

The Transpension Type: Technical Report

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August 20, 2020

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*Holds a PhD Fellowship from the Research Foundation - Flanders (FWO).

1 Introduction

The purpose of these notes is to give a categorical semantics for the transpension type [ND20], which is right adjoint to a potentially substructural dependent function type.

- In section 2 we discuss some prerequisites.
- In section 3, we define multipliers and discuss their properties.
- In section 4, we study how multipliers lift from base categories to presheaf categories.
- In section 5, we explain how typical presheaf modalities can be used in the presence of the transpension type.
- In section 6, we study commutation properties of prior modalities, substitution modalities and multiplier modalities.

2 Prerequisites

2.1 On adjoints

Lemma 2.1.1. Let $L \dashv R$.

- Natural transformations $LF \rightarrow G$ correspond to natural transformations $F \rightarrow RG$, naturally in F and G .
- Natural transformations $FR \rightarrow G$ correspond to natural transformations $F \rightarrow GL$, naturally in F and G .

Proof. The first statement is trivial.

To see the second statement, we send $\zeta : FR \rightarrow G$ to $\zeta L \circ F\eta : F \rightarrow GL$, and conversely $\theta : F \rightarrow GL$ to $G\epsilon \circ \theta R : FR \rightarrow G$. Naturality is clear. Mapping ζ to and fro, we get

$$G\epsilon \circ \zeta LR \circ F\eta R = \zeta \circ FR\epsilon \circ F\eta R = \zeta. \quad (1)$$

Mapping θ to and fro, we get

$$G\epsilon L \circ \theta RL \circ F\eta = G\epsilon L \circ GL\eta \circ \theta = \theta. \quad \square$$

Lemma 2.1.2. Assume 4 triples of adjoint functors: $E \dashv F \dashv G$ and $E' \dashv F' \dashv G'$ and $S_1 \dashv T_1 \dashv U_1$ and $S_2 \dashv T_2 \dashv U_2$ such that the following diagram commutes up to natural isomorphism:

$$\begin{array}{ccc} C_1 & \xrightarrow{F} & C_2 \\ T_1 \downarrow & & \downarrow T_2 \\ C'_1 & \xrightarrow{F'} & C'_2 \end{array} \quad (2)$$

Then we have

$$\begin{array}{lll} ES_2 \cong S_1E' & E'T_2 \rightarrow T_1E & \\ FS_1 \leftarrow S_2F' & F'T_1 \cong T_2F & FU_1 \rightarrow U_2F' \\ & G'T_2 \leftarrow T_1G & GU_2 \cong U_1G'. \end{array} \quad (3)$$

In fact, any one of these statements holds if only the adjoints used by that statement are given.

Proof. The central isomorphism is given. The other isomorphisms are obtained by taking the left/right adjoints of both hands of the original isomorphism. By picking one direction of the central isomorphism, we can step to the left/right/top/bottom by applying lemma 2.1.1. \square

Proposition 2.1.3. If a functor $R : C \rightarrow \widehat{\mathcal{W}}$ from a CwF C to a presheaf CwF $\widehat{\mathcal{W}}$ has a left adjoint L , then it is a weak CwF morphism.

Proof. We use the presheaf notations from [Nuy18] (section 2.3.1).

For $\Gamma \vdash_C T$ type, define $R\Gamma \vdash_{\widehat{\mathcal{W}}} RT$ type by

$$(W \triangleright_{\widehat{\mathcal{W}}} (RT)[\delta]) : \cong (LyW \vdash_C T[\varepsilon \circ L\delta]). \quad (4)$$

Naturality of this operation is easy to show, and the action of R on terms is given by $({}^Rt)[\delta] = t[\varepsilon \circ L\delta]$. \square

Definition 2.1.4. Given adjoint functors $L \dashv R$ such that R is a weak CwF morphism, and $A \in \text{Ty}(L\Gamma)$, we write $\langle R \mid A \rangle := (RA)[\eta] \in \text{Ty}(\Gamma)$.

Note that $\langle R \mid A[\varepsilon] \rangle = (RA)[R\varepsilon][\eta] = RA$.

2.2 Dependent ends and co-ends

We will use \forall and \exists to denote ends and co-ends as well as their dependent generalizations [Nuy20, §2.2.6-7]:

Definition 2.2.1. A **dependent end** of a functor $F : \text{Tw}(I) \rightarrow C$, somewhat ambiguously denoted $\forall i. F(i \xrightarrow{\text{id}} i)$, is a limit of F .

Definition 2.2.2. A **dependent co-end** of a functor $F : \text{Tw}(I)^{\text{op}} \rightarrow C$, somewhat ambiguously denoted $\exists i. F(i \xrightarrow{\text{id}} i)$, is a colimit of F .

Example 2.2.3. Assume a functor $G : C \rightarrow \mathcal{D}$. One way to denote the set of natural transformations $\text{Id}_C \rightarrow \text{Id}_C$ which map to the identity under G , is as

$$A := \forall (c \in C). \{ \chi : c \rightarrow c \mid G\chi = \text{id}_{Gc} \}.$$

In order to read the above as a dependent end, we must find a functor $G : \text{Tw}(C) \rightarrow \text{Set}$ such that $G(c \xrightarrow{\text{id}} c) = \{ \chi : c \rightarrow c \mid G\chi = \text{id}_{Gc} \}$. Clearly every covariant occurrence of c refers to the codomain of $(c \xrightarrow{\text{id}} c)$, whereas every contravariant occurrence refers to the domain. So when we apply G to a general object $(x \xrightarrow{\varphi} y)$ of $\text{Tw}(C)$, we should substitute x for every contravariant c and y for every covariant c . We can then throw in φ wherever this is necessary to keep things well-typed, as φ disappears anyway when $(x \xrightarrow{\varphi} y) = (c \xrightarrow{\text{id}} c)$. Thus, we get

$$G(x \xrightarrow{\varphi} y) = \{ \chi : x \rightarrow y \mid G\chi = G\varphi \}.$$

So we see that using a *dependent* end was necessary in order to mention id_c , as this generalizes to $\varphi : x \rightarrow y$ to which we do not have access in a non-dependent end.

An element of $v \in A$ is then a function

$$v : (c \in C) \rightarrow \{ \chi : c \rightarrow c \mid G\chi = \text{id}_{Gc} \}$$

such that, whenever $\varphi : x \rightarrow y$, we have $\varphi \circ v_x = v_y \circ \varphi$.

2.3 Presheaves

2.3.1 Notation

We use the presheaf notations from [Nuy18]. Concretely:

- The application of a presheaf $\Gamma \in \widehat{\mathcal{W}}$ to an object $W \in \mathcal{W}$ is denoted $W \Rightarrow \Gamma$.
- The restriction of $\gamma : W \Rightarrow \Gamma$ by $\varphi : V \rightarrow W$ is denoted $\gamma \circ \varphi$ or $\gamma\varphi$.
- The application of a presheaf morphism $\sigma : \Gamma \rightarrow \Delta$ to $\gamma : W \Rightarrow \Gamma$ is denoted $\sigma \circ \gamma$ or $\sigma\gamma$.
 - By naturality of σ , we have $\sigma \circ (\gamma \circ \varphi) = (\sigma \circ \gamma) \circ \varphi$.
- If $\Gamma \in \widehat{\mathcal{W}}$ and $T \in \text{Ty}(\Gamma)$ (also denoted $\Gamma \vdash T$ type), i.e. T is a presheaf over the category of elements \mathcal{W}/Γ , then we write the application of T to (W, γ) as $(W \triangleright T[\gamma])$ and $t \in (W \triangleright T[\gamma])$ as $W \triangleright t : T[\gamma]$.
 - By definition of type substitution in a presheaf CwF, we have $(W \triangleright T[\sigma][\gamma]) = (W \triangleright T[\sigma\gamma])$
- The restriction of $W \triangleright t : T[\gamma]$ by $\varphi : (V, \gamma \circ \varphi) \rightarrow (W, \gamma)$ is denoted as $W \triangleright t \langle \varphi \rangle : T[\gamma\varphi]$.
- If $t \in \text{tm}(\Gamma, T)$ (also denoted $\Gamma \vdash t : T$), then the application of t to (W, γ) is denoted $V \triangleright t[\gamma] : T[\gamma]$.
 - The naturality condition for terms is then expressed as $t[\gamma] \langle \varphi \rangle = t[\gamma\varphi]$.
 - By definition of term substitution in a presheaf CwF, we have $t[\sigma][\gamma] = t[\sigma\gamma]$.
- We omit applications of the isomorphisms $(W \Rightarrow \Gamma) \cong (\mathbf{y}W \rightarrow \Gamma)$ and $(W \triangleright T[\gamma]) \cong (\mathbf{y}W \vdash T[\gamma])$. This is not confusing: e.g. given $W \triangleright t : T[\gamma]$, the term $\mathbf{y}W \vdash t' : T[\gamma]$ is defined by $t'[\varphi] := t \langle \varphi \rangle$.

One advantage of these notations is that we can put presheaf cells in diagrams; we will use double arrows when doing so.

2.3.2 On the Yoneda-embedding

We consider the Yoneda-embedding $\mathbf{y} : \mathcal{W} \rightarrow \widehat{\mathcal{W}}$.

Proposition 2.3.1. A morphism $\varphi : V \rightarrow W$ in \mathcal{W} is:

- Mono if and only if $\mathbf{y}\varphi$ is mono,
- Split epi if and only if $\mathbf{y}\varphi$ is epi.

Proof. It is well-known that a presheaf morphism $\sigma : \Gamma \rightarrow \Delta$ is mono/epi if and only if $\sigma \circ \sqcup : (W \Rightarrow \Gamma) \rightarrow (W \Rightarrow \Delta)$ is injective/surjective for all W . Now $\mathbf{y}\varphi \circ \sqcup = \varphi \circ \sqcup$. So $\mathbf{y}\varphi$ is mono if and only if $\varphi \circ \sqcup$ is injective, which means φ is mono. On the other hand, $\mathbf{y}\varphi$ is epi if and only if $\varphi \circ \sqcup$ is surjective, which is the case precisely when id is in its image, and that exactly means that φ is split epi. \square

2.3.3 Lifting functors

Theorem 2.3.2. Any functor $F : \mathcal{V} \rightarrow \mathcal{W}$ gives rise to functors $F_! \dashv F^* \dashv F_*$, with a natural isomorphism $F_! \circ \mathbf{y} \cong \mathbf{y} \circ F : \mathcal{V} \rightarrow \widehat{\mathcal{W}}$. We will call $F_! : \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{W}}$ the **left lifting** of F to presheaves, $F^* : \widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{V}}$ the **central** and $F_* : \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{W}}$ the **right lifting**.¹ [Sta19]

¹The central and right liftings are also sometimes called the inverse image and direct image of F , but these are actually more general concepts and as such could perhaps cause confusion or unwanted connotations in some circumstances. The left-central-right terminology is very no-nonsense.

Proof. Using quantifier symbols for ends and co-ends, we can define:

$$\begin{aligned} W \Rightarrow F_! \Gamma &:= \exists V. (W \rightarrow FV) \times (V \Rightarrow \Gamma), \\ V \Rightarrow F^* \Delta &:= FV \Rightarrow \Delta \\ W \Rightarrow F_* \Gamma &:= \forall V. (FV \rightarrow W) \rightarrow (V \Rightarrow \Gamma) = (F^* y W \rightarrow \Gamma). \end{aligned}$$

By the co-Yoneda lemma, we have:

$$W \Rightarrow F_! y V = \exists V'. (W \rightarrow FV') \times (V' \rightarrow V) \cong (W \rightarrow FV) = (W \Rightarrow y FV),$$

i.e. $F_! y V \cong y FV$.

Adjointness also follows from applications of the Yoneda and co-Yoneda lemmas. \square

Notation 2.3.3. • We denote the cell $(V, \varphi, \gamma) : W \Rightarrow F_! \Gamma$ as $F_! \gamma \circ \varphi$. If we rename $F_!$, then we will also do so in this notation. We will further abbreviate $F_! \gamma \circ \text{id} = F_! \gamma$ and, if $\Gamma = yV$, also $F_! \text{id} \circ \varphi = \varphi$.

- If $\delta : FV \Rightarrow \Delta$, then we write $\alpha_F(\delta) : V \Rightarrow F^* \Delta$.
- If $\gamma : F^* y W \Rightarrow \Gamma$, then we write $\beta_F(\gamma) : W \Rightarrow F_* \Gamma$.

2.3.4 Dependent presheaf categories

Let \mathcal{W} be a category. Then $\widehat{\mathcal{W}}$ is a category with families (CwF). The following are standard notions:

Definition 2.3.4. For any $U \in \mathcal{W}$, the **category of slices over U** , denoted \mathcal{W}/U , has objects (W, ψ) where $W \in \mathcal{W}$ and $\psi : W \rightarrow U$ and the morphisms $(W, \psi) \rightarrow (W', \psi')$ are the morphisms $\chi : W \rightarrow W'$ such that $\psi' \circ \chi = \psi$.

Definition 2.3.5. For any $\Gamma \in \widehat{\mathcal{W}}$, the **category of elements** of Γ , denoted

$$\int_{\mathcal{W}} \Gamma \quad \text{or} \quad \mathcal{W}/\Gamma \tag{5}$$

has objects (W, γ) where $W \in \mathcal{W}$ and $\gamma : W \Rightarrow \Gamma$, and the morphisms $(W, \gamma) \rightarrow (W', \gamma')$ are the morphisms $\chi : W \rightarrow W'$ such that $\gamma' \circ \chi = \gamma$.

Clearly, we have an isomorphism $\mathcal{W}/U \cong \mathcal{W}/yU$ between the category of slices over U and the category of elements of yU .²

We will use type-theoretic notation to make statements about the CwF $\widehat{\mathcal{W}}$, e.g. $\Gamma \vdash \text{Ctx}$ means $\Gamma \in \widehat{\mathcal{W}}$ and $\Gamma \vdash T$ type means $T \in \text{Ty}(\Gamma)$. Now for any context or closed type $\Gamma \in \widehat{\mathcal{W}}$, there is another CwF $\widehat{\mathcal{W}}/\Gamma$. Statements about this category will also be denoted using type-theoretic notation, but prefixed with ' $\Gamma \mid$ '.

By unfolding the definitions of types and terms in a presheaf CwF, it is trivial to show that there is a correspondence — which we will treat as though it were the identity — between both CwFs:

- Contexts $\Gamma \mid \Theta \vdash \text{Ctx}$ correspond to types $\Gamma \vdash \Theta$ type which we will think of as telescopes $\Gamma. \Theta \vdash \text{Ctx}$,
- Substitutions $\Gamma \mid \sigma : \Theta \rightarrow \Theta'$ correspond to functions $\Gamma \vdash \sigma : \Theta \rightarrow \Theta'$, or equivalently to telescope substitutions $\text{id}_\Gamma. \sigma : \Gamma. \Theta \rightarrow \Gamma. \Theta'$,
- Types $\Gamma \mid \Theta \vdash T$ type correspond to types $\Gamma. \Theta \vdash T$ type,
- Terms $\Gamma \mid \Theta \vdash t : T$ correspond to terms $\Gamma. \Theta \vdash t : T$.

In summary, the pipe is equivalent to a dot.

Proposition 2.3.6. We have an equivalence of categories $\widehat{\mathcal{W}}/\Gamma \simeq \widehat{\mathcal{W}}/\Gamma$.

Proof. \rightarrow We map the presheaf $\Gamma \mid \Theta \vdash \text{Ctx}$ to the slice $(\Gamma. \Theta, \pi)$.

²Depending on pedantic details, we may even have $\mathcal{W}/U = \mathcal{W}/yU$.

← We map the slice (Δ, σ) to the preimage of σ , i.e. the presheaf σ^{-1} which sends (W, γ) to $\{\delta : W \Rightarrow \Delta \mid \sigma \circ \delta = \gamma\}$.

$\widehat{\mathcal{W}}/\Gamma$ We need a natural isomorphism $\eta : \forall \Theta. (\Gamma \mid \eta : \Theta \cong \pi^{-1})$. If $\theta : (W, \gamma) \Rightarrow \Theta$, then we define $\eta(\theta) = (\gamma, \theta) : W \Rightarrow \Gamma. \Theta$ and indeed we have $\pi \circ (\gamma, \theta) = \gamma$. This is clearly invertible.

$\widehat{\mathcal{W}}/\Gamma$ We need a natural isomorphism $\varepsilon : \forall (\Delta, \sigma). (\Gamma. \sigma^{-1}, \pi) \cong (\Delta, \sigma)$. Given $(\gamma, \delta) : W \Rightarrow \Gamma. \sigma^{-1}$ (i.e. we know $\sigma \circ \delta = \gamma$), we define $\varepsilon \circ (\gamma, \delta) = \delta : W \Rightarrow \Delta$. Then

$$\sigma \circ \varepsilon \circ (\gamma, \delta) = \sigma \circ \delta = \gamma = \pi \circ (\gamma, \delta), \quad (6)$$

so indeed we have a morphism in the slice category. It is inverted by sending $\delta : W \Rightarrow \Delta$ to $(\sigma \circ \delta, \delta) : W \Rightarrow \Gamma. \sigma^{-1}$. \square

Corollary 2.3.7. We have $\widehat{\mathcal{W}}/U \cong \widehat{\mathcal{W}}/yU \simeq \widehat{\mathcal{W}}/yU$. \square

2.3.5 Substitution and its adjoints

Definition 2.3.8. Given $U \in \mathcal{W}$, we write

- $\Sigma_U : \mathcal{W}/U \rightarrow \mathcal{W} : (W, \psi) \mapsto W$,
- $\Omega_U : \mathcal{W} \rightarrow \mathcal{W}/U : W \mapsto (W \times U, \pi_2)$ (if \mathcal{W} has cartesian products with U).

Proposition 2.3.9. If Ω_U exists, then $\Sigma_U \dashv \Omega_U$. We denote the unit as $\text{copy}_U : \text{Id} \rightarrow \Omega_U \Sigma_U$ and the co-unit as $\text{drop}_U : \Sigma_U \Omega_U \rightarrow \text{Id}$. \square

Proposition 2.3.10. If $U \rightarrow \top$ is split epi, then the functor Ω_U is faithful. If it is mono, then Ω_U is full.

Proof. To see faithfulness, we have some $v : \top \rightarrow U$, so that the action of Ω_U on morphisms can be inverted: $\varphi = \pi_1 \circ (\varphi \times U) \circ (\text{id}, v)$.

To see fullness, take slices (W_1, ψ_1) and (W_2, ψ_2) and a morphism $\varphi : W_1 \rightarrow W_2$. The fact that $U \rightarrow \top$ is mono just means that there is only a single morphism arriving in U . Then φ is also a morphism between the slices. \square

Definition 2.3.11. Given $\chi : W'_0 \rightarrow W_0$ in \mathcal{W} , we write

- $\Sigma^{\chi} : \mathcal{W}/W'_0 \rightarrow \mathcal{W}/W_0 : (W', \psi') \mapsto (W', \chi \circ \psi')$,
- $\Omega^{\chi} : \mathcal{W}/W_0 \rightarrow \mathcal{W}/W'_0$ for the functor that maps (W, ψ) to its pullback along χ (if \mathcal{W} has pullbacks along χ).

If $\chi = \pi_1 : W_0 \times U \rightarrow W_0$, we also write $\Sigma_U^{W_0} : \mathcal{W}/(W_0 \times U) \rightarrow \mathcal{W}/W_0$ and $\Omega_U^{W_0} : \mathcal{W}/W_0 \rightarrow \mathcal{W}/(W_0 \times U)$.

Proposition 2.3.12. If Ω^{χ} exists, then $\Sigma^{\chi} \dashv \Omega^{\chi}$. We denote the unit as $\text{copy}^{\chi} : \text{Id} \rightarrow \Omega^{\chi} \Sigma^{\chi}$ and the co-unit as $\text{drop}^{\chi} : \Sigma^{\chi} \Omega^{\chi} \rightarrow \text{Id}$. \square

Proposition 2.3.13. If χ is split epi, then Ω^{χ} is faithful. If χ is mono, then it is full.

Proof. To see faithfulness, we have some $v : W_0 \rightarrow W'_0$ such that $\chi \circ v = \text{id}$. Then the action of Ω^{χ} can be inverted: given $\varphi : (W_1, \psi_1) \rightarrow (W_2, \psi_2) \in \mathcal{W}/W_0$, we have

$$\varphi : W_1 \xrightarrow{(\text{id}, v \circ \psi_1)} W_1 \times_{W_0} W'_0 \xrightarrow{\varphi \times_{W_0} W'_0} W_2 \times_{W_0} W'_0 \xrightarrow{\pi_1} W_2. \quad (7)$$

To see fullness, take a morphism $\varphi : (W_1, \chi \circ \psi_1) \rightarrow (W_2, \chi \circ \psi_2)$. Then $\chi \circ \psi_2 \circ \varphi = \chi \circ \psi_1$. Because χ is mono, this implies that $\psi_2 \circ \varphi = \psi_1$, i.e. $\varphi : (W_1, \psi_1) \rightarrow (W_2, \psi_2)$. \square

Definition 2.3.14. Given $\sigma : \Psi' \rightarrow \Psi$ in $\widehat{\mathcal{W}}$, we write

- $\Sigma^{\sigma} : \mathcal{W}/\Psi' \rightarrow \mathcal{W}/\Psi : (W', \psi') \mapsto (W', \sigma \circ \psi')$,

- $\Omega^\sigma : \mathcal{W}/\Psi \rightarrow \mathcal{W}/\Psi'$ for the functor that maps (W, ψ) to its pullback along σ (if \mathcal{W} has pullbacks along σ), by which we mean a universal solution W' to the diagram

$$\begin{array}{ccc} W' & \xrightarrow{\quad} & W \\ \downarrow & & \downarrow \psi \\ \Psi' & \xrightarrow{\sigma} & \Psi. \end{array} \quad (8)$$

If $\sigma = \pi_1 : \Psi \times \Phi \rightarrow \Psi$, we also write $\Sigma_\Phi^\Psi : \mathcal{W}/(\Psi \times \Phi) \rightarrow \mathcal{W}/\Psi$ and $\Omega_\Phi^\Psi : \mathcal{W}/\Psi \rightarrow \mathcal{W}/(\Psi \times \Phi)$.

Proposition 2.3.15. If Ω^σ exists, then $\Sigma^\sigma \dashv \Omega^\sigma$. We denote the unit as $\text{copy}^\sigma : \text{Id} \rightarrow \Omega^\sigma \Sigma^\sigma$ and the co-unit as $\text{drop}^\sigma : \Sigma^\sigma \Omega^\sigma \rightarrow \text{Id}$. \square

Proposition 2.3.16. If σ is surjective, then Ω^σ is faithful. If σ is injective, then it is full.

Proof. If σ is surjective, then by the axiom of choice, there is at least a non-natural $f : \Psi \rightarrow \Psi'$ such that $\sigma \circ f = \text{id}$. The rest of the proof is as for proposition 2.3.13. \square

Definition 2.3.17. The functors $\Sigma^\sigma \dashv \Omega^\sigma$ give rise to four adjoint functors

$$\Sigma^\sigma \dashv \Omega^\sigma \dashv \Pi^\sigma \dashv \$^\sigma \quad (9)$$

between $\widehat{\mathcal{W}/\Psi}$ and $\widehat{\mathcal{W}/\Psi'}$, of which the first three exist if only Σ^σ exists.

The units and co-units will be denoted:

$$\begin{array}{ll} \text{copy}^\sigma : \text{Id} \rightarrow \Omega^\sigma \Sigma^\sigma & \text{drop}^\sigma : \Sigma^\sigma \Omega^\sigma \rightarrow \text{Id} \\ \text{const}^\sigma : \text{Id} \rightarrow \Pi^\sigma \Omega^\sigma & \text{app}^\sigma : \Omega^\sigma \Pi^\sigma \rightarrow \text{Id} \\ \text{reidx}^\sigma : \text{Id} \rightarrow \$^\sigma \Pi^\sigma & \text{unmerid}^\sigma : \Pi^\sigma \$^\sigma \rightarrow \text{Id} \end{array} \quad (10)$$

We remark that, if we read presheaves over \mathcal{W}/Ψ as types in context Ψ , then $\Omega^\sigma : \widehat{\mathcal{W}/\Psi} \rightarrow \widehat{\mathcal{W}/\Psi'}$ is the standard interpretation of substitution in a presheaf category. If $\sigma = \pi : \Psi.A \rightarrow \Psi$ is a weakening morphism, then $\Omega_A^\Psi := \Omega^\pi$ is the weakening substitution, $\Pi_A^\Psi := \Pi^\pi : \widehat{\mathcal{W}/\Psi}.A \rightarrow \widehat{\mathcal{W}/\Psi}$ is the standard interpretation of the Π -type and $\Sigma_A^\Psi := \Sigma^\pi : \widehat{\mathcal{W}/\Psi}.A \rightarrow \widehat{\mathcal{W}/\Psi}$ is isomorphic to the standard interpretation of the Σ -type.

Theorem 2.3.18. Given types $\Psi \vdash A, B$ type, the projections constitute a pullback diagram:

$$\begin{array}{ccc} \Psi.(A \times B) & \xrightarrow{\beta'} & \Psi.A \\ \alpha' \downarrow & & \downarrow \alpha \\ \Psi.B & \xrightarrow{\beta} & \Psi, \end{array} \quad (11)$$

and every pullback diagram in a presheaf category is isomorphic to a diagram of this form. We have the following commutation properties:

	Σ_B	Ω_B	Π_B	$\$B$
Σ_A	$\Sigma^\alpha \Sigma^{\beta'} \cong \Sigma^\beta \Sigma^{\alpha'}$	$\Sigma^{\alpha'} \Omega^{\beta'} \cong \Omega^\beta \Sigma^\alpha$	$\Sigma^\alpha \Pi^{\beta'} \rightarrow \Pi^\beta \Sigma^{\alpha'}$	
Ω_A	$\Omega^\alpha \Sigma^\beta \cong \Sigma^{\beta'} \Omega^{\alpha'}$	$\Omega^{\alpha'} \Omega^\beta \cong \Omega^{\beta'} \Omega^\alpha$	$\Omega^\alpha \Pi^{\beta'} \cong \Pi^{\beta'} \Omega^{\alpha'}$	$\Omega^{\alpha'} \$^\beta \rightarrow \$^{\beta'} \Omega^\alpha$
Π_A	$\Pi^\alpha \Sigma^{\beta'} \leftarrow \Sigma^\beta \Pi^{\alpha'}$	$\Pi^{\alpha'} \Omega^{\beta'} \cong \Omega^\beta \Pi^\alpha$	$\Pi^\alpha \Pi^{\beta'} \cong \Pi^{\beta'} \Pi^\alpha$	$\Pi^{\alpha'} \$^{\beta'} \cong \$^\beta \Pi^\alpha$
$\$A$		$\$^{\alpha'} \Omega^\beta \leftarrow \Omega^{\beta'} \$^\alpha$	$\$^\alpha \Pi^\beta \cong \Pi^{\beta'} \$^{\alpha'}$	$\$^{\alpha'} \$^\beta \cong \$^{\beta'} \$^\alpha$

(12)

where every statement holds if the mentioned functors exist.

Proof. In the base category, it is evident that $\Sigma^{\alpha}\Sigma^{\beta'} = \Sigma^{\beta}\Sigma^{\alpha'}$. By applying the functor \sqsubset^* , we obtain $\Omega^{\alpha'}|\Omega^{\beta}| = \Omega^{\beta'}|\Omega^{\alpha}|$, whence by lemma 2.1.2 the entire diagonal of the commutation table.

It is a well-known fact that Σ - and Π -types are respected by substitution, which gives us the isomorphisms for swapping Ω and either Σ or Π . Lemma 2.1.2 then gives the rest. \square

Theorem 2.3.19. Given $\sigma : \Psi' \rightarrow \Psi$, the following operations are invertible:

$$\frac{\Psi \mid \Sigma^{\sigma} \Gamma \vdash T \text{ type}}{\Psi' \mid \Gamma \vdash (\Omega^{\sigma} T)[\text{copy}^{\sigma}] \text{ type}} \quad \frac{\Psi \mid \Sigma^{\sigma} \Gamma \vdash t : T}{\Psi' \mid \Gamma \vdash (\Omega^{\sigma} t)[\text{copy}^{\sigma}] : (\Omega^{\sigma} T)[\text{copy}^{\sigma}]} \quad (13)$$

Proof. Note that T is a presheaf over $(\mathcal{W}/\Psi)/\Sigma^{\sigma} \Gamma$, and $(\Omega^{\sigma} T)[\text{copy}^{\sigma}]$ is a presheaf over $(\mathcal{W}/\Psi')/\Gamma$. We compare the objects of these categories:

$$\begin{aligned} & \text{Obj}((\mathcal{W}/\Psi)/\Sigma^{\sigma} \Gamma) \\ &= (W \in \mathcal{W}) \times (\psi : W \Rightarrow \Psi) \times \exists((W', \psi') \in \mathcal{W}/\Psi'. (\chi : (W, \psi) \rightarrow \Sigma^{\sigma}(W', \psi')) \times ((W', \psi') \Rightarrow \Gamma)) \\ &\cong (W \in \mathcal{W}) \times (\psi : W \Rightarrow \Psi) \times \exists W'. (\psi' : W' \Rightarrow \Psi') \times (\chi : (W, \psi) \rightarrow \Sigma^{\sigma}(W', \psi')) \times ((W', \psi') \Rightarrow \Gamma) \\ &\cong (W \in \mathcal{W}) \times (\psi : W \Rightarrow \Psi) \times \exists W'. (\psi' : W' \Rightarrow \Psi') \times (\chi : (W, \psi) \rightarrow (W', \sigma \circ \psi')) \times ((W', \psi') \Rightarrow \Gamma) \\ &\cong (W \in \mathcal{W}) \times \exists W'. (\psi' : W' \Rightarrow \Psi') \times (\chi : W \rightarrow W') \times ((W', \psi') \Rightarrow \Gamma) \\ &\quad \text{because } \chi \text{ is a slice morphism iff } \psi = \sigma \circ \psi' \circ \chi \\ &\cong (W \in \mathcal{W}) \times (\psi' : W \Rightarrow \Psi') \times ((W, \psi') \Rightarrow \Gamma) \\ &\cong \text{Obj}((\mathcal{W}/\Psi')/\Gamma). \end{aligned}$$

A similar consideration of the Hom-sets leads to the conclusion that both categories are isomorphic. Moreover, we remark that the isomorphism sends $((W, \psi'), \gamma)$ on the right to $((W, \sigma \circ \psi'), \Sigma^{\sigma} \gamma)$ on the left. When we consider the action of $(\Omega^{\sigma} T)[\text{copy}^{\sigma}]$ on $((W, \psi'), \gamma)$, we find:

$$\begin{aligned} ((W, \psi') \triangleright (\Omega^{\sigma} T)[\text{copy}^{\sigma}][\gamma]) &= (\Sigma^{\sigma}(W, \psi') \triangleright T[\Sigma^{\sigma} \gamma]) \\ &= ((W, \sigma \circ \psi') \triangleright T[\Sigma^{\sigma} \gamma]) \end{aligned}$$

In other words, the types T and $(\Omega^{\sigma} T)[\text{copy}^{\sigma}]$ are equal over an isomorphism of categories. Then certainly T can be retrieved from $(\Omega^{\sigma} T)[\text{copy}^{\sigma}]$. An identical argument works for terms. \square

3 Multipliers in the base category

3.1 Definition

Definition 3.1.1. Let \mathcal{W} be a category with terminal object \top . An object W is **spooky** if $() : W \rightarrow \top$ is not split epi. A category is **spooky** if it has a spooky object. Therefore, a category is non-spooky iff all morphisms to the terminal object are split epi.

Definition 3.1.2. Let \mathcal{W} be a category with terminal object \top . A **multiplier** for an object $U \in \mathcal{V}$ is a functor $\sqsubset \times U : \mathcal{W} \rightarrow \mathcal{V}$ such that $\top \times U \cong U$. This gives us a second projection $\pi_2 : \forall W. W \times U \rightarrow U$. We define the **fresh weakening functor** as $\lrcorner_U : \mathcal{W} \rightarrow \mathcal{V}/U : W \mapsto (W \times U, \pi_2)$.

We say that a multiplier is:

- **Endo** if it is an endofunctor (i.e. $\mathcal{V} = \mathcal{W}$), and in that case:
 - **Semicartesian** if it is copointed, i.e. if there is also a first projection $\pi_1 : \forall W. W \times U \rightarrow W$,
 - **3/4-cartesian** if it is a comonad, i.e. if there is additionally a ‘diagonal’ natural transformation $\sqsubset \times \delta : \forall W. W \times U \rightarrow (W \times U) \times U$ such that $\pi_1 \circ (W \times \delta) = (\pi_1 \times U) \circ (W \times \delta) = \text{id}$.
 - **Cartesian** if it is naturally isomorphic to the cartesian product with U ,

- **Cancellative** if \lrcorner_U is faithful, or equivalently (lemma 3.2.2) if $\sqcup \ltimes U$ is faithful,
- **Affine** if \lrcorner_U is full,
- **Non-spooky** if $\pi_2 : W \ltimes U \rightarrow U$ is always split epi, and in that case:
 - **Connection-free** if \lrcorner_U is essentially surjective on objects (V, ψ) such that ψ is split epi, i.e. if every such object in \mathcal{V}/U is isomorphic to some $\lrcorner_U W$.
 - A split epi slice (V, ψ) that is not in the image of \lrcorner_U even up to isomorphism, will be called a **connection** of the multiplier.
- **Quantifiable** if \lrcorner_U has a left adjoint $\exists_U : \mathcal{V}/U \rightarrow \mathcal{W}$. We denote the unit as $\text{copy}_U : \text{Id} \rightarrow \lrcorner_U \exists_U$ and the co-unit as $\text{drop}_U : \exists_U \lrcorner_U \rightarrow \text{Id}$.

3.2 Basic properties

Some readers may prefer to first consult some examples (section 3.3).

Proposition 3.2.1. For any multiplier, we have $(\sqcup \ltimes U) = \Sigma_U \circ \lrcorner_U$. □

Lemma 3.2.2. The functor $\sqcup \ltimes U$ is faithful if and only if \lrcorner_U is faithful.

Proof. We have $(\sqcup \ltimes U) = \Sigma_U \circ \lrcorner_U$ and $\Sigma_U : \mathcal{V}/U \rightarrow \mathcal{V}$ is faithful as is obvious from its definition. □

Proposition 3.2.3. A multiplier with a non-spooky domain is non-spooky.

Proof. The multiplier, as any functor, preserves split epimorphisms. □

Proposition 3.2.4. Cartesian multipliers are 3/4-cartesian, and 3/4-cartesian multipliers are semicartesian.

Proof. The functor $\sqcup \ltimes U$ is a comonad, and comonads are cointegrated by their co-unit. □

Proposition 3.2.5. Cartesian multipliers are quantifiable.

Proof. The left adjoint to $\lrcorner_U = \Omega_U$ is then given by $\exists_U(V, \varphi) = \Sigma_U(V, \varphi) = V$ (proposition 2.3.9). □

Proposition 3.2.6. Cartesian endomultipliers for non-spooky objects, are cancellative.

Non-spookiness is not required however: cancellative cartesian endomultipliers may be spooky (examples 3.3.4 and 3.3.6).

Proof. In this case, $\lrcorner_U = \Omega_U$ and $U \rightarrow \top$ is split epi, so this is part of proposition 2.3.10. □

Proposition 3.2.7. If an endomultiplier for U is both 3/4-cartesian and affine, then U is a terminal object. If the multiplier is moreover cartesian, then it is naturally isomorphic to the identity functor.

Proof. Consider the following diagram:

$$\begin{array}{ccc}
 \top \ltimes U & \xrightarrow{\top \ltimes \delta} & (\top \ltimes U) \ltimes U \\
 \searrow \pi_2 & & \swarrow \pi_2 \\
 & U &
 \end{array} \tag{14}$$

This is a morphism from $\top \ltimes \delta : \lrcorner_U \top \rightarrow \lrcorner_U(\top \ltimes U)$ and thus, by affinity, of the form $\lrcorner_U v$ for some $v : \top \rightarrow \top \ltimes U$. This means in particular that

$$\text{id}_{\top \ltimes U} = \pi_1 \circ (\top \ltimes \delta) = \pi_1 \circ (v \ltimes U) = v \circ \pi_1 : \top \ltimes U \rightarrow \top \ltimes U. \tag{15}$$

Composing on both sides with $\pi_2 : \top \ltimes U \cong U$, we find that $\text{id}_U = (\pi_2 \circ v) \circ (\pi_1 \circ \pi_2^{-1})$ factors over \top , which means exactly that $\pi_2 \circ v : \top \rightarrow U$ and $\pi_1 \circ \pi_2^{-1} : U \rightarrow \top$ constitute an isomorphism, i.e. U is terminal.

If $\sqcup \ltimes U$ is cartesian, then it is a cartesian product with a terminal object and therefore naturally isomorphic to the identity functor. □

3.3 Examples

Example 3.3.1 (Identity). The identity functor $W \times \top := W$ is an endomultiplier for \top .

It is cartesian, cancellative, affine, spooky iff \mathcal{W} is and otherwise connection-free, and quantifiable.

The functor $\perp_{\top} : \mathcal{W} \rightarrow \mathcal{W}/\top : W \mapsto (W, ())$ has a left adjoint $\exists_{\top} : \mathcal{W}/\top \rightarrow \mathcal{W} : (W, ()) \mapsto W$.

Example 3.3.2 (Cartesian product). Let \mathcal{W} be a category with finite products and $U \in \mathcal{W}$.

Then $\sqcup \times U$ is an endomultiplier for U .

It is cartesian, cancellative if (but not only if) U is non-spooky (proposition 3.2.6), affine if and only if $U \cong \top$ (proposition 3.2.7) and quantifiable (proposition 3.2.5). We do not consider spookiness for this general case.

The functor $\perp_U = \Omega_U : V \mapsto (V \times U, \pi_2)$ has a left adjoint $\exists_U = \Sigma_U : (W, \psi) \mapsto W$. Hence, we have $\exists_U \perp_U = \sqcup \times U$.

Example 3.3.3 (Affine cubes). Let \square^k be the category of affine non-symmetric k -ary cubes \mathbb{I}^n as used in [BCH14] (binary) or [BCM15] (unary). A morphism $\varphi : \mathbb{I}^m \rightarrow \mathbb{I}^n$ is a function $\sqcup \langle \varphi \rangle : \{i_1, \dots, i_n\} \rightarrow \{i_1 \dots i_m, 0, \dots, k-1\}$ such that $i \langle \varphi \rangle = j \langle \varphi \rangle \notin \{0, \dots, k-1\}$ implies $i = j$. We also write $\varphi = (i_1 \langle \varphi \rangle / i_1, \dots, i_n \langle \varphi \rangle / i_n)$. This category is spooky if and only if $k = 0$.

Consider the functor $\sqcup * \mathbb{I} : \square^k \rightarrow \square^k : \mathbb{I}^n \mapsto \mathbb{I}^{n+1}$, which is a multiplier for \mathbb{I} . It acts on morphisms $\varphi : \mathbb{I}^m \rightarrow \mathbb{I}^n$ by setting $\varphi * \mathbb{I} = (\varphi, i_{m+1}/i_{n+1})$.

It is straightforwardly seen to be semicartesian, not 3/4-cartesian, cancellative, affine, spooky iff $k = 0$ and connection-free when $k \neq 0$, and quantifiable.

The functor $\perp_{\mathbb{I}} : \mathbb{I}^n \mapsto (\mathbb{I}^{n+1}, (i_{n+1}/i_1))$ has a left adjoint the functor $\exists_{\mathbb{I}}$ which sends (\mathbb{I}^n, ψ) to \mathbb{I}^n if $i_1 \langle \psi \rangle \in \{0, \dots, k-1\}$ and to \mathbb{I}^{n-1} (by removing the variable $i_1 \langle \psi \rangle$ and renaming the next ones) otherwise. The action on morphisms is straightforwardly constructed.

In the case where $k = 2$, we can throw in an involution $\neg : \mathbb{I} \rightarrow \mathbb{I}$. This changes none of the above results, except that $i_1 \langle \psi \rangle$ may be the negation $\neg j$ of a variable j , in which case \exists_U removes the variable j .

Example 3.3.4 (Cartesian cubes). Let \boxtimes^k be the category of cartesian non-symmetric k -ary cubes \mathbb{I}^n . A morphism $\varphi : \mathbb{I}^m \rightarrow \mathbb{I}^n$ is any function $\sqcup \langle \varphi \rangle : \{i_1, \dots, i_n\} \rightarrow \{i_1 \dots i_m, 0, \dots, k-1\}$. This category is spooky if and only if $k = 0$.

Consider the functor $\sqcup \times \mathbb{I} : \boxtimes^k \rightarrow \boxtimes^k : \mathbb{I}^n \mapsto \mathbb{I}^{n+1}$, which is an endomultiplier for \mathbb{I} .

It is cartesian (hence non-affine and quantifiable with $\exists_{\mathbb{I}}(W, \psi) = W$), cancellative, spooky iff $k = 0$ and otherwise connection-free.

Again, involutions change none of the above results.

Example 3.3.5 (CCHM cubes). Let $\boxtimes_{\vee, \wedge, \neg}$ be the category of (binary) CCHM cubes [CCHM17]. What's special here is that we have morphisms $\vee, \wedge : \mathbb{I}^2 \rightarrow \mathbb{I}$ (as well as involutions). This category is not spooky.

Again, we consider the functor $\sqcup \times \mathbb{I} : \boxtimes_{\vee, \wedge, \neg} \rightarrow \boxtimes_{\vee, \wedge, \neg} : \mathbb{I}^n \mapsto \mathbb{I}^{n+1}$, which is an endomultiplier for \mathbb{I} .

It is cartesian (hence non-affine and quantifiable with $\exists_{\mathbb{I}}(W, \psi) = W$), cancellative, not spooky, and not connection-free (since (\mathbb{I}^2, \vee) and (\mathbb{I}^2, \wedge) are connections).

Example 3.3.6 (Clocks). Let \odot be the category of clocks, used as a base category in guarded type theory [BM20]. Its objects take the form $(i_1 : \odot_{k_1}, \dots, i_n : \odot_{k_n})$ where all $k_j \geq 0$. We can think of a variable of type \odot_k as representing a clock (i.e. a time dimension) paired up with a certificate that we do not care what happens after the time on this clock exceeds k . Correspondingly, we have a map $\odot_k \rightarrow \odot_{\ell}$ if $k \leq \ell$. These maps, together with weakening, exchange, and contraction, generate the category. The terminal object is $()$ and every other object is spooky.

Consider in this category the functor $\sqcup \times (i : \odot_k) : \odot \rightarrow \odot : W \mapsto (W, i : \odot_k)$, which is an endomultiplier for $(i : \odot_k)$.

It is cartesian (hence non-affine and quantifiable with $\exists_{(i : \odot_k)}(W, \psi) = W$), cancellative and spooky.

Example 3.3.7 (Twisting posets). Let \mathcal{P} be the category of finite non-empty posets and monotonic maps. This category is non-spooky.

Let $\mathbb{I} = \{0 < 1\}$ and let $W \times \mathbb{I} = (W^{\text{op}} \times \{0\}) \cup (W \times \{1\})$ with $(x, 0) < (y, 1)$ for all $x, y \in W$. This is an endomultiplier for \mathbb{I} .

It is easily seen to be: not semicartesian, cancellative, not affine, not spooky, not connection-free, and quantifiable.

The functor $\perp_{\mathbb{I}} : V \mapsto (V \ltimes \mathbb{I}, \pi_2)$ has a left adjoint $\exists_{\mathbb{I}} : (W, \psi) \mapsto \psi^{-1}(0)^{\text{op}} \uplus \psi^{-1}(1)$ where elements from different sides of the \uplus are incomparable.

We see this category as a candidate base category for directed type theory. The idea is that a cell over W is a commutative diagram in a category. A problem here is that a cell over a discrete poset such as $\{x, y\}$ where x and y are incomparable, should then be the same as a pair of cells over $\{x\}$ and $\{y\}$. This will require that we restrict from presheaves to sheaves, but that makes it notoriously difficult to model the universe [XE16]. One solution would be to restrict to totally ordered sets, but then we lose the left adjoint $\exists_{\mathbb{I}}$. We address this in example 3.3.8.

Example 3.3.8 (Affine twisted cubes). Let \boxtimes be the subcategory of \mathcal{P} whose objects are generated by \top and $\sqcup \ltimes \mathbb{I}$ (note that every object then also has an opposite since $\top^{\text{op}} = \top$ and $(V \ltimes \mathbb{I})^{\text{op}} \cong V \ltimes \mathbb{I}$), and whose morphisms are given by

- $(\varphi, 0) : \boxtimes(V, W \ltimes \mathbb{I})$ if $\varphi : \boxtimes(V, W^{\text{op}})$,
- $(\varphi, 1) : \boxtimes(V, W \ltimes \mathbb{I})$ if $\varphi : \boxtimes(V, W)$,
- $\varphi \ltimes \mathbb{I} : \boxtimes(V \ltimes \mathbb{I}, W \ltimes \mathbb{I})$ if $\varphi : \boxtimes(V, W)$,
- $() : \boxtimes(V, \top)$.

Note that this collection automatically contains all identities, composites, and opposites. It is isomorphic to Pinyo and Kraus's category of twisted cubes, as can be seen from the ternary representation of said category [PK19, def. 34]. This category is not spooky.

Again, we consider the functor $\sqcup \ltimes \mathbb{I} : \boxtimes \rightarrow \boxtimes$, which is well-defined by construction of \boxtimes and an endomultiplier for \mathbb{I} . It corresponds to Pinyo and Kraus's twisted prism functor.

It is: not semicartesian, cancellative, affine, not spooky, connection-free, and quantifiable.

The left adjoint to $\perp_{\mathbb{I}} : W \mapsto (W \ltimes \mathbb{I}, \pi_2)$ is now given by

$$\exists_{\mathbb{I}} : \begin{cases} (W, ((), 0)) & \mapsto W^{\text{op}} \\ (W, ((), 1)) & \mapsto W \\ (W \ltimes \mathbb{I}, () \ltimes \mathbb{I}) & \mapsto W, \end{cases} \quad (16)$$

with the obvious action on morphisms.

Example 3.3.9 (Embargoes). In order to define contextual fibrancy [BT17] internally, we need to be able to somehow put a sign in the context $\Gamma.!\Theta$ in order to be able to say: the type is fibrant over Θ in context Γ . We call this an embargo and say that Θ is embargoed whereas Γ is not. If C is the category of contexts, then $\Gamma.!\Theta$ can be seen as an object of the arrow category C^{\uparrow} , namely the arrow $\Gamma.\Theta \rightarrow \Gamma$.

If $C = \widehat{\mathcal{W}}$ happens to be a presheaf category, then we have an isomorphism of categories $H : \widehat{\mathcal{W}}^{\uparrow} \cong \widehat{\mathcal{W} \times \uparrow}$ where $\uparrow = \{\perp \rightarrow \top\}$. Under this isomorphism, we have $y(W, \top) \cong H(yW \xrightarrow{\text{id}} yW)$ which we think of as $yW.!\top$ and $y(W, \perp) \cong H(\perp \xrightarrow{\text{id}} yW)$ which we think of as $yW.!\perp$. Thus, forgetting the second component of (W, o) amounts to erasing the embargoed information. A (W, \top) -cell of $\Gamma.!\Theta$ is a W -cell of $\Gamma.\Theta$, i.e. a partly embargoed W -cell. We can extract the unembargoed information by restricting to (W, \perp) , as a (W, \perp) -cell of $\Gamma.!\Theta$ is just a W -cell of Γ .

There are 3 adjoint functors $\perp \dashv () \dashv \top$ between \uparrow and Point from which we obtain 3 adjoint functors $(\text{Id}, \perp) \dashv \pi_1 \dashv (\text{Id}, \top)$ between $\mathcal{W} \times \uparrow$ and \mathcal{W} . The rightmost functor $(\text{Id}, \top) : \mathcal{W} \rightarrow \mathcal{W} \times \uparrow$ is a multiplier for the terminal object $! := (\top, \top) \in \mathcal{W} \times \uparrow$, denoted $\sqcup \ltimes !$.

It is: not endo, cancellative, affine, spooky iff \mathcal{W} is and otherwise connection-free, and quantifiable.

In order to look at the left adjoint, note first that since $!$ is terminal, we have $(\mathcal{W} \times \uparrow)/! \cong \mathcal{W} \times \uparrow$ and clearly $\perp_!$ corresponds to (Id, \top) under this isomorphism. This functor is part of a chain of *three* adjoint functors $(\text{Id}, \perp) \dashv \pi_1 \dashv (\text{Id}, \top)$ so that the multiplier is not just quantifiable but $\exists_!$ even has a further left adjoint!

If $\sqcup \ltimes U : \mathcal{V} \rightarrow \mathcal{W}$ is a multiplier, then we can lift it to a multiplier $\sqcup \ltimes (U \ltimes !) : \mathcal{V} \times \uparrow \rightarrow \mathcal{W} \times \uparrow$ by applying it to the first component, i.e. $(W, o) \ltimes (U \ltimes !) = (W \ltimes U, o)$. The resulting multiplier inherits all properties in definition 3.1.2 from $\sqcup \ltimes U$, except that it is always spooky.

Example 3.3.10 (Enhanced embargoes). If $\sqcup \ltimes U$ is a semicartesian endomultiplier on \mathcal{W} , then we might want to apply it to an arrow $V \xrightarrow{\psi} W$ by sending it to $V \ltimes U \xrightarrow{\psi \circ \pi_1} W$. This operation is not definable on $\mathcal{W} \times \uparrow$, which only encodes identity arrows of the forms $W \rightarrow W$ (as (W, \top)) and $\perp \rightarrow W$ (as (W, \perp)). For this reason, we move to the comma category $\mathcal{W}_\perp / \mathcal{W}$ where \mathcal{W}_\perp is \mathcal{W} with a freely added initial object. This comma category has as its objects arrows $V \xrightarrow{\psi} W$ where $V \in \mathcal{W}_\perp$ and $W \in \mathcal{W}$. Morphisms are simply commutative squares. A $(V \xrightarrow{\psi} W)$ -cell is now a non-embargoed W -cell γ with embargoed information about $\gamma \circ \psi$.

We still have three adjoint functors $(\perp \rightarrow \sqcup) \dashv \text{Cod} \dashv \Delta$ where $\Delta W = (W \rightarrow W)$. Further right adjoints would be $\text{Dom} \dashv (\sqcup \rightarrow \top)$, but Dom is not definable as the domain might be \perp . We take $\Delta : W \mapsto (W \rightarrow W)$ as a multiplier for $\mathbf{!} := (\top \rightarrow \top)$, denoted $\sqcup \ltimes \mathbf{!} := \Delta$. For reasons that will become apparent later, we write $\mathbf{!} \sqrt{\sqcup} := (\sqcup \rightarrow \top)$. Note that a $(\mathbf{!} \sqrt{U})$ -cell is an unembargoed point with embargoed information about the degenerate U -cell on that point. E.g. in a context $\Gamma. \mathbf{!} \Theta$, an $(\mathbf{!} \sqrt{\mathbb{I}})$ -cell is exactly a path in Θ above a point in Γ , which is a concept that we need to quantify over when defining internal Kan fibrancy [BT17].

This multiplier is: not endo, cancellative, affine, spooky iff \mathcal{W} is and otherwise connection-free, and quantifiable.

Now we get back to our multiplier $\sqcup \ltimes U$ which we can still lift to $\sqcup \ltimes (U \ltimes \mathbf{!})$ by applying it to both domain and codomain, i.e. $(V \rightarrow W) \ltimes (U \ltimes \mathbf{!}) := (U \ltimes V \rightarrow U \ltimes W)$, where by convention $\perp \ltimes U = \perp$. It inherits all properties in definition 3.1.2 from $\sqcup \ltimes U$, except that it is always spooky.

If $\sqcup \ltimes U$ is semicartesian, then we can also lift it to $\sqcup \ltimes (\mathbf{!} \sqrt{U})$ by applying it only to the domain, i.e. $(V \rightarrow W) \ltimes (\mathbf{!} \sqrt{U}) = (V \ltimes U \rightarrow W)$. This again inherits all properties in definition 3.1.2 from $\sqcup \ltimes U$, except that it is always spooky.

Example 3.3.11 (Depth d cubes). Let \Box_d with $d \geq -1$ be the category of depth d cubes, used as a base category in degrees of relatedness [ND18, Nuy18].³ Its objects take the form $(i_1 : \langle k_1 \rangle, \dots, i_n : \langle k_n \rangle)$ where all $k_j \in \{0, \dots, d\}$. We have a map $\langle \ell \rangle \rightarrow \langle k \rangle$ if $k \leq \ell$. These maps, together with weakening, exchange, and contraction, generate the category. The terminal object is $()$ and the category is non-spooky.

Consider in this category the functor $\sqcup \ltimes (i : \langle k \rangle) : \Box_d \rightarrow \Box_d : W \mapsto (W, i : \langle k \rangle)$, which is an endomultiplier for $(i : \langle k \rangle)$.

It is cartesian (hence non-affine and quantifiable with $\exists_{(i : \langle k \rangle)}(W, \psi) = W$), cancellative, non-spooky and connection-free.

Example 3.3.12 (Erasure). Let $\text{Erase}_d = \{\top \leftarrow 0 \leftarrow 1 \leftarrow \dots \leftarrow d\}$ with $d \geq -1$. This category has cartesian products $m \times n = \max(m, n)$ and all non-terminal objects are spooky. We remark that $\widehat{\text{Erase}}_0$ is the Sierpiński topos.

We consider the endomultiplier $\sqcup \ltimes i : \text{Erase}_d \rightarrow \text{Erase}_d$.

It is cartesian (hence non-affine and quantifiable with $\exists_i(j, \psi) = j$), cancellative and spooky.

We believe that this base category is a good foundation for studying the semantics of erasure of irrelevant subterms in Degrees of Relatedness [ND18]. The idea is that, for a presheaf Γ , the set $\top \Rightarrow \Gamma$ is the set of elements, whereas the set $i \Rightarrow \Gamma$ is the set of elements considered up to i -relatedness, but also whose existence is only guaranteed by a derivation up to i -relatedness.

3.4 Properties

3.4.1 Functoriality

Definition 3.4.1. A **multiplier morphism** or **morphism multiplier** for $v : U \rightarrow U'$ is a natural transformation $\sqcup \ltimes v : \sqcup \ltimes U \rightarrow \sqcup \ltimes U'$ such that $\pi_2 \circ (\top \ltimes v) \circ \pi_2^{-1} = v : U \rightarrow U'$ (or equivalently $\pi_2 \circ (W \ltimes v) = v \circ \pi_2 : W \ltimes U \rightarrow U'$ for all W).

- A morphism of semicartesian multipliers is **semicartesian** if it is a morphism of copointed endofunctors, i.e. if $\pi_1 \circ (W \ltimes v) = \pi_1$,

³For $d = -1$, we get the point category. For $d = 0$, we get the category of binary cartesian cubes \Box^2 . For $d = 1$, we get the category of bridge/path cubes [NVD17, Nuy18].

- A morphism of 3/4-cartesian multipliers is **3/4-cartesian** if it is a monad morphism, i.e. if additionally $(W \ltimes \delta) \circ (W \ltimes v) = ((W \ltimes v) \ltimes v) \circ (W \ltimes \delta)$,
- A morphism of cartesian multipliers is **cartesian** if it is the cartesian product with v .

Proposition 3.4.2. A semicartesian morphism of cartesian multipliers, is cartesian.

Proof. We have $\pi_2 \circ (W \ltimes v) = v \circ \pi_2$ and $\pi_1 \circ (W \ltimes v) = \pi_1$. Hence, $(W \ltimes v) = (\pi_1, v \circ \pi_2) = W \times v$. \square

Proposition 3.4.3 (Functoriality). A multiplier morphism $\sqcup \ltimes v : \sqcup \ltimes U \rightarrow \sqcup \ltimes U'$ gives rise to a natural transformation $\Sigma^{/v} \circ \downarrow_U \rightarrow \downarrow_{U'}$. Hence, for quantifiable multipliers, we also have $\exists_{U'} \circ \Sigma^{/v} \rightarrow \exists_U$.

Proof. We have to show that for every $W \in \mathcal{W}$, we get $(W \ltimes U, v \circ \pi_2) \rightarrow (W \ltimes U', \pi_2)$. The morphism $W \ltimes v : W \ltimes U \rightarrow W \ltimes U'$ does the job. The second statement follows from lemma 2.1.1. \square

3.4.2 Quantification and kernel theorem

Theorem 3.4.4 (Quantification theorem). If $\sqcup \ltimes U$ is

1. cancellative, affine and quantifiable, then we have a natural isomorphism $\text{drop}_U : \exists_U \downarrow_U \cong \text{Id}$.
2. semi-cartesian, then we have:
 - (a) $\text{hide}_U : \Sigma_U \rightarrow \exists_U$ (if quantifiable),
 - (b) $\text{spoil}_U : \downarrow_U \rightarrow \Omega_U$ (if Ω_U exists),
 - (c) in any case $\Sigma_U \downarrow_U \rightarrow \text{Id}$.
3. 3/4-cartesian, then there is a natural transformation $\Sigma^{/\delta} \circ \downarrow_U \rightarrow \downarrow_{U \ltimes U}$, where we compose multipliers as in theorem 3.6.1.
4. cartesian, then we have:
 - (a) $\exists_U \cong \Sigma_U$,
 - (b) $\downarrow_U \cong \Omega_U$,
 - (c) $\exists_U \downarrow_U \cong \Sigma_U \Omega_U = (\sqcup \times U) \cong (\sqcup \ltimes U)$.

Moreover, these isomorphisms become equalities by choosing Σ_U and Ω_U wisely (both are defined only up to isomorphism).

Proof. 1. This is a standard fact of fully faithful right adjoints such as \downarrow_U .

2. By lemma 2.1.1, it is sufficient to prove $\Sigma_U \downarrow_U \rightarrow \text{Id}$. But $\Sigma_U \downarrow_U = (\sqcup \ltimes U)$, so this is exactly the statement that the multiplier is semicartesian.

3. This is a special case of proposition 3.4.3.

4. By uniqueness of the cartesian product, we have $\downarrow_U \cong \Omega_U$. Then the multiplier is quantifiable with $\exists_U \cong \Sigma_U$. The last point is now trivial. \square

Theorem 3.4.5 (Kernel theorem for non-spooky multipliers). If $\sqcup \ltimes U : \mathcal{W} \rightarrow \mathcal{V}$ is non-spooky, cancellative, affine and connection-free, then $\downarrow_U : \mathcal{W} \simeq \mathcal{V} // U$ is an equivalence of categories, where $\mathcal{V} // U$ is the full subcategory of \mathcal{V}/U whose objects are the split epimorphic slices.

Proof. By non-spookiness, \downarrow_U lands in $\mathcal{V} // U$. The other properties assert that \downarrow_U is fully faithful and essentially surjective as a functor $\mathcal{W} \rightarrow \mathcal{V} // U$. \square

Definition 3.4.6. If we are doing classical mathematics, or if $\sqcup \ltimes U$ is quantifiable, then we obtain an inverse functor, which we denote \ker_U .

The kernel theorem applies to examples 3.3.3 and 3.3.8.

We will use the kernel theorem in order to model Moulin's Φ -operator [Mou16], and to make the semantics of Moulin's Ψ -type [Mou16] unique up to isomorphism.

3.4.3 Dealing with spookiness

Since spooky multipliers do not guarantee that \perp_U produces split epi slices, we need to come up with a larger class of suitable epi-like morphisms to U before we can proceed.

Definition 3.4.7. Given a multiplier $\sqcup \ltimes U : \mathcal{W} \rightarrow \mathcal{V}$, we say that a morphism $\varphi : V \rightarrow U$ is **dimensionally split** if there is some $W \in \mathcal{W}$ such that $\pi_2 : W \ltimes U \rightarrow U$ factors over φ . If $\pi_2 = \varphi \circ \chi$, we say that χ is a **dimensional section** of φ . We write $\mathcal{V} // U$ for the full subcategory of \mathcal{V}/U of dimensionally split slices.

The non-spookiness condition for multipliers is automatically satisfied if we replace ‘split epi’ with ‘dimensionally split’:

Corollary 3.4.8. For any multiplier $\sqcup \ltimes U$, any projection $\pi_2 : W \ltimes U \rightarrow U$ is dimensionally split. \square

Proposition 3.4.9. Take a multiplier $\sqcup \ltimes U : \mathcal{W} \rightarrow \mathcal{V}$.

1. If $\varphi \circ \chi$ is dimensionally split, then so is φ .
2. The identity morphism $\text{id}_U : U \rightarrow U$ is dimensionally split.
3. If $\varphi : V \rightarrow U$ is dimensionally split and $\chi : V' \rightarrow V$ is split epi, then $\varphi \circ \chi : V' \rightarrow U$ is dimensionally split.
4. Every split epimorphism to U is dimensionally split.
5. If $\sqcup \ltimes U$ is non-spooky, then every dimensionally split morphism is split epi.

Proof. 1. If $\pi_2 : W \ltimes U \rightarrow U$ factors over $\varphi \circ \chi$, then it certainly factors over φ .

2. Since $\pi_2 : \top \ltimes U \rightarrow U$ factors over id_U .

3. Let φ' be a dimensional section of φ and χ' a section of χ . Then $\chi' \circ \varphi'$ is a dimensional section of $\varphi \circ \chi$.

4. From the previous two points, or (essentially by composition of the above reasoning) because if $\chi : U \rightarrow V$ is a section of $\varphi : V \rightarrow U$, then $\chi \circ \pi_2 : \top \ltimes U \rightarrow V$ is a dimensional section of φ .

5. If $\varphi : V \rightarrow U$ is dimensionally split, then some $\pi_2 : W \ltimes U \rightarrow U$ factors over φ . Since π_2 is split epi, id_U factors over π_2 and hence over φ , i.e. φ is split epi. \square

We can now extend the notions of connection and connection-freedom to spooky multipliers without changing their meaning for non-spooky multipliers:

Definition 3.4.10. We say that a multiplier $\sqcup \ltimes U : \mathcal{W} \rightarrow \mathcal{V}$ is **connection-free** if \perp_U is essentially surjective on $\mathcal{V} // U$, the full subcategory of \mathcal{V}/U of dimensionally split slices. A dimensionally split slice (V, ψ) that is not in the image of \perp_U even up to isomorphism, will be called a **connection** of the multiplier.

Theorem 3.4.11 (Kernel theorem). If a multiplier $\sqcup \ltimes U : \mathcal{W} \rightarrow \mathcal{V}$ is cancellative, affine and connection-free, then $\perp_U : \mathcal{W} \simeq \mathcal{V} // U$ is an equivalence of categories. \square

Example 3.4.12 (Identity). In the category \mathcal{W} with the identity multiplier $W \ltimes \top = W$, every morphism $W \rightarrow \top$ is dimensionally split. The multiplier is connection-free.

Example 3.4.13 (Nullary cubes). In the categories of nullary affine cubes \square^0 (example 3.3.3) and nullary cartesian cubes \square^0 (example 3.3.4), a morphism $\varphi : \mathbb{I}^n \rightarrow \mathbb{I}$ is dimensionally split if $i_1 \langle \varphi \rangle$ is a variable. The multipliers $\sqcup * \mathbb{I} : \square^0 \rightarrow \square^0$ and $\sqcup \times \mathbb{I} : \square^0 \rightarrow \square^0$ are connection-free.

Example 3.4.14 (Clocks). In the category of clocks \odot (example 3.3.6), a morphism $\varphi : V \rightarrow (i : \odot_k)$ is dimensionally split if $i \langle \varphi \rangle$ has clock type \odot_k . The multiplier $\sqcup \times (i : \odot_k)$ is connection-free.

Example 3.4.15 (Twisted cubes). In the category \boxtimes of twisted cubes (example 3.3.8), a morphism $\varphi : V \rightarrow \top \times \mathbb{I}$ is dimensionally split if it equals $\varphi = () \times \mathbb{I}$. The multiplier $\sqcup \times \mathbb{I}$ is connection-free.

Example 3.4.16 (Embargoes). For the embargo multiplier $\sqcup \times \mathbb{I} := (\text{Id}, \top) : \mathcal{W} \rightarrow \mathcal{W} \times \uparrow$ (example 3.3.9) for $\mathbb{I} := (\top, \top)$, a morphism $((), ()) : (W, o) \rightarrow \mathbb{I}$ is dimensionally split if $o = \top$. The multiplier $\sqcup \times \mathbb{I}$ is connection-free.

For $\sqcup \times (\mathbb{I} \times U)$, a morphism $(\varphi, ()) : (W, o) \rightarrow (U, \top) = (\mathbb{I} \sqrt{U})$ is dimensionally split if $\varphi : W \rightarrow U$ is dimensionally split for $\sqcup \times U$. Connection-freeness is then inherited from $\sqcup \times U$.

Example 3.4.17 (Enhanced embargoes). For the enhanced embargo multiplier $\sqcup \times \mathbb{I} : \mathcal{W} \rightarrow \mathcal{W}_{\perp} / \mathcal{W} : W \mapsto (W \rightarrow W)$ (example 3.3.10), a morphism $(V \rightarrow W) \rightarrow (\top \rightarrow \top) = \mathbb{I}$ is dimensionally split if $V \neq \perp$. This multiplier is generally not connection-free: since it only produces identity arrows, any dimensionally split non-identity arrow is a connection.

For $\sqcup \times (U \times \mathbb{I})$, a morphism $(V \rightarrow W) \rightarrow (U \rightarrow U) = (U \times \mathbb{I})$ is dimensionally split (with section $([], \chi) : (\perp \rightarrow W' \times U) \rightarrow (V \rightarrow W)$) if the morphism $W \rightarrow U$ is dimensionally split for $\sqcup \times U$ with section $\chi : W' \times U \rightarrow W$. The multiplier $\sqcup \times (U \times \mathbb{I})$ is generally not connection-free, as the domain part of a dimensionally split morphism could be anything.

For $\sqcup \times (\mathbb{I} \sqrt{U})$, any morphism $(V \rightarrow W) \rightarrow (U \rightarrow \top) = (\mathbb{I} \sqrt{U})$ is dimensionally split by

$$([], \text{id}) : (\perp \rightarrow W) \times (\mathbb{I} \sqrt{U}) = (\perp \rightarrow W) \rightarrow (V \rightarrow W). \quad (17)$$

This multiplier is therefore generally not connection-free.

To conclude, we have made the base category more complicated in order to be able to define the last multiplier, but as a tradeoff we now have connections to deal with.

Example 3.4.18 (Erasure). In the category Erase_d (example 3.3.12) with multiplier $\sqcup \times i$, all morphisms to i are dimensionally split. The multiplier is connection-free.

3.4.4 Boundaries

Definition 3.4.19. The boundary ∂U of a multiplier $\sqcup \times U : \mathcal{W} \rightarrow \mathcal{V}$ is a presheaf over \mathcal{V} such that the cells $V \Rightarrow \partial U$ are precisely the morphisms $V \rightarrow U$ that are *not* dimensionally split.

Proposition 3.4.20. If $\sqcup \times U$ is non-spooky, then ∂U is the largest strict subobject of yU .

Proof. Recall that if the multiplier is non-spooky, then dimensionally split and split epi are synonymous.

Clearly, $\partial U \subseteq yU$. Since $\text{id} : U \rightarrow U$ is split epi, we have $\partial U \subsetneq yU$. Now take another strict subobject $Y \subsetneq yU$. We show that $Y \subseteq \partial U$.

We start by showing that $\text{id} \notin U \Rightarrow Y$. Otherwise, every $\varphi \in V \Rightarrow yU$ would have to be a cell of Y as it is a restriction of id , which would imply $Y = yU$.

Now id is a restriction of any split epimorphism, so Y contains no split epimorphisms, i.e. $Y \subseteq \partial U$. \square

Example 3.4.21. In all the binary cube categories mentioned in section 3.3, $\partial \mathbb{I}$ is isomorphic to the constant presheaf of booleans.

For affine cubes, if we define a multiplier $\sqcup * \mathbb{I}^2$ in the obvious way, then $\partial \mathbb{I}^2$ is isomorphic to a colimit of four times $y\top$ and four times $y\perp$, i.e. a square without filler. For cartesian asymmetric cubes, the square also gains a diagonal. For symmetric cubes (with an involution $\neg : \mathbb{I} \rightarrow \mathbb{I}$), the other diagonal also appears.

3.5 Acting on slices

Definition 3.5.1. Given a multiplier $\sqcup \times U : \mathcal{W} \rightarrow \mathcal{V}$, we define

$$\mathcal{J}_U^{/W_0} : \mathcal{W}/W_0 \rightarrow \mathcal{V}/(W_0 \times U) : (W, \psi) \mapsto (W \times U, \psi \times U). \quad (18)$$

We say that $\sqcup \times U$ is:

- **Strongly cancellative** if for all W_0 , the functor $\mathcal{J}_U^{/W_0}$ is faithful,

- **Strongly affine** if for all W_0 , the functor $\downarrow_U^{W_0}$ is full,
- **Strongly connection-free** if for all W_0 , the functor $\downarrow_U^{W_0}$ is essentially surjective on slices $(V, \varphi) \in \mathcal{V}/(W_0 \times U)$ such that $\pi_2 \circ \varphi$ is dimensionally split,
 - We point out that the full subcategory of such slices is isomorphic to $(\mathcal{V} // U)/(W_0 \times U, \pi_2)$,
- **Strongly quantifiable** if for all W_0 , the functor $\downarrow_U^{W_0}$ has a left adjoint $\exists_U^{W_0} : \mathcal{V}/(W_0 \times U) \rightarrow \mathcal{W}/W_0$. We denote the unit as $\text{copy}_U^{W_0} : \text{Id} \rightarrow \downarrow_U^{W_0} \exists_U^{W_0}$ and the co-unit as $\text{drop}_U^{W_0} : \exists_U^{W_0} \downarrow_U^{W_0} \rightarrow \text{Id}$.

The above definition generalizes the functor \downarrow_U that we already had:

Proposition 3.5.2. The functor $\downarrow_U^\top : \mathcal{W}/\top \rightarrow \mathcal{V}/(\top \times U)$ is equal to $\downarrow_U : \mathcal{W} \rightarrow \mathcal{V}/U$ over the obvious isomorphisms between their domains and codomains. Hence, each of the strong properties implies the basic property. \square

Note that strong connection-freeness is well-defined:

Proposition 3.5.3. The functor $\downarrow_U^{W_0}$ factors over $(\mathcal{V} // U)/(W_0 \times U, \pi_2)$.

Proof. The functor $\downarrow_U^{W_0}$ sends (W, ψ) to $(W \times U, \psi \times U)$. Since $\pi_2 \circ (\psi \times U) = \pi_2$, it is dimensionally split. \square

Proposition 3.5.4. If $\sqcup \times U : \mathcal{W} \rightarrow \mathcal{V}$ is cancellative, then it is strongly cancellative.

Proof. Pick morphisms $\varphi, \chi : (W, \psi) \rightarrow (W', \psi')$ in \mathcal{W}/W_0 such that $\downarrow_U^{W_0} \varphi = \downarrow_U^V \chi$. Expanding the definition of $\downarrow_U^{W_0}$, we see that this means that $\varphi \times U = \chi \times U$, and hence $\varphi = \chi$ by cancellation of $\sqcup \times U$. \square

Proposition 3.5.5. If $\sqcup \times U : \mathcal{W} \rightarrow \mathcal{V}$ is cancellative and affine, then it is strongly affine.

Proof. Pick (W, ψ) and (W', ψ') in \mathcal{W}/W_0 , and a morphism $\chi : \downarrow_U^{W_0}(W, \psi) \rightarrow \downarrow_U^{W_0}(W', \psi')$. This amounts to a diagram:

$$\begin{array}{ccc}
 W \times U & \xrightarrow{\chi} & W' \times U \\
 \searrow \psi \times U & & \swarrow \psi' \times U \\
 & W_0 \times U & \\
 \swarrow \pi_2 & \downarrow \pi_2 & \searrow \pi_2 \\
 & U &
 \end{array} \tag{19}$$

i.e. a triangle in \mathcal{W}/U , the objects of which are in the image of $\downarrow_U : \mathcal{W} \rightarrow \mathcal{W}/U$. Then, by fullness (affinity) we get $\chi_0 : W \rightarrow W'$ such that $\downarrow_U \chi_0 = \chi$, which by faithfulness (cancellativity) makes the following diagram commute:

$$\begin{array}{ccc}
 W & \xrightarrow{\chi_0} & W' \\
 \searrow \psi & & \swarrow \psi' \\
 & W_0 &
 \end{array} \tag{20}$$

Then χ_0 is a morphism $\chi_0 : (W, \psi) \rightarrow (W', \psi')$ in \mathcal{W}/W_0 and $\downarrow_U^{W_0} \chi_0 = \chi$. \square

Proposition 3.5.6. If $\sqcup \times U : \mathcal{W} \rightarrow \mathcal{V}$ is affine and connection-free, then it is strongly connection-free.

Proof. Pick some $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$ such that $\pi_2 \circ \varphi : V \rightarrow U$ is dimensionally split. Because \lrcorner_U is essentially surjective on \mathcal{V}/U , there must be some $W \in \mathcal{W}$ such that $\iota : \lrcorner_U W = (W \ltimes U, \pi_2) \cong (V, \pi_2 \circ \varphi)$ as slices over U . Because \lrcorner_U is full, there is a morphism $\psi : W \rightarrow W_0$ such that $\psi \ltimes U = \varphi \circ \iota : W \ltimes U \rightarrow W_0 \ltimes U$. Thus, $\iota^{-1} : (V, \varphi) \cong (W \ltimes U, \psi \ltimes U) = \lrcorner_U^{/W_0}(W, \psi)$ as slices over $W_0 \ltimes U$.

$$\begin{array}{ccc}
 V & \xleftarrow[\cong]{\iota} & W \ltimes U \\
 \searrow \varphi & & \swarrow \psi \ltimes U \\
 & W_0 \ltimes U & \\
 \downarrow \pi_2 & & \swarrow \pi_2 \\
 & U &
 \end{array}
 \quad (21)$$

□

Example 3.5.7. In the category \square^k of k -ary cartesian cubes (example 3.3.4), the diagonal $\delta : \mathbb{I} \rightarrow \mathbb{I} \times \mathbb{I}$ has the property that $\pi_2 \circ \delta$ is split epi, but (\mathbb{I}, δ) is not in the image of $\lrcorner_{\mathbb{I}}^{/\mathbb{I}}$. Thus, $\sqcup \ltimes \mathbb{I}$ is not strongly connection-free, despite being connection-free.

Proposition 3.5.8. If $\sqcup \ltimes U : \mathcal{W} \rightarrow \mathcal{V}$ is quantifiable, then it is strongly quantifiable, with

$$\begin{aligned}
 \exists_U^{/W_0}(V, \varphi) &= (\exists_U(V, \pi_2 \circ \varphi), \text{drop}_U \circ \exists_U \varphi), \\
 \text{drop}_U^{/W_0}(W, \psi) &= \text{drop}_U W : \exists_U^{/W_0} \lrcorner_U^{/W_0}(W, \psi) \rightarrow (W, \psi), \\
 \text{copy}_U^{/W_0}(V, \varphi) &= \text{copy}_U(V, \pi_2 \circ \varphi) : (V, \varphi) \rightarrow \lrcorner_U^{/W_0} \exists_U^{/W_0}(V, \varphi).
 \end{aligned}$$

Proof. Pick $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$. Then we have, in \mathcal{V}/U , a morphism $\varphi : (V, \pi_2 \circ \varphi) \rightarrow (W_0 \ltimes U, \pi_2) = \lrcorner_U W_0$. Transposing this, we obtain

$$\text{drop}_U \circ \exists_U \varphi : \exists_U(V, \pi_2 \circ \varphi) \rightarrow W_0. \quad (22)$$

This is a slice over W_0 that we take as the definition of $\exists_U^{/W_0}(V, \varphi)$.

To prove adjointness, take a general morphism $\chi : \exists_U^{/W_0}(V, \varphi) \rightarrow (W, \psi)$ and note that there is a correspondence between diagrams of the following shape, where in the first step we apply $\exists_U \dashv \lrcorner_U$ and in the second step we use $(\mathcal{W}/U)/\lrcorner_U W_0 \cong \mathcal{W}/(W_0 \ltimes U)$:

$$\begin{array}{ccc}
 \exists_U(V, \pi_2 \circ \varphi) & \xrightarrow{\chi} & W \\
 \searrow \text{drop}_U \circ \exists_U \varphi & & \swarrow \psi \\
 & W_0 &
 \end{array}$$

$$\begin{array}{ccc}
 (V, \pi_2 \circ \varphi) & \xrightarrow{\lrcorner_U \chi \circ \text{copy}_U} & \lrcorner_U W \\
 \searrow \varphi & & \swarrow \lrcorner_U \psi \\
 & \lrcorner_U W_0 &
 \end{array}$$

$$(V, \varphi) \xrightarrow{\lrcorner_U \chi \circ \text{copy}_U} (W \ltimes U, \psi \ltimes U),$$

i.e. we get a morphism $(V, \varphi) \rightarrow \lrcorner_U^{/W_0}(W, \psi)$. Moreover, we note that the transposition action on morphisms is precisely that of $\exists_U \dashv \lrcorner_U$, and hence both adjunctions have the same unit and co-unit. □

Proposition 3.5.9 (Functoriality for slices). A morphism of multipliers $\sqcup \ltimes v : \sqcup \ltimes U \rightarrow \sqcup \ltimes U'$ gives rise to a natural transformation $\Sigma^{/W_0 \ltimes v} \circ \exists_U^{/W_0} \rightarrow \exists_{U'}^{/W_0}$. Hence, if both multipliers are quantifiable, we also get $\exists_{U'}^{/W_0} \circ \Sigma^{/W_0 \ltimes v} \rightarrow \exists_U^{/W_0}$.

Proof. For any $(W, \psi) \in \mathcal{W}/W_0$, we have to prove $(W \ltimes U, (W_0 \ltimes v) \circ (\psi \ltimes U)) \rightarrow (W \ltimes U', \psi \ltimes U')$. The morphism $W \ltimes v : W \ltimes U \rightarrow W \ltimes U'$ does the job. The second statement follows from lemma 2.1.1. \square

Theorem 3.5.10 (Strong quantification theorem). If $\sqcup \ltimes U$ is

1. cancellative, affine and quantifiable, then we have a natural isomorphism $\text{drop}_U^{/W_0} : \exists_U^{/W_0} \exists_U^{/W_0} \cong \text{Id}$.
2. semi-cartesian, then we have
 - (a) $\text{hide}_U^{/W_0} : \Sigma_U^{/W_0} \rightarrow \exists_U^{/W_0}$ (if quantifiable),
 - (b) $\text{spoil}_U^{/W_0} : \exists_U^{/W_0} \rightarrow \Omega_U^{/W_0}$ (if $\Omega_U^{/W_0}$ exists),
 - (c) in any case $\Sigma_U^{/W_0} \exists_U^{/W_0} \rightarrow \text{Id}$.
3. 3/4-cartesian, then there is a natural transformation $\Sigma^{/W_0 \ltimes \delta} \circ \exists_U^{/W_0} \rightarrow \exists_{U \ltimes U}^{/W_0}$, where we compose multipliers as in theorem 3.6.1.
4. cartesian, then we have natural isomorphisms:
 - (a) $\exists_U^{/W_0}(V, \varphi) \cong \Sigma_U^{/W_0}(V, \varphi) = (V, \pi_1 \circ \varphi)$,
 - (b) $\exists_U^{/W_0}(W, \psi) \cong \Omega_U^{/W_0}(W, \psi)$,
 - (c) $\exists_U^{/W_0} \exists_U^{/W_0}(W, \psi) \cong \Sigma_U^{/W_0} \Omega_U^{/W_0}(W, \psi) \cong (W \times U, \psi \circ \pi_1)$.

Moreover, these isomorphisms become equality if $\exists_U^{/W_0}$ is constructed as above from $\exists_U = \Sigma_U$, and $\Omega_U^{/W_0}(W, \psi)$ is chosen wisely. (Both functors are defined only up to isomorphism.)

Proof. 1. This is a standard fact about fully faithful right adjoints such as $\exists_U^{/W_0}$.

2. By lemma 2.1.1, it is sufficient to prove $\Sigma_U^{/W_0} \exists_U^{/W_0} \rightarrow \text{Id}$, and indeed we have $\pi_1 : \Sigma_U^{/W_0} \exists_U^{/W_0}(W, \psi) = (W \ltimes U, \pi_1 \circ (\psi \ltimes U)) = (W \ltimes U, \psi \circ \pi_1) \rightarrow (W, \psi)$.

3. This is a special case of proposition 3.5.9.

4. (a) Recall that $\exists_U(V, \varphi) \cong \Sigma_U(V, \varphi) = V$. The co-unit is essentially given by $\text{drop}_U = \pi_1 : W \times U \rightarrow W$. The construction of $\exists_U^{/W_0}$ then reveals that $\exists_U^{/W_0}(V, \varphi) \cong (V, \pi_1 \circ \varphi)$, which is the definition of $\Sigma_U^{/W_0}(V, \varphi)$.
- (b) This follows from the definitions.
- (c) We have

$$\exists_U^{/W_0} \exists_U^{/W_0}(W, \psi) = \exists_U^{/W_0}(W \times U, \psi \times U) \cong (W \times U, \pi_1 \circ (\psi \times U)) = (W \times U, \psi \circ \pi_1). \quad \square$$

Theorem 3.5.11 (Strong kernel theorem). If $\sqcup \ltimes U : \mathcal{W} \rightarrow \mathcal{V}$ is cancellative, affine and connection-free, then $\exists_U^{/W_0} : \mathcal{W}/W_0 \simeq (\mathcal{V} // U) / (W_0 \ltimes U, \pi_2)$ is an equivalence of categories.⁴ \square

⁴We use a slight abuse of notation by using $(\mathcal{V} // U) / (W_0 \ltimes U, \pi_2)$ as a subcategory of $\mathcal{V} / (W_0 \ltimes U)$.

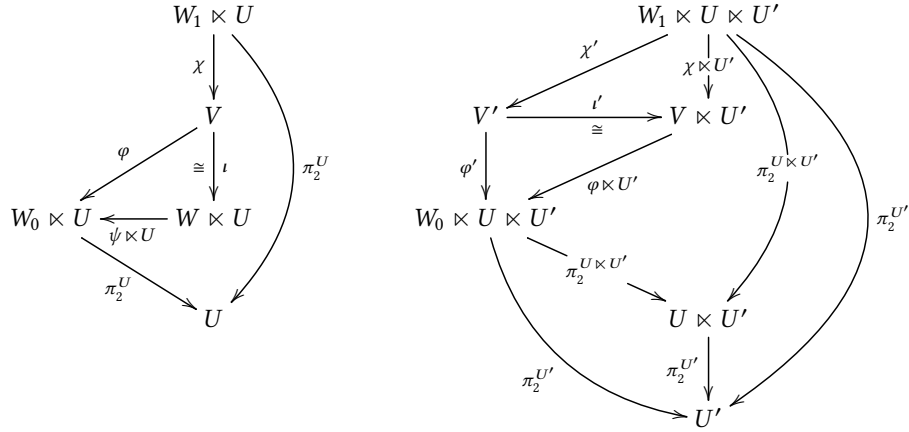
3.6 Composing multipliers

Theorem 3.6.1. If $\sqcup \ltimes U : \mathcal{W} \rightarrow \mathcal{V}$ is a multiplier for U and $\sqcup \ltimes U' : \mathcal{V} \rightarrow \mathcal{V}'$ is a multiplier for U' , then their composite $\sqcup \ltimes (U \ltimes U') := (\sqcup \ltimes U) \ltimes U'$ is a multiplier for $U \ltimes U'$.

1. The functor $\downarrow_{U \ltimes U'} : \mathcal{W} \rightarrow \mathcal{V}'/(U \ltimes U')$ equals $\downarrow_{U'}^U \circ \downarrow_U$.
2. The functor $\downarrow_{U \ltimes U'}^{W_0} : \mathcal{W} \rightarrow \mathcal{V}'/(U \ltimes U')$ equals $\downarrow_{U'}^{W_0 \ltimes U} \circ \downarrow_U^{W_0}$.
3. Assume both multipliers are endo. Then:
 - (a) The composite $\sqcup \ltimes (U \ltimes U')$ is semicartesian if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are semicartesian,
 - (b) The composite $\sqcup \ltimes (U \ltimes U')$ is 3/4-cartesian if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are 3/4-cartesian,
 - (c) The composite $\sqcup \ltimes (U \ltimes U')$ is cartesian if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are cartesian.
4. The composite $\sqcup \ltimes (U \ltimes U')$ is strongly cancellative if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are cancellative.
5. The composite $\sqcup \ltimes (U \ltimes U')$ is affine if $\sqcup \ltimes U$ is affine and $\sqcup \ltimes U'$ is strongly affine.
6. The composite $\sqcup \ltimes (U \ltimes U')$ is strongly affine if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are strongly affine.
7. The composite $\sqcup \ltimes (U \ltimes U')$ is connection-free if $\sqcup \ltimes U$ is connection-free and $\sqcup \ltimes U'$ is cancellative, affine and connection-free.
8. The composite $\sqcup \ltimes (U \ltimes U')$ is strongly connection-free if $\sqcup \ltimes U$ is strongly connection-free and $\sqcup \ltimes U'$ is cancellative, affine and connection-free.
9. The composite $\sqcup \ltimes (U \ltimes U')$ is strongly quantifiable if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are quantifiable, and in that case we have:
 - (a) $\exists_{U \ltimes U'} = \exists_U \circ \exists_{U'}^U$,
 - (b) $\exists_{U \ltimes U'}^{W_0} = \exists_U^{W_0} \circ \exists_{U'}^{W_0 \ltimes U}$.

Proof. Since $\top \ltimes U \cong U$, we see that $(\top \ltimes U) \ltimes U' \cong U \ltimes U'$, so the composite is indeed a multiplier for $U \ltimes U'$.

- 1-2. Follows from expanding the definitions.
3. (a) Copointed endofunctors compose.
(b) Comonads compose.
(c) By associativity of the cartesian product.
4. Cancellative multipliers are strongly cancellative (proposition 3.5.4), and the composite $\downarrow_{U \ltimes U'}^{W_0} = \downarrow_{U'}^{W_0} \downarrow_U^{W_0 \ltimes U'}$ of faithful functors is faithful.
- 5-6. Follows from the first two properties, since the composite of full functors is full.
7. Analogous to the next point.
8. Recall that the assumptions imply that $\sqcup \ltimes U'$ is strongly cancellative, strongly affine and strongly connection free.
Pick a slice $(V', \varphi') \in \mathcal{V}'/(W_0 \ltimes U \ltimes U')$ such that $\pi_2^{U \ltimes U'} \circ \varphi' : V' \rightarrow U \ltimes U'$ is dimensionally split with section $\chi' : W_1 \ltimes U \ltimes U' \rightarrow V'$. Then $\pi_2^{U'} \circ \pi_2^{U \ltimes U'} \circ \varphi' = \pi_2^{U'} \circ \varphi' : V' \rightarrow U'$ is also dimensionally split with the same section.
Because $\sqcup \ltimes U'$ is strongly connection-free, we find some $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$ such that $\iota' : (V', \varphi') \cong \downarrow_{U'}^{W_0 \ltimes U}(V, \varphi) \in \mathcal{V}'/(W_0 \ltimes U \ltimes U')$.



Because $\lrcorner_{U'}$ is fully faithful, the morphism $\iota' \circ \chi' : \lrcorner_{U'}(W_1 \times U) \rightarrow \lrcorner_{U'}V$ has a unique preimage $\chi : W_1 \times U \rightarrow V$ under $\lrcorner_{U'}$. The uniqueness of the inverse images implies that $\pi_2^U \circ \varphi \circ \chi = \pi_2^U : W_1 \times U \rightarrow U$, as the equation holds after applying $\lrcorner_{U'}$ (note that $\pi_2^{U \times U'} = \pi_2^U \times U'$).

Thus, we see that $\pi_2 \circ \varphi : V \rightarrow U$ is dimensionally split. Because $\sqcup \times U$ is strongly connection-free, we find some slice $(W, \psi) \in \mathcal{W}/W_0$ so that $\iota : (V, \varphi) \cong \lrcorner_U^{W_0}(W, \psi) \in \mathcal{V}/(W_0 \times U)$. We conclude that

$$(V', \varphi') \cong \lrcorner_{U'}^{W_0 \times U}(V, \varphi) \cong \lrcorner_{U'}^{W_0 \times U} \lrcorner_U^{W_0}(W, \psi) = \lrcorner_{U \times U'}^{W_0}(W, \psi). \quad (23)$$

9. Quantifiable multipliers are strongly quantifiable (proposition 3.5.8), and the composite of left adjoints is a left adjoint to the composite. \square

4 Multipliers and presheaves

Definition 4.0.1. Every multiplier $\sqcup \times U : \mathcal{W} \rightarrow \mathcal{V}$ gives rise to three adjoint endofunctors between $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{V}}$, which we will denote

$$(\sqcup \times \mathbf{y}U) \dashv (\mathbf{y}U \multimap \sqcup) \dashv (\mathbf{y}U \sqrt{\sqcup}). \quad (24)$$

Correspondingly, a morphism of multipliers $\sqcup \times v$ gives rise to natural transformations $\sqcup \times \mathbf{y}v, \mathbf{y}v \multimap \sqcup$ and $\mathbf{y}v \sqrt{\sqcup}$.

Note that the functor $\sqcup \times \mathbf{y}U : \widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{V}}$ is quite reminiscent of the Day-convolution.

4.1 Acting on elements

In section 3.5, we generalized $\lrcorner_U : \mathcal{W} \rightarrow \mathcal{V}/U$ to act on slices as $\lrcorner_U^{W_0} : \mathcal{W}/W_0 \rightarrow \mathcal{V}/(W_0 \times U)$. Here, we further generalize to a functor whose domain is the category of elements:

Definition 4.1.1. We define (using notation 2.3.3):

- $\lrcorner_U^\Psi : \mathcal{W}/\Psi \rightarrow \mathcal{V}/(\Psi \times \mathbf{y}U) : (W, \psi) \mapsto (W \times U, \psi \times \mathbf{y}U),$
- $\lrcorner_U^{\in \Psi} : (W \Rightarrow \Psi) \rightarrow \{\varphi : W \times U \Rightarrow \Psi \times \mathbf{y}U \mid \pi_2 \circ \varphi = \pi_2\} : \psi \mapsto \psi \times \mathbf{y}U.$

We say that $\sqcup \times U$ is:

- **Providently cancellative** if for all Ψ , the functor \lrcorner_U^Ψ is faithful,
- **Elementally cancellative** if for all Ψ , the natural transformation $\lrcorner_U^{\in \Psi}$ is componentwise injective,

- **Providently affine** if for all Ψ , the functor \downarrow_U^{Ψ} is full,
- **Elementally affine** if for all Ψ , the natural transformation \downarrow_U^{Ψ} is componentwise surjective,
- **Providently connection-free** if for all Ψ , the functor \downarrow_U^{Ψ} is essentially surjective on slices $(V, \varphi) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ such that $\pi_2 \circ \varphi$ is dimensionally split,
- **Providently quantifiable** if for all Ψ , the functor \downarrow_U^{Ψ} has a left adjoint $\exists_U^{\Psi} : \mathcal{V}/(\Psi \ltimes \mathbf{y}U) \rightarrow \mathcal{W}/\Psi$.
We denote the unit as $\text{copy}_U^{\Psi} : \text{Id} \rightarrow \downarrow_U^{\Psi} \exists_U^{\Psi}$ and the co-unit as $\text{drop}_U^{\Psi} : \exists_U^{\Psi} \downarrow_U^{\Psi} \rightarrow \text{Id}$.

This is indeed a generalization:

Proposition 4.1.2. The functor $\downarrow_U^{\mathbf{y}W_0} : \mathcal{W}/\mathbf{y}W_0 \rightarrow \mathcal{V}/(\mathbf{y}W_0 \ltimes \mathbf{y}U)$ is equal to $\downarrow_U^{W_0} : \mathcal{W}/W_0 \rightarrow \mathcal{V}/(W_0 \ltimes U)$ over the obvious isomorphisms between their domains and codomains. Hence, each of the provident notions implies the strong notion 3.5.1. Moreover, each of the elemental notions implies the basic notion.

Proof. Most of this is straightforward. To see the last claim, note that

$$\{\varphi : W \ltimes U \Rightarrow \mathbf{y}W_0 \ltimes \mathbf{y}U \mid \pi_2 \circ \varphi = \pi_2\} \cong ((W \ltimes U, \pi_2) \rightarrow (W_0 \ltimes U, \pi_2)) = (\downarrow_U W \rightarrow \downarrow_U W_0).$$

So if injectivity/surjectivity holds for all W_0 , then we can conclude that \downarrow_U is cancellative/affine. \square

Proposition 4.1.3. If $\sqsubset \ltimes U$ is cancellative, then it is providently cancellative.

Proof. Analogous to proposition 3.5.4. \square

Proposition 4.1.4. If $\sqsubset \ltimes U$ is cancellative, then it is elementally cancellative

Proof. We have

$$\begin{aligned} & \{\varphi : W \ltimes U \Rightarrow \Psi \ltimes \mathbf{y}U \mid \pi_2 \circ \varphi = \pi_2\} \\ & \cong \exists W_0. (\varphi' : W \ltimes U \rightarrow W_0 \ltimes U) \times (\psi : W_0 \Rightarrow \Psi) \times (\pi_2 \circ (\psi \ltimes \mathbf{y}U) \circ \varphi' = \pi_2) \\ & \cong \exists W_0. (\varphi' : W \ltimes U \rightarrow W_0 \ltimes U) \times (\psi : W_0 \Rightarrow \Psi) \times (\pi_2 \circ \varphi' = \pi_2) \\ & \cong \exists W_0. (\varphi' : \downarrow_U W \rightarrow \downarrow_U W_0) \times (\psi : W_0 \Rightarrow \Psi) \end{aligned} \quad (25)$$

and

$$(W \Rightarrow \Psi) \cong \exists W_0. (W \rightarrow W_0) \times (W_0 \Rightarrow \Psi). \quad (26)$$

Moreover, the action of \downarrow_U^{Ψ} sends (W_0, χ, ψ) in eq. (26) to $(W_0, \downarrow_U \chi, \psi)$ in eq. (25). Naively, one would say that this proves injectivity, but some care is required with the equality relation for co-ends. It might be that (W_0, χ, ψ) and (W_0, χ', ψ) are sent to the same object. This would mean that there exists a zigzag ζ from W_0 to itself such that the following diagrams commute:

$$\begin{array}{ccc} & \downarrow_U W_0 & \\ \swarrow \downarrow_U \chi & \uparrow \downarrow_U \zeta & \searrow \downarrow_U \chi' \\ \downarrow_U W & & \downarrow_U W_0 \end{array} \quad \begin{array}{ccc} W_0 & & \Psi \\ \swarrow \psi & & \uparrow \psi \\ \downarrow \zeta & & \downarrow \psi \\ W_0 & & \end{array} \quad (27)$$

Then by cancellativity of \downarrow_U , we see that the unique preimage of the left triangle also commutes and hence $\psi \circ \chi = \psi \circ \chi'$, so that $(W_0, \chi, \psi) = (W, \text{id}, \psi \circ \chi) = (W, \text{id}, \psi \circ \chi') = (W_0, \chi', \psi)$. \square

Proposition 4.1.5. If $\sqsubset \ltimes U$ is cancellative and affine, then it is providently affine.

Proof. Pick (W, ψ) and (W', ψ') in \mathcal{W}/Ψ and a morphism $\chi : \downarrow_U^\Psi(W, \psi) \rightarrow \downarrow_U^\Psi(W', \psi')$. Then we also have $\chi : \downarrow_U W \rightarrow \downarrow_U W'$ and by fullness, we find an inverse image $\chi_0 : W \rightarrow W'$ under \downarrow_U . By elemental cancellativity, we see that $\psi' \circ \chi_0 = \psi$, so that χ_0 is a morphism of slices $\chi_0 : (W, \psi) \rightarrow (W', \psi') \in \mathcal{W}/\Psi$ and $\downarrow_U^\Psi \chi_0 = \chi$. \square

Proposition 4.1.6. If $\sqcup \ltimes U$ is affine, then it is elementally affine.

Proof. In the proof of proposition 4.1.4, we saw that $\downarrow_U^{\in \Psi}$ essentially sends (W_0, χ, ψ_0) to $(W_0, \downarrow_U \chi, \psi_0)$. Then if $\downarrow_U \chi$ is full, it is immediate that this operation is surjective. \square

Proposition 4.1.7. If $\sqcup \ltimes U$ is strongly connection-free, then it is providently connection-free.

Proof. Pick a slice $(V, \varphi) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ such that $\pi_2 \circ \varphi$ is dimensionally split. By definition of $\sqcup \ltimes \mathbf{y}U$, there is some W_0 such that φ factors as $\varphi = (\psi^{W_0 \Rightarrow \Psi} \ltimes U) \circ \chi$. Clearly, $\pi_2 \circ \varphi = \pi_2 \circ \chi$ is dimensionally split. Hence, by strong connection-freeness, $(V, \chi) \cong \downarrow_U^{W_0}(W, \chi') \in \mathcal{V}/(W_0 \ltimes U)$ for some $(W, \chi') \in \mathcal{W}/W_0$. Then we also have $(V, \varphi) = (V, (\psi \ltimes \mathbf{y}U) \circ \chi) \cong \downarrow_U^\Psi(W, \psi \circ \chi')$. \square

Proposition 4.1.8. If $\sqcup \ltimes U$ is quantifiable, then it is providently quantifiable, with

$$\begin{aligned} \exists_U^\Psi(V, (\psi \ltimes \mathbf{y}U) \circ \varphi_0) &= \Sigma^\psi \exists_U^{W_0}(V, \varphi_0), \\ \text{drop}_U^\Psi(W, \psi) &= \text{drop}_U W, \\ \text{copy}_U^\Psi(V, \varphi) &= \text{copy}_U(V, \pi_2 \circ \varphi). \end{aligned}$$

Proof. Pick $(V, \varphi) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$. Then φ factors as $(\psi^{W_0 \Rightarrow \Psi} \ltimes \mathbf{y}U) \circ \varphi_0^{V \rightarrow W_0 \ltimes U}$. Then $(V, \varphi_0) \in \mathcal{V}/(W_0 \ltimes U)$ and hence $\exists_U^{W_0}(V, \varphi_0) \in \mathcal{W}/W_0$. We define

$$\begin{aligned} \exists_U^\Psi(V, \varphi) &:= \Sigma^\psi \exists_U^{W_0}(V, \varphi_0) \\ &= \Sigma^\psi(\exists_U(V, \pi_2 \circ \varphi_0), \text{drop}_U \circ \exists_U \varphi_0) \\ &= (\exists_U(V, \pi_2^{W_0 \ltimes U \rightarrow U} \circ \varphi_0), \psi \circ \text{drop}_U \circ \exists_U \varphi_0) \\ &= (\exists_U(V, \pi_2^{\Psi \ltimes \mathbf{y}U \rightarrow \mathbf{y}U} \circ \varphi), \psi \circ \text{drop}_U \circ \exists_U \varphi_0). \end{aligned}$$

We need to prove that this is well-defined, i.e. respects equality on the co-end that defines $V \Rightarrow \Psi \ltimes \mathbf{y}U$. To this end, assume that $\varphi = (\psi_0^{W_0 \Rightarrow \Psi} \ltimes \mathbf{y}U) \circ \varphi_0^{V \rightarrow W_0 \ltimes U} = (\psi_1^{W_1 \Rightarrow \Psi} \ltimes \mathbf{y}U) \circ \varphi_1^{V \rightarrow W_1 \ltimes U}$. This means there is a zigzag ζ from W_0 to W_1 such that the following triangles commute:

$$\begin{array}{ccc} & W_0 \ltimes U & \\ \varphi_0 \nearrow & & \searrow \psi_0 \\ V & & \Psi \\ \varphi_1 \searrow & & \nearrow \psi_1 \\ & W_1 \ltimes U & \end{array} \quad \begin{array}{ccc} & W_0 & \\ \zeta \nearrow & & \searrow \psi_0 \\ & & \Psi \\ \zeta \searrow & & \nearrow \psi_1 \\ & W_1 & \end{array} \quad (28)$$

By naturality of drop_U , we find that $\psi_0 \circ \text{drop}_U \circ \exists_U \varphi_0 = \psi_1 \circ \text{drop}_U \circ \exists_U \varphi_1$. We conclude that $\exists_U^\Psi(V, \varphi)$ is well-defined.

To prove adjointness, we first show how \downarrow_U^Ψ on the right can be turned into \exists_U^Ψ on the left. Pick $\chi : (V, \varphi) \rightarrow \downarrow_U^\Psi(W, \psi) = (W \ltimes U, \psi \ltimes \mathbf{y}U) : \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$. Then $\varphi = (\psi \ltimes \mathbf{y}U) \circ \chi$ so by definition, $\exists_U^\Psi(V, \varphi) = (\exists_U(V, \pi_2 \circ \varphi), \psi \circ \text{drop}_U \circ \exists_U \chi)$ which clearly factors over ψ , i.e. has a morphism $\text{drop}_U \circ \exists_U \chi : \exists_U^\Psi(V, \varphi) \rightarrow (W, \psi)$. If $\chi = \text{id}$, then we obtain the co-unit $\text{drop}_U^\Psi = \text{drop}_U \circ \exists_U \text{id} = \text{drop}_U$.

Next, we construct the unit $\text{copy}_U^\Psi : (V, \varphi) \rightarrow \downarrow_U^\Psi \exists_U^\Psi(V, \varphi)$. If $\varphi = (\psi \ltimes \mathbf{y}U) \circ \varphi_0$, then we have

$$\downarrow_U^\Psi \exists_U^\Psi(V, \varphi) = \downarrow_U^\Psi \Sigma^\psi \exists_U^{W_0}(V, \varphi_0)$$

$$= \Sigma^{\psi \times yU} \exists_U^{W_0} \exists_U^{W_0} (V, \varphi_0).$$

On the other hand, $(V, \varphi) = \Sigma^{\psi \times yU} (V, \varphi_0)$, so as the unit we can take $\text{copy}_U^{\Psi} = \Sigma^{\psi \times yU} \text{copy}_U^{W_0} = \text{copy}_U^{W_0} = \text{copy}_U$.

The adjunction laws are then inherited from $\exists_U \dashv \exists_U$. \square

Proposition 4.1.9 (Functoriality for elements). A morphism of multipliers $\sqcup \times v : \sqcup \times U \rightarrow \sqcup \times U'$ gives rise to a natural transformation $\Sigma^{\Psi \times yv} \circ \exists_U^{\Psi} \rightarrow \exists_{U'}^{\Psi}$. Hence, if both multipliers are quantifiable, we also get $\exists_{U'}^{\Psi} \circ \Sigma^{\Psi \times yv} \rightarrow \exists_U^{\Psi}$.

Proof. For any $(W, \psi) \in \mathcal{W}/\Psi$, we have to prove $(W \times U, (\Psi \times yv) \circ (\psi \times yU)) \rightarrow (W \times U', \psi \times yU')$. The morphism $W \times v : W \times U \rightarrow W \times U'$ does the job. The second statement follows from lemma 2.1.1. \square

Theorem 4.1.10 (Provident quantification theorem). If $\sqcup \times U$ is

1. cancellative, affine and quantifiable, then we have a natural isomorphism $\text{drop}_U^{\Psi} : \exists_U^{\Psi} \exists_U^{\Psi} \cong \text{Id}$.
2. semi-cartesian, then we have
 - (a) $\text{hide}_U^{\Psi} : \Sigma_U^{\Psi} \rightarrow \exists_U^{\Psi}$ (if quantifiable),
 - (b) $\text{spoil}_U^{\Psi} : \exists_U^{\Psi} \rightarrow \Omega_U^{\Psi}$ (if Ω_U^{Ψ} exists),
 - (c) in any case $\Sigma_U^{\Psi} \exists_U^{\Psi} \rightarrow \text{Id}$.
3. 3/4-cartesian, then there is a natural transformation $\Sigma^{\Psi \times y\delta} \circ \exists_U^{\Psi} \rightarrow \exists_{U \times U}^{\Psi}$.
4. cartesian, then we have natural isomorphisms:
 - (a) $\exists_U^{\Psi} (V, \varphi) \cong \Sigma_U^{\Psi} (V, \varphi) = (V, \pi_1 \circ \varphi)$,
 - (b) $\exists_U^{\Psi} (W, \psi) \cong \Omega_U^{\Psi} (W, \psi)$,
 - (c) $\exists_U^{\Psi} \exists_U^{\Psi} (W, \psi) \cong \Sigma_U^{\Psi} \Omega_U^{\Psi} (W, \psi) \cong (W \times yU, \psi \circ \pi_1)$.

Moreover, these isomorphisms become equality if \exists_U^{Ψ} is constructed as above from $\exists_U^{W_0} = \Sigma_U^{W_0}$, and $\Omega_U^{\Psi} (W, \psi)$ is chosen wisely. (Both functors are defined only up to isomorphism.)

Proof. 1. This is a standard fact about fully faithful right adjoints such as \exists_U^{Ψ} .

2. By lemma 2.1.1, it is sufficient to prove $\Sigma_U^{\Psi} \exists_U^{\Psi} \rightarrow \text{Id}$, and indeed we have $\pi_1 : \Sigma_U^{\Psi} \exists_U^{\Psi} (W, \psi) = (W \times U, \pi_1 \circ (\psi \times yU)) = (W \times U, \psi \circ \pi_1) \rightarrow (W, \psi)$.
3. This is a special case of proposition 4.1.9.
4. (a) Let $\varphi = (\psi \times yU) \circ \varphi_0$. Then we have

$$\begin{aligned} \exists_U^{\Psi} (V, \varphi) &= \Sigma^{\psi} \exists_U^{W_0} (V, \varphi_0) \\ &\cong \Sigma^{\psi} \Sigma_U^{W_0} (V, \varphi_0) \\ &= \Sigma^{\psi} (V, \pi_1 \circ \varphi_0) \\ &= (V, \psi \circ \pi_1 \circ \varphi_0) = (V, \pi_1 \circ (\psi \times yU) \circ \varphi_0) = (V, \pi_1 \circ \varphi). \end{aligned}$$

(b) This follows from the definitions.

(c) We have

$$\exists_U^{\Psi} \exists_U^{\Psi} (W, \psi) = \exists_U^{\Psi} (W \times U, \psi \times yU) \cong (W \times U, \pi_1 \circ (\psi \times yU)) \quad (29)$$

and of course $\pi_1 \circ (\psi \times yU) = \psi \circ \pi_1 : W \times U \rightarrow \Psi$. \square

Theorem 4.1.11 (Provident kernel theorem). If $\sqcup \times U : \mathcal{W} \rightarrow \mathcal{V}$ is cancellative, affine and connection-free, then $\exists_U^{\Psi} : \mathcal{W}/\Psi \simeq (\mathcal{V} // U) / (\Psi \times yU, \pi_2)$ is an equivalence of categories.⁵ \square

⁵We use a slight abuse of notation as $(\mathcal{V} // U) / (\Psi \times yU, \pi_2)$ is in fact neither a category of slices nor of elements.

4.2 Acting on presheaves

Proposition 4.2.1. The functor $\sqcup \ltimes yU : \widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{V}}$:

1. is a multiplier for yU ,
2. has the property that $\downarrow_{yU} : \widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{V}}/yU$ is naturally isomorphic to $(\downarrow_U)_! : \widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{V}}/U$ over the equivalence between their codomains,
3. has the property that the slice functor $\downarrow_{yU}^{/\Psi} : \widehat{\mathcal{W}}/\Psi \rightarrow \widehat{\mathcal{V}}/(\Psi \ltimes yU)$ is naturally isomorphic to the left lifting of the elements functor $(\downarrow_U^{/\Psi})_! : \widehat{\mathcal{W}}/\Psi \rightarrow \widehat{\mathcal{V}}/(\Psi \ltimes yU)$ over the equivalences between their domains and codomains,
4. is semicartesian if and only if $\sqcup \ltimes U$ is,
5. is 3/4-cartesian if and only if $\sqcup \ltimes U$ is,
6. is cartesian if and only if $\sqcup \ltimes U$ is,
7. is cancellative if one of the following conditions holds:
 - (a) $\sqcup \ltimes U$ is cancellative and affine,
 - (b) $\sqcup \ltimes U$ is cartesian and U is not spooky.

(see example 4.2.2 for a counterexample showing the necessity of either non-spookiness or affinity),
8. is strongly affine if $\sqcup \ltimes U$ is cancellative and affine,
9. is quantifiable if $\sqcup \ltimes U$ is, and \exists_{yU} is naturally isomorphic to $(\exists_U)_!$ over the equivalence $\widehat{\mathcal{V}}/U \simeq \widehat{\mathcal{V}}/yU$,
10. is strongly quantifiable if $\sqcup \ltimes U$ is quantifiable, and $\exists_{yU}^{/\Psi}$ is naturally isomorphic to $(\exists_U^{/\Psi})_!$ over the equivalences between their domain and codomain.

Proof. 1. Since $\top \ltimes yU \cong y\top \ltimes yU \cong y(\top \ltimes U) \cong yU$. We use, in order, that y preserves the terminal object, that $F_! \circ y \cong y \circ F$ (theorem 2.3.2) and that $\sqcup \ltimes U$ is a multiplier for U .

2. The functor $(\downarrow_U)_!$ sends a presheaf $\Gamma \in \widehat{\mathcal{W}}$ to the presheaf in $\widehat{\mathcal{V}}/U$ determined by

$$(V, \varphi) \Rightarrow (\downarrow_U)_! \Gamma = \exists W. ((V, \varphi) \rightarrow \downarrow_U W) \times (W \Rightarrow \Gamma). \quad (30)$$

On the other hand, $\downarrow_{yU} \Gamma$ is the slice $(\Gamma \ltimes yU, \pi_2) \in \widehat{\mathcal{V}}/yU$. Taking the preimage of π_2 (proposition 2.3.6), we get a presheaf $\Delta \in \widehat{\mathcal{V}}/U$ determined by

$$\begin{aligned} (V, \varphi) \Rightarrow \Delta &= \{(\gamma \ltimes yU) \circ \chi : V \Rightarrow \Gamma \ltimes yU \mid \pi_2 \circ (\gamma \ltimes yU) \circ \chi = \varphi\} \\ &= \{(\gamma \ltimes yU) \circ \chi : V \Rightarrow \Gamma \ltimes yU \mid \pi_2 \circ \chi = \varphi\} \\ &\cong \exists W. (\chi : V \rightarrow W \ltimes U) \times (\gamma : W \Rightarrow \Gamma) \times (\pi_2 \circ \chi = \varphi) \\ &\cong \exists W. (\chi : (V, \varphi) \rightarrow \downarrow_U W) \times (W \Rightarrow \Gamma). \end{aligned}$$

Indeed, we see that these functors are isomorphic.

3. The functor $(\downarrow_U^{/\Psi})_!$ sends a presheaf $\Psi \mid \Gamma \vdash \text{Ctx}$ over \mathcal{W}/Ψ to the presheaf $\Psi \ltimes yU \mid (\downarrow_U \Psi)_! \Gamma \vdash \text{Ctx}$ over $\mathcal{V}/(\Psi \ltimes yU)$ determined by:

$$(V, \varphi^{V \Rightarrow \Psi \ltimes yU}) \Rightarrow (\downarrow_U^{/\Psi})_! \Gamma = \exists (W, \psi^{W \Rightarrow \Psi}). ((V, \varphi) \rightarrow \downarrow_U^{/\Psi} (W, \psi)) \times ((W, \psi) \Rightarrow \Gamma). \quad (31)$$

On the other hand, $\downarrow_{yU}^{\Psi}(\Psi.\Gamma, \pi)$ is the slice $(\Psi.\Gamma \ltimes yU, \pi \ltimes yU) \in \widehat{\mathcal{V}}/(\Psi \ltimes yU)$. Taking the preimage of $\pi \ltimes yU$ (proposition 2.3.6), we get a presheaf $\Psi \ltimes yU \mid \Delta \vdash \text{Ctx}$ over $\mathcal{V}/(\Psi \ltimes yU)$ determined by

$$\begin{aligned} (V, \varphi^{V \Rightarrow \Psi \ltimes yU}) &\Rightarrow \Delta \\ &= \{(\psi.y \ltimes yU) \circ \chi : V \Rightarrow \Psi.\Gamma \ltimes yU \mid (\pi \ltimes yU) \circ (\psi.y \ltimes yU) \circ \chi = \varphi\} \\ &= \{(\psi.y \ltimes yU) \circ \chi : V \Rightarrow \Psi.\Gamma \ltimes yU \mid (\psi \ltimes yU) \circ \chi = \varphi\} \\ &\cong \exists W.(\chi : V \rightarrow W \ltimes U) \times (\psi : W \Rightarrow \Psi) \times (\gamma : (W, \psi) \Rightarrow \Gamma) \times ((\psi \ltimes yU) \circ \chi = \varphi) \\ &\cong \exists(W, \psi^{W \Rightarrow \Psi}).(\chi : (V, \varphi) \rightarrow \downarrow_U^{\Psi}(W, \psi)) \times (\gamma : (W, \psi) \Rightarrow \Gamma). \end{aligned}$$

Indeed, we see that these functors are isomorphic.

4. Assume that $\sqcup \ltimes U$ is semicartesian. It is immediate from the construction of $\sqcup_!$ that $\sqcup_!$ preserves natural transformations. Moreover, we have $\text{Id}_! \cong \text{Id}$, so we get $\pi_1 : (\sqcup \ltimes yU) \rightarrow \text{Id}$.

Conversely, assume that $\sqcup \ltimes yU$ is semicartesian. Then we have $y(\sqcup \ltimes U) \cong (y\sqcup \ltimes yU) \rightarrow y$. Since y is fully faithful, we have proven $(\sqcup \ltimes U) \rightarrow \text{Id}$.

5. Analogous to the previous point.
6. Assume that $\sqcup \ltimes U$ is cartesian. We apply the universal property of the cartesian product, and the co-Yoneda lemma:

$$\begin{aligned} V \Rightarrow (\Gamma \ltimes yU) &= \exists W.(V \rightarrow W \ltimes U) \times (W \Rightarrow \Gamma) \\ &\cong \exists W.(V \rightarrow W) \times (V \rightarrow U) \times (W \Rightarrow \Gamma) \\ &\cong (V \rightarrow U) \times (V \Rightarrow \Gamma) \\ &\cong V \Rightarrow \Gamma \ltimes yU. \end{aligned}$$

Conversely, if $\sqcup \ltimes yU$ is cartesian, we have

$$\begin{aligned} V \rightarrow W \ltimes U &= V \Rightarrow y(W \ltimes U) \\ &\cong V \Rightarrow yW \ltimes yU \\ &\cong (V \Rightarrow yW) \times (V \Rightarrow yU) \\ &\cong (V \rightarrow W) \times (V \rightarrow U) \cong V \rightarrow W \times U. \end{aligned}$$

7. The reasoning is different in both cases:

- (a) In this case, \downarrow_U is fully faithful, implying that $(\downarrow_U)_!$ (which is essentially \downarrow_{yU}) is also fully faithful, i.e. $\sqcup \ltimes yU$ is cancellative and affine.
- (b) Similar to proposition 3.2.6.

8. Since it is then cancellative and affine, as proven in the previous item.
9. We know that $(\exists_U)_! \dashv (\downarrow_U)_!$ so moving it through the natural transformation yields a left adjoint to \downarrow_{yU} .
10. By proposition 4.1.8, \exists_U^{Ψ} exists. We know that $(\exists_U^{\Psi})_! \dashv (\downarrow_U^{\Psi})_!$ so moving it through the natural transformation yields a left adjoint to \downarrow_{yU}^{Ψ} . \square

Example 4.2.2 (Non-cancellativity of $\sqcup \ltimes yU$ for spooky U). Consider the category of nullary cartesian cubes \square^0 (example 3.3.4) and let Γ be the terminal presheaf and $(\top \Rightarrow \Delta) = \text{Bool}$ and $(\mathbb{I}^n \Rightarrow \Delta) = \{\star\}$ for $n > 0$. Then $\Delta \ltimes y\mathbb{I}$ is the terminal presheaf. Hence, $\sqcup \ltimes y\mathbb{I}$ is not injective on morphisms $\Gamma \rightarrow \Delta$.

4.3 Four adjoint functors

Unlike the category of slices $\widehat{\mathcal{W}}/\Psi$, the equivalent category $\widehat{\mathcal{W}}/\Psi$ is a presheaf category and therefore immediately a model of dependent type theory. Therefore, we prefer to work with that category, and to use the corresponding functors:

Definition 4.3.1. The adjoint functors $\exists_U^{\Psi} \dashv \exists_U^{\Psi}$ give rise to four adjoint functors between presheaf categories over slice categories, which we denote

$$\exists_{yU}^{\Psi} \dashv \exists_{yU}^{\Psi} \dashv \forall_{yU}^{\Psi} \dashv \forall_{yU}^{\Psi}. \quad (32)$$

We call the fourth functor **transpension**.

The units and co-units will be denoted:

$$\begin{array}{ll} \text{copy}_{yU}^{\Psi} : \text{Id} \rightarrow \exists_{yU}^{\Psi} \exists_{yU}^{\Psi} & \text{drop}_{yU}^{\Psi} : \exists_{yU}^{\Psi} \exists_{yU}^{\Psi} \rightarrow \text{Id} \\ \text{const}_{yU}^{\Psi} : \text{Id} \rightarrow \forall_{yU}^{\Psi} \exists_{yU}^{\Psi} & \text{app}_{yU}^{\Psi} : \exists_{yU}^{\Psi} \forall_{yU}^{\Psi} \rightarrow \text{Id} \\ \text{reid}_{yU}^{\Psi} : \text{Id} \rightarrow \forall_{yU}^{\Psi} \forall_{yU}^{\Psi} & \text{unmerid}_{yU}^{\Psi} : \forall_{yU}^{\Psi} \forall_{yU}^{\Psi} \rightarrow \text{Id} \end{array} \quad (33)$$

For now, we define all of these functors only up to isomorphism, i.e. for the middle two we do not specify whether they arise as a left, central or right lifting.

Note that, if in a judgement $\Psi \mid \Gamma \vdash J$, we view the part before the pipe (\mid) as part of the context, then \exists_{yU}^{Γ} and \forall_{yU}^{Γ} bind a (substructural) variable of type yU , whereas \exists_{yU}^{Ψ} and \forall_{yU}^{Ψ} depend on one.

Corollary 4.3.2. The properties asserted by proposition 4.2.1 for \exists_U^{Ψ} also hold for \exists_{yU}^{Ψ} .

Proof. Follows from the fact that $\exists_{yU}^{\Psi} \cong (\exists_U^{\Psi})_!$, and the observation in proposition 4.2.1 that this functor in turn corresponds to \exists_U^{Ψ} . \square

Proposition 4.3.3 (Presheaf functoriality). A morphism of multipliers $\sqsubset \times v : \sqsubset \times U \rightarrow \sqsubset \times U'$ gives rise to natural transformations

- $\exists_{yU'}^{\Psi} \circ \Sigma^{\Psi \times yv} \rightarrow \exists_{yU}^{\Psi}$ (if quantifiable),
- $\Sigma^{\Psi \times yv} \circ \exists_{yU}^{\Psi} \rightarrow \exists_{yU'}^{\Psi}$ and $\exists_{yU}^{\Psi} \rightarrow \Omega^{\Psi \times yv} \circ \exists_{yU'}^{\Psi}$,
- $\forall_{yU'}^{\Psi} \rightarrow \forall_{yU}^{\Psi} \circ \Omega^{\Psi \times yv}$ and $\forall_{yU'}^{\Psi} \circ \Pi^{\Psi \times yv} \rightarrow \forall_{yU}^{\Psi}$,
- $\Pi^{\Psi \times yv} \circ \forall_{yU}^{\Psi} \rightarrow \forall_{yU'}^{\Psi}$,

Proof. Follows directly from proposition 4.1.9. \square

Proposition 4.3.4 (Presheaf quantification theorem). If $\sqsubset \times U$ is

1. cancellative and affine, then drop_{yU}^{Ψ} , const_{yU}^{Ψ} and $\text{unmerid}_{yU}^{\Psi}$ are natural isomorphisms.
2. semi-cartesian, then we have
 - (a) $\text{hide}_{yU}^{\Psi} : \Sigma_{yU}^{\Psi} \rightarrow \exists_{yU}^{\Psi}$ (if quantifiable),
 - (b) $\text{spoil}_{yU}^{\Psi} : \exists_{yU}^{\Psi} \rightarrow \Omega_{yU}^{\Psi}$,
 - (c) $\text{cospoil}_{yU}^{\Psi} : \Pi_{yU}^{\Psi} \rightarrow \forall_{yU}^{\Psi}$.
3. 3/4-cartesian, then we can apply proposition 4.3.3 to $\sqsubset \times \delta : \sqsubset \times U \rightarrow \sqsubset \times (U \times U)$.
4. cartesian, then we have natural isomorphisms:

- (a) $\exists_U^\Psi \cong \Sigma_U^\Psi$,
- (b) $\perp_U^\Psi \cong \Omega_U^\Psi$,
- (c) $\forall_U^\Psi \cong \Pi_U^\Psi$,
- (d) $\wp_U^\Psi \cong \wp_U^\Psi$ (between the functors of the same notation from definitions 2.3.17 and 4.3.1).

Equality is achieved for any pair of functors if they are lifted in the same way from functors that were equal in theorem 4.1.10.

- Proof.* 1. The fact that drop_{yU}^Ψ is an isomorphism, is a standard fact about fully faithful right adjoints such as \perp_{yU}^Ψ . This property then carries over to further adjoints.
2. By lemma 2.1.1, it is sufficient to prove $\Sigma_{yU}^\Psi \perp_{yU}^\Psi \rightarrow \text{Id}$, which follows immediately from $\pi_1 : \Sigma_U^\Psi \perp_U^\Psi \rightarrow \text{Id}$.
3. Of course we can.
4. This is an immediate corollary of theorem 3.5.10. \square

Proposition 4.3.5 (Fresh exchange). If $\Psi \mid \Gamma \vdash \text{Ctx}$, i.e. $\Gamma \in \widehat{\mathcal{W}/\Psi}$, then we have an isomorphism of slices (natural in Γ):

$$\begin{array}{ccc}
 (\Psi \ltimes yU) \cdot \perp_{yU}^\Psi \Gamma & \xrightarrow{\cong} & \Psi \cdot \Gamma \ltimes yU \\
 \pi \searrow & & \swarrow \pi \ltimes yU \\
 & \Psi \ltimes yU. &
 \end{array} \tag{34}$$

This proposition explains the meaning of \perp_{yU}^Γ : it is the type depending on a variable of type yU whose elements are required to be fresh for that variable, where the meaning of ‘fresh’ depends on the nature of the multiplier. If the multiplier is cartesian, then \perp_{yU}^Γ is clearly just weakening over yU .

Proof. The slice on the right is $\perp_{yU}^\Psi(\Psi \cdot \Gamma, \pi)$. By proposition 4.2.1, this is isomorphic to $\perp_{yU}^\Psi \Gamma$ over the equivalence from proposition 2.3.6 which sends Δ to $((\Psi \ltimes yU) \cdot \Delta, \pi)$. \square

4.4 Investigating the transpension functor

Definition 4.4.1. Write $\Psi \ltimes \partial U$ for the pullback

$$\begin{array}{ccc}
 \Psi \ltimes \partial U & \xrightarrow{\subseteq} & \Psi \ltimes yU \\
 \downarrow \pi_2 & & \downarrow \pi_2 \\
 \partial U & \xrightarrow{\subseteq} & yU.
 \end{array} \tag{35}$$

Definition 4.4.2. Write $(\in \partial U)$ for the inverse image of $\Psi \ltimes \partial U \subseteq \Psi \ltimes yU$, which is a presheaf over $\mathcal{W}/(\Psi \ltimes yU)$ such that $(\Psi \ltimes yU) \cdot (\in \partial U) \cong \Psi \ltimes \partial U$. We also write $(\in \partial U)$ for the inverse image of $\partial U \subseteq yU$. Finally, we write $\Sigma_{(\in \partial U)}^{\Psi \ltimes yU} \dashv \dots$ for the functors arising from $\Psi \ltimes \partial U \subseteq \Psi \ltimes yU$.

Theorem 4.4.3 (Poles of the transpension). For any non-spooky multiplier $\sqsubset \ltimes U : \mathcal{W} \rightarrow \mathcal{V}$, the functor

$\Omega_{(\in \partial U)}^{\Psi \ltimes yU} \circ \wp_{yU}^\Psi : \widehat{\mathcal{W}/\Psi} \rightarrow \widehat{\mathcal{V}/(\Psi \ltimes \partial U)}$ sends any presheaf to the terminal presheaf, i.e. $\Omega_{(\in \partial U)}^{\Psi \ltimes yU} \circ \wp_{yU}^\Psi = \top$.

Proof. We show that there is always a unique cell $(V, \varphi^{V \Rightarrow \Psi \ltimes \partial U}) \Rightarrow \Omega_{(\in \partial U)}^{\Psi \ltimes yU} \mathfrak{Y}_{yU}^{\Psi} \Gamma$. We have

$$\begin{aligned}
(V, \varphi^{V \Rightarrow \Psi \ltimes \partial U}) &\Rightarrow \Omega_{(\in \partial U)}^{\Psi \ltimes yU} \mathfrak{Y}_{yU}^{\Psi} \Gamma \\
&= \Sigma_{(\in \partial U)}^{\Psi \ltimes yU} (V, \varphi^{V \Rightarrow \Psi \ltimes \partial U}) \Rightarrow \mathfrak{Y}_{yU}^{\Psi} \Gamma \\
&= (V, \varphi^{V \Rightarrow \Psi \ltimes yU}) \Rightarrow \mathfrak{Y}_{yU}^{\Psi} \Gamma \\
&= \forall_{yU}^{\Psi} \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes yU}) \rightarrow \Gamma \\
&= \forall(W, \psi^{W \Rightarrow \Psi}). \left((W, \psi) \Rightarrow \forall_{yU}^{\Psi} \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes yU}) \right) \rightarrow ((W, \psi) \Rightarrow \Gamma) \\
&= \forall(W, \psi^{W \Rightarrow \Psi}). \left(\perp_U^{\Psi} (W, \psi) \Rightarrow \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes yU}) \right) \rightarrow ((W, \psi) \Rightarrow \Gamma) \\
&= \forall(W, \psi^{W \Rightarrow \Psi}). \left((W \ltimes U, \psi \ltimes yU) \rightarrow (V, \varphi^{V \Rightarrow \Psi \ltimes yU}) \right) \rightarrow ((W, \psi) \Rightarrow \Gamma),
\end{aligned}$$

and then we see that the last argument χ cannot exist. Indeed, suppose we have a commuting diagram

$$\begin{array}{ccc}
W \ltimes U & \xrightarrow{\chi} & V \\
\searrow \psi \ltimes yU & & \downarrow \varphi \\
& \Psi \ltimes yU & \xleftarrow{\cong} \Psi \ltimes \partial U \\
\searrow \pi & \downarrow \pi_2 & \downarrow \pi_2 \\
& yU & \xleftarrow{\cong} \partial U.
\end{array} \tag{36}$$

Then we see that $\pi_2 \circ \varphi : V \rightarrow U$ is dimensionally split with section χ but is also a cell of ∂U which means exactly that it is not dimensionally split. \square

The following theorem shows that dimensionally split morphisms are an interesting concept:

Theorem 4.4.4 (Boundary theorem). We have $\top \ltimes yU \mid (\in \partial U) \cong \mathfrak{Y}_{yU}^{\top} \perp \vdash \text{Ctx}$.

Proof. We prove this by characterizing the right hand side of the isomorphism. We have

$$\begin{aligned}
(V, \varphi^{V \Rightarrow \top \ltimes yU}) &\Rightarrow \mathfrak{Y}_{yU}^{\top} \perp \\
&= \forall_{yU}^{\top} \mathbf{y}(V, \varphi^{V \Rightarrow \top \ltimes yU}) \rightarrow \perp \\
&= \forall(W, ()^{W \Rightarrow \top}). ((W, ()) \Rightarrow \forall_{yU}^{\top} \mathbf{y}(V, \varphi^{V \Rightarrow \top \ltimes yU})) \rightarrow ((W, ()) \Rightarrow \perp) \\
&= \forall(W, ()^{W \Rightarrow \top}). ((W, ()) \Rightarrow \forall_{yU}^{\top} \mathbf{y}(V, \varphi^{V \Rightarrow \top \ltimes yU})) \rightarrow \emptyset \\
&= \forall(W, ()^{W \Rightarrow \top}). (\perp_U^{\top} (W, ()) \rightarrow (V, \varphi^{V \Rightarrow \top \ltimes yU})) \rightarrow \emptyset \\
&= \forall(W, ()^{W \Rightarrow \top}). ((W \ltimes U, () \ltimes U) \rightarrow (V, \varphi^{V \Rightarrow \top \ltimes yU})) \rightarrow \emptyset \\
&\cong \forall W. ((W \ltimes U, \pi_2) \rightarrow (V, \pi_2 \circ \varphi)) \rightarrow \emptyset \\
&\cong (\exists W. (W \ltimes U, \pi_2) \rightarrow (V, \pi_2 \circ \varphi)) \rightarrow \emptyset.
\end{aligned}$$

Clearly, the left hand side of the last line is inhabited if and only if $\pi_2 \circ \varphi$ is dimensionally split. Hence, there is a unique cell $(V, \varphi^{V \Rightarrow \top \ltimes yU}) \Rightarrow \mathfrak{Y}_{yU}^{\top} \perp$ if and only if $\pi_2 \circ \varphi$ is *not* dimensionally split, showing that $\mathfrak{Y}_{yU}^{\top} \perp$ is indeed isomorphic to $(\in \partial U)$. \square

Remark 4.4.5. In section 6.3 (theorem 6.3.1), we will see that unless the multiplier is cancellative and affine, the transpension type is not stable under substitution. Instead, for $\sigma : \Psi_1 \rightarrow \Psi_2$, we only have $\Omega^{\sigma \ltimes yU} \circ \mathfrak{Y}_{yU}^{\Psi_2} \rightarrow \mathfrak{Y}_{yU}^{\Psi_1} \circ \Omega^{\sigma}$. Writing $\omega : \Psi \rightarrow \top$, this gives us over $\Psi \ltimes yU$:

$$(\in \partial U) = \Omega^{\omega \ltimes yU} (\in \partial U) \cong \Omega^{\omega \ltimes yU} \mathfrak{Y}_{yU}^{\top} \perp \rightarrow \mathfrak{Y}_{yU}^{\Psi} \Omega^{\omega} \perp = \mathfrak{Y}_{yU}^{\Psi} \perp,$$

which we already knew from theorem 4.4.3.

On the other hand, $(\in \partial U)$ is stable under substitution. Hence, theorem 4.4.4 breaks if we replace \top with an arbitrary Ψ , unless $\sqsubset \times U$ is cancellative and affine.

Theorem 4.4.6 (Transpension elimination). Let $\sqsubset \times U : \mathcal{W} \rightarrow \mathcal{V}$ be a cancellative, affine and connection-free. Then we have

$$\begin{array}{c}
\Psi \times yU \mid \Gamma \vdash \text{Ctx} \\
\Psi \mid \forall_{yU}^{\Psi} \Gamma \vdash A \text{ type} \\
\Psi \times yU \mid \Gamma. \left\langle \check{\chi}_{yU}^{\Psi} \mid A \right\rangle \vdash B \text{ type} \\
\Psi \times \partial U \mid \Omega_{(\in \partial U)}^{\Psi \times \partial U} \Gamma \vdash b_{\partial} : \left(\Omega_{(\in \partial U)}^{\Psi \times \partial U} B \right) [(\text{id}, -)] \\
\Psi \mid \left(\forall_{yU}^{\Psi} \Gamma \right). A \vdash \check{b} : \left(\forall_{yU}^{\Psi} B \right) \left[\left(\pi, \left(\text{unmerid}_{yU}^{\Psi} \right)^{-1} (\xi) \right) \right] \\
\Psi \times \partial U \mid \Omega_{(\in \partial U)}^{\Psi \times \partial U} \check{\chi}_{yU}^{\Psi} \left(\left(\forall_{yU}^{\Psi} \Gamma \right). A \right) \vdash \Omega_{(\in \partial U)}^{\Psi \times \partial U} \left(\text{app}_{yU}^{\Psi} \left(\check{\chi}_{yU}^{\Psi} \check{b} \right) \right) = b_{\partial} \left[\Omega_{(\in \partial U)}^{\Psi \times \partial U} \left(\text{app}_{yU}^{\Psi} \circ \pi \right) \right] \\
: \left(\Omega_{(\in \partial U)}^{\Psi \times \partial U} B \right) [(\text{id}, -)] \left[\Omega_{(\in \partial U)}^{\Psi \times \partial U} \left(\text{app}_{yU}^{\Psi} \circ \pi \right) \right] \\
\hline
\Psi \times yU \mid \Gamma. \left\langle \check{\chi}_{yU}^{\Psi} \mid A \right\rangle \vdash b : B
\end{array} \tag{37}$$

and b reduces to b_{∂} and \check{b} if we apply to it the same functors and substitutions that have been applied to B in the types of b_{∂} and \check{b} .

(If the multiplier is not quantifiable, then $\check{\chi}_{yU}^{\Psi}$ may not be a CwF morphism, but the term $\text{app}_{yU}^{\Psi} \left(\check{\chi}_{yU}^{\Psi} \check{b} \right)$ is essentially a dependent transposition for the adjunction $\check{\chi}_{yU}^{\Psi} \dashv \forall_{yU}^{\Psi}$ which even exists if only the right adjoint is a CwF morphism [Nuy18]).

In words: if we want to eliminate an element of the transpension type, then we can do so by induction. We distinguish two cases and a coherence condition:

- In the first case (b_{∂}), we are on the boundary of U and the transpension type trivializes.
- In the second case, we are defining an action on cells that live over all of yU . In the transpension type, such cells are in 1-1 correspondence with cells of type A under the isomorphism $\text{unmerid}_{yU}^{\Psi} : \forall_{yU}^{\Psi} \check{\chi}_{yU}^{\Psi} \cong \text{Id}$.
- The boundary of the image of cells in the second case, must always be b_{∂} .

Note that right adjoint weak CwF morphisms such as $\check{\chi}_{yU}^{\Psi}$ give rise to a DRA by applying the CwF morphism and then substituting with the unit of the adjunction. As such, the transpension type is modelled by the DRA sending A to $\left\langle \check{\chi}_{yU}^{\Psi} \mid A \right\rangle = \left(\check{\chi}_{yU}^{\Psi} A \right) \left[\text{reidx}_{yU}^{\Psi} \right]$.

Proof. Well-formedness. We first show that the theorem is well-formed.

- The rule for Γ just assumes that Γ is a presheaf over $\mathcal{V}/(\Psi \times yU)$.
- Then $\forall_{yU}^{\Psi} \Gamma$ is a presheaf over \mathcal{W}/Ψ and we assume that A is a type in that context, i.e. a presheaf over the category of elements of $\forall_{yU}^{\Psi} \Gamma$.
- Then the DRA of $\check{\chi}_{yU}^{\Psi}$ applied to A is a type in context Γ . We assume that B is a type over the extended context.
- Being a central lifting, $\Omega_{(\in \partial U)}^{\Psi \times \partial U} B$ is a CwF morphism and can be applied to B , yielding a type in context

$$\Omega_{(\in \partial U)}^{\Psi \times \partial U} \left(\Gamma. \left(\check{\chi}_{yU}^{\Psi} A \right) \left[\text{reidx}_{yU}^{\Psi} \right] \right) = \Omega_{(\in \partial U)}^{\Psi \times \partial U} \Gamma. \left(\Omega_{(\in \partial U)}^{\Psi \times \partial U} \check{\chi}_{yU}^{\Psi} A \right) \left[\Omega_{(\in \partial U)}^{\Psi \times \partial U} \text{reidx}_{yU}^{\Psi} \right] \cong \Omega_{(\in \partial U)}^{\Psi \times \partial U} \Gamma. \top,$$

where the isomorphism is an application of theorem 4.4.3. The substitution $(\text{id}, _) = \pi^{-1}$ yields a type in context $\Omega_{(\in \partial U)}^{\Psi \times \partial U} \Gamma$. We assume that b_∂ has this type.

- Being a central lifting, \forall_{yU}^Ψ is a CwF morphism and can be applied to B , yielding a type in context

$$\forall_{yU}^\Psi \left(\Gamma. \left(\check{\forall}_{yU}^\Psi A \right) \left[\text{reidx}_{yU}^\Psi \right] \right) = \forall_{yU}^\Psi \Gamma. \left(\forall_{yU}^\Psi \check{\forall}_{yU}^\Psi A \right) \left[\forall_{yU}^\Psi \text{reidx}_{yU}^\Psi \right].$$

The natural transformation $(\text{unmerid}_{yU}^\Psi)^{-1}$ gives rise [Nuy18] to a function

$$(\text{unmerid}_{yU}^\Psi)^{-1} : A \rightarrow \left(\forall_{yU}^\Psi \check{\forall}_{yU}^\Psi A \right) [(\text{unmerid}_{yU}^\Psi)^{-1}]. \quad (38)$$

Now, by the adjunction laws, $\forall_{yU}^\Psi \text{reidx}_{yU}^\Psi \circ \text{unmerid}_{yU}^\Psi = \text{id}$, so

$$\forall_{yU}^\Psi \text{reidx}_{yU}^\Psi = \forall_{yU}^\Psi \text{reidx}_{yU}^\Psi \circ \text{unmerid}_{yU}^\Psi \circ (\text{unmerid}_{yU}^\Psi)^{-1} = (\text{unmerid}_{yU}^\Psi)^{-1}. \quad (39)$$

Then we have

$$(\text{unmerid}_{yU}^\Psi)^{-1} : A \rightarrow \left(\forall_{yU}^\Psi \check{\forall}_{yU}^\Psi A \right) \left[\forall_{yU}^\Psi \text{reidx}_{yU}^\Psi \right]. \quad (40)$$

Thus, we can substitute $\forall_{yU}^\Psi B$ with $(\pi, (\text{unmerid}_{yU}^\Psi)^{-1}(\xi))$, yielding a type in the desired context.

We assume that \mathring{b} has this type.

- In the coherence criterion, we have applied operations to b_∂ and \mathring{b} before equating them. We have to ensure that the resulting terms are well-typed in the given context and type.

- If we apply \downarrow_{yU}^Ψ to the term \mathring{b} , we get

$$\Psi \times yU \mid \left(\downarrow_{yU}^\Psi \forall_{yU}^\Psi \Gamma \right). \downarrow_{yU}^\Psi A \vdash \downarrow_{yU}^\Psi \mathring{b} : \left(\downarrow_{yU}^\Psi \forall_{yU}^\Psi B \right) \left[\downarrow_{yU}^\Psi \left(\pi, (\text{unmerid}_{yU}^\Psi)^{-1}(\xi) \right) \right].$$

If we subsequently apply app_{yU}^Ψ , we get

$$\Psi \times yU \mid \left(\downarrow_{yU}^\Psi \forall_{yU}^\Psi \Gamma \right). \downarrow_{yU}^\Psi A \vdash \text{app}_{yU}^\Psi \left(\downarrow_{yU}^\Psi \mathring{b} \right) : B \left[\text{app}_{yU}^\Psi \right] \left[\downarrow_{yU}^\Psi \left(\pi, (\text{unmerid}_{yU}^\Psi)^{-1}(\xi) \right) \right].$$

Next, we apply $\Omega_{(\in \partial U)}^{\Psi \times yU}$ and obtain something of type

$$\left(\Omega_{(\in \partial U)}^{\Psi \times yU} B \right) \left[\Omega_{(\in \partial U)}^{\Psi \times yU} \text{app}_{yU}^\Psi \right] \left[\Omega_{(\in \partial U)}^{\Psi \times yU} \downarrow_{yU}^\Psi \left(\pi, (\text{unmerid}_{yU}^\Psi)^{-1}(\xi) \right) \right].$$

Now if we look at the context of $\Omega_{(\in \partial U)}^{\Psi \times yU} B$, we see that the last type is the unit type by theorem 4.4.3, so the substitution applied to B is determined by its weakening. So we rewrite:

$$\begin{aligned} \dots &= \left(\Omega_{(\in \partial U)}^{\Psi \times yU} B \right) [(\text{id}, _)] [\pi] \left[\Omega_{(\in \partial U)}^{\Psi \times yU} \text{app}_{yU}^\Psi \right] \left[\Omega_{(\in \partial U)}^{\Psi \times yU} \downarrow_{yU}^\Psi \left(\pi, (\text{unmerid}_{yU}^\Psi)^{-1}(\xi) \right) \right] \\ &= \left(\Omega_{(\in \partial U)}^{\Psi \times yU} B \right) [(\text{id}, _)] \left[\Omega_{(\in \partial U)}^{\Psi \times yU} \text{app}_{yU}^\Psi \right] [\pi] \left[\Omega_{(\in \partial U)}^{\Psi \times yU} \downarrow_{yU}^\Psi \left(\pi, (\text{unmerid}_{yU}^\Psi)^{-1}(\xi) \right) \right] \\ &= \left(\Omega_{(\in \partial U)}^{\Psi \times yU} B \right) [(\text{id}, _)] \left[\Omega_{(\in \partial U)}^{\Psi \times yU} \text{app}_{yU}^\Psi \right] \left[\Omega_{(\in \partial U)}^{\Psi \times yU} \downarrow_{yU}^\Psi \left(\pi \circ \left(\pi, (\text{unmerid}_{yU}^\Psi)^{-1}(\xi) \right) \right) \right] \\ &= \left(\Omega_{(\in \partial U)}^{\Psi \times yU} B \right) [(\text{id}, _)] \left[\Omega_{(\in \partial U)}^{\Psi \times yU} \text{app}_{yU}^\Psi \right] \left[\Omega_{(\in \partial U)}^{\Psi \times yU} \downarrow_{yU}^\Psi \pi \right] \\ &= \left(\Omega_{(\in \partial U)}^{\Psi \times yU} B \right) [(\text{id}, _)] \left[\Omega_{(\in \partial U)}^{\Psi \times yU} \left(\text{app}_{yU}^\Psi \circ \pi \right) \right]. \end{aligned}$$

- It is immediate that the substitution applied to b_∂ yields the given type.

Soundness of the coherence criterion. Note that, if we apply to b the same reasoning that we applied to B to show well-formedness of the last 3 premises, we find that the coherence criterion does hold if b_∂ and \mathring{b} arise from a common b .

Completeness of the elimination clauses. We now show that b is fully determined by the b_∂ and \mathring{b} that can be derived from it. Afterwards, we will show that the given coherence condition is sufficient to make sure that b_∂ and \mathring{b} determine some b .

Note that B , being a type in a presheaf \mathbf{CwF} , is a presheaf over the category of elements of $\Gamma \cdot (\mathring{\mathcal{Q}}_{yU}^\Psi A) \left[\text{reid}_{yU}^\Psi \right]$. Hence it acts on cells

$$\left(V, \varphi^{V \Rightarrow \Psi \times yU}, \gamma^{(V, \varphi) \Rightarrow \Gamma}, a^{(V, \varphi, \gamma) \Rightarrow (\mathring{\mathcal{Q}}_{yU}^\Psi A) \left[\text{reid}_{yU}^\Psi \right]} \right).$$

Now we divide such cells in two classes: on-boundary cells (for which $\pi_2 \circ \varphi$ is *not* dimensionally split) and total cells (the others). As $\Omega_{(\in \partial U)}^{\Psi \times yU}$ is exactly the restriction of presheaves to the on-boundary cells, it is clear that b_∂ determines the action of b on those.

For total cells, note that the full subcategory of $\mathcal{V}/(\Psi \times yU)$ consisting of the total elements, is (by theorem 4.1.11) equivalent to \mathcal{W}/Ψ , with one direction given by \mathcal{J}_U^Ψ . Restriction to total cells is then given by the central lifting of that functor, being \mathcal{V}_{yU}^Ψ . Combined with the knowledge that $\mathcal{V}_{yU}^\Psi \mathring{\mathcal{Q}}_{yU}^\Psi \cong \text{Id}$ (theorem 4.1.10), this reveals that \mathring{b} determines the action of b on total cells.

Completeness of the coherence criterion. The action of a term on cells should be natural with respect to restriction. This is automatic when considered with respect to morphisms between cells that are either both total or both on-boundary. Moreover, there are no morphisms $\chi : (V, \varphi) \rightarrow (V', \varphi') : \mathcal{V}/(\Psi \times U)$ from a total cell to an on-boundary cell, since proposition 3.4.9 asserts that if $\pi_2 \circ \varphi' \circ \chi$ is dimensionally split, then so is $\pi_2 \circ \varphi'$. So we still need to prove naturality w.r.t. morphisms from on-boundary cells to total cells.

Let $\chi : (V, \varphi) \rightarrow (V', \varphi')$ be such a morphism. Then $(V', \varphi') \cong \mathcal{J}_U^\Psi(W, \psi) \cong \mathcal{J}_U^\Psi \mathcal{J}_U^\Psi(W, \psi) \cong \mathcal{J}_U^\Psi \mathcal{J}_U^\Psi(V', \varphi')$ by an isomorphism

$$\begin{aligned} & \mathcal{J}_U^\Psi \mathcal{J}_U^\Psi \iota^{-1} \circ \mathcal{J}_U^\Psi (\text{drop}_U^\Psi)^{-1} \circ \iota \\ &= \mathcal{J}_U^\Psi \mathcal{J}_U^\Psi \iota^{-1} \circ \text{copy}_U^\Psi \circ \iota \\ &= \text{copy}_U^\Psi \circ \iota^{-1} \circ \iota = \text{copy}_U^\Psi. \end{aligned}$$

Hence, by naturality, $\chi = (\text{copy}_U^\Psi)^{-1} \circ \text{copy}_U^\Psi \circ \chi = (\text{copy}_U^\Psi)^{-1} \circ \mathcal{J}_U^\Psi \mathcal{J}_U^\Psi \chi \circ \text{copy}_U^\Psi$. Thus, we have factored χ as an instance of the unit copy_U^Ψ followed by a morphism between total cells. This means it is sufficient to show naturality with respect to $\text{copy}_U^\Psi : (V, \varphi) \rightarrow \mathcal{J}_U^\Psi \mathcal{J}_U^\Psi(V, \varphi)$. (The cells of Γ and the transpension type available for (V', φ') carry over to $\mathcal{J}_U^\Psi \mathcal{J}_U^\Psi(V, \varphi)$ by restriction.)

Now the action of b on (V, φ) is given by the action of b_∂ on (V, φ) . Meanwhile, the action of b on $\mathcal{J}_U^\Psi \mathcal{J}_U^\Psi(V, \varphi)$ is given by the action of \mathring{b} on $\mathcal{J}_U^\Psi(V, \varphi)$, which is the action of $\mathcal{J}_U^\Psi b$ on (V, φ) . These have to correspond via $\text{copy}_U^\Psi : (V, \varphi) \rightarrow \mathcal{J}_U^\Psi \mathcal{J}_U^\Psi(V, \varphi)$, which corresponds via central lifting to the natural transformation app_U^Ψ on presheaves. This is exactly what happens in the coherence criterion: we use $\text{app}_U^\Psi : \mathcal{J}_U^\Psi \mathcal{V}_U^\Psi \rightarrow \text{Id}$ to bring b_∂ and \mathring{b} to the same context and type, and then equate them. Since b_∂ only exists on the boundary, we also have to restrict \mathring{b} to the boundary, but that's OK since we were interested in an on-boundary cell anyway. \square

Example 4.4.7 (Affine cubes). We instantiate theorem 4.4.6 for the multiplier $\sqcup * \mathbb{I} : \square^k \rightarrow \square^k$ (example 3.3.3). There, $\partial \mathbb{I}$ is essentially the constant presheaf with k elements. So b_∂ determines the images of the k poles of the transpension type. The term \mathring{b} determines the action on paths (for $k = 2$, for general k perhaps ‘webs’ is a better term), and the paths/webs of the transpension type are essentially the elements

of A . The coherence condition says that the image of such paths/webs should always have the endpoints given by b_∂ .

Example 4.4.8 (Clocks). We instantiate theorem 4.4.6 for the multiplier $\sqcup * (i : \odot_k)$ (example 3.3.6), where we adapt the base category to forbid diagonals: a morphism may use every variable of its domain at most once. The boundary $\partial(i : \odot_k)$ is isomorphic to $y(i : \odot_{k-1})$ if $k > 0$ and to the empty presheaf \perp if $k = 0$. So if we want to eliminate an element of the transpension type over $y(i : \odot_k)$, which means we have a clock and we don't care about what happens if the time exceeds k , then we need to handle two cases. The first case b_∂ says what happens if we don't even care what happens at timestamp k ; in which case the transpension type trivializes. Then, by giving \dot{b} , we say what happens at timestamp k and need to make sure that this is consistent with b_∂ . The elements of the transpension type at timestamp k are essentially the elements of A , which are fresh for the clock.

Example 4.4.9 (Embargoes). Recall that the multiplier $\sqcup \times \mathbf{!}$ sends $W \in \mathcal{W}$ to $(W, \top) \in \mathcal{W} \times \uparrow$, the Yoneda-embedding of which represents the arrow $yW \rightarrow yW$, i.e. $yW.\mathbf{!}.\top$ under the convention that $\Psi.\mathbf{!}.\Theta \cong (\Psi.\Theta \rightarrow \Psi)$. Its left lifting $\sqcup \times y\mathbf{!} : \widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{W} \times \uparrow}$, and $y\mathbf{!}$ is the terminal object, so that $\widehat{\mathcal{W} \times \uparrow} / \mathbf{!} \cong \widehat{\mathcal{W} \times \uparrow}$. We get 5 adjoint functors, of which we give here the action up to isomorphism:

$$\begin{array}{rclcl} & & \Psi & \mapsto & (\perp \rightarrow \Psi), \\ & \exists_{y\mathbf{!}} : & \Psi & \leftarrow & (\Psi.\Theta \rightarrow \Psi), \\ \sqcup \times y\mathbf{!} & \text{or} & \downarrow_{y\mathbf{!}} : & \Psi & \mapsto & (\Psi \rightarrow \Psi), \\ y\mathbf{!} \multimap \sqcup & \text{or} & \forall_{y\mathbf{!}} : & \Psi.\Theta & \leftarrow & (\Psi.\Theta \rightarrow \Psi), \\ y\mathbf{!} \sqrt{\sqcup} & \text{or} & \emptyset_{y\mathbf{!}} : & \Psi & \mapsto & (\Psi \rightarrow \top). \end{array}$$

The boundary of $y\mathbf{!}$ is $\partial\mathbf{!} \cong y(\top, \perp)$ which is isomorphic to the arrow $\perp \rightarrow \top$. Thus, we see:

- If, for some unknown embargo, we have information partly under that embargo, then we can only extract the unembargoed information,
- If information is fresh for an embargo, then it is unembargoed,
- If, for any embargo, we have information partly under that embargo, then we can extract the information,
- If information is transpended over an embargo, then it is completely embargoed.

So let us now instantiate theorem 4.4.6, which allows us to eliminate an element of the transpension type, i.e. essentially an element of $A \rightarrow \top$. The boundary case exists over the boundary $\perp \rightarrow \top$ and allows us to consider only the codomain of the arrow, i.e. the part of the context before the signpost, where A is trivial. The case \dot{b} then requires us to say how to act on embargoed data in a coherent way with what we already specified in b_∂ . The embargoed data is essentially an element of A .

5 Prior modalities

Many modalities arise as central or right liftings of functors between base categories [NVD17, ND18, Nuy18, BM20]. The following definition allows us to use such modalities even when part of the context is in front of a pipe.

Definition 5.0.1. A functor $G : \mathcal{W} \rightarrow \mathcal{W}'$ yields a functor $G^{/\Psi} : \mathcal{W}/\Psi \rightarrow \mathcal{W}'/G\mathbf{!}\Psi : (W, \psi) \mapsto (GW, G\mathbf{!}\psi)$. This in turn yields three adjoint functors between presheaf categories:

$$G_!^{\Psi} \dashv G^{\Psi|*} \dashv G_*^{\Psi}. \quad (41)$$

If a modality is both a right and a central lifting, then the following theorem relates the corresponding 'piped' modalities:

Theorem 5.0.2. If $G : \mathcal{W} \rightarrow \mathcal{W}'$ has a right adjoint $G \dashv S$, then we have

$$\begin{array}{c|c}
\begin{array}{c}
\Sigma^{\varepsilon_1} \circ G/S_! \Psi' \dashv S^{\Psi'} \\
\Sigma^{\varepsilon_1} \circ G_!^{S_! \Psi'} \dashv S_!^{\Psi'} \\
S_!^{\Psi'} \cong G^{S_! \Psi'} \circ \Omega^{\varepsilon_1} \dashv S^{\Psi'} \\
S^{\Psi'} \cong \Pi^{\varepsilon_1} \circ G_*^{S_! \Psi'} \dashv S_*^{\Psi'}
\end{array}
&
\begin{array}{c}
G^{\Psi} \dashv \Omega/\eta_! \circ S/G_! \Psi \\
G_!^{\Psi} \dashv \Omega/\eta_! \circ S_!^{G_! \Psi} \cong G^{\Psi|*} \\
G^{\Psi|*} \dashv S^{G_! \Psi|*} \circ \Pi^{\eta_!} \cong G_*^{\Psi} \\
G_*^{\Psi} \dashv \chi^{\eta_!} \circ S_*^{G_! \Psi}
\end{array}
\end{array} \quad (42)$$

assuming – where mentioned – that $\Omega/\eta_!$ and Ω/η_* exist.

Proof. For the left half of the table, we only prove the first line. The other adjunctions follow from the fact that $\sqsubset_!$, \sqsubset^* and \sqsubset_* are pseudofunctors, and the isomorphisms follow from uniqueness of the adjoint. We have a correspondence of diagrams

$$\begin{array}{ccc}
W & \xrightarrow{\quad} & SW' \\
\searrow \psi & & \swarrow S_! \psi' \\
& S_! \Psi' &
\end{array}
\quad
\begin{array}{ccc}
GW & \xrightarrow{\quad} & W' \\
\searrow \varepsilon_! \circ G_! \psi & & \swarrow \psi' \\
& \Psi' &
\end{array} \quad (43)$$

i.e. morphisms $(W, \psi) \rightarrow S^{\Psi'}(W', \psi') : \mathcal{W}/S_! \Psi'$ correspond to morphisms $\Sigma^{\varepsilon_1} G/S_! \Psi'(W, \psi) \rightarrow (W', \psi') : \mathcal{W}'/\Psi'$.

On the right side of the table, we can prove all adjunctions from the corresponding adjunction on the left. Again, the isomorphisms then follow from uniqueness of the adjoint. We illustrate this only for the first line. The left adjoint to $\Omega/\eta_! \circ S/G_! \Psi$ is $(\Sigma^{\varepsilon_1} \circ G/S_! G_! \Psi) \circ \Sigma/\eta_!$. We prove that this is equal to G^{Ψ} :

$$\begin{aligned}
& \Sigma^{\varepsilon_1} G/S_! G_! \Psi \Sigma/\eta_! (W, \psi : W \rightarrow \Psi) \\
&= \Sigma^{\varepsilon_1} G/S_! G_! \Psi (W, \eta_! \circ \psi : W \rightarrow S_! G_! \Psi) \\
&= \Sigma^{\varepsilon_1} (GW, G_! \eta_! \circ G_! \psi : GW \rightarrow G_! S_! G_! \Psi) \\
&= (GW, \varepsilon_! \circ G_! \eta_! \circ G_! \psi : GW \rightarrow G_! \Psi) = (GW, G_! \psi : GW \rightarrow G_! \Psi).
\end{aligned}$$

For the other lines, the required equation follows from the above one by left/central/right lifting. \square

6 Commutation rules

6.1 Substitution and substitution

See theorem 2.3.18.

6.2 Modality and substitution

Theorem 6.2.1. Assume a functor $G : \mathcal{W} \rightarrow \mathcal{W}'$ and a morphism $\sigma : \Psi_1 \rightarrow \Psi_2 : \widehat{\mathcal{W}}$. Then we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{W}/\Psi_1 & \xrightarrow{G/\Psi_1} & \mathcal{W}'/G_! \Psi_1 \\
\Sigma/\sigma \downarrow & & \downarrow \Sigma/G_! \sigma \\
\mathcal{W}/\Psi_2 & \xrightarrow{G/\Psi_2} & \mathcal{W}'/G_! \Psi_2
\end{array} \quad (44)$$

and hence

	$G_!$	G^*	G_*
Σ	$\Sigma^{G_! \sigma} G_!^{\Psi_1} \cong G_!^{\Psi_2} \Sigma \sigma $	$\Sigma \sigma G^{\Psi_1} \rightarrow G^{\Psi_2} \Sigma^{G_! \sigma} $	
Ω	$\Omega^{G_! \sigma} G_!^{\Psi_2} \leftarrow G_!^{\Psi_1} \Omega \sigma $	$\Omega \sigma G^{\Psi_2} \cong G^{\Psi_1} \Omega^{G_! \sigma} $	$\Omega^{G_! \sigma} G_*^{\Psi_2} \rightarrow G_*^{\Psi_1} \Omega \sigma $
Π		$\Pi \sigma G^{\Psi_1} \leftarrow G^{\Psi_2} \Pi^{G_! \sigma} $	$\Pi^{G_! \sigma} G_*^{\Psi_1} \cong G_*^{\Psi_2} \Pi \sigma $
$\$$			$\$^{G_! \sigma} G_*^{\Psi_2} \leftarrow G_*^{\Psi_1} \$ \sigma $

where every statement holds if the mentioned functors exist.

Proof. It is evident from the definitions that the given diagram commutes. Then by applying $\sqcup_!$, we find that $\Sigma^{G_! \sigma} |G_!^{\Psi_1}| \cong G_!^{\Psi_2} | \Sigma \sigma |$. The rest of the table then follows by lemma 2.1.2. \square

Remark 6.2.2. • If $\sigma = \pi : \Psi.A \rightarrow \Psi$, then this says something about weakening and the Σ - and Π -types over A .

- If $G_!$ moreover happens to be a CwF morphism, then this relates weakening and the Σ - and Π -types over A to those over $G_!A$.
- If $\sqcup \times U$ is a cartesian multiplier and we take $\sigma = \pi_1 : \Psi \times yU \rightarrow \Psi$, then by theorem 4.1.10, this says something about $\exists_{yU}^{\Psi|} \dashv \exists_{yU}^{\Psi|} \dashv \forall_{yU}^{\Psi|} \dashv \S_{yU}^{\Psi|}$.

6.3 Multiplier and substitution

If, in section 6.2, we take G equal to some multiplier $\sqcup \times U : \mathcal{W} \rightarrow \mathcal{V}$, then we have

$$G^{\Psi} = \exists_G^{\Psi}, \quad G_! = \sqcup \times yU, \quad G_!^{\Psi|} = \exists_U^{\Psi|}, \quad G^{\Psi|*} = \forall_U^{\Psi|}, \quad G_*^{\Psi|} = \exists_U^{\Psi|}. \quad (45)$$

This immediately yields two theorems:

Theorem 6.3.1. Assume a multiplier $\sqcup \times U : \mathcal{W} \rightarrow \mathcal{V}$ and a morphism $\sigma : \Psi_1 \rightarrow \Psi_2 : \widehat{\mathcal{W}}$. Write $\tau = \sigma \times yU$. Then we have:

	\exists	\exists	\forall	\exists
Σ	$\Sigma^{\sigma} \exists_U^{\Psi_1} \triangleleft^1 \exists_U^{\Psi_2} \Sigma^{\tau} $	$\Sigma^{\tau} \exists_U^{\Psi_1} \cong \exists_U^{\Psi_2} \Sigma^{\sigma} $	$\Sigma^{\sigma} \forall_U^{\Psi_1} \triangleright_1 \forall_U^{\Psi_2} \Sigma^{\tau} $	$\Sigma^{\tau} \exists_U^{\Psi_1} \triangleright_2 \exists_U^{\Psi_2} \Sigma^{\sigma} $
Ω	$\Omega^{\sigma} \exists_U^{\Psi_2} \triangleleft^2 \exists_U^{\Psi_1} \Omega^{\tau} $	$\Omega^{\tau} \exists_U^{\Psi_2} \triangleleft^1 \exists_U^{\Psi_1} \Omega^{\sigma} $	$\Omega^{\sigma} \forall_U^{\Psi_2} \cong \forall_U^{\Psi_1} \Omega^{\tau} $	$\Omega^{\tau} \exists_U^{\Psi_2} \triangleright_1 \exists_U^{\Psi_1} \Omega^{\sigma} $
Π		$\Pi^{\tau} \exists_U^{\Psi_1} \triangleleft^2 \exists_U^{\Psi_2} \Pi^{\sigma} $	$\Pi^{\sigma} \forall_U^{\Psi_1} \triangleleft^1 \forall_U^{\Psi_2} \Pi^{\tau} $	$\Pi^{\tau} \exists_U^{\Psi_1} \cong \exists_U^{\Psi_2} \Pi^{\sigma} $
\S			$\S^{\sigma} \forall_U^{\Psi_2} \triangleleft^2 \forall_U^{\Psi_1} \S^{\tau} $	$\S^{\tau} \exists_U^{\Psi_2} \triangleleft^1 \exists_U^{\Psi_1} \S^{\sigma} $

(46)

where every statement holds if the mentioned functors exist, and where

1. In general, \triangleleft^1 means \leftarrow , \triangleright_1 means \rightarrow and the other symbols mean nothing.
2. If $\sqcup \times U$ is quantifiable, then \triangleleft^1 upgrades to \cong and \triangleleft^2 upgrades to \leftarrow .
3. If $\sqcup \times U$ is cancellative and affine, then we have

$$\Sigma^{\sigma} | \forall_U^{\Psi_1} | \cong \forall_U^{\Psi_2} | \Sigma^{\tau} | : \widehat{\mathcal{V}/(\Psi_1 \times yU)} \rightarrow \widehat{\mathcal{W}/\Psi_2} \quad (47)$$

so that \triangleright_1 upgrades to \cong and \triangleright_2 upgrades to \rightarrow .

Proof. 1. The general case is a corollary of theorem 6.2.1 for $G = \sqcup \times U$.

2. To prove the quantifiable case, we show in the base category that $\Sigma^{\sigma} \exists_U^{\Psi_1} = \exists_U^{\Psi_2} \Sigma^{(\sigma \times yU)}$. We use the construction of \exists_U^{Ψ} in the proof of provident quantifiability (proposition 4.1.8). On one hand, we have:

$$\Sigma^{\sigma} \exists_U^{\Psi_1} (V, (\psi_1^{W_0 \Rightarrow \Psi_1} \times yU) \circ \varphi^{V \Rightarrow W_0 \times U}) = \Sigma^{\sigma} \Sigma^{\psi_1} \exists_U^{W_0} (V, \varphi) = \Sigma^{\sigma \circ \psi_1} \exists_U^{W_0} (V, \varphi).$$

On the other hand:

$$\exists_U^{\Psi_2} \Sigma^{(\sigma \times yU)} (V, (\psi_1^{W_0 \Rightarrow \Psi_1} \times yU) \circ \varphi^{V \Rightarrow W_0 \times U}) = \exists_U^{\Psi_2} (V, ((\sigma \circ \psi_1) \times yU) \circ \varphi) = \Sigma^{\sigma \circ \psi_1} \exists_U^{W_0} (V, \varphi).$$

3. We show that $\Sigma^\sigma | \forall_U^{\Psi_1} | \cong \forall_U^{\Psi_2} | \Sigma^\tau |$. Pick a presheaf Γ over $\mathcal{V}/(\Psi_1 \times \mathbf{y}U)$. On the one hand, we have:

$$\begin{aligned}
(W_2, \psi_2^{W_2 \Rightarrow \Psi_2}) &\Rightarrow \Sigma^\sigma | \forall_U^{\Psi_1} | \Gamma \\
&= \exists(W_1, \psi_1^{W_1 \Rightarrow \Psi_1}). \left(\theta : (W_2, \psi_2) \rightarrow \Sigma'^\sigma(W_1, \psi_1) \right) \times \left((W_1, \psi_1) \Rightarrow \forall_U^{\Psi_1} | \Gamma \right) \\
&= \exists(W_1, \psi_1^{W_1 \Rightarrow \Psi_1}). (\theta : (W_2, \psi_2) \rightarrow (W_1, \sigma \circ \psi_1)) \times ((W_1 \times U, \psi_1 \times \mathbf{y}U) \Rightarrow \Gamma) \\
&\cong \exists W_1, \psi_1^{W_1 \Rightarrow \Psi_1}, \theta^{W_2 \rightarrow W_1}. (\psi_2 = \sigma \circ \psi_1 \circ \theta) \times ((W_1 \times U, \psi_1 \times \mathbf{y}U) \Rightarrow \Gamma) \\
&\quad \text{We now absorb } \theta \text{ into } \psi_1: \\
&\cong \psi_1^{W_2 \Rightarrow \Psi_1}. (\psi_2 = \sigma \circ \psi_1) \times ((W_2 \times U, \psi_1 \times \mathbf{y}U) \Rightarrow \Gamma).
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
(W_2, \psi_2^{W_2 \Rightarrow \Psi_2}) &\Rightarrow \forall_U^{\Psi_2} | \Sigma^\tau | \Gamma \\
&= (W_2 \times U, \psi_2 \times \mathbf{y}U) \Rightarrow \Sigma^\tau | \Gamma \\
&= \exists(V_1, \varphi_1^{V_1 \Rightarrow \Psi_1 \times \mathbf{y}U}). \left(\omega : (W_2 \times U, \psi_2 \times \mathbf{y}U) \rightarrow \Sigma'^\tau(V_1, \varphi_1) \right) \times ((V_1, \varphi_1) \Rightarrow \Gamma) \\
&= \exists(V_1, \varphi_1^{V_1 \Rightarrow \Psi_1 \times \mathbf{y}U}). (\omega : (W_2 \times U, \psi_2 \times \mathbf{y}U) \rightarrow (V_1, (\sigma \times \mathbf{y}U) \circ \varphi_1)) \times ((V_1, \varphi_1) \Rightarrow \Gamma) \\
&\quad \text{We now deconstruct } \varphi_1 = (\psi_1 \times \mathbf{y}U) \circ \chi: \\
&\cong \exists V_1, W_1, \chi^{V_1 \rightarrow W_1 \times U}, \psi_1^{W_1 \Rightarrow \Psi_1}. \\
&\quad (\omega : (W_2 \times U, \psi_2 \times \mathbf{y}U) \rightarrow (V_1, ((\sigma \circ \psi_1) \times \mathbf{y}U) \circ \chi)) \times ((V_1, (\psi_1 \times \mathbf{y}U) \circ \chi) \Rightarrow \Gamma) \\
&\cong \exists V_1, W_1, \chi^{V_1 \rightarrow W_1 \times U}, \psi_1^{W_1 \Rightarrow \Psi_1}, \omega^{W_2 \times U \rightarrow V_1}. \\
&\quad (\psi_2 \times \mathbf{y}U = ((\sigma \circ \psi_1) \times \mathbf{y}U) \circ \chi \circ \omega) \times ((V_1, (\psi_1 \times \mathbf{y}U) \circ \chi) \Rightarrow \Gamma) \\
&\quad \text{We now absorb } \omega \text{ into } \chi: \\
&\cong \exists W_1, \psi_1^{W_1 \Rightarrow \Psi_1}, \chi^{W_2 \times U \rightarrow W_1 \times U}. (\psi_2 \times \mathbf{y}U = ((\sigma \circ \psi_1) \times \mathbf{y}U) \circ \chi) \times ((W_2 \times U, (\psi_1 \times \mathbf{y}U) \circ \chi) \Rightarrow \Gamma) \\
&\quad \text{Let } \chi = \downarrow_U^{\Psi_2} \theta : \downarrow_U^{\Psi_2}(W_2, \psi_2) \rightarrow \downarrow_U^{\Psi_2}(W_1, \sigma \circ \psi_1): \\
&\cong \exists W_1, \psi_1^{W_1 \Rightarrow \Psi_1}, \theta^{W_2 \rightarrow W_1}. (\psi_2 = \sigma \circ \psi_1 \circ \theta) \times ((W_2 \times U, ((\psi_1 \circ \theta) \times \mathbf{y}U)) \Rightarrow \Gamma) \\
&\quad \text{We now absorb } \theta \text{ into } \psi_1: \\
&\cong \psi_1^{W_2 \Rightarrow \Psi_1}. (\psi_2 = \sigma \circ \psi_1) \times ((W_2 \times U, (\psi_1 \times \mathbf{y}U)) \Rightarrow \Gamma)
\end{aligned}$$

This proves the isomorphism. The rest follows from lemma 2.1.2. \square

6.4 Multiplier and modality

Theorem 6.4.1. Assume a commutative diagram (up to natural isomorphism ν)

$$\begin{array}{ccc}
\mathcal{W} & \xrightarrow{G} & \mathcal{W}' \\
\sqcup \times U \downarrow & & \downarrow \sqcup \times U' \\
\mathcal{V} & \xrightarrow{F} & \mathcal{V}'
\end{array} \tag{48}$$

where $\sqcup \times U$ and $\sqcup \times U'$ are multipliers for U and U' , and G preserves the terminal object so that $U' \cong FU$.

Write δ for the isomorphism $\delta : F_!(\Psi \times \mathbf{y}U) \cong G_!\Psi \times \mathbf{y}U'$. Then Σ'^δ is a strictly invertible functor and hence we have

$$\Sigma^{\delta|} \cong \Omega^{\delta^{-1}|} \cong \Pi^{\delta|} \cong \breve{\Sigma}^{\delta^{-1}|} \quad \Sigma'^{\delta^{-1}|} \cong \Omega^{\delta|} \cong \Pi'^{\delta^{-1}|} \cong \breve{\Sigma}^{\delta|}, \tag{49}$$

where $\Omega^{\delta^{-1}|}$ is the strict inverse to $\Omega^{\delta|}$.

Then we have $\Sigma^{\delta^{-1}} \dashv_{U'}^{G_! \Psi} G^{\Psi} \cong F^{\Psi \times U} \dashv_U^{\Psi}$. This yields the following commutation table:

	$F_!, G_!$	F^*, G^*	F_*, G_*
\exists	$\exists_{U'}^{G_! \Psi} \Omega^{\delta^{-1}} F_!^{\Psi \times yU} \triangleright_1 G_!^{\Psi} \exists_U^{\Psi} $	$\exists_U^{\Psi} F^{\Psi *} \triangleright_2 G^{\Psi *} \exists_{U'}^{G_! \Psi} \Omega^{\delta^{-1}} $	
\dashv	$\Omega^{\delta} \dashv_{U'}^{G_! \Psi} G_!^{\Psi} \cong F_!^{\Psi \times yU} \dashv_U^{\Psi} $	$\dashv_U^{\Psi} G^{\Psi *} \triangleright_1 F^{\Psi \times yU *} \Omega^{\delta} \dashv_{U'}^{G_! \Psi} $	$\Omega^{\delta} \dashv_{U'}^{G_! \Psi} G_*^{\Psi} \triangleright_2 F_*^{\Psi \times yU} \dashv_U^{\Psi} $
\forall	$\forall_{U'}^{G_! \Psi} \Omega^{\delta^{-1}} F_!^{\Psi \times yU} \leftarrow G_!^{\Psi} \forall_U^{\Psi} $	$\forall_U^{\Psi} F^{\Psi *} \cong G^{\Psi *} \forall_{U'}^{G_! \Psi} \Omega^{\delta^{-1}} $	$\forall_{U'}^{G_! \Psi} \Omega^{\delta^{-1}} F_*^{\Psi \times yU} \triangleright_1 G_*^{\Psi} \forall_U^{\Psi} $
$\check{\exists}$		$\check{\exists}_U^{\Psi} G^{\Psi *} \leftarrow F^{\Psi \times yU *} \Omega^{\delta} \check{\exists}_{U'}^{G_! \Psi} $	$\Omega^{\delta} \check{\exists}_{U'}^{G_! \Psi} G_*^{\Psi} \cong F_*^{\Psi \times yU} \check{\exists}_U^{\Psi} $

where any statement holds if the mentioned functors exist, and where

1. In general, \triangleright_1 means \rightarrow and \triangleright_2 means nothing.
2. If $\sqsubset \times U$ and $\sqsubset \times U'$ are quantifiable and the morphism $\theta : \exists_{U'} \circ F^U \rightarrow G \circ \exists_U : \mathcal{V}/U \rightarrow \mathcal{W}'$ is invertible,⁶ then \triangleright_1 upgrades to \cong and \triangleright_2 upgrades to \rightarrow .

This applies in particular if $\mathcal{W} = \mathcal{V}$, $\mathcal{W}' = \mathcal{V}'$, $G = F$ preserves finite products and the multipliers are cartesian.

Proof. The isomorphism δ is obtained by applying $\sqsubset_!$ to $\nu : F(\sqsubset \times U) \rightarrow G\sqsubset \times U'$ and then applying the resulting natural transformation to Ψ . Since δ is an isomorphism, Σ^{δ} is a strictly invertible functor with inverse $\Sigma^{\delta^{-1}}$. Since \sqsubset^* is a 2-functor, $\Omega^{\delta|}$ is also strictly invertible with inverse $\Omega^{\delta^{-1}|}$. Because equivalences of categories are adjoint to their inverse, we get the chains of isomorphisms displayed.

1. The given commutation property in the base category follows immediately from the definitions and naturality of ν and its image under $\sqsubset_!$. The rest of the table then follows by lemma 2.1.2.
2. We show in the base category that $\exists_{U'}^{G_! \Psi} \Sigma^{\delta} F^{\Psi \times yU} \cong G^{\Psi} \exists_U^{\Psi} : \mathcal{V}/(\Psi \times yU) \rightarrow \mathcal{W}'/G_! \Psi$. Without loss of generality we assume that $FU = U'$. Pick some $(V, (\psi^{W_0 \Rightarrow \Psi} \times yU) \circ \varphi_0^{V \rightarrow W_0 \times U}) \in \mathcal{V}/(\Psi \times yU)$.

On one hand, we have:

$$\begin{aligned}
& \exists_{U'}^{G_! \Psi} \Sigma^{\delta} F^{\Psi \times yU} (V, (\psi \times yU) \circ \varphi_0) \\
&= \exists_{U'}^{G_! \Psi} \Sigma^{\delta} (FV, F_!(\psi \times yU) \circ F\varphi_0) \\
&= \exists_{U'}^{G_! \Psi} (FV, \nu_! \circ F_!(\psi \times yU) \circ F\varphi_0) \\
&= \exists_{U'}^{G_! \Psi} (FV, (G_! \psi \times yU') \circ \nu \circ F\varphi_0) \\
&= \Sigma^{G_! \psi} \exists_{U'}^{G^{W_0}} (FV, \nu \circ F\varphi_0) \\
&= \Sigma^{G_! \psi} (\exists_{U'} (FV, \pi_2 \circ \nu \circ F\varphi_0), \text{drop}_{U'} \circ \exists_{U'} (\nu \circ F\varphi_0)) \\
&= \Sigma^{G_! \psi} (\exists_{U'} (FV, \pi_2 \circ F\varphi_0), \text{drop}_{U'} \circ \exists_{U'} (\nu \circ F\varphi_0)).
\end{aligned}$$

On the other hand:

$$\begin{aligned}
& G^{\Psi} \exists_U^{\Psi} (V, (\psi \times yU) \circ \varphi_0) \\
&= G^{\Psi} \Sigma^{\psi} \exists_U^{W_0} (V, \varphi_0) \\
&= \Sigma^{G_! \psi} G^{W_0} \exists_U^{W_0} (V, \varphi_0) \\
&= \Sigma^{G_! \psi} G^{W_0} (\exists_U (V, \pi_2 \circ \varphi_0), \text{drop}_U \circ \exists_U \varphi_0) \\
&= \Sigma^{G_! \psi} (G\exists_U (V, \pi_2 \circ \varphi_0), G\text{drop}_U \circ G\exists_U \varphi_0) \\
&\cong \Sigma^{G_! \psi} (\exists_{U'} F^U (V, \pi_2 \circ \varphi_0), G\text{drop}_U \circ G\exists_U \varphi_0 \circ \theta) \\
&= \Sigma^{G_! \psi} (\exists_{U'} (FV, \pi_2 \circ F\varphi_0), G\text{drop}_U \circ \theta \circ \exists_{U'} F\varphi_0)
\end{aligned}$$

It remains to be shown that $G\text{drop}_U \circ \theta = \text{drop}_{U'} \circ \exists_{U'} \nu : \exists_{U'} F^U \dashv_U \rightarrow G$.

⁶This is a slight abuse of notation, as we know that $U' \cong FU$ but not that $U' = FU$.

But *the* morphism θ , which we have assumed to be an isomorphism, arises from ν as follows:

$$\theta : \exists_{U'} F/U \xrightarrow{\exists_{U'} F/U \text{ copy}_U \downarrow_U} \exists_{U'} F/U \downarrow_U \exists_U \xrightarrow{\exists_{U'} \nu \exists_U} \exists_{U'} \downarrow_U G \exists_U \xrightarrow{\text{drop}_{U'} G \exists_U} G \exists_U.$$

Then the following diagram does commute:

$$\begin{array}{ccccccc} & & & \theta \downarrow_U & & & \\ & \nearrow & & \searrow & \nearrow & & \\ \exists_{U'} F/U \downarrow_U & \xrightarrow{\exists_{U'} F/U \text{ copy}_U \downarrow_U} & \exists_{U'} F/U \downarrow_U \exists_U \downarrow_U & \xrightarrow{\exists_{U'} \nu \exists_U \downarrow_U} & \exists_{U'} \downarrow_U G \exists_U \downarrow_U & \xrightarrow{\text{drop}_{U'} G \exists_U \downarrow_U} & G \exists_U \downarrow_U \\ & \searrow \text{id} & \downarrow \exists_{U'} F/U \downarrow_U \text{ drop}_U & & \downarrow \exists_{U'} \downarrow_U G \text{ drop}_U & & \downarrow G \text{ drop}_U \\ & & \exists_{U'} F/U \downarrow_U & \xrightarrow{\exists_{U'} \nu} & \exists_{U'} \downarrow_U G & \xrightarrow{\text{drop}_{U'} G} & G \end{array}$$

□

Remark 6.4.2. In the above theorem, we think of F and G as similar functors; if we are dealing with endomultipliers, we will typically take $F = G$. The multipliers, however, will typically be different, as in general $U \not\cong FU$.

6.5 Multiplier and multiplier

Theorem 6.5.1. Assume we have a commutative diagram (up to natural isomorphism ν) of multipliers

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\sqcup \times U} & \mathcal{V} \\ \sqcup \times I \downarrow & & \downarrow \sqcup \times I' \\ \mathcal{W}' & \xrightarrow{\sqcup \times U'} & \mathcal{V}' \end{array}$$

Write $\sigma : \Psi \times \mathbf{y}I \times \mathbf{y}U' \cong \Psi \times \mathbf{y}U \times \mathbf{y}I'$. Then we have the following commutation table:

	\exists_I	\downarrow_I	\forall_I	\exists_I
\exists_U	$\exists_{\mathbf{y}U}^{\Psi } \exists_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \cong \exists_{\mathbf{y}I}^{\Psi } \exists_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \Omega \sigma $	$\exists_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \Omega \sigma \downarrow_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \triangleright_1 \downarrow_{\mathbf{y}I}^{\Psi } \exists_{\mathbf{y}U}^{\Psi }$	$\exists_{\mathbf{y}U}^{\Psi } \forall_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \triangleright_2 \forall_{\mathbf{y}I}^{\Psi } \exists_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \Omega \sigma $	$\exists_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \Omega \sigma \downarrow_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \triangleright_3 \downarrow_{\mathbf{y}I}^{\Psi } \exists_{\mathbf{y}U}^{\Psi }$
\downarrow_U	$\downarrow_{\mathbf{y}U}^{\Psi } \exists_{\mathbf{y}I}^{\Psi } \triangleleft^1 \exists_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \Omega \sigma^{-1} \downarrow_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I }$	$\Omega \sigma^{-1} \downarrow_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \downarrow_{\mathbf{y}I}^{\Psi } \cong \downarrow_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \downarrow_{\mathbf{y}U}^{\Psi }$	$\downarrow_{\mathbf{y}U}^{\Psi } \forall_{\mathbf{y}I}^{\Psi } \triangleright_1 \forall_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \Omega \sigma^{-1} \downarrow_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I }$	$\Omega \sigma^{-1} \downarrow_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \downarrow_{\mathbf{y}I}^{\Psi } \triangleright_2 \downarrow_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \downarrow_{\mathbf{y}U}^{\Psi }$
\forall_U	$\forall_{\mathbf{y}U}^{\Psi } \exists_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \triangleleft^2 \exists_{\mathbf{y}I}^{\Psi } \forall_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \Omega \sigma $	$\forall_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \Omega \sigma \downarrow_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \triangleleft^1 \downarrow_{\mathbf{y}I}^{\Psi } \forall_{\mathbf{y}U}^{\Psi }$	$\forall_{\mathbf{y}U}^{\Psi } \forall_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \cong \forall_{\mathbf{y}I}^{\Psi } \forall_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \Omega \sigma $	$\forall_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \Omega \sigma \downarrow_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \triangleleft_1 \downarrow_{\mathbf{y}I}^{\Psi } \forall_{\mathbf{y}U}^{\Psi }$
\exists_U	$\downarrow_{\mathbf{y}U}^{\Psi } \exists_{\mathbf{y}I}^{\Psi } \triangleleft^3 \exists_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \Omega \sigma^{-1} \downarrow_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I }$	$\Omega \sigma^{-1} \downarrow_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \downarrow_{\mathbf{y}I}^{\Psi } \triangleleft^2 \downarrow_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \downarrow_{\mathbf{y}U}^{\Psi }$	$\downarrow_{\mathbf{y}U}^{\Psi } \forall_{\mathbf{y}I}^{\Psi } \triangleleft^1 \forall_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \Omega \sigma^{-1} \downarrow_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I }$	$\Omega \sigma^{-1} \downarrow_{\mathbf{y}U'}^{\Psi \times \mathbf{y}I } \downarrow_{\mathbf{y}I}^{\Psi } \cong \downarrow_{\mathbf{y}I'}^{\Psi \times \mathbf{y}U } \downarrow_{\mathbf{y}U}^{\Psi }$

where every statement hold if the mentioned functors exist, and where

1. In general, \triangleright_1 means \rightarrow , \triangleleft^1 means \leftarrow and the other symbols mean nothing.
2. If $\sqcup \times U$ and $\sqcup \times U'$ are quantifiable and the morphism $\theta : \exists_{U'}^I \circ \downarrow_{I'}^U \rightarrow \downarrow_I \circ \exists_U : \mathcal{V}/U \rightarrow \mathcal{W}'$ is invertible,⁷ then \triangleright_1 upgrades to \cong and \triangleright_2 upgrades to \rightarrow .
3. If $\sqcup \times U$ and $\sqcup \times U'$ are cartesian and $\sqcup \times I = \sqcup \times I'$ preserves pullbacks, then \triangleright_1 upgrades to \cong and \triangleright_2 upgrades to \rightarrow .

⁷This is a slight abuse of notation, as we know that $I \times U' \cong U \times I'$ but not that $I \times U' = U \times I'$.

- (a) If $\sqcup \ltimes I$ and $\sqcup \ltimes I'$ are moreover affine and cancellative, then \triangleright_2 upgrades to \cong and \triangleright_3 upgrades to \rightarrow .
4. The symbols \triangleleft^i upgrade under symmetric conditions.
- Proof.* 1. In the base category, it is clear that $\Sigma/\sigma \downarrow_{U'}^{\Psi \ltimes yI} \downarrow_I^{\Psi} \cong \downarrow_{I'}^{\Psi \ltimes yU} \downarrow_U^{\Psi}$. Applying the 2-functor \sqcup^* yields the commutation law for \forall and hence, by lemma 2.1.2, the general case.

2. We want to invoke theorem 6.4.1 with $G = \sqcup \ltimes I : \mathcal{W} \rightarrow \mathcal{W}'$ and $F = \sqcup \ltimes I' : \mathcal{V} \rightarrow \mathcal{V}'$. However, this is not possible, as we do not know that $\sqcup \ltimes I$ preserves the terminal object. Instead, we take $G = \downarrow_I : \mathcal{W} \rightarrow \mathcal{W}'/I$ and $F = \downarrow_{I'} : \mathcal{V} \rightarrow \mathcal{V}'/I'$ which do preserve the terminal object. Instead of $\sqcup \ltimes U'$ we pass

$$\sqcup \ltimes \downarrow_{I'} U : \mathcal{W}'/I \xrightarrow{(\sqcup \ltimes U')/I} \mathcal{V}'/(I \ltimes U') \xrightarrow{\vee} \mathcal{V}'/(U \ltimes I') \xrightarrow{\pi_2} \mathcal{V}'/I'$$

which is a multiplier for $\downarrow_{I'} U$ whose $\downarrow_{\downarrow_{I'} U} : \mathcal{W}'/I \rightarrow (\mathcal{V}'/I')/\downarrow_{I'} U$ is essentially $\downarrow_{U'}^I : \mathcal{W}'/I \rightarrow \mathcal{V}'/(I \ltimes U')$ and hence whose $\exists_{\downarrow_{I'} U}$ is essentially $\exists_{U'}^I$. Then the property $\exists_{U'}^I \circ \downarrow_{I'}^U \cong \downarrow_I \circ \exists_U$ guarantees exactly $\exists_{\downarrow_{I'} U} \circ (\downarrow_{I'})^U \cong \downarrow_I \circ \exists_U$, which is the criterion found in theorem 6.4.1.

This adapted invocation of theorem 6.4.1 yields results about other functors than the ones mentioned in the current theorem. However, we have a general isomorphism $\mathcal{Z}/(yZ.\Xi) \cong (\mathcal{Z}/Z)/\Xi$ where $Z \in \mathcal{Z}$ and $\Xi \in \widehat{\mathcal{Z}/Z}$. Moreover, we have $yI.(\downarrow_I)_! \Psi \cong \Psi \ltimes yI$ and $yI'.(\downarrow_{I'})_! \Phi \cong \Phi \ltimes yI'$. Under the resulting strict isomorphism between the categories we want to talk about (such as $\mathcal{W}'/(I \ltimes U)$) and the categories we obtain results about (such as $(\mathcal{W}'/I)/(\downarrow_I)_! \Psi$), the functors that we want to talk about will be naturally equivalent to those that we obtain results about.

3. This is a special case of the previous point.

Alternatively, we can invoke theorem 6.3.1 with $\sigma = \pi_1 : \Psi \times U \rightarrow \Psi$ and $\tau = \pi_1 \ltimes I = \pi_1 \circ \sigma^{-1} : (\Psi \times U) \ltimes I \rightarrow \Psi \ltimes I$ and $\sqcup \ltimes I$ as the multiplier at hand.

- (a) This also follows from theorem 6.3.1.
4. By symmetry. □

7 Acknowledgements

We thank Jean-Philippe Bernardy, Lars Birkedal, Daniel Gratzer, Alex Kavvos, Magnus Baunsgaard Kristensen, Daniel Licata, Rasmus Ejlers Møgelberg and Andrea Vezzosi for relevant discussions. Special thanks to Dominique Devriese, whose pesky questions led to the current research.

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