HWA-11, Exercises: 1.9, 1.11, 1.15, 1.20, 2.3.
HWA-12, Exercises: 2.7, 2.8, 2.9, 2.10, 2.11.
HWA-13, Exercises: 3.1, 3.2, 3.4, 3.5, 3.6.

Lecture 1

Magic cloaks

Based on a lecture of Sergei Ivanov [13].

A Scattering data

Given a Riemannian manifold M denote by $\tau\colon \mathrm{S}M \to M$ its unit tangent bundle

$$\mathrm{S}M = \{\, \mathrm{v} \in \mathrm{T}\,M \;:\; |\mathrm{v}| = 1 \,\}\,.$$

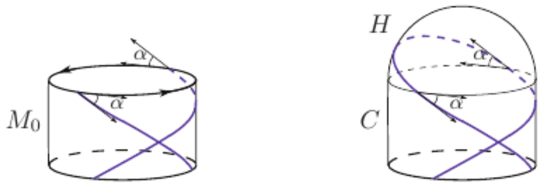
Recall that by *Liouville's theorem*, the geodesic flow φ^t preserves the natural volume on $\mathrm{S}M$ in its domain of definition. Denote by G the vector field on $\mathrm{S}M$ that defines the geodesic flow φ^t.

Suppose M has nonempty boundary ∂M; in other words, M is a closed region of an ambient Riemannian manifold bounded by a smooth hypersurface ∂M. Denote by $\partial_+\mathrm{S}M$ the set of unit vectors at points on ∂M that point in M; $\partial_+\mathrm{S}M$ is a bundle over ∂M with fibers formed by closed half-spheres. The set $\partial_+\mathrm{S}M$ is a subset of $\partial\mathrm{S}M$ that can be also defined as the closure of the subset at which the geodesic flow enters $\mathrm{S}M$.

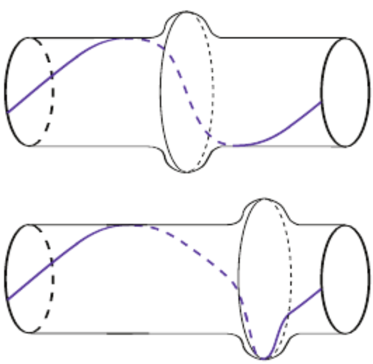
Consider a geodesic γ_{u} in the direction of a vector $\mathrm{u} \in \partial_+\mathrm{S}M$. Suppose that γ_{u} hits the boundary again. In other words, $\gamma_{\mathrm{u}}(t) = \tau\circ\varphi^t(\mathrm{u})$. Denote by $\ell(\mathrm{u})$ the first *hitting time*, so $\gamma_{\mathrm{u}}(\ell(\mathrm{u})) \in \partial M$. Note that in this case the vector $\mathrm{v} = -\gamma'(\ell)$ lies in $\partial_+\mathrm{S}M$. The map $s\colon \mathrm{u} \mapsto \mathrm{v}$ is defined if $\ell(\mathrm{u}) < \infty$; it is a partially defined involution on $\mathrm{S}M$ which we will call *scattering map*.

Suppose that M and $\bar M$ be two compact connected Riemannian manifolds with boundary such that a neighborhood of ∂M can be isometrically identified with a neighborhood of $\partial \bar M$, and moreover, the scattering maps in M and $\bar M$ are identical. In this case we say that M and $\bar{M}$ have identical *scattering data*. If in addition their hitting time functions coincide (that is, if $\ell(\mathrm{U}) \equiv \bar{\ell}(\mathrm{U})$), then we say that M and $\bar{M}$ have the identical *lens data*.

Notice that if a manifold contains a copy of a round hemisphere, then cutting it and cluing the opposite points of its boundary produces a manifold with identical scattering data. The lens data for the



constructed pair of manifolds are not identical. An example of nonisometric manifolds with identical lens data can be found among surfaces of revolution which look like cylinders with bumps on them that are shifted and otherwise look the same.



1.1. Exercise. *Check that the described examples have identical lens data.*

Construct a pair of nonisometric Riemannian metrics on the disc with identical lens data.

B Main theorem

The following theorem proved by Mikhael Gromov [10]; it is the main subject of this lecture.

1.2. Theorem. *Any connected compact region M of Euclidean space of dimension at least 2 cut by a smooth hypersurface is* scattering rigid*; that is, any manifold with scattering data identical to M is isometric to M.*

1.3. Corollary. *Suppose a Riemannian metric g on $\mathbb{R}^n$ coincides with Euclidean metric g_0 outside of a compact set K. Suppose that the complement $\gamma\backslash K$ of any nontrivial g-geodesic γ coincides with the complement of a line (as sets). Then $(\mathbb{R}^n, g)$ is isometric to the Euclidean space.*

Note that in the corollary we cannot claim that $g = g_0$. Indeed for any diffeomorphism $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ that is identical outside K the metric $g = \varphi_* g_0$ satisfies the assumption in the corollary. Clearly one can choose φ so that $g \neq g_0$.

C Identical lens data

1.4. Lemma. *Suppose that a manifold $\bar{M}$ has identical scattering data with a compact region M in Euclidean space of dimension at least 2. Then M and $\bar{M}$ have identical lens data.*

The Euclidean space can be exchanged to any complete Riemannian manifold with unique geodesic between any pair of points; the proof is the same.

Proof. Denote by W the complement of the interior of M in the Euclidean space E. Let us glue $\bar{M}$ to W along the isometry in the definition of scattering data. This way we obtain a complete Riemannian manifold $\bar{E}$ that is Euclidean outside of a $\bar{M}$.

Suppose that a smooth hypersurface Σ in E surrounds M. Then Σ cuts from E and $\bar{E}$ manifolds with identical scattering data. Moreover if $\bar{M}$ and M are not isometric, then the obtained pair is not isometric as well.

It follows that we can assume that M is a ball.[1] In this case any geodesic $\bar{\gamma}$ in $\bar{E}$ visits $\bar{M}$ at most once. In other words, if $\bar{\gamma}$ enters $\bar{M}$, then the complement $\bar{\gamma}\backslash\bar{M}$ has two connected components which are parts of a line γ in E.

Choose a unit-speed geodesic $\bar{\gamma}$ that visits $\bar{M}$. Let us include it in a smooth one-parameter family of unit-speed geodesics $\bar{\gamma}_\tau$ for $\tau \in [0, 1]$ so that $\bar{\gamma}_0$ does not visit $\bar{M}$ and $\bar{\gamma}_1 = \bar{\gamma}$.

[1] This is true for the proof of the lemma and theorem as well.

We can assume that the vector field $\bar{\mathrm{I}} = \frac{\partial}{\partial\tau}\bar\gamma_\tau(t)$ is orthogonal to $\bar{\mathrm{T}} = \frac{\partial}{\partial t}\bar\gamma_\tau(t)$ at every point $\bar\gamma_\tau(t_0)$ for some fixed t_0.

Observe that in this case $\langle\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle = 0$ at all points $\bar\gamma_\tau(t)$. Indeed

$$\begin{aligned}\tfrac{\partial}{\partial t}\langle\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle &= \bar{\mathrm{T}}\langle\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle = \\ &= \langle\nabla_{\bar{\mathrm{T}}}\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle + \cancel{\langle\bar{\mathrm{I}},\nabla_{\bar{\mathrm{T}}}\bar{\mathrm{T}}\rangle} = \\ &= \langle\nabla_{\bar{\mathrm{I}}}\bar{\mathrm{T}},\bar{\mathrm{T}}\rangle = \\ &= \tfrac12\cdot\bar{\mathrm{I}}\langle\bar{\mathrm{T}},\bar{\mathrm{T}}\rangle = \\ &= \tfrac{\partial}{\partial\tau}|\bar\gamma_\tau'(t)|^2 = \\ &= 0.\end{aligned}$$

That is, $\langle\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle$ does not depend on t. Since $\langle\bar{\mathrm{I}},\bar{\mathrm{T}}\rangle = 0$ at all points $\bar\gamma_\tau(t_0)$, the same holds for all points $\bar\gamma_\tau(t)$.

Consider a family of geodesics γ_τ in E that coincide with $\bar\gamma_\tau$ (as sets) outside of M. The same argument shows that $\langle\mathrm{I},\mathrm{T}\rangle = 0$ for the corresponding vector fields $\mathrm{I} = \frac{\partial}{\partial\tau}\gamma_\tau(t)$ and $\mathrm{T} = \frac{\partial}{\partial t}\gamma_\tau(t)$ at all points $\gamma_\tau(t)$.

By assumption, $\bar{\mathrm{T}} = \mathrm{T}$ and $\bar{\mathrm{I}} - \mathrm{I}$ is proprtional to T in W. It follows that $\bar{\mathrm{I}} = \mathrm{I}$ in W. Therefore $\gamma_\tau(t) = \bar\gamma_\tau(t)$ for any t and τ, provided that $\bar\gamma(t)\in W$. Whence the γ spends exactly the same time in M as $\bar\gamma$ spends in $\bar M$ and the lemma follows. □

Comments. The identity $\langle\mathrm{I},\mathrm{T}\rangle = 0$ is proved the same way as the *Gauss lemma*. The vector fields as I in the proof restricted to γ_τ describe a variation of a geodesics. These fields are called *Jacobi fields* along γ_τ; we will see them again.

1.5. Exercise. *Suppose $\bar M$ and $\bar E$ be as in the proof. Show that there is a universal upper bound on time that a unit-speed geodesic spends in $\bar M$.*

Hint: Show that the set of vectors $\mathrm{U}\in \mathrm{S}\bar E$ such that the arc $\gamma_{\mathrm{U}}|_{[0,T]}$ lies in $\bar M$ is open and closed; here $T = 2\cdot\operatorname{diam} M$ and γ_{U} is the geodesic defined by $\gamma_{\mathrm{U}}'(0) = \mathrm{U}$.

D Volume equality

1.6. Lemma. *Suppose that a manifold $\bar M$ has identical scattering data with a compact region M in Euclidean space of dimension at least 2. Then*

$$\operatorname{vol} M = \operatorname{vol}\bar M.$$

Proof. We will denote by $\bar\tau\colon \mathrm{S}\bar M \to \bar M$ the unit tangent bundle over $\bar M$ and by $\bar\varphi^t\colon \mathrm{S}\bar M \to \mathrm{S}\bar M$ its the geodesic flow.

Set $\bar\Omega = \bar\tau^{-1}(\bar M)$. Since geodesic flow preserves the volume, we get

$$\begin{aligned}\operatorname{vol}\mathbb{S}^{n-1}\cdot\operatorname{vol}(\bar M, g) &= \operatorname{vol}\bar\Omega = \\ &= \operatorname{vol}[\bar\varphi^t(\bar\Omega)].\end{aligned}$$

By 1.5, we can choose t so that $\tau(\mathrm{v}) \notin \bar M$ for any $\mathrm{v} \in \varphi^t(\bar\Omega)$.

Repeat the same construction for M. By 1.4 and 1.5, $\varphi^t(\Omega) = \bar\varphi^t(\bar\Omega)$. In particular,

$$\operatorname{vol}\Omega = \operatorname{vol}[\varphi^t(\Omega)] = \operatorname{vol}[\bar\varphi^t(\bar\Omega)] = \operatorname{vol}\bar\Omega.$$

whence the result follows. $\square$

E Santaló formula

Santaló formula is a simple corollary of Liouville's theorem — geodesic flow preserves the volume. It gives an expression for a volume of a Riemannian manifold with boundary in terms of hitting times of its geodesics. It provides a more direct proof of 1.6.

Suppose M is a Reimannian manifold with nonempty boundary ∂M. Recall that

- ◇ $\mathrm{S}M$ denotes the unit tangent bundle over M.
- ◇ φ^t denotes geodesic flow. In particular, if γ_{U} is the geodesic in M defined by $\gamma_{\mathrm{U}}'(0) = \mathrm{u}$, then $\gamma_{\mathrm{U}}'(t) = \varphi^t(\mathrm{u})$.
- ◇ Let $\ell\colon \mathrm{S}M \to [0,\infty]$ denoted the hitting time of γ_{U} in the boundary of M.
- ◇ $\partial_+\mathrm{S}M$ denotes by the bundle of unit vectors at points on ∂M that point in M. It can be defined as the closure of the subset of $\partial\mathrm{S}M$ at which the geodesic flow enters $\mathrm{S}M$.

1.7. Theorem. *Let M be an n-dimensional Riemannian manifold with nonempty boundary. Suppose that any unit-speed geodesic in M hits its boundary in finite time. Then for any smooth function $f\colon \mathrm{S}M \to \mathbb{R}$ the following identity holds:*

$$\int\limits_{\mathrm{w}\in\mathrm{S}M} f(\mathrm{w}) = \int\limits_{\mathrm{u}\in\partial_+\mathrm{S}M} \langle\mathrm{u},\mathrm{n}\rangle\cdot\int\limits_0^{\ell(\mathrm{u})} f\circ\varphi^t(\mathrm{u})\cdot dt,$$

where n denotes the unit vector field normal to ∂M that points inside M.

In particular, by taking $f \equiv 1$, we get

$$\operatorname{vol}\mathbb{S}^{n-1}\cdot\operatorname{vol}M = \int\limits_{\mathrm{u}\in\partial_+\mathrm{S}M} \ell(\mathrm{u})\cdot\langle\mathrm{u},\mathrm{n}\rangle.$$

Proof. Note that any vector $\mathrm{w}\in\mathrm{S}M$ can be uniquely described as $\varphi^t(\mathrm{u})$ for some $\mathrm{u}\in\partial_+\mathrm{S}M$ and $0\leqslant t\leqslant\ell(\mathrm{u})$. In other words $\mathrm{S}M$ can be identified with the subgraph

$$\Phi = \{\,(\mathrm{u},t)\in(\partial_+\mathrm{S}M)\times\mathbb{R} \,:\, 0\leqslant t\leqslant\ell(\mathrm{u})\,\}\,.$$

The subgraph Φ has two volume forms: the first, say ω, is the pull back of the volume form on $\mathrm{S}M$; the the second $\chi = dt\wedge\alpha$, where α is the volume form on $\partial_+\mathrm{S}M$.

Note that both forms are invariant with respect to shifts along $\mathbb{R}$. For ω it is true by Liouville's theorem; for χ it follows from the definition.

Set $r(\mathrm{v}) = \operatorname{dist}_{\partial M}\circ\tau(\mathrm{v})$; note that r is a smooth function near $\partial\mathrm{S}M$. Observe that $dr = \langle\mathrm{u},\mathrm{n}\rangle\cdot dt$ on $\partial\mathrm{S}M$. Extend α to $\mathrm{S}M$ arbitrary. Note that the equality $\omega = dr\wedge\alpha$ holds on $\partial\mathrm{S}M$. Whence

❶ $$\omega = \langle\mathrm{u},\mathrm{n}\rangle\cdot\chi$$

on $\partial\mathrm{S}M$. Since both forms are invariant with respect to vertical shifts, we get that ❶ holds everywhere in Φ. ☐

1.8. Exercise. *Construct two Reimannian metrics g_0 and g_1 on the disc $\mathbb{D}$ that coincide near the boundary and such that*

$$\operatorname{area}(\mathbb{D},g_0) > \operatorname{area}(\mathbb{D},g_1),$$

but

$$\ell_0(\mathrm{u}) < \ell_1(\mathrm{u}),$$

where $\ell_i(\mathrm{u})$ denotes hitting time of g_i-geodesic in the direction u; that is, $\ell_i(\mathrm{u})$ is the length of g_i-geodesic that starts at a point $p\in\partial\mathbb{D}$ in the direction $\mathrm{u}\in\partial_+\mathrm{S}\mathbb{D}$.

Why does this example not contradict the Santaló formula?

1.9. Exercise. *Denote by ω the volume form on $\mathrm{S}M$ and by G the vector field on $\mathrm{S}M$ that describes the geodesic flow. Given a function $f\colon\mathrm{S}M\to\mathbb{R}$, consider the function $F\colon\mathrm{S}M\to\mathbb{R}$ defined by*

$$F(\mathrm{u}) = -\int\limits_0^{\ell(\mathrm{u})} f\circ\varphi^t(\mathrm{u})\cdot dt.$$

Prove the Santaló formula applying Stokes' theorem to form $\iota_{\mathrm{G}}(F\cdot\omega)$.

F Differentiability of distance function

1.10. Theorem. *For any closed set A in a complete Riemannian manifold M and any point $x\notin A$ the differential d_xf of the distance function $f=\operatorname{dist}_A$ is defined if and only if there is a unique minimizing geodesic γ from x to A.*

Moreover, if $\mathrm{u}\in\mathrm{T}_x$ is the unit vector points in the direction of the unique geodesic γ, then

$$d_xf(\mathrm{w})=-\langle\mathrm{u},\mathrm{w}\rangle$$

for any $\mathrm{w}\in\mathrm{T}_x$; or, equivalently,

$$\nabla_xf=-\mathrm{u}.$$

Proof; only-if part. Choose

- ◇ a closed set A, a point $x\notin A$, and $\varepsilon>0$,
- ◇ a unit-speed minimizing geodesic γ from x to A,
- ◇ a smooth unit-speed curve α that such that $\alpha(0)=x$, and set $\mathrm{w}=\alpha'(0)$.

Observe that

$$\begin{aligned}|\gamma(\tfrac{t}{\varepsilon})-\alpha(t)|_M&=t\cdot\sqrt{\tfrac{1}{\varepsilon^2}-2\cdot\langle\mathrm{u},\mathrm{w}\rangle\cdot\tfrac{1}{\varepsilon}+1}+o(t)=\\&=\tfrac{1}{\varepsilon}\cdot t-\langle\mathrm{u},\mathrm{w}\rangle\cdot t+O(\varepsilon\cdot t).\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, the triangle inequality implies that

$$f\circ\alpha(t)\leqslant|p-x|+t\cdot\langle\mathrm{u},\mathrm{w}\rangle+o(t).$$

In particular,

$$(f\circ\alpha)'(0)=-\langle\mathrm{u},\mathrm{w}\rangle\qquad ❶$$

if the left hand side is defined.

Observe that if d_xf is defined, then $(f\circ\alpha)'(0)=d_xf(\mathrm{w})$. Therefore

$$d_xf(\mathrm{w})\leqslant-\langle\mathrm{u},\mathrm{w}\rangle$$

for any $\mathrm{w}\in\mathrm{T}_x$. Since both sides of the last inequality are linear, we get that the equality

$$d_xf(\mathrm{w})=-\langle\mathrm{u},\mathrm{w}\rangle$$

holds for any $\mathrm{w}\in\mathrm{T}_x$.

Suppose that γ_1 is another minimizing geodesic from x to A; set $\mathrm{u}_1 = \gamma_1'(0)$. If $d_x f$ is defined, then we have

$$-\langle \mathrm{u}, \mathrm{w}\rangle = d_x f(\mathrm{w}) = -\langle \mathrm{u}_1, \mathrm{w}\rangle;$$

that is, $\mathrm{u}_1 = \mathrm{u}$ and therefore $\gamma_1 = \gamma$.

If part. Suppose that γ is a unique geodesic from x to A. Choose α as above. For each t choose a minimizing geodesic γ_t from $\alpha(t)$ to A; Set $\mathrm{u}(t) = \gamma_t'(0)$ and $\mathrm{w}(t) = \alpha'(t)$.

Recall that f and α are Lipschitz. By Rademacher's theorem and ❶, we have that

$$(f \circ \alpha)'(t) \overset{a.e.}{=\!=} -\langle \mathrm{u}(t), \mathrm{w}(t)\rangle;$$

moreover

$$f \circ \alpha(\tau) - f \circ \alpha(0) = -\int\limits_0^\tau \langle \mathrm{u}(t), \mathrm{w}(t)\rangle \cdot t.$$

It remains to show that $\langle \mathrm{u}(t), \mathrm{w}(t)\rangle \to \langle \mathrm{u}, \mathrm{w}\rangle$ as $t \to 0$.

The latter follows if $\mathrm{u}(t) \to \mathrm{u}$ as $t \to 0$. Assume the contrary, then there is a sequence $t_n \to 0$ such that $\mathrm{u}(t_n)$ converges to a unit vector $\mathrm{v} \in \mathrm{T}_x$ that is distinct from u. The minimizing geodesics γ_{t_n} converge to a geodesic from x to A that runs in the direction v. Since $\mathrm{v} \neq \mathrm{u}$, this geodesic is distinct from γ — a contradiction. $\square$

1.11. Exercise. *Suppose that M is a compact Riemannian manifold with convex boundary ∂M; that is, any shortest path in M may only have its endpoints on ∂M. Assume that for any $p \in \partial M$ the function* dist_p *is differentiable on $\partial M \backslash \{p\}$.*

Prove the following statements:

(a) Any geodesic in M is minimizing.
(b) For any $p \in M$ the distance function dist_p *is differentiable in $M \backslash \{p\}$.*
(c) Show that M is homeomorphic to a ball.
(d) The restriction of the distance function to ∂M determines the lens data of M.

G Besicovitch inequality

The following theorem was proved by Abram Besicovitch [1].

1.12. Theorem. *Let g be a metric tensor on a unit n-dimensional cube $\square$. Suppose that the g-distances between the opposite facets of $\square$ are at least 1; that is, any Lipschitz curve that connects opposite faces has g-length at least 1. Then* $\mathrm{vol}(\square, g) \geqslant 1$.

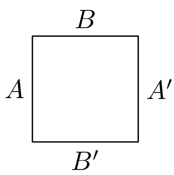
The following statement is assumed to be known.

1.13. Coarea inequality. *Let $f\colon \mathcal{M}\to\mathcal{N}$ be a locally Lipschitz map between n-dimensional Riemannian manifolds. Suppose that $|\mathrm{jac}_p f|\leqslant$ $\leqslant 1$ almost everywhere in A, then*

$$\operatorname{vol} A \geqslant \operatorname{vol}[f(A)].$$

Proof. We will consider the case $n=2$; the other cases are proved the same way.

Denote by A, A', and B, B' the opposite facets of the square $\square$. Consider two functions



$$\begin{aligned} f_A(x) &:= \min\{\operatorname{dist}_A(x)_g, 1\},\\ f_B(x) &:= \min\{\operatorname{dist}_B(x)_g, 1\}. \end{aligned}$$

Let $f\colon \square\to\square$ be the map with coordinate functions f_A and f_B; that is, $f(x) := (f_A(x), f_B(x))$.

Observe that f maps each face to itself. Indeed,

$$x\in A \quad\Longrightarrow\quad \operatorname{dist}_A(x)_g = 0 \quad\Longrightarrow\quad f_A(x)=0 \quad\Longrightarrow\quad f(x)\in A.$$

Similarly, if $x\in B$, then $f(x)\in B$. Further,

$$x\in A' \quad\Longrightarrow\quad \operatorname{dist}_A(x)_g \geqslant 1 \quad\Longrightarrow\quad f_A(x)=1 \quad\Longrightarrow\quad f(x)\in A'.$$

Similarly, if $x\in B'$, then $f(x)\in B'$.

Therefore

$$f_t(x) = t\cdot x + (1-t)\cdot f(x)$$

defines a homotopy of maps of the pair of spaces $(\square,\partial\square)$ from f to the identity map. It follows that degree of f is 1; that is, f sends the fundamental class of $(\square,\partial\square)$ to itself. In particular f is onto.

Suppose that Jacobian matrix $\mathrm{Jac}_p f$ of f is defined at $p\in\square$. Choose an orthonormal frame in T_p with respect to g and the standard frame in the target $\square$. Observe that the differentials $d_p f_A$ and $d_p f_B$ written in these frames are the rows of $\mathrm{Jac}_p f$. Evidently $|d_p f_A|\leqslant 1$ and $|d_p f_B|\leqslant 1$. Since the determinant of a matrix is the volume of the parallelepiped spanned on its rows, we get

$$|\mathrm{jac}_p f| \leqslant |d_p f_A|\cdot|d_p f_B| \leqslant 1.$$

Since $f\colon\square\to\square$ is a Lipschitz onto map, the area inequality (1.13) implies that

$$\operatorname{vol}(\square, g) \geqslant \operatorname{vol}\square = 1.$$

$\square$

The following theorem can be proved along the same lines.

1.14. Theorem. *Let (M,g) be an n-dimensional Riemannian manifold. Suppose that there is a degree 1 map from its boundary ∂M to the surface of n-dimensional cube $\square$; denote by $d_1,\dots,d_n$ the distances between the inverse images of pairs of opposite facets of $\square$ in ∂M. Then*

$$\operatorname{vol}(M,g) \geqslant d_1 \cdots d_n.$$

1.15. Exercise. *Suppose g is a metric tensor on a regular hexagon $\hexagon$ such that g-distances between the opposite sides are at least* 1. *Is there a positive lower bound on* $\operatorname{area}(\hexagon, g)$*?*

1.16. Exercise. *Let V be a compact set in the n-dimensional Euclidean space $\mathbb{E}^n$ bounded by a hypersurface Σ. Suppose g is a Riemannian metric on V such that*

$$|p-q|_g \geqslant |p-q|_{\mathbb{E}^n}$$

for any two points $p,q\in\Sigma$. Show that

$$\operatorname{vol}(V,g) \geqslant \operatorname{vol}(V)_{\mathbb{E}^n}.$$

H Equality case

1.17. Theorem. *Suppose that equality holds in 1.14, then* $\operatorname{vol}(M,g)$ *is isometric to the product* $[0,d_1]\times\cdots\times[0,d_n]$.

Proof. We will prove the 2-dimensional case, assuming that $d_1 = d_2 = 1$; the general case can be proved along the same lines. Let us use the same notation as in the proof of 1.12.

Consider the map $s\colon x\mapsto(\operatorname{dist}_A(x)_g, \operatorname{dist}_B(x)_g)$. From the proof of 1.12 we get that $\operatorname{Im} s \supset \square$. Observe that in the case of equality we have that $\operatorname{Im} s = \square$. Indeed, the same argument shows that

$$\operatorname{vol}(s^{-1}(\square), g) \geqslant \operatorname{vol}\square = 1.$$

The set $s^{-1}(\mathbb{R}^2\backslash\square)$ is an open subset of $\square$. If it is nonempty, then it has positive volume. In this case

$$\operatorname{vol}(\square, g) > \operatorname{vol}(s^{-1}(\square), g) \geqslant 1$$

— a contradiction.

Summarizing: there is a geodesic path of g-length 1 connecting any point on one face of the cube to a point on the opposite face.

Moreover, for any pair of opposite facets and a point $p \in \square$, there is a unique geodesic path of g-length 1 from one face to the other that passes thru p. The latter can be shown by cutting $\square$ into two rectangles by a level set of dist_A thru p, applying the above statement to both rectangles and taking the concatenation of the obtained geodesic paths with end at p. If such a path is not unique, then one could make a shortcut near p — a contradiction.

Let γ be such a geodesic path from A to A'. By 1.10, $\gamma'(t) = \nabla_{\gamma(t)}\mathrm{dist}_A$. Therefore dist_A is differentiable at every point $p \in \square$. It follows that the map s is differentiable.

Further, checking the equality case in each inequality in the proof of 1.12, we get that s is a bijection and the equalities

$$|d_p\mathrm{dist}_A| = 1, \quad |d_p\mathrm{dist}_B| = 1, \quad \text{and} \quad \langle d_p\mathrm{dist}_A, d_p\mathrm{dist}_B\rangle = 0$$

hold for almost all $p \in \square$. Since $d_p\mathrm{dist}_A$ and $d_p\mathrm{dist}_B$ are well defined, we get that the equalities hold everywhere. That is, s is an isometry. □

I Proof assembling

Proof of 1.2. Suppose that $\bar M$ and M have identical scattering data. By 1.4 $\bar M$ and M have identical lens data. Further, by 1.6 (or by Santaló formula 1.7), we have

$$\mathrm{vol}\, M = \mathrm{vol}\, \bar M.$$

Without loss of generality we may assume that M lies in a unit cube $\square$. Cut from $\square$ the manifold M and glue in $\bar M$ by the isometry provided by the definition of scattering data; denote the obtained modified cube by $\bar\square$. Note that the distances between points on the boundary of $\bar\square$ remain unchanged. The latter follows that distance is the length of a minimizing geodesic between a pair of points and the geodesics in $\square$ and $\bar\square$ behave exactly the same way and they spend exactly the same time in M and $\bar M$ respectively.

It follows that in the Besicovitch inequality, an equality holds for $\bar\square$. By 1.17, $\bar\square$ is isometric to $\square$; whence $\bar M$ is isometric to M. □

J More exercises

Two Riemannian metrics g_0 and g_1 on M are called *conformally equivalent* if there is a function λ such that $g_1 = \lambda^2 \cdot g_0$. In this case the function λ is called *conformal factor*. Note that for any g_0-unit-speed curve $\gamma\colon [a,b] \to M$ we have

$$\operatorname{length}_{g_1} \gamma = \int_a^b \lambda \circ \gamma(t) \cdot dt$$

1.18. Exercise. *Let g_0 be the canonical metric on the projective space $\mathbb{R}\mathrm{P}^n$; that is, $(\mathbb{R}\mathrm{P}^n, g_0)$ is isometric to the quotient space of the unit sphere $\mathbb{S}^n$ by central symmetry. Suppose that g_1 is conformally equivalent to g_0. Denote by ℓ_0 and ℓ_1 the* systoles *— the lengths of shortest noncontractible closed curves in $(\mathbb{R}\mathrm{P}^n, g_0)$ and $(\mathbb{R}\mathrm{P}^n, g_1)$ respectively (so $\ell_0 = \pi$). Show that*

$$\frac{\operatorname{vol}(\mathbb{R}\mathrm{P}^n, g_1)}{\ell_1^n} \geqslant \frac{\operatorname{vol}(\mathbb{R}\mathrm{P}^n, g_0)}{\ell_0^n}.$$

Hint: Use that geodesic flow preserves volume of the unit tangent bundle to rewrite the integral of conformal factor over $(\mathbb{R}\mathrm{P}^n, g_0)$ and interpret the result.

1.19. Definition. *A compact Riemannian manifold M with nonempty boundary ∂M is called* simple *if any geodesic in M is minimizing and its boundary is convex; that is, any shortest path in M may only have its endpoints on ∂M.*

Note that 1.11 provides a condition on a manifold with boundary that guarantees its simplicity.

1.20. Exercise. *Let (M, g_0) be a simple Riemannian manifold. Suppose that a conformally equivalent metric $g_1 = \lambda^2 \cdot g_0$ on M induce the same distances on the boundary ∂M; that is,*

$$|x - y|_{g_1} = |x - y|_{g_0}$$

for any $x, y \in \partial M$. Show that $\lambda \equiv 1$; that is, $g_1 = g_0$.

Hint: Apply 1.11 plus the Santaló formula 1.7 and argue similarly to 1.18.

1.21. Conjecture. *Let (M, g_0) be a simple Riemannian manifold and g_1 is another Reimannian metric on M. Suppose that the metric* *induced by g_1 on ∂M is at least as large as the metric induced by g_0; that is,*

$$|x-y|_{g_1} \geqslant |x-y|_{g_0}$$

for any $x, y \in \partial M$. Then

$$\operatorname{vol}(M, g_1) \geqslant \operatorname{vol}(M, g_0).$$

Let (M, g) be a Riemannian manifold. The Sasaki metric is a natural choice of Riemannian metric $\hat{g}$ on the total space of the tangent bundle $\tau\colon \mathrm{T}\,M \to M$ defined the following way:

Identify the tangent space $\mathrm{T}_u[\mathrm{T}\,M]$ for any $u \in \mathrm{T}_p\,M$ with the direct sum of vertical and horizontal subspaces $\mathrm{T}_p\,M \oplus \mathrm{T}_p\,M$. The projection of this splitting is defined by the differential $d\tau\colon \mathrm{T}\,\mathrm{T}\,M \to \mathrm{T}\,M$ and we assume that the velocity of a curve in $\mathrm{T}\,M$ formed by a parallel field along a curve in M is horizontal. Then $\mathrm{T}_u[\mathrm{T}\,M]$ is equipped with the metric $\hat{g}$ defined by

$$\hat{g}(X, Y) = g(X^V, Y^V) + g(X^H, Y^H),$$

where X^V and $X^H \in \mathrm{T}_p\,M$ denote the vertical and horizontal components of $X \in \mathrm{T}_u[\mathrm{T}\,M]$.

1.22. Exercise. *Let g be the canonical Riemannian metric on the sphere $\mathbb{S}^2$. Consider the tangent bundle $\mathrm{T}\,\mathbb{S}^2$ equipped with the induced Sasaki metric $\hat{g}$. Let S_R be the hypersurface in $\mathrm{T}\,\mathbb{S}^2$ of vectors with norm R; we assume that S_R is equipped with induced Riemannian metric.*

Show that $\operatorname{vol} S_R \to \infty$ *as $R \to \infty$, but* $\operatorname{diam} S_R$ *stays bounded for all R.*

K Remarks

The fact that not all manifolds are scattering rigid was pointed out by Christopher Croke [5]. More examples constructed by Christopher Croke and Bruce Kleiner [6].

Theorem 1.2 has a number of variations and generalizations. In particular an analog of this theorem holds in the following cases:

- ◇ For regions in 2-dimensional Riemannian manifolds with unique geodesic between any two points; proved by Leonid Pestov and Gunther Uhlmann [19].
- ◇ For regions in a round hemispheres; proved by René Michel [15].
- ◇ For regions in hyperbolic spaces; it follows from the result of Gérard Besson, Gilles Courtois, and Sylvestre Gallot [2].

- ◇ For regions in the product space $\mathbb{R} \times M$, where M is a Riemannian manifold with unique geodesic between any two points; proved by Christopher Croke and Bruce Kleiner [7].
- ◇ For small regions in any Riemannian manifold; proved by Dmitri Burago and Sergei Ivanov [3].

Lecture 2

Fundamental theorem

This lecture is based on a tiny piece from the book by Mikhael Gromov [11].

A Formulation

The name *Fundamental theorem of Riemannian geometry* can be used for two results: the theorem on existence and uniqueness of Levi-Civita connection on Riemannian manifold and the following theorem proved by John Nash [18]:

2.1. Fundamental theorem. *Any n-dimensional Riemannian manifold (M, g) admits a smooth length-preserving embedding into a Euclidean space of sufficiently large dimension q.*

We will prove this result modulo the so-called Nash–Moser implicit function theorem. We will assume that M is compact, but it is not all a principle assumption.

The dimension q can be found explicitly in terms of n. For example, any $q = 100{\cdot}n^2$ will do, but we will only show that there is q that depends on M.

B Induced metric

Recall that a field g of bilinear forms on the M is called *metric tensor.* A metric tenor g is called *Riemannian* if it is positive definite; that is, $g(\mathrm{v}, \mathrm{v}) > 0$ for any $\mathrm{v} \neq 0$.

Let $\boldsymbol{f}\colon M \to \mathbb{R}^q$ be a smooth map defined on a manifold M; here we consider $\mathbb{R}^q$ with standard Euclidean metric. We say that a metric tenor g is induced by $\boldsymbol{f}$ if

$$g(\mathrm{v},\mathrm{w}) = \langle \mathrm{v}\boldsymbol{f}, \mathrm{w}\boldsymbol{f}\rangle, \quad \text{or, equivalently} \quad g(\mathrm{v},\mathrm{w}) = \langle (d\boldsymbol{f})\mathrm{v}, (d\boldsymbol{f})\mathrm{w}\rangle$$

Note that $\boldsymbol{f}\colon (M,g) \to \mathbb{R}^q$ is length-preserving if and only if

$$g(\mathrm{v},\mathrm{v}) = |\mathrm{v}\boldsymbol{f}|^2$$

for any tangent vector v.

Recall that any bilinear form g completely determined by the corresponding quadratic form; that is, if we know $g(\mathrm{v},\mathrm{v})$ for any vector v, then we know $g(\mathrm{v},\mathrm{w})$ for any pair of vectors v,w. The latter is proved by the following identity:

$$g(\mathrm{v},\mathrm{w}) = \tfrac{1}{2}\cdot[g(\mathrm{v}+\mathrm{w},\mathrm{v}+\mathrm{w}) - g(\mathrm{v},\mathrm{v}) - g(\mathrm{w},\mathrm{w})].$$

Therefore $\boldsymbol{f}\colon (M,g) \to \mathbb{R}^q$ is length-preserving if and only if

$$g(\mathrm{v},\mathrm{w}) = \langle \mathrm{v}\boldsymbol{f}, \mathrm{w}\boldsymbol{f}\rangle.$$

Assume that $f_1,\dots,f_q$ are coordinate functions of $\boldsymbol{f}$. Then the latter identity can be written as

$$g = (df_1)^2 + \dots + (df_q)^2,$$

where $(df_i)^2$ is a shortcut for the metric tenor b_i defined by

$$b_i(\mathrm{v},\mathrm{w}) := df_i(\mathrm{v})\cdot df_i(\mathrm{w}) = (\mathrm{v}f_i)\cdot(\mathrm{w}f_i).$$

$$g = (df_1)^2 + \dots + (df_q)^2.$$

Let us show that the fundamental theorem can be reduced to the following statement (as always, in the compact case).

2.2. Reformulation. *For any Riemannian metric g on a compact smooth manifold M there are smooth functions $f_1,\dots,f_q\colon M \to \mathbb{R}$ such that*

❶ $$g = (df_1)^2 + \dots + (df_q)^2.$$

Proof of equivalence in the compact case. If $\boldsymbol{f}\colon (M,g) \to \mathbb{R}^q$ is a smooth length-preserving map, then, as it was shown above, g is induced by $\boldsymbol{f}$ and ❶ holds for the coordinate functions $f_1,\dots,f_q$ of $\boldsymbol{f}$.

Now, assume the reformulation (2.2) is proved. Consider a smooth embedding $\boldsymbol{h}\colon M \to \mathbb{R}^{2\cdot n+1}$ provided by the Whitney embedding theorem. Denote by g_0 the Riemannian metric on M induced by $\boldsymbol{h}$; that is,

$$g_0 = (dh_1)^2 + \dots + (dh_{2\cdot n+1})^2,$$

where $h_1, \dots, h_{2\cdot n+1}$ are coordinate functions of $\boldsymbol{h}$. Passing to an scaled embedding $\varepsilon\cdot\boldsymbol{h}$ for some small $\varepsilon > 0$, we can assume that $g > g_0$; that is, $\bar{g} = g - g_0$ is a Riemannian metric on M. (Here we used compactness of M, but *not* in an essential way.)

Applying the reformulation (2.2) to $(M, \bar{g})$ we get a smooth length-preserving immersion $\boldsymbol{f}\colon (M, \bar{g}) \to \mathbb{R}^q$. It remains to observe that the smooth embedding $M \to \mathbb{R}^{2\cdot n+1} \oplus \mathbb{R}^q$ defined by $x \mapsto (\boldsymbol{h}(x), \boldsymbol{f}(x))$ has induced metric tensor $g = g_0 + \bar{g}$; therefore it is length-preserving. $\square$

C Nash's twist

The following exercise is a weaker form of 2.2; it will play a key role in this section.

2.3. Exercise. *Show that for any Riemannian metric g on a smooth compact manifold M there are smooth functions*

$$\varphi_1, \dots, \varphi_q, f_1, \dots, f_q\colon M \to \mathbb{R}$$

such that

$$g = (\varphi_1)^2\cdot(df_1)^2 + \dots + (\varphi_q)^2\cdot(df_q)^2.$$

Let φ and f be smooth functions on a smooth manifold M. Given $\varepsilon > 0$, denote by $\mathbb{S}^1_\varepsilon$ the circle of radius ε in $\mathbb{R}^2$; consider an length-preserving map $\ell_\varepsilon\colon \mathbb{R} \to \mathbb{S}^1_\varepsilon$, say

$$\ell_\varepsilon(x) = (\varepsilon\cdot \cos \tfrac{x}{\varepsilon}, \varepsilon\cdot \sin \tfrac{x}{\varepsilon}).$$

Then the map $F\colon M \to \mathbb{R}^2$ defined by

$$F(x) = \varphi(x)\cdot(\ell_\varepsilon \circ f(x))$$

is called *Nash's twist* for the triple $(\varepsilon, \varphi, f)$.

Suppose that v is a tangent vector on M, then

$$\begin{aligned}\mathrm{v}F &= \mathrm{v}(\varphi\cdot\ell_\varepsilon \circ f) = \\ &= (\mathrm{v}\varphi)\cdot(\ell_\varepsilon \circ f) + \varphi\cdot(\ell'_\varepsilon \circ f)\cdot(\mathrm{v}f) = \\ &= d\varphi(\mathrm{v})\cdot(\ell_\varepsilon \circ f) + \varphi\cdot df(\mathrm{v})\cdot(\ell'_\varepsilon \circ f).\end{aligned}$$

Observe that $|\ell_\varepsilon| = \varepsilon$, $|\ell'_\varepsilon| = 1$, and $\ell_\varepsilon \perp \ell'_\varepsilon$.

$$\langle \mathrm{v}F, \mathrm{w}F\rangle = \varepsilon^2\cdot d\varphi(\mathrm{v})\cdot d\varphi(\mathrm{w}) + \varphi^2\cdot df(\mathrm{v})\cdot df(\mathrm{w}).$$

Whence we get the following:

2.4. Claim. *The metric tensor $\varphi^2 \cdot (df)^2 + \varepsilon^2 \cdot (d\varphi)^2$ is induced by a Nash's twist for $(\varepsilon, \varphi, f)$.*

2.5. Approximation theorem. *Let (M, g) be a compact Riemannian manifold. Then there are smooth functions $\varphi_1, \dots \varphi_q$ on M such that for any $\varepsilon > 0$ the metric tensor*

$$h = (d\varphi_1)^2 + \dots + (d\varphi_q)^2$$

the following condition holds:

For any $\varepsilon > 0$, the metric tensor $g + \varepsilon^2 \cdot h$ is induced by a smooth map $\boldsymbol{F}_\varepsilon \colon M \to \mathbb{R}^{2 \cdot q}$ to a Euclidean space.

Proof. Let $\varphi_1, f_1, \dots, \varphi_q, f_q$ be the functions on M provided by 2.3.

Choose $\varepsilon > 0$. Consider the Nash's twist F_i for each triple $(\varepsilon, \varphi_i, f_i)$. Denote by $\boldsymbol{F}_\varepsilon$ the map $M \to \mathbb{R}^{2 \cdot q}$ with pairs of coordinate functions as in $F_1, \dots, F_q$.

By 2.4, the metric tensor $g + \varepsilon^2 \cdot h$ is induced by $\boldsymbol{F}_\varepsilon$. $\square$

D Pseudoeuclidean case

Denote by $\mathbb{R}^{r,s}$ the pseudoeuclidean space with signature (r, s); that is the space $\mathbb{R}^{r+s}$ with scalar product defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 \cdot y_1 + \dots + x_s \cdot y_s - x_{s+1} \cdot y_{s+1} - \dots - x_{s+r} \cdot y_{s+r},$$

where x_i and y_i denote the coordinates of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^{r+s}$.

The induced metric tensor for a map to a pseudoeuclidean space can be defined the same way.

2.6. Theorem. *Any metric tensor g on a compact smooth manifold M is induced by a smooth map $\boldsymbol{f} \colon M \to \mathbb{R}^{r,s}$ for some positive integers r and s; in other words,*

$$g = (df_1)^2 + \dots + (df_r)^2 - (df_{r+1})^2 + \dots + (df_{r+s})^2$$

for some smooth functions $f_1, \dots, f_{r+s}$ on M.

Proof. Note that any metric tensor on M can be expressed as a difference of two Riemannian tenors. Therefore we can assume that g is Riemannian.

Suppose that $\boldsymbol{F}_\varepsilon \colon M \to \mathbb{R}^{2 \cdot q}$ and $\varphi_1, \dots, \varphi_q$ are provided by the approximation theorem (2.5). Consider the map $\boldsymbol{\varphi} \colon M \to \mathbb{R}^q$ with coordinate functions $\varphi_1, \dots, \varphi_q$.

Consider the map $\boldsymbol{f}\colon M\to\mathbb{R}^{2\cdot q}\oplus\mathbb{R}^q=\mathbb{R}^{2\cdot q,q}$ defined by $\boldsymbol{f}\colon x\mapsto$ $\mapsto(\boldsymbol{F}_\varepsilon(x),\varepsilon\cdot\boldsymbol{\varphi}(x))$. Its induced metric tensor is $g=g+\varepsilon\cdot h-\varepsilon\cdot h$. $\square$

2.7. Exercise. *Let $\boldsymbol{f}\colon(M,g)\to\mathbb{S}^{q-1}$ be a smooth length-preserving embedding. Construct a smooth length-preserving embedding of any conformally equivalent manifold into $\mathbb{R}^{q,1}$.*

That is, given a smooth positive function φ on M, construct a smooth map $\boldsymbol{F}\colon M\to\mathbb{R}^{q,1}$ with induced metric tensor $\varphi^2\cdot g$.

E Free maps

Let $\boldsymbol{f}\colon M\to\mathbb{R}^q$ be a smooth map defined on a smooth n-dimensional manifold M.

Recall that $\boldsymbol{f}$ is called *regular* if $d\boldsymbol{f}$ has rank n at each point. In other words, for any local coordinates $(x_1,\dots,x_n)$ on M all first order partial derivatives

$$\tfrac{\partial}{\partial x_1}\boldsymbol{f},\dots,\tfrac{\partial}{\partial x_n}\boldsymbol{f}$$

are linearly independent at each point $p\in M$.

A map $\boldsymbol{f}\colon M\to\mathbb{R}^q$ is called *free* if an analogous property holds for first *and second* partial derivatives; that is, if all $\frac{n\cdot(n+3)}{2}$ vectors

$$\tfrac{\partial}{\partial x_1}\boldsymbol{f},\dots,\tfrac{\partial}{\partial x_n}\boldsymbol{f},\tfrac{\partial^2}{\partial x_1^2}\boldsymbol{f},\tfrac{\partial^2}{\partial x_1\partial x_2}\boldsymbol{f},\dots,\tfrac{\partial^2}{\partial x_n^2}\boldsymbol{f}$$

are linearly independent at each point $p\in M$. Observe that any free map is regular.

2.8. Exercise. *Show that the definition of free map does not depend on the choice of local coordinates.*

2.9. Exercise. *Consider the (x,y)-plane $\mathbb{R}^2$. Let F_x, F_y, and F_{x+y} are Nash's twists $\mathbb{R}^2\to\mathbb{R}^2$ for the triples $(1,1,x)$, $(1,1,y)$, and $(1,1,x+$ $+y)$. Show that the map $\mathbb{R}^2\to\mathbb{R}^6=\mathbb{R}^2\oplus\mathbb{R}^2\oplus\mathbb{R}^2$ defined by $p\mapsto$ $\mapsto(F_x(p),F_y(p),F_{x+y}(p))$ is free.*

Generalize the statement to maps $\mathbb{R}^n\to\mathbb{R}^{n\cdot(n+1)}$.

2.10. Exercise. *Let $\boldsymbol{f}\colon M\hookrightarrow\mathbb{R}^q$ is a regular smooth embedding and $\boldsymbol{F}\colon\mathbb{R}^q\hookrightarrow\mathbb{R}^Q$ is a free smooth embedding. Show that the composition $\boldsymbol{F}\circ\boldsymbol{f}\colon M\hookrightarrow\mathbb{R}^Q$ is free.*

Use 2.9 to conclude that any smooth manifold admits a free embedding into a Euclidean space.

If $\boldsymbol{f}\colon M\to\mathbb{R}^q$ is a smooth embedding, then the smooth manifold M with its image $\boldsymbol{f}(M)$. If $\boldsymbol{f}$ is free, than we say that M is a *free submanifold.*

The space spanned by the first and second partial derivatives of $\boldsymbol{f}$ at p will be denoted by $\mathrm{T}^2_p = \mathrm{T}^2_p M$. The ortogonal complement of the tangent space T_p in T^2_p will be called *binormal space* and denoted by $\mathrm{BN}_p = \mathrm{BN}_p M$; in other words,

$$\mathrm{BN}_p = \mathrm{T}^2_p \cap \mathrm{N}_p,$$

where N_p denotes the normal space to M at p.

Recall that second fundamental form S is a field of symmetric quadratic forms on $\mathrm{T}\,M$ with values in $\mathrm{N}\,M$ that defined by

$$S(\mathrm{v}, \mathrm{w}) = \nabla_{\mathrm{v}} \mathrm{w} - \bar{\nabla}_{\mathrm{v}} \mathrm{w},$$

where ∇ and $\bar{\nabla}$ denote the Levi-Civita connection on M and the ambient manifold; in this particular case, $\bar{\nabla}$ is defined by the parallel translations on the Euclidean space. Observe that the values of S lie in binormal bundle $\mathrm{BN}M$.

2.11. Exercise. *Suppose that M is a free submanifold of $\mathbb{R}^q$. Show that for any metric tensor h on M there is a unique binormal field* N *such that*

$$h(\mathrm{v}, \mathrm{w}) = \langle S(\mathrm{v}, \mathrm{w}), \mathrm{N}\rangle$$

for any vector fields v, w on M.

Given h, consider the one-parameter family of maps $\boldsymbol{f}_t \colon M \to \mathbb{R}^q$ defined by

$$\boldsymbol{f}_t(p) = p + t \cdot \mathrm{N}(p);$$

let $g(t)$ be the metric tensor induced by $\boldsymbol{f}_t$. Show that $g'(0) = 2 \cdot h$.

The exercise says that a free embedding can be perturbed so that the induced metric tensor moves in a given direction h. Note that freeness of embedding is an *open condition*; namely, if M is a free compact submanifold then any C^2-close embedding of M is free as well. One may think that these two properties *easily* imply the following statement, but that is not at all easy; it is a consequence of a deep result — the so-called Nash–Moser theorem [16]. A simplified proof was obtained by Matthias Günther [12].

2.12. Perturbation theorem. *Let $\boldsymbol{f} \colon M \hookrightarrow \mathbb{R}^q$ be a free embedding, g is the Riemannian metric induced by $\boldsymbol{f}$ and h is another metric tensor on M. Then for any t sufficiently close to* 0, *there is a free embedding of $\boldsymbol{f}_t \colon M \hookrightarrow \mathbb{R}^q$ with induced metric tensor $g + t \cdot h$.*

Proof of 2.2 modulo the perturbation theorem. Choose a free embedding $\boldsymbol{f} \colon M \hookrightarrow \mathbb{R}^s$; it exists by 2.10. Denote by g_0 its induced metric.

Scaling down $\boldsymbol{f}$ if necessary, we can assume that $g > g_0$; that is the metric tensor $\bar{g} = g - g_0$ is Riemannian.

Applying the approximation theorem (2.5) we get a one parameter family of maps $\boldsymbol{F}_{\boldsymbol{\varepsilon}}\colon M \to \mathbb{R}^q$ with induced metrics $\bar{g} + \varepsilon^2{\cdot}h$ for a fixed metric tensor h.

By the perturbation theorem (2.12) there is a one parameter family of embedding $\boldsymbol{f}_t\colon M \to \mathbb{R}^s$ with induced metric $g_0 + t{\cdot}h$.

Choose sufficiently small $\varepsilon > 0$ so that $\boldsymbol{f}_t$ is defined for $t = -\varepsilon^2$. Consider the map $M \to \mathbb{R}^s \times \mathbb{R}^q$ defined by $x \mapsto (\boldsymbol{f}_t(x), \boldsymbol{F}_\varepsilon(x))$. Observe that the induced metric of this map is

$$g_0 + t{\cdot}h + \bar{g} + \varepsilon^2{\cdot}h = g.$$

□

F Remarks

Let us state another closely related result that shows a huge difference between C^1 and C^2 isometric embeddings. For example, it implies that the unit sphere admits C^1 length-preserving embedding into an arbitrarily small ball in Euclidean 3-space. There is no such C^2-embedding since Gauss curvature of the unit sphere is 1, but at an extremal point it must be at least $\frac{1}{r^2}$, where r is the radius of the ball.

2.13. Nash–Kuiper theorem. *Let (M, g) be a n-dimensional Riemannian manifold and $f\colon (M, g) \to \mathbb{R}^q$ be a short smooth regular map. Suppose that $q \geqslant n + 1$. Then for any $\varepsilon > 0$ there is an C^1-smooth length-preserving maps $f_\varepsilon\colon (M, g) \to \mathbb{R}^q$ that is ε close to f; that is, $|f_\varepsilon(x) - f(x)| < \varepsilon$ for any $x \in M$.*

Moreover if f is an embedding then we can assume that so is f_ε.

It was originally proved by John Nash [17] with the condition $q \geqslant n + 2$ and improved to $q \geqslant n + 1$ by Nicolaas Kuiper [14]. The original proof uses Nash's twist in a different way. Both papers are reader-friendly, it is better to start with the paper of Nash. One may also start with lectures by Allan Yashinski and the author [20] where related results were obtained using an alternative approach.

The discussed result formed a part of foundations of the so-called *homotopy principle*, or *h-principle*; an excellent introduction is given in the book by Yakov Eliashberg and Nikolai Mishachev [9].

Many related questions are open. For example, it is unknown if *a neighborhood of any point in 2-dimensional Riemannian manifold admits a smooth length-preserving embedding into* $\mathbb{R}^3$.

Lecture 3

Algebra of curvature

The curvature of a Riemannian manifold is described by a tensor, not just a number. This is one of the principle differences between differential geometry of surfaces and higher-dimensional differential geometry.

In this lecture we will give an outline of algebra related to curvature tensor. Most of the statements come without proofs, but everything can be proved by straightforward calculations (which are often tedious).

Most proofs can be found in [8, Chapters 4, 6] and [4, Chapter 3]

A Definition

Let X, Y, V, and W be vector field on a Riemannian manifold (M, g). Recall that ∇ denotes the Levi-Civita connection on M.

The *Riemannian curvature tensor* R is defined by[1]

$$\mathrm{R}(\mathrm{X},\mathrm{Y})\mathrm{V} = \nabla_{\mathrm{X}}\nabla_{\mathrm{Y}}\mathrm{V} - \nabla_{\mathrm{Y}}\nabla_{\mathrm{X}}\mathrm{V} - \nabla_{[\mathrm{X},\mathrm{Y}]}\mathrm{V}.$$

It has valence 4 — it takes 3 vectors in a tangent space of the manifold and returns another tangent vector. We do not need to specify covariance/contravariance type of the tensor since the Riemannian metric tensor identifies tangent and cotangent bundles.

The way we defined the curvature tensor, it depends linearly on vector fields. One has to show that R is a tensor — that is, to show that the vector $\mathrm{R}(\mathrm{X},\mathrm{Y})\mathrm{V}$ at the given point p depends only on the

[1] Many authors (for example do Carmo) define it with opposite sign. If you see notation Rm then most likely the sign is opposite. This convention fits better with an earlier convention that sphere has positive Gauss curvature.

tangent vectors at the point. The latter follows from the following identities:

$$\begin{aligned} f\cdot \mathrm{R}(\mathrm{X},\mathrm{Y})(\mathrm{V}) &= \mathrm{R}(f\cdot\mathrm{X},\mathrm{Y})(\mathrm{V}) = \\ &= \mathrm{R}(\mathrm{X},f\cdot\mathrm{Y})(\mathrm{V}) = \\ &= \mathrm{R}(\mathrm{X},\mathrm{Y})(f\cdot\mathrm{V}). \end{aligned}$$

for any vector fields X, Y, V and a function f. All of them can be proved by straightforward computations.

B Curvatuere transformation

For given tangent vectors X and Y at a point the linear map

$$\mathrm{R}(\mathrm{X},\mathrm{Y})\colon\ \mathrm{T}\to\mathrm{T}$$

is called *curvature transformation*; some authors prefer to denote it by $\mathrm{R}_{\mathrm{X},\mathrm{Y}}$. It has the following geometric meaning:

Let γ be the contour of small parallelogram spanned by vectors X and Y at a point p. Let us denote by $\iota_\gamma\colon\ \mathrm{T}_p\to\mathrm{T}_p$ the parallel transport along γ. Denote by a the area of parallelogram. Then

$$\iota_\gamma = \mathrm{id} + a\cdot \mathrm{R}(\mathrm{X},\mathrm{Y}) + o(a).$$

where id denotes the identity map on T_p.

C Symmetries

Set

$$\hat{\mathrm{R}}(\mathrm{X},\mathrm{Y},\mathrm{V},\mathrm{W}) := \langle \mathrm{R}(\mathrm{X},\mathrm{Y})\mathrm{V},\mathrm{W}\rangle.$$

Note that $\hat{\mathrm{R}}$ remembers everything about the curvature tenor R (assuming that metric tensor is known). The $\hat{\mathrm{R}}$-form of R is more convenient to describe the symmetries of curvature tensor:

❶
$$\begin{aligned} \hat{\mathrm{R}}(\mathrm{X},\mathrm{Y},\mathrm{V},\mathrm{W}) &= -\hat{\mathrm{R}}(\mathrm{Y},\mathrm{X},\mathrm{V},\mathrm{W}) = -\hat{\mathrm{R}}(\mathrm{X},\mathrm{Y},\mathrm{W},\mathrm{V}), \\ 0 &= \hat{\mathrm{R}}(\mathrm{X},\mathrm{Y},\mathrm{V},\mathrm{W}) + \hat{\mathrm{R}}(\mathrm{Y},\mathrm{V},\mathrm{X},\mathrm{W}) + \hat{\mathrm{R}}(\mathrm{V},\mathrm{X})\mathrm{Y},\mathrm{W}). \end{aligned}$$

The last identity is called the *algebraic Bianchi identity* or *first Bianchi identity* (and it was *not* discovered by Bianchi).

These identities can be proved by straightforward computations. Latter we will show that these symmetries provide a complete list; that is, given a tensor that satisfies these identiries, thare is a Riemannian manifold with such curvature tensor at some point.

The following equality follows from the main symmetries

❷
$$\hat{\mathrm{R}}(\mathrm{X},\mathrm{Y},\mathrm{V},\mathrm{W}) = \hat{\mathrm{R}}(\mathrm{V},\mathrm{W},\mathrm{X},\mathrm{Y}).$$

D Space of curvature tensors

Let us denote by $\mathrm{A}^4\,\mathrm{T}$ the space of all *algebraic* curvature tensors on T; that is, $\mathrm{A}^4\,\mathrm{T}$ all valence 4 tenors with the described symmetries ❶.

Given a Euclidean space E, we denote by $\mathrm{S}^n E$ and $\Lambda^n E$ the space of symmetric and antisymmetric tensors of valence n over E.

Note that ❷ and the first line in ❶ imply that

$$\mathrm{A}^4\,\mathrm{T} \subset \mathrm{S}^2\Lambda^2\,\mathrm{T}.$$

In other words, a curvature tensor can be discribed as a symmetric bilinear form on the space of bivectors $\Lambda^2\,\mathrm{T}$; or as the so-called *curvature operator* — a linear operator $\mathbf{R}\colon \Lambda^2\,\mathrm{T} \to \Lambda^2\,\mathrm{T}$ defined by[2]

$$\langle \mathbf{R}(\mathrm{X}\wedge\mathrm{Y}), \mathrm{V}\wedge\mathrm{W}\rangle = -\langle \mathrm{R}(\mathrm{X},\mathrm{Y})\mathrm{V}, \mathrm{W}\rangle.$$

The symmetry ❷ implies that $\mathbf{R}$ is self-adjoint; that is,

$$\langle \mathbf{R}\,\varphi, \psi\rangle = \langle \varphi, \mathbf{R}\,\psi\rangle$$

for any bivectors $\varphi,\psi \in \Lambda^2\,\mathrm{T}$.

The algebraic Bianchi identity implies that complete antisymmetrization of $\hat{\mathrm{R}}$ vanish. More precisely, if $\alpha\colon \mathrm{S}^2\Lambda^2\,\mathrm{T} \to \Lambda^4\,\mathrm{T}$ denotes the complete anysymmetrization then space of curvature tensors is the kernel of α.[3] The latter can be written as

$$\mathrm{A}^4\,\mathrm{T} = \mathrm{S}^2\Lambda^2\,\mathrm{T} \ominus \Lambda^4\,\mathrm{T} \quad \text{or} \quad \mathrm{A}^4\,\mathrm{T} = \mathrm{S}^2\Lambda^2\,\mathrm{T} \cap (\Lambda^4\,\mathrm{T})^\perp,$$

[2] If $\{\mathrm{E}_i\}$ is an orthonormal basis of T, then the scalar product on $\Lambda^2\,\mathrm{T}$ is defined by stating that it has an orthonormal basis $\{\mathrm{E}_i\wedge\mathrm{E}_j\}_{i>j}$.

Alternatively, the scalar product in $\Lambda^2\,\mathrm{T}$ can be also defined on simple bivectros $\mathrm{X}\wedge\mathrm{Y}$ and $\mathrm{V}\wedge\mathrm{W}$ by stating that

$$\langle \mathrm{X}\wedge\mathrm{Y}, \mathrm{V}\wedge\mathrm{W}\rangle = \langle \mathrm{X},\mathrm{V}\rangle\cdot\langle \mathrm{Y},\mathrm{W}\rangle - \langle \mathrm{X},\mathrm{W}\rangle\cdot\langle \mathrm{Y},\mathrm{V}\rangle$$

and extended linearly to whole $\Lambda^2\,\mathrm{T}$.

[3] As far as I see, the following property is absolutely useless, but it is funny. Given a curvature tenor R consider the tensor

$$\text{Я}(\mathrm{X},\mathrm{V},\mathrm{Y},\mathrm{W}) := \langle \mathrm{R}(\mathrm{X},\mathrm{Y})\mathrm{V},\mathrm{W}\rangle + \mathrm{R}(\mathrm{V},\mathrm{Y})\mathrm{X},\mathrm{W}\rangle.$$

Then the linear transformation $\mathrm{R} \to \text{Я}$ describes an isomorphism

$$\mathrm{S}^2\Lambda^2\,\mathrm{T} \ominus \Lambda^4\,\mathrm{T} \longleftrightarrow \mathrm{S}^2\mathrm{S}^2\,\mathrm{T} \ominus \mathrm{S}^4\,\mathrm{T}.$$

That is, curvature tensor can described as quadratic forms on quadratic forms that lie in the kernal of complete symmetrization $\sigma\colon \mathrm{S}^2\mathrm{S}^2\,\mathrm{T} \to \mathrm{S}^4\,\mathrm{T}$; in other words Я satisfies the following symmetries:

$$\begin{aligned}\text{Я}(\mathrm{X},\mathrm{V},\mathrm{Y},\mathrm{W}) = \text{Я}(\mathrm{V},\mathrm{X},\mathrm{Y},\mathrm{W}) = \text{Я}(\mathrm{X},\mathrm{V},\mathrm{W},\mathrm{Y}) &= \text{Я}(\mathrm{Y},\mathrm{W},\mathrm{X},\mathrm{V}),\\ 0 &= \text{Я}(\mathrm{X},\mathrm{X},\mathrm{X},\mathrm{X}).\end{aligned}$$

where $L^\perp$ stands for the orthogonal complement L in the Euclidean metric on $\mathrm{S}^2\Lambda^2\,\mathrm{T}$ induced from T.

If $n=\dim\mathrm{T}$ then the dimension of the space of curvature tensors over T can be easily calculated:

$$\dim(\mathrm{S}^2\Lambda^2\,\mathrm{T})-\dim(\Lambda^4\,\mathrm{T})=\binom{\binom{n}{2}+1}{2}-\binom{n}{4}=\frac{n^2\cdot(n^2-1)}{12}.$$

3.1. Exercise. *Suppose* $\mathbf{R}_\varphi\colon\Lambda^2\,\mathrm{T}\to\Lambda^2\,\mathrm{T}$ *is an othogonal projection to a 1-dimnsional subspace spanned by a bivector* $\varphi\in\Lambda^2\,\mathrm{T}$*. Show that* $\mathbf{R}$ *is a curvature operator if and only if* φ *is a simple bivector; that is, if* $\varphi=\mathrm{X}\wedge\mathrm{Y}$ *for some* $\mathrm{X},\mathrm{Y}\in\mathrm{T}$*.*

Show that in this case $\mathbf{R}$ *is the curvature operator of a direct product of a surface and a Euclidean space.*

Use the results in Section 3H to show that any algebraic curvature tensor can appear as a curvature tensor of a Riemannian manifold.

E Sectional curvature

Let p be a point in a Riemannian manifold (M,g). Choose a plane σ in the tangent space T_p. Consider a surface Σ in M sweeped by short geodesics from p in the directions on σ. The Gauss curvature of Σ at p is called *sectional curvature* and denoted by $\sec\sigma$; the plane σ is called *sectional direction.*

If the sectional direction σ is spanned by vectors X and Y, then

$$\begin{aligned}\sec\sigma&=\frac{\langle\mathrm{R}(\mathrm{X},\mathrm{Y})\mathrm{Y},\mathrm{X}\rangle}{|\mathrm{X}|^2\cdot|\mathrm{Y}|^2-\langle\mathrm{X},\mathrm{Y}\rangle^2}=\\&=\frac{K(\mathrm{X},\mathrm{Y})}{|\mathrm{X}\wedge\mathrm{Y}|^2};\end{aligned}$$

in the last expression we use shortcut

$$K(\mathrm{X},\mathrm{Y})=\langle\mathrm{R}(\mathrm{X},\mathrm{Y})\mathrm{Y},\mathrm{X}\rangle=-\langle\mathrm{R}(\mathrm{X},\mathrm{Y})\mathrm{X},\mathrm{Y}\rangle=\langle\mathbf{R}(\mathrm{X}\wedge\mathrm{Y}),\mathrm{X}\wedge\mathrm{Y}\rangle;$$

note that K is quadratic in both arguments.

The formula above implies that sectional curvature can be recovered from curvature tensor.

The curvature tensor can be expressed using sectional curvature as well. Indeed, once we know curvature in any sectional direction, we can use the above formula to find $K(\mathrm{X},\mathrm{Y})$ for any tangent vectors

$\mathrm{X}, \mathrm{Y} \in \mathrm{T}_p$. After that one could apply the following formula:

$$\begin{aligned}6\cdot \mathrm{R}(\mathrm{X},\mathrm{Y})\mathrm{V},\mathrm{W}\rangle = [&K(\mathrm{X}+\mathrm{V},\mathrm{Y}+\mathrm{W}) + K(\mathrm{X},\mathrm{W}) + K(\mathrm{Y},\mathrm{V}) - \\ &-K(\mathrm{X}+\mathrm{V},\mathrm{Y}) - K(\mathrm{X}+\mathrm{V},\mathrm{W}) - K(\mathrm{X},\mathrm{Y}+\mathrm{W}) - K(\mathrm{V},\mathrm{Y}+\mathrm{W})] - \\ &- [K(\mathrm{X}+\mathrm{W},\mathrm{Y}+\mathrm{V}) + K(\mathrm{Y},\mathrm{W}) + K(\mathrm{X},\mathrm{V}) \\ &-K(\mathrm{X}+\mathrm{W},\mathrm{Y}) - K(\mathrm{X}+\mathrm{W},\mathrm{V}) - K(\mathrm{X},\mathrm{Y}+\mathrm{V}) - K(\mathrm{W},\mathrm{Y}+\mathrm{V})].\end{aligned}$$

The formula is scary, but it is very similar to recovery of a symmetric bilinear form its quadratic form.

3.2. Exercise. *Let* $\dim \mathrm{T} = 3$. *Suppose that a a curvature tenor* $\mathrm{R} \in \mathrm{A}^4\,\mathrm{T}$ *has positive sectional curvature in all directions. Show that the corresponding curvature operator is positive; that is* $\langle \mathbf{R}\,\varphi, \varphi\rangle > 0$ *is for any* $\varphi \in \Lambda^2\,\mathrm{T}$.

Show that if $\dim \mathrm{T} = 4$, *then analogous statement does not hold.*

F Ricci decomposition

Let E_i be an orthonormal basis at a point p of Riemannian manifold. The so called *Ricci curvature tensor* is a linerar transformation, $\mathrm{Ric}\colon \mathrm{T}_p \to \mathrm{T}_p$ defined as

$$\langle \mathrm{Ric}\,\mathrm{X}, \mathrm{X}\rangle = \sum_i K(\mathrm{X}, \mathrm{E}_i).$$

Furthe the *scalar curvature* Sc is defined by

$$\mathrm{Sc} = \sum_{i,j} K(\mathrm{E}_i, \mathrm{E}_j) = \sum_j \langle \mathrm{Ric}\,\mathrm{E}_j, \mathrm{E}_j\rangle = 2\cdot \operatorname{trace} \mathbf{R}\,.$$

If $|\mathrm{X}| = 1$ then the value $\langle \mathrm{Ric}(\mathrm{X}), \mathrm{X}\rangle$ is called *Ricci curvature* in the direction X. For example n-dimensional unit sphere has sectional curvature 1, Ricci curvature $n-1$ in all directions and scalar curvature $n\cdot(n-1)$.

The action of orthogonal group $O(\mathrm{T})$ can be extended to $\mathrm{A}^4\,\mathrm{T}$. Evidently the kernels of $\mathrm{R} \to \mathrm{Ric}$ and $\mathrm{R} \to \mathrm{Sc}$ and their orthogonal complements are invariant with respect to this action. In other words, Ricci tenor Ric and scalar curvature Sc do not depend on the choice of the orthonormal basis E_i.

It turns out that these are the only subspaces of $\mathrm{A}^4\,\mathrm{T}$ that invariant with respect to $O(\mathrm{T})$. In other words, there is a decomposition

$$\mathrm{A}^4\,\mathrm{T} = U \oplus V \oplus W$$

where U, V, and W are the subspaces of $\mathrm{A}^4\,\mathrm{T}$ described the following way:

- ◇ W is the kernel of $\mathrm{R} \to \mathrm{Ric}$; the orthogonal projection to W of the curvature tensor is called its *Weil tensor*.
- ◇ V is the intersection of the kernel $\mathrm{R} \to \mathrm{Sc}$ and the orthogonal complement of W. The projection to V is completely determined by the *traceless Ricci tenor* $\mathrm{Ric}_0 = \mathrm{Ric} - \frac{1}{n}\cdot\mathrm{Sc}\cdot g$.
- ◇ U is a one-dimensional subset of curvature tenors with constant sectional curvatures; so the curvature operator is proportional to the identical operator.

G Curvature bounds

The following theorem is a classical result in Riemannian geometry with long history.

3.3. Quarter-pinched sphere theorem. *Suppose that (M, g) is a simply connected Riemannian manifold with sectional curvature strictly between 1 and 4 at each point. Then M is diffeomorphic to a sphere.*

It gives an example of the so called *local-to-global theorems*. Its assumption is a local property that is given by a curvature bounds at each point. The conclusion says something about global structure of manifold, in this example it says something about its topological type.

Riemannian geometry has other types of theorems[4], but nothing else makes Riemannian geometer nearly as happy as the local-to-global results.

Typically the local condition is given by restriction on curvature tensor of Riemannian manifold; that is, we specify a subset $\Omega \subset \mathrm{A}^4\,\mathrm{T}$ and assume that curvature tensor of a Riemannian manifold belongs to Ω at each point. Most of the time the set $\Omega \subset \mathrm{A}^4\,\mathrm{T}$ is open or at least has nonempty interior.[5]

It is reasonable to assume that Ω is connected. If $\dim \mathrm{T} = 2$, then $\dim \mathrm{A}^4\,\mathrm{T} = 1$; in this case we do not have much choice — we might consider curvature bounded above, or below, or from both sides. That what we did for surfaces.

Starting from dimension 3, things getting more complicated — it is better to assume that Ω is convex (or at least connected) plus it is invariant with respect to rotations of the tangent space; in other words Ω has to be invariant with respect to action of the orthonormal group $O(\mathrm{T})$ of the space T extended to $\mathrm{A}^4\,\mathrm{T}$.

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[4]for example *rigidity theorems* say that a Riemannian manifold with given property must be isometric to some known manifold (typically a round sphere).

[5]There are exceptions, for example Einstein manifolds defined by an equation on curvature tensor. But Einstein manifolds is on a half way from differential geometry to partial differential equations.

So starting from dimension 3 we have huge family of curvature conditions. The Ricci flow technique deals with few families of curvature bounds in the proofs. Couple of dozens of curvature bounds made it to a formulation a meaningful theorem. The champions seem to be upper, lower, and bilteral bounds on sectional curvature, lower bounds on Ricci curvature, and lower bounds on scalar curvature.

H Submanifolds

Suppose M is a submanifold in a Riemannian manifold $(\bar{M}, g)$. Recall that restricting g to the tnagent bundle over M produce a Riemannian metric on M. Let us denote by $\bar{\nabla}$ and ∇ the Levi-Civita connction on $\bar{M}$ and M respectively.

Recall that second fundamental form S of M is defined by

$$S(\mathrm{X}, \mathrm{Y}) = \nabla_{\mathrm{X}}\mathrm{Y} - \bar{\nabla}_{\mathrm{X}}\mathrm{Y};$$

it takes two tangent vectors on M and spits a normal vector; so is a tensor in $\mathrm{S}^2\,\mathrm{T}\,M \otimes \mathrm{N}\,M$ (here we identify tangent/cotangent normal/conormal bundles as usual).

Again straightforward computations show that that S is indeed a tensor — one has to show that

$$f{\cdot}S(\mathrm{X}, \mathrm{Y}) = S(f{\cdot}\mathrm{X}, \mathrm{Y}) = S(\mathrm{X}, f{\cdot}\mathrm{Y})$$

for any tangent vector fields X, Y, and a smooth function f on M.

The following formula gives a relation between curvature tensors R and $\bar{\mathrm{R}}$ of M and $\bar{M}$ respectively:

$$\begin{aligned}\langle \mathrm{R}(\mathrm{X}, \mathrm{Y})\mathrm{V}, \mathrm{W}\rangle = \langle \bar{\mathrm{R}}(\mathrm{X}, \mathrm{Y}), \mathrm{V}, \mathrm{W}\rangle +& \\ &+ \langle S(\mathrm{X}, \mathrm{W}), S(\mathrm{Y}, \mathrm{V})\rangle - \langle S(\mathrm{X}, \mathrm{V}), S(\mathrm{Y}, \mathrm{W})\rangle.\end{aligned}$$

In particular, using the shortcut $K(\mathrm{X}, \mathrm{Y}) = \langle \mathrm{R}(\mathrm{X}, \mathrm{Y})\mathrm{Y}, \mathrm{X}\rangle$, we can write

$$K(\mathrm{X}, \mathrm{Y}) = \bar{K}(\mathrm{X}, \mathrm{Y}) + \langle S(\mathrm{X}, \mathrm{X}), S(\mathrm{Y}, \mathrm{Y})\rangle - |S(\mathrm{X}, \mathrm{Y})|^2.$$

This is a generalization of the formula for Gauss curvature of surface in the Euclidean space:

$$K = \ell{\cdot}n - m^2,$$

where ℓ, m, and n are components of Hessian matrix at p in an othonormal basis. Indeed, if X, Y is an orthonormal frame at a point in a surface, then $K = K(\mathrm{X}, \mathrm{Y})$ and

$$S(\mathrm{X}, \mathrm{X}) = \ell{\cdot}\nu, \qquad S(\mathrm{X}, \mathrm{Y}) = m{\cdot}\nu, \qquad S(\mathrm{Y}, \mathrm{Y}) = n{\cdot}\nu,$$

where ν is a unit normal vector at p.

3.4. Exercise. *Show that there is a 4-dimensional Riemannian manfifold (M, g) such that no neighborhood of a point p in M admits a smooth length-preserving embedding in $\mathbb{R}^5$.*

Hint: Count dimensions of second fundamental form and curvature tensor at p and apply the formula.

3.5. Exercise. *Let M be a smooth submanifold of a Euclidean space. Assume* $\operatorname{codim} M = 2$. *Show that if sectional curvature of M is positive, then so is its curvature operator.*

An analogues statement for submanifolds of codimension 3 does not hold — try to guess why.

Hint: Show first that $\langle S(\mathrm{X}, \mathrm{X}), S(\mathrm{Y}, \mathrm{Y})\rangle > 0$ for any two nonvanishing tangent vectors X, Y at any point $p \in M$. Furhter, show and use that if $\operatorname{codim} M = 2$, then the normal space admits an orthonormal basic $\mathrm{U}_1, \mathrm{U}_2$ such that both quadratic forms $s_i(\mathrm{X}, \mathrm{Y}) := \langle S(\mathrm{X}, \mathrm{Y}), \mathrm{U}_i\rangle$ are positive definite.

I Submersions

Let $s\colon \bar M \to M$ be a submersion between smooth manifolds. Suppose that manifolds $\bar M$ and M are equipped with Riemannian metrics.

Suppose that $s(\bar p) = p$.

The kernel of the differential $ds\colon \mathrm{T}_{\bar p} \bar M \to \mathrm{T}_p M$ will be called *vertical subspace*; it will be denoted by $\mathrm{V}_{\bar p}$. The orthogonal complement of $\mathrm{V}_{\bar p}$ in $\mathrm{T}_{\bar p}$ will be called *horisontal subspace* at $\bar p$; it will be denoted by $\mathrm{H}_{\bar p}$.

The submersion s is called *Riemannian* if the restriction of ds to H is an isometry.

Note that $\mathrm{T}_{\bar p} = \mathrm{H}_{\bar p} \oplus \mathrm{V}_{\bar p}$. In particular, any tangent vector $\mathrm{X} \in \mathrm{T} \bar M$ can be spitted into its vertical and horizontal part denoted by X^{H} and X^{V}, so

$$\mathrm{X} = \mathrm{X}^{\mathrm{H}} + \mathrm{X}^{\mathrm{V}}.$$

By the definition of Riemannian submersion, for any vector $\mathrm{X} \in \mathrm{T}_p$ there is a unique vector $\bar{\mathrm{X}} \in \mathrm{H}_{\bar p}$ such that $\mathrm{X} = ds(\bar{\mathrm{X}})$.

The curvature of M can be found using the so called *O'Neil formula*:

$$K(\mathrm{X}, \mathrm{Y}) = \bar K(\bar{\mathrm{X}}, \bar{\mathrm{Y}}) + \tfrac{3}{4}\cdot|[\bar{\mathrm{X}}, \bar{\mathrm{Y}}]^{\mathrm{V}}|^2.$$

The notion of Riemannian submersion is a dual to submanifold ($\approx$ lenght-preserving immersion). The tensor A defined by

$$A_{\bar{p}}(\mathrm{X}, \mathrm{Y}) = [\bar{\mathrm{X}}, \bar{\mathrm{Y}}]^{\mathrm{V}}$$

is indeed a tensor in $\Lambda^2\mathrm{H}\otimes\mathrm{V}$ (it is proved the usual way). This tensor plays a role similar to the second fundamental form of a submanifold.

3.6. Exercise. *Let $\bar{M} \to M$ be a Riemannian submersion. Suppose that $\bar{M}$ has positive curvature operator at all points. Show that at any point of M there is a 4-vector η such that*

$$\langle \mathbf{R}\,\varphi, \varphi\rangle + \langle \eta, \varphi \wedge \varphi\rangle \geqslant 0$$

for any 2-vector φ.

Hint: Use Section 3D.

J Lie groups

Suppose G is a Lie group with biinvariant metric. By straightforward comutations, one gets the following identities for left-invariant vector fields on G:

$$\begin{aligned}
\nabla_{\mathrm{X}}\mathrm{Y} &= \tfrac{1}{2}\cdot[\mathrm{X}, \mathrm{Y}]\\
\langle \mathrm{R}(\mathrm{X}, \mathrm{Y})\mathrm{V}, \mathrm{W}\rangle &= \tfrac{1}{4}\cdot(\langle[\mathrm{X}, \mathrm{W}], [\mathrm{Y}, \mathrm{V}]\rangle - \langle[\mathrm{X}, \mathrm{V}], [\mathrm{Y}, \mathrm{W}]\rangle)\\
K(\mathrm{X}, \mathrm{Y}) &= \tfrac{1}{4}\cdot|[\mathrm{X}, \mathrm{Y}]|^2
\end{aligned}$$

Note that the first identity implies that $\nabla_{\mathrm{X}}\mathrm{X} = 0$. Therefore one-parameter subgroups in G are geodesics.

K Cheeger's trick

Suppose a Lie group G acts isometrically on a Riemannian manifold $M = (M, g)$. Suppose that a G admits a bi-invarinat metric (this is alway the case if G is compact).

Consider the diagonal action of G on the product $G \times M$; that is, $a\cdot(b, x) := (a\cdot b, a\cdot x)$ for any $a, b \in G$ and $x \in M$. Note that this action is isometric and free. Therefore the quotient map $(\lambda\cdot G)\times M \to (M, g_\lambda)$ is a Riemannian sumersion for some metric g_λ. This way we obtain a one parameter family g_λ of Riemannian metrics on M. Note that $g_\lambda \to g$ as $\lambda \to \infty$.

This procedure is called *Cheeger's trick* and the obtained family is called *Cheeger's deformation.* (Jeff Cheeger found number of applications of this trick, but it was invented earlier.)

By O'Nail formula, if (M, g) had a nonnegative (respectively positive) sectional curvature, then so does (M, g_λ) for any $\lambda > 0$.

L Berger spheres

Applying the Cheegers to the isometric action of $\mathbb{S}^1$ on $\mathbb{S}^3$ by complex multiplication one gets *Berger spheres* — a family of metrics on $\mathbb{S}^3$ with positive sectional curvature It is an important example in Riemannian geometry. (It could be compared to the Cantor set in analysis.)

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