Exercise 6.29

Let (x_A, y_A) and (x_B, y_B) be the coordinates of distinct points A and B in the Euclidean plane. Show that the line (AB) is the set of points with coordinates (x, y) such that

$$(x - x_A) \cdot (y_B - y_A) = (y - y_A) \cdot (x_B - x_A).$$

Solution. Without loss of generality, we can assume that $x_A \neq x_B$; otherwise switch x and y.

Denote by ℓ the set of points with coordinates (x, y) satisfying

$$(x - x_A) \cdot (y_B - y_A) = (y - y_A) \cdot (x_B - x_A).$$

Fix two points $(x, y), (x', y') \in \ell$. note that

$$(x - x') \cdot (y_B - y_A) = (y - y') \cdot (x_B - x_A)$$

Therefore

$$\begin{aligned}(x - x')^2 + (y - y')^2 &= \left(\frac{x - x'}{x_B - x_A}\right)^2 \cdot \left((x_B - x_A)^2 + (y_B - y_A)^2\right) = \\ &= \left(\frac{x - x'}{x_B - x_A}\right)^2 \cdot AB^2. \\ &= |f(x) - f(x')|^2.\end{aligned}$$

Where $f\colon \ell \to \mathbb{R}$ is defined as $(x, y) \mapsto \frac{AB}{|x_A - x_B|} \cdot x$. That is, f is distance preserving.

Note that given $x \in \mathbb{R}$ there is unique $y \in \mathbb{R}$ such that $(x, y) \in \ell$. It follows that the map $f\colon \ell \to \mathbb{R}$ is a bijection and therefore it is an isometry and therefore ℓ is a line.

Finally note that $A, B \in \ell$. By Axiom II, $\ell = (AB)$.