Extra credit problems

Math 485

0. Find a mistake or misprint in “Extra pearls”. (The score depends on the type of mistake.)

~~1.~~ *(solved)* Assume $d_1, \dots, d_p$ is a sequence of integers in a nonincreasing order. Show that it is multigraphic if and only if $d_p \geq 0$, the sum $d_1 + \dots + d_p$ is even and

$$d_1 \leq d_2 + \dots + d_p.$$

(A sequence of integers $d_1, \dots, d_p$ is called *multigraphic* if it appears as a sequence of degrees of a multigraph.)

~~2.~~ *(solved)* Assume that a sequence $d_1, \dots, d_p$ is graphic, $d_i \geq 1$ for each i and

$$d_1 + \dots + d_p \geq 2 \cdot (p - 1).$$

Show that there is a *connected* graph G with the degree sequence $d_1, \dots, d_p$.

~~3.~~ *(solved)* Show that in any connected graph G there is a vertex v such that $G - v$ is connected.

~~4.~~ *(solved)* Let G be a connected graph. Show that any two paths of maximum length in G have a common vertex.

5. Assume two trees R and S have the vertices $r_1, \dots, r_n$ and $s_1, \dots, s_n$ correspondingly. Assume that $R - r_i$ is isomorphic to $S - s_i$ for each i. Show that R is isomorphic to S.

~~6.~~ *(solved)* Let G be a critical graph and $\chi(G) = k+1$. Show that after removing any $k - 1$ edges from G the obtained graph remains connected.

~~7.~~ *(solved)* Assume both sequences $d_1, \dots, d_p$ and $d_1 - 1, \dots, d_p - 1$ are graphic. Show that there is a graph with a 1-factor and degree sequence $d_1, \dots, d_p$.

8. Show that any edge of cubic graph lies in an even number of Hamiltonian cycles.

~~9.~~ *(solved)* Assume that the edges of a complete graph are colored in two colors. Show that there is a Hamiltonian cycle which either monochromatic or consists of two monochromatic paths.

10. Assume that G is a connected cubic graph such that for any two vertices v, w of G there is an isomorphism $G \to G$ which sends v to w. Prove that G remains connected after removing any 2 edges.

~~11.~~ *(solved)* Let G be a connected graph with no triangles. Given any two vertices v, w in G, denote by $d(v, w)$ the length of a shortest path containing v to w.

Show that G is a tree if and only if

$$d(x, y) + d(v, w) = \max\{\, d(x, v) + d(y, w), d(x, w) + d(y, v)\,\}$$

for any 4 vertices x, y, v, w in G.

12. Understand the proof of Vizing's theorem (see the references for Theorem 2.2.2 in "Pearls" or find a proof elsewhere).

13. Suppose $p = r(m, n)$ is the Ramsey number for m and n.

Assume that G results from K_p by deleting a single edge. Show that G has a red/blue edge coloring with no red K_m and no blue K_n.

14. Assume that the real intervals $I_1, \dots, I_k$ have $2{\cdot}k$ different ends that are chosen randomly from $\{1, 2, \dots, 2{\cdot}k\}$. Consider the graph G with k vertexes such that ith vertex is connected to jth vertex if and only if I_i overlaps with I_j. Show that G has diameter ≤ 2 with probability at least $\frac{2}{3}$.

15. Understand the proof of Erdős–Lovász theorem (see the references for Theorem 2.1.5 in "Pearls", or Theorem 2.14 in "Extra pearls", or find a proof elsewhere).

~~16.~~ *(solved)* Let G be a graph with p vertexes. Show that G is a tree if and only if its chromatic polynomial is $x \cdot (x-1)^{p-1}$.

17. Let T be a spanning tree in a graph G with weighted edges. Show that T has minimal total weight if and only if the weight of any edge (a, b) in G is at least as large as the weight of any edge on the path from a to b in T.

18. In a group of people, for some fixed s and any k, any k girls like at least $k - s$ boys in total. Show that then all but s girls may get married on the boys they like.

19. Let M be a maximal planar map. Assume the vertexes of M are colored in 3 colors. Show that the number of the regions that get all three colors is even.

20. Let M be a map and G be its dual. Show that M and G have the same number of spanning trees.

21. Show that any map has a vertex of degree at most 3 or a country with at most 3 sides.