The Transpension Type: Technical Report

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Contents

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1 Introduction

The purpose of these notes is to give a categorical semantics for the transpension type, which is right adjoint to a potentially substructural dependent function type.

- In section 2 we discuss some prerequisites.
- In section 3, we define multipliers and discuss their properties.
- In section 4, we study how multipliers lift from base categories to presheaf categories.
- In section 5, we explain how typical presheaf modalities can be used in the presence of the transpension type.
- In section 6, we study commutation properties of prior modalities, substitution modalities and multiplier modalities.

2 Prerequisites

2.1 On Adjoints

Lemma 2.1.1. Let $L \dashv R$.

- Natural transformations $LF \to G$ correspond to natural transformations $F \to RG$.
- Natural transformations $FR \to G$ correspond to natural transformations $F \to GL$.

Proof. The first statement is trivial.

To see the second statement, we send $\zeta : FR \to G$ to $\zeta L \circ F\eta : F \to GL$, and conversely $\theta : F \to GL$ to $G\varepsilon \circ \theta R : FR \to G$. Mapping ζ to and fro, we get

$$G\varepsilon \circ \zeta LR \circ F\eta R = \zeta \circ FR\varepsilon \circ F\eta R = \zeta. \tag{1}$$

Mapping θ to and fro, we get

$$G\varepsilon L \circ \theta RL \circ F\eta = G\varepsilon L \circ GL\eta \circ \theta = \theta.$$

□

Lemma 2.1.2. Assume 4 triples of adjoint functors: $E \dashv F \dashv G$ and $E' \dashv F' \dashv G'$ and $S_1 \dashv T_1 \dashv U_1$ and $S_2 \dashv T_2 \dashv U_2$ such that the following diagram commutes up to natural isomorphism:

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "F"] \arrow[d, "T_1"'] & \mathcal{C}_2 \arrow[d, "T_2"] \\ \mathcal{C}'_1 \arrow[r, "F'"'] & \mathcal{C}'_2 \end{tikzcd} \tag{2}$$

Then we have

$$\begin{array}{ccc} ES_2 \cong S_1E' & E'T_2 \to T_1E & \\ FS_1 \leftarrow S_2F' & F'T_1 \cong T_2F & FU_1 \to U_2F' \\ & G'T_2 \leftarrow T_1G & GU_2 \cong U_1G'. \end{array} \tag{3}$$

In fact, any one of these statements holds if only the adjoints used by that statement are given.

Proof. The central isomorphism is given. The other isomorphisms are obtained by taking the left/right adjoints of both hands of the original isomorphism. By picking one direction of the central isomorphism, we can step to the left/right/top/bottom by applying lemma 2.1.1. □

Proposition 2.1.3. If a functor $R : \mathcal{C} \to \widehat{\mathcal{W}}$ from a CwF $\mathcal{C}$ to a presheaf CwF $\widehat{\mathcal{W}}$ has a left adjoint L, then it is a weak CwF morphism.

Proof. We use the presheaf notations from [Nuy18] (section 2.2.1).

For $\Gamma \vdash_{\mathcal{C}} T$ type, define $R\Gamma \vdash_{\widehat{\mathcal{W}}} RT$ type by

$$(W \rhd_{\widehat{\mathcal{W}}} (RT)[\delta\rangle) :\cong (L\mathbf{y}W \vdash_{\mathcal{C}} T[\varepsilon \circ L\delta]). \tag{4}$$

Naturality of this operation is easy to show, and the action of R on terms is given by $({}^{R}t)[\delta\rangle = t[\varepsilon \circ L\delta]$. □

Definition 2.1.4. Given adjoint functors $L \dashv R$ such that R is a weak CwF morphism, and $A \in \mathrm{Ty}(L\Gamma)$, we write $\langle R \mid A\rangle := (RA)[\eta] \in \mathrm{Ty}(\Gamma)$.

Note that $\langle R \mid A[\varepsilon]\rangle = (RA)[R\varepsilon][\eta] = RA$.

2.2 Presheaves

2.2.1 Notation

We use the presheaf notations from [Nuy18]. Concretely:

- The application of a presheaf $\Gamma \in \widehat{\mathcal{W}}$ to an object $W \in \mathcal{W}$ is denoted $W \Rightarrow \Gamma$.
- The restriction of $\gamma : W \Rightarrow \Gamma$ by $\varphi : V \to W$ is denoted $\gamma \circ \varphi$ or $\gamma\varphi$.
- The application of a presheaf morphism $\sigma : \Gamma \to \Delta$ to $\gamma : W \Rightarrow \Gamma$ is denoted $\sigma \circ \gamma$ or $\sigma\gamma$.
  - By naturality of σ, we have $\sigma \circ (\gamma \circ \varphi) = (\sigma \circ \gamma) \circ \varphi$.
- If $\Gamma \in \widehat{\mathcal{W}}$ and $T \in \mathrm{Ty}(\Gamma)$ (also denoted $\Gamma \vdash T$ type), i.e. T is a presheaf over the category of elements $\mathcal{W}/\Gamma$, then we write the application of T to (W, γ) as $(W \rhd T[\gamma\rangle)$ and $t \in (W \rhd T[\gamma\rangle)$ as $W \rhd t : T[\gamma\rangle$.
  - By definition of type substitution in a presheaf CwF, we have $(W \rhd T[\sigma][\gamma\rangle) = (W \rhd T[\sigma\gamma\rangle)$
- The restriction of $W \rhd t : T[\gamma\rangle$ by $\varphi : (V, \gamma \circ \varphi) \to (W, \gamma)$ is denoted as $W \rhd t\,\langle\varphi\rangle : T[\gamma\varphi\rangle$.
- If $t \in \mathrm{Tm}(\Gamma, T)$ (also denoted $\Gamma \vdash t : T$), then the application of t to (W, γ) is denoted $V \rhd t[\gamma\rangle : T[\gamma\rangle$.
  - The naturality condition for terms is then expressed as $t[\gamma\rangle\,\langle\varphi\rangle = t[\gamma\varphi\rangle$.
  - By definition of term substitution in a presheaf CwF, we have $t[\sigma][\gamma\rangle = t[\sigma\gamma\rangle$.
- We omit applications of the isomorphisms $(W \Rightarrow \Gamma) \cong (\mathbf{y}W \to \Gamma)$ and $(W \rhd T[\gamma\rangle) \cong (\mathbf{y}W \vdash T[\gamma])$. This is not confusing: e.g. given $W \rhd t : T[\gamma\rangle$, the term $\mathbf{y}W \vdash t' : T[\gamma]$ is defined by $t'[\varphi\rangle := t\,\langle\varphi\rangle$.

One advantage of these notations is that we can put presheaf cells in diagrams; we will use double arrows when doing so.

We will also denote ends and co-ends using $\forall$ and $\exists$.

2.2.2 On the Yoneda-embedding

We consider the Yoneda-embedding $\mathbf{y} : \mathcal{W} \to \widehat{\mathcal{W}}$.

Proposition 2.2.1. A morphism $\varphi : V \to W$ in $\mathcal{W}$ is:

- Mono if and only if $\mathbf{y}\varphi$ is mono,
- Split epi if and only if $\mathbf{y}\varphi$ is epi.

Proof. It is well-known that a presheaf morphism $\sigma : \Gamma \to \Delta$ is mono/epi if and only if $\sigma \circ \sqcup : (W \Rightarrow \Gamma) \to (W \Rightarrow \Delta)$ is injective/surjective for all W. Now $\mathbf{y}\varphi \circ \sqcup = \varphi \circ \sqcup$. So $\mathbf{y}\varphi$ is mono if and only if $\varphi \circ \sqcup$ is injective, which means φ is mono. On the other hand, $\mathbf{y}\varphi$ is epi if and only if $\varphi \circ \sqcup$ is surjective, which is the case precisely when id is in its image, and that exactly means that φ is split epi. □

2.2.3 Lifting functors

Theorem 2.2.2. Any functor $F : \mathcal{V} \to \mathcal{W}$ gives rise to functors $F_! \dashv F^* \dashv F_*$, with a natural isomorphism $F_! \circ \mathbf{y} \cong \mathbf{y} \circ F : \mathcal{V} \to \widehat{\mathcal{W}}$. We will call $F_! : \widehat{\mathcal{V}} \to \widehat{\mathcal{W}}$ the **left lifting** of F to presheaves, $F^* : \widehat{\mathcal{W}} \to \widehat{\mathcal{V}}$ the **central** and $F_* : \widehat{\mathcal{V}} \to \widehat{\mathcal{W}}$ the **right lifting**.[1] [Sta19]

Proof. Using quantifier symbols for ends and co-ends, we can define:

$$\begin{array}{rcl} W \Rightarrow F_!\Gamma & := & \exists V.(W \to FV) \times (V \Rightarrow \Gamma), \\ V \Rightarrow F^*\Delta & := & FV \Rightarrow \Delta \\ W \Rightarrow F_*\Gamma & := & \forall V.(FV \to W) \to (V \Rightarrow \Gamma) \quad = \quad (F^*\mathbf{y}W \to \Gamma). \end{array}$$

By the co-Yoneda lemma, we have:

$$W \Rightarrow F_!\mathbf{y}V \quad = \quad \exists V'.(W \to FV') \times (V' \to V) \quad \cong \quad (W \to FV) \quad = \quad (W \Rightarrow \mathbf{y}FV),$$

i.e. $F_!\mathbf{y}V \cong \mathbf{y}FV$.

Adjointness also follows from applications of the Yoneda and co-Yoneda lemmas. ☐

Notation 2.2.3.

- We denote the cell $(V, \varphi, \gamma) : W \Rightarrow F_!\Gamma$ as $F_!\gamma \circ \varphi$. If we rename $F_!$, then we will also do so in this notation. We will further abbreviate $F_!\gamma \circ \mathrm{id} = F_!\gamma$ and, if $\Gamma = \mathbf{y}V$, also $F_!\mathrm{id} \circ \varphi = \varphi$.
- If $\delta : FV \Rightarrow \Delta$, then we write $\alpha_F(\delta) : V \Rightarrow F^*\Delta$.
- If $\gamma : F^*\mathbf{y}W \Rightarrow \Gamma$, then we write $\beta_F(\gamma) : W \Rightarrow F_*\Gamma$.

2.2.4 Dependent presheaf categories

Let $\mathcal{W}$ be a category. Then $\widehat{\mathcal{W}}$ is a category with families (CwF). The following are standard notions:

Definition 2.2.4. For any $U \in \mathcal{W}$, the **category of slices over** U, denoted $\mathcal{W}/U$, has objects (W, ψ) where $W \in \mathcal{W}$ and $\psi : W \to U$ and the morphisms $(W, \psi) \to (W', \psi')$ are the morphisms $\chi : W \to W'$ such that $\psi' \circ \chi = \psi$.

Definition 2.2.5. For any $\Gamma \in \widehat{\mathcal{W}}$, the **category of elements** of Γ, denoted

$$\int_{\mathcal{W}} \Gamma \quad \text{or} \quad \mathcal{W}/\Gamma \tag{5}$$

has objects (W, γ) where $W \in \mathcal{W}$ and $\gamma : W \Rightarrow \Gamma$, and the morphisms $(W, \gamma) \to (W', \gamma')$ are the morphisms $\chi : W \to W'$ such that $\gamma' \circ \chi = \gamma$.

Clearly, we have an isomorphism $\mathcal{W}/U \cong \mathcal{W}/\mathbf{y}U$ between the category of slices over U and the category of elements of $\mathbf{y}U$.[2]

We will use type-theoretic notation to make statements about the CwF $\widehat{\mathcal{W}}$, e.g. $\Gamma \vdash \mathsf{Ctx}$ means $\Gamma \in \widehat{\mathcal{W}}$ and $\Gamma \vdash T\ \mathsf{type}$ means $T \in \mathrm{Ty}(\Gamma)$. Now for any context or closed type $\Gamma \in \widehat{\mathcal{W}}$, there is another CwF $\widehat{\mathcal{W}/\Gamma}$. Statements about this category will also be denoted using type-theoretic notation, but prefixed with '$\Gamma \mid$'.

By unfolding the definitions of types and terms in a presheaf CwF, it is trivial to show that there is a correspondence — which we will treat as though it were the identity — between both CwFs:

- Contexts $\Gamma \mid \Theta \vdash \mathsf{Ctx}$ correspond to types $\Gamma \vdash \Theta\ \mathsf{type}$ which we will think of as telescopes $\Gamma.\Theta \vdash \mathsf{Ctx}$,
- Substitutions $\Gamma \mid \sigma : \Theta \to \Theta'$ correspond to functions $\Gamma \vdash \sigma : \Theta \to \Theta'$, or equivalently to telescope substitutions $\mathrm{id}_\Gamma.\sigma : \Gamma.\Theta \to \Gamma.\Theta'$,

[1] The central and right liftings are also sometimes called the inverse image and direct image of F, but these are actually more general concepts and as such could perhaps cause confusion or unwanted connotations in some circumstances. The left-central-right terminology is very no-nonsense.

[2] Depending on pedantic details, we may even have $\mathcal{W}/U = \mathcal{W}/\mathbf{y}U$.

- Types $\Gamma \mid \Theta \vdash T$ type correspond to types $\Gamma.\Theta \vdash T$ type,
- Terms $\Gamma \mid \Theta \vdash t : T$ correspond to terms $\Gamma.\Theta \vdash t : T$.

In summary, the pipe is equivalent to a dot.

Proposition 2.2.6. We have an equivalence of categories $\widehat{\mathcal{W}/\Gamma} \simeq \widehat{\mathcal{W}}/\Gamma$.

Proof. $\rightarrow$ We map the presheaf $\Gamma \mid \Theta \vdash \mathrm{Ctx}$ to the slice $(\Gamma.\Theta, \pi)$.

$\leftarrow$ We map the slice (Δ, σ) to the preimage of σ, i.e. the presheaf σ^{-1} which sends (W, γ) to $\{\delta : W \Rightarrow \Delta \mid \sigma \circ \delta = \gamma\}$.

$\widehat{\mathcal{W}/\Gamma}$ We need a natural isomorphism $\eta : \forall \Theta.(\Gamma \mid \eta : \Theta \cong \pi^{-1})$. If $\theta : (W, \gamma) \Rightarrow \Theta$, then we define $\eta(\theta) = (\gamma, \theta) : W \Rightarrow \Gamma.\Theta$ and indeed we have $\pi \circ (\gamma, \theta) = \gamma$. This is clearly invertible.

$\widehat{\mathcal{W}}/\Gamma$ We need a natural isomorphism $\varepsilon : \forall(\Delta, \sigma).(\Gamma.\sigma^{-1}, \pi) \cong (\Delta, \sigma)$. Given $(\gamma, \delta) : W \Rightarrow \Gamma.\sigma^{-1}$ (i.e. we know $\sigma \circ \delta = \gamma$), we define $\varepsilon \circ (\gamma, \delta) = \delta : W \Rightarrow \Delta$. Then

$$\sigma \circ \varepsilon \circ (\gamma, \delta) = \sigma \circ \delta = \gamma = \pi \circ (\gamma, \delta), \tag{6}$$

so indeed we have a morphism in the slice category. It is inverted by sending $\delta : W \Rightarrow \Delta$ to $(\sigma \circ \delta, \delta) : W \Rightarrow \Gamma.\sigma^{-1}$. $\square$

Corollary 2.2.7. We have $\widehat{\mathcal{W}/U} \cong \widehat{\mathcal{W}/\mathrm{y}U} \simeq \widehat{\mathcal{W}}/\mathrm{y}U$. $\square$

2.2.5 Substitution and its adjoints

Definition 2.2.8. Given $U \in \mathcal{W}$, we write

- $\Sigma_U : \mathcal{W}/U \rightarrow \mathcal{W} : (W, \psi) \mapsto W$,
- $\Omega_U : \mathcal{W} \rightarrow \mathcal{W}/U : W \rightarrow (W \times U, \pi_2)$ (if $\mathcal{W}$ has cartesian products with U).

Proposition 2.2.9. If Ω_U exists, then $\Sigma_U \dashv \Omega_U$. We denote the unit as $\mathrm{copy}_U : \mathrm{Id} \rightarrow \Omega_U \Sigma_U$ and the co-unit as $\mathrm{drop}_U : \Sigma_U \Omega_U \rightarrow \mathrm{Id}$. $\square$

Proposition 2.2.10. If $U \rightarrow \top$ is split epi, then the functor Ω_U is faithful. If it is mono, then Ω_U is full.

Proof. To see faithfulness, we have some $v : \top \rightarrow U$, so that the action of Ω_U on morphisms can be inverted: $\varphi = \pi_1 \circ (\varphi \times U) \circ (\mathrm{id}, v)$.

To see fullness, take slices (W_1, ψ_1) and (W_2, ψ_2) and a morphism $\varphi : W_1 \rightarrow W_2$. The fact that $U \rightarrow \top$ is mono just means that there is only a single morphism arriving in U. Then φ is also a morphism between the slices. $\square$

Definition 2.2.11. Given $\chi : W_0' \rightarrow W_0$ in $\mathcal{W}$, we write

- $\Sigma^{/\chi} : \mathcal{W}/W_0' \rightarrow \mathcal{W}/W_0 : (W', \psi') \mapsto (W', \chi \circ \psi')$,
- $\Omega^{/\chi} : \mathcal{W}/W_0 \rightarrow \mathcal{W}/W_0'$ for the functor that maps (W, ψ) to its pullback along χ (if $\mathcal{W}$ has pullbacks along χ).

If $\chi = \pi_1 : W_0 \times U \rightarrow W_0$, we also write $\Sigma_U^{/W_0} : \mathcal{W}/(W_0 \times U) \rightarrow \mathcal{W}/W_0$ and $\Omega_U^{/W_0} : \mathcal{W}/W_0 \rightarrow \mathcal{W}/(W_0 \times U)$.

Proposition 2.2.12. If $\Omega^{/\chi}$ exists, then $\Sigma^{/\chi} \dashv \Omega^{/\chi}$. We denote the unit as $\mathrm{copy}^{/\chi} : \mathrm{Id} \rightarrow \Omega^{/\chi}\Sigma^{/\chi}$ and the co-unit as $\mathrm{drop}^{/\chi} : \Sigma^{/\chi}\Omega^{/\chi} \rightarrow \mathrm{Id}$. $\square$

Proposition 2.2.13. If χ is split epi, then $\Omega^{/\chi}$ is faithful. If χ is mono, then it is full.

Proof. To see faithfulness, we have some $\upsilon : W_0 \to W_0'$ such that $\chi \circ \upsilon = \mathrm{id}$. Then the action of $\Omega^{/\chi}$ can be inverted: given $\varphi : (W_1, \psi_1) \to (W_2, \psi_2) \in \mathcal{W}/W_0$, we have

$$\varphi : W_1 \xrightarrow{(\mathrm{id}, \upsilon \circ \psi_1)} W_1 \times_{W_0} W_0' \xrightarrow{\varphi \times_{W_0} W_0'} W_2 \times_{W_0} W_0' \xrightarrow{\pi_1} W_2. \tag{7}$$

To see fullness, take a morphism $\varphi : (W_1, \chi \circ \psi_1) \to (W_2, \chi \circ \psi_2)$. Then $\chi \circ \psi_2 \circ \varphi = \chi \circ \psi_1$. Because χ is mono, this implies that $\psi_2 \circ \varphi = \psi_1$, i.e. $\varphi : (W_1, \psi_1) \to (W_2, \psi_2)$. ☐

Definition 2.2.14. Given $\sigma : \Psi' \to \Psi$ in $\widehat{\mathcal{W}}$, we write

- $\Sigma^{/\sigma} : \mathcal{W}/\Psi' \to \mathcal{W}/\Psi : (W', \psi') \mapsto (W', \sigma \circ \psi')$,
- $\Omega^{/\sigma} : \mathcal{W}/\Psi \to \mathcal{W}/\Psi'$ for the functor that maps (W, ψ) to its pullback along σ (if $\mathcal{W}$ has pullbacks along σ), by which we mean a universal solution W' to the diagram

$$\begin{tikzcd} W' \arrow[r, dotted] \arrow[d, Rightarrow, dotted] & W \arrow[d, Rightarrow, "\psi"] \\ \Psi' \arrow[r, "\sigma"'] & \Psi. \end{tikzcd} \tag{8}$$

If $\sigma = \pi_1 : \Psi \times \Phi \to \Psi$, we also write $\Sigma^{/\Psi}_{\Phi} : \mathcal{W}/(\Psi \times \Phi) \to \mathcal{W}/\Psi$ and $\Omega^{/\Psi}_{\Phi} : \mathcal{W}/\Psi \to \mathcal{W}/(\Psi \times \Phi)$.

Proposition 2.2.15. If $\Omega^{/\sigma}$ exists, then $\Sigma^{/\sigma} \dashv \Omega^{/\sigma}$. We denote the unit as $\mathsf{copy}^{/\sigma} : \mathrm{Id} \to \Omega^{/\sigma}\Sigma^{/\sigma}$ and the co-unit as $\mathsf{drop}^{/\sigma} : \Sigma^{/\sigma}\Omega^{/\sigma} \to \mathrm{Id}$. ☐

Proposition 2.2.16. If σ is surjective, then $\Omega^{/\sigma}$ is faithful. If σ is injective, then it is full.

Proof. If σ is surjective, then by the axiom of choice, there is at least a non-natural $f : \Psi \to \Psi'$ such that $\sigma \circ f = \mathrm{id}$. The rest of the proof is as for proposition 2.2.13. ☐

Definition 2.2.17. The functors $\Sigma^{/\sigma} \dashv \Omega^{/\sigma}$ give rise to four adjoint functors

$$\Sigma^{\sigma|} \dashv \Omega^{\sigma|} \dashv \Pi^{\sigma|} \dashv \between^{\sigma|} \tag{9}$$

between $\widehat{\mathcal{W}/\Psi}$ and $\widehat{\mathcal{W}/\Psi'}$, of which the first three exist if only $\Sigma^{/\sigma}$ exists.

The units and co-units will be denoted:

$$\begin{array}{rcl@{\qquad\qquad}rcl} \mathsf{copy}^{\sigma|} & : & \mathrm{Id} \to \Omega^{\sigma|}\Sigma^{\sigma|} & \mathsf{drop}^{\sigma|} & : & \Sigma^{\sigma|}\Omega^{\sigma|} \to \mathrm{Id} \\ \mathsf{const}^{\sigma|} & : & \mathrm{Id} \to \Pi^{\sigma|}\Omega^{\sigma|} & \mathsf{app}^{\sigma|} & : & \Omega^{\sigma|}\Pi^{\sigma|} \to \mathrm{Id} \\ \mathsf{reidx}^{\sigma|} & : & \mathrm{Id} \to \between^{\sigma|}\Pi^{\sigma|} & \mathsf{unmerid}^{\sigma|} & : & \Pi^{\sigma|}\between^{\sigma|} \to \mathrm{Id} \end{array} \tag{10}$$

We remark that, if we read presheaves over $\mathcal{W}/\Psi$ as types in context Ψ, then $\Omega^{\sigma|} : \widehat{\mathcal{W}/\Psi} \to \widehat{\mathcal{W}/\Psi'}$ is the standard interpretation of substitution in a presheaf category. If $\sigma = \pi : \Psi.A \to \Psi$ is a weakening morphism, then $\Omega^{\Psi|}_A := \Omega^{\pi|}$ is the weakening substitution, $\Pi^{\Psi|}_A := \Pi^{\pi|} : \widehat{\mathcal{W}/\Psi.A} \to \widehat{\mathcal{W}/\Psi}$ is the standard interpretation of the Π-type and $\Sigma^{\Psi|}_A := \Sigma^{\pi|} : \widehat{\mathcal{W}/\Psi.A} \to \widehat{\mathcal{W}/\Psi}$ is isomorphic to the standard interpretation of the Σ-type.

Theorem 2.2.18. Given types $\Psi \vdash A, B\,\mathrm{type}$, the projections constitute a pullback diagram:

$$\begin{tikzcd} \Psi.(A \times B) \arrow[r, "\beta'"] \arrow[d, "\alpha'"'] & \Psi.A \arrow[d, "\alpha"] \\ \Psi.B \arrow[r, "\beta"'] & \Psi, \end{tikzcd} \tag{11}$$

and every pullback diagram in a presheaf category is isomorphic to a diagram of this form. We have the following commutation properties:

	Σ_B	Ω_B	Π_B	$\between_B$
Σ_A	$\Sigma^{\alpha\vert}\Sigma^{\beta'\vert} \cong \Sigma^{\beta\vert}\Sigma^{\alpha'\vert}$	$\Sigma^{\alpha'\vert}\Omega^{\beta'\vert} \cong \Omega^{\beta\vert}\Sigma^{\alpha\vert}$	$\Sigma^{\alpha\vert}\Pi^{\beta'\vert} \to \Pi^{\beta\vert}\Sigma^{\alpha'\vert}$	
Ω_A	$\Omega^{\alpha\vert}\Sigma^{\beta\vert} \cong \Sigma^{\beta'\vert}\Omega^{\alpha'\vert}$	$\Omega^{\alpha'\vert}\Omega^{\beta\vert} = \Omega^{\beta'\vert}\Omega^{\alpha\vert}$	$\Omega^{\alpha\vert}\Pi^{\beta\vert} \cong \Pi^{\beta'\vert}\Omega^{\alpha'\vert}$	$\Omega^{\alpha'\vert}\between^{\beta\vert} \to \between^{\beta'\vert}\Omega^{\alpha\vert}$
Π_A	$\Pi^{\alpha\vert}\Sigma^{\beta'\vert} \leftarrow \Sigma^{\beta\vert}\Pi^{\alpha'\vert}$	$\Pi^{\alpha'\vert}\Omega^{\beta'\vert} \cong \Omega^{\beta\vert}\Pi^{\alpha\vert}$	$\Pi^{\alpha\vert}\Pi^{\beta'\vert} \cong \Pi^{\beta\vert}\Pi^{\alpha'\vert}$	$\Pi^{\alpha'\vert}\between^{\beta'\vert} \cong \between^{\beta\vert}\Pi^{\alpha\vert}$
$\between_A$		$\between^{\alpha'\vert}\Omega^{\beta\vert} \leftarrow \Omega^{\beta'\vert}\between^{\alpha\vert}$	$\between^{\alpha\vert}\Pi^{\beta\vert} \cong \Pi^{\beta'\vert}\between^{\alpha'\vert}$	$\between^{\alpha'\vert}\between^{\beta\vert} \cong \between^{\beta'\vert}\between^{\alpha\vert}$

(12)

where every statement holds if the mentioned functors exist.

Proof. In the base category, it is evident that $\Sigma^{/\alpha}\Sigma^{/\beta'} = \Sigma^{/\beta}\Sigma^{/\alpha'}$. By applying the functor $\sqcup^*$, we obtain $\Omega^{\alpha'|}\Omega^{\beta|} = \Omega^{\beta'|}\Omega^{\alpha|}$, whence by lemma 2.1.2 the entire diagonal of the commutation table.

It is a well-known fact that Σ- and Π-types are respected by substitution, which gives us the isomorphisms for swapping Ω and either Σ or Π. Lemma 2.1.2 then gives the rest. $\square$

Theorem 2.2.19. Given $\sigma : \Psi' \to \Psi$, the following operations are invertible:

$$\frac{\Psi \mid \Sigma^{\sigma|}\Gamma \vdash T\ \mathsf{type}}{\Psi' \mid \Gamma \vdash (\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}]\ \mathsf{type}} \qquad \frac{\Psi \mid \Sigma^{\sigma|}\Gamma \vdash t : T}{\Psi' \mid \Gamma \vdash ({}^{\Omega^{\sigma|}}t)[\mathsf{copy}^{\sigma|}] : (\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}]} \tag{13}$$

Proof. Note that T is a presheaf over $(\mathcal{W}/\Psi)/\Sigma^{\sigma|}\Gamma$, and $(\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}]$ is a presheaf over $(\mathcal{W}/\Psi')/\Gamma$. We compare the objects of these categories:

$$\begin{aligned}
&\mathsf{Obj}((\mathcal{W}/\Psi)/\Sigma^{\sigma|}\Gamma)\\
&= (W \in \mathcal{W}) \times (\psi : W \Rightarrow \Psi) \times \exists((W', \psi') \in \mathcal{W}/\Psi'.(\chi : (W, \psi) \to \Sigma^{/\sigma}(W', \psi')) \times ((W', \psi') \Rightarrow \Gamma)\\
&\cong (W \in \mathcal{W}) \times (\psi : W \Rightarrow \Psi) \times \exists W'.(\psi' : W' \Rightarrow \Psi') \times (\chi : (W, \psi) \to \Sigma^{/\sigma}(W', \psi')) \times ((W', \psi') \Rightarrow \Gamma)\\
&\cong (W \in \mathcal{W}) \times (\psi : W \Rightarrow \Psi) \times \exists W'.(\psi' : W' \Rightarrow \Psi') \times (\chi : (W, \psi) \to (W', \sigma \circ \psi')) \times ((W', \psi') \Rightarrow \Gamma)\\
&\cong (W \in \mathcal{W}) \times \exists W'.(\psi' : W' \Rightarrow \Psi') \times (\chi : W \to W') \times ((W', \psi') \Rightarrow \Gamma)\\
&\qquad \text{because } \chi \text{ is a slice morphism iff } \psi = \sigma \circ \psi' \circ \chi\\
&\cong (W \in \mathcal{W}) \times (\psi' : W \Rightarrow \Psi') \times ((W, \psi') \Rightarrow \Gamma)\\
&\cong \mathsf{Obj}((\mathcal{W}/\Psi')/\Gamma).
\end{aligned}$$

A similar consideration of the Hom-sets leads to the conclusion that both categories are isomorphic. Moreover, we remark that the isomorphism sends $((W, \psi'), \gamma)$ on the right to $((W, \sigma \circ \psi'), \Sigma^{\sigma|}\gamma)$ on the left. When we consider the action of $(\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}]$ on $((W, \psi'), \gamma)$, we find:

$$\begin{aligned}
\Big((W, \psi') \rhd (\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}][\gamma\rangle\Big) &= \Big(\Sigma^{/\sigma}(W, \psi') \rhd T\Big[\Sigma^{/\sigma}\gamma\Big\rangle\Big)\\
&= \Big((W, \sigma \circ \psi') \rhd T\Big[\Sigma^{/\sigma}\gamma\Big\rangle\Big)
\end{aligned}$$

In other words, the types T and $(\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}]$ are equal over an isomorphism of categories. Then certainly T can be retrieved from $(\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}]$. An identical argument works for terms. $\square$

3 Multipliers in the base category

3.1 Definition

Definition 3.1.1. Let $\mathcal{W}$ be a category with terminal object $\top$. An object W is **spooky** if $() : W \to \top$ is not split epi. A category is **spooky** if it has a spooky object. Therefore, a category is non-spooky iff all morphisms to the terminal object are split epi.

Definition 3.1.2. Let $\mathcal{W}$ be a category with terminal object $\top$. A **multiplier** for an object $U \in \mathcal{V}$ is a functor $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ such that $\top \ltimes U \cong U$. This gives us a second projection $\pi_2 : \forall W.W \ltimes U \to U$. We define the **fresh weakening functor** as $Ⅎ_U : \mathcal{W} \to \mathcal{V}/U : W \mapsto (W \ltimes U, \pi_2)$.

We say that a multiplier is:

- **Endo** if it is an endofunctor (i.e. $\mathcal{V} = \mathcal{W}$), and in that case:
  - **Semicartesian** if it is copointed, i.e. if there is also a first projection $\pi_1 : \forall W.W \ltimes U \to W$,
  - **3/4-cartesian** if it is a comonad, i.e. if there is additionally a 'diagonal' natural transformation $\sqcup \ltimes \delta : \forall W.W \ltimes U \to (W \ltimes U) \ltimes U$ such that $\pi_1 \circ (W \ltimes \delta) = (\pi_1 \ltimes U) \circ (W \ltimes \delta) = \mathrm{id}$.
  - **Cartesian** if it is naturally isomorphic to the cartesian product with U,
- **Cancellative** if $Ⅎ_U$ is faithful, or equivalently (lemma 3.2.2) if $\sqcup \ltimes U$ is faithful,
- **Affine** if $Ⅎ_U$ is full,
- **Non-spooky** if $\pi_2 : W \ltimes U \to U$ is always split epi, and in that case:
  - **Connection-free** if $Ⅎ_U$ is essentially surjective on objects (V, ψ) such that ψ is split epi, i.e. if every such object in $\mathcal{V}/U$ is isomorphic to some $Ⅎ_U W$.
  - A split epi slice (V, ψ) that is not in the image of $Ⅎ_U$ even up to isomorphism, will be called a **connection** of the multiplier.
- **Quantifiable** if $Ⅎ_U$ has a left adjoint $\exists_U : \mathcal{V}/U \to \mathcal{W}$. We denote the unit as $\mathrm{copy}_U : \mathrm{Id} \to Ⅎ_U \exists_U$ and the co-unit as $\mathrm{drop}_U : \exists_U Ⅎ_U \to \mathrm{Id}$.

3.2 Basic properties

Some readers may prefer to first consult some examples (section 3.3).

Proposition 3.2.1. For any multiplier, we have $(\sqcup \ltimes U) = \Sigma_U \circ Ⅎ_U$. ☐

Lemma 3.2.2. The functor $\sqcup \ltimes U$ is faithful if and only if $Ⅎ_U$ is faithful.

Proof. We have $(\sqcup \ltimes U) = \Sigma_U \circ Ⅎ_U$ and $\Sigma_U : \mathcal{V}/U \to \mathcal{V}$ is faithful as is obvious from its definition. ☐

Proposition 3.2.3. A multiplier with a non-spooky domain is non-spooky.

Proof. The multiplier, as any functor, preserves split epimorphisms. ☐

Proposition 3.2.4. Cartesian multipliers are 3/4-cartesian, and 3/4-cartesian multipliers are semicartesian.

Proof. The functor $\sqcup \times U$ is a comonad, and comonads are copointed by their co-unit. ☐

Proposition 3.2.5. Cartesian multipliers are quantifiable.

Proof. The left adjoint to $Ⅎ_U = \Omega_U$ is then given by $\exists_U(V, \varphi) = \Sigma_U(V, \varphi) = V$ (proposition 2.2.9). ☐

Proposition 3.2.6. Cartesian endomultipliers for non-spooky objects, are cancellative.

Non-spookiness is not required however: cancellative cartesian endomultipliers may be spooky (examples 3.3.4 and 3.3.6).

Proof. In this case, $Ⅎ_U = \Omega_U$ and $U \to \top$ is split epi, so this is part of proposition 2.2.10. ☐

Proposition 3.2.7. If an endomultiplier for U is both 3/4-cartesian and affine, then U is a terminal object. If the multiplier is moreover cartesian, then it is naturally isomorphic to the identity functor.

Proof. Consider the following diagram:

$$\begin{tikzcd} \top \ltimes U \arrow[rr, "\top \ltimes \delta"] \arrow[dr, "\pi_2"'] & & (\top \ltimes U) \ltimes U \arrow[dl, "\pi_2"] \\ & U & \end{tikzcd} \tag{14}$$

This is a morphism from $\top \ltimes \delta : \exists_U \top \to \exists_U(\top \ltimes U)$ and thus, by affinity, of the form $\exists_U v$ for some $v : \top \to \top \ltimes U$. This means in particular that

$$\mathrm{id}_{\top \ltimes U} = \pi_1 \circ (\top \ltimes \delta) = \pi_1 \circ (v \ltimes U) = v \circ \pi_1 : \top \ltimes U \to \top \ltimes U. \tag{15}$$

Composing on both sides with $\pi_2 : \top \ltimes U \cong U$, we find that $\mathrm{id}_U = (\pi_2 \circ v) \circ (\pi_1 \circ \pi_2^{-1})$ factors over $\top$, which means exactly that $\pi_2 \circ v : \top \to U$ and $\pi_1 \circ \pi_2^{-1} : U \to \top$ constitute an isomorphism, i.e. U is terminal.

If $\sqcup \ltimes U$ is cartesian, then it is a cartesian product with a terminal object and therefore naturally isomorphic to the identity functor. □

3.3 Examples

Example 3.3.1 (Identity)**.** The identity functor $W \ltimes \top := W$ is an endomultiplier for $\top$.

It is cartesian, cancellative, affine, spooky iff $\mathcal{W}$ is and otherwise connection-free, and quantifiable.

The functor $\exists_\top : \mathcal{W} \to \mathcal{W}/\top : W \mapsto (W, ())$ has a left adjoint $\exists_\top : \mathcal{W}/\top \to \mathcal{W} : (W, ()) \mapsto W$.

Example 3.3.2 (Cartesian product)**.** Let $\mathcal{W}$ be a category with finite products and $U \in \mathcal{W}$.

Then $\sqcup \times U$ is an endomultiplier for U.

It is cartesian, cancellative if (but not only if) U is non-spooky (proposition 3.2.6), affine if and only if $U \cong \top$ (proposition 3.2.7) and quantifiable (proposition 3.2.5). We do not consider spookiness for this general case.

The functor $\exists_U = \Omega_U : V \mapsto (V \times U, \pi_2)$ has a left adjoint $\exists_U = \Sigma_U : (W, \psi) \mapsto W$. Hence, we have $\exists_U \exists_U = \sqcup \times U$.

Example 3.3.3 (Affine cubes)**.** Let $\square^k$ be the category of affine non-symmetric k-ary cubes $\mathbb{I}^n$ as used in [BCH14] (binary) or [BCM15] (unary). A morphism $\varphi : \mathbb{I}^m \to \mathbb{I}^n$ is a function $\sqcup \langle\varphi\rangle : \{i_1, \ldots, i_n\} \to \{i_1 \ldots i_m, 0, \ldots, k-1\}$ such that $i\langle\varphi\rangle = j\langle\varphi\rangle \notin \{0, \ldots, k-1\}$ implies $i = j$. We also write $\varphi = (i_1\langle\varphi\rangle / i_1, \ldots, i_n\langle\varphi\rangle / i_n)$. This category is spooky if and only if $k = 0$.

Consider the functor $\sqcup * \mathbb{I} : \square^k \to \square^k : \mathbb{I}^n \mapsto \mathbb{I}^{n+1}$, which is a multiplier for $\mathbb{I}$. It acts on morphisms $\varphi : \mathbb{I}^m \to \mathbb{I}^n$ by setting $\varphi * \mathbb{I} = (\varphi, i_{m+1}/i_{n+1})$.

It is straightforwardly seen to be semicartesian, not 3/4-cartesian, cancellative, affine, spooky iff $k = 0$ and connection-free when $k \neq 0$, and quantifiable.

The functor $\exists_\mathbb{I} : \mathbb{I}^n \mapsto (\mathbb{I}^{n+1}, (i_{n+1}/i_1))$ has a left adjoint the functor $\exists_\mathbb{I}$ which sends $(\mathbb{I}^n, \psi)$ to $\mathbb{I}^n$ if $i_1\langle\psi\rangle \in \{0, \ldots, k-1\}$ and to $\mathbb{I}^{n-1}$ (by removing the variable $i_1\langle\psi\rangle$ and renaming the next ones) otherwise. The action on morphisms is straightforwardly constructed.

In the case where $k = 2$, we can throw in an involution $\neg : \mathbb{I} \to \mathbb{I}$. This changes none of the above results, except that $i_1\langle\psi\rangle$ may be the negation $\neg j$ of a variable j, in which case $\exists_U$ removes the variable j.

Example 3.3.4 (Cartesian cubes)**.** Let $\boxslash^k$ be the category of cartesian non-symmetric k-ary cubes $\mathbb{I}^n$. A morphism $\varphi : \mathbb{I}^m \to \mathbb{I}^n$ is any function $\sqcup\langle\varphi\rangle : \{i_1, \ldots, i_n\} \to \{i_1 \ldots i_m, 0, \ldots, k-1\}$. This category is spooky if and only if $k = 0$.

Consider the functor $\sqcup \times \mathbb{I} : \boxslash^k \to \boxslash^k : \mathbb{I}^n \mapsto \mathbb{I}^{n+1}$, which is an endomultiplier for $\mathbb{I}$.

It is cartesian (hence non-affine and quantifiable with $\exists_\mathbb{I}(W, \psi) = W$), cancellative, spooky iff $k = 0$ and otherwise connection-free.

Again, involutions change none of the above results.

Example 3.3.5 (CCHM cubes). Let $\boxslash_{\vee,\wedge,\neg}$ be the category of (binary) CCHM cubes [CCHM17]. What's special here is that we have morphisms $\vee, \wedge : \mathbb{I}^2 \to \mathbb{I}$ (as well as involutions). This category is not spooky.

Again, we consider the functor $\sqcup \times \mathbb{I} : \boxslash_{\vee,\wedge,\neg} \to \boxslash_{\vee,\wedge,\neg} : \mathbb{I}^n \mapsto \mathbb{I}^{n+1}$, which is an endomultiplier for $\mathbb{I}$.

It is cartesian (hence non-affine and quantifiable with $\exists_{\mathbb{I}}(W, \psi) = W$), cancellative, not spooky, and not connection-free (since $(\mathbb{I}^2, \vee)$ and $(\mathbb{I}^2, \wedge)$ are connections).

Example 3.3.6 (Clocks). Let $\oplus$ be the category of clocks, used as a base category in guarded type theory [BM18]. Its objects take the form $(i_1 : \oplus_{k_1}, \ldots, i_n : \oplus_{k_n})$ where all $k_j \geq 0$. We can think of a variable of type $\oplus_k$ as representing a clock (i.e. a time dimension) paired up with a certificate that we do not care what happens after the time on this clock exceeds k. Correspondingly, we have a map $\oplus_k \to \oplus_\ell$ if $k \leq \ell$. These maps, together with weakening, exchange, and contraction, generate the category. The terminal object is $()$ and every other object is spooky.

Consider in this category the functor $\sqcup \times (i : \oplus_k) : \oplus \to \oplus : W \mapsto (W, i : \oplus_k)$, which is an endo multiplier for $(i : \oplus_k)$.

It is cartesian (hence non-affine and quantifiable with $\exists_{(i:\oplus_k)}(W, \psi) = W$), cancellative and spooky.

Example 3.3.7 (Twisting posets). Let $\mathcal{P}$ be the category of finite non-empty posets and monotonic maps. This category is non-spooky.

Let $\mathbb{I} = \{0 < 1\}$ and let $W \ltimes \mathbb{I} = (W^{\mathrm{op}} \times \{0\}) \cup (W \times \{1\})$ with $(x, 0) < (y, 1)$ for all $x, y \in W$. This is an endomultiplier for $\mathbb{I}$.

It is easily seen to be: not semicartesian, cancellative, not affine, not spooky, not connection-free, and quantifiable.

The functor $\exists_{\mathbb{I}} : V \mapsto (V \ltimes \mathbb{I}, \pi_2)$ has a left adjoint $\exists_{\mathbb{I}} : (W, \psi) \mapsto \psi^{-1}(0)^{\mathrm{op}} \uplus \psi^{-1}(1)$ where elements from different sides of the $\uplus$ are incomparable.

We see this category as a candidate base category for directed type theory. The idea is that a cell over W is a commutative diagram in a category. A problem here is that a cell over a discrete poset such as $\{x, y\}$ where x and y are incomparable, should then be the same as a pair of cells over $\{x\}$ and $\{y\}$. This will require that we restrict from presheaves to sheaves, but that makes it notoriously difficult to model the universe [XE16]. One solution would be to restrict to totally ordered sets, but then we lose the left adjoint $\exists_{\mathbb{I}}$. We address this in example 3.3.8.

Example 3.3.8 (Affine twisted cubes). Let $\bowtie$ be the subcategory of $\mathcal{P}$ whose objects are generated by $\top$ and $\sqcup \ltimes \mathbb{I}$ (note that every object then also has an opposite since $\top^{\mathrm{op}} = \top$ and $(V \ltimes \mathbb{I})^{\mathrm{op}} \cong V \ltimes \mathbb{I}$), and whose morphisms are given by

- $(\varphi, 0) : \bowtie(V, W \ltimes \mathbb{I})$ if $\varphi : \bowtie(V, W^{\mathrm{op}})$,
- $(\varphi, 1) : \bowtie(V, W \ltimes \mathbb{I})$ if $\varphi : \bowtie(V, W)$,
- $\varphi \ltimes \mathbb{I} : \bowtie(V \ltimes \mathbb{I}, W \ltimes \mathbb{I})$ if $\varphi : \bowtie(V, W)$,
- $() : \bowtie(V, \top)$.

Note that this collection automatically contains all identities, composites, and opposites. It is isomorphic to Pinyo and Kraus's category of twisted cubes, as can be seen from the ternary representation of said category [PK19, def. 34]. This category is not spooky.

Again, we consider the functor $\sqcup \ltimes \mathbb{I} : \bowtie \to \bowtie$, which is well-defined by construction of $\bowtie$ and an endomultiplier for $\mathbb{I}$. It corresponds to Pinyo and Kraus's twisted prism functor.

It is: not semicartesian, cancellative, affine, not spooky, connection-free, and quantifiable.

The left adjoint to $\exists_{\mathbb{I}} : W \mapsto (W \ltimes \mathbb{I}, \pi_2)$ is now given by

$$\exists_{\mathbb{I}} : \begin{cases} (W, ((), 0)) & \mapsto & W^{\mathrm{op}} \\ (W, ((), 1)) & \mapsto & W \\ (W \ltimes \mathbb{I}, () \ltimes \mathbb{I}) & \mapsto & W, \end{cases} \tag{16}$$

with the obvious action on morphisms.

Example 3.3.9 (Signposts). In order to define contextual fibrancy [BT17] internally, we need to be able to somehow put a signpost in the context $\Gamma.\mp.\Theta$ in order to be able to say: the type is fibrant over Θ in context Γ. If C is the category of contexts, then $\Gamma.\mp.\Theta$ can be seen as an object of the arrow category $C^{\uparrow}$, namely the arrow $\Gamma.\Theta \to \Gamma$.

If $C = \widehat{\mathcal{W}}$ happens to be a presheaf category, then we have an isomorphism of categories $H : \widehat{\mathcal{W}}^{\uparrow} \cong \widehat{\mathcal{W} \times \uparrow}$ where $\uparrow = \{\bot \to \top\}$. Under this isomorphism, we have $\mathbf{y}(W, \top) \cong H(\mathbf{y}W \xrightarrow{\mathrm{id}} \mathbf{y}W)$ which we think of as $\mathbf{y}W.\mp$ and $\mathbf{y}(W, \bot) \cong H(\bot \xrightarrow{[]} \mathbf{y}W)$ which we think of as $\mathbf{y}W.\mp.\bot$. Thus, forgetting the second component of (W, o) amounts to erasing whatever comes after the signpost.

There are 3 adjoint functors $\bot \dashv () \dashv \top$ between $\uparrow$ and Point from which we obtain 3 adjoint functors $(\mathrm{Id}, \bot) \dashv \pi_1 \dashv (\mathrm{Id}, \top)$ between $\mathcal{W} \times \uparrow$ and $\mathcal{W}$. The rightmost functor $(\mathrm{Id}, \top) : \mathcal{W} \to \mathcal{W} \times \uparrow$ is a multiplier for the terminal object $(\top, \top) \in \mathcal{W} \times \uparrow$.

It is: not endo, cancellative, affine, spooky iff $\mathcal{W}$ is and otherwise connection-free, and quantifiable.

In order to look at the left adjoint, note first that since $(\top, \top)$ is terminal, we have $(\mathcal{W} \times \uparrow)/(\top, \top) \cong \mathcal{W} \times \uparrow$ and clearly $\dashv_{(\top,\top)}$ corresponds to $(\mathrm{Id}, \top)$ under this isomorphism. This functor is part of a chain of *three* adjoint functors $(\mathrm{Id}, \bot) \dashv \pi_1 \dashv (\mathrm{Id}, \top)$ so that the multiplier is not just quantifiable but $\exists_{(\top,\top)}$ even has a further left adjoint!

If $\sqcup \ltimes U : \mathcal{V} \to \mathcal{W}$ is a multiplier, then we can lift it to a multiplier $\sqcup \ltimes (U, \top) : \mathcal{V} \times \uparrow \to \mathcal{W} \times \uparrow$ by applying it to the first component. The resulting multiplier inherits all properties in definition 3.1.2 from $\sqcup \ltimes U$, except that it is always spooky.

Example 3.3.10 (Signposts II). If $\sqcup \ltimes U$ is a semicartesian endomultiplier on $\mathcal{W}$, then we might want to apply it to an arrow $V \xrightarrow{\psi} W$ by sending it to $V \ltimes U \xrightarrow{\psi \circ \pi_1} W$. This operation is not definable on $\mathcal{W} \times \uparrow$, which only encodes identity arrows and arrows from $\bot$. For this reason, we move to the comma category $\mathcal{W}_\bot/\mathcal{W}$ where $\mathcal{W}_\bot$ is $\mathcal{W}$ with a freely added initial object. This comma category has as its objects arrows $V \xrightarrow{\psi} W$ where $V \in \mathcal{W}_\bot$ and $W \in \mathcal{W}$. Morphisms are simply commutative squares. We still have three adjoint functors $(\bot \to \sqcup) \dashv \mathrm{Cod} \dashv \Delta$. Furhter right adjoints would be $\mathrm{Dom} \dashv (\sqcup \to \top)$, but Dom is not definable as the domain might be $\bot$. We take $\Delta : W \mapsto (W \to W)$ as a multiplier for $(\top \to \top)$.

This multiplier is: not endo, cancellative, affine, spooky iff $\mathcal{W}$ is and otherwise connection-free, and quantifiable.

Now we get back to our multiplier $\sqcup \ltimes U$ which we can still lift to $\sqcup \ltimes \Delta U$ by applying it to both domain and codomain (where by convention $\bot \ltimes U = \bot$). It inherits all properties in definition 3.1.2 from $\sqcup \ltimes U$, except that it is always spooky.

If $\sqcup \ltimes U$ is semicartesian, then we can also lift it to $\sqcup \ltimes (U \to \top)$ by applying it only to the domain. This again inherits all properties in definition 3.1.2 from $\sqcup \ltimes U$, except that it is always spooky.

Example 3.3.11 (Depth d cubes). Let $\boxslash_d$ with $d \geq -1$ be the category of depth d cubes, used as a base category in degrees of relatedness [ND18, Nuy18].[3] Its objects take the form $(i_1 : (\!|k_1|\!), \ldots, i_n : (\!|k_n|\!))$ where all $k_j \in \{0, \ldots, d\}$. We have a map $(\!|\ell|\!) \to (\!|k|\!)$ if $k \leq \ell$. These maps, together with weakening, exchange, and contraction, generate the category. The terminal object is $()$ and the category is non-spooky.

Consider in this category the functor $\sqcup \times (i : (\!|k|\!)) : \boxslash_d \to \boxslash_d : W \mapsto (W, i : (\!|k|\!))$, which is an endomultiplier for $(i : (\!|k|\!))$.

It is cartesian (hence non-affine and quantifiable with $\exists_{(i:(\!|k|\!))}(W, \psi) = W$), cancellative, non-spooky and connection-free.

Example 3.3.12 (Erasure). Let $\mathsf{Erase}_d = \{\top \leftarrow 0 \leftarrow 1 \leftarrow \ldots \leftarrow d\}$ with $d \geq -1$. This category has cartesian products $m \times n = \max(m, n)$ and all non-terminal objects are spooky. We remark that $\widehat{\mathsf{Erase}_0}$ is the Sierpiński topos.

We consider the endomultiplier $\sqcup \times i : \mathsf{Erase}_d \to \mathsf{Erase}_d$.

It is cartesian (hence non-affine and quantifiable with $\exists_i(j, \psi) = j$), cancellative and spooky.

We believe that this base category is a good foundation for studying the semantics of erasure of irrelevant subterms in Degrees of Relatedness [ND18]. The idea is that, for a presheaf Γ, the set $\top \Rightarrow \Gamma$ is

[3] For $d = -1$, we get the point category. For $d = 0$, we get the category of binary cartesian cubes $\boxslash^2$. For $d = 1$, we get the category of bridge/path cubes [NVD17, Nuy18].

the set of elements, whereas the set $i \Rrightarrow \Gamma$ is the set of elements considered up to i-relatedness, but also whose existence is only guaranteed by a derivation up to i-relatedness.

3.4 Properties

3.4.1 Functoriality

Definition 3.4.1. A **multiplier morphism** or **morphism multiplier** for $v : U \to U'$ is a natural transformation $\sqcup \ltimes v : \sqcup \ltimes U \to \sqcup \ltimes U'$ such that $\pi_2 \circ (\top \ltimes v) \circ \pi_2^{-1} = v : U \to U'$ (or equivalently $\pi_2 \circ (W \ltimes v) = v \circ \pi_2 : W \ltimes U \to U'$ for all W).

- A morphism of semicartesian multipliers is **semicartesian** if it is a morphism of copointed endofunctors, i.e. if $\pi_1 \circ (W \ltimes v) = \pi_1$,
- A morphism of 3/4-cartesian multipliers is **3/4-cartesian** if it is a monad morphism, i.e. if additionally $(W \ltimes \delta) \circ (W \ltimes v) = ((W \ltimes v) \ltimes v) \circ (W \ltimes \delta)$,
- A morphism of cartesian multipliers is **cartesian** if it is the cartesian product with v.

Proposition 3.4.2. A semicartesian morphism of cartesian multipliers, is cartesian.

Proof. We have $\pi_2 \circ (W \ltimes v) = v \circ \pi_2$ and $\pi_1 \circ (W \ltimes v) = \pi_1$. Hence, $(W \ltimes v) = (\pi_1, v \circ \pi_2) = W \times v$. □

Proposition 3.4.3 (Functoriality)**.** A multiplier morphism $\sqcup \ltimes v : \sqcup \ltimes U \to \sqcup \ltimes U'$ gives rise to a natural transformation $\Sigma^{/v} \circ \exists\!\!\!\!\!\text{—}_U \to Ⅎ_{U'}$. Hence, for quantifiable multipliers, we also have $\exists_{U'} \circ \Sigma^{/v} \to \exists_U$.

Proof. We have to show that for every $W \in \mathcal{W}$, we get $(W \ltimes U, v \circ \pi_2) \to (W \ltimes U', \pi_2)$. The morphism $W \ltimes v : W \ltimes U \to W \ltimes U'$ does the job. The second statement follows from lemma 2.1.1. □

3.4.2 Quantification and kernel theorem

Theorem 3.4.4 (Quantification theorem)**.** If $\sqcup \ltimes U$ is

1. cancellative, affine and quantifiable, then we have a natural isomorphism $\mathsf{drop}_U : \exists_U Ⅎ_U \cong \mathrm{Id}$.
2. semi-cartesian, then we have:
   (a) $\mathsf{hide}_U : \Sigma_U \to \exists_U$ (if quantifiable),
   (b) $\mathsf{spoil}_U : Ⅎ_U \to \Omega_U$ (if Ω_U exists),
   (c) in any case $\Sigma_U Ⅎ_U \to \mathrm{Id}$.
3. 3/4-cartesian, then there is a natural transformation $\Sigma^{/\delta} \circ Ⅎ_U \to Ⅎ_{U \ltimes U}$, where we compose multipliers as in theorem 3.6.1.
4. cartesian, then we have:
   (a) $\exists_U \cong \Sigma_U$,
   (b) $Ⅎ_U \cong \Omega_U$,
   (c) $\exists_U Ⅎ_U \cong \Sigma_U \Omega_U = (\sqcup \times U) \cong (\sqcup \ltimes U)$.

   Moreover, these isomorphisms become equalities by choosing Σ_U and Ω_U wisely (both are defined only up to isomorphism).

Proof. 1. This is a standard fact of fully faithful right adjoints such as $Ⅎ_U$.

2. By lemma 2.1.1, it is sufficient to prove $\Sigma_U Ⅎ_U \to \mathrm{Id}$. But $\Sigma_U Ⅎ_U = (\sqcup \ltimes U)$, so this is exactly the statement that the multiplier is semicartesian.
3. This is a special case of proposition 3.4.3.

4. By uniqueness of the cartesian product, we have $\boxminus_U \cong \Omega_U$. Then the multiplier is quantifiable with $\exists_U \cong \Sigma_U$. The last point is now trivial. $\square$

Theorem 3.4.5 (Kernel theorem for non-spooky multipliers)**.** If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is non-spooky, cancellative, affine and connection-free, then $\boxminus_U : \mathcal{W} \simeq \mathcal{V}/\!/U$ is an equivalence of categories, where $\mathcal{V}/\!/U$ is the full subcategory of $\mathcal{V}/U$ whose objects are the split epimorphic slices.

Proof. By non-spookiness, $\boxminus_U$ lands in $\mathcal{V}/\!/U$. The other properties assert that $\boxminus_U$ is fully faithful and essentially surjective as a functor $\mathcal{W} \to \mathcal{V}/\!/U$. $\square$

Definition 3.4.6. If we are doing classical mathematics, or if $\sqcup \ltimes U$ is quantifiable, then we obtain an inverse functor, which we denote $\ker_U$.

The kernel theorem applies to examples 3.3.3 and 3.3.8.

We will use the kernel theorem in order to model Moulin's Φ-operator [Mou16], and to make the semantics of Moulin's Ψ-type [Mou16] unique up to isomorphism.

3.4.3 Dealing with spookiness

Since spooky multipliers do not guarantee that $\boxminus_U$ produces split epi slices, we need to come up with a larger class of suitable epi-like morphisms to U before we can proceed.

Definition 3.4.7. Given a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$, we say that a morphism $\varphi : V \to U$ is **dimensionally split** if there is some $W \in \mathcal{W}$ such that $\pi_2 : W \ltimes U \to U$ factors over φ. If $\pi_2 = \varphi \circ \chi$, we say that χ is a **dimensional section** of φ. We write $\mathcal{V}/\!/U$ for the full subcategory of $\mathcal{V}/U$ of dimensionally split slices.

The non-spookiness condition for multipliers is automatically satisfied if we replace 'split epi' with 'dimensionally split':

Corollary 3.4.8. For any multiplier $\sqcup \ltimes U$, any projection $\pi_2 : W \ltimes U \to U$ is dimensionally split. $\square$

Proposition 3.4.9. Take a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$.

1. If $\varphi \circ \chi$ is dimensionally split, then so is φ.
2. The identity morphism $\mathrm{id}_U : U \to U$ is dimensionally split.
3. If $\varphi : V \to U$ is dimensionally split and $\chi : V' \to V$ is split epi, then $\varphi \circ \chi : V' \to U$ is dimensionally split.
4. Every split epimorphism to U is dimensionally split.
5. If $\sqcup \ltimes U$ is non-spooky, then every dimensionally split morphism is split epi.

Proof.

1. If $\pi_2 : W \ltimes U \to U$ factors over $\varphi \circ \chi$, then it certainly factors over φ.
2. Since $\pi_2 : \top \ltimes U \to U$ factors over id_U.
3. Let φ' be a dimensional section of φ and χ' a section of χ. Then $\chi' \circ \varphi'$ is a dimensional section of $\varphi \circ \chi$.
4. From the previous two points, or (essentially by composition of the above reasoning) because if $\chi : U \to V$ is a section of $\varphi : V \to U$, then $\chi \circ \pi_2 : \top \ltimes U \to V$ is a dimensional section of φ.
5. If $\varphi : V \to U$ is dimensionally split, then some $\pi_2 : W \ltimes U \to U$ factors over φ. Since π_2 is split epi, id_U factors over π_2 and hence over φ, i.e. φ is split epi. $\square$

We can now extend the notions of connection and connection-freedom to spooky multipliers without changing their meaning for non-spooky multipliers:

Definition 3.4.10. We say that a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is **connection-free** if $\exists_U$ is essentially surjective on $\mathcal{V}/\!/U$, the full subcategory of $\mathcal{V}/U$ of dimensionally split slices. A dimensionally split slice (V, ψ) that is not in the image of $\exists_U$ even up to isomorphism, will be called a **connection** of the multiplier.

Theorem 3.4.11 (Kernel theorem)**.** If a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is cancellative, affine and connection-free, then $\exists_U : \mathcal{W} \simeq \mathcal{V}/\!/U$ is an equivalence of categories. □

Example 3.4.12 (Identity)**.** In the category $\mathcal{W}$ with the identity multiplier $W \ltimes \top = W$, every morphism $W \to \top$ is dimensionally split. The multiplier is connection-free.

Example 3.4.13 (Nullary cubes)**.** In the categories of nullary affine cubes $\square^0$ (example 3.3.3) and nullary cartesian cubes $\boxslash^0$ (example 3.3.4), a morphism $\varphi : \mathbb{I}^n \to \mathbb{I}$ is dimensionally split if $i_1 \langle \varphi \rangle$ is a variable. The multipliers $\sqcup * \mathbb{I} : \square^0 \to \square^0$ and $\sqcup \times \mathbb{I} : \boxslash^0 \to \boxslash^0$ are connection-free.

Example 3.4.14 (Clocks)**.** In the category of clocks $\circleddash$ (example 3.3.6), a morphism $\varphi : V \to (i : \circleddash_k)$ is dimensionally split if $i \langle \varphi \rangle$ has clock type $\circleddash_k$. The multiplier $\sqcup \times (i : \circleddash_k)$ is connection-free.

Example 3.4.15 (Twisted cubes)**.** In the category $\bowtie$ of twisted cubes (example 3.3.8), a morphism $\varphi : V \to \top \ltimes \mathbb{I}$ is dimensionally split if it equals $\varphi = () \ltimes \mathbb{I}$. The multiplier $\sqcup \ltimes \mathbb{I}$ is connection-free.

Example 3.4.16 (Signposts)**.** For the signpost multiplier $(\mathrm{Id}, \top)$ (example 3.3.9), a morphism $((), ()) : (W, o) \to (\top, \top)$ is dimensionally split if $o = \top$. The multiplier $(\mathrm{Id}, \top) : \mathcal{W} \to \mathcal{W} \times \uparrow$ is connection-free.

For $\sqcup \ltimes (U, \top)$, a morphism $(\varphi, ())(W, O) \to (U, \top)$ is dimensionally split if $\varphi : W \to U$ is dimensionally split for $\sqcup \ltimes U$. Connection-freedom is then inherited from $\sqcup \ltimes U$.

Example 3.4.17 (Signposts II)**.** For the enhanced signpost multiplier Δ (example 3.3.10), a morphism $(V \to W) \to (\top \to \top)$ is dimensionally split if $V \neq \bot$. The multiplier $\Delta : \mathcal{W} \to \mathcal{W}_\bot/\mathcal{W}$ is generally not connection-free: as Δ only produces identity arrows, any non-identity arrow is a connection.

For $\sqcup \ltimes \Delta U$, a morphism $(V \to W) \to (U \to U)$ is dimensionally split (with section $([], \chi) : (\bot \to W' \ltimes U) \to (V \to W)$) if the morphism $W \to U$ is dimensionally split for $\sqcup \ltimes U$ with section $\chi : W' \ltimes U \to W$. The multiplier $\sqcup \ltimes \Delta U$ is generally not connection-free, as the morphism between the domains could be anything.

For $\sqcup \ltimes (U \to \top)$, any morphism $(V \to W) \to (U \to \top)$ is dimensionally split by

$$([], \mathrm{id}) \quad : \quad (\bot \to W) \ltimes (U \to \top) \quad = \quad (\bot \to W) \quad \to \quad (V \to W). \tag{17}$$

This multiplier is therefore generally not connection-free.

To conclude, we have made the base category more complicated in order to be able to define the last multiplier, but as a tradeoff we now have connections to deal with.

Example 3.4.18 (Erasure)**.** In the category Erase_d (example 3.3.12) with multiplier $\sqcup \times i$, all morphisms to i are dimensionally split. The multiplier is connection-free.

3.4.4 Boundaries

Definition 3.4.19. The boundary ∂U of a a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is a presheaf over $\mathcal{V}$ such that the cells $V \Rightarrow \partial U$ are precisely the morphisms $V \to U$ that are *not* dimensionally split.

Proposition 3.4.20. If $\sqcup \ltimes U$ is non-spooky, then ∂U is the largest strict subobject of $\mathbf{y}U$.

Proof. Recall that if the multiplier is non-spooky, then dimensionally split and split epi are synonymous.

Clearly, $\partial U \subseteq \mathbf{y}U$. Since $\mathrm{id} : U \to U$ is split epi, we have $\partial U \subsetneq \mathbf{y}U$. Now take another strict subobject $\Upsilon \subsetneq \mathbf{y}U$. We show that $\Upsilon \subseteq \partial U$.

We start by showing that $\mathrm{id} \notin U \Rightarrow \Upsilon$. Otherwise, every $\varphi \in V \Rightarrow \mathbf{y}U$ would have to be a cell of Υ as it is a restriction of id, which would imply $\Upsilon = \mathbf{y}U$.

Now id is a restriction of any split epimorphism, so Υ contains no split epimorphisms, i.e. $\Upsilon \subseteq \partial U$. □

Example 3.4.21. In all the binary cube categories mentioned in section 3.3, $\partial\mathbb{I}$ is isomorphic to the constant presheaf of booleans.

For affine cubes, if we define a multiplier $\sqcup * \mathbb{I}^2$ in the obvious way, then $\partial\mathbb{I}^2$ is isomorphic to a colimit of four times $\mathbf{y}\mathbb{I}$ and four times $\mathbf{y}\top$, i.e. a square without filler. For cartesian asymmetric cubes, the square also gains a diagonal. For symmetric cubes (with an involution $\neg : \mathbb{I} \to \mathbb{I}$), the other diagonal also appears.

3.5 Acting on slices

Definition 3.5.1. Given a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$, we define

$$\Finv_U^{/W_0} : \mathcal{W}/W_0 \to \mathcal{V}/(W_0 \ltimes U) : (W, \psi) \mapsto (W \ltimes U, \psi \ltimes U). \tag{18}$$

We say that $\sqcup \ltimes U$ is:

- **Strongly cancellative** if for all W_0, the functor $\Finv_U^{/W_0}$ is faithful,
- **Strongly affine** if for all W_0, the functor $\Finv_U^{/W_0}$ is full,
- **Strongly connection-free** if for all W_0, the functor $\Finv_U^{/W_0}$ is essentially surjective on slices $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$ such that $\pi_2 \circ \varphi$ is dimensionally split,
  - We point out that the full subcategory of such slices is isomorphic to $(\mathcal{V}/\!/U)/(W_0 \ltimes U, \pi_2)$,
- **Strongly quantifiable** if for all W_0, the functor $\Finv_U^{/W_0}$ has a left adjoint $\exists_U^{/W_0} : \mathcal{V}/(W_0 \ltimes U) \to \mathcal{W}/W_0$. We denote the unit as $\mathrm{copy}_U^{/W_0} : \mathrm{Id} \to \Finv_U^{/W_0} \exists_U^{/W_0}$ and the co-unit as $\mathrm{drop}_U^{/W_0} : \exists_U^{/W_0} \Finv_U^{/W_0} \to \mathrm{Id}$.

The above definition generalizes the functor $\Finv_U$ that we already had:

Proposition 3.5.2. The functor $\Finv_U^{/\top} : \mathcal{W}/\top \to \mathcal{V}/(\top \ltimes U)$ is equal to $\Finv_U : \mathcal{W} \to \mathcal{V}/U$ over the obvious isomorphisms between their domains and codomains. Hence, each of the strong properties implies the basic property. ☐

Note that strong connection-freedom is well-defined:

Proposition 3.5.3. The functor $\Finv_U^{/W_0}$ factors over $(\mathcal{V}/\!/U)/(W_0 \ltimes U, \pi_2)$.

Proof. The functor $\Finv_U^{/W_0}$ sends (W, ψ) to $(W \ltimes U, \psi \ltimes U)$. Since $\pi_2 \circ (\psi \ltimes U) = \pi_2$, it is dimensionally split. ☐

Proposition 3.5.4. If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is cancellative, then it is strongly cancellative.

Proof. Pick morphisms $\varphi, \chi : (W, \psi) \to (W', \psi')$ in $\mathcal{W}/W_0$ such that $\Finv_U^{/W_0} \varphi = \Finv_U^{/V} \chi$. Expanding the definition of $\Finv_U^{/W_0}$, we see that this means that $\varphi \ltimes U = \chi \ltimes U$, and hence $\varphi = \chi$ by cancellation of $\sqcup \ltimes U$. ☐

Proposition 3.5.5. If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is cancellative and affine, then it is strongly affine.

Proof. Pick (W, ψ) and (W', ψ') in $\mathcal{W}/W_0$, and a morphism $\chi : \Finv_U^{/W_0}(W, \psi) \to \Finv_U^{/W_0}(W', \psi')$. This amounts to a diagram:

$$\begin{tikzcd}
W \ltimes U \arrow[rr, "\chi"] \arrow[dr, "\psi \ltimes U"'] \arrow[ddr, bend right, "\pi_2"'] & & W' \ltimes U \arrow[dl, "\psi' \ltimes U"] \arrow[ddl, bend left, "\pi_2"] \\
& W_0 \ltimes U \arrow[d, "\pi_2"] & \\
& U, &
\end{tikzcd} \tag{19}$$

i.e. a triangle in $\mathcal{W}/U$, the objects of which are in the image of $\Finv_U : \mathcal{W} \to \mathcal{W}/U$. Then, by fullness (affinity) we get $\chi_0 : W \to W'$ such that $\Finv_U \chi_0 = \chi$, which by faithfulness (cancellativity) makes the following diagram commute:

$$\begin{tikzcd}
W \arrow[rr, "\chi_0"] \arrow[dr, "\psi"'] & & W' \arrow[dl, "\psi'"] \\
& W_0 &
\end{tikzcd} \tag{20}$$

Then χ_0 is a morphism $\chi_0 : (W, \psi) \to (W', \psi')$ in $\mathcal{W}/W_0$ and $\Finv_U^{/W_0} \chi_0 = \chi$. ☐

Proposition 3.5.6. If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is affine and connection-free, then it is strongly connection-free.

Proof. Pick some $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$ such that $\pi_2 \circ \varphi : V \to U$ is dimensionally split. Because $\dashv_U$ is essentially surjective on $\mathcal{V}/\!/U$, there must be some $W \in \mathcal{W}$ such that $\iota : \dashv_U W = (W \ltimes U, \pi_2) \cong (V, \pi_2 \circ \varphi)$ as slices over U. Because $\dashv_U$ is full, there is a morphism $\psi : W \to W_0$ such that $\psi \ltimes U = \varphi \circ \iota : W \ltimes U \to W_0 \ltimes U$. Thus, $\iota^{-1} : (V, \varphi) \cong (W \ltimes U, \psi \ltimes U) = \dashv_U^{/W_0}(W, \psi)$ as slices over $W_0 \ltimes U$.

$$\begin{tikzcd}
V \arrow[dr, "\varphi"'] & & W \ltimes U \arrow[ll, "\iota"', "\cong"] \arrow[dl, dotted, "\psi \ltimes U"] \arrow[ddl, bend left, "\pi_2"] \\
& W_0 \ltimes U \arrow[d, "\pi_2"] & \\
& U &
\end{tikzcd} \tag{21}$$

□

Example 3.5.7. In the category $\boxslash^k$ of k-ary cartesian cubes (example 3.3.4), the diagonal $\delta : \mathbb{I} \to \mathbb{I} \times \mathbb{I}$ has the property that $\pi_2 \circ \delta$ is split epi, but $(\mathbb{I}, \delta)$ is not in the image of $\dashv_{\mathbb{I}}^{/\mathbb{I}}$. Thus, $\sqcup \ltimes \mathbb{I}$ is not strongly connection-free, despite being connection-free.

Proposition 3.5.8. If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is quantifiable, then it is strongly quantifiable, with

$$\begin{aligned}
\exists_U^{/W_0}(V, \varphi) &= (\exists_U(V, \pi_2 \circ \varphi), \mathsf{drop}_U \circ \exists_U \varphi), \\
\mathsf{drop}_U^{/W_0}(W, \psi) &= \mathsf{drop}_U W : \exists_U^{/W_0} \dashv_U^{/W_0}(W, \psi) \to (W, \psi), \\
\mathsf{copy}_U^{/W_0}(V, \varphi) &= \mathsf{copy}_U(V, \pi_2 \circ \varphi) : (V, \varphi) \to \dashv_U^{/W_0} \exists_U^{/W_0}(V, \varphi).
\end{aligned}$$

Proof. Pick $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$. Then we have, in $\mathcal{V}/U$, a morphism $\varphi : (V, \pi_2 \circ \varphi) \to (W_0 \ltimes U, \pi_2) = \dashv_U W_0$. Transposing this, we obtain

$$\mathsf{drop}_U \circ \exists_U \varphi : \exists_U(V, \pi_2 \circ \varphi) \to W_0. \tag{22}$$

This is a slice over W_0 that we take as the definition of $\exists_U^{/W_0}(V, \varphi)$.

To prove adjointness, take a general morphism $\chi : \exists_U^{/W_0}(V, \varphi) \to (W, \psi)$ and note that there is a correspondence between diagrams of the following shape, where in the first step we apply $\exists_U \dashv \dashv_U$ and in the second step we use $(\mathcal{W}/U)/\dashv_U W_0 \cong \mathcal{W}/(W_0 \ltimes U)$:

$$\begin{tikzcd}
\exists_U(V, \pi_2 \circ \varphi) \arrow[rr, "\chi"] \arrow[dr, "\mathsf{drop}_U \circ \exists_U \varphi"'] & & W \arrow[dl, "\psi"] \\
& W_0 &
\end{tikzcd}$$

$$\begin{tikzcd}
(V, \pi_2 \circ \varphi) \arrow[rr, "\dashv_U \chi \circ \mathsf{copy}_U"] \arrow[dr, "\varphi"'] & & \dashv_U W \arrow[dl, "\dashv_U \psi"] \\
& \dashv_U W_0 &
\end{tikzcd}$$

$$\begin{tikzcd}
(V, \varphi) \arrow[r, "\dashv_U \chi \circ \mathsf{copy}_U"] & (W \ltimes U, \psi \ltimes U),
\end{tikzcd}$$

i.e. we get a morphism $(V, \varphi) \to \dashv_U^{/W_0}(W, \psi)$. Moreover, we note that the transposition action on morphisms is precisely that of $\exists_U \dashv \dashv_U$, and hence both adjunctions have the same unit and co-unit. □

Proposition 3.5.9 (Functoriality for slices)**.** A morphism of multipliers $\sqcup \ltimes v : \sqcup \ltimes U \to \sqcup \ltimes U'$ gives rise to a natural transformation $\Sigma^{/W_0 \ltimes v} \circ \text{\reflectbox{$\exists$}}_U^{/W_0} \to \text{\reflectbox{$\exists$}}_{U'}^{/W_0}$. Hence, if both multipliers are quantifiable, we also get $\exists_{U'}^{/W_0} \circ \Sigma^{/W_0 \ltimes v} \to \exists_U^{/W_0}$.

Proof. For any $(W, \psi) \in \mathcal{W}/W_0$, we have to prove $(W \ltimes U, (W_0 \ltimes v) \circ (\psi \ltimes U)) \to (W \ltimes U', \psi \ltimes U')$. The morphism $W \ltimes v : W \ltimes U \to W \ltimes U'$ does the job. The second statement follows from lemma 2.1.1. $\square$

Theorem 3.5.10 (Strong quantification theorem)**.** If $\sqcup \ltimes U$ is

1. cancellative, affine and quantifiable, then we have a natural isomorphism $\mathsf{drop}_U^{/W_0} : \exists_U^{/W_0} \text{\reflectbox{$\exists$}}_U^{/W_0} \cong \mathrm{Id}$.
2. semi-cartesian, then we have
   (a) $\mathsf{hide}_U^{/W_0} : \Sigma_U^{/W_0} \to \exists_U^{/W_0}$ (if quantifiable),
   (b) $\mathsf{spoil}_U^{/W_0} : \text{\reflectbox{$\exists$}}_U^{/W_0} \to \Omega_U^{/W_0}$ (if $\Omega_U^{/W_0}$ exists),
   (c) in any case $\Sigma_U^{/W_0} \text{\reflectbox{$\exists$}}_U^{/W_0} \to \mathrm{Id}$.
3. 3/4-cartesian, then there is a natural transformation $\Sigma^{/W_0 \ltimes \delta} \circ \text{\reflectbox{$\exists$}}_U^{/W_0} \to \text{\reflectbox{$\exists$}}_{U \ltimes U}^{/W_0}$, where we compose multipliers as in theorem 3.6.1.
4. cartesian, then we have natural isomorphisms:
   (a) $\exists_U^{/W_0}(V, \varphi) \cong \Sigma_U^{/W_0}(V, \varphi) = (V, \pi_1 \circ \varphi)$,
   (b) $\text{\reflectbox{$\exists$}}_U^{/W_0}(W, \psi) \cong \Omega_U^{/W_0}(W, \psi)$,
   (c) $\exists_U^{/W_0} \text{\reflectbox{$\exists$}}_U^{/W_0}(W, \psi) \cong \Sigma_U^{/W_0} \Omega_U^{/W_0}(W, \psi) \cong (W \times U, \psi \circ \pi_1)$.

   Moreover, these isomorphisms become equality if $\exists_U^{/W_0}$ is constructed as above from $\exists_U = \Sigma_U$, and $\Omega_U^{/W_0}(W, \psi)$ is chosen wisely. (Both functors are defined only up to isomorphism.)

Proof.

1. This is a standard fact about fully faithful right adjoints such as $\text{\reflectbox{$\exists$}}_U^{/W_0}$.
2. By lemma 2.1.1, it is sufficient to prove $\Sigma_U^{/W_0} \text{\reflectbox{$\exists$}}_U^{/W_0} \to \mathrm{Id}$, and indeed we have $\pi_1 : \Sigma_U^{/W_0} \text{\reflectbox{$\exists$}}_U^{/W_0}(W, \psi) = (W \ltimes U, \pi_1 \circ (\psi \ltimes U)) = (W \ltimes U, \psi \circ \pi_1) \to (W, \psi)$.
3. This is a special case of proposition 3.5.9.
4. (a) Recall that $\exists_U(V, \varphi) \cong \Sigma_U(V, \varphi) = V$. The co-unit is essentially given by $\mathsf{drop}_U = \pi_1 : W \times U \to W$. The construction of $\exists_U^{/W_0}$ then reveals that $\exists_U^{/W_0}(V, \varphi) \cong (V, \pi_1 \circ \varphi)$, which is the definition of $\Sigma_U^{/W_0}(V, \varphi)$.
   (b) This follows from the definitions.
   (c) We have

$$\exists_U^{/W_0} \text{\reflectbox{$\exists$}}_U^{/W_0}(W, \psi) = \exists_U^{/W_0}(W \times U, \psi \times U) \cong (W \times U, \pi_1 \circ (\psi \times U)) = (W \times U, \psi \circ \pi_1). \quad \square$$

Theorem 3.5.11 (Strong kernel theorem)**.** If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is cancellative, affine and connection-free, then $\text{\reflectbox{$\exists$}}_U^{/W_0} : \mathcal{W}/W_0 \simeq (\mathcal{V}/\!/U)/(W_0 \ltimes U, \pi_2)$ is an equivalence of categories.[4] $\square$

[4] We use a slight abuse of notation by using $(\mathcal{V}/\!/U)/(W_0 \ltimes U, \pi_2)$ as a subcategory of $\mathcal{V}/(W_0 \ltimes U)$.

3.6 Composing multipliers

Theorem 3.6.1. If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is a multiplier for U and $\sqcup \ltimes U' : \mathcal{V} \to \mathcal{V}'$ is a multiplier for U', then their composite $\sqcup \ltimes (U \ltimes U') := (\sqcup \ltimes U) \ltimes U'$ is a multiplier for $U \ltimes U'$.

1. The functor $\dashv_{U \ltimes U'} : \mathcal{W} \to \mathcal{V}'/(U \ltimes U')$ equals $\dashv_{U'}^{/U} \circ \dashv_U$.
2. The functor $\dashv_{U \ltimes U'}^{/W_0} : \mathcal{W} \to \mathcal{V}'/(U \ltimes U')$ equals $\dashv_{U'}^{/W_0 \ltimes U} \circ \dashv_U^{/W_0}$.
3. Assume both multipliers are endo. Then:
   (a) The composite $\sqcup \ltimes (U \ltimes U')$ is semicartesian if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are semicartesian,
   (b) The composite $\sqcup \ltimes (U \ltimes U')$ is 3/4-cartesian if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are 3/4-cartesian,
   (c) The composite $\sqcup \ltimes (U \ltimes U')$ is cartesian if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are cartesian.
4. The composite $\sqcup \ltimes (U \ltimes U')$ is strongly cancellative if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are cancellative.
5. The composite $\sqcup \ltimes (U \ltimes U')$ is affine if $\sqcup \ltimes U$ is affine and $\sqcup \ltimes U'$ is strongly affine.
6. The composite $\sqcup \ltimes (U \ltimes U')$ is strongly affine if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are strongly affine.
7. The composite $\sqcup \ltimes (U \ltimes U')$ is connection-free if $\sqcup \ltimes U$ is connection-free and $\sqcup \ltimes U'$ is cancellative, affine and connection-free.
8. The composite $\sqcup \ltimes (U \ltimes U')$ is strongly connection-free if $\sqcup \ltimes U$ is strongly connection-free and $\sqcup \ltimes U'$ is cancellative, affine and connection-free.
9. The composite $\sqcup \ltimes (U \ltimes U')$ is strongly quantifiable if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are quantifiable, and in that case we have:
   (a) $\exists_{U \ltimes U'} = \exists_U \circ \exists_{U'}^{/U}$,
   (b) $\exists_{U \ltimes U'}^{/W_0} = \exists_U^{/W_0} \circ \exists_{U'}^{/W_0 \ltimes U}$.

Proof. Since $\top \ltimes U \cong U$, we see that $(\top \ltimes U) \ltimes U' \cong U \ltimes U'$, so the composite is indeed a multiplier for $U \ltimes U'$.

1-2. Follows from expanding the definitions.

3. (a) Copointed endofunctors compose.
   (b) Comonads compose.
   (c) By associativity of the cartesian product.

4. Cancellative multipliers are strongly cancellative (proposition 3.5.4), and the composite $\dashv_{U \ltimes U'}^{/W_0} = \dashv_{U'}^{/W_0} \dashv_U^{/W_0 \ltimes U'}$ of faithful functors is faithful.

5-6. Follows from the first two properties, since the composite of full functors is full.

7. Analogous to the next point.

8. Recall that the assumptions imply that $\sqcup \ltimes U'$ is strongly cancellative, strongly affine and strongly connection free.

   Pick a slice $(V', \varphi') \in \mathcal{V}'/(W_0 \ltimes U \ltimes U')$ such that $\pi_2^{U \ltimes U'} \circ \varphi' : V' \to U \ltimes U'$ is dimensionally split with section $\chi' : W_1 \ltimes U \ltimes U' \to V'$. Then $\pi_2^{U'} \circ \pi_2^{U \ltimes U'} \circ \varphi' = \pi_2^{U'} \circ \varphi' : V' \to U'$ is also dimensionally split with the same section.

   Because $\sqcup \ltimes U'$ is strongly connection-free, we find some $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$ such that $\iota' : (V', \varphi') \cong \dashv_{U'}^{/W_0 \ltimes U}(V, \varphi) \in \mathcal{V}'/(W_0 \ltimes U \ltimes U')$.

$$\begin{tikzcd}
 & W_1 \ltimes U \arrow[d, "\chi"'] \arrow[dd, bend left=40, "\pi_2^U"] \\
 & V \arrow[dl, "\varphi"'] \arrow[d, "\iota", "\cong"'] \\
W_0 \ltimes U \arrow[dr, "\pi_2^U"'] & W \ltimes U \arrow[l, "\psi \ltimes U"] \\
 & U
\end{tikzcd}
\qquad
\begin{tikzcd}
 & W_1 \ltimes U \ltimes U' \arrow[dl, "\chi'"'] \arrow[d, "\chi \ltimes U'"] \arrow[ddd, bend left, "\pi_2^{U \ltimes U'}"] \arrow[dddd, bend left=60, "\pi_2^{U'}"] \\
V' \arrow[r, "\iota'", "\cong"'] \arrow[d, "\varphi'"'] & V \ltimes U' \arrow[dl, "\varphi \ltimes U'"] \\
W_0 \ltimes U \ltimes U' \arrow[ddr, bend right, "\pi_2^{U'}"'] \arrow[dr, "\pi_2^{U \ltimes U'}"] & \\
 & U \ltimes U' \arrow[d, "\pi_2^{U'}"] \\
 & U'
\end{tikzcd}$$

Because $\exists_{U'}$ is fully faithful, the morphism $\iota' \circ \chi' : \exists_{U'}(W_1 \ltimes U) \to \exists_{U'} V$ has a unique preimage $\chi : W_1 \ltimes U \to V$ under $\exists_{U'}$. The uniqueness of the inverse images implies that $\pi_2^U \circ \varphi \circ \chi = \pi_2^U : W_1 \ltimes U \to U$, as the equation holds after applying $\exists_{U'}$ (note that $\pi_2^{U \ltimes U'} = \pi_2^U \ltimes U'$).

Thus, we see that $\pi_2 \circ \varphi : V \to U$ is dimensionally split. Because $\sqcup \ltimes U$ is strongly connection-free, we find some slice $(W, \psi) \in \mathcal{W}/W_0$ so that $\iota : (V, \varphi) \cong \exists_U^{/W_0}(W, \psi) \in \mathcal{V}/(W_0 \ltimes U)$. We conclude that

$$(V', \varphi') \cong \exists_{U'}^{/W_0 \ltimes U}(V, \varphi) \cong \exists_{U'}^{/W_0 \ltimes U} \exists_U^{/W_0}(W, \psi) = \exists_{U \ltimes U'}^{/W_0}(W, \psi). \tag{23}$$

9. Quantifiable multipliers are strongly quantifiable (proposition 3.5.8), and the composite of left adjoints is a left adjoint to the composite. ☐

4 Multipliers and presheaves

Definition 4.0.1. Every multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ gives rise to three adjoint endofunctors between $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{V}}$, which we will denote

$$(\sqcup \ltimes \mathbf{y}U) \dashv (\mathbf{y}U \multimap \sqcup) \dashv (\mathbf{y}U \surd \sqcup). \tag{24}$$

Correspondingly, a morphism of multipliers $\sqcup \ltimes v$ gives rise to natural transformations $\sqcup \ltimes \mathbf{y}v$, $\mathbf{y}v \multimap \sqcup$ and $\mathbf{y}v \surd \sqcup$.

Note that the functor $\sqcup \ltimes \mathbf{y}U : \widehat{\mathcal{W}} \to \widehat{\mathcal{V}}$ is quite reminiscent of the Day-convolution.

4.1 Acting on elements

In section 3.5, we generalized $\exists_U : \mathcal{W} \to \mathcal{V}/U$ to act on slices as $\exists_U^{/W_0} : \mathcal{W}/W_0 \to \mathcal{V}/(W_0 \ltimes U)$. Here, we further generalize to a functor whose domain is the category of elements:

Definition 4.1.1. We define (using notation 2.2.3):

- $\exists_U^{/\Psi} : \mathcal{W}/\Psi \to \mathcal{V}/(\Psi \ltimes \mathbf{y}U) : (W, \psi) \mapsto (W \ltimes U, \psi \ltimes \mathbf{y}U)$,
- $\exists_U^{\in\Psi} : (W \Rightarrow \Psi) \to \{\varphi : W \ltimes U \Rightarrow \Psi \ltimes \mathbf{y}U \mid \pi_2 \circ \varphi = \pi_2\} : \psi \mapsto \psi \ltimes \mathbf{y}U$.

We say that $\sqcup \ltimes U$ is:

- **Providently cancellative** if for all Ψ, the functor $\exists_U^{/\Psi}$ is faithful,
- **Elementally cancellative** if for all Ψ, the natural transformation $\exists_U^{\in\Psi}$ is componentwise injective,

- **Providently affine** if for all Ψ, the functor $\dashv_U^{/\Psi}$ is full,
- **Elementally affine** if for all Ψ, the natural transformation $\dashv_U^{\in\Psi}$ is componentwise surjective,
- **Providently connection-free** if for all Ψ, the functor $\dashv_U^{/\Psi}$ is essentially surjective on slices $(V, \varphi) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ such that $\pi_2 \circ \varphi$ is dimensionally split,
- **Providently quantifiable** if for all Ψ, the functor $\dashv_U^{/\Psi}$ has a left adjoint $\exists_U^{/\Psi} : \mathcal{V}/(\Psi \ltimes \mathbf{y}U) \to \mathcal{W}/\Psi$. We denote the unit as $\mathrm{copy}_U^{/\Psi} : \mathrm{Id} \to \dashv_U^{/\Psi} \exists_U^{/\Psi}$ and the co-unit as $\mathrm{drop}_U^{/\Psi} : \exists_U^{/\Psi} \dashv_U^{/\Psi} \to \mathrm{Id}$.

This is indeed a generalization:

Proposition 4.1.2. The functor $\dashv_U^{/\mathbf{y}W_0} : \mathcal{W}/\mathbf{y}W_0 \to \mathcal{V}/(\mathbf{y}W_0 \ltimes \mathbf{y}U)$ is equal to $\dashv_U^{/W_0} : \mathcal{W}/W_0 \to \mathcal{V}/(W_0 \ltimes U)$ over the obvious isomorphisms between their domains and codomains. Hence, each of the provident notions implies the strong notion 3.5.1. Moreover, each of the elemental notions implies the basic notion.

Proof. Most of this is straightforward. To see the last claim, note that

$$\{\varphi : W \ltimes U \Rightarrow \mathbf{y}W_0 \ltimes \mathbf{y}U \mid \pi_2 \circ \varphi = \pi_2\} \cong ((W \ltimes U, \pi_2) \to (W_0 \ltimes U, \pi_2)) = (\dashv_U W \to \dashv_U W_0).$$

So if injectivity/surjectivity holds for all W_0, then we can conclude that $\dashv_U$ is cancellative/affine. $\square$

Proposition 4.1.3. If $\sqcup \ltimes U$ is cancellative, then it is providently cancellative.

Proof. Analogous to proposition 3.5.4. $\square$

Proposition 4.1.4. If $\sqcup \ltimes U$ is cancellative, then it is elementally cancellative

Proof. We have

$$\begin{aligned}
&\{\varphi : W \ltimes U \Rightarrow \Psi \ltimes \mathbf{y}U \mid \pi_2 \circ \varphi = \pi_2\} \\
&\cong \exists W_0.(\varphi' : W \ltimes U \to W_0 \ltimes U) \times (\psi : W_0 \Rightarrow \Psi) \times (\pi_2 \circ (\psi \ltimes \mathbf{y}U) \circ \varphi' = \pi_2) \\
&\cong \exists W_0.(\varphi' : W \ltimes U \to W_0 \ltimes U) \times (\psi : W_0 \Rightarrow \Psi) \times (\pi_2 \circ \varphi' = \pi_2) \\
&\cong \exists W_0.(\varphi' : \dashv_U W \to \dashv_U W_0) \times (\psi : W_0 \Rightarrow \Psi)
\end{aligned} \tag{25}$$

and

$$(W \Rightarrow \Psi) \cong \exists W_0.(W \to W_0) \times (W_0 \Rightarrow \Psi). \tag{26}$$

Moreover, the action of $\dashv_U^{\in\Psi}$ sends (W_0, χ, ψ) in eq. (26) to $(W_0, \dashv_U \chi, \psi)$ in eq. (25). Naively, one would say that this proves injectivity, but some care is required with the equality relation for co-ends. It might be that (W_0, χ, ψ) and (W_0, χ', ψ) are sent to the same object. This would mean that there exists a zigzag ζ from W_0 to itself such that the following diagrams commute:

$$\begin{tikzcd}
& \dashv_U W_0 \arrow[dd, squiggly, -, "\dashv_U \zeta"] & W_0 \arrow[dd, squiggly, -, "\zeta"'] \arrow[dr, Rightarrow, "\psi"] & \\
\dashv_U W \arrow[ur, "\dashv_U \chi"] \arrow[dr, "\dashv_U \chi'"'] & & & \Psi. \\
& \dashv_U W_0 & W_0 \arrow[ur, Rightarrow, "\psi"'] &
\end{tikzcd} \tag{27}$$

Then by cancellativity of $\dashv_U$, we see that the unique preimage of the left triangle also commutes and hence $\psi \circ \chi = \psi \circ \chi'$, so that $(W_0, \chi, \psi) = (W, \mathrm{id}, \psi \circ \chi) = (W, \mathrm{id}, \psi \circ \chi') = (W_0, \chi', \psi)$. $\square$

Proposition 4.1.5. If $\sqcup \ltimes U$ is cancellative and affine, then it is providently affine.

Proof. Pick (W, ψ) and (W', ψ') in $\mathcal{W}/\Psi$ and a morphism $\chi : \reflectbox{$\exists$}_U^{/\Psi}(W, \psi) \to \reflectbox{$\exists$}_U^{/\Psi}(W', \psi')$. Then we also have $\chi : \reflectbox{$\exists$}_U W \to \reflectbox{$\exists$}_U W'$ and by fullness, we find an inverse image $\chi_0 : W \to W'$ under $\reflectbox{$\exists$}_U$. By elemental cancellativity, we see that $\psi' \circ \chi_0 = \psi$, so that χ_0 is a morphism of slices $\chi_0 : (W, \psi) \to (W', \psi') \in \mathcal{W}/\Psi$ and $\reflectbox{$\exists$}_U^{/\Psi} \chi_0 = \chi$. $\square$

Proposition 4.1.6. If $\sqcup \ltimes U$ is affine, then it is elementally affine.

Proof. In the proof of proposition 4.1.4, we saw that $\reflectbox{$\exists$}_U^{\in\Psi}$ essentially sends (W_0, χ, ψ_0) to $(W_0, \reflectbox{$\exists$}_U \chi, \psi_0)$. Then if $\reflectbox{$\exists$}_U \chi$ is full, it is immediate that this operation is surjective. $\square$

Proposition 4.1.7. If $\sqcup \ltimes U$ is strongly connection-free, then it is providently connection-free.

Proof. Pick a slice $(V, \varphi) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ such that $\pi_2 \circ \varphi$ is dimensionally split. By definition of $\sqcup \ltimes \mathbf{y}U$, there is some W_0 such that φ factors as $\varphi = (\psi^{W_0 \Rightarrow \Psi} \ltimes U) \circ \chi$. Clearly, $\pi_2 \circ \varphi = \pi_2 \circ \chi$ is dimensionally split. Hence, by strong connection-freedom, $(V, \chi) \cong \reflectbox{$\exists$}_U^{/W_0}(W, \chi') \in \mathcal{V}/(W_0 \ltimes U)$ for some $(W, \chi') \in \mathcal{W}/W_0$. Then we also have $(V, \varphi) = (V, (\psi \ltimes \mathbf{y}U) \circ \chi) \cong \reflectbox{$\exists$}_U^{/\Psi}(W, \psi \circ \chi')$. $\square$

Proposition 4.1.8. If $\sqcup \ltimes U$ is quantifiable, then it is providently quantifiable, with

$$\begin{aligned}
\exists_U^{/\Psi}(V, (\psi \ltimes \mathbf{y}U) \circ \varphi_0) &= \Sigma^{/\psi} \exists_U^{/W_0}(V, \varphi_0), \\
\mathsf{drop}_U^{/\Psi}(W, \psi) &= \mathsf{drop}_U W, \\
\mathsf{copy}_U^{/\Psi}(V, \varphi) &= \mathsf{copy}_U(V, \pi_2 \circ \varphi).
\end{aligned}$$

Proof. Pick $(V, \varphi) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$. Then φ factors as $(\psi^{W_0 \Rightarrow \Psi} \ltimes \mathbf{y}U) \circ \varphi_0^{V \to W_0 \ltimes U}$. Then $(V, \varphi_0) \in \mathcal{V}/(W_0 \ltimes U)$ and hence $\exists_U^{/W_0}(V, \varphi_0) \in \mathcal{W}/W_0$. We define

$$\begin{aligned}
\exists_U^{/\Psi}(V, \varphi) &:= \Sigma^{/\psi} \exists_U^{/W_0}(V, \varphi_0) \\
&= \Sigma^{/\psi}(\exists_U(V, \pi_2 \circ \varphi_0), \mathsf{drop}_U \circ \exists_U \varphi_0) \\
&= (\exists_U(V, \pi_2^{W_0 \ltimes U \to U} \circ \varphi_0), \psi \circ \mathsf{drop}_U \circ \exists_U \varphi_0) \\
&= (\exists_U(V, \pi_2^{\Psi \ltimes \mathbf{y}U \to \mathbf{y}U} \circ \varphi), \psi \circ \mathsf{drop}_U \circ \exists_U \varphi_0).
\end{aligned}$$

We need to prove that this is well-defined, i.e. respects equality on the co-end that defines $V \Rightarrow \Psi \ltimes \mathbf{y}U$. To this end, assume that $\varphi = (\psi_0^{W_0 \Rightarrow \Psi} \ltimes \mathbf{y}U) \circ \varphi_0^{V \to W_0 \ltimes U} = (\psi_1^{W_1 \Rightarrow \Psi} \ltimes \mathbf{y}U) \circ \varphi_1^{V \to W_1 \ltimes U}$. This means there is a zigzag ζ from W_0 to W_1 such that the following triangles commute:

$$\begin{tikzcd}
 & W_0 \ltimes U \arrow[dd, squiggly, no head, "\zeta \ltimes U"] & W_0 \arrow[dd, squiggly, no head, "\zeta"'] \arrow[dr, Rightarrow, "\psi_0"] & \\
V \arrow[ur, "\varphi_0"] \arrow[dr, "\varphi_1"'] & & & \Psi. \\
 & W_1 \ltimes U & W_1 \arrow[ur, Rightarrow, "\psi_1"'] &
\end{tikzcd} \tag{28}$$

By naturality of drop_U, we find that $\psi_0 \circ \mathsf{drop}_U \circ \exists_U \varphi_0 = \psi_1 \circ \mathsf{drop}_U \circ \exists_U \varphi_1$. We conclude that $\exists_U^{/\Psi}(V, \varphi)$ is well-defined.

To prove adjointness, we first show how $\reflectbox{$\exists$}_U^{/\Psi}$ on the right can be turned into $\exists_U^{/\Psi}$ on the left. Pick $\chi : (V, \varphi) \to \reflectbox{$\exists$}_U^{/\Psi}(W, \psi) = (W \ltimes U, \psi \ltimes \mathbf{y}U) : \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$. Then $\varphi = (\psi \ltimes \mathbf{y}U) \circ \chi$ so by definition, $\exists_U^{/\Psi}(V, \varphi) = (\exists_U(V, \pi_2 \circ \varphi), \psi \circ \mathsf{drop}_U \circ \exists_U \chi)$ which clearly factors over ψ, i.e. has a morphism $\mathsf{drop}_U \circ \exists_U \chi : \exists_U^{/\Psi}(V, \varphi) \to (W, \psi)$. If $\chi = \mathrm{id}$, then we obtain the co-unit $\mathsf{drop}_U^{/\Psi} = \mathsf{drop}_U \circ \exists_U \mathrm{id} = \mathsf{drop}_U$.

Next, we construct the unit $\mathsf{copy}_U^{/\Psi} : (V, \varphi) \to \reflectbox{$\exists$}_U^{/\Psi} \exists_U^{/\Psi}(V, \varphi)$. If $\varphi = (\psi \ltimes \mathbf{y}U) \circ \varphi_0$, then we have

$$\reflectbox{$\exists$}_U^{/\Psi} \exists_U^{/\Psi}(V, \varphi) = \reflectbox{$\exists$}_U^{/\Psi} \Sigma^{/\psi} \exists_U^{/W_0}(V, \varphi_0)$$

$$= \Sigma^{/\psi \ltimes \mathbf{y}U} \reflectbox{$\exists$}_U^{/W_0} \exists_U^{/W_0}(V, \varphi_0).$$

On the other hand, $(V, \varphi) = \Sigma^{/\psi \ltimes \mathbf{y}U}(V, \varphi_0)$, so as the unit we can take $\mathsf{copy}_U^{/\Psi} = \Sigma^{/\psi \ltimes \mathbf{y}U} \mathsf{copy}_U^{/W_0} = \mathsf{copy}_U^{/W_0} = \mathsf{copy}_U$.

The adjunction laws are then inherited from $\exists_U \dashv \reflectbox{$\exists$}_U$. $\square$

Proposition 4.1.9 (Functoriality for elements)**.** A morphism of multipliers $\sqcup \ltimes v : \sqcup \ltimes U \to \sqcup \ltimes U'$ gives rise to a natural transformation $\Sigma^{/\Psi \ltimes \mathbf{y}v} \circ \reflectbox{$\exists$}_U^{/\Psi} \to \reflectbox{$\exists$}_{U'}^{/\Psi}$. Hence, if both multipliers are quantifiable, we also get $\exists_{U'}^{/\Psi} \circ \Sigma^{/\Psi \ltimes \mathbf{y}v} \to \exists_U^{/\Psi}$.

Proof. For any $(W, \psi) \in \mathcal{W}/\Psi$, we have to prove $(W \ltimes U, (\Psi \ltimes \mathbf{y}v) \circ (\psi \ltimes \mathbf{y}U)) \to (W \ltimes U', \psi \ltimes \mathbf{y}U')$. The morphism $W \ltimes v : W \ltimes U \to W \ltimes U'$ does the job. The second statement follows from lemma 2.1.1. $\square$

Theorem 4.1.10 (Provident quantification theorem)**.** If $\sqcup \ltimes U$ is

1. cancellative, affine and quantifiable, then we have a natural isomorphism $\mathsf{drop}_U^{/\Psi} : \exists_U^{/\Psi} \reflectbox{$\exists$}_U^{/\Psi} \cong \mathrm{Id}$.
2. semi-cartesian, then we have
   (a) $\mathsf{hide}_U^{/\Psi} : \Sigma_U^{/\Psi} \to \exists_U^{/\Psi}$ (if quantifiable),
   (b) $\mathsf{spoil}_U^{/\Psi} : \reflectbox{$\exists$}_U^{/\Psi} \to \Omega_U^{/\Psi}$ (if $\Omega_U^{/\Psi}$ exists),
   (c) in any case $\Sigma_U^{/\Psi} \reflectbox{$\exists$}_U^{/\Psi} \to \mathrm{Id}$.
3. 3/4-cartesian, then there is a natural transformation $\Sigma^{/\Psi \ltimes \mathbf{y}\delta} \circ \reflectbox{$\exists$}_U^{/\Psi} \to \reflectbox{$\exists$}_{U \ltimes U}^{/\Psi}$.
4. cartesian, then we have natural isomorphisms:
   (a) $\exists_U^{/\Psi}(V, \varphi) \cong \Sigma_U^{/\Psi}(V, \varphi) = (V, \pi_1 \circ \varphi)$,
   (b) $\reflectbox{$\exists$}_U^{/\Psi}(W, \psi) \cong \Omega_U^{/\Psi}(W, \psi)$,
   (c) $\exists_U^{/\Psi} \reflectbox{$\exists$}_U^{/\Psi}(W, \psi) \cong \Sigma_U^{/\Psi} \Omega_U^{/\Psi}(W, \psi) \cong (W \times \mathbf{y}U, \psi \circ \pi_1)$.

   Moreover, these isomorphisms become equality if $\exists_U^{/\Psi}$ is constructed as above from $\exists_U^{/W_0} = \Sigma_U^{/W_0}$, and $\Omega_U^{/\Psi}(W, \psi)$ is chosen wisely. (Both functors are defined only up to isomorphism.)

Proof.

1. This is a standard fact about fully faithful right adjoints such as $\reflectbox{$\exists$}_U^{/\Psi}$.
2. By lemma 2.1.1, it is sufficient to prove $\Sigma_U^{/\Psi} \reflectbox{$\exists$}_U^{/\Psi} \to \mathrm{Id}$, and indeed we have $\pi_1 : \Sigma_U^{/\Psi} \reflectbox{$\exists$}_U^{/\Psi}(W, \psi) = (W \ltimes U, \pi_1 \circ (\psi \ltimes \mathbf{y}U)) = (W \ltimes U, \psi \circ \pi_1) \to (W, \psi)$.
3. This is a special case of proposition 4.1.9.
4. (a) Let $\varphi = (\psi \ltimes \mathbf{y}U) \circ \varphi_0$. Then we have

$$\begin{aligned}
\exists_U^{/\Psi}(V, \varphi) &= \Sigma^{/\psi} \exists_U^{/W_0}(V, \varphi_0) \\
&\cong \Sigma^{/\psi} \Sigma_U^{/W_0}(V, \varphi_0) \\
&= \Sigma^{/\psi}(V, \pi_1 \circ \varphi_0) \\
&= (V, \psi \circ \pi_1 \circ \varphi_0) = (V, \pi_1 \circ (\psi \ltimes \mathbf{y}U) \circ \varphi_0) = (V, \pi_1 \circ \varphi).
\end{aligned}$$

   (b) This follows from the definitions.
   (c) We have

$$\exists_U^{/\Psi} \reflectbox{$\exists$}_U^{/\Psi}(W, \psi) = \exists_U^{/\Psi}(W \times U, \psi \times \mathbf{y}U) \cong (W \times U, \pi_1 \circ (\psi \times \mathbf{y}U)) \tag{29}$$

   and of course $\pi_1 \circ (\psi \times \mathbf{y}U) = \psi \circ \pi_1 : W \times U \to \Psi$. $\square$

Theorem 4.1.11 (Provident kernel theorem)**.** If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is cancellative, affine and connection-free, then $\reflectbox{$\exists$}_U^{/\Psi} : \mathcal{W}/\Psi \simeq (\mathcal{V}/\!/U)/(\Psi \ltimes \mathbf{y}U, \pi_2)$ is an equivalence of categories.[5] $\square$

[5] We use a slight abuse of notation as $(\mathcal{V}/\!/U)/(\Psi \ltimes \mathbf{y}U, \pi_2)$ is in fact neither a category of slices nor of elements.

4.2 Acting on presheaves

Proposition 4.2.1. The functor $\sqcup \ltimes \mathbf{y}U : \widehat{\mathcal{W}} \to \widehat{\mathcal{V}}$:

1. is a multiplier for $\mathbf{y}U$,
2. has the property that $\Finv_{\mathbf{y}U} : \widehat{\mathcal{W}} \to \widehat{\mathcal{V}}/\mathbf{y}U$ is naturally isomorphic to $(\Finv_U)_! : \widehat{\mathcal{W}} \to \widehat{\mathcal{V}/U}$ over the equivalence between their codomains,
3. has the property that the slice functor $\Finv^{/\Psi}_{\mathbf{y}U} : \widehat{\mathcal{W}}/\Psi \to \widehat{\mathcal{V}}/(\Psi \ltimes \mathbf{y}U)$ is naturally isomorphic to the left lifting of the elements functor $(\Finv^{/\Psi}_U)_! : \widehat{\mathcal{W}/\Psi} \to \widehat{\mathcal{V}/(\Psi \ltimes \mathbf{y}U)}$ over the equivalences between their domains and codomains,
4. is semicartesian if and only if $\sqcup \ltimes U$ is,
5. is 3/4-cartesian if and only if $\sqcup \ltimes U$ is,
6. is cartesian if and only if $\sqcup \ltimes U$ is,
7. is cancellative if one of the following conditions holds:
   (a) $\sqcup \ltimes U$ is cancellative and affine,
   (b) $\sqcup \ltimes U$ is cartesian and U is not spooky.

   (see example 4.2.2 for a counterexample showing the necessity of either non-spookiness or affinity),
8. is strongly affine if $\sqcup \ltimes U$ is cancellative and affine,
9. is quantifiable if $\sqcup \ltimes U$ is, and $\exists_{\mathbf{y}U}$ is naturally isomorphic to $(\exists_U)_!$ over the equivalence $\widehat{\mathcal{V}/U} \simeq \widehat{\mathcal{V}}/\mathbf{y}U$,
10. is strongly quantifiable if $\sqcup \ltimes U$ is quantifiable, and $\exists^{/\Psi}_{\mathbf{y}U}$ is naturally isomorphic to $(\exists^{/\Psi}_U)_!$ over the equivalences between their domain and codomain.

Proof. 1. Since $\top \ltimes \mathbf{y}U \cong \mathbf{y}\top \ltimes \mathbf{y}U \cong \mathbf{y}(\top \ltimes U) \cong \mathbf{y}U$. We use, in order, that $\mathbf{y}$ preserves the terminal object, that $F_! \circ \mathbf{y} \cong \mathbf{y} \circ F$ (theorem 2.2.2) and that $\sqcup \ltimes U$ is a multiplier for U.

2. The functor $(\Finv_U)_!$ sends a presheaf $\Gamma \in \widehat{\mathcal{W}}$ to the presheaf in $\widehat{\mathcal{V}/U}$ determined by

$$(V, \varphi) \Rightarrow (\Finv_U)_!\Gamma \quad = \quad \exists W.((V, \varphi) \to \Finv_U W) \times (W \Rightarrow \Gamma). \tag{30}$$

On the other hand, $\Finv_{\mathbf{y}U}\Gamma$ is the slice $(\Gamma \ltimes \mathbf{y}U, \pi_2) \in \widehat{\mathcal{V}}/\mathbf{y}U$. Taking the preimage of π_2 (proposition 2.2.6), we get a presheaf $\Delta \in \widehat{\mathcal{V}/U}$ determined by

$$\begin{aligned} (V, \varphi) \Rightarrow \Delta \quad &= \quad \{(\gamma \ltimes \mathbf{y}U) \circ \chi : V \Rightarrow \Gamma \ltimes \mathbf{y}U \mid \pi_2 \circ (\gamma \ltimes \mathbf{y}U) \circ \chi = \varphi\} \\ &= \quad \{(\gamma \ltimes \mathbf{y}U) \circ \chi : V \Rightarrow \Gamma \ltimes \mathbf{y}U \mid \pi_2 \circ \chi = \varphi\} \\ &\cong \quad \exists W.(\chi : V \to W \ltimes U) \times (\gamma : W \Rightarrow \Gamma) \times (\pi_2 \circ \chi = \varphi) \\ &\cong \quad \exists W.(\chi : (V, \varphi) \to \Finv_U W) \times (W \Rightarrow \Gamma). \end{aligned}$$

Indeed, we see that these functors are isomorphic.

3. The functor $(\Finv^{/\Psi}_U)_!$ sends a presheaf $\Psi \mid \Gamma \vdash \mathrm{Ctx}$ over $\mathcal{W}/\Psi$ to the presheaf $\Psi \ltimes \mathbf{y}U \mid (\Finv_U \Psi)_!\Gamma \vdash \mathrm{Ctx}$ over $\mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ determined by:

$$(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \Rightarrow \left(\Finv^{/\Psi}_U\right)_! \Gamma \quad = \quad \exists (W, \psi^{W \Rightarrow \Psi}).((V, \varphi) \to \Finv^{/\Psi}_U (W, \psi)) \times ((W, \psi) \Rightarrow \Gamma). \tag{31}$$

On the other hand, $\dashv^{/\Psi}_{\mathbf{y}U}(\Psi.\Gamma, \pi)$ is the slice $(\Psi.\Gamma \ltimes \mathbf{y}U, \pi \ltimes \mathbf{y}U) \in \widehat{\mathcal{V}}/(\Psi \ltimes \mathbf{y}U)$. Taking the preimage of $\pi \ltimes \mathbf{y}U$ (proposition 2.2.6), we get a presheaf $\Psi \ltimes \mathbf{y}U \mid \Delta \vdash \mathsf{Ctx}$ over $\mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ determined by

$$\begin{aligned}
&(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \Rightarrow \Delta \\
&= \{(\psi.\gamma \ltimes \mathbf{y}U) \circ \chi : V \Rightarrow \Psi.\Gamma \ltimes \mathbf{y}U \mid (\pi \ltimes \mathbf{y}U) \circ (\psi.\gamma \ltimes \mathbf{y}U) \circ \chi = \varphi\} \\
&= \{(\psi.\gamma \ltimes \mathbf{y}U) \circ \chi : V \Rightarrow \Psi.\Gamma \ltimes \mathbf{y}U \mid (\psi \ltimes \mathbf{y}U) \circ \chi = \varphi\} \\
&\cong \exists W.(\chi : V \to W \ltimes U) \times (\psi : W \Rightarrow \Psi) \times (\gamma : (W, \psi) \Rightarrow \Gamma) \times ((\psi \ltimes \mathbf{y}U) \circ \chi = \varphi) \\
&\cong \exists (W, \psi^{W \Rightarrow \Psi}).(\chi : (V, \varphi) \to \dashv^{/\Psi}_{U}(W, \psi)) \times (\gamma : (W, \psi) \Rightarrow \Gamma).
\end{aligned}$$

Indeed, we see that these functors are isomorphic.

4. Assume that $\sqcup \ltimes U$ is semicartesian. It is immediate from the construction of $\sqcup_!$ that $\sqcup_!$ preserves natural transformations. Moreover, we have $\mathrm{Id}_! \cong \mathrm{Id}$, so we get $\pi_1 : (\sqcup \ltimes \mathbf{y}U) \to \mathrm{Id}$.

   Conversely, assume that $\sqcup \ltimes \mathbf{y}U$ is semicartesian. Then we have $\mathbf{y}(\sqcup \ltimes U) \cong (\mathbf{y}\sqcup \ltimes \mathbf{y}U) \to \mathbf{y}$. Since $\mathbf{y}$ is fully faithful, we have proven $(\sqcup \ltimes U) \to \mathrm{Id}$.

5. Analogous to the previous point.

6. Assume that $\sqcup \ltimes U$ is cartesian. We apply the universal property of the cartesian product, and the co-Yoneda lemma:

$$\begin{aligned}
V \Rightarrow (\Gamma \ltimes \mathbf{y}U) &= \exists W.(V \to W \ltimes U) \times (W \Rightarrow \Gamma) \\
&\cong \exists W.(V \to W) \times (V \to U) \times (W \Rightarrow \Gamma) \\
&\cong (V \to U) \times (V \Rightarrow \Gamma) \\
&\cong V \Rightarrow \Gamma \times \mathbf{y}U.
\end{aligned}$$

   Conversely, if $\sqcup \ltimes \mathbf{y}U$ is cartesian, we have

$$\begin{aligned}
V \to W \ltimes U &= V \Rightarrow \mathbf{y}(W \ltimes U) \\
&\cong V \Rightarrow \mathbf{y}W \ltimes \mathbf{y}U \\
&\cong (V \Rightarrow \mathbf{y}W) \times (V \Rightarrow \mathbf{y}U) \\
&\cong (V \to W) \times (V \to U) \cong V \to W \times U.
\end{aligned}$$

7. The reasoning is different in both cases:

   (a) In this case, $\dashv_U$ is fully faithful, implying that $(\dashv_U)_!$ (which is essentially $\dashv_{\mathbf{y}U}$) is also fully faithful, i.e. $\sqcup \ltimes \mathbf{y}U$ is cancellative and affine.

   (b) Similar to proposition 3.2.6.

8. Since it is then cancellative and affine, as proven in the previous item.

9. We know that $(\exists_U)_! \dashv (\dashv_U)_!$ so moving it through the natural transformation yields a left adjoint to $\dashv_{\mathbf{y}U}$.

10. By proposition 4.1.8, $\exists^{/\Psi}_U$ exists. We know that $(\exists^{/\Psi}_U)_! \dashv (\dashv^{/\Psi}_U)_!$ so moving it through the natural transformation yields a left adjoint to $\dashv^{/\Psi}_{\mathbf{y}U}$. □

Example 4.2.2 (Non-cancellativity of $\sqcup \times \mathbf{y}U$ for spooky U). Consider the category of nullary cartesian cubes $\boxslash^0$ (example 3.3.4) and let Γ be the terminal presheaf and $(\top \Rightarrow \Delta) = \mathsf{Bool}$ and $(\mathbb{I}^n \Rightarrow \Delta) = \{\star\}$ for $n > 0$. Then $\Delta \times \mathbf{y}\mathbb{I}$ is the terminal presheaf. Hence, $\sqcup \times \mathbf{y}\mathbb{I}$ is not injective on morphisms $\Gamma \to \Delta$.

4.3 Four adjoint functors

Unlike the category of slices $\widehat{\mathcal{W}}/\Psi$, the equivalent category $\widehat{\mathcal{W}/\Psi}$ is a presheaf category and therefore immediately a model of dependent type theory. Therefore, we prefer to work with that category, and to use the corresponding functors:

Definition 4.3.1. The adjoint functors $\exists_U^{/\Psi} \dashv Ⅎ_U^{/\Psi}$ give rise to four adjoint functors between presheaf categories over slice categories, which we denote

$$\exists_{\mathbf{y}U}^{\Psi|} \dashv Ⅎ_{\mathbf{y}U}^{\Psi|} \dashv \forall_{\mathbf{y}U}^{\Psi|} \dashv \between_{\mathbf{y}U}^{\Psi|}. \tag{32}$$

We call the fourth functor **transpension**.

The units and co-units will be denoted:

$$\begin{aligned}
\mathsf{copy}_{\mathbf{y}U}^{\Psi|} &: \mathrm{Id} \to Ⅎ_{\mathbf{y}U}^{\Psi|}\exists_{\mathbf{y}U}^{\Psi|} & \qquad \mathsf{drop}_{\mathbf{y}U}^{\Psi|} &: \exists_{\mathbf{y}U}^{\Psi|}Ⅎ_{\mathbf{y}U}^{\Psi|} \to \mathrm{Id} \\
\mathsf{const}_{\mathbf{y}U}^{\Psi|} &: \mathrm{Id} \to \forall_{\mathbf{y}U}^{\Psi|}Ⅎ_{\mathbf{y}U}^{\Psi|} & \qquad \mathsf{app}_{\mathbf{y}U}^{\Psi|} &: Ⅎ_{\mathbf{y}U}^{\Psi|}\forall_{\mathbf{y}U}^{\Psi|} \to \mathrm{Id} \\
\mathsf{reidx}_{\mathbf{y}U}^{\Psi|} &: \mathrm{Id} \to \between_{\mathbf{y}U}^{\Psi|}\forall_{\mathbf{y}U}^{\Psi|} & \qquad \mathsf{unmerid}_{\mathbf{y}U}^{\Psi|} &: \forall_{\mathbf{y}U}^{\Psi|}\between_{\mathbf{y}U}^{\Psi|} \to \mathrm{Id}
\end{aligned} \tag{33}$$

For now, we define all of these functors only up to isomorphism, i.e. for the middle two we do not specify whether they arise as a left, central or right lifting.

Note that, if in a judgement $\Psi \mid \Gamma \vdash J$, we view the part before the pipe ($|$) as part of the context, then $\exists_{\mathbf{y}U}^{\Gamma|}$ and $\forall_{\mathbf{y}U}^{\Gamma|}$ bind a (substructural) variable of type $\mathbf{y}U$, whereas $Ⅎ_{\mathbf{y}U}^{\Gamma|}$ and $\between_{\mathbf{y}U}^{\Gamma|}$ depend on one.

Corollary 4.3.2. The properties asserted by proposition 4.2.1 for $Ⅎ_{\mathbf{y}U}^{/\Psi}$ also hold for $Ⅎ_{\mathbf{y}U}^{\Psi|}$.

Proof. Follows from the fact that $Ⅎ_{\mathbf{y}U}^{\Psi|} \cong (Ⅎ_U^{/\Psi})_!$, and the observation in proposition 4.2.1 that this functor in turn corresponds to $Ⅎ_{\mathbf{y}U}^{/\Psi}$. □

Proposition 4.3.3 (Presheaf functoriality)**.** A morphism of multipliers $\sqcup \ltimes v : \sqcup \ltimes U \to \sqcup \ltimes U'$ gives rise to natural transformations

- $\exists_{\mathbf{y}U'}^{\Psi|} \circ \Sigma^{\Psi \ltimes \mathbf{y}v|} \to \exists_{\mathbf{y}U}^{\Psi|}$ (if quantifiable),
- $\Sigma^{\Psi \ltimes \mathbf{y}v|} \circ Ⅎ_{\mathbf{y}U}^{\Psi|} \to Ⅎ_{\mathbf{y}U'}^{\Psi|}$ and $Ⅎ_{\mathbf{y}U}^{\Psi|} \to \Omega^{\Psi \ltimes \mathbf{y}v|} \circ Ⅎ_{\mathbf{y}U'}^{\Psi|}$,
- $\forall_{\mathbf{y}U'}^{\Psi|} \to \forall_{\mathbf{y}U}^{\Psi|} \circ \Omega^{\Psi \ltimes \mathbf{y}v|}$ and $\forall_{\mathbf{y}U'}^{\Psi|} \circ \Pi^{\Psi \ltimes \mathbf{y}v|} \to \forall_{\mathbf{y}U}^{\Psi|}$,
- $\Pi^{\Psi \ltimes \mathbf{y}v|} \circ \between_{\mathbf{y}U}^{\Psi|} \to \between_{\mathbf{y}U'}^{\Psi|}$,

Proof. Follows directly from proposition 4.1.9. □

Proposition 4.3.4 (Presheaf quantification theorem)**.** If $\sqcup \ltimes U$ is

1. cancellative and affine, then $\mathsf{drop}_{\mathbf{y}U}^{\Psi|}$, $\mathsf{const}_{\mathbf{y}U}^{\Psi|}$ and $\mathsf{unmerid}_{\mathbf{y}U}^{\Psi|}$ are natural isomorphisms.
2. semi-cartesian, then we have
   (a) $\mathsf{hide}_{\mathbf{y}U}^{\Psi|} : \Sigma_{\mathbf{y}U}^{\Psi|} \to \exists_{\mathbf{y}U}^{\Psi|}$ (if quantifiable),
   (b) $\mathsf{spoil}_{\mathbf{y}U}^{\Psi|} : Ⅎ_{\mathbf{y}U}^{\Psi|} \to \Omega_{\mathbf{y}U}^{\Psi|}$,
   (c) $\mathsf{cospoil}_{\mathbf{y}U}^{\Psi|} : \Pi_{\mathbf{y}U}^{\Psi|} \to \forall_{\mathbf{y}U}^{\Psi|}$.
3. 3/4-cartesian, then we can apply proposition 4.3.3 to $\sqcup \ltimes \delta : \sqcup \ltimes U \to \sqcup \ltimes (U \ltimes U)$.
4. cartesian, then we have natural isomorphisms:

(a) $\exists_U^{\Psi|} \cong \Sigma_U^{\Psi|}$,

(b) $\Finv_U^{\Psi|} \cong \Omega_U^{\Psi|}$,

(c) $\forall_U^{\Psi|} \cong \Pi_U^{\Psi|}$,

(d) $\between_U^{\Psi|} \cong \between_U^{\Psi|}$ (between the functors of the same notation from definitions 2.2.17 and 4.3.1).

Equality is achieved for any pair of functors if they are lifted in the same way from functors that were equal in theorem 4.1.10.

Proof. 1. The fact that $\mathrm{drop}_{\mathbf{y}U}^{\Psi|}$ is an isomorphism, is a standard fact about fully faithful right adjoints such as $\Finv_{\mathbf{y}U}^{\Psi|}$. This property then carries over to further adjoints.

2. By lemma 2.1.1, it is sufficient to prove $\Sigma_{\mathbf{y}U}^{\Psi|} \Finv_{\mathbf{y}U}^{\Psi|} \to \mathrm{Id}$, which follows immediately from $\pi_1 : \Sigma_U^{/\Psi} \Finv_U^{/\Psi} \to \mathrm{Id}$.

3. Of course we can.

4. This is an immediate corollary of theorem 3.5.10. $\square$

Proposition 4.3.5 (Fresh exchange)**.** If $\Psi \mid \Gamma \vdash \mathsf{Ctx}$, i.e. $\Gamma \in \widehat{\mathcal{W}/\Psi}$, then we have an isomorphism of slices (natural in Γ):

$$\begin{tikzcd}
(\Psi \ltimes \mathbf{y}U).\Finv_{\mathbf{y}U}^{\Psi|}\Gamma \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & \Psi.\Gamma \ltimes \mathbf{y}U \arrow[dl, "\pi \ltimes \mathbf{y}U"] \\
& \Psi \ltimes \mathbf{y}U. &
\end{tikzcd} \tag{34}$$

This proposition explains the meaning of $\Finv_{\mathbf{y}U}^{/\Gamma}$: it is the type depending on a variable of type $\mathbf{y}U$ whose elements are required to be fresh for that variable, where the meaning of 'fresh' depends on the nature of the multiplier. If the multiplier is cartesian, then $\Finv_{\mathbf{y}U}^{/\Gamma}$ is clearly just weakening over $\mathbf{y}U$.

Proof. The slice on the right is $\Finv_{\mathbf{y}U}^{/\Psi}(\Psi.\Gamma, \pi)$. By proposition 4.2.1, this is isomorphic to $\Finv_{\mathbf{y}U}^{\Psi|}\Gamma$ over the equivalence from proposition 2.2.6 which sends Δ to $((\Psi \ltimes \mathbf{y}U).\Delta, \pi)$. $\square$

4.4 Investigating the transpension functor

Definition 4.4.1. Write $\Psi \ltimes \partial U$ for the pullback

$$\begin{tikzcd}
\Psi \ltimes \partial U \arrow[r, "\subseteq"] \arrow[d, "\pi_2"] & \Psi \ltimes \mathbf{y}U \arrow[d, "\pi_2"] \\
\partial U \arrow[r, "\subseteq"] & \mathbf{y}U.
\end{tikzcd} \tag{35}$$

Definition 4.4.2. Write $(\in \partial U)$ for the inverse image of $\Psi \ltimes \partial U \subseteq \Psi \ltimes \mathbf{y}U$, which is a presheaf over $\mathcal{W}/(\Psi \ltimes \mathbf{y}U)$ such that $(\Psi \ltimes \mathbf{y}U).(\in \partial U) \cong \Psi \ltimes \partial U$. We also write $(\in \partial U)$ for the inverse image of $\partial U \subseteq \mathbf{y}U$. Finally, we write $\Sigma_{(\in \partial U)}^{/\Psi \ltimes \mathbf{y}U} \dashv \dots$ for the functors arising from $\Psi \ltimes \partial U \subseteq \Psi \ltimes \mathbf{y}U$.

Theorem 4.4.3 (Poles of the transpension)**.** For any non-spooky multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$, the functor $\Omega_{(\in \partial U)}^{\Psi \ltimes \mathbf{y}U|} \circ \between_{\mathbf{y}U}^{\Psi|} : \widehat{\mathcal{W}/\Psi} \to \widehat{\mathcal{V}/(\Psi \ltimes \partial U)}$ sends any presheaf to the terminal presheaf, i.e. $\Omega_{(\in \partial U)}^{\Psi \ltimes \mathbf{y}U|} \circ \between_{\mathbf{y}U}^{\Psi|} = \top$.

Proof. We show that there is always a unique cell $(V, \varphi^{V \Rightarrow \Psi \ltimes \partial U}) \Rightarrow \Omega^{\Psi \ltimes \mathbf{y}U |}_{(\in \partial U)} \between^{\Psi |}_{\mathbf{y}U} \Gamma$. We have

$$
\begin{aligned}
&(V, \varphi^{V \Rightarrow \Psi \ltimes \partial U}) \Rightarrow \Omega^{\Psi \ltimes \mathbf{y}U |}_{(\in \partial U)} \between^{\Psi |}_{\mathbf{y}U} \Gamma \\
&= \Sigma^{/ \Psi \ltimes \mathbf{y}U}_{(\in \partial U)} (V, \varphi^{V \Rightarrow \Psi \ltimes \partial U}) \Rightarrow \between^{\Psi |}_{\mathbf{y}U} \Gamma \\
&= (V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \Rightarrow \between^{\Psi |}_{\mathbf{y}U} \Gamma \\
&= \forall^{\Psi |}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \to \Gamma \\
&= \forall (W, \psi^{W \Rightarrow \Psi}). \left((W, \psi) \Rightarrow \forall^{\Psi |}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \right) \to ((W, \psi) \Rightarrow \Gamma) \\
&= \forall (W, \psi^{W \Rightarrow \Psi}). \left(\exists^{/ \Psi}_{U} (W, \psi) \Rightarrow \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \right) \to ((W, \psi) \Rightarrow \Gamma) \\
&= \forall (W, \psi^{W \Rightarrow \Psi}). \left((W \ltimes U, \psi \ltimes \mathbf{y}U) \to (V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \right) \to ((W, \psi) \Rightarrow \Gamma),
\end{aligned}
$$

and then we see that the last argument χ cannot exist. Indeed, suppose we have a commuting diagram

$$
\begin{tikzcd}
W \ltimes U \arrow[rr, "\chi"] \arrow[dr, Rightarrow, "\psi \ltimes \mathbf{y}U"'] \arrow[ddr, Rightarrow, bend right, "\pi"'] & & V \arrow[d, Rightarrow, "\varphi"] \\
& \Psi \ltimes \mathbf{y}U \arrow[d, "\pi_2"] & \Psi \ltimes \partial U \arrow[l, "\supseteq"] \arrow[d, "\pi_2"] \\
& \mathbf{y}U & \partial U. \arrow[l, "\supseteq"]
\end{tikzcd}
\tag{36}
$$

Then we see that $\pi_2 \circ \varphi : V \to U$ is dimensionally split with section χ but is also a cell of ∂U which means exactly that it is not dimensionally split. ☐

The following theorem shows that dimensionally split morphisms are an interesting concept:

Theorem 4.4.4 (Boundary theorem)**.** We have $\top \ltimes \mathbf{y}U \mid (\in \partial U) \cong \between^{\top |}_{\mathbf{y}U} \bot \vdash \mathsf{Ctx}$.

Proof. We prove this by characterizing the right hand side of the isomorphism. We have

$$
\begin{aligned}
&(V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U}) \Rightarrow \between^{\top |}_{\mathbf{y}U} \bot \\
&= \forall^{\top |}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U}) \to \bot \\
&= \forall (W, ()^{W \Rightarrow \top}).((W, ()) \Rightarrow \forall^{\top |}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U})) \to ((W, ()) \Rightarrow \bot) \\
&= \forall (W, ()^{W \Rightarrow \top}).((W, ()) \Rightarrow \forall^{\top |}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U})) \to \varnothing \\
&= \forall (W, ()^{W \Rightarrow \top}).(\exists^{/ \top}_{U} (W, ()) \to (V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U})) \to \varnothing \\
&= \forall (W, ()^{W \Rightarrow \top}).((W \ltimes U, () \ltimes U) \to (V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U})) \to \varnothing \\
&\cong \forall W.((W \ltimes U, \pi_2) \to (V, \pi_2 \circ \varphi)) \to \varnothing \\
&\cong (\exists W.(W \ltimes U, \pi_2) \to (V, \pi_2 \circ \varphi)) \to \varnothing.
\end{aligned}
$$

Clearly, the left hand side of the last line is inhabited if and only if $\pi_2 \circ \varphi$ is dimensionally split. Hence, there is a unique cell $(V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U}) \Rightarrow \between^{\top |}_{\mathbf{y}U} \bot$ if and only if $\pi_2 \circ \varphi$ is *not* dimensionally split, showing that $\between^{\top |}_{\mathbf{y}U} \bot$ is indeed isomorphic to $(\in \partial U)$. ☐

Remark 4.4.5. In section 6.2 (theorem 6.2.1), we will see that unless the multiplier is cancellative and affine, the transpension type is not stable under substitution. Instead, for $\sigma : \Psi_1 \to \Psi_2$, we only have $\Omega^{\sigma \ltimes \mathbf{y}U |} \circ \between^{\Psi_2 |}_{\mathbf{y}U} \to \between^{\Psi_1 |}_{\mathbf{y}U} \circ \Omega^{\sigma |}$. Writing $\omega : \Psi \to \top$, this gives us over $\Psi \ltimes \mathbf{y}U$:

$$
(\in \partial U) \quad = \quad \Omega^{\omega \ltimes \mathbf{y}U |}(\in \partial U) \quad \cong \quad \Omega^{\omega \ltimes \mathbf{y}U |} \between^{\top |}_{\mathbf{y}U} \bot \quad \to \quad \between^{\Psi |}_{\mathbf{y}U} \Omega^{\omega |} \bot \quad = \quad \between^{\Psi |}_{\mathbf{y}U} \bot,
$$

which we already knew from theorem 4.4.3.

On the other hand, $(\in \partial U)$ *is* stable under substitution. Hence, theorem 4.4.4 breaks if we replace $\top$ with an arbitrary Ψ, unless $\sqcup \ltimes U$ is cancellative and affine.

Theorem 4.4.6 (Transpension elimination)**.** Let $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ be a cancellative, affine, connection-free and quantifiable multiplier. Then we have

$$\frac{\begin{array}{l}
\Psi \ltimes \mathbf{y}U \mid \Gamma \vdash \mathsf{Ctx} \\
\Psi \mid \forall^{\Psi|}_{\mathbf{y}U}\Gamma \vdash A\,\mathsf{type} \\
\Psi \ltimes \mathbf{y}U \mid \Gamma.\left\langle \between^{\Psi|}_{\mathbf{y}U} \middle| A \right\rangle \vdash B\,\mathsf{type} \\
\Psi \ltimes \partial U \mid \Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)}\Gamma \vdash b_\partial : \left(\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} B\right)[(\mathrm{id}, _)] \\
\Psi \mid \left(\forall^{\Psi|}_{\mathbf{y}U}\Gamma\right).A \vdash \mathring{b} : \left(\forall^{\Psi|}_{\mathbf{y}U} B\right)\left[\left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U}\right)^{-1}(\xi)\right)\right] \\
\Psi \ltimes \partial U \mid \Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \dashv^{\Psi|}_{\mathbf{y}U}\left(\left(\forall^{\Psi|}_{\mathbf{y}U}\Gamma\right).A\right) \vdash \Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)}\left(\mathsf{app}^{\Psi|}_{\mathbf{y}U}\left(\dashv^{\Psi|}_{\mathbf{y}U}\mathring{b}\right)\right) = b_\partial\left[\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)}\left(\mathsf{app}^{\Psi|}_{\mathbf{y}U} \circ \pi\right)\right] \\
\qquad : \left(\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} B\right)[(\mathrm{id}, _)]\left[\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)}\left(\mathsf{app}^{\Psi|}_{\mathbf{y}U} \circ \pi\right)\right]
\end{array}}{\Psi \ltimes \mathbf{y}U \mid \Gamma.\left\langle \between^{\Psi|}_{\mathbf{y}U} \middle| A \right\rangle \vdash b : B} \tag{37}$$

and b reduces to b_∂ and $\mathring{b}$ if we apply to it the same functors and substitutions that have been applied to B in the types of b_∂ and $\mathring{b}$.

In words: if we want to eliminate an element of the transpension type, then we can do so by induction. We distinguish two cases and a coherence condition:

- In the first case (b_∂), we are on the boundary of U and the transpension type trivializes.
- In the second case, we are defining an action on cells that live over all of $\mathbf{y}U$. In the transpension type, such cells are in 1-1 correspondence with cells of type A under the isomorphism $\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} : \forall^{\Psi|}_{\mathbf{y}U}\between^{\Psi|}_{\mathbf{y}U} \cong \mathrm{Id}$.
- The boundary of the image of cells in the second case, must always be b_∂.

Note that right adjoint weak CwF morphisms such as $\between^{\Psi|}_{\mathbf{y}U}$ give rise to a DRA by applying the CwF morphism and then substituting with the unit of the adjunction. As such, the transpension *type* is modelled by the DRA sending A to $\left\langle \between^{\Psi|}_{\mathbf{y}U} \middle| A \right\rangle = \left(\between^{\Psi|}_{\mathbf{y}U} A\right)\left[\mathsf{reidx}^{\Psi|}_{\mathbf{y}U}\right]$.

Proof. **Well-formedness.** We first show that the theorem is well-formed.

- The rule for Γ just assumes that Γ is a presheaf over $\mathcal{V}/(\Psi \ltimes \mathbf{y}U)$.
- Then $\forall^{\Psi|}_{\mathbf{y}U}\Gamma$ is a presheaf over $\mathcal{W}/\Psi$ and we assume that A is a type in that context, i.e. a presheaf over the category of elements of $\forall^{\Psi|}_{\mathbf{y}U}\Gamma$.
- Then the DRA of $\between^{\Psi|}_{\mathbf{y}U}$ applied to A is a type in context Γ. We assume that B is a type over the extended context.
- Being a central lifting, $\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} B$ is a CwF morphism and can be applied to B, yielding a type in context

  $$\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)}\left(\Gamma.\left(\between^{\Psi|}_{\mathbf{y}U} A\right)\left[\mathsf{reidx}^{\Psi|}_{\mathbf{y}U}\right]\right) = \Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)}\Gamma.\left(\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)}\between^{\Psi|}_{\mathbf{y}U} A\right)\left[\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)}\mathsf{reidx}^{\Psi|}_{\mathbf{y}U}\right] \cong \Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)}\Gamma.\top,$$

  where the isomorphism is an application of theorem 4.4.3. The substitution $(\mathrm{id}, \_) = \pi^{-1}$ yields a type in context $\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)}\Gamma$. We assume that b_∂ has this type.

- Being a central lifting, $\forall^{\Psi|}_{\mathbf{y}U}$ is a CwF morphism and can be applied to B, yielding a type in context

$$\forall^{\Psi|}_{\mathbf{y}U}\left(\Gamma.\left(\between^{\Psi|}_{\mathbf{y}U}A\right)\left[\mathsf{reidx}^{\Psi|}_{\mathbf{y}U}\right]\right) = \forall^{\Psi|}_{\mathbf{y}U}\Gamma.\left(\forall^{\Psi|}_{\mathbf{y}U}\between^{\Psi|}_{\mathbf{y}U}A\right)\left[\forall^{\Psi|}_{\mathbf{y}U}\mathsf{reidx}^{\Psi|}_{\mathbf{y}U}\right].$$

  The natural transformation $(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1}$ gives rise [Nuy18] to a function

$$(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1} : A \to \left(\forall^{\Psi|}_{\mathbf{y}U}\between^{\Psi|}_{\mathbf{y}U}A\right)[(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1}]. \tag{38}$$

  Now, by the adjunction laws, $\forall^{\Psi|}_{\mathbf{y}U}\mathsf{reidx}^{\Psi|}_{\mathbf{y}U} \circ \mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} = \mathrm{id}$, so

$$\forall^{\Psi|}_{\mathbf{y}U}\mathsf{reidx}^{\Psi|}_{\mathbf{y}U} = \forall^{\Psi|}_{\mathbf{y}U}\mathsf{reidx}^{\Psi|}_{\mathbf{y}U} \circ \mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} \circ (\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1} = (\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1}. \tag{39}$$

  Then we have

$$(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1} : A \to \left(\forall^{\Psi|}_{\mathbf{y}U}\between^{\Psi|}_{\mathbf{y}U}A\right)\left[\forall^{\Psi|}_{\mathbf{y}U}\mathsf{reidx}^{\Psi|}_{\mathbf{y}U}\right]. \tag{40}$$

  Thus, we can substitute $\forall^{\Psi|}_{\mathbf{y}U}B$ with $(\pi, (\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1}(\xi))$, yielding a type in the desired context. We assume that $\mathring{b}$ has this type.

- In the coherence criterion, we have applied operations to b_∂ and $\mathring{b}$ before equating them. We have to ensure that the resulting terms are well-typed in the given context and type.
  - If we apply $\dashv^{\Psi|}_{\mathbf{y}U}$ to the term $\mathring{b}$, we get

$$\Psi \ltimes \mathbf{y}U \mid \left(\dashv^{\Psi|}_{\mathbf{y}U}\forall^{\Psi|}_{\mathbf{y}U}\Gamma\right).\dashv^{\Psi|}_{\mathbf{y}U}A \vdash \dashv^{\Psi|}_{\mathbf{y}U}\mathring{b} : \left(\dashv^{\Psi|}_{\mathbf{y}U}\forall^{\Psi|}_{\mathbf{y}U}B\right)\left[\dashv^{\Psi|}_{\mathbf{y}U}\left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U}\right)^{-1}(\xi)\right)\right].$$

    If we subsequently apply $\mathsf{app}^{\Psi|}_{\mathbf{y}U}$, we get

$$\Psi \ltimes \mathbf{y}U \mid \left(\dashv^{\Psi|}_{\mathbf{y}U}\forall^{\Psi|}_{\mathbf{y}U}\Gamma\right).\dashv^{\Psi|}_{\mathbf{y}U}A \vdash \mathsf{app}^{\Psi|}_{\mathbf{y}U}\left(\dashv^{\Psi|}_{\mathbf{y}U}\mathring{b}\right) : B\left[\mathsf{app}^{\Psi|}_{\mathbf{y}U}\right]\left[\dashv^{\Psi|}_{\mathbf{y}U}\left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U}\right)^{-1}(\xi)\right)\right].$$

    Next, we apply $\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}$ and obtain something of type

$$\left(\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}B\right)\left[\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}\mathsf{app}^{\Psi|}_{\mathbf{y}U}\right]\left[\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}\dashv^{\Psi|}_{\mathbf{y}U}\left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U}\right)^{-1}(\xi)\right)\right].$$

    Now if we look at the context of $\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}B$, we see that the last type is the unit type by theorem 4.4.3, so the substitution applied to B is determined by its weakening. So we rewrite:

$$\begin{aligned}
\ldots &= \left(\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}B\right)[(\mathrm{id}, _)][\pi]\left[\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}\mathsf{app}^{\Psi|}_{\mathbf{y}U}\right]\left[\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}\dashv^{\Psi|}_{\mathbf{y}U}\left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U}\right)^{-1}(\xi)\right)\right] \\
&= \left(\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}B\right)[(\mathrm{id}, _)]\left[\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}\mathsf{app}^{\Psi|}_{\mathbf{y}U}\right][\pi]\left[\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}\dashv^{\Psi|}_{\mathbf{y}U}\left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U}\right)^{-1}(\xi)\right)\right] \\
&= \left(\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}B\right)[(\mathrm{id}, _)]\left[\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}\mathsf{app}^{\Psi|}_{\mathbf{y}U}\right]\left[\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}\dashv^{\Psi|}_{\mathbf{y}U}\left(\pi \circ \left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U}\right)^{-1}(\xi)\right)\right)\right] \\
&= \left(\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}B\right)[(\mathrm{id}, _)]\left[\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}\mathsf{app}^{\Psi|}_{\mathbf{y}U}\right]\left[\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}\dashv^{\Psi|}_{\mathbf{y}U}\pi\right] \\
&= \left(\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}B\right)[(\mathrm{id}, _)]\left[\Omega^{\Psi\ltimes\mathbf{y}U|}_{(\in\partial U)}\left(\mathsf{app}^{\Psi|}_{\mathbf{y}U} \circ \pi\right)\right].
\end{aligned}$$

  - It is immediate that the substitution applied to b_∂ yields the given type.

Soundness of the coherence criterion. Note that, if we apply to b the same reasoning that we applied to B to show well-formedness of the last 3 premises, we find that the coherence criterion does hold if b_∂ and $\mathring{b}$ arise from a common b.

Completeness of the elimination clauses. We now show that b is fully determined by the b_∂ and $\mathring{b}$ that can be derived from it. Afterwards, we will show that the given coherence condition is sufficient to make sure that b_∂ and $\mathring{b}$ determine some b.

Note that B, being a type in a presheaf CwF, is a presheaf over the category of elements of $\Gamma.\left(\between^{\Psi|}_{\mathrm{y}U} A\right)\left[\mathsf{reidx}^{\Psi|}_{\mathrm{y}U}\right]$. Hence it acts on cells

$$\left(V, \varphi^{V \Rightarrow \Psi \ltimes \mathrm{y}U}, \gamma^{(V,\varphi)\Rightarrow\Gamma}, a^{(V,\varphi,\gamma)\Rightarrow\left(\between^{\Psi|}_{\mathrm{y}U} A\right)\left[\mathsf{reidx}^{\Psi|}_{\mathrm{y}U}\right]}\right).$$

Now we divide such cells in two classes: on-boundary cells (for which $\pi_2 \circ \varphi$ is *not* dimensionally split) and total cells (the others). As $\Omega^{\Psi \ltimes \mathrm{y}U|}_{(\in \partial U)}$ is exactly the restriction of presheaves to the on-boundary cells, it is clear that b_∂ determines the action of b on those.

For total cells, note that the full subcategory of $\mathcal{V}/(\Psi \ltimes \mathrm{y}U)$ consisting of the total elements, is (by theorem 4.1.11) equivalent to $\mathcal{W}/\Psi$, with one direction given by $\dashv^{/\Psi}_U$. Restriction to total cells is then given by the central lifting of that functor, being $\forall^{\Psi|}_{\mathrm{y}U}$. Combined with the knowledge that $\forall^{\Psi|}_{\mathrm{y}U}\between^{\Psi|}_{\mathrm{y}U} \cong \mathrm{Id}$ (theorem 4.1.10), this reveals that $\mathring{b}$ determines the action of b on total cells.

Completeness of the coherence criterion. The action of a term on cells should be natural with respect to restriction. This is automatic when considered with respect to morphisms between cells that are either both total or both on-boundary. Moreover, there are no morphisms $\chi : (V, \varphi) \to (V', \varphi') : \mathcal{V}/(\Psi \ltimes U)$ from a total cell to an on-boundary cell, since proposition 3.4.9 asserts that if $\pi_2 \circ \varphi' \circ \chi$ is dimensionally split, then so is $\pi_2 \circ \varphi'$. So we still need to prove naturality w.r.t. morphisms from on-boundary cells to total cells.

Let $\chi : (V, \varphi) \to (V', \varphi')$ be such a morphism. Then $(V', \varphi') \cong_\iota \dashv^{/\Psi}_U (W, \psi) \cong \dashv^{/\Psi}_U \exists^{/\Psi}_U \dashv^{/\Psi}_U (W, \psi) \cong \dashv^{/\Psi}_U \exists^{/\Psi}_U (V', \varphi')$ by an isomorphism

$$\begin{aligned}
&\dashv^{/\Psi}_U \exists^{/\Psi}_U \iota^{-1} \circ \dashv^{/\Psi}_U (\mathsf{drop}^{/\Psi}_U)^{-1} \circ \iota \\
&= \dashv^{/\Psi}_U \exists^{/\Psi}_U \iota^{-1} \circ \mathsf{copy}^{/\Psi}_U \circ \iota \\
&= \mathsf{copy}^{/\Psi}_U \circ \iota^{-1} \circ \iota = \mathsf{copy}^{/\Psi}_U .
\end{aligned}$$

Hence, by naturality, $\chi = (\mathsf{copy}^{/\Psi}_U)^{-1} \circ \mathsf{copy}^{/\Psi}_U \circ \chi = (\mathsf{copy}^{/\Psi}_U)^{-1} \circ \dashv^{/\Psi}_U \exists^{/\Psi}_U \chi \circ \mathsf{copy}^{/\Psi}_U$. Thus, we have factored χ as an instance of the unit $\mathsf{copy}^{/\Psi}_U$ followed by a morphism between total cells. This means it is sufficient to show naturality with respect to $\mathsf{copy}^{/\Psi}_U : (V, \varphi) \to \dashv^{/\Psi}_U \exists^{/\Psi}_U (V, \varphi)$. (The cells of Γ and the transpension type available for (V', φ') carry over to $\dashv^{/\Psi}_U \exists^{/\Psi}_U (V, \varphi)$ by restriction.)

Now the action of b on (V, φ) is given by the action of b_∂ on (V, φ). Meanwhile, the action of b on $\dashv^{/\Psi}_U \exists^{/\Psi}_U (V, \varphi)$ is given by the action of $\mathring{b}$ on $\exists^{/\Psi}_U (V, \varphi)$, which is the action of $^{\dashv^{/\Psi}_U} b$ on (V, φ). These have to correspond via $\mathsf{copy}^{/\Psi}_U : (V, \varphi) \to \dashv^{/\Psi}_U \exists^{/\Psi}_U (V, \varphi)$, which corresponds via central lifting to the natural transformation $\mathsf{app}^{\Psi|}_U$ on presheaves. This is exactly what happens in the coherence criterion: we use $\mathsf{app}^{\Psi|}_U : \dashv^{\Psi|}_U \forall^{\Psi|}_U \to \mathrm{Id}$ to bring b_∂ and $\mathring{b}$ to the same context and type, and then equate them. Since b_∂ only exists on the boundary, we also have to restrict $\mathring{b}$ to the boundary, but that's OK since we were interested in an on-boundary cell anyway. $\square$

Example 4.4.7 (Affine cubes)**.** We instantiate theorem 4.4.6 for the multiplier $\sqcup * \mathbb{I} : \square^k \to \square^k$ (example 3.3.3). There, $\partial\mathbb{I}$ is essentially the constant presheaf with k elements. So b_∂ determines the images of the k poles of the transpension type. The term $\mathring{b}$ determines the action on paths (for $k = 2$, for general k perhaps 'webs' is a better term), and the paths/webs of the transpension type are essentially the elements of A. The coherence condition says that the image of such paths/webs should always have the endpoints given by b_∂.

Example 4.4.8 (Clocks)**.** We instantiate theorem 4.4.6 for the multiplier $\sqcup * (i : \text{◷}_k)$ (example 3.3.6), where we adapt the base category to forbid diagonals: a morphism may use every variable of its domain at most once. The boundary $\partial(i : \text{◷}_k)$ is isomorphic to $\mathbf{y}(i : \text{◷}_{k-1})$ if $k > 0$ and to the empty presheaf $\bot$ if $k = 0$. So if we want to eliminate an element of the transpension type over $\mathbf{y}(i : \text{◷}_k)$, which means we have a clock and we don't care about what happens if the time exceeds k, then we need to handle two cases. The first case b_∂ says what happens if we don't even care what happens at timestamp k; in which case the transpension type trivializes. Then, by giving $\mathring{b}$, we say what happens at timestamp k and need to make sure that this is consistent with b_∂. The elements of the transpension type at timestamp k are essentially the elements of A, which are fresh for the clock.

Example 4.4.9 (Signposts)**.** Recall that the functor $(\mathrm{Id}, \top)$ sends $W \in \mathcal{W}$ to $(W, \top)$, the Yoneda-embedding of which represents the arrow $\mathbf{y}W \to \mathbf{y}W$, i.e. $\mathbf{y}W.\text{🪧}.\top$ under the convention that $\Psi.\text{🪧}.\Theta \cong (\Psi.\Theta \to \Psi)$. We will write its left lifting as $\sqcup \ltimes \text{🪧} : \widehat{\mathcal{W}} \to \widehat{\mathcal{W} \times \uparrow}$, and $\text{🪧} := \top \ltimes \text{🪧} \cong \mathbf{y}(\top, \top)$ is the terminal object, so that $\widehat{\mathcal{W} \times \uparrow}/\text{🪧} \cong \widehat{\mathcal{W} \times \uparrow}$. We get 5 adjoint functors, of which we give here the action up to isomorphism:

$$
\begin{array}{ccccccccc}
 & & & & \Psi & \mapsto & \Psi.\text{🪧}.\bot & \text{i.e.} & \bot \to \Psi, \\
 & & \exists_{\text{🪧}} & : & \Psi & \leftarrow\!\shortmid & \Psi.\text{🪧}.\Theta & \text{i.e.} & \Psi.\Theta \to \Psi, \\
\sqcup \ltimes \text{🪧} & \text{or} & \text{⫞}_{\text{🪧}} & : & \Psi & \mapsto & \Psi.\text{🪧}.\top & \text{i.e.} & \Psi \to \Psi, \\
\text{🪧} \multimap \sqcup & \text{or} & \forall_{\text{🪧}} & : & \Psi.\Theta & \leftarrow\!\shortmid & \Psi.\text{🪧}.\Theta & \text{i.e.} & \Psi.\Theta \to \Psi, \\
\text{🪧} \surd \sqcup & \text{or} & \between_{\text{🪧}} & : & \Psi & \mapsto & \top.\text{🪧}.\Psi & \text{i.e.} & \Psi \to \top.
\end{array}
$$

The boundary of $(\top, \top)$ is $\partial(\top, \top) \cong \mathbf{y}(\top, \bot)$ which we could write as $\top.\text{🪧}.\bot$ or as the arrow $\bot \to \top$.

So let us now instantiate theorem 4.4.6, which allows us to eliminate an element of the transpension type, i.e. essentially an element of $A \to \top$. The boundary case exists over the boundary $\bot \to \top$ and allows us to consider only the codomain of the arrow, i.e. the part of the context before the signpost, where A is trivial. The case $\mathring{b}$ then requires us to say how to act on data beyond the signpost in a coherent way with what we already specified in b_∂. The elements beyond the signpost are essentially the elements of A, which are fresh for the signpost.

5 Prior modalities

Many modalities arise as central or right liftings of functors between base categories [NVD17, ND18, Nuy18, BM18]. The following definition allows us to use such modalities even when part of the context is in front of a pipe.

Definition 5.0.1. A functor $G : \mathcal{W} \to \mathcal{W}'$ yields a functor $G^{/\Psi} : \mathcal{W}/\Psi \to \mathcal{W}'/G_!\Psi : (W, \psi) \mapsto (GW, G_!\psi)$. This in turn yields three adjoint functors between presheaf categories:

$$
G_!^{\Psi|} \dashv G^{\Psi|*} \dashv G_*^{\Psi|}. \tag{41}
$$

If a modality is both a right and a central lifting, then the following theorem relates the corresponding 'piped' modalities:

Theorem 5.0.2. If $G : \mathcal{W} \to \mathcal{W}'$ has a right adjoint $G \dashv S$, then we have

$$
\begin{array}{ccccc|ccccc}
 & & \Sigma^{/\varepsilon_!} \circ G^{/S_!\Psi'} & \dashv & S^{/\Psi'} & G^{/\Psi} & \dashv & \Omega^{/\eta_!} \circ S^{/G_!\Psi} & & \\
\hline
 & & \Sigma^{\varepsilon_!|} \circ G_!^{S_!\Psi'|} & \dashv & S_!^{\Psi'|} & G_!^{\Psi|} & \dashv & \Omega^{\eta_!|} \circ S_!^{G_!\Psi|} & \cong & G^{\Psi|*} \\
S_!^{\Psi'|} & \cong & G^{S_!\Psi'|*} \circ \Omega^{\varepsilon_!|} & \dashv & S^{\Psi'|*} & G^{\Psi|*} & \dashv & S^{G_!\Psi|*} \circ \Pi^{\eta_!|} & \cong & G_*^{\Psi|} \\
S^{\Psi'|*} & \cong & \Pi^{\varepsilon_!|} \circ G_*^{S_!\Psi'|} & \dashv & S_*^{\Psi'|} & G_*^{\Psi|} & \dashv & \between^{\eta_!|} \circ S_*^{G_!\Psi|} & &
\end{array} \tag{42}
$$

assuming – where mentioned – that $\Omega^{/\eta_!}$ and $\Omega^{/\eta_!}$ exist.

Proof. For the left half of the table, we only prove the first line. The other adjunctions follow from the fact that $\sqcup_!$, $\sqcup^*$ and $\sqcup_*$ are pseudofunctors, and the isomorphisms follow from uniqueness of the adjoint. We have a correspondence of diagrams

$$
\begin{tikzcd}
W \arrow[rr] \arrow[dr, Rightarrow, "\psi"'] & & SW' \arrow[dl, Rightarrow, "S_!\psi'"] \\
& S_!\Psi' &
\end{tikzcd}
\qquad
\begin{tikzcd}
GW \arrow[rr] \arrow[dr, Rightarrow, "\varepsilon_! \circ G_!\psi"'] & & W' \arrow[dl, Rightarrow, "\psi'"] \\
& \Psi' &
\end{tikzcd}
\tag{43}
$$

i.e. morphisms $(W, \psi) \to S^{/\Psi'}(W', \psi') : \mathcal{W}/S_!\Psi'$ correspond to morphisms $\Sigma^{/\varepsilon_!}G^{/S_!\Psi'}(W, \psi) \to (W', \psi') : \mathcal{W}'/\Psi'$.

On the right side of the table, we can prove all adjunctions from the corresponding adjunction on the left. Again, the isomorphisms then follow from uniqueness of the adjoint. We illustrate this only for the first line. The left adjoint to $\Omega^{/\eta_!} \circ S^{/G_!\Psi}$ is $\left(\Sigma^{/\varepsilon_!} \circ G^{/S_!G_!\Psi}\right) \circ \Sigma^{/\eta_!}$. We prove that this is equal to $G^{/\Psi}$:

$$
\begin{aligned}
&\Sigma^{/\varepsilon_!}G^{/S_!G_!\Psi}\Sigma^{/\eta_!}(W, \psi : W \to \Psi) \\
&= \Sigma^{/\varepsilon_!}G^{/S_!G_!\Psi}(W, \eta_! \circ \psi : W \to S_!G_!\Psi) \\
&= \Sigma^{/\varepsilon_!}(GW, G_!\eta_! \circ G_!\psi : GW \to G_!S_!G_!\Psi) \\
&= (GW, \varepsilon_! \circ G_!\eta_! \circ G_!\psi : GW \to G_!\Psi) = (GW, G_!\psi : GW \to G_!\Psi).
\end{aligned}
$$

For the other lines, the required equation follows from the above one by left/central/right lifting. □

6 Commutation rules

6.1 Modality and substitution

Theorem 6.1.1. Assume a functor $G : \mathcal{W} \to \mathcal{W}'$ and a morphism $\sigma : \Psi_1 \to \Psi_2 : \widehat{\mathcal{W}}$. Then we have a commutative diagram

$$
\begin{tikzcd}
\mathcal{W}/\Psi_1 \arrow[r, "G^{/\Psi_1}"] \arrow[d, "\Sigma^{/\sigma}"'] & \mathcal{W}'/G_!\Psi_1 \arrow[d, "\Sigma^{/G_!\sigma}"] \\
\mathcal{W}/\Psi_2 \arrow[r, "G^{/\Psi_2}"'] & \mathcal{W}'/G_!\Psi_2
\end{tikzcd}
\tag{44}
$$

and hence

	$G_!$	G^*	G_*
Σ	$\Sigma^{G_!\sigma}\lvert G_!^{\Psi_1}\rvert \cong G_!^{\Psi_2}\lvert\Sigma^{\sigma}\rvert$	$\Sigma^{\sigma}\lvert G^{\Psi_1}\rvert^* \to G^{\Psi_2}\lvert^*\Sigma^{G_!\sigma}\rvert$	
Ω	$\Omega^{G_!\sigma}\lvert G_!^{\Psi_2}\rvert \leftarrow G_!^{\Psi_1}\lvert\Omega^{\sigma}\rvert$	$\Omega^{\sigma}\lvert G^{\Psi_2}\rvert^* \cong G^{\Psi_1}\lvert^*\Omega^{G_!\sigma}\rvert$	$\Omega^{G_!\sigma}\lvert G_*^{\Psi_2}\rvert \to G_*^{\Psi_1}\lvert\Omega^{\sigma}\rvert$
Π		$\Pi^{\sigma}\lvert G^{\Psi_1}\rvert^* \leftarrow G^{\Psi_2}\lvert^*\Pi^{G_!\sigma}\rvert$	$\Pi^{G_!\sigma}\lvert G_*^{\Psi_1}\rvert \cong G_*^{\Psi_2}\lvert\Pi^{\sigma}\rvert$
$\between$			$\between^{G_!\sigma}\lvert G_*^{\Psi_2}\rvert \leftarrow G_*^{\Psi_1}\lvert\between^{\sigma}\rvert$

where every statement holds if the mentioned functors exist.

Proof. It is evident from the definitions that the given diagram commutes. Then by applying $\sqcup_!$, we find the that $\Sigma^{G_!\sigma}\lvert G_!^{\Psi_1}\rvert \cong G_!^{\Psi_2}\lvert\Sigma^{\sigma}\rvert$. The rest of the table then follows by lemma 2.1.2. □

Remark 6.1.2.

- If $\sigma = \pi : \Psi.A \to \Psi$, then this says something about weakening and the Σ- and Π-types over A.
- If $G_!$ moreover happens to be a CwF morphism, then this relates weakening and the Σ- and Π-types over A to those over $G_!A$.
- If $\sqcup \times U$ is a cartesian multiplier and we take $\sigma = \pi_1 : \Psi \times \mathrm{y}U \to \Psi$, then by theorem 4.1.10, this says something about $\exists_{\mathrm{y}U}^{\Psi\rvert} \dashv \exists_{\mathrm{y}U}^{\Psi\rvert} \dashv \forall_{\mathrm{y}U}^{\Psi\rvert} \dashv \between_{\mathrm{y}U}^{\Psi\rvert}$.

6.2 Multiplier and substitution

If, in section 6.1, we take G equal to some multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$, then we have

$$G^{/\Psi} = Ⅎ_G^{/\Psi}, \qquad G_! = \sqcup \ltimes \mathbf{y}U, \qquad G_!^{\Psi|} = Ⅎ_U^{\Psi|}, \qquad G^{\Psi|*} = \forall_U^{\Psi|}, \qquad G_*^{\Psi|} = \between_U^{\Psi|}. \tag{45}$$

This immediately yields two theorems:

Theorem 6.2.1. Assume a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ and a morphism $\sigma : \Psi_1 \to \Psi_2 : \widehat{\mathcal{W}}$. Write $\tau = \sigma \ltimes \mathbf{y}U$. Then we have:

	$\exists$	$Ⅎ$	$\forall$	$\between$
Σ	$\Sigma^{\sigma\|}\exists_U^{\Psi_1\|} \lhd^1 \exists_U^{\Psi_2\|}\Sigma^{\tau\|}$	$\Sigma^{\tau\|}Ⅎ_U^{\Psi_1\|} \cong Ⅎ_U^{\Psi_2\|}\Sigma^{\sigma\|}$	$\Sigma^{\sigma\|}\forall_U^{\Psi_1\|} \rhd_1 \forall_U^{\Psi_2\|}\Sigma^{\tau\|}$	$\Sigma^{\tau\|}\between_U^{\Psi_1\|} \rhd_2 \between_U^{\Psi_2\|}\Sigma^{\sigma\|}$
Ω	$\Omega^{\sigma\|}\exists_U^{\Psi_2\|} \lhd^2 \exists_U^{\Psi_1\|}\Omega^{\tau\|}$	$\Omega^{\tau\|}Ⅎ_U^{\Psi_2\|} \lhd^1 Ⅎ_U^{\Psi_1\|}\Omega^{\sigma\|}$	$\Omega^{\sigma\|}\forall_U^{\Psi_2\|} \cong \forall_U^{\Psi_1\|}\Omega^{\tau\|}$	$\Omega^{\tau\|}\between_U^{\Psi_2\|} \rhd_1 \between_U^{\Psi_1\|}\Omega^{\sigma\|}$
Π		$\Pi^{\tau\|}Ⅎ_U^{\Psi_1\|} \lhd^2 Ⅎ_U^{\Psi_2\|}\Pi^{\sigma\|}$	$\Pi^{\sigma\|}\forall_U^{\Psi_1\|} \lhd^1 \forall_U^{\Psi_2\|}\Pi^{\tau\|}$	$\Pi^{\tau\|}\between_U^{\Psi_1\|} \cong \between_U^{\Psi_2\|}\Pi^{\sigma\|}$
$\between$			$\between^{\sigma\|}\forall_U^{\Psi_2\|} \lhd^2 \forall_U^{\Psi_1\|}\between^{\tau\|}$	$\between^{\tau\|}\between_U^{\Psi_2\|} \lhd^1 \between_U^{\Psi_1\|}\between^{\sigma\|}$

(46)

where every statement holds if the mentioned functors exist, and where

1. In general, $\lhd^1$ means $\leftarrow$, $\rhd_1$ means $\to$ and the other symbols mean nothing.
2. If $\sqcup \ltimes U$ is quantifiable, then $\lhd^1$ upgades to $\cong$ and $\lhd^2$ upgrades to $\leftarrow$.
3. If $\sqcup \ltimes U$ is cancellative and affine, then we have
$$\Sigma^{\sigma|}\forall_U^{\Psi_1|} \cong \forall_U^{\Psi_2|}\Sigma^{\tau|} : \widehat{\overbrace{\mathcal{V}/(\Psi_1 \ltimes \mathbf{y}U)}} \to \widehat{\mathcal{W}/\Psi_2} \tag{47}$$
so that $\rhd_1$ upgrades to $\cong$ and $\rhd_2$ upgrades to $\to$.

Proof. 1. The general case is a corollary of theorem 6.1.1 for $G = \sqcup \ltimes U$.

2. To prove the quantifiable case, we show in the base category that $\Sigma^{/\sigma}\exists_U^{/\Psi_1} = \exists_U^{/\Psi_2}\Sigma^{/(\sigma\ltimes\mathbf{y}U)}$. We use the construction of $\exists_U^{/\Psi}$ in the proof of provident quantifiability (proposition 4.1.8). On one hand, we have:
$$\Sigma^{/\sigma}\exists_U^{/\Psi_1}(V, (\psi_1^{W_0\Rightarrow\Psi_1} \ltimes \mathbf{y}U) \circ \varphi^{V\Rightarrow W_0\ltimes U}) = \Sigma^{/\sigma}\Sigma^{/\psi_1}\exists_U^{/W_0}(V,\varphi) = \Sigma^{/\sigma\circ\psi_1}\exists_U^{/W_0}(V,\varphi).$$
On the other hand:
$$\exists_U^{/\Psi_2}\Sigma^{/(\sigma\ltimes\mathbf{y}U)}(V, (\psi_1^{W_0\Rightarrow\Psi_1} \ltimes \mathbf{y}U) \circ \varphi^{V\Rightarrow W_0\ltimes U}) = \exists_U^{/\Psi_2}(V, ((\sigma\circ\psi_1) \ltimes \mathbf{y}U) \circ \varphi) = \Sigma^{/\sigma\circ\psi_1}\exists_U^{/W_0}(V,\varphi).$$

3. We show that $\Sigma^{\sigma|}\forall_U^{\Psi_1|} \cong \forall_U^{\Psi_2|}\Sigma^{\tau|}$. Pick a presheaf Γ over $\mathcal{V}/(\Psi_1 \ltimes \mathbf{y}U)$. On the one hand, we have:
$$\begin{aligned}
&(W_2, \psi_2^{W_2\Rightarrow\Psi_2}) \Rightarrow \Sigma^{\sigma|}\forall_{\mathbf{y}U}^{\Psi_1|}\Gamma \\
&= \exists(W_1, \psi_1^{W_1\Rightarrow\Psi_1}).\left(\theta : (W_2,\psi_2) \to \Sigma^{/\sigma}(W_1,\psi_1)\right) \times \left((W_1,\psi_1) \Rightarrow \forall_{\mathbf{y}U}^{\Psi_1|}\Gamma\right) \\
&= \exists(W_1, \psi_1^{W_1\Rightarrow\Psi_1}).(\theta : (W_2,\psi_2) \to (W_1, \sigma\circ\psi_1)) \times ((W_1 \ltimes U, \psi_1 \ltimes \mathbf{y}U) \Rightarrow \Gamma) \\
&\cong \exists W_1, \psi_1^{W_1\Rightarrow\Psi_1}, \theta^{W_2\to W_1}.(\psi_2 = \sigma\circ\psi_1\circ\theta) \times ((W_1 \ltimes U, \psi_1 \ltimes \mathbf{y}U) \Rightarrow \Gamma) \\
&\qquad \text{We now absorb } \theta \text{ into } \psi_1\text{:} \\
&\cong \psi_1^{W_2\Rightarrow\Psi_1}.(\psi_2 = \sigma\circ\psi_1) \times ((W_2 \ltimes U, \psi_1 \ltimes \mathbf{y}U) \Rightarrow \Gamma)\,.
\end{aligned}$$
On the other hand, we have:
$$(W_2, \psi_2^{W_2\Rightarrow\Psi_2}) \Rightarrow \forall_{\mathbf{y}U}^{\Psi_2|}\Sigma^{\tau|}\Gamma$$

$= (W_2 \ltimes U, \psi_2 \ltimes \mathbf{y}U) \Rightarrow \Sigma^{\tau|}\Gamma$
$= \exists(V_1, \varphi_1^{V_1 \Rightarrow \Psi_1 \ltimes \mathbf{y}U}). \Big(\omega : (W_2 \ltimes U, \psi_2 \ltimes \mathbf{y}U) \to \Sigma^{/\tau}(V_1, \varphi_1)\Big) \times ((V_1, \varphi_1) \Rightarrow \Gamma)$
$= \exists(V_1, \varphi_1^{V_1 \Rightarrow \Psi_1 \ltimes \mathbf{y}U}). (\omega : (W_2 \ltimes U, \psi_2 \ltimes \mathbf{y}U) \to (V_1, (\sigma \ltimes \mathbf{y}U) \circ \varphi_1)) \times ((V_1, \varphi_1) \Rightarrow \Gamma)$

We now deconstruct $\varphi_1 = (\psi_1 \ltimes \mathbf{y}U) \circ \chi$:

$\cong \exists V_1, W_1, \chi^{V_1 \to W_1 \ltimes U}, \psi_1^{W_1 \Rightarrow \Psi_1}.$
$(\omega : (W_2 \ltimes U, \psi_2 \ltimes \mathbf{y}U) \to (V_1, ((\sigma \circ \psi_1) \ltimes \mathbf{y}U) \circ \chi)) \times ((V_1, (\psi_1 \ltimes \mathbf{y}U) \circ \chi) \Rightarrow \Gamma)$
$\cong \exists V_1, W_1, \chi^{V_1 \to W_1 \ltimes U}, \psi_1^{W_1 \Rightarrow \Psi_1}, \omega^{W_2 \ltimes U \to V_1}.$
$(\psi_2 \ltimes \mathbf{y}U = ((\sigma \circ \psi_1) \ltimes \mathbf{y}U) \circ \chi \circ \omega) \times ((V_1, (\psi_1 \ltimes \mathbf{y}U) \circ \chi) \Rightarrow \Gamma)$

We now absorb ω into χ:

$\cong \exists W_1, \psi_1^{W_1 \Rightarrow \Psi_1}, \chi^{W_2 \ltimes U \to W_1 \ltimes U}.(\psi_2 \ltimes \mathbf{y}U = ((\sigma \circ \psi_1) \ltimes \mathbf{y}U) \circ \chi) \times ((W_2 \ltimes U, (\psi_1 \ltimes \mathbf{y}U) \circ \chi) \Rightarrow \Gamma)$

Let $\chi = \reflectbox{F}_U^{/\Psi_2}\theta : \reflectbox{F}_U^{/\Psi_2}(W_2, \psi_2) \to \reflectbox{F}_U^{/\Psi_2}(W_1, \sigma \circ \psi_1)$:

$\cong \exists W_1, \psi_1^{W_1 \Rightarrow \Psi_1}, \theta^{W_2 \to W_1}.(\psi_2 = \sigma \circ \psi_1 \circ \theta) \times ((W_2 \ltimes U, ((\psi_1 \circ \theta) \ltimes \mathbf{y}U)) \Rightarrow \Gamma)$

We now absorb θ into ψ_1:

$\cong \psi_1^{W_2 \Rightarrow \Psi_1}.(\psi_2 = \sigma \circ \psi_1) \times ((W_2 \ltimes U, (\psi_1 \ltimes \mathbf{y}U)) \Rightarrow \Gamma)$

This proves the isomorphism. The rest follows from lemma 2.1.2. □

6.3 Multiplier and modality

Theorem 6.3.1. Assume a commutative diagram (up to natural isomorphism ν)

$$\begin{tikzcd} \mathcal{W} \arrow[r, "G"] \arrow[d, "\sqcup \ltimes U"'] & \mathcal{W}' \arrow[d, "\sqcup \ltimes U'"] \\ \mathcal{V} \arrow[r, "F"'] & \mathcal{V}' \end{tikzcd} \tag{48}$$

where $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are multipliers for U and U', and G preserves the terminal object so that $U' \cong FU$.

Write δ for the isomorphism $\delta : F_!(\Psi \ltimes \mathbf{y}U) \cong G_!\Psi \ltimes \mathbf{y}U'$. Then $\Sigma^{/\delta}$ is a strictly invertible functor and hence we have

$$\Sigma^{\delta|} \cong \Omega^{\delta^{-1}|} \cong \Pi^{\delta|} \cong \between^{\delta^{-1}|} \qquad \Sigma^{\delta^{-1}|} \cong \Omega^{\delta|} \cong \Pi^{\delta^{-1}|} \cong \between^{\delta|}, \tag{49}$$

where $\Omega^{\delta^{-1}|}$ is the strict inverse to $\Omega^{\delta|}$.

Then we have $\Sigma^{/\delta^{-1}} \reflectbox{F}_{U'}^{/G_!\Psi} G^{/\Psi} \cong F^{/\Psi \ltimes U} \reflectbox{F}_U^{/\Psi}$. This yields the following commutation table:

	$F_!, G_!$	F^*, G^*	F_*, G_*
$\exists$	$\exists_{U'}^{G_!\Psi\|}\Omega^{\delta^{-1}\|}F_!^{\Psi \ltimes \mathbf{y}U\|} \rhd_1 G_!^{\Psi\|}\exists_U^{\Psi\|}$	$\exists_U^{\Psi\|}F^{\Psi\|*} \rhd_2 G^{\Psi\|*}\exists_{U'}^{G_!\Psi\|}\Omega^{\delta^{-1}\|}$	
$\reflectbox{F}$	$\Omega^{\delta\|}\reflectbox{F}_{U'}^{G_!\Psi\|}G_!^{\Psi\|} \cong F_!^{\Psi \ltimes \mathbf{y}U\|}\reflectbox{F}_U^{\Psi\|}$	$\reflectbox{F}_U^{\Psi\|}G^{\Psi\|*} \rhd_1 F^{\Psi \ltimes \mathbf{y}U\|*}\Omega^{\delta\|}\reflectbox{F}_{U'}^{G_!\Psi\|}$	$\Omega^{\delta\|}\reflectbox{F}_{U'}^{G_!\Psi\|}G_*^{\Psi\|} \rhd_2 F_*^{\Psi \ltimes \mathbf{y}U\|}\reflectbox{F}_U^{\Psi\|}$
$\forall$	$\forall_{U'}^{G_!\Psi\|}\Omega^{\delta^{-1}\|}F_!^{\Psi \ltimes \mathbf{y}U\|} \leftarrow G_!^{\Psi\|}\forall_U^{\Psi\|}$	$\forall_U^{\Psi\|}F^{\Psi\|*} \cong G^{\Psi\|*}\forall_{U'}^{G_!\Psi\|}\Omega^{\delta^{-1}\|}$	$\forall_{U'}^{G_!\Psi\|}\Omega^{\delta^{-1}\|}F_*^{\Psi \ltimes \mathbf{y}U\|} \rhd_1 G_*^{\Psi\|}\forall_U^{\Psi\|}$
$\between$		$\between_U^{\Psi\|}G^{\Psi\|*} \leftarrow F^{\Psi \ltimes \mathbf{y}U\|*}\Omega^{\delta\|}\between_{U'}^{G_!\Psi\|}$	$\Omega^{\delta\|}\between_{U'}^{G_!\Psi\|}G_*^{\Psi\|} \cong F_*^{\Psi \ltimes \mathbf{y}U\|}\between_U^{\Psi\|}$

where any statement holds if the mentioned functors exist, and where

1. In general, $\rhd_1$ means $\longrightarrow$ and $\rhd_2$ means nothing.
2. If $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are quantifiable and the morphism $\theta : \exists_{U'} \circ F^{/U} \to G \circ \exists_U : \mathcal{V}/U \to \mathcal{W}'$ is invertible,[6] then $\rhd_1$ upgrades to $\cong$ and $\rhd_2$ upgrades to $\longrightarrow$.

[6] This is a slight abuse of notation, as we know that $U' \cong FU$ but not that $U' = FU$.

This applies in particular if $\mathcal{W} = \mathcal{V}$, $\mathcal{W}' = \mathcal{V}'$, $G = F$ preserves finite products and the multipliers are cartesian.

Proof. The isomorphism δ is obtained by applying $\sqcup_!$ to $\nu : F(\sqcup \ltimes U) \to G\sqcup \ltimes U'$ and then applying the resulting natural transformation to Ψ. Since δ is an isomorphism, $\Sigma^{/\delta}$ is a strictly invertible functor with inverse $\Sigma^{/\delta^{-1}}$. Since $\sqcup^*$ is a 2-functor, $\Omega^{\delta|}$ is also strictly invertible with inverse $\Omega^{\delta^{-1}|}$. Because equivalences of categories are adjoint to their inverse, we get the chains of isomorphisms displayed.

1. The given commutation property in the base category follows immediately from the definitions and naturality of ν and its image under $\sqcup_!$. The rest of the table then follows by lemma 2.1.2.

2. We show in the base category that $\exists_{U'}^{/G_!\Psi}\Sigma^{/\delta}F^{/\Psi\ltimes \mathbf{y}U} \cong G^{/\Psi}\exists_U^{/\Psi} : \mathcal{V}/(\Psi \ltimes \mathbf{y}U) \to \mathcal{W}'/G_!\Psi$. Without loss of generality we assume that $FU = U'$. Pick some $(V, (\psi^{W_0 \Rightarrow \Psi} \ltimes \mathbf{y}U) \circ \varphi_0^{V \to W_0 \ltimes U}) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$.

   On one hand, we have:

$$
\begin{aligned}
&\exists_{U'}^{/G_!\Psi}\Sigma^{/\nu_!}F^{/\Psi\ltimes \mathbf{y}U}(V, (\psi \ltimes \mathbf{y}U) \circ \varphi_0) \\
&= \exists_{U'}^{/G_!\Psi}\Sigma^{/\nu_!}(FV, F_!(\psi \ltimes \mathbf{y}U) \circ F\varphi_0) \\
&= \exists_{U'}^{/G_!\Psi}(FV, \nu_! \circ F_!(\psi \ltimes \mathbf{y}U) \circ F\varphi_0) \\
&= \exists_{U'}^{/G_!\Psi}(FV, (G_!\psi \ltimes \mathbf{y}U') \circ \nu \circ F\varphi_0) \\
&= \Sigma^{/G_!\psi}\exists_{U'}^{/GW_0}(FV, \nu \circ F\varphi_0) \\
&= \Sigma^{/G_!\psi}(\exists_{U'}(FV, \pi_2 \circ \nu \circ F\varphi_0), \mathsf{drop}_{U'} \circ \exists_{U'}(\nu \circ F\varphi_0)) \\
&= \Sigma^{/G_!\psi}(\exists_{U'}(FV, \pi_2 \circ F\varphi_0), \mathsf{drop}_{U'} \circ \exists_{U'}(\nu \circ F\varphi_0)).
\end{aligned}
$$

   On the other hand:

$$
\begin{aligned}
&G^{/\Psi}\exists_U^{/\Psi}(V, (\psi \ltimes \mathbf{y}U) \circ \varphi_0) \\
&= G^{/\Psi}\Sigma^{/\psi}\exists_U^{/W_0}(V, \varphi_0) \\
&= \Sigma^{/G_!\psi}G^{/W_0}\exists_U^{/W_0}(V, \varphi_0) \\
&= \Sigma^{/G_!\psi}G^{/W_0}(\exists_U(V, \pi_2 \circ \varphi_0), \mathsf{drop}_U \circ \exists_U\varphi_0) \\
&= \Sigma^{/G_!\psi}(G\exists_U(V, \pi_2 \circ \varphi_0), G\mathsf{drop}_U \circ G\exists_U\varphi_0) \\
&\cong \Sigma^{/G_!\psi}(\exists_{U'}F^{/U}(V, \pi_2 \circ \varphi_0), G\mathsf{drop}_U \circ G\exists_U\varphi_0 \circ \theta) \\
&= \Sigma^{/G_!\psi}(\exists_{U'}(FV, \pi_2 \circ F\varphi_0), G\mathsf{drop}_U \circ \theta \circ \exists_{U'}F\varphi_0)
\end{aligned}
$$

   It remains to be shown that $G\mathsf{drop}_U \circ \theta = \mathsf{drop}_{U'} \circ \exists_{U'}\nu : \exists_{U'}F^{/U}\reflectbox{$\exists$}_U \to G$.

   But *the* morphism θ, which we have assumed to be an isomorphism, arises from ν as follows:

$$
\theta : \exists_{U'}F^{/U} \xrightarrow{\exists_{U'}F^{/U}\mathsf{copy}_U} \exists_{U'}F^{/U}\reflectbox{$\exists$}_U\exists_U \xrightarrow{\exists_{U'}\nu\exists_U} \exists_{U'}\reflectbox{$\exists$}_{U'}G\exists_U \xrightarrow{\mathsf{drop}_{U'}G\exists_U} G\exists_U.
$$

   Then the following diagram does commute:

$$
\begin{tikzcd}
\exists_{U'}F^{/U}\reflectbox{$\exists$}_U \arrow[r, "\exists_{U'}F^{/U}\mathsf{copy}_U\reflectbox{$\exists$}_U"] \arrow[dr, "\mathrm{id}"'] \arrow[rrr, bend left, "\theta\reflectbox{$\exists$}_U"] & \exists_{U'}F^{/U}\reflectbox{$\exists$}_U\exists_U\reflectbox{$\exists$}_U \arrow[r, "\exists_{U'}\nu\exists_U\reflectbox{$\exists$}_U"] \arrow[d, "\exists_{U'}F^{/U}\reflectbox{$\exists$}_U\mathsf{drop}_U"'] & \exists_{U'}\reflectbox{$\exists$}_{U'}G\exists_U\reflectbox{$\exists$}_U \arrow[r, "\mathsf{drop}_{U'}G\exists_U\reflectbox{$\exists$}_U"] \arrow[d, "\exists_{U'}\reflectbox{$\exists$}_{U'}G\mathsf{drop}_U"] & G\exists_U\reflectbox{$\exists$}_U \arrow[d, "G\mathsf{drop}_U"] \\
& \exists_{U'}F^{/U}\reflectbox{$\exists$}_U \arrow[r, "\exists_{U'}\nu"'] & \exists_{U'}\reflectbox{$\exists$}_{U'}G \arrow[r, "\mathsf{drop}_{U'}G"'] & G
\end{tikzcd}
$$

□

Remark 6.3.2. In the above theorem, we think of F and G as similar functors; if we are dealing with endomultipliers, we will typically take $F = G$. The multipliers, however, will typically be different, as in general $U \not\cong FU$.

6.4 Multiplier and multiplier

Theorem 6.4.1. Assume we have a commutative diagram (up to natural isomorphism ν) of multipliers

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "\sqcup \ltimes U"] \arrow[d, "\sqcup \ltimes I"'] & \mathcal{V} \arrow[d, "\sqcup \ltimes I'"] \\
\mathcal{W}' \arrow[r, "\sqcup \ltimes U'"'] & \mathcal{V}'.
\end{tikzcd}$$

Write $\sigma : \Psi \ltimes \mathbf{y}I \ltimes \mathbf{y}U' \cong \Psi \ltimes \mathbf{y}U \ltimes \mathbf{y}I'$. Then we have the following commutation table:

	$\exists_I$	$\dashv_I$	$\forall_I$	$\between_I$
$\exists_U$	$\exists^{\Psi\|}_{\mathbf{y}U} \exists^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \cong \exists^{\Psi\|}_{\mathbf{y}I} \exists^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \Omega^{\sigma\|}$	$\exists^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \Omega^{\sigma\|} \dashv^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \rhd_1 \dashv^{\Psi\|}_{\mathbf{y}I} \exists^{\Psi\|}_{\mathbf{y}U}$	$\exists^{\Psi\|}_{\mathbf{y}U} \forall^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \rhd_2 \forall^{\Psi\|}_{\mathbf{y}I} \exists^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \Omega^{\sigma\|}$	$\exists^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \Omega^{\sigma\|} \between^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \rhd_3 \between^{\Psi\|}_{\mathbf{y}I} \exists^{\Psi\|}_{\mathbf{y}U}$
$\dashv_U$	$\dashv^{\Psi\|}_{\mathbf{y}U} \exists^{\Psi\|}_{\mathbf{y}I} \lhd^1 \exists^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \Omega^{\sigma^{-1}\|} \dashv^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'}$	$\Omega^{\sigma^{-1}\|} \dashv^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \dashv^{\Psi\|}_{\mathbf{y}I} \cong \dashv^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \dashv^{\Psi\|}_{\mathbf{y}U}$	$\dashv^{\Psi\|}_{\mathbf{y}U} \forall^{\Psi\|}_{\mathbf{y}I} \rhd_1 \forall^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \Omega^{\sigma^{-1}\|} \dashv^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'}$	$\Omega^{\sigma^{-1}\|} \dashv^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \between^{\Psi\|}_{\mathbf{y}I} \rhd_2 \between^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \dashv^{\Psi\|}_{\mathbf{y}U}$
$\forall_U$	$\forall^{\Psi\|}_{\mathbf{y}U} \exists^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \lhd^2 \exists^{\Psi\|}_{\mathbf{y}I} \forall^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \Omega^{\sigma\|}$	$\forall^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \Omega^{\sigma\|} \dashv^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \lhd^1 \dashv^{\Psi\|}_{\mathbf{y}I} \forall^{\Psi\|}_{\mathbf{y}U}$	$\forall^{\Psi\|}_{\mathbf{y}U} \forall^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \cong \forall^{\Psi\|}_{\mathbf{y}I} \forall^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \Omega^{\sigma\|}$	$\forall^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \Omega^{\sigma\|} \between^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \rhd_1 \between^{\Psi\|}_{\mathbf{y}I} \forall^{\Psi\|}_{\mathbf{y}U}$
$\between_U$	$\between^{\Psi\|}_{\mathbf{y}U} \exists^{\Psi\|}_{\mathbf{y}I} \lhd^3 \exists^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \Omega^{\sigma^{-1}\|} \between^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'}$	$\Omega^{\sigma^{-1}\|} \between^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \dashv^{\Psi\|}_{\mathbf{y}I} \lhd^2 \dashv^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \between^{\Psi\|}_{\mathbf{y}U}$	$\between^{\Psi\|}_{\mathbf{y}U} \forall^{\Psi\|}_{\mathbf{y}I} \lhd^1 \forall^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \Omega^{\sigma^{-1}\|} \between^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'}$	$\Omega^{\sigma^{-1}\|} \between^{\Psi \ltimes \mathbf{y}I\|}_{\mathbf{y}U'} \between^{\Psi\|}_{\mathbf{y}I} \cong \between^{\Psi \ltimes \mathbf{y}U\|}_{\mathbf{y}I'} \between^{\Psi\|}_{\mathbf{y}U}$

where every statement hold if the mentioned functors exist, and where

1. In general, $\rhd_1$ means $\to$, $\lhd^1$ means $\leftarrow$ and the other symbols mean nothing.
2. If $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are quantifiable and the morphism $\theta : \exists^{/I}_{U'} \circ \dashv^{/U}_{I'} \to \dashv_I \circ \exists_U : \mathcal{V}/U \to \mathcal{W}'$ is invertible,[7] then $\rhd_1$ upgrades to $\cong$ and $\rhd_2$ upgrades to $\to$.
3. If $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are cartesian and $\sqcup \ltimes I = \sqcup \ltimes I'$ preserves pullbacks, then $\rhd_1$ upgrades to $\cong$ and $\rhd_2$ upgrades to $\to$.
   (a) If $\sqcup \ltimes I$ and $\sqcup \ltimes I'$ are moreover affine and cancellative, then $\rhd_2$ upgrades to $\cong$ and $\rhd_3$ upgrades to $\to$.
4. The symbols $\lhd^i$ upgrade under symmetric conditions.

Proof. 1. In the base category, it is clear that $\Sigma^{/\sigma} \dashv^{/\Psi \ltimes \mathbf{y}I}_{U'} \dashv^{/\Psi}_{I} \cong \dashv^{/\Psi \ltimes \mathbf{y}U}_{I'} \dashv^{/\Psi}_{U}$. Applying the 2-functor $\sqcup^*$ yields the commutation law for $\forall$ and hence, by lemma 2.1.2, the general case.

2. We want to invoke theorem 6.3.1 with $G = \sqcup \ltimes I : \mathcal{W} \to \mathcal{W}'$ and $F = \sqcup \ltimes I' : \mathcal{V} \to \mathcal{V}'$. However, this is not possible, as we do not know that $\sqcup \ltimes I$ preserves the terminal object. Instead, we take $G = \dashv_I : \mathcal{W} \to \mathcal{W}'/I$ and $F = \dashv_{I'} : \mathcal{V} \to \mathcal{V}'/I'$ which do preserve the terminal object. Instead of $\sqcup \ltimes U'$ we pass

$$\sqcup \ltimes \dashv_{I'} U : \mathcal{W}'/I \xrightarrow{(\sqcup \ltimes U')^{/I}} \mathcal{V}'/(I \ltimes U') \xrightarrow{\nu} \mathcal{V}'/(U \ltimes I') \xrightarrow{\pi_2} \mathcal{V}'/I'$$

which is a multiplier for $\dashv_{I'} U$ whose $\dashv_{\dashv_{I'} U} : \mathcal{W}'/I \to (\mathcal{V}'/I')/\dashv_{I'} U$ is essentially $\dashv^{/I}_{U'} : \mathcal{W}'/I \to \mathcal{V}'/(I \ltimes U')$ and hence whose $\exists_{\dashv_{I'} U}$ is essentially $\exists^{/I}_{U'}$. Then the property $\exists^{/I}_{U'} \circ \dashv^{/U}_{I'} \cong \dashv_I \circ \exists_U$ guarantees exactly $\exists_{\dashv_{I'} U} \circ (\dashv_{I'})^{/U} \cong \dashv_I \circ \exists_U$, which is the criterion found in theorem 6.3.1.

This adapted invocation of theorem 6.3.1 yields results about other functors than the ones mentioned in the current theorem. However, we have a general isomorphism $\mathcal{Z}/(\mathbf{y}Z.\Xi) \cong (\mathcal{Z}/Z)/\Xi$

[7] This is a slight abuse of notation, as we know that $I \ltimes U' \cong U \ltimes I'$ but not that $I \ltimes U' = U \ltimes I'$.

where $Z \in \mathcal{Z}$ and $\Xi \in \widehat{\mathcal{Z}/Z}$. Moreover, we have $\mathbf{y}I.(\exists_I)_!\Psi \cong \Psi \ltimes \mathbf{y}I$ and $\mathbf{y}I'.(\exists_{I'})_!\Phi \cong \Phi \ltimes \mathbf{y}I'$. Under the resulting strict isomorphism between the categories we want to talk about (such as $\mathcal{W}'/(\Psi \ltimes I)$) and the categories we obtain results about (such as $(\mathcal{W}'/I)/(\exists_I)_!\Psi$), the functors that we want to talk about will be naturally equivalent to those that we obtain results about.

3. This is a special case of the previous point.

   Alternatively, we can invoke theorem 6.2.1 with $\sigma = \pi_1 : \Psi \times U \to \Psi$ and $\tau = \pi_1 \ltimes I = \pi_1 \circ \sigma^{-1} : (\Psi \times U) \ltimes I \to \Psi \ltimes I$ and $\sqcup \ltimes I$ as the multiplier at hand.

   (a) This also follows from theorem 6.2.1.

4. By symmetry. □

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