Extra pearls for MATH 485

Contents

Preface

I used the topics below in addition the book "Pearls in graph theory" written by Nora Hartsfield and Gerhard Ringel [3]. I tried to keep clarity and simplicity on the same level.

Hope that someone will find it useful for something.

Chapter 1

Ramsey numbers continued

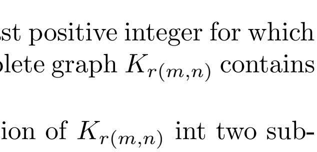
Lower bounds

Recall that Ramsey number $r(m,n)$ is a least positive integer for which every blue-red coloring of edges in the complete graph $K_{r(m,n)}$ contains a blue K_m or a red K_n.

Equivalently, every for any decomposition of $K_{r(m,n)}$ int two subgraphs G and H either G contains a copy of K_m or H contains a copy of K_n.

Therefore, to get a lower bound on $r(m,n) \geqslant s+1$, it is sufficient to decompose K_s into two subraphs with no isomorphic copy of K_m in the first one and no isomorphic copy of K_n in the second one.

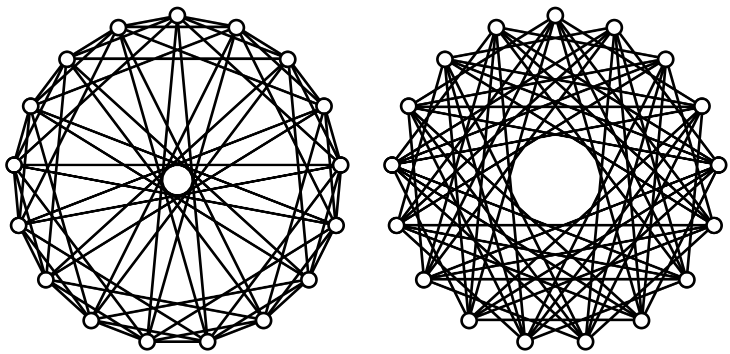
For example, the subgraphs in the decomposition of K_5 on the diagram has no monochromatic triangles; the later implies that $r(3,3) \geqslant 6$. We showed already that for any decomposition of K_6 into two subgraphs, one of the subgraphs has a triangle; that is $r(3,3) = 6$.



Similarly, to show that $r(3,4) \geqslant 9$, we need to construct a decomposition of K_8 in to two subgraphs G and H such that G contains no triangle K_3 and H contains no K_4.

Similarly, to show that $r(4,4) \geqslant 18$, we need to construct a decomposition of K_{17} in to two subgraphs with no K_4. (In fact, $r(4,4) = 18$, but we are not going to prove it.) This is much harder, the corresponding decomposition is given on the following diagram.

The constructed decomposition is rationally symmetric; the first subgraph contains the chords of angle lengths 1, 2, 4, and 8 and the second to all the cords of angle lengths 3, 5, 6 and 7.



1.1. Exercise. *Show that*

(a) In the decomposition of K_8 above, the left graph contains no triangle and the right graph contains no K_4.

(b) In the decomposition of K_{17} above, both graph contain no no K_4.

Hint: In each cases, fix one vertex v and draw the subgraph induced by the vertexes connected to v.

For larger values m and n the problem of finding the exact lower bound for $r(m,n)$ is quickly becomes too hard. Even getting a reasonable rough estimate is challenging. In the next section we will show how to obtain such estimate using probability.

Probabilistic method

The probabilistic method makes possible to prove the existence of graphs with certain properties without constructing them explicitly. The idea is to show that if one randomly chooses a graph from a specified class, then probability that the result is of the needed property is more than zero. The latter implies that a graph with needed property exists.

Despite that this method of proof uses probability, the final conclusion is determined for certain, without any possible error.

This method finds applications in many areas of mathematics; not only in graph theory. It was introduced by Paul Erdős.

Recall that $\binom{n}{m}$ denotes the *binomial coefficient*; that is, m and n are integers, $n \geqslant 0$ and

$$\binom{n}{m} = \frac{n!}{m! \cdot (n-m)!}$$

if $0 \leqslant m \leqslant n$ and $\binom{n}{m} = 0$ otherwise.

The number $\binom{n}{m}$ plays an important role in combinatorics — it gives the number of ways that m objects can be chosen from among n different objects.

1.2. Theorem. *Assume that the inequality*

$$\binom{N}{n} < 2^{\binom{n}{2}-1}$$

holds for a pair of positive integers N and n. Then $r(n,n) > N$.

Proof. We need to show that the complete graph K_N admits a coloring of edges in red and blue such that it has no monochromatic subgraph isomorphic to K_n.

Let us color the edges randomly — color each edge independently with probability $\frac{1}{2}$ in red and otherwise in blue.

Fix a set S of n vertexes. Define the variable $X(S)$ to be 1 if every edge amongst the n vertexes is the same color, and 0 otherwise. Note that the number of monochromatic n-subgraphs in K_N is the sum of $X(S)$ over all possible subsets.

Note that the expected value of $X(S)$ is simply the probability that all of the $\binom{n}{2} = \frac{n \cdot (n-1)}{2}$ edges in S are the same color. The probability that all the edges with the ends in S are blue is $1/2^{\binom{n}{2}}$ and with the same probability all edges are red. Since these two possibilities exclude each other the expected value of $X(S)$ is $2/2^{\binom{n}{2}}$.

This holds for any n-vertex subset S of the vertexes of K_N. The total number of such subsets is $\binom{N}{n}$. Therefore the expected value for the sum of $X(S)$ over all S is

$$X = 2 \cdot \binom{N}{n} / 2^{\binom{n}{2}}.$$

Assume that $X < 1$. Note that at least in one coloring suppose to have at most X complete monochromatic n-subgraphs. Since this number has to be an integer, at least one coloring must have no complete monochromatic n-subgraphs.

Therefore if $\binom{N}{n} < 2^{\binom{n}{2}-1}$, then there is a coloring K_N without monochromatic n-subgraphs. Hence the statement follows. $\square$

The following corollary implies that the function $n \mapsto r(n,n)$ grows at lest exponentially.

1.3. Corollary. *$r(n,n) > \frac{1}{8}\cdot 2^{\frac{n}{2}}$ for all positive integers $n \geqslant 2$.*

Proof. Set $N = \lfloor \frac{1}{8}\cdot 2^{\frac{n}{2}} \rfloor$; that is, N is the largest integer $\leqslant \frac{1}{8}\cdot 2^{\frac{n}{2}}$.

Note that

$$2^{\binom{n}{2}-1} > (2^{\frac{n-3}{2}})^n \geqslant N^n.$$

and

$$\binom{N}{n} = \frac{N\cdot(N-1)\cdots(N-n+1)}{n!} < N^n.$$

Therefore

$$\binom{N}{n} < 2^{\binom{n}{2}-1}.$$

By Theorem 1.2, we get $r(n,n) > N$. □

In the following exercise, mimic the proof of Theorem 1.2, very rough estimates will do the job.

1.4. Exercise. *By random coloring we will understand a coloring edges of a given graph in red and blue such that each edge is colored independently in red or blue with equal chances.*

Assume the edges of the complete bigraph $K_{100,100}$ is colored randomly. Show that probability that $K_{100,100}$ is monochromatic is less than $\frac{1}{10^{2500}}$.

Show that the number of different subgraphs in $K_{10^{10},10^{10}}$ isomorphic to $K_{100,100}$ is less than 10^{2000}.

Assume the edges of the complete bigraph $K_{10^{10},10^{10}}$ is colored randomly. Show that the expected number of monochromatic subgraphs isomorphic to $K_{100,100}$ in $K_{10^{10},10^{10}}$ is less than 1.

Conclude that the complete bigraph $K_{10^{10},10^{10}}$ admits an edge coloring in two colors such that it contains no monochromatic $K_{100,100}$.

Counting proof

In this section we will repeat the proof of Theorem 1.2 using a different language, without use of probability. We do this to affirm that probabilistic method provides real proof, without any possible error.

In principle, any probabilistic proof admits such translation, but in most cases, the the translation is less intuitive.

Proof of 1.2. The graph K_N has $\binom{N}{2}$ edges. Each edge can be colored in blue or red therefore the total number of different colorings is

$$\Omega = 2^{\binom{N}{2}}.$$

Fix a subgraph isomorphic to K_n in K_N. Note that this graph is red in $\Omega/2^{\binom{n}{2}}$ different colorings and yet in $\Omega/2^{\binom{n}{2}}$ different colorings this subgraph is blue.

There are $\binom{N}{n}$ different subgraphs isomorphic to K_n in K_N. Therefore the total number of monochromatic K_n's in all the colorings is

$$M = \binom{N}{n} \cdot \Omega \cdot 2/2^{\binom{n}{2}}.$$

If $M < \Omega$, then by the pigeonhole principle, there is a coloring with no monochromatic K_n. Hence the result. $\square$

Remarks

Note that probabilistic method is not nonconstructive — often when the existence of a certain graph is probed by probabilistic method, it is still uncontrollably hard to describe a concrete example.

While the presented example gives an idea about using the method, a typical proof with probabilistic method, is somewhat more involved. The following theorem is more advanced, but it gives a better illustration for probabilistic method.

1.5. Theorem. *Given a positive integer g and k there is a graph G with girth at least g and chromatic number at least k.*

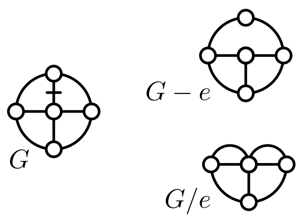
Reads the chapter "Probability makes counting easy" in [1] if you learn more about this method.

Chapter 2

Deletion and contraction

Definitions

Let e be an edge in the pseudograph G. Denote by $G-e$ the pseudograph obtained from G by deleting e, and by G/e the pseudograph obtained from G by contraction the edge e to a point; see the diagram.



If G is a graph then so is $G-e$. On the other hand, G/e might have double edges but no loops; therefore G/e is a multigraph.

If G is a multigraph then so is $G-e$. If e is a double edge in G then G/e might have loops; therefore G/e is a pseudograph in general.

Chromatic polynomial

Assume we want to count different colorings of the graph G into k colors. Denote by $\chi(G,k)$ the number of colorings of the graph G in k colors such that the ends of each edge get different colors.

2.1. Exercise. *Assume that a graph G has two connected components H_1 and H_2. Show that*

$$\chi(G,k)=\chi(H_1,k)\cdot\chi(H_2,k)$$

for any k.

2.2. Exercise. *Show that for any integer $n \geqslant 3$,*

$$\chi(W_n, k+1) = (k+1)\cdot\chi(C_n, k),$$

where W_n denotes the wheel with n spokes and C_n is the cycle of length n.

2.3. Deletion-minus-contraction formula. *For any edge e in the pseudograph G we have*

$$\chi(G, k) = \chi(G - e, k) - \chi(G/e, k).$$

Proof. Indeed, the admissible colorings of $G - e$ can be divided into two groups: (1) those where the ends of the edge e get different colors — these remain to be valid colorings of G and (2) those where the ends of e get the same color — each of such colorings corresponds to unique coloring of G/e. Hence

$$\chi(G - e, k) = \chi(G, k) + \chi(G/e, k),$$

which is equivalent to the deletion-minus-contraction formula. □

Note that if the pseudograph G has loops then $\chi(G, k) = 0$ for any k. Indeed in an admissible coloring the ends of loop should get different colors, which is impossible.

One can see also it from the deletion-minus-contraction formula. Note that if e is a loop in G, then $G/e = G - e$; therefore $\chi(G - e, k) = \chi(G/e, k)$ and

$$\chi(G, k) = \chi(G - e, k) - \chi(G/e, k) = 0.$$

Similarly, removing a double edge from a pseudograph G does not change $\chi(G, k) = 0$. Indeed, if e is an edge of G which has a parallel edge f then in G/e the edge f becomes a loop. Therefore $\chi(G/e, k) = 0$ for any k and by deletion-minus-contraction formula we get

$$\chi(G, k) = \chi(G - e, k).$$

The same identity can be seen directly — any admisible coloring of $G - e$ is also admissible in G — since the ends of f get different colors, so does e.

It follows that the problem of finding $\chi(G, k)$ can be reduced to the case when G is a graph.

Recall that polynomial P of k is an expression of the following type

$$P(k) = a_0 + a_1\cdot k + \cdots + a_n\cdot k^n,$$

with constants $a_0, \dots, a_n$, which are called *coefficients* of the polynomial. If $a_n \neq 0$, it is called *leading coefficient* of P; in this case n is the degree of P. If the leading coefficient is 1 then the polynomial is called *monic.*

2.4. Theorem. *For any fixed pseudograph G, the function $\chi(G,k)$ is a polynomial of k with integer coefficients; this polynomial is called* chromatic polynomial *of the graph G.*

Moreover, if G has a loop then $\chi(G,k) \equiv 0$; otherwise the polynomial monic and has degree p, where p is the number of vertexes in G.

Proof. We will use the induction on total number of edges in the pseudograph and the deletion-minus-contraction formula.

As the base case, consider the null graph N_p; that is the graph with p vertexes and no edges. Since N_p has no edges, any coloring of N_p is admissible. We have k choices for each of n vertexes therefore

$$\chi(N_p, k) = k^p.$$

In particular, the function $k \mapsto \chi(N_p, k)$ is given by monic polynomial of degree p with integer coefficients.

Assume that the first statement holds for all pseudographs with at most $q-1$ edges. Fix a graph G with q edges. Applying the deletion-minus-contraction formula, we get

$$\chi(G,k) = \chi(G-e,k) - \chi(G/e,k).$$

Note that the pseudographs $G-e$ and G/e have $q-1$ edges. By induction hypothesis, $\chi(G-e,k) = P(k)$ and $\chi(G/e,k) = Q(k)$ for some polynomials P and Q with integere coefficients. Hence the same holds for their difference $\chi(G,k) = P(k) - Q(k)$.

If G has a loop then $\chi(G,k) = 0$ as G has no valid colorings. The remaining case is also proved by induction; we assume that the characteristic polynomial of any multigraph G with at most $q-1$ edges and at most p vertexes is monic of degree p.

Fix a multigraph G with p vertexes and q edges. Note that $G-e$ is a multigraph with p vertexes, in particular its chromatic polynomial P is monic of degree p. Further the pseudograph G/e has $p-1$ vertexes, and its characteristic polynomial is either zero or has degree $p-1$. In both cases $P-Q$ is a monic polynomial of degree p. □

2.5. Exercise. *Use induction and deletion-minus-contraction formula to show that*

(a) $\chi(P_q, k) = k\cdot(k-1)^q$ *where P_q denotes the path with q edges;*

(b) $\chi(T,k) = k\cdot(k-1)^q$ *for any tree* T *with* q *edges;*
(c) $\chi(C_p,k) = (k-1)^p + (-1)^p\cdot(k-1)$ *for the cycle* C_p *of length* p*.*

2.6. Exercise. *Show that*

$$\chi(K_p,k) = k\cdot(k-1)\cdots(k-p+1).$$

Note that for any graph G with p vertexes we have

$$\chi(K_p,k) \leqslant \chi(G,k) \leqslant \chi(N_p,k)$$

for any k. Since

$$\chi(K_p,k) = k\cdot(k-1)\cdots(k-p+1),$$
$$\chi(N_p,k) = k^p,$$

it follows that chromatic polynomial of G is monic of degree p. It gives an alternative way to prove the second statement in Theorem 2.4.

2.7. Exercise. *Construct a pair of nonisomorphic graphs with equal chromatic polynomials.*

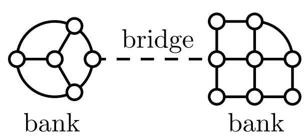
Number of spanning trees

Recall that $t(G)$ denotes the number of spanning trees in the pseudograph G.

An edge e in a connected graph G is called *bridge*, if deletion of this edge makes it disconnected; that is, the remaining graph has two connected components which are called *banks*.

2.8. Exercise. *Assume that the graph* G *contains a bridge between banks* H_1 *and* H_2*. Show that*

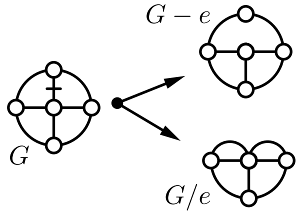
$$t(G) = t(H_1)\cdot t(H_2).$$



2.9. Theorem. *Let* e *be an edge in the pseudograph* G*. Assume* e *is not a loop, then the following identity holds*

$$t(G) = t(G-e) + t(G/e), \qquad (*)$$

The identity $(*)$ is called *deletion-plus-contraction formula.* Often it is convenient to write the identity $(*)$ using a diagram as on the picture; the edge e is marked on the diagram.



Proof. Note that the spanning trees of G can be subdivided into two groups — (1) those which contain the edge e and (2) those which do not. For the trees in the first group, contraction of e to a point gives a spanning tree in G/e, while the trees in the second group are also spanning trees in $G - e$.

Moreover, both of the described correspondences are one-to-one. Hence the formula follows. □

Note that a spanning tree can not have loops. Therefore if we remove all loops from the pseudograph, then the number of spanning trees remains unchanged. In other words, for any loop e the following identity holds

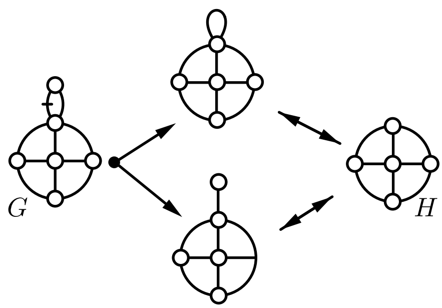
$$t(G) = t(G - e).$$

From the deletion-plus-contraction formula we can deduce few other useful identities. For example, assume that the graph G has an end vertex w (that is $\deg w = 1$). If we remove the vertex w and its edge from G, then in obtained graph $G - w$ the number of spanning trees remians unchanged; that is

$$(**) \qquad t(G) = t(G - w).$$

Indeed, denote by e the only edge incident to w. Note that the graph $G - e$ is not connected, since the vertex w is isolated. Therefore $t(G - e) = 0$. On the other hand $G/e = G - w$ therefore $(*)$ implies $(**)$.

On the diagrams, we will use two-sided arrow "$\leftrightarrow$" for the graphs with equal number of the spanning trees. For example, from the discussed identities we can draw the diagram, which in particular implies the following identity:



$$t(G) = 2 \cdot t(H).$$

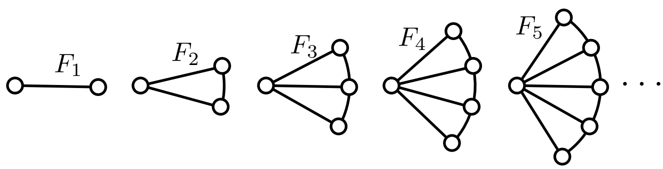
The deletion-plus-contraction formula gives an algorithm to calculate the value $t(G)$ for given pseudograph G. Indeed, for any edge e, both graphs $G - e$ and G/e have smaller number of edges. That is, deletion-plus-contraction formula reduce the problem of finding number of the trees to simpler graphs; applying this formula few times we can reduce the question to a collection of graphs the answer is evident for each.

Fans and their relatives

Recall that Fibonacci numbers f_n are defined using the recursive identity $f_{n+1} = f_n + f_{n-1}$ with $f_1 = f_2 = 1$. The sequence of Fibonacci numbers starts as

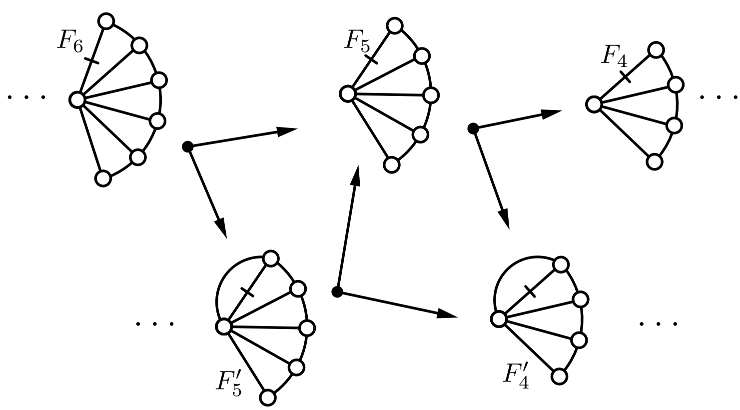
$$1, 1, 2, 3, 5, 8, 13, \ldots$$

The graphs of the following type are called *fans*; a fan with $n+1$ vertex will be denoted by F_n.



2.10. Theorem. $t(F_n) = f_{2\cdot n}$.

Proof. Applying the delection-plus-contraction formula, we can draw the following infinite diagram. In addition to the fans F_n we use its variations F'_n, which differ from F_n by one double edge.



Set $a_n = t(F_n)$ and $a'_n = t(F'_n)$. From the diagram we get the following two recurcive relations:

$$\begin{aligned} a_{n+1} &= a'_n + a_n, \\ a'_n &= a_n + a'_{n-1}. \end{aligned}$$

That is, in the sequence

$$a_1, a'_1, a_2, a'_2, a_3 \dots$$

every number starting from a_2 is sum of previous two.

Further note that F_1 has two vertexes connected by unique edge, and F'_1 has two vertexes connected by double edge. Hence $a_1 = 1 = f_2$ and $a'_1 = 2 = f_3$ and therefore

$$a_n = f_{2\cdot n}$$

for any n. □

Comments. We can deduce a recursive relation for a_n, without using a'_n:

$$\begin{aligned} a_{n+1} &= a'_n + a_n = \\ &= 2\cdot a_n + a'_{n-1} = \\ &= 3\cdot a_n - a_{n-1}. \end{aligned}$$

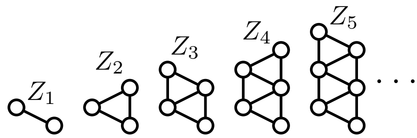
For the sequences defined by the *linear recursion* as above are called *constant-recursive sequences*. The general term of such sequence can be expressed by a closed formula, see [5]. In our case it is

$$a_n = \tfrac{1}{\sqrt{5}}\cdot\left((\tfrac{3+\sqrt{5}}{2})^n - (\tfrac{3-\sqrt{5}}{2})^n\right).$$

Since a_n is integer and $0 < \tfrac{1}{\sqrt{5}}\cdot(\tfrac{3-\sqrt{5}}{2})^n < 1$ for any $n \geqslant 1$ a shorter formula can be written using the floor function $x \mapsto \lfloor x \rfloor$

$$a_n = \left\lfloor \tfrac{1}{\sqrt{5}}\cdot(\tfrac{3+\sqrt{5}}{2})^n \right\rfloor.$$

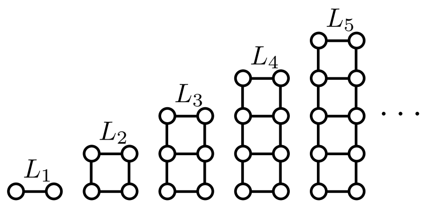
2.11. Exercise. *Consider the sequence of zig-zag graphs Z_n of the following type:*



Show that $t(Z_n) = f_{2\cdot n}$ *for any* n.

Hint: Use the induction on n and/or mimic the proof above.

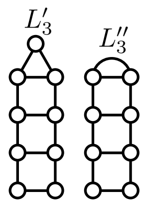
2.12. Exercise. *Let us denote by* b_n *the number of spanning trees in the* n-step ladder L_n*; that is, in the graph of the following type:*



Apply the method we used for fans F_n *to show that the sequence* b_n *satisfies the following linear recurcive relation*

$$b_{n+1} = 4\cdot b_n - b_{n-1}.$$

Hint: To construct the recusive relation, in addition to the ladders L_n you will need two of its analogs L'_n and L''_n shown on the diagram.

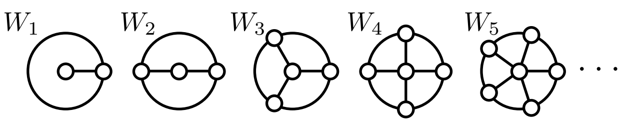


Note that $b_1 = 1 \;\; b_2 = 4$; applying the exercise we could calculate first numbers of the sequence (b_n):

$$1, 4, 15, 56, 209, 780, 2911, \ldots$$

The following exercise is analogous, but more complicated.

2.13. Advanced exercise. *Recall that a wheel* W_n *is the graph of following type:*



Show the sequence $c_n = t(W_n)$ *satifies the following recurcive relation*

$$c_{n+1} = 4\cdot c_n - 4\cdot c_{n-1} + c_{n-2}.$$

Using the exercise above and applying induction one can show that

$$c_n = f_{2\cdot n+1} + f_{2\cdot n-1} - 2 = l_{2\cdot n} - 2$$

for any n; the numbers $l_n = f_{n+1} + f_{n-1}$ are called *Lucas numbers*; they pop up in combinatorics as often as Fibonacci numbers.

Final remarks

The proof of recurrent relation above is taken from [7]; this problem is also discussed in the classical book [4].

Chapter 3

Matrix theorem

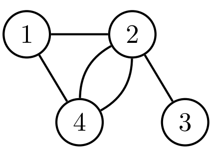
Adjacency matrix

Let us describe a way to encode the given multigraph G with p vertexes by an $p{\times}p$ matrix. First, enumerate the vertexes of the multigraph by numbers from 1 to p; such multigraph will be called *labeled.* Consider the matrix $A = A_G$ with the component $a_{i,j}$ equal to the number of edges from i-th vertex to the j-th vertex of G.

This matrix A is called *adjacency matrix* of G. Note that A is *symmetric*; that is, $a_{i,j} = a_{j,i}$ for any pair i, j. Also, the diagonal components of A vanish; that is $a_{i,i} = 0$ for any i.

For example, the labeled multigraph G shown on the diagram, we get the following adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$$



3.1. Exercise. *Let A be the* adjacency matrix *of a labeled multigraph. Show that the components $b_{i,j}$ of the n-th power A^n is the number of walks of length n in the graph from vertex i to vertex j.*

Hint: Use induction on n.

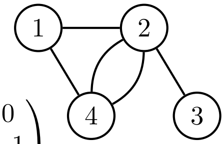
Kirchhoff minor

We are interested in the number of spanning trees in a pseudograph G. Since the loops do not change the the number of spanning trees, we can remove all of them. In other words we can (and will) always assume that G is a multigraph.

First, we will use the $p\times p$ adjacency matrix $A = A_G$ of G to construct $(p-1)\times(p-1)$-matrix $K = K_G$.

1. Revert the signs of the components of A and exchange the zeros on the diagonal by the degrees of the corresponding vertexes. (The matrix A' is called *Kirchhoff matrix*, *Laplacian matrix* or *admittance matrix* of the graph G.)
2. Delete from A' the last column and the last row; the obtained matrix $K = K_G$ will be called *Kirchoff minor* of the labeled pseudograph G.

The labeled multigraph G on the diagram has the following Kirchhoff matrix and Kirchoff minor:



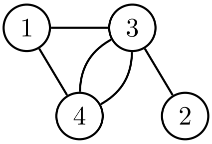
$$A' = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -1 & -2 \\ 0 & -1 & 1 & 0 \\ -1 & -2 & 0 & 3 \end{pmatrix}, \quad K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

3.2. Exercise. *Show that in the obtained matrix A' the sum of components in each row or column vanishes. Conclude that*

$$\det A' = 0.$$

Relabeling. Let us understand what happens with Kirchhoff minor and its determinant as we swap two labels distinct from the last one.

For example, assume we swap the labels 2 and 3 in the graph above, we get an other labeling shown on the diagram. Then the corresponding Kirchhoff minor will be



$$K' = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 4 \end{pmatrix}$$

which is obtained from K by swapping columns 2 and 3 following by swapping rows 2 and 3.

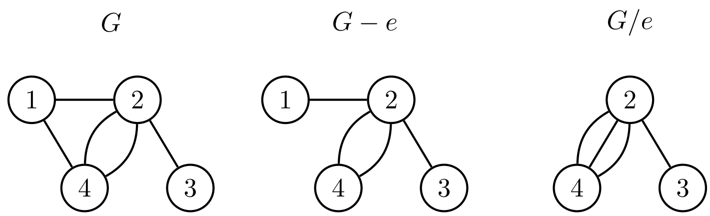
Note that swapping a pair of columns or rows changes the sign of determinant. Therefore swapping one pair of rows and one pair of columns does not change the determinant. The same holds in general; let us summarize it:

3.3. Observation. *Assume G is a labeled graph with p vertexes and K_G is its Kirchhoff minor. If we swap two labels $i, j < p$ then corresponding Kirchhoff minor K'_G can be obtained from K_G by swapping columns i and j following by swapping rows i and j. In particular,*

$$\det K'_G = \det K_G.$$

Deletion and contraction. Next let us understand what happens with Kirchhoff minor if we delete or contract an edge in the labeled multigraph. (If after contraction of an edge we get loops, we remove it; this way we obtain a multigraph.).

Assume edge e connects first and last vertex of labeled multigraph G as in the following example:



Note that deleting e only reduce the corner component of K_G by one, while contracting it removes first row and column. That is, since

$$K_G = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

we have

$$K_{G-e} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad K_{G/e} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}.$$

Again, the same holds in general, let us summarize it in the following observation.

3.4. Observation. *Assume e is an edge of labeled multigraph G between the first and last vertex then and K_G is the Kirchhoff minor of G/e. Then*

(a) to obtain the Kirchhoff minor K_{G-e} of $G - e$ it is sufficient to subtract 1 from the corner element with index (1,1) of K_G;

(b) to obtain the Kirchhoff minor $K_{G/e}$ of G/e it is to remove first row and column in K_G

In particular by cofactor expansion of determinant we get

$$\det K_G = \det K_{G-e} + \det K_{G/e}$$

Note that the last formula reminds deletion-plus-contraction formula. This is the key observation in the proof of matrix theorem below.

Matrix theorem

3.5. Matrix theorem. *Let K be the Kirchoff minor of labeled multigraph G with at least two vertexes. Then*

$$t(G) = \det K, \tag{***}$$

where $t(G)$ denotes the number of spanning trees in G.

Proof. As usual we denote by p and q the number of vertexes and edges in G. We will use induction on the number $p+q$; we will always assume that $p \geqslant 2$.

Note that if $p = 2$ then $t(G) = q$. Further note that $K_G = (q)$; that is, K_G is a $1{\times}1$ matrix with single component q. In particular $\det K_G = q$ and therefore the formula (∗∗∗) holds if $p = 2$. We will use it as a part of the base.

Denote by d the degree of the last vertex in G.

Assume $d = 0$. Then G is not connected and therefore $t(G) = 0$. On the other hand, the sum in each row of K_G vanish. Hence the sum of all columns in K_G vanish; in particular, the columns in K_G are linearly dependent and hence $\det K_G = 0$.

In particular if $q = 0$ then $d = 0$ hence in this case formula (∗∗∗) holds if $q = 0$. This is the second part of the induction base.

If $d > 0$, then without loss of generality, we may assume that the first and last vertexes of G are adjacent, otherwise permute pair of labels $i, j < p$ and apply Observation 3.3. Denote by e the edge between the first and last vertex.

Note that the total number number of vertexes and edges in the pseudographs $G-e$ and G/e are smaller than $p+q$; hence by induction hypothesis the matrix theorem holds for these two pseudographs.

Applying deletion-plus-contraction formula, the induction hypothesis and Observation 3.4, we get

$$\begin{aligned} t(G) &= t(G-e) + t(G/e) = \\ &= \det K_{G-e} + \det K_{G/e} = \\ &= \det K_G. \end{aligned}$$

Which proves the induction step. □

3.6. Exercise. *Fix a labeling for each of the following graphs, find its Kirchoff's minor and use matrix theorem to find the number of spanning trees.*

(a) $t(K_{3,3})$*;*

(b) $t(W_6)$*;*

(c) $t(Q_3)$*.*

(You can use `http://matrix.reshish.com/determinant.php`*, or any other matrix calculator.)*

Calculation of determinants

In this section we recall key properties of determinant which will be used in the next section.

Let M be an $n{\times}n$-matrix; that is a table $n{\times}n$, filled with numbers which are called *components of the matrix*. The determinat $\det M$ is a polynomial of the n^2 components of M, which satisfies the following conditions:

1. The unit matrix has determinant 1; that is,

$$\det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = 1.$$

2. If we multiply each component of one of the rows of the matrix M multiply by a number λ, then for the obtained matrix M', we have

$$\det M' = \lambda{\cdot} \det M.$$

3. If one of the rows in the matrix M add (or subtract) term-by-term to an other row, then the obtained matrix M' has the same determinant

$$\det M' = \det M.$$

These three conditions define determinant in a unique way. We will not give a proof of the statement, it is not evident and not complicated and soon or later you will have to learn it, if it is not done already.

3.7. Exercise. *Show that the following property follows from the properties above.*

4. If we permute two rows in the matrix M then the obtained matrix M' will have determinant of opposite sign; that is,

$$\det M' = -\det M.$$

The determinant of $n \times n$-matrix can be written explicatively as a sum of $n!$ terms. For example,

$$a_1 \cdot b_2 \cdot c_3 + a_2 \cdot b_3 \cdot c_1 + a_3 \cdot b_1 \cdot c_2 - a_3 \cdot b_2 \cdot c_1 - a_2 \cdot b_1 \cdot c_3 - a_1 \cdot b_3 \cdot c_2$$

is the determinant of the matrix

$$M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

However, the properties described above give a more convinient and faster way to calculate the determinant, especially for larger values n.

Let us show it on one example which will be needed in the next section:

$$\begin{aligned}
\det \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} &= \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} = \\
&= \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} = \\
&= 5^3 \cdot \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
&= 5^3 \cdot \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
&= 5^3.
\end{aligned}$$

Let us describe what we used on each line above:

1. property 3 three times — we add to the first row each of the remaining rows;
2. property 3 three times — we add first row to the each of the remaining three rows;

3. property 2 three times;
4. property 3 three times — we subtract from the first row the remaining three rows;
5. property 1.

Cayley formula

Recall that the *complete graph* is the graph where each pair of vertexes is connected by an edge; complete graph with p vertexes is denoted by K_p.

Note that every vertex of K_p has degree $p-1$. Therefore the Kirchhoff minor $K = K_{K_p}$ in the matrix formula (∗∗∗) for K_p is the following $(p-1) \times (p-1)$-matrix:

$$K = \begin{pmatrix} p-1 & -1 & \cdots & -1 \\ -1 & p-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p-1 \end{pmatrix}.$$

The argument given in the end of the previous section admits a direct generalization and it implies that

$$\det K = p^{p-2}$$

and therefore

$$t(K_p) = p^{p-2}$$

for any p.

This identity is called *Cayley formula.*

Final remarks

There is strong connection between counting spanning of the given graph and calculations of currents in an electric chain.

Assume that the graph G describes an electic chain; each edge has resistance one Ohm. A battery is connected to the vertexes a and b. The total current between these vertexes is I Ampere. Assume we need to calculate the current thru edge e.

Fix an orientation of e. Note that the spanning trees of G can be subdivided into the following three groups: (1) those where the edged e appears on the (necessary unique) path from a to b with positive orientation, (2) those where the edged e appears on the path from a to b with negative orientation, (3) those which the edged e do not appear on the path from a to b. Denote by t_+, t_- and t_0 the number of the trees in each group. Clearly

$$t(G) = t_+ + t_- + t_0.$$

The current I_e along e can be calculated using the following formula:

$$I_e = \frac{t_+ - t_-}{t(G)} \cdot I.$$

This statement can be proved by checking Kirchoff rules for the currents calculated by this formula.

There are many other applications of Kirchoff rules to graph theory For example, in [6], they were used to prove the Eulers's formula

$$p - q + r = 2,$$

where p, q r denotes the number of vertexes, edges and regions of in a plane drawing of graph.

The Kirchoff rules and the *delection-plus-contraction* formula were used in the solution of the so called *squaring the square problem.* The history of this problem and its solution discussed in [2, Chapter 17].

Few proofs of Cayley formula, inculding Prüfer's code are given in [1, Chapter 30].

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