

# Extra credit problems

Math 485

0. Find a mistake or misprint in the “Extra pearls”.

1- (solved) Assume that the sequence  $d_1, \dots, d_p$  is graphic,  $d_i \geq 1$  for each  $i$  and

$$d_1 + \dots + d_p \geq 2 \cdot (p - 1).$$

Show that there is a connected graph  $G$  with the degree sequence  $d_1, \dots, d_p$ .

2. Assume  $d_1, \dots, d_p$  is a sequence of integers in a nonincreasing order. Show that it is multigraphic if and only if  $d_p \geq 0$ , the sum  $d_1 + \dots + d_p$  is even and

$$d_1 \leq d_2 + \dots + d_p.$$

(A sequence of integers  $d_1, \dots, d_p$  is called *multigraphic* if it appears as a sequence of degrees of a multigraph.)

3- (solved) Show that in any connected graph  $G$  there is a vertex  $v$  such that  $G - v$  is connected.

4. Let  $G$  be a critical graph and  $\chi(G) = k + 1$ . Show after removing any  $k - 1$  edges from  $G$  the obtained graph remains connected.

5. Assume both sequences  $d_1, \dots, d_p$  and  $d_1 - 1, \dots, d_p - 1$  are graphic. Show that there is a graph with a 1-factor and with the degree sequence  $d_1, \dots, d_p$ .

6- (solved) Show that any 4-regular graph has a 2-factor.

7. Show that any edge of cubic graph lies in an even number of Hamiltonian cycles.

8- (solved) Let  $G$  be a connected graph. Show that any two paths of maximum length in  $G$  have a common vertex.

9- (solved) Assume that the edges of a complete graph are colored in two colors. Show that there is a Hamiltonian cycle which either monochromatic or consists of two monochromatic paths.

10. Assume two trees  $R$  and  $S$  have the vertices  $r_1, \dots, r_n$  and  $s_1, \dots, s_n$  correspondingly. Assume that  $R - r_i$  is isomorphic to  $S - s_i$  for each  $i$ . Show that  $R$  is isomorphic to  $S$ .

11. Assume that  $G$  is 3-regular connected graph such that for any two vertices  $v, w$  of  $G$  there is an isomorphism  $G \rightarrow G$  which sends  $v$  to  $w$ . Prove that  $G$  remains connected after removing any 2 edges.

12. Let  $G$  be a connected graph. Given any two vertices  $v, w$  in  $G$ , denote by  $d(v, w)$  the length of a shortest path containing  $v$  to  $w$ .

Assume that  $G$  has no triangles and

$$d(x, y) + d(v, w) = \max\{d(x, v) + d(y, w), d(x, w) + d(y, v)\}$$

for any 4 vertices  $x, y, v, w$  in  $G$ . Show that  $G$  is a tree.

13. Suppose  $p = r(m, n)$  is the Ramsey number for  $m$  and  $n$ .

Assume that  $G$  results from  $K_p$  by deleting a single edge. Show that  $G$  has a red/blue edge coloring with no red  $K_m$  and no blue  $K_n$ .

14. Solve 2.13 from “Extra pearls”.

~~15. Let  $T$  be a spanning tree in a graph  $G$  with weighted edges. Show that  $T$  has minimal total weight if and only if the weight of any edge  $(a, b)$  in  $G$  is at least as large as the weight of any edge on the path from  $a$  to  $b$  in  $T$ .~~

16. In a group of people, for some fixed  $s$  and any  $k$ , any  $k$  girls like at least  $k - s$  boys in total. Show that then all but  $s$  girls may get married on the boys they like.

16. Let  $M$  be a connected planar map and  $G$  be its dual. Show that  $M$  and  $G$  have the same number of spanning trees.