Module 1 review

Probability theory

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Seminar Overview

1. Quiz

2. Problems, problems, problems, ...

Quiz

A video rental estimates that annual expenditures of members on rentals follow a normal distribution with mean \$100. It was also found that 10% of all members spend more than \$130 in a year. What percentage of members spends more than \$140 in a year?

Problem 1

Suppose $X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$, $Y \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ and $\rho(X, Y) = 0.8$.
Find the joint distribution of X and Y.

Problem 2

Suppose random variables X and Y have joint normal distribution.

1. If X and Y are standard normal, and $\mathsf{P}(X + Y > 1.96) = 0.025$, what is the correlation between X and Y ?
2. If X and Y are independent, what is $\mathsf{P}(X > 1.96 \mid |Y| > 1.96)$?

Problem 3

Consider two random variables X and Y. The joint probabilities for each values pair are given by the following table.

	$X = 0$	$X = 1$	$X = 2$
$Y = 0$	0.20	0.15	0.10
$Y = 1$	0.15	0.10	0.05
$Y = 2$	0.10	0.05	0.00
$Y = 3$	0.10	0.00	0.00

1. Let $Z = \min(X, Y)$, and $W = |Y - X|$. Find $\mathsf{E}(Z), \mathsf{E}(W)$.
2. Find $\mathsf{E}(Z \mid W \leq 1)$.
3. Find $\mathsf{Cov}(Z, W)$.

Problem 4

There are 2 bowls with white and black balls:

$$A : 1 \text{ black and } 3 \text{ white}; \qquad B : 1 \text{ black and } 5 \text{ white}.$$

You choose 2 balls with the following procedure. First you choose at random one ball from the bowl B and put it into the bowl A. After that you choose two balls from the bowl A at random. Let X be the number of black balls from the two chosen.

1. Find p.m.f. of X.
2. Find $\mathsf{E}(X)$, $\mathsf{V}(X)$.

Problem 5

Four fair coins are tossed at random. The coins coming down heads up the first time are all tossed again.

1. What is the probability that finally there are two heads?
2. What is the probability that there were three heads after the first tosses if there are two heads at the end?

Problem 6

To reduce theft, suppose a company proposes to screen its workers with a lie-detector test that has been proved correct 90% of the time (for guilty subjects, and also for innocent subjects). The company will fire all the workers who fail the test. Suppose also that 5% of workers steal from time to time.

1. Of the fired workers, what proportion would actually be innocent?
2. Of the remaining workers not fired, what proportion would actually be guilty?

Problem 7
Statement

Two antibiotics are available as treatment for a common ear infection.

- Antibiotic A is known to effectively cure the infection 60 percent of the time. Treatment with antibiotic A costs \$50.
- Antibiotic B is known to effectively cure the infection 90 percent of the time. Treatment with antibiotic B costs \$80.

The antibiotics work independently of one another. Both antibiotics can be safely administered to children. A health insurance company intends to recommend one of the two plans of treatment for children with this ear infection.

- Plan I: Treat with antibiotic A first. If it is not effective then treat with antibiotic B.
- Plan II: Treat with antibiotic B first. If it is not effective then treat with antibiotic A.

1. If a doctor treats a child with an ear infection using plan I, what is the probability that the child will be cured? If a doctor treats a child with an ear infection using plan II, what is the probability that the child will be cured?
2. Compute the expected cost per child when plan I is used for treatment. Compute the expected cost per child when plan II is used for treatment.
3. Based on results from parts (1) and (2) which plan would you recommend?

Problem 8

Suppose there are three assets with returns X_1, X_2, and X_3. It is known that the returns are uncorrelated and their means and standard deviations are:

$$\mu_1 = 0.10, \mu_2 = 0.05, \mu_3 = 0.02,$$

$$\sigma_1 = 0.40, \sigma_2 = 0.20, \sigma_3 = 0.05.$$

Find the “optimal” portfolio with mean $\mu = 0.06$ (“optimal” means smallest variance).

Problem 9

Let $X \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$ be a normal random variable with p.d.f. $f_1(x)$, and $Y \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$ be a normal random variable with p.d.f. $f_2(x)$.

Consider the function $g(x) = \dfrac{1}{2} f_1(x) + \dfrac{1}{2} f_2(x)$.

1. Demonstrate, that $g(x)$ is p.d.f. of some random variable Z.
2. Find expected value $\mathsf{E}(Z)$ and variance $\mathsf{V}(Z)$ if $\mu_1 = 10, \sigma_1^2 = 9; \mu_2 = 16, \sigma_2^2 = 16$.

Problem 10

Consider two random variables X and Y. They both take the values $-1, 0$ and 1. The joint probabilities for each pair are given by the following table, with parameter $\theta \in \mathbb{R}$.

	$X = -1$	$X = 0$	$X = 1$
$Y = -1$	$0.1 + \theta$	0.1	$0.3 + 3\theta$
$Y = 0$	$0.1 + \theta$	$0.2 - 6\theta$	0
$Y = 1$	$0.1 + \theta$	0.1	0

1. What is the range of values the parameter θ can take for the above table to correspond to a probability table?
2. Calculate the marginal distributions and the expected values of X and Y.
3. Define $U = \min(X, Y)$. Calculate the covariance of U and X.
4. Are there any values of θ so that U and X are independent random variables? Explain your answer.

Look at the time!