

Extra credit problems

Math 485

0. Find a mistake or misprint in “Extra pearls”. (The score depends on the type of mistake.)

1. (*solved*) Assume d_1, \dots, d_p is a sequence of integers in a nonincreasing order. Show that it is multigraphic if and only if $d_p \geq 0$, the sum $d_1 + \dots + d_p$ is even and

$$d_1 \leq d_2 + \dots + d_p.$$

(A sequence of integers d_1, \dots, d_p is called *multigraphic* if it appears as a sequence of degrees of a multigraph.)

2. (*solved*) Assume that a sequence d_1, \dots, d_p is graphic, $d_i \geq 1$ for each i and

$$d_1 + \dots + d_p \geq 2 \cdot (p - 1).$$

Show that there is a *connected* graph G with the degree sequence d_1, \dots, d_p .

3. (*solved*) Show that in any connected graph G there is a vertex v such that $G - v$ is connected.

4. (*solved*) Let G be a connected graph. Show that any two paths of maximum length in G have a common vertex.

5. Assume two trees R and S have the vertices r_1, \dots, r_n and s_1, \dots, s_n correspondingly. Assume that $R - r_i$ is isomorphic to $S - s_i$ for each i . Show that R is isomorphic to S .

6. (*solved*) Let G be a critical graph and $\chi(G) = k + 1$. Show that after removing any $k - 1$ edges from G the obtained graph remains connected.

7. (*solved*) Assume both sequences d_1, \dots, d_p and $d_1 - 1, \dots, d_p - 1$ are graphic. Show that there is a graph with a 1-factor and degree sequence d_1, \dots, d_p .

8. (*solved*) Show that any edge of cubic graph lies in an even number of Hamiltonian cycles.

9. (*solved*) Assume that the edges of a complete graph are colored in two colors. Show that there is a Hamiltonian cycle which either monochromatic or consists of two monochromatic paths.

~~10.~~ Assume that G is a connected cubic graph such that for any two vertices v, w of G there is an isomorphism $G \rightarrow G$ which sends v to w . Prove that G remains connected after removing any 2 edges.

~~11.~~ (*solved*) Let G be a connected graph with no triangles. Given any two vertices v, w in G , denote by $d(v, w)$ the length of a shortest path containing v to w .

Show that G is a tree if and only if

$$d(x, y) + d(v, w) = \max\{d(x, v) + d(y, w), d(x, w) + d(y, v)\}$$

for any 4 vertices x, y, v, w in G .

~~12.~~ Understand the proof of Vizing's theorem (see the references for Theorem 2.2.2 in "Pearls" or find a proof elsewhere).

~~13.~~ Suppose $p = r(m, n)$ is the Ramsey number for m and n .

Assume that G results from K_p by deleting a single edge. Show that G has a red/blue edge coloring with no red K_m and no blue K_n .

~~14.~~ Assume that the real intervals I_1, \dots, I_k have $2 \cdot k$ different ends that are chosen randomly from $\{1, 2, \dots, 2 \cdot k\}$. Consider the graph G with k vertexes such that i th vertex is connected to j th vertex if and only if I_i overlaps with I_j . Show that G has diameter ≤ 2 with probability at least $\frac{2}{3}$.

~~15.~~ Understand the proof of Erdős–Lovász theorem (see the references for Theorem 2.1.5 in "Pearls", or Theorem 2.14 in "Extra pearls", or find a proof elsewhere).

~~16.~~ (*solved*) Let G be a graph with p vertexes. Show that G is a tree if and only if its chromatic polynomial is $x \cdot (x - 1)^{p-1}$.

~~17.~~ Let T be a spanning tree in a graph G with weighted edges. Show that T has minimal total weight if and only if the weight of any edge (a, b) in G is at least as large as the weight of any edge on the path from a to b in T .

~~18.~~ In a group of people, for some fixed s and any k , any k girls like at least $k - s$ boys in total. Show that then all but s girls may get married on the boys they like.

~~19.~~ Let M be a maximal planar map. Assume the vertexes of M are colored in 3 colors. Show that the number of the regions that get all three colors is even.

~~20.~~ Let M be a map and G its dual. Show that M and G have the same number of spanning trees.

~~21.~~ Show that any map has a vertex of degree at most 3 or a country with at most 3 sides.