Every Modality is a Relative Right Adjoint

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EuroProofNet WG6 Meeting Vienna, Austria April 24, 2023 Let $R: \mathscr{C} \to \mathscr{D}$ be a functor.

$$au: \Gamma o \Gamma' @ \mathscr{C}$$

$$\Gamma \vdash T \text{ type } @ \mathscr{C}$$

$$\frac{\tau : \Gamma \to \Gamma' @ \mathscr{C}}{R\tau : R\Gamma \to R\Gamma' @ \mathscr{D}} \qquad \frac{\Gamma \vdash T \text{ type } @ \mathscr{C}}{R\Gamma \vdash RT \text{ type } @ \mathscr{D}} \qquad \frac{\Gamma \vdash t : T @ \mathscr{C}}{R\Gamma \vdash Rt : RT @ \mathscr{D}}$$

$$\frac{?}{\Delta \vdash RT \text{ type}}$$

Let $R: \mathscr{C} \to \mathscr{D}$ be a CwF morphism.

Ok, so how do we check

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$$\triangle \vdash RT$$
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We check $\Gamma \vdash T$ type $@ \mathscr{C}$ and substitute with $\sigma : \Delta \to R\Gamma$

BUT: Don't bother the user. Synthesize Γ and σ

 $\Gamma \in \mathscr{C}$ should be the **universal** context Γ such that $\sigma : \Delta \to R\Gamma$ exists.

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+ some sensible laws $\sim L - R$

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MTT [GKNB21] is parametrized by a 2-category:

- modes *p*, *q*, *r*, . . .
- modalities $\mu : p \rightarrow q$

• (2-cells $\alpha: \mu \Rightarrow v$).

Semantics

- [p] is a (often presheaf) category modelling all of DTT,
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[Shulman, March 2023]

Categorically Adds locks to contexts **cofreely**.

Morally Defines locks by induction on syntactic context formation.
These approximate the left adjoint.

- Subsumes MTT without modifications.
- \Longrightarrow We can still **internally** prove that $\langle \mu \mid \rangle$ preserves limits. This is also assumed in the **model**.

[Shu23, assumption 4.1]

Our solution (WIP):

Categorically Move to **copresheaf** category.

Morally Move to metaprogramming with continuations.

- \bigcirc $\langle \mu \mid \rangle$ does not need to:
 - be a DRA,
 - preserve limits,
 - or even be applicative.
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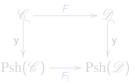
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$$Psh(\mathscr{C}) = [\mathscr{C}^{op}, Set]$$

$$\begin{array}{l} \text{Swap \& curry Hom}: \mathscr{C}^{\text{op}} \times \mathscr{C} \to \text{Set} \\ \text{to get } \textbf{y}: \mathscr{C} \to \text{Psh}(\mathscr{C}): \Gamma \mapsto \text{Hom}(-,\Gamma) \end{array}$$

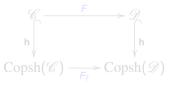
Functor $F : \mathscr{C} \to \mathscr{D}$ yields $F_! \dashv F^* \dashv F_* : \operatorname{Psh}(\mathscr{C}) \to \operatorname{Psh}(\mathscr{D})$ where $F_!$ extends F:



Copresheaves:

$$\operatorname{\mathsf{Copsh}}(\mathscr{C}) = \operatorname{\mathsf{Psh}}(\mathscr{C}^{\operatorname{\mathsf{op}}})^{\operatorname{\mathsf{op}}} = [\mathscr{C}, \operatorname{\mathsf{Set}}]^{\operatorname{\mathsf{op}}}$$

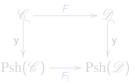
Curry $\operatorname{Hom}^{\operatorname{op}}: \mathscr{C} \times \mathscr{C}^{\operatorname{op}} o \operatorname{Set}^{\operatorname{op}}$ to $\operatorname{get} \mathbf{h}: \mathscr{C} o \operatorname{Copsh}(\mathscr{C}): \Gamma \mapsto \operatorname{Hom}(\Gamma, -)$ sending Γ to its copresheaf of continuations



$$\operatorname{Psh}(\mathscr{C}) = [\mathscr{C}^{\operatorname{op}},\operatorname{\mathsf{Set}}]$$

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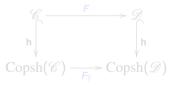
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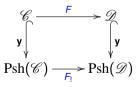
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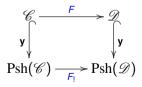
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$$\begin{array}{ccc}
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h & & & \downarrow h \\
\operatorname{Copsh}(\mathscr{C}) & \xrightarrow{F_2} & \operatorname{Copsh}(\mathscr{D})
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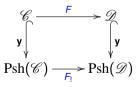
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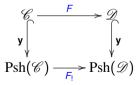
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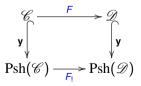
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\mathscr{C} & \xrightarrow{F} & \mathscr{D} \\
h & & & h \\
\text{Copsh}(\mathscr{C}) & \xrightarrow{F_2} & \text{Copsh}(\mathscr{D})
\end{array}$$

$$Psh(\mathscr{C}) = [\mathscr{C}^{op}, Set]$$

Swap & curry $\operatorname{Hom}:\mathscr{C}^{\operatorname{op}}\times\mathscr{C}\to\operatorname{Set}$ to $\operatorname{get}\mathbf{y}:\mathscr{C}\to\operatorname{Psh}(\mathscr{C}):\Gamma\mapsto\operatorname{Hom}(-,\Gamma)$

Functor $F : \mathscr{C} \to \mathscr{D}$ yields $F_! \dashv F^* \dashv F_* : Psh(\mathscr{C}) \to Psh(\mathscr{D})$ where $F_!$ extends F:



Copresheaves:

$$\operatorname{Copsh}(\mathscr{C}) = \operatorname{Psh}(\mathscr{C}^{\operatorname{op}})^{\operatorname{op}} \\
 = [\mathscr{C}, \operatorname{\mathsf{Set}}]^{\operatorname{op}} = [\mathscr{C}^{\operatorname{op}}, \operatorname{\mathsf{Set}}^{\operatorname{op}}]$$

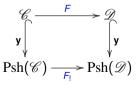
Curry $\operatorname{Hom}^{\operatorname{op}}: \mathscr{C} \times \mathscr{C}^{\operatorname{op}} \to \operatorname{Set}^{\operatorname{op}}$ to get $\mathbf{h}: \mathscr{C} \to \operatorname{Copsh}(\mathscr{C}): \Gamma \mapsto \operatorname{Hom}(\Gamma, -)$ sending Γ to its copresheaf of continuations.

$$\begin{array}{ccc}
\mathscr{C} & \xrightarrow{F} & \mathscr{D} \\
h & & & h \\
\text{Copsh}(\mathscr{C}) & \xrightarrow{F_2} & \text{Copsh}(\mathscr{D})
\end{array}$$

$$Psh(\mathscr{C}) = [\mathscr{C}^{op}, Set]$$

 $\label{eq:Swap & curry Hom : $\mathscr{C}^{op} \times \mathscr{C} \to Set$ to get $\mathbf{y}: \mathscr{C} \to Psh(\mathscr{C}): \Gamma \mapsto Hom(-,\Gamma)$ }$

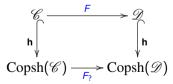
Functor $F : \mathscr{C} \to \mathscr{D}$ yields $F_! \dashv F^* \dashv F_* : Psh(\mathscr{C}) \to Psh(\mathscr{D})$ where $F_!$ extends F:



Copresheaves:

$$\operatorname{Copsh}(\mathscr{C}) = \operatorname{Psh}(\mathscr{C}^{\operatorname{op}})^{\operatorname{op}} \\
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Curry $\operatorname{Hom}^{\operatorname{op}}: \mathscr{C} \times \mathscr{C}^{\operatorname{op}} \to \operatorname{Set}^{\operatorname{op}}$ to get $\mathbf{h}: \mathscr{C} \to \operatorname{Copsh}(\mathscr{C}): \Gamma \mapsto \operatorname{Hom}(\Gamma, -)$ sending Γ to its copresheaf of continuations.



Presheaves:

$\operatorname{Hom}_{\mathscr{D}}(\mathsf{F}\Delta,\mathsf{\Gamma})$

$$\cong$$
 yF $\triangle \rightarrow$ y Γ

$$\cong$$
 $F_! \mathbf{y} \Delta \to \mathbf{y} \Gamma$

$$\cong$$
 y $\triangle \rightarrow F^*$ y \lceil

$$= \operatorname{Hom}_{\mathscr{D}}(-,\Delta) \to \operatorname{Hom}_{\mathscr{D}}(F-,\Gamma)$$

This is a left-relative adjunction:

$$F_y \dashv F^*y$$

Copresheaves:

$$\operatorname{Hom}_{\mathscr{D}}(\Gamma, F\Delta)$$

$$\cong$$
 h $\Gamma \rightarrow$ h $F\Delta$

$$\cong$$
 h $\Gamma \rightarrow F_{?}$ h Δ

$$\cong$$
 $F^{\circ}h\Gamma \rightarrow h\Delta$

$$=\operatorname{Hom}_{\mathscr{D}}(\Delta,-) o \operatorname{Hom}_{\mathscr{D}}(\Gamma,F-)$$

$$F^{\circ}h \dashv_{h} F$$

$$\mathbf{h}\Gamma, \overline{\mathbf{A}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle @ \operatorname{Copsh}(\mathscr{C})$$
$$\Gamma \vdash \operatorname{mod}_{\mu}^{\mathbf{h}} t : \langle \mu \mid T \rangle @ \mathscr{D}$$

where
$$\left \lceil \overline{\mathbf{A}}_{\mu} \right
ceil = \left \lceil \mu \right
ceil^\circ$$

Presheaves:

$$\operatorname{Hom}_{\mathscr{D}}(F\Delta,\Gamma)$$

$$\cong$$
 y $F\Delta \rightarrow y\Gamma$

$$\cong$$
 $F_! y \triangle \rightarrow y \Gamma$

$$\cong$$
 y $\triangle \rightarrow F^*$ y Γ

$$= \operatorname{Hom}_{\mathscr{D}}(-,\Delta) \to \operatorname{Hom}_{\mathscr{D}}(F-,\Gamma)$$

This is a left-relative adjunction:

$$F_y \dashv F^*y$$

Copresheaves:

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 $F^{\circ}h\Gamma \rightarrow h\Delta$

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$$F^{\circ}h\dashv_{h}F$$

$$\mathbf{h}\Gamma, \overline{\mathbf{A}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle @ \operatorname{Copsh}(\mathscr{C})$$
$$\Gamma \vdash \operatorname{mod}_{\mu}^{\mathbf{h}} t : \langle \mu \mid T \rangle @ \mathscr{D}$$

where
$$\left \lceil \overline{\mathbf{A}}_{\mu} \right
ceil = \left \lceil \mu \right
ceil^\circ$$

Presheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\varnothing}(F\Delta,\Gamma) \\
\cong & \mathsf{y}F\Delta \to \mathsf{y}\Gamma \\
\cong & F_{!}\mathsf{y}\Delta \to \mathsf{y}\Gamma \\
\cong & \mathsf{y}\Delta \to F^{*}\mathsf{y}\Gamma
\end{array}$$

This is a left-relative adjunction:

$$F_y \dashv F^*y$$

Copresheaves:

$$\begin{array}{ll} \operatorname{Hom}_{\mathscr{D}}(\Gamma, F\Delta) \\ \cong & \mathsf{h}\Gamma \to \mathsf{h}F\Delta \\ \cong & \mathsf{h}\Gamma \to F_? \mathsf{h}\Delta \\ \cong & F^\circ \mathsf{h}\Gamma \to \mathsf{h}\Delta \end{array}$$

$$= & \operatorname{Hom}_{\mathscr{D}}(\Lambda, -) \to \operatorname{Hom}_{\mathscr{D}}(\Gamma, F-1) = 0$$

$$F^{\circ}h \dashv_{h} F$$

$$\mathbf{h}\Gamma, \overline{\mathbf{\Delta}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle @ \operatorname{Copsh}(\mathscr{C})$$
$$\Gamma \vdash \operatorname{mod}_{\mu}^{\mathbf{h}} t : \langle \mu \mid T \rangle @ \mathscr{D}$$

where
$$\left[\!\left[ar{m{\Box}}_{\!m{\mu}}
ight]\!\right] = \left[\!\left[m{\mu}
ight]\!\right]^\circ$$

Presheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\varnothing}(F\Delta,\Gamma) \\
\cong & \mathsf{y}F\Delta \to \mathsf{y}\Gamma \\
\cong & F_{!}\mathsf{y}\Delta \to \mathsf{y}\Gamma \\
\cong & \mathsf{y}\Delta \to F^{*}\mathsf{y}\Gamma \\
= & \operatorname{Hom}_{\varnothing}(-,\Delta) \to \operatorname{Hom}_{\varnothing}(F-,\Gamma)
\end{array}$$

This is a left-relative adjunction

$$F_y \dashv F^*y$$

Copresheaves:

$$\begin{array}{ll} \operatorname{Hom}_{\mathscr{D}}(\Gamma, F\Delta) \\ \cong & \mathsf{h}\Gamma \to \mathsf{h}F\Delta \\ \cong & \mathsf{h}\Gamma \to F_{?}\mathsf{h}\Delta \\ \cong & F^{\circ}\mathsf{h}\Gamma \to \mathsf{h}\Delta \end{array}$$

$$= & \operatorname{Hom}_{\mathscr{D}}(\Delta, -) \to \operatorname{Hom}_{\mathscr{D}}(\Gamma, F-)$$

$$F^{\circ}h\dashv_{h}F$$

$$\mathbf{h}\Gamma, \overline{\mathbf{\Delta}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle @ \operatorname{Copsh}(\mathscr{C})$$
$$\Gamma \vdash \operatorname{mod}_{\mu}^{\mathbf{h}} t : \langle \mu \mid T \rangle @ \mathscr{D}$$

where
$$\left \lceil \overline{\mathbf{A}}_{\mu} \right
ceil = \left \lfloor \mu \right
floor^{\circ}$$

Presheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(F\Delta,\Gamma) \\
\cong & \mathsf{y}F\Delta \to \mathsf{y}\Gamma \\
\cong & F_{!}\mathsf{y}\Delta \to \mathsf{y}\Gamma \\
\cong & \mathsf{y}\Delta \to F^{*}\mathsf{y}\Gamma \\
= & \operatorname{Hom}_{\mathscr{D}}(-,\Delta) \to \operatorname{Hom}_{\mathscr{D}}(F-,\Gamma)
\end{array}$$

This is a left-relative adjunction $F_{\mathbf{v}} \dashv F^* \mathbf{y}$

Copresheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(\Gamma, F\Delta) \\
\cong & \mathsf{h}\Gamma \to \mathsf{h}F\Delta \\
\cong & \mathsf{h}\Gamma \to F_{?}\mathsf{h}\Delta \\
\cong & F^{\circ}\mathsf{h}\Gamma \to \mathsf{h}\Delta
\end{array}$$

$$= \operatorname{Hom}_{\mathscr{D}}(\Delta, -) \to \operatorname{Hom}_{\mathscr{D}}(\Gamma, F-)$$

$$\begin{array}{c} \mathbf{h}\Gamma, \overline{\underline{\mathbf{A}}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle @ \operatorname{Copsh}(\mathscr{C}) \\ \Gamma \vdash \operatorname{mod}^{\mathbf{h}}_{\mu} t : \langle \mu \mid T \rangle @ \mathscr{D} \end{array}$$

where
$$\left[\!\left[\overline{\mathbf{A}}_{\mu} \right]\!\right] = \left[\!\left[\mu \right]\!\right]^{\circ}$$

Presheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(F\Delta,\Gamma) \\
\cong & \mathsf{y}F\Delta \to \mathsf{y}\Gamma \\
\cong & F_{!}\mathsf{y}\Delta \to \mathsf{y}\Gamma \\
\cong & \mathsf{y}\Delta \to F^{*}\mathsf{y}\Gamma \\
= & \operatorname{Hom}_{\mathscr{D}}(-,\Delta) \to \operatorname{Hom}_{\mathscr{D}}(F-,\Gamma)
\end{array}$$

This is a left-relative adjunction:

$$F_y \dashv F^*y$$

Copresheaves:

$$\begin{array}{ll} \operatorname{Hom}_{\mathscr{D}}(\Gamma, F\Delta) \\ \cong & \mathsf{h}\Gamma \to \mathsf{h}F\Delta \\ \cong & \mathsf{h}\Gamma \to F_{?}\mathsf{h}\Delta \\ \cong & F^{\circ}\mathsf{h}\Gamma \to \mathsf{h}\Delta \end{array}$$

$$= & \operatorname{Hom}_{\mathscr{D}}(\Delta, -) \to \operatorname{Hom}_{\mathscr{D}}(\Gamma, F-)$$

$$\mathbf{h}\Gamma, \overline{\mathbf{\Delta}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle @ \operatorname{Copsh}(\mathscr{C})$$
$$\Gamma \vdash \operatorname{mod}_{\mu}^{\mathbf{h}} t : \langle \mu \mid T \rangle @ \mathscr{D}$$

where
$$\left[\!\left[\overline{\mathbf{A}}_{\mu}\right]\!\right] = \left[\!\left[\mu\right]\!\right]^{\circ}$$

Presheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(F\Delta,\Gamma) \\
\cong & \mathsf{y}F\Delta \to \mathsf{y}\Gamma \\
\cong & F_{!}\mathsf{y}\Delta \to \mathsf{y}\Gamma \\
\cong & \mathsf{y}\Delta \to F^{*}\mathsf{y}\Gamma \\
= & \operatorname{Hom}_{\mathscr{D}}(-,\Delta) \to \operatorname{Hom}_{\mathscr{D}}(F-,\Gamma)
\end{array}$$

This is a left-relative adjunction:

$$F_y \dashv F^*y$$

Copresheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(\Gamma, F\Delta) \\
\cong & \mathsf{h}\Gamma \to \mathsf{h}F\Delta \\
\cong & \mathsf{h}\Gamma \to F_{?}\mathsf{h}\Delta \\
\cong & F^{\circ}\mathsf{h}\Gamma \to \mathsf{h}\Delta
\end{array}$$

$$= & \operatorname{Hom}_{\mathscr{D}}(\Delta, -) \to \operatorname{Hom}_{\mathscr{D}}(\Gamma, F-)$$

$$F^{\circ}h\dashv_{\mathsf{h}}F$$

$$\mathbf{h}\Gamma, \overline{\mathbf{A}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle @ \operatorname{Copsh}(\mathscr{C})$$
$$\Gamma \vdash \operatorname{mod}_{\mu}^{\mathbf{h}} t : \langle \mu \mid T \rangle @ \mathscr{D}$$

where
$$\left[\!\left[\overline{\mathbf{A}}_{\mu} \right]\!\right] = \left[\!\left[\mu \right]\!\right]^{\circ}$$

Presheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(F\Delta,\Gamma) \\
\cong & \mathsf{y}F\Delta \to \mathsf{y}\Gamma \\
\cong & F_{!}\mathsf{y}\Delta \to \mathsf{y}\Gamma \\
\cong & \mathsf{y}\Delta \to F^{*}\mathsf{y}\Gamma \\
= & \operatorname{Hom}_{\mathscr{D}}(-,\Delta) \to \operatorname{Hom}_{\mathscr{D}}(F-,\Gamma)
\end{array}$$

This is a left-relative adjunction:

$$F_{\mathbf{y}} \dashv F^*\mathbf{y}$$

Copresheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(\Gamma, F\Delta) \\
\cong & \mathsf{h}\Gamma \to \mathsf{h}F\Delta \\
\cong & \mathsf{h}\Gamma \to F_{?}\mathsf{h}\Delta \\
\cong & F^{\circ}\mathsf{h}\Gamma \to \mathsf{h}\Delta
\end{array}$$

$$= & \operatorname{Hom}_{\mathscr{D}}(\Delta, -) \to \operatorname{Hom}_{\mathscr{D}}(\Gamma, F-)$$

$$\frac{\mathbf{h}\Gamma,\overline{\underline{\mathbf{A}}}_{\mu}\vdash t:\langle\mathbf{h}\mid T\rangle @\operatorname{Copsh}(\mathscr{C})}{\Gamma\vdash \operatorname{mod}^{\mathbf{h}}_{\mu}t:\langle\mu\mid T\rangle @\mathscr{D}}$$

where
$$\left \lceil \overline{\mathbf{\Delta}}_{\mu} \right
ceil = \left \lfloor \mu \right
floor^\circ$$

Presheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(F\Delta,\Gamma) \\
\cong & \mathsf{y}F\Delta \to \mathsf{y}\Gamma \\
\cong & F_{!}\mathsf{y}\Delta \to \mathsf{y}\Gamma \\
\cong & \mathsf{y}\Delta \to F^{*}\mathsf{y}\Gamma \\
= & \operatorname{Hom}_{\mathscr{D}}(-,\Delta) \to \operatorname{Hom}_{\mathscr{D}}(F-,\Gamma)
\end{array}$$

This is a left-relative adjunction:

$$F_y \dashv F^*y$$

Copresheaves:

$$\begin{array}{ll}
\text{Hom}_{\mathscr{D}}(\Gamma, F\Delta) \\
\cong & \mathsf{h}\Gamma \to \mathsf{h}F\Delta \\
\cong & \mathsf{h}\Gamma \to F_{?}\mathsf{h}\Delta \\
\cong & F^{\circ}\mathsf{h}\Gamma \to \mathsf{h}\Delta
\end{array}$$

$$= & \text{Hom}_{\mathscr{D}}(\Delta, -) \to \text{Hom}_{\mathscr{D}}(\Gamma, F-)$$

$$F^{\circ}h\dashv_{h}F$$

$$\mathbf{h}\Gamma, \overline{\mathbf{\Delta}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle @ \operatorname{Copsh}(\mathscr{C})$$
$$\Gamma \vdash \operatorname{mod}_{\mu}^{\mathbf{h}} t : \langle \mu \mid T \rangle @ \mathscr{D}$$

where
$$\left[\!\left[\overline{\mathbf{A}}_{\mu} \right]\!\right] = \left[\!\left[\mu \right]\!\right]^{\circ}$$

Presheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(F\Delta,\Gamma) \\
\cong & \mathsf{y}F\Delta \to \mathsf{y}\Gamma \\
\cong & F_{!}\mathsf{y}\Delta \to \mathsf{y}\Gamma \\
\cong & \mathsf{y}\Delta \to F^{*}\mathsf{y}\Gamma \\
= & \operatorname{Hom}_{\mathscr{D}}(-,\Delta) \to \operatorname{Hom}_{\mathscr{D}}(F-,\Gamma)
\end{array}$$

This is a left-relative adjunction:

$$F_y \dashv F^*y$$

Copresheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(\Gamma, F\Delta) \\
\cong & \mathsf{h}\Gamma \to \mathsf{h}F\Delta \\
\cong & \mathsf{h}\Gamma \to F_? \mathsf{h}\Delta \\
\cong & F^{\circ}\mathsf{h}\Gamma \to \mathsf{h}\Delta
\end{array}$$

$$= & \operatorname{Hom}_{\mathscr{D}}(\Delta, -) \to \operatorname{Hom}_{\mathscr{D}}(\Gamma, F-)$$

This is a right-relative adjunction: $F \circ \mathbf{h} \dashv_{\mathbf{h}} F$

$$\begin{array}{c} \mathbf{h} \Gamma, \overline{\underline{\mathbf{A}}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle @ \operatorname{Copsh}(\mathscr{C}) \\ \Gamma \vdash \operatorname{mod}^{\mathbf{h}}_{\mu} t : \langle \mu \mid T \rangle @ \mathscr{D} \end{array}$$

where
$$\left[\!\left[\overline{\mathbf{\Delta}}_{\mu} \right]\!\right] = \left[\!\left[\boldsymbol{\mu} \right]\!\right]^{\circ}$$

Presheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(F\Delta,\Gamma) \\
\cong & \mathsf{y}F\Delta \to \mathsf{y}\Gamma \\
\cong & F_{!}\mathsf{y}\Delta \to \mathsf{y}\Gamma \\
\cong & \mathsf{y}\Delta \to F^{*}\mathsf{y}\Gamma \\
= & \operatorname{Hom}_{\mathscr{D}}(-,\Delta) \to \operatorname{Hom}_{\mathscr{D}}(F-,\Gamma)
\end{array}$$

This is a left-relative adjunction:

$$F_y \dashv F^*y$$

Copresheaves:

$$\begin{array}{ll} \operatorname{Hom}_{\mathscr{D}}(\Gamma, F\Delta) \\ \cong & \mathsf{h}\Gamma \to \mathsf{h}F\Delta \\ \cong & \mathsf{h}\Gamma \to F_{?}\mathsf{h}\Delta \\ \cong & F^{\circ}\mathsf{h}\Gamma \to \mathsf{h}\Delta \end{array}$$

$$= & \operatorname{Hom}_{\mathscr{D}}(\Delta, -) \to \operatorname{Hom}_{\mathscr{D}}(\Gamma, F-)$$

This is a right-relative adjunction: $F \circ h \rightarrow F$

$$\frac{\mathbf{h}\Gamma,\overline{\mathbf{A}}_{\mu}\vdash t:\langle\mathbf{h}\mid T\rangle @\operatorname{Copsh}(\mathscr{C})}{\Gamma\vdash \operatorname{mod}_{\mu}^{\mathbf{h}}t:\langle\mu\mid T\rangle @\mathscr{D}}$$

where
$$\left[\!\left[\overline{\mathbf{A}}_{\mu}\right]\!\right] = \left[\!\left[\mu\right]\!\right]^{\circ}$$

Presheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(F\Delta,\Gamma) \\
\cong & \mathsf{y}F\Delta \to \mathsf{y}\Gamma \\
\cong & F_{!}\mathsf{y}\Delta \to \mathsf{y}\Gamma \\
\cong & \mathsf{y}\Delta \to F^{*}\mathsf{y}\Gamma \\
= & \operatorname{Hom}_{\mathscr{D}}(-,\Delta) \to \operatorname{Hom}_{\mathscr{D}}(F-,\Gamma)
\end{array}$$

This is a left-relative adjunction:

$$F_{\mathbf{y}} \dashv F^*\mathbf{y}$$

Copresheaves:

$$\begin{array}{ll}
\operatorname{Hom}_{\mathscr{D}}(\Gamma, F\Delta) \\
\cong & \mathsf{h}\Gamma \to \mathsf{h}F\Delta \\
\cong & \mathsf{h}\Gamma \to F_? \mathsf{h}\Delta \\
\cong & F^\circ \mathsf{h}\Gamma \to \mathsf{h}\Delta \\
= & \operatorname{Hom}_{\mathscr{D}}(\Delta, -) \to \operatorname{Hom}_{\mathscr{D}}(\Gamma, F-)
\end{array}$$

$$F^{\circ}$$
h $\dashv_{h} F$

$$\frac{\mathbf{h}\Gamma,\overline{\underline{\mathbf{A}}}_{\mu}\vdash t:\langle\mathbf{h}\mid T\rangle @\operatorname{Copsh}(\mathscr{C})}{\Gamma\vdash \operatorname{mod}^{\mathbf{h}}_{\mu}t:\langle\mu\mid T\rangle @\mathscr{D}}$$

where
$$\left \lceil \overline{\mathbf{A}}_{\mu} \right
ceil = \left \lfloor \mu \right
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Presheaves:

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\operatorname{Hom}_{\mathscr{D}}(F\Delta,\Gamma) \\
\cong & \mathsf{y}F\Delta \to \mathsf{y}\Gamma \\
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$$F^{\circ}h \dashv_{h} F$$

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$$\Gamma \vdash \operatorname{\mathsf{mod}}_{\mu}^{\mathbf{h}} t : \langle \mu \mid T \rangle @ \mathscr{D}$$

where
$$\left[\!\left[\overline{\mathbf{A}}_{\boldsymbol{\mu}} \right]\!\right] = \left[\!\left[\boldsymbol{\mu} \right]\!\right]^{\circ}$$

As of this point, things are going downhill.

Thoughts & ideas appreciated.

$$\frac{\mathbf{h}\Gamma, \overline{\mathbf{\Delta}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle}{\Gamma \vdash \mathsf{mod}_{\mu}^{\mathbf{h}} \, t : \langle \mu \mid T \rangle}$$

In non-pathological situations:

- h is never a DRA,
- h never preserves limits,

$$\langle \mathbf{h} \mid A \times B \rangle \xrightarrow{\cong} \langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid B \rangle$$

h is never applicative

$$\langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid A \to C \rangle \to \langle \mathbf{h} \mid A \times (A \to C) \rangle$$

→ h is an MTT-unsupportive sediment.

To use a variable:

$$\frac{\mathbf{h}(\Gamma, \mathbf{v} \mid \mathbf{x} : T), \overline{\mathbf{\Delta}}_{\mu} \vdash ? : \langle \mathbf{h} \mid T \rangle}{\Gamma, \mathbf{v} \mid \mathbf{x} : T \vdash \mathsf{mod}_{\mu}^{\mathbf{h}} ? : \langle \mu \mid T \rangle}$$

we need

$$\mu^{\circ} h v \rightarrow h$$
 $\cong h v \rightarrow \mu_{?} h$
 $\cong h v \rightarrow h \mu$
 $\cong v \rightarrow \mu_{?}$

$$\frac{\mathbf{h}\Gamma, \overline{\mathbf{\Delta}}_{\boldsymbol{\mu}} \vdash t : \langle \mathbf{h} \mid T \rangle}{\Gamma \vdash \mathsf{mod}^{\mathbf{h}}_{\boldsymbol{\mu}} t : \langle \boldsymbol{\mu} \mid T \rangle}$$

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$$\frac{\mathbf{h}\Gamma, \overline{\mathbf{\Delta}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle}{\Gamma \vdash \mathsf{mod}^{\mathbf{h}}_{\mu} \, t : \langle \underline{\mu} \mid T \rangle}$$

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- h never preserves limits,

$$\langle \mathbf{h} \mid A \times B \rangle \xrightarrow{\ncong} \langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid B \rangle$$

h is never applicative.

$$\langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid A \rightarrow C \rangle \nrightarrow \langle \mathbf{h} \mid A \times (A \rightarrow C) \rangle$$

 \sim **h** is an MTT-unsupportive sediment.

To use a variable:

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- h is never a DRA,
- h never preserves limits,

$$\langle \mathbf{h} \mid A \times B \rangle \xrightarrow{\cong} \langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid B \rangle$$

h is never applicative.

$$\langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid A \rightarrow C \rangle \nrightarrow \langle \mathbf{h} \mid A \times (A \rightarrow C) \rangle$$

 \sim **h** is an MTT-unsupportive sediment.

To use a variable:

$$\frac{\mathbf{h}(\Gamma, \mathbf{v} \mid \mathbf{x} : T), \overline{\mathbf{\Delta}}_{\mu} \vdash ? : \langle \mathbf{h} \mid T \rangle}{\Gamma, \mathbf{v} \mid \mathbf{x} : T \vdash \mathsf{mod}_{\mu}^{\mathbf{h}} ? : \langle \mu \mid T \rangle}$$

we need

$$\mu^{\circ} h v \rightarrow h$$
 $\cong h v \rightarrow \mu_{?} h$
 $\cong h v \rightarrow h \mu$
 $\cong v \rightarrow u$

$$\frac{\mathbf{h}\Gamma, \overline{\mathbf{A}}_{\mu} \vdash t : \langle \mathbf{h} \mid T \rangle}{\Gamma \vdash \mathsf{mod}^{\mathbf{h}}_{\mu} t : \langle \mu \mid T \rangle}$$

In non-pathological situations:

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$Copsh(\mathscr{C})$ is a CwF.

Giraud CwF structure [Gir65, BCMMPS20]

Every category ${\mathscr D}$ with \top and pullbacks is a CwF:

- ullet Contexts and substitutions: ${\mathscr D}$
- $T \in \mathrm{Ty}(\Gamma)$:

Δ | | | | |

- Substitution
- Context extension

However, $Copsh(\mathscr{C})$ has:

- No Π-types!So no library functions
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Possible solution: Move to Psh(Copsh(C)). (Is this getting out of hand?

 \mathfrak{D} This is 2LTT for Copsh(\mathscr{C}). [ACKS17/23]

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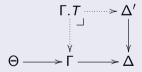
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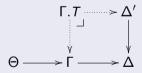
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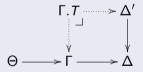
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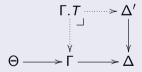
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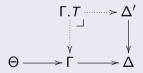
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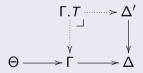
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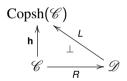
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We do not always need copresheaves.

It doesn't have to be a relative right adjoint along h.

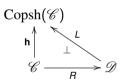


$$\operatorname{Hom}_{\operatorname{Copsh}(\mathscr{C})}(\mathit{Ld},\mathsf{h}\mathit{c})\cong \operatorname{Hom}_{\mathscr{D}}(\mathit{d},\mathit{R}\mathit{c})$$



$$\operatorname{Hom}_{\mathscr{C}'}(\mathit{Ld},\mathit{Jc})\cong \operatorname{Hom}_{\mathscr{D}}(\mathit{d},\mathit{Rc})$$

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 $\operatorname{Hom}_{\operatorname{Copsh}(\mathscr{C})}(\mathit{Ld},\mathsf{h}\mathit{c})\cong \operatorname{Hom}_{\mathscr{D}}(\mathit{d},\mathit{R}\mathit{c})$



Container functor

$$FY = \Sigma(s:S).(Ps \rightarrow Y)$$

$$(X \to FY) \cong$$

 $\Sigma(f: X \to S).((x: X) \times P(fx) \to Y)$

Parametric right adjoint (PRA

Functor $F: \mathcal{C} \to \mathcal{D}$ such that $F^{/\top}: \mathcal{C} \cong \mathcal{C}/\top \to \mathcal{D}/F\top$ is right adjoint.

 $\operatorname{Hom}_{\mathscr{D}}(X, FY)$ $\cong \Sigma(f : \operatorname{Hom}_{\mathscr{D}}(X, F\top)).\operatorname{Hom}_{\mathscr{D}/F\top}((X, f), F/\top Y)$ $\cong \Sigma(f : \operatorname{Hom}_{\mathscr{D}}(X, F\top)).\operatorname{Hom}_{\mathscr{C}}(L(X, f), Y)$

Right multi-adjoint

PRA without referring to ⊤

Relative right adjoint

$$C'$$

$$J \downarrow \qquad L$$

$$C \xrightarrow{R} \mathcal{D}$$

Container functor

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PRA without referring to \top

Relative right adjoint

$$\begin{array}{c} \mathcal{C}' \\ \downarrow \\ \mathcal{C} \xrightarrow{B} \mathcal{D} \end{array}$$

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 $\cong \operatorname{Hom}_{\operatorname{Cart}(\mathscr{C})}(\prod_{f:\operatorname{Hom}_{\mathscr{Q}}(X,F\top)}[L(X,f)],[Y])$

Right multi-adjoint

PRA without referring to T.

Relative right adjoint

$$C'$$

$$C \xrightarrow{R} D$$

Container functor

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 $(X \rightarrow FY) \cong$

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Right multi-adjoint

PRA without referring to T

Relative right adjoint

Container functors \subseteq PRAs \subseteq Right multi-adjoints \subseteq Relative right adjoints

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\cong \Sigma(f:\operatorname{Hom}_{\mathscr{D}}(X,F\top)).\operatorname{Hom}_{\mathscr{C}}(L(X,f),Y)
\cong \operatorname{Hom}_{\operatorname{Cat}(\mathscr{D})}(\Pi_{L}\operatorname{Hom}_{\mathscr{D}}(X,F\top)[L(X,f)],[Y])$$

Right multi-adjoint

PRA without referring to \top .

Relative right adjoint

 $\operatorname{Hom}_{\mathscr{C}'}(Ld,Jc) \cong \operatorname{Hom}_{\mathscr{D}}(d,Rc)$

Container functors \subseteq PRAs \subseteq Right multi-adjoints \subseteq Relative right adjoints

Container functor

$$egin{aligned} extit{FY} &= \Sigma(s:S).(extit{P}\,s
ightarrow Y) \ (extit{X}
ightarrow extit{FY}) \cong \end{aligned}$$

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 $\operatorname{Hom}_{\mathscr{C}'}(Ld,Jc) \cong \operatorname{Hom}_{\mathscr{D}}(d,Rc)$

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 $\Gamma \vdash s : S$

 $\Gamma, p: Ps \vdash a: A$

 $\Gamma \vdash (s, \lambda p.a) : \Sigma(s : S).(Ps \rightarrow A)$

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$$\Gamma \vdash s : \langle F \mid T \rangle$$

 $\Gamma \mid s \vdash a : A$
 $\Gamma \vdash mod_{-}(s,a) : \langle F \mid A \rangle$

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We want MTT for non-right-adjoint modalities:

- Shulman has a (categorified) syntactic solution for limit-preserving modalities.
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 - Unclear if usable.
- Does anyone need this generality?

Thanks!

We want MTT for non-right-adjoint modalities:

- Shulman has a (categorified) syntactic solution for limit-preserving modalities.
- There may be a semantic solution via Copsh(\$\mathcal{C}\$) or Psh(Copsh(\$\mathcal{C}\$)).
- We lack guidance from relevant examples (most examples are at least PRAs).
 - Unclear if usable.
- Does anyone need this generality?

Thanks!

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