

Extra pearls for MATH 485

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Chapter 1

Ramsey numbers continued

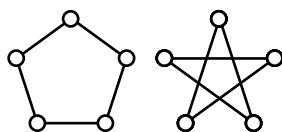
Lower bounds

Recall that Ramsey number $r(m, n)$ is a least positive integer for which every blue-red coloring of edges in the complete graph $K_{r(m, n)}$ contains a blue K_m or a red K_n .

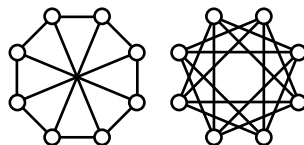
Equivalently, every for any decomposition of $K_{r(m, n)}$ into two subgraphs G and H either G contains a copy of K_m or H contains a copy of K_n .

Therefore, to get a lower bound on $r(m, n) \geq s + 1$, it is sufficient to decompose K_s into two subgraphs with no isomorphic copy of K_m in the first one and no isomorphic copy of K_n in the second one.

For example, the subgraphs in the decomposition of K_5 on the diagram has no monochromatic triangles; the later implies that $r(3, 3) \geq 6$. We showed already that for any decomposition of K_6 into two subgraphs, one of the subgraphs has a triangle; that is $r(3, 3) = 6$.



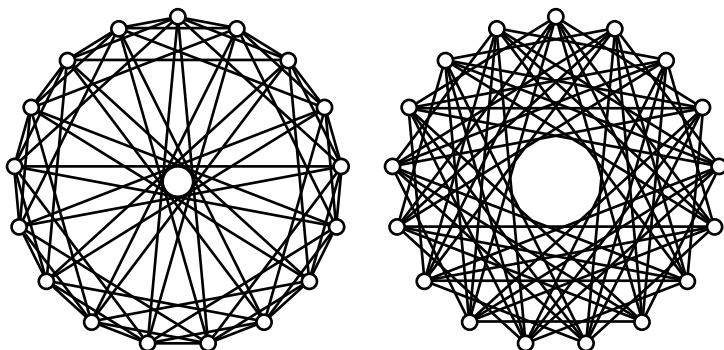
Similarly, to show that $r(3, 4) \geq 9$, we need to construct a decomposition of K_8 into two subgraphs G and H such that G contains no triangle K_3 and H contains no K_4 .



Similarly, to show that $r(4, 4) \geq 18$, we need to construct a decomposition of K_{17} into two subgraphs with no K_4 . (In fact, $r(4, 4) = 18$,

but we are not going to prove it.) This is much harder, the corresponding decomposition is given on the following diagram.

The constructed decomposition is rationally symmetric; the first subgraph contains the chords of angle lengths 1, 2, 4, and 8 and the second to all the cords of angle lengths 3, 5, 6 and 7.



1.1. Exercise. *Show that*

- (a) *In the decomposition of K_8 above, the left graph contains no triangle and the right graph contains no K_4 .*
 - (b) *In the decomposition of K_{17} above, both graph contain no K_4 .*
- (Hint: In each cases, fix one vertex v and draw the subgraph induced by the vertexes connected to v .)*

For larger values m and n the problem of finding the exact lower bound for $r(m, n)$ is quickly becomes too hard. Even getting a reasonable rough estimate is challenging. In the next section we will show how to obtain such estimate using probability.

Probabilistic method

The probabilistic method makes possible to prove the existence of graphs with certain properties without constructing them explicitly. The idea is to show that if one randomly chooses a graph from a specified class, then probability that the result is of the needed property is more than zero. The latter implies that a graph with needed property exists.

Despite that this method of proof uses probability, the final conclusion is determined for certain, without any possible error.

This method finds applications in many areas of mathematics; not only in graph theory. It was introduced by Paul Erdős.

Recall that $\binom{n}{m}$ denotes the *binomial coefficient*; that is, m and n are integers, $n \geq 0$ and

$$\binom{n}{m} = \frac{n!}{m! \cdot (n-m)!}$$

if $0 \leq m \leq n$ and $\binom{n}{m} = 0$ otherwise.

The number $\binom{n}{m}$ plays an important role in combinatorics — it gives the number of ways that m objects can be chosen from among n different objects.

1.2. Theorem. *Assume that the inequality*

$$\binom{N}{n} < 2^{\binom{n}{2}-1}$$

holds for a pair of positive integers N and n . Then $r(n, n) > N$.

Proof. We need to show that the complete graph K_N admits a coloring of edges in red and blue such that it has no monochromatic subgraph isomorphic to K_n .

Let us color the edges randomly — color each edge independently with probability $\frac{1}{2}$ in red and otherwise in blue.

Fix a set S of n vertexes. Define the variable $X(S)$ to be 1 if every edge amongst the n vertexes is the same color, and 0 otherwise. Note that the number of monochromatic n -subgraphs in K_N is the sum of $X(S)$ over all possible subsets.

Note that the expected value of $X(S)$ is simply the probability that all of the $\binom{n}{2} = \frac{n \cdot (n-1)}{2}$ edges in S are the same color. The probability that all the edges with the ends in S are blue is $1/2^{\binom{n}{2}}$ and with the same probability all edges are red. Since these two possibilities exclude each other the expected value of $X(S)$ is $2/2^{\binom{n}{2}}$.

This holds for any n -vertex subset S of the vertexes of K_N . The total number of such subsets is $\binom{N}{n}$. Therefore the expected value for the sum of $X(S)$ over all S is

$$X = 2 \cdot \binom{N}{n} / 2^{\binom{n}{2}}.$$

Assume that $X < 1$. Note that at least in one coloring suppose to have at most X complete monochromatic n -subgraphs. Since this number has to be an integer, at least one coloring must have no complete monochromatic n -subgraphs.

Therefore if $\binom{N}{n} < 2^{\binom{n}{2}-1}$, then there is a coloring K_N without monochromatic n -subgraphs. Hence the statement follows. \square

The following corollary implies that the function $n \mapsto r(n, n)$ grows at least exponentially.

1.3. Corollary. $r(n, n) > \frac{1}{8} \cdot 2^{\frac{n}{2}}$ for all positive integers $n \geq 2$.

Proof. Set $N = \lfloor \frac{1}{8} \cdot 2^{\frac{n}{2}} \rfloor$; that is, N is the largest integer $\leq \frac{1}{8} \cdot 2^{\frac{n}{2}}$.

Note that

$$2^{\binom{n}{2}-1} > (2^{\frac{n-3}{2}})^n \geq N^n.$$

and

$$\binom{N}{n} = \frac{N \cdot (N-1) \cdots (N-n+1)}{n!} < N^n.$$

Therefore

$$\binom{N}{n} < 2^{\binom{n}{2}-1}.$$

By Theorem 1.2, we get $r(n, n) > N$. □

In the following exercise, mimic the proof of Theorem 1.2, very rough estimates will do the job.

1.4. Exercise. *By random coloring we will understand a coloring edges of a given graph in red and blue such that each edge is colored independently in red or blue with equal chances.*

Assume the edges of the complete bigraph $K_{100,100}$ is colored randomly. Show that probability that $K_{100,100}$ is monochromatic is less than $\frac{1}{10^{2500}}$.

Show that the number of different subgraphs in $K_{10^{10},10^{10}}$ isomorphic to $K_{100,100}$ is less than 10^{2000} .

Assume the edges of the complete bigraph $K_{10^{10},10^{10}}$ is colored randomly. Show that the expected number of monochromatic subgraphs isomorphic to $K_{100,100}$ in $K_{10^{10},10^{10}}$ is less than 1.

Conclude that the complete bigraph $K_{10^{10},10^{10}}$ admits an edge coloring in two colors such that it contains no monochromatic $K_{100,100}$.

Counting proof

In this section we will repeat the proof of Theorem 1.2 using a different language, without use of probability. We do this to affirm that probabilistic method provides real proof, without any possible error.

In principle, any probabilistic proof admits such translation, but in most cases, the translation is less intuitive.

Proof of 1.2. The graph K_N has $\binom{N}{2}$ edges. Each edge can be colored in blue or red therefore the total number of different colorings is

$$\Omega = 2^{\binom{N}{2}}.$$

Fix a subgraph isomorphic to K_n in K_N . Note that this graph is red in $\Omega/2^{\binom{n}{2}}$ different colorings and yet in $\Omega/2^{\binom{n}{2}}$ different colorings this subgraph is blue.

There are $\binom{N}{n}$ different subgraphs isomorphic to K_n in K_N . Therefore the total number of monochromatic K_n 's in all the colorings is

$$M = \binom{N}{n} \cdot \Omega \cdot 2/2^{\binom{n}{2}}.$$

If $M < \Omega$, then by the pigeonhole principle, there is a coloring with no monochromatic K_n . Hence the result. \square

Remarks

Note that probabilistic method is not nonconstructive — often when the existence of a certain graph is probed by probabilistic method, it is still uncontrollably hard to describe a concrete example.

The following theorem gives an other classical application of probabilistic method.

1.5. Theorem. *Given a positive integer g and k there is a graph G with girth at least g and chromatic number at least k .*

To learn more about this method, reads the chapter “Probability makes counting easy” in [1].

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