Solutions

Quiz 1

1. Discuss the existence and uniqueness of the following initial value problem:

$$\dot{x} = x^{1/3}; \quad x(0) = 0.$$

Solution. The function $x \mapsto x^{1/3}$ is continuous therefore equation has local solution for any initial data.

Note that $x(t) = 0$ is a solution and

$$x(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ (\frac{2}{3} \cdot t)^{3/2} & \text{if } t > 0 \end{cases}$$

is also a solution. The latter is evident for $t < 0$ and

$$\dot{x} = \tfrac{3}{2} \cdot (\tfrac{2}{3} \cdot t)^{1/2} \cdot \tfrac{2}{3} = x^{1/3}$$

for $t \geq 0$.

Hence there is no uniqueness.

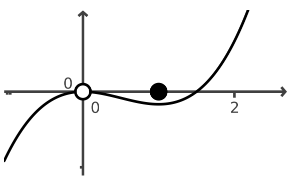
Comment. The function $x \mapsto x^{1/3}$ is not Lipschitz at 0, otherwise local uniqueness would follow we could use Picard's theorem.

Quiz 2

1. For the following vector field, plot the potential function $V(x)$ and identify all the equilibrium points and their stability.

$$\dot{x} = x(1 - x).$$

Solution.



Quiz 3

1. Consider the equation $\dot{x} = rx + x^3$, where $r > 0$ is fixed. Show that $x(t) \to \pm\infty$ in finite time, starting from any initial condition $x_0 \neq 0$.

Solution. Since $f(x) = rx + x^3$ is an odd function, it is sufficient to consider the case $x_0 > 0$.

Since $r > 0$, we have $f(x) > x^3 > 0$ for $x > 0$. Therefore it is sufficient to show that the solution of

$$\dot{x} = x^3$$

escapes to ∞ in finite time for any initial condition $x_0 > 0$.

Solving the equation we get

$$x(t) = \frac{1}{(\frac{1}{x_0^2} - 2t)^{1/2}};$$

the solution approach ∞ as $t \to \frac{2}{x_0^2}$

Quiz 4

1. For the following flow on the circle

$$\dot{\theta} = \mu \cos\theta + \sin(2 \cdot \theta),$$

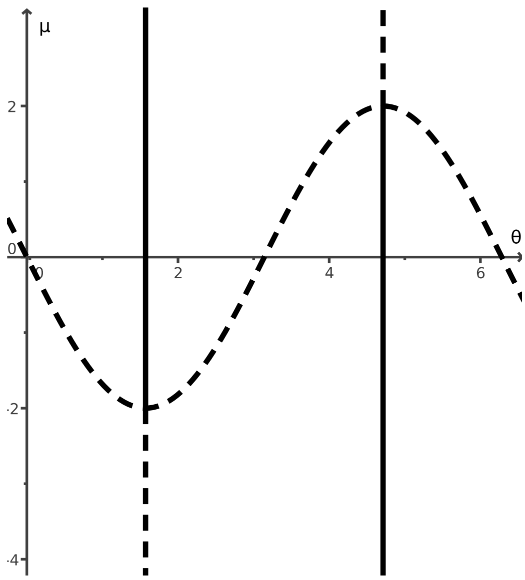
draw the phase portrait, classify the bifurcations that occur as μ varies, and find all the bifurcation values of μ.

Solution.

$$\mu \cos\theta + \sin(2 \cdot \theta) = 0$$

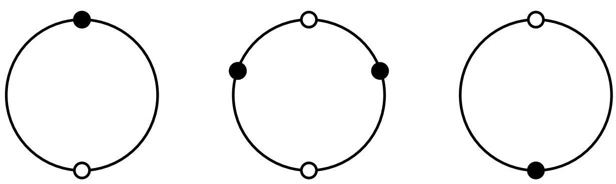
if and only if

$$\theta = \pm\frac{\pi}{2} \pmod{\pi} \quad \text{or} \quad \mu = -2\sin\theta.$$



We have two (subcritical) pitchfork bifurcations at $\mu = \pm 2$.

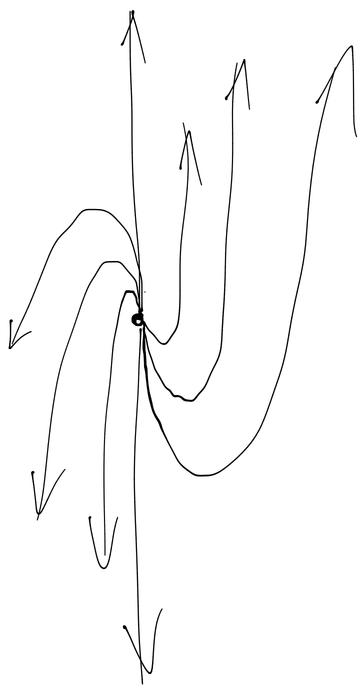
The following diagram shows the behavior of the flow fro $\mu \geq 2$, $|\mu| < 2$ and $\mu \leq -2$ correspondingly.



2. Sketch some typical trajectories of the linear system

$$\begin{cases} \dot{x} = x, \\ \dot{y} = x + y. \end{cases}$$

Solution. The matrix is $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$; both eigenvalues are 1 and it has only one eigenvector $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. So, it is unstable degenerate node. Typical trajectories should be go like this:



General solution: $x(t) = x_0 \cdot e^t$, $y(t) = (x_0 \cdot t + y_0) \cdot e^t$.