Quizzes

Quiz 1

1. Prove or give a counterexample: If $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{v}$, then $\mathbf{w} = \mathbf{0}$.
Solution:
If $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{v}$ then

$$\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w} = 0.$$

Therefore $\|\mathbf{w}\| = 0$ and $\mathbf{w} = \mathbf{0}$.

2. Assume $\|\mathbf{v}\| = \|\mathbf{w}\|$. Show that $(\mathbf{v} + \mathbf{w}) \perp (\mathbf{v} - \mathbf{w})$.

Solution:

$$\begin{aligned}
(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} &= \\
&= \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2 = \\
&= 0.
\end{aligned}$$

Therefore $(\mathbf{v} + \mathbf{w}) \perp (\mathbf{v} - \mathbf{w})$.

Quiz 2

1. If $\mathbf{v}$ and $\mathbf{w}$ are unit vectors in $\mathbb{R}^3$, under what condition(s) would $\mathbf{v} \times \mathbf{w}$ also be a unit vector in $\mathbb{R}^3$? Justify your answer.

Solution: Denote by α the angle between $\mathbf{v}$ and $\mathbf{w}$. Then

$$\begin{aligned}
\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \, \| \times \mathbf{w}\| \sin \alpha &= \\
&= \sin \alpha
\end{aligned}$$

Therefore

$$\|\mathbf{v} \times \mathbf{w}\| \iff \sin \alpha = 1 \iff \alpha = \tfrac{\pi}{2} \iff \mathbf{v} \perp \mathbf{w}.$$

2. Find the distance d from the origin to the plane $x + 2y + 3z = 4$.

Solution: Vector $\mathbf{n} = (1, 2, 3)$ is normal to the plane and the point with the position vector $\mathbf{x}_0 = (4, 0, 0)$ belongs to the plane. The origin has position vector $\mathbf{0} = (0, 0, 0)$. Therefore

$$d = \frac{\mathbf{n} \cdot (\mathbf{x}_0 - \mathbf{0})}{\|\mathbf{n}\|} =$$

$$= \frac{4}{\sqrt{1 + 4 + 9}} =$$

$$= \frac{4}{\sqrt{14}}.$$

Quiz 3

1. Determine if the given equation describes a sphere. If so, find its radius and center.

$$2x^2 + 2y^2 + 2z^2 + 4x + 4y + 4z + 44 = 0.$$

Solution: The equation can be rewritten as

$$(x + 2)^2 + (y + 2)^2 + (z + 2)^2 = -10.$$

It has no solutions since the left hand side is nonnegative and right hand side is negative; in particular it does not describe a sphere

2. Assume $\rho(t) = t$, $\theta(t) = t$ and $\varphi(t) = t$ be the spherical coordinates of the paricle, $0 < t < \pi$. Find the position vector $\mathbf{r}(t)$ of the particle in the Cartesian coordinates and its velocity vector $\mathbf{v}(t) = \mathbf{r}'(t)$.

Solution:

$$\mathbf{r}(t) = (t \sin t \cos t, t \sin^2 t, t \cos t).$$

Therefore

$$\mathbf{v}(t) = (\sin t \cos t + t(\cos^2 t - \sin^2 t), \sin^2 t + 2t \sin t \cos t, \cos t - t \sin t).$$

Quiz 4

1. Calculate the arc length of $\mathbf{f}(t) = (3 \cos 2t, 3 \sin 2t, 3t)$ over the interval $[0, \pi/2]$.

2. Let $\mathbf{g}(s)$ be a smooth curve with arc length parametrization and $\kappa(s)$ be its curvature. Show that

$$\mathbf{g}'''(s) \cdot \mathbf{g}'(s) = -\kappa(s)^2$$

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for any s.

Solution:

$$\|\mathbf{g}\,'(s)\| = 1,$$
$$\mathbf{g}\,'(s) \cdot \mathbf{g}\,'(s) = 1, \quad \text{take the derivative of both sides}$$
$$2\mathbf{g}\,''(s) \cdot \mathbf{g}\,'(s) = 0, \quad \text{once more}$$
$$\mathbf{g}\,'''(s) \cdot \mathbf{g}\,'(s) + \mathbf{g}\,''(s) \cdot \mathbf{g}\,''(s) = 0, \quad \text{since} \quad \kappa(s) = \|\mathbf{g}\,''(s)\|,$$
$$\mathbf{g}\,'''(s) \cdot \mathbf{g} = -\kappa^2(s).$$

Quiz 5

1. Parametrize the curve $\mathbf{f}(t) = (\cos 3t, \sin 3t, 4t)$, $t \in [0, \infty)$ by arc length.

2. Recall that the pedal curve is traced by the orthogonal projection of a fixed point P on the tangent lines of a given curve $\mathbf{f}(t)$.

Write a parametric expression for the pedal curve for the unit circle $\mathbf{f}(t) = (\cos t, \sin t)$ and the point $P = (1, 0)$.

Solution: The projection of P to the tangent line is the sum of $\mathbf{f}(t)$ and the projection $\mathbf{w}(t)$ of $(1, 0) - \mathbf{f}(t)$ in the direction of $\mathbf{f}'(t)$. Since $\|\mathbf{f}'(t)\| = 1$ we have
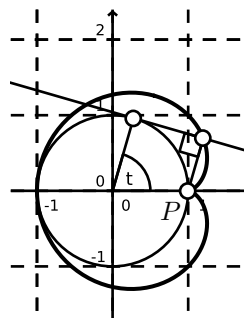
$$\mathbf{w}(t) = [((1,0) - \mathbf{f}(t)) \cdot \mathbf{f}'(t)]\mathbf{f}'(t).$$

Therefore

$$\mathbf{h}(t) = \mathbf{f}(t) + \mathbf{w}(t) =$$
$$= \mathbf{f}(t) + [((1,0) - \mathbf{f}(t)) \cdot \mathbf{f}'(t)]\mathbf{f}'(t)$$

is a parametric equation for the pedal curve.

It remains to do the calculations:

$$\mathbf{f}(t) = (\cos t, \sin t)$$
$$\mathbf{f}'(t) = (-\sin t, \cos t)$$
$$((1,0) - \mathbf{f}(t)) \cdot \mathbf{f}'(t) = -\sin t$$
$$\mathbf{h}(t) = (\cos t + \sin^2 t, \sin t - \sin t \cos t).$$

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Quiz 6

1. Compute the gradient of

$$f(x, y, z) = xyz.$$

2. Find the equation of the tangent plane to the surface

$$xyz = 6$$

at the point $P = (1, 2, 3)$.
Solution: $f(x, y, z) = xyz$, $\nabla f(x, y, z) = (yz, xz, xy)$. Therefore

$$\nabla f(1, 2, 3) = (6, 3, 2)$$

and the equation is

$$6(x - 1) + 3(y - 2) + 2(z - 3) = 0.$$

Quiz 7

1. Find three positive numbers x, y, z whose sum is 10 such that $x^2 y^2 z$ is a maximum.

2. Prove that if (x_0, y_0) is a minimum point for a smooth function $f(x, y)$, then the tangent plane to the surface $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is parallel to the xy-plane.

Solution: Let $z_0 = f(x_0, y_0)$ and $g(x, y, z) = f(x, y) - z$, so

$$g(x, y, z) = 0$$

is the equation for our surface.

The tangent plane to the surface at (x_0, y_0, z_0) is

$$(x - x_0, y - y_0, z - z_0) \cdot \nabla g(x_0, y_0, z_0) = 0$$

or

$$(x - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial x}(x_0, y_0) - (z - z_0) = 0$$

Since (x_0, y_0) is a minimum point

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) = 0.$$

Therefore the equation of tangent plane boils down to $z = z_0$ which is an equation of horizontal plane.

Bicycle

Assume that the trajectory of the back wheel of an ideal bicycle is given by smooth plane curve $\mathbf{b}(t)$, here t denotes time. Write an expression for the trajectory of the front wheel $\mathbf{f}(t)$. Show that the speed of the back wheel can not exceed the speed of the front wheel. We assume that in the ideal bicycle the distance from back wheel and front wheel is fixed, let us denote it by R and the back wheel always moves in the direction to the front wheel.

Solution: Note that $\mathbf{u}(t) = \mathbf{b}'(t)/\|\mathbf{b}'(t)\|$ is the unit vector in the direction from $\mathbf{b}(t)$ to $\mathbf{f}(t)$. Therefore

$$\mathbf{f}(t) = \mathbf{b}(t) + R\mathbf{u}(t) =$$
$$= \mathbf{b}(t) + R\frac{\mathbf{b}'(t)}{\|\mathbf{b}'(t)\|}.$$

and

$$\mathbf{f}'(t) = \mathbf{b}'(t) + R\mathbf{u}'(t).$$

Since $\|\mathbf{u}(t)\| = 1$ we have $\mathbf{u}'(t) \perp \mathbf{u}(t) \parallel \mathbf{b}'(t)$. By Pythagorean theorem,

$$\|\mathbf{f}'(t)\|^2 = \|\mathbf{b}'(t)\|^2 + R^2\|\mathbf{u}'(t)\|^2.$$

In particular,
$$\|\mathbf{f}'(t)\| \geqslant \|\mathbf{b}'(t)\|$$

for any t.