Math 312, Extra Credit Problems

These problems are hard and interesting. The solutions should
be presents orally before April 14. It might improve your score,
but should be used for fun. Only the first solution will be graded.

1a. Show that the field of all numbers of the form $a+b\cdot\sqrt{2}$ for $a, b \in \mathbb{Q}$ admits a nonstandard ordereing (that is, not the one coming from the standard ordering of real numbers).

~~1b.~~ *(solved)* Show that the field of rational numbers admits unique a ordering.

~~1c.~~ *(solved)* Show that any finite field admits no ordering.

2. Let x, y be positive real numbers such that $\frac{x}{y}$ is irrational. Show that the set

$$\{\, m\cdot x + n\cdot y \mid m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$.

3. Let us define θ-set as the union of a circle with one of its chords. Show that any collection of disjoint θ sets in the plane is countable.

~~4.~~ *(solved)* Prove that any sequence contains a monotone subsequence.

~~5.~~ *(solved)* Let a_n be a converging sequence and

$$b_n = \frac{1}{n} \cdot \sum_{m=1}^{n} a_m.$$

Prove that b_n is converging and

$$\lim b_n = \lim a_n.$$

~~6.~~ *(solved)* Prove that

$$\sum_{n=1}^{\infty} 2^{-2^n} \notin \mathbb{Q}.$$

~~7.~~ *(solved)* Let a_n be a decreasing sequence and $\lim a_n = 0$ prove that

$$\sum_{n=1}^{\infty} a_n \cdot \sin n$$

converge.

8*. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n + \sin n}$$

converges.

~~9.~~ *(solved)* Let $\{x_n\}$ be a decreasing sequence such that $\sum x_n$ converges. Show that

$$\lim_{n\to\infty} n{\cdot}x_n = 0.$$

~~10*a*.~~ *(solved)* Assume that for the function $f\colon \mathbb{R} \to \mathbb{R}$ the identity

$$f(x+y) = f(x) + f(y)$$

holds for any $x, y \in \mathbb{R}$ and f is continuous at 0. Show that $f(x) = a \cdot x$ for some $a \in \mathbb{R}$.

~~10*b*.~~ *(solved)*. Assume that for a monotone function $f\colon \mathbb{R} \to \mathbb{R}$ the identity

$$f(x+y) = f(x) + f(y)$$

holds for any $x, y \in \mathbb{R}$. Show that $f(x) = a \cdot x$ for some $a \in \mathbb{R}$.

11. Construct a continuous function $f\colon [0,1] \to [0,1]$ which takes every value $[0,1]$ infinitely many times.