Homomorphic Encryption for Developers

Unlocking data privacy with powerful cryptographic techniques

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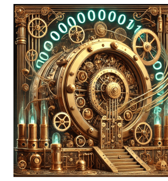
0.1 Introduction



Imagine you're building a healthcare app that needs to analyze patient data stored in the cloud. Since the data is sensitive, you encrypt it before sending it. However, every time you need to analyze the data, you have to decrypt it, which means the data is exposed and creates a security risk.

This is the main problem with traditional encryption systems like RSA[1] and AES[2]. They protect data while it's stored or sent, but as soon as you need to use the data, you have to decrypt it. It's like keeping money in a safe but needing to take it out every time you want to count it. This fundamental limitation makes it challenging to keep sensitive information secure throughout its lifecycle, especially as more applications rely on cloud computing, where the need for remote processing is common.

Homomorphic encryption (HE) aims to solve this problem by allowing data to remain encrypted even while it's being processed. It promises to make the cloud much safer for storing and analyzing data, which could have far-reaching impacts on healthcare, finance, and many other fields. Imagine being able to calculate the average income of a group of people without ever knowing how much any individual earns, that's the promise of HE.

0.1.1 The challenge with data security

Even when data is encrypted and stored in the cloud, there are still some risks:

1. Metadata exposure: Even if the data is encrypted, cloud providers can still see some information:
    - When the data is accessed.
    - How much data is being processed.
    - Patterns of usage that could reveal some details.

    Metadata may not contain the actual content of the data, but it can still provide insights that compromise privacy. For instance, frequent access to a medical record could imply a serious health condition, even if the actual diagnosis remains encrypted.

2. Trust issues: Cloud providers or intermediaries who have access to encryption keys could:
    - Access decrypted data when it's being processed.
    - Keep metadata even after the service ends.
    - Create privacy risks by storing information about data access, which could help them infer details even if the data itself is never fully decrypted

[1] The RSA algorithm is named after its inventors: Rivest, Shamir, and Adleman, who developed it in 1977. It is a widely-used asymmetric encryption method that relies on the computational difficulty of factoring large integers, currently enabling secure data transmission with a public key for encryption and a private key for decryption. Quantum computers can use Shor's algorithm to factor integers exponentially faster than classical algorithms, making RSA effectively insecure against quantum attacks. See: Rivest, R. L., Shamir, A., & Adleman, L. (1978). **A method for obtaining digital signatures and public-key cryptosystems**. *Communications of the ACM*, 21(2), 120–126. DOI, and Shor, P. W. (1994). **Algorithms for quantum computation: Discrete logarithms and factoring**. *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, 124–134. IEEE. DOI

[2] The AES algorithm (Advanced Encryption Standard) is a symmetric encryption standard established by the National Institute of Standards and Technology (NIST) in 2001, based on the Rijndael cipher designed by Joan Daemen and Vincent Rijmen. It is widely used for secure data encryption due to its speed and robustness. See: Daemen, J., & Rijmen, V. (1998). **Advanced Encryption Standard (AES) (FIPS PUB 197)**. *Federal Information Processing Standards Publications*. National Institute of Standards and Technology (NIST). Download. AES relies on the computational difficulty of brute-forcing keys, which requires trying all possible key combinations. Quantum computers can use **Grover's algorithm**, which provides a quadratic speedup for searching through possible keys. Instead of taking 2^n steps to brute-force an n-bit key, Grover's algorithm reduces it to approximately $2^{n/2}$ steps. This means that AES-128 (128-bit keys) would have the equivalent security of a 64-bit key against a quantum computer, making it potentially vulnerable. AES-256 is considered quantum-resistant for the foreseeable future because Grover's algorithm would reduce its effective strength to 2^{128}, which is still computationally infeasible. See: UK National Cyber Security Centre. **On the practical cost of Grover's algorithm for AES key recovery**. *Fifth PQC Standardization Conference*. Download

These issues highlight the importance of removing the need to trust third parties. HE can help solve this problem by ensuring that data remains encrypted, even when it's being analyzed.

0.1.2 Computing on encrypted data

Let's say Alice has some data m, and Bob has a function f. Alice wants to know the answer to $f(m)$:

- Traditional approach: Alice has to share m with Bob.

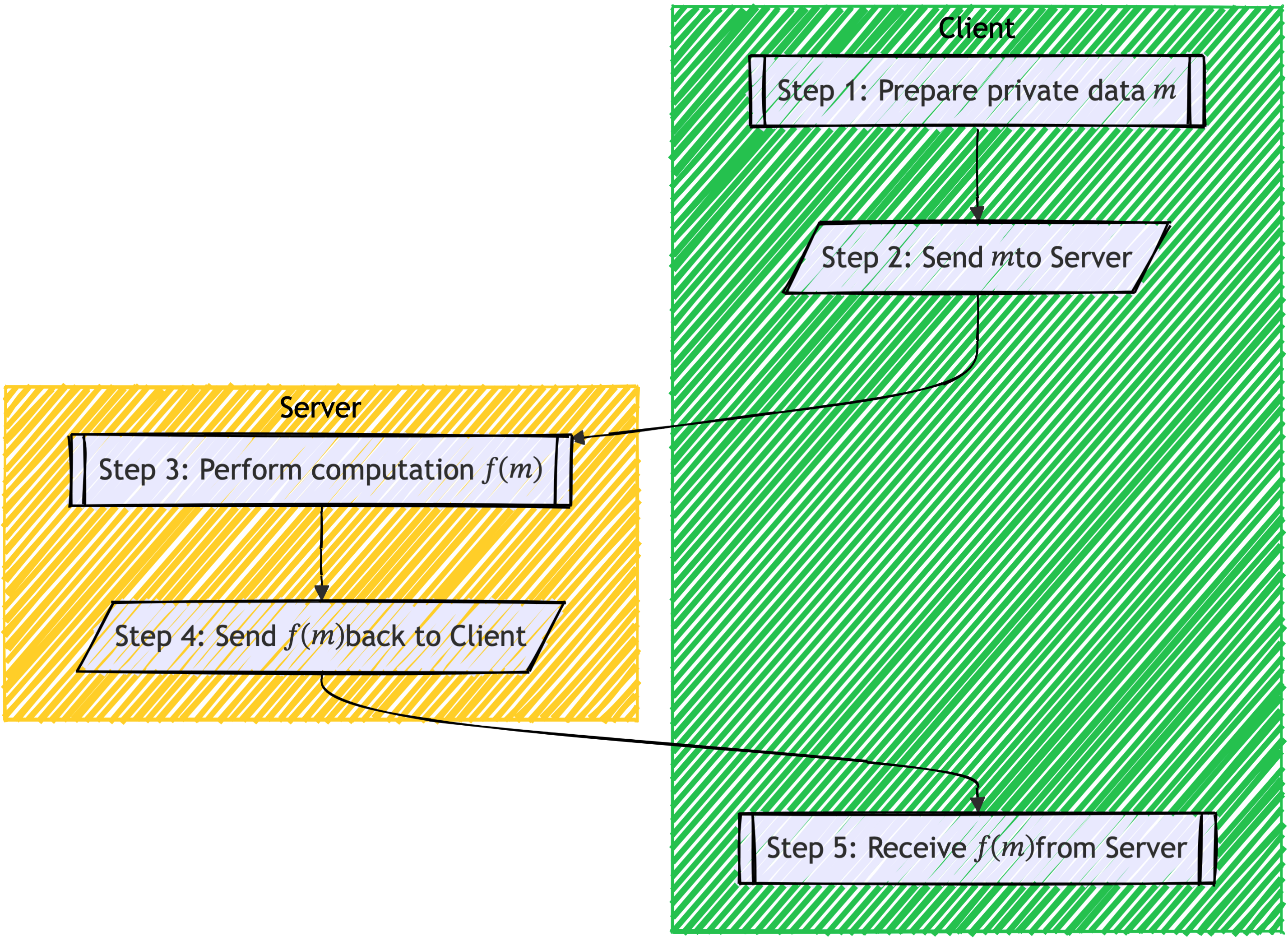
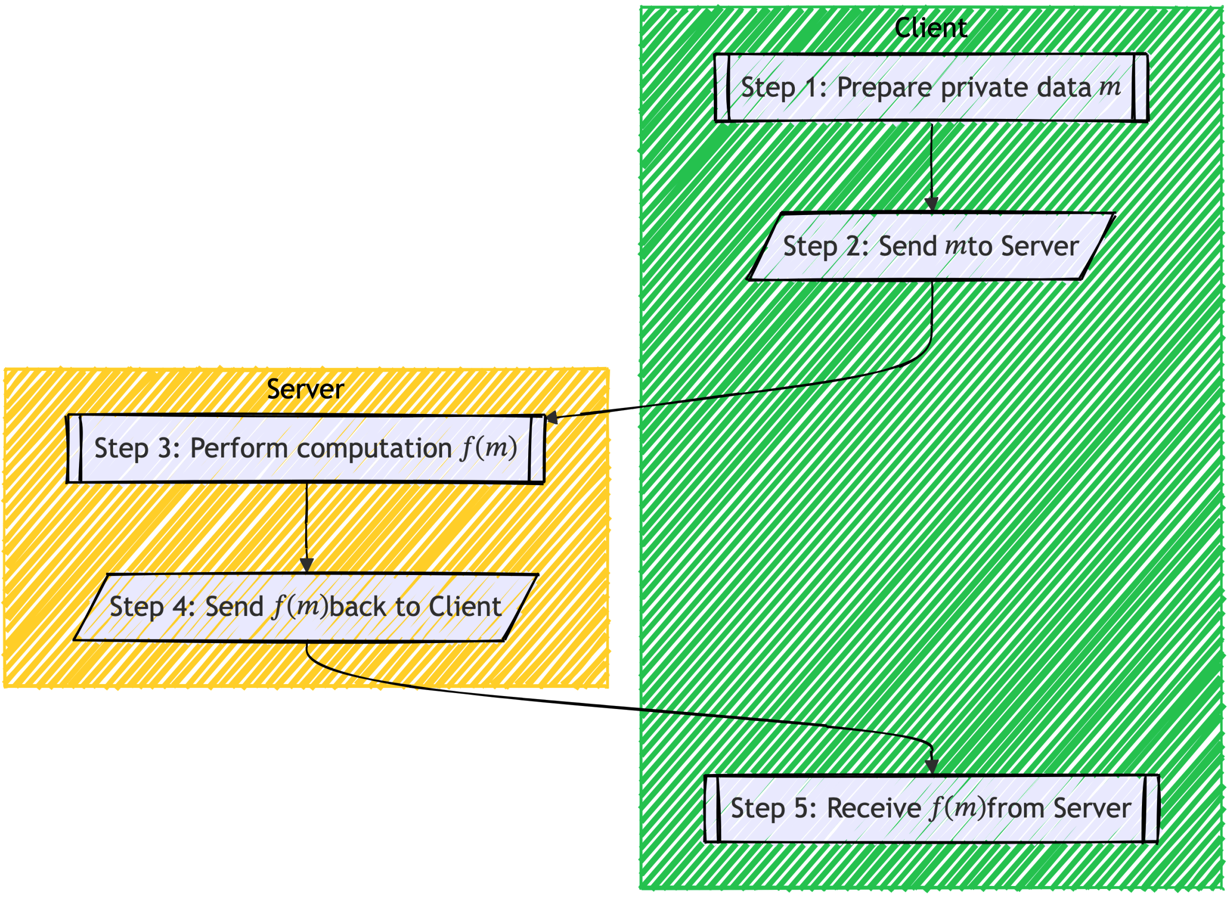


Figure 1: A simple client-server scenario for the traditional approach, where C is Client (Alice) and S is Server (Bob)



- HE approach: Alice sends an encrypted version of m to Bob, and Bob does the calculations on the encrypted data.

In traditional encryption, encrypted data can't be processed in any useful way. HE is different, because it keeps the relationships between numbers, even when they're encrypted. Here's a simple example:

- Let's say you have two numbers, a and b.
- You encrypt them to get $Enc(a)$ and $Enc(b)$.
- With HE, you can add $Enc(a)$ and $Enc(b)$ and get an encrypted result that, when decrypted, gives you $a + b$.

This means you can perform calculations on encrypted data without having to decrypt it first. The ability to compute on encrypted data without decryption is what makes HE so revolutionary. In essence, it allows data to stay secure throughout its entire lifecycle, from collection to storage to processing.

HE works by using complex mathematical operations that preserve the structure of the data even when it's encrypted. The mathematics behind this is quite advanced, involving abstract algebra and number theory. These mathematical techniques ensure that operations such as addition and multiplication can be performed on the encrypted data in a way that yields correct results when decrypted.

0.1.3 Semantic security and controlled malleability

HE is possible thanks to two key cryptographic concepts: **semantic security** and **controlled malleability**. While these might sound technical, they're not too hard to understand when broken down.

First, let's talk about semantic security. This property ensures that encrypted data reveals absolutely

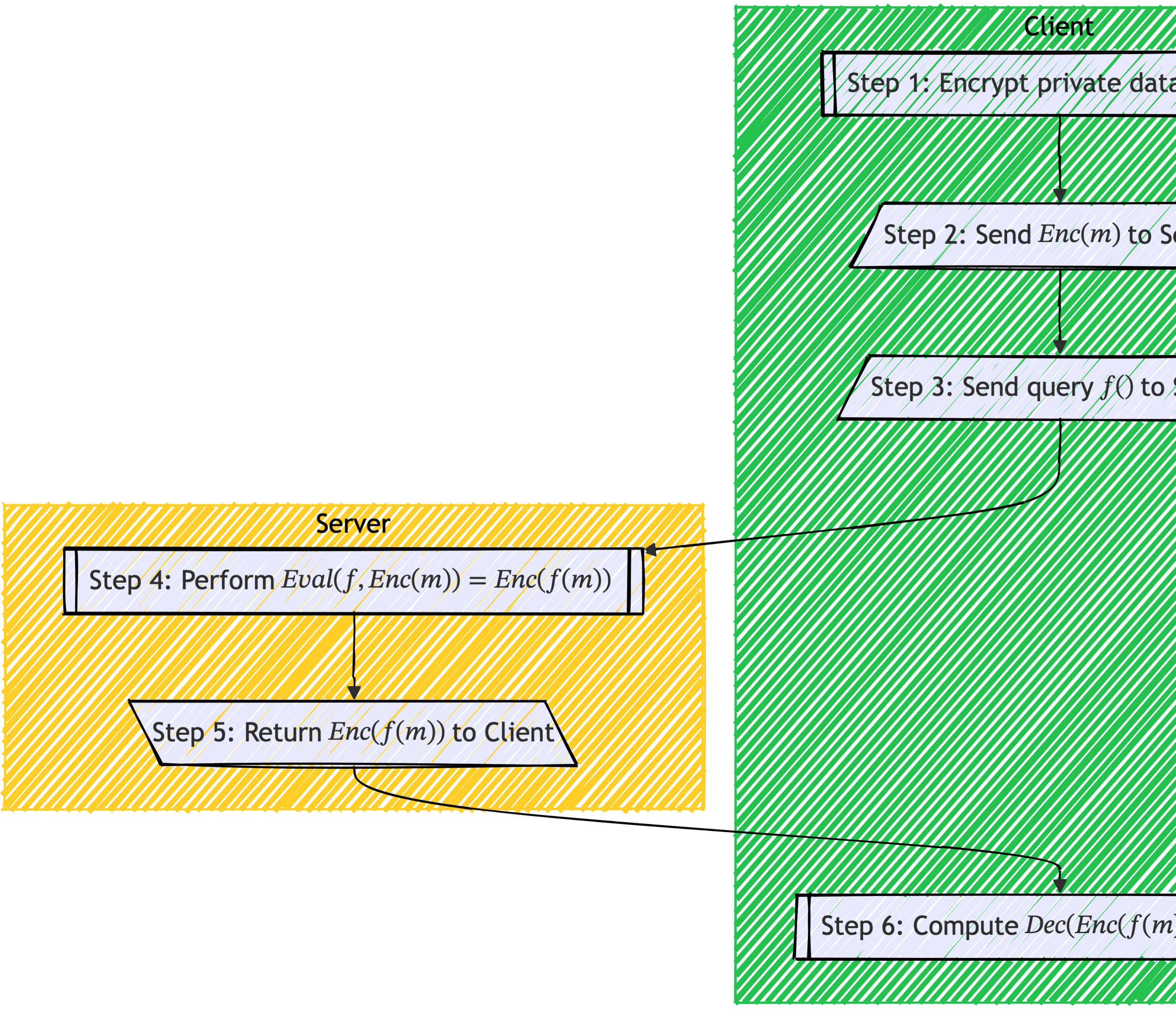


Figure 2: A simple client-server HE scenario, where C is Client (Alice) and S is Server (Bob)

nothing about the original data. For example, even if you encrypt the same message twice, the results will look completely different every time, like writing a note and hiding it in different locked boxes that look unique each time. This randomness makes it impossible for someone to guess the original message just by looking at the encrypted result. Semantic security is a cornerstone of most modern encryption schemes, such as AES for secure data storage and RSA for transmitting confidential messages over the internet. In these systems, semantic security ensures that an attacker cannot deduce the plaintext, even if they intercept encrypted messages.

Now, let's look at controlled malleability. Normally, encryption schemes are designed to prevent any modification of encrypted data. For example, in secure messaging or financial transactions, tampering with ciphertexts could lead to corruption or malicious alterations. This is why many encryption schemes aim to be non-malleable, ensuring ciphertexts cannot be manipulated in any meaningful way. However, some cryptographic protocols intentionally use a controlled form of malleability. For instance:

- RSA encryption supports a basic level of malleability, enabling certain transformations (e.g., multiplying ciphertexts) that correspond to transformations on the plaintext. This is leveraged in digital signatures and secure voting systems.
- Secure Multi-Party Computation (SMC) uses malleable properties to allow multiple parties to jointly compute a function over their inputs without revealing them to each other.

HE takes controlled malleability a step further by enabling a rich set of mathematical operations, such as additions and multiplications, to be performed directly on encrypted data. This means that encrypted data can be actively processed, opening up new possibilities for secure computation without exposing sensitive information.

By combining semantic security with controlled malleability, HE represents a powerful new paradigm in cryptography. While semantic security ensures that the original data remains completely hidden, controlled malleability allows computations on that hidden data in a secure and predictable way. Together, these concepts extend the boundaries of what encryption can achieve, enabling privacy-preserving technologies that go far beyond the limitations of traditional cryptographic schemes.

0.1.4 Types of HE

HE encompasses various schemes, each with distinct capabilities, applications, and a shared mathematical heritage that connects their evolution. These different types of HE have progressively built on one another, with each advancement adding new capabilities while maintaining foundational principles rooted in number theory and algebra.

1. **Partially Homomorphic Encryption (PHE):**
    - PHE supports a single type of operation, either addition or multiplication, on encrypted data, which offers high efficiency due to its limited operational scope.
    - Applications: Ideal for scenarios requiring only one type of computation. For instance, PHE is utilized in secure voting systems, where votes are encrypted and then aggregated (added) without decryption, ensuring voter privacy and data integrity.
    - Historical context: The concept of PHE dates back to 1978 with the introduction of the RSA algorithm, which supports multiplicative homomorphism. Subsequent schemes, such as the Paillier cryptosystem introduced in 1999, provided additive homomorphism, allowing for the addition of encrypted values. These early approaches laid the mathematical foundation for later, more complex forms of HE. The development of RSA was also a part of broader cryptographic breakthroughs in public-key cryptography, which fundamentally changed secure communication by allowing encryption without pre-shared keys.
    - Some notable examples:
        - RSA: Supports multiplication as the homomorphic operation.
        - Paillier: Addition.

- ElGamal: Multiplication.
- Goldwasser-Micali: XOR.
- Okamoto-Uchiyama: Addition.

2. **Somewhat Homomorphic Encryption (SWHE):**
   - SWHE enables both addition and multiplication operations but only up to a certain depth or number of operations. It balances between operational flexibility and computational efficiency, making it suitable for applications with limited computational requirements.
   - Applications: SWHE is applied in secure data aggregation, where a limited number of operations are performed on encrypted data to compute aggregate statistics without exposing individual data points.
   - Historical context: SWHE schemes emerged as researchers sought to extend the capabilities of PHE. By building on the foundational mathematics of PHE, these schemes introduced the ability to perform both additive and multiplicative operations, though with certain limitations. This progression marked an important step towards achieving fully HE. The development of SWHE was influenced by lattice-based cryptography, which also played a crucial role in providing security against quantum computing attacks, linking SWHE to advances in post-quantum cryptography.

3. **Fully Homomorphic Encryption (FHE):**
   - FHE allows an unlimited number of both addition and multiplication operations on encrypted data. While computationally intensive, FHE provides the most comprehensive functionality, enabling complex computations on encrypted datasets.
   - Applications: FHE is particularly valuable in privacy-preserving data processing, such as performing machine learning algorithms on encrypted medical records, allowing for data analysis without compromising patient confidentiality.
   - Historical context: The concept of FHE was first realized by Craig Gentry in 2009[3], marking a significant advancement in cryptography. Gentry's construction built upon the principles and challenges addressed by PHE and SWHE, demonstrating that it was possible to perform arbitrary computations on encrypted data without decryption. This breakthrough opened new avenues for secure data processing, rooted in the same mathematical lineage that began with PHE. Gentry's work was heavily influenced by the concept of ideal lattices and the use of bootstrapping, which allowed for refreshing encrypted data, a concept that is closely related to error correction techniques used in coding theory. FHE also contributed to advancements in multi-party computation and secure function evaluation, highlighting its relationship with other cryptographic fields focused on secure collaborative computing.

0.1.5 How HE enhances private computing

HE can be combined with other privacy techniques to keep data secure while still being able to use it. These techniques are independent but can work well together with HE to achieve privacy goals:

- **Differential Privacy (DP)**: DP[4] is a method that ensures individual data points in a dataset can't be identified, even if the results are analyzed multiple times. By adding noise to the output, DP protects people's privacy while still allowing useful insights to be gained from the data. HE can be combined with DP to keep data encrypted during analysis, while DP adds another layer of privacy. For example, a healthcare company could use HE to compute encrypted patient data and add DP to ensure that the output does not compromise individual identities.

[3] Gentry, C. (2009). **Fully homomorphic encryption using ideal lattices**. *Proceedings of the 41st Annual ACM Symposium on Theory of Computing*, 169–178. DOI

[4] Dwork, C., & Roth, A. (2014). **The algorithmic foundations of differential privacy**. *Foundations and Trends in Theoretical Computer Science*, 9(3–4), 211–407. DOI

- **Secure Multi-Party Computation (SMC)**: SMC[5] allows several parties to jointly compute a result from their inputs without revealing those inputs to each other. HE is often used in SMC to make sure the data stays encrypted throughout the computation. This way, everyone can contribute without giving up their private data. For example, multiple banks could jointly analyze data to detect fraud patterns without sharing individual customer information.
- **Zero-Knowledge Proofs (ZKPs)**: ZKPs[6] are a way to prove that a statement is true without revealing any other information beyond the fact that the statement is true. ZKPs can be combined with HE to verify computations on encrypted data without revealing any sensitive information. This is particularly useful in scenarios like blockchain, where privacy and verification are both important. For instance, ZKPs could allow someone to prove they have enough funds for a transaction without revealing their exact account balance.

0.1.6 Applications of HE

0.1.6.1 Public cloud services

Imagine a giant digital library that many people share—this is essentially what a public cloud service is. Services like Google Drive, Dropbox, Microsoft Azure, or any Software as a Service (SaaS) application, such as email platforms, social networks, or collaboration tools, are examples where many users store and process their data in the same place. It's like having your personal locker in a public gym—while you have your private space, you're still using a shared facility. The more "layers" or services your data interacts with, the greater the privacy risks become, as each layer can potentially expose your data to further vulnerabilities.

The challenge with public clouds is keeping your information private while still being able to use all the helpful features they offer. Think about it like this: you want to ask someone to count your money, but you don't want them to see how much you have. That's where HE comes in: it lets the cloud service work with your data without actually seeing what's in it.

Public cloud services are used for various purposes, including data storage, file sharing, and running applications remotely. The privacy challenge in public cloud services is significant, as many users want the benefits of powerful processing without sacrificing the confidentiality of their data. HE offers a groundbreaking solution, allowing computations to be performed while the data remains encrypted. This means users can get useful insights and results from their data without exposing it to the cloud provider or any unauthorized third party.

HE enables users to make the most of public cloud services without giving up their privacy. For example, organizations can store and process customer information, health records, and financial data without ever exposing sensitive information. This capability makes public cloud services more secure and suitable for a wide range of applications involving confidential data. Additionally, HE can help governments, businesses, and individuals alike to harness the full potential of cloud-based services without the fear of privacy breaches.

Moreover, HE provides a way for SaaS applications like email platforms and social networks to perform useful functions on user data while maintaining privacy. For instance, an email service could filter spam emails or provide automated categorization features without actually accessing the content of your emails. Similarly, a social network could analyze user preferences to deliver targeted content or enhance user experience, all while keeping personal data fully encrypted.

When using SaaS applications, data often passes through multiple "layers" of services, each adding to the potential privacy risks. These layers could involve data storage, processing, and analysis, all of which need to be handled with the utmost care. HE mitigates these risks by ensuring that data is encrypted throughout its entire journey—from storage to computation. This makes public cloud services and SaaS platforms much safer environments for processing sensitive information, as the data remains encrypted at every stage.

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[5] Yao, A. C. (1982). **Protocols for secure computations**. *23rd annual symposium on foundations of computer science (SFCS 1982)* (pp. 160-164). IEEE. DOI

[6] Goldwasser, S., Micali, S., & Rackoff, C. (1989). **The knowledge complexity of interactive proof systems**. *SIAM Journal on computing*, 18(1), 186-208. DOI

Real-world examples:

- Navigation apps: Helps you find your way without revealing where you are. Imagine telling someone, "I'm somewhere in New York" and getting directions without revealing your exact street corner. The privacy benefit is that your location stays secret while still getting accurate directions. HE allows navigation services to process your location data while keeping the exact coordinates hidden, ensuring your privacy while still providing efficient route guidance. This is especially important for users who are concerned about sharing their real-time location with third parties.
- Health monitoring Devices: Your smartwatch or fitness tracker can process your health data securely. It's like having a doctor analyze your health charts while they're in a sealed envelope. You get health insights while keeping your personal metrics private. Imagine that a health service aggregates data from thousands of users' fitness trackers to find patterns in sleep quality. HE allows this analysis while keeping every user's specific sleep data private, so the service can improve recommendations without compromising privacy. This means that even if the cloud service processes millions of health records, individual users' data remains secure and confidential.
- Personal finance: Gets insights from your data without exposing the details. Similar to having someone tell you if your spending is normal for your age group without seeing your actual purchases. You learn from your data while keeping it confidential. A budgeting app could use HE to compare a user's spending habits against aggregate data to provide personalized recommendations, all while keeping individual transactions encrypted and secure. For instance, the app could analyze spending trends, identify areas for improvement, and suggest budgeting strategies—all without ever accessing your raw financial data in a readable form.
- Email filtering: Modern email services often use filters to identify spam, categorize messages, and even detect potential phishing attacks. With HE, these services can perform all of these operations without having to read the content of your emails. This ensures that your private messages remain confidential while still benefiting from advanced filtering and organizational features. Imagine an email provider categorizing your emails into folders such as Promotions, Social, and Primary—all without actually knowing what the emails say.
- Social networks: Social media platforms often use algorithms to suggest content based on user behavior. With HE, these platforms can analyze user interactions, such as likes, comments, and shares, to provide tailored content recommendations, all while keeping user behavior encrypted. For example, if a social network wants to recommend friends or content, it can do so based on encrypted data, ensuring that your activity and preferences are kept private.
- Collaboration tools: SaaS collaboration tools like document editors or project management software can use HE to provide enhanced features while keeping user data private. Imagine multiple users collaborating on a shared document, HE can ensure that the document remains encrypted while allowing authorized users to make edits and comments. This is crucial for businesses that need to ensure confidentiality while leveraging the benefits of cloud-based collaboration.

HE represents a transformative approach to data privacy, particularly in the context of public cloud services and SaaS applications. However, as the usage of digital services continues to expand, the potential for data misuse also grows, posing significant risks to both individuals and companies. Data can be weaponized for malicious purposes, from targeted disinformation to financial exploitation, and traditional privacy measures, such as DP, may not be sufficient to fully protect sensitive information in these evolving digital landscapes. DP, while effective at masking individual contributions in datasets, often relies on the careful calibration of privacy budgets and noise, which can degrade utility or be insufficient against sophisticated attacks like reconstruction or linkage attacks, where adversaries can leverage external datasets to infer private information. HE, on the other hand, offers a promising solution by enabling computation on encrypted data without ever exposing it, providing a stronger safeguard against these emerging threats.

0.1.6.2 Private cloud computing

Private cloud computing provides organizations with greater control over their data and infrastructure compared to public cloud environments. This model is particularly suitable for handling sensitive information but requires a sophisticated, defense-in-depth approach to maintain data privacy and security throughout its lifecycle.

Private clouds are often employed by organizations that need to comply with stringent regulatory requirements, such as those related to healthcare, finance, or government operations. These regulations, including standards like HIPAA, GDPR, NIS2, and PCI DSS, mandate strict data protection protocols and require demonstrable security controls and audit trails.

Despite the advantages of private clouds, they remain susceptible to various threats across different layers of the technology stack. Infrastructure layer threats include software vulnerabilities in virtualization platforms, hypervisors, or orchestration tools, which can lead to risks such as privilege escalation or remote code execution (RCE). Hardware vulnerabilities, such as side-channel attacks exploiting cache timing, power analysis, or electromagnetic emanations, also pose significant risks. Physical security concerns, such as cold boot attacks and DMA attacks, along with supply chain vulnerabilities in hardware components or firmware, further complicate the security landscape.

Network layer threats include attacks such as ARP poisoning, VLAN hopping, and compromises of software-defined networking (SDN) controllers. Weaknesses in virtual network functions (VNFs) and east-west traffic attacks between workloads within the cloud are also notable vulnerabilities.

Application layer threats involve issues like API security vulnerabilities, container escape risks that allow attackers to move from containers to host systems, weaknesses in securing microservice interactions, and data leakage through application logic flaws.

Human and operational threats are also significant. Configuration drift and misconfigurations can lead to gradual deviation from secure states, while inadequate privilege management and insider threats (both malicious and unintentional) can compromise security. Operational security failures, such as lapses in maintaining secure practices, are also critical factors that must be addressed.

To mitigate these risks, organizations need a comprehensive, multi-layered security strategy that implements defense-in-depth through multiple complementary technologies. HE serves as one critical component within this broader security architecture, particularly for protecting data confidentiality during processing. Various cryptographic and security measures work together as follows:

- In the foundational security layer, hardware security modules (HSMs) are used for key management, providing secure storage and handling of cryptographic keys which are crucial for HE operations. Trusted platform modules (TPMs) ensure boot integrity, establishing a trusted baseline for secure operations, which is essential for protecting the integrity of encrypted data processed using HE. Secure boot and measured boot processes protect the system from boot-level attacks, creating a secure foundation for any HE-related operations. Physical security controls and monitoring provide physical safeguards for cloud hardware, preventing physical attacks that could compromise the hardware used to perform HE computations.
- In the network security layer, microsegmentation with zero-trust principles limits lateral movement within the network, ensuring that even if an attacker gains access, they cannot reach the nodes performing HE computations. Virtual network encryption ensures data confidentiality across virtual networks, which complements HE by protecting data during transit, even before or after HE-based processing. Network access control with 802.1x enforces authentication for devices on the network, preventing unauthorized devices from accessing data that may be encrypted using HE. SDN security, involving the separation of control and data planes, helps mitigate vulnerabilities within SDN environments, providing a secure pathway for the data to be processed using HE without risking exposure.
- For data in transit, Transport Layer Security (TLS) 1.3 with perfect forward secrecy protects data from interception, while IPsec provides network-level encryption, ensuring that data remains secure during transmission before and after HE operations. For data at rest, AES-256 encryption with secure key management protects stored data from unauthorized access, complementing HE by providing

strong encryption when data is not actively being processed. Format-preserving encryption is used for structured data, allowing for HE-based operations to occur without altering the structure of sensitive datasets, which is particularly useful for preserving data integrity while performing encrypted computations.

- For data in use, HE is combined with Trusted Execution Environments (TEEs)[7] for enhanced data protection during processing. TEEs provide a secure, isolated hardware environment for executing sensitive operations, protecting against unauthorized access by ensuring that data and computations are shielded from other processes on the system. HE further enhances this by keeping the data encrypted even within the TEE, ensuring that even if the secure environment is compromised, the data remains confidential.
- SMC is also employed for collaborative computations without revealing individual inputs. Advanced integrations include using HE with Intel SGX for secure computation spaces, hybrid HE-MPC protocols for efficient distributed computing, and memory encryption with AMD SEV or Intel TDX for enhanced data protection.
- HE can also be integrated with Attribute-Based Encryption (ABE)[8] to allow fine-grained access control, ensuring that data access is granted only to users with specific attributes or roles. Identity-Based Encryption (IBE)[9] simplifies key management by allowing public keys to be derived from unique user identifiers, reducing the complexity of certificate distribution (Boneh, D., & Franklin, M., 2001). ZKPs provide anonymous authentication, allowing users to prove their identity or access rights without revealing any underlying sensitive information. By combining these techniques, HE ensures that data remains encrypted throughout its lifecycle while still allowing flexible and secure access management, simplified key handling, and privacy-preserving authentication.

This layered approach ensures that HE is not deployed in isolation but rather as part of a comprehensive security architecture where each component strengthens the overall security posture. The combination of these technologies provides defense-in-depth while addressing specific threats at each layer of the infrastructure.

Real-world examples:

- Medical research: HE, when combined with AES-256 encryption and TEEs, allows hospitals to study patient data while maintaining privacy. Within the private cloud, patient data is securely stored using AES-256 encryption and processed within TEEs, while HE allows computations on encrypted data without decryption. For example, doctors can analyze medical images with patient details encrypted and isolated, enabling researchers to identify important patterns without seeing individual patient information. When data needs to be shared across institutions, SMC is used to ensure data privacy, thereby identifying effective treatments and new drug opportunities while ensuring patient privacy.
- Financial services: In the private cloud, financial institutions store customer data encrypted using AES-256 and conduct computations using HE combined with TEEs. TLS ensures data confidentiality when it moves in and out of the private cloud. HE, in combination with TLS for data in transit and TEEs for processing, helps financial institutions process banking information while keeping account details secret. Banks can use HE to assess loan applications by running risk analyses on encrypted financial data within TEEs, enabling automated decision-making without exposing customers' financial histories. This combination ensures data remains confidential throughout its lifecycle, from transmission to analysis.
- Defense sector: Within a private cloud environment, sensitive defense-related data is encrypted with AES-256 and processed securely using HE and TEEs. For example, a remote-controlled drone can

[7] McKeen, F., Alexandrovich, I., Berenzon, A., Rozas, C., Shafi, H., Shanbhogue, V., & Savagaonkar, U. R. (2013). **Innovative instructions and software model for isolated execution**. *Proceedings of the 2nd International Workshop on Hardware and Architectural Support for Security and Privacy (HASP)*. https://doi.org/10.1145/2487726.2488368

[8] Goyal, V., Pandey, O., Sahai, A., & Waters, B. (2006). **Attribute-based encryption for fine-grained access control of encrypted data**. *Proceedings of the 13th ACM Conference on Computer and Communications Security (CCS)*, 89-98. DOI

[9] Boneh, D., & Franklin, M. (2001). **Identity-based encryption from the Weil pairing**. *SIAM Journal on Computing*, 32(3), 586-615. DOI

perform target calculations using HE while ensuring that even if intercepted, the encrypted data and computations remain confidential, safeguarding operational integrity. Logistics data can also be analyzed collaboratively among trusted partners using SMPC without revealing the underlying sensitive information, ensuring data privacy and safeguarding national security interests. TLS and IPsec are used to protect data that enters or exits the private cloud, ensuring that no sensitive information is exposed during transmission.

0.1.6.3 Blockchain technology

Blockchain technology can be thought of as a digital ledger that everyone can see—like a giant spreadsheet that tracks transactions. The challenge is: how do you keep certain details private on this public ledger? It's similar to wanting to tell people you bought something without revealing how much you paid for it.

Blockchain technology is known for its transparency and security, which are useful for verifying transactions. However, this transparency also creates a privacy challenge. To address this, HE, ZKPs, and SMC are employed to protect sensitive information while maintaining the integrity and verifiability of blockchain data.

0.1.6.3.1 HE, ZKPs, and SMC

HE ensures that sensitive information remains protected throughout the process. In blockchain systems, this is crucial for maintaining privacy without compromising the ability to verify data integrity. For example, HE can be used to perform operations on encrypted transaction details, such as calculating total transaction amounts or processing smart contract conditions, enabling stakeholders to verify outcomes without seeing the underlying sensitive data. In privacy-focused Layer 2 solutions on Ethereum, HE can be applied to compute transaction fees or aggregate user balances in encrypted form, maintaining both privacy and scalability. Similarly, in blockchain-based supply chain systems, HE enables participants to encrypt transaction details before adding them to the blockchain, ensuring that sensitive information (like pricing or quantities) remains hidden while the overall process can still be verified by stakeholders. This privacy-preserving transparency is crucial in competitive environments, allowing stakeholders to verify product provenance without exposing confidential business information.

ZKPs are leveraged in blockchain to enhance privacy by allowing parties to prove that certain statements are true without revealing specific information. In supply chain scenarios, ZKPs can prove that specific procedures were followed or quality standards were met without disclosing proprietary details. This ensures compliance while maintaining confidentiality. In digital identity verification, ZKPs allow individuals to prove attributes of their identity (such as being of legal age) without exposing their full identity or birthdate, ensuring privacy and compliance.

SMC is leveraged to enable collaborative decision-making or data aggregation on the blockchain without exposing individual inputs. This is particularly useful in decentralized finance (DeFi) platforms or voting mechanisms within decentralized governance systems. For instance, in Decentralized Autonomous Organizations (DAOs), SMC allows members to collectively compute outcomes (such as voting results) while keeping individual votes private, ensuring both transparency and privacy in the decision-making process.

Both HE and ZKPs aim to preserve privacy while proving computation correctness. They are often used together to enhance privacy in blockchain systems. For instance, HE can encrypt inputs while ZKPs prove the correctness of computations on these encrypted inputs. zk-SNARKs (Zero-Knowledge Succinct Non-Interactive Arguments of Knowledge) can also be used to prove the correct execution of homomorphic operations, providing efficient and verifiable computations. Hybrid protocols that combine HE and ZKPs create efficient, private smart contracts where the correctness of encrypted computations is guaranteed without revealing sensitive information.

SMC and HE are complementary technologies for performing private computations on blockchain. HE can be integrated within SMC protocols to reduce the number of communication rounds required, leading to more efficient computations. Hybrid protocols that combine FHE and SMC provide improved performance and security in blockchain applications. For example, SMC and HE are used together in threshold cryptography implementations to enable secure collaborative decision-making and private data aggregation, while ensuring sensitive information remains confidential.

0.1.6.3.2 Other cryptographic techniques

The following cryptographic techniques share a common foundation in supporting privacy-preserving, hidden but verifiable computations on blockchain. These methods are often combined to enhance privacy, security, and efficiency in blockchain systems:

- **Commitment schemes and HE**: A commitment scheme[10] is a cryptographic protocol that allows one party to commit to a chosen value while keeping it hidden from others, with the ability to reveal the value later. It ensures both secrecy and the ability to verify the commitment, which is essential for many blockchain applications. Commitment schemes and HE support hidden but verifiable computations on blockchain. Homomorphic commitments allow computations to be performed on committed values without revealing them, which can be combined with HE for verifiable encrypted computations. This combination is particularly useful in confidential transaction protocols, where participants need to commit to transaction values while still allowing certain operations to be verified.
- **Threshold cryptography and HE**: Threshold cryptography[11] is a cryptographic approach in which a secret is divided into multiple parts, and a predefined number (or threshold) of those parts is required to reconstruct the secret. This approach ensures security by distributing control among several parties, reducing the risk of a single point of failure. In blockchain, threshold cryptography can be used for distributed key generation, ensuring that no single entity has full access to sensitive information, thereby enhancing security and resilience in systems like multi-signature wallets or decentralized voting. HE shares common mathematical foundations with threshold cryptography. Threshold Fully Homomorphic Encryption (TFHE)[12] schemes allow distributed key generation and secure computations among multiple parties without revealing individual contributions. Multi-key HE is another application, enabling secure distributed computations while ensuring privacy. These techniques can also be used for shared decryption of homomorphically processed data, ensuring that no single participant can access the data in its entirety.
- **Ring signatures and HE**: A ring signature[13] is a type of digital signature that allows a member of a group to sign a message on behalf of the group, without revealing which specific member signed it. This provides anonymity for the signer while still proving that they are part of the group. HE and ring signatures are used together to support privacy-preserving operations on blockchain. For example, they can be combined to develop privacy-preserving voting schemes where votes are encrypted using HE, while ring signatures provide anonymity. They can also be used in anonymous credential systems where user attributes are encrypted, supporting confidential transactions without revealing individual identities.

0.1.6.3.3 Real-world applications

The integration of advanced cryptographic techniques into blockchain technology enables various real-world applications that enhance privacy, security, and transparency. Below are examples of how these techniques are used in practice:

- Supply chain management (Ethereum-based systems): In blockchain-based supply chain systems, HE can keep transaction details private while allowing stakeholders to verify the authenticity and origin of goods. For example, in a global supply chain where manufacturers, suppliers, and logistics providers contribute information about a product's journey, HE ensures that while the overall process can be

[10] Brassard, G., Chaum, D., & Crépeau, C. (1988). **Minimum disclosure proofs of knowledge**. *Journal of Computer and System Sciences*, 37(2), 156-189. DOI

[11] Desmedt, Y. (1994). **Threshold cryptography**. *European Transactions on Telecommunications*, 5(4), 449-457. DOI

[12] Asharov, G., Jain, A., López-Alt, A., Tromer, E., Vaikuntanathan, V., & Wichs, D. (2012). **Multiparty computation with low communication, computation and interaction via threshold FHE**. *Advances in Cryptology–EUROCRYPT 2012* (pp. 483-501). Springer, Berlin, Heidelberg. DOI

[13] Rivest, R. L., Shamir, A., & Tauman, Y. (2001). **How to leak a secret**. *International Conference on the Theory and Application of Cryptology and Information Security* (pp. 552-565). Springer, Berlin, Heidelberg. DOI

verified, no sensitive information (like supplier pricing or quantities) is exposed to unauthorized parties. ZKPs further enhance privacy by allowing parties to prove they followed specific procedures or met quality standards without disclosing proprietary details. These technologies ensure compliance and transparency while maintaining competitive confidentiality.

- Digital identity verification (Algorand blockchain): HE is used to allow individuals to prove aspects of their identity without revealing unnecessary information. For instance, a person can prove they are of legal drinking age without revealing their birthdate using a blockchain-based identity verification system. ZKPs are also used in this scenario to validate identity attributes securely, ensuring privacy while maintaining compliance with regulations.
- Decentralized marketplace transactions (Ethereum Layer 2 solutions): Buyers and sellers in a decentralized marketplace can use HE to conduct transactions privately, keeping details like transaction amounts or account balances confidential. For example, a user buying digital art can make payments using HE, ensuring that neither the marketplace nor any third parties can access their financial details.
- Real estate transactions via Smart Contracts (Hyperledger Fabric): In a real estate transaction conducted through a smart contract, HE can be used to keep payment amounts and identities confidential while executing securely on the blockchain. This ensures compliance with local regulations while maintaining privacy for both buyers and sellers.
- Luxury goods supply chain (VeChain): A luxury goods manufacturer may use blockchain to track the journey of products from factory to retailer. HE would keep sensitive details like supplier pricing confidential while providing proof of authenticity to consumers. For example, a watch manufacturer might leverage HE to ensure that authenticity data is available to buyers while keeping internal processes private.
- Age verification for digital services (Cardano blockchain): Using HE, a user can prove they are above the legal age to access age-restricted products without revealing their full identity. A blockchain-based gaming platform could use HE to verify users' ages while protecting personal data from exposure.
- National election voting system (Tezos blockchain): In a national election using blockchain, HE keeps voter identities and preferences confidential while allowing an accurate vote count. Voters can cast their ballots online through a secure blockchain-based voting system, ensuring that individual privacy is maintained while the results remain transparent and trustworthy.
- DAO voting (Ethereum-based DAOs): In DAOs where members vote on proposals, HE allows each vote to remain encrypted while ensuring accuracy in vote counting. This is particularly useful for DAOs managing decentralized funds, where members vote on fund allocation without revealing individual preferences.

0.1.6.4 Secure data operations

Secure data operations involve multiple organizations working together with their data while keeping individual information private. In Federated Learning[14] (FL) scenarios, where multiple entities (e.g., hospitals, financial institutions) collaborate on training a machine learning model without sharing raw data, HE and DP play crucial roles in preserving privacy throughout the process. Each participant retains control of their

[14] FL, introduced by McMahan et al. (2017), represents a paradigm shift in machine learning by enabling model training on decentralized data. This approach was developed to address the growing privacy concerns and regulatory requirements around data protection while maintaining the benefits of large-scale machine learning. The key innovation of FL lies in its "bring the code to the data" rather than "bring the data to the code" approach. In the FL framework, instead of collecting raw data from users' devices, the model itself travels to where the data resides. Local models are trained on individual devices, and only model updates are shared with the central server, never the raw data. The paper defined the federated averaging (FedAvg) algorithm, which remains the foundation for many modern FL systems. The authors demonstrated that their approach could train deep neural networks using unbalanced and non-IID (Independent and Identically Distributed) data distributed across millions of mobile devices. See: McMahan, H. B., Moore, E., Ramage, D., Hampson, S., & y Arcas, B. A. (2017). **Communication-efficient learning of deep networks from decentralized data**. *Proceedings of the 20th International Conference on Artificial Intelligence and Statistics (AISTATS)*. Download

data and only shares encrypted model updates or contributions, which are combined to produce a global model.

DP works in tandem with HE to ensure privacy by adding random noise to the final results. This makes it difficult to determine if an individual's data is part of the dataset or not. Imagine you are trying to guess the favorite fruit of a group of people, but you cannot be certain about any single person's choice because a bit of randomness is added to their answers. This randomness helps protect individual privacy while still allowing you to make general conclusions about the group.

- Adding noise: DP adds noise to the final results of computations so that individual contributions are hidden. This noise is carefully controlled to strike a balance between privacy and accuracy.
- Privacy budget: The privacy budget, represented by ϵ, controls how much noise is added. A smaller ϵ means more noise and greater privacy, but less accurate results. Conversely, a larger ϵ means less noise, resulting in more accuracy but reduced privacy.
- Mathematical definition: DP ensures that the results of computations are nearly identical, regardless of whether an individual is included in the dataset. This is achieved through the privacy budget ϵ, which limits the amount of information that can be inferred about any single data point. The smaller the value of ϵ, the stronger the privacy protection, as it reduces the likelihood that an individual's data can be distinguished in the output.

The integration of HE and DP technologies creates a multi-layered privacy framework that enhances privacy at different stages of the data lifecycle:

1. Initial data protection: Each participating organization encrypts its data using HE, ensuring the raw data remains secure even during computations. For instance, in FL, each hospital encrypts patient data so that it never leaves the hospital in a readable form.
2. Secure computation: Using HE, model updates are computed directly on encrypted data. For example, in training a machine learning model, HE allows hospitals to calculate model updates without decrypting patient data. All computations are performed while the data is encrypted, ensuring no sensitive information is exposed.
3. Privacy-preserving output: After computations, DP adds controlled noise to the model updates to prevent inference attacks. The privacy budget ϵ is tracked across training iterations to ensure cumulative privacy loss remains acceptable, meaning that the privacy of individual data points is still maintained.

0.1.6.4.1 Privacy budget management

The privacy budget management becomes more sophisticated when combining HE and DP. Advanced composition theorems help manage privacy loss in repeated operations:

- Basic composition: Every time a query is made on the data, some privacy is lost. Basic composition means that the total privacy loss simply adds up for each query.
- Advanced composition: Privacy loss grows more slowly (with the square root of the number of queries), which helps limit the total loss.
- Moments accountant: This technique provides even tighter privacy control, especially for scenarios like machine learning, where many computations need to be performed. It allows the privacy budget to be managed more efficiently.

In practice, organizations can achieve strong privacy guarantees while still getting useful results. For example, with , there is strong privacy protection with only a small chance of leaking information, and the resulting analysis typically has an error of 1-10%, which is acceptable for most real-world uses.

0.1.6.4.2 Advanced protocols

The combination of HE and DP also enables advanced protocols, such as:

- Private Set Intersection with DP guarantees: Imagine two organizations wanting to compare customer lists without revealing all their data to each other. Private Set Intersection allows them to find common customers while using DP to ensure no extra information is leaked.
- Secure aggregation: Multiple parties can contribute encrypted data, and the aggregate result can be computed without revealing the individual contributions. DP ensures that even if the aggregate result is shared, the privacy of individual contributors is preserved.
- Privacy-preserving machine learning: This approach allows models to be trained using data from different organizations while ensuring data privacy. HE ensures data is never decrypted, while DP guarantees that the trained model does not reveal any individual's data.

When implementing these techniques, several practical considerations must be addressed:

- Performance optimization:
  - Batching homomorphic operations: Performing many homomorphic operations together can make them more efficient, helping to manage the increased computational cost of using HE.
  - Optimizing noise addition: Adding noise carefully helps maintain data utility while preserving privacy.
  - Managing computational overhead: HE and DP both introduce computational complexity. Efficiently managing this overhead is critical to make these privacy-preserving techniques practical.
- Security parameters:
  - Key size selection: Choosing the right key size for HE is important. Larger keys provide stronger security but also increase computational cost.
  - Noise parameter tuning: DP requires careful tuning of noise parameters to ensure privacy without losing too much accuracy.
  - Privacy budget allocation: Allocating the privacy budget effectively helps balance the level of privacy protection with the need for accurate results.
- Protocol design:
  - Communication efficiency: In FL, communication efficiency is crucial since participants need to exchange encrypted model updates.
  - Error handling: Noise and ciphertext expansion can introduce errors, which need to be managed to ensure accurate results.
  - Protocol composition: Combining different privacy-preserving techniques requires careful protocol design to maintain privacy guarantees throughout complex workflows.

These technical foundations enable organizations to implement robust privacy-preserving data operations while maintaining precise control over privacy guarantees and computational efficiency. The framework provides mathematical certainty about privacy protection while enabling valuable data analysis and collaboration.

0.1.6.4.3 Real-world applications

- Joint medical research: Multiple hospitals can use HE and DP to collaborate on research involving sensitive patient data, such as detecting trends in rare diseases. Each hospital encrypts its patient records, and encrypted datasets are analyzed together to identify emerging health issues without compromising patient confidentiality. After computation, DP ensures that individual patient contributions are hidden by adding noise to the model updates, ensuring privacy. For example, HE can be used to detect early indicators of a rare genetic disorder by combining encrypted datasets from various hospitals, while DP prevents any single patient's data from being identified in the final results.

- Corporate surveys: HE can be used to perform privacy-preserving surveys across companies in a specific industry to compare salary ranges or employee satisfaction without sharing individual responses. Each company's data is encrypted before submission, and the combined analysis reveals industry trends while keeping each company's data private. DP is used to add noise to the aggregated survey results, ensuring that individual responses cannot be inferred, even if someone tries to analyze the outputs in detail.
- Financial fraud detection consortium: Banks can collaborate to detect fraud patterns by sharing encrypted transaction records. HE allows the encrypted data to be analyzed collectively to identify unusual patterns across multiple institutions. DP is applied to the final aggregated fraud detection results to ensure that no single bank's customer data can be inferred from the analysis. For instance, encrypted datasets can be used to spot potential fraud schemes involving cross-bank transactions without compromising any bank's customer data.
- Government resource auctions: Governments can use HE in auctions for spectrum licenses or natural resources. Participants submit their bids in an encrypted form, ensuring that their bidding strategy is kept secret. DP adds an additional layer of privacy by ensuring that even the aggregated bidding data cannot reveal individual bidding strategies. Only the winning bid is revealed at the end, preserving fairness and confidentiality throughout the auction process.
- Collaborative pharmaceutical research: Pharmaceutical companies can collaborate to analyze clinical trial data securely. HE allows them to combine and analyze encrypted datasets from multiple trials, enhancing the ability to identify effective treatments faster. DP adds noise to the outputs, ensuring that the results cannot be traced back to any individual patient in the clinical trials. This helps companies work together on drug development without exposing sensitive patient data.
- Cross-border health data analysis: During public health crises, different countries' health agencies can use HE to securely share and analyze encrypted health data. For example, during a pandemic, agencies can combine encrypted data on infection rates, hospital capacity, and resources needed. DP ensures that the final combined results maintain privacy, so that individual contributions from specific regions cannot be identified, ensuring coordinated responses while maintaining privacy across borders.
- Collaborative risk assessment for insurance: Insurance companies can share encrypted claims data to develop better risk models that help predict and price insurance products. HE allows insurers to perform calculations on encrypted claims data, and DP adds noise to the resulting models, preventing any individual customer's claims data from being exposed. For instance, multiple insurers can securely collaborate to build risk prediction models for natural disasters while keeping individual customer claims data confidential.

0.1.6.5 Private Information Retrieval

Private Information Retrieval (PIR) is a cryptographic technique that allows a client to retrieve data from a large database held by a server without revealing which specific piece of data is being requested. More formally, PIR ensures that the query sent by the client does not leak any information to the server about the data being retrieved, while still enabling the server to provide the correct response.

PIR is especially useful in situations where privacy is crucial, such as when accessing large public databases or confidential corporate data. It allows users to perform queries without revealing their interests or compromising their privacy. This ensures that sensitive information remains confidential, even when interacting with third-party databases, thereby enhancing both security and user trust.

HE has had a profound impact on the evolution of PIR, particularly by enabling more efficient and practical implementations of single-server PIR schemes. HE allows computation on encrypted data without revealing the underlying plaintext, which means a server can process queries directly on encrypted requests, ensuring that the data and the query both remain confidential. This approach significantly improves the efficiency and security of PIR, as it removes the need for multiple non-colluding servers and allows for privacy-preserving data retrieval with a single server setup.

The integration of HE into PIR protocols leverages its ability to perform arithmetic operations on encrypted data, enabling the server to respond to client queries without ever decrypting them. This not only enhances the privacy guarantees but also makes PIR more scalable and practical in real-world applications. By using HE, single-server PIR implementations can efficiently compute responses to encrypted queries, minimizing computational overhead while maintaining strong privacy protections. In practice, tools like Microsoft's SEAL library incorporate HE, specifically Ring Learning With Errors (Ring-LWE), to implement these capabilities.

PIR implementations generally follow two main approaches. The first is the Chor-Goldreich-Kushilevitz (CGK)[15] scheme for information-theoretic PIR, which provides unconditional security by distributing the database across multiple non-colluding servers. The second approach uses HE and lattice-based methods for computational PIR, which rely on cryptographic assumptions and typically operate with a single server. These lattice-based approaches leverage mathematical structures called lattices to create secure encryption schemes that allow efficient query processing while maintaining privacy.

The use of HE has fundamentally transformed single-server PIR, making it a more viable and efficient solution for privacy-preserving data retrieval. This combination of theoretical approaches and practical implementations has made PIR increasingly applicable across a wide range of privacy-sensitive scenarios, including its use in Private Set Intersection (PSI). The significance of HE cannot be overstated, as it not only strengthens privacy guarantees in PIR but also paves the way for other advanced cryptographic constructions, ultimately broadening the scope and utility of secure data retrieval solutions.

One notable example of PIR in action is its integration with Private Set Intersection (PSI)[16]. PSI allows two or more parties to find common elements in their datasets without revealing any additional information beyond the intersection itself. For instance, two companies may wish to identify common customers without sharing their entire customer lists. By leveraging PIR, each party can retrieve information about the intersection privately, ensuring that no non-intersecting data is exposed. This approach is particularly valuable in scenarios where maintaining the confidentiality of the datasets is crucial, such as in healthcare collaborations or financial partnerships.

Real-world examples:

- Patent database retrieval: A client can request a specific record from a large patent database without revealing which one they need. The client sends an encrypted index of the record, and the server processes this to return the encrypted result. For example, researchers can use PIR to access specific patents in the US patent database for a project without revealing which patents they are interested in. This ensures that sensitive intellectual property research remains confidential.
- Medical information retrieval: PIR allows a patient to retrieve a specific medical record from a hospital database without the hospital knowing which record was requested. For example, a patient undergoing treatment for a sensitive condition can use PIR to retrieve specific medical records without revealing their interest to the hospital staff, thereby ensuring full confidentiality. This approach is especially beneficial for patients dealing with stigmatized conditions, allowing them to maintain privacy while managing their health.

[15] The Chor-Goldreich-Kushilevitz (CGK) scheme is an information-theoretic approach to Private Information Retrieval (PIR). It was proposed by researchers Benny Chor, Oded Goldreich, Eyal Kushilevitz, and Madhu Sudan. The CGK scheme ensures that a client can retrieve data from a database without revealing any information about which data is being requested. This method achieves unconditional privacy, meaning the privacy guarantee does not depend on computational assumptions but rather on the architecture of the system. In the CGK scheme, the database is replicated across multiple non-colluding servers. The client sends specially crafted queries to each server, ensuring that no single server learns which data is being retrieved. As long as the servers do not collude with each other, the client's privacy is preserved. The approach offers perfect privacy, but it requires the assumption that multiple servers are involved and that they do not share information about their interactions with the client. The CGK scheme is significant in scenarios where high privacy guarantees are required, but it comes with the practical limitation of needing multiple non-colluding servers, which may not always be feasible in real-world applications. See: Chor, B., Goldreich, O., Kushilevitz, E., & Sudan, M. (1998). **Private information retrieval**. *Journal of the ACM (JACM)*, 45(6), 965-981. DOI.

[16] Freedman, M. J., Nissim, K., & Pinkas, B. (2004). **Efficient private matching and set intersection**. *International Conference on the Theory and Applications of Cryptographic Techniques (EUROCRYPT)*, 3027, 1-19. DOI. This reference covers foundational work on PSI, introducing efficient protocols for private set intersection and private matching.

- Corporate data retrieval: Employees of a company can retrieve records from a confidential database without revealing which record they are looking for. For instance, an employee working on a confidential project could use PIR to access specific internal documents without revealing the nature of their query to the IT team, ensuring that confidential research remains secure. This is particularly important for organizations in competitive industries, where safeguarding project details and proprietary research is essential.
- Academic research collaboration: PIR enables multiple research institutions to collaboratively access sensitive datasets while maintaining the confidentiality of each request. For example, researchers studying sensitive health data across different universities can use PIR to collaborate on a large-scale study while maintaining privacy regarding their specific research interests.
- Customer support information retrieval: Customer service representatives can use PIR to access specific customer records without revealing which record is being accessed to unauthorized personnel. For instance, a representative could retrieve a customer's previous support history without the support platform's backend knowing which customer record was accessed. This helps maintain the privacy of sensitive customer information.
- E-commerce product information: PIR allows buyers to access specific product details from a large e-commerce catalog without revealing which product they are interested in. For instance, a user researching a high-value item can retrieve product information without revealing their interest, thereby preventing targeted marketing or price manipulation by the platform.
- Government records access: PIR enables citizens to access certain public records without the government knowing which specific record is being accessed. For example, a journalist researching a sensitive topic can use PIR to access specific government documents without revealing their focus, ensuring freedom of information while maintaining confidentiality.
- Intellectual property research: Legal teams or corporations can search through a database of patents or trademarks without revealing the specific intellectual property they are researching. For instance, during early stages of product development, a company can use PIR to verify patent details without competitors learning about their research interests, thus maintaining strategic confidentiality.
- Human resources record access: HR personnel can access specific employee records without revealing which record they are interested in to other departments or unauthorized personnel. For example, during an internal audit, an HR manager might need to review sensitive records without exposing which employees are being audited, ensuring privacy and avoiding unnecessary speculation.
- Legal document retrieval: Law firms often need to access specific legal documents from a shared database without disclosing which document they are searching for, especially during cases involving multiple parties. For instance, during a merger or acquisition, legal teams can use PIR to access critical contract details without tipping off competing firms about their focus, keeping negotiations confidential.
- Supply chain data access: PIR allows manufacturers to access specific supply chain information from a shared logistics database without revealing their focus to other stakeholders. For example, a car manufacturer may verify part availability without revealing to suppliers which model they are currently prioritizing, thereby maintaining competitive confidentiality.
- Market analysis for financial institutions: Financial analysts may need to retrieve specific market data from a large dataset without revealing which data points they are interested in. By using PIR, analysts can query the database and obtain encrypted results without disclosing their market focus. For example, an investment firm researching emerging markets can access key economic indicators without revealing their specific interests, thereby maintaining a competitive edge.

0.1.7 Beyond HE

HE is a powerful tool in cryptography that has the potential to revolutionize data privacy. It allows computations to be carried out on encrypted data without requiring access to the original plaintext. This capability has significant implications for secure data processing, enabling cloud-based services to perform calculations on sensitive information while preserving privacy. However, despite its transformative possibilities, HE comes with several limitations and challenges that must be addressed before it can be widely adopted in practical applications.

Below, we outline some of the challenges and constraints associated with HE, providing a deeper understanding of its current limitations and the efforts needed to overcome them.

0.1.7.1 Challenges

1. Encrypted output: While HE allows for arbitrary computations on encrypted data, the outcome of these computations is still encrypted. This means that the result is only useful to someone with the secret key to decrypt it. For example, if a cloud server performs a complex computation on encrypted health records, the resulting encrypted output cannot be interpreted without the corresponding decryption key. This presents a challenge for practical implementations, as it requires data owners to perform decryption locally to understand the results. In contrast, other techniques like obfuscation and functional encryption enable certain types of encrypted computations where the output is directly accessible in plaintext. These techniques can be more practical in situations where immediate interpretation of results is required. Another drawback of the encrypted output is the lack of flexibility for collaboration. In many use cases, organizations need to share the results of computations with multiple stakeholders who may not have access to the decryption key. This means that HE, by default, limits the ease of sharing processed information unless additional mechanisms for key distribution are implemented. As a result, using HE often necessitates careful planning around how decryption keys are managed and shared, which can introduce additional security concerns. Managing key distribution securely while ensuring accessibility is an ongoing area of research in the field of cryptography.
2. Single key requirement: To perform computations on encrypted data, all inputs must be encrypted using the same key. This constraint limits scenarios where data from multiple sources, encrypted with different keys, needs to be jointly processed. For instance, in a scenario where multiple healthcare providers wish to collaborate on a dataset of encrypted patient records, each provider's data must be encrypted with the same key for joint analysis to be possible. This presents a significant barrier to collaboration, as coordinating the use of a single encryption key across multiple entities introduces security and logistical challenges. Addressing this limitation often requires the use of advanced key management techniques or trusted intermediaries, which can complicate the overall system architecture. Techniques like SMC can sometimes be used alongside HE to facilitate joint computations without sharing a common key, but these solutions tend to increase computational overhead and complexity. Moreover, the need for a single key also raises concerns about key compromise—if the key is exposed, all encrypted data becomes vulnerable, making key security a critical aspect of using HE in real-world applications. Researchers are actively exploring methods to allow computations on data encrypted with different keys, such as through key homomorphism or the use of proxy re-encryption. These approaches aim to enable interoperability between datasets encrypted with different keys, thereby enhancing the practicality of HE for collaborative applications. However, these methods are still in their experimental stages and are not yet widely adopted in mainstream cryptographic systems.
3. No integrity guarantees: HE allows for computations on encrypted data, but it does not provide a mechanism to verify that the computations were performed correctly. In other words, there is no inherent way to confirm if the resulting ciphertext is genuinely the outcome of the intended computation or if it is simply a new encryption of an unrelated value. This lack of integrity verification is a significant limitation, particularly in scenarios where the correctness of the computation is critical, such as financial transactions or medical data analysis. Without integrity guarantees, there is a risk that a malicious server could manipulate the computation process, resulting in incorrect outputs without detection. For instance, if a cloud provider intentionally or unintentionally alters the computation on encrypted financial records, the resulting encrypted output could be incorrect, leading to potential financial losses for the data owner. To address this issue, additional cryptographic tools such as ZKPs can be used in combination with HE to provide assurance that computations were performed correctly. ZKPs allow one party to prove to another that a computation was executed as expected without revealing any

information about the input data. By integrating ZKPs with HE, it is possible to create a system where the server can provide verifiable proof that it performed the computation correctly. However, adding ZKPs to the process increases computational complexity and may impact performance, making it important to balance the need for integrity with the computational resources available. Another approach to ensuring the integrity of computations is the use of blockchain technology. By recording the steps of the computation on a blockchain, it is possible to create a transparent and tamper-resistant log that can be audited by all parties involved. This method, while promising, also introduces additional overhead and requires careful consideration of scalability, especially when dealing with large volumes of data.

0.1.7.2 Future directions

In addition to the limitations outlined above, HE faces several other challenges that need to be addressed to make it more practical for widespread use. These challenges include:

1. Performance overheads: HE is computationally intensive compared to traditional encryption methods. Performing even basic operations on encrypted data can require significantly more processing power and time. FHE, which supports arbitrary computations, is particularly demanding and often impractical for real-time applications due to its high computational costs. Researchers are working on optimizing FHE schemes to reduce these performance overheads, but significant progress is still needed before they can be used in everyday applications. Advances such as bootstrapping optimizations and hardware acceleration are being explored to mitigate these challenges.

2. Large ciphertext sizes: Encrypted data under HE schemes tends to be much larger than the original plaintext data. This increase in data size, known as ciphertext expansion, can lead to storage and bandwidth issues, particularly when dealing with large datasets. For example, encrypting a simple medical record using FHE can result in a ciphertext that is several orders of magnitude larger than the original record. This makes storage and transmission of encrypted data more challenging, especially in environments with limited resources. Researchers are investigating techniques like compression schemes and more efficient ciphertext representations to reduce the overhead associated with HE.

3. Complexity of implementation: Implementing HE is complex and requires a deep understanding of advanced mathematics and cryptographic principles. This complexity makes it difficult for developers to integrate HE into their applications without specialized knowledge. To address this barrier, researchers and developers are working on creating libraries and tools that simplify the use of HE, making it more accessible to non-experts. However, there is still a long way to go before these tools are as user-friendly as traditional encryption libraries. Efforts like Microsoft SEAL, PALISADE, and other open-source libraries are helping bridge this gap, but more work is needed to make HE adoption mainstream.

4. Lack of standardization: Another challenge with HE is the lack of standardization across different implementations. Currently, there are multiple HE schemes, each with its unique properties, trade-offs, and performance characteristics. This fragmentation makes it difficult for developers and organizations to choose the right scheme for their needs and complicates interoperability between systems using different HE protocols. Ongoing efforts by organizations such as the HomomorphicEncryption.org community aim to create standardized benchmarks and guidelines to help users navigate the complexities of HE and choose the most suitable options for their use cases.

5. Key management and distribution: The effective management of encryption keys is a critical factor in ensuring the security of HE systems. As discussed earlier, HE often requires a single key to encrypt all data inputs, making key distribution a complex challenge, particularly in collaborative environments. If the key is compromised, all encrypted data becomes vulnerable. Key rotation mechanisms, secure key storage solutions, and the development of multi-key HE are all areas of active research to address these key management challenges. Proxy re-encryption and distributed key generation are also being explored as potential solutions to facilitate secure key sharing across different entities without compromising security.

6. Scalability issues: HE can be difficult to scale, especially for applications requiring large-scale data

processing, such as big data analytics or machine learning. The computational overhead and increased data sizes make scaling HE to handle vast amounts of information a considerable challenge. Researchers are exploring the use of hybrid cryptographic solutions, where HE is combined with other privacy-preserving techniques like DP and SMC, to achieve a balance between scalability and privacy. These hybrid approaches can potentially make HE more viable for large-scale, real-time applications by distributing the computational burden and reducing latency.

0.2 Foundations of HE

Having explored the high-level benefits and use cases of HE, from cloud-based analytics to privacy-preserving computations, we now turn to the mathematical underpinnings that make these capabilities possible. In the sections ahead, we'll look at the abstract algebra concepts that allow encrypted values to behave almost like ordinary data, so that addition, multiplication, and even more complex operations can be carried out securely.

We'll begin by examining homomorphisms, the structural "translations" that keep arithmetic consistent between plaintext and ciphertext. Next, we'll see how groups, rings, and fields fit into HE, and why functional completeness—the ability to both add and multiply encrypted data—is a cornerstone of fully homomorphic schemes. Finally, we'll dissect the main building blocks of a typical HE setup (KeyGen, Enc, Dec, and Eval) and learn how noise management, parameter selection, and other technical considerations come together to deliver robust security without sacrificing too much performance. By grounding ourselves in these foundational concepts, we'll gain a clearer sense of how HE works, equipping us to better understand why it is so transformative for secure data processing.

0.2.1 Homomorphisms

Homomorphisms are an important concept in abstract algebra, referring to a function between two algebraic structures that preserves the operations of those structures. Simply put, if we have two sets, each with their own operations, a homomorphism ensures that operations performed on elements of the first set correspond directly to the operations on their mapped elements in the second set.

Let's break this down with a simple analogy. Imagine we have two different languages, but both languages describe similar actions. A homomorphism is like a translation between these languages that ensures the meaning of sentences is preserved. If you take an action in the first language, the translation will represent the same action in the second language. The structure remains consistent.

Consider two sets of numbers, A and B, where B is derived from A using a homomorphism function. If we take two numbers, 3 and 5, from A and add them to get 8, the homomorphism ensures that their images in B, 6 and 10, also add up to give the corresponding result, which is 16.

In formal terms, let A be represented by elements $a_1, a_2 \in A$, and B by their corresponding images under the homomorphism $f : AoB$. If $a_1 = 3$ and $a_2 = 5$, then:

$$a_1 + a_2 = 8$$

Applying the homomorphism f:

$$f(a_1) = 6, \quad f(a_2) = 10$$

Thus:

$$f(a_1 + a_2) = f(a_1) + f(a_2) = 6 + 10 = 16$$

This demonstrates how the homomorphism preserves the operation between the sets.

0.2.2 Group properties in homomorphism

In abstract algebra, a set S and an operation "$\star$" that combines any two elements a and b to form another element $a \star b$ qualifies as a group if the following properties hold:

- Closure: For all $a, b \in S$, the result of $a \star b$ is also in S. Example: Consider the set of integers $\mathbb{Z}$ under addition $+$. If $a = 3$ and $b = 5$, then $a + b = 8 \in \mathbb{Z}$. The result is also an integer, demonstrating closure.
- Associativity: For all $a, b, c \in S$, $(a \star b) \star c = a \star (b \star c)$. Example: In the set of integers $(\mathbb{Z}, +)$, addition is associative. For any integers $a = 3$, $b = 5$, and $c = 2$, we have:

$$(a + b) + c = (3 + 5) + 2 = 8 + 2 = 10$$

$$a + (b + c) = 3 + (5 + 2) = 3 + 7 = 10$$

  Thus, $(a + b) + c = a + (b + c)$, which shows that addition is associative.
- Identity element: There exists an element $e \in S$ such that $e \star a = a \star e = a$ for all $a \in S$. Example: In $(\mathbb{Z}, +)$, the identity element is 0, as $0 + a = a + 0 = a$ for any integer a. For instance, $0 + 5 = 5$ and $5 + 0 = 5$.
- Inverse element: For each element $a \in S$, there exists an element $b \in S$ such that $a \star b = b \star a = e$. Example: In $(\mathbb{Z}, +)$, the inverse of an element a is $-a$, since $a + (-a) = (-a) + a = 0$, where 0 is the identity element. For example, the inverse of 5 is -5, because:

$$5 + (-5) = 0$$

These properties ensure consistency and predictability in operations involving homomorphisms, making them a crucial aspect of algebraic structures.

0.2.3 HE scheme

An encryption scheme is called **homomorphic** over an operation $\star$ if it supports the following property:

$$Enc(m_1) \star Enc(m_2) = Enc(m_1 \star m_2), \quad \forall m_1, m_2 \in M$$

where Enc is the encryption algorithm, and M is the set of all possible messages. This property means that performing the operation on encrypted data yields the same result as performing the operation on the plaintexts and then encrypting the outcome.

Let's make this more concrete with a simple example. Suppose we have two numbers, $m_1 = 5$ and $m_2 = 3$, and we want to add them, that is $\star$ represents addition. Normally, we would calculate $5 + 3 = 8$. In a HE scheme, instead of adding 5 and 3 directly, we first encrypt them:

$$Enc(5), \quad Enc(3)$$

If the encryption scheme is homomorphic over addition, we can add these encrypted values directly:

$$Enc(5) + Enc(3) = Enc(8)$$

After computing on the encrypted values, we can decrypt the result to get the sum:

$$Dec(Enc(8)) = 8$$

0.2.4 Functional completeness

Imagine you have a secret message inside a locked box, and you want someone else to be able to perform some calculations on it without unlocking the box. HE allows this kind of magic. But how much can they really do with the box still locked?

Functional completeness is a fancy way of saying that if we can perform just two basic kinds of calculations on our locked message, then we can actually compute *anything*. These two basic calculations are **addition** and **multiplication**.

Think of these as building blocks, like Lego pieces. With just addition and multiplication, you can build any mathematical function you want. It's a bit like how you only need a few types of Lego pieces to build a spaceship, a car, or even a whole castle. Addition and multiplication are enough to recreate every possible calculation.

In fact, even in the world of computers and logic, every complicated decision or process can be broken down into combinations of simpler pieces. For example, **XOR** (which acts like addition without carrying over numbers) and **AND** (which acts like multiplication) are the Lego pieces of digital logic. If an encryption system allows you to perform these two operations, you can calculate any kind of logical operation on encrypted data—without ever seeing the original secret message.

It is also worth noting that **NAND** or **NOR** gates alone can form a complete basis for Boolean logic. This means that, just like addition and multiplication, NAND or NOR are also sufficient to represent any Boolean function. This is an interesting parallel to the completeness of addition and multiplication in HE.

This is why addition and multiplication are so powerful. They are enough to make the encryption scheme fully homomorphic, meaning it can perform *any* kind of computation on encrypted data, keeping the secrets locked up but still allowing useful work to be done. In simple terms, if you can add and multiply, you can do it all!

0.2.4.1 Formal definition

To formally understand functional completeness in HE, it's important to start with the algebraic foundations. So, let's come back to the definition of homomorphism as a structure-preserving map between two algebraic structures, such as groups, rings, or fields. This means that the operations defined in one structure are preserved under the mapping to the other structure. In the context of HE, the homomorphism property allows operations to be carried out on encrypted data that mirror operations on the plaintext.

Group is a set equipped with an operation that satisfies closure, associativity, has an identity element, and where every element has an inverse. When we extend these properties to include additional operations like multiplication, we get rings and fields, which have more complex properties. In a **ring**, both addition and multiplication are defined, but not every element necessarily has a multiplicative inverse. In a **field**, every non-zero element has a multiplicative inverse, making it a richer structure.

In HE, we work with these algebraic structures because they provide the foundation for well-defined operations on encrypted data. The key operations, namely addition and multiplication, are defined over these structures in a way that ensures they behave predictably and securely. When we say that an encryption scheme is homomorphic, we mean that it allows addition and multiplication to be performed on encrypted values, and the result, when decrypted, matches what would have been obtained if the operations were performed directly on the plaintext values.

Functional completeness, in this context, refers to the ability of an encryption scheme to support arbitrary computations on encrypted data by leveraging both addition and multiplication. These two operations are fundamental because they form a functionally complete set over finite fields. The latter distinction is important because functional completeness is well-defined in finite fields due to properties like closure under addition and multiplication. In infinite fields, however, the same guarantees may not hold, and constructing certain functions can be more challenging. The finite nature ensures that every combination of addition and multiplication stays within the set, which is a crucial requirement for functional completeness in encryption schemes. Moreover, since physical computations in digital systems are inherently carried out over finite fields, this limitation does not affect practical applications. As a matter of fact, digital systems use finite representations (such as bits), and operations are performed over well-defined finite fields like **GF(2)**[17].

To clarify the connection with the previous discussion, consider **Boolean circuits**, which are a model of computation used in computer science to represent logical functions. A Boolean circuit consists of logic gates such as XOR and AND, which can be seen as parallels to addition and multiplication, and can be combined to represent any possible computation, as supported by the concept of Turing completeness, which states that any computation can be performed given sufficient resources and the right set of operations, such as XOR and AND, which were previously introduced as parallels to addition and multiplication, and together are sufficient to represent any computable function.

Similarly, in arithmetic circuits, addition and multiplication serve as the fundamental operations. By chaining these operations together, we can construct any polynomial function. The ability to construct polynomial functions is significant because, according to the **Stone-Weierstrass theorem**[18], any continuous function can be approximated by a polynomial to any desired degree of accuracy. This means that by constructing polynomial functions, we can represent a wide range of complex computations, including those needed for encryption and data processing. In the context of HE, this enables us to perform arbitrary functions on encrypted data, ultimately allowing powerful and flexible operations while preserving data privacy. In the context of HE, this means we can perform a wide variety of operations on encrypted data, ultimately allowing us to evaluate arbitrary functions while preserving data privacy. This explanation builds on the algebraic foundations discussed earlier, demonstrating how addition and multiplication form a functionally complete set for building complex computations, both in logical and arithmetic contexts.

In HE, an encryption scheme is said to be **fully homomorphic** if it supports both addition and multiplication on encrypted data, without needing to decrypt it. This property allows for the evaluation of any arithmetic circuit or Boolean circuit on encrypted data, effectively enabling arbitrary computation while preserving the confidentiality of the original data.

For instance, in the context of Boolean logic, XOR can be represented by addition (without carry), and AND can be represented by multiplication. These two gates are sufficient to build any Boolean function, making them functionally complete. Therefore, an encryption scheme that supports homomorphic addition and multiplication can evaluate any Boolean function, making it a FHE scheme.

0.2.4.2 Relevance in HE

The concept of functional completeness is crucial because it determines the power and flexibility of a HE scheme. If an encryption scheme can only support addition or only multiplication, it is called **PHE**. Such schemes can perform useful but limited computations, like adding encrypted numbers together or multiplying them by a constant. However, they cannot handle more complex functions that require a combination of both operations.

Examples of partially HE schemes include **RSA**, which is **multiplicatively homomorphic**, and **Paillier**, which is **additively homomorphic**. These schemes allow for specific types of computations on encrypted data but lack the flexibility of fully HE.

A **FHE** scheme, on the other hand, allows for arbitrary computations on encrypted data. This means that any function, no matter how complex, can be evaluated while the data remains encrypted.

0.2.5 Symmetric vs. asymmetric HE

HE schemes can be broadly categorized into **symmetric** and **asymmetric** types, each with unique characteristics and use cases. Symmetric HE uses the same key for both encryption and decryption, while asymmetric HE uses different keys for encryption and decryption.

[17] GF(2), or Galois Field of order 2, is a finite field consisting of just two elements: usually represented as 0 and 1. See: van Lint, J. H., & Wilson, R. M. (2001). **A Course in Combinatorics** (2nd ed.). Cambridge University Press. DOI

[18] Rudin, W. (1976). **Principles of Mathematical Analysis** (3rd ed.). McGraw-Hill. ISBN: 978-0070856134

0.2.5.1 Symmetric HE

In symmetric HE, the same secret key is used for both encryption and decryption. This approach is often simpler to implement and is computationally efficient compared to asymmetric schemes. Imagine you and your friend have the same combination lock. You can lock up a message, and your friend can unlock it using the same combination. In symmetric HE, the same secret key is used to lock (encrypt) and unlock (decrypt) the data, even when performing computations.

Symmetric HE schemes are generally faster and require less computational power than asymmetric ones. This is because symmetric algorithms tend to have simpler key structures, leading to more efficient operations. The key size in symmetric HE schemes can be smaller while maintaining an equivalent level of security compared to asymmetric systems.

One of the primary challenges of symmetric HE is key management. If multiple users need access to the encrypted data, the secret key must be shared securely, which can be challenging, especially in distributed environments. In multi-user scenarios, symmetric encryption poses a security risk since all parties must share the same key. If any user mishandles the key, the entire system's security is compromised.

Symmetric HE is most suitable for use cases where there is a trusted environment, such as a single user encrypting their own data for secure local processing or a tightly controlled group where the key can be securely shared.

0.2.5.2 Asymmetric HE

Asymmetric HE schemes utilize a pair of keys: a **public key** for encryption and a **private key** for decryption. The public key can be shared openly, allowing anyone to encrypt data, but only the holder of the corresponding private key can decrypt it. Imagine you have a special mailbox with a slot that anyone can drop letters into (public key) but only you have the key to open the mailbox and read the letters (private key). Asymmetric HE works similarly, allowing anyone to encrypt data, but only the intended recipient can decrypt it.

Asymmetric HE schemes are generally more computationally intensive than symmetric ones. The key structures and encryption/decryption algorithms tend to be more complex, leading to slower performance. To achieve a similar level of security, asymmetric keys need to be larger compared to symmetric keys, which can increase storage and processing requirements.

Asymmetric HE is ideal for scenarios involving multiple users, such as cloud computing, where data needs to be encrypted by many users but only decrypted by a trusted party. The use of a public key enables easy data sharing without compromising the security of the private key. Since the public key is openly distributed, any number of users can encrypt data, making asymmetric HE more scalable for environments involving many participants.

0.2.6 Key components of an HE scheme

An HE scheme is fundamentally characterized by four essential operations: $KeyGen$, Enc, Dec, and $Eval$. These components work together to enable secure computation on encrypted data while maintaining the critical property of homomorphism. Let's explore each component in detail and understand their mathematical foundations, practical implications, and the subtle nuances involved.

The four core components that make HE functional are discussed in detail below:

- The $KeyGen$ algorithm is the foundation of any HE scheme's security. It generates the cryptographic keys necessary for the system's operation.

  Imagine that $KeyGen$ is like creating a secure lock and key for a treasure chest. If the lock is too simple, it might be easy for a thief to pick it, compromising the security of the chest. On the other hand, if the lock is extremely complex, it might take a very long time to make and might even be difficult for the rightful owner to use efficiently. In HE, $KeyGen$ works in a similar way: it needs to create a key that is strong enough to keep attackers out but also practical enough for users to operate.

The security parameter λ is like deciding how sophisticated the lock should be, higher values make it harder for unauthorized access but require more effort and resources to manage.

- For symmetric HE: $k \leftarrow KeyGen(1^\lambda)$, where λ is the security parameter, and k is the secret key.
- For asymmetric HE: $(pk, sk) \leftarrow KeyGen(1^\lambda)$, where pk is the public key and sk is the secret key. The security parameter λ determines the computational hardness of breaking the encryption scheme. Larger values of λ provide stronger security but increase computational overhead. The key generation process typically involves:
  * Generation of random numbers: Random numbers are generated from a specified distribution, such as uniform, Gaussian, or discrete Gaussian distributions. For example, uniform distributions ensure equal likelihood across a range, while Gaussian distributions are used to introduce controlled randomness with a specific mean and standard deviation. Discrete Gaussian distributions, common foundational in HE, particularly in lattice-based cryptography, add noise with precision suitable for cryptographic operations. The randomness is crucial for ensuring that every generated key is unique and unpredictable.
  * Mathematical operations: Complex mathematical operations are used based on the scheme's underlying hardness assumptions. For example, mathematical frameworks in HE, such as Ring-LWE (Learning With Errors) and NTRU (Nth degree Truncated Polynomial Ring) are foundational in HE. These frameworks define problems that are computationally infeasible to solve without specific secret information (such as the private or secret key generated during the $KeyGen$ phase), ensuring the security of the encryption scheme. These assumptions make it computationally infeasible for an attacker to derive the private key from the public key or ciphertexts, as they are based on the hardness of specific mathematical problems (e.g., Ring-LWE or NTRU). These problems require secret information, such as the private or secret key generated during the $KeyGen$ phase, to be solvable within a practical timeframe.
  * Parameter generation: Generation of additional parameters is often required for homomorphic evaluation, such as relinearization keys in some schemes. These parameters help to maintain efficiency and support specific operations like multiplication without a significant increase in ciphertext size. Relinearization keys simplify the increased complexity that occurs after a ciphertext multiplication by "recompressing" the resulting ciphertext into a manageable size and form, ensuring efficient further computations. Without these keys, ciphertexts could grow exponentially, making further evaluations impractical.

The robustness of the $KeyGen$ function directly impacts the overall security of the HE scheme. It must ensure that the generated keys meet the desired security standards while balancing the computational resources required for efficient operation.

- The Enc encryption function transforms plaintext messages into ciphertexts. In HE schemes, this process must preserve the algebraic structure that enables homomorphic operations:
  - For symmetric HE: $c \leftarrow Enc(k, m)$, where m is the plaintext message, k is the secret key, and c is the resulting ciphertext.
  - For asymmetric HE: $c \leftarrow Enc(pk, m)$, where pk is the public key.

  Key characteristics of the encryption process include:
  - Addition of random noise: The encryption process introduces random noise into the plaintext to ensure semantic security, making it computationally difficult for an adversary to distinguish between different ciphertexts. While this noise is critical for maintaining security, it must be carefully controlled to avoid excessive growth that can disrupt subsequent computations. Effective noise management ensures that the ciphertext remains usable for homomorphic operations without compromising security.
  - Message embedding: The message is embedded into a structured mathematical framework, such

as polynomial rings[19] or lattice-based constructions. This embedding serves two purposes: first, to secure the message against unauthorized access, and second, to enable efficient computations on encrypted data. By embedding the message in a way that retains the necessary algebraic properties, the encryption scheme supports operations like addition and multiplication directly on ciphertexts.

The encryption step is not just about securing the data but also ensuring that the encrypted data can still participate in meaningful computations. This dual requirement makes HE distinct from conventional encryption methods.

- The *Dec* decryption function recovers the original plaintext from the ciphertext:
  - For symmetric HE: $m \leftarrow Dec(k, c)$.
  - For asymmetric HE: $m \leftarrow Dec(sk, c)$.

  Critical aspects of the decryption process include:

  - Noise removal: Decryption involves removing the noise added during encryption. This is achieved by leveraging the secret key or decryption algorithm to isolate the original message from the noisy ciphertext. Noise levels must remain below a threshold defined by the scheme's parameters; otherwise, the decryption process may fail, yielding incorrect results. Techniques like modulus alignment[20] or parameter scaling[21] are often used to ensure the noise is adequately suppressed during decryption.
  - Extraction from mathematical structure: The message is extracted from its embedded mathematical structure. This involves interpreting the ciphertext within the mathematical framework it was transformed into during encryption (e.g., polynomial rings or lattice structures). Decryption uses the secret key to reverse this transformation by applying the inverse operations in the specified algebraic domain. This process isolates the plaintext while ensuring that noise and other artifacts are accounted for, reconstructing the original message accurately.
  - Error handling: Error handling is crucial for situations where noise growth has exceeded acceptable bounds. When the noise level is too high, the decryption process may fail, indicating that the homomorphic operations performed exceeded the scheme's limitations.
  - Integrity verification: Decryption must ensure that the recovered message is the exact original plaintext without any alterations. This process involves verifying the correctness of the decryption by checking the consistency of the output with the encryption parameters and the intended operations performed during evaluation. Integrity verification is essential to detect and prevent errors introduced during encryption, evaluation, or decryption. This may include confirming that noise levels remained within permissible thresholds and that no tampering or corruption of ciphertext occurred throughout the process.

  The computational efficiency and correctness of the decryption process are vital for the practical usability of an HE scheme. It must accurately recover the plaintext without compromising security.

- The *Eval* evaluation function is the distinguishing feature of HE, enabling computation on encrypted data. For a function f and ciphertexts $c_1, c_2, \dots, c_n$:

[19] A *polynomial ring* is an algebraic structure where the elements are polynomials, and it is closed under addition and multiplication. This means that adding or multiplying two polynomials within the ring always results in another polynomial within the same ring, making it an ideal framework for cryptographic operations like those leveraged in HE.

[20] The *modulus* serves as the upper limit for the arithmetic space within which operations are performed. When ciphertexts undergo operations like addition or multiplication, their noise increases, and the resulting values may exceed the modulus. Modulus alignment scales the ciphertext back to a compatible modulus, ensuring that it remains within the arithmetic boundaries required by the HE scheme and enabling accurate decryption.

[21] *Parameter scaling* involves adjusting specific parameters, such as scaling factors or precision levels. For example, plaintexts are often scaled (multiplied) by a large constant before encryption to ensure sufficient precision during operations. This scaling factor helps maintain accuracy when performing computations on ciphertexts. However, if not properly adjusted, it can lead to noise accumulation or overflow errors.

$$c_{result} \leftarrow Eval(eval\_key, f, c_1, \dots, c_n)$$

The *Eval* function must satisfy the homomorphic property:

$$Dec(sk, Eval(eval\_key, f, c_1, \dots, c_n)) = f(Dec(sk, c_1), \dots, Dec(sk, c_n))$$

In simple terms, the equation says that if you evaluate a function on encrypted data, then decrypt the result, you will get the same outcome as if you had evaluated the function directly on the unencrypted data. Terms explanation:

– $Eval(eval\_key, f, c_1, \dots, c_n)$ represents applying the function f to the encrypted values (ciphertexts) $c_1, c_2, \dots, c_n$ using an evaluation key.

– $Dec(sk, Eval(...))$ means that you decrypt the output of this evaluated ciphertext using the secret key sk.

– $f(Dec(sk, c_1), \dots, Dec(sk, c_n))$ represents applying the function f to the original plaintext values that were encrypted.

Key considerations for the evaluation function include:

1. Correctness: The evaluation must preserve the relationship between the function applied to plaintexts and the function applied to ciphertexts (here, $\boxplus$ and $\boxdot$ represent homomorphic addition and multiplication operations):

– For addition: $Dec(c_1 \boxplus c_2) = Dec(c_1) + Dec(c_2)$.

– For multiplication: $Dec(c_1 \boxdot c_2) = Dec(c_1) \times Dec(c_2)$.

2. Noise management: Each homomorphic operation increases the noise level in the ciphertext due to the mathematical transformations applied during evaluation. Noise control is essential to ensure that computations remain accurate and ciphertexts decrypt correctly. Techniques like modulus switching and bootstrapping are employed to manage noise. Modulus switching reduces noise by scaling down ciphertexts to a smaller modulus, aligning them with parameters such as the scaling factor, which determines how plaintexts are encoded into ciphertexts. Bootstrapping, on the other hand, resets the noise entirely by re-encrypting the ciphertext and refreshing its parameters, such as the modulus. These approaches ensure that noise levels remain within tolerable limits, enabling accurate decryption and supporting further computations. In particular:

– Addition operations: During addition, the noise levels from the input ciphertexts combine, leading to a linear increase in noise. Modulus switching can be used here to prevent the combined noise from exceeding tolerable limits, ensuring that further operations can still be performed without requiring bootstrapping.

– Multiplication operations: Multiplication causes a more significant challenge due to exponential noise growth. This is because the interaction of ciphertext terms amplifies the noise and can quickly surpass the allowable threshold. Bootstrapping is particularly crucial in these cases to reset noise levels after one or more multiplications, enabling further computations without risking decryption failure.

3. Ciphertext format preservation: The output ciphertext must maintain a consistent format to support further homomorphic computations without interruptions. Ensuring that the ciphertext remains in a format compatible with the scheme's parameters prevents issues in subsequent operations, including decryption at the end of the computation process. To achieve this:

– To support further operations, the ciphertext's structure must remain aligned with the homomorphic scheme's requirements. Techniques like modulus switching or parameter adjustments during evaluation help preserve this format, enabling seamless execution of complex computations.

- Proper size management is also crucial to maintain efficiency. Modulus switching not only helps align the ciphertext's format but also prevents excessive size growth. Without such controls, ciphertext expansion can render the scheme impractical for real-world applications.

4. Performance considerations: Efficient evaluation requires careful attention to several performance-related factors. These considerations ensure the scheme remains practical for real-world applications while balancing computational overhead and security.

- Circuit depth optimization: In leveled HE schemes, the number of operations is limited by the depth of the computational circuit. Reducing circuit depth minimizes noise growth and improves efficiency. Techniques such as modulus switching (introduced earlier) and parameter optimization are often used to manage this depth effectively, ensuring operations stay within the allowed limits.

- Memory management: Homomorphic operations often result in ciphertext expansion, where the size of ciphertexts increases with each operation. This can lead to significant memory demands, especially when working with large datasets or deep circuits. Efficient memory management strategies, such as controlling ciphertext growth through size management techniques (e.g., modulus alignment), are crucial to maintaining performance.

- Computational complexity: Different homomorphic operations have varying computational costs. Addition is computationally inexpensive and introduces manageable noise, while multiplication is more resource-intensive and causes exponential noise growth. Techniques like bootstrapping, already seen, play a key role in managing this complexity by resetting noise levels and enabling further computations.

5. Special evaluation keys: They play a crucial role in extending the functionality of HE schemes by enabling specific operations and improving efficiency during evaluation. These keys are generated during the $KeyGen$ phase and are used as follows:

- Relinearization keys: After multiplication, the resulting ciphertext may have increased complexity. Relinearization keys are used to simplify the ciphertext, making it more manageable for subsequent operations.

- Rotation keys: These keys enable operations that involve rotating encrypted vectors, which are essential in applications like matrix multiplication or encrypted machine learning. They facilitate secure transformations within encrypted data while preserving homomorphic properties.

- Bootstrapping keys: In FHE schemes, bootstrapping keys are indispensable for managing noise. They refresh noisy ciphertexts by re-encrypting them, resetting noise levels, and allowing unlimited operations without risking decryption failure.

The evaluation function is what sets HE apart from traditional encryption schemes. It allows encrypted data to be processed without compromising security, making it highly suitable for scenarios where data privacy is critical. In practice, there may be many different computations required, which means multiple $Eval$ operations might be needed. Each $Eval$ operation allows a specific computation to be performed on the encrypted data, such as addition, multiplication, or other custom functions, without revealing the underlying data. Additionally, $Eval$ operations can be composed, meaning that the result of one evaluation can be used as input for another.

0.2.6.1 Security and functionality properties

The interaction between these components must satisfy several security and functionality properties:

- Semantic security: The encryption process (Enc) ensures that ciphertexts reveal no information about the plaintexts, even if intercepted by an adversary. This is achieved through the addition of random noise during Enc, which obfuscates the relationship between the plaintext and ciphertext. The security of semantic encryption relies on hardness assumptions such as Ring-LWE or NTRU, ensuring that decryption without the secret key is computationally infeasible.

- Compactness: In the evaluation process ($Eval$), the size of ciphertexts should remain independent of the complexity of the function f. Techniques like modulus switching and relinearization, introduced during evaluation, ensure compactness by controlling ciphertext growth and preserving efficiency. Without these techniques, ciphertexts could grow exponentially, rendering the scheme impractical.
- Circuit privacy: To preserve the confidentiality of the computation, the evaluated ciphertext must not reveal information about the function f applied during $Eval$. Bootstrapping and parameter adjustments obscure the internal operations, ensuring that proprietary algorithms or sensitive computations remain private.
- Noise growth bounds: Every homomorphic operation increases noise in ciphertexts. Noise management techniques, such as modulus switching and bootstrapping, introduced in $Eval$, provide clear bounds on noise growth. These bounds define the maximum circuit depth that can be evaluated before decryption becomes unreliable. Effective noise control is vital for ensuring that computations on ciphertexts can be completed successfully.
- Efficiency: The efficiency of HE schemes depends on balancing computational overhead, noise management, and security. Operations like addition, which introduce linear noise, are computationally inexpensive. However, multiplication, which introduces exponential noise, requires careful management through relinearization and bootstrapping. The choice of parameters during $KeyGen$, such as key length and modulus size[22], directly impacts the trade-offs between efficiency and security.

Understanding these components and their properties is crucial for leveraging HE effectively in theory and practice. These considerations highlight the interplay between the core HE operations and their implications:

- Implementing HE schemes correctly: Each operation, from key generation ($KeyGen$) to evaluation ($Eval$), requires careful implementation to maintain the scheme's homomorphic properties. For instance, noise management techniques like modulus switching and bootstrapping must be integrated seamlessly to ensure the scheme's reliability.
- Choosing appropriate parameters: Selecting suitable parameters—such as key length, modulus size, and noise bounds—is vital for balancing security and efficiency. These parameters, determined during the $KeyGen$ phase, dictate the computational depth, noise tolerance, and performance of the scheme.
- Optimizing performance: Real-world applications demand efficiency. Optimizing the encryption (Enc), evaluation ($Eval$), and decryption (Dec) processes ensures the scheme remains practical. Techniques like relinearization keys and circuit depth optimization are indispensable for achieving computational feasibility.
- Ensuring security guarantees: Security guarantees like semantic security, circuit privacy, and integrity verification must hold throughout the computation. This requires consistent adherence to the principles of noise management and compactness during every stage of the HE workflow.
- Designing efficient protocols: HE enables secure protocols by leveraging its unique properties. Applications like encrypted database queries or privacy-preserving machine learning benefit from advanced evaluation capabilities, such as rotation keys for vector manipulations or bootstrapping keys for resetting noise.

0.3 Homomorphism on the RSA

To understand how HE schemes work, we will explore the RSA algorithm. First, we must review the essential mathematical concepts behind encryption schemes. These include number theory, group theory, field theory, probability and statistics, complexity theory, which form the foundation for modern cryptography.

[22] In HE schemes, modulus size and key length are essential cryptographic parameters and FHE and SWHE schemes rely heavily on them. *Modulus size*, which refers to the number size used in modular arithmetic, impacts both security and computational overhead—larger sizes increase security but require more processing power. *Key length*, typically measured in bits, determines the strength of encryption; longer keys offer higher security by increasing the complexity of brute-force attacks. These parameters are carefully selected to balance security, performance, and the computational depth of operations, especially in FHE schemes where extensive computations on encrypted data are possible.

0.3.1 Number theory

0.3.1.1 Primes and factorization

The journey into number theory starts with the basic building blocks of integers and their relationships, such as divisibility and the idea of prime numbers.

> **Definition (*Set of integers*)**: The set of integers is denoted as $\mathbb{Z} = ..., -2, -1, 0, 1, 2, ...$

> **Definition (*Divisibility*)**: Two integers a and b are divisible if there exists an integer c such that $b = a \cdot c$. When true, this relationship is written as $a \mid b$.

For example, $6 \mid 18$ holds because $18 = 6 \cdot 3$. If $a \nmid b$, then a does not divide b.

> **Theorem (*Division Algorithm*)**: For any integers a and $b > 0$, there exist unique integers q (quotient) and r (remainder) such that:
>
> $$a = q \cdot b + r, \quad 0 \leq r < b$$

For example, dividing 17 by 5 gives $q = 3$ and $r = 2$, as $17 = 3 \cdot 5 + 2$.

Prime numbers are the building blocks of integers:

> **Definition (*Integer primality*)**: A number $p > 1$ is prime if its only divisors are 1 and p.

For instance, 7 is prime, while 12 is composite because $12 = 2 \cdot 6$.

> **Theorem (*Fundamental Theorem of Arithmetic*)**: Every integer $n > 1$ can be uniquely expressed as a product of prime powers:
>
> $$n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}$$
>
> where p_i are distinct primes, i and e_i are positive integers, and k is the number of distinct prime factors of n.

For example, $84 = 2^2 \cdot 3 \cdot 7$ demonstrates this principle.

> **Remark**: It is straightforward to compute the product of two large prime numbers p and q. However, the reverse operation, determining the original prime factors from their product $n = p \cdot q$, is computationally difficult. This difficulty arises from the lack of efficient algorithms for factorizing large integers. The best-known algorithms, such as the General Number Field Sieve (GNFS)[23], have exponential time complexity for large inputs. This asymmetry makes factoring infeasible within a reasonable timeframe as the bit length of p and q increases. Moreover, the factors p and q are typically chosen to be large primes of similar bit length to avoid simple heuristics or optimizations. This problem is so significant that it has its own name, the Integer Factorization Problem (denoted $ Fact[n] $), and it underpins the security of many public-key cryptosystems, including RSA, ensuring that decrypting or compromising encrypted data without the private key remains practically impossible.

0.3.1.2 Greatest common divisor

To explore relationships between numbers, we often need their greatest common divisor, e.g. to simplify a fraction or to synchronize cycles.

> **Definition (*Greatest common divisor, GCD*)**: The greatest common divisor of two integers a and b, denoted $\gcd(a, b)$, is the largest integer dividing both a and b.

For example, $\gcd(12, 18) = 6$.

[23] Lenstra, A. K., & Lenstra, H. W. (1993). **The development of the number field sieve** (Vol. 1554). *Springer-Verlag.* DOI

Definition (*Relatively primality of integers*): Two integers are relatively prime if their GCD is 1.

Finding the GCD is efficient with the Euclidean algorithm:

Theorem (*Euclidean Algorithm*): The GCD of two integers a and b, where at least one is nonzero, can be computed using the recursive relation:

$$\gcd(a, b) = \gcd(b, a \mod b).$$

This recursive formula stems from the property of divisors:

$$\gcd(a, b) = \gcd(b, a - q \cdot b)$$

where q is the quotient when a is divided by b. Since $a - q \cdot b = a \mod b$, the recursion simplifies to:

$$\gcd(a, b) = \gcd(b, r)$$

where $r = a \mod b$.

For example, consider the integers 385 and 364. Using the Euclidean Algorithm:

$$\begin{aligned} \gcd(385, 364) &= \gcd(364, 385 \mod 364) = \gcd(364, 21), \\ \gcd(364, 21) &= \gcd(21, 364 \mod 21) = \gcd(21, 7), \\ \gcd(21, 7) &= \gcd(7, 21 \mod 7) = \gcd(7, 0) = 7. \end{aligned}$$

Thus, $\gcd(385, 364) = 7$.

The Euclidean Algorithm can be applied to any integers, positive or negative, as long as at least one of the integers is nonzero. The process uses the relationship $\gcd(a, b) = \gcd(b, a \mod b)$, where the modulus operation $a \mod b$ always returns a remainder r satisfying $0 \leq r < |b|$. This means the algorithm effectively reduces to positive remainders during the recursion, even if a or b starts as a negative number.

Example with negative integers, $\gcd(-48, 18)$:

$$\begin{aligned} \gcd(-48, 18) &= \gcd(18, -48 \mod 18) = \gcd(18, 12), \\ \gcd(18, 12) &= \gcd(12, 18 \mod 12) = \gcd(12, 6), \\ \gcd(12, 6) &= \gcd(6, 12 \mod 6) = \gcd(6, 0) = 6. \end{aligned}$$

The sign of the integers does not affect the result, as $\gcd(-a, b) = \gcd(a, b)$.

The GCD is not only useful for determining divisibility but also plays a key role in finding linear combinations of integers. This is formalized in Bézout's Identity:

Theorem (*Bézout's Identity*): For any integers a and b, there exist integers x and y such that:

$$\gcd(a, b) = ax + by.$$

The integers x and y are called Bézout coefficients.

These coefficients are not unique; for any integer k, another pair (x', y') can be generated as:

$$\begin{aligned} x' &= x + k \cdot \frac{b}{\gcd(a, b)}, \\ y' &= y - k \cdot \frac{a}{\gcd(a, b)}. \end{aligned}$$

The **Extended Euclidean Algorithm** builds upon the Euclidean Algorithm to compute the Bézout coefficients x and y. It works by tracing back the remainders obtained during the GCD computation:

Algorithm (*Extended Euclidean Algorithm*):

1. Input integers a and b.
2. Initialize r_0, r_1, s_0, s_1, t_0, t_1, i:
    1. $r_0 = a$, $r_1 = b$
    2. $s_0 = 1$, $s_1 = 0$
    3. $t_0 = 0$, $t_1 = 1$
    4. $i = 1$.
3. While $r_i \neq 0$:
    1. Compute quotient $q = r_{i-1} \div r_i$
    2. $r_{i+1} = r_{i-1} - q \times r_i$
    3. $s_{i+1} = s_{i-1} - q \times s_i$
    4. $t_{i+1} = t_{i-1} - q \times t_i$
    5. $i = i + 1$.
4. Return GCD, x, and y where $ax + by = GCD(a, b)$:
    1. $GCD = r_{i-1}$
    2. $(x, y) = (s_{i-1}, t_{i-1})$.

Below it is shown how the Extended Euclidean Algorithm works step by step with $a = 48$ and $b = 18$:

1. Initialize:
    - $r_0 = 48$, $r_1 = 18$
    - $s_0 = 1$, $s_1 = 0$
    - $t_0 = 0$, $t_1 = 1$
    - $i = 1$.
2. First iteration ($i = 1$):
    - $q = r_0 \div r_1 = 48 \div 18 = 2$ (quotient)
    - $r_2 = r_0 - q \times r_1 = 48 - 2 \times 18 = 12$
    - $s_2 = s_0 - q \times s_1 = 1 - 2 \times 0 = 1$
    - $t_2 = t_0 - q \times t_1 = 0 - 2 \times 1 = -2$.
3. Second iteration ($i = 2$):
    - $q = r_1 \div r_2 = 18 \div 12 = 1$
    - $r_3 = r_1 - q \times r_2 = 18 - 1 \times 12 = 6$
    - $s_3 = s_1 - q \times s_2 = 0 - 1 \times 1 = -1$
    - $t_3 = t_1 - q \times t_2 = 1 - 1 \times (-2) = 3$.
4. Third iteration ($i = 3$):
    - $q = r_2 \div r_3 = 12 \div 6 = 2$
    - $r_4 = r_2 - q \times r_3 = 12 - 2 \times 6 = 0$
    - $s_4 = s_2 - q \times s_3 = 1 - 2 \times (-1) = 3$
    - $t_4 = t_2 - q \times t_3 = -2 - 2 \times 3 = -8$.
5. Since $r_4 = 0$, we stop and return:
    - $GCD = r_3 = 6$
    - $x = s_3 = -1$
    - $y = t_3 = 3$.

Therefore:

- $GCD(48, 18) = 6$.
- The coefficients are $x = -1$ and $y = 3$.
- We can verify: $48 \times (-1) + 18 \times 3 = -48 + 54 = 6$.

So the equation $ax + by = GCD(a, b)$ is satisfied: $48(-1) + 18(3) = 6$.

This identity is critical in RSA for computing modular inverses, which rely on finding such coefficients.

0.3.2 Modular arithmetic

Modular arithmetic is the backbone of cryptographic systems like RSA, enabling secure and efficient encryption, decryption, and key exchange. By confining computations to equivalence classes, modular arithmetic limits operations to a manageable finite set of remainders $\{0, 1, \dots, n-1\}$. This reduction simplifies calculations with large numbers, allowing consistent and efficient arithmetic even when working with very large exponents or products, as is typical in cryptography. For example, modular exponentiation uses this confinement to ensure that intermediate computations remain bounded and practical, avoiding the inefficiencies of dealing with massive numbers directly.

0.3.2.1 Congruence

> **Definition (*Congruence*)**: For integers a, b, and n with $n > 0$, a is congruent to b modulo n, written $a \equiv b \pmod{n}$, if $n \mid (a - b)$.

For example, $23 \equiv 8 \pmod{5}$ because $5 \mid (23 - 8)$.

This congruence partitions integers into **congruence classes** modulo n, grouping numbers that share the same remainder when divided by n. These equivalence classes reduce infinitely many integers to a manageable finite set.

> **Theorem (*Equivalence relation*)**: Congruence modulo n satisfies the three fundamental properties of an equivalence relation:
>
> 1. Reflexivity: $a \equiv a \pmod{n}$, since $n \mid (a - a) = 0$.
> 2. Symmetry: If $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$, because $n \mid (a - b)$ implies $n \mid (b - a)$.
> 3. Transitivity: If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$, as $n \mid (a - b)$ and $n \mid (b - c)$ imply $n \mid (a - c)$.

These properties ensure that modular arithmetic forms a consistent framework for mathematical operations.

> **Theorem (*Congruence and remainders*)**: From the Division Algorithm, we have $r = a \mod b$, meaning $a \equiv b \pmod{n}$ if and only if a and b share the same remainder when divided by n. Moreover, both a and b are congruent to that common remainder:
>
> $$a \equiv b \pmod{n} \implies a \mod n = b \mod n.$$

This relationship provides a computational foundation for modular arithmetic.

Every integer modulo n can be uniquely represented by a remainder within a specific range. This principle is foundational to modular arithmetic, as it ensures that each congruence class has a single canonical representative. The following theorem formalizes this idea:

> **Theorem (*Unique representation*)**: For $n \geq 2$, every integer is congruent modulo n to exactly one element of the set $\{0, 1, 2, \dots, n-1\}$.

The notion of congruence naturally leads to the concept of a congruence class, which groups integers that share the same remainder when divided by n. These classes partition the set of integers into distinct subsets, each representing one equivalence class under congruence modulo n.

> **Definition (*Congruence class*)**: A congruence class modulo n, denoted $[a]_n$, is the set of integers equivalent to $a \pmod{n}$:
>
> $$[a]_n = \{a + kn \mid k \in \mathbb{Z}\}.$$

These classes partition $\mathbb{Z}$ into n disjoint subsets, which together form the set $\mathbb{Z}_n$, the set of equivalence classes modulo n. Each subset corresponds to a unique remainder in $\{0, 1, \dots, n-1\}$.

For example, modulo 3, the congruence classes are:

- $[0]_3 = \{..., -3, 0, 3, 6, ...\}$,
- $[1]_3 = \{..., -2, 1, 4, 7, ...\}$,
- $[2]_3 = \{..., -1, 2, 5, 8, ...\}$.

Thus, $\mathbb{Z}_3 = \{[0]_3, [1]_3, [2]_3\}$, representing all possible congruence classes modulo 3.

The concept of a congruence class provides a structured way to organize integers under modulo n. Each congruence class contains infinitely many integers that share the same modular properties. To simplify working with these classes, it is common to choose specific representatives for computations. The following definitions introduce the two most commonly used representatives:

> **Definition (*Least positive representative*)**: The least positive representative of a congruence class modulo n is the smallest nonnegative integer in the class, given by $a \mod n$.

For example, consider modulo 5:

- For $[7]_5$, the least positive representative is $7 \mod 5 = 2$.
- For $[-11]_5$, the least positive representative is $-11 \mod 5 = 4$.

> **Definition (*Least magnitude representative*)**: The least magnitude representative of a congruence class modulo n minimizes $|r|$, where $-n/2 < r \leq n/2$.

Again, for modulo 5:

- For $[7]_5$, the least magnitude representative is 2, as $-5/2 < 2 \leq 5/2$.
- For $[-11]_5$, the least magnitude representative is -1, as $-5/2 < -1 \leq 5/2$.

These representatives are key to simplifying modular arithmetic calculations and ensuring consistent results.

0.3.2.2 Addition and multiplication

In modular arithmetic, fundamental operations like addition and multiplication follow specific rules that maintain consistency within the modular system. These rules are formalized in the following theorem:

> **Theorem (Modular addition and multiplication)**: For integers a and b:
>
> $$\begin{aligned} (a+b) \mod n &= ((a \mod n) + (b \mod n)) \mod n, \\ (a \cdot b) \mod n &= ((a \mod n) \cdot (b \mod n)) \mod n. \end{aligned}$$

When comparing $\mathbb{Z}$ (integers) and $\mathbb{Z}_n$ (integers modulo n), we find both similarities and key differences in their algebraic properties:

1. **Similarities**:
    - Both have well-defined addition and multiplication operations.
    - Zero has no multiplicative inverse in both systems.
    - 1 (and -1 in $\mathbb{Z}$ or its equivalent $n-1$ in $\mathbb{Z}_n$) always has a multiplicative inverse.
2. **Differences**:
    - In $\mathbb{Z}$, only ± 1 have multiplicative inverses.
    - In $\mathbb{Z}_n$, any element a where $\gcd(a, n) = 1$ has a multiplicative inverse.
    - $\mathbb{Z}$ is infinite, while $\mathbb{Z}_n$ has exactly n elements.
    - All operations in $\mathbb{Z}_n$ are bounded by n, while operations in $\mathbb{Z}$ can grow indefinitely.

This distinction in multiplicative inverses makes $\mathbb{Z}_n$ particularly useful in applications like cryptography, where invertible elements are crucial for encryption and decryption operations.

0.3.2.3 Modular exponentiation

Modular exponentiation is a key operation in cryptography, enabling efficient computation of powers modulo a number. This operation is central to cryptographic systems like RSA, where large exponentiations are common.

> **Definition (*Modular exponentiation*)**: Modular exponentiation computes $a^b \mod n$, where a is the base, b is the exponent, and n is the modulus.

Direct computation is impractical for large b, so efficient algorithms like **square-and-multiply** are used:

> **Algorithm (*Right-to-left square-and-multiply algorithm*)**:
>
> 1. Input integers a, b, and n where a is the base, b is the exponent, and n is the modulus.
> 2. Convert b to its binary representation:
> 3. Input integer b.
> 4. Initialize $binary\_representation = []$.
> 5. While $b > 0$: 1. Append $b \mod 2$ to $binary\_representation$ 2. Update $b = b//2$.
> 6. Initialize $reversed\_representation = []$.
> 7. For each bit in $binary\_representation$, starting from the last element, append the bit to $reversed\_representation$.
> 8. Initialize $result = 1$.
> 9. For each bit m in $reversed\_representation$:
>   1. $result = (result \cdot result) \mod n$.
>   2. If $m == 1$, then $result = (result \cdot a) \mod n$.
> 10. Return $result$, which is $a^b \mod n$.

The alternative lef-to-right approach is obtained omitting steps 2.4 and 2.5, then computing step 4 on $binary\_representation$.

Let's compute $3^{13} \mod 7$:

1. Input integers $a = 3$, $b = 13$, and $n = 7$.
2. Initialize $binary\_representation = []$.
3. While $b > 0$:
    1. Append $13 \mod 2 = 1$, $binary\_representation = [1]$.
    2. Update $b = 13//2 = 6$.
    3. Append $6 \mod 2 = 0$, $binary\_representation = [1, 0]$.
    4. Update $b = 6//2 = 3$.
    5. Append $3 \mod 2 = 1$, $binary\_representation = [1, 0, 1]$.
    6. Update $b = 3//2 = 1$.
    7. Append $1 \mod 2 = 1$, $binary\_representation = [1, 0, 1, 1]$.
    8. Update $b = 1//2 = 0$.
4. $reversed\_representation = [1, 1, 0, 1]$.
5. Initialize $result = 1$.
6. First iteration ($m = 1$):
    1. $result = (result \cdot result) \mod 7 = (1 \cdot 1) \mod 7 = 1$.
    2. $result = (result \cdot a) \mod 7 = (1 \cdot 3) \mod 7 = 3$.
7. Second iteration ($m = 1$):

   1. $result = (result \cdot result) \mod 7 = (3 \cdot 3) \mod 7 = 2$.
   2. $result = (result \cdot a) \mod 7 = (2 \cdot 3) \mod 7 = 6$.

8. Third iteration ($m = 0$):

   1. $result = (result \cdot result) \mod 7 = (6 \cdot 6) \mod 7 = 1$.
   2. No multiplication since $m = 0$.

9. Fourth iteration ($m = 1$):

   1. $result = (result \cdot result) \mod 7 = (1 \cdot 1) \mod 7 = 1$.
   2. $result = (result \cdot a) \mod 7 = (1 \cdot 3) \mod 7 = 3$.

We get $result = 3$, so $3^{13} \mod 7 = 3$.

0.3.2.4 Modular inverse

The modular inverse is a fundamental concept in number theory and cryptography. It is essential for solving modular equations, whose solution is determined within a given modulus n, meaning the values satisfy the equation in terms of congruence relations.

> **Definition (*Modular inverse*)**: The modular inverse of an integer a modulo n, denoted as $a^{-1} \pmod n$, is an integer x such that:
>
> $$a^{-1} \pmod n = a \cdot x \equiv 1 \pmod n.$$

The modular inverse relies on several fundamental principles in number theory, including conditions for existence, efficient computation methods, and connections to primality tests. Below we will outline these key theorems, algorithms, and applications.

> **Theorem (*Existence of modular inverse*)**: An integer a has a modular inverse modulo n if and only if $\gcd(a, n) = 1$. If the modular inverse exists, it is unique modulo n.

A proof sketch can be given leveraging the Bézout's Identity, because if $\gcd(a, n) = 1$, then there exist integers x and y such that:

$$ax + ny = 1.$$

Taking this equation modulo n, we get:

$$ax \equiv 1 \pmod n$$

proving that x is the modular inverse of a modulo n. The uniqueness follows from the properties of congruence classes.

The modular inverse can be computed using the Extended Euclidean Algorithm. This algorithm builds on the general method of finding the greatest common divisor (GCD) while also determining the coefficients that satisfy Bézout's Identity. Here, it is specialized to calculate the modular inverse by assuming $\gcd(a, n) = 1$. The steps are given in the following algorithm:

> **Algorithm (*Modular inverse via Extended Euclidean Algorithm*)**:
>
> 1. Input integers a and n, where $\gcd(a, n) = 1$.
> 2. Initialize:
>   1. $r_0 = n$, $r_1 = a$ (remainder terms)
>   2. Coefficients for n: $s_0 = 1$, $s_1 = 0$ (coefficients for n)
>   3. Coefficients for a: $t_0 = 0$, $t_1 = 1$ (coefficients for a).
> 3. While $r_1 \neq 0$:
>   1. $q = \lfloor r_0 / r_1 \rfloor$ (quotient)
>   2. $r_2 = r_0 - q \cdot r_1$

    3. $s_2 = s_0 - q \cdot s_1$
    4. $t_2 = t_0 - q \cdot t_1$
    5. $r_0 = r_1$, $r_1 = r_2$, $s_0 = s_1$, $s_1 = s_2$, $t_0 = t_1$, $t_1 = t_2$.
4. Return $a^{-1} \pmod{n}$: $a^{-1} \pmod{n} = t_0 \mod n$.

Or defining a function EEA for the Extended Euclidean Algorithm as EEA : $(a, b) \to (\text{GCD}(a, b), x, y)$, where x is the Bézout coefficient for a:

Algorithm (*Modular inverse via Extended Euclidean Algorithm*):

1. Input integers a and n, where $\gcd(a, n) = 1$.
2. Call EEA : $(a, n) \to (\text{GCD}(a, n), x, y)$.
3. If $\text{GCD}(a, n) \neq 1$ then return "No modular inverse exists".
4. Return $a^{-1} \pmod{n}$: $a^{-1} \pmod{n} = x \mod n$.

Fermat's Little Theorem enables efficient computation of modular inverses and serves as a basis for primality testing:

Theorem (*Fermat's Little Theorem*): If p is a prime number and a is an integer such that $p \nmid a$, then:

$$a^{p-1} \equiv 1 \pmod{p}.$$

To find the modular inverse, we rewrite a^{p-1} as:

$$a^{p-1} = a \cdot a^{p-2}$$

Substituting this into Fermat's Little Theorem gives:

$$a \cdot a^{p-2} \equiv 1 \pmod{p}$$

By definition, the modular inverse a^{-1} satisfies:

$$a \cdot a^{-1} \equiv 1 \pmod{p}$$

Comparing this with the result above, we conclude that a^{p-2} must be the modular inverse of a modulo p;

$$a^{-1} \equiv a^{p-2} \pmod{p}.$$

This theorem provides a more efficient way to compute modular inverses when n is prime compared to the Extended Euclidean Algorithm.

Remark: $p - 1$ is the smallest exponent satisfying $a^{p-1} \equiv 1 \pmod{p}$ for prime p.

By Fermat's Little Theorem, for any integer a such that $p \nmid a$:

$$a^{p-1} \equiv 1 \pmod{p}.$$

Assume, for contradiction, that there exists a smaller positive integer $k < p - 1$ such that:

$$a^k \equiv 1 \pmod{p}.$$

If $a^k \equiv 1 \pmod{p}$, then we can write $p - 1$ using the Division Algorithm:

$$p - 1 = q \cdot k + r,$$

where q and r are integers, and $0 \leq r < k$.

Substituting into a^{p-1}, this simplifies to:

$$a^{p-1} = a^{q \cdot k + r} = (a^k)^q \cdot a^r.$$

Since $a^k \equiv 1 \pmod{p}$, we get:

$$(a^k)^q \equiv 1^q \equiv 1 \pmod{p}.$$

Thus:

$$a^{p-1} \equiv a^r \pmod{p}.$$

By Fermat's Little Theorem, $a^{p-1} \equiv 1 \pmod{p}$, so:

$$a^r \equiv 1 \pmod{p}.$$

However, $r < k$, contradicting the assumption that k is the smallest positive integer such that $a^k \equiv 1 \pmod{p}$. Hence, no such smaller $k < p-1$ exists, and $p-1$ must be the smallest exponent satisfying $a^{p-1} \equiv 1 \pmod{p}$.

Equivalently, if $a^{p-1} \not\equiv 1 \pmod{p}$ for some a with $\gcd(a,p) = 1$, then p is composite. However, the converse of Fermat's Little Theorem is not true: if $a^{n-1} \equiv 1 \pmod{n}$ for all a with $\gcd(a,n) = 1$, then n is not necessarily a prime. In other words, Fermat's Little Theorem is effective for disproving primality (when the congruence fails), it is insufficient for proving it.

Numbers that satisfy Fermat's Little Theorem are defined as **Carmichael numbers**:

1. 561: $561 = 3 \cdot 11 \cdot 17$.
2. 1105: $1105 = 5 \cdot 13 \cdot 17$.
3. 1729: $1729 = 7 \cdot 13 \cdot 19$.
4. 2465: $2465 = 5 \cdot 17 \cdot 29$.
5. 2821: $2821 = 7 \cdot 13 \cdot 31$.

In 1994, it was proven by Alford, Granville, and Pomerance[24] that there are infinitely many Carmichael numbers. However, they become increasingly sparse as numbers grow larger.

Defining the function Square-and-multiply : $(a,b,n) \to a^b \mod n$, we can apply the following test for primality to n, choosing as many a as possible, where $1 < a < n$:

Algorithm (*Fermat primality test*):

1. Input integers n and a.
2. Call Square-and-multiply : $(a, n-1, n) \to a^{n-1} \mod n$.
3. If $a^{n-1} \not\equiv 1 \pmod{n}$ then "n is composite" else "n is likely prime".

As an alternative to Fermat primality test, there is a brute-force approach for determining whether a number n is prime by dividing n by smaller prime numbers up to $\sqrt{n}$:

Algorithm (*Trial division primality test*):

1. Input integer $n > 1$.
2. If $n = 2$, return "n is prime".
3. If $n \mod 2 = 0$, return "n is composite".
4. For d where $d = 2k+1$ and $1 \leq k \leq \lfloor \sqrt{n}/2 \rfloor$:
    1. If $n \mod d = 0$ then return "n is composite".

[24] Alford, W. R., Granville, A., & Pomerance, C. (1994). **There are infinitely many Carmichael numbers**. *Annals of Mathematics*, 139(3), 703–722. DOI

5. Return "n is prime".

The test involves at most $\sqrt{n}$ divisions, making it computationally expensive for large n, having a time complexity of $\mathcal{O}(\sqrt{n})$, assuming that a single modulo operation is $\mathcal{O}(1)$. The algorithm becomes more efficient when using a precomputed list of primes up to $\sqrt{n}$, skipping unnecessary checks for non-prime divisors.

For very large n, the size of n impacts the complexity of each modulo operation. If n has b bits, then the modulo operation takes $\mathcal{O}(b^2)$ time using simple arithmetic or $\mathcal{O}(b \log b)$ with optimized algorithms. In such cases, the overall complexity becomes $\mathcal{O}(\sqrt{n} \cdot \text{modulo complexity})$.

A more refined algorithm is the Miller-Rabin primality test[25], which is much faster and more robust than trial division and the basic Fermat test.

0.3.2.5 Euler's theorem

Euler's theorem is a fundamental result in number theory that generalizes Fermat's little theorem. It provides a condition for modular exponentiation when the base and modulus are coprime.

> **Definition (*Euler totient function*)**: Let n be a positive integer, the Euler totient function, denoted as $\phi(n)$, counts the number of positive integers less than n that are relatively prime to n[26]:
>
> $$\phi(n) = \#\{a \in \mathbb{Z} : 1 \leq a < n, \gcd(a, n) = 1\}.$$

Properties of the Euler totient function:

1. If p is a prime number, then:
$$\phi(p) = p - 1.$$

2. If n has the prime factorization $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, then $\phi(n)$ is given by:
$$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

3. The totient function is **multiplicative**, meaning that if m and n are coprime, then:
$$\phi(mn) = \phi(m)\phi(n).$$

> **Theorem (*Euler*)**: Let a and n be coprime integers (i.e., $\gcd(a, n) = 1$). Then:
>
> $$a^{\phi(n)} \equiv 1 \pmod{n}.$$

This theorem generalizes Fermat's little theorem, which is a special case when n is prime, where $\phi(p) = p-1$ and thus:

$$a^{p-1} \equiv 1 \pmod{p}.$$

Euler's theorem is widely used in cryptographic algorithms, particularly in the RSA encryption scheme, where it is employed to compute modular inverses efficiently. The theorem allows us to find the modular inverse of a modulo n when $\gcd(a, n) = 1$, using:

$$a^{-1} \equiv a^{\phi(n)-1} \pmod{n}.$$

For example, to compute $3^{\phi(25)} \mod 25$:

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[25] Miller, G. L. (1976). **Riemann's hypothesis and tests for primality**. *Journal of Computer and System Sciences*, 13(3), 300–317. DOI. Rabin, M. O. (1980). **Probabilistic algorithm for testing primality**. *Journal of Number Theory*, 12(1), 128–138. DOI

[26] The symbol # denotes the **cardinality** (size) of a set, which represents the number of elements in that set. It is commonly used in combinatorics and number theory. An alternative notation for the cardinality of a set S is $|S|$, which is more prevalent in set theory.

- First, calculate $\phi(25)$:
$$\phi(25) = 25\left(1 - \frac{1}{5}\right) = 25 \times \frac{4}{5} = 20.$$

- Then,
$$3^{20} \equiv 1 \pmod{25}.$$

Thus, using Euler's theorem, we can directly conclude that any power of 3 raised to 20 will be congruent to 1 modulo 25.

0.3.2.6 Carmichael's theorem

Carmichael's theorem refines Euler's theorem by defining the smallest exponent that guarantees modular exponentiation behaves predictably for all coprime bases. This exponent is given by the **Carmichael function**, denoted as $\lambda(n)$.

> **Theorem (*Carmichael*)**: Let n be a positive integer. The function $\lambda(n)$ is the smallest integer such that:
>
> $$a^{\lambda(n)} \equiv 1 \pmod{n}$$
>
> for all $a \in \mathbb{Z}_n^*$.

This function provides a stricter condition than Euler's theorem and guarantees that for any integer a coprime to n, the smallest exponent e for which $a^e \equiv 1 \pmod{n}$ is always a divisor of $\lambda(n)$. In other words, the values of e that satisfy this condition must be factors of $\lambda(n)$. By definition, $\lambda(n)$ is also always a divisor of $\phi(n)$:
$$\lambda(n) \mid \phi(n).$$

To compute $\lambda(n)$, we use the least common multiple (lcm) function, which determines the smallest positive integer that is divisible by a given set of numbers.

> **Definition (*least common multiple)***: The lcm of two integers a and b, denoted as $\operatorname{lcm}(a, b)$, is the smallest positive integer that is a multiple of both a and b:
>
> $$\operatorname{lcm}(a, b) = \frac{|a \cdot b|}{\gcd(a, b)},$$
>
> where $\gcd(a, b)$ is the greatest common divisor of a and b.

This concept extends naturally to multiple integers, allowing for an efficient computation of $\lambda(n)$ when n has multiple prime factors.

Example: consider $n = 18$, which has the prime factorization $n = 2 \times 3^2$ and we compute:

$$\lambda(2) = 1, \quad \lambda(3^2) = \phi(3^2) = 3.$$

Since n consists of relatively prime factors, we use the least common multiple:

$$\lambda(18) = \operatorname{lcm}(\lambda(2), \lambda(3^2)) = \operatorname{lcm}(1, 3) = 3.$$

This tells us that for any integer a coprime to 18, the smallest exponent satisfying $a^e \equiv 1 \pmod{18}$ must be a multiple of 3.

To compute $\lambda(n)$ efficiently for any integer n, we apply the following structured approach, which relies on prime power properties and the least common multiple:

1. If n is a power of a single prime, p^e, we compute $\lambda(n)$ as follows:
    - When p is an **odd prime** or $e \leq 2$, $\lambda(p^e)$ is simply $\phi(p^e)$, the Euler totient function.

- When $p = 2$ and $e \geq 3$, the exponent is halved:

$$\lambda(p^e) = \frac{1}{2}\phi(p^e).$$

  This accounts for the behavior of powers of 2 in modular arithmetic, ensuring that exponentiation remains consistent with Carmichael's theorem.

2. If n is a product of multiple **pairwise relatively prime numbers** $n_1, n_2, ..., n_r$, the Carmichael function is computed using the least common multiple:

$$\lambda(n) = \text{lcm}(\lambda(n_1), ..., \lambda(n_r)).$$

   This ensures that $\lambda(n)$ is compatible with each individual modulus, making it the smallest exponent that satisfies $a^{\lambda(n)} \equiv 1 \pmod{n}$ for all coprime bases a.

3. If n is given in its **prime factorized form**:

$$n = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r},$$

   then $\lambda(n)$ is computed as:

$$\lambda(n) = \text{lcm}(\lambda(p_1^{e_1}), ..., \lambda(p_r^{e_r})).$$

   This approach ensures that we first compute λ for each prime power individually (using the prime power rule) and then combine the results using the least common multiple.

By following these structured steps, we can efficiently compute $\lambda(n)$ for any integer n, making it a practical function for number theory and cryptographic applications.

Now, let's introduce numbers that pass Fermat's primality test despite being composite:

Definition (*Carmichael number*): a Carmichael number is a composite number n that satisfies:

$$a^{n-1} \equiv 1 \pmod{n},$$

for all a coprime to n.

A number is Carmichael if and only if $\lambda(n)$ divides $n - 1$. For example, consider:

$$\lambda(1105) = \text{lcm}(\lambda(5), \lambda(13), \lambda(17)) = \text{lcm}(4, 12, 16) = 48.$$

Since 48 divides $1105 - 1 = 1104$, this confirms that 1105 is a Carmichael number.

Carmichael numbers are important in cryptography because they can deceive certain primality tests, making them crucial in designing secure encryption algorithms.

0.3.2.7 Generators

A **generator** is a number that, when multiplied by itself multiple times (using modular arithmetic), cycles through many or all possible values before repeating. This cycle length is called the **multiplicative order** of the number. In simple terms, it tells us how long it takes for the number to "reset" back to 1 when repeatedly multiplied by itself modulo n.

For example, if we take 3 and multiply it repeatedly modulo 7:

$$3^1 \equiv 3 \pmod{7}, \quad 3^2 \equiv 9 \equiv 2 \pmod{7}, \quad 3^3 \equiv 6 \pmod{7}, \quad 3^4 \equiv 4 \pmod{7}, \quad 3^5 \equiv 5 \pmod{7}, \quad 3^6 \equiv 1 \pmod{7}.$$

Here, the number 3 cycles through all possible values before repeating, making it a generator modulo 7.

This concept is useful in cryptography because some security systems rely on the fact that **finding how many times you need to multiply a number to get back to 1 (the order) is hard to figure out**. This is used in encryption methods like **Diffie-Hellman key exchange**, which helps people securely share secret keys over public networks.

More formally:

Definition (*Multiplicative order*): given a positive integer n and an element $a \in \mathbb{Z}_n^*$, the multiplicative order of a, denoted as $\text{ord}_n(a)$, is the smallest integer $e > 1$ such that:

$$a^e \equiv 1 \pmod{n}.$$

Properties of the multiplicative order:

1. The order of a always divides $\phi(n)$, a consequence of Euler's theorem.
2. For any integer i, $a^i \equiv 1 \pmod{n}$ if and only if $\text{ord}_n(a) \mid i$.

Definition (*Generator*): an element $g \in \mathbb{Z}_n^*$ is called a generator (or a primitive root) of $\mathbb{Z}_n^*$ if its order is maximal, meaning:

$$\text{ord}_n(g) = \phi(n).$$

This implies that g can produce all elements of $\mathbb{Z}_n^*$ through exponentiation.

A generator a of $\mathbb{Z}_n^*$ remains a generator under exponentiation if and only if the exponent i is chosen correctly, as stated in the following theorem.

Theorem (*Generator preservation*): if a is a generator of $\mathbb{Z}_n^*$, then for any integer i, the element $b \equiv a^i \pmod{n}$ is also a generator of $\mathbb{Z}_n^*$ if and only if:

$$\gcd(i, \phi(n)) = 1.$$

This property is essential in cryptographic protocols such as Diffie-Hellman key exchange and RSA encryption, where security relies on the fact that, while it is easy to compute exponentiation modulo n, finding the original exponent i given only the result $a^i \mod n$ (a problem known as the **discrete logarithm problem**) is computationally difficult.

To illustrate the concept of a generator, consider $\mathbb{Z}_{10}^*$, which consists of the elements:

$$\mathbb{Z}_{10}^* = \{1, 3, 7, 9\}.$$

The totient function gives $\phi(10) = 4$, so a generator g must satisfy $\text{ord}_{10}(g) = 4$.

Checking powers of 3 modulo 10:

$$3^1 \equiv 3 \pmod{10}, \quad 3^2 \equiv 9 \pmod{10}, \quad 3^3 \equiv 7 \pmod{10}, \quad 3^4 \equiv 1 \pmod{10}.$$

Since the order of 3 is 4, it is a generator of $\mathbb{Z}_{10}^*$.

0.3.2.8 Chinese Remainder Theorem

The **Chinese Remainder Theorem (CRT)** is a fundamental result in number theory that provides a way to solve systems of simultaneous congruences when the moduli are pairwise relatively prime.

Theorem (*Chinese Remainder*): let $n_1, n_2, \dots, n_k$ be pairwise relatively prime positive integers, then for any given integers $r_1, r_2, \dots, r_k$, the system of congruences:

$$x \equiv r_i \pmod{n_i}, \quad \text{for } i = 1, \dots, k,$$

has a unique solution modulo $n = n_1 n_2 \cdots n_k$.

The solution is given by:

$$x \equiv \sum_{i=1}^{k} r_i \cdot c_i \cdot m_i \pmod{n},$$

where $m_i = \frac{n}{n_i}$ and c_i is the modular inverse of m_i modulo n_i, satisfying $c_i m_i \equiv 1 \pmod{n_i}$.

Solving systems with CRT algorithm:

1. Compute $n = n_1 n_2 \cdots n_k$.
2. For each i, compute $m_i = n/n_i$.
3. Compute the modular inverse $c_i \equiv m_i^{-1} \pmod{n_i}$.
4. Compute $x = \sum_{i=1}^{k} r_i \cdot c_i \cdot m_i$ and reduce modulo n.

For example, solve the system:

$$x \equiv 4 \pmod 9, \quad x \equiv 7 \pmod{13}, \quad x \equiv 2 \pmod{17}.$$

Since 9, 13, and 17 are pairwise relatively prime, we compute: - $n = 9 \times 13 \times 17 = 1989$. - $m_1 = 1989/9 = 221$, $m_2 = 1989/13 = 153$, $m_3 = 1989/17 = 117$. - Compute the modular inverses: - $c_1 = 221^{-1} \equiv 4 \pmod 9$. - $c_2 = 153^{-1} \equiv 12 \pmod{13}$. - $c_3 = 117^{-1} \equiv 10 \pmod{17}$. - Compute x:

$$x \equiv (4 \times 4 \times 221 + 7 \times 12 \times 153 + 2 \times 10 \times 117) \pmod{1989}.$$

Evaluating, we find $x \equiv 8776 \equiv 418 \pmod{1989}$.

Thus, the unique solution modulo 1989 is $x \equiv 418 \pmod{1989}$.

This demonstrates the power of the CRT in reconstructing values from modular congruences.

0.3.2.9 Quadratic residues

Definition (*Quadratic residue*): a number $a \in \mathbb{Z}_n^*$ is a quadratic residue modulo n if there exists an integer x such that:

$$a \equiv x^2 \pmod n.$$

Otherwise, a is called a **quadratic non-residue** modulo n.

This theorem allows us to efficiently determine whether a number is a quadratic residue:

Theorem(*Euler's criterion*): Let p be an odd prime and $a \in \mathbb{Z}_p^*$. Then:

- If a is a quadratic residue modulo p:

$$a^{(p-1)/2} \equiv 1 \pmod p.$$

- If a is a quadratic non-residue modulo p:

$$a^{(p-1)/2} \equiv -1 \pmod p.$$

Definition (*Legendre symbol*): the **Legendre symbol** is a function that determines whether an integer a is a quadratic residue modulo an odd prime p; it is defined as:

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } p \mid a, \\ 1, & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo } p. \end{cases}$$

Using Euler's criterion, we compute:

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod p.$$

Then, we can state:

Theorem (*Properties of the Legendre symbol*): let p be an odd prime and a, b be integers, the Legendre symbol satisfies the following properties:

1. $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod p$ (Euler's criterion).
2. If $a \equiv b \pmod p$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

3. $\left(\frac{a \cdot b}{p}\right) = \left(\frac{a}{p}\right) \times \left(\frac{b}{p}\right)$ (Multiplicative property) .
4. $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$.
5. If p and q are odd primes (Law of quadratic reciprocity) :
   - If $p \equiv 1 \pmod 4$ or $q \equiv 1 \pmod 4$, then $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$.
   - If $p \equiv q \equiv 3 \pmod 4$, then $\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right)$.

As an example, for $p = 19$, determine whether $a = 11$ is a quadratic residue:

$$11^{(19-1)/2} = 11^9 \equiv -1 \pmod{19}.$$

Since the result is -1, 11 is a quadratic non-residue modulo 19.

The **Jacobi symbol** generalizes the Legendre symbol for odd composite moduli:

> **Definition (*Jacobi symbol*)**:
> $$\left(\frac{a}{n}\right) = \prod_{i=1}^{r} \left(\frac{a}{p_i}\right)^{e_i},$$
> where $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ is the prime factorization of n.

The Jacobi symbol shares properties with the Legendre symbol but does not definitively indicate whether a is a quadratic residue modulo n.

If n is an odd composite integer, determining whether a with $\left(\frac{a}{n}\right) = 1$ is a quadratic residue modulo n is called the **Quadratic Residuosity Problem (QR)**. This problem is computationally difficult without knowing the factorization of n, linking it to cryptographic security.

The QP is central to probabilistic encryption schemes such as the Goldwasser-Micali cryptosystem, where the difficulty of distinguishing quadratic residues from non-residues provides semantic security. It is also relevant in zero-knowledge proofs and commitment schemes, where proving knowledge of a square root modulo n can be done without revealing the value itself. By leveraging the hardness of the QR problem, cryptographic systems can achieve stronger security guarantees, making it an essential tool in modern cryptography.

0.3.2.10 Higher-order residues

> **Definition (*rth residue modulo*)**: an integer $a \in \mathbb{Z}_n^*$ is called an **rth residue modulo** n if there exists an integer $x \in \mathbb{Z}_n^*$ such that:
> $$a \equiv x^r \pmod n.$$
> If no such x exists, then a is called an **rth non-residue** modulo n.

> **Lemma (*Structure of higher-order residues*)**: 1. The set of rth residues modulo n that are relatively prime to n forms a subgroup of $\mathbb{Z}_n^*$. 2. Each rth residue modulo n has the same number of rth roots.

Determining whether an element is an rth residue modulo n is known as the **Higher Residuosity Problem (HRP)**. When n is composite and its factorization is unknown, this problem is computationally difficult, making it useful in cryptographic settings. A special case of the HRP occurs when r is replaced by n and n is replaced by n^2, where $n = pq$ is a product of two distinct odd primes. This version is called the **Composite Residuosity Problem (CRP)** and is used in cryptographic protocols such as Paillier encryption.

> **Lemma (*Residue completeness condition*)**: if $\gcd(r, \phi(n)) = 1$, then every integer in $\mathbb{Z}_n^*$ is an rth residue modulo n.

0.3.2.11 Residue classes

Definition (*Residue class*): For fixed integers r, n, y with $y \in \mathbb{Z}_n^*$, an element $w \in \mathbb{Z}_n^*$ is said to belong to a residue class if it can be expressed as:

$$w \equiv y^m \cdot u^r \pmod{n},$$

for some integer m and some $u \in \mathbb{Z}_n^*$. The residue class of w is denoted as:

$$RC[m] = \{w \in \mathbb{Z}_n^* : w \equiv y^m u^r \pmod{n} \text{ for some } u \in \mathbb{Z}_n^*\}.$$

In particular, $RC[0]$ represents the set of rth residues modulo n.

Lemma (*Addition and inversion in residue classes*): 1. If $w_1 \in RC[m_1]$ and $w_2 \in RC[m_2]$, then $w_1 \cdot w_2 \in RC[m_1 + m_2]$. 2. If $w \in RC[m]$, then $w^{-1} \in RC[-m]$.

The problem of determining the residue class of a given w is conjectured to be computationally difficult and is known as the **Residue Class Problem (RCP)**. A special case arises when n is composite, known as the **Composite Residuosity Class Problem (CRP)**, forming the basis of secure cryptographic schemes.

A fundamental question in this context is determining the number of distinct rth roots a given residue has. This is particularly important in cryptographic applications, where knowing the structure of these roots can influence security guarantees. The following theorem establishes a precise condition under which an rth residue has exactly r distinct roots:

Theorem (*Uniqueness and count of rth roots*): Let $y \in \mathbb{Z}_n^*$ be an rth residue modulo n. If $r \mid \phi(n)$ and $\gcd(r, \phi(n)/r) = 1$, then y has exactly r distinct rth roots.

Residue classes provide a structured way to categorize elements of $\mathbb{Z}_n^*$ based on their power relationships, enabling cryptographic operations such as **trapdoor functions**, which allow for efficient decryption while keeping encryption computationally difficult, and **homomorphic encryption** schemes, which enable computations on encrypted data without needing decryption. These concepts are foundational in privacy-preserving cryptographic protocols, such as the **Paillier cryptosystem**, which relies on the Composite Residuosity Problem for encryption, **RSA-based voting schemes**, which utilize quadratic residues for secure tallying, and **homomorphic encryption frameworks** like **ElGamal encryption**, which allow operations on encrypted data without decryption. These methods are crucial in secure voting systems, digital signatures, and confidential data processing.

0.3.2.12 Random number generators

In cryptographic applications, particularly homomorphic encryption, random numbers are essential for security. A function RANDINT(a, b) is defined to return a uniformly selected integer from the range $[a, b]$. Ensuring unpredictability in random numbers is a fundamental challenge in cryptographic design.

Random number generators (RNGs) are categorized into: - **True Random Number Generators (TRNGs)**: Based on physical processes such as thermal noise or electronic circuit randomness, offering high security against prediction. - **Deterministic Random Number Generators (DRNGs)**: Algorithmic methods that produce sequences from an initial seed, commonly used in cryptographic protocols.

A DRNGs is fast and efficient but can be predictable if its starting value (seed) is not chosen securely. In contrast, a TRNG relies on physical processes to generate randomness, making it more secure but often slower and requiring specialized hardware. To balance speed and security, many systems use a hybrid approach, where a TRNG provides an initial high-quality seed, and a DRNG expands it to generate more random values efficiently.

When generating a cryptographic key, it's important to use a secure random number generator (RNG) to ensure unpredictability. A common approach is to use a **Cryptographically Secure Pseudorandom Number Generator (CSPRNG)**, which expands a small amount of highly unpredictable data (called a **seed**) into a long sequence of random values.

A **high-entropy seed** means the initial data used to start the generator is difficult to guess, coming from unpredictable sources like hardware noise, mouse movements, or system timings.

One well-known approach is the **Fortuna**[27] algorithm, a security-focused random number generator that works as follows:

1. **Collect random data** from multiple sources, such as user input timings, network activity, or hardware randomness.
2. **Mix the collected data** using a cryptographic hash function to update an internal state securely.
3. **Generate random values** using a block cipher (e.g., AES in counter mode) to ensure strong randomness.
4. **Periodically refresh the seed** to prevent attackers from predicting future random outputs.

This method ensures that even if part of the system state is exposed, the generated numbers remain secure and unpredictable.

For cryptographic security, DRNGs should satisfy: 1. Uniform distribution: ensuring statistical randomness. 2. Independence: ensuring no correlation between outputs. 3. Unpredictability: preventing attackers from inferring future values.

Secure choices for transition functions include cryptographic hash functions and block ciphers, ensuring resistance to attacks. Well-known cryptographic DRNGs include also:

- **Yarrow**[28]: used in macOS for secure randomness.
- **NIST SP 800-90A DRBG**[29]: a standardized family of deterministic random bit generators.

These RNGs play a crucial role in encryption schemes, key generation, digital signatures, and secure multiparty computation.

0.3.3 Group theory

Cryptography frequently leverages **groups**, which are sets endowed with an operation that behaves in a mathematically predictable way. This predictability makes them ideal for building secure protocols, ranging from Diffie–Hellman key exchange to advanced homomorphic encryption. In this chapter, we outline core definitions and theorems that underpin these constructions.

0.3.3.1 Definition

A group $(G, \star)$ is a set G equipped with a binary operation "$\star$" satisfying four key properties:

1. **Closure**: For any $a, b \in G$, the result of $a \star b$ is still in G.
2. **Associativity**: For any $a, b, c \in G$, $(a \star b) \star c = a \star (b \star c)$.
3. **Identity element**: There exists an element $e \in G$ such that $e \star a = a \star e = a$ for all $a \in G$.
4. **Inverse element**: Each $a \in G$ has an element a^{-1} satisfying $a \star a^{-1} = a^{-1} \star a = e$.

A classic example is the set of integers $\mathbb{Z}$ under addition $(+)$:
- 0 is the identity (adding zero changes nothing).
- Every integer n has an inverse $-n$.

[27] Fortuna is a CSPRNG designed by cryptographers Bruce Schneier and Niels Ferguson, introduced in their 2003 book Practical Cryptography. It is named after the Roman goddess of chance, Fortuna. Fortuna is designed to be a secure PRNG that can also accept random inputs from analog sources, enhancing its security. It has been adopted in systems like FreeBSD's /dev/random since version 11 and in Apple's operating systems since early 2020. See *Schneier on Security* blog post.

[28] Kelsey, J., Schneier, B., & Ferguson, N. (1999). **Yarrow-160: Notes on the design and analysis of the Yarrow cryptographic pseudorandom number generator**. *Selected Areas in Cryptography*, 13–33. DOI.

[29] Barker, E., & Kelsey, J. (2015). **Recommendation for Random Number Generation Using Deterministic Random Bit Generators**. *SP 800-90A Rev. 1. National Institute of Standards and Technology. DOI

In **multiplicative** notation—common in cryptography—a group might be a set of "invertible" numbers modulo n. For instance, $\mathbb{Z}_p^*$ (integers $\{1, 2, \dots, p-1\}$ under multiplication mod p) forms a group if p is prime.

> **Why it matters**: Groups let us "move around" elements predictably. For a cryptosystem, that often means repeatedly applying an operation—e.g., exponentiation mod p—without leaving the safety of the group.

0.3.4 2. Multiplicative vs. Additive Notation

Groups used in cryptography often come in two flavors:

- **Multiplicative groups**, denoted $(G, \times)$. We write the identity as 1 and the inverse of g as g^{-1}. An example is $\mathbb{Z}_p^*$, the nonzero integers mod p.

- **Additive groups**, denoted $(G, +)$. We write the identity as 0 and the inverse of a as $-a$. An example is $(\mathbb{Z}, +)$ or $(\mathbb{Z}_n, +)$.

Both notations obey the same fundamental group axioms. Cryptographic schemes switch between them depending on context; for instance, elliptic-curve cryptography usually adopts additive notation, while classical $\mathbb{Z}_p^*$ cryptography uses multiplicative.

0.3.5 3. Order of a Group and Order of an Element

1. **Order of a group** $|G|$ (or $\mathrm{ord}(G)$) is the number of elements in G when G is finite. In cryptographic contexts, we often choose large prime-order groups to avoid certain attacks.
2. **Order of an element** $g \in G$ is the smallest positive integer k such that

$$g^k = e \quad \text{(multiplicative)},$$

   or

$$k \cdot g = 0 \quad \text{(additive)},$$

   where e is the identity. If no finite k satisfies this, we say the element has infinite order (common in some infinite groups, but for cryptography we focus on finite ones).

   Why it matters: The element's order dictates cycles in exponentiation. For instance, in $\mathbb{Z}_p^*$, exponentiating an element g repeatedly will eventually loop back to 1. Cryptosystems like Diffie–Hellman rely on the difficulty of reversing such exponentiations (the "discrete logarithm problem").

0.4 12. Homomorphisms

A **group homomorphism** from one group $(G, \star)$ to another $(H, \circ)$ is a map

$$f : G \to H$$

that **preserves** the group operation. Formally, for all $x, y \in G$,

$$f(x \star y) = f(x) \circ f(y).$$

In other words, performing the operation in G first—then applying f—has the same effect as applying f *individually* to each of x and y, and then using the operation in H. If f is also bijective (one-to-one and onto), we call it an **isomorphism**, and say G and H are *isomorphic*.

0.4.1 12.1 Illustrative Examples

Below are several examples that highlight different ways homomorphisms appear:

Example 1: Reduction mod n

- **Groups**: $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$.
- **Map**: $f(x) = x \bmod n$.

Check homomorphism:

1. In $\mathbb{Z}$, the operation is integer addition $(+)$.
2. In $\mathbb{Z}_n$, the operation is addition modulo n.
3. For any $x, y \in \mathbb{Z}$,

$$f(x+y) = (x+y) \bmod n \quad \text{and} \quad f(x) \ + \ f(y) = (x \bmod n) \ + \ (y \bmod n).$$

   Since $(x+y) \bmod n = (x \bmod n) + (y \bmod n) \bmod n$, the two sides match in $\mathbb{Z}_n$.

Hence f respects addition and is a group homomorphism. This is an extremely common map in modular arithmetic.

Example 2: Exponential map exp

- **Groups**: $(\mathbb{R}, +)$ and $(\mathbb{R}_{>0}, \times)$.
- **Map**: $f(x) = e^x$.

Check homomorphism:

1. $\mathbb{R}$ under "+" is an additive group; $\mathbb{R}_{>0}$ under "$\times$" is multiplicative.
2. For real numbers x, y,

$$f(x+y) = e^{x+y} = e^x \cdot e^y = f(x) \ \times \ f(y).$$

Hence exp is a homomorphism from an additive group to a multiplicative group. While exp is *not* typically used directly as a "cryptographic" function (because $\mathbb{R}$ is infinite-precision), the idea of mapping addition to multiplication is the same principle that shows up in exponentiation mod p.

Example 3: Encryption as a Homomorphism

Suppose we have a cryptosystem $\Pi = (P, C, K, E, D)$ where:

- P = Plaintext space (a group with operation $\star$).
- C = Ciphertext space (a group with operation $\odot$).
- K = Key space, from which we draw a key k.
- E_k = Encryption function.
- D_k = Decryption function.

If Π is **additively homomorphic**, then for any plaintexts $m_1, m_2 \in P$,

$$D_k(E_k(m_1) \odot E_k(m_2)) \ = \ m_1 \ \star \ m_2.$$

Such a map E_k is effectively a group homomorphism (up to decryption). Concretely:

1. **Additively homomorphic**:
   $\star$ = "+" in plaintext, $\odot$ = some "$\oplus$" or "+ mod N" in ciphertext.
   - **Paillier encryption** is a classic example that supports $E(m_1 + m_2) \equiv E(m_1) \cdot E(m_2)$.

2. **Multiplicatively homomorphic**:
   $\star$ = "$\times$" in plaintext, $\odot$ = "$\times$" mod n in ciphertext.
   - **RSA** in raw form is multiplicatively homomorphic over $\mathbb{Z}_n^*$.

   Real cryptosystems often require additional mechanisms to prevent malicious manipulations. But the fundamental "operation preserved under encryption" is precisely a homomorphism property.

0.4.2 12.2 Partial vs. Full Homomorphisms

- A **partially homomorphic** scheme supports only one operation homomorphically (e.g., *either* addition *or* multiplication). RSA and Paillier each provide one such property in their simplest form.

- A **somewhat homomorphic** scheme supports both addition and multiplication but only for a limited number of times.

- A **fully homomorphic** scheme (FHE) allows *arbitrary* sequences of additions and multiplications on ciphertexts without decryption. This is far more advanced and underpins privacy-preserving computation in the cloud.

Why it matters: Homomorphisms formalize how operations on plaintexts get mirrored by operations on ciphertexts—an essential idea in cryptography whenever we want to manipulate data without decrypting it first.

0.4.3 12.3 Homomorphisms and Isomorphisms

If a homomorphism f is also **injective** (no two distinct elements in G map to the same element in H) *and* **surjective** (every element in H is hit by something in G), then f is an **isomorphism**. In this scenario, the groups G and H are structurally identical for practical purposes—just written in different "languages." An isomorphism means:

$$f(x \star y) \;=\; f(x) \circ f(y), \quad \text{and} \quad f^{-1} \text{ exists.}$$

Example: $\exp : (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \times)$ is a group isomorphism if we allow all real numbers. Indeed, ln is its inverse map. This underlies many exponentiation-based protocols in cryptography, except we usually replace $\mathbb{R}$ with a finite group $\mathbb{Z}_p^*$.

0.4.4 Key Takeaways

1. **Core Definition**: Homomorphisms preserve a group's structure across a mapping, ensuring "operation before mapping" equals "mapped elements' operation."

2. **Cryptographic Relevance**:
   - Partially homomorphic schemes (like RSA, Paillier) rely on a single operation being preserved.

   - Fully homomorphic encryption extends this to all operations (addition & multiplication), enabling computations on encrypted data.

3. **Concrete Illustrations**:
   - **Reduction mod** n shows how integers $\mathbb{Z}$ project down to $\mathbb{Z}_n$.

   - **Exponential** unites an additive domain with a multiplicative codomain.

   - **Encryption** as a group homomorphism captures the essence of "privacy-preserving" transformations.

Homomorphisms thus tie together abstract algebra and the practical design of encryption protocols. Their ability to "translate" operations from one set to another—without destroying structure—is precisely what makes partially or fully homomorphic encryption feasible, and what makes so many cryptographic primitives both powerful and elegant.

1 The RSA Cryptosystem

1.1 Introduction

The RSA cryptosystem, introduced in 1977 by Ronald Rivest, Adi Shamir, and Leonard Adleman, was the first practical public-key encryption scheme. Prior to its development, encryption relied on symmetric-key cryptography, which required secure key exchanges between communicating parties. RSA fundamentally changed cryptographic security by enabling secure communication without requiring a pre-established shared secret.

RSA's security is rooted in the computational difficulty of factoring large composite numbers, a problem for which no efficient classical algorithm exists. This difficulty ensures that, with sufficiently large key sizes, RSA remains secure against adversarial decryption attempts. As a result, RSA has become a foundational algorithm for modern cryptographic applications, including secure online transactions, digital signatures, and encrypted communications.

This document introduces RSA as a stepping stone to understanding **Partially Homomorphic Encryption (PHE)** and, ultimately, **Fully Homomorphic Encryption (FHE)**. By examining RSA's encryption and homomorphic properties, we establish the necessary cryptographic foundations for more advanced encryption paradigms that allow computations on encrypted data.

1.2 1. Key Generation

The RSA key generation process consists of the following steps:

1. Select two large prime numbers, p and q, ensuring they are sufficiently large for security.
2. Compute the modulus:
$$n = p \times q.$$
3. Compute Euler's totient function:
$$\varphi(n) = (p-1) \times (q-1).$$
4. Choose an encryption exponent e such that:
$$1 < e < \varphi(n), \quad \gcd(e, \varphi(n)) = 1.$$
   A common choice for e is 65537 due to its efficiency and security properties.
5. Compute the decryption exponent d as the modular inverse of e modulo $\varphi(n)$:
$$d \equiv e^{-1} \mod \varphi(n).$$
6. The public key is (n, e), while the private key is d.

The security of RSA depends on the difficulty of factoring n. If p and q are sufficiently large, brute-force factorization remains computationally infeasible.

1.3 2. Encryption and Decryption

1.3.1 2.1 Encryption

To encrypt a message m, first convert it to an integer such that $0 < m < n$. Then compute:

$$c = m^e \mod n.$$

1.3.2 2.2 Decryption

To decrypt a ciphertext c, compute:

$$m = c^d \mod n.$$

This restores the original plaintext message m.

1.4 3. RSA as a Partially Homomorphic Encryption (PHE) Scheme

RSA exhibits **Partially Homomorphic Encryption (PHE)** properties because it supports homomorphic multiplication. The key operations in a PHE scheme, as defined in homomorphic encryption, include:

1. **Key Generation (KeyGen):**
   The algorithm generates a key pair:

   $$(pk, sk) \leftarrow \text{KeyGen}(1^\lambda),$$

   where \$ pk = (n, e) \$ is the public key, and \$ sk = d \$ is the private key.

2. **Encryption (Enc):**
   A plaintext message \$ m \$ is encrypted using the public key:

   $$c \leftarrow \text{Enc}(pk, m) = m^e \mod n.$$

   This produces a ciphertext \$ c \$, which can be transmitted securely.

3. **Decryption (Dec):**
   A ciphertext \$ c \$ is decrypted using the private key:

   $$m \leftarrow \text{Dec}(sk, c) = c^d \mod n.$$

   This retrieves the original plaintext \$ m \$.

4. **Evaluation (Eval):**
   RSA supports homomorphic multiplication, meaning that multiplying two ciphertexts results in a ciphertext corresponding to the product of the plaintexts:

   $$\text{Eval}(f, c_1, c_2) = c_1 \times c_2 \mod n.$$

   This ensures:

   $$\text{Dec}(sk, \text{Eval}(f, c_1, c_2)) = (m_1 \times m_2) \mod n.$$

Because RSA does not support homomorphic addition, it is classified as a **Partially Homomorphic Encryption (PHE)** system rather than a Fully Homomorphic Encryption (FHE) system.

1.5 4. Security Considerations

1.5.1 4.1 Factoring-Based Security

RSA's security is predicated on the **Integer Factorization Problem (IFP)**. The computational infeasibility of factoring large semiprimes (products of two large primes) ensures that an adversary cannot efficiently derive the private key from the public key.

1.5.2 4.2 Attacks on RSA

While RSA is secure under proper implementation, certain vulnerabilities exist: - **Small Exponent Attacks:** Choosing a small encryption exponent (e.g., $e = 3$) can lead to predictable ciphertexts and facilitate decryption. - **Padding Oracle Attacks:** RSA without proper padding (e.g., OAEP) is susceptible to chosen-ciphertext attacks. - **Quantum Threats:** Shor's algorithm, if implemented on a sufficiently powerful quantum computer, could break RSA by efficiently factoring n.

1.5.3 4.3 Digital Signatures

RSA is widely used for **digital signatures**, ensuring message authenticity and integrity: 1. The sender computes a signature:

$$S = m^d \mod n.$$

2. The recipient verifies authenticity by computing:

$$S^e \mod n = m.$$

Digital signatures prevent tampering and enable non-repudiation.

1.6 5. Conclusion

RSA remains one of the most widely adopted cryptographic algorithms. Its significance extends beyond encryption to authentication and secure communications. However, its reliance on the hardness of factorization means that evolving computational capabilities, particularly quantum computing, may necessitate a shift toward alternative cryptographic approaches, such as lattice-based encryption. As we explore more advanced encryption schemes, including Fully Homomorphic Encryption (FHE), understanding RSA provides a critical foundation for grasping the principles of secure computation on encrypted data.