TRANSPENSION: THE RIGHT ADJOINT TO THE PI-TYPE

ANDREAS NUYTS* AND DOMINIQUE DEVRIESE°

*KU Leuven, Belgium
URL: anuyts.github.io

°Vrije Universiteit Brussel, Belgium
e-mail address: dominique.devriese@vub.be

Abstract. Presheaf models of dependent type theory have been successfully applied to model HoTT, parametricity, and directed, guarded and nominal type theory. There has been considerable interest in internalizing aspects of these presheaf models, either to make the resulting language more expressive, or in order to carry out further reasoning internally, allowing greater abstraction and sometimes automated verification. While the constructions of presheaf models largely follow a common pattern, approaches towards internalization do not. Throughout the literature, various internal presheaf operators ($\surd$, Φ/extent, Ψ/Gel, Glue, Weld, mill, the strictness axiom and locally fresh names) can be found and little is known about their relative expressivenes. Moreover, some of these require that variables whose type is a shape (representable presheaf, e.g. an interval) be used affinely.

We propose a novel type former, the transpension type, which is right adjoint to universal quantification over a shape. Its structure resembles a dependent version of the suspension type in HoTT. We give general typing rules and a presheaf semantics in terms of base category functors dubbed multipliers. Structural rules for shape variables and certain aspects of the transpension type depend on characteristics of the multiplier. We demonstrate how the transpension type and the strictness axiom can be combined to implement all and improve some of the aforementioned internalization operators (without formal claim in the case of locally fresh names).

1. Introduction and Related Work

1.1. **The Power of Presheaves.** Presheaf semantics [Hof97, HS97] are an excellent tool for modelling relational preservation properties of (dependent) type theory. They have been applied to parametricity (which is about preservation of relations) [AGJ14, BCM15, ND18a, NVD17], univalent type theory (preservation of equivalences) [BCH14, CMS20, CCHM17, Hub16, KLV12, Ort18, OP18], directed type theory (preservation of morphisms), guarded type theory (preservation of the stage of advancement of computation) [BM18] and even

Key words and phrases: dependent type theory, presheaf models, modal type theory, homotopy type theory, parametricity, directed type theory, guarded type theory.

*Holds a PhD Fellowship from the Research Foundation - Flanders (FWO).

Preprint submitted to
Logical Methods in Computer Science

combinations thereof [BBC+19, CH20, RS17, WL20].[1] The presheaf models cited almost all follow a common pattern: First one chooses a suitable base category $\mathcal{W}$. The presheaf category over $\mathcal{W}$ is automatically a model of dependent type theory with important basic type formers [Hof97] as well as a tower of universes [HS97]. Next, one identifies a suitable notion of fibrancy and replaces or supplements the existing type judgement $\Gamma \vdash T\ \mathsf{type}$ with one that classifies fibrant types:

HoTT: For homotopy type theory (HoTT, [Uni13]), one considers Kan fibrant types, i.e. presheaves in which edges can be composed and inverted as in an ∞-groupoid. The precise definition may differ in different treatments.

Parametricity: For parametric type theory, one considers discrete types [AGJ14, CH20, ND18a, NVD17]: essentially those that satisfy Reynolds' identity extension property [Rey83] which states that homogeneously related objects are equal.

Directed: In directed type theory, one may want to consider Segal, covariant, discrete and Rezk types [RS17] and possibly also Conduché types [Gir64, Nuy18b][Nuy20a, ex. 8.1.27].

Guarded: In guarded type theory, one considers clock-irrelevant types [BM18]: types A such that any non-dependent function $\textcircled{\scriptsize 1} \to A$ from the clock type, is constant.

Nominal: Nominal type theory [Che12, PMD15] can be modelled in the Schanuel topos [Pit14].

To the extent possible, one subsequently proves that the relevant notions of fibrancy are closed under basic type formers, so that we can restrict to fibrant types and still carry out most of the familiar type-theoretic reasoning and programming. Special care is required for the universe: it is generally straightforward to adapt the standard Hofmann-Streicher universe to classify only fibrant types, but the universe of fibrant types is in general not automatically fibrant itself. In earlier work on parametricity with Vezzosi [NVD17, ND18a], we made the universe discrete by modifying its presheaf structure and introduced a parametric modality in order to use that universe. In contrast, Atkey et al. [AGJ14] and Cavallo and Harper [CH20] simply accept that their universes of discrete types are not discrete. In guarded type theory, Bizjak et al. [BGC+16] let the universe depend on a collection of in-scope clock variables lest the clock-indexed later modality $\triangleright : \forall(\kappa : \textcircled{\scriptsize 1}).\mathsf{U}_\Delta \to \mathsf{U}_\Delta$ (where $\kappa \in \Delta$) be non-dependent and therefore constant (not clock-indexed) by clock-irrelevance of $\mathsf{U}_\Delta \to \mathsf{U}_\Delta$ [BM18].

1.2. **Internalizing the Power of Presheaves.** Purely metatheoretic results about type theory certainly have their value. Parametricity, for instance, has originated and proven its value as a metatheoretic technique for reasoning about programs. However, with dependent type theory being not only a programming language but also a logic, it is preferrable to formulate results about it within the type system, rather than outside it. We highlight two particular motivations for doing so: to enlarge the end user's toolbox, and to be able to prove internally that a type is fibrant.

[1] We omit models that are not explicitly structured as presheaf models [AHH18, LH11, Nor19].

Enlarging the end user's toolbox. One motivation for internalizing metatheorems is to enlarge the toolbox of the end user of the proof assistant. If this is the only goal, then we can prove the desired results in the model on pen and paper and then internalize them ad hoc with an axiom with or without computation rules.

HoTT: Book HoTT [Uni13] simply postulates the univalence axiom without computational behaviour, as justified e.g. by the model of Kan-fibrant simplicial sets [KLV12].

CCHM cubical type theory [CCHM17] provides the Glue type, which comes with introduction, elimination, β- and η-rules and which turns the univalence axiom into a theorem with computational behaviour. It also contains CCHM-Kan-fibrancy of all types as an axiom, in the form of the CCHM-Kan composition operator, with decreed computational behaviour that is defined by induction on the type.

Parametricity: Bernardy, Coquand and Moulin [BCM15, Mou16] (henceforth: Moulin et al.) internalize their (unary, but generalizable to k-ary) cubical set model of parametricity using two combinators Φ and Ψ [Mou16], a.k.a. extent and Gel [CH20]. Φ internalizes the presheaf structure of the function type, and Ψ that of the universe.

The combinator Φ and at first sight also Ψ require that the cubical set model lacks diagonals. Indeed, to construct a value over the primitive interval, Φ and Ψ each take one argument for every endpoint and one argument for the edge as a whole. Nested use of these combinators, e.g. to create a square, will take $(k+1)^2$ arguments for k^2 vertices, $2k$ sides and 1 square as a whole but none for specifying the diagonal. For this reason, Moulin et al.'s type system enforces a form of *affine* use of interval variables.

In earlier work with Vezzosi [NVD17], we have internalized parametricity instead using the Glue type [CCHM17] and its dual Weld. Later on, we added a primitive mill [ND18b] for swapping Weld and $\Pi(i : \mathbb{I})$. These operations are sound in presheaves over any base category where we can multiply with $\mathbb{I}$, and therefore strictly less expressive than Φ which is not. Discreteness of all types was internalized as a non-computing *path degeneracy* axiom.

Directed: Weaver and Licata [WL20] use a bicubical set model to show that directed HoTT [RS17] can be soundly extended with a directed univalence *axiom*.

Guarded: In guarded type theory [BM18], one axiomatizes Löb induction and clock-irrelevance.

Nominal: One version of nominal type theory [PMD15] provides the locally fresh name abstraction $\nu(i : \mathbb{I})$ which introduces a name but requires a body that is fresh for the name (i.e. we do not get to use it) and which can be used anywhere (i.e. the goal type remains the same after we abstract over a fresh name). This would be rather useless, were it not that we are allowed to *capture* the fresh name (see section 8). Bridging the formal gap between the syntactic phenomenon of name capture and the denotational presheaf semantics is beyond the scope of this paper, which is why we make no formal claims regarding nominal type theory but instead demonstrate the relevance of the transpension type by example (section 8).

Internalizing fibrancy proofs. Another motivation to internalize aspects of presheaf categories, is for building parts of the model inside the type theory, thus abstracting away certain categorical details such as the very definition of presheaves, and for some type systems enabling automatic verification of proofs. Given the common pattern in models described in

the previous section, it is particularly attractive to try and define fibrancy and prove results about it internally.

In the context of HoTT, Orton and Pitts [Ort18, OP18] study CCHM-Kan-fibrancy [CCHM17] in a type theory extended with a set of axioms, of which all but one serve to characterize the interval and the notion of cofibration. One axiom, *strictness*, provides a type former Strict for strictifying partial isomorphisms, which exists in every presheaf category. In order to prove fibrancy of the universe, Licata et al. postulate an "amazing right adjoint" $\mathbb{I} \surd \sqcup$ to the non-dependent path functor $\mathbb{I} \to \sqcup$ [LOPS18, Ort18], which indeed exists in presheaves over cartesian base categories if $\mathbb{I}$ is representable. Since $\mathbb{I} \surd \sqcup$ and its related axioms are global operations (only applicable to closed terms, unless you want to open Pandora's box as we do in the current paper), they keep everything sound by introducing a judgemental comonadic *global* modality $\flat$.

Orton et al.'s formalization [LOPS18, Ort18, OP18] is only what we call *meta-internal*: the argument is internalized to *some* type theory which still only serves as a metatheory of the type system of interest. Ideally, we would define and prove fibrancy of types *within* the type theory of interest, which we call *auto-internal*. Such treatments exist of discrete types in parametricity [CH20], and discrete, fibrewise-Segal and Rezk types in directed type theory [RS17], but not yet for covariant, Segal or Kan fibrant types due to the need to consider paths in the context $\mathbb{I} \to \Gamma$.

1.3. **The Transpension Type.** What is striking about the previous section is that, while most authors have been able to solve their own problems, a common approach is completely absent. We have encountered Φ and Ψ [Mou16], the amazing right adjoint $\surd$ [LOPS18], Glue [CCHM17, NVD17], Weld [NVD17], mill [ND18b] and the strictness axiom [OP18]. We have also seen that Φ and Ψ presently require an affine base category, and that $\surd$ presently requires the global modality $\flat$.

The goal of the current paper is to develop a smaller collection of internal primitives that impose few restrictions on the choice of base category and allow the internal construction of the aforementioned operators when sound. To this end, we introduce the **transpension** type former $\between i : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Gamma, i : \mathbb{I})$ which in cartesian settings is right adjoint to $\Pi(i : \mathbb{I}) : \mathrm{Ty}(\Gamma, i : \mathbb{I}) \to \mathrm{Ty}(\Gamma)$ and is therefore not a quantifier binding i, but a coquantifier that *depends* on it. Using the transpension and Strict, we can construct Φ (when sound), Ψ, $\surd$ and Glue. Given a type former for certain pushouts, we can also construct Weld and mill, and perform reasoning akin to what is made possible by capturing locally fresh names.

The transpension coquantifier $\between(u : \mathbb{U}) : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Gamma, u : \mathbb{U})$ is part of a sequence of adjoints $\Sigma\, u \dashv \Omega\, u \dashv \Pi\, u \dashv \between u$, preceded by the Σ-type, weakening and the Π-type. Adjointness of the first three is provable from the structural rules of type theory. However, it is not immediately clear how to add typing rules for a further adjoint. Birkedal et al. [BCM$^+$20] explain how to add a single modality that has a left adjoint in the semantics. If we want to have two or more adjoint modalities internally, then we can use a multimodal type system such as MTT [GKNB20b, GKNB20a]. Each modality in MTT needs a semantic left adjoint, so we can only internalize $\Omega\, u$, $\Pi\, u$ and $\between u$. A drawback which we accept, is that $\Omega\, u$ and $\Pi\, u$ become modalities which are a bit more awkward to deal with than ordinary weakening and Π-types.

1.4. **Contributions.** Our central contribution is to reduce the plethora of interal presheaf operators in the literature to only a few operations.

- To this end, we introduce the **transpension type** $⟆(u : \mathbb{U})$, right adjoint to $\Pi(u : \mathbb{U})$, with typing rules built on *extensional* MTT [GKNB20b, GKNB20a]. We explain how it is reminiscent of the suspension type from HoTT [Uni13].
- More generally, the transpension type can be right adjoint to any quantifier-like operation $\forall(u : \mathbb{U})$ which need neither respect the exchange rule, nor weakening or contraction. In this setting, we also introduce the **fresh weakening** coquantifier $\exists(u : \mathbb{U})$, which is left adjoint to $\forall(u : \mathbb{U})$ and therefore coincides with weakening $\Omega(u : \mathbb{U})$ in cartesian settings.
- We provide a categorical semantics for $⟆(u : \mathbb{U})$ in almost any presheaf category $\mathrm{Psh}(\mathcal{W})$ over base category $\mathcal{W}$, for almost any representable object $\mathbb{U} = \mathbf{y}U$, $U \in \mathcal{W}$. To accommodate non-cartesian variables, our system is not parametrized by a representable object $\mathbb{U} = \mathbf{y}U$, but by an arbitrary endofunctor $\sqcup \ltimes U$ on $\mathcal{W}$: the **multiplier**. We introduce **criteria** for characterizing the multiplier – viz. semi-cartesian, cartesian, cancellative, affine, connection-free and quantifiable – which we use as requirements for internal type theoretic features. We identify a complication dubbed **spookiness** in certain models (most notably in guarded type theory), and define dimensionally split morphisms (a generalization of split epimorphisms) in order to include spooky models. We exhibit relevant multipliers in base categories found in the literature (section 4.3.1).
- We show that **all general presheaf internalization operators** that we are aware of – viz. Φ/extent (when sound), Ψ/Gel [Mou16, BCM15], the amazing right adjoint $\surd$ [LOPS18], Glue [CCHM17, NVD17], Weld [NVD17], mill [ND18b] and (with no formal claim) locally fresh names – can be **recovered** from just the transpension type, the strictness axiom and pushouts along $\mathsf{snd} : \varphi \times A \to A$ where $\varphi : \mathsf{Prop}$. In the process, some of these operators can be **improved**: We generalize Ψ from affine to arbitrary multipliers, including cartesian ones and we justify $\mathbb{U} \surd \sqcup$ without a global modality and get proper computation rules for it. Moreover, since our system provides an operation $🔒_{⟆\,u}$ for quantifying over contexts, we take a step towards auto-internalizing Orton et al.'s work [LOPS18, Ort18, OP18]. When Φ is not sound (e.g. in cartesian or non-connection-free settings), we suggest the internal notion of **transpensive** types to retain some of its power. Finally, a form of higher dimensional pattern matching is enabled by exposing $\forall(u : \mathbb{U})$ internally as a left adjoint.
- In a technical report [Nuy20b], we investigate how the modalities introduced in this paper commute with each other, and with prior modalities (i.e. those already present before adding the transpension type). We also consider natural transformations between modalities (called 2-cells in MTT) arising from natural transformations between multipliers.

While MTT [GKNB20b, GKNB20a] satisfies canonicity[2], decidable type-checking will at least require a computational understanding of the mode theory, and of some new typing rules that we add to MTT. For this reason, we build on extensional MTT [GKNB20a], and defer decidability and canonicity to future work.

Overview of the paper. In section 2, we demonstrate in a simple setting how the transpension type resembles the suspension type from HoTT and how it allows for higher-dimensional pattern matching. In section 3, we give a brief overview of the typing rules of MTT and

[2]The current state of affairs is that MTT extended with a single typing rule satisfies canonicity. That rule is not compatible with the model used in this paper, but Gratzer et al. are working on a canonicity theorem for MTT proper.

Affine shape variables:

$$\frac{\Gamma\,\mathsf{ctx}}{\Gamma, u : \mathbb{U}\,\mathsf{ctx}} \qquad \frac{\sigma : \Gamma \to \Gamma'}{(\sigma, u/v) : (\Gamma, u : \mathbb{U}) \to (\Gamma', v : \mathbb{U})} \qquad \frac{\sigma : \Gamma \to \Gamma'}{\sigma : (\Gamma, u : \mathbb{U}) \to \Gamma'}$$

Affine function type:

$$\frac{\Gamma, u : \mathbb{U} \vdash A\,\mathsf{type}}{\Gamma \vdash \forall\, u.A\,\mathsf{type}} \qquad \frac{\Gamma, u : \mathbb{U} \vdash a : A}{\Gamma \vdash \lambda u.a : \forall\, u.A} \qquad \frac{\Gamma \vdash f : \forall\, u.A}{\Gamma, u : \mathbb{U}, \delta : \Delta \vdash f\, u : A}$$

Telescope quantification:

$$\begin{array}{lcl} \forall\, u.() & = & () \\ \forall\, u.(\delta : \Delta, x : A) & = & (\forall\, u.(\delta : \Delta)), x|_{u=\sqcup} : \forall\, u.(A[\delta|_{u=u}/\delta]) \\ \forall\, u.(\delta : \Delta, v : \mathbb{U}) & = & (\forall\, u.(\delta : \Delta)), v : \mathbb{U} \end{array}$$

Transpension type:

$$\frac{\begin{array}{l}\Gamma, u : \mathbb{U} \vdash (\delta : \Delta)\,\mathsf{telescope} \\ \Gamma, \forall\, u.(\delta : \Delta) \vdash A\,\mathsf{type}\end{array}}{\Gamma, u : \mathbb{U}, \delta : \Delta \vdash \wr(u : \mathbb{U}).A\,\mathsf{type}} \qquad \frac{\Gamma, u : \mathbb{U} \vdash t : \wr\, u.A}{\begin{array}{l}\Gamma \vdash \mathsf{unmer}(u.t) : A \\ \text{where } \mathsf{unmer}(u.\mathsf{mer}\, u\, a) = a\end{array}}$$

$$\frac{\Gamma, \forall\, u.(\delta : \Delta) \vdash a : A}{\begin{array}{l}\Gamma, u : \mathbb{U}, \delta : \Delta \vdash \mathsf{mer}\, u\, a : \wr(u : \mathbb{U}).A \\ \text{where } \mathsf{mer}\, u\, (\mathsf{unmer}(v.t[v/u, \delta|_{u=v}/\delta])) = t\end{array}}$$

Figure 1: Selection of typing rules for a naïve transpension type.

introduce our notation using *ticks*. In section 4, we define the mode theory on which we will instantiate MTT and highlight some important modalities. The instantiation of MTT is the type system we are after, but it is next to unreadable for humans, so in section 5, we define an informal notation that is *much* less precise but more readable. In section 6, we supplement our MTT instance with a few specialized typing rules. In section 7, we investigate the structure of the transpension type. In section 8, we explain how to recover known internal presheaf operators. We conclude in section 9.

2. A Naïve Transpension Type

In this section, for purposes of demonstration, we present simplified typing rules for the transpension. Using these, we will already be able to exhibit the transpension as similar to a dependent version of the suspension in HoTT [Uni13]. Moreover, in order to showcase how the transpension type allows us to internalize the presheaf structure of other types, we will demonstrate a technique which we call higher-dimensional pattern matching and which has already been demonstrated by Pitts [Pit14] in nominal type theory using locally fresh names.

2.1. **Typing rules.** To do this, we first present, in fig. 1, typing rules for the transpension type in a very simple setting: a type system with affine shape variables $u : \mathbb{U}$. Variables to the left of u are understood to be fresh for u; variables introduced after u may be substituted with terms depending on u. In particular, we have no contraction $(w/u, w/v) : (w : \mathbb{U}) \to (u, v : \mathbb{U})$,

while exchange $(x : A, u : \mathbb{U}) \to (u : \mathbb{U}, x : A)$ only works in one direction. This is enforced by the special substitution rules for shape variables.

The rule for $f\,u$ requires that the function f be fresh for u, i.e. that f depend only on variables to the left of u [BCM15, Mou16].

Additionally, the system contains a transpension type $\wr\, u.A$ over $\mathbb{U}$, with more unusual rules. When checking the type $\wr(u : \mathbb{U}).A$ in context $(\Gamma, u : \mathbb{U}, y : B)$, the part A will be checked in a modified context [BV17, ND19], were $y : B$ (which potentially depends on u) will change type, becoming a function $y|_{u=\sqcup} : \forall\, u.B$ that can be applied to $v : \mathbb{U}$ yielding $y|_{u=v} : B[v/u]$. In other words, we get hold of the dependency of y on u. The *meridian* constructor $\mathsf{mer}\, u\, a$ is checked in a similar way, and is modelled by transposition for the adjunction $\forall\, u \dashv \wr\, u$. We remark that both $\wr\, u.A$ and $\mathsf{mer}\, u\, a$ depend on u, whereas A and a do not, so in a way the transpension lifts data to a higher dimension, turning points into $\mathbb{U}$-cells. The elimination rule takes data again to a lower dimension: it turns a dependent $\mathbb{U}$-cell in the transpension into a point in A. This is the co-unit of the adjunction. The β- and η-rules internalize the adjunction laws.

These typing rules are sound in certain presheaf categories (those where $\mathbb{U}$ is cancellative and affine, see definition 4.1), but are unsatisfactory in several respects. First, we have no story for substitutions which exist in cubical type systems such as $(0/i) : \Gamma \to (\Gamma, i : \mathbb{I})$ [BCM15, BCH14, CCHM17] or $(j \wedge k/i) : (\Gamma, j, k : \mathbb{I}) \to (\Gamma, i : \mathbb{I})$ [CCHM17], as there is no formation rule for $\wr\, 0.A$ or $\wr\, (j \wedge k).A$. Secondly, in non-affine generalizations, the transpension is not stable under substitution of the shape variables preceding u (cf. the commutation properties for transpension and substitution in [Nuy20b]). In order to obtain a better behaved type system, in the rest of the paper we will rely on MTT, which we briefly summarize in section 3.

2.2. **Poles.** We can still try to get a grasp on $\wr\, 0.A$, however. In general we have $T[0/i] \cong (\forall\, i.(i = 0) \to T)$. For $T = \wr\, i.A$, the latter type is inhabited by $\lambda i.\lambda e.\mathsf{mer}\, i\, (\lightning\, (e|_{i=1}))$, since $e|_{i=1}$ proves $1 = 0$. Moreover, using the η-rule, we can show that this is the only element.

Thus we see that the transpension type essentially consists of meridians $(i : \mathbb{I}) \to \wr\, i.T$ for all $t : T$ which are all equal when $i = 0$ or $i = 1$. This makes the transpension type quite reminiscent of a dependent version of the suspension type from HoTT [Uni13], although the quantification of the context is obviously a distinction.

2.3. **Higher-dimensional pattern matching.** Given two types $A, B : \mathsf{U}$, higher-dimensional pattern matching allows us to construct a function $\Gamma \vdash f : (\forall\, u.A \uplus B) \to (\forall\, u.A) \uplus (\forall\, u.B)$. This function expresses that any $\mathbb{U}$-shaped cell in the coproduct type $A \uplus B$ must be either a cell in A or a cell in B. In that sense, it exposes the presheaf structure of the coproduct type $A \uplus B$ and other colimit constructions. We can define f as follows (and generalization to $A, B : \forall\, u.\mathsf{U}$ is straightforward):

$$f\ \hat{c} = \mathsf{unmer}\left(u.\mathsf{case}\ \hat{c}\ u\ \mathsf{of}\left\{\begin{array}{lcl} \mathsf{inl}\, a & \mapsto & \mathsf{mer}\, u\, (\mathsf{inl}\, (\lambda v.a|_{u=v})) \\ \mathsf{inr}\, b & \mapsto & \mathsf{mer}\, u\, (\mathsf{inr}\, (\lambda v.b|_{u=v})) \end{array}\right\}\right)$$

The argument to $\mathsf{unmer}(u.\sqcup)$ should be of type $\wr\, u.((\forall\, u.A) \uplus (\forall\, u.B))$. Interestingly, since we have u in scope, we can apply $\hat{c} : \forall\, u.\, (A \uplus B)$ to it, and pattern match to decide which case we are in. Both cases are analogous; in the first case, a variable $a : A$ is brought in scope, so we are in context $(\Gamma, \hat{c} : \forall\, u.A \uplus B, u : \mathbb{U}, a : A)$. We then use the constructor $\mathsf{mer}\, u\, \sqcup$, which again removes u from scope and turns $a : A$ into a function $a|_{u=\sqcup} : \forall\, u.A$.

Then we trivially finish the proof by writing $\mathsf{inl}\,(\lambda v.a|_{u=v})$, where we have η-expanded $a|_{u=\llcorner\lrcorner}$ mainly for facilitating further narrative. In summary, between unmer and mer, we had temporary access to a variable $u : \mathbb{U}$ which allowed us to pattern-match on $\hat{c}\,u$. More conceptually, we can say that the transpension allows us to temporarily work with $\hat{c}$ as if it were a lower-dimensional value of type $A \uplus B$. We will see in section 8 how similar ideas can be used to implement other internalization operators. Interestingly, this construction of f using the transpension also comes with suitable computational behaviour. When we evaluate $f\,(\lambda u.\mathsf{inl}(\hat{a}\,u))$, then $\hat{a}\,u$ is substituted for a. Next, $(\hat{a}\,u)|_{u=v}$ simplifies to $\hat{a}\,v$, so we can η-contract $\lambda v.\hat{a}\,v$, and β-reduce unmer, which yields $\mathsf{inl}\,\hat{a}$ as expected.

3. Multimode Type Theory

As announced, we will rely on the extensional version of Gratzer et al.'s multimode and multimodal dependent type system MTT [GKNB20b, GKNB20a] in order to frame the transpension and its left adjoints as modal operators. We refer to the original work for details, but in this section we highlight some important aspects of MTT and introduce our notation using *ticks*.

3.1. **The mode theory.** MTT is parametrized by a *mode theory*, which is a strict 2-category whose objects, morphisms and 2-cells we will refer to as **modes**, **modalities** and, well, **2-cells** respectively. Semantically, every mode p will correspond to an entire model of dependent type theory $[\![p]\!]$. A modality $\mu : p \to q$ will consist of a functor $[\![🔒_\mu]\!] : [\![q]\!] \to [\![p]\!]$ and an operation $[\![\mu]\!]$ that is almost a dependent right adjoint (DRA [BCM+20]) to $[\![🔒_\mu]\!]$; for all our purposes it will be an actual DRA and even one arising from a weak CwF morphism [BCM+20, lemma 17][Nuy18a]. A 2-cell $\alpha : \mu \Rightarrow \nu$ is interpreted as a natural transformation $[\![\alpha \downarrow]\!] : [\![🔒_\nu]\!] \to [\![🔒_\mu]\!]$ and hence also gives rise to an appropriate transformation $[\![\alpha]\!] : [\![\mu]\!] \to [\![\nu]\!]$.

3.2. **Judgement forms.** The judgement forms of MTT are listed in fig. 2. All forms are annotated with a mode p which specifies in what category they are to be interpreted. Every judgement form also has a corresponding equality judgement, which is respected by everything as the typing rules are to be read as a specification of a generalized algebraic theory (GAT [Car86]). The statements $p\,\mathsf{mode}$, $\mu : p \to q$ and $\alpha : \mu \Rightarrow \nu$ are simply requirements about the mode theory. This means we give no syntax or equality rules for modalities and 2-cells: these are fixed by the choice of mode theory.

3.3. **Typing rules.** The typing rules are listed in Figures 2 to 4. We will discuss these in turn. As every mode corresponds to a model of all of dependent type theory (DTT), we can import all the usual typing rules of DTT and apply them at any given fixed mode. We assume that DTT has a **universe** à la Coquand with mutually inverse encoding and decoding operations (which we will henceforth suppress), and we ignore cumulativity-related hassle, referring to Gratzer et al. [GKNB20b] for details.

The formation and introduction rules of **modal types** $\langle \mu \mid A \rangle$ in fig. 3 (which should help to understand the more basic rules) work by transposition: we apply the left adjoint (in the form of a lock) to the context of their argument. As such, they behave like DRAs, but their elimination rule is weaker, so we call them weak DRAs.

Judgement forms:

$p \mid \Gamma\,\mathsf{ctx}$	Γ is a context at mode p,
$p \mid \sigma : \Gamma \to \Delta$	σ is a simultaneous substitution from Γ to Δ at mode p,
$p \mid \Gamma \vdash T\,\mathsf{type}$	T is a type in context Γ at mode p,
$p \mid \Gamma \vdash t : T$	t has type T in context Γ at mode p.

Prerequisites:

Basic rules of dependent type theory (including all desired types) at each mode p, e.g.:

CTX-EMPTY
$$\frac{q\,\mathsf{mode}}{q \mid \cdot\,\mathsf{ctx}}$$

CTX-EXT
$$\frac{q \mid \Gamma \vdash T\,\mathsf{type}}{q \mid \Gamma.T\,\mathsf{ctx}}$$

UNI
$$\frac{q \mid \Gamma\,\mathsf{ctx} \quad \ell \in \mathbb{N}}{q \mid \Gamma \vdash \mathsf{U}^q_\ell\,\mathsf{type}_{\ell+1}}$$

UNI:ELIM
$$\frac{q \mid \Gamma \vdash t : \mathsf{U}^q_\ell}{q \mid \Gamma \vdash \mathsf{El}(t)\,\mathsf{type}_\ell} \quad \text{where } \mathsf{El}(\ulcorner T \urcorner) = T$$

UNI:INTRO
$$\frac{q \mid \Gamma \vdash T\,\mathsf{type}_\ell}{q \mid \Gamma \vdash \ulcorner T \urcorner : \mathsf{U}^q_\ell} \quad \text{where } \ulcorner \mathsf{El}(t) \urcorner = t$$

Context locking:

Note: $\downarrow^{\mathsf{m}}_{\mathsf{m}}$ is shorthand for $1\downarrow^{\mathsf{m}}_{\mathsf{m}}$.

LOCK
$$\frac{q \mid \Gamma\,\mathsf{ctx} \quad \mu : p \to q}{p \mid \Gamma, \mathbb{a}^{\mathsf{m}}_{\mu}\,\mathsf{ctx}} \quad \text{where } \Gamma = (\Gamma, \mathbb{a}^{\bullet}_{1}),\ (\Gamma, \mathbb{a}^{\mathsf{n}}_{\nu}, \mathbb{a}^{\mathsf{m}}_{\mu}) = (\Gamma, \mathbb{a}^{\mathsf{nm}}_{\nu\circ\mu})$$

LOCK:FMAP
$$\frac{q \mid \sigma : \Gamma \to \Delta \qquad \mu, \nu : p \to q \qquad \alpha : \mu \Rightarrow \nu}{p \mid (\sigma, \alpha\downarrow^{\mathsf{m}}_{\mathsf{n}}) : (\Gamma, \mathbb{a}^{\mathsf{n}}_{\nu}) \to (\Delta, \mathbb{a}^{\mathsf{m}}_{\mu})}$$
where $\sigma = (\sigma, \downarrow^{\bullet}_{\bullet})$ $\quad (\sigma, \alpha'\downarrow^{\mathsf{m}'}_{\mathsf{n}'}, \alpha\downarrow^{\mathsf{m}}_{\mathsf{n}}) = (\sigma, (\alpha' \star \alpha)\downarrow^{\mathsf{m}'\mathsf{m}}_{\mathsf{n}'\mathsf{n}})$
$1 = (1, \downarrow^{\mathsf{m}}_{\mathsf{m}})$ $\quad (\sigma, \alpha\downarrow^{\mathsf{m}}_{\mathsf{n}}) \circ (\tau, \beta\downarrow^{\mathsf{n}}_{\mathsf{o}}) = (\sigma \circ \tau, (\beta \circ \alpha)\downarrow^{\mathsf{m}}_{\mathsf{o}})$

Modal context extension:

We consider the non-modal rule CTX-EXT and its introduction, elimination and computation rules as a special case of CTX-MODEXT for $p = q$ and $\mu = 1$.

CTX-MODEXT
$$\frac{q \mid \Gamma\,\mathsf{ctx} \quad \mu : p \to q \qquad p \mid \Gamma, \mathbb{a}^{\mathsf{m}}_{\mu} \vdash T\,\mathsf{type}}{q \mid \Gamma, \mu \vartriangleright x :^{\mathsf{m}} T\,\mathsf{ctx}}$$

CTX-MODEXT:INTRO
$$\frac{q \mid \sigma : \Gamma \to \Delta \qquad \mu : p \to q \qquad p \mid \Gamma, \mathbb{a}^{\mathsf{m}}_{\mu} \vdash t : T[\sigma]}{q \mid (\sigma, t/x\downarrow_{\mathsf{m}}) : \Gamma \to (\Delta, \mu \vartriangleright x :^{\mathsf{m}} T)}$$
where $\tau = ((x/\oslash) \circ \tau, (x\downarrow_{\mathsf{m}})[\tau]/x\downarrow_{\mathsf{m}})$
$(\sigma, t/x\downarrow_{\mathsf{m}}) \circ \rho = (\sigma \circ \rho, t[\rho, 1\downarrow^{\mathsf{m}}_{\mathsf{m}}]/x\downarrow_{\mathsf{m}})$

CTX-MODEXT:WKN
$$\frac{q \mid \Gamma\,\mathsf{ctx} \qquad \mu : p \to q \qquad p \mid \Gamma, \mathbb{a}^{\mathsf{m}}_{\mu} \vdash T\,\mathsf{type}}{q \mid (x/\oslash) : (\Gamma, \mu \vartriangleright x :^{\mathsf{m}} T) \to \Gamma} \quad \text{where } (x/\oslash) \circ (\sigma, t/x\downarrow_{\mathsf{m}}) = \sigma$$

CTX-MODEXT:VAR
$$\frac{q \mid \Gamma\,\mathsf{ctx} \qquad \mu : p \to q \qquad p \mid \Gamma, \mathbb{a}^{\mathsf{m}}_{\mu} \vdash T\,\mathsf{type} \quad \alpha : \mu \Rightarrow \mathsf{locks}(\Delta)}{q \mid \Gamma, \mu \vartriangleright x :^{\mathsf{m}} T, \Delta \vdash x\,\alpha\downarrow_{\mathsf{ticks}(\Delta)} : T[\alpha\downarrow^{\mathsf{m}}_{\mathsf{ticks}(\Delta)}]}$$
where $x\,\alpha\downarrow_{\mathsf{ticks}(\Delta)}[\mathrm{id}_\Gamma, t/x\downarrow_{\mathsf{m}}, \mathrm{id}_\Delta] = t[\alpha\downarrow^{\mathsf{m}}_{\mathsf{ticks}(\Delta)}]$
$x\,\alpha\downarrow_{\mathsf{ticks}(\Delta)}[\beta\downarrow^{\mathsf{ticks}(\Delta)}_{\mathsf{ticks}(\Theta)}] = x\,(\beta \circ \alpha)\downarrow_{\mathsf{ticks}(\Theta)}$

Locks in a telescope:

$\mathsf{locks}(\cdot) = 1$ $\quad \mathsf{locks}(\Delta, \mathbb{a}^{\mathsf{m}}_{\mu}) = \mathsf{locks}(\Delta) \circ \mu$ $\quad \mathsf{locks}(\Delta, \mu \vartriangleright x :^{\mathsf{m}} T) = \mathsf{locks}(\Delta)$
$\mathsf{ticks}(\cdot) = \bullet$ $\quad \mathsf{ticks}(\Delta, \mathbb{a}^{\mathsf{m}}_{\mu}) = \mathsf{ticks}(\Delta)\mathsf{m}$ $\quad \mathsf{ticks}(\Delta, \mu \vartriangleright x :^{\mathsf{m}} T) = \mathsf{ticks}(\Delta)$

Figure 2: Judgement forms and structural rules of MTT [GKNB20b][Nuy20a, fig. 5.5].

WDRA

$$\frac{\mu : p \to q \qquad p \mid \Gamma, \locked{}{}{\mu}{\mathsf{m}} \vdash A\,\mathsf{type}_\ell}{q \mid \Gamma \vdash \langle \mu \mid^{\mathsf{m}} A\rangle\,\mathsf{type}_\ell}$$

WDRA:INTRO

$$\frac{\mu : p \to q \qquad p \mid \Gamma, \locked{}{}{\mu}{\mathsf{m}} \vdash a : A}{q \mid \Gamma \vdash \mathsf{mod}^{\mathsf{m}}_{\mu}\, a : \langle \mu \mid^{\mathsf{m}} A\rangle}$$

WDRA:ELIM

$$\frac{\begin{array}{l}\mu : p \to q \qquad \nu : q \to r \\ q \mid \Gamma, \locked{}{}{\nu}{\mathsf{n}} \vdash \hat{a} : \langle \mu \mid^{\mathsf{m}} A\rangle \\ r \mid \Gamma, \nu \vartriangleright \hat{x} :^{\mathsf{n}} \langle \mu \mid^{\mathsf{m}} A\rangle \vdash C\,\mathsf{type} \\ r \mid \Gamma, \nu \circ \mu \vartriangleright x :^{\mathsf{nm}} A \vdash c : C[\mathsf{mod}^{\mathsf{m}}_{\mu}\, x{\downarrow_{\mathsf{nm}}}/\hat{x}{\downarrow_{\mathsf{n}}}]\end{array}}{\begin{array}{l}r \mid \Gamma \vdash \mathsf{let}^{\mathsf{n}}_{\nu}\,(\mathsf{mod}^{\mathsf{m}}_{\mu}\, x{\downarrow_{\mathsf{nm}}} = \hat{a})\,\mathsf{in}\,c : C[\hat{a}/\hat{x}{\downarrow_{\mathsf{n}}}] \\ \text{where } \mathsf{let}^{\mathsf{n}}_{\nu}\,(\mathsf{mod}^{\mathsf{m}}_{\mu}\, x{\downarrow_{\mathsf{nm}}} = \mathsf{mod}^{\mathsf{m}}_{\mu}\, a)\,\mathsf{in}\,c = c[a/x{\downarrow_{\mathsf{nm}}}]\end{array}}$$

Figure 3: Typing rules for MTT's modal types (weak DRAs) [GKNB20b][Nuy20a, fig. 5.6].

MODPI

$$\frac{p \mid \Gamma, \locked{}{}{\mu}{\mathsf{m}} \vdash A\,\mathsf{type}_\ell \qquad \mu : p \to q \qquad q \mid \Gamma, \mu \vartriangleright x :^{\mathsf{m}} A \vdash B\,\mathsf{type}_\ell}{q \mid \Gamma \vdash (\mu \vartriangleright x :^{\mathsf{m}} A) \to B\,\mathsf{type}_\ell}$$

MODPI:INTRO

$$\frac{\mu : p \to q \qquad q \mid \Gamma, \mu \vartriangleright x :^{\mathsf{m}} A \vdash b : B}{\begin{array}{l}q \mid \Gamma \vdash \lambda(\mu \vartriangleright x).b : (\mu \vartriangleright x :^{\mathsf{m}} A) \to B \\ \text{where } \lambda(\mu \vartriangleright x).(f \cdot^{\mathsf{m}} x{\downarrow_{\mathsf{m}}}) = f\end{array}}$$

MODPI:ELIM

$$\frac{q \mid \Gamma \vdash f : (\mu \vartriangleright x :^{\mathsf{m}} A) \to B \qquad p \mid \Gamma, \locked{}{}{\mu}{\mathsf{m}} \vdash a : A \qquad \mu : p \to q}{\begin{array}{l}q \mid \Gamma \vdash f \cdot^{\mathsf{m}} a : B[a/x] \\ \text{where } (\lambda(\mu \vartriangleright x).b) \cdot^{\mathsf{m}} a = b[a/x{\downarrow_{\mathsf{m}}}]\end{array}}$$

Figure 4: Typing rules for MTT's modal Π-types [GKNB20b][Nuy20a, fig. 5.7] (of which non-modal Π-types are a special case for $p = q$ and $\mu = 1$).

In order to disambiguate in certain situations, we choose to enrich the original MTT syntax [GKNB20b, GKNB20a] by annotating locks with a name, which we call **ticks** [Nuy20a, §5.3] after Bahr et al. [BGM17]. So $\langle \mu \mid^{\mathsf{m}} A\rangle$ and $\mathsf{mod}^{\mathsf{m}}_{\mu}\, a$ will both bind a tick m.

Going back to fig. 2, **context formation** starts with the empty context which exists at any mode, and proceeds by adding locks and variables.

Adding **locks** is strictly functorial: it preserves identity and composition of modalities (and we take the liberty to use strings like $\mathtt{nm}$ and $\bullet$ as ticks). In fact, it is strictly 2-functorial: it also has an action on 2-cells (producing substitutions between locked contexts) that preserves identity and composition of 2-cells. It is also strictly bifunctorial: we can combine a substitution and a 2-cell to a substitution between locked contexts. If the 2-cell is the identity and we are just renaming locks, then we write $\downarrow^{\mathsf{m}}_{\mathsf{m}'}$ for $1_\mu{\downarrow^{\mathsf{m}}_{\mathsf{m}'}}$.[3]

A **modal variable** $\mu \vartriangleright x :^{\mathsf{m}} T$ is essentially the same as a non-modal variable $\hat{x} : \langle \mu \mid^{\mathsf{m}} T\rangle$ (which in turn is shorthand for $1 \vartriangleright \hat{x} :^{\bullet} \langle \mu \mid^{\mathsf{m}} T\rangle$), but the judgemental modal annotation allows direct access to a term of type A through the variable rule. Hence, the type T is checked the same way as it would be in $\langle \mu \mid^{\mathsf{m}} T\rangle$. Terms substituted for a modal variable x

[3] Just as a variable is a placeholder for another program to be *pre*composed, a tick is a placeholder for another 2-cell to be *post*composed. Thus, it may seem silly that our notation cleanly separates ticks from 2-cells, and we could write $\mathsf{n} \circ \alpha/\mathsf{m}$ instead of $\alpha{\downarrow^{\mathsf{m}}_{\mathsf{n}}}$. While mathematically more elegant, we found this to be completely unreadable.

are also checked in the locked context, and therefore term substitution binds a tick. The basic **variable** rule would be $\Gamma, \mu \mathbin{\vartriangleright} x :^{\mathsf{m}} T, \mathbb{\Large 🔒}^{\mathsf{m}}_{\mu} \vdash x\downarrow_{\mathsf{m}} : T$. A 2-cell $\alpha : \mu \Rightarrow \nu$ takes the variable to a context with $\mathbb{\Large 🔒}^{\mathsf{n}}_{\nu}$ instead of $\mathbb{\Large 🔒}^{\mathsf{m}}_{\mu}$. By functoriality of locks, $\mathbb{\Large 🔒}^{\mathsf{n}}_{\nu}$ may be split into multiple locks, and by weakening we may insert types in between. Building in these substitutions, we get the general variable rule which features an arbitrary telescope Δ with $\alpha : \mu \Rightarrow \mathsf{locks}(\Delta)$.

Modal elimination (fig. 3) uses a let-syntax to turn the modal type into a judgemental annotation on a variable. Modal **function type** formation and introduction (fig. 4) are by simple abstraction. Modal function application binds a tick.

3.4. **Results.** We highlight some results about MTT that are relevant in the current paper.

Proposition 3.1. *We have* $\langle 1 \mid^{\bullet} A\rangle \cong A$ *and* $\langle \nu \circ \mu \mid^{\mathsf{nm}} A\rangle \cong \langle \nu \mid^{\mathsf{n}} \langle \mu \mid^{\mathsf{m}} A\rangle\rangle$.

Proposition 3.2. *For any 2-cell* $\alpha : \mu \Rightarrow \nu$*, we have* $\langle \mu \mid^{\mathsf{m}} A\rangle \to \langle \nu \mid^{\mathsf{n}} A[\alpha\downarrow^{\mathsf{m}}_{\mathsf{n}}]\rangle$.

Proposition 3.3. *If* $\kappa \dashv \mu$ *internal to the mode theory, there is a function* $\mathsf{prmod}_{\mu} : (\kappa \mathbin{\vartriangleright}^{\mathbb{k}} \langle \mu \mid^{\mathsf{m}} A\rangle) \to A[\varepsilon\downarrow^{\mathbb{k}\mathsf{m}}_{\bullet}]$*, satisfying a* β*- and (thanks to extensionality) an* η*-law. Combined with these rules,* prmod_{μ} *is equally expressive as the* let*-eliminator for* $\langle \mu \mid \sqcup \rangle$.

Proposition 3.4. *If* $\kappa \dashv \mu$ *internal to the mode theory, there is an isomorphism of contexts* $\sigma = (x\,\eta\downarrow_{\mathsf{m}\mathbb{k}}/y\downarrow_{\mathbb{k}}) : (\Gamma, x : A, \mathbb{\Large 🔒}^{\mathsf{m}}_{\mu}) \cong (\Gamma, \mathbb{\Large 🔒}^{\mathsf{m}}_{\mu}, \kappa \mathbin{\vartriangleright} y :^{\mathbb{k}} A[\eta\downarrow^{\bullet}_{\mathsf{m}\mathbb{k}}])$ *with inverse* $\sigma^{-1} = (\mathrm{id}_{\Gamma}, \eta\downarrow^{\bullet}_{\mathsf{m}\mathbb{k}}, y\downarrow_{\mathbb{k}}/x\downarrow_{\bullet}, \downarrow^{\mathsf{m}}_{\mathsf{m}'}) \circ (\mathrm{id}_{(\Gamma, \mathbb{\Large 🔒}^{\mathsf{m}}_{\mu}, y)}, \varepsilon\downarrow^{\mathbb{k}\mathsf{m}'}_{\bullet})$*. Correspondingly, given* B *in the latter context, there is an isomorphism of types* $((x : A) \to \langle \mu \mid^{\mathsf{m}} B[\sigma]\rangle) \cong \big\langle \mu \mid^{\mathsf{m}} (\kappa \mathbin{\vartriangleright} y :^{\mathbb{k}} A[\eta\downarrow^{\bullet}_{\mathsf{m}\mathbb{k}}]) \to B\big\rangle$.

4. A Mode Theory for Shape (Co)quantification

For space reasons, we assume that there are no prior modalities, i.e. that the type system to which we wish to add a transpension type is non-modal in the sense that it has a single mode and only the identity modality. Prior modalities are considered in the technical report [Nuy20b]. We assume that this single prior mode is modelled by the presheaf category $\mathrm{Psh}(\mathcal{W})$.

4.1. **Shape Contexts.** A first complication is that the modalities $\Omega\, u \dashv \Pi\, u \dashv \between u$ all bind or depend on a variable, a phenomenon which is not supported by MTT. However, the following trick solves this problem. Assume we have in the prior system a context Ξ modelled by a presheaf over $\mathcal{W}$. Then the presheaves $\mathrm{Psh}(\mathcal{W}/\Xi)$ over the category of elements $\mathcal{W}/\Xi$ of the presheaf Ξ are also a model of dependent type theory. Denoting the judgements of the latter system with a prefix $\Xi \mid$, it happens to be the case that judgements $\Xi \mid \Gamma \vdash J$ (i.e. $\Gamma \vdash J$ in $\mathrm{Psh}(\mathcal{W}/\Xi)$) have precisely the same meaning as judgements $\Xi.\Gamma \vdash J$ in $\mathrm{Psh}(\mathcal{W})$ (for a suitable but straightforward translation of J). Thus, we will group together all shape variables (variables for which we want a transpension type) in a **shape context** Ξ in front of the typing context. Our judgements will then take the form $\Xi \mid \Gamma \vdash J$. This allows us to frame Ξ as the *mode* of the judgement.

4.2. **Mode Theory.** For simplicity, we take a highly general mode theory and will then only be able to say interesting things about specific modes, modalities and 2-cells. In practice, and especially in implementations, one will want to select a more syntactic subtheory right away.

As **modes**, we take the set of all small presheaves over $\mathcal{W}$, which we think of as **shape contexts**. The mode Ξ is modelled in $\mathrm{Psh}(\mathcal{W}/\Xi)$. As **modalities** $\mu : \Xi_1 \to \Xi_2$, we take all functors $[\![🔒_\mu]\!] : \mathrm{Psh}(\mathcal{W}/\Xi_2) \to \mathrm{Psh}(\mathcal{W}/\Xi_1)$ which have a right adjoint μ that is then automatically a weak CwF morphism [Nuy20b][Nuy20a, thm. 6.4.1] and gives rise to a DRA [BCM+20, lemma 17][Nuy18a]. As **2-cells** $\alpha : \mu \Rightarrow \nu$, we take all natural transformations.

4.3. **Shapes and Multipliers.** We now proceed by highlighting some interesting modes, modalities, and 2-cells. To begin with, we fix a collection of shapes. We associate to each shape $\mathbb{U}$ a functor $\sqcup \ltimes U : \mathcal{W} \to \mathcal{W}$ which extends to a functor $\sqcup \ltimes \mathbf{y}U : \mathrm{Psh}(\mathcal{W}) \to \mathrm{Psh}(\mathcal{W})$.[4] Internally, we will use shape variables to increase human readability and to reduce the heaviness of notation for shape substitutions, writing $(\Xi, u : \mathbb{U})$ for the shape context $\Xi \ltimes \mathbf{y}U$. However, in a fully elaborate syntax these variables would be redundant.

Of course, if we model shape context extension with $u : \mathbb{U}$ by an *arbitrary* functor, then we will not be able to prove many results. Depending on the properties of the functor, the variable u will behave like an affine one or like a cartesian one (or perhaps neither) and the Φ-combinator will or will not be sound for $\mathbb{U}$. For this reason, we introduce some criteria that help us classify shapes:

Definition 4.1. Assume $\mathcal{W}$ has a terminal object $\top$. A **multiplier** for an object U is a functor $\sqcup \ltimes U : \mathcal{W} \to \mathcal{W}$ such that $\top \ltimes U \cong U$.[5] This gives us a natural second projection $\pi_2 : (\sqcup \ltimes U) \to U$. We define the **fresh weakening functor** to the slice category as $\nexists_U : \mathcal{W} \to \mathcal{W}/U : W \mapsto (W \ltimes U, \pi_2)$. We say that a multiplier (as well as its shape) is:

- **Semicartesian** if it is copointed, i.e. has a first projection $\pi_1 : (\sqcup \ltimes U) \to \mathrm{Id}$,
- **Cartesian** if it is naturally isomorphic to the cartesian product with U,
- **Cancellative** if $\nexists_U$ is faithful,
- **Affine** if $\nexists_U$ is full,
- **Connection-free** if $\nexists_U$ is essentially surjective on objects (V, ψ) such that ψ is dimensionally split (definition 4.3),
- **Quantifiable** if $\nexists_U$ has a left adjoint $\exists_U : \mathcal{W}/U \to \mathcal{W}$.

Variables of shape $\mathbb{U}$ admit weakening if and only if the multiplier is semicartesian, and exchange and contraction if (but not only if) it is cartesian. Weakening, exchange and contraction are all shape substitutions, which will be internalized as modalities.

A non-trivial multiplier cannot be both affine and cartesian:

Proposition 4.2. *If a multiplier is affine and cartesian, then it is the identity functor.* [Nuy20b]

Definition 4.3. A morphism $\psi : V \to U$ is called **dimensionally split** (w.r.t. $\sqcup \ltimes U$) if there is some W such that $\pi_2 : W \ltimes U \to U$ factors over ψ. We define the **boundary** ∂U as the subpresheaf of the Yoneda-embedding $\mathbf{y}U$ consisting of those morphisms that are

[4] Both $\sqcup \ltimes U$ and $\sqcup \ltimes \mathbf{y}U$ are to be regarded as single-character symbols, i.e. $\ltimes$ in itself is meaningless.

[5] In the technical report [Nuy20b], we generalize beyond endofunctors.

not dimensionally split, and we define $\sqcup \ltimes \partial U$ by pullback.[6] We also write $(\Xi, u : \partial\mathbb{U})$ for $\Xi \ltimes \partial U$.

In most popular base categories, all morphisms to $\top$ are split epi. Being dimensionally split is then equivalent to being split epi. We call a category **spooky** if some morphism to $\top$ is not split epi. The notion of dimensionally split morphisms lets us consider the boundary and connection-freedom (a requirement for modelling Φ) also in spooky base categories, where the output of $\exists_U$ may not be split epi.

4.3.1. *Examples.* Let us look at some examples of multipliers. Their properties are listed in fig. 5. Most properties are easy to verify, so we omit the proofs.

Example 4.4 (Identity)**.** The identity functor on an arbitrary category $\mathcal{W}$ is an endomultiplier for $\top$. It is quantifiable, with $\exists_\top : \mathcal{W}/\top \to \mathcal{W} : (W, ()) \mapsto W$.

Example 4.5 (Cartesian product)**.** Let $\mathcal{W}$ be a category with finite products and $U \in \mathcal{W}$. Then $\sqcup \times U$ is an endomultiplier for U, which is affine if and only if it is the identity (i.e. U is terminal, cf. proposition 4.2 and example 4.4). It is quantifiable with $\exists_U : \mathcal{W}/U \to \mathcal{W} : (W, \psi) \mapsto W$. Hence, we have $\exists_U \exists_U = \sqcup \times U$.

Definition 4.6. Let RG be the category generated by the following diagram and equations:

$$\mathbb{N} \underset{\mathbf{t}}{\overset{\mathbf{s}}{\underset{\longleftarrow \mathbf{r} \longrightarrow}{\rightleftarrows}}} \mathbb{I} \qquad \begin{aligned} \mathbf{r} \circ \mathbf{s} &= 1_{\mathbb{N}}, \\ \mathbf{r} \circ \mathbf{t} &= 1_{\mathbb{N}}. \end{aligned}$$

A presheaf over RG is a reflexive graph. More generally, let ${}^k\mathsf{RG}$ be the category generated by the following:

$$\mathbb{N} \overset{\mathbf{e}_0, \ldots, \mathbf{e}_{k-1}}{\underset{\longleftarrow \mathbf{r} \longrightarrow}{\rightrightarrows}} \mathbb{I} \qquad \mathbf{r} \circ \mathbf{e}_i = 1_{\mathbb{N}}.$$

Example 4.7 (Cartesian cubes)**.** Let ${}^a\mathsf{Cube}$ be the category of cartesian a-ary cubes. It is the free cartesian monoidal category with same terminal object over ${}^a\mathsf{RG}$. Concretely:

- Its objects take the form $(i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I})$ (the names are desugared to de Bruijn indices, i.e. the objects are really just natural numbers),
- Its morphisms $(i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I}) \to (j_1 : \mathbb{I}, \ldots, j_m : \mathbb{I})$ are arbitrary functions
$$\varphi : \{j_1, \ldots, j_m\} \to \{i_1, \ldots, i_n\} \cup \{0, \ldots, a-1\} : j \mapsto j\langle\varphi\rangle.$$
We also write $\varphi = (j_1\langle\varphi\rangle/j_1, \ldots, j_m\langle\varphi\rangle/j_m)$. If a variable i is not used, we may write $i/\oslash$ to emphasize this.

This category is spooky if and only if $a = 0$. On this category, we consider the multiplier $\sqcup \times (i : \mathbb{I})$, which is an instance of example 4.5.

Example 4.8 (Affine cubes)**.** Let ${}^a\mathsf{Cube}_\square$ be the category of affine a-ary cubes as used in [BCH14] (binary) or [BCM15] (unary). It is the free monoidal category with (same) terminal unit over ${}^a\mathsf{RG}$. Concretely:

- Objects are as in ${}^a\mathsf{Cube}$,

[6] I.e. $\Xi \ltimes \partial U$ is the pullback of $\Xi \ltimes \mathbf{y}U \to \mathbf{y}U \supseteq \partial U$. It is the subpresheaf of $\Xi \ltimes \partial U$ of all cells ψ such that $\pi_2 \circ \psi$ is *not* dimensionally split.

Example	Base category	Multiplier	Spooky category	Weakening/ Semicartesian	Exchange	Contraction	Cartesian	Cancellative	Affine	Connection-free	Quantifiable
4.4	$\mathcal{W}$	Id	?	✔	✔	✔	✔	✔	✔	✔	✔
4.5	$\mathcal{W}$	$(\sqcup \times U) \ncong \mathrm{Id}$	?	✔	✔	✔	✔	?	✖	?	✔
4.7	${}^a\mathsf{Cube}$	$\sqcup \times (i : \mathbb{I})$	$a = 0$	✔	✔	✔	✔	✔	✖	✔	✔
4.8	${}^a\mathsf{Cube}_\square$	$\sqcup * (i : \mathbb{I})$	$a = 0$	✔	✔	✖	✖	✔	✔	✔	✔
4.9	CCHM	$\sqcup \times (i : \mathbb{I})$	✖	✔	✔	✔	✔	✔	✖	✖	✔
4.10	DCube_d	$\sqcup \times (i : (\!|k|\!))$	✖	✔	✔	✔	✔	✔	✖	✔	✔
4.11	Clock	$\sqcup \times (i : \circledcirc_k)$	✔	✔	✔	✔	✔	✔	✖	✔	✔
4.12	TwCube	$\sqcup \ltimes \mathbb{I}$	✖	✖	✖	✖	✖	✔	✔	✔	✔
4.13	n	$\min(\sqcup, i)$	✔	✔	✔	✔	✔	✔	✖	✔	✔
4.14	Simplex	$\sqcup \uplus_< [0]$	✖	✔	✖	✔	✖	✔	✔	✖	✔
4.15	${}^2\mathsf{Cube}_\bot$	$\sqcup \times \bot$	✖	✔	✔	✔	✔	✖	✖	✔	✔

Figure 5: Some interesting multipliers and their properties.

- Morphisms are as in ${}^a\mathsf{Cube}$ such that if $j\langle\varphi\rangle = k\langle\varphi\rangle \notin \{0, \dots, a-1\}$, then $j = k$. This rules out diagonal maps.

This category is spooky if and only if $a = 0$. On this category, we the functor $\sqcup * (i : \mathbb{I}) : W \mapsto (W, i : \mathbb{I})$, which is a multiplier for $(i : \mathbb{I})$. This functor is quantifiable with $\exists_{(i:\mathbb{I})}((W, j : \mathbb{I}), (j/i)) = W$ and $\exists_{(i:\mathbb{I})}(W, (\varepsilon/i)) = W$ for each of the a endpoints ε.

In the nullary case, ${}^0\mathsf{Cube}_\square$ is the base category of the Schanuel topos, which is equivalent to the category of nominal sets [Pit13]. In that case, $\exists_{(i:\mathbb{I})}$ is not just left adjoint to $\dashv_{(i:\mathbb{I})}$, but in fact an inverse and hence also right adjoint. This is in line with the fact that in nominal type theory [PMD15], there is a single name quantifier which can be read as either existential or universal quantification.

Example 4.9 (CCHM cubes)**.** Let CCHM be the category of CCHM cubes [CCHM17]. Its objects are as in ${}^2\mathsf{Cube}$ and its morphisms are functions from $\{j_1, \dots, j_m\}$ to the free de Morgan algebra over $\{i_1, \dots, i_n\}$. We again consider $\sqcup \times (i : \mathbb{I})$, another instance of example 4.5. Connection-freedom is now violated by $(j \vee k/i), (j \wedge k/i) : (j : \mathbb{I}, k : \mathbb{I}) \to (i : \mathbb{I})$.

Example 4.10 (Depth d cubes)**.** Let DCube_d with $d \geq -1$ be the category of depth d cubes, used as a base category in degrees of relatedness [ND18a, Nuy18a]. This is a generalization of the category of binary cartesian cubes Cube, where instead of typing every dimension with *the* interval $\mathbb{I}$, we type them with the k-interval $(\!|k|\!)$, where $k \in \{0, \dots, d\}$ is called the degree of relatedness of the edge. Its objects take the form $(i_1 : (\!|k_1|\!), \dots, i_n : (\!|k_n|\!))$. Conceptually, we have a map $(\!|k|\!) \to (\!|\ell|\!)$ if $k \geq \ell$. Thus, morphisms $\varphi : (i_1 : (\!|k_1|\!), \dots, i_n : (\!|k_n|\!)) \to (j_1 : (\!|\ell_1|\!), \dots, j_m : (\!|\ell_m|\!))$ send every variable $j : (\!|\ell|\!)$ of the codomain to $j\langle\varphi\rangle$, which is either 0, 1 or a variable $i : (\!|k|\!)$ of the domain such that $k \geq \ell$.

- If $d = -1$, then there is only one object () and only the identity morphism, i.e. we have the point category.
- If $d = 0$, we just get Cube.

- If $d = 1$, we obtain the category of bridge/path cubes $\mathsf{BPCube} := \mathsf{DCube}_1$. We write $\mathbb{P}$ for $(\!|0|\!)$ (the path interval) and $\mathbb{B}$ for $(\!|1|\!)$ (the bridge interval). Bridge/path cubical sets are used as a model for parametric quantifiers [NVD17, Nuy18a].

On this category, we consider $\sqcup \times (i : (\!|k|\!))$, which is another instance of example 4.5.

Example 4.11 (Clocks)**.** Let Clock be the category of clocks, used as a base category in guarded type theory [BM18]. It is the free cartesian category over ω. Concretely:

- Its objects take the form $(i_1 : \clubsuit_{k_1}, \ldots, i_n : \clubsuit_{k_n})$ where all $k_j \geq 0$. We can think of a variable of type $\clubsuit_k$ as representing a clock (i.e. a time dimension) paired up with a certificate that we do not care what happens after the time on this clock exceeds k.
- Correspondingly, we should have a map $\clubsuit_k \to \clubsuit_\ell$ if $k \leq \ell$. So the morphisms $\varphi : (i_1 : \clubsuit_{k_1}, \ldots, i_n : \clubsuit_{k_n}) \to (j_1 : \clubsuit_{\ell_1}, \ldots, j_m : \clubsuit_{\ell_m})$ are functions that send (de Bruijn) variables $j : \clubsuit_\ell$ of the codomain to a variable $j\langle\varphi\rangle : \clubsuit_k$ of the domain such that $k \leq \ell$.

This category is spooky. On this category we consider $\sqcup \times (i : \clubsuit_k)$, which is another instance of example 4.5.

Example 4.12 (Twisted cubes)**.** Pinyo and Kraus's category of twisted cubes TwCube [PK19] can be described as a subcategory of the category of non-empty finite linear orders (or, if you want, of its skeletalization: the category of simplices $\mathsf{Simplex}$). On these categories, we can define a functor $\sqcup \ltimes \mathbb{I}$ such that $W \ltimes \mathbb{I} = W^{\mathsf{op}} \uplus_< W$, where we consider elements from the left smaller than those from the right. Now TwCube is the subcategory of $\mathsf{Simplex}$ whose objects are generated by $\top$ and $\sqcup \ltimes \mathbb{I}$ (note that every object then also has an opposite since $\top^{\mathsf{op}} = \top$ and $(V \ltimes \mathbb{I})^{\mathsf{op}} \cong V \ltimes \mathbb{I}$), and whose morphisms are given by

- $(\varphi, 0) : \mathrm{Hom}_{\mathsf{TwCube}}(V, W \ltimes \mathbb{I})$ for all $\varphi : \mathrm{Hom}_{\mathsf{TwCube}}(V, W^{\mathsf{op}})$,
- $(\varphi, 1) : \mathrm{Hom}_{\mathsf{TwCube}}(V, W \ltimes \mathbb{I})$ for all $\varphi : \mathrm{Hom}_{\mathsf{TwCube}}(V, W)$,
- $\varphi \ltimes \mathbb{I} : \mathrm{Hom}_{\mathsf{TwCube}}(V \ltimes \mathbb{I}, W \ltimes \mathbb{I})$ for all $\varphi : \mathrm{Hom}_{\mathsf{TwCube}}(V, W)$,
- $() : \mathrm{Hom}_{\mathsf{TwCube}}(V, \top)$.

Note that this collection automatically contains all identities, composites, and opposites. Isomorphism to Pinyo and Kraus's category of twisted cubes can be seen from their ternary representation [PK19, def. 34]. We now consider the multiplier $\sqcup \ltimes \mathbb{I} : \mathsf{TwCube} \to \mathsf{TwCube}$, which Pinyo and Kraus call the twisted prism functor. It is quantifiable with

$$\exists_{\mathbb{I}} : \begin{cases} (W, ((), 0)) & \mapsto & W^{\mathsf{op}} \\ (W, ((), 1)) & \mapsto & W \\ (W \ltimes \mathbb{I}, () \ltimes \mathbb{I}) & \mapsto & W, \end{cases} \tag{4.1}$$

with the obvious action on morphisms.

Example 4.13 (Finite ordinals)**.** In the base category ω of the topos of trees, used in guarded type theory [BMSS12], a cartesian product is given by $i \times j = \min(i, j)$. However, this category lacks a terminal object. Instead, on the subcategory n, which is endowed with the same cartesian product, we consider the multiplier $\sqcup \times i$, which is again an instance of example 4.5.

Example 4.14 (Simplices)**.** In the category of simplices $\mathsf{Simplex}$, we consider the functor $\sqcup \ltimes [1] := \sqcup \uplus_< [0]$, which adds a maximal element to a linear order. This is a multiplier for $[1]$ and is quantifiable with $\exists_{[1]}(W, \psi) = \psi^{-1}(0)$.

Example 4.15 (Non-cancellativity)**.** Let ${}^2\mathsf{Cube}_\perp$ be the category of binary cubes extended with an initial object. We consider the cartesian product $\sqcup \times \perp$ which sends everything to $\perp$.

4.4. **Modalities for Substitution.** A substitution $\sigma : \Xi_1 \to \Xi_2$ gives rise to a functor $\Sigma^{/\sigma} : \mathcal{W}/\Xi_1 \to \mathcal{W}/\Xi_2$ and hence [Sta19] to a triple of adjoint functors $\Sigma\,\sigma \dashv \Omega\,\sigma \dashv \Pi\,\sigma$ between the presheaf categories, where $\Omega\,\sigma$ has the exact same semantics as ordinary substitution. By consequence, if σ is a weakening substitution, then $\Sigma\,\sigma$ and $\Pi\,\sigma$ have semantics isomorphic to those of ordinary Σ- and Π-types.

The two functors $\Omega\,\sigma$ and $\Pi\,\sigma$ give rise to DRAs and can be internalized as MTT modalities. We denote these as[7] $\Omega\,\sigma$ or $\Omega(\xi_1 : \Xi_1, \xi_2 = \xi_2[\sigma]) : (\xi_2 : \Xi_2) \to (\xi_1 : \Xi_1)$ and as $\Pi\,\sigma$ or $\Pi(\xi_1 : \Xi_1, \xi_2 = \xi_2[\sigma]) : (\xi_1 : \Xi_1) \to (\xi_2 : \Xi_2)$. For example, in cubical type theory, we get $\Omega(i = 0) : (\Xi, i : \mathbb{I}) \to \Xi$ with right adjoint $\Pi(i = 0)$, and for semicartesian $\mathbb{U}$, we get $\Omega(u : \mathbb{U}) : \Xi \to (\Xi, u : \mathbb{U})$ with right adjoint $\Pi(u : \mathbb{U})$.

The Π-modality is strictly functorial, whereas the Ω-modality is pseudofunctorial: we need explicit 2-cells witnessing $\Omega\,\tau \circ \Omega\,\sigma \cong \Omega(\tau \circ \sigma)$.[8] These modalities are adjoint internally by virtue of the 2-cells for the unit $\mathsf{const}_\sigma : 1 \Rightarrow \Pi\,\sigma \circ \Omega\,\sigma$ and co-unit $\mathsf{app}_\sigma : \Omega\,\sigma \circ \Pi\,\sigma \Rightarrow 1$. If σ introduces variables, then the codomain of the co-unit may be a variable renaming that is semantically the identity, e.g. $\mathsf{app}_{(v/u:\mathbb{U})} : \Omega(v : \mathbb{U}) \circ \Pi(u : \mathbb{U}) \Rightarrow \Omega(v : \mathbb{U}, u = v)$.

Example 4.16. If $\mathbb{U}$ is cartesian, then there is a diagonal substitution $(w/u, w/v) : (\Xi, w : \mathbb{U}) \to (\Xi, u, v : \mathbb{U})$. Writing

$$\alpha = 1_{\Pi\,u} \star 1_{\Pi\,v} \star \mathsf{const}_{(w,u=w,v=w)} : \Pi\,u \circ \Pi\,v \Rightarrow \\ \Pi\,u \circ \Pi\,v \circ \Pi(w, u = w, v = w) \circ \Omega(w, u = w, v = w) = \Pi\,w \circ \Omega(w, u = w, v = w),$$

this allows us to type the naïvely typed function $\lambda f.\lambda w.f\,w\,w : (\Pi\,u.\Pi\,v.A) \to \Pi\,w.A[w/u, w/v]$ as

$$\langle \Pi(u : \mathbb{U}) \mid^{\mathsf{p}_u} \langle \Pi(v : \mathbb{U}) \mid^{\mathsf{p}_v} A\rangle\rangle \to \left\langle \Pi(w : \mathbb{U}) \mid^{\mathsf{p}_w} \left\langle \Omega(w : \mathbb{U}, u = w, v = w) \mid^{\mathsf{o}} A[\alpha{\downarrow}^{\mathsf{p}_u \mathsf{p}_v}_{\mathsf{p}_w \mathsf{o}}]\right\rangle\right\rangle.$$

Remark 4.17. The reframing of shape substitutions as a modality, has the annoying consequence that substitution no longer reduces. However, both $\langle \Omega\,\sigma \mid \sqcup\rangle$ and $\mathsf{mod}_{\Omega\,\sigma}$ are semantically an ordinary substitution.[9] Thus, we could add computation rules such as:

$$\begin{array}{rclcrcl} \langle \Omega\,\sigma \mid^{\mathsf{o}} A \times B\rangle & = & \langle \Omega\,\sigma \mid^{\mathsf{o}} A\rangle \times \langle \Omega\,\sigma \mid^{\mathsf{o}} B\rangle, & \qquad & \langle \Omega\,\sigma \mid^{\mathsf{o}} \mathsf{U}\rangle & = & \mathsf{U}, \\ \mathsf{mod}^{\mathsf{o}}_{\Omega\,\sigma}\,(a, b) & = & (\mathsf{mod}^{\mathsf{o}}_{\Omega\,\sigma}\,a, \mathsf{mod}^{\mathsf{o}}_{\Omega\,\sigma}\,b), & & \mathsf{mod}^{\mathsf{o}}_{\Omega\,\sigma}\,A & = & \langle \Omega\,\sigma \mid^{\mathsf{o}} A\rangle. \end{array}$$

This is fine in an extensional type system, but would not play well with the β-rule for modal types in an intensional system. Indeed, β-reduction for $\langle \Omega\,\sigma \mid^{\mathsf{o}} A\rangle$ requires a solution to the following problem: when $\hat{a} = \mathsf{mod}^{\mathsf{o}}_{\Omega\,\sigma}\,a$ definitionally, then we need to be able to infer a up to definitional equality from $\hat{a}$.

4.5. **Modalities for (Co)quantification.** The fresh weakening functor $\exists_U : \mathcal{W} \to \mathcal{W}/U$ generalizes to a functor $\exists_U^{/\Xi} : \mathcal{W}/\Xi \to \mathcal{W}/(\Xi \ltimes \mathbf{y}U)$ between categories of elements. Assuming that the multiplier is quantifiable,[10] there is a left adjoint $\exists_U^{/\Xi} \dashv \exists_U^{/\Xi}$ [Nuy20b]. These two give rise to four adjoint functors $\exists(u : \mathbb{U}) \dashv \exists(u : \mathbb{U}) \dashv \forall(u : \mathbb{U}) \dashv \between(u : \mathbb{U})$ between the

[7] By the second notation, we mean that we declare new variables (with their type if there is space) and write $u = t$ when we substitute t for u.

[8] This is because we defined the modality μ via $[\![\fbox{$\mu$}]\!]$ and $[\![\fbox{$\Pi\,\sqcup$}]\!] = \Omega\,\sqcup$ is strictly functorial in the model while $[\![\fbox{$\Omega\,\sqcup$}]\!] = \Sigma\,\sqcup$ is merely pseudofunctorial. However, Gratzer et al. [GKNB20a] have a strictification theorem for models of MTT which can strictify the isomorphism.

[9] Not along $\sigma : \Xi_1 \to \Xi_2$, but along $\sigma.\eta_{\Sigma\,\sigma \dashv \Omega\,\sigma} : \Xi_1.\Gamma \cong \Xi_2.\Sigma\,\sigma\,\Gamma$, which happens to be an isomorphism.

[10] We have not encountered any interesting examples where this is not the case.

presheaf categories. The latter three give rise to DRAs and can be internalized as MTT modalities.

Here, $\exists(u:\mathbb{U})$, $\fatslash(u:\mathbb{U})$ and $\forall(u:\mathbb{U})$ are the substructural counterparts to the cartesian (co)quantifiers $\Sigma(u:\mathbb{U})$, $\Omega(u:\mathbb{U})$ and $\Pi(u:\mathbb{U})$; and $\talloblong(u:\mathbb{U})$ is the transpension coquantifier. Reusing symbols for the cartesian counterparts, we denote the units and co-units of these adjunctions as

$$\mathsf{const}_{(u:\mathbb{U})} : 1 \Rightarrow \forall(u:\mathbb{U}) \circ \fatslash(u:\mathbb{U}) \qquad \mathsf{app}_{(u:\mathbb{U})} : \fatslash(u:\mathbb{U}) \circ \forall(u:\mathbb{U}) \Rightarrow 1$$
$$\mathsf{reidx}_{(u:\mathbb{U})} : 1 \Rightarrow \talloblong(u:\mathbb{U}) \circ \forall(u:\mathbb{U}) \quad \mathsf{unmer}_{(u:\mathbb{U})} : \forall(u:\mathbb{U}) \circ \talloblong(u:\mathbb{U}) \Rightarrow 1$$

Again, we also write $\mathsf{app}_{(v/u:\mathbb{U})} : \fatslash(v:\mathbb{U}) \circ \forall(u:\mathbb{U}) \Rightarrow \Omega(v:\mathbb{U}, u = v)$ and $\mathsf{reidx}_{(v/u:\mathbb{U})} : \Omega(v:\mathbb{U}, u = v) \Rightarrow \talloblong(v:\mathbb{U}) \circ \forall(u:\mathbb{U})$ to handle shape variable renamings that are semantically the identity.

Theorem 4.18 (Quantification)**.** [Nuy20b] *If the multiplier is*

- *cancellative and affine, then the substructural* $\mathsf{const}_{(u:\mathbb{U})}$ *and* $\mathsf{unmer}_{(u:\mathbb{U})}$ *are natural isomorphisms,*
- *semi-cartesian (so that* $\Omega(u:\mathbb{U})$ *and* $\Pi(u:\mathbb{U})$ *exist), then we have* $\mathsf{spoil}_{(u:\mathbb{U})} : \fatslash(u:\mathbb{U}) \Rightarrow \Omega(u:\mathbb{U})$ *and* $\mathsf{cospoil}_{(u:\mathbb{U})} : \Pi(u:\mathbb{U}) \Rightarrow \forall(u:\mathbb{U})$*,*
- *cartesian, then we can soundly identify* $\fatslash(u:\mathbb{U}) = \Omega(u:\mathbb{U})$ *and* $\forall(u:\mathbb{U}) = \Pi(u:\mathbb{U})$*.*

To understand the difference between $\fatslash(u:\mathbb{U})$ and $\Omega(u:\mathbb{U})$, we can compare to the naïve system (section 2) or Moulin et al.'s system [BCM15, Mou16]. There, shape variables $u:\mathbb{U}$ are part of the context, and variables to the left of u are fresh for u. Here, shape variables are all in the shape context, but we use $\fatslash\, u$ and $\Omega\, u$ (as well as the semantically identical $🔒_{\forall\, u}$ and $🔒_{\Pi\, u}$) to keep track of whether or not other variables should be fresh for u. Thus, in terms of the naïve system, $\forall(u:\mathbb{U})$ introduces u at the end of Γ (marking all variables fresh with $🔒_{\forall\, u}$), and $\Pi(u:\mathbb{U})$ introduces u in front of Γ (marking them non-fresh with $🔒_{\Pi\, u}$). The naïve system allows exchanging shape and other variables in one direction only, which is represented here by the generally non-invertible 2-cells spoil and cospoil. We emphasize that the word 'fresh' needs to be taken with a grain of salt: only for cancellative and affine $\mathbb{U}$ does invertibility of const_u guarantee that something fresh for u does not depend on u.

Example 4.19. As an instance of proposition 3.3, we obtain $\mathsf{prmod}_{\forall(v/u:\mathbb{U})} : (\fatslash(v:\mathbb{U}) \mathbin{|^{\mathsf{f}}} \langle \forall(u:\mathbb{U}) \mathbin{|^{\mathsf{a}}} A\rangle) \to \left\langle \Omega(v:\mathbb{U}, u = v) \mathbin{|^{\bullet}} A[\mathsf{app}_{v/u}{\downarrow}^{\mathsf{f}\mathsf{a}}_{\bullet}] \right\rangle$, which says that a non-cartesian function g with naïve type $g : \forall\, u.A$ can be applied to $v:\mathbb{U}$ provided that g is fresh for v.

Example 4.20 (Higher-dimensional pattern matching)**.** As in section 2, we create a function $f : \langle \forall(u:\mathbb{U}) \mathbin{|^{\mathsf{a}}} A \uplus B\rangle \to \langle \forall(u:\mathbb{U}) \mathbin{|^{\mathsf{a}}} A\rangle \uplus \langle \forall(u:\mathbb{U}) \mathbin{|^{\mathsf{a}}} B\rangle$, namely:

$$f\, \hat{c} = \mathsf{prmod}_{\talloblong\, u} \cdot^{\mathsf{a}} \mathsf{case}\,(\mathsf{prmod}_{\forall\, u} \cdot^{\mathsf{o}} (\hat{c}\, \mathsf{const}_u{\downarrow}_{\mathsf{a}\mathsf{o}}))\ \mathsf{of} \left\{ \begin{array}{l} \mathsf{inl}\, a \mapsto \mathsf{mod}^{\mathsf{t}}_{\talloblong\, u}\,(\mathsf{inl}\,(\mathsf{mod}^{\mathsf{a}'}_{\forall\, u}\,(a\, \mathsf{reidx}_u{\downarrow}_{\mathsf{t}\mathsf{a}'}))) \\ \mathsf{inr}\, b \mapsto \mathsf{mod}^{\mathsf{t}}_{\talloblong\, u}\,(\mathsf{inr}\,(\mathsf{mod}^{\mathsf{a}'}_{\forall\, u}\,(b\, \mathsf{reidx}_u{\downarrow}_{\mathsf{t}\mathsf{a}'}))) \end{array} \right\}.$$

We see that $\mathsf{mod}_{\forall\, u}$ and $\mathsf{prmod}_{\forall\, u}$ correspond to shape abstraction and application in the naïve system, $\mathsf{prmod}_{\talloblong\, u}$ to unmer, and $\mathsf{mod}_{\talloblong\, u}$ to mer. The annotation const_u is an explicit weakening over $u:\mathbb{U}$, whereas reidx_u replaces an explicit reindexing $|_{u=u}$.

5. An Informal Notation for Human Readers

At this point, we can instantiate MTT with the mode theory defined in the previous section 4. This is the type system that we will be working in, but direct usage of the notations from MTT (section 3) will be a bit obscure. For example, in section 2 we denoted quantification of a context using $\forall$. In the MTT instance, we will instead use a lock for the right adjoint to $\forall$, i.e. we will write $🔒_{\langle\!(\, u}$. Similarly, shape abstraction will not be denoted using a λ, but instead as $\mathsf{mod}_{\forall\, u}\, a$, and application will be denoted as $\mathsf{prmod}_{\forall\, u}\, f$.

We will actually use these notations, with the intention of making clear which MTT concepts we are using, because using an instance of MTT is precisely our solution to the syntactic complications involved in creating a syntax more similar to that of section 2. However, the MTT notation gets in the way of intuition and even readability. For this reason, we additionally introduce an alternative notation, which is purely informal (MTT being the formal notation).

Figure 6 lists the rule WDRA:INTRO in formal and informal notation for each of the available modalities, thus implicitly listing the informal notations for locks (LOCK), modal types (weak DRAs, WDRA) and their introduction rules (WDRA:INTRO). The introduction rule for $\forall(u : \mathbb{U}).A$ is in line with the abstraction rule in fig. 1: the premise's context is extended with a variable u, and all other variables are taken to be fresh for u. In the naïve system, we expressed this by putting u to the right of Γ; here, we wrap Γ in $\beth\, u$.

The modality $\Pi(i = 0)$ abstracts over the assumption that $i = 0$. So inside it, we get to use that assumption. By consequence, we no longer need i as it just means 0, so it disappears from the context. Its left adjoint $\Omega(i = 0)$ does the converse: when entering that modality, we define i to be 0 and subsequently forget that $i = 0$. Thus, within $\Omega(i = 0)$, we have i in scope; outside it, we do not, as we can just use 0. The quantifier $\Sigma(i = 0)$ retains this information so that Γ still makes sense, but to the right of the turnstile we do not have syntactic access to that information anymore.

It should be noted that, since Σ does not preserve the empty context, Ω has a different meaning when applied to contexts or to types. When applied to a context, it is an actual substitution of the shape context. When applied to a type, it is semantically still a substitution, but between two isomorphic shape-and-type contexts $(\Xi, u : \mathbb{U}).\Gamma$ and $\Xi.(\Sigma(u : \mathbb{U}).\Gamma)$. A similar remark holds for $\exists(u : \mathbb{U})$ and $\beth\, u$, with the additional note that when we apply $\beth\, u$ to a type, we transfer the responsibility of asserting freshness for u from the context (which hid u away via $\exists\, u$) to the type.

For cancellative and affine multipliers, we invoke the quantification theorem 4.18 to extract the fresh part of the context from the quantifier. Then the introduction rule for $\langle\!(\, u.A$ is in line with the meridian rule in fig. 1: u disappears from the context and the non-fresh part of the context (in the naïve system: the part to the right of u) is quantified over:

$$\frac{\Xi \mid \Gamma, 🔒^{a}_{\forall\, u}, \Delta, 🔒^{t}_{\langle\!(u:\mathbb{U})} \vdash a : A}{\Xi, u : \mathbb{U} \mid \Gamma, 🔒^{a}_{\forall\, u}, \Delta \vdash \mathsf{mod}^{t}_{\langle\!(\, u}\, a : \langle\!(\, u \mid^{t} A\rangle} \quad \rightsquigarrow \quad \frac{\Xi \mid \Gamma, \forall(u : \mathbb{U}).\Delta \vdash a : A}{\Xi, u : \mathbb{U} \mid \beth\, u.\Gamma, \Delta \vdash \mathsf{mer}\, u\, a : \langle\!(\, u.A}$$

The introduction rule for $\beth\, u.A$ looks similar, but uses existential quantification:

$$\frac{\Xi \mid \Gamma, 🔒^{a}_{\forall\, u}, \Delta, 🔒^{f}_{\exists(u:\mathbb{U})} \vdash a : A}{\Xi, u : \mathbb{U} \mid \Gamma, 🔒^{a}_{\forall\, u}, \Delta \vdash \mathsf{mod}^{f}_{\beth\, u}\, a : \langle\beth\, u \mid^{f} A\rangle} \quad \rightsquigarrow \quad \frac{\Xi \mid \Gamma, \exists(u : \mathbb{U}).\Delta \vdash a : A}{\Xi, u : \mathbb{U} \mid \beth\, u.\Gamma, \Delta \vdash \mathsf{fresh}\, u\, a : \beth\, u.A}$$

Introduction rules for modal types:

Weakening and substitution:

$$\frac{\Xi \mid \Gamma, \text{🔒}^{\mathsf{f}}_{\lrcorner(u:\mathbb{U})} \vdash a : A}{\Xi, u:\mathbb{U} \mid \Gamma \vdash \mathsf{mod}^{\mathsf{f}}_{\lrcorner\, u}\, a : \langle \lrcorner\, u \mid^{\mathsf{f}} A\rangle} \rightsquigarrow \frac{\Xi \mid \exists(u:\mathbb{U}).\Gamma \vdash a : A}{\Xi, u:\mathbb{U} \mid \Gamma \vdash \mathsf{fresh}\, u\, a : \lrcorner u.A}$$

$$\frac{\Xi \mid \Gamma, \text{🔒}^{\mathsf{o}}_{\Omega(u:\mathbb{U})} \vdash a : A}{\Xi, u:\mathbb{U} \mid \Gamma \vdash \mathsf{mod}^{\mathsf{o}}_{\Omega\, u}\, a : \langle \Omega\, u \mid^{\mathsf{o}} A\rangle} \rightsquigarrow \frac{\Xi \mid \Sigma(u:\mathbb{U}).\Gamma \vdash a : A}{\Xi, u:\mathbb{U} \mid \Gamma \vdash \mathsf{wkn}\, u\, a : \Omega\, u.A}$$

$$\frac{\Xi, i:\mathbb{I} \mid \Gamma, \text{🔒}^{\mathsf{o}}_{\Omega(i=0)} \vdash a : A}{\Xi \mid \Gamma \vdash \mathsf{mod}^{\mathsf{o}}_{\Omega(i=0)}\, a : \langle \Omega(i=0) \mid^{\mathsf{o}} A\rangle} \rightsquigarrow \frac{\Xi, i:\mathbb{I} \mid \Sigma(i=0).\Gamma \vdash a : A}{\Xi \mid \Gamma \vdash \mathsf{wkn}\, (i=0)\, a : \Omega(i=0).A}$$

Universal quantification:

$$\frac{\Xi, u:\mathbb{U} \mid \Gamma, \text{🔒}^{\mathsf{a}}_{\forall\, u} \vdash a : A}{\Xi \mid \Gamma \vdash \mathsf{mod}^{\mathsf{a}}_{\forall(u:\mathbb{U})}\, a : \langle \forall(u:\mathbb{U}) \mid^{\mathsf{a}} A\rangle} \rightsquigarrow \frac{\Xi, u:\mathbb{U} \mid \lrcorner u.\Gamma \vdash a : A}{\Xi \mid \Gamma \vdash \lambda u.a : \forall(u:\mathbb{U}).A}$$

$$\frac{\Xi, u:\mathbb{U} \mid \Gamma, \text{🔒}^{\mathsf{p}}_{\Pi\, u} \vdash a : A}{\Xi \mid \Gamma \vdash \mathsf{mod}^{\mathsf{p}}_{\Pi(u:\mathbb{U})}\, a : \langle \Pi(u:\mathbb{U}) \mid^{\mathsf{p}} A\rangle} \rightsquigarrow \frac{\Xi, u:\mathbb{U} \mid \Omega\, u.\Gamma \vdash a : A}{\Xi \mid \Gamma \vdash \lambda u.a : \Pi(u:\mathbb{U}).A}$$

$$\frac{\Xi \mid \Gamma, \text{🔒}^{\mathsf{p}}_{\Pi(i=0)} \vdash a : A}{\Xi, i:\mathbb{I} \mid \Gamma \vdash \mathsf{mod}^{\mathsf{p}}_{\Pi(i=0)}\, a : \langle \Pi(i=0) \mid^{\mathsf{p}} A\rangle} \rightsquigarrow \frac{\Xi \mid \Omega(i=0).\Gamma \vdash a : A}{\Xi, i:\mathbb{I} \mid \Gamma \vdash \lambda\_.a : \Pi(i=0).A}$$

Transpension:

$$\frac{\Xi \mid \Gamma, \text{🔒}^{\mathsf{t}}_{\lozenge(u:\mathbb{U})} \vdash a : A}{\Xi, u:\mathbb{U} \mid \Gamma \vdash \mathsf{mod}^{\mathsf{t}}_{\lozenge\, u}\, a : \langle \lozenge\, u \mid^{\mathsf{t}} A\rangle} \rightsquigarrow \frac{\Xi \mid \forall(u:\mathbb{U}).\Gamma \vdash a : A}{\Xi, u:\mathbb{U} \mid \Gamma \vdash \mathsf{mer}\, u\, a : \lozenge u.A}$$

Simplifications based on the quantification theorem 4.18:

- Cancellative and affine multipliers:
  - Write $\Gamma, \exists(u:\mathbb{U}).\Delta$ instead of $\exists(u:\mathbb{U}).(\lrcorner u.\Gamma, \Delta)$, and Γ instead of $\exists(u:\mathbb{U}).\lrcorner u.\Gamma$,
  - Write $\Gamma, \forall(u:\mathbb{U}).\Delta$ instead of $\forall(u:\mathbb{U}).(\lrcorner u.\Gamma, \Delta)$, and Γ instead of $\forall(u:\mathbb{U}).\lrcorner u.\Gamma$.
- Cartesian multipliers:
  - Write Σ, Ω, Π instead of $\exists, \lrcorner, \forall$.
- Usage of parentheses:
  (Co)quantifiers range until the next comma, e.g. $\forall(u:\mathbb{U}).\Gamma, x:A$ means $(\forall(u:\mathbb{U}).\Gamma), x:A$.

Figure 6: Informal notation for locks and modal types.

The existential quantifier makes the variables in Δ syntactically unavailable (unless they bear a modal annotation $\lrcorner u$), so a can really only depend on the fresh parts of the context of $\mathsf{fresh}\, u\, a$.

For cartesian multipliers, we invoke the quantification theorem 4.18 to further simplify the notation.

Figure 7 lists the informal notation for modal function application and for projections out of the modal types (proposition 3.3). This is not particularly exciting, we just need to make sure that we bind all the variables we introduce in the shape context.

Figure 8 lists notations for substitutions arising from 2-cells of the mode theory. We see that $\mathsf{const}_u{\downarrow}$ (semantically the co-unit drop of $\exists(u:\mathbb{U}) \dashv \lrcorner u$ or $\Sigma(u:\mathbb{U}) \dashv \Omega\, u$) is akin to weakening: it discards the variable $u:\mathbb{U}$ which is the first component of what can be thought of as a non-dependent pair type. In the case of a cancellative and affine multiplier, this first component is completely hidden, so forgetting it is an isomorphism. Conversely,

Modal function application:

$$f \cdot^{\mathsf{f}}_{\Box\, u} a \rightsquigarrow f\, a \qquad f \cdot^{\mathsf{a}}_{\forall\, u} a \rightsquigarrow f\,(u.a) \qquad f \cdot^{\mathsf{t}}_{\flat\, u} a \rightsquigarrow f\, a$$
$$f \cdot^{\mathsf{o}}_{\Omega\, u} a \rightsquigarrow f\, a \qquad f \cdot^{\mathsf{p}}_{\Pi\, u} a \rightsquigarrow f\,(u.a)$$
$$f \cdot^{\mathsf{o}}_{\Omega(i=0)} a \rightsquigarrow f\,(i.a) \qquad f \cdot^{\mathsf{p}}_{\Pi(i=0)} a \rightsquigarrow f\,(\_.a)$$

Modal projections:

$$\mathsf{prmod}_{\forall\, u} \cdot^{\mathsf{f}}_{\Box\, u} f \rightsquigarrow f\, u \qquad \mathsf{prmod}_{\flat\, u} \cdot^{\mathsf{a}}_{\forall\, u} t \rightsquigarrow \mathsf{unmer}\,(u.t)$$
$$\mathsf{prmod}_{\Pi\, u} \cdot^{\mathsf{o}}_{\Omega\, u} f \rightsquigarrow f\, u$$
$$\mathsf{prmod}_{\Pi(i=0)} \cdot^{\mathsf{o}}_{\Omega(i=0)} f \rightsquigarrow f\,(i = 0)$$

Figure 7: Informal notation for modal functions.

$\mathsf{app}_u\!\downarrow$ (semantically the unit copy of $\exists(u : \mathbb{U}) \dashv \Box\, u$ or $\Sigma(u : \mathbb{U}) \dashv \Omega\, u$) is akin to contraction: it operates in a shape context where u is available, and substitutes a copy of u for v.

The substitution $\mathsf{reidx}_u\!\downarrow$ (semantically the co-unit app of $\Box\, u \dashv \forall(u : \mathbb{U})$ or $\Omega\, u \dashv \Pi(u : \mathbb{U})$) operates in a shape context where u is available and applies the affine function $\gamma|_{v=\sqcup} : \forall\, v.\Gamma$, which is fresh for u, to u. Conversely, $\mathsf{unmer}_u\!\downarrow$ (semantically the unit const of $\Box\, u \dashv \forall(u : \mathbb{U})$ or $\Omega\, u \dashv \Pi(u : \mathbb{U})$) creates what can be thought of as a non-dependent function, namely the constant one. In the case of a cancellative and affine multiplier, the output of the function must be completely fresh for the input, so constant functions are the only ones possible and hence the operation is an isomorphism.

Finally, we treat $\mathsf{spoil}_u\!\downarrow$ (semantically $\mathsf{hide} : \Sigma(u : \mathbb{U}) \to \exists(u : \mathbb{U})$) and $\mathsf{cospoil}_u\!\downarrow$ (semantically $\mathsf{spoil} : \Box\, u \to \Omega\, u$) as implicit coercions.

As an example, note that if we write example 4.20 in the informal notation, then we get very close again to the function as defined in section 2.

6. Additional Typing Rules

In this section, we add a few extensions to MTT in order to reason about boundaries (definition 4.3) in the *type* theory, rather than in the shape theory, and in order to recover all known presheaf operators in section 8.

6.1. **Subobject classifier.** We add a universe of propositions (semantically the subobject classifier) $\mathsf{Prop} : \mathsf{U}_0$, with implicit encoding and decoding operations à la Coquand. This universe is closed under logical operators and weak DRAs [Nuy20a, §6.5]. This is necessary to talk about Ψ and Φ. We identify all proofs of the same proposition.

6.2. **Boundary predicate.** We add a predicate $\Xi, u : \mathbb{U} \mid \cdot \vdash u \in \partial\mathbb{U} : \mathsf{Prop}$ corresponding in the model to the subobject $(\Xi, u : \partial\mathbb{U}) \subseteq (\Xi, u : \mathbb{U})$. A naïve introduction rule would be $\Xi, u : \partial\mathbb{U} \mid \cdot \vdash \_ : \langle \Omega(u \in \partial\mathbb{U}) \mid^{\mathsf{o}} u \in \partial\mathbb{U} \rangle$. As all our modalities are proper DRAs [BCM+20] as opposed to the weaker concepts required by the general model of MTT, the modal introduction rule is invertible in the model, so we may as well take $\Xi, u : \mathbb{U} \mid \cdot, \blacklozenge^{\mathsf{o}}_{\Omega(u \in \partial\mathbb{U})} \vdash \mathsf{on}\partial\!\downarrow_{\mathsf{o}} : u \in \partial\mathbb{U}$ as an introduction rule. If we absorb a substitution into these rules, we get

$$\frac{\Xi, u : \mathbb{U} \mid \Gamma\ \mathsf{ctx}}{\Xi, u : \mathbb{U} \mid \Gamma \vdash u \in \partial\mathbb{U}\ \mathsf{prop}}, \qquad \frac{\Xi, u : \mathbb{U} \mid \Gamma, \Delta\ \mathsf{ctx} \quad \alpha : \Omega(u \in \partial\mathbb{U}) \Rightarrow \mathsf{locks}(\Delta)}{\Xi, u : \mathbb{U} \mid \Gamma, \Delta \vdash \mathsf{on}\partial\, \alpha\!\downarrow_{\mathsf{ticks}(\Delta)} : u \in \partial\mathbb{U}}.$$

Units becoming co-units:

$$\begin{array}{lllllll} & \Xi & | & \mathsf{const}_u{\downarrow}^{\bullet}_{\mathsf{af}} & : & (\Gamma, \blacksquare^{\mathsf{a}}_{\forall\, u}, \blacksquare^{\mathsf{f}}_{\exists\, u}) & \to & \Gamma \\ \rightsquigarrow & \Xi & | & () & : & \exists(u:\mathbb{U}).\exists\, u.\Gamma & \to & \Gamma \end{array}$$

$$\begin{array}{lllllll} & \Xi & | & \mathsf{const}_u{\downarrow}^{\bullet}_{\mathsf{po}} & : & (\Gamma, \blacksquare^{\mathsf{p}}_{\Pi\, u}, \blacksquare^{\mathsf{o}}_{\Omega\, u}) & \to & \Gamma \\ \rightsquigarrow & \Xi & | & () & : & \Sigma(u:\mathbb{U}).\Omega\, u.\Gamma & \to & \Gamma \end{array}$$

$$\begin{array}{lllllll} & \Xi, i:\mathbb{I} & | & \mathsf{const}_{(i=0)}{\downarrow}^{\bullet}_{\mathsf{po}} & : & (\Gamma, \blacksquare^{\mathsf{p}}_{\Pi(i=0)}, \blacksquare^{\mathsf{o}}_{\Omega(i=0)}) & \to & \Gamma \\ \rightsquigarrow & \Xi, i:\mathbb{I} & | & () & : & \Sigma(i=0).\Omega(j=0).\Gamma[j/i] & \to & \Gamma \end{array}$$

$$\begin{array}{lllllll} & \Xi, u:\mathbb{U} & | & \mathsf{reidx}_u{\downarrow}^{\bullet}_{\mathsf{ta}} & : & (\Gamma, \blacksquare^{\mathsf{t}}_{\Diamond\, u}, \blacksquare^{\mathsf{a}}_{\forall\, u}) & \to & \Gamma \\ \rightsquigarrow & \Xi, u:\mathbb{U} & | & (\gamma|_{v=u}/\gamma) & : & \exists\, u.\forall(v:\mathbb{U}).\Gamma & \to & \Gamma[u/v] \end{array}$$

Co-units becoming units:

$$\begin{array}{lllllll} & \Xi, u:\mathbb{U} & | & \mathsf{app}_u{\downarrow}^{\mathsf{fa}}_{\bullet} & : & \Gamma & \to & (\Gamma, \blacksquare^{\mathsf{f}}_{\exists\, u}, \blacksquare^{\mathsf{a}}_{\forall\, u}) \\ \rightsquigarrow & \Xi, u:\mathbb{U} & | & (u/v) & : & \Gamma[u/v] & \to & \exists\, u.\exists(v:\mathbb{U}).\Gamma \end{array}$$

$$\begin{array}{lllllll} & \Xi, u:\mathbb{U} & | & \mathsf{app}_u{\downarrow}^{\mathsf{op}}_{\bullet} & : & \Gamma & \to & (\Gamma, \blacksquare^{\mathsf{o}}_{\Omega\, u}, \blacksquare^{\mathsf{p}}_{\Pi\, u}) \\ \rightsquigarrow & \Xi, u:\mathbb{U} & | & (u/v) & : & \Gamma[u/v] & \to & \Omega\, u.\Sigma(v:\mathbb{U}).\Gamma \end{array}$$

$$\begin{array}{lllllll} & \Xi & | & \mathsf{app}_{(i=0)}{\downarrow}^{\mathsf{op}}_{\bullet} & : & \Gamma & \to & (\Gamma, \blacksquare^{\mathsf{o}}_{\Omega(i=0)}, \blacksquare^{\mathsf{p}}_{\Pi(i=0)}) \\ \rightsquigarrow & \Xi & | & () & : & \Gamma & \to & \Omega(i=0).\Sigma(i=0).\Gamma \end{array}$$

$$\begin{array}{lllllll} & \Xi & | & \mathsf{unmer}_u{\downarrow}^{\mathsf{at}}_{\bullet} & : & \Gamma & \to & (\Gamma, \blacksquare^{\mathsf{a}}_{\forall\, u}, \blacksquare^{\mathsf{t}}_{\Diamond\, u}) \\ \rightsquigarrow & \Xi & | & (\lambda\_.\gamma/\gamma|_{u=\sqcup}) & : & \Gamma & \to & \forall(u:\mathbb{U}).\exists\, u.\Gamma \end{array}$$

Affine and cancellative multipliers:

$$\begin{array}{lllllll} & \Xi & | & \mathsf{const}_u{\downarrow}^{\bullet}_{\mathsf{af}} & : & (\Gamma, \blacksquare^{\mathsf{a}}_{\forall\, u}, \blacksquare^{\mathsf{f}}_{\exists\, u}) & \cong & \Gamma \\ \rightsquigarrow & \Xi & | & () & : & \Gamma & \cong & \Gamma \end{array}$$

$$\begin{array}{lllllll} & \Xi & | & \mathsf{unmer}_u{\downarrow}^{\mathsf{at}}_{\bullet} & : & \Gamma & \cong & (\Gamma, \blacksquare^{\mathsf{a}}_{\forall\, u}, \blacksquare^{\mathsf{t}}_{\Diamond\, u}) \\ \rightsquigarrow & \Xi & | & () & : & \Gamma & \cong & \Gamma \end{array}$$

Semicartesian multipliers:

$$\begin{array}{lllllll} & \Xi & | & \mathsf{spoil}{\downarrow}^{\mathsf{f}}_{\mathsf{o}} & : & (\Gamma, \blacksquare^{\mathsf{o}}_{\exists\, u}) & \to & (\Gamma, \blacksquare^{\mathsf{f}}_{\Omega\, u}) \\ \rightsquigarrow & \Xi & | & () & : & \Sigma(u:\mathbb{U}).\Gamma & \to & \exists(u:\mathbb{U}).\Gamma \end{array}$$

$$\begin{array}{lllllll} & \Xi, u:\mathbb{U} & | & \mathsf{cospoil}{\downarrow}^{\mathsf{p}}_{\mathsf{a}} & : & (\Gamma, \blacksquare^{\mathsf{a}}_{\forall\, u}) & \to & (\Gamma, \blacksquare^{\mathsf{p}}_{\Pi\, u}) \\ \rightsquigarrow & \Xi, u:\mathbb{U} & | & () & : & \exists\, u.\Gamma & \to & \Omega\, u.\Gamma \end{array}$$

Figure 8: Informal notation for 2-cell substitutions.

An elimination rule could build a function $(\_ : u \in \partial\mathbb{U}) \to A\ \_$ from $\Pi(u \in \partial\mathbb{U}) \circ \Omega(u \in \partial\mathbb{U})$ applied to A, but it is not clear how to formulate the β-rule. Instead, we will eliminate to the transpension type (theorem 7.1).

6.3. **Strictness axiom.** The strictness axiom [OP18] allows to extend a partial type T to a total type if T is isomorphic to a total type A, effectively strictifying the isomorphism:

$$\frac{\Xi \mid \Gamma \vdash \varphi : \mathsf{Prop} \qquad \Xi \mid \Gamma \vdash A : \mathsf{U}_\ell \qquad \Xi \mid \Gamma, \_ : \varphi \vdash T : \mathsf{U}_\ell \qquad \Xi \mid \Gamma, \_ : \varphi \vdash i : A \cong T}{\begin{array}{l}\Xi \mid \Gamma \vdash \mathsf{Strict}\{A \cong (\varphi \mathbin{?} T \mathbin{;} i)\} : \mathsf{U}_\ell \qquad \Xi \mid \Gamma \vdash \mathsf{strict}\{\varphi \mathbin{?} i\} : A \cong \mathsf{Strict}\{A \cong (\varphi \mathbin{?} T \mathbin{;} i)\} \\ \text{where } \Gamma, \_ : \varphi \vdash \mathsf{Strict}\{A \cong (\varphi \mathbin{?} T \mathbin{;} i)\} = T : \mathsf{U}_\ell \qquad \Gamma, \_ : \varphi \vdash \mathsf{strict}\{\varphi \mathbin{?} i\} = i : A \cong T\end{array}}$$

7. Investigating the Transpension Type

In this section, we investigate the structure of the transpension type.

7.1. **Poles.** Our first observation is that on the boundary, the transpension type is trivial. Let $\top : \Xi_1 \to \Xi_2$ be the modality, between any two shape contexts, which maps any presheaf to the terminal presheaf. We clearly have $\top \circ \mu = \top$ for any μ, but also $\mu \circ \top \cong \top$ because all internal modalities are right adjoints and therefore preserve the terminal object.

Theorem 7.1 (Pole)**.** *We have $\Omega(u \in \partial\mathbb{U}) \circ \fatslash(u : \mathbb{U}) \cong \top$. We can thus postulate a term $\Xi, u : \mathbb{U} \mid \_ : u \in \partial\mathbb{U} \vdash \mathsf{pole} : \langle \fatslash(u : \mathbb{U}) \mid^{\mathsf{t}} T \rangle$ for any Ξ and T, with an η-rule $\Xi, u : \mathbb{U} \mid \_ : u \in \partial\mathbb{U} \vdash t = \mathsf{pole} : \langle \fatslash(u : \mathbb{U}) \mid^{\mathsf{t}} T \rangle$.*

Sketch of proof. The left adjoints $\forall(u : \mathbb{U}) \circ \Sigma(u \in \partial\mathbb{U})$ and $\bot$ are isomorphic because $\forall(u : \mathbb{U}).(u \in \partial\mathbb{U})$ is false. We give a full proof in the technical report [Nuy20b]. □

Definition 4.3 of the boundary relied on the notion of dimensional splitness. The following result shows that it was a good one: the transpension is *only* trivial on the boundary:

Theorem 7.2. *In the model, we have $\cdot, u : \mathbb{U} \mid \Gamma \vdash (u \in \partial\mathbb{U}) \cong \langle \fatslash(u : \mathbb{U}) \mid^{\mathsf{t}} \mathsf{Empty} \rangle$.* [Nuy20b]

7.2. **Meridians.** As all our modalities are proper DRAs [BCM+20], the modal introduction rule is invertible in the model. This immediately shows that sections[11] of the transpension type

$$\begin{array}{ll}\Xi \mid \Gamma \vdash f : \langle \forall(u : \mathbb{U}) \mid^{\mathsf{a}} \langle \fatslash(u : \mathbb{U}) \mid^{\mathsf{t}} T \rangle \rangle & \text{(formal MTT)} \\ \Xi \mid \Gamma \vdash f : \forall(u : \mathbb{U}).\fatslash\, u.T & \text{(informal)}\end{array}$$

(which we call meridians) are in 1-1 correspondence with terms

$$\begin{array}{ll}\Xi \mid \Gamma, \blacklozenge^{\mathsf{a}}_{\forall(u:\mathbb{U})}, \blacklozenge^{\mathsf{t}}_{\fatslash(u:\mathbb{U})} \vdash t : T & \text{(formal MTT)} \\ \Xi \mid \forall(u : \mathbb{U}).\shortdownarrow u.\Gamma \vdash t : T & \text{(informal).}\end{array}$$

If it were not for the locking of the context, this characterization in terms of poles and meridians would make the transpension type look quite similar to a dependent version of the suspension type in HoTT [Uni13], whence our choice of name. If $\mathbb{U}$ is cancellative and affine, then the locks can actually be ignored (theorem 4.18). In any case, proposition 3.3 tells us that the let-rule for $\fatslash(u : \mathbb{U})$ has the same power as

$$\begin{array}{ll}\mathsf{prmod}_{\fatslash(u:\mathbb{U})} : (\forall(u : \mathbb{U}) \shortrightarrow^{\mathsf{a}} \langle \fatslash(u : \mathbb{U}) \mid^{\mathsf{t}} T \rangle) \to T[\mathsf{unmer}_u \downarrow^{\mathsf{a}\mathsf{t}}_{\bullet}] & \text{(formal MTT)} \\ \mathsf{unmer} : (\forall(u : \mathbb{U}) \shortrightarrow \fatslash\, u.T) \to T[\lambda\_.\gamma/\gamma|_{u=\sqcup}] & \text{(informal)}\end{array}$$

[11] By a section of a dependent type, we mean a dependent function with the same domain as the type.

In formal MTT syntax:

TRANSP:ELIM

$$\frac{\begin{array}{l}\Xi, u : \mathbb{U} \mid \Gamma\,\mathsf{ctx} \\ \Xi \mid \Gamma, \blacksquare^{\mathfrak{t}}_{\lozenge u} \vdash A\,\mathsf{type} \\ \Xi, u : \mathbb{U} \mid \Gamma, r : \langle \lozenge u \mid^{\mathfrak{t}} A\rangle \vdash C\,\mathsf{type} \\ \Xi, u : \mathbb{U} \mid \Gamma, \_ : u \in \partial\mathbb{U} \vdash c_{\mathsf{pole}} : C[\mathsf{pole}/r] \\ \Xi, u : \mathbb{U} \mid \Gamma, \blacksquare^{\mathfrak{t}}_{\lozenge u}, x : A, \blacksquare^{\mathfrak{a}}_{\forall u} \vdash c_{\mathsf{merid}} : C\left[\mathsf{reidx}_u{\downarrow}^{\bullet}_{\mathfrak{t}\mathfrak{a}}, \mathsf{mod}^{\mathfrak{t}'}_{\lozenge u}\,(x\,\mathsf{unmer}_u^{-1}{\downarrow}_{\mathfrak{a}\mathfrak{t}'})/r\right] \\ \Xi, u : \mathbb{U} \mid \Gamma, \blacksquare^{\mathfrak{t}}_{\lozenge u}, x : A, \blacksquare^{\mathfrak{a}}_{\forall u}, \_ : u \in \partial\mathbb{U} \vdash c_{\mathsf{merid}} = c_{\mathsf{pole}}[\mathsf{reidx}_u{\downarrow}^{\bullet}_{\mathfrak{t}\mathfrak{a}}] : C[\mathsf{reidx}_u{\downarrow}^{\bullet}_{\mathfrak{t}\mathfrak{a}}, \mathsf{pole}/r] \\ \Xi, u : \mathbb{U} \mid \Gamma \vdash t : \langle \lozenge_u \mid^{\mathfrak{t}} A\rangle\end{array}}{\begin{array}{l}\Xi, u : \mathbb{U} \mid \Gamma \vdash c := \mathsf{case}\,t\,\mathsf{of}\left\{\ \mathsf{pole} \mapsto c_{\mathsf{pole}} \mid \mathsf{merid}(\mathfrak{t}, x, \mathfrak{a}) \mapsto c_{\mathsf{merid}}\ \right\} : C[t/r] \\ \text{where } c[\mathsf{pole}/t] = c_{\mathsf{pole}} \\ \qquad \Delta, \blacksquare^{\mathfrak{a}'}_{\forall u} \vdash c = c_{\mathsf{merid}}[\mathsf{prmod}_{\lozenge u}{}^{\cdot^{\mathfrak{a}''}}\,(t[\mathsf{reidx}_u{\downarrow}^{\bullet}_{\mathfrak{t}\mathfrak{a}''}])/x, {\downarrow}^{\mathfrak{a}}_{\mathfrak{a}}][(\mathsf{unmer}_u \star 1_{\forall u}){\downarrow}^{\mathfrak{a}'\mathfrak{t}\mathfrak{a}}_{\mathfrak{a}'}]\end{array}}$$

In informal notation:

$$\frac{\begin{array}{l}\Xi, u : \mathbb{U} \mid \Gamma\,\mathsf{ctx} \\ \Xi \mid \forall(u : \mathbb{U}).\Gamma \vdash A\,\mathsf{type} \\ \Xi, u : \mathbb{U} \mid \Gamma, r : \lozenge u.A \vdash C\,\mathsf{type} \\ \Xi, u : \mathbb{U} \mid \Gamma, \_ : u \in \partial\mathbb{U} \vdash c_{\mathsf{pole}} : C[\mathsf{pole}/r] \\ \Xi, v : \mathbb{U} \mid \lrcorner v.(\forall(u : \mathbb{U}).\Gamma, x : A) \vdash c_{\mathsf{merid}} : C[\gamma|_{u=v}/\gamma, \mathsf{mer}\,v\,x/r] \\ \Xi, v : \mathbb{U} \mid \lrcorner v.(\forall(u : \mathbb{U}).\Gamma, x : A), \_ : v \in \partial\mathbb{U} \vdash c_{\mathsf{merid}} = c_{\mathsf{pole}}[\gamma|_{u=v}/\gamma] : C[\gamma|_{u=v}/\gamma, \mathsf{pole}/r] \\ \Xi, u : \mathbb{U} \mid \Gamma \vdash t : \lozenge u.A\end{array}}{\begin{array}{l}\Xi, u : \mathbb{U} \mid \Gamma \vdash c := \mathsf{case}\,t\,\mathsf{of}\left\{\ \mathsf{pole} \mapsto c_{\mathsf{pole}} \mid \mathsf{mer}\,v\,x \mapsto c_{\mathsf{merid}}\ \right\} : C[t/r] \\ \text{where } c[\mathsf{pole}/t] = c_{\mathsf{pole}} \\ \qquad \lrcorner u.\Delta \vdash c = c_{\mathsf{merid}}[u/v, \mathsf{unmer}\,(w.t[w/u])/x]\end{array}}$$

Figure 9: Transpension elimination by pattern matching (sound if $\mathbb{U}$ is cancellative, affine and connection-free).

which extracts meridians. If $\mathbb{U}$ is cancellative and affine, then unmer_u is invertible (theorem 4.18) and we can also straightforwardly *create* meridians from elements of $T[\mathsf{unmer}_u{\downarrow}^{\mathfrak{a}\mathfrak{t}}_{\bullet}]$.

7.3. **Pattern matching.** The eliminator $\mathsf{prmod}_{\lozenge(u:\mathbb{U})}$ is only capable of eliminating *sections* of the transpension type. Sometimes we can eliminate locally by pattern matching:

Theorem 7.3. *If $\mathbb{U}$ is cancellative, affine and connection-free, then the rule* TRANSP:ELIM *in fig. 9 is sound* [Nuy20b].

We get a context Γ depending on $u : \mathbb{U}$, a type A depending on sections of Γ (recall that $\blacksquare_{\lozenge u}$ is semantically $\forall u$), a type C depending on u and $\langle \lozenge u \mid A\rangle$, and an argument t of type $\langle \lozenge u \mid A\rangle$. To obtain a value of type C, we need to give an action c_{pole} on the boundary, where t is necessarily pole (pole theorem 7.1), and a compatible action on sections of the transpension type, i.e. meridians, which are essentially elements of A (quantification theorem 4.18), to sections of C (the quantifier has been brought to the left as $\blacksquare_{\forall u}$). Thanks to connection-freedom, we know that everything that is not a section[12], is on the boundary, so this suffices. The computation rule for meridians is in a non-general context and needs

[12] or a 'dimensional section' in case of spooky base categories

to be forcibly closed under substitution (we have not found a better way to phrase this computation rule, but neither do we exclude that there exists one).

8. Recovering Known Operators

In this section, we explain how to recover the amazing right adjoint $\surd$ [LOPS18], Moulin et al.'s Φ and Ψ combinators [BCM15, Mou16] and Glue [CCHM17, NVD17], Weld [NVD17] and mill [ND18b] from the transpension, the strictness axiom [OP18] and certain pushouts.

8.1. **The amazing right adjoint $\surd$.** Licata et al. [LOPS18] use presheaves over a cartesian base category of cubes and introduce $\surd$ as the right adjoint to the non-dependent exponential $\mathbb{I} \to \sqcup$. We generalize to semicartesian systems and look for a right adjoint to $\mathbb{U} \multimap \sqcup$, which decomposes as substructural quantification after cartesian weakening $\forall(u : \mathbb{U}) \circ \Omega(u : \mathbb{U})$. Then the right adjoint is obviously $\surd_{\mathbb{U}} := \Pi(u : \mathbb{U}) \circ \between(u : \mathbb{U})$. The type constructor has type $\langle \surd_{\mathbb{U}} \mid \sqcup \rangle : (\surd_{\mathbb{U}} \shortmid^{\mathsf{r}} \mathsf{U}_\ell) \to \mathsf{U}_\ell$ and the transposition rule is as in proposition 3.4. This is an improvement in two ways: First, we have computation rules, so that we do not need to postulate functoriality of $\surd_{\mathbb{U}}$ and invertibility of transposition. Secondly, we have no need for a global sections modality $\flat$. Instead, we use the modality $\surd_{\mathbb{U}}$ to escape Licata et al.'s no-go theorems.

Our overly general mode theory does contain a global sections modality $\flat : () \to ()$ acting in the empty shape context, and we can use this to recover Licata et al.'s axioms for the amazing right adjoint. Let us write $\mathsf{spconst}_u : 1 \Rightarrow \forall\, u \circ \Omega\, u$ and $\mathsf{spunmer}_u : \Pi\, u \circ \between\, u \Rightarrow 1$ for the 2-cells built by transposition from either spoil_u or $\mathsf{cospoil}_u$, which are each other's transposite. The following are isomorphisms:

$$\kappa := \mathsf{spconst}_u \star 1_\flat : \flat \cong \forall\, u \circ \Omega\, u \circ \flat,$$
$$\zeta := 1_\flat \circ \mathsf{spunmer}_u : \flat \circ \Pi\, u \circ \between\, u \cong \flat.$$

For κ, this is intuitively clear from the fact that we are considering $\mathbb{U}$-cells in a discrete presheaf produced by $\flat$. For ζ, this is similarly clear after taking the left adjoints:

$$\mathsf{spconst}_u \star 1_\int : \int \cong \forall\, u \circ \Omega\, u \circ \int$$

where $\int \dashv \flat$ is the connected components functor which also produces discrete presheaves. Write $\varepsilon : \flat \Rightarrow 1$ for the co-unit of the comonad $\flat$. We can define

$$\mathbb{U} \surd \sqcup : (\flat \shortmid^{\flat} \mathsf{U}) \to \mathsf{U} \qquad \mathbb{U} \multimap \sqcup : \mathsf{U} \to \mathsf{U}$$
$$\mathbb{U} \surd^{\flat} A = \left\langle \Pi\, u \circ \between\, u \mid^{\mathsf{pt}} A[\zeta^{-1}{\downarrow^{\flat}_{\mathsf{bpt}}}][\varepsilon{\downarrow^{\flat}_{\bullet}}, {\downarrow^{\mathsf{pt}}_{\mathsf{pt}}}] \right\rangle \qquad \mathbb{U} \multimap A = \langle \forall\, u \circ \Omega\, u \mid^{\mathsf{a}\circ} A[\mathsf{spconst}_u{\downarrow^{\bullet}_{\mathsf{a}\circ}}] \rangle.$$

Let two global types $\Gamma, \text{🔒}^{\flat}_{\flat} \vdash A, B\ \mathsf{type}$ be given. Applying the non-dependent version of proposition 3.4 to $\forall\, u \circ \Omega\, u \dashv \surd_{\mathbb{U}} = \Pi\, u \circ \between\, u$ yields:

$$\begin{aligned}
&\left(A[\varepsilon{\downarrow^{\flat}_{\bullet}}] \to \mathbb{U} \surd^{\flat} B \right) \\
&\cong \left\langle \surd_{\mathbb{U}} \mid^{\mathsf{pt}} (\forall\, u \circ \Omega\, u \shortmid^{\mathsf{a}\circ} A[\varepsilon{\downarrow^{\flat}_{\bullet}}][\mathsf{const}_u{\downarrow^{\bullet}_{\mathsf{pt}}}][{\downarrow^{\mathsf{p}}_{\mathsf{p}}}, \mathsf{reidx}_u{\downarrow^{\bullet}_{\mathsf{ta}}}, {\downarrow^{\mathsf{o}}_{\mathsf{o}}}]) \to B[\zeta^{-1}{\downarrow^{\flat}_{\mathsf{bpt}}}][\varepsilon{\downarrow^{\flat}_{\bullet}}, {\downarrow^{\mathsf{pt}}_{\mathsf{pt}}}] \right\rangle \\
&= \left\langle \surd_{\mathbb{U}} \mid^{\mathsf{pt}} (\forall\, u \circ \Omega\, u \shortmid^{\mathsf{a}\circ} A[\mathsf{spconst}_u{\downarrow^{\bullet}_{\mathsf{a}\circ}}])[\zeta^{-1}{\downarrow^{\flat}_{\mathsf{bpt}}}][\varepsilon{\downarrow^{\flat}_{\bullet}}, {\downarrow^{\mathsf{pt}}_{\mathsf{pt}}}] \to B[\zeta^{-1}{\downarrow^{\flat}_{\mathsf{bpt}}}][\varepsilon{\downarrow^{\flat}_{\bullet}}, {\downarrow^{\mathsf{pt}}_{\mathsf{pt}}}] \right\rangle \\
&= \mathbb{U} \surd^{\flat} ((\mathbb{U} \multimap A) \to B).
\end{aligned}$$

Binary and destrictified reformulation of the original Φ-rule [BCM15, Mou16]**:**

$$\frac{\begin{array}{l}\Delta, i : \mathbb{I} \vdash B\,\mathsf{type}\\ \Delta, i : \mathbb{I}, y : B \vdash C\,\mathsf{type}\\ \Delta, y : B[\epsilon/i] \vdash c_\epsilon : C[\epsilon/i] \qquad (\epsilon \in \{0,1\})\\ \Delta, h : \forall(i : \mathbb{I}).B, i : \mathbb{I} \vdash c : C[h\,i/b]\\ \Delta, h : \forall(i : \mathbb{I}).B \vdash c[\epsilon/i] = c_\epsilon[h\,\epsilon/b] : C[h\,\epsilon/b] \quad (\epsilon \in \{0,1\})\\ \Delta, i : \mathbb{I} \vdash b : B\end{array}}{\begin{array}{l}\Delta, i : \mathbb{I} \vdash \Phi_i\,c_0\,c_1\,c\,b : C[b/y]\\ \text{where } \Phi_\epsilon\,c_0\,c_1\,c\,b = c_\epsilon \qquad (\epsilon \in \{0,1\})\\ \qquad\quad \Phi_i\,c_0\,c_1\,c\,b = c[\lambda i.b/h]\end{array}}$$

In formal MTT syntax:

PHI

$$\frac{\begin{array}{l}\Xi, u : \mathbb{U} \mid \Gamma \vdash B\,\mathsf{type}\\ \Xi, u : \mathbb{U} \mid \Gamma, y : B \vdash C\,\mathsf{type}\\ \Xi, u : \mathbb{U} \mid \Gamma, y : B, \_ : u \in \partial\mathbb{U} \vdash c_\partial : C\\ \Xi, u : \mathbb{U} \mid \Gamma, y : B, \fatbslash^{\mathsf{t}}_{\lozenge\,u}, \fatbslash^{\mathsf{a}}_{\forall\,u} \vdash c : C[\mathsf{reidx}_u{\downarrow}^{\bullet}_{\mathsf{t}\mathsf{a}}]\\ \Xi, u : \mathbb{U} \mid \Gamma, y : B, \fatbslash^{\mathsf{t}}_{\lozenge\,u}, \fatbslash^{\mathsf{a}}_{\forall\,u}, \_ : u \in \partial\mathbb{U} \vdash c = c_\partial[\mathsf{reidx}_u{\downarrow}^{\bullet}_{\mathsf{t}\mathsf{a}}] : C[\mathsf{reidx}_u{\downarrow}^{\bullet}_{\mathsf{t}\mathsf{a}}]\\ \Xi, u : \mathbb{U} \mid \Gamma \vdash b : B\end{array}}{\begin{array}{l}\Xi, u : \mathbb{U} \mid \Gamma \vdash \Phi_u\,(y.c_\partial)\,(y.\mathsf{t}.\mathsf{a}.c)\,b : C[b/y]\\ \text{where } \Gamma, \_ : u \in \partial\mathbb{U} \vdash \Phi_u\,(y.c_\partial)\,(y.\mathsf{t}.\mathsf{a}.c)\,b = c_\partial[b/y]\\ \qquad \Delta, \fatbslash^{\mathsf{a}'}_{\forall\,u} \vdash (\Phi_u\,(y.c_\partial)\,(y.\mathsf{t}.\mathsf{a}.c)\,b) = c[b/y, {\downarrow}^{\mathsf{t}\mathsf{a}}_{\mathsf{t}\mathsf{a}}][(\mathsf{unmer}_u \star 1_{\forall u}){\downarrow}^{\mathsf{a}'\,\mathsf{t}\mathsf{a}}_{\mathsf{a}'}]\end{array}}$$

In informal notation:

$$\frac{\begin{array}{l}\Xi, u : \mathbb{U} \mid \Gamma \vdash B\,\mathsf{type}\\ \Xi, u : \mathbb{U} \mid \Gamma, y : B \vdash C\,\mathsf{type}\\ \Xi, u : \mathbb{U} \mid \Gamma, y : B, \_ : u \in \partial\mathbb{U} \vdash c_\partial : C\\ \Xi, v : \mathbb{U} \mid \lrcorner\, v.\forall(u : \mathbb{U}).(\Gamma, y : B) \vdash c : C[v/u, \gamma|_{u=v}/\gamma, y|_{u=v}/y]\\ \Xi, v : \mathbb{U} \mid \lrcorner\, v.\forall(u : \mathbb{U}).(\Gamma, y : B), \_ : v \in \partial\mathbb{U} \vdash\\ \qquad\quad c = c_\partial[v/u, \gamma|_{u=v}/\gamma, y|_{u=v}/y] : C[v/u, \gamma|_{u=v}/\gamma, y|_{u=v}/y]\\ \Xi, u : \mathbb{U} \mid \Gamma \vdash b : B\end{array}}{\begin{array}{l}\Xi, u : \mathbb{U} \mid \Gamma \vdash \Phi_u\,(y.c_\partial)\,(y.v.c)\,b : C[b/y]\\ \text{where } \Gamma, \_ : u \in \partial\mathbb{U} \vdash \Phi_u\,(y.c_\partial)\,(y.v.c)\,b = c_\partial[b/y]\\ \qquad \lrcorner\, u.\Delta \vdash \Phi_u\,(y.c_\partial)\,(y.v.c)\,b = c[u/v, \lambda\_.b/y|_{v=\sqcup}]\end{array}}$$

Figure 10: The Φ-rule for $(y : B) \to C$ (sound if $\mathbb{U}$ is cancellative, affine and connection-free).

Equality of the substitutions applied to A is proven by transposing $\forall\,u \circ \Omega\,u$ to the left as $\Pi\,u \circ \lozenge\,u$. Then the unit $\mathsf{const}_u \circ (1_{\Pi\,u} \star \mathsf{reidx}_u \star 1_{\Omega\,u}) : 1 \Rightarrow \Pi\,u \circ \lozenge\,u \circ \forall\,u \circ \Omega\,u$ becomes $1_{\Pi\,u \circ \lozenge\,u}$, leaving just $\varepsilon : \flat \Rightarrow 1$, whereas $\mathsf{spconst}_u : 1 \Rightarrow \forall\,u \circ \Omega\,u$ becomes $\mathsf{spunmer}_u : \Pi\,u \circ \lozenge\,u \Rightarrow 1$ and cancels against ζ^{-1}, again leaving just ε. Applying the $\flat$ modality to both sides of the isomorphism and using ζ yields transposition functions as given by Licata et al. [LOPS18].

8.2. **The Φ-combinator.** In fig. 10, we *state* Moulin et al.'s Φ-rule [BCM15, Mou16]; both a slight reformulation adapted to the naïve system (section 2) and the rule PHI adapted to the current system in formal and informal notation. For types B and C (depending on

$u : \mathbb{U}$), the combinator allows us to define functions of naïve type $\forall\, u.(y : B\, u) \to C\, u\, y$ from an action c_∂ on the boundary (in Moulin et al.'s work: on every endpoint of the interval) and a compatible action c on sections $\forall\, u.B\, u$. Given a section of B (recall that $🔒_{◊\, u}$ is semantically $\forall\, u$), c provides a section of C (but the quantification has been moved to the left as $🔒_{\forall\, u}$), compatible with c_∂ on the boundary. The result is a term of type C which matches the given actions on the boundary and on sections. Again, the computation rule for sections is in a non-general context and needs to be forcibly closed under substitution.

The main difference with Moulin et al.'s formulation is that they require that $i : \mathbb{I}$ be the last variable before y, i.e. it comes after Γ. Here, this would mean that Γ is fresh for $u : \mathbb{U}$, i.e. $\Gamma = (\Delta, 🔒_{\forall\, u})$ or informally $\Gamma = \beth u.\Delta$. In that case, because Moulin et al.'s system is cancellative and affine, the informal notation of the context of c then becomes $\beth v.(\Delta, \forall(u : \mathbb{U}).(y : B))$, which corresponds more closely to Moulin et al.'s rules. The greater generality of our Φ-rule means that there is in fact no reason *not* to absorb B into Γ.

We remark that if $C = \langle ◊\, u \mid^{\mathsf{t}} D \rangle$ and $\mathbb{U}$ is cancellative and affine, then the Φ-rule follows from the introduction rule of the transpension type by pole theorem 7.1 and quantification theorem 4.18. Indeed, we can then define:

$$\Phi_u\, (y.c_\partial)\, (y.\mathsf{t}.\mathsf{a}.c)\, y := \mathsf{mod}^{\mathsf{t}}_{◊\, u}\, (\mathsf{prmod}_{◊\, u}\, \cdot^{\mathsf{a}}\, c) : \langle ◊\, u \mid^{\mathsf{t}} D \rangle$$
$$\Phi_u\, (y.c_\partial)\, (y.v.c)\, y := \mathsf{merid}\, u\, (\mathsf{unmer}\, (v.c)) : ◊\, u.D.$$

Theorem 8.1. *If $\mathbb{U}$ is cancellative, affine and connection-free, then the Φ-rule (fig. 10) is sound and indeed derivable from* TRANSP:ELIM *for all B and C.*

Proof. Absorb B into Γ and use the case-eliminator for $\langle ◊\, u \mid^{\mathsf{t}} \mathsf{Unit} \rangle$ (theorem 7.3):

$$\Phi_u\, (y.c_\partial)\, (y.\mathsf{t}.\mathsf{a}.c)\, y := \mathsf{case}\, (\mathsf{mod}^{\mathsf{t}}_{◊\, u}\, \mathsf{unit})\, \mathsf{of} \left\{ \begin{array}{lcl} \mathsf{pole} & \mapsto & c_\partial \\ \mathsf{merid}(\mathsf{t}, \_, \mathsf{a}) & \mapsto & c \end{array} \right\}$$
$$\Phi_u\, (y.c_\partial)\, (y.v.c)\, y := \mathsf{case}\, (\mathsf{merid}\, u\, \mathsf{unit})\, \mathsf{of} \left\{ \begin{array}{lcl} \mathsf{pole} & \mapsto & c_\partial \\ \mathsf{merid}\, v\, \_ & \mapsto & c \end{array} \right\} \qquad \square$$

8.3. **The Ψ-type.** Moulin et al.'s Ψ-combinator constructs an edge in the universe with endpoints A_ϵ from a relation $R : \bigtimes_\epsilon A_\epsilon \to \mathsf{U}$. In fig. 11, we adapt the typing rules, replacing the concept of endpoints with the boundary. The formation rule takes a type A that exists on the boundary, and a type R depending on sections of A (with $🔒_{◊\, u}$ being $\forall\, u$). It yields a Ψ-type which equals A on the boundary. The introduction rule is only necessary when we are not on the boundary (otherwise we can just coerce elements of A). It requires an element of A on the boundary and, for every section of Γ (which need not exist), a proof that the resulting boundary section satisfies R. The elimination rule sends sections of the Ψ-type (the quantification has been brought to the left as $🔒_{\forall\, u}$) to proofs that the boundary part satisfies R. We have β- and η-rules. In cancellative, affine and connection-free systems, the Φ-rule yields a pattern-matching eliminator. Again, the main difference with Moulin et al. is that Γ need not be fresh for u.

The Ψ-type can be implemented by strictifying the following using Strict:

$$\Psi_u\, A\, (\alpha.\mathsf{t}.R) :\cong (\alpha : (\_ : u \in \partial U) \to A) \times \langle ◊\, u \mid^{\mathsf{t}} R \rangle,$$

The fact that it is isomorphic to A on the boundary follows from the pole theorem 7.1.

Binary and destrictified reformulation of the original Ψ-type [BCM15, Mou16]**:**

$$\frac{\Delta \vdash A_\epsilon\,\mathsf{type}\quad(\epsilon\in\{0,1\})\qquad \Delta, x_0 : A_0, x_1 : A_1 \vdash R\,\mathsf{type}}{\Delta, i : \mathbb{I} \vdash \Psi_i\, A_0\, A_1\, (x_0.x_1.R)\,\mathsf{type}}$$
$$\text{where } \Psi_\epsilon\, A_0\, A_1\, R = A_\epsilon \quad (\epsilon\in\{0,1\})$$

$$\frac{\Delta \vdash a_\epsilon : A_\epsilon\quad(\epsilon\in\{0,1\})\qquad \Delta \vdash r : R[a_0/x_0, a_1/x_1]}{\Delta, i : \mathbb{I} \vdash \mathsf{in}\Psi_i\, a_0\, a_1\, r : \Psi_i\, A_0\, A_1\, (x_0.x_1.R)}$$
$$\text{where } \mathsf{in}\Psi_\epsilon\, a_0\, a_1\, r = a_\epsilon \quad (\epsilon\in\{0,1\})$$
$$q = \mathsf{in}\Psi_i\, a_0\, a_1\, (\mathsf{out}\Psi(j.q[j/i]))$$

$$\frac{\Delta, i : \mathbb{I} \vdash q : \Psi_i\, A_0\, A_1\, (x_0.x_1.R)}{\Delta \vdash \mathsf{out}\Psi(i.q) : R[q[0/i]/x_0, q[1/i]/x_1]}$$
$$\text{where } \mathsf{out}\Psi(i.\mathsf{in}\Psi_i\, a_0\, a_1\, r) = r$$

In formal MTT syntax:

PSI
$$\frac{\Xi, u : \mathbb{U} \mid \Gamma, \_ : u \in \partial\mathbb{U} \vdash A\,\mathsf{type}\qquad \Xi \mid \Gamma, \alpha : (\_ : u \in \partial\mathbb{U}) \to A, \text{🔒}^{\mathsf{t}}_{\emptyset(u:\mathbb{U})} \vdash R\,\mathsf{type}}{\Xi, u : \mathbb{U} \mid \Gamma \vdash \Psi_u\, A\, (\alpha.\mathsf{t}.R)\,\mathsf{type}}$$
$$\text{where } \_ : u \in \partial\mathbb{U} \vdash \Psi_u\, A\, (\alpha.\mathsf{t}.R) = A$$

PSI:INTRO
$$\frac{\Xi, u : \mathbb{U} \mid \Gamma, \_ : u \in \partial\mathbb{U} \vdash a : A\qquad \Xi \mid \Gamma, \text{🔒}^{\mathsf{t}}_{\emptyset(u:\mathbb{U})} \vdash r : R[\lambda\_.a/\alpha, \downarrow^{\mathsf{t}}_{\mathsf{t}}]}{\Xi, u : \mathbb{U} \mid \Gamma \vdash \mathsf{in}\Psi_u\, a\, (\mathsf{t}.r) : \Psi_u\, A\, (\alpha.\mathsf{t}.R)}$$
$$\text{where } \_ : u \in \partial\mathbb{U} \vdash \mathsf{in}\Psi_u\, a\, (\mathsf{t}.r) = a$$
$$q = \mathsf{in}\Psi_u\, q\, (\mathsf{t}.\mathsf{out}\Psi \cdot^{\mathsf{a}} q[\mathsf{reidx}_u{\downarrow}^{\bullet}_{\mathsf{ta}}])$$

PSI:ELIM
$$\frac{\Xi, u : \mathbb{U} \mid \Delta, \text{🔒}^{\mathsf{a}}_{\forall(u:\mathbb{U})} \vdash q : \Psi_u\, A\, (\alpha.\mathsf{t}.R)}{\Xi \mid \Delta \vdash \mathsf{out}\Psi \cdot^{\mathsf{a}} q : R[\lambda\_.q/\alpha, \downarrow^{\mathsf{t}}_{\mathsf{t}}][\mathsf{unmer}_u{\downarrow}^{\mathsf{a}\,\mathsf{t}}_{\bullet}]}$$
$$\text{where } \mathsf{out}\Psi \cdot^{\mathsf{a}} \mathsf{in}\Psi_u\, a\, (\mathsf{t}.r) = r[\mathsf{unmer}_u{\downarrow}^{\mathsf{a}\,\mathsf{t}}_{\bullet}]$$

In informal notation:

$$\frac{\Xi, u : \mathbb{U} \mid \Gamma, \_ : u \in \partial\mathbb{U} \vdash A\,\mathsf{type}\qquad \Xi \mid \forall(u : \mathbb{U}).(\Gamma, \alpha : (\_ : u \in \partial\mathbb{U}) \to A) \vdash R\,\mathsf{type}}{\Xi, u : \mathbb{U} \mid \Gamma \vdash \Psi_u\, A\, (\alpha.R)\,\mathsf{type}}$$
$$\text{where } \_ : u \in \partial\mathbb{U} \vdash \Psi_u\, A\, (\alpha.R) = A$$

$$\frac{\Xi, u : \mathbb{U} \mid \Gamma, \_ : u \in \partial\mathbb{U} \vdash a : A\qquad \Xi \mid \forall(u : \mathbb{U}).\Gamma \vdash r : R[\lambda u.\lambda\_.a/\alpha|_{u=\sqcup}]}{\Xi, u : \mathbb{U} \mid \Gamma \vdash \mathsf{in}\Psi_u\, a\, r : \Psi_u\, A\, (\alpha.R)}$$
$$\text{where } \_ : u \in \partial\mathbb{U} \vdash \mathsf{in}\Psi_u\, a\, r = a$$
$$q = \mathsf{in}\Psi_u\, a\, (\mathsf{out}\Psi(v.q[v/u, \gamma|_{u=v}/\gamma]))$$

$$\frac{\Xi, u : \mathbb{U} \mid \exists u.\Delta \vdash q : \Psi_u\, A\, (\alpha.R)}{\Xi \mid \Delta \vdash \mathsf{out}\Psi(u.q) : R[\lambda\_.\delta/\delta|_{u=\sqcup}, \lambda u.\lambda\_.q/\alpha|_{u=\sqcup}]}$$
$$\text{where } \mathsf{out}\Psi(u.\mathsf{in}\Psi_u\, a\, r) = r[\lambda\_.\delta/\delta|_{u=\sqcup}]$$

Figure 11: Typing rules for the Ψ-type.

8.4. **Transpensivity.** The Φ-rule is extremely powerful but not available in all systems. However, when the codomain C is a Ψ-type, then the $\mathsf{in}\Psi$-rule is actually quite similar to the Φ-rule. As such, we take an interest in types that are very Ψ-like. We have a monad (idempotent if $\mathbb{U}$ is cancellative and affine)

$$\bar{\Psi}_u A := \Psi_u\, A\, (\alpha.\mathsf{t}.\langle \forall\, u \mid^{\mathsf{a}} A\, \mathsf{ext}\{u \in \partial\mathbb{U}\ ?\ \alpha\ \_\}[\mathsf{reidx}_u{\downarrow}^{\bullet}_{\mathsf{ta}}]\rangle)$$
$$\bar{\Psi}_u A := \Psi_u\, A\, (\alpha.\forall\, v.A\, \mathsf{ext}\{u \in \partial\mathbb{U}\ ?\ \alpha\ \_\}[v/u, \gamma|_{u=v}/\gamma])$$

where $A\,\mathsf{ext}\{\varphi\ ?\ a\}$ is the type of elements of A that are equal to a when φ holds:

$$A\,\mathsf{ext}\{\varphi\ ?\ a\} := (x : A) \times ((\_ : \varphi) \to (x \equiv_A a)).$$

Definition 8.2. A type is **transpensive over** u if it is a monad-algebra for $\bar{\Psi}_u$.

For cancellative, affine and connection-free multipliers, Φ entails that all types are transpensive. For other systems, many interesting types will still be transpensive, allowing to eliminate *to* them in a Φ-like way.

8.5. **Glue, Weld, mill.** $\mathsf{Glue}\{A \leftarrow (\varphi\ ?\ T\ ;\ f)\}$ and $\mathsf{Weld}\{A \to (\varphi\ ?\ T\ ;\ g)\}$ are similar to Strict but extend unidirectional functions. Orton and Pitts [OP18] already show that Glue [CCHM17, NVD17] can be implemented by strictifying a pullback along $A \to (\varphi \to A)$ [ND18b] which is definable internally using a Σ-type. Dually, Weld [NVD17] can be implemented if there is a type former for pushouts along $\varphi \times A \to A$ where $\varphi : \mathsf{Prop}$ [Nuy20a, §6.3.3], which is sound in all presheaf categories.

Finally, mill [ND18b] states that $\forall(u : \mathbb{U})$ preserves Weld and is provable by higher-dimensional pattern matching. In informal notation:

$$f : (\forall\, u.\mathsf{Weld}\{A \to (\varphi\ ?\ T\ ;\ g)\}) \to \mathsf{Weld}\{\forall\, u.A \to (\forall\, u.\varphi\ ?\ \forall\, u.T\ ;\ g \circ \llcorner\lrcorner)\}$$

$$f\ \hat{w} = \mathsf{unmer}\left(u.\mathsf{case}\,\hat{w}\,u\,\mathsf{of}\left\{\begin{array}{lcl} \varphi\ ?\ t & \mapsto & \mathsf{mer}\,u\,(\lambda v.t|_{u=v}) \\ \mathsf{weld}(\varphi\ ?\ g)\,a & \mapsto & \mathsf{mer}\,u\,(\mathsf{weld}(\forall\, u.\varphi\ ?\ g \circ \llcorner\lrcorner)\,(\lambda v.a|_{u=v})) \end{array}\right\}\right).$$

In the first clause, we are asserted that φ holds so that the left hand Weld-type equals T, and we are given $t : T$. Then by entering $\mathsf{mer}\,u$, the proof of φ in the context becomes a proof of $\forall\, u.\varphi$, so that the right hand Weld-type equals $\forall\, u.T$. In the second clause, we get an element $a : A$ and can proceed as in section 2.3. When φ holds, the weld-constructor reduces to g and both clauses are required to match. This is intuitively clearly the case.

8.6. **Locally fresh names.** Nominal type theory is modelled in the Schanuel topos [Pit14] which is a subcategory of $\mathrm{Psh}({}^0\mathsf{Cube}_\square)$ (example 4.8). As fibrancy is not considered in this chapter, we will work directly in $\mathrm{Psh}({}^0\mathsf{Cube}_\square)$. Names can be modelled using the multiplier $\llcorner\lrcorner * (i : \mathbb{I})$. Interestingly, the fresh weakening functor $\lrcorner_{(i:\mathbb{I})}$ is then *inverse* to its left adjoint $\exists_{(i:\mathbb{I})}$. By consequence, we get $\exists\, i \cong \forall\, i$ (the fresh name quantifier) and $\lrcorner\, i \cong \rangle\!\langle\, i$.

Moreover, by the quantification theorem 4.18, we have $\forall\, i \circ \lrcorner\, i \cong 1$. An element of $\langle \forall\, i \circ \lrcorner\, i \mid A\rangle$ is created by first abstracting over i, then making sure that the body is fresh for i. This is exactly how locally fresh name abstraction $\nu(i : \mathbb{I})$ works, and the fact that $\forall\, i \circ \lrcorner\, i \cong 1$ allows us to use it anywhere.

Consider Pitts's implementation of higher dimensional pattern matching [Pit14]:

$$f : (\forall i.X \uplus Y) \to (\forall i.X) \uplus (\forall i.Y)$$

$$f\ \hat{c} = \nu(i : \mathbb{I}).\mathsf{case}\,\hat{c}\,i\,\mathsf{of}\left\{\begin{array}{lcl} \mathsf{inl}\,a & \mapsto & \mathsf{inl}\,(\lambda i.a) \\ \mathsf{inr}\,b & \mapsto & \mathsf{inr}\,(\lambda i.b) \end{array}\right\}$$

The right to use ν is derived from the isomorphism $\mathsf{const}_i^{-1} : \forall\, i \circ \lrcorner\, i \cong 1$. Recalling that $\lrcorner\, i \cong \rangle\!\langle\, i$, this is equivalently to $\mathsf{unmer}_i : \forall\, i \circ \rangle\!\langle\, i \cong 1$. Then ν itself translates to abstraction over i, and instead of making the body fresh for i, we can transpend over it, allowing us to view a and b as functions, justifying the variable capture in nominal type theory. The example thus translates precisely to what we did in section 2.3 and example 4.20.

9. Conclusion

To summarize, the transpension type can be defined in a broad class of presheaf models and generalizes previous internalization operators. For now, we only present an extensional type system without an algorithmic typing judgement. The major hurdles towards producing an intensional version with decidable type-checking, are the following:

- We need to decide equality of 2-cells. Solutions may exist in the literature on higher-dimensional rewriting.
- If we want the substitution modality to reduce (remark 4.17), we need to solve the following problem: when $\hat{a} = \mathsf{mod}^{\circ}_{\Omega\,\sigma}\, a$ definitionally, then we need to infer a up to definitional equality from $\hat{a}$ in order to β-reduce the let-rule for $\langle \Omega\, \sigma \mid A \rangle$.
- We need a syntax-directed way to close the section computation rules of Φ (fig. 10) and transpension elimination (section 7) under substitution.
- We need to decide whether a proposition is true. This problem has been dealt with in special cases, e.g. in implementations of cubical type theory [VMA19].

References

[AGJ14] Robert Atkey, Neil Ghani, and Patricia Johann. A relationally parametric model of dependent type theory. In *Principles of Programming Languages*, 2014. doi:10.1145/2535838.2535852.

[AHH18] Carlo Angiuli, Kuen-Bang Hou (Favonia), and Robert Harper. Cartesian Cubical Computational Type Theory: Constructive Reasoning with Paths and Equalities. In Dan Ghica and Achim Jung, editors, *Computer Science Logic (CSL 2018)*, volume 119 of *LIPIcs*, pages 6:1–6:17, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. URL: http://drops.dagstuhl.de/opus/volltexte/2018/9673, doi:10.4230/LIPIcs.CSL.2018.6.

[BBC+19] Lars Birkedal, Aleš Bizjak, Ranald Clouston, Hans Bugge Grathwohl, Bas Spitters, and Andrea Vezzosi. Guarded cubical type theory. *Journal of Automated Reasoning*, 63(2):211–253, 8 2019. doi:10.1007/s10817-018-9471-7.

[BCH14] Marc Bezem, Thierry Coquand, and Simon Huber. A Model of Type Theory in Cubical Sets. In *19th International Conference on Types for Proofs and Programs (TYPES 2013)*, volume 26, pages 107–128, Dagstuhl, Germany, 2014. URL: http://drops.dagstuhl.de/opus/volltexte/2014/4628, doi:10.4230/LIPIcs.TYPES.2013.107.

[BCM15] Jean-Philippe Bernardy, Thierry Coquand, and Guilhem Moulin. A presheaf model of parametric type theory. *Electron. Notes in Theor. Comput. Sci.*, 319:67 – 82, 2015. doi:http://dx.doi.org/10.1016/j.entcs.2015.12.006.

[BCM+20] Lars Birkedal, Ranald Clouston, Bassel Mannaa, Rasmus Ejlers Møgelberg, Andrew M. Pitts, and Bas Spitters. Modal dependent type theory and dependent right adjoints. *Mathematical Structures in Computer Science*, 30(2):118138, 2020. doi:10.1017/S0960129519000197.

[BGC+16] Aleš Bizjak, Hans Bugge Grathwohl, Ranald Clouston, Rasmus Ejlers Møgelberg, and Lars Birkedal. Guarded dependent type theory with coinductive types. In *FOSSACS '16*, 2016. doi:10.1007/978-3-662-49630-5_2.

[BGM17] Patrick Bahr, Hans Bugge Grathwohl, and Rasmus Ejlers Møgelberg. The clocks are ticking: No more delays! In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*, pages 1–12, 2017. doi:10.1109/LICS.2017.8005097.

[BM18] Aleš Bizjak and Rasmus Ejlers Møgelberg. Denotational semantics for guarded dependent type theory. *CoRR*, abs/1802.03744, 2018. URL: http://arxiv.org/abs/1802.03744, arXiv:1802.03744.

[BMSS12] Lars Birkedal, Rasmus Mgelberg, Jan Schwinghammer, and Kristian Stvring. First steps in synthetic guarded domain theory: step-indexing in the topos of trees. *Logical Methods in Computer Science*, 8(4), 2012. doi:10.2168/LMCS-8(4:1)2012.

[BV17] Jean-Philippe Bernardy and Andrea Vezzosi. Parametric application. Private communication, 2017.

[Car86] John Cartmell. Generalised algebraic theories and contextual categories. *Ann. Pure Appl. Logic*, 32:209–243, 1986. doi:10.1016/0168-0072(86)90053-9.

[CCHM17] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: A constructive interpretation of the univalence axiom. *FLAP*, 4(10):3127–3170, 2017. URL: http://www.cse.chalmers.se/~simonhu/papers/cubicaltt.pdf.

[CH20] Evan Cavallo and Robert Harper. Internal parametricity for cubical type theory. In *28th EACSL Annual Conference on Computer Science Logic, CSL 2020, January 13-16, 2020, Barcelona, Spain*, pages 13:1–13:17, 2020. doi:10.4230/LIPIcs.CSL.2020.13.

[Che12] James Cheney. A dependent nominal type theory. *Log. Methods Comput. Sci.*, 8(1), 2012. doi:10.2168/LMCS-8(1:8)2012.

[CMS20] Evan Cavallo, Anders Mörtberg, and Andrew W Swan. Unifying Cubical Models of Univalent Type Theory. In Maribel Fernández and Anca Muscholl, editors, *Computer Science Logic (CSL 2020)*, volume 152 of *LIPIcs*, pages 14:1–14:17, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. URL: https://drops.dagstuhl.de/opus/volltexte/2020/11657, doi:10.4230/LIPIcs.CSL.2020.14.

[Gir64] Jean Giraud. Méthode de la descente. *Bull. Soc. Math. Fr., Suppl., Mém.*, 2:115, 1964. doi:10.24033/msmf.2.

[GKNB20a] Daniel Gratzer, Alex Kavvos, Andreas Nuyts, and Lars Birkedal. Type theory à la mode. Pre-print, 2020. URL: https://anuyts.github.io/files/mtt-techreport.pdf.

[GKNB20b] Daniel Gratzer, G. A. Kavvos, Andreas Nuyts, and Lars Birkedal. Multimodal dependent type theory. In Holger Hermanns, Lijun Zhang, Naoki Kobayashi, and Dale Miller, editors, *LICS '20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science, Saarbrücken, Germany, July 8-11, 2020*, pages 492–506. ACM, 2020. doi:10.1145/3373718.3394736.

[Hof97] Martin Hofmann. *Syntax and Semantics of Dependent Types*, chapter 4, pages 79–130. Cambridge University Press, 1997.

[HS97] Martin Hofmann and Thomas Streicher. Lifting grothendieck universes. Unpublished note, 1997. URL: https://www2.mathematik.tu-darmstadt.de/~streicher/NOTES/lift.pdf.

[Hub16] Simon Huber. *Cubical Interpretations of Type Theory*. PhD thesis, University of Gothenburg, Sweden, 2016. URL: http://www.cse.chalmers.se/~simonhu/misc/thesis.pdf.

[KLV12] Chris Kapulkin, Peter LeFanu Lumsdaine, and Vladimir Voevodsky. The simplicial model of univalent foundations. 2012. Preprint, http://arxiv.org/abs/1211.2851.

[LH11] Daniel R. Licata and Robert Harper. 2-dimensional directed type theory. *Electr. Notes Theor. Comput. Sci.*, 276:263–289, 2011. doi:10.1016/j.entcs.2011.09.026.

[LOPS18] Daniel R. Licata, Ian Orton, Andrew M. Pitts, and Bas Spitters. Internal universes in models of homotopy type theory. In *3rd International Conference on Formal Structures for Computation and Deduction, FSCD 2018, July 9-12, 2018, Oxford, UK*, pages 22:1–22:17, 2018. doi:10.4230/LIPIcs.FSCD.2018.22.

[Mou16] Guilhem Moulin. *Internalizing Parametricity*. PhD thesis, Chalmers University of Technology, Sweden, 2016. URL: publications.lib.chalmers.se/records/fulltext/235758/235758.pdf.

[ND18a] Andreas Nuyts and Dominique Devriese. Degrees of relatedness: A unified framework for parametricity, irrelevance, ad hoc polymorphism, intersections, unions and algebra in dependent type theory. In *Logic in Computer Science (LICS) 2018, Oxford, UK, July 09-12, 2018*, pages 779–788, 2018. doi:10.1145/3209108.3209119.

[ND18b] Andreas Nuyts and Dominique Devriese. Internalizing Presheaf Semantics: Charting the Design Space. In *Workshop on Homotopy Type Theory / Univalent Foundations*, 2018. URL: https://hott-uf.github.io/2018/abstracts/HoTTUF18_paper_1.pdf.

[ND19] Andreas Nuyts and Dominique Devriese. Dependable atomicity in type theory. In *TYPES*, 2019.

[Nor19] Paige Randall North. Towards a directed homotopy type theory. *Proceedings of the Thirty-Fifth Conference on the Mathematical Foundations of Programming Semantics, MFPS 2019, London, UK, June 4-7, 2019*, pages 223–239, 2019. doi:10.1016/j.entcs.2019.09.012.

[Nuy18a] Andreas Nuyts. Presheaf models of relational modalities in dependent type theory. *CoRR*, abs/1805.08684, 2018. arXiv:1805.08684.

[Nuy18b] Andreas Nuyts. Robust notions of contextual fibrancy. In *Workshop on Homotopy Type Theory / Univalent Foundations*, 2018. URL: https://hott-uf.github.io/2018/abstracts/HoTTUF18_paper_2.pdf.

[Nuy20a] Andreas Nuyts. *Contributions to Multimode and Presheaf Type Theory*. PhD thesis, KU Leuven, Belgium, 8 2020. URL: https://anuyts.github.io/files/phd.pdf.

[Nuy20b] Andreas Nuyts. The transpension type: Technical report. Pre-print, 2020. URL: https://anuyts.github.io/files/transpension-techreport.pdf.

[NVD17] Andreas Nuyts, Andrea Vezzosi, and Dominique Devriese. Parametric quantifiers for dependent type theory. *PACMPL*, 1(ICFP):32:1–32:29, 2017. URL: http://doi.acm.org/10.1145/3110276, doi:10.1145/3110276.

[OP18] Ian Orton and Andrew M. Pitts. Axioms for modelling cubical type theory in a topos. *Logical Methods in Computer Science*, 14(4), 2018. doi:10.23638/LMCS-14(4:23)2018.

[Ort18] Ian Orton. *Cubical Models of Homotopy Type Theory - An Internal Approach*. PhD thesis, University of Cambridge, 2018.

[Pit13] A. M. Pitts. *Nominal Sets: Names and Symmetry in Computer Science*, volume 57 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2013.

[Pit14] Andrew Pitts. Nominal sets and dependent type theory. In *TYPES*, 2014. URL: https://www.irif.fr/~letouzey/types2014/slides-inv3.pdf.

[PK19] Gun Pinyo and Nicolai Kraus. From cubes to twisted cubes via graph morphisms in type theory. *CoRR*, abs/1902.10820, 2019. URL: http://arxiv.org/abs/1902.10820, arXiv:1902.10820.

[PMD15] Andrew M. Pitts, Justus Matthiesen, and Jasper Derikx. A dependent type theory with abstractable names. *Electronic Notes in Theoretical Computer Science*, 312:19 – 50, 2015. Ninth Workshop on Logical and Semantic Frameworks, with Applications (LSFA 2014). URL: http://www.sciencedirect.com/science/article/pii/S1571066115000079, doi:https://doi.org/10.1016/j.entcs.2015.04.003.

[Rey83] John C. Reynolds. Types, abstraction and parametric polymorphism. In *IFIP Congress*, pages 513–523, 1983.

[RS17] E. Riehl and M. Shulman. A type theory for synthetic ∞-categories. *ArXiv e-prints*, May 2017. arXiv:1705.07442.

[Sta19] The Stacks Project Authors. Stacks project. http://stacks.math.columbia.edu, 2019. Tags 00VC and 00XF.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. http://homotopytypetheory.org/book, IAS, 2013.

[VMA19] Andrea Vezzosi, Anders Mörtberg, and Andreas Abel. Cubical Agda: a dependently typed programming language with univalence and higher inductive types. *PACMPL*, 3(ICFP):87:1–87:29, 2019. doi:10.1145/3341691.

[WL20] Matthew Z. Weaver and Daniel R. Licata. A constructive model of directed univalence in bicubical sets. In Holger Hermanns, Lijun Zhang, Naoki Kobayashi, and Dale Miller, editors, *LICS '20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science, Saarbrücken, Germany, July 8-11, 2020*, pages 915–928. ACM, 2020. doi:10.1145/3373718.3394794.