The Transpension Type: Technical Report

Andreas Nuyts*
Vrije Universiteit Brussel, Belgium

August 31, 2021

Contents

*Carried out most of this research at imec-DistriNet, KU Leuven, holding a PhD Fellowship from the Research Foundation - Flanders (FWO).

1 Introduction

The purpose of these notes is to give a categorical semantics for the transpension type [ND20], which is right adjoint to a potentially substructural dependent function type.

- In section 2 we discuss some prerequisites.
- In section 3, we define multipliers and discuss their properties.
- In section 4, we study how multipliers lift from base categories to presheaf categories.
- In section 5, we explain how typical presheaf modalities can be used in the presence of the transpension type.
- In section 6, we study commutation properties of prior modalities, substitution modalities and multiplier modalities.

2 Prerequisites

2.1 On adjoints

Lemma 2.1.1. Let $L \dashv R$.

- Natural transformations $LF \to G$ correspond to natural transformations $F \to RG$, naturally in F and G.
- Natural transformations $FR \to G$ correspond to natural transformations $F \to GL$, naturally in F and G.

Proof. The first statement is trivial.

To see the second statement, we send $\zeta : FR \to G$ to $\zeta L \circ F\eta : F \to GL$, and conversely $\theta : F \to GL$ to $G\varepsilon \circ \theta R : FR \to G$. Naturality is clear. Mapping ζ to and fro, we get

$$G\varepsilon \circ \zeta LR \circ F\eta R = \zeta \circ FR\varepsilon \circ F\eta R = \zeta. \tag{1}$$

Mapping θ to and fro, we get

$$G\varepsilon L \circ \theta RL \circ F\eta = G\varepsilon L \circ GL\eta \circ \theta = \theta. \qquad \square$$

Lemma 2.1.2. Assume 4 triples of adjoint functors: $E \dashv F \dashv G$ and $E' \dashv F' \dashv G'$ and $S_1 \dashv T_1 \dashv U_1$ and $S_2 \dashv T_2 \dashv U_2$ such that the following diagram commutes up to natural isomorphism:

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "F"] \arrow[d, "T_1"'] & \mathcal{C}_2 \arrow[d, "T_2"] \\ \mathcal{C}'_1 \arrow[r, "F'"'] & \mathcal{C}'_2 \end{tikzcd} \tag{2}$$

Then we have

$$\begin{array}{lll} ES_2 \cong S_1 E' & E'T_2 \to T_1 E & \\ FS_1 \leftarrow S_2 F' & F'T_1 \cong T_2 F & FU_1 \to U_2 F' \\ & G'T_2 \leftarrow T_1 G & GU_2 \cong U_1 G'. \end{array} \tag{3}$$

In fact, any one of these statements holds if only the adjoints used by that statement are given.

Proof. The central isomorphism is given. The other isomorphisms are obtained by taking the left/right adjoints of both hands of the original isomorphism. By picking one direction of the central isomorphism, we can step to the left/right/top/bottom by applying lemma 2.1.1. ☐

Proposition 2.1.3. If a functor $R : \mathcal{C} \to \widehat{\mathcal{W}}$ from a CwF $\mathcal{C}$ to a presheaf CwF $\widehat{\mathcal{W}}$ has a left adjoint L, then it is a weak CwF morphism.

Proof. We use the presheaf notations from [Nuy18] (section 2.3.1).

For $\Gamma \vdash_{\mathcal{C}} T$ type, define $R\Gamma \vdash_{\widehat{\mathcal{W}}} RT$ type by

$$(W \rhd_{\widehat{\mathcal{W}}} (RT)[\delta\rangle) :\cong (L\mathbf{y}W \vdash_{\mathcal{C}} T[\varepsilon \circ L\delta]). \tag{4}$$

Naturality of this operation is easy to show, and the action of R on terms is given by $({}^{R}t)[\delta\rangle = t[\varepsilon \circ L\delta]$. ☐

Definition 2.1.4. Given adjoint functors $L \dashv R$ such that R is a weak CwF morphism, and $A \in \mathrm{Ty}(L\Gamma)$, we write $\langle R \,|\, A\rangle := (RA)[\eta] \in \mathrm{Ty}(\Gamma)$.

Note that $\langle R \,|\, A[\varepsilon]\rangle = (RA)[R\varepsilon][\eta] = RA$.

2.2 Dependent ends and co-ends

We will use $\forall$ and $\exists$ to denote ends and co-ends as well as their dependent generalizations [Nuy20, §2.2.6-7]:

Definition 2.2.1. A **dependent end** of a functor $F : \mathsf{Tw}(\mathcal{I}) \to \mathcal{C}$, somewhat ambiguously denoted $\forall i.F(i \xrightarrow{\mathrm{id}} i)$, is a limit of F.

Definition 2.2.2. A **dependent co-end** of a functor $F : \mathsf{Tw}(\mathcal{I})^{\mathsf{op}} \to \mathcal{C}$, somewhat ambiguously denoted $\exists i.F(i \xrightarrow{\mathrm{id}} i)$, is a colimit of F.

Example 2.2.3. Assume a functor $G : \mathcal{C} \to \mathcal{D}$. One way to denote the set of natural transformations $\mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ which map to the identity under G, is as

$$A := \forall(c \in \mathcal{C}).\, \{\chi : c \to c \,|\, G\chi = \mathrm{id}_{Gc}\}\,.$$

In order to read the above as a dependent end, we must find a functor $G : \mathsf{Tw}(\mathcal{C}) \to \mathsf{Set}$ such that $G(c \xrightarrow{\mathrm{id}} c) = \{\chi : c \to c \,|\, G\chi = \mathrm{id}_{Gc}\}$. Clearly every covariant occurrence of c refers to the codomain of $(c \xrightarrow{\mathrm{id}} c)$, whereas every contravariant occurrence refers to the domain. So when we apply G to a general object $(x \xrightarrow{\varphi} y)$ of $\mathsf{Tw}(\mathcal{C})$, we should substitute x for every contravariant c and y for every covariant c. We can then throw in φ wherever this is necessary to keep things well-typed, as φ disappears anyway when $(x \xrightarrow{\varphi} y) = (c \xrightarrow{\mathrm{id}} c)$. Thus, we get

$$G(x \xrightarrow{\varphi} y) = \{\chi : x \to y \,|\, G\chi = G\varphi\}\,.$$

So we see that using a *dependent* end was necessary in order to mention id_c, as this generalizes to $\varphi : x \to y$ to which we do not have access in a non-dependent end.

An element of $\nu \in A$ is then a function

$$\nu : (c \in \mathcal{C}) \to \{\chi : c \to c \,|\, G\chi = \mathrm{id}_{Gc}\}$$

such that, whenever $\varphi : x \to y$, we have $\varphi \circ \nu_x = \nu_y \circ \varphi$.

2.3 Presheaves

2.3.1 Notation

We use the presheaf notations from [Nuy18]. Concretely:

- The application of a presheaf $\Gamma \in \widehat{\mathcal{W}}$ to an object $W \in \mathcal{W}$ is denoted $W \Rightarrow \Gamma$.
- The restriction of $\gamma : W \Rightarrow \Gamma$ by $\varphi : V \to W$ is denoted $\gamma \circ \varphi$ or $\gamma\varphi$.
- The application of a presheaf morphism $\sigma : \Gamma \to \Delta$ to $\gamma : W \Rightarrow \Gamma$ is denoted $\sigma \circ \gamma$ or $\sigma\gamma$.
  - By naturality of σ, we have $\sigma \circ (\gamma \circ \varphi) = (\sigma \circ \gamma) \circ \varphi$.
- If $\Gamma \in \widehat{\mathcal{W}}$ and $T \in \mathrm{Ty}(\Gamma)$ (also denoted $\Gamma \vdash T\,\mathsf{type}$), i.e. T is a presheaf over the category of elements $\mathcal{W}/\Gamma$, then we write the application of T to (W, γ) as $(W \rhd T[\gamma\rangle)$ and $t \in (W \rhd T[\gamma\rangle)$ as $W \rhd t : T[\gamma\rangle$.
  - By definition of type substitution in a presheaf CwF, we have $(W \rhd T[\sigma][\gamma\rangle) = (W \rhd T[\sigma\gamma\rangle)$
- The restriction of $W \rhd t : T[\gamma\rangle$ by $\varphi : (V, \gamma \circ \varphi) \to (W, \gamma)$ is denoted as $W \rhd t\,\langle\varphi\rangle : T[\gamma\varphi\rangle$.
- If $t \in \mathrm{Tm}(\Gamma, T)$ (also denoted $\Gamma \vdash t : T$), then the application of t to (W, γ) is denoted $V \rhd t[\gamma\rangle : T[\gamma\rangle$.
  - The naturality condition for terms is then expressed as $t[\gamma\rangle\,\langle\varphi\rangle = t[\gamma\varphi\rangle$.
  - By definition of term substitution in a presheaf CwF, we have $t[\sigma][\gamma\rangle = t[\sigma\gamma\rangle$.
- We omit applications of the isomorphisms $(W \Rightarrow \Gamma) \cong (\mathbf{y}W \to \Gamma)$ and $(W \rhd T[\gamma\rangle) \cong (\mathbf{y}W \vdash T[\gamma])$. This is not confusing: e.g. given $W \rhd t : T[\gamma\rangle$, the term $\mathbf{y}W \vdash t' : T[\gamma]$ is defined by $t'[\varphi\rangle := t\,\langle\varphi\rangle$.

One advantage of these notations is that we can put presheaf cells in diagrams; we will use double arrows when doing so.

2.3.2 On the Yoneda-embedding

We consider the Yoneda-embedding $\mathbf{y} : \mathcal{W} \to \widehat{\mathcal{W}}$.

Proposition 2.3.1. A morphism $\varphi : V \to W$ in $\mathcal{W}$ is:

- Mono if and only if $\mathbf{y}\varphi$ is mono,
- Split epi if and only if $\mathbf{y}\varphi$ is epi.

Proof. It is well-known that a presheaf morphism $\sigma : \Gamma \to \Delta$ is mono/epi if and only if $\sigma \circ \sqcup : (W \Rightarrow \Gamma) \to (W \Rightarrow \Delta)$ is injective/surjective for all W. Now $\mathbf{y}\varphi \circ \sqcup = \varphi \circ \sqcup$. So $\mathbf{y}\varphi$ is mono if and only if $\varphi \circ \sqcup$ is injective, which means φ is mono. On the other hand, $\mathbf{y}\varphi$ is epi if and only if $\varphi \circ \sqcup$ is surjective, which is the case precisely when id is in its image, and that exactly means that φ is split epi. ☐

2.3.3 Lifting functors

Theorem 2.3.2. Any functor $F : \mathcal{V} \to \mathcal{W}$ gives rise to functors $F_! \dashv F^* \dashv F_*$, with a natural isomorphism $F_! \circ \mathbf{y} \cong \mathbf{y} \circ F : \mathcal{V} \to \widehat{\mathcal{W}}$. We will call $F_! : \widehat{\mathcal{V}} \to \widehat{\mathcal{W}}$ the **left lifting** of F to presheaves, $F^* : \widehat{\mathcal{W}} \to \widehat{\mathcal{V}}$ the **central** and $F_* : \widehat{\mathcal{V}} \to \widehat{\mathcal{W}}$ the **right lifting**.[1][2] [Sta19]

[1] The central and right liftings are also sometimes called the inverse image and direct image of F, but these are actually more general concepts and as such could perhaps cause confusion or unwanted connotations in some circumstances. The left-central-right terminology is very no-nonsense.

[2] From the construction, it is evident that F^* is precomposition with F and hence, by definition of Kan extension, $F_!$ and F_* are the left and right Kan extensions of F.

Proof. Using quantifier symbols for ends and co-ends, we can define:

$$\begin{array}{rcll} W \Rightarrow F_!\Gamma & := & \exists V.(W \to FV) \times (V \Rightarrow \Gamma), & \\ V \Rightarrow F^*\Delta & := & FV \Rightarrow \Delta & \\ W \Rightarrow F_*\Gamma & := & \forall V.(FV \to W) \to (V \Rightarrow \Gamma) & = \quad (F^*\mathbf{y}W \to \Gamma). \end{array}$$

By the co-Yoneda lemma, we have:

$$W \Rightarrow F_!\mathbf{y}V \quad = \quad \exists V'.(W \to FV') \times (V' \to V) \quad \cong \quad (W \to FV) \quad = \quad (W \Rightarrow \mathbf{y}FV),$$

i.e. $F_!\mathbf{y}V \cong \mathbf{y}FV$.

Adjointness also follows from applications of the Yoneda and co-Yoneda lemmas. ☐

Notation 2.3.3.

- We denote the cell $(V, \varphi, \gamma) : W \Rightarrow F_!\Gamma$ as $F_!\gamma \circ \varphi$. If we rename $F_!$, then we will also do so in this notation. We will further abbreviate $F_!\gamma \circ \mathrm{id} = F_!\gamma$ and, if $\Gamma = \mathbf{y}V$, also $F_!\mathrm{id} \circ \varphi = \varphi$.
- If $\delta : FV \Rightarrow \Delta$, then we write $\alpha_F(\delta) : V \Rightarrow F^*\Delta$.
- If $\gamma : F^*\mathbf{y}W \Rightarrow \Gamma$, then we write $\beta_F(\gamma) : W \Rightarrow F_*\Gamma$.

Proposition 2.3.4. A functor $F : \mathcal{V} \to \mathcal{W}$ is fully faithful if and only if $F_!$ is fully faithful.

Proof. To see the implication from left to right: It is a standard fact of adjoint functors [nLa21a] that the left adjoint $F_!$ is fully faithful if and only if $\eta : \Gamma \to F^*F_!\Gamma$ is a natural isomorphism. If F is fully faithful, then we can apply the co-Yoneda lemma:

$$(V \Rightarrow F^*F_!\Gamma) = (\exists V'.(FV \to FV') \times (V' \Rightarrow \Gamma)) \cong (\exists V'.(V \to V') \times (V' \Rightarrow \Gamma)) \cong (V \Rightarrow \Gamma)$$

i.e. $F^*F_!\Gamma \cong \Gamma$ and it is straightforward to see that this isomorphism is indeed the co-unit.

The implication from right to left is straightforward. By full faithfulness of $\mathbf{y}$ and by theorem 2.3.2 we have

$$\begin{array}{r} (\mathbf{y}U \to \mathbf{y}V) \cong (U \to V), \\ (F_!\mathbf{y}U \to F_!\mathbf{y}V) \cong (\mathbf{y}FU \to \mathbf{y}FV) \cong (FU \to FV). \end{array}$$

☐

2.3.4 Dependent presheaf categories

Let $\mathcal{W}$ be a category. Then $\widehat{\mathcal{W}}$ is a category with families (CwF). The following are standard notions:

Definition 2.3.5. For any $U \in \mathcal{W}$, the **category of slices over** U, denoted $\mathcal{W}/U$, has objects (W, ψ) where $W \in \mathcal{W}$ and $\psi : W \to U$ and the morphisms $(W, \psi) \to (W', \psi')$ are the morphisms $\chi : W \to W'$ such that $\psi' \circ \chi = \psi$.

Definition 2.3.6. For any $\Gamma \in \widehat{\mathcal{W}}$, the **category of elements** of Γ, denoted

$$\int_{\mathcal{W}} \Gamma \quad \text{or} \quad \mathcal{W}/\Gamma \tag{5}$$

has objects (W, γ) where $W \in \mathcal{W}$ and $\gamma : W \Rightarrow \Gamma$, and the morphisms $(W, \gamma) \to (W', \gamma')$ are the morphisms $\chi : W \to W'$ such that $\gamma' \circ \chi = \gamma$.

Clearly, we have an isomorphism $\mathcal{W}/U \cong \mathcal{W}/\mathbf{y}U$ between the category of slices over U and the category of elements of $\mathbf{y}U$.[3]

We will use type-theoretic notation to make statements about the CwF $\widehat{\mathcal{W}}$, e.g. $\Gamma \vdash \mathsf{Ctx}$ means $\Gamma \in \widehat{\mathcal{W}}$ and $\Gamma \vdash T\,\mathsf{type}$ means $T \in \mathrm{Ty}(\Gamma)$. Now for any context or closed type $\Gamma \in \widehat{\mathcal{W}}$, there is another CwF $\widehat{\mathcal{W}/\Gamma}$. Statements about this category will also be denoted using type-theoretic notation, but prefixed with '$\Gamma \mid$'.

By unfolding the definitions of types and terms in a presheaf CwF, it is trivial to show that there is a correspondence — which we will treat as though it were the identity — between both CwFs:

[3]Depending on pedantic details, we may even have $\mathcal{W}/U = \mathcal{W}/\mathbf{y}U$.

- Contexts $\Gamma \mid \Theta \vdash \mathsf{Ctx}$ correspond to types $\Gamma \vdash \Theta\,\mathsf{type}$ which we will think of as telescopes $\Gamma.\Theta \vdash \mathsf{Ctx}$,
- Substitutions $\Gamma \mid \sigma : \Theta \to \Theta'$ correspond to functions $\Gamma \vdash \sigma : \Theta \to \Theta'$, or equivalently to telescope substitutions $\mathrm{id}_\Gamma.\sigma : \Gamma.\Theta \to \Gamma.\Theta'$,
- Types $\Gamma \mid \Theta \vdash T\,\mathsf{type}$ correspond to types $\Gamma.\Theta \vdash T\,\mathsf{type}$,
- Terms $\Gamma \mid \Theta \vdash t : T$ correspond to terms $\Gamma.\Theta \vdash t : T$.

In summary, the pipe is equivalent to a dot.

Proposition 2.3.7. We have an equivalence of categories $\widehat{\mathcal{W}/\Gamma} \simeq \widehat{\mathcal{W}}/\Gamma$.

Proof. $\to$ We map the presheaf $\Gamma \mid \Theta \vdash \mathsf{Ctx}$ to the slice $(\Gamma.\Theta, \pi)$.

$\leftarrow$ We map the slice (Δ, σ) to the preimage of σ, i.e. the presheaf σ^{-1} which sends (W, γ) to $\{\delta : W \Rightarrow \Delta \mid \sigma \circ \delta = \gamma\}$.

$\widehat{\mathcal{W}/\Gamma}$ We need a natural isomorphism $\eta : \forall \Theta.(\Gamma \mid \eta : \Theta \cong \pi^{-1})$. If $\theta : (W, \gamma) \Rightarrow \Theta$, then we define $\eta(\theta) = (\gamma, \theta) : W \Rightarrow \Gamma.\Theta$ and indeed we have $\pi \circ (\gamma, \theta) = \gamma$. This is clearly invertible.

$\widehat{\mathcal{W}}/\Gamma$ We need a natural isomorphism $\varepsilon : \forall(\Delta, \sigma).(\Gamma.\sigma^{-1}, \pi) \cong (\Delta, \sigma)$. Given $(\gamma, \delta) : W \Rightarrow \Gamma.\sigma^{-1}$ (i.e. we know $\sigma \circ \delta = \gamma$), we define $\varepsilon \circ (\gamma, \delta) = \delta : W \Rightarrow \Delta$. Then

$$\sigma \circ \varepsilon \circ (\gamma, \delta) = \sigma \circ \delta = \gamma = \pi \circ (\gamma, \delta), \tag{6}$$

so indeed we have a morphism in the slice category. It is inverted by sending $\delta : W \Rightarrow \Delta$ to $(\sigma \circ \delta, \delta) : W \Rightarrow \Gamma.\sigma^{-1}$. $\square$

Corollary 2.3.8. We have $\widehat{\mathcal{W}/U} \cong \widehat{\mathcal{W}/\mathbf{y}U} \simeq \widehat{\mathcal{W}}/\mathbf{y}U$. $\square$

2.3.5 Substitution and its adjoints

Definition 2.3.9. Given $U \in \mathcal{W}$, we write

- $\Sigma_U : \mathcal{W}/U \to \mathcal{W} : (W, \psi) \mapsto W$,
- $\Omega_U : \mathcal{W} \to \mathcal{W}/U : W \to (W \times U, \pi_2)$ (if $\mathcal{W}$ has cartesian products with U).

Proposition 2.3.10. If Ω_U exists, then $\Sigma_U \dashv \Omega_U$. We denote the unit as $\mathsf{copy}_U : \mathrm{Id} \to \Omega_U \Sigma_U$ and the co-unit as $\mathsf{drop}_U : \Sigma_U \Omega_U \to \mathrm{Id}$. $\square$

Proposition 2.3.11.
1. If $U \to \top$ is split epi, then the functor Ω_U is faithful.
2. (Not used). If $U \to \top$ is mono, then Σ_U is full.[4]

Proof.
1. We have some $\upsilon : \top \to U$, so that the action of Ω_U on morphisms sending $\varphi \mapsto \varphi \times U$ can be inverted: $\varphi = \pi_1 \circ (\varphi \times U) \circ (\mathrm{id}, \upsilon)$.
2. Take slices (W_1, ψ_1) and (W_2, ψ_2) and a morphism $\varphi : W_1 \to W_2$. The fact that $U \to \top$ is mono just means that there is only a single morphism arriving in U. Then φ is also a morphism between the slices. $\square$

Definition 2.3.12. Given $\chi : W_0' \to W_0$ in $\mathcal{W}$, we write

- $\Sigma^{/\chi} : \mathcal{W}/W_0' \to \mathcal{W}/W_0 : (W', \psi') \mapsto (W', \chi \circ \psi')$,
- $\Omega^{/\chi} : \mathcal{W}/W_0 \to \mathcal{W}/W_0'$ for the functor that maps (W, ψ) to its pullback along χ (if $\mathcal{W}$ has pullbacks along χ).

[4] An earlier version asserted fullness of Ω_U instead, but proved the current theorem.

If $\chi = \pi_1 : W_0 \times U \to W_0$, we also write $\Sigma_U^{/W_0} : \mathcal{W}/(W_0 \times U) \to \mathcal{W}/W_0$ and $\Omega_U^{/W_0} : \mathcal{W}/W_0 \to \mathcal{W}/(W_0 \times U)$.

Proposition 2.3.13. If $\Omega^{/\chi}$ exists, then $\Sigma^{/\chi} \dashv \Omega^{/\chi}$. We denote the unit as $\mathsf{copy}^{/\chi} : \mathrm{Id} \to \Omega^{/\chi}\Sigma^{/\chi}$ and the co-unit as $\mathsf{drop}^{/\chi} : \Sigma^{/\chi}\Omega^{/\chi} \to \mathrm{Id}$. □

Proposition 2.3.14 (Ultimately not used)**.**

1. If χ is split epi, then $\Omega^{/\chi}$ is faithful.
2. If χ is mono, then $\Sigma^{/\chi}$ is full.[5]

Proof.

1. We have some $\upsilon : W_0 \to W_0'$ such that $\chi \circ \upsilon = \mathrm{id}$. Then the action of $\Omega^{/\chi}$ on morphisms sending $\varphi \mapsto \varphi \times_{W_0} W_0'$ can be inverted: given $\varphi : (W_1, \psi_1) \to (W_2, \psi_2) \in \mathcal{W}/W_0$, we have

$$\varphi : W_1 \xrightarrow{(\mathrm{id}, \upsilon \circ \psi_1)} W_1 \times_{W_0} W_0' \xrightarrow{\varphi \times_{W_0} W_0'} W_2 \times_{W_0} W_0' \xrightarrow{\pi_1} W_2. \tag{7}$$

2. Take a morphism $\varphi : (W_1, \chi \circ \psi_1) \to (W_2, \chi \circ \psi_2)$. Then $\chi \circ \psi_2 \circ \varphi = \chi \circ \psi_1$. Because χ is mono, this implies that $\psi_2 \circ \varphi = \psi_1$, i.e. $\varphi : (W_1, \psi_1) \to (W_2, \psi_2)$. □

Definition 2.3.15. Given $\sigma : \Psi' \to \Psi$ in $\widehat{\mathcal{W}}$, we write

- $\Sigma^{/\sigma} : \mathcal{W}/\Psi' \to \mathcal{W}/\Psi : (W', \psi') \mapsto (W', \sigma \circ \psi')$,
- $\Omega^{/\sigma} : \mathcal{W}/\Psi \to \mathcal{W}/\Psi'$ for the functor that maps (W, ψ) to its pullback along σ (if $\mathcal{W}$ has pullbacks along σ), by which we mean a universal solution W' to the diagram

$$\begin{tikzcd} W' \arrow[r, dotted] \arrow[d, Rightarrow, dotted] & W \arrow[d, Rightarrow, "\psi"] \\ \Psi' \arrow[r, "\sigma"'] & \Psi. \end{tikzcd} \tag{8}$$

If $\sigma = \pi_1 : \Psi \times \Phi \to \Psi$, we also write $\Sigma_\Phi^{/\Psi} : \mathcal{W}/(\Psi \times \Phi) \to \mathcal{W}/\Psi$ and $\Omega_\Phi^{/\Psi} : \mathcal{W}/\Psi \to \mathcal{W}/(\Psi \times \Phi)$.

Proposition 2.3.16. If $\Omega^{/\sigma}$ exists, then $\Sigma^{/\sigma} \dashv \Omega^{/\sigma}$. We denote the unit as $\mathsf{copy}^{/\sigma} : \mathrm{Id} \to \Omega^{/\sigma}\Sigma^{/\sigma}$ and the co-unit as $\mathsf{drop}^{/\sigma} : \Sigma^{/\sigma}\Omega^{/\sigma} \to \mathrm{Id}$. □

Proposition 2.3.17 (Not used)**.**

1. If σ is surjective, then $\Omega^{/\sigma}$ is faithful.
2. If σ is injective, then $\Sigma^{/\sigma}$ is full.[6]

Proof.

1. If σ is surjective, then by the axiom of choice, there is at least a non-natural $f : \Psi \to \Psi'$ such that $\sigma \circ f = \mathrm{id}$. The rest of the proof is as for proposition 2.3.14.
2. Same as for proposition 2.3.14. □

Definition 2.3.18. The functors $\Sigma^{/\sigma} \dashv \Omega^{/\sigma}$ give rise to four adjoint functors

$$\Sigma^{\sigma|} \dashv \Omega^{\sigma|} \dashv \Pi^{\sigma|} \dashv \$^{\sigma|} \tag{9}$$

between $\widehat{\mathcal{W}/\Psi}$ and $\widehat{\mathcal{W}/\Psi'}$, of which the first three exist if only $\Sigma^{/\sigma}$ exists.

The units and co-units will be denoted:

$$\begin{array}{rcl@{\qquad\qquad}rcl} \mathsf{copy}^{\sigma|} & : & \mathrm{Id} \to \Omega^{\sigma|}\Sigma^{\sigma|} & \mathsf{drop}^{\sigma|} & : & \Sigma^{\sigma|}\Omega^{\sigma|} \to \mathrm{Id} \\ \mathsf{const}^{\sigma|} & : & \mathrm{Id} \to \Pi^{\sigma|}\Omega^{\sigma|} & \mathsf{app}^{\sigma|} & : & \Omega^{\sigma|}\Pi^{\sigma|} \to \mathrm{Id} \\ \mathsf{reidx}^{\sigma|} & : & \mathrm{Id} \to \$^{\sigma|}\Pi^{\sigma|} & \mathsf{unmerid}^{\sigma|} & : & \Pi^{\sigma|}\$^{\sigma|} \to \mathrm{Id} \end{array} \tag{10}$$

[5] An earlier version asserted fullness of $\Omega^{/\chi}$ instead, but proved the current theorem.

[6] An earlier version asserted fullness of $\Omega^{/\sigma}$ instead, but proved the current theorem.

We remark that, if we read presheaves over $\mathcal{W}/\Psi$ as types in context Ψ, then $\Omega^{\sigma|} : \widehat{\mathcal{W}/\Psi} \to \widehat{\mathcal{W}/\Psi'}$ is the standard interpretation of substitution in a presheaf category. If $\sigma = \pi : \Psi.A \to \Psi$ is a weakening morphism, then $\Omega^{\Psi|}_A := \Omega^{\pi|}$ is the weakening substitution, $\Pi^{\Psi|}_A := \Pi^{\pi|} : \widehat{\mathcal{W}/\Psi.A} \to \widehat{\mathcal{W}/\Psi}$ is isomorphic to the standard interpretation of the Π-type and $\Sigma^{\Psi|}_A := \Sigma^{\pi|} : \widehat{\mathcal{W}/\Psi.A} \to \widehat{\mathcal{W}/\Psi}$ is isomorphic to the standard interpretation of the Σ-type.

Theorem 2.3.19. Given types $\Psi \vdash A, B$ type, the projections constitute a pullback diagram:

$$\begin{tikzcd} \Psi.(A \times B) \arrow[r, "\beta'"] \arrow[d, "\alpha'"'] & \Psi.A \arrow[d, "\alpha"] \\ \Psi.B \arrow[r, "\beta"'] & \Psi, \end{tikzcd} \tag{11}$$

and every pullback diagram in a presheaf category is isomorphic to a diagram of this form. We have the following commutation properties:

	Σ_B	Ω_B	Π_B	$\$_B$
Σ_A	$\Sigma^{\alpha\|}\Sigma^{\beta'\|} \cong \Sigma^{\beta\|}\Sigma^{\alpha'\|}$	$\Sigma^{\alpha'\|}\Omega^{\beta'\|} \cong \Omega^{\beta\|}\Sigma^{\alpha\|}$	$\Sigma^{\alpha\|}\Pi^{\beta'\|} \to \Pi^{\beta\|}\Sigma^{\alpha'\|}$	
Ω_A	$\Omega^{\alpha\|}\Sigma^{\beta\|} \cong \Sigma^{\beta'\|}\Omega^{\alpha'\|}$	$\Omega^{\alpha'\|}\Omega^{\beta\|} = \Omega^{\beta'\|}\Omega^{\alpha\|}$	$\Omega^{\alpha\|}\Pi^{\beta\|} \cong \Pi^{\beta'\|}\Omega^{\alpha'\|}$	$\Omega^{\alpha'\|}\$^{\beta\|} \to \$^{\beta'\|}\Omega^{\alpha\|}$
Π_A	$\Pi^{\alpha\|}\Sigma^{\beta'\|} \leftarrow \Sigma^{\beta\|}\Pi^{\alpha'\|}$	$\Pi^{\alpha'\|}\Omega^{\beta'\|} \cong \Omega^{\beta\|}\Pi^{\alpha\|}$	$\Pi^{\alpha\|}\Pi^{\beta'\|} \cong \Pi^{\beta\|}\Pi^{\alpha'\|}$	$\Pi^{\alpha'\|}\$^{\beta'\|} \cong \$^{\beta\|}\Pi^{\alpha\|}$
$\$_A$		$\$^{\alpha'\|}\Omega^{\beta\|} \leftarrow \Omega^{\beta'\|}\$^{\alpha\|}$	$\$^{\alpha\|}\Pi^{\beta\|} \cong \Pi^{\beta'\|}\$^{\alpha'\|}$	$\$^{\alpha'\|}\$^{\beta\|} \cong \$^{\beta'\|}\$^{\alpha\|}$

(12)

where every statement holds if the mentioned functors exist.

Proof. In the base category, it is evident that $\Sigma^{/\alpha}\Sigma^{/\beta'} = \Sigma^{/\beta}\Sigma^{/\alpha'}$. By applying the functor $\sqcup^*$, we obtain $\Omega^{\alpha'|}\Omega^{\beta|} = \Omega^{\beta'|}\Omega^{\alpha|}$, whence by lemma 2.1.2 the entire diagonal of the commutation table.

It is a well-known fact that Σ- and Π-types are respected by substitution, which gives us the isomorphisms for swapping Ω and either Σ or Π. Lemma 2.1.2 then gives the rest. $\square$

Theorem 2.3.20. Given $\sigma : \Psi' \to \Psi$, the following operations are invertible:

$$\frac{\Psi \mid \Sigma^{\sigma|}\Gamma \vdash T \text{ type}}{\Psi' \mid \Gamma \vdash (\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}] \text{ type}} \qquad \frac{\Psi \mid \Sigma^{\sigma|}\Gamma \vdash t : T}{\Psi' \mid \Gamma \vdash ({}^{\Omega^{\sigma|}}t)[\mathsf{copy}^{\sigma|}] : (\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}]} \tag{13}$$

Proof. Note that T is a presheaf over $(\mathcal{W}/\Psi)/\Sigma^{\sigma|}\Gamma$, and $(\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}]$ is a presheaf over $(\mathcal{W}/\Psi')/\Gamma$. We compare the objects of these categories:

$$\begin{aligned}
&\mathsf{Obj}((\mathcal{W}/\Psi)/\Sigma^{\sigma|}\Gamma)\\
&= (W \in \mathcal{W}) \times (\psi : W \Rightarrow \Psi) \times \exists((W', \psi') \in \mathcal{W}/\Psi'.(\chi : (W, \psi) \to \Sigma^{/\sigma}(W', \psi')) \times ((W', \psi') \Rightarrow \Gamma)\\
&\cong (W \in \mathcal{W}) \times (\psi : W \Rightarrow \Psi) \times \exists W'.(\psi' : W' \Rightarrow \Psi') \times (\chi : (W, \psi) \to \Sigma^{/\sigma}(W', \psi')) \times ((W', \psi') \Rightarrow \Gamma)\\
&\cong (W \in \mathcal{W}) \times (\psi : W \Rightarrow \Psi) \times \exists W'.(\psi' : W' \Rightarrow \Psi') \times (\chi : (W, \psi) \to (W', \sigma \circ \psi')) \times ((W', \psi') \Rightarrow \Gamma)\\
&\cong (W \in \mathcal{W}) \times \exists W'.(\psi' : W' \Rightarrow \Psi') \times (\chi : W \to W') \times ((W', \psi') \Rightarrow \Gamma)\\
&\qquad \text{because } \chi \text{ is a slice morphism iff } \psi = \sigma \circ \psi' \circ \chi\\
&\cong (W \in \mathcal{W}) \times (\psi' : W \Rightarrow \Psi') \times ((W, \psi') \Rightarrow \Gamma)\\
&\cong \mathsf{Obj}((\mathcal{W}/\Psi')/\Gamma).
\end{aligned}$$

A similar consideration of the Hom-sets leads to the conclusion that both categories are isomorphic. Moreover, we remark that the isomorphism sends $((W, \psi'), \gamma)$ on the right to $((W, \sigma \circ \psi'), \Sigma^{\sigma|}\gamma)$ on the left. When we consider the action of $(\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}]$ on $((W, \psi'), \gamma)$, we find:

$$\Big((W, \psi') \rhd (\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}][\gamma\rangle\Big) = \Big(\Sigma^{/\sigma}(W, \psi') \rhd T\Big[\Sigma^{\sigma|}\gamma\Big\rangle\Big)$$

$$= \left((W, \sigma \circ \psi') \rhd T\left[\Sigma^{\sigma|} \gamma \right\rangle \right)$$

In other words, the types T and $(\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}]$ are equal over an isomorphism of categories. Then certainly T can be retrieved from $(\Omega^{\sigma|}T)[\mathsf{copy}^{\sigma|}]$. An identical argument works for terms. □

2.3.6 Reconstructing right adjoints

Proposition 2.3.21. Given a left adjoint functor $L : \widehat{\mathcal{W}} \to \mathcal{C}$, we can construct a right adjoint $R_L : \mathcal{C} \to \widehat{\mathcal{W}}$ without using the axiom of choice.

Proof. Define $(W \Rightarrow R_L\Gamma) := (L\mathbf{y}W \to \Gamma)$. As a matter of notational hygiene, write $\alpha_L : (L\mathbf{y}W \to \Gamma) \to (W \Rightarrow R_L\Gamma)$ for the identity function. Define restriction by $\alpha_L(\gamma) \circ \varphi = \alpha_L(\gamma \circ L\mathbf{y}\varphi)$ and the functorial action by $R_L\sigma \circ \alpha_L(\gamma) = \alpha_L(\sigma \circ \gamma)$. This is a well-defined presheaf functor.

Now we show that $L \dashv R_L$. Since L is a left adjoint, it has a right adjoint R. We have natural isomorphisms

$$(W \Rightarrow R_L\Gamma) = (L\mathbf{y}W \to \Gamma) \cong (\mathbf{y}W \to R\Gamma) \cong (W \Rightarrow R\Gamma)$$

so that R_L is naturally isomorphic to R and indeed right adjoint to L. □

3 Multipliers in the base category

3.1 Definition

Definition 3.1.1. Let $\mathcal{W}$ be a category with terminal object $\top$. An object W is **spooky** if $() : W \to \top$ is not split epi. A category is **spooky** if it has a spooky object. Therefore, a category is non-spooky iff all morphisms to the terminal object are split epi.

Definition 3.1.2. Let $\mathcal{W}$ be a category with terminal object $\top$. A **multiplier** for an object $U \in \mathcal{V}$ is a functor $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ such that $\top \ltimes U \cong U$. This gives us a second projection $\pi_2 : \forall W. W \ltimes U \to U$. We define the **fresh weakening functor** as $\exists_U : \mathcal{W} \to \mathcal{V}/U : W \mapsto (W \ltimes U, \pi_2)$.

We say that a multiplier is:

- **Endo** if it is an endofunctor (i.e. $\mathcal{V} = \mathcal{W}$), and in that case:
  - **Semicartesian** if it is copointed, i.e. if there is also a first projection $\pi_1 : \forall W. W \ltimes U \to W$,
  - **3/4-cartesian** if it is a comonad, i.e. if there is additionally a 'diagonal' natural transformation $\sqcup \ltimes \delta : \forall W. W \ltimes U \to (W \ltimes U) \ltimes U$ such that $\pi_1 \circ (W \ltimes \delta) = (\pi_1 \ltimes U) \circ (W \ltimes \delta) = \mathrm{id}$.
  - **Cartesian** if it is naturally isomorphic to the cartesian product with U,
- **Cancellative** if $\exists_U$ is faithful, or equivalently (lemma 3.2.2) if $\sqcup \ltimes U$ is faithful,
- **Affine** if $\exists_U$ is full,
- **Non-spooky** if $\pi_2 : W \ltimes U \to U$ is always split epi, and in that case:
  - **Connection-free** if $\exists_U$ is essentially surjective on objects (V, ψ) such that ψ is split epi, i.e. if every such object in $\mathcal{V}/U$ is isomorphic to some $\exists_U W$.
  - A split epi slice (V, ψ) that is not in the image of $\exists_U$ even up to isomorphism, will be called a **connection** of the multiplier.
- **Quantifiable** if $\exists_U$ has a left adjoint $\exists_U : \mathcal{V}/U \to \mathcal{W}$, i.e. if $\sqcup \ltimes U$ is a *local right adjoint* (a.k.a. parametric right adjoint) [nLa21b]. We denote the unit as $\mathsf{copy}_U : \mathrm{Id} \to \exists_U \exists_U$ and the co-unit as $\mathsf{drop}_U : \exists_U \exists_U \to \mathrm{Id}$.

3.2 Basic properties

Some readers may prefer to first consult some examples (section 3.3).

Proposition 3.2.1. For any multiplier, we have $(\sqcup \ltimes U) = \Sigma_U \circ \dashv_U$. $\square$

Lemma 3.2.2. The functor $\sqcup \ltimes U$ is faithful if and only if $\dashv_U$ is faithful.

Proof. We have $(\sqcup \ltimes U) = \Sigma_U \circ \dashv_U$ and $\Sigma_U : \mathcal{V}/U \to \mathcal{V}$ is faithful as is obvious from its definition. $\square$

Proposition 3.2.3. A multiplier with a non-spooky domain is non-spooky.

Proof. The multiplier, as any functor, preserves split epimorphisms. $\square$

Proposition 3.2.4. Cartesian multipliers are 3/4-cartesian, and 3/4-cartesian multipliers are semicartesian.

Proof. The functor $\sqcup \times U$ is a comonad, and comonads are copointed by their co-unit. $\square$

Proposition 3.2.5. Cartesian multipliers are quantifiable.

Proof. The left adjoint to $\dashv_U = \Omega_U$ is then given by $\exists_U(V, \varphi) = \Sigma_U(V, \varphi) = V$ (proposition 2.3.10). $\square$

Proposition 3.2.6. Cartesian endomultipliers for non-spooky objects, are cancellative.

Non-spookiness is not required however: cancellative cartesian endomultipliers may be spooky (examples 3.3.4 and 3.3.6).

Proof. In this case, $\dashv_U = \Omega_U$ and $U \to \top$ is split epi, so this is part of proposition 2.3.11. $\square$

Proposition 3.2.7. If an endomultiplier for U is both 3/4-cartesian and affine, then U is a terminal object. If the multiplier is moreover cartesian, then it is naturally isomorphic to the identity functor.

Proof. Consider the following diagram:

$$\begin{tikzcd}
\top \ltimes U \arrow[rr, "\top \ltimes \delta"] \arrow[dr, "\pi_2"'] & & (\top \ltimes U) \ltimes U \arrow[dl, "\pi_2"] \\
& U &
\end{tikzcd} \tag{14}$$

This is a morphism from $\top \ltimes \delta : \dashv_U \top \to \dashv_U(\top \ltimes U)$ and thus, by affinity, of the form $\dashv_U \upsilon$ for some $\upsilon : \top \to \top \ltimes U$. This means in particular that

$$\mathrm{id}_{\top \ltimes U} = \pi_1 \circ (\top \ltimes \delta) = \pi_1 \circ (\upsilon \ltimes U) = \upsilon \circ \pi_1 : \top \ltimes U \to \top \ltimes U. \tag{15}$$

Composing on both sides with $\pi_2 : \top \ltimes U \cong U$, we find that $\mathrm{id}_U = (\pi_2 \circ \upsilon) \circ (\pi_1 \circ \pi_2^{-1})$ factors over $\top$, which means exactly that $\pi_2 \circ \upsilon : \top \to U$ and $\pi_1 \circ \pi_2^{-1} : U \to \top$ constitute an isomorphism, i.e. U is terminal.

If $\sqcup \ltimes U$ is cartesian, then it is a cartesian product with a terminal object and therefore naturally isomorphic to the identity functor. $\square$

3.3 Examples

Example 3.3.1 (Identity). The identity functor $W \ltimes \top := W$ is an endomultiplier for $\top$.

It is cartesian, cancellative, affine, spooky iff $\mathcal{W}$ is and otherwise connection-free, and quantifiable.

The functor $\exists_\top : \mathcal{W} \to \mathcal{W}/\top : W \mapsto (W, ())$ has a left adjoint $\exists_\top : \mathcal{W}/\top \to \mathcal{W} : (W, ()) \mapsto W$.

Example 3.3.2 (Cartesian product). Let $\mathcal{W}$ be a category with finite products and $U \in \mathcal{W}$.

Then $\sqcup \times U$ is an endomultiplier for U.

It is cartesian, cancellative if (but not only if) U is non-spooky (proposition 3.2.6), affine if and only if $U \cong \top$ (proposition 3.2.7) and quantifiable (proposition 3.2.5). We do not consider spookiness for this general case.

The functor $\exists_U = \Omega_U : V \mapsto (V \times U, \pi_2)$ has a left adjoint $\exists_U = \Sigma_U : (W, \psi) \mapsto W$. Hence, we have $\exists_U \exists_U = \sqcup \times U$.

Example 3.3.3 (Affine cubes). Let $\square^k$ be the category of affine non-symmetric k-ary cubes $\mathbb{I}^n$ as used in [BCH14] (binary) or [BCM15] (unary). A morphism $\varphi : \mathbb{I}^m \to \mathbb{I}^n$ is a function $\sqcup \langle\varphi\rangle : \{i_1, \ldots, i_n\} \to \{i_1 \ldots i_m, 0, \ldots, k-1\}$ such that $i\langle\varphi\rangle = j\langle\varphi\rangle \notin \{0, \ldots, k-1\}$ implies $i = j$. We also write $\varphi = (i_1\langle\varphi\rangle / i_1, \ldots, i_n\langle\varphi\rangle / i_n)$. This category is spooky if and only if $k = 0$.

Consider the functor $\sqcup * \mathbb{I} : \square^k \to \square^k : \mathbb{I}^n \mapsto \mathbb{I}^{n+1}$, which is a multiplier for $\mathbb{I}$. It acts on morphisms $\varphi : \mathbb{I}^m \to \mathbb{I}^n$ by setting $\varphi * \mathbb{I} = (\varphi, i_{m+1}/i_{n+1})$.

It is straightforwardly seen to be semicartesian, not 3/4-cartesian, cancellative, affine, spooky iff $k = 0$ and connection-free when $k \neq 0$, and quantifiable.

The functor $\exists_\mathbb{I} : \mathbb{I}^n \mapsto (\mathbb{I}^{n+1}, (i_{n+1}/i_1))$ has a left adjoint the functor $\exists_\mathbb{I}$ which sends $(\mathbb{I}^n, \psi)$ to $\mathbb{I}^n$ if $i_1\langle\psi\rangle \in \{0, \ldots, k-1\}$ and to $\mathbb{I}^{n-1}$ (by removing the variable $i_1\langle\psi\rangle$ and renaming the next ones) otherwise. The action on morphisms is straightforwardly constructed.

In the case where $k = 2$, we can throw in an involution $\neg : \mathbb{I} \to \mathbb{I}$. This changes none of the above results, except that $i_1\langle\psi\rangle$ may be the negation $\neg j$ of a variable j, in which case $\exists_U$ removes the variable j.

Example 3.3.4 (Cartesian cubes). Let $\boxslash^k$ be the category of cartesian non-symmetric k-ary cubes $\mathbb{I}^n$. A morphism $\varphi : \mathbb{I}^m \to \mathbb{I}^n$ is any function $\sqcup\langle\varphi\rangle : \{i_1, \ldots, i_n\} \to \{i_1 \ldots i_m, 0, \ldots, k-1\}$. This category is spooky if and only if $k = 0$.

Consider the functor $\sqcup \times \mathbb{I} : \boxslash^k \to \boxslash^k : \mathbb{I}^n \mapsto \mathbb{I}^{n+1}$, which is an endomultiplier for $\mathbb{I}$.

It is cartesian (hence non-affine and quantifiable with $\exists_\mathbb{I}(W, \psi) = W$), cancellative, spooky iff $k = 0$ and otherwise connection-free.

Again, involutions change none of the above results.

Example 3.3.5 (CCHM cubes). Let $\boxslash_{\vee,\wedge,\neg}$ be the category of (binary) CCHM cubes [CCHM17]. What's special here is that we have morphisms $\vee, \wedge : \mathbb{I}^2 \to \mathbb{I}$ (as well as involutions). This category is not spooky.

Again, we consider the functor $\sqcup \times \mathbb{I} : \boxslash_{\vee,\wedge,\neg} \to \boxslash_{\vee,\wedge,\neg} : \mathbb{I}^n \mapsto \mathbb{I}^{n+1}$, which is an endomultiplier for $\mathbb{I}$.

It is cartesian (hence non-affine and quantifiable with $\exists_\mathbb{I}(W, \psi) = W$), cancellative, not spooky, and not connection-free (since $(\mathbb{I}^2, \vee)$ and $(\mathbb{I}^2, \wedge)$ are connections).

Example 3.3.6 (Clocks). Let $\circleddash$ be the category of clocks, used as a base category in guarded type theory [BM20]. Its objects take the form $(i_1 : \circleddash_{k_1}, \ldots, i_n : \circleddash_{k_n})$ where all $k_j \geq 0$. We can think of a variable of type $\circleddash_k$ as representing a clock (i.e. a time dimension) paired up with a certificate that we do not care what happens after the time on this clock exceeds k. Correspondingly, we have a map $\circleddash_k \to \circleddash_\ell$ if $k \leq \ell$. These maps, together with weakening, exchange, and contraction, generate the category. The terminal object is $()$ and every other object is spooky.

Consider in this category the functor $\sqcup \times (i : \circleddash_k) : \circleddash \to \circleddash : W \mapsto (W, i : \circleddash_k)$, which is an endomultiplier for $(i : \circleddash_k)$.

It is cartesian (hence non-affine and quantifiable with $\exists_{(i:\circleddash_k)}(W, \psi) = W$), cancellative and spooky.

Example 3.3.7 (Twisting posets). Let $\mathcal{P}$ be the category of finite non-empty posets and monotonic maps. This category is non-spooky.

Let $\mathbb{I} = \{0 < 1\}$ and let $W \ltimes \mathbb{I} = (W^{\mathsf{op}} \times \{0\}) \cup (W \times \{1\})$ with $(x, 0) < (y, 1)$ for all $x, y \in W$. This is an endomultiplier for $\mathbb{I}$.

It is easily seen to be: not semicartesian, cancellative, not affine, not spooky, not connection-free, and quantifiable.

The functor $\dashv_{\mathbb{I}} : V \mapsto (V \ltimes \mathbb{I}, \pi_2)$ has a left adjoint $\exists_{\mathbb{I}} : (W, \psi) \mapsto \psi^{-1}(0)^{\mathsf{op}} \uplus \psi^{-1}(1)$ where elements from different sides of the $\uplus$ are incomparable.

We see this category as a candidate base category for directed type theory. The idea is that a cell over W is a commutative diagram in a category. A problem here is that a cell over a discrete poset such as $\{x, y\}$ where x and y are incomparable, should then be the same as a pair of cells over $\{x\}$ and $\{y\}$. This will require that we restrict from presheaves to sheaves, but that makes it notoriously difficult to model the universe [XE16]. One solution would be to restrict to totally ordered sets, but then we lose the left adjoint $\exists_{\mathbb{I}}$. We address this in example 3.3.8.

Example 3.3.8 (Affine twisted cubes)**.** Let $\bowtie$ be the subcategory of $\mathcal{P}$ whose objects are generated by $\top$ and $\sqcup \ltimes \mathbb{I}$ (note that every object then also has an opposite since $\top^{\mathsf{op}} = \top$ and $(V \ltimes \mathbb{I})^{\mathsf{op}} \cong V \ltimes \mathbb{I}$), and whose morphisms are given by

- $(\varphi, 0) : \bowtie(V, W \ltimes \mathbb{I})$ if $\varphi : \bowtie(V, W^{\mathsf{op}})$,
- $(\varphi, 1) : \bowtie(V, W \ltimes \mathbb{I})$ if $\varphi : \bowtie(V, W)$,
- $\varphi \ltimes \mathbb{I} : \bowtie(V \ltimes \mathbb{I}, W \ltimes \mathbb{I})$ if $\varphi : \bowtie(V, W)$,
- $() : \bowtie(V, \top)$.

Note that this collection automatically contains all identities, composites, and opposites. It is isomorphic to Pinyo and Kraus's category of twisted cubes, as can be seen from the ternary representation of said category [PK20, def. 34]. This category is not spooky.

Again, we consider the functor $\sqcup \ltimes \mathbb{I} : \bowtie \to \bowtie$, which is well-defined by construction of $\bowtie$ and an endomultiplier for $\mathbb{I}$. It corresponds to Pinyo and Kraus's twisted prism functor.

It is: not semicartesian, cancellative, affine, not spooky, connection-free, and quantifiable.

The left adjoint to $\dashv_{\mathbb{I}} : W \mapsto (W \ltimes \mathbb{I}, \pi_2)$ is now given by

$$\exists_{\mathbb{I}} : \begin{cases} (W, ((), 0)) & \mapsto & W^{\mathsf{op}} \\ (W, ((), 1)) & \mapsto & W \\ (W \ltimes \mathbb{I}, () \ltimes \mathbb{I}) & \mapsto & W, \end{cases} \tag{16}$$

with the obvious action on morphisms.

Example 3.3.9 (Embargoes)**.** In order to define contextual fibrancy [BT17] internally, we need to be able to somehow put a sign in the context $\Gamma.\mathbf{!}.\Theta$ in order to be able to say: the type is fibrant over Θ in context Γ. We call this an embargo and say that Θ is embargoed whereas Γ is not. If $\mathcal{C}$ is the category of contexts, then $\Gamma.\mathbf{!}.\Theta$ can be seen as an object of the arrow category $\mathcal{C}^{\uparrow}$, namely the arrow $\Gamma.\Theta \to \Gamma$.

If $\mathcal{C} = \widehat{\mathcal{W}}$ happens to be a presheaf category, then we have an isomorphism of categories $H : \widehat{\mathcal{W}}^{\uparrow} \cong \widehat{\mathcal{W} \times \uparrow}$ where $\uparrow = \{\perp \to \top\}$. Under this isomorphism, we have $\mathbf{y}(W, \top) \cong H(\mathbf{y}W \xrightarrow{\mathrm{id}} \mathbf{y}W)$ which we think of as $\mathbf{y}W.\mathbf{!}.\top$ and $\mathbf{y}(W, \perp) \cong H(\perp \xrightarrow{[]} \mathbf{y}W)$ which we think of as $\mathbf{y}W.\mathbf{!}.\perp$. Thus, forgetting the second component of (W, o) amounts to erasing the embargoed information. A $(W, \top)$-cell of $\Gamma.\mathbf{!}.\Theta$ is a W-cell of $\Gamma.\Theta$, i.e. a partly embargoed W-cell. We can extract the unembargoed information by restricting to $(W, \perp)$, as a $(W, \perp)$-cell of $\Gamma.\mathbf{!}.\Theta$ is just a W-cell of Γ.

There are 3 adjoint functors $\perp \dashv () \dashv \top$ between $\uparrow$ and Point from which we obtain 3 adjoint functors $(\mathrm{Id}, \perp) \dashv \pi_1 \dashv (\mathrm{Id}, \top)$ between $\mathcal{W} \times \uparrow$ and $\mathcal{W}$. The rightmost functor $(\mathrm{Id}, \top) : \mathcal{W} \to \mathcal{W} \times \uparrow$ is a multiplier for the terminal object $\mathbf{!} := (\top, \top) \in \mathcal{W} \times \uparrow$, denoted $\sqcup \ltimes \mathbf{!}$.

It is: not endo, cancellative, affine, spooky iff $\mathcal{W}$ is and otherwise *not* connection-free as every non-identity arrow is a connection, and quantifiable.

In order to look at the left adjoint, note first that since $\mathbf{!}$ is terminal, we have $(\mathcal{W} \times \uparrow)/\mathbf{!} \cong \mathcal{W} \times \uparrow$ and clearly $\dashv_{\mathbf{!}}$ corresponds to $(\mathrm{Id}, \top)$ under this isomorphism. This functor is part of a chain of *three* adjoint functors $(\mathrm{Id}, \bot) \dashv \pi_1 \dashv (\mathrm{Id}, \top)$ so that the multiplier is not just quantifiable but $\exists_{\mathbf{!}}$ even has a further left adjoint!

If $\sqcup \ltimes U : \mathcal{V} \to \mathcal{W}$ is a multiplier, then we can lift it to a multiplier $\sqcup \ltimes (U \ltimes \mathbf{!}) : \mathcal{V} \times \uparrow \to \mathcal{W} \times \uparrow$ by applying it to the first component, i.e. $(W, o) \ltimes (U \ltimes \mathbf{!}) = (W \ltimes U, o)$. The resulting multiplier inherits all properties in definition 3.1.2 from $\sqcup \ltimes U$, except that it is always spooky.

Example 3.3.10 (Enhanced embargoes)**.** If $\sqcup \ltimes U$ is a semicartesian endomultiplier on $\mathcal{W}$, then we might want to apply it to an arrow $V \xrightarrow{\psi} W$ by sending it to $V \ltimes U \xrightarrow{\psi \circ \pi_1} W$. This operation is not definable on $\mathcal{W} \times \uparrow$, which only encodes identity arrows of the forms $W \to W$ (as $(W, \top)$) and $\bot \to W$ (as $(W, \bot)$). For this reason, we move to the comma category $\mathcal{W}_\bot / \mathcal{W}$ where $\mathcal{W}_\bot$ is $\mathcal{W}$ with a freely added initial object. This comma category has as its objects arrows $V \xrightarrow{\psi} W$ where $V \in \mathcal{W}_\bot$ and $W \in \mathcal{W}$. Morphisms are simply commutative squares. A $(V \xrightarrow{\psi} W)$-cell is now a non-embargoed W-cell γ with embargoed information about $\gamma \circ \psi$.

We still have three adjoint functors $(\bot \to \sqcup) \dashv \mathrm{Cod} \dashv \Delta$ where $\Delta W = (W \to W)$. Further right adjoints would be $\mathrm{Dom} \dashv (\sqcup \to \top)$, but Dom is not definable as the domain might be $\bot$. We take $\Delta : W \mapsto (W \to W)$ as a multiplier for $\mathbf{!} := (\top \to \top)$, denoted $\sqcup \ltimes \mathbf{!} := \Delta$. For reasons that will become apparent later, we write $\mathbf{!} \surd \sqcup := (\sqcup \to \top)$. Note that a $(\mathbf{!} \surd U)$-cell is an unembargoed point with embargoed information about the degenerate U-cell on that point. E.g. in a context $\Gamma.\mathbf{!}.\Theta$, an $(\mathbf{!} \surd \mathbb{I})$-cell is exactly a path in Θ above a point in Γ, which is a concept that we need to quantify over when defining internal Kan fibrancy [BT17].

This multiplier is: not endo, cancellative, affine, spooky iff $\mathcal{W}$ is and otherwise connection-free, and quantifiable.

Now we get back to our multiplier $\sqcup \ltimes U$ which we can still lift to $\sqcup \ltimes (U \ltimes \mathbf{!})$ by applying it to both domain and codomain, i.e. $(V \to W) \ltimes (U \ltimes \mathbf{!}) := (U \ltimes V \to U \ltimes W)$, where by convention $\bot \ltimes U = \bot$. It inherits all properties in definition 3.1.2 from $\sqcup \ltimes U$, except that it is always spooky.

If $\sqcup \ltimes U$ is semicartesian, then we can also lift it to $\sqcup \ltimes (\mathbf{!} \surd U)$ by applying it only to the domain, i.e. $(V \to W) \ltimes (\mathbf{!} \surd U) = (V \ltimes U \to W)$. This again inherits all properties in definition 3.1.2 from $\sqcup \ltimes U$, except that it is always spooky, and that quantifiability can only be inherited if $\mathcal{W}$ has pushouts. In that case, we have

$$\exists_{(\mathbf{!} \surd U)}(W_1 \xrightarrow{\psi} W_2, (\psi_1, ())) = (\exists_U(W_1, \psi 1) \to W_2 \uplus_{W_1} \exists_U(W_1, \psi 1)). \tag{17}$$

Here, the morphism $W_1 = \Sigma_U(W_1, \psi_1) \to \exists_U(W_1, \psi 1)$ is an instance of the natural transformation $\mathsf{hide}_U : \Sigma_U \to \exists_U$ obtained by lemma 2.1.1 from $\pi_1 : \sqcup \ltimes U = \Sigma_U \Finv_U \to \mathrm{Id}$ (theorem 3.4.4). Indeed, given a morphism of slices $(\chi_1, \chi_2) : (W_1 \xrightarrow{\psi} W_2, (\psi_1, ())) \to \Finv_{(\mathbf{!} \surd U)}(V_1 \xrightarrow{\varphi} V_2)$, i.e.

$$\begin{tikzcd}
 & & V_1 \ltimes U \arrow[r, "\pi_1"] \arrow[ddl, "\pi_2"] & V_1 \arrow[r, "\varphi"] & V_2 \arrow[ddl] \\
W_1 \arrow[urr, "\chi_1"] \arrow[r, "\psi"'] \arrow[dr, "\psi_1"'] & W_2 \arrow[urrr, "\chi_2"] \arrow[drr] & & & \\
 & U \arrow[rr] & & \top &
\end{tikzcd} \tag{18}$$

we get a commutative diagram (the upper right square commutes by construction of hide_U)

$$\begin{tikzcd}
\exists_U(W_1,\psi_1) \arrow[r, "\exists_U \chi_1"] & \exists_U \text{Ǝ}_U V_1 \arrow[r, "\mathsf{drop}_U"] & V_1 \arrow[d, equal] \\
\Sigma_U(W_1,\psi_1) \arrow[u, "\mathsf{hide}_U"] \arrow[r, "\Sigma_U \chi_1"] \arrow[d, equal] & \Sigma_U \text{Ǝ}_U V_1 \arrow[u, "\mathsf{hide}_U"'] \arrow[d, equal] & {} \arrow[d, equal] \\
W_1 \arrow[r, "\chi_1"] \arrow[d, "\psi"'] & V_1 \ltimes U \arrow[r, "\pi_1"] & V_1 \arrow[d, "\varphi"] \\
W_2 \arrow[rr, "\chi_2"] & & V_2
\end{tikzcd} \tag{19}$$

so the top horizontal line, which is the transpose of χ_1, is a well-typed first component of the transpose of (χ_1, χ_2), while the three horizontal lines together constitute an arrow from the pushout to V_2 which is a well-typed second component. Conversely, given $(\omega_1, \omega_2) : \exists_{(\mathbf{!}\sqrt{U})}(W_1 \xrightarrow{\psi} W_2, (\psi_1, ())) \to (V_1 \xrightarrow{\varphi} V_2)$, i.e. (unwrapping the pushout)

$$\begin{tikzcd}
W_1 \arrow[r, "\mathsf{hide}_U"] \arrow[d, "\psi"'] & \exists_U(W_1,\psi_1) \arrow[r, "\omega_1"] & V_1 \arrow[d, "\varphi"] \\
W_2 \arrow[rr, "\chi_2"'] & & V_2
\end{tikzcd} \tag{20}$$

we can take the transpose of ω_1 as a first component and χ_2 as a second component of the transpose of (ω_1, ω_2). It remains to show that these form a commutative diagram with $\psi : W_1 \to W_2$ and $\varphi \circ \pi_1 : V_1 \ltimes U \to V_2$. But we have a commutative diagram

$$\begin{tikzcd}
W_1 \arrow[r, equal] \arrow[dd, equal] & \Sigma_U(W_1,\psi_1) \arrow[r, "\Sigma_U \mathsf{copy}_U"] \arrow[d, "\mathsf{hide}_U"] & \Sigma_U \text{Ǝ}_U \exists_U(W_1,\psi_1) \arrow[r, "\Sigma_U \text{Ǝ}_U \omega_1"] \arrow[d, "\mathsf{hide}_U"] & \Sigma_U \text{Ǝ}_U V_1 \arrow[r, equal] \arrow[d, "\mathsf{hide}_U"] & V_1 \ltimes U \arrow[d, "\pi_1"] \\
& \exists_U(W_1,\psi_1) \arrow[r, "\exists_U \mathsf{copy}_U"'] \arrow[d, equal] & \exists_U \text{Ǝ}_U \exists_U(W_1,\psi_1) \arrow[r, "\exists_U \text{Ǝ}_U \omega_1"'] & \exists_U \text{Ǝ}_U V_1 \arrow[r, "\mathsf{drop}_U"'] & V_1 \arrow[d, equal] \\
W_1 \arrow[r, "\mathsf{hide}_U"'] & \exists_U(W_1,\psi_1) \arrow[rrr, "\omega_1"'] & & & V_1
\end{tikzcd}$$

which can be pasted on top of the previous one to settle the matter. Finally, it is surprisingly easy to verify that the transposition operations just defined are mutually inverse.

Example 3.3.11 (Depth d cubes)**.** Let $\boxslash_d$ with $d \geq -1$ be the category of depth d cubes, used as a base category in degrees of relatedness [ND18, Nuy18].[7] Its objects take the form $(i_1 : \llparenthesis k_1 \rrparenthesis, \ldots, i_n : \llparenthesis k_n \rrparenthesis)$ where all $k_j \in \{0, \ldots, d\}$. Conceptually, we have a map $\llparenthesis k \rrparenthesis \to \llparenthesis \ell \rrparenthesis$ if $k \geq \ell$. Thus, morphisms $\varphi : (i_1 : \llparenthesis k_1 \rrparenthesis, \ldots, i_n : \llparenthesis k_n \rrparenthesis) \to (j_1 : \llparenthesis \ell_1 \rrparenthesis, \ldots, j_m : \llparenthesis \ell_m \rrparenthesis)$ send every variable $j : \llparenthesis \ell \rrparenthesis$ of the codomain to a value $j \langle \varphi \rangle$, which is either 0, 1 or a variable $i : \llparenthesis k \rrparenthesis$ of the domain such that $k \geq \ell$. The terminal object is $()$ and the category is non-spooky.

Consider in this category the functor $\sqcup \times (i : \llparenthesis k \rrparenthesis) : \boxslash_d \to \boxslash_d : W \mapsto (W, i : \llparenthesis k \rrparenthesis)$, which is an endomultiplier for $(i : \llparenthesis k \rrparenthesis)$.

It is cartesian (hence non-affine and quantifiable with $\exists_{(i:\llparenthesis k \rrparenthesis)}(W, \psi) = W$), cancellative, non-spooky and connection-free.

[7] For $d = -1$, we get the point category. For $d = 0$, we get the category of binary cartesian cubes $\boxslash^2$. For $d = 1$, we get the category of bridge/path cubes [NVD17, Nuy18].

Example 3.3.12 (Erasure). Let $\mathsf{Erase}_d = \{\top \leftarrow 0 \leftarrow 1 \leftarrow \ldots \leftarrow d\}$ with $d \geq -1$. This category has cartesian products $m \times n = \max(m, n)$ and all non-terminal objects are spooky. We remark that $\widehat{\mathsf{Erase}_0}$ is the Sierpiński topos.

We consider the endomultiplier $\sqcup \times i : \mathsf{Erase}_d \to \mathsf{Erase}_d$.

It is cartesian (hence non-affine and quantifiable with $\exists_i(j, \psi) = j$), cancellative and spooky.

We believe that this base category is a good foundation for studying the semantics of erasure of irrelevant subterms in Degrees of Relatedness [ND18]. The idea is that, for a presheaf Γ, the set $\top \Rightarrow \Gamma$ is the set of elements, whereas the set $i \Rightarrow \Gamma$ is the set of elements considered up to i-relatedness, but also whose existence is only guaranteed by a derivation up to i-relatedness.

3.4 Properties

3.4.1 Functoriality

Definition 3.4.1. A **multiplier morphism** or **morphism multiplier** for $v : U \to U'$ is a natural transformation $\sqcup \ltimes v : \sqcup \ltimes U \to \sqcup \ltimes U'$ such that $\pi_2 \circ (\top \ltimes v) \circ \pi_2^{-1} = v : U \to U'$ (or equivalently $\pi_2 \circ (W \ltimes v) = v \circ \pi_2 : W \ltimes U \to U'$ for all W).

- A morphism of semicartesian multipliers is **semicartesian** if it is a morphism of copointed endofunctors, i.e. if $\pi_1 \circ (W \ltimes v) = \pi_1$,
- A morphism of 3/4-cartesian multipliers is **3/4-cartesian** if it is a monad morphism, i.e. if additionally $(W \ltimes \delta) \circ (W \ltimes v) = ((W \ltimes v) \ltimes v) \circ (W \ltimes \delta)$,
- A morphism of cartesian multipliers is **cartesian** if it is the cartesian product with v.

Proposition 3.4.2. A semicartesian morphism of cartesian multipliers, is cartesian.

Proof. We have $\pi_2 \circ (W \ltimes v) = v \circ \pi_2$ and $\pi_1 \circ (W \ltimes v) = \pi_1$. Hence, $(W \ltimes v) = (\pi_1, v \circ \pi_2) = W \times v$. ☐

Proposition 3.4.3 (Functoriality). A multiplier morphism $\sqcup \ltimes v : \sqcup \ltimes U \to \sqcup \ltimes U'$ gives rise to a natural transformation $\Sigma^{/v} \circ \text{Ⅎ}_U \to \text{Ⅎ}_{U'}$. Hence, for quantifiable multipliers, we also have $\exists_{U'} \circ \Sigma^{/v} \to \exists_U$.

Proof. We have to show that for every $W \in \mathcal{W}$, we get $(W \ltimes U, v \circ \pi_2) \to (W \ltimes U', \pi_2)$. The morphism $W \ltimes v : W \ltimes U \to W \ltimes U'$ does the job. The second statement follows from lemma 2.1.1. ☐

3.4.2 Quantification and kernel theorem

Theorem 3.4.4 (Quantification theorem). If $\sqcup \ltimes U$ is

1. cancellative, affine and quantifiable, then we have a natural isomorphism $\mathsf{drop}_U : \exists_U \text{Ⅎ}_U \cong \mathrm{Id}$.
2. semi-cartesian, then we have:
   (a) $\mathsf{hide}_U : \Sigma_U \to \exists_U$ (if quantifiable),
   (b) $\mathsf{spoil}_U : \text{Ⅎ}_U \to \Omega_U$ (if Ω_U exists),
   (c) in any case $\Sigma_U \text{Ⅎ}_U \to \mathrm{Id}$.
3. 3/4-cartesian, then there is a natural transformation $\Sigma^{/\delta} \circ \text{Ⅎ}_U \to \text{Ⅎ}_{U \ltimes U}$, where we compose multipliers as in theorem 3.6.1.
4. cartesian, then we have:
   (a) $\exists_U \cong \Sigma_U$,
   (b) $\text{Ⅎ}_U \cong \Omega_U$,
   (c) $\exists_U \text{Ⅎ}_U \cong \Sigma_U \Omega_U = (\sqcup \times U) \cong (\sqcup \ltimes U)$.

   Moreover, these isomorphisms become equalities by choosing $\exists_U$ and Ω_U wisely (both are defined only up to isomorphism).

Proof. 1. This is a standard fact of fully faithful right adjoints such as $\dashv_U$.

2. By lemma 2.1.1, it is sufficient to prove $\Sigma_U \dashv_U \to \mathrm{Id}$. But $\Sigma_U \dashv_U = (\sqcup \ltimes U)$, so this is exactly the statement that the multiplier is semicartesian.

3. This is a special case of proposition 3.4.3.

4. By uniqueness of the cartesian product, we have $\dashv_U \cong \Omega_U$. Then the multiplier is quantifiable with $\exists_U \cong \Sigma_U$. The last point is now trivial. $\square$

Theorem 3.4.5 (Kernel theorem for non-spooky multipliers)**.** If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is non-spooky, cancellative, affine and connection-free, then $\dashv_U : \mathcal{W} \simeq \mathcal{V}/\!/U$ is an equivalence of categories, where $\mathcal{V}/\!/U$ is the full subcategory of $\mathcal{V}/U$ whose objects are the split epimorphic slices.

Proof. By non-spookiness, $\dashv_U$ lands in $\mathcal{V}/\!/U$. The other properties assert that $\dashv_U$ is fully faithful and essentially surjective as a functor $\mathcal{W} \to \mathcal{V}/\!/U$. $\square$

Definition 3.4.6. If we are doing classical mathematics, or if $\sqcup \ltimes U$ is quantifiable, then we obtain an inverse functor, which we denote $\ker_U$.

The kernel theorem applies to examples 3.3.1, 3.3.3 and 3.3.8 to 3.3.10.
We will use the kernel theorem in theorem 4.4.6 on transpension elimination, a dependent eliminator for the transpension type from which we can build a dependent eliminator for BCM's Ψ-type and prove BCM's Φ-rule [Mou16, BCM15].

3.4.3 Dealing with spookiness

Since spooky multipliers do not guarantee that $\dashv_U$ produces split epi slices, we need to come up with a larger class of suitable epi-like morphisms to U before we can proceed.

Definition 3.4.7. Given a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$, we say that a morphism $\varphi : V \to U$ is **dimensionally split** if there is some $W \in \mathcal{W}$ such that $\pi_2 : W \ltimes U \to U$ factors over φ. The other factor χ such that $\pi_2 = \varphi \circ \chi$ will be called a **dimensional section** of φ. We write $\mathcal{V}/\!/U$ for the full subcategory of $\mathcal{V}/U$ of dimensionally split slices.

The non-spookiness condition for multipliers is automatically satisfied if we replace 'split epi' with 'dimensionally split':

Corollary 3.4.8. For any multiplier $\sqcup \ltimes U$, any projection $\pi_2 : W \ltimes U \to U$ is dimensionally split. $\square$

Proposition 3.4.9. Take a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$.

1. If $\varphi \circ \chi$ is dimensionally split, then so is φ.
2. The identity morphism $\mathrm{id}_U : U \to U$ is dimensionally split.
3. If $\varphi : V \to U$ is dimensionally split and $\chi : V' \to V$ is split epi, then $\varphi \circ \chi : V' \to U$ is dimensionally split.
4. Every split epimorphism to U is dimensionally split.
5. If $\sqcup \ltimes U$ is non-spooky, then every dimensionally split morphism is split epi.

Proof. 1. If $\pi_2 : W \ltimes U \to U$ factors over $\varphi \circ \chi$, then it certainly factors over φ.

2. Since $\pi_2 : \top \ltimes U \to U$ factors over id_U.

3. Let φ' be a dimensional section of φ and χ' a section of χ. Then $\chi' \circ \varphi'$ is a dimensional section of $\varphi \circ \chi$.

4. From the previous two points, or (essentially by composition of the above reasoning) because if $\chi : U \to V$ is a section of $\varphi : V \to U$, then $\chi \circ \pi_2 : \top \ltimes U \to V$ is a dimensional section of φ.

5. If $\varphi : V \to U$ is dimensionally split, then some $\pi_2 : W \ltimes U \to U$ factors over φ. Since π_2 is split epi, id_U factors over π_2 and hence over φ, i.e. φ is split epi. $\square$

We can now extend the notions of connection and connection-freedom to spooky multipliers without changing their meaning for non-spooky multipliers:

Definition 3.4.10. We say that a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is **connection-free** if $\exists_U$ is essentially surjective on $\mathcal{V}//U$, the full subcategory of $\mathcal{V}/U$ of dimensionally split slices. A dimensionally split slice (V, ψ) that is not in the image of $\exists_U$ even up to isomorphism, will be called a **connection** of the multiplier.

Note that a multiplier is connection-free if every dimensionally split slice has an *invertible* dimensional section.

Theorem 3.4.11 (Kernel theorem)**.** If a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is cancellative, affine and connection-free, then $\exists_U : \mathcal{W} \simeq \mathcal{V}//U$ is an equivalence of categories. $\square$

Example 3.4.12 (Identity)**.** In the category $\mathcal{W}$ with the identity multiplier $W \ltimes \top = W$, every morphism $W \to \top$ is dimensionally split with id_W as an invertible dimensional section. The multiplier is connection-free.

Example 3.4.13 (Nullary cubes)**.** In the categories of nullary affine cubes $\square^0$ (example 3.3.3) and nullary cartesian cubes $\boxslash^0$ (example 3.3.4), a morphism $\varphi : \mathbb{I}^n \to \mathbb{I}$ is dimensionally split if $i_1 \langle \varphi \rangle$ is a variable. The multipliers $\sqcup * \mathbb{I} : \square^0 \to \square^0$ and $\sqcup \times \mathbb{I} : \boxslash^0 \to \boxslash^0$ are connection-free.

Example 3.4.14 (Clocks)**.** In the category of clocks $\circleddash$ (example 3.3.6), a morphism $\varphi : V \to (i : \circleddash_k)$ is dimensionally split if $i \langle \varphi \rangle$ has clock type $\circleddash_k$. The multiplier $\sqcup \times (i : \circleddash_k)$ is connection-free.

Example 3.4.15 (Twisted cubes)**.** In the category $\bowtie$ of twisted cubes (example 3.3.8), a morphism $\varphi : V \to \top \ltimes \mathbb{I}$ is dimensionally split if it equals $\varphi = () \ltimes \mathbb{I}$, with the identity as an invertible dimensional section. The multiplier $\sqcup \ltimes \mathbb{I}$ is connection-free.

Example 3.4.16 (Embargoes)**.** For the embargo multiplier $\sqcup \ltimes \mathbf{!} := (\mathrm{Id}, \top) : \mathcal{W} \to \mathcal{W} \times \uparrow$ (example 3.3.9) for $\mathbf{!} := (\top, \top)$, a morphism $((), ()) : (W, o) \to \mathbf{!}$ is dimensionally split if $o = \top$, with the identity as an invertible dimensional section. The multiplier $\sqcup \ltimes \mathbf{!}$ is connection-free.

For $\sqcup \ltimes (\mathbf{!} \ltimes U) : (W, o) \mapsto (W \ltimes U, o)$, a morphism $(\varphi, ()) : (W, o) \to (U, \top) = (\mathbf{!} \ltimes U)$ is dimensionally split if $\varphi : W \to U$ is dimensionally split for $\sqcup \ltimes U$. If $\chi : W' \ltimes U \to W$ is a dimensional section for φ, then $(\chi, \mathrm{id}_o) : (W' \ltimes U, o) \to (W, o)$ is a dimensional section for $(\varphi, ())$. Connection-freedom is then inherited from $\sqcup \ltimes U$.

Example 3.4.17 (Enhanced embargoes)**.** For the enhanced embargo multiplier $\sqcup \ltimes \mathbf{!} : \mathcal{W} \to \mathcal{W}_\perp/\mathcal{W} : W \mapsto (W \to W)$ (example 3.3.10), a morphism $(V \xrightarrow{\varphi} W) \to (\top \to \top) = \mathbf{!}$ is dimensionally split if $V \neq \perp$, with dimensional section $(\mathrm{id}_V, \varphi) : (V \to V) \to (V \xrightarrow{\varphi} W)$. This multiplier is generally not connection-free: since it only produces identity arrows, any dimensionally split non-identity arrow is a connection.

For $\sqcup \ltimes (U \ltimes \mathbf{!}) : (V \to W) \mapsto (V \ltimes U \to W \ltimes U)$, a morphism $(V \to W) \to (U \to U) = (U \ltimes \mathbf{!})$ is dimensionally split (with section $([], \chi) : (\perp \to W' \ltimes U) \to (V \to W)$) if the morphism $W \to U$ is dimensionally split for $\sqcup \ltimes U$ with section $\chi : W' \ltimes U \to W$. The multiplier $\sqcup \ltimes (U \ltimes \mathbf{!})$ is generally not connection-free, as the domain part of a dimensionally split morphism could be anything.

For $\sqcup \ltimes (\mathbf{!} \surd U) : (V \to W) \mapsto (V \ltimes U \to W)$, any morphism $(V \to W) \to (U \to \top) = (\mathbf{!} \surd U)$ is dimensionally split by

$$([], \mathrm{id}) \quad : \quad (\perp \to W) \ltimes (\mathbf{!} \surd U) \quad = \quad (\perp \to W) \quad \to \quad (V \to W). \tag{21}$$

This multiplier is therefore generally not connection-free.

To conclude, we have made the base category more complicated in order to be able to define the latter multiplier, but as a trade-off we now have connections to deal with.

Example 3.4.18 (Erasure)**.** In the category Erase_d (example 3.3.12) with multiplier $\sqcup \times i$, all morphisms to i are dimensionally split with the identity as an invertible dimensional section. The multiplier is connection-free.

3.4.4 Boundaries

Definition 3.4.19. The boundary ∂U of a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is a presheaf over $\mathcal{V}$ such that the cells $V \Rightarrow \partial U$ are precisely the morphisms $V \to U$ that are *not* dimensionally split.

This is a valid presheaf by proposition 3.4.9.

Proposition 3.4.20. If $\sqcup \ltimes U$ is non-spooky, then ∂U is the largest strict subobject of $\mathbf{y}U$.

Proof. Recall that if the multiplier is non-spooky, then dimensionally split and split epi are synonymous.

Clearly, $\partial U \subseteq \mathbf{y}U$. Since $\mathrm{id} : U \to U$ is split epi, we have $\partial U \subsetneq \mathbf{y}U$. Now take another strict subobject $\Upsilon \subsetneq \mathbf{y}U$. We show that $\Upsilon \subseteq \partial U$.

We start by showing that $\mathrm{id} \notin U \Rightarrow \Upsilon$. Otherwise, every $\varphi \in V \Rightarrow \mathbf{y}U$ would have to be a cell of Υ as it is a restriction of id, which would imply $\Upsilon = \mathbf{y}U$.

Now id is a restriction of any split epimorphism, so Υ contains no split epimorphisms, i.e. $\Upsilon \subseteq \partial U$. □

Example 3.4.21. In all the binary cube categories mentioned in section 3.3, $\partial\mathbb{I}$ is isomorphic to the constant presheaf of booleans.

For affine cubes, if we define a multiplier $\sqcup * \mathbb{I}^2$ in the obvious way, then $\partial\mathbb{I}^2$ is isomorphic to a colimit of four times $\mathbf{y}\mathbb{I}$ and four times $\mathbf{y}\top$, i.e. a square without filler. For cartesian asymmetric cubes, the square also gains a diagonal. For symmetric cubes (with an involution $\neg : \mathbb{I} \to \mathbb{I}$), the other diagonal also appears.

3.5 Acting on slices

Definition 3.5.1. Given a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$, we define

$$\exists_U^{/W_0} : \mathcal{W}/W_0 \to \mathcal{V}/(W_0 \ltimes U) : (W, \psi) \mapsto (W \ltimes U, \psi \ltimes U). \tag{22}$$

We say that $\sqcup \ltimes U$ is:

- **Strongly cancellative** if for all W_0, the functor $\exists_U^{/W_0}$ is faithful,
- **Strongly affine** if for all W_0, the functor $\exists_U^{/W_0}$ is full,
- **Indirectly strongly connection-free** (obsolete[8]) if for all W_0, the functor $\exists_U^{/W_0}$ is essentially surjective on slices $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$ such that $\pi_2 \circ \varphi$ is dimensionally split,
  - We point out that the full subcategory of such slices is isomorphic to $(\mathcal{V}/\!/U)/(W_0 \ltimes U, \pi_2)$,
- **Directly strongly connection-free** if for all W_0, the functor $\exists_U^{/W_0}$ is essentially surjective on slices $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$ such that $\varphi : V \to W_0 \ltimes U$ is directly dimensionally split:
  - We say that $\varphi : V \to W_0 \ltimes U$ is **directly dimensionally split** with direct dimensional section $\chi : W \ltimes U \to V$ if $\varphi \circ \chi$ is of the form $\psi \ltimes U$. The section can alternatively be presented as a morphism of slices $\chi : \exists_U^{/W_0}(W, \psi) \to (V, \varphi)$.
  - We denote the full subcategory of directly dimensionally split slices as $\mathcal{V}/\!/W_0 \ltimes U$.
  - A directly dimensionally split slice $(V, \psi) \in \mathcal{V}/W_0 \ltimes U$ that is not in the image of $\exists_U^{/W_0}$ even up to isomorphism, will be called a **direct connection** of the multiplier.

[8] This was the original notion of connection-freedom, as referred to in [Nuy20], but it leads to a boundary predicate (definition 4.4.2) that is respected by the substitution functor $\Omega^{\Psi|}_{\mathbf{y}U}$. This is in contrast to the transpension type, which in general is not respected by the substitution functor (it is if the multiplier is cancellative and affine, see theorem 6.3.1). As a result, with this notion of indirect strong connection-freedom, the boundary theorem 4.4.4 can only be stated over $\top \ltimes \mathbf{y}U$ as opposed to general $\Psi \ltimes \mathbf{y}U$. For this reason, we prefer the notion of *direct strong connection-freedom* below.

- **Strongly quantifiable** if for all W_0, the functor $\dashv_U^{/W_0}$ has a left adjoint $\exists_U^{/W_0} : \mathcal{V}/(W_0 \ltimes U) \to \mathcal{W}/W_0$. We denote the unit as $\mathsf{copy}_U^{/W_0} : \mathrm{Id} \to \dashv_U^{/W_0} \exists_U^{/W_0}$ and the co-unit as $\mathsf{drop}_U^{/W_0} : \exists_U^{/W_0} \dashv_U^{/W_0} \to \mathrm{Id}$.

The above definition generalizes the functor $\dashv_U$ that we already had:

Proposition 3.5.2. The functor $\dashv_U^{/\top} : \mathcal{W}/\top \to \mathcal{V}/(\top \ltimes U)$ is equal to $\dashv_U : \mathcal{W} \to \mathcal{V}/U$ over the obvious isomorphisms between their domains and codomains. Hence, each of the strong properties implies the basic property. (Both notions of strong connection-freedom imply basic connection-freedom.) $\square$

Note that both notions of strong connection-freedom are well-defined:

Proposition 3.5.3.

1. (Obsolete.) The functor $\dashv_U^{/W_0}$ factors over $(\mathcal{V}/\!/U)/(W_0 \ltimes U, \pi_2)$.
2. The functor $\dashv_U^{/W_0}$ factors over $\mathcal{V}/\!/W_0 \ltimes U$.
3. For every directly dimensionally split morphism $\varphi : V \to W_0 \ltimes U$, we have that $\pi_2 \circ \varphi$ is dimensionally split, i.e. there is a functor $\mathcal{V}/\!/W_0 \ltimes U \to (\mathcal{V}/\!/U)/(W_0 \ltimes U, \pi_2)$. Hence, indirect strong connection-freedom implies direct strong connection-freedom.

Proof.

1. The functor $\dashv_U^{/W_0}$ sends (W, ψ) to $(W \ltimes U, \psi \ltimes U)$. Since $\pi_2 \circ (\psi \ltimes U) = \pi_2$, it is dimensionally split with the identity as a section.
2. The identity is a strong dimensional section.
3. Let $\varphi : V \to W_0 \ltimes U$ be directly dimensionally split with section χ, i.e. $\varphi \circ \chi = \psi \ltimes U$. Then $\pi_2 \circ \varphi \circ \chi = \pi_2 \circ \psi \ltimes U = \pi_2$, so $\pi_2 \circ \varphi$ is dimensionally split with section χ. $\square$

Proposition 3.5.4. If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is cancellative, then it is strongly cancellative.

Proof. Pick morphisms $\varphi, \chi : (W, \psi) \to (W', \psi')$ in $\mathcal{W}/W_0$ such that $\dashv_U^{/W_0} \varphi = \dashv_U^{/V} \chi$. Expanding the definition of $\dashv_U^{/W_0}$, we see that this means that $\varphi \ltimes U = \chi \ltimes U$, and hence $\varphi = \chi$ by cancellation of $\sqcup \ltimes U$. $\square$

Proposition 3.5.5. If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is cancellative and affine, then it is strongly affine.

Proof. Pick (W, ψ) and (W', ψ') in $\mathcal{W}/W_0$, and a morphism $\chi : \dashv_U^{/W_0}(W, \psi) \to \dashv_U^{/W_0}(W', \psi')$. This amounts to a diagram:

$$\begin{tikzcd}
W \ltimes U \arrow[rr, "\chi"] \arrow[dr, "\psi \ltimes U"'] \arrow[ddr, bend right, "\pi_2"'] & & W' \ltimes U \arrow[dl, "\psi' \ltimes U"] \arrow[ddl, bend left, "\pi_2"] \\
& W_0 \ltimes U \arrow[d, "\pi_2"] & \\
& U, &
\end{tikzcd} \tag{23}$$

i.e. a triangle in $\mathcal{W}/U$, the objects of which are in the image of $\dashv_U : \mathcal{W} \to \mathcal{W}/U$. Then, by fullness (affinity) we get $\chi_0 : W \to W'$ such that $\dashv_U \chi_0 = \chi$, which by faithfulness (cancellativity) makes the following diagram commute:

$$\begin{tikzcd}
W \arrow[rr, "\chi_0"] \arrow[dr, "\psi"'] & & W' \arrow[dl, "\psi'"] \\
& W_0 &
\end{tikzcd} \tag{24}$$

Then χ_0 is a morphism $\chi_0 : (W, \psi) \to (W', \psi')$ in $\mathcal{W}/W_0$ and $\dashv_U^{/W_0} \chi_0 = \chi$. $\square$

Proposition 3.5.6. 1. (Obsolete.) If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is affine and connection-free, then it is indirectly strongly connection-free.

2. If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is affine and connection-free, then it is directly strongly connection-free.

Proof. 1. Pick some $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$ such that $\pi_2 \circ \varphi : V \to U$ is dimensionally split. Because $\Finv_U$ is essentially surjective on $\mathcal{V}//U$, there must be some $W \in \mathcal{W}$ such that $\iota : \Finv_U W = (W \ltimes U, \pi_2) \cong (V, \pi_2 \circ \varphi)$ as slices over U. Because $\Finv_U$ is full, there is a morphism $\psi : W \to W_0$ such that $\psi \ltimes U = \varphi \circ \iota : W \ltimes U \to W_0 \ltimes U$. Thus, $\iota^{-1} : (V, \varphi) \cong (W \ltimes U, \psi \ltimes U) = \Finv_U^{/W_0}(W, \psi)$ as slices over $W_0 \ltimes U$.

$$\begin{tikzcd}
V \arrow[dr, "\varphi"'] & & W \ltimes U \arrow[ll, "\iota"', "\cong"] \arrow[dl, dotted, "\psi \ltimes U"] \arrow[ddl, bend left, "\pi_2"] \\
& W_0 \ltimes U \arrow[d, "\pi_2"] & \\
& U &
\end{tikzcd} \tag{25}$$

2. Pick some $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$ that is directly dimensionally split. Then $\pi_2 \circ \varphi$ is dimensionally split, so we can proceed as in the previous point. $\square$

Example 3.5.7. In the category $\boxslash^k$ of k-ary cartesian cubes (example 3.3.4), the diagonal $\delta : \mathbb{I} \to \mathbb{I} \times \mathbb{I}$ has the property that $\pi_2 \circ \delta$ is split epi, but $(\mathbb{I}, \delta)$ is not in the image of $\Finv_{\mathbb{I}}^{/\mathbb{I}}$. Thus, $\sqcup \ltimes \mathbb{I}$ is not *indirectly* strongly connection-free, despite being connection-free.

Proposition 3.5.8. If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is quantifiable, then it is strongly quantifiable, with

$$\begin{aligned}
\exists_U^{/W_0}(V, \varphi) &= (\exists_U(V, \pi_2 \circ \varphi), \mathsf{drop}_U \circ \exists_U \varphi), \\
\mathsf{drop}_U^{/W_0}(W, \psi) &= \mathsf{drop}_U W : \exists_U^{/W_0} \Finv_U^{/W_0}(W, \psi) \to (W, \psi), \\
\mathsf{copy}_U^{/W_0}(V, \varphi) &= \mathsf{copy}_U(V, \pi_2 \circ \varphi) : (V, \varphi) \to \Finv_U^{/W_0} \exists_U^{/W_0}(V, \varphi).
\end{aligned}$$

Proof. Pick $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$. Then we have, in $\mathcal{V}/U$, a morphism $\varphi : (V, \pi_2 \circ \varphi) \to (W_0 \ltimes U, \pi_2) = \Finv_U W_0$. Transposing this, we obtain

$$\mathsf{drop}_U \circ \exists_U \varphi : \exists_U(V, \pi_2 \circ \varphi) \to W_0. \tag{26}$$

This is a slice over W_0 that we take as the definition of $\exists_U^{/W_0}(V, \varphi)$.

To prove adjointness, take a general morphism $\chi : \exists_U^{/W_0}(V, \varphi) \to (W, \psi)$ and note that there is a correspondence between diagrams of the following shape, where in the first step we apply $\exists_U \dashv \Finv_U$ and in the second step we use $(\mathcal{W}/U)/\Finv_U W_0 \cong \mathcal{W}/(W_0 \ltimes U)$:

$$\begin{tikzcd}
\exists_U(V, \pi_2 \circ \varphi) \arrow[rr, "\chi"] \arrow[dr, "\mathsf{drop}_U \circ \exists_U \varphi"'] & & W \arrow[dl, "\psi"] \\
& W_0 &
\end{tikzcd}$$

$$\begin{tikzcd}
(V, \pi_2 \circ \varphi) \arrow[rr, "\Finv_U \chi \circ \mathsf{copy}_U"] \arrow[dr, "\varphi"'] & & \Finv_U W \arrow[dl, "\Finv_U \psi"] \\
& \Finv_U W_0 &
\end{tikzcd}$$

$$\begin{tikzcd}
(V, \varphi) \arrow[r, "\Finv_U \chi \circ \mathsf{copy}_U"] & (W \ltimes U, \psi \ltimes U),
\end{tikzcd}$$

i.e. we get a morphism $(V,\varphi) \to \reflectbox{$\exists$}_U^{/W_0}(W,\psi)$. Moreover, we note that the transposition action on morphisms is precisely that of $\exists_U \dashv \reflectbox{$\exists$}_U$, and hence both adjunctions have the same unit and co-unit. $\square$

Proposition 3.5.9 (Functoriality for slices)**.** A morphism of multipliers $\sqcup \ltimes v : \sqcup \ltimes U \to \sqcup \ltimes U'$ gives rise to a natural transformation $\Sigma^{/W_0 \ltimes v} \circ \reflectbox{$\exists$}_U^{/W_0} \to \reflectbox{$\exists$}_{U'}^{/W_0}$. Hence, if both multipliers are quantifiable, we also get $\exists_{U'}^{/W_0} \circ \Sigma^{/W_0 \ltimes v} \to \exists_U^{/W_0}$.

Proof. For any $(W,\psi) \in \mathcal{W}/W_0$, we have to prove $(W \ltimes U, (W_0 \ltimes v) \circ (\psi \ltimes U)) \to (W \ltimes U', \psi \ltimes U')$. The morphism $W \ltimes v : W \ltimes U \to W \ltimes U'$ does the job. The second statement follows from lemma 2.1.1. $\square$

Theorem 3.5.10 (Strong quantification theorem)**.** If $\sqcup \ltimes U$ is

1. cancellative, affine and quantifiable, then we have a natural isomorphism $\mathsf{drop}_U^{/W_0} : \exists_U^{/W_0} \reflectbox{$\exists$}_U^{/W_0} \cong \mathrm{Id}$.
2. semi-cartesian, then we have
   (a) $\mathsf{hide}_U^{/W_0} : \Sigma_U^{/W_0} \to \exists_U^{/W_0}$ (if quantifiable),
   (b) $\mathsf{spoil}_U^{/W_0} : \reflectbox{$\exists$}_U^{/W_0} \to \Omega_U^{/W_0}$ (if $\Omega_U^{/W_0}$ exists),
   (c) in any case $\Sigma_U^{/W_0} \reflectbox{$\exists$}_U^{/W_0} \to \mathrm{Id}$.
3. 3/4-cartesian, then there is a natural transformation $\Sigma^{/W_0 \ltimes \delta} \circ \reflectbox{$\exists$}_U^{/W_0} \to \reflectbox{$\exists$}_{U \ltimes U}^{/W_0}$, where we compose multipliers as in theorem 3.6.1.
4. cartesian, then we have natural isomorphisms:
   (a) $\exists_U^{/W_0}(V,\varphi) \cong \Sigma_U^{/W_0}(V,\varphi) = (V, \pi_1 \circ \varphi)$,
   (b) $\reflectbox{$\exists$}_U^{/W_0}(W,\psi) \cong \Omega_U^{/W_0}(W,\psi)$,
   (c) $\exists_U^{/W_0} \reflectbox{$\exists$}_U^{/W_0}(W,\psi) \cong \Sigma_U^{/W_0} \Omega_U^{/W_0}(W,\psi) \cong (W \times U, \psi \circ \pi_1)$.

   Moreover, these isomorphisms become equality if $\exists_U^{/W_0}$ is constructed as above from $\exists_U = \Sigma_U$, and $\Omega_U^{/W_0}(W,\psi)$ is chosen wisely. (Both functors are defined only up to isomorphism.)

Proof.

1. This is a standard fact about fully faithful right adjoints such as $\reflectbox{$\exists$}_U^{/W_0}$.
2. By lemma 2.1.1, it is sufficient to prove $\Sigma_U^{/W_0} \reflectbox{$\exists$}_U^{/W_0} \to \mathrm{Id}$, and indeed we have $\pi_1 : \Sigma_U^{/W_0} \reflectbox{$\exists$}_U^{/W_0}(W,\psi) = (W \ltimes U, \pi_1 \circ (\psi \ltimes U)) = (W \ltimes U, \psi \circ \pi_1) \to (W,\psi)$.
3. This is a special case of proposition 3.5.9.
4. (a) The isomorphism is obtained from the next point by uniqueness of adjoints. We prove the equality if $\exists_U = \Sigma_U$. The co-unit is then given by $\mathsf{drop}_U = \pi_1 : W \times U \to W$. The construction of $\exists_U^{/W_0}$ then reveals that $\exists_U^{/W_0}(V,\varphi) = (V, \pi_1 \circ \varphi)$, which is the definition of $\Sigma_U^{/W_0}(V,\varphi)$.
   (b) This follows from the definitions.
   (c) We have

$$\exists_U^{/W_0} \reflectbox{$\exists$}_U^{/W_0}(W,\psi) = \exists_U^{/W_0}(W \times U, \psi \times U) \cong (W \times U, \pi_1 \circ (\psi \times U)) = (W \times U, \psi \circ \pi_1). \;\square$$

Theorem 3.5.11 (Strong kernel theorem)**.** If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is cancellative, affine and connection-free, then

1. (Obsolete.) $\reflectbox{$\exists$}_U^{/W_0} : \mathcal{W}/W_0 \simeq (\mathcal{V}/\!/U)/(W_0 \ltimes U, \pi_2)$ is an equivalence of categories,[9]
2. $\reflectbox{$\exists$}_U^{/W_0} : \mathcal{W}/W_0 \simeq \mathcal{V}/\!/W_0 \ltimes U$ is an equivalence of categories. $\square$

[9] We use a slight abuse of notation by using $(\mathcal{V}/\!/U)/(W_0 \ltimes U, \pi_2)$ as a subcategory of $\mathcal{V}/(W_0 \ltimes U)$.

3.6 Composing multipliers

Theorem 3.6.1. If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is a multiplier for U and $\sqcup \ltimes U' : \mathcal{V} \to \mathcal{V}'$ is a multiplier for U', then their composite $\sqcup \ltimes (U \ltimes U') := (\sqcup \ltimes U) \ltimes U'$ is a multiplier for $U \ltimes U'$.

1. The functor $\exists_{U \ltimes U'} : \mathcal{W} \to \mathcal{V}'/(U \ltimes U')$ equals $\exists_{U'}^{/U} \circ \exists_U$.
2. The functor $\exists_{U \ltimes U'}^{/W_0} : \mathcal{W} \to \mathcal{V}'/(U \ltimes U')$ equals $\exists_{U'}^{/W_0 \ltimes U} \circ \exists_U^{/W_0}$.
3. Assume both multipliers are endo. Then:
   (a) The composite $\sqcup \ltimes (U \ltimes U')$ is semicartesian if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are semicartesian,
   (b) ~~The composite $\sqcup \ltimes (U \ltimes U')$ is 3/4-cartesian if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are 3/4-cartesian,~~
   (c) The composite $\sqcup \ltimes (U \ltimes U')$ is cartesian if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are cartesian.
4. The composite $\sqcup \ltimes (U \ltimes U')$ is strongly cancellative if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are cancellative.
5. The composite $\sqcup \ltimes (U \ltimes U')$ is affine if $\sqcup \ltimes U$ is affine and $\sqcup \ltimes U'$ is strongly affine.
6. The composite $\sqcup \ltimes (U \ltimes U')$ is strongly affine if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are strongly affine.
7. The composite $\sqcup \ltimes (U \ltimes U')$ is connection-free if $\sqcup \ltimes U$ is connection-free and $\sqcup \ltimes U'$ is cancellative, affine and connection-free.
8. (a) (Obsolete). The composite $\sqcup \ltimes (U \ltimes U')$ is indirectly strongly connection-free if $\sqcup \ltimes U$ is indirectly strongly connection-free and $\sqcup \ltimes U'$ is cancellative, affine and connection-free.
   (b) The composite $\sqcup \ltimes (U \ltimes U')$ is directly strongly connection-free if $\sqcup \ltimes U$ is directly strongly connection-free and $\sqcup \ltimes U'$ is cancellative, affine and connection-free.
9. The composite $\sqcup \ltimes (U \ltimes U')$ is strongly quantifiable if $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are quantifiable, and in that case we have:
   (a) $\exists_{U \ltimes U'} = \exists_U \circ \exists_{U'}^{/U}$,
   (b) $\exists_{U \ltimes U'}^{/W_0} = \exists_U^{/W_0} \circ \exists_{U'}^{/W_0 \ltimes U}$.

Proof. Since $\top \ltimes U \cong U$, we see that $(\top \ltimes U) \ltimes U' \cong U \ltimes U'$, so the composite is indeed a multiplier for $U \ltimes U'$.

1-2. Follows from expanding the definitions.

3. (a) Copointed endofunctors compose.
   (b) ~~Comonads compose.~~ They most certainly do not!
   (c) By associativity of the cartesian product.
4. Cancellative multipliers are strongly cancellative (proposition 3.5.4), and the composite $\exists_{U \ltimes U'}^{/W_0} = \exists_{U'}^{/W_0} \exists_U^{/W_0 \ltimes U'}$ of faithful functors is faithful.

5-6. Follows from the first two properties, since the composite of full functors is full.

7. Analogous to the next point.
8. Recall that the assumptions imply that $\sqcup \ltimes U'$ is strongly cancellative, strongly affine and strongly indirectly and directly connection free.

(a) Pick a slice $(V', \varphi') \in \mathcal{V}'/(W_0 \ltimes U \ltimes U')$ such that $\pi_2^{U \ltimes U'} \circ \varphi' : V' \to U \ltimes U'$ is dimensionally split with section $\chi' : W_1 \ltimes U \ltimes U' \to V'$. Then $\pi_2^{U'} \circ \pi_2^{U \ltimes U'} \circ \varphi' = \pi_2^{U'} \circ \varphi' : V' \to U'$ is also dimensionally split with the same section.

Because $\sqcup \ltimes U'$ is indirectly strongly connection-free, we find some $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$ such that $\iota' : (V', \varphi') \cong \exists_{U'}^{/W_0 \ltimes U}(V, \varphi) \in \mathcal{V}'/(W_0 \ltimes U \ltimes U')$.

$$\begin{tikzcd}
 & W_1 \ltimes U \arrow[d, "\chi"] \arrow[dd, bend left=60, phantom] \arrow[ddd, bend left=50, "\pi_2^U"] \\
 & V \arrow[dl, "\varphi"'] \arrow[d, "\iota", "\cong"'] \\
W_0 \ltimes U \arrow[dr, "\pi_2^U"'] & W \ltimes U \arrow[l, "\psi \ltimes U"] \\
 & U
\end{tikzcd}
\qquad
\begin{tikzcd}
 & W_1 \ltimes U \ltimes U' \arrow[dl, "\chi'"'] \arrow[d, "\chi \ltimes U'"] \arrow[ddd, bend left=30, "\pi_2^{U \ltimes U'}"] \arrow[dddd, bend left=50, "\pi_2^{U'}"] \\
V' \arrow[r, "\iota'", "\cong"'] \arrow[d, "\varphi'"'] & V \ltimes U' \arrow[dl, "\varphi \ltimes U'"] \\
W_0 \ltimes U \ltimes U' \arrow[ddr, "\pi_2^{U \ltimes U'}"] \arrow[dddr, bend right, "\pi_2^{U'}"'] & \\
 & \\
 & U \ltimes U' \arrow[d, "\pi_2^{U'}"'] \\
 & U'
\end{tikzcd}$$

Because $\exists_{U'}$ is fully faithful, the morphism $\iota' \circ \chi' : \exists_{U'}(W_1 \ltimes U) \to \exists_{U'}V$ has a unique preimage $\chi : W_1 \ltimes U \to V$ under $\exists_{U'}$. The uniqueness of the inverse images implies that $\pi_2^U \circ \varphi \circ \chi = \pi_2^U : W_1 \ltimes U \to U$, as the equation holds after applying $\exists_{U'}$ (note that $\pi_2^{U \ltimes U'} = \pi_2^U \ltimes U'$).

Thus, we see that $\pi_2 \circ \varphi : V \to U$ is dimensionally split. Because $\sqcup \ltimes U$ is indirectly strongly connection-free, we find some slice $(W, \psi) \in \mathcal{W}/W_0$ so that $\iota : (V, \varphi) \cong \exists_U^{/W_0}(W, \psi) \in \mathcal{V}/(W_0 \ltimes U)$. We conclude that

$$(V', \varphi') \cong \exists_{U'}^{/W_0 \ltimes U}(V, \varphi) \cong \exists_{U'}^{/W_0 \ltimes U} \exists_U^{/W_0}(W, \psi) = \exists_{U \ltimes U'}^{/W_0}(W, \psi). \tag{27}$$

(b) Pick a slice $(V', \varphi') \in \mathcal{V}'/(W_0 \ltimes U \ltimes U')$ that is directly dimensionally split for the composite multiplier with section $\chi' : W_1 \ltimes U \ltimes U' \to V'$, composing to $\varphi' \circ \chi' = \psi_1 \ltimes U \ltimes U'$. Then φ' is also directly dimensionally split for $\sqcup \ltimes U'$.

Because $\sqcup \ltimes U'$ is directly strongly connection-free, we find some $(V, \varphi) \in \mathcal{V}/(W_0 \ltimes U)$ such that $\iota' : (V', \varphi') \cong \exists_{U'}^{/W_0 \ltimes U}(V, \varphi) \in \mathcal{V}'/(W_0 \ltimes U \ltimes U')$.

$$\begin{tikzcd}
 & W_1 \ltimes U \arrow[d, "\chi"] \arrow[ddl, bend left=80, "\psi_1 \ltimes U"] \\
 & V \arrow[dl, "\varphi"'] \arrow[d, "\iota", "\cong"'] \\
W_0 \ltimes U & W \ltimes U \arrow[l, "\psi \ltimes U"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & W_1 \ltimes U \ltimes U' \arrow[dl, "\chi'"'] \arrow[d, "\chi \ltimes U'"] \arrow[ddl, bend left=80, "\psi_1 \ltimes U \ltimes U'"] \\
V' \arrow[r, "\iota'", "\cong"'] \arrow[d, "\varphi'"'] & V \ltimes U' \arrow[dl, "\varphi \ltimes U'"] \\
W_0 \ltimes U \ltimes U' &
\end{tikzcd}$$

Because $\exists_{U'}$ is fully faithful, the morphism $\iota' \circ \chi' : \exists_{U'}(W_1 \ltimes U) \to \exists_{U'}V$ has a unique preimage $\chi : W_1 \ltimes U \to V$ under $\exists_{U'}$. The uniqueness of the inverse images implies that $\varphi \circ \chi = \psi_1 \ltimes U : W_1 \ltimes U \to U$, as the equation holds after applying $\exists_{U'}$.

Thus, we see that $\varphi : V \to U$ is directly dimensionally split. Because $\sqcup \ltimes U$ is directly strongly connection-free, we find some slice $(W, \psi) \in \mathcal{W}/W_0$ so that $\iota : (V, \varphi) \cong \exists_U^{/W_0}(W, \psi) \in \mathcal{V}/(W_0 \ltimes U)$. We conclude that

$$(V', \varphi') \cong \exists_{U'}^{/W_0 \ltimes U}(V, \varphi) \cong \exists_{U'}^{/W_0 \ltimes U} \exists_U^{/W_0}(W, \psi) = \exists_{U \ltimes U'}^{/W_0}(W, \psi). \tag{28}$$

9. Quantifiable multipliers are strongly quantifiable (proposition 3.5.8), and the composite of left adjoints is a left adjoint to the composite. ☐

4 Multipliers and presheaves

Definition 4.0.1. Every multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ gives rise to three adjoint endofunctors between $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{V}}$ via theorem 2.3.2, which we will denote

$$(\sqcup \ltimes \mathbf{y}U) \dashv (\mathbf{y}U \multimap \sqcup) \dashv (\mathbf{y}U \surd \sqcup). \tag{29}$$

Correspondingly, a morphism of multipliers $\sqcup \ltimes \upsilon$ gives rise to natural transformations $\sqcup \ltimes \mathbf{y}\upsilon$, $\mathbf{y}\upsilon \multimap \sqcup$ and $\mathbf{y}\upsilon \surd \sqcup$.

We will not actually be using the latter two of these functors, although they can be retrieved at least up to isomorphism from the functors in definitions 2.3.18 and 4.3.1 via the equation $\sqcup \ltimes U = \Sigma_U \exists_U$.

Note that the functor $\sqcup \ltimes \mathbf{y}U : \widehat{\mathcal{W}} \to \widehat{\mathcal{V}}$ is quite reminiscent of the Day-convolution.

4.1 Acting on elements

In section 3.5, we generalized $\exists_U : \mathcal{W} \to \mathcal{V}/U$ to act on slices as $\exists_U^{/W_0} : \mathcal{W}/W_0 \to \mathcal{V}/(W_0 \ltimes U)$. Here, we further generalize to a functor whose domain is the category of elements:

Definition 4.1.1. We define (using notation 2.3.3):

- $\exists_U^{/\Psi} : \mathcal{W}/\Psi \to \mathcal{V}/(\Psi \ltimes \mathbf{y}U) : (W, \psi) \mapsto (W \ltimes U, \psi \ltimes \mathbf{y}U)$,
- $\exists_U^{\in\Psi} : (W \Rightarrow \Psi) \to \{\varphi : W \ltimes U \Rightarrow \Psi \ltimes \mathbf{y}U \mid \pi_2 \circ \varphi = \pi_2\} : \psi \mapsto \psi \ltimes \mathbf{y}U$.

We say that $\sqcup \ltimes U$ is:

- **Providently cancellative** if for all Ψ, the functor $\exists_U^{/\Psi}$ is faithful,
- **Elementally cancellative** if for all Ψ, the natural transformation $\exists_U^{\in\Psi}$ is componentwise injective,
- **Providently affine** if for all Ψ, the functor $\exists_U^{/\Psi}$ is full,
- **Elementally affine** if for all Ψ, the natural transformation $\exists_U^{\in\Psi}$ is componentwise surjective,
- **Indirectly providently connection-free** (obsolete[10]) if for all Ψ, the functor $\exists_U^{/\Psi}$ is essentially surjective on elements $(V, \varphi) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ such that $\pi_2 \circ \varphi$ is dimensionally split,
- **Directly providently connection-free** if for all Ψ, the functor $\exists_U^{/\Psi}$ is essentially surjective on elements $(V, \varphi) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ such that $\varphi : V \to \Psi \ltimes \mathbf{y}U$ is directly dimensionally split:
  - We say that $\varphi : V \Rightarrow \Psi \ltimes \mathbf{y}U$ is **directly dimensionally split** with direct dimensional section $\chi : W \ltimes U \to V$ if $\varphi \circ \chi$ is of the form $\psi \ltimes \mathbf{y}U$. The section can alternatively be presented as a morphism of elements $\chi : \exists_U^{/\Psi}(W, \psi) \to (V, \varphi)$.
  - We denote the full subcategory of directly dimensionally split elements as $\mathcal{V}/\!/\Psi \ltimes \mathbf{y}U$.

[10] see definition 3.5.1

– A directly dimensionally split element $(V, \psi) \in \mathcal{V}/\Psi \ltimes \mathbf{y}U$ that is not in the image of $\Finv_U^{/\Psi}$ even up to isomorphism, will be called a **direct connection** of the multiplier.

- **Providently quantifiable** if for all Ψ, the functor $\Finv_U^{/\Psi}$ has a left adjoint $\exists_U^{/\Psi} : \mathcal{V}/(\Psi \ltimes \mathbf{y}U) \to \mathcal{W}/\Psi$. We denote the unit as $\mathsf{copy}_U^{/\Psi} : \mathrm{Id} \to \Finv_U^{/\Psi} \exists_U^{/\Psi}$ and the co-unit as $\mathsf{drop}_U^{/\Psi} : \exists_U^{/\Psi} \Finv_U^{/\Psi} \to \mathrm{Id}$.

This is indeed a generalization:

Proposition 4.1.2. The functor $\Finv_U^{/\mathbf{y}W_0} : \mathcal{W}/\mathbf{y}W_0 \to \mathcal{V}/(\mathbf{y}W_0 \ltimes \mathbf{y}U)$ is equal to $\Finv_U^{/W_0} : \mathcal{W}/W_0 \to \mathcal{V}/(W_0 \ltimes U)$ over the obvious isomorphisms between their domains and codomains. Hence, each of the provident notions implies the strong notion 3.5.1. Moreover, each of the elemental notions implies the basic notion.

Proof. Most of this is straightforward after extracting the construction of the isomorphism $\mathbf{y}W_0 \ltimes \mathbf{y}U \cong \mathbf{y}(W_0 \ltimes U)$ from the proof of theorem 2.3.2. To see the last claim, note that

$$\{\varphi : W \ltimes U \Rightarrow \mathbf{y}W_0 \ltimes \mathbf{y}U \mid \pi_2 \circ \varphi = \pi_2\} \cong ((W \ltimes U, \pi_2) \to (W_0 \ltimes U, \pi_2)) = (\Finv_U W \to \Finv_U W_0).$$

So if injectivity/surjectivity holds for all W and W_0, then we can conclude that $\Finv_U$ is cancellative/affine. ☐

Proposition 4.1.3. If $\sqcup \ltimes U$ is cancellative, then it is providently cancellative.

Proof. Analogous to proposition 3.5.4. ☐

Proposition 4.1.4. If $\sqcup \ltimes U$ is cancellative and affine, then it is elementally cancellative

Proof. We have

$$\begin{aligned}
&\{\varphi : W \ltimes U \Rightarrow \Psi \ltimes \mathbf{y}U \mid \pi_2 \circ \varphi = \pi_2\} \\
&\cong \exists W_0.(\varphi' : W \ltimes U \to W_0 \ltimes U) \times (\psi : W_0 \Rightarrow \Psi) \times (\pi_2 \circ (\psi \ltimes \mathbf{y}U) \circ \varphi' = \pi_2) \\
&\cong \exists W_0.(\varphi' : W \ltimes U \to W_0 \ltimes U) \times (\psi : W_0 \Rightarrow \Psi) \times (\pi_2 \circ \varphi' = \pi_2) \\
&\cong \exists W_0.(\varphi' : \Finv_U W \to \Finv_U W_0) \times (\psi : W_0 \Rightarrow \Psi)
\end{aligned} \tag{30}$$

and

$$(W \Rightarrow \Psi) \cong \exists W_0.(W \to W_0) \times (W_0 \Rightarrow \Psi). \tag{31}$$

Moreover, the action of $\Finv_U^{\in\Psi}$ sends (W_0, χ, ψ) in eq. (31) to $(W_0, \Finv_U \chi, \psi)$ in eq. (30). Naively, one would say that this proves injectivity, but some care is required with the equality relation for co-ends. It might be that (W_0, χ, ψ) and (W_0, χ', ψ) are sent to the same object. This would mean that there exists a zigzag ζ from W_0 to itself and jagwise morphisms $\Finv_U W \to \Finv_U \zeta$ (a priori not necessarily in the image of $\Finv_U$ which is why we need affinity) and jagwise cells $\zeta \Rightarrow \Psi$ such that the following diagrams commute:

$$\begin{tikzcd}
 & \Finv_U W_0 \arrow[dd, squiggly, no head, "\Finv_U \zeta"] & W_0 \arrow[dd, squiggly, no head, "\zeta"'] \arrow[dr, Rightarrow, "\psi"] & \\
\Finv_U W \arrow[ur, "\Finv_U \chi"] \arrow[dr, "\Finv_U \chi'"'] \arrow[r, "\cdots"] & {} & {} \arrow[r, Rightarrow, "\cdots"] & \Psi. \\
 & \Finv_U W_0 & W_0 \arrow[ur, Rightarrow, "\psi"'] &
\end{tikzcd} \tag{32}$$

Then by full faithfulness of $\Finv_U$, we see that the unique preimage of the left triangle exists and also commutes and hence $\psi \circ \chi = \psi \circ \chi'$, so that $(W_0, \chi, \psi) = (W, \mathrm{id}, \psi \circ \chi) = (W, \mathrm{id}, \psi \circ \chi') = (W_0, \chi', \psi)$. ☐

Proposition 4.1.5. If $\sqcup \ltimes U$ is cancellative and affine, then it is providently affine.

Proof. Pick (W, ψ) and (W', ψ') in $\mathcal{W}/\Psi$ and a morphism $\chi : \exists_U^{/\Psi}(W, \psi) \to \exists_U^{/\Psi}(W', \psi')$. Then we also have $\chi : \exists_U W \to \exists_U W'$ and by fullness, we find an inverse image $\chi_0 : W \to W'$ under $\exists_U$. By elemental cancellativity, we see that $\psi' \circ \chi_0 = \psi$, so that χ_0 is a morphism of slices $\chi_0 : (W, \psi) \to (W', \psi') \in \mathcal{W}/\Psi$ and $\exists_U^{/\Psi} \chi_0 = \chi$. $\square$

Proposition 4.1.6. If $\sqcup \ltimes U$ is affine, then it is elementally affine.

Proof. In the proof of proposition 4.1.4, we saw that $\exists_U^{\in\Psi}$ essentially sends (W_0, χ, ψ_0) to $(W_0, \exists_U \chi, \psi_0)$. Then if $\exists_U \chi$ is full, it is immediate that this operation is surjective. $\square$

Proposition 4.1.7.
1. (Obsolete.) If $\sqcup \ltimes U$ is indirectly strongly connection-free, then it is indirectly providently connection-free.
2. If $\sqcup \ltimes U$ is affine and connection-free, then it is directly providently connection-free.

Proof.
1. Pick a slice $(V, \varphi) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ such that $\pi_2 \circ \varphi$ is dimensionally split. By definition of $\sqcup \ltimes \mathbf{y}U$, there is some W_0 such that φ factors as $\varphi = (\psi^{W_0 \Rightarrow \Psi} \ltimes \mathbf{y}U) \circ \chi$. Clearly, $\pi_2 \circ \varphi = \pi_2 \circ \chi$ is dimensionally split. Hence, by indirect strong connection-freedom, $(V, \chi) \cong \exists_U^{/W_0}(W, \chi') \in \mathcal{V}/(W_0 \ltimes U)$ for some $(W, \chi') \in \mathcal{W}/W_0$. Then we also have $(V, \varphi) = (V, (\psi \ltimes \mathbf{y}U) \circ \chi) \cong \exists_U^{/\Psi}(W, \psi \circ \chi')$.
2. Pick some $(V, \varphi) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ that is directly dimensionally split. Then $\pi_2 \circ \varphi$ is dimensionally split. Because $\exists_U$ is essentially surjective on $\mathcal{V}//U$, there must be some $W \in \mathcal{W}$ such that $\iota : \exists_U W = (W \ltimes U, \pi_2) \cong (V, \pi_2 \circ \varphi)$ as slices over U. By elemental affineness, there is a cell $\psi : W \Rightarrow \Psi$ such that $\psi \ltimes \mathbf{y}U = \varphi \circ \iota : W \ltimes U \Rightarrow \Psi \ltimes \mathbf{y}U$. Thus, $\iota^{-1} : (V, \varphi) \cong (W \ltimes U, \psi \ltimes \mathbf{y}U) = \exists_U^{/\Psi}(W, \psi)$ as slices over $\Psi \ltimes \mathbf{y}U$.

$$\begin{tikzcd}
V \arrow[dr, Rightarrow, "\varphi"'] & & W \ltimes U \arrow[ll, "\iota"', "\cong"] \arrow[dl, Rightarrow, dotted, "\psi \ltimes U"] \arrow[ddl, Rightarrow, bend left, "\pi_2"] \\
& \Psi \ltimes \mathbf{y}U \arrow[d, "\pi_2"] & \\
& \mathbf{y}U &
\end{tikzcd} \tag{33}$$

$\square$

Proposition 4.1.8. If $\sqcup \ltimes U$ is quantifiable, then it is providently quantifiable, with

$$\exists_U^{/\Psi}(V, (\psi \ltimes \mathbf{y}U) \circ \varphi_0) = \Sigma^{/\psi} \exists_U^{/W_0}(V, \varphi_0),$$
$$\mathsf{drop}_U^{/\Psi}(W, \psi) = \mathsf{drop}_U W,$$
$$\mathsf{copy}_U^{/\Psi}(V, \varphi) = \mathsf{copy}_U(V, \pi_2 \circ \varphi).$$

Proof. Pick $(V, \varphi) \in \mathcal{V}/(\Psi \ltimes \mathbf{y}U)$. Then φ factors as $(\psi^{W_0 \Rightarrow \Psi} \ltimes \mathbf{y}U) \circ \varphi_0^{V \to W_0 \ltimes U}$. Then $(V, \varphi_0) \in \mathcal{V}/(W_0 \ltimes U)$ and hence $\exists_U^{/W_0}(V, \varphi_0) \in \mathcal{W}/W_0$. We define

$$\begin{aligned}
\exists_U^{/\Psi}(V, \varphi) &:= \Sigma^{/\psi} \exists_U^{/W_0}(V, \varphi_0) \\
&= \Sigma^{/\psi}(\exists_U(V, \pi_2 \circ \varphi_0), \mathsf{drop}_U \circ \exists_U \varphi_0) \\
&= (\exists_U(V, \pi_2^{W_0 \ltimes U \to U} \circ \varphi_0), \psi \circ \mathsf{drop}_U \circ \exists_U \varphi_0) \\
&= (\exists_U(V, \pi_2^{\Psi \ltimes \mathbf{y}U \to \mathbf{y}U} \circ \varphi), \psi \circ \mathsf{drop}_U \circ \exists_U \varphi_0).
\end{aligned}$$

We need to prove that this is well-defined, i.e. respects equality on the co-end that defines $V \Rightarrow \Psi \ltimes \mathbf{y}U$. To this end, assume that $\varphi = (\psi_0^{W_0 \Rightarrow \Psi} \ltimes \mathbf{y}U) \circ \varphi_0^{V \to W_0 \ltimes U} = (\psi_1^{W_1 \Rightarrow \Psi} \ltimes \mathbf{y}U) \circ \varphi_1^{V \to W_1 \ltimes U}$. This means there are a zigzag ζ from W_0 to W_1, jagwise morphisms $V \to \zeta \ltimes U$ and jagwise cells $\zeta \Rightarrow \Psi$ such that the following triangles commute:

$$\begin{tikzcd} & W_0 \ltimes U \arrow[dd, squiggly, "\zeta \ltimes U"] \\ V \arrow[ur, "\varphi_0"] \arrow[r, "\cdots"] \arrow[dr, "\varphi_1"'] & {} \\ & W_1 \ltimes U \end{tikzcd} \qquad \begin{tikzcd} W_0 \arrow[dd, squiggly, "\zeta"'] \arrow[dr, Rightarrow, "\psi_0"] & \\ {} \arrow[r, Rightarrow, "\cdots"] & \Psi. \\ W_1 \arrow[ur, Rightarrow, "\psi_1"'] & \end{tikzcd} \tag{34}$$

By naturality of π_2, we find that $(V, \pi_2 \circ \varphi_0) = (V, \pi_2 \circ \varphi_1) \in \mathcal{V}/U$. By naturality of drop_U, we find that $\psi_0 \circ \mathsf{drop}_U \circ \exists_U \varphi_0 = \psi_1 \circ \mathsf{drop}_U \circ \exists_U \varphi_1 : (V, \pi_2 \circ \varphi_0) = (V, \pi_2 \circ \varphi_1) \Rightarrow \Psi$. We conclude that $\exists_U^{/\Psi}(V, \varphi)$ is well-defined.

To prove adjointness, we first show how $\reflectbox{$\exists$}_U^{/\Psi}$ on the right can be turned into $\exists_U^{/\Psi}$ on the left. Pick a morphism $\chi : (V, \varphi) \to \reflectbox{$\exists$}_U^{/\Psi}(W, \psi) = (W \ltimes U, \psi \ltimes \mathbf{y}U)$ in $\mathcal{V}/(\Psi \ltimes \mathbf{y}U)$. Then $\varphi = (\psi \ltimes \mathbf{y}U) \circ \chi$ so by definition, $\exists_U^{/\Psi}(V, \varphi) = (\exists_U(V, \pi_2 \circ \varphi), \psi \circ \mathsf{drop}_U \circ \exists_U \chi)$ which clearly factors over ψ, i.e. has a morphism $\mathsf{drop}_U \circ \exists_U \chi : \exists_U^{/\Psi}(V, \varphi) \to (W, \psi)$. If $\chi = \mathrm{id}$, then we obtain the co-unit $\mathsf{drop}_U^{/\Psi} = \mathsf{drop}_U \circ \exists_U \mathrm{id} = \mathsf{drop}_U$.

Next, we construct the unit $\mathsf{copy}_U^{/\Psi} : (V, \varphi) \to \reflectbox{$\exists$}_U^{/\Psi} \exists_U^{/\Psi}(V, \varphi)$. If $\varphi = (\psi \ltimes \mathbf{y}U) \circ \varphi_0$, then we have

$$\begin{aligned} \reflectbox{$\exists$}_U^{/\Psi} \exists_U^{/\Psi}(V, \varphi) &= \reflectbox{$\exists$}_U^{/\Psi} \Sigma^{/\psi} \exists_U^{/W_0}(V, \varphi_0) \\ &= \Sigma^{/\psi \ltimes \mathbf{y}U} \reflectbox{$\exists$}_U^{/W_0} \exists_U^{/W_0}(V, \varphi_0). \end{aligned}$$

On the other hand, $(V, \varphi) = \Sigma^{/\psi \ltimes \mathbf{y}U}(V, \varphi_0)$, so as the unit we can take $\mathsf{copy}_U^{/\Psi} = \Sigma^{/\psi \ltimes \mathbf{y}U} \mathsf{copy}_U^{/W_0} = \mathsf{copy}_U^{/W_0} = \mathsf{copy}_U$.

The adjunction laws are then inherited from $\exists_U \dashv \reflectbox{$\exists$}_U$. ☐

Proposition 4.1.9 (Functoriality for elements)**.** A morphism of multipliers $\sqcup \ltimes \upsilon : \sqcup \ltimes U \to \sqcup \ltimes U'$ gives rise to a natural transformation $\Sigma^{/\Psi \ltimes \mathbf{y}\upsilon} \circ \reflectbox{$\exists$}_U^{/\Psi} \to \reflectbox{$\exists$}_{U'}^{/\Psi}$. Hence, if both multipliers are quantifiable, we also get $\exists_{U'}^{/\Psi} \circ \Sigma^{/\Psi \ltimes \mathbf{y}\upsilon} \to \exists_U^{/\Psi}$.

Proof. For any $(W, \psi) \in \mathcal{W}/\Psi$, we have to prove $(W \ltimes U, (\Psi \ltimes \mathbf{y}\upsilon) \circ (\psi \ltimes \mathbf{y}U)) \to (W \ltimes U', \psi \ltimes \mathbf{y}U')$. The morphism $W \ltimes \upsilon : W \ltimes U \to W \ltimes U'$ does the job. The second statement follows from lemma 2.1.1. ☐

Theorem 4.1.10 (Provident quantification theorem)**.** If $\sqcup \ltimes U$ is

1. cancellative, affine and quantifiable, then we have a natural isomorphism $\mathsf{drop}_U^{/\Psi} : \exists_U^{/\Psi} \reflectbox{$\exists$}_U^{/\Psi} \cong \mathrm{Id}$.
2. semi-cartesian, then we have
   (a) $\mathsf{hide}_U^{/\Psi} : \Sigma_U^{/\Psi} \to \exists_U^{/\Psi}$ (if quantifiable),
   (b) $\mathsf{spoil}_U^{/\Psi} : \reflectbox{$\exists$}_U^{/\Psi} \to \Omega_U^{/\Psi}$ (if $\Omega_U^{/\Psi}$ exists),
   (c) in any case $\Sigma_U^{/\Psi} \reflectbox{$\exists$}_U^{/\Psi} \to \mathrm{Id}$.
3. 3/4-cartesian, then there is a natural transformation $\Sigma^{/\Psi \ltimes \mathbf{y}\delta} \circ \reflectbox{$\exists$}_U^{/\Psi} \to \reflectbox{$\exists$}_{U \ltimes U}^{/\Psi}$.
4. cartesian, then we have natural isomorphisms:
   (a) $\exists_U^{/\Psi}(V, \varphi) \cong \Sigma_U^{/\Psi}(V, \varphi) = (V, \pi_1 \circ \varphi)$,
   (b) $\reflectbox{$\exists$}_U^{/\Psi}(W, \psi) \cong \Omega_U^{/\Psi}(W, \psi)$,
   (c) $\exists_U^{/\Psi} \reflectbox{$\exists$}_U^{/\Psi}(W, \psi) \cong \Sigma_U^{/\Psi} \Omega_U^{/\Psi}(W, \psi) \cong (W \times \mathbf{y}U, \psi \circ \pi_1)$.

Moreover, these isomorphisms become equality if $\exists_U^{/\Psi}$ is constructed as above from $\exists_U^{/W_0} = \Sigma_U^{/W_0}$, and $\Omega_U^{/\Psi}(W, \psi)$ is chosen wisely. (Both functors are defined only up to isomorphism.)

Proof. 1. This is a standard fact about fully faithful right adjoints such as $\Finv_U^{/\Psi}$.

2. By lemma 2.1.1, it is sufficient to prove $\Sigma_U^{/\Psi} \Finv_U^{/\Psi} \to \mathrm{Id}$, and indeed we have $\pi_1 : \Sigma_U^{/\Psi} \Finv_U^{/\Psi}(W, \psi) = (W \ltimes U, \pi_1 \circ (\psi \ltimes \mathbf{y}U)) = (W \ltimes U, \psi \circ \pi_1) \to (W, \psi)$.

3. This is a special case of proposition 4.1.9.

4. (a) Let $\varphi = (\psi \ltimes \mathbf{y}U) \circ \varphi_0$. Then we have

$$\begin{aligned}
\exists_U^{/\Psi}(V, \varphi) &= \Sigma^{/\psi} \exists_U^{/W_0}(V, \varphi_0) \\
&\cong \Sigma^{/\psi} \Sigma_U^{/W_0}(V, \varphi_0) \\
&= \Sigma^{/\psi}(V, \pi_1 \circ \varphi_0) \\
&= (V, \psi \circ \pi_1 \circ \varphi_0) = (V, \pi_1 \circ (\psi \ltimes \mathbf{y}U) \circ \varphi_0) = (V, \pi_1 \circ \varphi).
\end{aligned}$$

   (b) This follows from the definitions.

   (c) We have

$$\exists_U^{/\Psi} \Finv_U^{/\Psi}(W, \psi) = \exists_U^{/\Psi}(W \times U, \psi \times \mathbf{y}U) \cong (W \times U, \pi_1 \circ (\psi \times \mathbf{y}U)) \tag{35}$$

   and of course $\pi_1 \circ (\psi \times \mathbf{y}U) = \psi \circ \pi_1 : W \times U \to \Psi$. □

Theorem 4.1.11 (Provident kernel theorem)**.** If $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ is cancellative, affine and connection-free, then

1. (Obsolete.) $\Finv_U^{/\Psi} : \mathcal{W}/\Psi \simeq (\mathcal{V}/\!/U)/(\Psi \ltimes \mathbf{y}U, \pi_2)$ is an equivalence of categories.[11]
2. $\Finv_U^{/\Psi} : \mathcal{W}/\Psi \simeq \mathcal{V}/\!/\Psi \ltimes \mathbf{y}U$ is an equivalence of categories. □

4.2 Acting on presheaves

Proposition 4.2.1. The functor $\sqcup \ltimes \mathbf{y}U : \widehat{\mathcal{W}} \to \widehat{\mathcal{V}}$:

1. is a multiplier for $\mathbf{y}U$,
2. has the property that $\Finv_{\mathbf{y}U} : \widehat{\mathcal{W}} \to \widehat{\mathcal{V}}/\mathbf{y}U$ is naturally isomorphic to $(\Finv_U)_! : \widehat{\mathcal{W}} \to \widehat{\mathcal{V}/U}$ over the equivalence between their codomains,
3. has the property that the slice functor $\Finv_{\mathbf{y}U}^{/\Psi} : \widehat{\mathcal{W}}/\Psi \to \widehat{\mathcal{V}}/(\Psi \ltimes \mathbf{y}U)$ is naturally isomorphic to the left lifting of the elements functor $(\Finv_U^{/\Psi})_! : \widehat{\mathcal{W}/\Psi} \to \widehat{\mathcal{V}/(\Psi \ltimes \mathbf{y}U)}$ over the equivalences between their domains and codomains,
4. is semicartesian if and only if $\sqcup \ltimes U$ is,
5. is 3/4-cartesian if and only if $\sqcup \ltimes U$ is,
6. is cartesian if and only if $\sqcup \ltimes U$ is,
7. is cancellative and affine if and only if $\sqcup \ltimes U$ is cancellative and affine,
8. is strongly cancellative and strongly affine if and only if $\sqcup \ltimes U$ is providently cancellative and providently affine,

[11] We use a slight abuse of notation as $(\mathcal{V}/\!/U)/(\Psi \ltimes \mathbf{y}U, \pi_2)$ is in fact neither a category of slices nor of elements.

9. is quantifiable if $\sqcup \ltimes U$ is quantifiable, and $\exists_{\mathbf{y}U}$ is naturally isomorphic to $(\exists_U)_!$ over the equivalence $\widehat{\mathcal{V}/U} \simeq \widehat{\mathcal{V}}/\mathbf{y}U$,

10. is strongly quantifiable if $\sqcup \ltimes U$ is providently quantifiable, and $\exists_{\mathbf{y}U}^{/\Psi}$ is naturally isomorphic to $(\exists_U^{/\Psi})_!$ over the equivalences between their domain and codomain.

Proof. 1. Since $\top \ltimes \mathbf{y}U \cong \mathbf{y}\top \ltimes \mathbf{y}U \cong \mathbf{y}(\top \ltimes U) \cong \mathbf{y}U$. We use, in order, that $\mathbf{y}$ preserves the terminal object, that $F_! \circ \mathbf{y} \cong \mathbf{y} \circ F$ (theorem 2.3.2) and that $\sqcup \ltimes U$ is a multiplier for U.

2. The functor $(\exists_U)_!$ sends a presheaf $\Gamma \in \widehat{\mathcal{W}}$ to the presheaf in $\widehat{\mathcal{V}/U}$ determined by

$$(V, \varphi) \Rightarrow (\exists_U)_! \Gamma \quad = \quad \exists W.((V, \varphi) \to \exists_U W) \times (W \Rightarrow \Gamma). \tag{36}$$

On the other hand, $\exists_{\mathbf{y}U}\Gamma$ is the slice $(\Gamma \ltimes \mathbf{y}U, \pi_2) \in \widehat{\mathcal{V}}/\mathbf{y}U$. Taking the preimage of π_2 (proposition 2.3.7), we get a presheaf $\Delta \in \widehat{\mathcal{V}/U}$ determined by

$$\begin{aligned}
(V, \varphi) \Rightarrow \Delta \quad &= \quad \{(\gamma \ltimes \mathbf{y}U) \circ \chi : V \Rightarrow \Gamma \ltimes \mathbf{y}U \mid \pi_2 \circ (\gamma \ltimes \mathbf{y}U) \circ \chi = \varphi\} \\
&= \quad \{(\gamma \ltimes \mathbf{y}U) \circ \chi : V \Rightarrow \Gamma \ltimes \mathbf{y}U \mid \pi_2 \circ \chi = \varphi\} \\
&\cong \quad \exists W.(\chi : V \to W \ltimes U) \times (\gamma : W \Rightarrow \Gamma) \times (\pi_2 \circ \chi = \varphi) \\
&\cong \quad \exists W.(\chi : (V, \varphi) \to \exists_U W) \times (W \Rightarrow \Gamma).
\end{aligned}$$

Indeed, we see that these functors are isomorphic.

3. The functor $(\exists_U^{/\Psi})_!$ sends a presheaf $\Psi \mid \Gamma \vdash \mathsf{Ctx}$ over $\mathcal{W}/\Psi$ to the presheaf $\Psi \ltimes \mathbf{y}U \mid (\exists_U \Psi)_! \Gamma \vdash \mathsf{Ctx}$ over $\mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ determined by:

$$(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \Rightarrow \left(\exists_U^{/\Psi}\right)_! \Gamma \quad = \quad \exists (W, \psi^{W \Rightarrow \Psi}).((V, \varphi) \to \exists_U^{/\Psi}(W, \psi)) \times ((W, \psi) \Rightarrow \Gamma). \tag{37}$$

On the other hand, $\exists_{\mathbf{y}U}^{/\Psi}(\Psi.\Gamma, \pi)$ is the slice $(\Psi.\Gamma \ltimes \mathbf{y}U, \pi \ltimes \mathbf{y}U) \in \widehat{\mathcal{V}}/(\Psi \ltimes \mathbf{y}U)$. Taking the preimage of $\pi \ltimes \mathbf{y}U$ (proposition 2.3.7), we get a presheaf $\Psi \ltimes \mathbf{y}U \mid \Delta \vdash \mathsf{Ctx}$ over $\mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ determined by

$$\begin{aligned}
&(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \Rightarrow \Delta \\
&= \{(\psi.\gamma \ltimes \mathbf{y}U) \circ \chi : V \Rightarrow \Psi.\Gamma \ltimes \mathbf{y}U \mid (\pi \ltimes \mathbf{y}U) \circ (\psi.\gamma \ltimes \mathbf{y}U) \circ \chi = \varphi\} \\
&= \{(\psi.\gamma \ltimes \mathbf{y}U) \circ \chi : V \Rightarrow \Psi.\Gamma \ltimes \mathbf{y}U \mid (\psi \ltimes \mathbf{y}U) \circ \chi = \varphi\} \\
&\cong \exists W.(\chi : V \to W \ltimes U) \times (\psi : W \Rightarrow \Psi) \times (\gamma : (W, \psi) \Rightarrow \Gamma) \times ((\psi \ltimes \mathbf{y}U) \circ \chi = \varphi) \\
&\cong \exists (W, \psi^{W \Rightarrow \Psi}).(\chi : (V, \varphi) \to \exists_U^{/\Psi}(W, \psi)) \times (\gamma : (W, \psi) \Rightarrow \Gamma).
\end{aligned}$$

Indeed, we see that these functors are isomorphic.

4. Assume that $\sqcup \ltimes U$ is semicartesian. It is immediate from the construction of $\sqcup_!$ that $\sqcup_!$ preserves natural transformations. Moreover, we have $\mathrm{Id}_! \cong \mathrm{Id}$, so we get $\pi_1 : (\sqcup \ltimes \mathbf{y}U) \to \mathrm{Id}$.

   Conversely, assume that $\sqcup \ltimes \mathbf{y}U$ is semicartesian. Then we have $\mathbf{y}(\sqcup \ltimes U) \cong (\mathbf{y}\sqcup \ltimes \mathbf{y}U) \to \mathbf{y}$. Since $\mathbf{y}$ is fully faithful, we have proven $(\sqcup \ltimes U) \to \mathrm{Id}$.

5. Analogous to the previous point.

6. Assume that $\sqcup \ltimes U$ is cartesian. We apply the universal property of the cartesian product, and the co-Yoneda lemma:

$$\begin{aligned}
V \Rightarrow (\Gamma \ltimes \mathbf{y}U) &= \exists W.(V \to W \ltimes U) \times (W \Rightarrow \Gamma) \\
&\cong \exists W.(V \to W) \times (V \to U) \times (W \Rightarrow \Gamma) \\
&\cong (V \to U) \times (V \Rightarrow \Gamma)
\end{aligned}$$

$$\cong V \Rightarrow \Gamma \times \mathbf{y}U.$$

Conversely, if $\textvisiblespace \ltimes \mathbf{y}U$ is cartesian, we have

$$\begin{aligned} V \to W \ltimes U &= V \Rightarrow \mathbf{y}(W \ltimes U) \\ &\cong V \Rightarrow \mathbf{y}W \ltimes \mathbf{y}U \\ &\cong (V \Rightarrow \mathbf{y}W) \times (V \Rightarrow \mathbf{y}U) \\ &\cong (V \to W) \times (V \to U) \cong V \to W \times U. \end{aligned}$$

7. This follows from point 2 and proposition 2.3.4.
8. This follows from point 3 and proposition 2.3.4.
9. We know that $(\exists_U)_! \dashv (\Finv_U)_!$ so moving it through the natural transformation yields a left adjoint to $\Finv_{\mathbf{y}U}$.
10. By proposition 4.1.8, $\exists_U^{/\Psi}$ exists. We know that $(\exists_U^{/\Psi})_! \dashv (\Finv_U^{/\Psi})_!$ so moving it through the natural transformation yields a left adjoint to $\Finv_{\mathbf{y}U}^{/\Psi}$. □

4.3 Four adjoint functors

Unlike the category of slices $\widehat{\mathcal{W}}/\Psi$, the equivalent category $\widehat{\mathcal{W}/\Psi}$ is a presheaf category and therefore immediately a model of dependent type theory. Therefore, we prefer to work with that category, and to use the corresponding functors:

Definition 4.3.1. The adjoint functors $\exists_U^{/\Psi} \dashv \Finv_U^{/\Psi}$ give rise to four adjoint functors between presheaf categories over slice categories, which we denote

$$\exists_{\mathbf{y}U}^{\Psi|} \dashv \Finv_{\mathbf{y}U}^{\Psi|} \dashv \forall_{\mathbf{y}U}^{\Psi|} \dashv \between_{\mathbf{y}U}^{\Psi|}. \tag{38}$$

We call the fourth functor **transpension**.

The units and co-units will be denoted:

$$\begin{array}{rcl@{\qquad\qquad}rcl} \mathsf{copy}_{\mathbf{y}U}^{\Psi|} & : & \mathrm{Id} \to \Finv_{\mathbf{y}U}^{\Psi|} \exists_{\mathbf{y}U}^{\Psi|} & \mathsf{drop}_{\mathbf{y}U}^{\Psi|} & : & \exists_{\mathbf{y}U}^{\Psi|} \Finv_{\mathbf{y}U}^{\Psi|} \to \mathrm{Id} \\ \mathsf{const}_{\mathbf{y}U}^{\Psi|} & : & \mathrm{Id} \to \forall_{\mathbf{y}U}^{\Psi|} \Finv_{\mathbf{y}U}^{\Psi|} & \mathsf{app}_{\mathbf{y}U}^{\Psi|} & : & \Finv_{\mathbf{y}U}^{\Psi|} \forall_{\mathbf{y}U}^{\Psi|} \to \mathrm{Id} \\ \mathsf{reidx}_{\mathbf{y}U}^{\Psi|} & : & \mathrm{Id} \to \between_{\mathbf{y}U}^{\Psi|} \forall_{\mathbf{y}U}^{\Psi|} & \mathsf{unmerid}_{\mathbf{y}U}^{\Psi|} & : & \forall_{\mathbf{y}U}^{\Psi|} \between_{\mathbf{y}U}^{\Psi|} \to \mathrm{Id} \end{array} \tag{39}$$

For now, we define all of these functors only up to isomorphism, i.e. for the middle two we do not specify whether they arise as a left, central or right lifting.

Note that, if in a judgement $\Psi \mid \Gamma \vdash J$, we view the part before the pipe ($|$) as part of the context, then $\exists_{\mathbf{y}U}^{\Gamma|}$ and $\forall_{\mathbf{y}U}^{\Gamma|}$ bind a (substructural) variable of type $\mathbf{y}U$, whereas $\Finv_{\mathbf{y}U}^{\Gamma|}$ and $\between_{\mathbf{y}U}^{\Gamma|}$ depend on one.

It is worth mentioning that, since $\textvisiblespace \ltimes U = \Sigma_U \Finv_U$, the functors in definition 4.0.1 can be (essentially) retrieved as

$$\textvisiblespace \ltimes \mathbf{y}U = \Sigma_{\mathbf{y}U}^{\top|} \Finv_{\mathbf{y}U}^{\top|} \quad\dashv\quad \mathbf{y}U \multimap \textvisiblespace = \forall_{\mathbf{y}U}^{\top|} \Omega_{\mathbf{y}U}^{\top|} \quad\dashv\quad \mathbf{y}U \surd \textvisiblespace = \Pi_{\mathbf{y}U}^{\top|} \between_{\mathbf{y}U}^{\top|}. \tag{40}$$

Corollary 4.3.2. The properties asserted by proposition 4.2.1 for $\Finv_{\mathbf{y}U}^{/\Psi}$ also hold for $\Finv_{\mathbf{y}U}^{\Psi|}$.

Proof. Follows from the fact that $\Finv_{\mathbf{y}U}^{\Psi|} \cong (\Finv_U^{/\Psi})_!$, and the observation in proposition 4.2.1 that this functor in turn corresponds to $\Finv_{\mathbf{y}U}^{/\Psi}$. □

Proposition 4.3.3 (Presheaf functoriality). A morphism of multipliers $\textvisiblespace \ltimes v : \textvisiblespace \ltimes U \to \textvisiblespace \ltimes U'$ gives rise to natural transformations

- $\exists^{\Psi|}_{\mathbf{y}U'} \circ \Sigma^{\Psi \ltimes \mathbf{y}v|} \to \exists^{\Psi|}_{\mathbf{y}U}$ (if quantifiable),
- $\Sigma^{\Psi \ltimes \mathbf{y}v|} \circ \Finv^{\Psi|}_{\mathbf{y}U} \to \Finv^{\Psi|}_{\mathbf{y}U'}$ and $\Finv^{\Psi|}_{\mathbf{y}U} \to \Omega^{\Psi \ltimes \mathbf{y}v|} \circ \Finv^{\Psi|}_{\mathbf{y}U'}$,
- $\forall^{\Psi|}_{\mathbf{y}U'} \to \forall^{\Psi|}_{\mathbf{y}U} \circ \Omega^{\Psi \ltimes \mathbf{y}v|}$ and $\forall^{\Psi|}_{\mathbf{y}U'} \circ \Pi^{\Psi \ltimes \mathbf{y}v|} \to \forall^{\Psi|}_{\mathbf{y}U}$,
- $\Pi^{\Psi \ltimes \mathbf{y}v|} \circ \between^{\Psi|}_{\mathbf{y}U} \to \between^{\Psi|}_{\mathbf{y}U'}$,

Proof. Follows directly from proposition 4.1.9. $\square$

Proposition 4.3.4 (Presheaf quantification theorem)**.** If $\sqcup \ltimes U$ is

1. cancellative and affine, then $\mathsf{drop}^{\Psi|}_{\mathbf{y}U}$, $\mathsf{const}^{\Psi|}_{\mathbf{y}U}$ and $\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U}$ are natural isomorphisms.
2. semi-cartesian, then we have
   (a) $\mathsf{hide}^{\Psi|}_{\mathbf{y}U} : \Sigma^{\Psi|}_{\mathbf{y}U} \to \exists^{\Psi|}_{\mathbf{y}U}$ (if quantifiable),
   (b) $\mathsf{spoil}^{\Psi|}_{\mathbf{y}U} : \Finv^{\Psi|}_{\mathbf{y}U} \to \Omega^{\Psi|}_{\mathbf{y}U}$,
   (c) $\mathsf{cospoil}^{\Psi|}_{\mathbf{y}U} : \Pi^{\Psi|}_{\mathbf{y}U} \to \forall^{\Psi|}_{\mathbf{y}U}$.
3. 3/4-cartesian, then we can apply proposition 4.3.3 to $\sqcup \ltimes \delta : \sqcup \ltimes U \to \sqcup \ltimes (U \ltimes U)$.
4. cartesian, then we have natural isomorphisms:
   (a) $\exists^{\Psi|}_{\mathbf{y}U} \cong \Sigma^{\Psi|}_{\mathbf{y}U}$,
   (b) $\Finv^{\Psi|}_{\mathbf{y}U} \cong \Omega^{\Psi|}_{\mathbf{y}U}$,
   (c) $\forall^{\Psi|}_{\mathbf{y}U} \cong \Pi^{\Psi|}_{\mathbf{y}U}$,
   (d) $\between^{\Psi|}_{\mathbf{y}U} \cong \$^{\Psi|}_{\mathbf{y}U}$ (if $\Omega^{/\Psi}_{U}$ exists).

   Equality is achieved for any pair of functors if they are lifted in the same way from functors that were equal in theorem 4.1.10.

Proof.
1. The fact that $\mathsf{drop}^{\Psi|}_{\mathbf{y}U}$ is an isomorphism, is a standard fact about fully faithful right adjoints such as $\Finv^{\Psi|}_{\mathbf{y}U}$. This property then carries over to further adjoints.
2. By lemma 2.1.1, it is sufficient to prove $\Sigma^{\Psi|}_{\mathbf{y}U} \Finv^{\Psi|}_{\mathbf{y}U} \to \mathrm{Id}$, which follows immediately from $\pi_1 : \Sigma^{/\Psi}_{U} \Finv^{/\Psi}_{U} \to \mathrm{Id}$.
3. Of course we can.
4. This is an immediate corollary of theorem 3.5.10. $\square$

Proposition 4.3.5 (Fresh exchange)**.** If $\Psi \mid \Gamma \vdash \mathsf{Ctx}$, i.e. $\Gamma \in \widehat{\mathcal{W}/\Psi}$, then we have an isomorphism of slices (natural in Γ):

$$\begin{tikzcd} (\Psi \ltimes \mathbf{y}U).\Finv^{\Psi|}_{\mathbf{y}U}\Gamma \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & \Psi.\Gamma \ltimes \mathbf{y}U \arrow[dl, "\pi \ltimes \mathbf{y}U"] \\ & \Psi \ltimes \mathbf{y}U. & \end{tikzcd} \tag{41}$$

This proposition explains the meaning of $\Finv^{/\Gamma}_{\mathbf{y}U}$: it is the type depending on a variable of type $\mathbf{y}U$ whose elements are required to be fresh for that variable, where the meaning of 'fresh' depends on the nature of the multiplier. If the multiplier is cartesian, then $\Finv^{/\Gamma}_{\mathbf{y}U}$ is clearly just weakening over $\mathbf{y}U$.

Proof. The slice on the right is $\Finv^{/\Psi}_{\mathbf{y}U}(\Psi.\Gamma, \pi)$. By proposition 4.2.1, this is isomorphic to $\Finv^{\Psi|}_{\mathbf{y}U}\Gamma$ over the equivalence from proposition 2.3.7 which sends Δ to $((\Psi \ltimes \mathbf{y}U).\Delta, \pi)$. $\square$

4.4 Investigating the transpension functor

Definition 4.4.1. 1. We define the **indirect boundary** $\Psi \ltimes \partial U$ as the pullback

$$\begin{tikzcd} \Psi \ltimes \partial U \arrow[r, "\subseteq"] \arrow[d, "\pi_2"] & \Psi \ltimes \mathbf{y}U \arrow[d, "\pi_2"] \\ \partial U \arrow[r, "\subseteq"] & \mathbf{y}U, \end{tikzcd} \tag{42}$$

i.e. the subpresheaf of $\Psi \ltimes \mathbf{y}U$ consisting of all cells φ such that $\pi_2 \circ \varphi$ is *not* dimensionally split.

2. We define the **direct boundary**, also denoted $\Psi \ltimes \partial U$, as the subpresheaf of $\Psi \ltimes \mathbf{y}U$ consisting of all cells φ that are *not* directly dimensionally split.

If φ is directly dimensionally split, then $\pi_2 \circ \varphi$ is dimensionally split, so the indirect boundary is a subpresheaf of the direct boundary.

Definition 4.4.2. For either notion of boundary, write $(\in \partial U)$ for the inverse image of $\Psi \ltimes \partial U \subseteq \Psi \ltimes \mathbf{y}U$, which is a presheaf over $\mathcal{W}/(\Psi \ltimes \mathbf{y}U)$ such that $(\Psi \ltimes \mathbf{y}U).(\in \partial U) \cong \Psi \ltimes \partial U$. We also write $(\in \partial U)$ for the inverse image of $\partial U \subseteq \mathbf{y}U$. Finally, we write $\Sigma^{/\Psi \ltimes \mathbf{y}U}_{(\in \partial U)} \dashv \ldots$ for the functors arising from $\Psi \ltimes \partial U \subseteq \Psi \ltimes \mathbf{y}U$.

Theorem 4.4.3 (Poles of the transpension)**.** For either notion of boundary and any multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$, the functor $\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \circ \between^{\Psi|}_{\mathbf{y}U} : \widehat{\mathcal{W}/\Psi} \to \widehat{\mathcal{V}/(\Psi \ltimes \partial U)}$ sends any presheaf to the terminal presheaf, i.e. $\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \circ \between^{\Psi|}_{\mathbf{y}U} = \top$.

Proof. We show that there is always a unique cell $(V, \varphi^{V \Rightarrow \Psi \ltimes \partial U}) \Rightarrow \Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \between^{\Psi|}_{\mathbf{y}U} \Gamma$. We have

$$\begin{aligned}
&(V, \varphi^{V \Rightarrow \Psi \ltimes \partial U}) \Rightarrow \Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \between^{\Psi|}_{\mathbf{y}U} \Gamma \\
&= \Sigma^{/\Psi \ltimes \mathbf{y}U}_{(\in \partial U)} (V, \varphi^{V \Rightarrow \Psi \ltimes \partial U}) \Rightarrow \between^{\Psi|}_{\mathbf{y}U} \Gamma \\
&= (V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \Rightarrow \between^{\Psi|}_{\mathbf{y}U} \Gamma \\
&= \forall^{\Psi|}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \to \Gamma \\
&= \forall (W, \psi^{W \Rightarrow \Psi}). \left((W, \psi) \Rightarrow \forall^{\Psi|}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U})\right) \to ((W, \psi) \Rightarrow \Gamma) \\
&= \forall (W, \psi^{W \Rightarrow \Psi}). \left(\exists^{/\Psi}_{U} (W, \psi) \Rightarrow \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U})\right) \to ((W, \psi) \Rightarrow \Gamma) \\
&= \forall (W, \psi^{W \Rightarrow \Psi}). \left((W \ltimes U, \psi \ltimes \mathbf{y}U) \to (V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U})\right) \to ((W, \psi) \Rightarrow \Gamma),
\end{aligned}$$

and then we see that the last argument χ cannot exist. Indeed, suppose we have a commuting diagram (where the dotted part only applies in the indirect setting)

$$\begin{tikzcd} W \ltimes U \arrow[rr, "\chi"] \arrow[dr, Rightarrow, "\psi \ltimes \mathbf{y}U"'] \arrow[ddr, dotted, bend right, "\pi"'] & & V \arrow[d, Rightarrow, "\varphi"] \\ & \Psi \ltimes \mathbf{y}U \arrow[d, dotted, "\pi_2"] & \Psi \ltimes \partial U \arrow[l, "\supseteq"] \arrow[d, dotted, "\pi_2"] \\ & \mathbf{y}U & \partial U. \arrow[l, dotted, "\supseteq"] \end{tikzcd} \tag{43}$$

indirect boundary Then we see that $\pi_2 \circ \varphi : V \to U$ is dimensionally split with section χ but is also a cell of ∂U which means exactly that it is not dimensionally split.

direct boundary Then we see that $\varphi : V \Rightarrow \Psi \ltimes \mathbf{y}U$ is directly dimensionally split with section χ but it is also a cell of $\Psi \ltimes \partial U$ which means exactly that it is not directly dimensionally split. $\square$

The following theorem shows that dimensionally split morphisms are an interesting concept:

Theorem 4.4.4 (Boundary theorem)**.** 1. (Obsolete.) Using the indirect boundary, we have

$$\top \ltimes \mathbf{y}U \mid (\in \partial U) \cong \between^{\top|}_{\mathbf{y}U} \bot \vdash \mathsf{Ctx}$$

and more generally

$$\Psi \ltimes \mathbf{y}U \mid (\in \partial U) \cong \Omega^{() \ltimes \mathbf{y}U|} \between^{\top|}_{\mathbf{y}U} \bot \vdash \mathsf{Ctx}.$$

2. Using the direct boundary, we have

$$\Psi \ltimes \mathbf{y}U \mid (\in \partial U) \cong \between^{\Psi|}_{\mathbf{y}U} \bot \vdash \mathsf{Ctx}.$$

Proof. 1. We prove the first statement by characterizing the right hand side of the isomorphism. We have

$$\begin{aligned}
&(V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U}) \Rightarrow \between^{\top|}_{\mathbf{y}U} \bot \\
&= \forall^{\top|}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U}) \to \bot \\
&= \forall (W, ()^{W \Rightarrow \top}).((W, ()) \Rightarrow \forall^{\top|}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U})) \to ((W, ()) \Rightarrow \bot) \\
&= \forall (W, ()^{W \Rightarrow \top}).((W, ()) \Rightarrow \forall^{\top|}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U})) \to \varnothing \\
&= \forall (W, ()^{W \Rightarrow \top}).(\exists^{/\top}_{U} (W, ()) \to (V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U})) \to \varnothing \\
&= \forall (W, ()^{W \Rightarrow \top}).((W \ltimes U, () \ltimes U) \to (V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U})) \to \varnothing \\
&\cong \forall W.((W \ltimes U, \pi_2) \to (V, \pi_2 \circ \varphi)) \to \varnothing \\
&\cong (\exists W.(W \ltimes U, \pi_2) \to (V, \pi_2 \circ \varphi)) \to \varnothing.
\end{aligned}$$

Clearly, the left hand side of the last line is inhabited if and only if $\pi_2 \circ \varphi$ is dimensionally split. Hence, there is a unique cell $(V, \varphi^{V \Rightarrow \top \ltimes \mathbf{y}U}) \Rightarrow \between^{\top|}_{\mathbf{y}U} \bot$ if and only if $\pi_2 \circ \varphi$ is *not* dimensionally split, showing that $\between^{\top|}_{\mathbf{y}U} \bot$ is indeed isomorphic to $(\in \partial U)$.

The second statement follows from applying $\Omega^{() \ltimes \mathbf{y}U|}$ to both sides of the first statement and observing that, being defined by pullback, the indirect boundary predicate is preserved by the substitution functor.

2. We prove this by characterizing the right hand side of the isomorphism. We have

$$\begin{aligned}
&(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \Rightarrow \between^{\Psi|}_{\mathbf{y}U} \bot \\
&= \forall^{\Psi|}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \to \bot \\
&= \forall (W, \psi^{W \Rightarrow \Psi}).((W, \psi) \Rightarrow \forall^{\Psi|}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U})) \to ((W, \psi) \Rightarrow \bot) \\
&= \forall (W, \psi^{W \Rightarrow \Psi}).((W, \psi) \Rightarrow \forall^{\Psi|}_{\mathbf{y}U} \mathbf{y}(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U})) \to \varnothing \\
&= \forall (W, \psi^{W \Rightarrow \Psi}).(\exists^{/\Psi}_{U} (W, \psi) \to (V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U})) \to \varnothing \\
&= \forall (W, \psi^{W \Rightarrow \Psi}).((W \ltimes U, \psi \ltimes U) \to (V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U})) \to \varnothing \\
&\cong (\exists (W, \psi^{W \Rightarrow \Psi}).(W \ltimes U, \psi \ltimes U) \to (V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U})) \to \varnothing.
\end{aligned}$$

Clearly, the left hand side of the last line is inhabited if and only if φ is directly dimensionally split. Hence, there is a unique cell $(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}) \Rightarrow \between^{\Psi|}_{\mathbf{y}U} \bot$ if and only if φ is *not* directly dimensionally split, showing that $\between^{\Psi|}_{\mathbf{y}U} \bot$ is indeed isomorphic to $(\in \partial U)$. $\square$

Remark 4.4.5. In section 6.3 (theorem 6.3.1), we will see that unless the multiplier is cancellative and affine, the transpension type is not stable under substitution. Instead, for $\sigma : \Psi_1 \to \Psi_2$, we only have $\Omega^{\sigma \ltimes \mathbf{y}U|} \circ \between^{\Psi_2|}_{\mathbf{y}U} \to \between^{\Psi_1|}_{\mathbf{y}U} \circ \Omega^{\sigma|}$.

Instantiating this with $\sigma = () : \Psi \to \top$ and applying both hands to $\bot$, which is preserved by the substitution functor, we find $\Omega^{() \ltimes \mathbf{y}U|} \between^{\top|}_{\mathbf{y}U} \bot \to \between^{\Psi|}_{\mathbf{y}U} \bot$, i.e. the indirect boundary predicate implies the direct boundary predicate.

Since the transpension type *is* stable under substitution for cancellative and affine multipliers, we can conclude that for those multipliers, both notions of boundary coincide.

Theorem 4.4.6 (Transpension elimination)**.** Let $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ be a cancellative, affine and connection-free. Then we have[12]

$$\frac{\begin{array}{l}
\Psi \ltimes \mathbf{y}U \mid \Gamma \vdash \mathsf{Ctx} \\
\Psi \mid \forall^{\Psi|}_{\mathbf{y}U} \Gamma \vdash A \,\mathsf{type} \\
\Psi \ltimes \mathbf{y}U \mid \Gamma . \left\langle \between^{\Psi|}_{\mathbf{y}U} \,\middle|\, A \right\rangle \vdash B \,\mathsf{type} \\
\Psi \ltimes \partial U \mid \Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \Gamma \vdash b_\partial : \left(\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} B \right) [(\mathrm{id}, _)] \\
\Psi \mid \left(\forall^{\Psi|}_{\mathbf{y}U} \Gamma \right) . A \vdash \mathring{b} : \left(\forall^{\Psi|}_{\mathbf{y}U} B \right) \left[\left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} \right)^{-1} (\xi) \right) \right] \\
\Psi \ltimes \partial U \mid \Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \exists^{\Psi|}_{\mathbf{y}U} \left(\left(\forall^{\Psi|}_{\mathbf{y}U} \Gamma \right) . A \right) \vdash \Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \left(\mathsf{app}^{\Psi|}_{\mathbf{y}U} \left(\exists^{\Psi|}_{\mathbf{y}U} \mathring{b} \right) \right) = b_\partial \left[\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \left(\mathsf{app}^{\Psi|}_{\mathbf{y}U} \circ \pi \right) \right] \\
\qquad : \left(\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} B \right) [(\mathrm{id}, _)] \left[\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \left(\mathsf{app}^{\Psi|}_{\mathbf{y}U} \circ \pi \right) \right]
\end{array}}{\Psi \ltimes \mathbf{y}U \mid \Gamma . \left\langle \between^{\Psi|}_{\mathbf{y}U} \,\middle|\, A \right\rangle \vdash b : B} \tag{44}$$

and b reduces to b_∂ and $\mathring{b}$ if we apply to it the same functors and substitutions that have been applied to B in the types of b_∂ and $\mathring{b}$.

(If the multiplier is not quantifiable, then $\exists^{\Psi|}_{\mathbf{y}U}$ may not be a CwF morphism, but the term $\mathsf{app}^{\Psi|}_{\mathbf{y}U} \left(\exists^{\Psi|}_{\mathbf{y}U} \mathring{b} \right)$ is essentially a dependent transposition for the adjunction $\exists^{\Psi|}_{\mathbf{y}U} \dashv \forall^{\Psi|}_{\mathbf{y}U}$ which even exists if only the right adjoint is a CwF morphism [Nuy18]).

In words: if we want to eliminate an element of the transpension type, then we can do so by induction. We distinguish two cases and a coherence condition:

- In the first case (b_∂), we are on the boundary of U and the transpension type trivializes.
- In the second case, we are defining an action on cells that live over all of $\mathbf{y}U$. In the transpension type, such cells are in 1-1 correspondence with cells of type A under the isomorphism $\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} : \forall^{\Psi|}_{\mathbf{y}U} \between^{\Psi|}_{\mathbf{y}U} \cong \mathrm{Id}$.
- The boundary of the image of cells in the second case, must always be b_∂.

Note that right adjoint weak CwF morphisms such as $\between^{\Psi|}_{\mathbf{y}U}$ give rise to a DRA by applying the CwF morphism and then substituting with the unit of the adjunction. As such, the transpension *type* is modelled by the DRA sending A to $\left\langle \between^{\Psi|}_{\mathbf{y}U} \,\middle|\, A \right\rangle = \left(\between^{\Psi|}_{\mathbf{y}U} A \right) \left[\mathsf{reidx}^{\Psi|}_{\mathbf{y}U} \right]$.

Proof. **Well-formedness.** We first show that the theorem is well-formed.

- The rule for Γ just assumes that Γ is a presheaf over $\mathcal{V}/(\Psi \ltimes \mathbf{y}U)$.
- Then $\forall^{\Psi|}_{\mathbf{y}U} \Gamma$ is a presheaf over $\mathcal{W}/\Psi$ and we assume that A is a type in that context, i.e. a presheaf over the category of elements of $\forall^{\Psi|}_{\mathbf{y}U} \Gamma$.

[12] regardless of the notion of boundary, as these coincide for cancellative and affine multipliers (remark 4.4.5); we do not even have to distinguish cases in the proof as we will simply apply the appropriate version of the kernel theorem 4.1.11.

- Then the DRA of $\between^{\Psi|}_{\mathbf{y}U}$ applied to A is a type in context Γ. We assume that B is a type over the extended context.

- Being a central lifting, $\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)}$ is a CwF morphism and can be applied to B, yielding a type in context

$$\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \left(\Gamma.\left(\between^{\Psi|}_{\mathbf{y}U} A \right) \left[\mathsf{reidx}^{\Psi|}_{\mathbf{y}U} \right] \right) = \Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \Gamma.\left(\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \between^{\Psi|}_{\mathbf{y}U} A \right) \left[\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \mathsf{reidx}^{\Psi|}_{\mathbf{y}U} \right] \cong \Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \Gamma.\top,$$

where the isomorphism is an application of theorem 4.4.3. The substitution $(\mathrm{id}, \_) = \pi^{-1}$ yields a type in context $\Omega^{\Psi \ltimes \partial U|}_{(\in \partial U)} \Gamma$. We assume that b_∂ has this type.

- Being a central lifting, $\forall^{\Psi|}_{\mathbf{y}U}$ is a CwF morphism and can be applied to B, yielding a type in context

$$\forall^{\Psi|}_{\mathbf{y}U} \left(\Gamma.\left(\between^{\Psi|}_{\mathbf{y}U} A \right) \left[\mathsf{reidx}^{\Psi|}_{\mathbf{y}U} \right] \right) = \forall^{\Psi|}_{\mathbf{y}U} \Gamma.\left(\forall^{\Psi|}_{\mathbf{y}U} \between^{\Psi|}_{\mathbf{y}U} A \right) \left[\forall^{\Psi|}_{\mathbf{y}U} \mathsf{reidx}^{\Psi|}_{\mathbf{y}U} \right].$$

The natural transformation $(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1}$ gives rise [Nuy18] to a function

$$(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1} : A \to \left(\forall^{\Psi|}_{\mathbf{y}U} \between^{\Psi|}_{\mathbf{y}U} A \right) [(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1}]. \tag{45}$$

Now, by the adjunction laws, $\forall^{\Psi|}_{\mathbf{y}U} \mathsf{reidx}^{\Psi|}_{\mathbf{y}U} \circ \mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} = \mathrm{id}$, so

$$\forall^{\Psi|}_{\mathbf{y}U} \mathsf{reidx}^{\Psi|}_{\mathbf{y}U} = \forall^{\Psi|}_{\mathbf{y}U} \mathsf{reidx}^{\Psi|}_{\mathbf{y}U} \circ \mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} \circ (\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1} = (\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1}. \tag{46}$$

Then we have

$$(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1} : A \to \left(\forall^{\Psi|}_{\mathbf{y}U} \between^{\Psi|}_{\mathbf{y}U} A \right) \left[\forall^{\Psi|}_{\mathbf{y}U} \mathsf{reidx}^{\Psi|}_{\mathbf{y}U} \right]. \tag{47}$$

Thus, we can substitute $\forall^{\Psi|}_{\mathbf{y}U} B$ with $(\pi, (\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U})^{-1}(\xi))$, yielding a type in the desired context. We assume that $\mathring{b}$ has this type.

- In the coherence criterion, we have applied operations to b_∂ and $\mathring{b}$ before equating them. We have to ensure that the resulting terms are well-typed in the given context and type.

  - If we apply $\dashv^{\Psi|}_{\mathbf{y}U}$ to the term $\mathring{b}$, we get

$$\Psi \ltimes \mathbf{y}U \mid \left(\dashv^{\Psi|}_{\mathbf{y}U} \forall^{\Psi|}_{\mathbf{y}U} \Gamma \right).\dashv^{\Psi|}_{\mathbf{y}U} A \vdash {}^{\dashv^{\Psi|}_{\mathbf{y}U}} \mathring{b} : \left(\dashv^{\Psi|}_{\mathbf{y}U} \forall^{\Psi|}_{\mathbf{y}U} B \right) \left[\dashv^{\Psi|}_{\mathbf{y}U} \left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} \right)^{-1} (\xi) \right) \right].$$

  If we subsequently apply $\mathsf{app}^{\Psi|}_{\mathbf{y}U}$, we get

$$\Psi \ltimes \mathbf{y}U \mid \left(\dashv^{\Psi|}_{\mathbf{y}U} \forall^{\Psi|}_{\mathbf{y}U} \Gamma \right).\dashv^{\Psi|}_{\mathbf{y}U} A \vdash \mathsf{app}^{\Psi|}_{\mathbf{y}U} \left({}^{\dashv^{\Psi|}_{\mathbf{y}U}} \mathring{b} \right) : B \left[\mathsf{app}^{\Psi|}_{\mathbf{y}U} \right] \left[\dashv^{\Psi|}_{\mathbf{y}U} \left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} \right)^{-1} (\xi) \right) \right].$$

  Next, we apply $\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)}$ and obtain something of type

$$\left(\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} B \right) \left[\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \mathsf{app}^{\Psi|}_{\mathbf{y}U} \right] \left[\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \dashv^{\Psi|}_{\mathbf{y}U} \left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} \right)^{-1} (\xi) \right) \right].$$

  Now if we look at the context of $\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} B$, we see that the last type is the unit type by theorem 4.4.3, so the substitution applied to B is determined by its weakening. So we rewrite:

$$\begin{aligned} \ldots &= \left(\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} B \right) [(\mathrm{id}, \_)][\pi] \left[\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \mathsf{app}^{\Psi|}_{\mathbf{y}U} \right] \left[\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \dashv^{\Psi|}_{\mathbf{y}U} \left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} \right)^{-1} (\xi) \right) \right] \\ &= \left(\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} B \right) [(\mathrm{id}, \_)] \left[\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \mathsf{app}^{\Psi|}_{\mathbf{y}U} \right] [\pi] \left[\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \dashv^{\Psi|}_{\mathbf{y}U} \left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U} \right)^{-1} (\xi) \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \left(\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} B\right) [(\mathrm{id}, \_)] \left[\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \mathsf{app}^{\Psi|}_{\mathbf{y}U}\right] \left[\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \Finv^{\Psi|}_{\mathbf{y}U} \left(\pi \circ \left(\pi, \left(\mathsf{unmerid}^{\Psi|}_{\mathbf{y}U}\right)^{-1} (\xi)\right)\right)\right] \\
&= \left(\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} B\right) [(\mathrm{id}, \_)] \left[\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \mathsf{app}^{\Psi|}_{\mathbf{y}U}\right] \left[\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \Finv^{\Psi|}_{\mathbf{y}U} \pi\right] \\
&= \left(\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} B\right) [(\mathrm{id}, \_)] \left[\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)} \left(\mathsf{app}^{\Psi|}_{\mathbf{y}U} \circ \pi\right)\right].
\end{aligned}
$$

– It is immediate that the substitution applied to b_∂ yields the given type.

Soundness of the coherence criterion. Note that, if we apply to b the same reasoning that we applied to B to show well-formedness of the last 3 premises, we find that the coherence criterion does hold if b_∂ and $\mathring{b}$ arise from a common b.

Completeness of the elimination clauses. We now show that b is fully determined by the b_∂ and $\mathring{b}$ that can be derived from it. Afterwards, we will show that the given coherence condition is sufficient to make sure that b_∂ and $\mathring{b}$ determine some b.

Note that B, being a type in a presheaf CwF, is a presheaf over the category of elements of $\Gamma.\left(\between^{\Psi|}_{\mathbf{y}U} A\right)\left[\mathsf{reidx}^{\Psi|}_{\mathbf{y}U}\right]$. Hence it acts on cells

$$
\left(V, \varphi^{V \Rightarrow \Psi \ltimes \mathbf{y}U}, \gamma^{(V,\varphi) \Rightarrow \Gamma}, a^{(V,\varphi,\gamma) \Rightarrow \left(\between^{\Psi|}_{\mathbf{y}U} A\right)\left[\mathsf{reidx}^{\Psi|}_{\mathbf{y}U}\right]}\right).
$$

Now we divide such cells in two classes: on-boundary cells (for which (V, φ) is on the boundary) and total cells (the others). As $\Omega^{\Psi \ltimes \mathbf{y}U|}_{(\in \partial U)}$ is exactly the restriction of presheaves to the on-boundary cells, it is clear that b_∂ determines the action of b on those.

For total cells, note that the full subcategory of $\mathcal{V}/(\Psi \ltimes \mathbf{y}U)$ consisting of the total elements, is (by theorem 4.1.11) equivalent to $\mathcal{W}/\Psi$, with one direction given by $\Finv^{/\Psi}_U$. Restriction to total cells is then given by the central lifting of that functor, being $\forall^{\Psi|}_{\mathbf{y}U}$. Combined with the knowledge that $\forall^{\Psi|}_{\mathbf{y}U} \between^{\Psi|}_{\mathbf{y}U} \cong \mathrm{Id}$ (theorem 4.1.10), this reveals that $\mathring{b}$ determines the action of b on total cells.

Completeness of the coherence criterion. The action of a term on cells should be natural with respect to restriction. This is automatic when considered with respect to morphisms between cells that are either both total or both on-boundary. Moreover, there are no morphisms $\chi : (V, \varphi) \to (V', \varphi') : \mathcal{V}/(\Psi \ltimes U)$ from a total cell to an on-boundary cell, since the boundary is a well-defined presheaf. So we still need to prove naturality w.r.t. morphisms from on-boundary cells to total cells.

Let $\chi : (V, \varphi) \to (V', \varphi')$ be such a morphism. Then $(V', \varphi') \cong_\iota \Finv^{/\Psi}_U (W, \psi) \cong \Finv^{/\Psi}_U \exists^{/\Psi}_U \Finv^{/\Psi}_U (W, \psi) \cong \Finv^{/\Psi}_U \exists^{/\Psi}_U (V', \varphi')$ by an isomorphism

$$
\begin{aligned}
&\Finv^{/\Psi}_U \exists^{/\Psi}_U \iota^{-1} \circ \Finv^{/\Psi}_U (\mathsf{drop}^{/\Psi}_U)^{-1} \circ \iota \\
&= \Finv^{/\Psi}_U \exists^{/\Psi}_U \iota^{-1} \circ \mathsf{copy}^{/\Psi}_U \circ \iota \\
&= \mathsf{copy}^{/\Psi}_U \circ \iota^{-1} \circ \iota = \mathsf{copy}^{/\Psi}_U.
\end{aligned}
$$

Hence, by naturality, $\chi = (\mathsf{copy}^{/\Psi}_U)^{-1} \circ \mathsf{copy}^{/\Psi}_U \circ \chi = (\mathsf{copy}^{/\Psi}_U)^{-1} \circ \Finv^{/\Psi}_U \exists^{/\Psi}_U \chi \circ \mathsf{copy}^{/\Psi}_U$. Thus, we have factored χ as an instance of the unit $\mathsf{copy}^{/\Psi}_U$ followed by a morphism between total cells. This means it is sufficient to show naturality with respect to $\mathsf{copy}^{/\Psi}_U : (V, \varphi) \to \Finv^{/\Psi}_U \exists^{/\Psi}_U (V, \varphi)$. (The cells of Γ and the transpension type available for (V', φ') carry over to $\Finv^{/\Psi}_U \exists^{/\Psi}_U (V, \varphi)$ by restriction.)

Now the action of b on (V, φ) is given by the action of b_∂ on (V, φ). Meanwhile, the action of b on $\Finv^{/\Psi}_U \exists^{/\Psi}_U (V, \varphi)$ is given by the action of $\mathring{b}$ on $\exists^{/\Psi}_U (V, \varphi)$, which is the action of ${}^{\Finv^{/\Psi}_U}\mathring{b}$ on (V, φ). These have to correspond via $\mathsf{copy}^{/\Psi}_U : (V, \varphi) \to \Finv^{/\Psi}_U \exists^{/\Psi}_U (V, \varphi)$, which corresponds via central lifting to the natural transformation $\mathsf{app}^{\Psi|}_{\mathbf{y}U}$ on presheaves. This is exactly what happens in the coherence criterion: we use $\mathsf{app}^{\Psi|}_{\mathbf{y}U} : \Finv^{\Psi|}_{\mathbf{y}U} \forall^{\Psi|}_{\mathbf{y}U} \to \mathrm{Id}$ to bring b_∂ and $\mathring{b}$ to the same context and type, and then equate them. Since b_∂ only exists on the boundary, we also have to restrict $\mathring{b}$ to the boundary, but that's OK since we were interested in an on-boundary cell anyway. $\square$

Example 4.4.7 (Affine cubes)**.** We instantiate theorem 4.4.6 for the multiplier $\sqcup * \mathbb{I} : \square^k \to \square^k$ (example 3.3.3). There, $\partial\mathbb{I}$ is essentially the constant presheaf with k elements. So b_∂ determines the images of the k poles of the transpension type. The term $\overset{\circ}{b}$ determines the action on paths (for $k = 2$, for general k perhaps 'webs' is a better term), and the paths/webs of the transpension type are essentially the elements of A. The coherence condition says that the image of such paths/webs should always have the endpoints given by b_∂.

Example 4.4.8 (Clocks)**.** We instantiate theorem 4.4.6 for the multiplier $\sqcup * (i : \text{◷}_k)$ (example 3.3.6), where we adapt the base category to forbid diagonals: a morphism may use every variable of its domain at most once. The boundary $\partial(i : \text{◷}_k)$ is isomorphic to $\mathbf{y}(i : \text{◷}_{k-1})$ if $k > 0$ and to the empty presheaf $\bot$ if $k = 0$. So if we want to eliminate an element of the transpension type over $\mathbf{y}(i : \text{◷}_k)$, which means we have a clock and we don't care about what happens if the time exceeds k, then we need to handle two cases. The first case b_∂ says what happens if we don't even care what happens at timestamp k; in which case the transpension type trivializes. Then, by giving $\overset{\circ}{b}$, we say what happens at timestamp k and need to make sure that this is consistent with b_∂. The elements of the transpension type at timestamp k are essentially the elements of A, which are fresh for the clock.

Example 4.4.9 (Embargoes)**.** Recall that the multiplier $\sqcup \ltimes \mathbf{!}$ sends $W \in \mathcal{W}$ to $(W, \top) \in \mathcal{W} \times \uparrow$, the Yoneda-embedding of which represents the arrow $\mathbf{y}W \to \mathbf{y}W$, i.e. $\mathbf{y}W.\mathbf{!}.\top$ under the convention that $\Psi.\mathbf{!}.\Theta$ denotes $(\Psi.\Theta \to \Psi)$. Its left lifting is $\sqcup \ltimes \mathbf{y!} : \widehat{\mathcal{W}} \to \widehat{\mathcal{W} \times \uparrow}$, and $\mathbf{y!}$ is the terminal object, so that $\widehat{\mathcal{W} \times \uparrow}/\mathbf{y!} \cong \widehat{\mathcal{W} \times \uparrow}$. We get 5 adjoint functors, of which we give here the action up to isomorphism:

$$\begin{array}{rcrcccr}
 & & & & \Psi & \mapsto & (\bot \to \Psi), \\
 & & \exists_{\mathbf{y!}} & : & \Psi & \leftarrow\!\shortmid & (\Psi.\Theta \to \Psi), \\
\sqcup \ltimes \mathbf{y!} & \text{or} & \reflectbox{$\exists$}_{\mathbf{y!}} & : & \Psi & \mapsto & (\Psi \to \Psi), \\
\mathbf{y!} \multimap \sqcup & \text{or} & \forall_{\mathbf{y!}} & : & \Psi.\Theta & \leftarrow\!\shortmid & (\Psi.\Theta \to \Psi), \\
\mathbf{y!} \surd \sqcup & \text{or} & \between_{\mathbf{y!}} & : & \Psi & \mapsto & (\Psi \to \top).
\end{array}$$

The boundary of $\mathbf{y!}$ is $\partial\mathbf{!} \cong \mathbf{y}(\top, \bot)$ which is isomorphic to the arrow $\bot \to \top$. Thus, we see:

$\exists_{\mathbf{y!}}$ If, for some unknown embargo, we have information partly under that embargo, then we can only extract the unembargoed information,

$\reflectbox{$\exists$}_{\mathbf{y!}}$ If information is fresh for an embargo, then it is unembargoed,

$\forall_{\mathbf{y!}}$ If, for any embargo, we have information partly under that embargo, then we can extract the information,

$\between_{\mathbf{y!}}$ If information is transpended over an embargo, then it is completely embargoed.

Perhaps the above is more intuitive if we think of an embargo as a key or a password.

So let us now instantiate theorem 4.4.6, which allows us to eliminate an element of the transpension type, i.e. essentially an element of $A \to \top$. The boundary case exists over the boundary $\bot \to \top$ and allows us to consider only the codomain of the arrow, i.e. the part of the context before the embargo, where the transpension type is trivial. The case $\overset{\circ}{b}$ then requires us to say how to act on embargoed data in a coherent way with what we already specified in b_∂. The embargoed data is essentially an element of A.

5 Prior modalities

Many modalities arise as central or right liftings of functors between base categories [NVD17, ND18, Nuy18, BM20]. The following definition allows us to use such modalities even when part of the context is in front of a pipe.

Definition 5.0.1. A functor $G : \mathcal{W} \to \mathcal{W}'$ yields a functor $G^{/\Psi} : \mathcal{W}/\Psi \to \mathcal{W}'/G_!\Psi : (W, \psi) \mapsto (GW, G_!\psi)$. This in turn yields three adjoint functors between presheaf categories:

$$G_!^{\Psi|} \dashv G^{\Psi|*} \dashv G_*^{\Psi|}. \tag{48}$$

If a modality is both a right and a central lifting, then the following theorem relates the corresponding 'piped' modalities:

Theorem 5.0.2. If $G : \mathcal{W} \to \mathcal{W}'$ has a right adjoint $G \dashv S$, then we have

$$
\begin{array}{ccccc|ccccc}
 & & \Sigma^{/\varepsilon_!} \circ G^{/S_!\Psi'} & \dashv & S^{/\Psi'} & G^{/\Psi} & \dashv & \Omega^{/\eta_!} \circ S^{/G_!\Psi} & & \\
\hline
 & & \Sigma^{\varepsilon_!|} \circ G_!^{S_!\Psi'|} & \dashv & S_!^{\Psi'|} & G_!^{\Psi|} & \dashv & \Omega^{\eta_!|} \circ S_!^{G_!\Psi|} & \cong & G^{\Psi|*} \\
S_!^{\Psi'|} & \cong & G^{S_!\Psi'|*} \circ \Omega^{\varepsilon_!|} & \dashv & S^{\Psi'|*} & G^{\Psi|*} & \dashv & S^{G_!\Psi|*} \circ \Pi^{\eta_!|} & \cong & G_*^{\Psi|} \\
S^{\Psi'|*} & \cong & \Pi^{\varepsilon_!|} \circ G_*^{S_!\Psi'|} & \dashv & S_*^{\Psi'|} & G_*^{\Psi|} & \dashv & \$^{\eta_!|} \circ S_*^{G_!\Psi|} & &
\end{array}
\tag{49}
$$

assuming – where mentioned – that $\Omega^{/\eta_!}$ exists.

Proof. For the left half of the table, we only prove the first line. The other adjunctions follow from the fact that $\sqcup_!$, $\sqcup^*$ and $\sqcup_*$ are pseudofunctors, and the isomorphisms follow from uniqueness of the adjoint. We have a correspondence of diagrams

$$
\begin{tikzcd}
W \arrow[rr] \arrow[dr, Rightarrow, "\psi"'] & & SW' \arrow[dl, Rightarrow, "S_!\psi'"] \\
 & S_!\Psi' &
\end{tikzcd}
\qquad
\begin{tikzcd}
GW \arrow[rr] \arrow[dr, Rightarrow, "\varepsilon_! \circ G_!\psi"'] & & W' \arrow[dl, Rightarrow, "\psi'"] \\
 & \Psi' &
\end{tikzcd}
\tag{50}
$$

i.e. morphisms $(W, \psi) \to S^{/\Psi'}(W', \psi') : \mathcal{W}/S_!\Psi'$ correspond to morphisms $\Sigma^{/\varepsilon_!} G^{/S_!\Psi'}(W, \psi) \to (W', \psi') : \mathcal{W}'/\Psi'$.

On the right side of the table, we similarly only need to prove the first line, and we prove it from the first line on the left side. The left adjoint to $\Omega^{/\eta_!} \circ S^{/G_!\Psi}$ is $\left(\Sigma^{/\varepsilon_!} \circ G^{/S_!G_!\Psi}\right) \circ \Sigma^{/\eta_!}$. We prove that this is equal to $G^{/\Psi}$:

$$
\begin{aligned}
&\Sigma^{/\varepsilon_!} G^{/S_!G_!\Psi} \Sigma^{/\eta_!}(W, \psi : W \to \Psi) \\
&= \Sigma^{/\varepsilon_!} G^{/S_!G_!\Psi}(W, \eta_! \circ \psi : W \to S_!G_!\Psi) \\
&= \Sigma^{/\varepsilon_!}(GW, G_!\eta_! \circ G_!\psi : GW \to G_!S_!G_!\Psi) \\
&= (GW, \varepsilon_! \circ G_!\eta_! \circ G_!\psi : GW \to G_!\Psi) = (GW, G_!\psi : GW \to G_!\Psi). \qquad \square
\end{aligned}
$$

6 Commutation rules

6.1 Substitution and substitution

See theorem 2.3.19.

6.2 Modality and substitution

Theorem 6.2.1. Assume a functor $G : \mathcal{W} \to \mathcal{W}'$ and a morphism $\sigma : \Psi_1 \to \Psi_2 : \widehat{\mathcal{W}}$. Then we have a commutative diagram

$$
\begin{tikzcd}
\mathcal{W}/\Psi_1 \arrow[r, "G^{/\Psi_1}"] \arrow[d, "\Sigma^{/\sigma}"'] & \mathcal{W}'/G_!\Psi_1 \arrow[d, "\Sigma^{/G_!\sigma}"] \\
\mathcal{W}/\Psi_2 \arrow[r, "G^{/\Psi_2}"'] & \mathcal{W}'/G_!\Psi_2
\end{tikzcd}
\tag{51}
$$

and hence

	$G_!$	G^*	G_*
Σ	$\Sigma^{G_!\sigma\|} G_!^{\Psi_1\|} \cong G_!^{\Psi_2\|} \Sigma^{\sigma\|}$	$\Sigma^{\sigma\|} G^{\Psi_1\|*} \to G^{\Psi_2\|*} \Sigma^{G_!\sigma\|}$	
Ω	$\Omega^{G_!\sigma\|} G_!^{\Psi_2\|} \leftarrow G_!^{\Psi_1\|} \Omega^{\sigma\|}$	$\Omega^{\sigma\|} G^{\Psi_2\|*} = G^{\Psi_1\|*} \Omega^{G_!\sigma\|}$	$\Omega^{G_!\sigma\|} G_*^{\Psi_2\|} \to G_*^{\Psi_1\|} \Omega^{\sigma\|}$
Π		$\Pi^{\sigma\|} G^{\Psi_1\|*} \leftarrow G^{\Psi_2\|*} \Pi^{G_!\sigma\|}$	$\Pi^{G_!\sigma\|} G_*^{\Psi_1\|} \cong G_*^{\Psi_2\|} \Pi^{\sigma\|}$
$\$$			$\$^{G_!\sigma\|} G_*^{\Psi_2\|} \leftarrow G_*^{\Psi_1\|} \$^{\sigma\|}$

where every statement holds if the mentioned functors exist.

Proof. It is evident from the definitions that the given diagram commutes. Then by applying $\sqcup^*$, we find the that $\Omega^{\sigma|}G^{\Psi_2|*} = G^{\Psi_1|*}\Omega^{G_!\sigma|}$. The rest of the table then follows by lemma 2.1.2. $\square$

Remark 6.2.2.
- If $\sigma = \pi : \Psi.A \to \Psi$, then this says something about weakening and the Σ- and Π-types over A.
- If $G_!$ moreover happens to be a CwF morphism, then this relates weakening and the Σ- and Π-types over A to those over $G_! A$.
- If $\sqcup \times U$ is a cartesian multiplier and we take $\sigma = \pi_1 : \Psi \times \mathbf{y}U \to \Psi$, then by theorem 4.1.10, this says something about $\exists^{\Psi|}_{\mathbf{y}U} \dashv \reflectbox{$\exists$}^{\Psi|}_{\mathbf{y}U} \dashv \forall^{\Psi|}_{\mathbf{y}U} \dashv \between^{\Psi|}_{\mathbf{y}U}$.

6.3 Multiplier and substitution

If, in section 6.2, we take G equal to some multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$, then we have

$$G^{/\Psi} = \reflectbox{$\exists$}^{/\Psi}_U, \qquad G_! = \sqcup \ltimes \mathbf{y}U, \qquad G^{\Psi|}_! = \reflectbox{$\exists$}^{\Psi|}_{\mathbf{y}U}, \qquad G^{\Psi|*} = \forall^{\Psi|}_{\mathbf{y}U}, \qquad G^{\Psi|}_* = \between^{\Psi|}_{\mathbf{y}U}. \tag{52}$$

This immediately yields the general case of the following theorem:

Theorem 6.3.1. Assume a multiplier $\sqcup \ltimes U : \mathcal{W} \to \mathcal{V}$ and a morphism $\sigma : \Psi_1 \to \Psi_2$ in $\widehat{\mathcal{W}}$. Write $\tau = \sigma \ltimes \mathbf{y}U$. Then we have:

	$\exists$	$\reflectbox{$\exists$}$	$\forall$	$\between$
Σ	$\Sigma^{\sigma\|}\exists^{\Psi_1\|}_{\mathbf{y}U} \lhd^1 \exists^{\Psi_2\|}_{\mathbf{y}U}\Sigma^{\tau\|}$	$\Sigma^{\tau\|}\reflectbox{$\exists$}^{\Psi_1\|}_{\mathbf{y}U} \cong \reflectbox{$\exists$}^{\Psi_2\|}_{\mathbf{y}U}\Sigma^{\sigma\|}$	$\Sigma^{\sigma\|}\forall^{\Psi_1\|}_{\mathbf{y}U} \rhd_1 \forall^{\Psi_2\|}_{\mathbf{y}U}\Sigma^{\tau\|}$	$\Sigma^{\tau\|}\between^{\Psi_1\|}_{\mathbf{y}U} \rhd_2 \between^{\Psi_2\|}_{\mathbf{y}U}\Sigma^{\sigma\|}$
Ω	$\Omega^{\sigma\|}\exists^{\Psi_2\|}_{\mathbf{y}U} \lhd^2 \exists^{\Psi_1\|}_{\mathbf{y}U}\Omega^{\tau\|}$	$\Omega^{\tau\|}\reflectbox{$\exists$}^{\Psi_2\|}_{\mathbf{y}U} \lhd^1 \reflectbox{$\exists$}^{\Psi_1\|}_{\mathbf{y}U}\Omega^{\sigma\|}$	$\Omega^{\sigma\|}\forall^{\Psi_2\|}_{\mathbf{y}U} = \forall^{\Psi_1\|}_{\mathbf{y}U}\Omega^{\tau\|}$	$\Omega^{\tau\|}\between^{\Psi_2\|}_{\mathbf{y}U} \rhd_1 \between^{\Psi_1\|}_{\mathbf{y}U}\Omega^{\sigma\|}$
Π	$\Pi^{\sigma\|}\exists^{\Psi_1\|}_{\mathbf{y}U} \lhd^3 \exists^{\Psi_2\|}_{\mathbf{y}U}\Pi^{\tau\|}$	$\Pi^{\tau\|}\reflectbox{$\exists$}^{\Psi_1\|}_{\mathbf{y}U} \lhd^2 \reflectbox{$\exists$}^{\Psi_2\|}_{\mathbf{y}U}\Pi^{\sigma\|}$	$\Pi^{\sigma\|}\forall^{\Psi_1\|}_{\mathbf{y}U} \lhd^1 \forall^{\Psi_2\|}_{\mathbf{y}U}\Pi^{\tau\|}$	$\Pi^{\tau\|}\between^{\Psi_1\|}_{\mathbf{y}U} \cong \between^{\Psi_2\|}_{\mathbf{y}U}\Pi^{\sigma\|}$
$\$$		$\$^{\tau\|}\reflectbox{$\exists$}^{\Psi_2\|}_{\mathbf{y}U} \lhd^3 \reflectbox{$\exists$}^{\Psi_1\|}_{\mathbf{y}U}\$^{\sigma\|}$	$\$^{\sigma\|}\forall^{\Psi_2\|}_{\mathbf{y}U} \lhd^2 \forall^{\Psi_1\|}_{\mathbf{y}U}\$^{\tau\|}$	$\$^{\tau\|}\between^{\Psi_2\|}_{\mathbf{y}U} \lhd^1 \between^{\Psi_1\|}_{\mathbf{y}U}\$^{\sigma\|}$

(53)

where every statement holds if the mentioned functors exist, and where

1. In general, $\lhd^1$ means $\leftarrow$, $\rhd_1$ means $\to$ and the other symbols mean nothing.
2. If $\sqcup \ltimes U$ is quantifiable, then $\lhd^1$ upgades to $\cong$ and $\lhd^2$ upgrades to $\leftarrow$.
3. If $\sqcup \ltimes U$ is cartesian (hence quantifiable), then $\lhd^1$ and $\lhd^2$ upgade to $\cong$ and $\lhd^3$ upgrades to $\leftarrow$.
4. If $\sqcup \ltimes U$ is cancellative and affine, then we have

$$\Sigma^{\sigma|}\forall^{\Psi_1|}_{\mathbf{y}U} \cong \forall^{\Psi_2|}_{\mathbf{y}U}\Sigma^{\tau|} : \widehat{\overbrace{\mathcal{V}/(\Psi_1 \ltimes \mathbf{y}U)}} \to \widehat{\mathcal{W}/\Psi_2} \tag{54}$$

   so that $\rhd_1$ upgrades to $\cong$ and $\rhd_2$ upgrades to $\to$.

Proof.
1. The general case is a corollary of theorem 6.2.1 for $G = \sqcup \ltimes U$.
2. To prove the quantifiable case, we show in the base category that $\Sigma^{/\sigma}\exists^{/\Psi_1}_U = \exists^{/\Psi_2}_U \Sigma^{/(\sigma \ltimes \mathbf{y}U)}$. We use the construction of $\exists^{/\Psi}_U$ in the proof of provident quantifiability (proposition 4.1.8). On one hand, we have:

$$\Sigma^{/\sigma}\exists^{/\Psi_1}_U(V, (\psi_1^{W_0 \Rightarrow \Psi_1} \ltimes \mathbf{y}U) \circ \varphi^{V \Rightarrow W_0 \ltimes U}) = \Sigma^{/\sigma}\Sigma^{/\psi_1}\exists^{/W_0}_U(V, \varphi) = \Sigma^{/\sigma \circ \psi_1}\exists^{/W_0}_U(V, \varphi).$$

   On the other hand:

$$\begin{aligned}\exists^{/\Psi_2}_U \Sigma^{/(\sigma \ltimes \mathbf{y}U)}(V, (\psi_1^{W_0 \Rightarrow \Psi_1} \ltimes \mathbf{y}U) \circ \varphi^{V \Rightarrow W_0 \ltimes U}) &= \exists^{/\Psi_2}_U(V, ((\sigma \circ \psi_1) \ltimes \mathbf{y}U) \circ \varphi) \\ &= \Sigma^{/\sigma \circ \psi_1}\exists^{/W_0}_U(V, \varphi).\end{aligned}$$

3. This follows from theorem 2.3.19.

4. We show that $\Sigma^{\sigma|}\forall^{\Psi_1|}_{\mathbf{y}U} \cong \forall^{\Psi_2|}_{\mathbf{y}U}\Sigma^{\tau|}$. Pick a presheaf Γ over $\mathcal{V}/(\Psi_1 \ltimes \mathbf{y}U)$. On the one hand, we have:

$$\begin{aligned}
&(W_2, \psi_2^{W_2 \Rightarrow \Psi_2}) \Rightarrow \Sigma^{\sigma|}\forall^{\Psi_1|}_{\mathbf{y}U}\Gamma \\
&= \exists(W_1, \psi_1^{W_1 \Rightarrow \Psi_1}).\left(\theta : (W_2, \psi_2) \to \Sigma^{/\sigma}(W_1, \psi_1)\right) \times \left((W_1, \psi_1) \Rightarrow \forall^{\Psi_1|}_{\mathbf{y}U}\Gamma\right) \\
&= \exists(W_1, \psi_1^{W_1 \Rightarrow \Psi_1}).(\theta : (W_2, \psi_2) \to (W_1, \sigma \circ \psi_1)) \times ((W_1 \ltimes U, \psi_1 \ltimes \mathbf{y}U) \Rightarrow \Gamma) \\
&\cong \exists W_1, \psi_1^{W_1 \Rightarrow \Psi_1}, \theta^{W_2 \to W_1}.(\psi_2 = \sigma \circ \psi_1 \circ \theta) \times ((W_1 \ltimes U, \psi_1 \ltimes \mathbf{y}U) \Rightarrow \Gamma) \\
&\qquad \text{We now absorb } \theta \text{ into } \psi_1\text{:} \\
&\cong \psi_1^{W_2 \Rightarrow \Psi_1}.(\psi_2 = \sigma \circ \psi_1) \times ((W_2 \ltimes U, \psi_1 \ltimes \mathbf{y}U) \Rightarrow \Gamma)\,.
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
&(W_2, \psi_2^{W_2 \Rightarrow \Psi_2}) \Rightarrow \forall^{\Psi_2|}_{\mathbf{y}U}\Sigma^{\tau|}\Gamma \\
&= (W_2 \ltimes U, \psi_2 \ltimes \mathbf{y}U) \Rightarrow \Sigma^{\tau|}\Gamma \\
&= \exists(V_1, \varphi_1^{V_1 \Rightarrow \Psi_1 \ltimes \mathbf{y}U}).\left(\omega : (W_2 \ltimes U, \psi_2 \ltimes \mathbf{y}U) \to \Sigma^{/\tau}(V_1, \varphi_1)\right) \times ((V_1, \varphi_1) \Rightarrow \Gamma) \\
&= \exists(V_1, \varphi_1^{V_1 \Rightarrow \Psi_1 \ltimes \mathbf{y}U}).(\omega : (W_2 \ltimes U, \psi_2 \ltimes \mathbf{y}U) \to (V_1, (\sigma \ltimes \mathbf{y}U) \circ \varphi_1)) \times ((V_1, \varphi_1) \Rightarrow \Gamma) \\
&\qquad \text{We now deconstruct } \varphi_1 = (\psi_1 \ltimes \mathbf{y}U) \circ \chi\text{:} \\
&\cong \exists V_1, W_1, \chi^{V_1 \to W_1 \ltimes U}, \psi_1^{W_1 \Rightarrow \Psi_1}. \\
&\qquad (\omega : (W_2 \ltimes U, \psi_2 \ltimes \mathbf{y}U) \to (V_1, ((\sigma \circ \psi_1) \ltimes \mathbf{y}U) \circ \chi)) \times ((V_1, (\psi_1 \ltimes \mathbf{y}U) \circ \chi) \Rightarrow \Gamma) \\
&\cong \exists V_1, W_1, \chi^{V_1 \to W_1 \ltimes U}, \psi_1^{W_1 \Rightarrow \Psi_1}, \omega^{W_2 \ltimes U \to V_1}. \\
&\qquad (\psi_2 \ltimes \mathbf{y}U = ((\sigma \circ \psi_1) \ltimes \mathbf{y}U) \circ \chi \circ \omega) \times ((V_1, (\psi_1 \ltimes \mathbf{y}U) \circ \chi) \Rightarrow \Gamma) \\
&\qquad \text{We now absorb } \omega \text{ into } \chi\text{:} \\
&\cong \exists W_1, \psi_1^{W_1 \Rightarrow \Psi_1}, \chi^{W_2 \ltimes U \to W_1 \ltimes U}.(\psi_2 \ltimes \mathbf{y}U = ((\sigma \circ \psi_1) \ltimes \mathbf{y}U) \circ \chi) \times ((W_2 \ltimes U, (\psi_1 \ltimes \mathbf{y}U) \circ \chi) \Rightarrow \Gamma) \\
&\qquad \text{Let } \chi = \Finv_U^{/\Psi_2}\theta : \Finv_U^{/\Psi_2}(W_2, \psi_2) \to \Finv_U^{/\Psi_2}(W_1, \sigma \circ \psi_1)\text{:} \\
&\cong \exists W_1, \psi_1^{W_1 \Rightarrow \Psi_1}, \theta^{W_2 \to W_1}.(\psi_2 = \sigma \circ \psi_1 \circ \theta) \times ((W_2 \ltimes U, ((\psi_1 \circ \theta) \ltimes \mathbf{y}U)) \Rightarrow \Gamma) \\
&\qquad \text{We now absorb } \theta \text{ into } \psi_1\text{:} \\
&\cong \psi_1^{W_2 \Rightarrow \Psi_1}.(\psi_2 = \sigma \circ \psi_1) \times ((W_2 \ltimes U, (\psi_1 \ltimes \mathbf{y}U)) \Rightarrow \Gamma)
\end{aligned}$$

This proves the isomorphism. The rest follows from lemma 2.1.2. □

6.4 Multiplier and modality

Theorem 6.4.1. Assume a commutative diagram (up to natural isomorphism $\nu : F(\sqcup \ltimes U) \cong G\sqcup \ltimes U'$)

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "G"] \arrow[d, "\sqcup \ltimes U"'] & \mathcal{W}' \arrow[d, "\sqcup \ltimes U'"] \\
\mathcal{V} \arrow[r, "F"'] & \mathcal{V}'
\end{tikzcd} \tag{55}$$

where $\sqcup \ltimes U$ and $\sqcup \ltimes U'$ are multipliers for U and U'.

Then $\Sigma^{/\nu_!}$ is a strictly invertible functor and hence we have

$$\Sigma^{\nu_!|} \cong \Omega^{\nu_!^{-1}|} \cong \Pi^{\nu_!|} \cong \between^{\nu_!^{-1}|} \qquad \Sigma^{\nu_!^{-1}|} \cong \Omega^{\nu_!|} \cong \Pi^{\nu_!^{-1}|} \cong \between^{\nu_!|}, \tag{56}$$

where $\Omega^{\nu_!^{-1}|}$ is the strict inverse to $\Omega^{\nu_!|}$.

Then we have $\Sigma^{/\nu_!^{-1}} \text{\reflectbox{$\exists$}}_{U'}^{/G_!\Psi} G^{/\Psi} \cong F^{/\Psi \ltimes U} \text{\reflectbox{$\exists$}}_U^{/\Psi}$. This yields the following commutation table:

	$F_!, G_!$	F^*, G^*	F_*, G_*
$\exists$	$\exists_{\mathbf{y}U'}^{G_!\Psi\|} \Omega^{\nu_!^{-1}\|} F_!^{\Psi\ltimes\mathbf{y}U\|} \rhd_1 G_!^{\Psi\|} \exists_{\mathbf{y}U}^{\Psi\|}$	$\exists_{\mathbf{y}U}^{\Psi\|} F^{\Psi\|*} \rhd_2 G^{\Psi\|*} \exists_{\mathbf{y}U'}^{G_!\Psi\|} \Omega^{\nu_!^{-1}\|}$	
$\text{\reflectbox{$\exists$}}$	$\Omega^{\nu_!\|} \text{\reflectbox{$\exists$}}_{\mathbf{y}U'}^{G_!\Psi\|} G_!^{\Psi\|} \cong F_!^{\Psi\ltimes\mathbf{y}U\|} \text{\reflectbox{$\exists$}}_{\mathbf{y}U}^{\Psi\|}$	$\text{\reflectbox{$\exists$}}_{\mathbf{y}U}^{\Psi\|} G^{\Psi\|*} \rhd_1 F^{\Psi\ltimes\mathbf{y}U\|*} \Omega^{\nu_!\|} \text{\reflectbox{$\exists$}}_{\mathbf{y}U'}^{G_!\Psi\|}$	$\Omega^{\nu_!\|} \text{\reflectbox{$\exists$}}_{\mathbf{y}U'}^{G_!\Psi\|} G_*^{\Psi\|} \rhd_2 F_*^{\Psi\ltimes\mathbf{y}U\|} \text{\reflectbox{$\exists$}}_{\mathbf{y}U}^{\Psi\|}$
$\forall$	$\forall_{\mathbf{y}U'}^{G_!\Psi\|} \Omega^{\nu_!^{-1}\|} F_!^{\Psi\ltimes\mathbf{y}U\|} \leftarrow G_!^{\Psi\|} \forall_{\mathbf{y}U}^{\Psi\|}$	$\forall_{\mathbf{y}U}^{\Psi\|} F^{\Psi\|*} \cong G^{\Psi\|*} \forall_{\mathbf{y}U'}^{G_!\Psi\|} \Omega^{\nu_!^{-1}\|}$	$\forall_{\mathbf{y}U'}^{G_!\Psi\|} \Omega^{\nu_!^{-1}\|} F_*^{\Psi\ltimes\mathbf{y}U\|} \rhd_1 G_*^{\Psi\|} \forall_{\mathbf{y}U}^{\Psi\|}$
$\between$		$\between_{\mathbf{y}U}^{\Psi\|} G^{\Psi\|*} \leftarrow F^{\Psi\ltimes\mathbf{y}U\|*} \Omega^{\nu_!\|} \between_{\mathbf{y}U'}^{G_!\Psi\|}$	$\Omega^{\nu_!\|} \between_{\mathbf{y}U'}^{G_!\Psi\|} G_*^{\Psi\|} \cong F_*^{\Psi\ltimes\mathbf{y}U\|} \between_{\mathbf{y}U}^{\Psi\|}$

where any statement holds if the mentioned functors exist, and where

1. In general, $\rhd_1$ means $\to$ and $\rhd_2$ means nothing.
2. If $\mathcal{W} = \mathcal{V}$, $\mathcal{W}' = \mathcal{V}'$, $F = G$, both multipliers are cartesian and ν respects the first projection, i.e. $\pi_1 \circ \nu = G\pi_1$, then $\rhd_1$ upgrades to $\cong$ and $\rhd_2$ upgrades to $\to$. Note that in this case we have $GU \cong G(\top \times U) \cong_\nu G\top \times U'$.

Remark 6.4.2. In the above theorem, we think of F and G as similar functors; if we are dealing with endomultipliers, we will typically take $F = G$. The multipliers, however, will typically be different, as in general $U \not\cong FU$.

Proof. Since $\nu_!$ is an isomorphism, $\Sigma^{/\nu_!}$ is a strictly invertible functor with inverse $\Sigma^{/\nu_!^{-1}}$. Since $\sqcup^*$ is a 2-functor, $\Omega^{\nu_!|}$ is also strictly invertible with inverse $\Omega^{\nu_!^{-1}|}$. Because equivalences of categories are adjoint to their inverse, we get the chains of isomorphisms displayed.

1. The given commutation property in the base category follows immediately from the definitions and naturality of ν and its image under $\sqcup_!$. The rest of the table then follows by lemma 2.1.2.
2. We invoke theorem 6.2.1 with $\sigma = \pi_2 : \Psi \times \mathbf{y}U \to \Psi$. This yields $\Omega_{\mathbf{y}U}^{\Psi|} G^{\Psi|*} = G^{\Psi\times\mathbf{y}U|*} \Omega^{G_!\pi_2|}$. Now $G_!\pi_2 = \pi_2 \circ \nu_!$ so we can rewrite this to $\Omega_{\mathbf{y}U}^{\Psi|} G^{\Psi|*} = G^{\Psi\times\mathbf{y}U|*} \Omega^{\nu_!|} \Omega_{\mathbf{y}U'}^{G_!\Psi|}$. The rest of the table then follows by lemma 2.1.2. $\square$

6.5 Multiplier and multiplier

Theorem 6.5.1. Assume we have a commutative diagram (up to natural isomorphism $\nu : \sqcup \ltimes U \ltimes I \cong \sqcup \ltimes J \ltimes U'$) of multipliers

$$\begin{tikzcd} \mathcal{W} \arrow[r, "\sqcup\ltimes J"] \arrow[d, "\sqcup\ltimes U"'] & \mathcal{W}' \arrow[d, "\sqcup\ltimes U'"] \\ \mathcal{V} \arrow[r, "\sqcup\ltimes I"'] & \mathcal{V}'. \end{tikzcd} \tag{57}$$

Then we have the commutation table given in fig. 1 where every statement holds if the mentioned functors exist, and where

1. In general, $\rhd_1$ means $\to$, $\lhd^1$ means $\leftarrow$ and the other symbols mean nothing.
2. If $\mathcal{W} = \mathcal{W}'$, $\mathcal{V} = \mathcal{V}'$, $\sqcup \ltimes U = \sqcup \ltimes U'$, the multipliers $\sqcup \ltimes J$ and $\sqcup \ltimes I$ are cartesian and $(\pi_1 \ltimes U) \circ \nu = \pi_1 : (\sqcup \ltimes U) \times I \to \sqcup \ltimes U$, then $\lhd^1$ upgrades to $\cong$ and $\lhd^2$ upgrades to $\leftarrow$.
   (a) If moreover $\sqcup \ltimes U$ is cancellative and affine, then $\lhd^2$ upgrades to $\cong$ and $\lhd^3$ upgrades to $\leftarrow$.
3. The symbols $\rhd_i$ upgrade under symmetric conditions.

Proof.

1. In the base category, it is clear that $\text{\reflectbox{$\exists$}}_{U'}^{/\Psi\ltimes\mathbf{y}J} \text{\reflectbox{$\exists$}}_J^{/\Psi} \cong \Sigma^{/\nu_!} \text{\reflectbox{$\exists$}}_I^{/\Psi\ltimes\mathbf{y}U} \text{\reflectbox{$\exists$}}_U^{/\Psi}$. Applying the 2-functor $\sqcup^*$ yields the commutation law for $\forall$ and hence, by lemma 2.1.2, the general case.

2. We invoke theorem 6.3.1 with $\sigma = \pi_1 : \Psi \times \mathbf{y}J \to \Psi$. This yields

$$\Omega^{\Psi|}_{\mathbf{y}J} \forall^{\Psi|}_{\mathbf{y}U} = \forall^{\Psi \times \mathbf{y}J|}_{\mathbf{y}U} \Omega^{\pi_1 \ltimes \mathbf{y}U|} : \overbrace{\widehat{\mathcal{V}/\Psi \ltimes \mathbf{y}U}} \to \overbrace{\widehat{\mathcal{W}/\Psi \times \mathbf{y}J}}.$$

Now $\pi_1 \ltimes \mathbf{y}U = \pi_1 \circ \nu_!^{-1}$ so we can rewrite this to

$$\Omega^{\Psi|}_{\mathbf{y}J} \forall^{\Psi|}_{\mathbf{y}U} = \forall^{\Psi \times \mathbf{y}J|}_{\mathbf{y}U} \Omega^{\nu_!^{-1}|} \Omega^{\Psi \ltimes \mathbf{y}U|}_{\mathbf{y}I} : \overbrace{\widehat{\mathcal{V}/\Psi \ltimes \mathbf{y}U}} \to \overbrace{\widehat{\mathcal{W}/\Psi \times \mathbf{y}J}}.$$

The rest then follows by lemma 2.1.2.

   (a) Also follows from the same invocation of theorem 6.3.1.

3. By symmetry. □

7 Acknowledgements

We thank Jean-Philippe Bernardy, Lars Birkedal, Daniel Gratzer, Alex Kavvos, Magnus Baunsgaard Kristensen, Daniel Licata, Rasmus Ejlers Møgelberg and Andrea Vezzosi for relevant discussions. Special thanks to Dominique Devriese, whose pesky questions led to the current research.

References

[BCH14] Marc Bezem, Thierry Coquand, and Simon Huber. A Model of Type Theory in Cubical Sets. In *19th International Conference on Types for Proofs and Programs (TYPES 2013)*, volume 26, pages 107–128, Dagstuhl, Germany, 2014. URL: `http://drops.dagstuhl.de/opus/volltexte/2014/4628`, `doi:10.4230/LIPIcs.TYPES.2013.107`.

[BCM15] Jean-Philippe Bernardy, Thierry Coquand, and Guilhem Moulin. A presheaf model of parametric type theory. *Electron. Notes in Theor. Comput. Sci.*, 319:67 – 82, 2015. `doi:http://dx.doi.org/10.1016/j.entcs.2015.12.006`.

[BM20] Ales Bizjak and Rasmus Ejlers Møgelberg. Denotational semantics for guarded dependent type theory. *Math. Struct. Comput. Sci.*, 30(4):342–378, 2020. `doi:10.1017/S0960129520000080`.

[BT17] Simon Boulier and Nicolas Tabareau. Model structure on the universe in a two level type theory. Working paper or preprint, 2017. URL: `https://hal.archives-ouvertes.fr/hal-01579822`.

[CCHM17] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: A constructive interpretation of the univalence axiom. *FLAP*, 4(10):3127–3170, 2017. URL: `http://collegepublications.co.uk/ifcolog/?00019`.

[Mou16] Guilhem Moulin. *Internalizing Parametricity.* PhD thesis, Chalmers University of Technology, Sweden, 2016.

[ND18] Andreas Nuyts and Dominique Devriese. Degrees of relatedness: A unified framework for parametricity, irrelevance, ad hoc polymorphism, intersections, unions and algebra in dependent type theory. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*, pages 779–788, 2018. `doi:10.1145/3209108.3209119`.

[ND20] Andreas Nuyts and Dominique Devriese. Transpension: The right adjoint to the pi-type. *CoRR*, abs/2008.08533, 2020. URL: `https://arxiv.org/abs/2008.08533`, `arXiv:2008.08533`.

	$\exists_I, \exists_J$	$\Finv_I, \Finv_J$	$\forall_I, \forall_J$	$\between_I, \between_J$
$\exists_U$	$\exists^{\Psi\|}_{\mathbf{y}U}\exists^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I} \cong \exists^{\Psi\|}_{\mathbf{y}J}\exists^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\Omega^{\nu_!^{-1}\|}$	$\exists^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\Omega^{\nu_!^{-1}\|}\Finv^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I} \rhd_1 \Finv^{\Psi\|}_{\mathbf{y}J}\exists^{\Psi\|}_{\mathbf{y}U}$	$\exists^{\Psi\|}_{\mathbf{y}U}\forall^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I} \rhd_2 \forall^{\Psi\|}_{\mathbf{y}J}\exists^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\Omega^{\nu_!^{-1}\|}$	$\exists^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\Omega^{\nu_!^{-1}\|}\between^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I} \rhd_3 \between^{\Psi\|}_{\mathbf{y}J}\exists^{\Psi\|}_{\mathbf{y}U}$
$\Finv_U$	$\Finv^{\Psi\|}_{\mathbf{y}U}\exists^{\Psi\|}_{\mathbf{y}J} \lhd^1 \exists^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I}\Omega^{\nu_!\|}\Finv^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}$	$\Omega^{\nu_!\|}\Finv^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\Finv^{\Psi\|}_{\mathbf{y}J} \cong \Finv^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I}\Finv^{\Psi\|}_{\mathbf{y}U}$	$\Finv^{\Psi\|}_{\mathbf{y}U}\forall^{\Psi\|}_{\mathbf{y}J} \rhd_1 \forall^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I}\Omega^{\nu_!\|}\Finv^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}$	$\Omega^{\nu_!\|}\Finv^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\between^{\Psi\|}_{\mathbf{y}J} \rhd_2 \between^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I}\Finv^{\Psi\|}_{\mathbf{y}U}$
$\forall_U$	$\forall^{\Psi\|}_{\mathbf{y}U}\exists^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I} \lhd^2 \exists^{\Psi\|}_{\mathbf{y}J}\forall^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\Omega^{\nu_!^{-1}\|}$	$\forall^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\Omega^{\nu_!^{-1}\|}\Finv^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I} \lhd^1 \Finv^{\Psi\|}_{\mathbf{y}J}\forall^{\Psi\|}_{\mathbf{y}U}$	$\forall^{\Psi\|}_{\mathbf{y}U}\forall^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I} \cong \forall^{\Psi\|}_{\mathbf{y}J}\forall^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\Omega^{\nu_!^{-1}\|}$	$\forall^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\Omega^{\nu_!^{-1}\|}\between^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I} \rhd_1 \between^{\Psi\|}_{\mathbf{y}J}\forall^{\Psi\|}_{\mathbf{y}U}$
$\between_U$	$\between^{\Psi\|}_{\mathbf{y}U}\exists^{\Psi\|}_{\mathbf{y}J} \lhd^3 \exists^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I}\Omega^{\nu_!\|}\between^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}$	$\Omega^{\nu_!\|}\between^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\Finv^{\Psi\|}_{\mathbf{y}J} \lhd^2 \Finv^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I}\between^{\Psi\|}_{\mathbf{y}U}$	$\between^{\Psi\|}_{\mathbf{y}U}\forall^{\Psi\|}_{\mathbf{y}J} \lhd^1 \forall^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I}\Omega^{\nu_!\|}\between^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}$	$\Omega^{\nu_!\|}\between^{\Psi\ltimes\mathbf{y}J\|}_{\mathbf{y}U'}\between^{\Psi\|}_{\mathbf{y}J} \cong \between^{\Psi\ltimes\mathbf{y}U\|}_{\mathbf{y}I}\between^{\Psi\|}_{\mathbf{y}U}$

Figure 1: Commutation table for 2 multipliers (theorem 6.5.1).

[nLa21a] nLab authors. adjoint functor, August 2021. [Online; consulted revision 107]. URL: http://ncatlab.org/nlab/show/adjoint%20functor.

[nLa21b] nLab authors. parametric right adjoint, August 2021. [Online; consulted revision 15]. URL: http://ncatlab.org/nlab/show/parametric%20right%20adjoint.

[Nuy18] Andreas Nuyts. Presheaf models of relational modalities in dependent type theory. *CoRR*, abs/1805.08684, 2018. arXiv:1805.08684.

[Nuy20] Andreas Nuyts. *Contributions to Multimode and Presheaf Type Theory*. PhD thesis, KU Leuven, Belgium, 8 2020. URL: https://anuyts.github.io/files/phd.pdf.

[NVD17] Andreas Nuyts, Andrea Vezzosi, and Dominique Devriese. Parametric quantifiers for dependent type theory. *PACMPL*, 1(ICFP):32:1–32:29, 2017. URL: http://doi.acm.org/10.1145/3110276, doi:10.1145/3110276.

[PK20] Gun Pinyo and Nicolai Kraus. From Cubes to Twisted Cubes via Graph Morphisms in Type Theory. In Marc Bezem and Assia Mahboubi, editors, *25th International Conference on Types for Proofs and Programs (TYPES 2019)*, volume 175 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 5:1–5:18, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. URL: https://drops.dagstuhl.de/opus/volltexte/2020/13069, doi:10.4230/LIPIcs.TYPES.2019.5.

[Sta19] The Stacks Project Authors. Stacks project. http://stacks.math.columbia.edu, 2019. Tags 00VC and 00XF.

[XE16] Chuangjie Xu and Martín Escardó. Universes in sheaf models. Unpublished note, 2016.