Hypotheses. Types I and II errors

Statistics

Anton Afanasev

Higher School of Economics

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Seminar Overview

1. Quiz

2. Introduction to hypotheses testing

3. Types I and II errors

Quiz

Suppose that X is a random observation from a uniform distribution on the interval $(0, \delta)$, where $\delta > 1$ and that one wants to estimate $\theta = \mathsf{P}(X > 1) = 1 - 1/\delta$. Consider the following estimator T of θ:

$$T = \begin{cases} 1, & X > 1, \\ 0, & X \leq 1. \end{cases}$$

1. Is T an unbiased estimator?
2. Find the mean squared error of the estimator.

Problem 1

The coin was tossed 10 times, and 8 heads were observed. Can the coin be considered fair? (Use significance level $\alpha = 0.05$.)

Problem 2

Junior researcher Angela is presenting her half-year project about speed of blood clotting in front of the Head of her laboratory. That speed is normally distributed with population standard deviation $\sigma = 3$ minutes. Angela is very nervous and in the very responsible moment she has forgotten the resultant value of true population speed μ – it's either 6 or 9 minutes. Presentation slide claims that they had 16 observations and the sample mean is 7 minutes, so Angela assumes that the correct value is 6.

1. Find the critical value $\overline{x}_{\text{crit}}$ of sample mean for a hypothesis $H_0 : \mu = 6$, which would guarantee that it's true within 95% confidence level.
2. Find the p-value for the hypothesis H_0.
3. Preserving confidence level from part (1), find the probability of Type II error, using aforementioned alternative $H_1 : \mu = 9$.

Types I and II errors

	H_0 is true	H_1 is true
H_0 is not rejected	confidence level $1-\alpha$	Type II error β
H_0 is rejected	Type I error α	power of the test $1-\beta$

Table: Probabilities in hypotheses testing

- α is also called a significance level of the test.
- Power of the test as a function of β is often denoted as $K(\beta)$.
- The dependency of Type II error β on parameters of population is graphed as OCC – operating characteristic curve.

Problem 3

A firm manufacturing memory chips found that if everything was going right, 10% of them were defective. If the production process was in trouble then 40% were defective. Firm's quality control office tests four memory chips each hour. If two or more of four were defective, production would be shut down to look for trouble.

1. What is Type I and Type II errors here?
2. What is probability that production will be shut down if everything was going right?
3. What is the probability of the missed alarm?

Problem 3

4. Suppose, you have additional information:
   (a) Production goes “out of control” about 10% of hours.
   (b) Testing of one chip costs \$10.
   (c) Missed alarm costs \$10000.
   (d) False alarm costs \$2000.

   Calculate the expected total cost associated with faulty production. Is it better to use another decision rule for detecting the trouble (shut the production if at least one of four tested chips is defective)?

5. New manager suggested testing 100 chips and shutting down production if more than 25 were defective. Calculate expected total cost associated with faulty production for this (number 3) decision rule. Compare with the first two.

Problem 4

You have a coin with $\mathsf{P}(\text{tail}) = p$. You test the null hypothesis that the coin is a fair one. You flip the coin 5 times and if number of tails is 2 or 3, you do not reject the null hypothesis, otherwise you suppose the coin is biased.

1. What is significance level of the test?
2. Plot the OCC function and the power function of the test.
3. Find values of the power function at the $p = 0.3$ and $p = 0.7$.

Problem 5

Random variable X has normal distribution $\mathcal{N}\left(\mu, \sigma^2\right)$. Let σ be equal to 25 and sample size be equal to 100. You test null hypothesis $\mathrm{H}_0 : \mu = 100$ against the alternative hypothesis $\mathrm{H}_1 : \mu < 100$. Significance level of the test is 10%. You reject the null hypothesis if $\overline{X} < c$. Plot the power function of the test.

Problem 6

Let X be $\mathcal{N}\left(\mu, 10^2\right)$. To test $\mathrm{H}_0 : \mu = 80$ against alternative $\mathrm{H}_1 : \mu > 80$ the critical region $\overline{x} > x_c = 83$ was chosen for the sample of size $n = 25$.

1. What is the power function $K(\mu)$ of this test?
2. What is the significance level of this test?
3. What are the values $K(80), K(83), K(86)$?
4. Sketch the graph of the power function and the OCC function.
5. What is the p-value corresponding to $\overline{x} = 83.41$?

Problem 7

Let X be a Bernoulli random variable: $\mathsf{P}(X = 1) = p$ and $\mathsf{P}(X = 0) = 1 - p$. We would like to test the null hypothesis $\mathrm{H}_0 : p \leq 0.4$ against the alternative hypothesis $\mathrm{H}_1 : p > 0.4$. For the test statistic use $Y = \sum\limits_{i=1}^{n} X_i$, where $X_1, X_2, \ldots, X_n$ is a random sample of size n from this Bernoulli distribution. Let the critical region be of the form $C = \{y : y \geq c\}$. (It means that if $Y \in C$ then the hypothesis H_0 is rejected and H_1 is accepted.)

1. Let $n = 100$. On the same set of axes, sketch the graphs of the power function corresponding to the three critical regions: $C_1 = \{y : y \geq 40\}$, $C_2 = \{y : y \geq 50\}$, $C_3 = \{y : y \geq 60\}$. Use normal approximation to compute the probabilities.
2. Let $C = \{y : y \geq 0.45n\}$. On the same set of axes, sketch the graphs of the power function corresponding to the three samples of size 10, 100, and 1000.

Look at the time!