Every Modality is a Relative Right Adjoint

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Let $R : \mathscr{C} \to \mathscr{D}$ be a functor.

$$\frac{\Gamma\ \mathrm{ctx} \mathbin{@} \mathscr{C}}{R\Gamma\ \mathrm{ctx} \mathbin{@} \mathscr{D}} \qquad \frac{\tau : \Gamma \to \Gamma' \mathbin{@} \mathscr{C}}{R\tau : R\Gamma \to R\Gamma' \mathbin{@} \mathscr{D}} \qquad \frac{\Gamma \vdash T\ \mathrm{type} \mathbin{@} \mathscr{C}}{R\Gamma \vdash RT\ \mathrm{type} \mathbin{@} \mathscr{D}} \qquad \frac{\Gamma \vdash t : T \mathbin{@} \mathscr{C}}{R\Gamma \vdash Rt : RT \mathbin{@} \mathscr{D}}$$

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$$\frac{?}{\Delta \vdash RT\ \mathrm{type}}$$

We check $\Gamma \vdash T\ \mathrm{type} \mathbin{@} \mathscr{C}$ and substitute with $\sigma : \Delta \to R\Gamma$.
BUT: Don't bother the user. Synthesize Γ and σ.

$\Gamma \in \mathscr{C}$ should be the **universal** context Γ such that $\sigma : \Delta \to R\Gamma$ exists.
I.e. if $\sigma' : \Delta \to R\Gamma'$ then we should have $\Gamma \to \Gamma'$.

+ some sensible laws $\rightsquigarrow L \dashv R$.

Let $R : \mathscr{C} \to \mathscr{D}$ be a CwF morphism.

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Multimodal TT (MTT)

MTT [GKNB21] is parametrized by a **2-category**:

- modes $p, q, r, \dots$
- modalities $\mu : p \to q$

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- (2-cells $\alpha : \mu \Rightarrow \nu$).

Semantics:

- $[\![p]\!]$ is a (often presheaf) category modelling all of DTT,
- $[\![\mu]\!]$ is a (weak) dependent right adjoint (DRA) [BCMMPS20] to $[\![\text{🔒}_\mu]\!]$,

Note: If codomain $\mathscr{D}$ is democratic, then DRA = right adjoint that is a CwF morphism.

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— *Mike Shulman, HoTT mailing list, Dec 1, 2022 (emphases are ours)*

- Valid concern: We can **internally** prove that MTT modalities preserve limits, e.g. $\langle \mu \mid A \times B \rangle \cong \langle \mu \mid A \rangle \times \langle \mu \mid B \rangle$.
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Multimodal Adjoint Type Theory (MATT):

[Shulman, March 2023]

Categorically Adds locks to contexts **cofreely**.

Morally Defines locks by induction on **syntactic context formation**. These **approximate** the left adjoint.

☺ $\langle \mu \mid - \rangle$ need not be a DRA.

☺ Subsumes MTT without modifications.

☹ $\Rightarrow$ We can still **internally** prove that $\langle \mu \mid - \rangle$ preserves limits. This is also assumed in the **model**. [Shu23, assumption 4.1]

Our solution (WIP):

Categorically Move to **copresheaf** category.

Morally Move to **metaprogramming with continuations**.

☺ $\langle \mu \mid - \rangle$ does not need to:

- be a DRA,
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☹ $\Rightarrow$ We can still **internally** prove that $\langle \mu \mid - \rangle$ preserves limits. This is also assumed in the **model**.
[Shu23, assumption 4.1]

Our solution (WIP):

Categorically: Move to **copresheaf** category.

Morally: Move to **metaprogramming with continuations**.

☺ $\langle \mu \mid - \rangle$ does not need to:

- be a DRA,
- preserve limits,
- or even be applicative.

☹ Modifies MTT.

☹ In particular, $\langle \mu \mid - \rangle$ may not:

- be a DRA,
- preserve limits,
- or even be applicative.

Great/Terrible!

Presheaves:
$\mathrm{Psh}(\mathscr{C}) = [\mathscr{C}^{\mathsf{op}}, \mathsf{Set}]$

Swap & curry $\mathrm{Hom} : \mathscr{C}^{\mathsf{op}} \times \mathscr{C} \to \mathsf{Set}$
to get $\mathbf{y} : \mathscr{C} \to \mathrm{Psh}(\mathscr{C}) : \Gamma \mapsto \mathrm{Hom}(-, \Gamma)$

Functor $F : \mathscr{C} \to \mathscr{D}$ yields
$F_! \dashv F^* \dashv F_* : \mathrm{Psh}(\mathscr{C}) \to \mathrm{Psh}(\mathscr{D})$
where $F_!$ extends F:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, hook, "\mathbf{y}"'] & \mathscr{D} \arrow[d, hook, "\mathbf{y}"] \\
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Copresheaves:
$\mathrm{Copsh}(\mathscr{C}) = \mathrm{Psh}(\mathscr{C}^{\mathsf{op}})^{\mathsf{op}}$
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Curry $\mathrm{Hom}^{\mathsf{op}} : \mathscr{C} \times \mathscr{C}^{\mathsf{op}} \to \mathsf{Set}^{\mathsf{op}}$
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Every functor is a left/right-relative left/right adjoint

Presheaves:

$$\begin{aligned}
& \mathrm{Hom}_{\mathscr{D}}(F\Delta, \Gamma) \\
\cong\ & \mathbf{y}F\Delta \to \mathbf{y}\Gamma \\
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=\ & \mathrm{Hom}_{\mathscr{D}}(-, \Delta) \to \mathrm{Hom}_{\mathscr{D}}(F-, \Gamma)
\end{aligned}$$

This is a left-relative adjunction:
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This is a left-relative adjunction:
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As of this point,
things are going downhill.

Thoughts & ideas appreciated.

So surely, **h** is well-behaved?

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In non-pathological situations:

- **h** is **never** a DRA,
- **h** **never** preserves limits,

  $\langle \mathbf{h} \mid A \times B \rangle \xrightarrow{\not\cong} \langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid B \rangle$
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  $\langle \mathbf{h} \mid A \rangle \times \langle \mathbf{h} \mid A \to C \rangle \to \langle \mathbf{h} \mid A \times (A \to C) \rangle$

$\leadsto$ **h** is an MTT-unsupportive sediment.

To use a variable:

$$\frac{\mathbf{h}(\Gamma, \nu \mid x : T), \overline{\text{🔒}}_\mu \vdash ? : \langle \mathbf{h} \mid T \rangle}{\Gamma, \nu \mid x : T \vdash \mathsf{mod}^{\mathbf{h}}_\mu\, ? : \langle \mu \mid T \rangle}$$

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$\mathrm{Copsh}(\mathscr{C})$ is a CwF.

Giraud CwF structure [Gir65, BCMMPS20]

Every category $\mathscr{D}$ with $\top$ and pullbacks is a CwF:

- Contexts and substitutions: $\mathscr{D}$
- $T \in \mathrm{Ty}(\Gamma)$:

$$\begin{tikzcd} & \Gamma.T & \Delta' \arrow[d] \\ \Theta & \Gamma \arrow[r] & \Delta \end{tikzcd}$$

- Substitution
- Context extension

However, $\mathrm{Copsh}(\mathscr{C})$ has:

- No Π-types!
  So no library functions!
- (We have co-exponentials.)
  $(A_E \to B) \cong (A \to E \uplus B)$
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Possible solution:
Move to $\mathrm{Psh}(\mathrm{Copsh}(\mathscr{C}))$.
(Is this getting out of hand?)

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Every category $\mathscr{D}$ with $\top$ and pullbacks is a CwF:

- Contexts and substitutions: $\mathscr{D}$
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However, $\mathrm{Copsh}(\mathscr{C})$ has:

- No Π-types!
  So no library functions!
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Possible solution:
Move to $\mathrm{Psh}(\mathrm{Copsh}(\mathscr{C}))$.
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It doesn't have to be a relative right adjoint along **h**.

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Container functors ⊆ PRAs ⊆ Right multi-adjoints ⊆ Relative right adjoints

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$$FY = \Sigma(s : S).(P\,s \to Y)$$

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PRA without referring to $\top$.

Relative right adjoint

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