

Confidence intervals. Part I

Statistics

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- Motivation

- Definition

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④ Confidence intervals for population variance

- Population mean is unknown

- Population mean is known

Please find maximum likelihood estimation of θ using sample X_1, \dots, X_n , generated from a normal distribution with parameters:

- ① $\mu = 0, \sigma^2 = \theta^2,$
- ② $\mu = \theta, \sigma^2 = 2\theta.$

What good of a point estimate?

Example

- Let $X \sim \mathcal{N}(0, 9)$.
- As a result of experiment, we have a sample with 3 observations:

$$x_1 = 4, \quad x_2 = 5, \quad x_3 = 6.$$

- Our best point estimate of $\mu = 0$ is:

$$\bar{x} = \frac{4 + 5 + 6}{3} = 5,$$

which is pretty far away from real μ .

- In order to give a quantitative perspective on how confident we are that the resulting value of a point estimate is close to real parameter, **interval estimation** is introduced.

Confidence intervals

- Let X_1, \dots, X_n be a random sample from a population with parameter θ .

Definition

Confidence interval for θ with confidence level $1 - \alpha$ is a pair of random variables $L(X_1, \dots, X_n)$ and $U(X_1, \dots, X_n)$, such that:

$$\mathbf{P}(L(X_1, \dots, X_n) \leq \theta \leq U(X_1, \dots, X_n)) = 1 - \alpha.$$

- α – significance level.
- Sometimes confidence interval for θ with confidence level $1 - \alpha$ is defined as

$$\mathbf{P}(L(X_1, \dots, X_n) \leq \theta \leq U(X_1, \dots, X_n)) \geq 1 - \alpha,$$

which is used when probabilities are partially identified, e.g. discrete cases.

How to find confidence intervals?

Algorithm

- 1 Determine, which point estimator you will be using:
 - $\mu \rightarrow \bar{X}$ (mean),
 - $\sigma^2 \rightarrow S^2$ (variance),
 - $p \rightarrow \hat{P}$ (proportion),
 - $\theta \rightarrow \hat{\theta}$ (arbitrary parameter).

Make sure you know the distribution of that point estimator.

- 2 Find a pivot function h , which depends **only** on the sample and estimated parameter. It must have a **table distribution**:

$$h(X_1, \dots, X_n; \theta) \sim \mathcal{N}(0, 1), \chi_k^2, t_k, F_{p,k}, \text{ etc.}$$

- 3 Constrain pivot function with critical values:

$$\mathbf{P} \left(x_{1-\alpha/2} \leq h(X_1, \dots, X_n; \theta) \leq x_{\alpha/2} \right) = 1 - \alpha.$$

- 4 Express θ in the inequality above.

Confidence interval for population mean μ

Population variance σ^2 is known

- Let X_1, \dots, X_n be a random sample from a population with mean μ and variance σ^2 . The value of σ^2 is known.
- Pivot function in this case is the standardized \bar{X} :

$$h(X_1, \dots, X_n; \mu) = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \stackrel{\text{CLT}}{\sim} \mathcal{N}(0, 1).$$

- Confidence interval of μ with confidence level $1 - \alpha$ then:

$$\mathbf{P} \left(-z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2} \right) = 1 - \alpha.$$

$$\mathbf{P} \left(\bar{X} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \right) = 1 - \alpha.$$

Confidence interval for population mean μ

Population variance σ^2 is known

- $(1 - \alpha) \cdot 100\%$ confidence interval for μ can be written as:

$$CI_{1-\alpha}(\mu) = \bar{X} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}.$$

- $\frac{\sigma}{\sqrt{n}}$ is called standard error of \bar{X} :

$$S.E.(\bar{X}) = \frac{\sigma}{\sqrt{n}}.$$

- Overall, for symmetric distributions of pivot functions, confidence intervals have the following view:

$$CI_{1-\alpha}(\text{Param}) = \text{Point Est.} \pm \text{Crit. Value} \left(\frac{\alpha}{2} \right) \times S.E.(\text{Point Est.}).$$

Confidence interval for population mean μ

Population variance σ^2 is known

- Let's simulate experiments to calculate sample mean and see, how confidence intervals are constructed.
- Refer to the 1st block in the link:

Confidence intervals for μ

- Common misconception:
 - This is **NOT** a correct interpretation of confidence intervals:
“ $1 - \alpha$ is a probability that μ belongs to CI”.
 - This is:
“ $1 - \alpha$ is a probability that CI contains μ ”,
since μ is a constant, while CI itself is a random variable.

Problem 1

Manager of a restaurant wants to estimate the mean amount μ that a visitor spends for a lunch. A sample contains 36 visitors. Sample mean is $\bar{x} = \$3.60$. Manager knows that the standard deviation for one visitor is \$0.72. Find the confidence level corresponding to the interval (\$3.5; \$3.7).

Problem 2

A college admission officer for an *MBA* program has determined that historically candidates have undergraduate grade point averages that are normally distributed with standard deviation 0.45. A random sample of twenty-five applications from the current year is taken, yielding a sample mean grade average of 2.90.

- 1 Find a 95% confidence interval for the population mean.
- 2 Based on these sample results, a statistician computes for the population mean a confidence interval running from 2.81 to 2.99. Find the probability content associated with this interval.

Confidence interval for population mean μ

Population variance σ^2 is unknown

- Let X_1, \dots, X_n be a random sample from a population, distributed as $\mathcal{N}(\mu, \sigma^2)$. The value of σ^2 is unknown.
- Old pivot function has unknown parameter:

$$h(X_1, \dots, X_n; \mu) = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}.$$

- Let's substitute σ with its estimate $S = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2}$:

$$h(X_1, \dots, X_n; \mu) = \frac{\bar{X} - \mu}{S / \sqrt{n}}.$$

- Both \bar{X} and S are random variables $\Rightarrow h(X_1, \dots, X_n; \mu)$ is not normally distributed.

Confidence interval for population mean μ

Population variance σ^2 is unknown

- Fisher's lemma:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2.$$

- Applying Fisher's lemma to $h(X_1, \dots, X_n; \mu)$:

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}.$$

- Confidence interval of μ with confidence level $1 - \alpha$ then:

$$\mathbf{P} \left(-t_{n-1; \alpha/2} \leq \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq t_{n-1; \alpha/2} \right) = 1 - \alpha.$$

$$\mathbf{P} \left(\bar{X} - t_{n-1; \alpha/2} \cdot \frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{n-1; \alpha/2} \cdot \frac{S}{\sqrt{n}} \right) = 1 - \alpha.$$

Confidence interval for population mean μ

Population variance σ^2 is unknown

- $(1 - \alpha) \cdot 100\%$ confidence interval for μ can be written as:

$$CI_{1-\alpha}(\mu) = \bar{X} \pm t_{n-1; \alpha/2} \cdot \frac{S}{\sqrt{n}}.$$

- $\frac{S}{\sqrt{n}}$ is called estimated standard error of \bar{X} :

$$\text{E.S.E.}(\bar{X}) = \frac{S}{\sqrt{n}}.$$

- If n is large ($n > 30$), we can use z -values, instead of t -values:

$$t_{n-1; \alpha/2} \xrightarrow{n \rightarrow \infty} z_{\alpha/2} \quad \text{and} \quad S \xrightarrow{n \rightarrow \infty} \sigma.$$

- For simulations refer to the 2nd block in the link:

Confidence intervals for μ

Problem 3

A random sample of 5 observations from a normal distribution with mean μ and variance σ^2 gives a sample mean 100. An independent random sample of size 10 from the same population has sample variance 9. Find a 90% confidence interval for the population mean.

\bar{X} and S^2 are independent

- Assuming that sample X_1, \dots, X_n is derived from a normal population, statistics \bar{X} and S^2 are independent.
- This allows us to state that

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1},$$

since numerator Z and denominator $\sqrt{\chi^2}$ should be independent to create a t -distribution.

\bar{X} and S^2 are independent

Proof

- Let's consider vector of residuals:

$$\mathbf{X} = (X_1 - \bar{X} \quad \dots \quad X_n - \bar{X})^\top.$$

- Each component of \mathbf{X} is uncorrelated with \bar{X} :

$$\text{Cov}(X_j - \bar{X}, \bar{X}) = \text{Cov}(X_j, \bar{X}) - \text{Cov}(\bar{X}, \bar{X}) = \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0,$$

thus making \bar{X} and \mathbf{X} uncorrelated.

- If components of multivariate normal distribution are uncorrelated \implies they are independent.
- \bar{X} and \mathbf{X} are independent as components of vector $(\bar{X} \quad \mathbf{X})^\top$.
- Inherently, \bar{X} is independent of $\mathbf{X}^\top \mathbf{X} = (n-1)S^2$.

Uncorrelatedness in multivariate normal distribution

- On the example of bivariate case.

Let vector $(X \ Y)^\top$ be bivariate normal with joint p.d.f.:

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right),$$

where $\rho = \text{Corr}(X, Y)$.

- Let $\rho = 0$:

$$\begin{aligned} f(x, y) &= \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left(-\frac{1}{2} \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right) = \\ &= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{1}{2} \left(\frac{x-\mu_X}{\sigma_X}\right)^2\right] \cdot \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{1}{2} \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] = \\ &= f_X(x) \cdot f_Y(y). \end{aligned}$$

- X and Y are independent by definition.

Accuracy of estimation

- Accuracy of estimation e is a half-width of the corresponding confidence interval.
- For μ with known σ^2 :

$$CI_{1-\alpha}(\mu) = \bar{X} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}},$$

$$e = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}.$$

- How many observations required to get the accuracy e :

$$z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \leq e,$$

$$n \geq \frac{z_{\alpha/2}^2 \cdot \sigma^2}{e^2}.$$

Problem 4

The reaction time of a patient to a certain stimulus is known to have a standard deviation of 0.05 seconds. How large a sample of measurements must a psychologist take in order to be 95% confident and 99% confident, respectively, that the error in the estimate of the mean reaction time will not exceed 0.01 seconds?

One-sided confidence intervals

- Let's constrain a pivot function with only one value. Critical value should separate α probability in a distribution tail for a confidence interval with level $1 - \alpha$.
- On the example of interval for μ with σ^2 unknown:

$$P\left(\frac{\bar{X} - \mu}{S/\sqrt{n}} < t_{n-1; \alpha}\right) = 1 - \alpha.$$

$$P\left(\mu > \bar{X} - t_{n-1; \alpha} \cdot \frac{S}{\sqrt{n}}\right) = 1 - \alpha.$$

$$CI_{1-\alpha}(\mu) = \left(\bar{X} - t_{n-1; \alpha} \cdot \frac{S}{\sqrt{n}}; +\infty\right).$$

- One-sided interval with upper bound similarly:

$$CI_{1-\alpha}(\mu) = \left(-\infty; \bar{X} + t_{n-1; \alpha} \cdot \frac{S}{\sqrt{n}}\right).$$

Problem 5

During the Friday night shift, $n = 28$ mints were selected at random from a production line and weighted. They had average weight of $\bar{x} = 21.45$ grams and $s = 0.31$ gram. Give the lower endpoint of a 90% one-sided confidence interval for μ , the mean weight of all mints.

Problem 6

- 1 A student constructed two 95% confidence intervals for unknown parameter θ : $(-\infty; 4.2)$ and $(0.5; \infty)$. What could be the confidence of the interval $(0.5; 4.2)$?
- 2 A student constructed two 95% confidence intervals for unknown parameter θ : $(-5; 4.2)$ and $(0.5; 7)$. What could be the confidence of the interval $(0.5; 4.2)$?

Problem 6

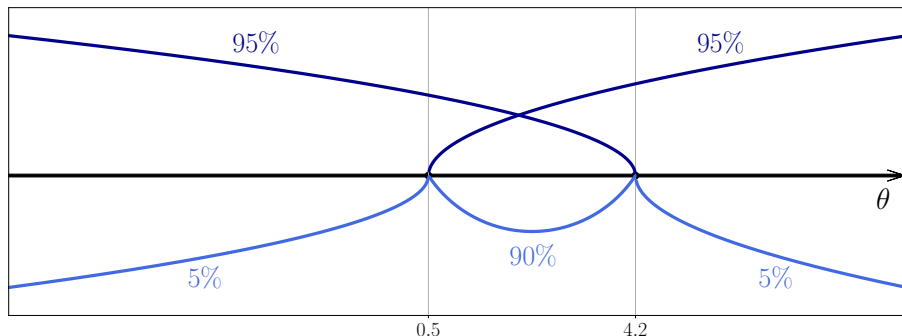


Figure: Intersection of infinite confidence intervals.

Problem 6

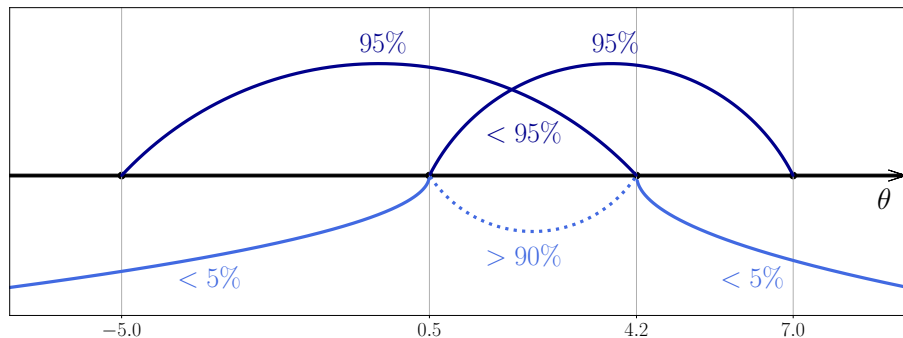


Figure: Intersection of finite confidence intervals.

Confidence interval for population variance σ^2

Population mean μ is unknown

- Let X_1, \dots, X_n be a random sample from a population, distributed as $\mathcal{N}(\mu, \sigma^2)$. The value of μ is unknown.
- Unbiased estimator of σ^2 is:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2.$$

- Pivot function is given by Fisher's lemma:

$$h(X_1, \dots, X_n; \sigma^2) = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2.$$

- Confidence interval of σ^2 with confidence level $1 - \alpha$ then:

$$\mathbf{P} \left(\chi_{n-1; 1-\alpha/2}^2 \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi_{n-1; \alpha/2}^2 \right) = 1 - \alpha.$$

Confidence interval for population variance σ^2

Population mean μ is unknown

$$P\left(\frac{1}{\chi_{n-1; 1-\alpha/2}^2} \geq \frac{\sigma^2}{(n-1)S^2} \geq \frac{1}{\chi_{n-1; \alpha/2}^2}\right) = 1 - \alpha.$$

$$P\left(\frac{(n-1)S^2}{\chi_{n-1; \alpha/2}^2} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi_{n-1; 1-\alpha/2}^2}\right) = 1 - \alpha.$$

- $(1 - \alpha) \cdot 100\%$ confidence interval for σ^2 can be written as:

$$CI_{1-\alpha}(\sigma^2) = \left(\frac{(n-1)S^2}{\chi_{n-1; \alpha/2}^2}; \frac{(n-1)S^2}{\chi_{n-1; 1-\alpha/2}^2}\right).$$

- For simulations refer to the 1st block in the link:

Confidence intervals for σ^2

Problem 7

A manufacturer bonds a plastic coating to a metal surface. A random sample of nine observations on the thickness of this coating is taken from a week's output. The sample thickness (in millimeters) were as follows:

19.8 21.2 18.6 20.4 21.6 19.8 19.9 20.3 20.8

Assuming that the population distribution is normal, find a 90% confidence interval for the population variance.

Confidence interval for population variance σ^2

Population mean μ is known

- Let X_1, \dots, X_n be a random sample from a population, distributed as $\mathcal{N}(\mu, \sigma^2)$. The value of μ is known.
- Unbiased estimator of σ^2 is:

$$\varsigma^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2.$$

- Pivot function:

$$h(X_1, \dots, X_n; \sigma^2) = \frac{n\varsigma^2}{\sigma^2} \sim \chi_n^2.$$

- Confidence interval of σ^2 with confidence level $1 - \alpha$ then:

$$\mathbf{P} \left(\chi_{n; 1-\alpha/2}^2 \leq \frac{n\varsigma^2}{\sigma^2} \leq \chi_{n; \alpha/2}^2 \right) = 1 - \alpha.$$

Confidence interval for population variance σ^2

Population mean μ is known

$$P\left(\frac{n\varsigma^2}{\chi_{n; \alpha/2}^2} \leq \sigma^2 \leq \frac{n\varsigma^2}{\chi_{n; 1-\alpha/2}^2}\right) = 1 - \alpha.$$

- $(1 - \alpha) \cdot 100\%$ confidence interval for σ^2 can be written as:

$$CI_{1-\alpha}(\sigma^2) = \left(\frac{n\varsigma^2}{\chi_{n; \alpha/2}^2}; \frac{n\varsigma^2}{\chi_{n; 1-\alpha/2}^2}\right).$$

- For simulations refer to the 2nd block in the link:

Confidence intervals for σ^2

Problem 8

Let 10.1, 9.7 be a sample from the normal population $X \sim \mathcal{N}(10, \sigma^2)$.
Let 20.1, 19.5, 20.4 be a sample from the normal population
 $Y \sim \mathcal{N}(20, \sigma^2)$. Find a two-sided 90% confidence interval for the
population standard deviation σ .



That's all Folks