A Lock Calculus for Multimode Type Theory

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Abstract

Multimode type theory (MTT) is parametrized by a *mode theory*: a 2-category whose objects, morphisms and 2-cells serve as internal modes, modalities and 2-cells. So far, this mode theory has remained in the metatheory, with syntactic modal type and term formers being indexed by metatheoretic gadgets. Building a syntactic lock calculus on top of the mode theory has several advantages: the modal aspects of substitution take the form of more familiar syntactical substitutions, the lock operation on contexts can be axiomatized as *pseudo*functorial so that models of MTT no longer need to be strict(ified), and we can have internal mode, modality and 2-cell polymorphism with intensional 2-cell equality.

Notation 1. Throughout the abstract, we let p, q, r, s stand for modes, μ, ν, ρ for modalities, α for 2-cells, $\mathfrak{m}, \mathfrak{n}, \mathfrak{o}$ for lock variables, $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}$ for lock terms, and $\mathfrak{S}, \mathfrak{T}$ for lock substitutions.

Lock variables and lock terms Before we consider what a lock calculus for MTT [GKNB21] should look like, we first modify the original MTT notation so that locks can be referred to via lock variables, which can be substituted with lock terms:[1]

Original	$\Gamma, 🔒_\mu$	$\Gamma, \mu \mid x : T$	x_α	$\langle \mu \mid T \rangle$	$\mathsf{mod}_\mu\, t$
Named	$\Gamma, \mathfrak{m} : 🔒_\mu$	$\Gamma, x : \{\mathfrak{m} : 🔒_\mu\} T$	$x(🔑_\alpha(\mathfrak{n}))$	$(\mathfrak{m} : 🔒_\mu) \to T$	$\lambda(\mathfrak{m} : 🔒_\mu).t$

Original	$\mathsf{let}_\nu\, (\mathsf{mod}_\mu\, x = s) \,\mathsf{in}\, t$
Named	$\mathsf{let}\, (\mathfrak{n} : 🔒_\nu \vdash \lambda(\mathfrak{m} : 🔒_\mu).x(\mathfrak{n})(\mathfrak{m}) = s) \,\mathsf{in}\, t$

This notation makes several inference rules look familiar or at least reasonable:

$$\frac{\Gamma, \mathfrak{m} : 🔒_\mu \vdash T \,\mathsf{type} \mathbin{@} p \qquad \mu : p \to q}{\Gamma, x : \{\mathfrak{m} : 🔒_\mu\} T \,\mathsf{ctx} \mathbin{@} q} \qquad \frac{\Gamma, \mathfrak{m} : 🔒_\mu \vdash t : T \mathbin{@} p \qquad \mu : p \to q}{\Gamma \vdash \lambda(\mathfrak{m} : 🔒_\mu).t : (\mathfrak{m} : 🔒_\mu) \to T \mathbin{@} q}$$

$$\frac{\dfrac{\alpha : \mu \Rightarrow \nu : p \to q}{\mathfrak{n} : 🔒_\nu \vdash 🔑_\alpha(\mathfrak{n}) : 🔒_\mu \mathbin{@} q \to p}}{\Gamma, x : \{\mathfrak{m} : 🔒_\mu\} T, \mathfrak{n} : 🔒_\nu \vdash x(🔑_\alpha(\mathfrak{n})) : T[🔑_\alpha(\mathfrak{n})/\mathfrak{m}] \mathbin{@} p}$$

In the intermediate step of the last rule, we already encounter a novel *lock term* $🔑_\alpha(\mathfrak{n})$.

Lock calculus The lock calculus LC has the following judgement forms, and we give their intuitive and semantical meaning:[2]

$\Psi \,\mathsf{ltele} \mathbin{@} q \to p$	Ψ is a lock telescope $q \to p$	$[\![\Psi]\!]$ is a functor $[\![q]\!] \to [\![p]\!]$.
$\Psi \vdash \mathfrak{t} : 🔒_\mu$	$\mathfrak{t}$ is a lock term in ctx. Ψ	$[\![\mathfrak{t}]\!] : [\![\Psi]\!] \to [\![🔒_\mu]\!]$ is a nat. transf.
$\Psi \vdash e : \mathfrak{s} = \mathfrak{t} : 🔒_\mu \mathbin{@} q \to p$	e proves intensional equality	$[\![\mathfrak{s}]\!] = [\![\mathfrak{t}]\!]$.
$\Psi \vdash \mathfrak{T} : \Phi \mathbin{@} q \to p$	$\mathfrak{T}$ is a lock subst. from Ψ to Φ	$[\![\mathfrak{T}]\!] : [\![\Psi]\!] \to [\![\Phi]\!]$ is a nat. transf.
$\Psi \vdash e : \mathfrak{S} = \mathfrak{T} : \Phi \mathbin{@} q \to p$	e proves intensional equality	$[\![\mathfrak{S}]\!] = [\![\mathfrak{T}]\!]$.

The origin of locks and lock terms remains the external mode theory:

$$\frac{}{()\,\mathsf{ltele}} \qquad \frac{\Psi \,\mathsf{ltele} \mathbin{@} r \to q \qquad \mu : p \to q}{\Psi, \mathfrak{m} : 🔒_\mu \,\mathsf{ltele} \mathbin{@} r \to p}$$

[1] This is an improvement of the tick notation proposed in [Nuy20, ch. 5.3].

[2] In the style of generalized algebraic theories [Car86, Car78], we omit judgments for definitional equality.

$$\frac{\mu : p \to q}{\mathfrak{m} : \text{🔒}_{\mu} \vdash \mathfrak{m} : \text{🔒}_{\mu} \mathbin{@} q \to p} \qquad \frac{\alpha : \mu \Rightarrow \nu : p \to q \qquad \Psi \vdash \mathfrak{t} : \text{🔒}_{\nu}}{\Psi \vdash \text{🔑}_{\alpha}(\mathfrak{t}) : \text{🔒}_{\mu} \mathbin{@} q \to p}.$$

Lock substitutions arise from terms and compose horizontally. Vertical composition of substitution is a special case of general substitution, which actually works by syntactical substitution of lock terms for lock variables, and should extend to MTT:

$$\frac{\Psi \vdash \mathfrak{t} : \text{🔒}_{\mu}}{\Psi \vdash (\mathfrak{t}/\mathfrak{m}) : (\mathfrak{m} : \text{🔒}_{\mu})} \qquad \frac{\Psi' \vdash \mathfrak{S} : \Phi' \mathbin{@} r \to q \qquad \Psi \vdash \mathfrak{T} : \Phi \mathbin{@} q \to p}{\Psi', \Psi \vdash \mathfrak{S}, \mathfrak{T} : \Phi', \Phi \mathbin{@} r \to p} \qquad \frac{\Psi \vdash J \qquad \Phi \vdash \mathfrak{S} : \Psi}{\Phi \vdash J[\mathfrak{S}]}$$

Identity and composite locks MTT originally has strict equality rules $(\Gamma, \text{🔒}_{\mathrm{id}}) = \Gamma$ and $(\Gamma, \text{🔒}_{\mu\circ\nu}) = (\Gamma, \text{🔒}_{\mu}, \text{🔒}_{\nu})$. We wish to turn these into natural isomorphisms, and we start by doing so in the lock calculus. The lock calculus serves to be the internal language of the mode theory, which can be any 2-category. As 2-categories are the horizontal categorification (a.k.a. oidification) [nLa23] of (non-symmetric) monoidal categories, we can draw inspiration from existing calculi for those [JM10, §2.1][Shu16, §2.4.2]. We get constructors for identity and composite locks:

$$\frac{}{\vdash () : \text{🔒}_{\mathrm{id}} \mathbin{@} p \to p} \qquad \frac{\Phi \vdash \mathfrak{s} : \text{🔒}_{\nu} \mathbin{@} r \to q \qquad \Psi \vdash \mathfrak{t} : \text{🔒}_{\mu} \mathbin{@} q \to p}{\Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}) : \text{🔒}_{\nu\circ\mu} \mathbin{@} r \to p}$$

These are eliminated using a let-expression. However, in order to be able to intensionally prove naturality of this let-expression w.r.t. its target lock, we will also provide the let expression for intensional equality. In order to be able to split composite and remove identity locks in MTT without the context equations given above, we will even provide the let-expression for MTT terms:[3]

$$\frac{\Phi, \mathfrak{n} : \text{🔒}_{\nu}, \mathfrak{m} : \text{🔒}_{\mu}, \Psi \vdash \mathfrak{u} : \text{🔒}_{\rho} \mathbin{@} s \to o \qquad \Xi \vdash \mathfrak{t} : \text{🔒}_{\nu\circ\mu} \mathbin{@} r \to p}{\Phi, \Xi, \Psi \vdash \mathsf{let}\,((\mathfrak{n}, \mathfrak{m}) = \mathfrak{t})\,\mathsf{in}\,\mathfrak{u} : \text{🔒}_{\rho} \mathbin{@} s \to o}$$

$$\frac{\begin{array}{c}\Phi, \mathfrak{o} : \text{🔒}_{\nu\circ\mu}, \Psi \vdash \mathfrak{u}, \mathfrak{v} : \text{🔒}_{\rho} \mathbin{@} s \to o \qquad \Xi \vdash \mathfrak{t} : \text{🔒}_{\nu\circ\mu} \mathbin{@} r \to p \\ \Phi, \mathfrak{n} : \text{🔒}_{\nu}, \mathfrak{m} : \text{🔒}_{\mu}, \Psi \vdash e : \mathfrak{u}[(\mathfrak{n}, \mathfrak{m})/\mathfrak{o}] = \mathfrak{v}[(\mathfrak{n}, \mathfrak{m})/\mathfrak{o}] : \text{🔒}_{\rho} \mathbin{@} s \to o\end{array}}{\Phi, \Xi, \Psi \vdash \mathsf{let}\,((\mathfrak{n}, \mathfrak{m}) = \mathfrak{t})\,\mathsf{in}\,e : \mathfrak{u}[\mathfrak{t}/\mathfrak{o}] = \mathfrak{v}[\mathfrak{t}/\mathfrak{o}] : \text{🔒}_{\rho} \mathbin{@} s \to o}$$

$$\frac{\begin{array}{c}\Gamma, \mathfrak{o} : \text{🔒}_{\nu\circ\mu}, \Psi \vdash A\,\mathsf{type} \mathbin{@} o \qquad \Xi \vdash \mathfrak{t} : \text{🔒}_{\nu\circ\mu} \mathbin{@} r \to p \\ \Gamma, \mathfrak{n} : \text{🔒}_{\nu}, \mathfrak{m} : \text{🔒}_{\mu}, \Psi \vdash a : A[(\mathfrak{n}, \mathfrak{m})/\mathfrak{o}] \mathbin{@} o\end{array}}{\Gamma, \Xi, \Psi \vdash \mathsf{let}\,((\mathfrak{n}, \mathfrak{m}) = \mathfrak{t})\,\mathsf{in}\,a : A[\mathfrak{t}/\mathfrak{o}] \mathbin{@} o}$$

These let-expressions β-reduce definitionally when $\mathfrak{t}$ is actually a pair. There are similar rules for 🔒_{id}. We can now admit pseudofunctorial models by interpreting the introduction and elimination rules for 🔒_{id} and $\text{🔒}_{\nu\circ\mu}$ via the unitors and compositors of the model.

The metamode We have not yet delivered on our promise to allow internal mode, modality and 2-cell polymorphism, nor have we explained how to use intensional 2-cell equality. In each of these cases, we need to step outside the mode theory to reason about it, and we need a type system for that, so that we can quantify and use a J-rule. To this end, we include another copy of MLTT – to be modelled in a category of sufficiently big sets – called the *metamode* (whose type assignment will be denoted with $a :: A$), and prefix every MTT and LC judgement with a metamode context Θ. (Dependent) types of modes, modalities and 2-cells exist in the metamode and their behaviour depends on the choice of mode theory. When MTT or LC requires a mode, modality or 2-cell, we can take a metamode term in context Θ. Additionally, there will be metamode types of lock and MTT terms, lock substitutions and intensional LC equality (with J-rule):[4]

$$\frac{\Theta \mid \cdot \vdash t : T \mathbin{@} p}{\Theta \vdash \ulcorner t \urcorner :: \mathsf{Tm}_p(T)} \qquad \frac{\Theta \mid \Psi \vdash \mathfrak{t} : \text{🔒}_{\mu} \mathbin{@} q \to p}{\Theta \vdash \ulcorner \mathfrak{t} \urcorner :: \mathsf{LTm}_{q\to p}(\Psi, \mu)} \qquad \frac{\Theta \mid \Psi \vdash e : \mathfrak{s} = \mathfrak{t} : \text{🔒}_{\mu} \mathbin{@} q \to p}{\Theta \vdash \ulcorner e \urcorner :: \mathsf{LTE}_{q\to p}(\Psi, \mu, \mathfrak{s}, \mathfrak{t})}$$

$$\frac{\Theta \vdash t :: \mathsf{Tm}_p(T)}{\Theta \mid \Gamma \vdash \llcorner t \lrcorner : T \mathbin{@} p} \qquad \frac{\Theta \vdash t :: \mathsf{LTm}_{q\to p}(\Psi, \mu)}{\Theta \mid \Psi \vdash \llcorner t \lrcorner : \text{🔒}_{\mu} \mathbin{@} q \to p} \qquad \frac{\Theta \vdash e : \mathsf{LTE}_{q\to p}(\Psi, \mu, \mathfrak{s}, \mathfrak{t})}{\Theta \mid \Psi \vdash \llcorner e \lrcorner : \mathfrak{s} = \mathfrak{t} : \text{🔒}_{\mu} \mathbin{@} q \to p}$$

[3] In MTT, the telescope to the right of Ξ can be quantified over, so we may assume it to be empty or more generally a lock telescope.

[4] In MTT, the entire context can be quantified over, so we may assume it to be empty.

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