Extra credit problems

Math 427

0. Find a mistake or misprint in the book. (The score depends on the type of mistake.)

1. Describe all the motions of the Manhattan plane.

2. Construct a metric space $\mathcal{X}$ and a distance preserving map $f\colon \mathcal{X} \to \mathcal{X}$ that is not a motion of $\mathcal{X}$.

3. Note that the following quantity

$$\tilde{\measuredangle} ABC = \left[\begin{array}{lll} \pi & \text{if} & \measuredangle ABC = \pi \\ -\measuredangle ABC & \text{if} & \measuredangle ABC < \pi \end{array}\right.$$

can serve as the angle measure; that is, the axioms hold if one changes everywhere $\measuredangle$ to $\tilde{\measuredangle}$.
(a). Show that $\measuredangle$ and $\tilde{\measuredangle}$ are the only possible angle measures on the plane.
(b). Show that without Axiom IIIc, this is not longer true.

~~4.~~ *(solved)* Let M be the midpoint of the side $[AB]$ of $\triangle ABC$ and M' be the midpoint of the side $[A'B']$ of $\triangle A'B'C'$. Assume $C'A' = CA$, $C'B' = CB$, and $C'M' = CM$. Prove that $\triangle A'B'C' \cong \triangle ABC$.

~~5.~~ *(solved)* Show that a composition of three reflections in the sides of a nondegenerate triangle does not have a fixed point.

~~6.~~ *(solved)* Consider triangle $\triangle ABC$ with $D \in (AC)$ such that $(BD) \perp (AC)$, and points N and M such that $AN = DC$, $CM = AD$, $(AN) \perp (AB)$ and $(CM) \perp (BC)$.

Prove that M and N are equidistant form B.

~~7.~~ *(solved)* Lines ℓ and m are tangent to two circles of radiuses r and R on such a way the circles are on one side of ℓ and on different sides of m. Let A and B be tangential points of ℓ and Q be the point of intersection ℓ and m. Show that

$$QA{\cdot}QB = R{\cdot}r.$$

8. Given two parallel lines ℓ and m and a point P, use only ruler to construct the line through P parallel to ℓ and m. (You can play with the java applet "Third parallel line" on anton-petrunin.github.io/birkhoff/car/.)

9. Let ABC be a nondegenerate triangele and $A' \in (BC)$, $B' \in (CA)$, $C' \in$ $\in (AB)$ be the points such that

$$2{\cdot}\measuredangle AA'B \equiv 2{\cdot}\measuredangle BB'C \equiv 2{\cdot}\measuredangle CC'A \equiv \tfrac{2}{3}{\cdot}\pi.$$

Show that the triangle formed by the lines (BB'), (CC'), and (AA'), is congruent to $\triangle ABC$.

10. Construct a triangle with the given perimeter, base, and the opposite angle (You can play with the java applet "Triangle with given base, perimeter and angle.html" on anton-petrunin.github.io/birkhoff/car/.)

11. Two points A and B lie on one side of a line ℓ. Two points M and N are chosen on ℓ such that $AM + BM$ is minimal and $AN = BN$. Show that points A, B, M and N lie on one circle.

12. Let Γ be a circle with the center O and A and C be two different points on Γ. For any third point P of the circle let X and Y be the midpoints of the segments AP and CP. Finally, let H be the orthocenter of the triangle OXY. Prove that the position of the point H does not depend on the choice of P.

13. Suppose D and E lie on the same side from (AC), $(AE) \parallel (CD)$, and $AB = BC$. Let K be the intersection of the bisectors of the angles EAB and BCD. Prove that $(BK) \parallel (AE)$.

14. Give a ruler-and-compass construction of a circle or a line which perpendicular to each of three given circles. (You may assume any two of three given circles do not intersect. Play with the java applet "Perpendicular to 3 circles.html" on anton-petrunin.github.io/birkhoff/car/.)

15. Give a ruler-and-compass construction of an inscribed quadrilateral with given sides.

16. Show that a neutral plane is Euclidean if and only if it has a rectangle.

17. Let $ABCDE$ be a regular right-angled pentagon in the hyperbolic plane; that is,

$$AB_h = BC_h = CD_h = DE_h = EA_h$$

and

$$\measuredangle_h ABC = \measuredangle_h BCD = \measuredangle_h CDE = \measuredangle_h DEA = \measuredangle_h EAB = \pm\tfrac{\pi}{2}.$$

Find its side AB_h.