

HWA-11, Exercises: 1.9, 1.11, 1.15, 1.20, 2.3.

HWA-12, Exercises: 2.7, 2.8, 2.9, 2.10, 2.11.



# Lecture 1

## Magic cloaks

Based on a lecture of Sergei Ivanov [11].

### A Scattering data

Given a Riemannian manifold  $M$  denote by  $\tau: SM \rightarrow M$  its unit tangent bundle

$$SM = \{v \in TM : |v| = 1\}.$$

Recall that by *Liouville's theorem*, the geodesic flow  $\varphi^t$  preserves the natural volume on  $SM$  in its domain of definition. Denote by  $g$  the vector field on  $SM$  that defines the geodesic flow  $\varphi^t$ .

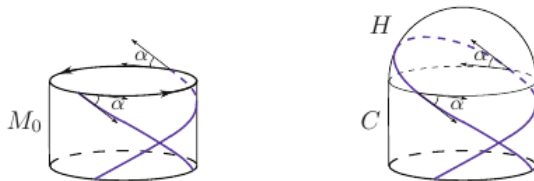
Suppose  $M$  has nonempty boundary  $\partial M$ ; in other words,  $M$  is a closed region of an ambient Riemannian manifold bounded by a smooth hypersurface  $\partial M$ . Denote by  $\partial_+ SM$  the set of unit vectors at points on  $\partial M$  that point in  $M$ ;  $\partial_+ SM$  is a bundle over  $\partial M$  with fibers formed by closed half-spheres. The set  $\partial_+ SM$  is a subset of  $\partial SM$  that can be also defined as the closure of the subset at which the geodesic flow enters  $SM$ .

Consider a geodesic  $\gamma_u$  in the direction of a vector  $u \in \partial_+ SM$ . Suppose that  $\gamma_u$  hits the boundary again. In other words,  $\gamma_u(t) = \tau \circ \varphi^t(u)$ . Denote by  $\ell(u)$  the first *hitting time*, so  $\gamma_u(\ell(u)) \in \partial M$ . Note that in this case the vector  $v = -\gamma'(\ell)$  lies in  $\partial_+ SM$ . The map  $s: u \mapsto v$  is defined if  $\ell(u) < \infty$ ; it is a partially defined involution on  $SM$  which we will call *scattering map*.

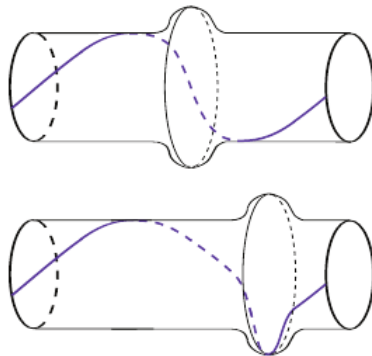
Suppose that  $M$  and  $\bar{M}$  be two compact connected Riemannian manifolds with boundary such that a neighborhood of  $\partial M$  can be isometrically identified with a neighborhood of  $\partial \bar{M}$ , and moreover, the scattering maps in  $M$  and  $\bar{M}$  are identical. In this case we say

that  $M$  and  $\bar{M}$  have identical *scattering data*. If in addition their hitting time functions coincide (that is, if  $\ell(u) \equiv \bar{\ell}(u)$ ), then we say that  $M$  and  $\bar{M}$  have the identical *lens data*.

Notice that if a manifold contains a copy of a round hemisphere, then cutting it and cluing the opposite points of its boundary produces a manifold with identical scattering data. The lens data for the



constructed pair of manifolds are not identical. An example of nonisometric manifolds with identical lens data can be found among surfaces of revolution which look like cylinders with bumps on them that are shifted and otherwise look the same.



**1.1. Exercise.** Check that the described examples have identical lens data.

Construct a pair of nonisometric Riemannian metrics on the disc with identical lens data.

## B Main theorem

The following theorem proved by Mikhael Gromov [8]; it is the main subject of this lecture.

**1.2. Theorem.** *Any connected compact region  $M$  of Euclidean space of dimension at least 2 cut by a smooth hypersurface is scattering rigid; that is, any manifold with scattering data identical to  $M$  is isometric to  $M$ .*

**1.3. Corollary.** *Suppose a Riemannian metric  $g$  on  $\mathbb{R}^n$  coincides with Euclidean metric  $g_0$  outside of a compact set  $K$ . Suppose that the complement  $\gamma \setminus K$  of any nontrivial  $g$ -geodesic  $\gamma$  coincides with the complement of a line (as sets). Then  $(\mathbb{R}^n, g)$  is isometric to the Euclidean space.*

Note that in the corollary we cannot claim that  $g = g_0$ . Indeed for any diffeomorphism  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$  that is identical outside  $K$  the metric  $g = \varphi_* g_0$  satisfies the assumption in the corollary. Clearly one can choose  $\varphi$  so that  $g \neq g_0$ .

## C Identical lens data

**1.4. Lemma.** *Suppose that a manifold  $\bar{M}$  has identical scattering data with a compact region  $M$  in Euclidean space of dimension at least 2. Then  $M$  and  $\bar{M}$  have identical lens data.*

The Euclidean space can be exchanged to any complete Riemannian manifold with unique geodesic between any pair of points; the proof is the same.

*Proof.* Denote by  $W$  the complement of the interior of  $M$  in the Euclidean space  $E$ . Let us glue  $\bar{M}$  to  $W$  along the isometry in the definition of scattering data. This way we obtain a complete Riemannian manifold  $\bar{E}$  that is Euclidean outside of a  $\bar{M}$ .

Suppose that a smooth hypersurface  $\Sigma$  in  $E$  surrounds  $M$ . Then  $\Sigma$  cuts from  $E$  and  $\bar{E}$  manifolds with identical scattering data. Moreover if  $\bar{M}$  and  $M$  are not isometric, then the obtained pair is not isometric as well.

It follows that we can assume that  $M$  is a ball.<sup>1</sup> In this case any geodesic  $\bar{\gamma}$  in  $\bar{E}$  visits  $\bar{M}$  at most once. In other words, if  $\bar{\gamma}$  enters  $\bar{M}$ , then the complement  $\bar{\gamma} \setminus \bar{M}$  has two connected components which are parts of a line  $\gamma$  in  $E$ .

Choose a unit-speed geodesic  $\bar{\gamma}$  that visits  $\bar{M}$ . Let us include it in a smooth one-parameter family of unit-speed geodesics  $\bar{\gamma}_\tau$  for  $\tau \in [0, 1]$  so that  $\bar{\gamma}_0$  does not visit  $\bar{M}$  and  $\bar{\gamma}_1 = \bar{\gamma}$ .

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<sup>1</sup>This is true for the proof of the lemma and theorem as well.

We can assume that the vector field  $\bar{\mathbf{I}} = \frac{\partial}{\partial \tau} \bar{\gamma}_\tau(t)$  is orthogonal to  $\bar{\mathbf{T}} = \frac{\partial}{\partial t} \bar{\gamma}_\tau(t)$  at every point  $\bar{\gamma}_\tau(t_0)$  for some fixed  $t_0$ .

Observe that in this case  $\langle \bar{\mathbf{I}}, \bar{\mathbf{T}} \rangle = 0$  at all points  $\bar{\gamma}_\tau(t)$ . Indeed

$$\begin{aligned} \frac{\partial}{\partial t} \langle \bar{\mathbf{I}}, \bar{\mathbf{T}} \rangle &= \bar{\mathbf{T}} \langle \bar{\mathbf{I}}, \bar{\mathbf{T}} \rangle = \\ &= \langle \nabla_{\bar{\mathbf{T}}} \bar{\mathbf{I}}, \bar{\mathbf{T}} \rangle + \langle \bar{\mathbf{I}}, \nabla_{\bar{\mathbf{T}}} \bar{\mathbf{T}} \rangle = \\ &= \langle \nabla_{\bar{\mathbf{I}}} \bar{\mathbf{T}}, \bar{\mathbf{T}} \rangle = \\ &= \frac{1}{2} \cdot \bar{\mathbf{I}} \langle \bar{\mathbf{T}}, \bar{\mathbf{T}} \rangle = \\ &= \frac{\partial}{\partial \tau} |\bar{\gamma}'_\tau(t)|^2 = \\ &= 0. \end{aligned}$$

That is,  $\langle \bar{\mathbf{I}}, \bar{\mathbf{T}} \rangle$  does not depend on  $t$ . Since  $\langle \bar{\mathbf{I}}, \bar{\mathbf{T}} \rangle = 0$  at all points  $\bar{\gamma}_\tau(t_0)$ , the same holds for all points  $\bar{\gamma}_\tau(t)$ .

Consider a family of geodesics  $\gamma_\tau$  in  $E$  that coincide with  $\bar{\gamma}_\tau$  (as sets) outside of  $M$ . The same argument shows that  $\langle \mathbf{I}, \mathbf{T} \rangle = 0$  for the corresponding vector fields  $\mathbf{I} = \frac{\partial}{\partial \tau} \gamma_\tau(t)$  and  $\mathbf{T} = \frac{\partial}{\partial t} \gamma_\tau(t)$  at all points  $\gamma_\tau(t)$ .

By assumption,  $\bar{\mathbf{T}} = \mathbf{T}$  and  $\bar{\mathbf{I}} - \mathbf{I}$  is proportional to  $\mathbf{T}$  in  $W$ . It follows that  $\bar{\mathbf{I}} = \mathbf{I}$  in  $W$ . Therefore  $\gamma_\tau(t) = \bar{\gamma}_\tau(t)$  for any  $t$  and  $\tau$ , provided that  $\bar{\gamma}(t) \in W$ . Whence the  $\gamma$  spends exactly the same time in  $M$  as  $\bar{\gamma}$  spends in  $\bar{M}$  and the lemma follows.  $\square$

*Comments.* The identity  $\langle \mathbf{I}, \mathbf{T} \rangle = 0$  is proved the same way as the *Gauss lemma*. The vector fields as  $\mathbf{I}$  in the proof restricted to  $\gamma_\tau$  describe a variation of a geodesics. These fields are called *Jacobi fields* along  $\gamma_\tau$ ; we will see them again.

**1.5. Exercise.** Suppose  $\bar{M}$  and  $\bar{E}$  be as in the proof. Show that there is a universal upper bound on time that a unit-speed geodesic spends in  $\bar{M}$ .

*Hint:* Show that the set of vectors  $\mathbf{U} \in S\bar{E}$  such that the arc  $\gamma_{\mathbf{U}}|_{[0,T]}$  lies in  $\bar{M}$  is open and closed; here  $T = 2 \cdot \text{diam } M$  and  $\gamma_{\mathbf{U}}$  is the geodesic defined by  $\gamma'_{\mathbf{U}}(0) = \mathbf{U}$ .

## D Volume equality

**1.6. Lemma.** Suppose that a manifold  $\bar{M}$  has identical scattering data with a compact region  $M$  in Euclidean space of dimension at least 2. Then

$$\text{vol } M = \text{vol } \bar{M}.$$

*Proof.* We will denote by  $\bar{\tau}: S\bar{M} \rightarrow \bar{M}$  the unit tangent bundle over  $\bar{M}$  and by  $\bar{\varphi}^t: S\bar{M} \rightarrow S\bar{M}$  its the geodesic flow.

Set  $\bar{\Omega} = \bar{\tau}^{-1}(\bar{M})$ . Since geodesic flow preserves the volume, we get

$$\begin{aligned} \text{vol } S^{n-1} \cdot \text{vol}(\bar{M}, g) &= \text{vol } \bar{\Omega} = \\ &= \text{vol}[\bar{\varphi}^t(\bar{\Omega})]. \end{aligned}$$

By 1.5, we can choose  $t$  so that  $\tau(v) \notin \bar{M}$  for any  $v \in \varphi^t(\bar{\Omega})$ .

Repeat the same construction for  $M$ . By 1.4 and 1.5,  $\varphi^t(\Omega) = \bar{\varphi}^t(\bar{\Omega})$ . In particular,

$$\text{vol } \Omega = \text{vol}[\varphi^t(\Omega)] = \text{vol}[\bar{\varphi}^t(\bar{\Omega})] = \text{vol } \bar{\Omega}.$$

whence the result follows.  $\square$

## E Santaló formula

Santaló formula is a simple corollary of Liouville's theorem — geodesic flow preserves the volume. It gives an expression for a volume of a Riemannian manifold with boundary in terms of hitting times of its geodesics. It provides a more direct proof of 1.6.

Suppose  $M$  is a Riemannian manifold with nonempty boundary  $\partial M$ . Recall that

- ◇  $SM$  denotes the unit tangent bundle over  $M$ .
- ◇  $\varphi^t$  denotes geodesic flow. In particular, if  $\gamma_u$  is the geodesic in  $M$  defined by  $\gamma'_u(0) = u$ , then  $\gamma'_u(t) = \varphi^t(u)$ .
- ◇ Let  $\ell: SM \rightarrow [0, \infty]$  denoted the hitting time of  $\gamma_u$  in the boundary of  $M$ .
- ◇  $\partial_+ SM$  denotes by the bundle of unit vectors at points on  $\partial M$  that point in  $M$ . It can be defined as the closure of the subset of  $\partial SM$  at which the geodesic flow enters  $SM$ .

**1.7. Theorem.** *Let  $M$  be an  $n$ -dimensional Riemannian manifold with nonempty boundary. Suppose that any unit-speed geodesic in  $M$  hits its boundary in finite time. Then for any smooth function  $f: SM \rightarrow \mathbb{R}$  the following identity holds:*

$$\int_{w \in SM} f(w) = \int_{u \in \partial_+ SM} \langle u, N \rangle \cdot \int_0^{\ell(u)} f \circ \varphi^t(u) \cdot dt,$$

where  $N$  denotes the unit vector field normal to  $\partial M$  that points inside  $M$ .

In particular, by taking  $f \equiv 1$ , we get

$$\text{vol } \mathbb{S}^{n-1} \cdot \text{vol } M = \int_{u \in \partial_+ SM} \ell(u) \cdot \langle u, N \rangle.$$

*Proof.* Note that any vector  $w \in SM$  can be uniquely described as  $\varphi^t(u)$  for some  $u \in \partial_+ SM$  and  $0 \leq t \leq \ell(u)$ . In other words  $SM$  can be identified with the subgraph

$$\Phi = \{ (u, t) \in (\partial_+ SM) \times \mathbb{R} : 0 \leq t \leq \ell(u) \}.$$

The subgraph  $\Phi$  has two volume forms: the first, say  $\omega$ , is the pull back of the volume form on  $SM$ ; the second  $\chi = dt \wedge \alpha$ , where  $\alpha$  is the volume form on  $\partial_+ SM$ .

Note that both forms are invariant with respect to shifts along  $\mathbb{R}$ . For  $\omega$  it is true by Liouville's theorem; for  $\chi$  it follows from the definition.

Set  $r(v) = \text{dist}_{\partial M} \circ \tau(v)$ ; note that  $r$  is a smooth function near  $\partial SM$ . Observe that  $dr = \langle u, N \rangle \cdot dt$  on  $\partial SM$ . Extend  $\alpha$  to  $SM$  arbitrary. Note that the equality  $\omega = dr \wedge \alpha$  holds on  $\partial SM$ . Whence

$$\textbf{1} \quad \omega = \langle u, N \rangle \cdot \chi$$

on  $\partial SM$ . Since both forms are invariant with respect to vertical shifts, we get that **1** holds everywhere in  $\Phi$ .  $\square$

**1.8. Exercise.** Construct two Riemannian metrics  $g_0$  and  $g_1$  on the disc  $\mathbb{D}$  that coincide near the boundary and such that

$$\text{area}(\mathbb{D}, g_0) > \text{area}(\mathbb{D}, g_1),$$

but

$$\ell_0(u) < \ell_1(u),$$

where  $\ell_i(u)$  denotes hitting time of  $g_i$ -geodesic in the direction  $u$ ; that is,  $\ell_i(u)$  is the length of  $g_i$ -geodesic that starts at a point  $p \in \partial \mathbb{D}$  in the direction  $u \in \partial_+ S\mathbb{D}$ .

Why does this example not contradict the Santaló formula?

**1.9. Exercise.** Denote by  $\omega$  the volume form on  $SM$  and by  $G$  the vector field on  $SM$  that describes the geodesic flow. Given a function  $f: SM \rightarrow \mathbb{R}$ , consider the function  $F: SM \rightarrow \mathbb{R}$  defined by

$$F(u) = - \int_0^{\ell(u)} f \circ \varphi^t(u) \cdot dt.$$

Prove the Santaló formula applying Stokes' theorem to form  $\iota_G(F \cdot \omega)$ .



## F Differentiability of distance function

**1.10. Theorem.** *For any closed set  $A$  in a complete Riemannian manifold  $M$  and any point  $x \notin A$  the differential  $d_x f$  of the distance function  $f = \text{dist}_A$  is defined if and only if there is a unique minimizing geodesic  $\gamma$  from  $x$  to  $A$ .*

*Moreover, if  $u \in T_x$  is the unit vector points in the direction of the unique geodesic  $\gamma$ , then*

$$d_x f(w) = -\langle u, w \rangle$$

*for any  $w \in T_x$ ; or, equivalently,*

$$\nabla_x f = -u.$$

*Proof; only-if part.* Choose

- ◊ a closed set  $A$ , a point  $x \notin A$ , and  $\varepsilon > 0$ ,
- ◊ a unit-speed minimizing geodesic  $\gamma$  from  $x$  to  $A$ ,
- ◊ a smooth unit-speed curve  $\alpha$  that such that  $\alpha(0) = x$ , and set  $w = \alpha'(0)$ .

Observe that

$$\begin{aligned} |\gamma(\tfrac{t}{\varepsilon}) - \alpha(t)|_M &= t \cdot \sqrt{\tfrac{1}{\varepsilon^2} - 2 \cdot \langle u, w \rangle \cdot \tfrac{1}{\varepsilon} + 1} + o(t) = \\ &= \tfrac{1}{\varepsilon} \cdot t - \langle u, w \rangle \cdot t + O(\varepsilon \cdot t). \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, the triangle inequality implies that

$$f \circ \alpha(t) \leq |p - x| + t \cdot \langle u, w \rangle + o(t).$$

In particular,

$$\textcircled{1} \quad (f \circ \alpha)'(0) = -\langle u, w \rangle$$

if the left hand side is defined.

Observe that if  $d_x f$  is defined, then  $(f \circ \alpha)'(0) = d_x f(w)$ . Therefore

$$d_x f(w) \leq -\langle u, w \rangle$$

for any  $w \in T_x$ . Since both sides of the last inequality are linear, we get that the equality

$$d_x f(w) = -\langle u, w \rangle$$

holds for any  $w \in T_x$ .

Suppose that  $\gamma_1$  is another minimizing geodesic from  $x$  to  $A$ ; set  $u_1 = \gamma_1'(0)$ . If  $d_x f$  is defined, then we have

$$-\langle u, w \rangle = d_x f(w) = -\langle u_1, w \rangle;$$

that is,  $u_1 = u$  and therefore  $\gamma_1 = \gamma$ .

*If part.* Suppose that  $\gamma$  is a unique geodesic from  $x$  to  $A$ . Choose  $\alpha$  as above. For each  $t$  choose a minimizing geodesic  $\gamma_t$  from  $\alpha(t)$  to  $A$ ; Set  $u(t) = \gamma_t'(0)$  and  $w(t) = \alpha'(t)$ .

Recall that  $f$  and  $\alpha$  are Lipschitz. By Rademacher's theorem and **1**, we have that

$$(f \circ \alpha)'(t) \stackrel{a.e.}{=} -\langle u(t), w(t) \rangle;$$

moreover

$$f \circ \alpha(\tau) - f \circ \alpha(0) = - \int_0^\tau \langle u(t), w(t) \rangle \cdot t.$$

It remains to show that  $\langle u(t), w(t) \rangle \rightarrow \langle u, w \rangle$  as  $t \rightarrow 0$ .

The latter follows if  $u(t) \rightarrow u$  as  $t \rightarrow 0$ . Assume the contrary, then there is a sequence  $t_n \rightarrow 0$  such that  $u(t_n)$  converges to a unit vector  $v \in T_x$  that is distinct from  $u$ . The minimizing geodesics  $\gamma_{t_n}$  converge to a geodesic from  $x$  to  $A$  that runs in the direction  $v$ . Since  $v \neq u$ , this geodesic is distinct from  $\gamma$  — a contradiction.  $\square$

**1.11. Exercise.** Suppose that  $M$  is a compact Riemannian manifold with convex boundary  $\partial M$ ; that is, any shortest path in  $M$  may only have its endpoints on  $\partial M$ . Assume that for any  $p \in \partial M$  the function  $\text{dist}_p$  is differentiable on  $\partial M \setminus \{p\}$ .

Prove the following statements:

- (a) Any geodesic in  $M$  is minimizing.
- (b) For any  $p \in M$  the distance function  $\text{dist}_p$  is differentiable in  $M \setminus \{p\}$ .
- (c) Show that  $M$  is homeomorphic to a ball.
- (d) The restriction of the distance function to  $\partial M$  determines the lens data of  $M$ .

## G Besicovitch inequality

The following theorem was proved by Abram Besicovitch [1].

**1.12. Theorem.** Let  $g$  be a metric tensor on a unit  $n$ -dimensional cube  $\square$ . Suppose that the  $g$ -distances between the opposite facets of  $\square$  are at least 1; that is, any Lipschitz curve that connects opposite faces has  $g$ -length at least 1. Then  $\text{vol}(\square, g) \geq 1$ .

The following statement is assumed to be known.

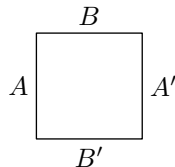
**1.13. Coarea inequality.** *Let  $f: \mathcal{M} \rightarrow \mathcal{N}$  be a locally Lipschitz map between  $n$ -dimensional Riemannian manifolds. Suppose that  $|\text{jac}_p f| \leq 1$  almost everywhere in  $A$ , then*

$$\text{vol } A \geq \text{vol}[f(A)].$$

*Proof.* We will consider the case  $n = 2$ ; the other cases are proved the same way.

Denote by  $A$ ,  $A'$ , and  $B$ ,  $B'$  the opposite facets of the square  $\square$ . Consider two functions

$$\begin{aligned} f_A(x) &:= \min\{\text{dist}_A(x)_g, 1\}, \\ f_B(x) &:= \min\{\text{dist}_B(x)_g, 1\}. \end{aligned}$$



Let  $f: \square \rightarrow \square$  be the map with coordinate functions  $f_A$  and  $f_B$ ; that is,  $f(x) := (f_A(x), f_B(x))$ .

Observe that  $f$  maps each face to itself. Indeed,

$$x \in A \implies \text{dist}_A(x)_g = 0 \implies f_A(x) = 0 \implies f(x) \in A.$$

Similarly, if  $x \in B$ , then  $f(x) \in B$ . Further,

$$x \in A' \implies \text{dist}_A(x)_g \geq 1 \implies f_A(x) = 1 \implies f(x) \in A'.$$

Similarly, if  $x \in B'$ , then  $f(x) \in B'$ .

Therefore

$$f_t(x) = t \cdot x + (1 - t) \cdot f(x)$$

defines a homotopy of maps of the pair of spaces  $(\square, \partial\square)$  from  $f$  to the identity map. It follows that degree of  $f$  is 1; that is,  $f$  sends the fundamental class of  $(\square, \partial\square)$  to itself. In particular  $f$  is onto.

Suppose that Jacobian matrix  $\text{Jac}_p f$  of  $f$  is defined at  $p \in \square$ . Choose an orthonormal frame in  $T_p$  with respect to  $g$  and the standard frame in the target  $\square$ . Observe that the differentials  $d_p f_A$  and  $d_p f_B$  written in these frames are the rows of  $\text{Jac}_p f$ . Evidently  $|d_p f_A| \leq 1$  and  $|d_p f_B| \leq 1$ . Since the determinant of a matrix is the volume of the parallelepiped spanned on its rows, we get

$$|\text{jac}_p f| \leq |d_p f_A| \cdot |d_p f_B| \leq 1.$$

Since  $f: \square \rightarrow \square$  is a Lipschitz onto map, the area inequality (1.13) implies that

$$\text{vol}(\square, g) \geq \text{vol } \square = 1.$$

□

The following theorem can be proved along the same lines.

**1.14. Theorem.** *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold. Suppose that there is a degree 1 map from its boundary  $\partial M$  to the surface of  $n$ -dimensional cube  $\square$ ; denote by  $d_1, \dots, d_n$  the distances between the inverse images of pairs of opposite facets of  $\square$  in  $\partial M$ . Then*

$$\text{vol}(M, g) \geq d_1 \cdots d_n.$$

**1.15. Exercise.** *Suppose  $g$  is a metric tensor on a regular hexagon  $\bigcirc$  such that  $g$ -distances between the opposite sides are at least 1. Is there a positive lower bound on  $\text{area}(\bigcirc, g)$ ?*

**1.16. Exercise.** *Let  $V$  be a compact set in the  $n$ -dimensional Euclidean space  $\mathbb{E}^n$  bounded by a hypersurface  $\Sigma$ . Suppose  $g$  is a Riemannian metric on  $V$  such that*

$$|p - q|_g \geq |p - q|_{\mathbb{E}^n}$$

*for any two points  $p, q \in \Sigma$ . Show that*

$$\text{vol}(V, g) \geq \text{vol}(V)_{\mathbb{E}^n}.$$

## H Equality case

**1.17. Theorem.** *Suppose that equality holds in 1.14, then  $\text{vol}(M, g)$  is isometric to the product  $[0, d_1] \times \cdots \times [0, d_n]$ .*

*Proof.* We will prove the 2-dimensional case, assuming that  $d_1 = d_2 = 1$ ; the general case can be proved along the same lines. Let us use the same notation as in the proof of 1.12.

Consider the map  $s: x \mapsto (\text{dist}_A(x)_g, \text{dist}_B(x)_g)$ . From the proof of 1.12 we get that  $\text{Im } s \supset \square$ . Observe that in the case of equality we have that  $\text{Im } s = \square$ . Indeed, the same argument shows that

$$\text{vol}(s^{-1}(\square), g) \geq \text{vol } \square = 1.$$

The set  $s^{-1}(\mathbb{R}^2 \setminus \square)$  is an open subset of  $\square$ . If it is nonempty, then it has positive volume. In this case

$$\text{vol}(\square, g) > \text{vol}(s^{-1}(\square), g) \geq 1$$

— a contradiction.

Summarizing: there is a geodesic path of  $g$ -length 1 connecting any point on one face of the cube to a point on the opposite face.

Moreover, for any pair of opposite facets and a point  $p \in \square$ , there is a unique geodesic path of  $g$ -length 1 from one face to the other that passes thru  $p$ . The latter can be shown by cutting  $\square$  into two rectangles by a level set of  $\text{dist}_A$  thru  $p$ , applying the above statement to both rectangles and taking the concatenation of the obtained geodesic paths with end at  $p$ . If such a path is not unique, then one could make a shortcut near  $p$  — a contradiction.

Let  $\gamma$  be such a geodesic path from  $A$  to  $A'$ . By 1.10,  $\gamma'(t) = \nabla_{\gamma(t)} \text{dist}_A$ . Therefore  $\text{dist}_A$  is differentiable at every point  $p \in \square$ . It follows that the map  $s$  is differentiable.

Further, checking the equality case in each inequality in the proof of 1.12, we get that  $s$  is a bijection and the equalities

$$|d_p \text{dist}_A| = 1, \quad |d_p \text{dist}_B| = 1, \quad \text{and} \quad \langle d_p \text{dist}_A, d_p \text{dist}_B \rangle = 0$$

hold for almost all  $p \in \square$ . Since  $d_p \text{dist}_A$  and  $d_p \text{dist}_B$  are well defined, we get that the equalities hold everywhere. That is,  $s$  is an isometry.  $\square$

## I Proof assembling

*Proof of 1.2.* Suppose that  $\bar{M}$  and  $M$  have identical scattering data. By 1.4  $\bar{M}$  and  $M$  have identical lens data. Further, by 1.6 (or by Santaló formula 1.7), we have

$$\text{vol } M = \text{vol } \bar{M}.$$

Without loss of generality we may assume that  $M$  lies in a unit cube  $\square$ . Cut from  $\square$  the manifold  $M$  and glue in  $\bar{M}$  by the isometry provided by the definition of scattering data; denote the obtained modified cube by  $\bar{\square}$ . Note that the distances between points on the boundary of  $\bar{\square}$  remain unchanged. The latter follows that distance is the length of a minimizing geodesic between a pair of points and the geodesics in  $\square$  and  $\bar{\square}$  behave exactly the same way and they spend exactly the same time in  $M$  and  $\bar{M}$  respectively.

It follows that in the Besicovitch inequality, an equality holds for  $\bar{\square}$ . By 1.17,  $\bar{\square}$  is isometric to  $\square$ ; whence  $\bar{M}$  is isometric to  $M$ .  $\square$

## J More exercises

Two Riemannian metrics  $g_0$  and  $g_1$  on  $M$  are called *conformally equivalent* if there is a function  $\lambda$  such that  $g_1 = \lambda^2 \cdot g_0$ . In this case the function  $\lambda$  is called *conformal factor*. Note that for any  $g_0$ -unit-speed curve  $\gamma: [a, b] \rightarrow M$  we have

$$\text{length}_{g_1} \gamma = \int_a^b \lambda \circ \gamma(t) \cdot dt$$

**1.18. Exercise.** Let  $g_0$  be the canonical metric on the projective space  $\mathbb{RP}^n$ ; that is,  $(\mathbb{RP}^n, g_0)$  is isometric to the quotient space of the unit sphere  $\mathbb{S}^n$  by central symmetry. Suppose that  $g_1$  is conformally equivalent to  $g_0$ . Denote by  $\ell_0$  and  $\ell_1$  the systoles — the lengths of shortest noncontractible closed curves in  $(\mathbb{RP}^n, g_0)$  and  $(\mathbb{RP}^n, g_1)$  respectively (so  $\ell_0 = \pi$ ). Show that

$$\frac{\text{vol}(\mathbb{RP}^n, g_1)}{\ell_1^n} \geq \frac{\text{vol}(\mathbb{RP}^n, g_0)}{\ell_0^n}.$$

*Hint:* Use that geodesic flow preserves volume of the unit tangent bundle to rewrite the integral of conformal factor over  $(\mathbb{RP}^n, g_0)$  and interpret the result.

**1.19. Definition.** A compact Riemannian manifold  $M$  with nonempty boundary  $\partial M$  is called *simple* if any geodesic in  $M$  is minimizing and its boundary is convex; that is, any shortest path in  $M$  may only have its endpoints on  $\partial M$ .

Note that 1.11 provides a condition on a manifold with boundary that guarantees its simplicity.

**1.20. Exercise.** Let  $(M, g_0)$  be a simple Riemannian manifold. Suppose that a conformally equivalent metric  $g_1 = \lambda^2 \cdot g_0$  on  $M$  induce the same distances on the boundary  $\partial M$ ; that is,

$$|x - y|_{g_1} = |x - y|_{g_0}$$

for any  $x, y \in \partial M$ . Show that  $\lambda \equiv 1$ ; that is,  $g_1 = g_0$ .

*Hint:* Apply 1.11 plus the Santaló formula 1.7 and argue similarly to 1.18.

**1.21. Conjecture.** Let  $(M, g_0)$  be a simple Riemannian manifold and  $g_1$  is another Riemannian metric on  $M$ . Suppose that the metric

induced by  $g_1$  on  $\partial M$  is at least as large as the metric induced by  $g_0$ ; that is,

$$|x - y|_{g_1} \geq |x - y|_{g_0}$$

for any  $x, y \in \partial M$ . Then

$$\text{vol}(M, g_1) \geq \text{vol}(M, g_0).$$

Let  $(M, g)$  be a Riemannian manifold. The Sasaki metric is a natural choice of Riemannian metric  $\hat{g}$  on the total space of the tangent bundle  $\tau: TM \rightarrow M$  defined the following way:

Identify the tangent space  $T_u[TM]$  for any  $u \in T_p M$  with the direct sum of vertical and horizontal subspaces  $T_p M \oplus T_p M$ . The projection of this splitting is defined by the differential  $d\tau: TTM \rightarrow TM$  and we assume that the velocity of a curve in  $TM$  formed by a parallel field along a curve in  $M$  is horizontal. Then  $T_u[TM]$  is equipped with the metric  $\hat{g}$  defined by

$$\hat{g}(X, Y) = g(X^V, Y^V) + g(X^H, Y^H),$$

where  $X^V$  and  $X^H \in T_p M$  denote the vertical and horizontal components of  $X \in T_u[TM]$ .

**1.22. Exercise.** Let  $g$  be the canonical Riemannian metric on the sphere  $S^2$ . Consider the tangent bundle  $TS^2$  equipped with the induced Sasaki metric  $\hat{g}$ . Let  $S_R$  be the hypersurface in  $TS^2$  of vectors with norm  $R$ ; we assume that  $S_R$  is equipped with induced Riemannian metric.

Show that  $\text{vol } S_R \rightarrow \infty$  as  $R \rightarrow \infty$ , but  $\text{diam } S_R$  stays bounded for all  $R$ .

## K Remarks

The fact that not all manifolds are scattering rigid was pointed out by Christopher Croke [4]. More examples constructed by Christopher Croke and Bruce Kleiner [5].

Theorem 1.2 has a number of variations and generalizations. In particular an analog of this theorem holds in the following cases:

- ◊ For regions in 2-dimensional Riemannian manifolds with unique geodesic between any two points; proved by Leonid Pestov and Gunther Uhlmann [17].
- ◊ For regions in a round hemispheres; proved by René Michel [13].
- ◊ For regions in hyperbolic spaces; it follows from the result of Gérard Besson, Gilles Courtois, and Sylvestre Gallot [2].

- ◇ For regions in the product space  $\mathbb{R} \times M$ , where  $M$  is a Riemannian manifold with unique geodesic between any two points; proved by Christopher Croke and Bruce Kleiner [6].
- ◇ For small regions in any Riemannian manifold; proved by Dmitri Burago and Sergei Ivanov [3].



# Lecture 2

## Fundamental theorem

This lecture is based on a tiny piece from the book by Mikhael Gromov [9].

### A Formulation

The name *Fundamental theorem of Riemannian geometry* can be used for two results: the theorem on existence and uniqueness of Levi-Civita connection on Riemannian manifold and the following theorem proved by John Nash [16]:

**2.1. Fundamental theorem.** *Any  $n$ -dimensional Riemannian manifold  $(M, g)$  admits a smooth length-preserving embedding into a Euclidean space of sufficiently large dimension  $q$ .*

We will prove this result modulo the so-called Nash–Moser implicit function theorem. We will assume that  $M$  is compact, but it is not all a principle assumption.

The dimension  $q$  can be found explicitly in terms of  $n$ . For example, any  $q = 100 \cdot n^2$  will do, but we will only show that there is  $q$  that depends on  $M$ .

### B Induced metric

Recall that a field  $g$  of bilinear forms on the  $M$  is called *metric tensor*. A metric tensor  $g$  is called *Riemannian* if it is positive definite; that is,  $g(v, v) > 0$  for any  $v \neq 0$ .

Let  $f: M \rightarrow \mathbb{R}^q$  be a smooth map defined on a manifold  $M$ ; here we consider  $\mathbb{R}^q$  with standard Euclidean metric. We say that a metric

tenor  $g$  is induced by  $\mathbf{f}$  if

$$g(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}\mathbf{f}, \mathbf{w}\mathbf{f} \rangle, \quad \text{or, equivalently} \quad g(\mathbf{v}, \mathbf{w}) = \langle (d\mathbf{f})\mathbf{v}, (d\mathbf{f})\mathbf{w} \rangle$$

Note that  $\mathbf{f}: (M, g) \rightarrow \mathbb{R}^q$  is length-preserving if and only if

$$g(\mathbf{v}, \mathbf{v}) = |\mathbf{v}\mathbf{f}|^2$$

for any tangent vector  $\mathbf{v}$ .

Recall that any bilinear form  $g$  completely determined by the corresponding quadratic form; that is, if we know  $g(\mathbf{v}, \mathbf{v})$  for any vector  $\mathbf{v}$ , then we know  $g(\mathbf{v}, \mathbf{w})$  for any pair of vectors  $\mathbf{v}, \mathbf{w}$ . The latter is proved by the following identity:

$$g(\mathbf{v}, \mathbf{w}) = \frac{1}{2} \cdot [g(\mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}) - g(\mathbf{v}, \mathbf{v}) - g(\mathbf{w}, \mathbf{w})].$$

Therefore  $\mathbf{f}: (M, g) \rightarrow \mathbb{R}^q$  is length-preserving if and only if

$$g(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}\mathbf{f}, \mathbf{w}\mathbf{f} \rangle.$$

Assume that  $f_1, \dots, f_q$  are coordinate functions of  $\mathbf{f}$ . Then the latter identity can be written as

$$g = (df_1)^2 + \dots + (df_q)^2,$$

where  $(df_i)^2$  is a shortcut for the metric tenor  $b_i$  defined by

$$b_i(\mathbf{v}, \mathbf{w}) := df_i(\mathbf{v}) \cdot df_i(\mathbf{w}) = (\mathbf{v}f_i) \cdot (\mathbf{w}f_i).$$

$$g = (df_1)^2 + \dots + (df_q)^2.$$

Let us show that the fundamental theorem can be reduced to the following statement (as always, in the compact case).

**2.2. Reformulation.** *For any Riemannian metric  $g$  on a compact smooth manifold  $M$  there are smooth functions  $f_1, \dots, f_q: M \rightarrow \mathbb{R}$  such that*

❶ 
$$g = (df_1)^2 + \dots + (df_q)^2.$$

*Proof of equivalence in the compact case.* If  $\mathbf{f}: (M, g) \rightarrow \mathbb{R}^q$  is a smooth length-preserving map, then, as it was shown above,  $g$  is induced by  $\mathbf{f}$  and ❶ holds for the coordinate functions  $f_1, \dots, f_q$  of  $\mathbf{f}$ .

Now, assume the reformulation (2.2) is proved. Consider a smooth embedding  $\mathbf{h}: M \rightarrow \mathbb{R}^{2 \cdot n+1}$  provided by the Whitney embedding theorem. Denote by  $g_0$  the Riemannian metric on  $M$  induced by  $\mathbf{h}$ ; that is,

$$g_0 = (dh_1)^2 + \dots + (dh_{2 \cdot n+1})^2,$$

where  $h_1, \dots, h_{2 \cdot n+1}$  are coordinate functions of  $\mathbf{h}$ . Passing to an scaled embedding  $\varepsilon \cdot \mathbf{h}$  for some small  $\varepsilon > 0$ , we can assume that  $g > g_0$ ; that is,  $\bar{g} = g - g_0$  is a Riemannian metric on  $M$ . (Here we used compactness of  $M$ , but *not* in an essential way.)

Applying the reformulation (2.2) to  $(M, \bar{g})$  we get a smooth length-preserving immersion  $\mathbf{f}: (M, \bar{g}) \rightarrow \mathbb{R}^q$ . It remains to observe that the smooth embedding  $M \rightarrow \mathbb{R}^{2 \cdot n+1} \oplus \mathbb{R}^q$  defined by  $x \mapsto (\mathbf{h}(x), \mathbf{f}(x))$  has induced metric tensor  $g = g_0 + \bar{g}$ ; therefore it is length-preserving.  $\square$

## C Nash's twist

The following exercise is a weaker form of 2.2; it will play a key role in this section.

**2.3. Exercise.** *Show that for any Riemannian metric  $g$  on a smooth compact manifold  $M$  there are smooth functions*

$$\varphi_1, \dots, \varphi_q, f_1, \dots, f_q: M \rightarrow \mathbb{R}$$

*such that*

$$g = (\varphi_1)^2 \cdot (df_1)^2 + \dots + (\varphi_q)^2 \cdot (df_q)^2.$$

Let  $\varphi$  and  $f$  be smooth functions on a smooth manifold  $M$ . Given  $\varepsilon > 0$ , denote by  $\mathbb{S}_\varepsilon^1$  the circle of radius  $\varepsilon$  in  $\mathbb{R}^2$ ; consider an length-preserving map  $\ell_\varepsilon: \mathbb{R} \rightarrow \mathbb{S}_\varepsilon^1$ , say

$$\ell_\varepsilon(x) = (\varepsilon \cdot \cos \frac{x}{\varepsilon}, \varepsilon \cdot \sin \frac{x}{\varepsilon}).$$

Then the map  $F: M \rightarrow \mathbb{R}^2$  defined by

$$F(x) = \varphi(x) \cdot (\ell_\varepsilon \circ f(x))$$

is called *Nash's twist* for the triple  $(\varepsilon, \varphi, f)$ .

Suppose that  $v$  is a tangent vector on  $M$ , then

$$\begin{aligned} vF &= v(\varphi \cdot \ell_\varepsilon \circ f) = \\ &= (v\varphi) \cdot (\ell_\varepsilon \circ f) + \varphi \cdot (\ell'_\varepsilon \circ f) \cdot (vf) = \\ &= d\varphi(v) \cdot (\ell_\varepsilon \circ f) + \varphi \cdot df(v) \cdot (\ell'_\varepsilon \circ f). \end{aligned}$$

Observe that  $|\ell_\varepsilon| = \varepsilon$ ,  $|\ell'_\varepsilon| = 1$ , and  $\ell_\varepsilon \perp \ell'_\varepsilon$ .

$$\langle vF, wF \rangle = \varepsilon^2 \cdot d\varphi(v) \cdot d\varphi(w) + \varphi^2 \cdot df(v) \cdot df(w).$$

Whence we get the following:

**2.4. Claim.** *The metric tensor  $\varphi^2 \cdot (df)^2 + \varepsilon^2 \cdot (d\varphi)^2$  is induced by a Nash's twist for  $(\varepsilon, \varphi, f)$ .*

**2.5. Approximation theorem.** *Let  $(M, g)$  be a compact Riemannian manifold. Then there are smooth functions  $\varphi_1, \dots, \varphi_q$  on  $M$  such that for any  $\varepsilon > 0$  the metric tensor*

$$h = (d\varphi_1)^2 + \dots + (d\varphi_q)^2$$

*the following condition holds:*

*For any  $\varepsilon > 0$ , the metric tensor  $g + \varepsilon^2 \cdot h$  is induced by a smooth map  $\mathbf{F}_\varepsilon: M \rightarrow \mathbb{R}^{2 \cdot q}$  to a Euclidean space.*

*Proof.* Let  $\varphi_1, f_1, \dots, \varphi_q, f_q$  be the functions on  $M$  provided by 2.3.

Choose  $\varepsilon > 0$ . Consider the Nash's twist  $F_i$  for each triple  $(\varepsilon, \varphi_i, f_i)$ . Denote by  $\mathbf{F}_\varepsilon$  the map  $M \rightarrow \mathbb{R}^{2 \cdot q}$  with pairs of coordinate functions as in  $F_1, \dots, F_q$ .

By 2.4, the metric tensor  $g + \varepsilon^2 \cdot h$  is induced by  $\mathbf{F}_\varepsilon$ . □

## D Pseudoeuclidean case

Denote by  $\mathbb{R}^{r,s}$  the pseudoeuclidean space with signature  $(r, s)$ ; that is the space  $\mathbb{R}^{r+s}$  with scalar product defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 \cdot y_1 + \dots + x_s \cdot y_s - x_{s+1} \cdot y_{s+1} - \dots - x_{s+r} \cdot y_{s+r},$$

where  $x_i$  and  $y_i$  denote the coordinates of vectors  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^{r+s}$ .

The induced metric tensor for a map to a pseudoeuclidean space can be defined the same way.

**2.6. Theorem.** *Any metric tensor  $g$  on a compact smooth manifold  $M$  is induced by a smooth map  $\mathbf{f}: M \rightarrow \mathbb{R}^{r,s}$  for some positive integers  $r$  and  $s$ ; in other words,*

$$g = (df_1)^2 + \dots + (df_r)^2 - (df_{r+1})^2 + \dots + (df_{r+s})^2$$

*for some smooth functions  $f_1, \dots, f_{r+s}$  on  $M$ .*

*Proof.* Note that any metric tensor on  $M$  can be expressed as a difference of two Riemannian tensors. Therefore we can assume that  $g$  is Riemannian.

Suppose that  $\mathbf{F}_\varepsilon: M \rightarrow \mathbb{R}^{2 \cdot q}$  and  $\varphi_1, \dots, \varphi_q$  are provided by the approximation theorem (2.5). Consider the map  $\varphi: M \rightarrow \mathbb{R}^q$  with coordinate functions  $\varphi_1, \dots, \varphi_q$ .

Consider the map  $\mathbf{f}: M \rightarrow \mathbb{R}^{2 \cdot q} \oplus \mathbb{R}^q = \mathbb{R}^{2 \cdot q, q}$  defined by  $\mathbf{f}: x \mapsto (\mathbf{F}_\varepsilon(x), \varepsilon \cdot \varphi(x))$ . Its induced metric tensor is  $g = g + \varepsilon \cdot h - \varepsilon \cdot h$ .  $\square$

**2.7. Exercise.** Let  $\mathbf{f}: (M, g) \rightarrow \mathbb{S}^{q-1}$  be a smooth length-preserving embedding. Construct a smooth length-preserving embedding of any conformally equivalent manifold into  $\mathbb{R}^{q,1}$ .

That is, given a smooth positive function  $\varphi$  on  $M$ , construct a smooth map  $\mathbf{F}: M \rightarrow \mathbb{R}^{q,1}$  with induced metric tensor  $\varphi^2 \cdot g$ .

## E Free maps

Let  $\mathbf{f}: M \rightarrow \mathbb{R}^q$  be a smooth map defined on a smooth  $n$ -dimensional manifold  $M$ .

Recall that  $\mathbf{f}$  is called *regular* if  $d\mathbf{f}$  has rank  $n$  at each point. In other words, for any local coordinates  $(x_1, \dots, x_n)$  on  $M$  all first order partial derivatives

$$\frac{\partial}{\partial x_1} \mathbf{f}, \dots, \frac{\partial}{\partial x_n} \mathbf{f}$$

are linearly independent at each point  $p \in M$ .

A map  $\mathbf{f}: M \rightarrow \mathbb{R}^q$  is called *free* if an analogous property holds for first and second partial derivatives; that is, if all  $\frac{n \cdot (n+3)}{2}$  vectors

$$\frac{\partial}{\partial x_1} \mathbf{f}, \dots, \frac{\partial}{\partial x_n} \mathbf{f}, \frac{\partial^2}{\partial x_1^2} \mathbf{f}, \frac{\partial^2}{\partial x_1 \partial x_2} \mathbf{f}, \dots, \frac{\partial^2}{\partial x_n^2} \mathbf{f}$$

are linearly independent at each point  $p \in M$ . Observe that any free map is regular.

**2.8. Exercise.** Show that the definition of free map does not depend on the choice of local coordinates.

**2.9. Exercise.** Consider the  $(x, y)$ -plane  $\mathbb{R}^2$ . Let  $F_x$ ,  $F_y$ , and  $F_{x+y}$  are Nash's twists  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  for the triples  $(1, 1, x)$ ,  $(1, 1, y)$ , and  $(1, 1, x+y)$ . Show that the map  $\mathbb{R}^2 \rightarrow \mathbb{R}^6 = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$  defined by  $p \mapsto (F_x(p), F_y(p), F_{x+y}(p))$  is free.

Generalize the statement to maps  $\mathbb{R}^n \rightarrow \mathbb{R}^{n \cdot (n+1)}$ .

**2.10. Exercise.** Let  $\mathbf{f}: M \hookrightarrow \mathbb{R}^q$  is a regular smooth embedding and  $\mathbf{F}: \mathbb{R}^q \hookrightarrow \mathbb{R}^Q$  is a free smooth embedding. Show that the composition  $\mathbf{F} \circ \mathbf{f}: M \hookrightarrow \mathbb{R}^Q$  is free.

Use 2.9 to conclude that any smooth manifold admits a free embedding into a Euclidean space.

If  $\mathbf{f}: M \rightarrow \mathbb{R}^q$  is a smooth embedding, then the smooth manifold  $M$  with its image  $\mathbf{f}(M)$ . If  $\mathbf{f}$  is free, then we say that  $M$  is a *free submanifold*.

The space spanned by the first and second partial derivatives of  $\mathbf{f}$  at  $p$  will be denoted by  $T_p^2 = T_p^2 M$ . The orthogonal complement of the tangent space  $T_p$  in  $T_p^2$  will be called *binormal space* and denoted by  $BN_p = BN_p M$ ; in other words,

$$BN_p = T_p^2 \cap N_p,$$

where  $N_p$  denotes the normal space to  $M$  at  $p$ .

Recall that second fundamental form  $S$  is a field of symmetric quadratic forms on  $TM$  with values in  $NM$  that defined by

$$S(v, w) = \nabla_v w - \bar{\nabla}_v w,$$

where  $\nabla$  and  $\bar{\nabla}$  denote the Levi-Civita connection on  $M$  and the ambient manifold; in this particular case,  $\bar{\nabla}$  is defined by the parallel translations on the Euclidean space. Observe that the values of  $S$  lie in binormal bundle  $BNM$ .

**2.11. Exercise.** *Suppose that  $M$  is a free submanifold of  $\mathbb{R}^q$ . Show that for any metric tensor  $h$  on  $M$  there is a unique binormal field  $N$  such that*

$$h(v, w) = \langle S(v, w), N \rangle$$

for any vector fields  $v, w$  on  $M$ .

Given  $h$ , consider the one-parameter family of maps  $\mathbf{f}_t: M \rightarrow \mathbb{R}^q$  defined by

$$\mathbf{f}_t(p) = p + t \cdot N(p);$$

let  $g(t)$  be the metric tensor induced by  $\mathbf{f}_t$ . Show that  $g'(0) = h$ .

The exercise says that a free embedding can be perturbed so that the induced metric tensor moves in a given direction  $h$ . Note that freeness of embedding is an *open condition*; namely, if  $M$  is a free compact submanifold then any  $C^2$ -close embedding of  $M$  is free as well. One may think that these two properties *easily* imply the following statement, but that is not at all easy; it is a consequence of a deep result — the so-called Nash–Moser theorem [14]. A simplified proof was obtained by Matthias Günther [10].

**2.12. Perturbation theorem.** *Let  $\mathbf{f}: M \hookrightarrow \mathbb{R}^q$  be a free embedding,  $g$  is the Riemannian metric induced by  $\mathbf{f}$  and  $h$  is another metric tensor on  $M$ . Then for any  $t$  sufficiently close to 0, there is a free embedding of  $\mathbf{f}_t: M \hookrightarrow \mathbb{R}^q$  with induced metric tensor  $g + t \cdot h$ .*

*Proof of 2.2 modulo the perturbation theorem.* Choose a free embedding  $\mathbf{f}: M \hookrightarrow \mathbb{R}^s$ ; it exists by 2.10. Denote by  $g_0$  its induced metric.

Scaling down  $\mathbf{f}$  if necessary, we can assume that  $g > g_0$ ; that is the metric tensor  $\bar{g} = g - g_0$  is Riemannian.

Applying the approximation theorem (2.5) we get a one parameter family of maps  $\mathbf{F}_\varepsilon: M \rightarrow \mathbb{R}^q$  with induced metrics  $\bar{g} + \varepsilon^2 \cdot h$  for a fixed metric tensor  $h$ .

By the perturbation theorem (2.12) there is a one parameter family of embedding  $\mathbf{f}_t: M \rightarrow \mathbb{R}^s$  with induced metric  $g_0 + t \cdot h$ .

Choose sufficiently small  $\varepsilon > 0$  so that  $\mathbf{f}_t$  is defined for  $t = -\varepsilon^2$ . Consider the map  $M \rightarrow \mathbb{R}^s \times \mathbb{R}^q$  defined by  $x \mapsto (\mathbf{f}_t(x), \mathbf{F}_\varepsilon(x))$ . Observe that the induced metric of this map is

$$g_0 + t \cdot h + \bar{g} + \varepsilon^2 \cdot h = g.$$

□

## F Remarks

Let us state another closely related result that shows a huge difference between  $C^1$  and  $C^2$  isometric embeddings. For example, it implies that the unit sphere admits  $C^1$  length-preserving embedding into an arbitrarily small ball in Euclidean 3-space. There is no such  $C^2$ -embedding since Gauss curvature of the unit sphere is 1, but at an extremal point it must be at least  $\frac{1}{r^2}$ , where  $r$  is the radius of the ball.

**2.13. Nash–Kuiper theorem.** *Let  $(M, g)$  be a  $n$ -dimensional Riemannian manifold and  $f: (M, g) \rightarrow \mathbb{R}^q$  be a short smooth regular map. Suppose that  $q \geq n + 1$ . Then for any  $\varepsilon > 0$  there is an  $C^1$ -smooth length-preserving maps  $f_\varepsilon: (M, g) \rightarrow \mathbb{R}^q$  that is  $\varepsilon$  close to  $f$ ; that is,  $|f_\varepsilon(x) - f(x)| < \varepsilon$  for any  $x \in M$ .*

*Moreover if  $f$  is an embedding then we can assume that so is  $f_\varepsilon$ .*

It was originally proved by John Nash [15] with the condition  $q \geq n + 2$  and improved to  $q \geq n + 1$  by Nicolaas Kuiper [12]. The original proof uses Nash’s twist in a different way. Both papers are reader-friendly, it is better to start with the paper of Nash. One may also start with lectures by Allan Yashinski and the author [18] where related results were obtained using an alternative approach.

The discussed result formed a part of foundations of the so-called *homotopy principle*, or *h-principle*; an excellent introduction is given in the book by Yakov Eliashberg and Nikolai Mishachev [7].

Many related questions are open. For example, it is unknown if a neighborhood of any point in 2-dimensional Riemannian manifold admits a smooth length-preserving embedding into  $\mathbb{R}^3$ .





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