

## Math 312, Extra Credit Problems

These problems are hard and interesting. The solutions should be presented orally before April 14. It might improve your score, but should be used for fun. Only the first solution will be graded.

1a. Show that the field of all numbers of the form  $a + b\sqrt{2}$  for  $a, b \in \mathbb{Q}$  admits a nonstandard ordering (that is, not the one coming from the standard ordering of real numbers).

1b. Show that the field of rational numbers admits a unique ordering.

1c. Show that any finite field admits no ordering.

2. Let  $x, y$  be positive real numbers such that  $\frac{x}{y}$  is irrational. Show that the set

$$\{mx + ny \mid m, n \in \mathbb{Z}\}$$

is dense in  $\mathbb{R}$ .

3. Let us define  $\theta$ -set as the union of a circle with one of its chords. Show that any collection of disjoint  $\theta$  sets in the plane is countable.

~~4.~~ (*solved*) Prove that any sequence contains a monotone subsequence.

5. Let  $a_n$  be a converging sequence and

$$b_n = \frac{1}{n} \sum_{m=1}^n a_m.$$

Prove that  $b_n$  is converging and

$$\lim b_n = \lim a_n.$$

6. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{2^{2^n}} \notin \mathbb{Q}.$$

7. Let  $a_n$  be a decreasing sequence and  $\lim a_n = 0$  prove that

$$\sum_{n=1}^{\infty} a_n \cdot \sin n$$

converge.

8. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n + \sin n}$$

converge.