-. Let ϕ be a flow on a Riemannian manifold (M, g) generated by a vector field X. Assume that ϕ^t is an isometry for each t. Show that the equality

$$\langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 0$$

holds for any two vector fields Y and Z on M.

Comment. A field X as above is called a *Killing field.*

-. Let $\hat{M}$ be a submanifold of a Riemannian manifold M. Note that $\hat{M}$ comes with a metric tensor, induced from M; in particular, $\hat{M}$ is a Riemannian manifold. Denote by ∇ and $\hat{\nabla}$ the Levi-Civita connections on M and $\hat{M}$ respectively.

Show that for any vector fields $\hat{X}$ and $\hat{Y}$ on $\hat{M}$ we have that

$$S(\hat{X}, \hat{Y}) = \hat{\nabla}_{\hat{X}} \hat{Y} - \nabla_{\hat{X}} \hat{Y}$$

is perpendicular to $\hat{M}$; that is, $\langle S(\hat{X}, \hat{Y}), \hat{W}\rangle \equiv 0$ for any tangent vector field $\hat{W}$ on $\hat{M}$.

Show that for any $p \in \hat{M}$, the value $[S(\hat{X}, \hat{Y})](p)$ depends only on $\hat{X}(p)$ and $\hat{Y}(p)$.

Comment. S can be called *second fundamental form* of $\hat{M}$ in M; it is a bilinear form on the tangent bundle of $\hat{M}$ with values in the normal bundle of $\hat{M}$ in M. Usually second fundamental form is defined for hypersurfaces; it has values in $\mathbb{R}$ — for hypersurfaces it can be identified with the normal bundle (locally).

-. A function f on a Riemannian manifold is called convex if for any geodesic γ, the composition $f \circ \gamma$ is a convex real-to-real function. Assume a complete Riemannian manifold (M, g) admits a nonconstant convex function. Show that $\operatorname{vol}(M, g) = \infty$.

Comment. A Riemannian manifolds is complete if the corresponding metric space is complete. This condition is equivalent to the fact that there is a both-side infinite geodesic in any direction.

-. Let M and N be complete m-dimensional simply connected Riemannian manifolds, and $f\colon M \to N$ a smooth map such that

$$|df(V)| \geq |V|$$

for any tangent vector V of M. Show that f is a diffeomorphism.

-. Let (M, g) be a closed n-dimensional Riemannian manifold and $f\colon M \to \mathbb{R}$ is a 1-Lipschitz function. Consider equally distributed random vector V with norm at most 1; denote by $\tau(V)$ its base point. Show that the expected value of

$$|f(\exp(V) - f(\tau(V))| \leq \varepsilon_n,$$

where ε_n depends only on the dimension n and $\varepsilon_n \to 0$ as $n \to \infty$.

Comment. This problem is an example in the Riemannian world of the so-called *concentration of measure phenomenon.*

Prove first the following: Suppose V is an equally distributed random vector in $\mathbb{R}^n$ such that that $|V| \leq 1$. Show that the expected absolute value of its first coordinate converges to 0 as $n \to \infty$.