Varation formulation with Material Tensor

\# We will be using Linear Elasticity module for begining

\# PSD Variational formulati

Lagrange firs

- assume finte elemet space $V^h = \{ v^h \in L^2(\Omega^h),\ v^h \in \mathbb{P}_1 \}$
- assume vector FE space $\mathbb{V}^h = V^h \times V^h \times V^h$

$$\boxed{\begin{aligned}&\text{Search } \bar{U} \in \mathbb{V}^h \\ &\int_\Omega \lambda \nabla \cdot \vec{U}\, \nabla \cdot \vec{v} + \int_\Omega 2\mu\, \varepsilon(\bar{U}) : \varepsilon(\bar{v}) + \int_\Omega \vec{F}_b^T \cdot \vec{v} - \int_{\partial\Omega} \vec{t}^T \cdot \vec{v} + \underbrace{\{B/C\}}_{\text{Dirichlet}} = 0 \qquad \forall \bar{v} \in \mathbb{V}^h\end{aligned}}$$

trial function $\vec{U} = \{u_i\}_{i=1}^n \in \mathbb{V}^h$

test function $\bar{V} = \{v_i\}_{i=1}^n \in \mathbb{V}^h$

domain $\Omega \in \mathbb{R}^n$ $\quad n = 2, 3$ for 2D or 3D

λ, μ are Lame parameters $(\lambda, \mu) \in \mathbb{R}^+$

$$\nabla \cdot \bar{U} = \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right)$$

Strain $\varepsilon(\bar{U}) = \left[\frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial y}, \frac{1}{\sqrt{2}} \left(\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} \right) \right]^T$

traction $\vec{t} = [t_1, t_2, t_3]^T \qquad t_i \in \mathbb{R} : \Omega^h$

body force $\vec{F} = [F_1, F_2, F_3]^T \quad F_i \in \mathbb{R} : \Omega^h$

$\mathbb{P}_1$ is piecewise Liner finete elemet

$\mathbb{P}_1 = \{ v \in H^1(\Omega) \mid \forall K \in \Omega^h \}$

- So far what is discussed is what we currently have with PSD

$\boxed{\text{Introducing the new variation formultion.}}$

- We need a FE space built on top of Quadrature elements to store the material tensor

$$\mathbb{Q}^h = \{ q \in L^2(\Omega) \mid \forall K \in \Omega^h : \alpha_K \in \mathbb{R} : q|_K = \alpha_K \}$$

degrees of free dame are quadrature points of cell

- $\mathbb{Q}^h = Q^h \times Q^h \times \dots Q^h = \{Q^h_i\}_{i=1}^{D}$ $\quad D = 3^2$ in 2D, $D = 6^2$ in 3D

- Note that since material tensor is symmetric we use
  D= 6 in 2D
  D= 21 in 3D

Variational form is

$$\boxed{\int_{\Omega} \varepsilon(\bar{u}) \cdot \underline{\underline{\{q^h_i\}_{i=1}^{D}}} : \varepsilon(\bar{v}) + \int_{\Omega} \mathbb{F}^T \cdot \bar{v} - \int_{\partial\Omega} \mathbb{t}^T \cdot v + \underbrace{\{B/C\}}_{\text{Dirichlet}} = 0 \quad \forall (u,v) \in V^h, \ \forall q \in \mathbb{Q}^h}$$

Mt = Material tensor $\begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ " & 2\mu+\lambda & 0 \\ " & " & \mu \end{bmatrix}$ in 2D

- With $q[0] = 2\mu+\lambda$ $\quad q[1] = \lambda$ $\quad q[2] = 0$
  $q[3] = 2\mu+\lambda$ $\quad q[4] = 0$
  $q[5] = \mu$

$$\varepsilon(\bar{u}) = \left[\frac{d u_1}{\partial x}, \frac{\partial u_2}{\partial y}, \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} \right]^T$$