Extra pearls for MATH 485

Contents

Chapter 1

Probabilistic method

The probabilistic method makes possible to prove the existence of graphs with certain properties without constructing them explicitly. The idea is to show that if one randomly chooses a graph from a specified class, then probability that the result is of the needed property is more than zero. The latter implies that a graph with needed property exists.

Despite that this method of proof uses probability, the final conclusion is determined for certain, without any possible error.

This method finds applications in many areas of mathematics; not only in graph theory. It was introduced by Paul Erdős.

1.1 A lower bound for Ramsey number

Recall that Ramsey number $r(m,n)$ is a least positive integer for which every blue-red coloring of edges in the complete graph $K_{r(m,n)}$ contains a blue K_m or a red K_n.

Recall that $\binom{n}{m}$ denotes the *binomial coefficient*; that is, m and n are integers, $n \geqslant 0$ and

$$\binom{n}{m} = \frac{n!}{m! \cdot (n-m)!}$$

if $0 \leqslant m \leqslant n$ and $\binom{n}{m} = 0$ otherwise.

The number $\binom{n}{m}$ plays an important role in combinatorics — it gives the number of ways that m objects can be chosen from among n different objects.

1.1. Theorem. *Assume that the inequality*

$$\binom{N}{n} < 2^{\binom{n}{2}-1}$$

holds for a pair of positive integers N and n. Then $r(n,n) > N$.

Proof. We need to show that the complete graph K_N admits a coloring of edges in red and blue such that it has no monochromatic subgraph isomorphic to K_n.

Let us color the edges randomly — color each edge independently with probability $\frac{1}{2}$ in red and otherwise in blue.

Fix a set S of n vertexes. Define the variable $X(S)$ to be 1 if every edge amongst the n vertexes is the same color, and 0 otherwise. Note that the number of monochromatic n-subgraphs in K_N is the sum of $X(S)$ over all possible subsets.

Note that the expected value of $X(S)$ is simply the probability that all of the $\binom{n}{2} = \frac{n\cdot(n-1)}{2}$ edges in S are the same color. The probability that all the edges with the ends in S are blue is $1/2^{\binom{n}{2}}$ and with the same probability all edges are red. Since these two possibilities exclude each other the expected value of $X(S)$ is $2/2^{\binom{n}{2}}$.

This holds for any n-vertex subset S of the vertexes of K_N. The total number of such subsets is $\binom{N}{n}$. Therefore the expected value for the sum of $X(S)$ over all S is

$$X = 2\cdot\binom{N}{n}/2^{\binom{n}{2}}.$$

Assume that $X < 1$. Note that at least in one coloring suppose to have at most X complete monochromatic n-subgraphs. Since this number has to be an integer, at least one coloring must have no complete monochromatic n-subgraphs.

Therefore if $\binom{N}{n} < 2^{\binom{n}{2}-1}$, then there is a coloring K_N without monochromatic n-subgraphs. Hence the statement follows. $\square$

The following corollary implies that the function $n \mapsto r(n,n)$ grows at lest exponentially.

1.2. Corollary. $r(n,n) > \frac{1}{8}\cdot 2^{\frac{n}{2}}$ *for all positive integers $n \geqslant 2$.*

Proof. Set $N = \lceil \frac{1}{8}\cdot 2^{\frac{n}{2}} \rceil$; that is, N is the largest integer $\leqslant \frac{1}{8}\cdot 2^{\frac{n}{2}}$.

Note that

$$2^{\binom{n}{2}-1} > (2^{\frac{n-3}{2}})^n \geqslant N^n.$$

and

$$\binom{N}{n} = \frac{N\cdot(N-1)\cdots(N-n+1)}{n!} < N^n.$$

Therefore

$$\binom{N}{n} < 2^{\binom{n}{2}-1}.$$

By Theorem 1.1, we get $r(n,n) > N$. $\square$

In the following exercise, mimic the proof of Theorem 1.1, very rough estimates will do the job.

1.3. Exercise. *By random coloring we will understand a coloring edges of a given graph in red and blue such that each edge is colored independently in red or blue with equal chances.*

Assume the edges of the complete bigraph $K_{100,100}$ is colored randomly. Show that probability that $K_{100,100}$ is monochromatic is less than $\frac{1}{10^{2500}}$.

Show that the number of different subgraphs in $K_{10^{10},10^{10}}$ isomorphic to $K_{100,100}$ is less than 10^{2000}.

Assume the edges of the complete bigraph $K_{10^{10},10^{10}}$ is colored randomly. Show that the expected number of monochromatic subgraphs isomorphic to $K_{100,100}$ in $K_{10^{10},10^{10}}$ is less than 1.

Conclude that the complete bigraph $K_{10^{10},10^{10}}$ admits an edge coloring in two colors such that it contains no monochromatic $K_{100,100}$.