Methods of estimation

Probability theory. Statistics

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Seminar Overview

Quiz

Two measurements of the side of the square were produced. Suppose the two measurements X_1 and X_2 are two independent random variables with mean a and variance σ^2. The true length of the side of the square is a. Find MSE for the following estimator of the area of the square: $X_1 X_2$.

Moments of a random variable

- Moment of a random variable – quantitative measure, which describes the shape of p.d.f.'s graph independently of translation.
- k^{th} raw moment of a random variable X:

$$\mu_k = \mathsf{E}\left(X^k\right).$$

- k^{th} central moment of a random variable X:

$$\widetilde{\mu}_k = \mathsf{E}\left((X - \mathsf{E}(X))^k\right).$$

Example

- $\mu_1 = \mathsf{E}(X)$ – expected value,
- $\widetilde{\mu}_2 = \mathsf{V}(X)$ – variance.

- If the p.d.f. of X represents density, then $\mathsf{E}(X)$ is the center of mass, and $\mathsf{V}(X)$ is the rotational inertia.

Problem 1

Come up with any linear p.d.f. (non-uniform)

1. Can you conclude which parameter is greater without calculations – mean or median?
2. Calculate them explicitly.
3. Discuss the result. Why p.d.f. is unbalanced in terms of “mass” around its median?

Higher-order moments

- Skewness shows the asymmetry of a p.d.f.:

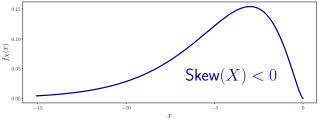
$$\mathsf{Skew}(X) = \frac{\widetilde{\mu}_3}{\sigma^3}.$$

- Excess kurtosis shows the sharpness of a p.d.f. peak:

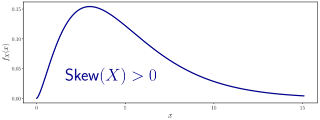
$$\mathsf{Kurt}(X) = \frac{\widetilde{\mu}_4}{\sigma^4} - 3.$$

- Shift -3 is used to manipulate the excess kurtosis of standard normal distribution to be 0, since the sharpness of a peak is estimated with a reference to that of $\mathcal{N}(0, 1)$.
- In order to make correct comparisons of two distributions with excess kurtosis, their variance should be identical.
- If $\mathsf{Kurt}(X) > 0$ – distribution peak is sharper than standard normal's one, if $\mathsf{Kurt}(X) < 0$ – distribution peak is smoother.

Higher-order moments



(a) Negatively skewed.



(b) Positively skewed.

Figure: P.d.f.-s of random variables with opposite skewness.

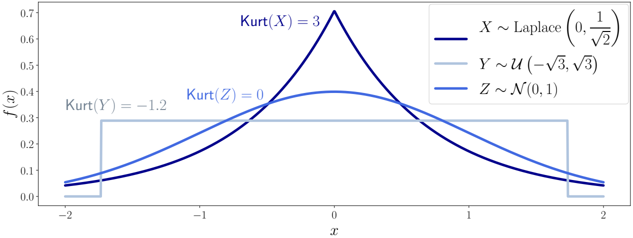


Figure: Comparison of excess kurtosis for Laplace X, uniform Y and normal Z distributions with zero mean and variance 1.

Laplace distribution

- Laplace distribution of $X \iff X \sim \text{Laplace}(\mu, b)$, where μ is a location of the peak, and b is a scale parameter.
- P.d.f.:
$$f_X(x) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}.$$
- C.d.f.:
$$F_X(x) = \frac{1}{2} + \frac{1}{2} \cdot \text{sign}(x-\mu) \cdot \left(1 - e^{-\frac{|x-\mu|}{b}}\right).$$
- Mean:
$$\mathsf{E}(X) = \mu.$$
- Variance:
$$\mathsf{V}(X) = 2b^2.$$

Moment-generating function

Definition

Moment-generating function of a random variable X:

$$\mathsf{M}_X(t) = \mathsf{E}\left(e^{tX}\right).$$

- In continuous case it's a bilateral Laplace transform of a p.d.f. $f_X(x)$ with parameter $-t$:

$$\mathsf{M}_X(t) = \int\limits_{-\infty}^{\infty} f_X(x) e^{-(-t)x} dx.$$

- $\mathsf{M}_X(t)$ is used to acquire raw moments μ_k:

$$\mu_k = \mathsf{E}\left(X^k\right) = \frac{d^k}{dt^k} \mathsf{M}_X(t) \bigg|_{t=0}.$$

Moment-generating function

- If $X_1, \ldots, X_n$ are independent random variables, then:

$$\mathsf{M}_{\sum\limits_{i=1}^{n} X_i}(t) = \prod_{i=1}^{n} \mathsf{M}_{X_i}(t).$$

- For linear transformation $\alpha X + \beta$, where $\alpha, \beta \in \mathbb{R}$:

$$\mathsf{M}_{\alpha X + \beta}(t) = e^{\beta t} \cdot \mathsf{M}_X(\alpha t).$$

- Collection of μ_k from $k = 0$ to $k = \infty$ uniquely determines the distribution, if p.d.f. is defined on bounded range. Please, refer to
  1. Hausdorff moment problem – support on $[0, 1]$. Always unique.
  2. Stieltjes moment problem – support on $[0, \infty)$. Requires sufficient condition for uniqueness.
  3. Hamburger moment problem – support on $(-\infty, \infty)$. Requires sufficient condition for uniqueness.

Method of moments

- Let $X_1, \ldots, X_n$ be a random sample from $F_X(x;\ \theta_1, \ldots, \theta_m)$.
- k^{th} sample moment:

$$M_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k = \overline{X^k}.$$

- Method of moments (MM) to find estimators of $\theta_1, \ldots, \theta_m$:

$$\begin{aligned} \mu_1\Big|_{\theta_1=\widehat{\theta}_1,\ldots,\theta_m=\widehat{\theta}_m} &= M_1, \\ \mu_2\Big|_{\theta_1=\widehat{\theta}_1,\ldots,\theta_m=\widehat{\theta}_m} &= M_2, \\ &\ldots \\ \mu_m\Big|_{\theta_1=\widehat{\theta}_1,\ldots,\theta_m=\widehat{\theta}_m} &= M_m. \end{aligned}$$

Problem 2

Let $\{X_1, \ldots, X_n\}$ be a random sample from a Bin (m, π) distribution, with both m and π unknown. Find the method of moments estimators for m – the number of trials, and π – the probability of success.

Problem 3

Suppose that we have a random sample $\{X_1, \ldots, X_n\}$ from a uniform distribution. Find the method of moments estimator of θ if

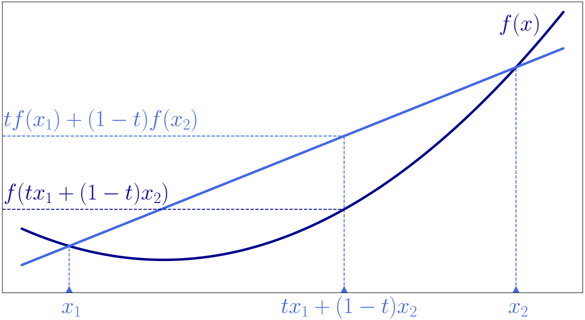
1. $X \sim \mathcal{U}(0, \theta)$,
2. $X \sim \mathcal{U}(-\theta, \theta)$.

Jensen's inequality

Jensen's inequality for 2 points x_1 and x_2

If f is a convex function, then

$$\forall t \in [0,1]: \qquad f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2),$$



Jensen's inequality

- Increasing number of splits, such that $\sum_{i=1}^{n} t_i = 1$:

$$f(t_1 x_1 + t_2 x_2 + \ldots + t_n x_n) \leq t_1 f(x_1) + t_2 f(x_2) + \ldots + t_n f(x_n),$$

- Giving t_i meaning of probabilities, Jensen's inequality becomes following:

Jensen's inequality (probabilistic statement)

If f is a convex function, then for any random variable X

$$f\left(\mathsf{E}(X)\right) \leq \mathsf{E}\left(f(X)\right).$$

- If f is concave, inequality is inverted.
- Equality is achieved when f is not strictly convex or concave, in other words only when f is linear.

Problem 3

- Results of Part (2):
  - $\widehat{\theta^2} = 3\overline{X^2}$,
  - $\widehat{\theta} = \sqrt{3\overline{X^2}}$.
- While $\widehat{\theta^2}$ is unbiased:

$$\mathsf{E}\left(\widehat{\theta^2}\right) = \mathsf{E}\left(3\overline{X^2}\right) = 3 \cdot \frac{1}{n} \cdot \mathsf{E}\left(n \cdot X^2\right) = 3 \cdot \frac{\theta^2}{3} = \theta^2,$$

  $\widehat{\theta}$ is NOT unbiased.
- Since square root $f(x) = \sqrt{x}$ is a strictly concave function, according to Jensen's inequality:

$$\mathsf{E}\left(\sqrt{3\overline{X^2}}\right) < \sqrt{\mathsf{E}\left(3\overline{X^2}\right)} = \sqrt{\theta^2} = \theta.$$

- Thus, $\widehat{\theta}$ underestimates true parameter θ.

Ordinary least squares

- Linear regression model of n observations $\{\mathbf{x}_i, y_i\}_{i=1}^n$:

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip} + \varepsilon_i,$$

  where $\mathbf{x}_i = \begin{pmatrix} x_{i1} & x_{i2} & \cdots & x_{ip} \end{pmatrix}^\top$ – vector of p regressors (independent variables), y_i – regressand (response variable), ε_i – noise / error / other influences on y_i.
- In matrix notation:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

  where $\mathbf{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^\top$ – vector of regressands,
  $\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_p \end{pmatrix}^\top$ – vector of unknown parameters,
  $\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n \end{pmatrix}^\top$ – vector of noise / errors,
  $\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix}$ – matrix of regressors.

Ordinary least squares

- Goal of regression: to find such estimate $\widehat{\boldsymbol{\beta}}$ that it will be as close as possible to the real vector $\boldsymbol{\beta}$.
- Ordinary least squares (OLS) is a method to find $\widehat{\boldsymbol{\beta}}$ via quadratic minimization of $\|\boldsymbol{\varepsilon}\|_2^2 = \varepsilon_1^2 + \varepsilon_2^2 + \ldots + \varepsilon_n^2$.

$$\widehat{\boldsymbol{\beta}}_{\text{OLS}} = \arg\min_{\boldsymbol{\beta}} \|\boldsymbol{\varepsilon}\|^2 = \arg\min_{\boldsymbol{\beta}} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|^2.$$

- This minimization problem has known solution:

$$\widehat{\boldsymbol{\beta}}_{\text{OLS}} = \left(\mathbf{X}^\top \mathbf{X}\right)^{-1} \mathbf{X}^\top \mathbf{y}.$$

Theorem (Gauss-Markov)

$\widehat{\boldsymbol{\beta}}_{\text{OLS}}$ has the lowest sample variance within the class of linear unbiased estimators, if $\forall i, j \in \overline{1, n} \quad \mathsf{E}(\varepsilon_i) = 0, \mathsf{V}(\varepsilon_i) = \sigma^2$ *and* $\mathsf{Cov}(\varepsilon_i, \varepsilon_j) \overset{i \neq j}{=} 0$.

Problem 4

Suppose that you are given observations y_1, y_2, y_3 and y_4 such that:

$$\begin{aligned}
y_1 &= \alpha + \beta + \varepsilon_1, \\
y_2 &= -\alpha + \beta + \varepsilon_2, \\
y_3 &= \alpha - \beta + \varepsilon_3, \\
y_4 &= -\alpha - \beta + \varepsilon_4.
\end{aligned}$$

The variables $\varepsilon_i, i \in \{1, 2, 3, 4\}$, are independent and normally distributed with mean 0 and variance σ^2.

1. Find the least squares estimators of the parameters α and β.
2. Verify that the least squares estimators in (a) are unbiased.
3. Find the variance of the least squares estimator of the parameter α.

Maximum likelihood estimator

- Let $X_1, \dots, X_n$ be a random sample from $F_X(x;\ \theta)$.
- Probability that all those observations happened is given by joint p.d.f. (or p.m.f. in discrete case):

$$f(X_1, X_2, \dots, X_n;\ \theta).$$

- Probability that we have sample $X_1, \dots, X_n$ given some particular θ is the same with probability that we have the same θ given that sample $X_1, \dots, X_n$.
  The latter is called likelihood function $\mathcal{L}(\theta)$:

$$\mathcal{L}(\theta) = \mathcal{L}(\theta;\ X_1, \dots, X_n) = f(X_1, X_2, \dots, X_n;\ \theta).$$

- Due to independence of X_i in sample:

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} f(X_i;\ \theta).$$

Maximum likelihood estimator

- Maximum likelihood estimator (MLE) is an estimate of parameter θ, which maximizes likelihood function $\mathcal{L}(\theta)$:

$$\widehat{\theta}_{\text{MLE}} = \arg\max_{\theta} \mathcal{L}(\theta).$$

- Likelihood function is built as a product $\prod$, which is pretty hard to take a derivative from.
  Log-likelihood $l(\theta)$ is introduced:

$$l(\theta) = \ln \mathcal{L}(\theta),$$

  which converts products to sums.
  Since the logarithm is a monotonic function:

$$\arg\max_{\theta} l(\theta) = \arg\max_{\theta} \mathcal{L}(\theta).$$

- Invariance principle of the MLE:

$$\phi = g(\theta) \qquad \Longrightarrow \qquad \widehat{\phi}_{\text{MLE}} = g\left(\widehat{\theta}_{\text{MLE}}\right).$$

Problem 5

Let $\{X_1, \ldots, X_n\}$ be a random sample from $\text{Exp}(\lambda)$ distribution. Find the MLE of λ.

Problem 6

A random sample $\{X_1, X_2, \ldots, X_n\}$ is drawn from the following probability distribution:

$$p(x; \lambda) = \frac{\lambda^{2x} e^{-\lambda^2}}{x!}$$

and $p(x; \lambda) = 0$ for all other values of x, with $\lambda > 0$.

1. Derive the maximum likelihood estimator of λ.
2. State the maximum likelihood estimator of $\theta = \lambda^3$.

Problem 7

Suppose that X is a discrete random variable with the following probability mass function:

x	0	1	2	3
$\mathsf{P}_X(x)$	$\frac{2\theta}{3}$	$\frac{\theta}{3}$	$\frac{2(1-\theta)}{3}$	$\frac{1-\theta}{3}$

where $0 \leq \theta \leq 1$ is a parameter. The following 10 independent observations were taken from such a distribution:

$$(3, 0, 2, 1, 3, 2, 1, 0, 2, 1).$$

What is the maximum likelihood estimate of θ.

Problem 8

Let $\{X_1, \ldots, X_n\}$ be a random sample from $\mathcal{U}(0, \theta)$ distribution. Find the MLE of θ.

Gamma distribution

- Gamma distribution of $X \iff X \sim \text{Gamma}(\alpha, \beta)$, where α is a shape parameter, and β is a rate parameter.
- P.d.f.:

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \cdot I_{\{x>0\}},$$

where $\Gamma(\alpha)$ is a generalization of the factorial to non-integers:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \qquad \Re(\alpha) > 0.$$

Its factorial-like behaviour is expressed via property of recurrence:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

- Mean: $\mathsf{E}(X) = \dfrac{\alpha}{\beta}$.
- Variance: $\mathsf{V}(X) = \dfrac{\alpha}{\beta^2}$.

Problem 9

Suppose that independent observations X and Y are taken from distributions Gamma $\left(a, \frac{1}{\eta}\right)$ and Gamma $\left(b, \frac{1}{\eta}\right)$ respectively, where both a and b are known and positive.

1. Find the maximum likelihood estimator (MLE) of η.
2. Show that the MLE of η is unbiased and find its variance.
3. Compare the MLE with the alternative estimator

$$\widehat{H} = \frac{1}{2}\left(\frac{X}{a} + \frac{Y}{b}\right).$$

   Which one is better?

Problem 10

Find maximum likelihood estimator of parameter θ from sample $\{X_1, \ldots, X_n\}$ with Laplace$(\theta, 1)$ distribution and p.d.f.:

$$f(x) = \frac{1}{2}e^{-|x-\theta|}.$$

Look at the time!