

# PROOFS FOR "OFF-THE-GRID MULTI-PITCH ESTIMATION USING OPTIMAL TRANSPORT"

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## 1. PROOF

We aim to solve the problem

$$\underset{\mathbf{M} \in \mathbb{R}_+^{F \times P}, \omega_0 \in \mathbb{R}^P}{\text{minimize}} \quad \langle \mathbf{C}_{\omega_0}, \mathbf{M} \rangle + \gamma \|\hat{\mathbf{r}} - \mathbf{A}\mathbf{M}\mathbf{1}\|_2^2 + \epsilon D(\mathbf{M}), \quad (1)$$

where,  $D(\mathbf{M}) = \sum_{f,p} [\mathbf{M}]_{f,p} \log[\mathbf{M}]_{f,p} - [\mathbf{M}]_{f,p} + 1$  is the entropic regularization term. To this end, we employ a block-coordinate descent scheme, where the blocks correspond to  $\mathbf{M}$  and  $\omega_0$ . By fixating  $\omega_0$ , we get the problem

$$\underset{\mathbf{M} \in \mathbb{R}_+^{F \times P}}{\text{minimize}} \quad \langle \mathbf{C}_{\omega_0}, \mathbf{M} \rangle + \gamma \|\hat{\mathbf{r}} - \mathbf{A}\mathbf{M}\mathbf{1}\|_2^2 + \epsilon D(\mathbf{M}), \quad (2)$$

For this, the following proposition holds.

**Proposition 1.** *For a given  $\mathbf{C}_{\omega_0}$ , the dual problem of (2) is*

$$\underset{\lambda \in \mathbb{R}^T}{\text{maximize}} \quad \lambda^T \hat{\mathbf{r}} - \epsilon \exp\left(\frac{1}{\epsilon} \lambda^T \mathbf{A}\right) (\mathbf{K}\mathbf{1}_P) - \frac{1}{4\gamma} \|\lambda\|_2^2 + \epsilon FP$$

where  $\mathbf{K} = \exp(-\frac{1}{\epsilon} \mathbf{C}_{\omega_0})$ . The optimal transport plan is then given by inserting the maximizing  $\lambda$  in

$$\mathbf{M} = \text{diag}\left(\exp\left(\frac{1}{\epsilon} \lambda^T \mathbf{A}\right)\right) \mathbf{K} \text{diag}(\mathbf{1}_P). \quad (3)$$

Furthermore, the maximizing  $\lambda$  is the unique root of the equation

$$\mathbf{A} \left( \exp\left(\frac{1}{\epsilon} \lambda^T \mathbf{A}\right) \odot \mathbf{K}\mathbf{1}_P \right) - \hat{\mathbf{r}} + \frac{1}{2\gamma} \lambda = 0, \quad (4)$$

where  $\odot$  denotes elementwise multiplication.

*Proof.* From (2) we form the equivalent problem, given by

$$\underset{\mathbf{M} \in \mathbb{R}_+^{F \times P}, \Delta \in \mathbb{R}^T}{\text{minimize}} \quad \langle \mathbf{C}_{\omega_0}, \mathbf{M} \rangle + \gamma \|\Delta\|_2^2 + \epsilon D(\mathbf{M}), \quad (5)$$

$$\text{s.t.} \quad \hat{\mathbf{r}} = \Delta + \mathbf{A}\mathbf{M}\mathbf{1}. \quad (6)$$

The Lagrangian is then given by

$$\mathcal{L}(\mathbf{M}, \Delta, \lambda) = \langle \mathbf{C}_{\omega_0}, \mathbf{M} \rangle + \gamma \|\Delta\|_2^2 + \epsilon D(\mathbf{M}) + \lambda^T (\hat{\mathbf{r}} - \Delta - \mathbf{A}\mathbf{M}\mathbf{1}). \quad (7)$$

This Lagrangian is strongly convex in  $\mathbf{M}$  and  $\Delta$  with a unique minimizer

$$\mathbf{M} = \text{diag}\left(\exp\left(\frac{1}{\epsilon} \lambda^T \mathbf{A}\right)\right) \mathbf{K} \text{diag}(\mathbf{1}_P), \quad (8)$$

$$\Delta = \frac{\lambda}{2\gamma}, \quad (9)$$

where  $\mathbf{K} = \exp(-\frac{1}{\epsilon} \mathbf{C}_{\omega_0})$ . Inserted into the Lagrangian yields the dual problem

$$\underset{\lambda \in \mathbb{R}^T}{\text{maximize}} \quad \lambda^T \hat{\mathbf{r}} - \epsilon \exp\left(\frac{1}{\epsilon} \lambda^T \mathbf{A}\right) (\mathbf{K}\mathbf{1}_P) - \frac{1}{4\gamma} \|\lambda\|_2^2 + \epsilon FP.$$

Differentiation with respect to  $\lambda$ , set to 0, then yields

$$\mathbf{A} \left( \exp\left(\frac{1}{\epsilon} \lambda^T \mathbf{A}\right) \odot \mathbf{K}\mathbf{1}_P \right) - \hat{\mathbf{r}} + \frac{1}{2\gamma} \lambda = 0. \quad (10)$$

□

### 1.1. Solve for $\omega_0$

We now fixate  $\lambda$  and solve for  $\omega_0$ . The problem in (1) then becomes

$$\underset{\omega_0 \in \mathbb{R}^P}{\text{minimize}} \quad \langle \mathbf{C}_{\omega_0}, \mathbf{M} \rangle, \quad (11)$$

where each element of  $\mathbf{C}_{\omega_0}$  is given by

$$c(\omega_f, \omega_0^{(p)}) = \left( \frac{\omega_f}{\omega_0^{(p)}} - \hat{h}(\omega_f, \hat{\omega}_0^{(p)}) \right)^2, \quad (12)$$

where  $\hat{\omega}_0^{(p)}$  is our prior estimate for pitch  $p$ . Substituting this yields the minimization problem

$$\underset{\omega_0 \in \mathbb{R}^P}{\text{minimize}} \quad \sum_{p=1}^P \sum_{f=1}^F \left( \frac{\omega_f}{\omega_0^{(p)}} - \hat{h}(\omega_f, \hat{\omega}_0^{(p)}) \right)^2 [\mathbf{M}]_{f,p},$$

where  $[\mathbf{M}]_{f,p}$  is the element of  $[\mathbf{M}]$  that corresponds to frequency  $f$  and pitch  $p$ . Differentiation with respect to  $\omega_0^{(p)}$  and setting the result to 0 yields

$$\sum_{f=1}^F -2 \left( \frac{\omega_f}{\omega_0^{(p)}} - \hat{h}(\omega_f, \hat{\omega}_0^{(p)}) \right) \left( \frac{\omega_f}{(\omega_0^{(p)})^2} \right) [\mathbf{M}]_{f,p} = 0,$$

which, when solved for  $\omega_0^{(p)}$ , finally yields the desired result,

$$\omega_0^{(p)} = \frac{\sum_{f=1}^F \omega_f^2 [\mathbf{M}]_{f,p}}{\sum_{f=1}^F \hat{h}(\omega_f, \hat{\omega}_0^{(p)}) \omega_f [\mathbf{M}]_{f,p}}. \quad (13)$$

### 1.2. Interval

We want to show that for an estimate  $\hat{\omega}_0$  of a harmonic signal of harmonic order  $H$  and with fundamental frequency  $\omega_{0,min}$ , such that

$$\frac{-1}{2H+1} \omega_{0,min} \leq \hat{\omega}_0 - \omega_{0,min} \leq \frac{1}{2H-1} \omega_{0,min}, \quad (14)$$

a local quadratic approximation of the objective function in the transport problem, minimized by  $\omega_{0,min}$ , is obtained, using the ground-cost function

$$c(\omega_f, \omega_0^{(p)}) = \min_{h \in \llbracket H \rrbracket} \left( \frac{\omega_f}{\omega_0^{(p)}} - h \right)^2. \quad (15)$$

*Proof.* Consider a harmonic signal with fundamental frequency  $\omega_{0,min}$ , of harmonic order  $H$ . By finding the minimizing  $h$  in (15) for each non-zero spectral component of the harmonic series for  $\omega_{0,min}$ , it may be noted that a local quadratic approximation of the objective function is obtained. The non-zero components of the harmonic series are all integer multiples of the fundamental frequency, given by  $h\omega_{0,min}$ . By letting

$$\hat{\omega}_0 = \omega_{0,min} + \Delta,$$

where  $\Delta \in \mathbb{R}$ , the approximation for each harmonic  $h$  of the pitch is given by

$$h\hat{\omega}_0 = h(\omega_{0,min} + \Delta).$$

We here note that the deviation from  $h\omega_{0,min}$  increases linearly by the harmonic order. Thus, we only need to look at the highest harmonic,  $H$ . Inserting this into the estimate of the harmonic number grants

$$\hat{h}(H\omega_{0,min}, (\omega_{0,min} + \Delta)) = \operatorname{argmin}_{h \in \llbracket H \rrbracket} \left( \frac{H\omega_{0,min}}{(\omega_{0,min} + \Delta)} - h \right)^2.$$

What we now wish to obtain is for the optimal  $h$  to be  $H$ , which is achieved when both

$$(q - H)^2 \leq (q - (H + 1))^2, \quad (16)$$

and

$$(q - H)^2 \leq (q - (H - 1))^2, \quad (17)$$

hold, where  $q = \frac{H\omega_{0,min}}{(\omega_{0,min} + \Delta)}$ . We first look at (16), where we expand the terms and get

$$0 \leq -2q + 2H + 1,$$

which yields

$$q \leq H + \frac{1}{2}.$$

Substituting  $q$  gives

$$\frac{H\omega_{0,min}}{(\omega_{0,min} + \Delta)} \leq H + \frac{1}{2},$$

which when solved for  $\Delta$  leads to the inequality

$$\Delta \geq \frac{1}{2H + 1} \omega_{0,min},$$

which holds when  $\omega_{0,min} + \Delta > 0$ . With similar calculations, we solve (17), granting the upper bound

$$\Delta \leq \frac{1}{2H - 1} \omega_{0,min},$$

which also holds when  $\omega_{0,min} + \Delta > 0$ . By substituting  $\Delta = \hat{\omega}_0 - \omega_{0,min}$ , we finally get the desired interval,

$$\frac{-1}{2H + 1} \omega_{0,min} \leq \hat{\omega}_0 - \omega_{0,min} \leq \frac{1}{2H - 1} \omega_{0,min}. \quad (18)$$

Finally, it may be noted that for the true harmonic number of each spectral component of the signal, the ground-cost in (12) is 0 for  $\omega_{0,min}$ .  $\square$