

# Extra credit problems

Math 427

0. Find a mistake or misprint in the book. (The score depends on the type of mistake.)

1. Describe all the motions of the Manhattan plane.

2. Construct a metric space  $\mathcal{X}$  and a distance preserving map  $f: \mathcal{X} \rightarrow \mathcal{X}$  that is not a motion of  $\mathcal{X}$ .

3. Note that the following quantity

$$\tilde{\angle}ABC = \begin{cases} \pi & \text{if } \angle ABC = \pi \\ -\angle ABC & \text{if } \angle ABC < \pi \end{cases}$$

can serve as the angle measure; that is, the axioms hold if one changes everywhere  $\angle$  to  $\tilde{\angle}$ .

(a). Show that  $\angle$  and  $\tilde{\angle}$  are the only possible angle measures on the plane.

(b). Show that without Axiom IIIc, this is not longer true.

4. (solved) Let  $M$  be the midpoint of the side  $[AB]$  of  $\triangle ABC$  and  $M'$  be the midpoint of the side  $[A'B']$  of  $\triangle A'B'C'$ . Assume  $C'A' = CA$ ,  $C'B' = CB$ , and  $C'M' = CM$ . Prove that  $\triangle A'B'C' \cong \triangle ABC$ .

5. (solved) Show that a composition of three reflections in the sides of a nondegenerate triangle does not have a fixed point.

6. (solved) Consider triangle  $\triangle ABC$  with  $D \in (AC)$  such that  $(BD) \perp (AC)$ , and points  $N$  and  $M$  such that  $AN = DC$ ,  $CM = AD$ ,  $(AN) \perp (AB)$  and  $(CM) \perp (BC)$ .

Prove that  $M$  and  $N$  are equidistant from  $B$ .

7. Lines  $\ell$  and  $m$  are tangent to two circles of radii  $r$  and  $R$  on such a way the circles are on one side of  $\ell$  and on different sides of  $m$ . Let  $A$  and  $B$  be tangential points of  $\ell$  and  $Q$  be the point of intersection  $\ell$  and  $m$ . Show that

$$QA \cdot QB = R \cdot r.$$

8. Given two parallel lines  $\ell$  and  $m$  and a point  $P$ , use only ruler to construct the line through  $P$  parallel to  $\ell$  and  $m$ . (You can play with the java applet “Third parallel line” on [anton-petrinin.github.io/birkhoff/car/](http://anton-petrinin.github.io/birkhoff/car/).)

9. Let  $ABC$  be a nondegenerate triangle and  $A' \in (BC)$ ,  $B' \in (CA)$ ,  $C' \in (AB)$  be the points such that

$$2 \cdot \angle AA'B \equiv 2 \cdot \angle BB'C \equiv 2 \cdot \angle CC'A \equiv \frac{2}{3}\pi.$$

Show that the triangle formed by the lines  $(BB')$ ,  $(CC')$ , and  $(AA')$ , is congruent to  $\triangle ABC$ .

10. Construct a triangle with the given perimeter, base, and the opposite angle (You can play with the java applet “Triangle with given base, perimeter and angle.html” on [anton-petrinin.github.io/birkhoff/car/](http://anton-petrinin.github.io/birkhoff/car/).)

11. Two points  $A$  and  $B$  lie on one side of a line  $\ell$ . Two points  $M$  and  $N$  are chosen on  $\ell$  such that  $AM + BM$  is minimal and  $AN = BN$ . Show that points  $A$ ,  $B$ ,  $M$  and  $N$  lie on one circle.

12. Let  $\Gamma$  be a circle with the center  $O$  and  $A$  and  $C$  be two different points on  $\Gamma$ . For any third point  $P$  of the circle let  $X$  and  $Y$  be the midpoints of the segments  $AP$  and  $CP$ . Finally, let  $H$  be the orthocenter of the triangle  $OXY$ . Prove that the position of the point  $H$  does not depend on the choice of  $P$ .

13. Suppose  $D$  and  $E$  lie on the same side from  $(AC)$ ,  $(AE) \parallel (CD)$ , and  $AB = BC$ . Let  $K$  be the intersection of the bisectors of the angles  $A$  and  $C$ . Prove that  $(BK) \parallel (AE)$ .

14. Give a ruler-and-compass construction of a circle or a line which perpendicular to each of three given circles. (You may assume any two of three given circles do not intersect.)