duration equivalent values of MBS holdings. From the start of QE1 in November 2008 to the end of QE4 in March 2022, the 10-year duration equivalent of the Fed's bond portfolio increased by a factor of 17, from \$0.5 trillion to \$8.5 trillion.

In the aftermath of the 2007-2008 financial crisis, the Fed implemented QE1 to stabilize

[15]See https://www.newyorkfed.org/markets/soma-holdings.

[16]Over the period between November 2008 and March 2022, the average for the Bloomberg LP's measure of MBS duration is 4.58 years. Over the same period, the average for the Bloomberg LP's measure of Treasury duration is 5.9 years.

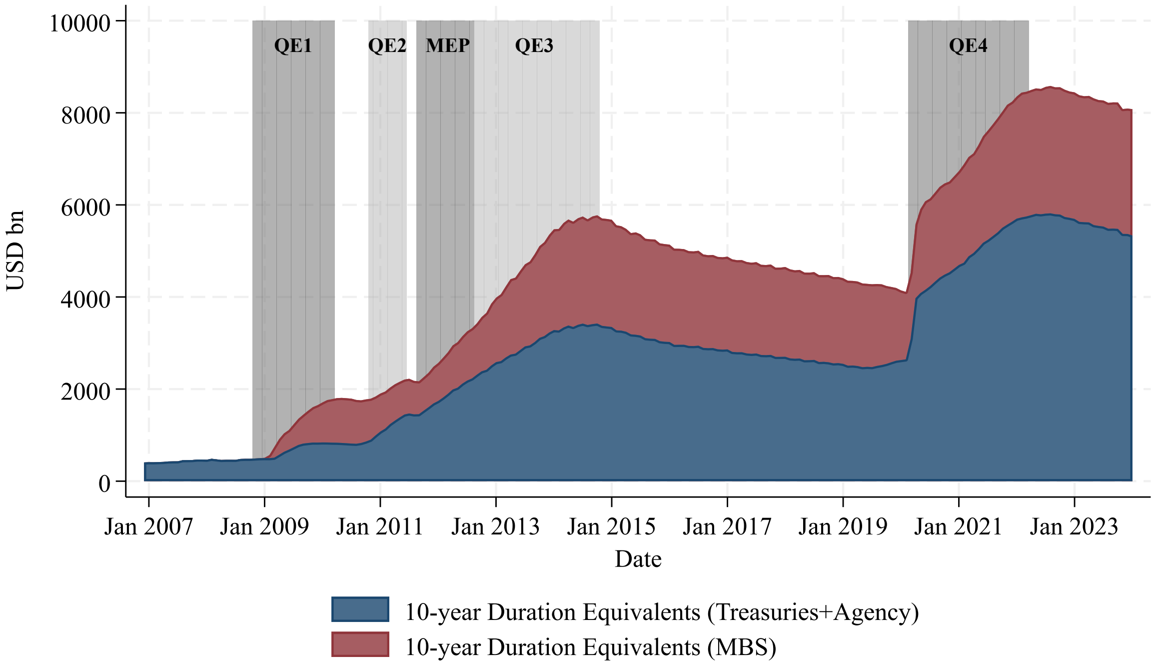


Figure 2: Size of the Federal Reserve's Bond Portfolio. This figure plots the evolution of the size of the Federal Reserve's bond portfolio measured in 10-year duration equivalents. The blue area depicts the 10-year duration equivalent values of Treasury and federal agency debt holdings, and the red area the 10-year duration equivalent values of MBS holdings. Shaded areas denote the periods of each specific asset purchase program.

financial markets, increase liquidity, and stimulate economic growth when conventional monetary policy tools were deemed insufficient. QE1 took place between November 2008 and March 2010 with the purchase of $175 billion in federal agency debt, $1.25 trillion in MBS, and $300 billion in long-term Treasury securities. During this period, the duration-adjusted size of the Federal Reserve's bond portfolio increased from $0.5 trillion to $1.8 trillion.

In November 2010, in response to concerns about low inflation and high unemployment, the Federal Reserve initiated QE2 to further stimulate the economy. From November 2010 to June 2011, the second round included $600 billion in long-term Treasury securities. During this period, the duration-adjusted size of the Fed's bond portfolio increased from $1.8 trillion to $2.2 trillion.

The MEP was launched in 2011 to further reduce longer-term interest rates without expanding the size of the central bank's balance sheet. From September 2011 through December 2012, the MEP included purchases of $667 billion in Treasury securities with remaining maturities of 6 years to 30 years, offset by sales of $634 billion in Treasury securities with remaining maturities of 3 years or less and $33 billion of Treasury security redemptions. From September 2011 through August 2012, the duration-adjusted size of the Fed's bond portfolio increased from \$2.2 trillion to \$3.3 trillion.[17]

The Fed launched QE3 in September 2012 as an open-ended program to reduce long-term interest rates, supporting the housing market's recovery and stimulating overall economic growth. In total, from September 2012 to October 2014, the Fed purchased \$790 billion in Treasury securities and \$823 billion in agency MBS. During this period, the duration-adjusted size of the Fed's bond portfolio increased from \$3.3 trillion to \$5.8 trillion.

In response to the economic repercussions of the COVID-19 pandemic, in March 2020, the Fed launched an unprecedented round of quantitative easing (QE4) to stabilize financial markets, lower long-term interest rates, and support economic recovery. From March 2020 to March 2022, cumulative purchases exceeded \$4.6 trillion in agency MBS and long-term Treasury securities, surpassing the combined total of previous QE programs. During this period, the duration-adjusted size of the Fed's bond portfolio more than doubled, increasing from \$4.1 trillion to \$8.5 trillion.

3.3 Preliminaries

Our ultimate objective is to measure the change in expected tax deadweight losses (left-hand side of equation (31)) for each QE program and compare it with estimates of the output effect of QE from the literature. In this section, we detail the steps needed to bring the expected deadweight loss equation to the data. The first step is to extend our model to account for infinite periods and assume that taxes are levied from both households and bondholders. Then, the period-0 objective of the government becomes

$$\max_{D,S} \ \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{\Lambda_t}{\Lambda_0} \left(Y_t - \frac{\xi(\tau_t)}{Y_t} \right) \right], \tag{38}$$

where $\xi(\tau_t) = \frac{\alpha}{2}(\tau_t)^2$ represents the deadweight losses from taxes. We normalize $\xi(\tau_t)$ by aggregate output $Y_t = y_t$ such that marginal losses are stationary and a function of the tax rate: $\xi'(\tau_t)/Y_t = \alpha\tau_t/Y_t$. We define the change in expected tax deadweight losses from a QE program as

$$\Delta^Q L_0 \equiv \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{\Lambda_t}{\Lambda_0} \frac{\xi\left(\tau_t^Q\right)}{Y_t} \right] - \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{\Lambda_t}{\Lambda_0} \frac{\xi\left(\tau_t^{nQ}\right)}{Y_t} \right], \tag{39}$$

[17]In the quantitative exercise below, we assume that MEP ends in August 2012 since QE3 starts in September 2012.

where τ_t^Q are taxes levied by the Treasury at time t and τ_t^{nQ} are the counterfactual taxes the Treasury would have levied in the absence of a QE program. The operator $\Delta^Q x_t$ takes the difference between the quantity x_t under the QE program and the counterfactual without the QE program so that $\Delta^Q \tau_t = \tau_t^Q - \tau_t^{nQ}$.

The government starts a QE program at date 0 by purchasing zero-coupon bonds $B_0(n)$ that pay \$1 in $n \leq N$ periods in quantity $\Delta^Q B_0(n)$ and borrowing 1-period zero-coupon bonds $B_0(1)$ in quantity $-\Delta^Q B_0(1) = \sum_{n=2}^N \Delta^Q B_0(n) \times p_0(n)/p_0(1)$ to finance the portfolio. During every subsequent period, the government maintains the maturity structure of the portfolio constant.[18]

Second, to simplify our quantification exercise, we make the conservative restriction that the government raises taxes at once at the end of the QE program instead of smoothing taxes over time. This restriction is conservative because it corresponds to the largest possible deadweight loss for a given level of central bank losses. It also conforms with the observation that fiscal consolidations often occur in short periods and through exceptional measures (Leigh, Pescatori, Devries, and Guajardo, 2011). This implies that the government must roll over short-term debt to finance the QE program instead of raising taxes during the QE program and accounts for losses (gains) at the end of the program by raising (reducing) taxes. Thus, the government borrows

$$-\Delta^Q B_t(1) = -\frac{\Delta^Q B_{t-1}(1)}{p_t(1)} - \Delta^Q B_{t-1}(2) + \sum_{n=2}^N \frac{p_t(n)}{p_t(1)} \left(\Delta^Q B_0(n) - \Delta^Q B_{t-1}(n+1)\right) \tag{40}$$

1-year zero-coupon bonds $B_t(1)$ in every period. At time T, the government unwinds the QE portfolio with returns

$$R_T^Q = \sum_{n=1}^N p_T(n) \Delta^Q B_T(n). \tag{41}$$

It then raises (reduces) taxes to account for the losses (profits), $\Delta^Q \tau_T = -R_T^Q$, whereas no taxes are raised in other periods, $\Delta^Q \tau_t = 0$ for all $t \neq T$.

Proposition 5 (Upper Bound to Deadweight Losses). *If $\Delta^Q \tau_t = 0$ for all $t \neq T$, then*

$$\Delta^Q L_0 = \alpha p_0(T) \left(\frac{1}{2} \mathrm{Cov}_0^{\mathbb{Q}} \left[\frac{R_T^Q}{Y_T}, R_T^Q \right] - \mathrm{Cov}_0^{\mathbb{Q}} \left[\frac{\tau_T^{nQ}}{Y_T}, R_T^Q \right] \right), \tag{42}$$

[18] That is, the government purchases $\Delta^Q B_0(n) - \Delta^Q B_{t-1}(n+1)$ additional n-year zero-coupon bonds $B_t(n)$ for all $n > 1$, such that $\Delta^Q B_t(n) = \Delta^Q B_0(n)$ for every period t.

where $\mathbb{Q}$ is the risk-neutral measure. Additionally, if $\Lambda_T/\Lambda_0 \leq 1$, then $\Delta^Q L_0 \leq \overline{\Delta^Q L_0}$, where $\overline{\Delta^Q L_0}$ is defined as

$$\overline{\Delta^Q L_0} \equiv \frac{\alpha}{2}\sqrt{\mathrm{Var}_0\left[\frac{R_T^Q}{Y_T}\right]\mathbb{E}_0\left[\left(R_T^Q\right)^2\right]} + \alpha\sqrt{\mathrm{Var}_0\left[\frac{\tau_T^{nQ}}{Y_T}\right]\mathbb{E}_0\left[\left(R_T^Q\right)^2\right]}. \tag{43}$$

In Proposition 5, we construct a measure for the deadweight losses as equation (42), which highlights the two sources of increased tax volatility induced by a QE program: (i) the volatility in QE-induced taxes that covers R_T^Q and (ii) the covariance between QE-induced taxes and taxes in the counterfactual without the QE program. The deadweight losses are scaled by $1/Y_T$, which captures the increase in the government's ability to collect taxes when the economy grows.

Precisely estimating this measure empirically would require taking a stance on which asset pricing model is relevant to welfare evaluation, which is not straightforward given the existence of market segmentations. Instead, we mostly rely on an upper bound measure of deadweight losses under the assumption that the time discount rate dominates the state-dependent variation in marginal utilities in the stochastic discount factor—that is, $\Lambda_T/\Lambda_0 \leq 1$. We argue that this is a reasonable assumption given the positive growth rate of the economy and a large time horizon of T for QE programs. The benefit of this approach is that it remains agnostic regarding the underlying model driving agents' stochastic discount factors. For completeness, we also report a measure for the expected deadweight loss of QE using equation (42) under the assumption that agents are risk-neutral.

3.4 Measurements

To calculate the cost of QE portfolios in equation (42) as well as the upper bound in equation (43), we construct QE portfolios and use a term structure model to forecast the returns on those portfolios for any holding period.

Construction of QE Portfolios For each QE program, the initial QE portfolio is a long-short portfolio with a net worth of 0, which consists of long-term bond purchases $\Delta^Q B_0(n)$ funded by 1-year bonds in quantity $-\Delta^Q B_0(1)$.[19] We calculate net purchases associated with each QE program as the cumulative change in the Fed's SOMA holdings

[19]While in practice, these portfolios are funded using reserves, term structure models are notoriously inaccurate to predict the short end of the yield curve. For this reason, we use 1-year bond rates instead of shorter maturity rates.

	QE1	QE2	MEP	QE3	QE4
Panel A: Face Value					
Treasury and Agency	3.09	4.86	-0.22	4.50	13.13
MBS	7.19	-0.93	-0.26	5.08	5.37
All	10.27	3.93	-0.48	9.59	18.50
Panel B: 10-Year Dur. Equiv.					
Treasury and Agency	2.37	3.83	4.85	6.94	12.40
MBS	5.53	-0.73	5.69	7.83	5.07
All	7.90	3.10	10.54	14.77	17.47

Table 1: Size of the Fed's Quantitative Easing Programs. This table reports the size in terms of net purchases for QE programs conducted by the Fed, in percentage of GDP at the end of the purchase phase, Y_0. Net purchases associated with each QE program are computed as the cumulative change in the Fed's SOMA holdings during the purchase phase. The 10-year duration equivalent is computed by $\sum_{t=1}^{30} t \cdot Pay_t/10$, where Pay_t is the principal and coupon payment accrued in t years. We normalize net purchases with nominal GDP at the year-end following each purchase phase.

during the purchase phase of each QE program announced by the Fed. Purchases include the Fed's holdings of Treasury debt, agency debt, and MBS.[20]

Table 1 reports total face value, total payment value, and the 10-year duration equivalent value of total payments from purchased assets. As detailed in Section 3.2, a mix of Treasury, agency, and MBS (net) purchases contribute to the size of the initial QE portfolio for QE1, QE3, and QE4, while only Treasury purchases contribute to the size of the initial QE2 portfolio.[21] The MEP, which swapped short-term for long-term Treasuries to extend the maturity of the Fed's holdings without expanding its balance sheet, has a size close to zero when measured in face value but is the third largest program after QE4 and QE3 when converted to a 10-year duration equivalent value.

To keep track of the QE portfolio and how much the consolidated government needs to borrow short-term to finance it, $-\Delta^Q B_t(1)$, we convert each bond in the portfolio into its

[20]Although our model does not explicitly feature agency debt and MBS, the purchase of agency debt and MBS from the central bank would be equivalent to the purchase of Treasuries in the model if agency debt and MBS (i) have the same interest rate exposures as Treasuries, and (ii) are held by bondholders, such that the central bank also reallocates interest rate risk from bondholders to households through agency debt and MBS purchases.

[21]Small differences between the size of programs (measured in face value terms) and the description of purchase amounts in Section 3.2 arise because part of the asset purchase amounts was used to replace maturing securities in the portfolios.

zero-coupon bond equivalent.[22] We round up the maturity of each zero-coupon bond in the QE portfolio to the nearest larger integer year and calculate $\Delta^Q B_t(1)$ using equation (40) and the term structure model detailed below. Since the maximum maturity of the predicted yield curve is 15 years due to data limitation, we winsorize the long end of payments in the QE portfolio at 15-year maturity while keeping the maturity-weighted size the same. For example, if the QE portfolio holds 30-year zero-coupon bonds with face value \$1, we consider instead a holding of 15-year zero-coupon bonds with \$2 face value.[23]

Term Structure Forecasts For each QE program, we train a 3-factor arbitrage-free term structure model using U.S. yield curve term structure data from November 1971 to the month in which the QE program purchase phase ended. We use the methodology of Hamilton and Wu (2012) to ensure global convergence and identify latent factors using historical data on 1-month, 1-year, and 5-year yields and estimate the rest of the parameters using data on the 15-year yield so that the model is exactly identified. We then generate 1,000 random samples of future factor paths starting from the end of the QE program purchase phase and calculate corresponding paths for the yield curve up to 15 years using the estimated structural parameters.

Figure 3 depicts the distribution of forecasts for 1-year yield and 10-year yield at a horizon of 10 years for QE1 and QE4. Our median forecast over a horizon of 10 years is slow-moving and persistent. For the 1-year yield, the probability of a larger than 4% deviation from the median forecast is lower than 10%, and the probability of a more than 8% deviation from the median forecast is lower than 1%. For the 10-year yield, the probability of a larger than 4% deviation from the median forecast is lower than 10%, and the probability of a more than 6% deviation is lower than 1%. Due to the unexpectedly fast pace of fed funds rate hikes following the end of the QE4 program, the following actual 1-year yield is close to the 99th percentile of the forecast. The 10-year yield, by contrast, is more stable and mostly remains below the 90 percentile of the forecast.[24]

Because of the lack of consensus in the literature on how to account for the presence of the ZLB in term structure models as well as the German experience that indicates the ZLB does not seem to be binding for long maturities, we do not impose a ZLB on

[22]We break down each bond in the QE portfolio into its cash flows—both coupon payments and the principal repayment at maturity—and treat each individual cash flow as a separate zero-coupon bond.

[23]In Appendix B.1, we estimate the term structure model with available data starting from November 1985 (instead of November 1971), the date from which the entire yield curve up to 30 years becomes available. This exercise yields smaller cost estimates because those do not account for high and volatile interest rates from the 1970s to the early 1980s.

[24]Note that, given the long maturity of the QE programs, the movement of the 10-year yield is significantly more impactful on the QE portfolio returns, which we compute below.

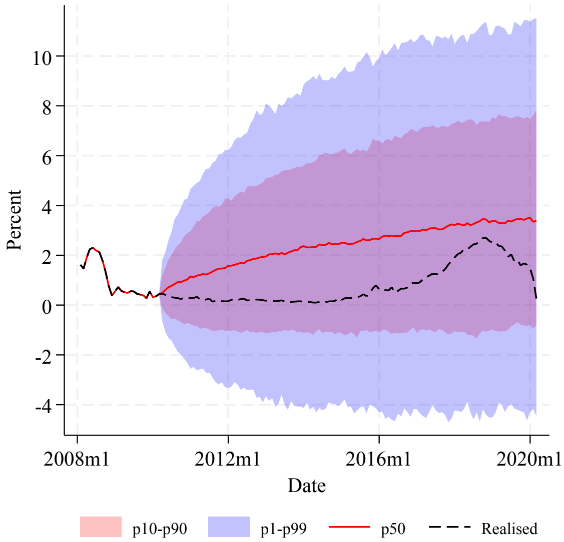
	QE1	QE2	MEP	QE3	QE4
$\text{Cov}_0(R_T^Q/Y_T, R_T^Q/Y_0)$	0.03	0.01	0.05	0.07	0.10
$\text{Cov}_0(\tau_T^{nQ}/Y_T, R_T^Q/Y_0)$	-0.01	-0.01	-0.02	-0.02	-0.02
$\text{Var}_0(R_T^Q/Y_T)$	0.02	0.00	0.03	0.05	0.06
$\mathbb{E}_0[(R_T^Q/Y_0)^2]$	0.06	0.01	0.08	0.15	0.21
$\text{Var}_0(\tau_T^{nQ}/Y_T)$	0.04	0.04	0.05	0.04	0.05

Table 2: Estimates of Cost Components. This table reports estimates of the components in the expected cost measures for all QE programs, in percentage of GDP at the end of the purchase phase, Y_0. Each QE portfolio is assumed to be held for $T = 10$ years, and the second moments are calculated using the distribution of corresponding variables from 1,000 yield curve paths.

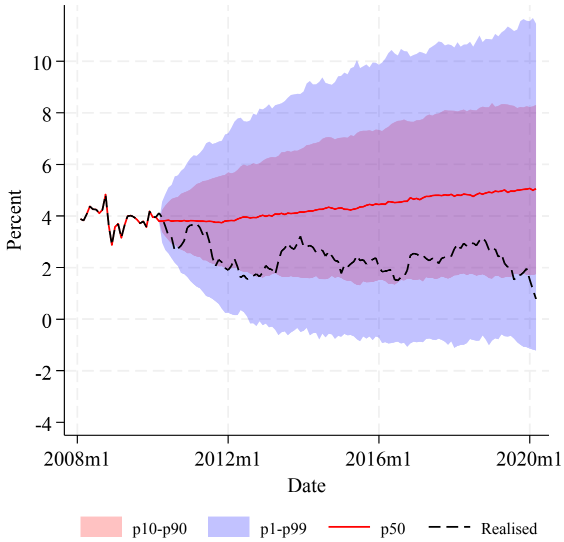
our term structure model. This assumption also results in more conservative estimates of the expected deadweight losses from QE because, as shown in equation (42), they are a function of QE portfolio return volatility and, hence, yield dispersion.

QE Portfolio Returns We calculate 1,000 corresponding paths of QE portfolio returns based on the 1,000 paths of the yield curve according to equation (41). In Figure 4, we plot the distribution of portfolio returns for a holding period of $T = 10$ years for QE1 and QE4 as a percentage of nominal GDP Y_0 at the fiscal year end following the program's purchase phase. QE portfolios earn the term premium in net, which is, on average, positive for all programs we consider. On the other hand, the variation in QE portfolio returns across scenarios is large due to the considerable size of QE purchases as well as interest risk exposure from the long-term bond position and upward-sloping term structures.

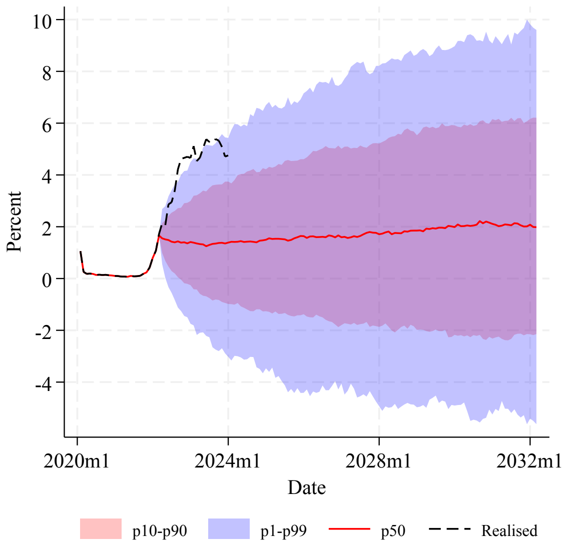
Second Moments In Table 2, we provide estimates of the components of QE portfolio cost (42) and the upper bound (43) for each QE program, which we normalize by GDP at the end of the purchase phase Y_0. $\mathbb{E}_0[(R_T^Q/Y_0)^2]$ is calculated directly from the 1,000 paths of QE portfolio returns. For macro variables, including tax rates τ_T^{nQ}/Y_T and nominal GDP Y_T, we build a forecasting model based on the latent factors of the term structure model and then generate 1,000 paths for the variables using 1,000 paths of latent factors. We calculate conditional second moments $\text{Cov}_0(\tau_T^{nQ}/Y_T, R_T^Q/Y_0)$, $\text{Cov}_0(R_T^Q/Y_T, R_T^Q/Y_0)$, $\text{Var}_0(\tau_T^{nQ}/Y_T)$, and $\text{Var}_0(R_T^Q/Y_T)$ accordingly. Details of those time-series models are relegated to Appendix B.2.



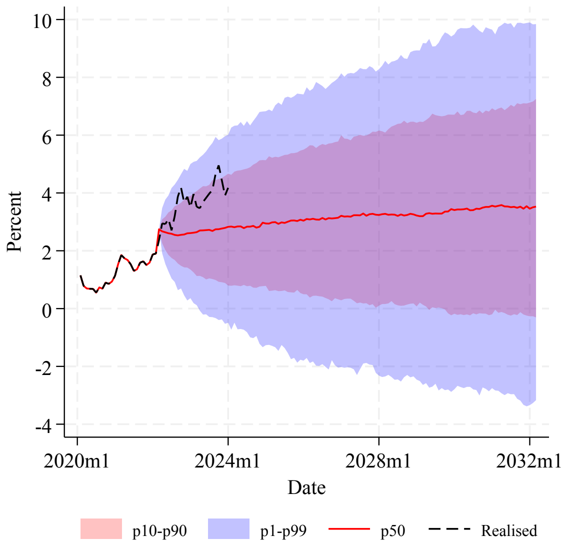
(a) 1-year yields forecasts from QE1



(b) 10-year yields forecasts from QE1

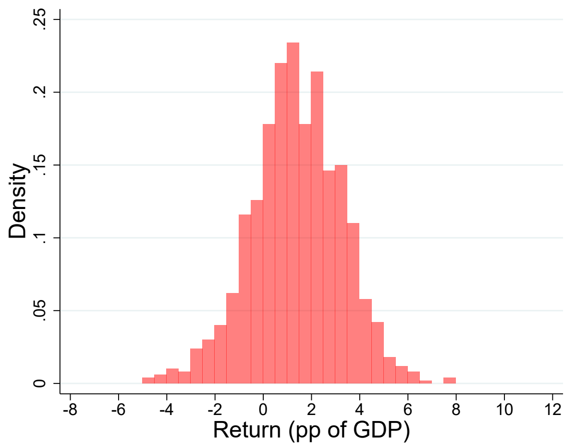


(c) 1-year yields forecasts from QE4

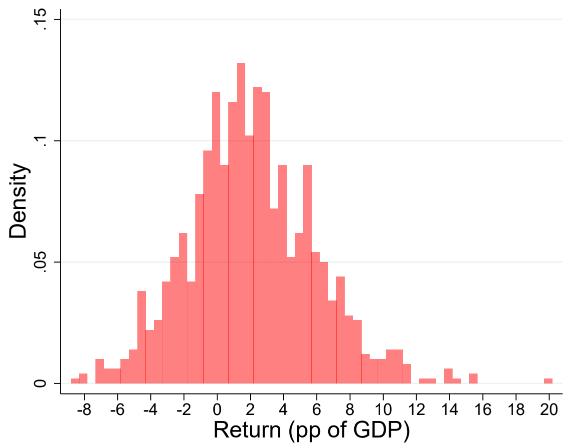


(d) 10-year yields forecasts from QE4

Figure 3: Term Structure Forecasts from QE1 and QE4. This figure plots 10-year forecasts of 1-year and 10-year yields for QE1 and QE4 and compares them with realized yields. For each QE program, we train a 3-factor term structure model using yield curve data up to 15-year yield from November 1971 to the month when QE purchases were finished. We generate 1,000 random paths for the yield curve and compute 1st, 10th, 50th, 90th, and 99th percentile for 1-year and 10-year yields.



(a) QE1



(b) QE4

Figure 4: Distribution of QE Portfolio Returns. This figure plots the distribution of QE portfolio returns after a holding period of 10 years for QE1 and QE4. For each QE program, the return path of the QE portfolio are computed using Equation (41) for each of the 1,000 yield curve returns forecasts.

Deadweight Losses In our baseline parameterization, we use Saez, Slemrod, and Giertz's (2012) estimate of the marginal deadweight losses, $\xi'(\tau_T)/Y_T = \alpha\tau_T/Y_T = 0.38$. This study estimates the marginal deadweight loss in the mid-range of the elasticity of taxable income literature for the top 1% income cutoff, where the marginal tax rate is 42.5% as of 2009. Thus, this number is in the upper range of the estimates of marginal deadweight loss for taxes because the national average marginal tax rate is lower than the rate for this income percentile. Together with a tax rate of 25%, this yields $\alpha = 1.52$.

Output Effect To measure the output effect for each QE program implemented by the Fed, we rely on 18 articles surveyed by Fabo, Jančoková, Kempf, and Pástor (2021). For each of the surveyed articles that jointly estimate the output effect of multiple QE programs, we assign an estimate to each program by breaking down the total estimated effect proportionally to the size of each program, measured in 10-year duration equivalents. After trimming the distribution of size-adjusted estimates below the 5th percentile and above the 95th percentile, we average the estimates for QE1, QE2, and QE3 separately. To obtain an output effect for MEP and QE4, which are not studied in these articles, we multiply the average size-adjusted effect across all estimates by the respective size of the MEP and QE4 programs. See Appendix B.3 for an exhaustive list of the studies considered and more details on their aggregation.

Fabo, Jančoková, Kempf, and Pástor (2021) show that articles written by central bank researchers report larger QE effects than those written by academics. Both estimates agree on a positive output effect of QE. However, there is a large disagreement on the magnitude of this effect: For QE1, the average output effect estimate from all articles is 0.37% of GDP but 0.19% of GDP for the articles by only academics. We refrain from taking a position on which estimate of the output effect is more accurate and, instead, report both.

3.5 Cost-Benefit Analysis

In this section, we compare our QE cost estimates to the benefits estimated in the literature.

	QE1	QE2	MEP	QE3	QE4	All
Panel A: Cost						
$\mathbb{P}$-measure Cost	0.03	0.01	0.06	0.07	0.08	0.25
Upper-Bound Cost	0.10	0.04	0.13	0.18	0.23	0.69
Panel B: Benefit						
Output Effect (All)	0.37	0.27	0.63	0.86	1.23	3.36
Output Effect (Academia)	0.19	0.04	0.18	0.44	0.34	1.19

Table 3: The Trade-Off of QE Programs. This table reports our baseline calculation of the trade-off of QE programs the Fed conducted, in percentage of GDP at the end of the purchase phase, Y_0. The first 5 columns present the trade-off for each QE program separately, and the last column summarizes the trade-off by aggregating across all QE programs. QE portfolios are assumed to be held for $T = 10$ years, and the distribution of losses is calculated using 1,000 yield curve forecasts. The $\mathbb{P}$-measure cost is calculated using Equation (42) assuming that agents are risk neutral. The upper bound cost is calculated using Equation (43). Output effects are calculated by averaging the estimates of articles surveyed in Fabo, Jančoková, Kempf, and Pástor (2021).

Baseline Estimates Table 3 reports our measures of the cost and benefit of each QE program. The $\mathbb{P}$-measure cost is calculated using Equation (42) assuming that agents are risk neutral, and the upper-bound cost is calculated using Equation (43). All estimates of the expected cost in deadweight losses are smaller than the output effect estimates across all QE programs conducted by the Fed. In particular, QE3 and QE4 report the highest estimates for the upper-bound measure at 0.18% and 0.23% of GDP, respectively, owing to the recent uptick in interest volatility. Those are, however, compensated by larger output effects of 0.86% and 1.23% of GDP, respectively. QE2 appears as the program with the lowest beneficiary margin with an upper-bound cost estimate equal to the output effect estimated by academic researchers of 0.04% of GDP.

Cumulatively across the five programs, we estimate that QE programs have increased expected deadweight losses of, respectively, 0.25% of GDP ($70 bn.) under a risk-neutral assumption and 0.69% of GDP ($195 bn.) for the upper bound measure. Both of those measures fall below the output benefit of QE as estimated by previous literature of 3.36% of GDP when including all studies and 1.19% of GDP when excluding those by central bank researchers. These results suggest that QE programs initiated by the Fed represented positive net present value projects when initiated. Although our analysis is conclusive in this respect, a lack of number of QE experiments implies that we can only measure the aggregate effects of QE but not its marginal impact. It is, therefore, possible that a smaller or a larger program would have resulted in a higher net welfare gain as per equation (36).

Alternative Parameterization We also calculate the upper bound $\overline{\Delta^Q L_0}$ for deadweight losses from QE portfolios under alternative parametrization for the tax loss convexity α and portfolio holding period T and compare these with output effect estimates in Table 4. Compared with a holding period of 10 years, when QE portfolios are held for a period as short as $T = 5$ years, the corresponding deadweight losses decrease by around 30% since the cumulative risk in portfolio return decreases. According to our model, the longer the program lasts, the higher the rollover risk exposure and, hence, the expected deadweight losses. Therefore, if we assume QE portfolios are held for $T = 15$ years instead of $T = 10$, expected deadweight losses across all five programs increase by around 30%. Still, the cumulative magnitude remains smaller than the average output effect estimates, both when including and excluding studies from central bank researchers. This conclusion, however, does not hold for all QE programs separately when considering the output effect estimated after excluding studies by central bank researchers. In particular, when increasing the convexity parameter α to 2 or increasing the QE time horizon T to 15, the estimated upper bound for expected deadweight loss cost from QE2 becomes larger (0.05% of GDP) than the estimated output benefit of QE estimated by academic researchers (0.04% of GDP).

3.6 Scenario Analysis

In this section, we complement our main analysis by estimating the magnitude of losses for paths of interest rates under several adverse scenarios. Instead of assessing *expected* losses, this exercise computes *realized* losses for a given yield curve path. This procedure is akin to the stress tests traditionally applied to banks and other financial institutions by regulators.

	QE1	QE2	MEP	QE3	QE4	All
Panel A: Upper-Bound Cost						
$\alpha = 1.52, T = 5$	0.06	0.02	0.08	0.11	0.14	0.42
$\alpha = 1.52, T = 10$	0.10	0.04	0.13	0.18	0.23	0.69
$\alpha = 1.52, T = 15$	0.14	0.05	0.17	0.24	0.29	0.88
$\alpha = 1, T = 10$	0.07	0.02	0.09	0.12	0.15	0.45
$\alpha = 2, T = 10$	0.13	0.05	0.18	0.24	0.31	0.90
Panel B: Benefit						
Output Effect (All)	0.37	0.27	0.63	0.86	1.23	3.36
Output Effect (Academia)	0.19	0.04	0.18	0.44	0.34	1.19

Table 4: Upper-Bound Cost under Alternative Parameterization. This table reports the upper-bound cost for QE programs the Fed conducted under alternative parameterization, in percentage of GDP at the end of the purchase phase, Y_0. Each QE portfolio is assumed to be held for $T = 10$ years and the distribution of losses is calculated using 1,000 yield curve paths.

For a given yield curve path, losses in the QE portfolio are transformed into economic losses through two channels: transfers to foreigners and deadweight losses from tax distortions. The first channel arises because the Fed purchases assets partly from foreign investors who, in the absence of such purchases, would have suffered losses when the U.S. interest rate rises. Assuming that the Fed purchases a proportion ϕ of its portfolio from foreign bondholders, the time-T value of net transfers to foreign investors $\Delta^Q F_T$ takes the following form:[25]

$$\Delta^Q F_T = -\phi R_T^Q. \tag{44}$$

As in the previous section, we assume that QE portfolio losses are all covered by taxes in period T. Therefore, the corresponding realized deadweight losses can be written as

$$\Delta^Q L_T = \frac{\xi\left(\tau_T^Q\right) - \xi\left(\tau_T^{nQ}\right)}{Y_T} = -\xi'\left(\tau_T^{nQ}\right)\frac{R_T^Q}{Y_T} + \frac{1}{2}\xi''\left(\tau_T^{nQ}\right)R_T^Q\frac{R_T^Q}{Y_T}. \tag{45}$$

We sum up the transfers to foreign investors and the increase in deadweight losses to get

[25]Note that these transfers to foreigners are 0 in expectation since $\mathbb{E}_0^{\mathbb{Q}}[R_T^Q] = 0$, and therefore absent from our measures of *expected* deadweight loss costs in the previous section.

an estimate of total realized economic losses due to QE:

$$\Delta^Q TC_T = \Delta^Q F_T + \Delta^Q L_T = -\left(\phi + \frac{\xi'\left(\tau_T^{nQ}\right)}{Y_T} - \frac{1}{2}\frac{\xi''\left(\tau_T^{nQ}\right) R_T^Q}{Y_T}\right) R_T^Q, \tag{46}$$

where $\xi'\left(\tau_T^{nQ}\right) = \alpha\tau_T^{nQ}$ and $\xi''\left(\tau_T^{nQ}\right) = \alpha$. We assume ϕ is equal to the proportion of foreign holdings in treasury securities before any QE program. For each QE program, we calculate the distribution of total welfare costs to the economy at a horizon of $T = 10$ years using the distribution of portfolio returns from Section 3.4 and compare adverse realizations with the estimated output effect.

Table 5 reports QE portfolio losses, tax deadweight losses, and transfers to foreign investor losses in percentage of GDP at the 95th percentile (Panel A) and the 75th percentile (Panel B) of portfolio losses as well as for the realized losses made by the Fed on its various QE portfolios (Panel C) for a holding period of $T = 10$ years. As can be observed by comparing Panel A to Panel D, at the 95th percentile, all programs result in large economic losses, which dominate the estimated output effect. At the 75th percentile (Panel B), however, the output effect dominates for all programs except MEP, resulting in net welfare gains when cumulating across all programs. For the actual realized path of interest rates (Panel C), the first three QE programs resulted in net portfolio gains, while the last two ended with net portfolio losses due to the interest rate spike since 2022. Cumulatively, across the five QE programs, our estimates indicate that the Fed generated a net economic gain of 2.31% of GDP through reductions in transfers to foreign investors and tax deadweight losses. When adding up the positive output effect of QE as estimated by academic researchers (1.19%), this figure increases to a total ex-post realized gain of 3.5% of GDP across all its programs. Panel E presents break-even percentiles, defined as the percentile in total economic loss realizations at which a given QE program breaks even. When using academic researchers' studies for the output effect, these break-even percentiles are in the 83rd to 84th range for all programs except for MEP, which stands at the 70th percentile.

4 Conclusion

This paper proposes a framework to evaluate QE programs through a trade-off between rollover risk and output stimulation. Such a trade-off emerges naturally from a simple 3-period model with market segmentation, tax distortions, nominal frictions, and a ZLB. The model also allows us to characterize the optimal QE size as equalizing the marginal increase in expected deadweight losses from QE to its marginal output gain.

	QE1	QE2	MEP	QE3	QE4	All
Panel A: Cost at 95th Percentile						
QE Portfolio Losses $-R_T^Q$	1.88	0.95	3.37	2.95	4.32	13.47
Tax Deadweight Losses $\Delta^Q L_T$	0.73	0.36	1.31	1.16	1.72	5.29
Transfer to Foreign Investors $\Delta^Q F_T$	1.08	0.58	2.08	1.87	1.68	7.28
Total Economic Losses $\Delta^Q TC_T$	1.81	0.94	3.39	3.02	3.41	12.57
Panel B: Cost at 75th Percentile						
QE Portfolio Losses $-R_T^Q$	-0.23	-0.01	0.96	-0.15	0.41	0.98
Tax Deadweight Losses $\Delta^Q L_T$	-0.09	-0.01	0.37	-0.06	0.16	0.38
Transfer to Foreign Investors $\Delta^Q F_T$	-0.13	-0.01	0.59	-0.10	0.16	0.52
Total Economic Losses $\Delta^Q TC_T$	-0.22	-0.01	0.96	-0.16	0.32	0.89
Panel C: Realized Cost						
QE Portfolio Losses $-R_T^Q$	-2.61	-1.00	-0.22	0.76	0.73	-2.34
Tax Deadweight Losses $\Delta^Q L_T$	-0.95	-0.38	-0.08	0.29	0.28	-0.84
Transfer to Foreign Investors $\Delta^Q F_T$	-1.49	-0.61	-0.14	0.48	0.29	-1.47
Total Economic Losses $\Delta^Q TC_T$	-2.45	-0.99	-0.22	0.78	0.57	-2.31
Panel D: Realized Benefit						
Output Effect (All)	0.37	0.27	0.63	0.86	1.23	3.36
Output Effect (Academia)	0.19	0.04	0.18	0.44	0.34	1.19
Panel E: Breakeven Percentile						
Breakeven Percentile (All)	83.20	83.10	70.90	84.10	83.90	81.04
Breakeven Percentile (Academia)	81.10	76.90	62.20	81.30	75.40	75.38

Table 5: Cost-Benefit Analysis under Adverse Scenarios. This table reports 10-year QE portfolio costs under adverse scenarios, in percentage of GDP at the end of the purchase phase, Y_0. Panel A (Panel B) reports the decomposition of costs at the 95th percentile (the 75th percentile) of the distribution for each variable across the 1,000 yield curve simulations after 10 years. Panel C reports the decomposition of costs using the historical realized yield curve. For QE3 and QE4, we use the realized cost as of 2023, as the programs started less than 10 years ago. Tax deadweight losses, transfers to foreigners, and total welfare losses are calculated using equation (45), (44), and (46), respectively. The break-even percentile reports the percentile realizations at which the QE program breaks even.

A quantitative evaluation of this trade-off applying term structure modeling to estimate the distribution of returns for the Fed's QE portfolio suggests that these programs had a positive net present value at their origination. These conclusions are reached under a monetary dominance assumption that fiscal policy will react by adjusting taxes in the future to make up for any central bank losses. We leave it to future research to relax these assumptions and study how inflation, default probabilities, and sovereign spreads may endogenously react to those dynamics.

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Appendices

A Proofs

A.1 Proof of Proposition 1

The government's problem in period 0 is given by

$$V_0^g = \mathbb{E}_0 \left[\sum_{t=0}^{2} \frac{\Lambda_t}{\Lambda_0} \left(y_t - \frac{\alpha}{2} \frac{\tau_t^2}{1-\theta} \right) \right], \tag{47}$$

subject to the budget constraint

$$G_0 = \tau_0 + B_0^S p_0^S + B_0^L p_0^L. \tag{48}$$

From the solution of the period-1 problem, we know

$$\tau_1(s) = \tau_2(s) = \frac{B_0^S + p_1^S(s) B_0^L}{1 + p_1^S(s)} \quad \text{and} \quad B_1^S(s) = \frac{B_0^S - B_0^L}{1 + p_1^S(s)}. \tag{49}$$

Since the ZLB is not binding, $y_t = a_t$. Substituting in all constraints and taking the first-order derivatives with respect to B_0^S and B_0^L yield

$$\mathbb{E}_0 \left[-\tau_0 p_0^S + \frac{\Lambda_1(s)}{\Lambda_0} \tau_1(s) \frac{1}{1 + p_1^S(s)} + \frac{\Lambda_2(s)}{\Lambda_0} \tau_2(s) \frac{1}{1 + p_1^S(s)} \right] = 0, \tag{50}$$

$$\mathbb{E}_0 \left[-\tau_0 p_0^L + \frac{\Lambda_1(s)}{\Lambda_0} \tau_1(s) \frac{p_1^S(s)}{1 + p_1^S(s)} + \frac{\Lambda_2(s)}{\Lambda_0} \tau_2(s) \frac{p_1^S(s)}{1 + p_1^S(s)} \right] = 0. \tag{51}$$

Combining the government's first-order conditions, bondholders' first-order conditions and market clearing conditions gives

$$\tau_0 = \tau_1(s) = \tau_2(s) = \frac{1}{1 + p_0^S + p_0^L} G_0, \tag{52}$$

$$B_0^S = B_0^L = \frac{1}{1 + p_0^S + p_0^L} G_0. \tag{53}$$

A.2 Proof of Lemma 2

From the the first-order condition for B_0^S, we get

$$p_0^S = \beta \mathbb{E}_0 \left[\left(\frac{c_1^b(s)}{c_0^b} \right)^{-\gamma} \right]. \tag{54}$$

Given the market clearing conditions for consumption goods, we have

$$c_0^b = y_0 - \frac{1-\phi}{\theta}G_0 + \frac{1-\phi}{\theta}\tau_0, \tag{55}$$

$$c_1^b(s) = y_1(s) + \frac{1-\phi}{\theta}\tau_1(s). \tag{56}$$

Then,

$$p_0^S = \beta\mathbb{E}_0\left[\left(\frac{\theta y_1(s) + (1-\phi)\tau_1(s)}{\theta y_0 - (1-\phi)G_0 + (1-\phi)\tau_0}\right)^{-\gamma}\right]. \tag{57}$$

When the ZLB is binding, $p_0^S = 1$. Since period-1 output $y_1(s)$ is always maximized, $y_1(s) = a_1(s)$. From the solution to the period-1 problem,

$$\tau_1(s) = \frac{B_0^S + p_1^S(s)B_0^L}{1+p_1^S(s)} = D\left(\frac{-R_1^c(s)}{1+p_1^S(s)}(S-S^\star) + \frac{1}{p_0^S + p_0^L}\right). \tag{58}$$

Then, we can rewrite equation (57) as

$$y_0 = \frac{1-\phi}{\theta}D + \left(\beta\mathbb{E}_0\left[\left(a_1(s) + \frac{1-\phi}{\theta}D\left[\frac{-R_1^c(s)}{1+p_1^S(s)}(S-S^\star) + \frac{1}{p_0^S+p_0^L}\right]\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}. \tag{59}$$

Since the aggregate output can never exceed productivity, $y_0 \leq a_0$.

A.3 Proof of Lemma 3

If the ZLB is binding,

$$y_0 = \frac{1-\phi}{\theta}D + \left(\beta\mathbb{E}_0\left[\left(a_1(s) + \frac{1-\phi}{\theta}D\left[\frac{-R_1^c(s)}{1+p_1^S(s)}(S-S^\star) + \frac{1}{p_0^S+p_0^L}\right]\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}. \tag{60}$$

Taking the partial derivative of aggregate output with respect to D and S yields

$$\frac{\partial y_0}{\partial D} = \frac{1-\phi}{\theta} + \frac{1-\phi}{\theta}\beta\mathbb{E}_0\left[\left(\frac{c_1^b(s)}{c_0^b}\right)^{-(\gamma+1)} \frac{\frac{S}{p_0^S} + \frac{1-S}{p_0^L}p_1^S(s)}{1+p_1^S(s)}\right] \tag{61}$$

and

$$\frac{\partial y_0}{\partial S} = \frac{1-\phi}{\theta}\beta\mathbb{E}_0\left[\left(\frac{c_1^b(s)}{c_0^b}\right)^{-(\gamma+1)} \frac{-R_1^c(s)}{1+p_1^S(s)}\right]D, \tag{62}$$

where we have simplified the expression with $c_0^b = y_0 - \frac{1-\phi}{\theta}(G_0 - \tau_0)$, $c_1^b(s) = a_1(s) + \frac{1-\phi}{\theta}\tau_1(s)$, $\tau_0 = G_0 - D$, and $\tau_1(s) = D\left[\frac{-R_1^c(s)}{1+p_1^S(s)}(S - S^\star) + \frac{1}{p_0^S + p_0^L}\right]$.

A.4 Proof of Proposition 2

We take the first-order derivative with respect to S in the reformulated problem in Lemma 1 and get

$$\frac{\alpha}{1-\theta}D^2 m\left(S - \frac{p_0^S}{p_0^S + p_0^L}\right) = \frac{\partial y_0}{\partial S}. \tag{63}$$

Given an inner solution, the optimal short-term bond share is thereby

$$S = \frac{p_0^S}{p_0^S + p_0^L} + \frac{1-\theta}{\alpha m D}\frac{1}{D}\frac{\partial y_0}{\partial S}, \tag{64}$$

and the optimal size of QE is

$$Q = D(S - S^\star) = \frac{1-\theta}{\alpha m}\frac{1}{D}\frac{\partial y_0}{\partial S}. \tag{65}$$

If $y_0 \geq a_0$ in the equilibrium characterized by the first-order condition above, the inner solution is not admissible. To also account for the corner solutions, we consider the Karush–Kuhn–Tucker conditions:

$$\frac{\alpha}{1-\theta}D^2 m(S - S^\star) - (1-\mu)\frac{\partial y_0}{\partial S} = 0, \tag{66}$$

where $\mu \geq 0$ is the Lagrange multiplier and $\mu(y_0 - a_0) = 0$. Therefore, the optimal size of QE is

$$Q = (1-\mu)\frac{1-\theta}{\alpha m}\frac{1}{D}\frac{\partial y_0}{\partial S}. \tag{67}$$

We define $\overline{Q}$ as the minimum size of QE pushing the economy out of the ZLB, that is, the solution to the following implicit function:

$$\left(a_0 - \frac{1-\phi}{\theta}D\right)^{-\gamma} - \beta\mathbb{E}_0\left[\left(a_1(s) + \frac{1-\phi}{\theta}\frac{-R_1^c(s)}{1+p_1^S(s)}Q + \frac{1-\phi}{\theta}\frac{D}{p_0^S + p_0^L}\right)^{-\gamma}\right] = 0. \tag{68}$$

Since $\mu(y_0 - a_0) = 0$, $Q = \overline{Q}$ when $\mu > 0$. Remember $\frac{1}{D}\frac{\partial y_0}{\partial S} > 0$. Then, $Q = \overline{Q} = (1-\mu)\frac{1-\theta}{\alpha m}\frac{1}{D}\frac{\partial y_0}{\partial S} < \frac{1-\theta}{\alpha m}\frac{1}{D}\frac{\partial y_0}{\partial S}$ when $\mu > 0$, and $Q = \frac{1-\theta}{\alpha m}\frac{1}{D}\frac{\partial y_0}{\partial S}$ when $\mu = 0$. We have $y_0 \leq a_0$ when $\mu = 0$, and thus $Q = \frac{1-\theta}{\alpha m}\frac{1}{D}\frac{\partial y_0}{\partial S} \leq \overline{Q}$ since $\frac{1}{D}\frac{\partial y_0}{\partial S} > 0$. In summary, $Q = \min\{\frac{1-\theta}{\alpha m}\frac{1}{D}\frac{\partial y_0}{\partial S}, \overline{Q}\}$.

A.5 Proof of Proposition 3

We plug the solution to the period-1 problem as well as the government's budget constraint into the period-0 objective, where y_1, y_2 terms are omitted because they are exogenous. The objective is then reformulated as

$$\mathbb{E}_0 \left\{ y_0 - \frac{\alpha}{2(1-\theta)} \left[(G_0 - p_0^S B_0^S - p_0^L B_0^L - M)^2 + \frac{\Lambda_1(s)}{\Lambda_0} \frac{(B_0^S + p_1^S(s)(B_0^L + M))^2}{1 + p_1^S(s)} \right] \right\}, \tag{69}$$

where we have used the Euler equation $p_1^S(s) = \frac{\Lambda_2(s)}{\Lambda_1(s)}$ to simplify the expression. Define $\tilde{B}_0^L = B_0^L + M$, $\tilde{D} = p_0^S B_0^S + p_0^L \tilde{B}_0^L, \tilde{S} = p_0^S B_0^S / \tilde{D}$, and $\tilde{G}_0 = G_0 - M(1 - p_0^L)$. The objective becomes

$$y_0 - \frac{\alpha}{2(1-\theta)} \left[(\tilde{G}_0 - \tilde{D})^2 + \tilde{D}^2 \left(m(\tilde{S} - \frac{p_0^S}{p_0^S + p_0^L})^2 + \frac{1}{p_0^S + p_0^L} \right) \right], \tag{70}$$

where

$$m = \mathbb{E}_0 \left[\frac{\Lambda_1(s)}{\Lambda_0(1 + p_1^S(s))} \left(\frac{1}{p_0^S} - \frac{p_1^S(s)}{p_0^L} \right)^2 \right] = \mathbb{E}_0 \left[\frac{\Lambda_1(s)}{\Lambda_0(1 + p_1^S(s))} \left(-R_1^c(s)\right)^2 \right]. \tag{71}$$

The government's problem here is isomorphic to the problem in Lemma 1. Therefore, Proposition 3 follows Proposition 2.

A.6 Proof of Proposition 4

Since the problem with currency is isomorphic to the problem without currency, we have $\tau_1(s) = \frac{B_0^S + p_1^S(s)(B_0^L + M)}{1 + p_1^S(s)}$, and $\tau_0 = G_0 - D$. Since QE is financed by currency, we have $p_0^L B_0^L + M = D - p_0^S B_0^S$, where D and B_0^S are held constant. When the ZLB is binding, we get

$$y_0 = \frac{1-\phi}{\theta} D + \left(\beta \mathbb{E}_0 \left[\left(a_1(s) + \frac{1-\phi}{\theta} \frac{B_0^S + p_1^S(s) \left[\frac{D - p_0^S B_0^S}{p_0^L} + M \left(1 - \frac{1}{p_0^L} \right) \right]}{1 + p_1^S(s)} \right)^{-\gamma} \right] \right)^{-\frac{1}{\gamma}} \tag{72}$$

and output effect of QE financed by currency issuance is given by

$$\frac{\partial y_0}{\partial M} = \frac{1-\phi}{\theta} \mathbb{E}_0 \left[\beta \left(\frac{c_1^b(s)}{c_0^b} \right)^{-(\gamma+1)} \frac{p_1^S(s)}{1 + p_1^S(s)} \left(1 - \frac{1}{p_0^L} \right) \right] < 0. \tag{73}$$

A.7 Proof of Proposition 5

In our modified model, the expression for time-0 expected total deadweight losses is

$$L_0 = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{\Lambda_t}{\Lambda_0} \frac{\xi(\tau_t)}{Y_t} \right], \tag{74}$$

where $\xi(\tau_t) = \frac{\alpha}{2}(\tau_t)^2$ are the tax deadweight losses. We denote time t taxes τ_t^Q, and the counterfactual taxes in the absence of QE programs τ_t^{nQ}. The contribution of QE to expected total deadweight losses can be expressed as:

$$\Delta^Q L_0 = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{\Lambda_t}{\Lambda_0} \frac{\xi(\tau_t^Q) - \xi(\tau_t^{nQ})}{Y_t} \right]. \tag{75}$$

We define $\Delta^Q \tau_t = \tau_t^Q - \tau_t^{nQ}$ where $\Delta^Q \tau_t$ is the contribution of QE on taxes. We expand the difference in the deadweight losses $\xi(\tau_t^Q) - \xi(\tau_t^{nQ})$ to second order around the counterfactual level of taxes τ_t^{nQ}, which is exact for $\xi(\tau_t) = \frac{\alpha}{2}(\tau_t)^2$:

$$\Delta^Q L_0 = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{\Lambda_t}{\Lambda_0} \left(\frac{\xi'(\tau_t^{nQ})}{Y_t} \Delta^Q \tau_t + \frac{\xi''(\tau_t^{nQ})}{2} \frac{\Delta^Q \tau_t}{Y_t} \Delta^Q \tau_t \right) \right], \tag{76}$$

where $\xi'(\tau_t^{nQ}) = \alpha \tau_t^{nQ}$, $\xi''(\tau_t^{nQ}) = \alpha$. Then

$$\Delta^Q L_0 = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{\Lambda_t}{\Lambda_0} \left(\alpha \frac{\tau_t^{nQ}}{Y_t} \Delta^Q \tau_t + \frac{\alpha}{2} \frac{\Delta^Q \tau_t}{Y_t} \Delta^Q \tau_t \right) \right]. \tag{77}$$

Since we assume $\Delta^Q \tau_t = -R_T^Q$ and $\Delta^Q \tau_t = 0$ for any $t \neq T$, the expression for $\Delta^Q L_0$ can be rewritten as

$$\Delta^Q L_0 = -\alpha \mathbb{E}_0 \left[\frac{\Lambda_T}{\Lambda_0} \frac{\tau_T^{nQ}}{Y_T} R_T^Q \right] + \frac{\alpha}{2} \mathbb{E}_0 \left[\frac{\Lambda_T}{\Lambda_0} \frac{R_T^Q}{Y_t} R_T^Q \right]. \tag{78}$$

We can write nn equivalent expression for equation (78) using the $\mathbb{Q}$-measure:

$$\Delta^Q L_0 = \alpha \cdot p_0(T) \left(-\mathrm{Cov}_0^{\mathbb{Q}} \left[\frac{\tau_T^{nQ}}{Y_T}, R_T^Q \right] + \frac{1}{2} \mathrm{Cov}_0^{\mathbb{Q}} \left[\frac{R_T^Q}{Y_T}, R_T^Q \right] \right), \tag{79}$$

where $p_0(T)$ is the time-0 price of zero-coupon long-term bond that pays \$1 at time T. To facilitate our quantification exercise, we normalize the deadweight loss by Y_0:

$$\frac{\Delta^Q L_0}{Y_0} = \alpha \cdot p_0(T) \left(-\mathrm{Cov}_0^{\mathbb{Q}} \left[\frac{\tau_T^{nQ}}{Y_T}, \frac{R_T^Q}{Y_0} \right] + \frac{1}{2} \mathrm{Cov}_0^{\mathbb{Q}} \left[\frac{R_T^Q}{Y_T}, \frac{R_T^Q}{Y_0} \right] \right). \tag{80}$$

To derive the upper bound, first notice that

$$\mathbb{E}_0\left[\frac{\Lambda_T}{\Lambda_0}\frac{\tau_T^{nQ}}{Y_T}R_T^Q\right]=\text{Cov}_0\left[\frac{\tau_T^{nQ}}{Y_T},\frac{\Lambda_T}{\Lambda_0}R_T^Q\right]\leq\sqrt{\text{Var}_0\left[\frac{\tau_T^{nQ}}{Y_T}\right]\text{Var}_0\left[\frac{\Lambda_T}{\Lambda_0}R_T^Q\right]} \tag{81}$$

because $\mathbb{E}_0[\Lambda_T R_T^Q]=0$ and a correlation is lower than 1. Similarly,

$$\mathbb{E}_0\left[\frac{\Lambda_T}{\Lambda_0}\frac{R_T^Q}{Y_T}R_T^Q\right]\leq\sqrt{\text{Var}_0\left[\frac{R_T^Q}{Y_T}\right]\text{Var}_0\left[\frac{\Lambda_T}{\Lambda_0}R_T^Q\right]}. \tag{82}$$

Assume the time discount dominates for a reasonably long holding period T for QE portfolios such that $\frac{\Lambda_T}{\Lambda_0}\leq 1$. Under the assumption, we have

$$\text{Var}_0\left[\frac{\Lambda_T}{\Lambda_0}R_T^Q\right]=\mathbb{E}_0\left[\left(\frac{\Lambda_T}{\Lambda_0}R_T^Q\right)^2\right]+\left(\mathbb{E}_0\left[\frac{\Lambda_T}{\Lambda_0}R_T^Q\right]\right)^2\leq\mathbb{E}_0\left[\left(R_T^Q\right)^2\right]. \tag{83}$$

Therefore,

$$\begin{aligned}\Delta^Q L_0=&-\alpha\mathbb{E}_0\left[\frac{\Lambda_T}{\Lambda_0}\frac{\tau_T^{nQ}}{Y_T}R_T^Q\right]+\frac{\alpha}{2}\mathbb{E}_0\left[\frac{\Lambda_T}{\Lambda_0}\frac{R_T^Q}{Y_T}R_T^Q\right]\\=&-\alpha\text{Corr}_0\left[\frac{\tau_T^{nQ}}{Y_T},\frac{\Lambda_T}{\Lambda_0}R_T^Q\right]\sqrt{\text{Var}_0\left[\frac{\tau_T^{nQ}}{Y_T}\right]\text{Var}_0\left[\frac{\Lambda_T}{\Lambda_0}R_T^Q\right]}\\&+\frac{\alpha}{2}\text{Corr}_0\left[\frac{R_T^Q}{Y_T},\frac{\Lambda_T}{\Lambda_0}R_T^Q\right]\sqrt{\text{Var}_0\left[\frac{R_T^Q}{Y_T}\right]\text{Var}_0\left[\frac{\Lambda_T}{\Lambda_0}R_T^Q\right]}\\\leq&\alpha\sqrt{\text{Var}_0\left[\frac{\tau_T^{nQ}}{Y_T}\right]\mathbb{E}_0\left[(R_T^Q)^2\right]}+\frac{\alpha}{2}\sqrt{\text{Var}_0\left[\frac{R_T^Q}{Y_T}\right]\mathbb{E}_0\left[(R_T^Q)^2\right]}.\end{aligned} \tag{84}$$

B Relegated Quantitative Results

B.1 Results with the Yield Curve Sample Starting in 1985

QE Portfolio Returns Due to limited access to yield data at long maturity before 1985, we winsorize the QE purchases at 15-year maturity while keeping the maturity-weight size unchanged in our main analysis. In this section, we restrict our analysis to the period starting in November 1985, when the yield curve data up to 30-year maturity is available. Figure 5 compares the term structure prediction from the shorter sample with the prediction in our main analysis. Table 6 calculates the $\mathbb{P}$-measure cost estimate and the upper bound of the cost and compares them with the output effect. As can be observed, the portfolio return volatility is smaller when only the post-1985 sample is used due to the reduced interest rate in-sample volatility.

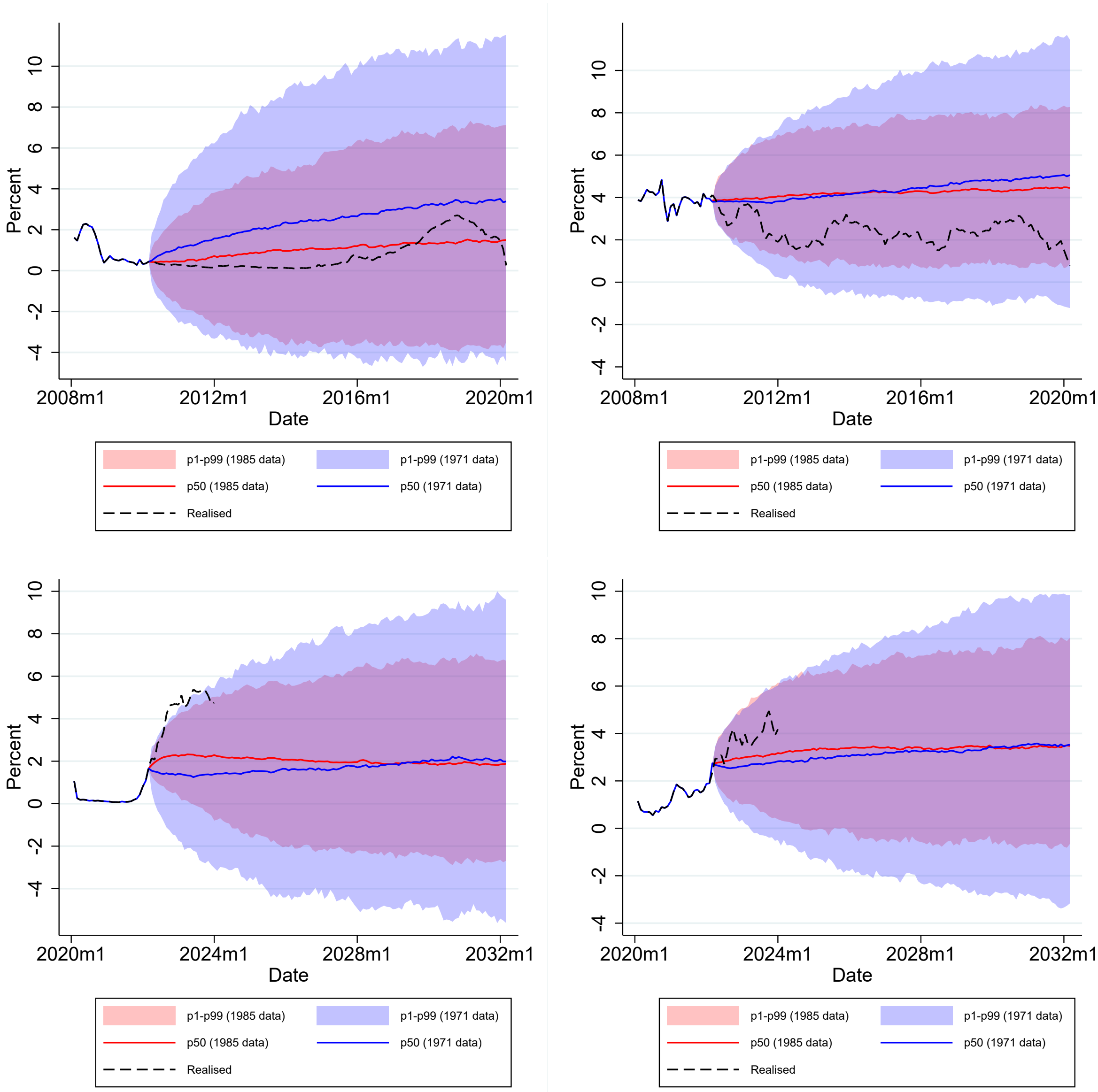

Figure 5: Term Structure Forecast with Different Training Period. This figure compares 10-year forecasts of 1-year and 10-year yield for QE1 and QE4 from term structure models trained with different samples. For each QE program, we train two 3-factor term structure models using two samples of the yield curve data. The first sample includes yields with maturity of up to 15 years, starting from November 1971 and ending on the month when QE purchases were finished; the second sample includes yields with maturity of up to 30 years, starting from November 1985 and ending on the month when QE purchases were finished. We generate 1,000 random paths for the yield curve for each term structure model and compute the 1st, 50th, and 99th percentile for 1-year and 10-year yields.

Cost-Benefit Analysis The key components of the costs are estimated in a similar step with the new term structure model, and we calculate the expected cost of QE portfolios using equation (42) as well as its upper bound using equation (43) for each QE program. In our baseline parameterization, we use Saez, Slemrod, and Giertz's (2012) estimate of the marginal deadweight losses, $\xi'(\tau_T)/Y_T = \alpha\tau_T/Y_T = 0.38$, which yields $\alpha = 1.52$ with a tax rate of 25%. The results are reported in Panel A of Table 6, in comparison with the output effect estimates reported in Panel B of Table 6. Compared to our baseline analysis with the term structure data starting from 1971, the estimates of expected costs are reduced.

	QE1	QE2	MEP	QE3	QE4	All
Panel A: The Cost of QE						
$\mathbb{P}$-measure Cost	0.01	0.00	0.01	0.01	0.02	0.06
Upper-Bound Cost	0.14	0.04	0.06	0.10	0.10	0.43
Panel B: The Benefit of QE						
Output Effect (All)	0.37	0.27	0.63	0.86	1.23	3.36
Output Effect (Academia)	0.19	0.04	0.18	0.44	0.34	1.19

Table 6: The Trade-off of QE Programs Estimated with a Shorter Sample. This table reports our baseline calculation of the trade-off of QE programs the Fed conducted, in percentage of GDP at the end of the purchase phase, Y_0. The first 5 columns present the trade-off for each QE program separately, and the last column summarizes the trade-off by aggregating across all QE programs. QE portfolios are assumed to be held for $T = 10$ years, and the distribution of losses is calculated using 1,000 yield curve forecasts generated with the term structure model trained with yield-curve data starting in 1985. The $\mathbb{P}$-measure cost is calculated using Equation (42) assuming that agents are risk neutral. The upper-bound cost of the QE portfolio is calculated using Equation (43). Output effects are calculated by averaging the estimates of articles surveyed in Fabo, Jančoková, Kempf, and Pástor (2021).

B.2 Details of Cost Estimation

We estimate the expected cost of QE portfolios for each QE program according to two different formulas. First, we estimate the normalized cost of QE portfolio directly according to

$$\frac{\Delta^Q L_0}{Y_0} = \alpha \cdot p_0(T)\left[-\mathrm{Cov}_0\left(\frac{\tau_T^{nQ}}{Y_T}, \frac{R_T^Q}{Y_0}\right) + \frac{1}{2}\mathrm{Cov}_0\left(\frac{R_T^Q}{Y_T}, \frac{R_T^Q}{Y_0}\right)\right], \tag{85}$$

where we abstract from the risk-neutral Q measure and estimate the P-measure counterpart instead by assuming agents are risk-neutral. Secondly, we estimate the upper bound for the normalized QE portfolio cost according to

$$\overline{\frac{\Delta^Q L_0}{Y_0}} \equiv \alpha\sqrt{\mathrm{Var}_0(\frac{\tau_T^{nQ}}{Y_T})\mathbb{E}_0\left[\left(\frac{R_T^Q}{Y_0}\right)^2\right]} + \frac{\alpha}{2}\sqrt{\mathrm{Var}_0(\frac{R_T^Q}{Y_T})\mathbb{E}_0\left[\left(\frac{R_T^Q}{Y_0}\right)^2\right]}. \tag{86}$$

Our target is to estimate several key components of the costs conditional on information up to time 0 for each QE program, that is, the end of the corresponding purchase phase. In Section 3.4, we generate 1,000 factor paths as well as the corresponding yield curves and QE portfolios returns. We make use of the latent variables we identify to also generate 1,000 paths of macro variables we are interest in so that we can estimate the conditional covariance between them and QE portfolio returns. Specifically, we assume the following data generating process for $X_t \in \{log(Y_t/Y_{t-1}), \tau_t/Y_t\}$, the log GDP growth and the tax rate:

$$X_t = \alpha + \sum_i \beta^i F_t^i + \epsilon_t, \tag{87}$$

where ϵ_t is possibly serially correlated, and F_t^i is the i-th factor we identify. We specify an AR(1) process for the error: $\epsilon_t = \rho\epsilon_{t-1} + \eta_t$, where $\mathbb{E}_{t-1}[\eta_t] = 0$. For each QE program, we estimate the model parameters using tax rate series, GDP growth series and latent factors from November 1971 to the end of QE program purchase phase. We then calculate the corresponding 1,000 paths for X_t based on the 1,000 factor paths generated. The key components $\mathrm{Var}_0(\tau_T^{nQ}/Y_T)$, $\mathrm{Cov}_0(\tau_T^{nQ}/Y_T, R_T^Q)$, $\mathrm{Cov}_0(R_T^Q/Y_T, R_T^Q/Y_0)$, $\mathrm{Var}_0(R_T^Q/Y_T)$ and $\mathbb{E}_0[(R_T^Q/Y_0)^2]$ are calculated using the realizations of all variables in the 1,000 paths.

B.3 Output Effect Measurements

To compare our cost estimates to the output benefit of QE, we rely on the 18 papers surveyed by Fabo et al. (2021) that study QE programs in the US. Our measure of the output effect is based on the cumulative measure reported for each paper in Appendix Table A.4 of Fabo et al. (2021). Those measures are taken at the end of the period studied by the authors and converted into a percentage of GDP in the base year.

Six of these studies cover at least two QE programs from the Fed. For each of these articles, we create one observation per program. For each study-program observation, we multiply the study's estimate by the share of the program-specific size in the total size of programs studied, where the size is measured in 10-year duration equivalent. This leaves us with 28 paper-program observations.

We exclude study-program observations where the effect (normalized by program size) is either lower than the 5th percentile or higher than the 95th percentile of the normalized effect distribution across papers. This procedure leaves us with 24 paper-program observations. We report the 24 paper-program observations in Appendix Table 7. The average (median) cumulative output effect per 1% of 10-year duration equivalent to GDP is 0.08 (0.04) percentage points of GDP. The standard deviation is 0.10 percentage points of GDP. Fabo et al. (2021) show that central bank authors report larger QE effects on both output and inflation than academic ones. Of the 24 paper-program observations, 7 are from papers with no authors affiliated with central banks. For this sample, the average (median) cumulative output effect per 1% of 10-year duration equivalent to GDP is 0.02 (0.02) percentage points of GDP. The standard deviation is 0.01 percentage points of GDP.

The literature surveyed by Fabo et al. (2021) only studies QE1, QE2, and QE3. To obtain an output effect for MEP and QE4, we first compute the average normalized effect across all 24 (7 for the restricted sample) study-program observations per 1% program size of 10-year duration equivalent to GDP, which is 0.08 pp of GDP. We then extrapolate the output effect for MEP and QE4 by multiplying this average normalized effect by the respective size of the MEP and QE4 programs.

Reference	Central Bank Affiliation	QE Program	Output Effect, GDP %	Output Effect per 10-y Dur. Equiv. pp of GDP, GDP %
Balatti et al. (2017)	0	LSAP1	0.13	0.02
Balatti et al. (2017)	0	LSAP2	0.05	0.02
Baumeister and Benati (2013)	1	LSAP1	0.98	0.14
Chen et al. (2012)	1	LSAP2	0.07	0.03
Chung et al. (2012)	1	LSAP1	0.40	0.06
Dahlahaus et al. (2018)	1	LSAP1	0.88	0.12
Dahlahaus et al. (2018)	1	LSAP2	0.30	0.12
Dahlahaus et al. (2018)	1	LSAP3	1.62	0.12
Del Negro et al. (2017)	1	LSAP1	0.26	0.04
Del Negro et al. (2017)	1	LSAP2	0.09	0.04
Engen et al. (2015)	1	LSAP1	0.28	0.04
Engen et al. (2015)	1	LSAP2	0.09	0.04
Engen et al. (2015)	1	LSAP3	0.52	0.04
Falagiarda (2014)	0	LSAP2	0.01	0.00
Fuhrer and Olivei (2011)	1	LSAP2	0.78	0.32
Gambacorta et al. (2014)	1	LSAP1	0.12	0.02
Gertler and Karadi (2013)	1	LSAP1	0.01	0.00
Haldane et al. (2016)	1	LSAP2	0.94	0.39
Hesse et al. (2018)	1	LSAP2	0.54	0.22
Popescu (2015)	0	LSAP2	0.04	0.02
Weale and Wieladek (2016)	1	LSAP1	0.34	0.05
Wu and Xia (2016)	0	LSAP1	0.24	0.03
Wu and Xia (2016)	0	LSAP2	0.08	0.03
Wu and Xia (2016)	0	LSAP3	0.44	0.03

Table 7: QE studies from Fabo et al. (2021). This table presents the effects of the QE program studied on cumulative output for each study-program observation in our sample. We report the effects on the output level in percent. Standardized effects are obtained by dividing the total program effects reported here by the size of each program through a 10-year duration equivalent to GDP.

Online Appendix

OA.1 Dispersed Tax Incidence

In this section, we relax our assumption that households bear all tax liabilities. We assume instead the tax incidence is as follows: a proportion of ψ is collected from domestic bondholders, and $1-\psi$ is collected from households. Then $\theta\tau_t^b + (1-\theta)\tau_t^h = \tau_t$, and $\theta\tau_t^b = \psi\tau_t$.

We start from the budget constraints of all market participants. For the government, we have

$$G_0 = \tau_0 + p_0^S B_0^S + p_0^L B_0^L \tag{1}$$
$$0 = \tau_1(s) - B_0^S + p_1^S(s) B_1^S(s) \tag{2}$$
$$0 = \tau_2(s) - B_1^S(s) - B_0^L. \tag{3}$$

For households, we have

$$(1-\theta)c_0^h = (1-\theta)y_0 - (1-\theta)\tau_0^h - (1-\theta)\xi(\tau_0^h) \tag{4}$$
$$(1-\theta)c_1^h(s) = (1-\theta)y_1(s) - (1-\theta)\tau_1^h(s) - (1-\theta)\xi(\tau_1^h(s)) \tag{5}$$
$$(1-\theta)c_2^h(s) = (1-\theta)y_2 - (1-\theta)\tau_2^h(s) - (1-\theta)\xi(\tau_2^h(s)). \tag{6}$$

For domestic bondholders,

$$\theta c_0^b = \theta y_0 - \phi(p_0^S B_0^S + p_0^L B_0^L) - \theta\tau_0^b - \theta\xi(\tau_0^b) \tag{7}$$
$$\theta c_1^b(s) = \theta y_1(s) - \phi(-B_0^S + p_1^S(s) B_1^S(s)) - \theta\tau_1^b(s) - \theta\xi(\tau_1^b(s)) \tag{8}$$
$$\theta c_2^b(s) = \theta y_2 - \phi(-B_1^S(s) - B_0^L) - \theta\tau_2^b(s) - \theta\xi(\tau_2^b(s)), \tag{9}$$

and for foreign bondholders

$$c_0^f = -\phi(p_0^S B_0^S + p_0^L B_0^L) \tag{10}$$
$$c_1^f(s) = -\phi((-B_0^S + p_1^S(s) B_1^S(s)) \tag{11}$$
$$c_2^f(s) = -\phi(-B_1^S(s) - B_0^L). \tag{12}$$

We combine the budget constraints for government, households, domestic bondholders and express τ_t^b, τ_t^h as a function of τ_t:

$$c_0^b = y_0 - \frac{1-\phi}{\theta}G_0 + \frac{1-\phi-\psi}{\theta}\tau_0 - \xi\left(\frac{\psi}{\theta}\tau_0\right) \tag{13}$$

$$c_1^b(s) = y_1(s) + \frac{1-\phi-\psi}{\theta}\tau_1(s) - \xi\left(\frac{\psi}{\theta}\tau_1(s)\right) \tag{14}$$

$$c_2^b(s) = y_2 + \frac{1-\phi-\psi}{\theta}\tau_2(s) - \xi\left(\frac{\psi}{\theta}\tau_2(s)\right). \tag{15}$$

Under our new specification, the total deadweight loss in period t is expressed as

$$\theta\xi(\tau_t^b) + (1-\theta)\xi(\tau_t^h) = \theta\xi\left(\frac{\psi}{\theta}\tau_t\right) + (1-\theta)\xi\left(\frac{1-\psi}{1-\theta}\tau_t\right). \tag{16}$$

We take $\xi(\tau) = \frac{\alpha}{2}\tau^2$ as before. Then we have

$$\theta\xi(\tau_t^b) + (1-\theta)\xi(\tau_t^h) = \frac{\alpha}{2}\left[\frac{\psi^2}{\theta} + \frac{(1-\psi)^2}{1-\theta}\right](\tau_t)^2. \tag{17}$$

Domestic bondholders' problem is the same as before, to which the solution is still charaterized by Euler Equations:

$$p_0^S = \beta\mathbb{E}_0\left[\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma}\right] = \mathbb{E}_0\left[\frac{\Lambda_1(s)}{\Lambda_0}\right] \tag{18}$$

$$p_0^L = \beta^2\mathbb{E}_0\left[\left(\frac{c_2^b(s)}{c_0^b}\right)^{-\gamma}\right] = \mathbb{E}_0\left[\frac{\Lambda_2(s)}{\Lambda_0}\right] \tag{19}$$

$$p_1^S(s) = \beta\left(\frac{c_2^b(s)}{c_1^b(s)}\right)^{-\gamma} = \frac{\Lambda_2(s)}{\Lambda_1(s)}. \tag{20}$$

Next, we solve the government's problem. The objective of the government is the same as before: it maximizes the sum of discounted aggregate output as a price taker. We assume no demand recession in periods 1 and 2 as before, that is, $y_2 = a_2$, and the period-1 interest rate is set exogenously such that $y_1(s) = a_1(s)$.

For the government, the period-1 problem is:

$$\max_{\{\tau_1(s),\tau_2(s),B_1^S(s)\}}\left\{\left(a_1(s) + \frac{\Lambda_2(s)}{\Lambda_1(s)}a_2\right) - \frac{\alpha}{2}\left[\frac{\psi^2}{\theta} + \frac{(1-\psi)^2}{1-\theta}\right]\left((\tau_1(s))^2 + \frac{\Lambda_2(s)}{\Lambda_1(s)}(\tau_2(s))^2\right)\right\}, \tag{21}$$

subject to the budget constraints:

$$0 = \tau_1(s) + B_1^S(s)p_1^S(s) - B_0^S \tag{22}$$

$$0 = \tau_2(s) - B_1^S(s) - B_0^L. \tag{23}$$

First-order condition is

$$\tau_1(s)(-p_1^S(s)) + \frac{\Lambda_2(s)}{\Lambda_1(s)}\tau_2(s) = 0. \tag{24}$$

Then the period-1 problem solution is

$$\tau_1(s) = \tau_2(s) = \frac{B_0^S + p_1^S(s)B_0^L}{1 + p_1^S(s)}. \tag{25}$$

Define $D = p_0^S B_0^S + p_0^L B_0^L$, $S = p_0^S B_0^S / D$,

$$\tau_1(s) = \tau_2(s) = D\frac{S/p_0^S + p_1^S(s)(1-S)/p_0^L}{1 + p_1^S(s)} \tag{26}$$

$$\tau_0 = G_0 - (p_0^S B_0^S + p_0^L B_0^L) = G_0 - D. \tag{27}$$

The government's period-0 objective is

$$\mathbb{E}_0\left[y_0 + \frac{\Lambda_1(s)}{\Lambda_0}a_1(s) + \frac{\Lambda_2(s)}{\Lambda_0}a_2\right] \tag{28}$$

$$-\frac{\alpha}{2}(\frac{\psi^2}{\theta} + \frac{(1-\psi)^2}{1-\theta})\left\{\mathbb{E}_0\left[(\tau_0)^2 + \frac{\Lambda_1(s)}{\Lambda_0}(\tau_1(s))^2 + \frac{\Lambda_2(s)}{\Lambda_0}(\tau_2(s))^2\right]\right\}. \tag{29}$$

We ignore $a_1(s), a_2$ in the objective because they are deterministic, and substitute the expression for $\tau_0, \tau_1(s), \tau_2(s)$ as well as the Euler Equation into the objective. The objective eventually takes the following form:

$$y_0 - \frac{\alpha}{2}\left(\frac{\psi^2}{\theta} + \frac{(1-\psi)^2}{1-\theta}\right)\left\{(G_0 - D)^2 + D^2\left[m(S - \frac{p_0^S}{p_0^S + p_0^L})^2 + \frac{1}{p_0^S + p_0^L}\right]\right\}, \tag{30}$$

where

$$m = \mathbb{E}_0\left[\left(\frac{R_1^c(s)}{1 + p_1^S(s)}\right)^2\left(\frac{\Lambda_1(s)}{\Lambda_0} + \frac{\Lambda_2(s)}{\Lambda_0}\right)\right]. \tag{31}$$

We take the first-order condition with respect to S:

$$\frac{\partial y_0}{\partial S} - \alpha\left(\frac{\psi^2}{\theta} + \frac{(1-\psi)^2}{1-\theta}\right) D^2 m(S - S^\star) = 0. \tag{32}$$

The new multiplier in marginal cost $\frac{\psi^2}{\theta} + \frac{(1-\psi)^2}{1-\theta}$ reflects the smoothness in tax distribution across agents. We next calculate the output effect $\partial y_0/\partial S$. From domestic bondholders' Euler Equation,

$$p_0^S = \beta \mathbb{E}_0\left[\left(\frac{a_1(s) + \frac{1-\phi-\psi}{\theta}\tau_1(s) - \frac{\alpha}{2}(\frac{\psi}{\theta}\tau_1(s))^2}{y_0 - \frac{1-\phi-\psi}{\theta}(G_0 - \tau_0) - \frac{\alpha}{2}(\frac{\psi}{\theta}\tau_0)^2}\right)^{-\gamma}\right], \tag{33}$$

which pins down the domestic bondholders' (and therefore aggregate) demand y_0. When the ZLB binds, p_0^S is constrained at 1. We can find the output effect:

$$\frac{\partial y_0}{\partial S} = \frac{\beta}{p_0^S}\mathbb{E}_0\left[\left(\frac{c_1^b(s)}{c_0^b}\right)^{-(\gamma+1)}\left[\frac{1-\phi-\psi}{\theta} - \alpha(\frac{\psi}{\theta}\tau_1(s))\frac{\psi}{\theta}\right]\frac{\partial \tau_1(s)}{\partial S}\right] \tag{34}$$

$$= \frac{D}{p_0^S}\frac{1-\phi-\psi}{\theta}\mathbb{E}_0\left[\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-(\gamma+1)}\frac{-R_1^c(s)}{1+p_1^S(s)}\right] \tag{35}$$

$$- \frac{D}{p_0^S}\frac{\psi}{\theta}\mathbb{E}_0\left[\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-(\gamma+1)}\frac{-R_1^c(s)}{1+p_1^S(s)}\alpha(\frac{\psi}{\theta}\tau_1(s))\right]. \tag{36}$$

Here $\frac{1-\phi-\psi}{\theta}$ is marginal decrease in domestic bondholders' net exposure to the interest rate risk, our regular term; the second term comes from the change in deadweight loss bondholders need to pay, because part of taxes are collected from them. Since deadweight loss is monotonically increasing in taxes, it inherits the cyclicality of taxes. Therefore, when QE reduces the risk exposure of domestic bondholders' bond return by purchasing positive-beta long-term bonds from them, it also reduces the risk exposure of deadweight loss. As deadweight loss reduces consumption, the aggregate effect of QE on domestic bondholders' consumption will be smaller than that in the bond return, and the output effect will be smaller or even negative.

OA.2 Sign of the Output Effect

In our model, domestic bondholders consume

$$c_0^b = y_0 - \frac{1-\phi}{\theta} D \tag{37}$$

$$c_1^b(s) = a_1(s) + \frac{1-\phi}{\theta}\tau_1(s) > a_1(s) \tag{38}$$

$$c_2^b(s) = a_2 + \frac{1-\phi}{\theta}\tau_2(s) > a_2, \tag{39}$$

where taxes (equal to bondholders' consumption from the bond portfolio) are

$$\tau_1(s) = \tau_2(s) = \frac{B_0^S + p_1^S(s) B_0^L}{1 + p_1^S(s)} = D\left[\frac{-R_1^c(s)}{1+p_1^S(s)}(S - S^\star) + \frac{1}{p_0^S + p_0^L}\right], \tag{40}$$

where $R_1^c(s) = p_1^S(s)/p_0^L - 1/p_0^S$ and $S^\star = p_0^S/(p_0^S + p_0^L)$. We decompose the output effect in Lemma 3 as

$$\frac{\partial y_0}{\partial S} = \frac{1-\phi}{\theta}\beta \mathbb{E}_0\left[\left(\frac{c_1^b(s)}{c_0^b}\right)^{-(\gamma+1)} \frac{-R_1^c(s)}{1+p_1^S(s)}\right] = \frac{1-\phi}{\theta}(IS + PS), \tag{41}$$

where

$$IS = \mathbb{E}_0\left[\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-(\gamma+1)}\right]\mathbb{E}_0\left[\frac{-R_1^c(s)}{1+p_1^S(s)}\right], \tag{42}$$

$$PS = \mathrm{Cov}_0\left[\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-(\gamma+1)}, \frac{-R_1^c(s)}{1+p_1^S(s)}\right] \tag{43}$$

represent the intertemporal-substitution and precautionary-saving term, respectively.

The fundamental shock is productivity shock $a_1(s)$, dependent on exogenous state s. Without loss of generality, we assume $a_1(s)$ is differentiable in s, and $\frac{\partial a_1(s)}{\partial s} > 0$. a_2 is constant. We define $\zeta = \frac{4\theta}{\gamma(1-\phi)(1/p_0^S + 1/p_0^L)} > 0$, and consider equilibria with $D(S - S^\star) > 0$. We summarize the sign of the output effect components with the following lemmas:

Lemma OA.1. *Given $\exists s, a_1(s) > a_2$, a sufficient condition for $p_1^S(s)$ increasing in s is*

$$\frac{D(S-S^\star)}{a_2} \leq \zeta \frac{1}{1 - a_2/\max\{a_1(s)\}}. \tag{44}$$

If $\forall s, a_1(s) \leq a_2$, $p_1^S(s)$ is always increasing in s.

Lemma OA.2. *A sufficient condition for* $\frac{\partial c_1^b(s)}{\partial s} > 0$ *is*

$$\frac{D(S - S^\star)}{a_2} \leq \zeta, \tag{45}$$

under which equation (44)*, the condition in equation* (OA.1) *also holds, and* $PS > 0$*.*

Lemma OA.3. *A sufficient condition for* $IS < 0$ *is*

$$\frac{D(S - S^\star)}{a_2} \leq \zeta \left[1 - \left[\frac{4}{p_0^S(1 + \max\{p_1^S(s)\})} \beta \left(\max \left\{ \frac{c_1^b(s)}{c_0^b} \right\} \right)^{-\gamma} \right]^{-1} \right]. \tag{46}$$

Lemma OA.4. *A sufficient condition for a positive output effect,* $IS + PS > 0$*, is: equation* (45)*, the condition in Lemma* OA.2*.*

The details are given below. First, we establish the relationship between equilibrium variables and the fundamental shock in our model. We can calculate the following derivatives

$$\begin{aligned} \frac{\partial c_1^b(s)}{\partial s} &= \frac{\partial a_1(s)}{\partial s} - \frac{1-\phi}{\theta} D(S - S^\star) \frac{1/p_0^S + 1/p_0^L}{(1 + p_1^S(s))^2} \frac{\partial p_1^S(s)}{\partial s} \\ \frac{\partial c_2^b(s)}{\partial s} &= -\frac{1-\phi}{\theta} D(S - S^\star) \frac{1/p_0^S + 1/p_0^L}{(1 + p_1^S(s))^2} \frac{\partial p_1^S(s)}{\partial s}. \end{aligned}$$

Denote $H(s) = \frac{1-\phi}{\theta} D(S - S^\star) \frac{1/p_0^S + 1/p_0^L}{(1+p_1^S(s))^2}$. We continue to calculate

$$\begin{aligned} \frac{\partial}{\partial s}\left(\frac{c_2^b(s)}{c_1^b(s)}\right) &= \frac{1}{(c_1^b(s))^2} \left[c_1^b(s) \left(-H(s) \frac{\partial p_1^S(s)}{\partial s} \right) - c_2^b(s) \left(\frac{\partial a_1(s)}{\partial s} - H(s) \frac{\partial p_1^S(s)}{\partial s} \right) \right], \\ \frac{\partial p_1^S(s)}{\partial s} &= -\gamma \beta \left(\frac{c_2^b(s)}{c_1^b(s)} \right)^{-(\gamma+1)} \frac{\partial}{\partial s} \left(\frac{c_2^b(s)}{c_1^b(s)} \right) \\ &= -\gamma \frac{p_1^S(s)}{c_1^b(s) c_2^b(s)} \left[c_1^b(s) \left(-H(s) \frac{\partial p_1^S(s)}{\partial s} \right) - c_2^b(s) \left(\frac{\partial a_1(s)}{\partial s} - H(s) \frac{\partial p_1^S(s)}{\partial s} \right) \right]. \end{aligned}$$

Then

$$\frac{\partial p_1^S(s)}{\partial s} = \gamma \frac{p_1^S(s)}{c_1^b(s) c_2^b(s)} \left[1 + \gamma H(s) p_1^S(s) \frac{c_2^b(s) - c_1^b(s)}{c_1^b(s) c_2^b(s)} \right]^{-1} c_2^b(s) \frac{\partial a_1(s)}{\partial s}.$$

We substitute the expression of $\frac{\partial p_1^S(s)}{\partial s}$ into $\frac{\partial c_1^b(s)}{\partial s}$, $\frac{\partial c_2^b(s)}{\partial s}$. We have

$$\begin{aligned}
\frac{\partial c_1^b(s)}{\partial s} &= \frac{\partial a_1(s)}{\partial s} - H(s)\left[\gamma \frac{p_1^S(s)}{c_1^b(s)c_2^b(s)}\left[1+\gamma H(s)p_1^S(s)\frac{c_2^b(s)-c_1^b(s)}{c_1^b(s)c_2^b(s)}\right]^{-1} c_2^b(s)\frac{\partial a_1(s)}{\partial s}\right] \\
&= \left(1-\gamma H(s)\frac{p_1^S(s)}{c_2^b(s)}\right)\left[1+\gamma H(s)p_1^S(s)\frac{c_2^b(s)-c_1^b(s)}{c_1^b(s)c_2^b(s)}\right]^{-1}\frac{\partial a_1(s)}{\partial s} \\
\frac{\partial c_2^b(s)}{\partial s} &= -\gamma H(s)\frac{p_1^S(s)}{c_1^b(s)}\left[1+\gamma H(s)p_1^S(s)\frac{c_2^b(s)-c_1^b(s)}{c_1^b(s)c_2^b(s)}\right]^{-1}\frac{\partial a_1(s)}{\partial s},
\end{aligned}$$

and

$$\frac{\partial}{\partial s}\left(\frac{c_2^b(s)}{c_1^b(s)}\right) = -\frac{c_2^b(s)}{(c_1^b(s))^2}\left[1+\gamma H(s)p_1^S(s)\frac{c_2^b(s)-c_1^b(s)}{c_1^b(s)c_2^b(s)}\right]^{-1}\frac{\partial a_1(s)}{\partial s}.$$

Our target is to find a set of parameters such that $c_1^b(s), p_1^S(s)$ increasing in s, which means consumption $c_1^b(s)$ is dominated by the fundamental shock $a_1(s)$, and long-term bonds have positive beta and positive term premium. To achieve that, we need the following set of conditions:

$$1+\gamma H(s)p_1^S(s)\frac{c_2^b(s)-c_1^b(s)}{c_1^b(s)c_2^b(s)} > 0 \tag{47}$$

$$1-\gamma H(s)\frac{p_1^S(s)}{c_2^b(s)} > 0, \tag{48}$$

where Equation (48) ensures $\frac{\partial c_1^b(s)}{\partial s} > 0$, and Equation (47) ensures $\frac{\partial p_1^S(s)}{\partial s} > 0$. We focus on the equilibria where $D(S-S^\star) > 0$ such that $H(s) = \frac{1-\phi}{\theta}D(S-S^\star)\frac{1/p_0^S+1/p_0^L}{(1+p_1^S(s))^2} > 0$, which is consistent with the equilibrium with a positive output effect.

For Equation (47), an equivalent expression is

$$1+\gamma H(s)p_1^S(s)\frac{c_2^b(s)-c_1^b(s)}{c_1^b(s)c_2^b(s)} = 1+\gamma H(s)p_1^S(s)\frac{a_2-a_1(s)}{c_1^b(s)c_2^b(s)}.$$

If $\forall s$, $a_1(s) \le a_2$, Equation (47) is always true, so we only consider scenarios where $\exists s, a_1(s) > a_2$. Since

$$\begin{aligned}
&1+\gamma H(s)p_1^S(s)\frac{a_2-a_1(s)}{c_1^b(s)c_2^b(s)} > 1+\gamma H(s)p_1^S(s)\frac{a_2-a_1(s)}{a_1(s)a_2} \\
>&1+\gamma\frac{1-\phi}{\theta}\frac{D(S-S^\star)}{a_2}\frac{1/p_0^S+1/p_0^L}{4}\frac{a_2-a_1(s)}{a_1(s)},
\end{aligned}$$

a sufficient condition for equation (47) is

$$\frac{D(S-S^{\star})}{a_2} \leq \frac{4\theta}{\gamma(1-\phi)(1/p_0^S+1/p_0^L)}\left(1-\frac{a_2}{\max\{a_1(s)\}}\right)^{-1}.$$

On the other hand, since

$$1+\gamma H(s)p_1^S(s)\frac{c_2^b(s)-c_1^b(s)}{c_1^b(s)c_2^b(s)} > 1-\gamma H(s)p_1^S(s)\frac{1}{c_2^b(s)},$$

equation (47) is satisfied if equation (48) is satisfied. Now we only consider the sufficient condition for equation (48). Since

$$\begin{aligned}
&1-\gamma H(s)p_1^S(s)\frac{1}{c_2^b(s)} > 1-\gamma H(s)p_1^S(s)\frac{1}{a_2} = 1-\gamma\frac{1-\phi}{\theta}\frac{D(S-S^{\star})}{a_2}\frac{1/p_0^S+1/p_0^L}{(1+p_1^S(s))^2/p_1^S(s)} \\
=&1-\gamma\frac{1-\phi}{\theta}\frac{D(S-S^{\star})}{a_2}\frac{1/p_0^S+1/p_0^L}{1/p_1^S(s)+2+p_1^S(s)} \geq 1-\gamma\frac{1-\phi}{\theta}\frac{D(S-S^{\star})}{a_2}\frac{1/p_0^S+1/p_0^L}{4},
\end{aligned}$$

a sufficient condition for equation (48) is

$$\frac{D(S-S^{\star})}{a_2} \leq \frac{4\theta}{\gamma(1-\phi)\,(1/p_0^S+1/p_0^L)}. \tag{49}$$

$D(S-S^{\star})/a_2$, the deviation from the tax-smoothing maturity scaled by the relative debt size, governs the magnitude of the shock in bond portfolio value for any given fundamental shock. If $D(S-S^{\star})/a_2$ is small enough such that bond portfolio value changes are small compared to the fundamental risk $a_1(s)$, consumption is always driven by $a_1(s)$ and mean-reverting such that $p_1^S(s)$ is positively correlated with $a_1(s)$.

Once equation (49) is satisfied, we have $\frac{\partial c_1^b(s)}{\partial s} > 0$, $\frac{\partial}{\partial s}\left(\frac{c_2^b(s)}{c_1^b(s)}\right) < 0$, $\frac{\partial p_1^S(s)}{\partial s} > 0$. $c_1^b(s), c_2^b(s)$ are positive, so the sign of derivative is also determined for all monotonic transformation, for example, $\frac{\partial}{\partial s}\left(\left(\frac{c_2^b(s)}{c_1^b(s)}\right)^{-\gamma}\right) > 0$, $\frac{\partial}{\partial s}\left(\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma}\right) < 0$. We introduce the following theorem, which is Theorem 2.14 of Boucheron et al. (2013):

Theorem 1 (Chebyshev's association inequality). *For any two continuously differentiable functions $f(X), g(X)$ where $f'(X) > 0, g'(X) > 0$, we have*

$$Cov[f(X), g(X)] > 0, \tag{50}$$

where X is a nonconstant random variable.

Using this theorem, we have

$$\mathrm{Cov}_0\left[\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma}, \beta\left(\frac{c_2^b(s)}{c_1^b(s)}\right)^{-\gamma}\right] < 0,$$

so

$$\begin{aligned}\mathbb{E}_0\left[p_0^L - p_0^S p_1^S(s)\right] &= \mathbb{E}_0\left[\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma}\beta\left(\frac{c_2^b(s)}{c_1^b(s)}\right)^{-\gamma}\right] \\ &\quad - \mathbb{E}_0\left[\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma}\right]\mathbb{E}_0\left[\beta\left(\frac{c_2^b(s)}{c_1^b(s)}\right)^{-\gamma}\right] \\ &= \mathrm{Cov}_0\left[\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma}, \beta\left(\frac{c_2^b(s)}{c_1^b(s)}\right)^{-\gamma}\right] < 0. \end{aligned} \tag{51}$$

We confirm that long-term bonds have positive term premium in this case.

Having established the pattern of consumption and interest rates, we start to discuss the sign of the output effect. First we consider $\frac{R_1^c(s)}{1+p_1^S(s)}$, that is the key term in $\frac{\partial c_1^b(s)}{\partial S}$. We have

$$\frac{\partial}{\partial s}\left(\frac{-R_1^c(s)}{1+p_1^S(s)}\right) = -\frac{1/p_0^S + 1/p_0^L}{(1+p_1^S(s))^2}\frac{\partial p_1^S(s)}{\partial s} < 0,$$

therefore

$$PS = \mathrm{Cov}_0\left[\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-(\gamma+1)}, \frac{-R_1^c(s)}{1+p_1^S(s)}\right] > 0. \tag{52}$$

We next compute $\mathbb{E}_0[\frac{-R_1^c(s)}{1+p_1^S(s)}]$, which determines the sign of IS. We decompose it using the definition of covariance as well as the expression of the term premium in equation (51), which gives us

$$\mathbb{E}_0\left[\frac{-R_1^c(s)}{1+p_1^S(s)}\right] = \frac{1}{p_0^L}\mathrm{Cov}_0\left[\frac{1}{p_0^S}\mathbb{E}_0\left[\frac{1}{1+p_1^S(s)}\right]\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma} - \frac{1}{1+p_1^S(s)}, p_1^S(s)\right].$$

Then we can calculate

$$
\begin{aligned}
&\frac{\partial}{\partial s}\left(\frac{1}{p_0^S}\mathbb{E}_0\left[\frac{1}{1+p_1^S(s)}\right]\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma}-\frac{1}{1+p_1^S(s)}\right)\\
=&-\gamma\left[1+\gamma H(s)p_1^S(s)\frac{c_2^b(s)-c_1^b(s)}{c_1^b(s)c_2^b(s)}\right]^{-1}\frac{p_1^S(s)}{c_1^b(s)(1+p_1^S(s))^2}\\
&\left[\frac{(1+p_1^S(s))^2}{p_0^S p_1^S(s)}\mathbb{E}_0\left[\frac{1}{1+p_1^S(s)}\right]\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma}\left(1-\gamma H(s)\frac{p_1^S(s)}{c_2^b(s)}\right)-1\right]\frac{\partial a_1(s)}{\partial s}.
\end{aligned}
$$

Therefore, given $\left[1+\gamma H(s)p_1^S(s)\frac{c_2^b(s)-c_1^b(s)}{c_1^b(s)c_2^b(s)}\right]^{-1}>0$, $IS<0$ if

$$
\frac{(1+p_1^S(s))^2}{p_0^S p_1^S(s)}\mathbb{E}_0\left[\frac{1}{1+p_1^S(s)}\right]\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma}\left(1-\gamma H(s)\frac{p_1^S(s)}{c_2^b(s)}\right)>1.
$$

Since

$$
\begin{aligned}
&\frac{(1+p_1^S(s))^2}{p_0^S p_1^S(s)}\mathbb{E}_0\left[\frac{1}{1+p_1^S(s)}\right]\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma}\left(1-\gamma H(s)\frac{p_1^S(s)}{c_2^b(s)}\right)\\
>&\frac{4}{p_0^S}\mathbb{E}_0\left[\frac{1}{1+p_1^S(s)}\right]\beta\left(\frac{c_1^b(s)}{c_0^b}\right)^{-\gamma}\left(1-\gamma H(s)\frac{p_1^S(s)}{a_2}\right)\\
>&\frac{4}{p_0^S(1+\max\{p_1^S(s)\})}\beta\left(\max\{\frac{c_1^b(s)}{c_0^b}\}\right)^{-\gamma}\left(1-\gamma H(s)\frac{p_1^S(s)}{a_2}\right),
\end{aligned}
$$

a sufficient condition for $IS<0$ is

$$
\frac{D(S-S^\star)}{a_2}\leq\frac{4\theta}{\gamma(1-\phi)(1/p_0^S+1/p_0^L)}\left[1-\left[\frac{4\beta\left(\max\{c_1^b(s)/c_0^b\}\right)^{-\gamma}}{p_0^S(1+\max\{p_1^S(s)\})}\right]^{-1}\right]. \tag{53}
$$

Note that under this condition, $\left[1+\gamma H(s)p_1^S(s)\frac{c_2^b(s)-c_1^b(s)}{c_1^b(s)c_2^b(s)}\right]^{-1}>0$ is satisfied, because

$$
1-\left[\frac{4}{p_0^S(1+\max\{p_1^S(s)\})}\beta\left(\max\{\frac{c_1^b(s)}{c_0^b}\}\right)^{-\gamma}\right]^{-1}<1<\left(1-\frac{a_2}{\max\{a_1(s)\}}\right)^{-1}.
$$

Finally, we discuss $IS + PS$.

$$IS + PS = \mathbb{E}_0 \left[\beta \left(\frac{c_1^b(s)}{c_0^b} \right)^{-(\gamma+1)} \frac{-R_1^c(s)}{1 + p_1^S(s)} \right]$$
$$= \mathrm{Cov}_0 \left[\beta \left(\frac{c_1^b(s)}{c_0^b} \right)^{-\gamma} (-R_1^c(s)), \left(\frac{c_1^b(s)}{c_0^b} \right)^{-1} \frac{1}{1 + p_1^S(s)} \right].$$

We calculate the partial derivatives with respect to s for the two terms in the covariance above. After substituting in the expressions for $\partial c_1^b(s)/\partial s$ and $\partial p_1^S(s)/\partial s$, we have

$$\frac{\partial}{\partial s} \left(\left(\frac{c_1^b(s)}{c_0^b} \right)^{-1} \frac{1}{1 + p_1^S(s)} \right)$$
$$= \Bigg[- \left(\frac{c_1^b(s)}{c_0^b} \right)^{-2} \frac{1}{1 + p_1^S(s)} \frac{1}{c_0^b} \left(1 - \gamma H(s) \frac{p_1^S(s)}{c_2^b(s)} \right) \left[1 + \gamma H(s) p_1^S(s) \frac{c_2^b(s) - c_1^b(s)}{c_1^b(s) c_2^b(s)} \right]^{-1}$$
$$- \left(\frac{c_1^b(s)}{c_0^b} \right)^{-1} \frac{\gamma p_1^S(s) c_1^b(s) c_2^b(s)}{(1 + p_1^S(s))^2} \left[1 + \gamma H(s) p_1^S(s) \frac{c_2^b(s) - c_1^b(s)}{c_1^b(s) c_2^b(s)} \right]^{-1} c_2^b(s) \Bigg] \frac{\partial a_1(s)}{\partial s} < 0$$

given $1 - \gamma H(s) \frac{p_1^S(s)}{c_2^b(s)} > 0$ and $1 + \gamma H(s) p_1^S(s) \frac{c_2^b(s) - c_1^b(s)}{c_1^b(s) c_2^b(s)} > 0$, and

$$\frac{\partial}{\partial s} \left(\beta \left(\frac{c_1^b(s)}{c_0^b} \right)^{-\gamma} (-R_1^c(s)) \right) = -\gamma \beta \left(\frac{c_1^b(s)}{c_0^b} \right)^{-(\gamma+1)} \frac{1}{c_0^b} \left[1 + \gamma H(s) p_1^S(s) \frac{c_2^b(s) - c_1^b(s)}{c_1^b(s) c_2^b(s)} \right]^{-1}$$
$$\left[\frac{1}{p_0^S} + \gamma H(s) \frac{p_1^S(s)}{c_2^b(s)} R_1^c(s) \right] \frac{\partial a_1(s)}{\partial s}.$$

The sign is controlled by $\frac{1}{p_0^S} + \gamma H(s) \frac{p_1^S(s)}{c_2^b(s)} R_1^c(s)$. If $\frac{1}{p_0^S} + \gamma H(s) \frac{p_1^S(s)}{c_2^b(s)} R_1^c(s) > 0$, we have $IS + PS > 0$. This is always true if $R_1^c(s) \geq 0$, so we only consider the scenarios where $R_1^c(s) < 0, \exists s$. Since

$$\frac{1}{p_0^S} + \gamma H(s) \frac{p_1^S(s)}{c_2^b(s)} R_1^c(s) > \frac{1}{p_0^S} + \gamma H(s) \frac{p_1^S(s)}{a_2} \min \{R_1^c(s)\}$$
$$> \frac{1}{p_0^S} + \gamma \frac{1 - \phi}{\theta} \frac{D(S - S^\star)}{a_2} \frac{1/p_0^S + 1/p_0^L}{4} \min \{R_1^c(s)\},$$

a sufficient condition for a positive output effect is

$$\frac{D(S - S^\star)}{a_2} \leq \frac{4\theta}{\gamma(1 - \phi)(1/p_0^S + 1/p_0^L)} \frac{1}{p_0^S \max \{-R_1^c(s)\}}. \tag{54}$$

However, because $p_0^S \max\{-R_1^c(s)\} = \max\{1 - p_0^L p_1^S(s)\} < 1$,

$$\frac{1}{p_0^S \max\{-R_1^c(s)\}} > 1, \tag{55}$$

and the sufficient condition for $1+\gamma H(s) p_1^S(s) \frac{c_2^b(s) - c_1^b(s)}{c_1^b(s) c_2^b(s)} > 0$ is also sufficient for a positive output effect.