Math 429, Extra Credit Problems

These problems are hard and interesting. The solutions should be presents orally before April 14. It might improve your score, but should be used for fun. Only the first solution will be graded.

1. Construct a topology $\mathcal{T}$ on $\mathbb{R}$ such that a function $f\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing if and only if it is continuous for the topology $\mathcal{T}$.

2. Find three disjoint open sets in the real line which have the same nonempty boundary.

3. How many pairwise distinct sets can one obtain from of a single set by using the operators closure and interior?

4. Construct a continuous function $f\colon [0,1] \to [0,1]$ such that f takes every value in $[0,1]$ an infinite number of times.

5. Describe all the homeomorphisms from the Sierpinski triangle to it self.

6. Prove that $\mathbb{R}^3\backslash S^1$ is homeomorphic to $\mathbb{R}^3\backslash(\ell \cup \{p\})$, where ℓ is a straight line and $p \notin \ell$ is a point in $\mathbb{R}^3$.

7. Show that any nonempty open star-shaped set in $\mathbb{R}^2$ is homeomorphic to the open disc.

8. Construct two functions $\mathbb{R} \to \mathbb{R}$, one is closed but not continuous, and the other is open but not continuous.

9. Construct a bounded open set Ω in $\mathbb{R}^2$ such that its boundary $\partial\Omega$ is totally path-diconnected; that is, no pair of distinct points $x, y \in \partial\Omega$ can be connected by a path in $\partial\Omega$.

10. Show that any connected finite space is path connected.

11. Let X be a path connected space, consider space

$$Z = X \times X/\sim,$$

where $(x, y) \sim (y, x)$. Prove that $\pi_1(Z)$ is abelian.

12. Prove that any two spaces with the same homotopy type can be embedded as deformation retracts in the same topological space.

13. Let X be a topological space, $U, V \subset X$ two open subsets. Prove that if $U \cup V$ and $U \cap V$ are simply connected, then so are U and V.

14. Let a topological space X be the union of two open path-connected sets U and V. Prove that if $U \cap V$ has at least three connected components, then the fundamental group of X is non-Abelian.