Lax-Idempotent 2-Monads, Degrees of Relatedness, and Multilevel Type Theory

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Parametricity and Degrees of Relatedness Fourty years ago, Reynolds [Rey83] formulated his model of relational parametricity for (predicative [Lei91, Rey84]) System F. This was later reorganized as a model of System Fω and dependent type theory in reflexive graphs [Atk12, AGJ14], which evolved further into a cubical model [BCM15, NVD17] in order to support parametricity w.r.t. proof-relevant relations, as well as internal iterated parametricity.

Most accounts of parametricity for dependent type theory do not satisfy Reynolds' identity extension lemma – a.k.a. discreteness – for large types [AGJ14, BCM15, CH20]. The lemma and the discreteness condition express that homogeneous (i.e. non-dependent) graph edges are reflexive (i.e. constant), so that the edge relation in general can be understood as heterogeneous equality. The fact that identity extension can be satisfied for small types [AGJ14] actually has little to do with size; the reason is simply that discrete types are closed under all the usual type formers, except for the universe, which can be excluded by requiring smallness. This coupling between universe level (a device for safeguarding predicativity) and discreteness breaks down upon introducing HITs [Uni13, ch. 6] with edge constructors, or a discrete truncation, or certain modal types. Therefore, it is better to introduce an orthogonal stratification. In the type system RelDTT for Degrees of Relatedness [ND18], universes are annotated by a level as well as a *depth* $p \geq -1$ indicating the relational complexity of the types they classify. The idea is that a type of depth p is equipped with $p+1$ proof-relevant reflexive relations (as well as equality and 'true')

$$x = y \quad \Rightarrow \qquad x \frown_0 y \quad \Rightarrow \quad x \frown_1 y \quad \Rightarrow \quad \dots \quad \Rightarrow \quad x \frown_p y \qquad \Rightarrow \quad \top,$$

where discrete types must satisfy the identity extension lemma w.r.t. $\frown_0$, making $\frown_0$ a notion of heterogeneous equality. Data types such as Bool and Nat have depth 0, whereas the (discrete) universe of depth p types has depth $p+1$. RelDTT can in fact be seen as an instance of multimode type theory (MTT) [GKNB21] instantiated on a specific *mode theory* DoR [Nuy20, §9.3]. The objects of the 2-category DoR are the depths $p \geq -2$, which serve as the modes of the theory. Modalities $\mu : p \to q$ are specified monotone functions $- \cdot \mu : \{0 \leq \ldots \leq q\} \to \{= \leq 0 \leq \ldots \leq p \leq \top\}$, denoted as $\langle 0 \cdot \mu, \ldots, q \cdot \mu \rangle$.[1,2,3] The modal type $\langle \mu \mid A \rangle$ is then conceptually the same type as A, but the ith relation of $\langle \mu \mid A \rangle$ is the $(i \cdot \mu)$th relation of A. Modal functions are (at least semantically) functions whose domain is a modal type, so that $e : x \frown_{i \cdot \mu} y$ is sent to $f(e) : f(x) \frown_i f(y)$. The identity function is then continuous ($\mathbf{con} : p \to p$ with $i \cdot \mathbf{con} = i$), while polymorphic functions may be parametric ($\mathbf{par} : p+1 \to p$ with $i \cdot \mathbf{par} = i+1$) or ad hoc ($\mathbf{hoc} : q \to p$ with $i \cdot \mathbf{hoc} = (=)$). Algebras depend on their structure via the structural modality $\mathbf{str} : p \to p+1$ such that $\mathbf{par} \circ \mathbf{str} = \mathbf{con}$.

Depth p types were originally modelled in presheaves over the category DCube_p of *depth p cubes*, which is the free cartesian-category-with-terminal-object-$\top$ over the diagram

$$\begin{tikzcd} \top \arrow[r, shift left] \arrow[r, shift right] & (\!|0|\!) \arrow[r] & \ldots \arrow[r] & (\!|p|\!). \end{tikzcd}$$

This specialized model proved soundness of RelDTT, but the fact that it was constructed specifically for this occasion was at odds with the claim that RelDTT and its semantics explain the existence and behaviour of many modalities found implicitly or explicitly throughout the literature.

[1] The domain becomes empty if $q = -1$. By convention, the depth -2 is a freely added strict initial object.

[2] By a combinatorial argument, DoR is isomorphic to the simplex 2-category.

[3] Actually, RelDTT internalizes only the morphisms in DoR that have a left adjoint in DoR (and does not use depth -2), which for the original model was an unnecessary restriction inspired by the perceived need for a left division operation respecting context extension. For the general model below, we need to apply the same restriction so as to ensure the existence of a left adjoint operation on contexts for every internal modality.

Multilevel Type Theory Two-level type theories (2LTT) [Voe13, ACK16, ACKS17] are type systems built on top of another type system (called the *inner* system; in any treatment that I am aware of this is immediately specialized to HoTT) by internalizing aspects of its metatheory (such as extensional equality in the HoTT application) which can then be reasoned about in the *outer* system. Annenkov et al. give a general model for 2LTT: if the inner system is modelled in a category $\mathcal{C}$ (potentially its category of syntactic contexts and substitutions) then the outer system can be modelled in the presheaf category $\mathrm{Psh}(\mathcal{C})$, which contains $\mathcal{C}$ via the Yoneda-embedding. We can of course iterate this idea, which we call multilevel type theory: viewing the outer system as the inner one, we can add a further system modelled in $\mathrm{Psh}(\mathrm{Psh}(\mathcal{C}))$. This construction exhibits properties reminiscent of RelDTT. Given two objects $c, d \in \mathrm{Obj}(\mathcal{C})$, we can either first embed them in $\mathrm{Psh}(\mathcal{C})$ and then take the coproduct, yielding $\mathbf{y}c \uplus \mathbf{y}d$, or the other way around, yielding $\mathbf{y}(c \uplus d)$, and we have $\mathbf{y}c \uplus \mathbf{y}d \to \mathbf{y}(c \uplus d)$. We can view objects of $\mathrm{Psh}(\mathcal{C})$ as objects of $\mathcal{C}$ equipped with a further, weakest, relation, and $\mathbf{y}$ can be viewed as a codiscrete embedding. Therefore, we propose to alternatively model depth p types of RelDTT in $\mathrm{Psh}^{p+2}(\mathcal{C})$ for any category $\mathcal{C}$.

Lax-Idempotent 2-Monads It is well-known that $\mathrm{Psh} : \mathsf{Cat} \to \mathsf{Cat}$ sends a category to its free cocompletion; it is then unsurprising that Psh has the structure of a (weak) 2-monad. In fact, Psh is a prototypical instance of the more general concept of a (weak) *lax-idempotent 2-monad*:

Definition 1. A (strict) 2-monad $(\mathbf{M}, \eta, \mu)$ on a (strict) 2-category $\mathbf{C}$ is **lax-idempotent** if it satisfies one of many equivalent properties [nLa23b, nLa23c, Koc95] including:

- The equality $\mathrm{id} = \mu \circ \mathbf{M}\eta$ is the unit of an adjunction $\mathbf{M}\eta \dashv \mu$,
- The equality $\mu \circ \eta\mathbf{M} = \mathrm{id}$ is the co-unit of an adjunction $\mu \dashv \eta\mathbf{M}$.

Recall that any functor $F : \mathcal{C} \to \mathcal{D}$ gives rise to a triple of adjoint functors $F_! \dashv F^* \dashv F_*$ [nLa23a]. Here, $\mathrm{Psh}\, F := F_! : \mathrm{Psh}(\mathcal{C}) \to \mathrm{Psh}(\mathcal{D})$ is taken to be the (pseudo)functorial action of Psh. The Yoneda-embedding $\eta := \mathbf{y} : \mathrm{Id} \to \mathrm{Psh}$ is (pseudo)natural and serves as the unit of the 2-monad. It turns out that the adjoint triple obtained for $F = \mathbf{y} : \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ is exactly $\mathrm{Psh}\,\eta \dashv \mu \dashv \eta\,\mathrm{Psh}$, so that Psh is indeed a (weak) lax-idempotent 2-monad.

Iterated applications of $\mathbf{M}$ generate long chains of adjoint morphisms, e.g.

$$\mathbf{MMM}\eta \dashv \mathbf{MM}\mu \dashv \mathbf{MM}\eta\mathbf{M} \dashv \mathbf{M}\mu\mathbf{M} \dashv \mathbf{M}\eta\mathbf{MM} \dashv \mu\mathbf{MM} \dashv \eta\mathbf{MMM} : \mathbf{M}^3\mathcal{C} \to \mathbf{M}^4\mathcal{C}.$$

We find similar chains in DoR,[4] listing the modalities that insert or remove 1 relation into/from the stack, which clearly generate all the morphisms of DoR:

$$\langle =, 0, 1\rangle \dashv \langle 1, 2\rangle \dashv \langle 0, 0, 1\rangle \dashv \langle 0, 2\rangle \dashv \langle 0, 1, 1\rangle \dashv \langle 0, 1\rangle \dashv \langle 0, 1, \top\rangle : 1 \to 2.$$

Proposition 2. The following defines is a lax-idempotent monad on the mode theory DoR:

$$\begin{array}{ll}
\mathbf{M}\,p = p+1, & \eta : p \to p+1 \\
i \cdot \mathbf{M}(\mu : p \to q) = \begin{cases} \quad (=) & \text{if } p = -2, \\ \text{else} \quad p+1 & \text{if } i = q+1, \\ \text{else} \quad p+1 & \text{if } i \cdot \mu = \top, \\ \text{else} \quad i \cdot \mu, & \end{cases} & \begin{array}{l} \eta = \langle 0, \ldots, p, \top\rangle \\ \\ \mu : p+1 \to p \\ \mu = \langle 0, \ldots, p\rangle \end{array}
\end{array}$$

Theorem 3 (WIP)**.** The 2-category DoR is *freely* generated by the object -2 and the existence of a strict lax-idempotent 2-monad $(\mathbf{M}, \eta, \mu)$.

Sketch of proof. By analysis of string diagram representations of 1-morphisms [nLa23d, nLa23e]. The relations $0, \ldots, p$ correspond to regions enclosed between two strings, while $=$ and $\top$ are the outer regions. A coherence property can be proven for commuting 2-cells around μ, thus establishing uniqueness of 2-cells. □

This makes RelDTT the internal language of a strict lax-idempotent 2-monad $\mathbf{M} : \mathsf{Cat} \to \mathsf{Cat}$ acting on a single category, assuming that $\mathbf{M}$ produces CwFs [Dyb96] and that all generated right adjoints are CwF morphisms (which is the case for Psh [Nuy20, thm. 6.4.1]). Modulo strictification, we can instantiate $\mathbf{M}$ with Psh, as desired.

[4] In fact, picking $\mathcal{C} = -2$ and $\mathbf{M}$ as in proposition 2, the chains for $\mathbf{M}$ specialize to the ones on DoR.

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