Extra pearls for MATH 485

Contents

Chapter 1

Ramsey numbers continued

Lower bounds

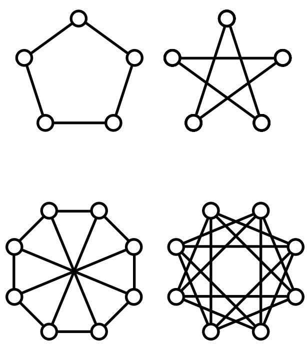
Recall that Ramsey number $r(m,n)$ is a least positive integer for which every blue-red coloring of edges in the complete graph $K_{r(m,n)}$ contains a blue K_m or a red K_n.

Equivalently, for any decomposition of $K_{r(m,n)}$ into two subgraphs G and H either G contains a copy of K_m or H contains a copy of K_n.

Therefore, in order to show that

$$r(m,n) \geqslant s+1,$$

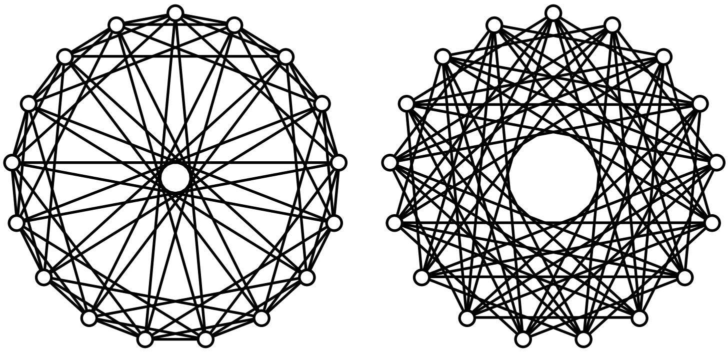
it is sufficient to decompose K_s into two subgraphs with no isomorphic copy of K_m in the first one and no isomorphic copy of K_n in the second one.



For example, the subgraphs in the decomposition of K_5 on the diagram has no monochromatic triangles; the latter implies that $r(3,3) \geqslant 6$. We showed already that for any decomposition of K_6 into two subgraphs, one of the subgraphs has a triangle; that is, $r(3,3) = 6$.

Similarly, to show that $r(3,4) \geqslant 9$, we need to construct a decomposition of K_8 in to two subgraphs G and H such that G contains no triangle K_3 and H contains no K_4.

Similarly, to show that $r(4,4) \geqslant 18$, we need to construct a decomposition of K_{17} in to two subgraphs with no K_4. (In fact, $r(4,4) = 18$, but we are not going to prove it.) The corresponding decomposition is given on the following diagram.



The constructed decomposition is rationally symmetric; the first subgraph contains the chords of angle lengths 1, 2, 4, and 8 and the second to all the cords of angle lengths 3, 5, 6 and 7.

1.1. Exercise. *Show that*

(a) *In the decomposition of K_8 above, the left graph contains no triangle and the right graph contains no K_4.*

(b) *In the decomposition of K_{17} above, both graph contain no no K_4.*

Hint: In each cases, fix one vertex v and draw the subgraph induced by the vertexes connected to v.

For larger values m and n the problem of finding the exact lower bound for $r(m,n)$ is quickly becomes too hard. Even getting a reasonable rough estimate is challenging. In the next section we will show how to obtain such estimate using probability.

Probabilistic method

The probabilistic method makes possible to prove the existence of graphs with certain properties without constructing them explicitly. The idea is to show that if one randomly chooses a graph or its coloring from a specified class, then probability that the result is of the needed property is more than zero. The latter implies that a graph with needed property exists.

Despite that this method of proof uses probability, the final conclusion is determined for certain, without any possible error.

Recall that $\binom{n}{m}$ denotes the *binomial coefficient*; that is, m and n are integers, $n \geqslant 0$ and

$$\binom{n}{m} = \frac{n!}{m! \cdot (n-m)!}$$

if $0 \leqslant m \leqslant n$ and $\binom{n}{m} = 0$ otherwise.

The number $\binom{n}{m}$ plays an important role in combinatorics — it gives the number of ways that m objects can be chosen from among n different objects.

1.2. Theorem. *Assume that the inequality*

$$\binom{N}{n} < 2^{\binom{n}{2}-1}$$

holds for a pair of positive integers N and n. Then $r(n,n) > N$.

Proof. We need to show that the complete graph K_N admits a coloring of edges in red and blue such that it has no monochromatic subgraph isomorphic to K_n.

Let us color the edges randomly — color each edge independently with probability $\frac{1}{2}$ in red and otherwise in blue.

Fix a set S of n vertexes. Define the variable $X(S)$ to be 1 if every edge amongst the n vertexes is the same color, and 0 otherwise. Note that the number of monochromatic n-subgraphs in K_N is the sum of $X(S)$ over all possible subsets.

Note that the expected value of $X(S)$ is simply the probability that all of the $\binom{n}{2} = \frac{n \cdot (n-1)}{2}$ edges in S are the same color. The probability that all the edges with the ends in S are blue is $1/2^{\binom{n}{2}}$ and with the same probability all edges are red. Since these two possibilities exclude each other the expected value of $X(S)$ is $2/2^{\binom{n}{2}}$.

This holds for any n-vertex subset S of the vertexes of K_N. The total number of such subsets is $\binom{N}{n}$. Therefore the expected value for the sum of $X(S)$ over all S is

$$X = 2 \cdot \binom{N}{n} / 2^{\binom{n}{2}}.$$

Assume that $X < 1$. Note that at least in one coloring suppose to have at most X complete monochromatic n-subgraphs. Since this number has to be an integer, at least one coloring must have no complete monochromatic n-subgraphs.

Therefore if $\binom{N}{n} < 2^{\binom{n}{2}-1}$, then there is a coloring K_N without monochromatic n-subgraphs. Hence the statement follows. $\square$

The following corollary implies that the function $n \mapsto r(n,n)$ grows at lest exponentially.

1.3. Corollary. *$r(n,n) > \frac{1}{8}\cdot 2^{\frac{n}{2}}$ for all positive integers $n \geqslant 2$.*

Proof. Set $N = \lfloor \frac{1}{8}\cdot 2^{\frac{n}{2}} \rfloor$; that is, N is the largest integer $\leqslant \frac{1}{8}\cdot 2^{\frac{n}{2}}$.

Note that

$$2^{\binom{n}{2}-1} > (2^{\frac{n-3}{2}})^n \geqslant N^n.$$

and

$$\binom{N}{n} = \frac{N\cdot(N-1)\cdots(N-n+1)}{n!} < N^n.$$

Therefore

$$\binom{N}{n} < 2^{\binom{n}{2}-1}.$$

By Theorem 1.2, we get $r(n,n) > N$. □

In the following exercise, mimic the proof of Theorem 1.2, very rough estimates will do the job.

1.4. Exercise. *By random coloring we will understand a coloring edges of a given graph in red and blue such that each edge is colored independently in red or blue with equal chances.*

Assume the edges of the complete bigraph $K_{100,100}$ is colored randomly. Show that probability that $K_{100,100}$ is monochromatic is less than $\frac{1}{10^{2500}}$.

Show that the number of different subgraphs in $K_{10^{10},10^{10}}$ isomorphic to $K_{100,100}$ is less than 10^{2000}.

Assume the edges of the complete bigraph $K_{10^{10},10^{10}}$ is colored randomly. Show that the expected number of monochromatic subgraphs isomorphic to $K_{100,100}$ in $K_{10^{10},10^{10}}$ is less than 1.

Conclude that the complete bigraph $K_{10^{10},10^{10}}$ admits an edge coloring in two colors such that it contains no monochromatic $K_{100,100}$.

Counting proof

In this section we will repeat the proof of Theorem 1.2 using a different language, without use of probability. We do this to affirm that probabilistic method provides real proof, without any possible error.

In principle, any probabilistic proof admits such translation, but in most cases, the the translation is less intuitive.

Proof of 1.2. The graph K_N has $\binom{N}{2}$ edges. Each edge can be colored in blue or red therefore the total number of different colorings is

$$\Omega = 2^{\binom{N}{2}}.$$

Fix a subgraph isomorphic to K_n in K_N. Note that this graph is red in $\Omega/2^{\binom{n}{2}}$ different colorings and yet in $\Omega/2^{\binom{n}{2}}$ different colorings this subgraph is blue.

There are $\binom{N}{n}$ different subgraphs isomorphic to K_n in K_N. Therefore the total number of monochromatic K_n's in all the colorings is

$$M = \binom{N}{n} \cdot \Omega \cdot 2/2^{\binom{n}{2}}.$$

If $M < \Omega$, then by the pigeonhole principle, there is a coloring with no monochromatic K_n. Hence the result. □

Typical properties

More involved examples of proofs based on the probabilistic method deal with *typical properties* of random graphs.

To describe the concept, let us consider the following *random process* which generates graph G_p with p vertexes.

Fix a positive integer p. Consider a graph G_p with the vertexes labeled by $1, \dots, p$, where every edge in G_p exists with probability $\frac{1}{2}$.

Note that the described process depends only on p and as a result we can get any graph on p vertexes with the same probability $2^{-\binom{p}{2}}$.

Fix a property of a graph (for example connectedness) and let α_p be the probability that G_p has this property. We say that property is *typical* if $\alpha_p \to 1$ as $p \to \infty$.

1.5. Exercise. *Show that random graphs are typically have diameter 2. That is, the probability that G_p is has diameter 2 converges to 1 as $p \to \infty$.*

Note that from the exercise above, it follows that in the described random process the random graphs are *typically connected.*

The following theorem gives a deeper illustration for probabilistic method with use of typical properties.

1.6. Theorem. *Given a positive integer g and k there is a graph G with girth at least g and chromatic number at least k.*

A proof can be found in the chapter "Probability makes counting easy" in [1]; read it if you want to learn more about this method.

Remarks

The probabilistic method finds applications in many areas of mathematics; not only in graph theory. It was introduced by Paul Erdős.

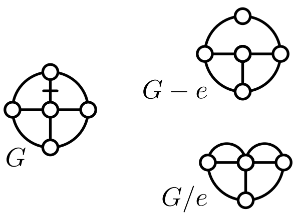
Note that probabilistic method is not nonconstructive — often when the existence of a certain graph is probed by probabilistic method, it is still uncontrollably hard to describe a concrete example.

Chapter 2

Deletion and contraction

Definitions

Let e be an edge in the pseudograph G. Denote by $G-e$ the pseudograph obtained from G by deleting e, and by G/e the pseudograph obtained from G by contraction the edge e to a point; see the diagram.



Assume G is a graph; that is, G has no loops and no parallel edges. Then so is $G-e$, but G/e might have parallel edges but no loops; that is, G/e is a multigraph.

If G is a multigraph then so is $G-e$. If the edge e is parallel to f in G, then f in G/e becomes a loop; that is, G/e is a pseudograph in general.

Chromatic polynomial

Denote by $\chi(G,k)$ the number of colorings of the graph G in k colors such that the ends of each edge get different colors.

2.1. Exercise. *Assume that a graph G has two connected components H_1 and H_2. Show that*

$$\chi(G,k)=\chi(H_1,k)\cdot\chi(H_2,k)$$

for any k.

2.2. Exercise. *Show that for any integer $n \geqslant 3$,*

$$\chi(W_n, k+1) = (k+1)\cdot\chi(C_n, k),$$

where W_n denotes the wheel with n spokes and C_n is the cycle of length n.

2.3. Deletion-minus-contraction formula. *For any edge e in the pseudograph G we have*

$$\chi(G, k) = \chi(G - e, k) - \chi(G/e, k).$$

Proof. The valid colorings of $G-e$ can be divided into two groups: (1) those where the ends of the edge e get different colors — these remain to be valid colorings of G and (2) those where the ends of e get the same color — each of such colorings corresponds to unique coloring of G/e. Hence

$$\chi(G - e, k) = \chi(G, k) + \chi(G/e, k),$$

which is equivalent to the deletion-minus-contraction formula. □

Note that if the pseudograph G has loops then $\chi(G, k) = 0$ for any k. Indeed in a valid coloring the ends of loop should get different colors, which is impossible.

The latter can be also proved using the deletion-minus-contraction formula. Indeed, if e is a loop in G, then $G/e = G - e$; therefore $\chi(G - e, k) = \chi(G/e, k)$ and

$$\chi(G, k) = \chi(G - e, k) - \chi(G/e, k) = 0.$$

Similarly, removing a parallel edge from a pseudograph G does not change $\chi(G, k) = 0$. Indeed, if e is an edge of G which has a parallel edge f then in G/e the edge f becomes a loop. Therefore $\chi(G/e, k) = 0$ for any k and by deletion-minus-contraction formula we get

$$\chi(G, k) = \chi(G - e, k).$$

The same identity can be seen directly — any admisible coloring of $G-e$ is also admissible in G — since the ends of f get different colors, so does e.

It follows that the problem of finding $\chi(G, k)$ can be reduced to the case when G is a graph.

Recall that polynomial P of k is an expression of the following type

$$P(k) = a_0 + a_1\cdot k + \dots + a_n\cdot k^n,$$

with constants $a_0, \dots, a_n$, which are called *coefficients* of the polynomial. If $a_n \neq 0$, it is called *leading coefficient* of P; in this case n is the degree of P. If the leading coefficient is 1 then the polynomial is called *monic*.

2.4. Theorem. *For any pseudograph G, there is a polynomial with integer coefficients P such that*

$$\chi(G, k) = P(k)$$

for any k. The polynomial P is called chromatic polynomial *of the graph G.*

Moreover, if G has a loop then $\chi(G, k) \equiv 0$; otherwise P is monic and has degree p, where p is the number of vertexes in G.

The deletion-minus-contraction formula will play the central role in the following proof.

Proof. As usual, denote by p and q the number of vertexes and edges in G.

To prove the first part, we will use the induction on q.

As the base case, consider the null graph N_p; that is, the graph with p vertexes and no edges. Since N_p has no edges, any coloring of N_p is admissible. We have k choices for each of n vertexes therefore

$$\chi(N_p, k) = k^p.$$

In particular, the function $k \mapsto \chi(N_p, k)$ is given by monic polynomial of degree p with integer coefficients.

Assume that the first statement holds for all pseudographs with at most $q-1$ edges. Fix a graph G with q edges. Applying the deletion-minus-contraction formula, we get

$$\chi(G, k) = \chi(G - e, k) - \chi(G/e, k).$$

Note that the pseudographs $G - e$ and G/e have $q-1$ edges. By induction hypothesis, $\chi(G - e, k) = P(k)$ and $\chi(G/e, k) = Q(k)$ for some polynomials P and Q with integere coefficients. Hence the same holds for their difference $\chi(G, k) = P(k) - Q(k)$.

To prove the second part, we also use the induction.

First note that if G has a loop then $\chi(G, k) = 0$ as G has no valid colorings.

Assume that the chromatic polynomial of any multigraph G with at most $q-1$ edges and at most p vertexes is monic of degree p.

Fix a multigraph G with p vertexes and q edges. Note that $G - e$ is a multigraph with p vertexes and $q-1$ edges. By the assumption, its chromatic polynomial P is monic of degree p.

Further the pseudograph G/e has $p-1$ vertexes, and its characteristic polynomial Q is either vanish or has degree $p-1$. In both cases $P-Q$ is a monic polynomial of degree p. It remains to apply the deletion-minus-contraction formula,

$$\chi(G,k) = P(k) - Q(k).$$

□

2.5. Exercise. *Use induction and deletion-minus-contraction formula to show that*

(a) $\chi(P_q,k) = k\cdot(k-1)^q$ *where* P_q *denotes the path with* q *edges;*

(b) $\chi(T,k) = k\cdot(k-1)^q$ *for any tree* T *with* q *edges;*

(c) $\chi(C_p,k) = (k-1)^p + (-1)^p\cdot(k-1)$ *for the cycle* C_p *of length* p*.*

2.6. Exercise. *Show that*

$$\chi(K_p,k) = k\cdot(k-1)\cdots(k-p+1).$$

Note that for any graph G with p vertexes we have

$$\chi(K_p,k) \leqslant \chi(G,k) \leqslant \chi(N_p,k)$$

for any k. Since

$$\begin{aligned}\chi(K_p,k) &= k\cdot(k-1)\cdots(k-p+1),\\ \chi(N_p,k) &= k^p,\end{aligned}$$

it follows that chromatic polynomial of G is monic of degree p. It gives an alternative way to prove the second statement in Theorem 2.4.

2.7. Exercise. *Construct a pair of nonisomorphic graphs with equal chromatic polynomials.*

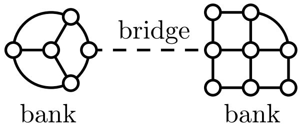
Number of spanning trees

Recall that $s(G)$ denotes the number of spanning trees in the pseudograph G.

An edge e in a connected graph G is called *bridge*, if deletion of this edge makes it disconnected; that is, the remaining graph has two connected components which are called *banks*.

2.8. Exercise. *Assume that the graph G contains a bridge between banks H_1 and H_2. Show that*

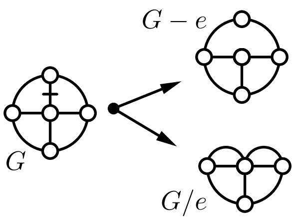
$$s(G) = s(H_1)\cdot s(H_2).$$



2.9. Deletion-plus-contraction formula. *Let e be an edge in the pseudograph G. Assume e is not a loop, then the following identity holds*

$$s(G) = s(G-e) + s(G/e), \tag{$*$}$$

Often it is convenient to write the identity $(*)$ using a diagram as on the picture; the edge e is marked on the diagram.



Proof. Note that the spanning trees of G can be subdivided into two groups — (1) those which contain the edge e and (2) those which do not. For the trees in the first group, contraction of e to a point gives a spanning tree in G/e, while the trees in the second group are also spanning trees in $G-e$.

Moreover, both of the described correspondences are one-to-one. Hence the formula follows. □

Note that a spanning tree can not have loops. Therefore if we remove all loops from the pseudograph, then the number of spanning trees remains unchanged. In other words, for any loop e the following identity holds

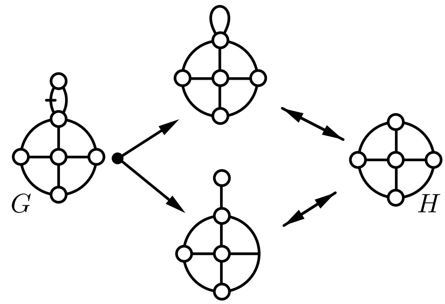
$$s(G) = s(G-e).$$

From the deletion-plus-contraction formula we can deduce few other useful identities. For example, assume that the graph G has an end vertex w (that is, $\deg w = 1$). If we remove the vertex w and its edge from G, then in obtained graph $G-w$ the number of spanning trees remains unchanged; that is,

$$s(G) = s(G-w). \tag{$**$}$$

Indeed, denote by e the only edge incident to w. Note that the graph $G-e$ is not connected, since the vertex w is isolated. Therefore $s(G-e) = 0$. On the other hand $G/e = G-w$ therefore $(*)$ implies $(**)$.

On the diagrams, we will use two-sided arrow "↔" for the graphs with equal number of the spanning trees. For example, from the discussed identities we can draw the diagram, which in particular implies the following identity:



$$s(G) = 2{\cdot}s(H).$$

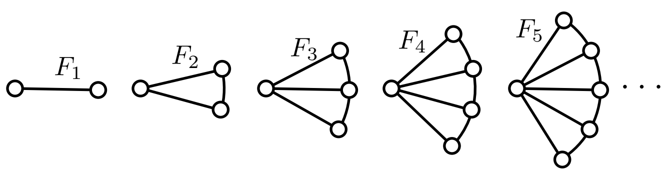
The deletion-plus-contraction formula gives an algorithm to calculate the value $s(G)$ for given pseudograph G. Indeed, for any edge e, both graphs $G - e$ and G/e have smaller number of edges. That is, deletion-plus-contraction formula reduce the problem of finding number of the trees to simpler graphs; applying this formula few times we can reduce the question to a collection of graphs the answer is evident for each. In the next section we will show how it works.

Fans and their relatives

Recall that Fibonacci numbers f_n are defined using the recursive identity $f_{n+1} = f_n + f_{n-1}$ with $f_1 = f_2 = 1$. The sequence of Fibonacci numbers starts as

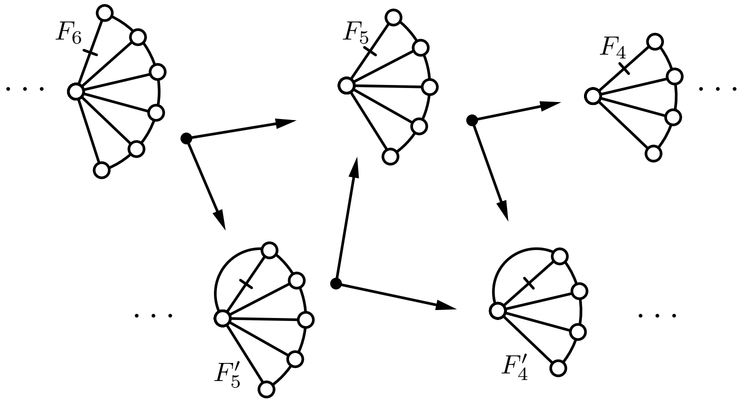
$$1, 1, 2, 3, 5, 8, 13, \dots$$

The graphs of the following type are called *fans*; a fan with $n + 1$ vertex will be denoted by F_n.



2.10. Theorem. $s(F_n) = f_{2\cdot n}$.

Proof. Applying the delection-plus-contraction formula, we can draw the following infinite diagram. In addition to the fans F_n we use its variations F'_n, which differ from F_n by an extra parallel edge.



Set $a_n = s(F_n)$ and $a'_n = s(F'_n)$. From the diagram we get the following two recurcive relations:

$$a_{n+1} = a'_n + a_n,$$
$$a'_n = a_n + a'_{n-1}.$$

That is, in the sequence

$$a_1, a'_1, a_2, a'_2, a_3 \dots$$

every number starting from a_2 is sum of previous two.

Further note that F_1 has two vertexes connected by unique edge, and F'_1 has two vertexes connected by a pair of parallel edges. Hence $a_1 = 1 = f_2$ and $a'_1 = 2 = f_3$ and therefore

$$a_n = f_{2\cdot n}$$

for any n. $\square$

Comments. We can deduce a recursive relation for a_n, without using a'_n:

$$\begin{aligned} a_{n+1} &= a'_n + a_n = \\ &= 2\cdot a_n + a'_{n-1} = \\ &= 3\cdot a_n - a_{n-1}. \end{aligned}$$

For the sequences defined by the *linear recursion* as above are called *constant-recursive sequences*. The general term of such sequence can be expressed by a closed formula, see [9]. In our case it is

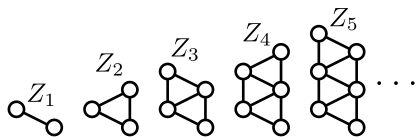
$$a_n = \tfrac{1}{\sqrt{5}}\cdot\left(\left(\tfrac{3+\sqrt{5}}{2}\right)^n - \left(\tfrac{3-\sqrt{5}}{2}\right)^n\right).$$

Since a_n is integer and $0 < \frac{1}{\sqrt{5}}\cdot(\frac{3-\sqrt{5}}{2})^n < 1$ for any $n \geqslant 1$ a shorter formula can be written

$$a_n = \left\lfloor \tfrac{1}{\sqrt{5}}\cdot(\tfrac{3+\sqrt{5}}{2})^n \right\rfloor,$$

where $\lfloor x \rfloor$ denotes floor of x; that is, $\lfloor x \rfloor$ is the maximal integer such that $\lfloor x \rfloor \leqslant x$.

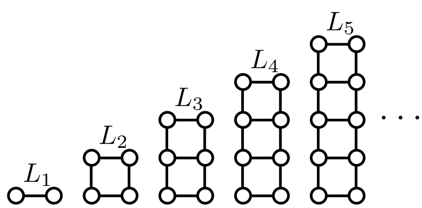
2.11. Exercise. *Consider the sequence of zig-zag graphs Z_n of the following type:*



Show that $s(Z_n) = f_{2\cdot n}$ for any n.

Hint: Use the induction on n and/or mimic the proof of Theorem 2.10.

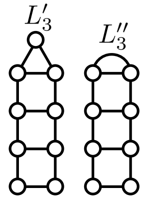
2.12. Exercise. *Let us denote by b_n the number of spanning trees in the n-*step ladder *L_n; that is, in the graph of the following type:*



Apply the method we used for fans F_n to show that the sequence b_n satisfies the following linear recurcive relation

$$b_{n+1} = 4\cdot b_n - b_{n-1}.$$

Hint: To construct the recusive relation, in addition to the ladders L_n you will need two of its analogs L'_n and L''_n shown on the diagram.

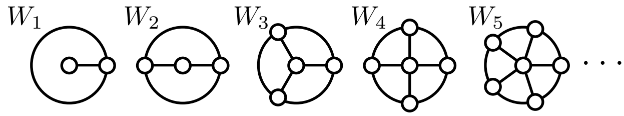


Note that $b_1 = 1$ и $b_2 = 4$; applying the exercise we could calculate first numbers of the sequence (b_n):

$$1, 4, 15, 56, 209, 780, 2911, \ldots$$

The following exercise is analogous, but more complicated.

2.13. Advanced exercise. *Recall that a wheel W_n is the graph of following type:*



Show that the sequence $c_n = s(W_n)$ *satifies the following recurcive relation*

$$c_{n+1} = 4{\cdot}c_n - 4{\cdot}c_{n-1} + c_{n-2}.$$

Using the exercise above and applying induction one can show that

$$c_n = f_{2\cdot n+1} + f_{2\cdot n-1} - 2 = l_{2\cdot n} - 2$$

for any n; the numbers $l_n = f_{n+1} + f_{n-1}$ are called *Lucas numbers*; they pop up in combinatorics as often as Fibonacci numbers.

Remarks

The Kirchoff rules and the *delection-plus-contraction* formula were used in the solution of the so called *squaring the square problem.* The history of this problem and its solution are discussed in [4, Chapter 17].

The proof of recurrent relation above is taken from [12]; this problem is also discussed in the classical book [8].

Chapter 3

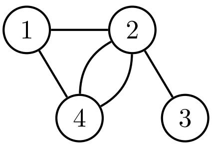
Matrix theorem

Adjacency matrix

Let us describe a way to encode the given multigraph G with p vertexes by an $p \times p$ matrix. First, enumerate the vertexes of the multigraph by numbers from 1 to p; such multigraph will be called *labeled.* Consider the matrix $A = A_G$ with the component $a_{i,j}$ equal to the number of edges from i-th vertex to the j-th vertex of G.

This matrix A is called *adjacency matrix* of G. Note that A is *symmetric*; that is, $a_{i,j} = a_{j,i}$ for any pair i, j. Also, the diagonal components of A vanish; that is, $a_{i,i} = 0$ for any i.

For example, for the labeled multigraph G shown on the diagram, we get the following adjacency matrix



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}.$$

3.1. Exercise. *Let A be the* adjacency matrix *of a labeled multigraph. Show that the components $b_{i,j}$ of the n-th power A^n is the number of walks of length n in the graph from vertex i to vertex j.*

Hint: Use induction on n.

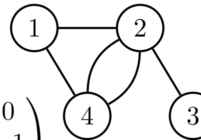
Kirchhoff minor

In this section we construct a special matrix, called *Kirchoff minor*, associated with a pseudograph and discuss its basic properties. This matrix will be used in the next section in a formula for the number of spanning trees in a pseudograph G. Since the loops do not change the the number of spanning trees, we can remove all of them. In other words we can (and will) always assume that G is a multigraph.

Fix a multigraph G and consider its adjacency matrix $A = A_G$; it is a $p{\times}p$ symmetric matrix with zeros on the diagonal.

1. Revert the signs of the components of A and exchange the zeros on the diagonal by the degrees of the corresponding vertexes. (The matrix A' is called *Kirchhoff matrix*, *Laplacian matrix* or *admittance matrix* of the graph G.)
2. Delete from A' the last column and the last row; the obtained matrix $M = M_G$ will be called *Kirchoff minor* of the labeled pseudograph G.

For example, the labeled multigraph G on the diagram has the following Kirchhoff matrix and Kirchoff minor:



$$A' = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -1 & -2 \\ 0 & -1 & 1 & 0 \\ -1 & -2 & 0 & 3 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

3.2. Exercise. *Show that in any Kirchhoff matrix A' the sum of components in each row or column vanishes. Conclude that*

$$\det A' = 0.$$

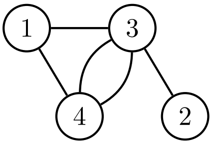
3.3. Exercise. *Draw a labeled pseudograph with following Kirchoff minor:*

$$\begin{pmatrix} 4 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & 0 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 4 \end{pmatrix}$$

3.4. Exercise. *Show that the sum of all components in every column of Kirchhoff minor is nonnegative.*

Moreover the the sum of all components in i-th column vanish if and only if i-th vertex is not adjacent to the last one.

Relabeling. Let us understand what happens with Kirchhoff minor and its determinant as we swap two labels distinct from the last one.



For example, if we swap the labels 2 and 3 in the graph above, we get an other labeling shown on the diagram. Then the corresponding Kirchhoff minor will be

$$M' = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 4 \end{pmatrix}$$

which is obtained from M by swapping columns 2 and 3 following by swapping rows 2 and 3.

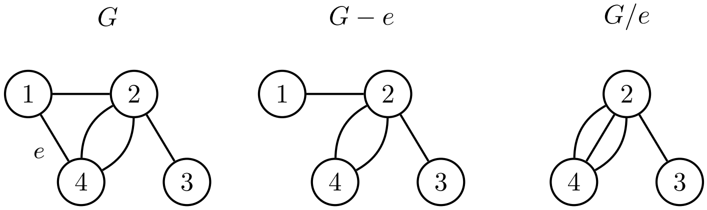
Note that swapping a pair of columns or rows changes the sign of determinant. Therefore swapping one pair of rows and one pair of columns does not change the determinant. The same holds in general; let us summarize it:

3.5. Observation. *Assume G is a labeled graph with p vertexes and M_G is its Kirchhoff minor. If we swap two labels $i, j < p$ then corresponding Kirchhoff minor M'_G can be obtained from M_G by swapping columns i and j following by swapping rows i and j. In particular,*

$$\det M'_G = \det M_G.$$

Deletion and contraction. Next let us understand what happens with Kirchhoff minor if we delete or contract an edge in the labeled multigraph. (If after contraction of an edge we get loops, we remove it; this way we obtain a multigraph.).

Assume edge e connects first and last vertex of labeled multigraph G as in the following example:



Note that deleting e only reduce the corner component of M_G by one, while contracting it removes first row and column. That is, since

$$M_G = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

we have

$$M_{G-e} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad M_{G/e} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}.$$

Again, the same holds in general, let us summarize it in the following observation.

3.6. Observation. *Assume e is an edge of labeled multigraph G between the first and last vertex then and M_G is the Kirchhoff minor of G. Then*

(a) the Kirchhoff minor M_{G-e} of $G-e$ can be obtained from M_G by subtracting 1 from the corner element with index (1,1);

(b) the Kirchhoff minor $M_{G/e}$ of G/e can be obtained from M_G by removing the first row and the first column in M_G.

In particular by cofactor expansion of determinant we get

$$\det M_G = \det M_{G-e} + \det M_{G/e}$$

Note that the last formula reminds deletion-plus-contraction formula. This is the key observation in the proof of the matrix theorem; see the next section.

Matrix theorem

3.7. Matrix theorem. *Let M be the Kirchoff minor of labeled multigraph G with at least two vertexes. Then*

$$s(G) = \det M, \tag{$***$}$$

where $s(G)$ denotes the number of spanning trees in G.

Proof. As usual we denote by p and q the number of vertexes and edges in G. We will use induction on the number $p+q$; we will always assume that $p \geqslant 2$.

The base case will include two cases $p = 2$ and $q = 0$.

Assume $p = 2$; that it G has two vetexes and q parallel edges. Clearly $s(G) = q$. Further note that $M_G = (q)$; that is, the Kirchhoff minor M_G is a $1{\times}1$ matrix with single component q. In particular $\det M_G = q$ and therefore the formula ($***$) holds if $p = 2$. It proves the statement in the first base case $p = 2$.

Denote by d the degree of the last vertex in G.

Assume $d = 0$. Then G is not connected and therefore $s(G) = 0$. On the other hand, the sum in each row of M_G vanish (compare to Exercise 3.4). Hence the sum of all columns in M_G vanish; in particular, the columns in M_G are linearly dependent and hence $\det M_G = 0$. In particular if $q = 0$ then $d = 0$ hence in this case formula $(***)$ holds if $q = 0$. It proves the statement in the second base case $q = 0$.

It remains to consider the case $d > 0$. In this case we may assume that the first and last vertexes of G are adjacent, otherwise permute pair of labels 1 and some $j < p$ and apply Observation 3.5. Denote by e the edge between the first and last vertex.

Note that the total number number of vertexes and edges in the pseudographs $G-e$ and G/e are smaller than $p+q$; hence by induction hypothesis the matrix theorem holds for these two pseudographs.

Applying deletion-plus-contraction formula, the induction hypothesis and Observation 3.6, we get

$$\begin{aligned} s(G) &= s(G-e) + s(G/e) = \\ &= \det M_{G-e} + \det M_{G/e} = \\ &= \det M_G. \end{aligned}$$

Which proves the induction step. □

3.8. Exercise. *Fix a labeling for each of the following graphs, find its Kirchoff's minor and use matrix theorem to find the number of spanning trees.*

(a) $s(K_{3,3})$*;*

(b) $s(W_6)$*;*

(c) $s(Q_3)$*.*

(You can use `http://matrix.reshish.com/determinant.php`*, or any other matrix calculator.)*

Calculation of determinants

In this section we recall key properties of determinant which will be used in the next section.

Let M be an $n{\times}n$-matrix; that is, a table $n{\times}n$, filled with numbers which are called *components of the matrix.* The determinat $\det M$ is a polynomial of the n^2 components of M, which satisfies the following conditions:

1. The unit matrix has determinant 1; that is,

$$\det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = 1.$$

2. If we multiply each component of one of the rows of the matrix M multiply by a number λ, then for the obtained matrix M', we have

$$\det M' = \lambda \cdot \det M.$$

3. If one of the rows in the matrix M add (or subtract) term-by-term to an other row, then the obtained matrix M' has the same determinant

$$\det M' = \det M.$$

These three conditions define determinant in a unique way. We will not give a proof of the statement, it is not evident and not complicated and soon or later you will have to learn it, if it is not done already.

3.9. Exercise. *Show that the following property follows from the properties above.*

4. If we permute two rows in the matrix M then the obtained matrix M' will have determinant of opposite sign; that is,

$$\det M' = -\det M.$$

The determinant of $n \times n$-matrix can be written explicitly as a sum of $n!$ terms. For example,

$$a_1 \cdot b_2 \cdot c_3 + a_2 \cdot b_3 \cdot c_1 + a_3 \cdot b_1 \cdot c_2 - a_3 \cdot b_2 \cdot c_1 - a_2 \cdot b_1 \cdot c_3 - a_1 \cdot b_3 \cdot c_2$$

is the determinant of the matrix

$$M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

However, the properties described above give a more convinient and faster way to calculate the determinant, especially for larger values n.

Let us show it on one example which will be needed in the next section:

$$\begin{aligned}
\det\begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} &= \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} = \\
&= \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} = \\
&= 5^3\cdot\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
&= 5^3\cdot\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
&= 5^3.
\end{aligned}$$

Let us describe what we used on each line above:

1. property 3 three times — we add to the first row each of the remaining rows;
2. property 3 three times — we add first row to the each of the remaining three rows;
3. property 2 three times;
4. property 3 three times — we subtract from the first row the remaining three rows;
5. property 1.

Cayley formula

Recall that the *complete graph* is the graph where each pair of vertexes is connected by an edge; complete graph with p vertexes is denoted by K_p.

Note that every vertex of K_p has degree $p-1$. Therefore the Kirchhoff minor $M = M_{K_p}$ in the matrix formula $(***)$ for K_p is the following $(p-1)\times(p-1)$-matrix:

$$M = \begin{pmatrix} p-1 & -1 & \cdots & -1 \\ -1 & p-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p-1 \end{pmatrix}.$$

The argument given in the end of the previous section admits a direct generalization:

$$\det \begin{pmatrix} p-1 & -1 & \cdots & -1 \\ -1 & p-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p-1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & p-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p-1 \end{pmatrix} =$$

$$= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p \end{pmatrix} =$$

$$= p^{p-2}.$$

That is,

$$\det M = p^{p-2}.$$

Therefore, applying the matrix theorem, we get the following:

3.10. Cayley formula. *The number of spanning trees in the complete graph K_p is p^{p-2}; that is,*

$$s(K_p) = p^{p-2}.$$

3.11. Exercise. *Show that $s(K_{m,n}) = m^{n-1} \cdot n^{m-1}$.*

Remarks

There is strong connection between counting spanning of the given graph and calculations of currents in an electric chain.

Assume that the graph G describes an electic chain; each edge has resistance one Ohm. A battery is connected to the vertexes a and b. The total current between these vertexes is I Ampere. Assume we need to calculate the current thru edge e.

Fix an orientation of e. Note that the spanning trees of G can be subdivided into the following three groups: (1) those where the edged

e appears on the (necessary unique) path from a to b with positive orientation, (2) those where the edge e appears on the path from a to b with negative orientation, (3) those which the edge e do not appear on the path from a to b. Denote by t_+, t_- and t_0 the number of the trees in each group. Clearly

$$s(G) = t_+ + t_- + t_0.$$

The current I_e along e can be calculated using the following formula:

$$I_e = \frac{t_+ - t_-}{s(G)} \cdot I.$$

This statement can be proved by checking Kirchoff rules for the currents calculated by this formula.

There are many other applications of Kirchoff rules to graph theory For example, in [11], they were used to prove the Eulers's formula

$$p - q + r = 2,$$

where p, q и r denotes the number of vertexes, edges and regions of in a plane drawing of graph.

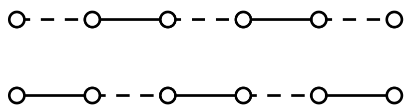
Few proofs of Cayley formula, inculding Prüfer's code are given in [1, Chapter 30].

Chapter 4

Alternated paths continued

Assume G be a bigraph and M is a matching in G. Recall that a path P in G is called M-*alternated* if the edges in P alternate between edges from M and edges not from M. If the path P connects two unmatched vertexes of G then it is called M-*augmenting*.

If there is an M-augmenting path P then the matching M can be improved by deleting from M the edges in P and adding the remaining edges of P. On the diagrams we denote the edges in M by solid lines and the remaining edges by dashed lines; the following diagram gives an example of the improvement.



In particular we get the following.

4.1. Observation. *Assume G be a bigraph and M is a maximal matching in G. Then G has no M-augmenting path.*

This observation plays the central role in the Hungarian algorithm. In this chapter we will show more ways to use this observation and its analogs. The following two exercises will be useful.

4.2. Exercise. *Let M be a matching in a bigraph G. Show that any M-augmenting path connects vertexes from the opposite parts of the bigraph.*

4.3. Exercise. *Let M be a maximal matching in a bigraph G. Assume two unmatched vertexes l and r lie in the opposite parts of G. Show that no pair of M-alternated paths starting from l and r can have a common vertex.*

Marriage theorem

Assume that G be a bigraph and S is a set of its vertexes. We say that a matching M of G *covers* S if any vertex in S is incident to an edge in M.

Given a set of vertexes W in a graph G, the set W' of all vertexes adjacent to at least one of vertexes in W will be called the *set of neighbors* of W. Note that if G is a bigraph and W lies in the left part then W' lies in the right part.

The following theorem was proved by Philip Hall in [5].

4.4. Marriage theorem. *Let G be a bigraph with the left and right parts L and R. Then G has a matching which covers L if and only if for any subset $W \subset L$ the set $W' \subset R$ of all neighbors of W contains at least as many vertexes as W; that is,*

$$|W'| \geqslant |W|.$$

Proof. Note that if there is a matching M covering L then for any set $W \subset L$ the set W' of its neighbors includes the vertexes matched with W. In particular,

$$|W'| \geqslant |W|;$$

it proves the "only if" part.

Consider a maximal matching M of G; to prove the "if" part it is sufficient to show that M covers L. Assume the contrary; that is, there is a vertex w in L which is not incident to any edge in M.

Consider the maximal set S of vertexes in G which are reachable from w by a M-alternated paths. Denote by W and W' the set of left and right vertexes in S correspondingly.

Since S is maximal, W' is the set of neighbors of W. According to Observation 4.1, the matching M provides a bijection between $W - w$ and W'. In particular,

$$|W| = |W'| + 1;$$

the latter contradicts the assumption. □

4.5. Exercise. *Assume G is a r-regular bigraph; $r \geqslant 1$. Show that*

(a) G admits a 1-factor;

(b) the edge chromatic number of G is r; in other words, G can be decomposed into 1-factors.

Remark. If $r = 2^n$ for an integer $n \geqslant 1$, then G in the exercise above has an Euler's circuit. Note that the total number of edges in G is even, so we can deltete all odd edges from the circuit. The obtained graph G' is regular with degree 2^{n-1}. Repeating the described procedure recursively n times, we will end up at 1-factor of G.

There is a tricky way to make this idea working for arbitrary r, not necessary a power of 2; it is discovered by Noga Alon, see [2] and also [7].

4.6. Exercise. *Children from 25 countries, 10 kids from each, decided to stand in a rectangular formation with 25 rows of 10 children in each row. Show that you can always choose one child from each row so that all 25 of them will be from different countries.*

4.7. Exercise. *The sons of the king divided the kingdom between each other into 23 parts of equal area — one for each son. Later a new son was born and the king proposed a new subdivision into 24 equal parts. Show that each of 23 older sons can choose a part in the new subdivision which overlaps with his old part.*

4.8. Exercise. *A table $n{\times}n$ filled with nonnegative numbers. Assume that the sum in each column and each row is 1. Show that one can choose n cells with positive numbers which do not share columns and rows.*

4.9. Advanced exercise. *In a group of people, for some fixed s and any k, any k girls like at least $k - s$ boys in total. Show that then all but s girls may get married on the boys they like.*

Vertex covers

A set S of vertexes in a graph is called *vertex cover* if any edge is incident to at least one of the vertexes in S.

The following theorem was discovered by Dénes Kőnig [10] and independently by Jenő Egerváry.

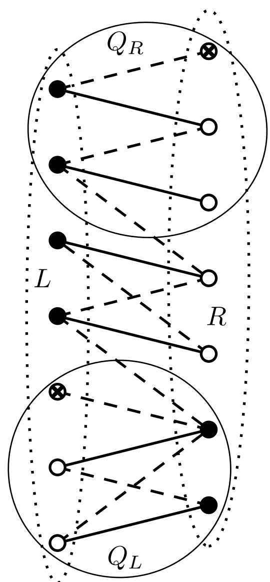
4.10. Theorem. *In any bigraph, the number of edges in a maximal matching equals the number of vertexes in a minimal vertex cover.*

On the following diagram, a maximal matching is marked by solid lines; the remaining edges of the graph are marked by dashed lines; the vertexes of constructed cover are marked by in black and the remaining vertexes in white; the only unmatched vertexes are marked by a cross.

Proof. Fix a bigraph G; denote by L and R its left and right part. Let M be a maximal matching in G.

Assume S is a vertex cover. Then for any edge m in M is incident to at least one vertex in S. Therefore

$$|S| \geqslant |M|;$$



that is, the number of vertexes in S is at least as large as the number of edges in any matching M. It remains to construct a vertex cover S such that $|S| = |M|$.

Denote by U_L and U_R the set of left and right unmatched vertexes. Denote by Q_L and Q_R the set of vertexes in G which can be reached by M-alternated paths starting from U_L and from U_R. Note that according to Exercise 4.3, Q_L and Q_R do not overlap. Further note that if two vertexes are matched then they both lie in Q_L or in Q_R or neither.

Let m be an edge of M. Note that both of the vertexes of m lie in Q_L or in Q_R or neither.

For each edge m in M, include in S the right of m if it connects vertexes in Q_L and left vertex otherwise. Since S is constructed by taking exactly one vertex incident to each edge of M, we have

$$|S| = |M|.$$

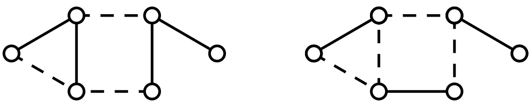
Fix and edge e in G. If e comes out of Q_L has both vertexes in S; if e connects vertexes in Q_L then the right vertex of e is in S; if e connects vertexes outside Q_L then the left vertex of e is in S. Hence S is a cover. □

4.11. Exercise. *On the chess board few squares are marked. Show that the minimal number of ranks and files which cover all marked squares is the same as the maximal number of rooks on the marked squares which do not threaten each other.*

Edge cover

A collection of edges N in a graph is called *edge cover* if every vertex is incident with at least one of the edges in N.

On he following diagram two edge cover of the same graph are marked in solid lines. The second cover is minimal.



4.12. Exercise. *Show that a minimal edge cover of any graph contains no paths of length 3 and no triangle.*

4.13. Exercise. *Assume that a minimal edge cover of a connected bigraph G contains n edges and a maximal matching of G contains m edges. Show that $m+n$ is the total number of vertexes of G.*

Minimal cut

Here is an other theorem which is proved using nearly the same idea.

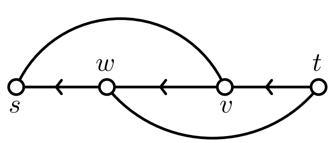
4.14. Min-cut theorem. *Let s and t be two vertexes in a digraph G. Then the maximal number of oriented paths from s to t which do not have common edges equals to the minimal number of edges one can remove from G so that there will be no oriented path from s to t.*

Proof. Denote by m the maximal number of oriented paths from s to t which do not have common edges and by n the minimal number of edges one can remove from G to make s disconnected from t.

Let $P_1, \dots, P_m$ be a maximal collection of paths from s to t which have no common edges. Note that to make s and t disconnected, we have to cut at least one edge from each path P_i; therefore $n \geqslant m$.

Consider the new orientation on G where each path P_i is oriented backwards — from t to s.

Consider the set S of the vertexes which are reachable from s by oriented paths for this new orientation.



Assume S contains t; that is, there is a path Q from s to t which can move along P_i only backwards. (Further the path Q will be used the same way as the augmenting path in the proof of marriage theorem.)

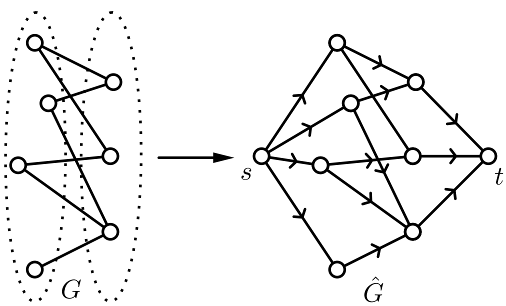
Assume Q overlaps with some of $P_1, \dots, P_m$. Without loss of generality, we can assume that Q first overlaps with P_1 — assume it meets P_i at the vertex v and leaves it at the vertex w. Let us modify the paths Q and P_i the following way: Instead of the path P_i consider the path P_i' which goes along Q from s to v and after that goes along P_i along to t. Instead of the path Q consider the trail Q' which goes along P_i from s to w and after that goes along Q to t.

The constructed trail Q' may repeat some vertex on P_1. In this case we discard a maximal circuit from Q' to obtain a genuine path, which we will still denote by Q'.

Note that the obtained collection of paths $Q', P_1', P_2 \dots, P_m$ satisfies the same conditions as the original collection, but it has smaller number of overlaps. Therefore repeating it several times produce $m+1$ paths without overlaps — a contradiction.[1]

It follows that $S \not\ni t$. In this case, all edges which connect S to the remaining vertexes of G are are oriented toward to S. That is, every such edge which comes out of S in the original orientation belongs to one of the paths P_i.

Moreover for each path P_i there is only one such edge; in other words if a path P_i leaves S then it can not come back. Otherwise S could be made larger by moving backwards along P_i. Therefore cutting one edge in each paths P_i makes impossible to leave S. In particular we can cut m edges leving no oriented path from s and t disconnected; that is, $n \leqslant m$. □



Let G be a bigraph. Let us add to G two vertexes s and t so that s is connected to each vertex in the left part of G and t is connected

[1]The described process has the following physical interpretation. Think of each path $P_1, \dots, P_m$ and Q as of watter pipelines from s to t. At each overlap of Q with the remaining paths the watter runs opposite direction, so we can cut the overlapping edges and connect the open ends of the pipes to each other while keeping the watter flow unchanged. As the result, we get $m + 1$ pipes form s to t with no common edges and possibly some cycles which we can discard.

to each vertex in the right part of G and orient the graph from left to right. Denote the obtained digraph by $\hat{G}$.

4.15. Advanced exercise. *Give an other proof of the marriage theorem for a bigraph G, applying the min-cut theorem to $\hat{G}$.*

Remarks

An extensive overview of the marriage theorem and its relatives is given by Alexander Evnin in [3], which I recommend to everyone who can read Russian.

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