Lecture 1

Magic cloaks

This lecture is based on a lecture of Sergei Ivanov [5].

A Visually identical manifolds

Given a Riemannian manifold M denote by $\mathrm{S}M$ its unit tangent bundle

$$\mathrm{S}M = \{\, \mathrm{v} \in \mathrm{T}M \,:\, |v| = 1 \,\}.$$

Recall that geodesic flow φ^t preserves the natural volume on $\mathrm{S}M$.

Suppose M has nonempty boundary ∂M; in other words, M is a closed region of a ambient Riemannian manifold bounded by a smooth hypersurface ∂M. Denote by $\mathrm{S}^+\partial M$ the set of unit vectors at points on ∂M that point in M; $\mathrm{S}^+\partial M$ is a bundle over ∂M with fibers formed by closed half-spheres.

Consider a geodesic γ_{u} in the direction of a vector $\mathrm{u} \in \mathrm{S}^+\partial M$. Suppose that γ_{u} hits the boundary again; denote by ℓ_{u} the first *hitting time*, so $\gamma_{\mathrm{u}}(\ell_{\mathrm{u}}) \in \partial M$. Note that in this case $\mathrm{v} = -\gamma'(\ell) \in \mathrm{S}^+\partial M$. Note that $\mathrm{u} \mapsto \mathrm{v}$ describes a partially defined involution on $\mathrm{S}M$ which we will call *return map*.

Suppose that M and $\bar{M}$ be two compact connected Riemannian manifolds with boundary such that a neighborhood of ∂M can be isometrically identified with a neighborhood of $\partial \bar{M}$, and moreover, the return maps in M and $\bar{M}$ are identical. In this case we say that M and $\bar{M}$ are *visually identical.*

Notice that a complement of a small ball in the standard sphere $\mathbb{S}^n$ is visually identical to the complement of identical ball in the projective space $\mathbb{R}\mathrm{P}^n = \mathbb{S}^n/\mathbb{Z}_2$.

Note that the hitting time functions for the constructed pair of manifolds are not identical. No examples of nonisometric visually identical manifolds with identical hitting time seems to be known.

1.1. Exercise. *Construct a pair of diffeomorphic visually identical Riemannian manifolds.*

Hint: Attach two examples as above by smooth tubes to you favorite Riemannian manifold.

1.2. Theorem. *Let M be a connected compact region of Euclidean space of dimension at least 2 cut by a smooth hypersurface. Then any visually identical manifold to M is isometric to M.*

1.3. Corollary. *Suppose a Riemannian metric g on $\mathbb{R}^n$ coincides with Euclidean metric g_0 outside of a compact set K. Suppose that that the complement $\gamma\backslash K$ of any g-geodesic γ coincides with the complement of a line. Then $(\mathbb{R}^n, g)$ is isometric to the Euclidean space.*

Note that one cannot claim that $g = g_0$. Indeed for any diffeomorphism $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ that is identical outside K the metric $g = \varphi_* g$ satisfies the assumption in the corollary. Clearly one can choose φ so that $\varphi_* g_0 \neq g_0$.

B Hitting time

Recall that hitting time function $\mathrm{u} \mapsto \ell(\mathrm{u})$ returns the time when the geodesic γ_{u} first hits the boundary.

1.4. Lemma. *Suppose that a manifold $\bar{M}$ is visually identical to a compact region M in Euclidean space of dimension at least 2. Then M and $\bar{M}$ have identical hitting time functions.*

Proof. Note that we can cut M from the Euclidean space E and glue $\bar{M}$ back in along the isometry in the definition of visually identical manifolds. This way we obtain a complete Riemannian manifold $\bar{E}$ that is Euclidean outside of a $\bar{M}$.

Suppose that a smooth hypersurface Σ in E surrounds M. Then Σ cuts from E and $\bar{E}$ visually identical manifolds. Moreover if $\bar{M}$ is not isometric to M, then the obtained pair of visually identical manifolds are not isometric as well.

It follows that we can assume that M is a ball. (This is true for the proof of the lemma and theorem as well.)

In this case any geodesic $\bar{\gamma}$ in $\bar{E}$ visits $\bar{M}$ at most once. In other words, if $\bar{\gamma}$ enters $\bar{M}$, then the complement $\bar{\gamma}\backslash\bar{M}$ has two connected components which are parts of a line γ in E.

Choose a unit-speed geodesic $\bar\gamma$ that visits $\bar M$. Let include it in a smooth one-parameter family of unit-speed geodescis $\bar\gamma_\tau$ for $\tau\in[0,1]$ so that $\bar\gamma_0$ does not visit $\bar M$ and $\bar\gamma_1=\bar\gamma$.

We can assume that the vector field $\bar{\mathrm{I}}=\tfrac{\partial}{\partial\tau}\gamma_\tau(t)$ is orthogonal to $\bar{\mathrm{T}}=\tfrac{\partial}{\partial t}\gamma_\tau(t)$ at every point $\gamma_\tau(t_0)$ for some fixed t_0.

Observe that in this case $\langle\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle=0$ at all points $\bar\gamma_\tau(t)$. Indeed

$$\begin{aligned}\tfrac{\partial}{\partial t}\langle\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle &= \bar{\mathrm{T}}\langle\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle=\\ &=\langle\nabla_{\bar{\mathrm{T}}}\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle+\cancel{\langle\bar{\mathrm{I}},\nabla_{\bar{\mathrm{T}}}\bar{\mathrm{T}}\rangle}=\\ &=\langle\nabla_{\bar{\mathrm{I}}}\bar{\mathrm{T}},\bar{\mathrm{T}}\rangle=\\ &=\tfrac12\cdot\bar{\mathrm{I}}\langle\bar{\mathrm{T}},\bar{\mathrm{T}}\rangle=\\ &=\tfrac{\partial}{\partial\tau}|\bar\gamma'_\tau(t)|^2=\\ &=0.\end{aligned}$$

That is $\langle\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle$ does not depend on t. Since $\langle\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle=0$ at all points $\gamma_\tau(t_0)$, the same holds for all points $\gamma_\tau(t)$.

Consider a family of geodesics γ_τ in E that coincide with $\bar\gamma_\tau$ (as sets) outside of M. The same argument shows that $\langle\mathrm{I},\mathrm{T}\rangle=0$ at all points $\gamma_\tau(t)$. It follows that $\bar{\mathrm{I}}=\mathrm{I}$ outside of $\bar M$ and M correspondingly. Therefore $\gamma_\tau(t)=\bar\gamma(t)$ for any t and τ, provided that $\bar\gamma(t)\notin\bar M$. Whence the γ spends exactly the same time in M as $\bar\gamma$ spends in $\bar M$ and the lemma follows. □

Comment. The vector fields I as in the proof restricted to γ_τ are called *Jacobi fields* along γ_τ; we will see them again.

1.5. Exercise. *Suppose $\bar M$ and $\bar E$ be as in the proof. Show that there is a universal upper bound on time that a unit-speed geodesic spends in $\bar M$.*

Hint: Show that the set of vectors $\mathrm{u}\in\mathrm{S}\bar E$ such that $\gamma_{\mathrm{u}}([0,T])\subset\bar M$ is open and closed; here $T=2\cdot\operatorname{diam}M$ and γ_{u} is the geodesic defined by $\gamma'_{\mathrm{u}}(0)=\mathrm{u}$.

C Volume equality

1.6. Lemma. *Suppose that M and $\bar M$ are visually identical. Then*

$$\operatorname{vol}M=\operatorname{vol}\bar M.$$

Proof. Recall that $\bar s\colon\mathrm{S}\bar M\to\bar M$ denotes the unit tangent bundle over $\bar M$ and $\bar\varphi^t\colon\mathrm{S}\bar M\to\mathrm{S}\bar M$ denotes the geodesic flow.

Set $\bar\Omega=\bar s^{-1}(\bar M)$. Since geodesic flow preserves the volume, we get

$$\begin{aligned}\operatorname{vol}\mathbb{S}^{n-1}\cdot\operatorname{vol}(\bar M,g)&=\operatorname{vol}\bar\Omega=\\&=\operatorname{vol}[\bar\varphi^t(\bar\Omega)].\end{aligned}$$

By 1.5, we can choose t so that $s(\mathrm{v})\notin\bar M$ for any $\mathrm{v}\in\varphi^t(\bar\Omega)$.

Repeat the same construction for M. By 1.4, the set $\varphi^t(\Omega)=$ $=\bar\varphi^t(\bar\Omega)$. In particular,

$$\operatorname{vol}\Omega=\operatorname{vol}[\varphi^t(\Omega)]=\operatorname{vol}[\bar\varphi^t(\bar\Omega)=\operatorname{vol}\bar\Omega.$$

whence the result follows. □

D Santalo formula

Santalo formula gives an expression for a volume of a Riemannian manifold with boundary in terms of hitting times of its geodesics. It provides a more direct proof of 1.6.

Suppose M is a Reimannian manifold with nonempty boundary ∂M. Recall that

- ◇ $\mathrm{S}M$ denotes the unit tangent bundle of a Reimannian manifold M.
- ◇ $\mathrm{S}^+\partial M$ denotes by the set of unit vectors at points on ∂M that point in M.
- ◇ $\ell\colon\mathrm{S}^+\partial M\to[0,\infty]$ denoted the hitting time.

1.7. Theorem. *Let M be an n-dimensional Riemannian manifold with nonempty boundary. Suppose that any unit-speed geodesic in M hits its boundary in finite time. Then for any smooth function $f\colon\mathrm{S}M$ the following identity holds:*

$$\int\limits_{\mathrm{w}\in\mathrm{S}M}f(\mathrm{w})=\int\limits_{\mathrm{u}\in\mathrm{S}^+\partial M}\int\limits_0^{\ell(\mathrm{u})}f[\gamma_{\mathrm{u}}'(t)]\cdot dt.$$

In particular, by taking $f\equiv1$, we get

$$\operatorname{vol}\mathbb{S}^{n-1}\cdot\operatorname{vol}M=\int\limits_{\mathrm{u}\in\mathrm{S}^+\partial M}\ell(\mathrm{u}).$$

1.8. Exercise. *Construct two Reimannian metrics g_0 and g_1 on the disc $\mathbb{D}$ that coincide near the boundary and such that*

$$\operatorname{area}(\mathbb{D},g_0)>\operatorname{area}(\mathbb{D},g_1),$$

but

$$\ell_0(\xi) < \ell_1(\xi),$$

where $\ell_i(\xi)$ denotes hitting time of g_i-geodesic in the direction ξ; that is, $\ell_i(\xi)$ is the length of g_i-geodesic that starts at a point $p \in \partial\mathbb{D}$ in the direction ξ.

Why does this example not contradict Santalo's formula?

E More exercises

Two Riemannian metrics g_0 and g_1 on M are called *conformally equivalent* if there is a function λ such that $g_1 = \lambda^2 \cdot g_0$. In this case the function λ is called *conformal factor*. Note that for any g_0-unit-speed curve $\gamma\colon [a,b] \to M$ we have

$$\operatorname{length}_{g_1}\gamma = \int_a^b \lambda\circ\gamma(t)\cdot dt$$

1.9. Exercise. *Let g_0 be the canonical metric on the projective space $\mathbb{R}\mathrm{P}^n$; that is, $(\mathbb{R}\mathrm{P}^n, g_0)$ is isometric to the quotient space of the unit sphere $\mathbb{S}^n$ by central symmetry. Suppose that g_1 is conformally equivalent to g_0. Denote by ℓ_0 and ℓ_1 the length of shortest noncontractible closed curves in $(\mathbb{R}\mathrm{P}^n, g_0)$ and $(\mathbb{R}\mathrm{P}^n, g_1)$ respectively (so $\ell_0 = \pi$). Show that*

$$\frac{\operatorname{vol}(\mathbb{R}\mathrm{P}^n, g_1)}{\ell_1^n} \geqslant \frac{\operatorname{vol}(\mathbb{R}\mathrm{P}^n, g_0)}{\ell_0^n}.$$

Hint: Use that geodesic flow preserves volume of the unit tangent bundle to rewrite the integral of conformal factor over $(\mathbb{R}\mathrm{P}^n, g_0)$ and interpret the result.

Let (M, g) be a Riemannian manifold. The Sasaki metric is a natural choice of Riemannian metric $\hat g$ on the total space of the tangent bundle $\tau\colon \mathrm{T}M \to M$ defined the following way:

Identify the tangent space $\mathrm{T}_u[\mathrm{T}M]$ for any $u \in \mathrm{T}_pM$ with the direct sum of vertical and horizontal subspaces $\mathrm{T}_pM \oplus \mathrm{T}_pM$. The projection of this splitting is defined by the differential $d\tau\colon \mathrm{TT}M \to \mathrm{T}M$ and we assume that the velocity of a curve in $\mathrm{T}M$ formed by a parallel field along a curve in M is horizontal. Then $\mathrm{T}_u[\mathrm{T}M]$ is equipped with the metric $\hat g$ defined by

$$\hat g(X,Y) = g(X^V, Y^V) + g(X^H, Y^H),$$

where X^V and $X^H \in \mathrm{T}_pM$ denote the vertical and horizontal components of $X \in \mathrm{T}_u[\mathrm{T}M]$.

1.10. Exercise. *Let g be the canonical Riemannian metric on the sphere $\mathbb{S}^2$. Consider the tangent bundle $\mathrm{T}\mathbb{S}^2$ equipped with the induced Sasaki metric $\hat{g}$. Let S_R be the hypersurface in $\mathrm{T}\mathbb{S}^2$ of vectors with norm R; we assume that S_R is equipped with induced Riemannian metric.*

Show that $\operatorname{vol} S_R \to \infty$ as $R \to \infty$, but $\operatorname{diam} S_R$ stays bounded for all R.

F Remarks

Theorem 1.2 proved by Mikhael Gromov [4]. It has a number of variations and generalizations. In particular an analog of this theorem holds in the following cases:

- ◇ For 2-dimensional Riemannian manifolds with unique geodesic between any two points [7].
- ◇ In hyperbolic space [1] and for regions in a round hemisphere [6].
- ◇ If M is a Riemannian manifold with unique geodesic between any two points, then the theorem holds for the product $\mathbb{R} \times M$ [3].
- ◇ For any Riemannian manifold, provided that the metric tensor is modified to in a sufficiently small region [2].

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