Fall Midterm review. Joint continuous distributions

Probability theory

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Seminar Overview

1. Quiz
2. Fall Midterm revision
3. Joint continuous distributions
4. Complex distributions
   - Composite distributions
   - Variables transformations

Quiz

1. What was the hardest problem in the Midterm?
2. How would you estimate overall difficulty of the Midterm?

Problem 1

Here, you have a joint probability distribution of two discrete random variable

$X \setminus Y$	$Y=-1$	$Y=0$	$Y=1$
$X=0$	0.2	0.1	0.3
$X=1$	0.2	0.1	c

1. Find a coefficient c?
2. Evaluate $\mathsf{E}(X)$ and $\mathsf{E}\left(X^2\right)$
3. Evaluate $\mathsf{E}(Y)$ and $\mathsf{E}\left(Y^2\right)$
4. Evaluate $\mathsf{Corr}(X, Y)$
5. Evaluate $\mathsf{Cov}(X, Y \mid X > 0)$

Problem 2

Call center of the company has 3 members of staff: Maria, Anna and Jack. When a client rings his call is randomly allocated to one of three operators. Number of calls during the day is n.

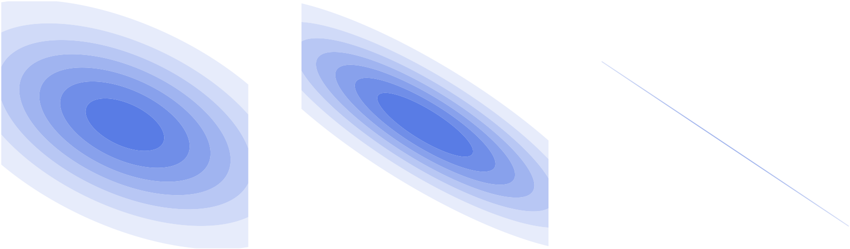
1. Find a probability that both Maria and Anna will answer at least one call during the day.
2. Operators sometimes don't know how to meet client's request: Maria with probability 0.1, Anna with probability 0.15, Jack with probability 0.3. In this case operator calls supervisor. If it is known that situation was escalated to supervisor what is the probability that call was answered by Anna?

Problem 3

Two random variables are given: $X \sim \mathcal{N}(0, 9)$ and $Y \sim \mathcal{N}(0, 4)$. $\mathsf{Corr}(X, Y) = -1$.

1. Evaluate $\mathsf{P}(X > 1)$
2. Evaluate $\mathsf{P}(2X + Y > 3)$
3. Evaluate $\mathsf{P}(2X + Y > 3 \mid X = 1)$
4. Evaluate $\mathsf{P}(|X| > 2)$
5. Evaluate $\mathsf{P}((Y + 1)^2 < 2)$

Problem 3



(a) $\rho = -\frac{1}{2}$ (b) $\rho = -\frac{5}{6}$ (c) $\rho \to -1$

Figure: P.d.f. of bivariate normal distribution with $\sigma_X = 3$ and $\sigma_Y = 2$.

Problem 4

The factory packs tea into packs with “100 gr.” written on them. The actual weight of a pack of tea has a normal distribution with a mean of 100 gr. and a standard deviation of 2 gr. Before sending to customers, packs are weighed. Packs weighing less than 97 gr. are rejected.

1. How many packs from a batch of 250 pieces will be rejected on average?
2. What is the probability that a randomly picked tea pack’s weight is below 97 gr.
3. Given the result of (2) you repeat this experiment 5 times. You may assume that each of experiment is independent. Please evaluate a probability that at least two tea pack appears to be less than 97 gr.

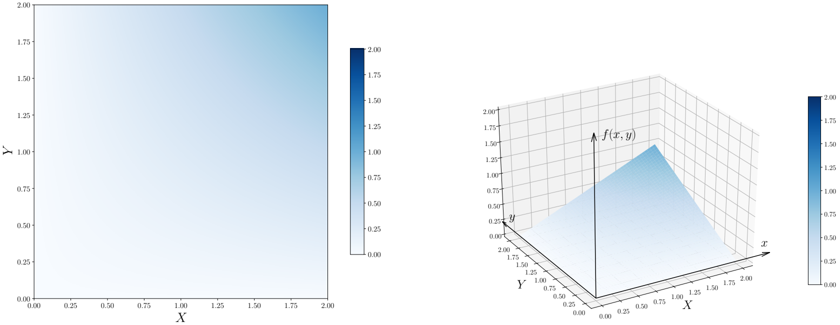
Problem 5

Let two random variables have joint p.d.f.:

$$f(x,y) = \begin{cases} cxy, & 0 < x, y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find c, marginal p.d.f. f_X, marginal p.d.f. f_Y. Are X and Y independent?
2. Find $\mathsf{Cov}(X, Y)$.
3. Let $g(x) = \mathsf{E}(Y \mid X = x)$. Find $g(x)$.
4. Find $\mathsf{P}\left(X^2 > Y^2\right)$.

Problem 5



(a) top view

(b) side view

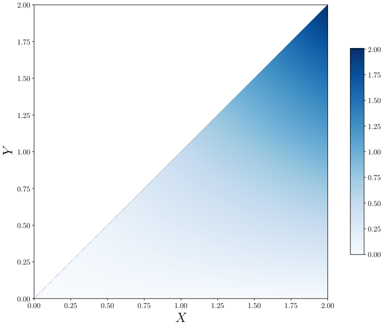
Figure: P.d.f. $f(x,y) = \dfrac{1}{4}xy \cdot I_{\{0<x,y<2\}}$.

Problem 6

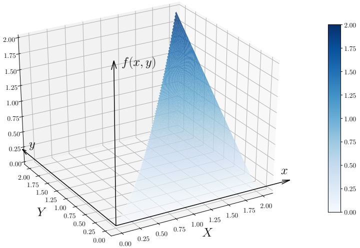
Same as in **Problem 5**, but joint p.d.f. is:

$$f(x,y) = \begin{cases} cxy, & 0 < y < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 6



(a) top view



(b) side view

Figure: P.d.f. $f(x, y) = \frac{1}{2}xy \cdot I_{\{0<y<x<2\}}$.

Problem 7

X is a random variable with p.d.f.

$$f(x) = \begin{cases} 0, & x \le 0, \\ \frac{1}{4}, & 0 < x \le 1, \\ ax - a, & 1 < x \le 2, \\ \frac{1}{4}, & 2 < x \le 3, \\ 0, & x > 3. \end{cases}$$

1. Find c.d.f.
2. Find $\mathsf{P}\left(X > \frac{3}{2}\right)$.
3. Find $\mathsf{P}\left(X < \frac{5}{2} \,\middle|\, X > 1\right)$.
4. Find $\mathsf{E}(X)$.

Problem 7

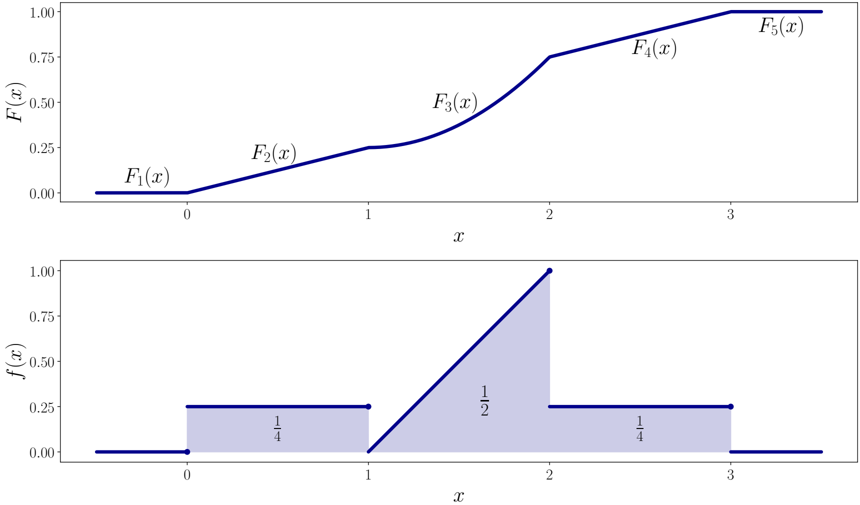


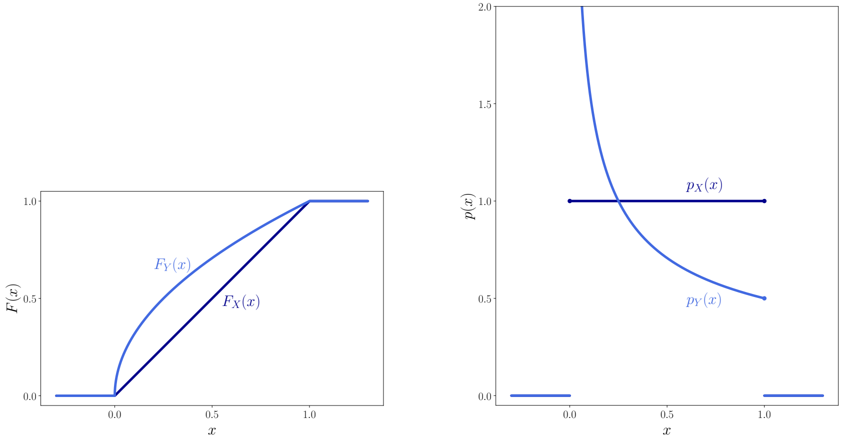
Figure: C.d.f. and p.d.f. of piecewise linear X.

Problem 8

Let X has uniform distribution $\mathcal{U}(0, 1)$. $Y = X^2$.

1. Find c.d.f. of Y.
2. Find p.d.f. of Y.

Problem 8



(a) C.d.f. (b) P.d.f.

Figure: $X \sim \mathcal{U}(0,1)$ and $Y = X^2$.

Change of variable in p.d.f.

- If $Y = g(X)$, where function g is strictly monotonic, then p.d.f.-s:

$$f_Y(y) = \left|\frac{dg^{-1}(y)}{dy}\right| \cdot f_X(g^{-1}(y)),$$

where absolute value is required for strictly decreasing g, since in that case $F_Y(y) = 1 - F_X(g^{-1}(y))$.

Example

- Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Z = \dfrac{X-\mu}{\sigma} = g(X)$. $X = g^{-1}(Z) = \mu + \sigma Z$.
- Applying change of variable formula:

$$f_Z(z) = \left|\frac{d(\mu+\sigma z)}{dz}\right| \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\mu+\sigma z-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

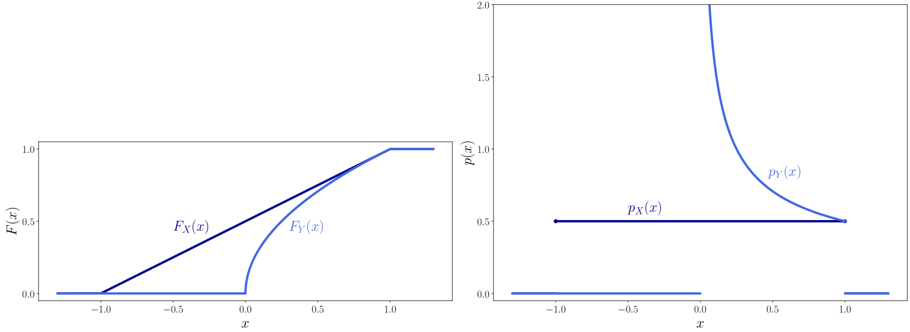
- Usable only if supports of X and $g(X)$ are identical.

Problem 9

Let X has uniform distribution $\mathcal{U}(-1, 1)$. $Y = X^2$.

1. Find c.d.f. of Y.
2. Find p.d.f. of Y.

Problem 9



(a) C.d.f.

(b) P.d.f.

Figure: $X \sim \mathcal{U}(-1, 1)$ and $Y = X^2$.

Problem 10

Let X, Y are independent random variables, distributed uniformly on $[0, 2]$. Let $W = \max\{X, Y\}$, $L = \min\{X, Y\}$.

1. Find c.d.f. of W.
2. Find p.d.f. of W.
3. Find $\mathsf{E}(W)$, $\mathsf{P}(W < \mathsf{E}(W))$.
4. Find $\mathsf{E}(L)$.

Problem 10

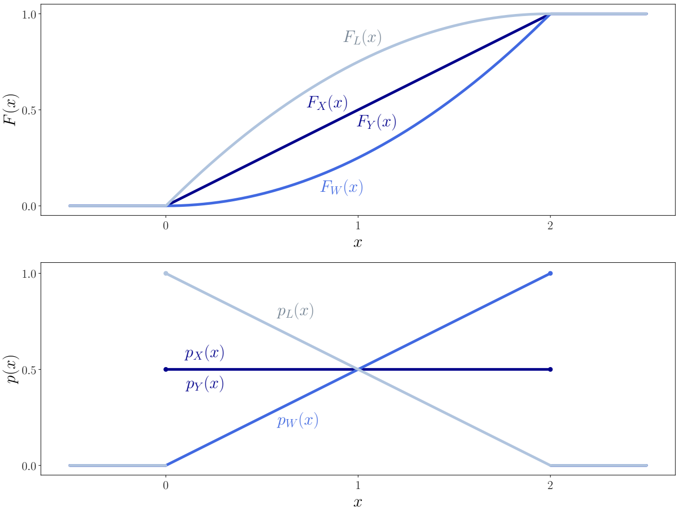


Figure: C.d.f. and p.d.f. of X, Y, $W = \max(X, Y)$ and $L = \min(X, Y)$.

Problem 11

Let X be a random variable with uniform distribution on the interval $[0, \pi]$. Find p.d.f. of random variables:

1. $Y = \sin X$,
2. $Z = X^3$.

Problem 11

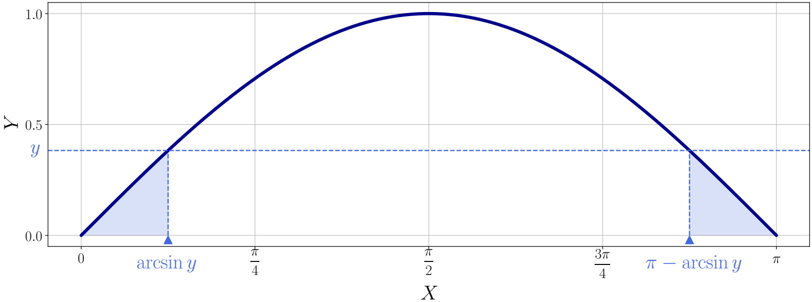


Figure: Calculation of $\mathsf{P}(\sin X \leq y)$.

Problem 11

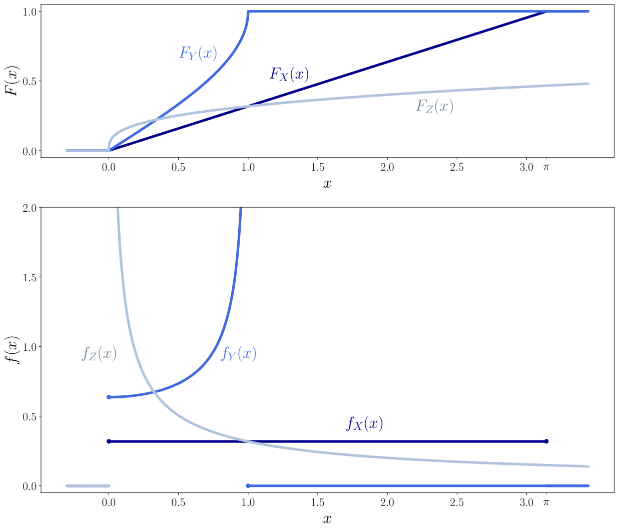


Figure: C.d.f. and p.d.f. of $X \sim \mathcal{U}(0, \pi)$, $Y = \sin X$ and $Z = X^3$.

Look at the time!