Homework assignments

Due 2023-03-02: 1.6, 1.7, 1.9, 1.10, 1.14.

Due 2023-03-09: 1.13, 2.3, 2.6, 2.8, 2.9.

Due 2023-03-16: 3.5, 3.7, 3.8, 3.10, 3.14.

Due 2023-03-23: 3.16, 3.17, 3.19, 4.7, 4.8.

Contents

Lecture 1

Alexandrov embedding theorem

This lecture contains selected material from Alexandrov's book [3].

We give a proof of the Cauchy theorem and then modify it to prove the Alexandrov uniqueness theorem. Further, we sketch a proof of the Alexandrov embedding theorem.

A Cauchy theorem

Further, surfaces of convex polyhedrons will be considered with intrinsic metric; it is defined as the length of a shortest path on the surface between points. Shortest paths parametrized by arclength will be called geodesics; this term has a slightly different meaning in Riemannian geometry.

1.1. Theorem. *Let K and K' be two non-degenerate convex polyhedrons in $\mathbb{E}^3$; denote their surfaces by P and P'. Suppose there is an isometry $P \to P'$ that sends each face of K to a face of K'. Then K is congruent to K'.*

Proof. Consider the graph Γ formed by the edges of K; the edges of K' form the same graph.

For an edge e in Γ, denote by α_e and α'_e the corresponding dihedral angles in K and K' respectively. Mark e by plus if $\alpha_e < \alpha'_e$ and by minus if $\alpha_e > \alpha'_e$.

Now remove from Γ everything which was not marked; that is, leave only the edges marked by $(+)$ or $(-)$ and their endpoints.

Note that the theorem follows if Γ is an empty graph; assume the contrary.

The graph Γ is embedded into P, which is homeomorphic to the sphere. In particular, the edges coming from one vertex have a natural cyclic order. Given a vertex v of Γ, count the number of sign changes around v; that is, the number of consequent pairs edges with different signs.

1.2. Local lemma. *For any vertex of* Γ *the number of sign changes is at least* 4.

In other words, at each vertex of Γ, one can choose 4 edges marked by $(+)$, $(-)$, $(+)$, $(-)$ in the same cyclical order. Note that the local lemma contradicts the following.

1.3. Global lemma. *Let* Γ *be a nonempty subgraph of the graph formed by the edges of a convex polyhedron. Then it is impossible to mark all of the edges of* Γ *by* $(+)$ *or* $(-)$ *such that the number of sign changes around each vertex of* Γ *is at least* 4.

It remains to prove these two lemmas. □

B Local lemma

Next lemma is the main ingredient in our proof of the local lemma.

1.4. Arm lemma. *Assume that* $A = [a_0 a_1 \dots a_n]$ *is a convex polygon in* $\mathbb{E}^2$ *and* $A' = [a'_0 a'_1 \dots a'_n]$ *be a polygonal line in* $\mathbb{E}^3$ *such that*

$$|a_i - a_{i+1}| = |a'_i - a'_{i+1}|$$

for any $i \in \{0, \dots, n-1\}$ *and*

$$\measuredangle a_i \leqslant \measuredangle a'_i$$

for each $i \in \{1, \dots, n-1\}$. *Then*

$$|a_0 - a_n| \leqslant |a'_0 - a'_n|$$

and equality holds if and only if A *is congruent to* A'.

One may view the polygonal lines $[a_0 a_1 \dots a_n]$ and $[a'_0 a'_1 \dots a'_n]$ as a robot's arm in two positions. The arm lemma states that when the arm opens, the distance between the shoulder and tips of the fingers increases.

1.5. Exercise. *Show that the arm lemma does not hold if instead of the convexity, one only the local convexity; that is, if you go along the polygonal line* $a_0 a_1 \dots a_n$, *then you only turn left.*

1.6. Exercise. *Suppose $A = [a_1 \dots a_n]$ and $A' = [a'_1 \dots a'_n]$ be non-congruent convex plane polygons with equal corresponding sides. Mark each vertex a_i with plus (minus) if the interior angle of A at a_i is smaller (respectively bigger) than the interior angle of A' at a'_i. Show that there are at least 4 sign changes around A.*

Give an example showing the statement does not hold without assuming convexity.

In the proof, we will use the following exercise which is the triangle inequality angles (or the spherical triangle inequality).

1.7. Exercise. *Let w_1, w_2, w_3 be unit vectors in $\mathbb{E}^3$. Denote by $\theta_{i,j}$ the angle between the vectors v_i and v_j. Show that*

$$\theta_{1,3} \leqslant \theta_{1,2} + \theta_{2,3}$$

and in case of equality, the vectors w_1, w_2, w_3 lie in a plane.

Proof. We will view $\mathbb{E}^2$ as the xy-plane in $\mathbb{E}^3$; so both A and A' lie in $\mathbb{E}^3$. Let a_m be the vertex of A that lies on the maximal distance to the line $(a_0 a_n)$.

Let us shift indexes of a_i and a'_i down by m, so that

$$\begin{array}{ccccc} a_{-m} := a_0, & \dots & a_0 := a_m, & \dots & a_k := a_n, \\ a'_{-m} := a'_0, & \dots & a'_0 := a'_m, & \dots & a'_k := a'_n, \end{array}$$

where $k = n - m$. (Here the symbol ":=" means an assignment as in programming.)

Without loss of generality, we may assume that

- ◇ $a_0 = a'_0$ and they both coincide with the origin $(0,0,0) \in \mathbb{E}^3$;
- ◇ all a_i lie in the xy-plane and the x-axis is parallel to the line $(a_{-m} a_k)$;
- ◇ the angle $\measuredangle a'_0$ lies in xy-plane and contains the angle $\measuredangle a_0$ inside and the directions to a'_{-1},a_{-1}, a_1 and a'_1 from a_0 appear in the same cyclic order.

Denote by x_i and x'_i the projections of a_i and a'_i to the x-axis. We can assume in addition that $x_k \geqslant x_{-m}$. In this case,

$$|a_k - a_{-m}| = x_k - x_{-m}.$$

Since the projection is a distance non-expanding, we also have

$$|a'_k - a'_{-m}| \geqslant x'_k - x'_{-m}.$$

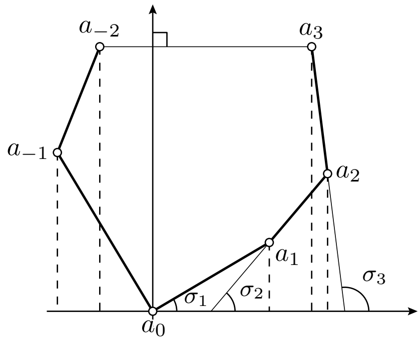
Therefore it is sufficient to show that

$$x'_k - x'_{-m} \geqslant x_k - x_{-m}.$$

The latter holds if

❶ $x'_i - x'_{i-1} \geqslant x_i - x_{i-1}.$

for each i. It remains to prove ❶.



Let us assume that $i > 0$; the case $i \leqslant 0$ is similar. Denote by σ_i (σ'_i) the angle between the vector $w_i = a_i - a_{i-1}$ (respectively $w'_i = a'_i - a'_{i-1}$) and the x-axis. Note that

❷
$$\begin{aligned} x_i - x_{i-1} &= |a_i - a_{i-1}| \cdot \cos\sigma_i, \\ x'_i - x'_{i-1} &= |a_i - a_{i-1}| \cdot \cos\sigma'_i \end{aligned}$$

for each $i > 0$. By construction $\sigma_1 \geqslant \sigma'_1$. Note that $\measuredangle(w_{i-1}, w_i) = \pi - \measuredangle a_i$. From convexity of $[a_1 a_1 \ldots a_i]$, we have

$$\sigma_i = \sigma_1 + (\pi - \measuredangle a_1) + \cdots + (\pi - \measuredangle a_i)$$

for any $i > 0$. Since $\measuredangle(w'_{i-1}, w'_i) = \pi - \measuredangle a'_i$, applying 1.7 several times, we get

$$\sigma'_i \leqslant \sigma'_1 + (\pi - \measuredangle a'_1) + \cdots + (\pi - \measuredangle a'_i).$$

Since $\measuredangle a'_j \geqslant \measuredangle a_j$ for each j, we get $\sigma'_i \leqslant \sigma_i$, and therefore

$$\cos\sigma'_i \geqslant \cos\sigma_i$$

Applying ❷, we get ❶.

In the case of equality, we have $\sigma_i = \sigma'_i$, which implies $\measuredangle a_i = \measuredangle a'_i$ for each i. This also implies that all a'_i lie in xy-plane. The latter easily follows from the equality case in 1.7. □

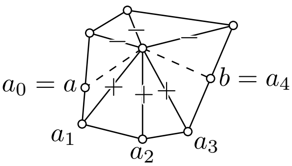
Proof of the local lemma (1.2). Assume that the local lemma does not hold at the vertex v of Γ. Cut from P a small pyramid Δ with the vertex v. One can choose two points a and b on the base of Δ so that on one side of the segments $[va]$ and $[vb]$ we have only pluses and on the other side only minuses.

The base of Δ has two polygonal lines with ends at a and b. Choose the one that has only pluses; denote it by $a_0 a_1 \ldots a_n$; so $a = a_0$ and $b = a_n$. Denote by $a'_0 a'_1 \ldots a'_n$ the corresponding line in P'; let $a' = a'_0$ and $b' = a'_n$.

Since each marked edge passing thru a_i has a $(+)$ on it or nothing, we have

$$\measuredangle a_i \leqslant \measuredangle a_i'$$

for each i.



1.8. Exercise. *Prove the last statement.*

By the construction we have $|a_i - a_{i-1}| = |a_i' - a_{i-1}'|$ for all i. By the arm lemma (1.4), we get

❸ $|a - b| \leqslant |a' - b'|$.

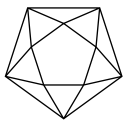
Swap K and K' and repeat the same construction. We get

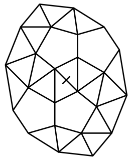
❹ $|a - b| \geqslant |a' - b'|$.

The claims ❸ and ❹ together imply $|a-b| = |a'-b'|$. The equality case in the arm lemma implies that no edge at v is marked; that is, v is not a vertex of Γ — a contradiction. □

From the proof, it follows that the local lemma is indeed local — it works for two noncongruent convex polyhedral angles with equal corresponding faces. Use this observation to solve the following exercise.

1.9. Exercise. *Consider two polyhedral discs in $\mathbb{E}^3$ glued from regular polygons by the rule on the diagrams. Assume that each disc is part of a surface of a convex polyhedron.*





(a) The first configuration is rigid; that is, one can not fix the position of the pentagon and continuously move the remaining 5 vertices in a new position so that each triangle moves by a one-parameter family of isometries of $\mathbb{E}^3$.

(b) Show that the second configuration has a rotational symmetry with the axis passing thru the midpoint of the marked edge.

C Global lemma

The proof of the global lemma is based on counting the sign changes in two ways; first while moving around each vertex of Γ and second while moving around each of the regions separated by Γ on the surface P. If two edges are adjacent at a vertex, then they are also adjacent in a region. The converse is true as well. Therefore, both countings give the same number.

It is instructive to do the next exercise before diving into the proof.

1.10. Exercise. *Try to mark the edges of an octahedron by pluses and minuses such that there would be 4 sign changes at each vertex.*

Show that this is impossible.

Proof of 1.3. We can assume that Γ is connected; that is, one can get from any vertex to any other vertex by walking along edges. (If not, pass to a connected component of Γ.)

Denote by k and l the number of vertices and edges in Γ. Denote by m the number of regions that Γ cuts from P. Since Γ is connected, each region is homeomorphic to an open disc.

1.11. Exercise. *Prove the last statement.*

Now we can apply Euler's formula

❶ $$k - l + m = 2.$$

Denote by s the total number of sign changes in Γ for all vertices. By the local lemma (1.2), we have

❷ $$4{\cdot}k \leqslant s.$$

Let us get an upper bound on s by counting the number of sign changes when you go around each region. Denote by m_n the number of regions bounded by n edges; if an edge appears twice when it is counted twice. Note that each region is bounded by at least 3 edges; therefore

❸ $$m = m_3 + m_4 + m_5 + \dots$$

Counting edges and using the fact that each edge belongs to exactly two regions, we get

$$2{\cdot}l = 3{\cdot}m_3 + 4{\cdot}m_4 + 5{\cdot}m_5 + \dots$$

Combining this with Euler's formula (❶), we get

❹ $$4{\cdot}k = 8 + 2{\cdot}m_3 + 4{\cdot}m_4 + 6{\cdot}m_5 + 8{\cdot}m_6 + \dots$$

Observe that the number of sign changes in n-gon regions has to be even and $\leqslant n$. Therefore

❺ $$s \leqslant 2{\cdot}m_3 + 4{\cdot}m_4 + 4{\cdot}m_5 + 6{\cdot}m_6 + \dots$$

Clearly, ❷ and ❺ contradict ❹. □

D Uniqueness

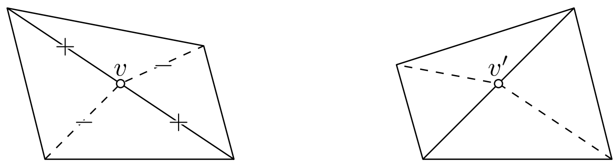
Alexandrov's uniqueness theorem states that the conclusion of the Cauchy theorem (1.1) still holds without the face-to-face assumption.

1.12. Theorem. *Any two convex polyhedrons in* $\mathbb{E}^3$ *with isometric surfaces are congruent.*

Moreover any isometry between surfaces of convex polyhedrons can be extended to an isometry of the whole $\mathbb{E}^3$*.*

Needed modifications in the proof of 1.1. Suppose $\iota\colon P \to P'$ be an isometry between surfaces of K and K'. Mark in P all the edges of K and all the inverse images of edges in K'; further, these will be called fake edges. The marked lines divide P into convex polygons, and the restriction of ι to each polygon is a rigid motion. These polygons play the role of faces in the proof above.

A vertex of the obtained graph can be a vertex of K or it can be a fake vertex; that is, it might be an intersection of an edge and a fake edge.



For the first type of vertex, the local lemma can be proved the same way. For a fake vertex v, it is easy to see that both parts of the edge coming thru v are marked with minus while both of the fake edges at v are marked with plus. Therefore, the local lemma holds for the fake vertices as well.

What remains in the proof needs no modifications. □

1.13. Exercise. *Let* K *be a convex polyhedron in* $\mathbb{E}^3$*; denote by* P *its surface. Show that each isometry* $\iota\colon P \to P$*, can be extended to an isometry of* $\mathbb{E}^3$*.*

E Existence

Let P be a surface with a polyhedral metric. The curvature of a point $p \in P$ is defined as $2\cdot\pi - \theta$, where θ is the total angle around p.

1.14. Exercise. *Suppose* P *is the surface of a convex polyhedron. Show that* P *is homeomorphic to the sphere, and it has nonnegative curvature at every point.*

1.15. Exercise. *Assume that the surface of a nonregular tetrahedron T has curvature π at each of its vertices. Show that*

(a) all faces of T are congruent;

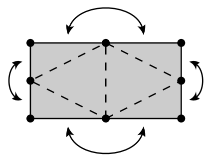
(b) the line passing thru midpoints of opposite edges of T intersects these edges at right angles.

Alexandrov's theorem states that the converse holds if one includes in the consideration twice covered polygons. In other words, we assume that a polyhedron can degenerate to a plane polygon; in this case, its surface is defined as two copies of the polygon glued along the boundaries. Intuitively, one can regard these copies as different sides of the polygon. To get from one side to the other one has to cross the boundary of the polygon.

1.16. Theorem. *A polyhedral metric on the sphere is isometric to the surface of a convex polyhedron (possibly degenerate) if and only if it has nonnegative curvature at each point.*

By 1.12, a convex polyhedron is completely defined by the intrinsic metric of its surface. By 1.16, it follows that knowing the metric we could find the position of the edges. However, in practice, it is not easy to do.

For example, the surface glued from a rectangle as shown on the diagram defines a tetrahedron. Some of the glued lines appear inside facets of the tetrahedron and some edges (dashed lines) do not follow the sides of the rectangle.



Space of polyhedrons. Let us denote by $\mathbf{K}$ the space of all convex polyhedrons in the Euclidean space, including polyhedrons that degenerate to a plane polygon. Polyhedra in $\mathbf{K}$ will be considered up to a motion of the space, and the whole space $\mathbf{K}$ will be considered with the natural topology (so far an intuitive meaning of closeness of two polyhedrons should be sufficient).

Further, denote by $\mathbf{K}_n$ the polyhedrons in $\mathbf{K}$ with exactly n vertices. Since any polyhedron has at least 3 vertices, the space $\mathbf{K}$ admits a subdivision into a countable number of subsets $\mathbf{K}_3, \mathbf{K}_4, \ldots$

Space of polyhedral metrics. The space of polyhedral metrics on the sphere with nonnegative curvature will be denoted by $\mathbf{P}$. The metrics in $\mathbf{P}$ will be considered up to an isometry, and the whole space $\mathbf{P}$ will be equipped with the natural topology (again, an intuitive meaning of closeness of two metrics is sufficient).

A point on the sphere with positive curvature will be called an essential vertex. The subset of $\mathbf{P}$ of all metrics with exactly n essential vertices will be denoted by $\mathbf{P}_n$. It is easy to see that any metric in $\mathbf{P}$ has at least 3 essential vertices. Therefore $\mathbf{P}$ is subdivided into countably many subsets $\mathbf{P}_3, \mathbf{P}_4, \dots$

From a polyhedron to its surface. By 1.14, passing from a polyhedron to its surface defines a map

$$\iota\colon \mathbf{K} \to \mathbf{P}.$$

Note that the number of vertices of a polyhedron is equal to the number of essential vertices on its surface. In other words, $\iota(\mathbf{K}_n) \subset \mathbf{P}_n$ for any $n \geqslant 3$.

Using the introduced notation, we can unite 1.12 and 1.16 in the following statement.

1.17. Reformulation. *For any integer $n \geqslant 3$, the map ι induces a bijection between $\mathbf{K}_n$ and $\mathbf{P}_n$.*

The proof is based on a construction of a one-parameter family of polyhedrons that starts at an arbitrary polyhedron and ends at a polyhedron with its surface isometric to the given one. This type of argument is called the continuity method; it is often used in the theory of differential equations.

Sketch. By 1.12, the map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is injective. Let us prove that it is surjective.

1.18. Lemma. *For any integer $n \geqslant 3$, the space $\mathbf{P}_n$ is connected.*

The proof of this lemma is not complicated, but it requires ingenuity; it can be done by the direct construction of a one-parameter family of metrics in $\mathbf{P}_n$ that connects two given metrics. Such a family can be obtained by a sequential application of the following construction and its inverse.

Let $P \in \mathbf{P}_n$. Suppose v and w are essential vertices in P. Let us cut P along a geodesic from v to w. Note that the geodesic cannot pass thru an essential vertex of P. Further, note that there is a three-parameter family of patches that can be used to patch the cut so that the obtained metric remains in $\mathbf{P}_n$; in particular, the obtained metric has exactly n essential vertices (after the patching, the vertices v and w may become inessential).

1.19. Lemma. *The map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is open, that is, it maps any open set in $\mathbf{K}_n$ to an open set in $\mathbf{P}_n$.*

In particular, for any $n \geqslant 3$, the image $\iota(\mathbf{K}_n)$ is open in $\mathbf{P}_n$.

This statement is very close to the so-called invariance of domain theorem; the latter states that a continuous injective map between manifolds of the same dimension is open.

Recall that ι is injective. The proof of the invariance of domain theorem can be adapted to our case since both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n-6)$-dimensional and both look like manifolds, altho, formally speaking, they are *not* manifolds. In a more technical language, $\mathbf{K}_n$ and $\mathbf{P}_n$ have the natural structure of $(3{\cdot}n-6)$-dimensional orbifolds, and the map ι respects the orbifold structure.

We will only show that both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n-6)$-dimensional.

Choose $K\in\mathbf{K}_n$. Note that K is uniquely determined by the $3{\cdot}n$ coordinates of its n vertices. We can assume that the first vertex is the origin, the second has two vanishing coordinates and the third has one vanishing coordinate; therefore, all polyhedrons in $\mathbf{K}_n$ that lie sufficiently close to K can be described by $3{\cdot}n-6$ parameters. If K has no symmetries, then this description can be made one-to-one; in this case, a neighborhood of K in $\mathbf{K}_n$ is a $(3{\cdot}n-6)$-dimensional manifold. If K has a nontrivial symmetry group, then this description is not one-to-one but it does not have an impact on the dimension of $\mathbf{K}_n$.

The case of polyhedral metrics is analogous. We need to construct a subdivision of the sphere into plane triangles using only essential vertices. By Euler's formula, there are exactly $3{\cdot}n-6$ edges in this subdivision. Note that the lengths of edges completely describe the metric, and slight changes in these lengths produce a metric with the same property. Again, if P has no symmetries, then this description is one-to-one.

1.20. Lemma. *The map $\iota\colon\mathbf{K}_n\to\mathbf{P}_n$ is closed; that is, the image of a closed set in $\mathbf{K}_n$ is closed in $\mathbf{P}_n$.*

In particular, for any $n\geqslant 3$, the set $\iota(\mathbf{K}_n)$ is closed in $\mathbf{P}_n$.

Choose a closed set Z in $\mathbf{K}_n$. Denote by $\bar{Z}$ the closure of Z in $\mathbf{K}$; note that $Z=\mathbf{K}_n\cap\bar{Z}$. Assume $K_1,K_2,\dots\in Z$ is a sequence of polyhedrons that converges to a polyhedron $K_\infty\in\bar{Z}$. Note that $\iota(K_n)$ converges to $\iota(K_\infty)$ in $\mathbf{P}$. In particular, $\iota(\bar{Z})$ is closed in $\mathbf{P}$.

Since $\iota(\mathbf{K}_n)\subset\mathbf{P}_n$ for any $n\geqslant 3$, we have $\iota(Z)=\iota(\bar{Z})\cap\mathbf{P}_n$; that is, $\iota(Z)$ is closed in $\mathbf{P}_n$.

Summarizing, $\iota(\mathbf{K}_n)$ is a nonempty closed and open set in $\mathbf{P}_n$, and $\mathbf{P}_n$ is connected for any $n\geqslant 3$. Therefore, $\iota(\mathbf{K}_n)=\mathbf{P}_n$; that is, $\iota\colon\mathbf{K}_n\to\mathbf{P}_n$ is surjective. $\square$

F Comments

In Euclid's Elements, solids were called equal if the same holds for their faces, but no proof was given. Adrien-Marie Legendre became interested in this problem towards the end of the 18th century. He discussed it with his colleague Joseph-Louis Lagrange, who suggested this problem to Augustin-Louis Cauchy in 1813; soon he proved it [12]. This theorem is included in many popular books [1, 13, 22].

The observation that the face-to-face condition can be removed was made by Alexandr Alexandrov [4].

Arm lemma. Original Cauchy's proof [12] also used a version of the arm lemma, but its proof contained a small mistake (corrected in one century).

Our proof of the arm lemma is due to Stanisław Zaremba. This and a couple of other proofs can be found in the letters between him and Isaac Schoenberg [20].

The following variation of the arm lemma makes sense for nonconvex spherical polygons. It is due to Viktor Zalgaller [26]. It can be used instead of the standard arm lemma.

1.21. Another arm lemma. *Let $A = [a_1 \dots a_n]$ and $A' = [a'_1 \dots \dots a'_n]$ be two spherical n-gons (not necessarily convex). Assume that A lies in a half-sphere, the corresponding sides of A and A' are equal and each angle of A is at least the corresponding angle in A'. Then A is congruent to A'.*

Global lemma. A more visual proof of the global lemma is given in [3, II §1.3].

Existence theorem. This theorem was proved by Alexandr Alexandrov [4]. Our sketch is taken from [15]; a complete proof is nicely written in [3]. A very different proof was found by Yuri Volkov in his thesis [24]; it uses a deformation of three-dimensional polyhedral space.

Lecture 2

CBB: definition

A Distances and geodesics

Distances. The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$. The function $(x,y)\mapsto|x-y|_{\mathcal{X}}$ is called metric; it has to meet the following conditions for any three points $x,y,z\in\mathcal{X}$:

(a) $|x-y|_{\mathcal{X}}\geqslant 0$,

(b) $|x-y|_{\mathcal{X}}=0 \iff x=y$,

(c) $|x-y|_{\mathcal{X}}=|y-x|_{\mathcal{X}}$,

(d) $|x-y|_{\mathcal{X}}+|y-z|_{\mathcal{X}}\geqslant|x-z|_{\mathcal{X}}$.

Geodesics. Let $\mathbb{I}$ be a real interval. A distance-preserving map γ from $\mathbb{I}$ to a metric space $\mathcal{X}$ is called a geodesic[1]; in other words, $\gamma\colon\mathbb{I}\to\mathcal{X}$ is a geodesic if

$$|\gamma(s)-\gamma(t)|_{\mathcal{X}}=|s-t|$$

for any pair $s,t\in\mathbb{I}$.

If $\gamma\colon[a,b]\to\mathcal{X}$ is a geodesic such that $p=\gamma(a)$, $q=\gamma(b)$, then we say that γ is a geodesic from p to q. In this case, the image of γ is denoted by $[pq]$, and, with abuse of notations, we also call it a geodesic. We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$.

[1]Others call it differently: *shortest path*, *minimizing geodesic*. Also, note that the meaning of the term *geodesic* is different from what is used in Riemannian geometry, altho they are closely related.

In general, a geodesic from p to q need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we assume that we have chosen such geodesic.

Geodesic path. A geodesic path is a geodesic with constant-speed parameterization by the unit interval $[0,1]$.

Geodesic space. A metric space is called geodesic if any pair of its points can be joined by a geodesic.

B Triangles, hinges, and angles

Triangles. Given a triple of points p,q,r in a metric space $\mathcal{X}$, a choice of geodesics $([qr],[rp],[pq])$ will be called a triangle; we will use the short notation $[pqr]=[pqr]_{\mathcal{X}}=([qr],[rp],[pq])$.

Given a triple $p,q,r\in\mathcal{X}$ there may be no triangle $[pqr]$ simply because one of the pairs of these points cannot be joined by a geodesic. Also, many different triangles with these vertices may exist, any of which can be denoted by $[pqr]$. If we write $[pqr]$, it means that we have chosen such a triangle.

Model triangles. Given three points p,q,r in a metric space $\mathcal{X}$, let us define its model triangle $[\tilde p\tilde q\tilde r]$ (briefly, $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$) to be a triangle in the Euclidean plane $\mathbb{E}^2$ such that

$$|\tilde p-\tilde q|_{\mathbb{E}^2}=|p-q|_{\mathcal{X}},\qquad |\tilde q-\tilde r|_{\mathbb{E}^2}=|q-r|_{\mathcal{X}},\qquad |\tilde r-\tilde p|_{\mathbb{E}^2}=|r-p|_{\mathcal{X}}.$$

The same way we can define the hyperbolic and the spherical model triangles $\tilde\triangle(pqr)_{\mathbb{H}^2}$, $\tilde\triangle(pqr)_{\mathbb{S}^2}$ in the Lobachevsky plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2$. In the latter case, the model triangle is said to be defined if in addition

$$|p-q|+|q-r|+|r-p|<2\cdot\pi.$$

In this case, the model triangle again exists and is unique up to an isometry of $\mathbb{S}^2$.

Model angles. If $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$ and $|p-q|,|p-r|>0$, the angle measure of $[\tilde p\tilde q\tilde r]$ at $\tilde p$ will be called the model angle of the triple p, q, r and will be denoted by $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{E}^2}$. The same way we define $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{H}^2}$ and $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{S}^2}$; in the latter case, we assume in addition that the model triangle $\tilde\triangle(pqr)_{\mathbb{S}^2}$ is defined.

We may use the notation $\tilde\measuredangle(p\,{}^q_r)$ if it is evident which of the model spaces $\mathbb{H}^2$, $\mathbb{E}^2$ or $\mathbb{S}^2$ is meant.

Hinges. Let $p,x,y\in\mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair of geodesics $([px],[py])$ will be called a hinge and will be denoted by $[p\,{}^x_y]=([px],[py])$.

C Baby Toponogov

Recall that polyhedral space is a geodesic space that admits a finite triangulation such that each simplex is isometric to a simplex in a Euclidean space. If, in addition, it is homeomorphic to a surface (without boundary), then it is called a polyhedral surface. A point on a polyhedral surface with nonzero curvature is called an essential vertex. Any other point on the surface will be called regular. Note that *any regular point has a neighborhood that is isometric to an open set in the Euclidean plane.*

2.1. Exercise. *Let P be a non-negatively curved polyhedral surface.*

(a) Show that a geodesic in P cannot pass thru an essential vertex.

(b) Show that if two geodesics in P intersect at two points, then these are the endpoints for both geodesics.

The next theorem gives a global geometric property of non-negatively curved polyhedral surfaces.

Given a hinge $[p\,{}^x_y]$ in a non-negatively curved polyhedral surface P, denote by $\measuredangle[p\,{}^x_y]$ the minimal angle that the hinge cuts from P at p. (Soon we will give a more general definition of $\measuredangle[p\,{}^x_y]$; see 3B.)

2.2. Theorem. *Let P be a polyhedral surface. Assume P has non-negative curvature at each point (see 1E). Then*

$$\measuredangle[p\,{}^x_y] \geqslant \tilde\measuredangle(p\,{}^x_y)$$

for any hinge $[p\,{}^x_y]$ in P.

The following exercise will be used in the proof.

2.3. Exercise. *Let $f\colon[0,\ell]\to\mathbb{R}$ be a continuous function such that for any $t\in\,]0,\ell[$ there is a linear function h that locally supports f from above; that is, $h(t_0)=f(t_0)$, and there is $\varepsilon>0$ such that $h(t)\geqslant f(t)$ if $|t-t_0|<\varepsilon$. Show that f is concave.*

Proof. Let $[pxy]$ be a triangle in P and let $[\tilde p\tilde x\tilde y]$ be the model triangle of $[pxy]$. Set $\ell=|x-y|_P=|\tilde x-\tilde y|_{\mathbb{E}^2}$.

Denote by $\gamma(t)$ and $\tilde\gamma(t)$ the geodesics $[xy]$ and $[\tilde x\tilde y]$ parametrized by length starting from x and $\tilde x$, respectively. Observe that it is sufficient to show that

$$|p-\gamma(t)|\leqslant|\tilde p-\tilde\gamma(t)| \qquad ❶$$

for any t in $[0,\ell]$.

We may assume that p is a regular point; otherwise, move it slightly and apply approximation.

From the cosine law, we get that the function

$$\tilde f(t) = |\tilde p - \tilde\gamma(t)|^2 - t^2$$

is linear. Consider the function

$$f(t) = |p - \gamma(t)|^2 - t^2.$$

Note that $f(0) = \tilde f(0)$, $f(\ell) = \tilde f(\ell)$, and the inequality ❶ is equivalent to

$$f(t) \geqslant \tilde f(t). \tag{❷}$$

By Jensen's inequality, ❷ holds if f is concave.

By 2.1, $\gamma(t_0)$ is regular. Since p is regular, a geodesic $[p\gamma(t)]$ contains only regular points. Therefore for small $\varepsilon > 0$, the ε-neighborhood of $[p\gamma(t)]$, say Ω, contains only regular points. We may assume that Ω is homeomorphic to a disc; in this case, there is a locally distance preserving embedding $\iota\colon \Omega \to \mathbb{E}^2$. Note the image $\iota[p\gamma(t)]$ is a line segment that and $\iota(\Omega)$ is the ε-neighborhood of $\iota[p\gamma(t)]$ in $\mathbb{E}^2$; in particular, $\iota(\Omega)$ is convex. Thus $\iota(\Omega)$ contains a triangle with base $\iota[\gamma(t_0 - \varepsilon)\ \gamma(t_0 + \varepsilon)]$ and vertex $\iota(p)$.

Clearly, for any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ we have

$$|\iota(p) - \iota(\gamma(t))| \geqslant |p - \gamma(t)|.$$

Note that the function

$$h(t) = |\iota(p) - \iota(\gamma(t))|^2 - t^2$$

is linear. From above, h supports f locally at t_0. It remains to apply 2.3. $\square$

D Definition

2.4. Definition. *A metric space $\mathcal{X}$ has nonnegative curvature in the sense of Alexandrov (briefly, $\mathcal{X} \in \mathrm{CBB}(0)$) if the inequality*

$$\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^y_z)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^z_x)_{\mathbb{E}^2} \leqslant 2\cdot\pi \tag{❶}$$

holds for any quadruple $p, x, y, z \in \mathcal{X}$ such that p is distinct from x, y, and z.

The inequality ❶ is called $\mathrm{CBB}(0)$ comparison for the quadruple p, x, y, z. If instead of $\mathbb{E}^2$, we use $\mathbb{S}^2$ or $\mathbb{H}^2$, then we get the definition of $\mathrm{CBB}(1)$ and $\mathrm{CBB}(-1)$ comparisons.

(More generally, one may apply this definition to $\mathbb{M}^2(\kappa)$ — the model plane of curvature κ, defined as follows: $\mathbb{M}(0) = \mathbb{E}^2$, if $\kappa > 0$, then $\mathbb{M}(\kappa)$ is the sphere of radius $\frac{1}{\sqrt{\kappa}}$ and if $\kappa < 0$, then it is Lobachevsky plane rescaled by factor $\frac{1}{\sqrt{-\kappa}}$. This way we define $\mathrm{CBB}(\kappa)$ *comparison for any real κ.)*

While this definition can be applied to any metric space, it is usually applied to geodesic spaces (or, at least, length spaces that will be defined later).

2.5. Exercise. *Show that Euclidean space $\mathbb{E}^n$ is* $\mathrm{CBB}(0)$.

2.6. Exercise. *Show that a polyhedral surface is* $\mathrm{CBB}(0)$ *if and only if it has nonnegative curvature in the sense of 1E.*

The following theorem generalizes 1.12 and 1.16.

2.7. Theorem. *A metric space $\mathcal{X}$ is isometric to the surface of a convex body in the Euclidean space if and only if $\mathcal{X}$ is a geodesic* $\mathrm{CBB}(0)$ *space that is homeomorphic to $\mathbb{S}^2$.*

Moreover, $\mathcal{X}$ determines the convex body up to congruence.

As before, a convex body can degenerate to a plane figure F; in this case, its surface is defined as two copies of F glued along the boundary.

The main part is due to Alexadr Alexandrov [7]; its proof is an application of 1.16 together with approximation. The last part is very difficult; it was proved by Aleksei Pogorelov [19].

Eventually, we will prove the only-if part of the theorem, which is the simplest part of the theorem; it requires only 1.14 which is the only-if part of 1.16. To do this we will need to introduce the convergence of subsets in Euclidean space (Hausdorff convergence) and convergence of metric spaces (Gromov–Hausdorff convergence); it will be done in the next lecture.

2.8. Exercise. *Show that a metric space $\mathcal{X}$ is* $\mathrm{CBB}(0)$ *if and only if for any quadruple of points $p, x_1, x_2, x_3 \in \mathcal{X}$ there is a quadruple of points $q, y_1, y_2, y_3 \in \mathbb{E}^2$ such that*

$$|p - x_i|_{\mathcal{X}} \geqslant |q - y_i|_{\mathbb{E}^2} \quad \textit{and} \quad |x_i - x_j|_{\mathcal{X}} \leqslant |y_i - y_j|_{\mathbb{E}^2}$$

for all i and j.

2.9. Exercise. *Show that $\mathbb{R}^2$ with metric induced by a norm is* $\mathrm{CBB}(0)$ *if and only if it is isometric to the Euclidean plane $\mathbb{E}^2$.*

E Four-point metric spaces

Let us give a more conceptual way to think about the comparison inequality in 2.4 and an analogous inequality for upper-curvature bound that will appear later.

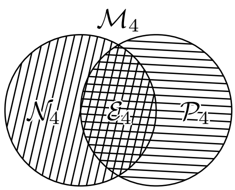
Consider the space $\mathcal{M}_4$ of all isometry classes of 4-point metric spaces. Each element in $\mathcal{M}_4$ can be described by 6 numbers — the distances between all 6 pairs of its points, say $\ell_{i,j}$ for $1 \leqslant i < j \leqslant 4$ modulo permutations of the index set $(1,2,3,4)$. These 6 numbers are subject to 12 triangle inequalities; that is,

$$\ell_{i,j} + \ell_{j,k} \geqslant \ell_{i,k}$$

holds for all i, j and k, where we assume that $\ell_{j,i} = \ell_{i,j}$ and $\ell_{i,i} = 0$.

The space $\mathcal{M}_4$ comes with topology. It can be defined as a quotient of the cone in $\mathbb{R}^6$ by permutations of the 4 points of the space.

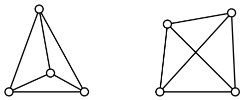
Consider the subset $\mathcal{E}_4 \subset \mathcal{M}_4$ of all isometry classes of 4-point metric spaces that admit isometric embeddings into Euclidean space.



2.10. Advanced exercise. *The complement $\mathcal{M}_4 \backslash \mathcal{E}_4$ has two connected components.*

Let us denote one of the components by $\mathcal{P}_4$ and the other by $\mathcal{N}_4$. Here $\mathcal{P}$ and $\mathcal{N}$ stand for positive and negative curvature because spheres have no quadruples of type $\mathcal{N}_4$ and Lobachevsky space has no quadruples of type $\mathcal{P}_4$.

A metric space that has no quadruples of points of type $\mathcal{P}_4$ or $\mathcal{N}_4$ respectively is called CAT(0) and CBB(0).



Let us describe the subdivision into $\mathcal{P}_4$, $\mathcal{E}_4$, and $\mathcal{N}_4$ intuitively. Imagine that you move out of $\mathcal{E}_4$ — your path is a one-parameter family of 4-point metric spaces. The last thing you see in $\mathcal{E}_4$ is one of the two plane configurations shown on the diagram. If you see the left configuration then you move into $\mathcal{N}_4$; if it is the one on the right, then you move into $\mathcal{P}_4$. More degenerate pictures can be avoided; for example, a triangle with a point on a side. From such a configuration one may move in $\mathcal{N}_4$ and $\mathcal{P}_4$ (as well as come back to $\mathcal{E}_4$).

Here is an exercise, solving which would force you to rebuild a considerable part of Alexandrov geometry. It might be helpful to spend some time thinking about this exercise before proceeding.

2.11. Advanced exercise. *Assume $\mathcal{X}$ is a geodesic space, containing only quadruples of type $\mathcal{E}_4$. Show that $\mathcal{X}$ is isometric to a convex set in a Hilbert space.*

In the definition above, instead of Euclidean space, one can take Lobachevsky space of curvature -1. In this case, one obtains the definition of spaces with curvature bounded above or below by -1 (CAT(-1) or CBB(-1)).

To define spaces with curvature bounded above or below by 1 (CAT(1) or CBB(1)), one has to take the unit 3-sphere and specify that only the quadruples of points such that each of the four triangles has perimeter less than $2\cdot\pi$ are checked.

F Comments

The first synthetic description of curvature is due to Abraham Wald [25]; it was given in a lone publication on a "coordinateless description of Gauss surfaces" published in 1936. In 1941, similar definitions were rediscovered by Alexandr Alexandrov [8].

In Alexandrov's work, the first applications of this approach were given. Mainly: the main part of 2.7 [4, 5] and the gluing theorem [6], which gave a flexible tool to modify non-negatively curved metrics on a sphere. These two results together formed the foundation of the branch of geometry now called Alexandrov geometry; they gave a very intuitive geometric tool to study embeddings and bending of surfaces in Euclidean space and changed the subject dramatically.

In particular, the existence of bending of a large spherical dome (sphere with a small disc removed) easily follows from these two theorems; moreover, it provides an intuitive description of such bending that can be extended to a closed convex surface.

Lecture 3

CBB: first steps

In this lecture, we start to study metric spaces that satisfy CBB comparison [see 2.4]. Most of the covered material will not be used further, it served as a motivation for CBB comparison.

A Quotients and submetries

3.1. Theorem. *Assume that group G acts isometrically on a* CBB(0) *space $\mathcal{L}$ and has closed orbits. Then the quotient space $\mathcal{L}/G$ is* CBB(0).

Proof. Denote by $\sigma\colon \mathcal{L} \to \mathcal{L}/G$ the quotient map.

Fix a quadruple of points $p, x_1, x_2, x_3 \in \mathcal{L}/G$. Choose an arbitrary $\hat p \in \mathcal{L}$ such that $\sigma(\hat p) = p$. Note that we can choose the points $\hat x_1, \hat x_2, \hat x_3 \in \mathcal{L}$ such that $\sigma(\hat x_i) = x_i$ and

$$|p - x_i|_{\mathcal{L}/G} \lesseqgtr |\hat p - \hat x_i|_{\mathcal{L}} \pm \delta$$

for all i and any fixed $\delta > 0$.

Given $\varepsilon > 0$, the value δ can be chosen in such a way that the inequality

$$\text{❶} \qquad \tilde\measuredangle(p\,{}^{x_i}_{x_j}) < \tilde\measuredangle(\hat p\,{}^{\hat x_i}_{\hat x_j}) + \varepsilon$$

holds for all i and j.

By CBB(0) comparison in $\mathcal{L}$, we have

$$\tilde\measuredangle(\hat p\,{}^{\hat x_1}_{\hat x_2}) + \tilde\measuredangle(\hat p\,{}^{\hat x_2}_{\hat x_3}) + \tilde\measuredangle(\hat p\,{}^{\hat x_3}_{\hat x_1}) \leqslant 2{\cdot}\pi.$$

Applying ❶, we get

$$\tilde\measuredangle(p\,{}^{x_1}_{x_2}) + \tilde\measuredangle(p\,{}^{x_2}_{x_3}) + \tilde\measuredangle(p\,{}^{x_3}_{x_1}) < 2\cdot\pi + 3\cdot\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we have

$$\tilde\measuredangle(p\,{}^{x_1}_{x_2}) + \tilde\measuredangle(p\,{}^{x_2}_{x_3}) + \tilde\measuredangle(p\,{}^{x_3}_{x_1}) \leqslant 2\cdot\pi;$$

that is, the CBB(0) comparison holds for this quadruple in $\mathcal{L}/G$. $\square$

A map $\sigma\colon \mathcal{X} \to \mathcal{Y}$ between the metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is called a submetry if

$$\sigma(\mathrm{B}(p,r)_{\mathcal{X}}) = \mathrm{B}(\sigma(p),r)_{\mathcal{Y}}$$

for any $p \in \mathcal{X}$ and $r \geqslant 0$.

Suppose G and $\mathcal{L}$ are as in 3.1. Observe that the quotient map $\sigma\colon \mathcal{L} \to \mathcal{L}/G$ is a submetry. Moreover, the proof above works for any submetry. Therefore we get the following.

3.2. Generalization. *Let $\sigma\colon \mathcal{L} \to \mathcal{M}$ be a submetry. Suppose $\mathcal{L}$ is a* CBB(0) *space, then so is $\mathcal{M}$.*

3.3. Advanced exercise. *Let G be a compact Lie group with a bi-invariant Riemannian metric. Show that G is isometric to a quotient of the Hilbert space by isometric group action.*

Conclude that $G \in$ CBB(0)*.*

B Angles

The angle measure of a hinge $[p\,{}^x_y]$ is defined as the following limit

$$\measuredangle[p\,{}^x_y] = \lim_{\bar x,\bar y\to p} \tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x \in\,]px]$ and $\bar y \in\,]py]$.

Note that if $\measuredangle[p\,{}^x_y]$ is defined, then

$$0 \leqslant \measuredangle[p\,{}^x_y] \leqslant \pi.$$

3.4. Exercise. *Suppose that in the above definition, one uses spherical or hyperbolic model angles instead of Euclidean. Show that it does not change the value $\measuredangle[p\,{}^x_y]$.*

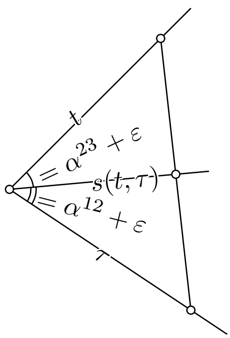
3.5. Exercise. *Give an example of a hinge $[p\,{}^x_y]$ in a metric space with an undefined angle $\measuredangle[p\,{}^x_y]$.*

3.6. Triangle inequality for angles. *Let $[px_1]$, $[px_2]$, and $[px_3]$ be three geodesics in a metric space. If all of the angles $\alpha_{ij} = \measuredangle[p\,{}^{x_i}_{x_j}]$ are defined then they satisfy the triangle inequality:*

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23}.$$

Proof. Since $\alpha_{13} \leqslant \pi$, we can assume that $\alpha_{12} + \alpha_{23} < \pi$. Denote by γ_i the unit-speed parametrization of $[px_i]$ from p to x_i. Given any $\varepsilon > 0$, for all sufficiently small $t, \tau, s \in \mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned} |\gamma_1(t) - \gamma_3(\tau)| &\leqslant |\gamma_1(t) - \gamma_2(s)| + |\gamma_2(s) - \gamma_3(\tau)| < \\ &< \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha_{12} + \varepsilon)} + \\ &\quad + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\alpha_{23} + \varepsilon)} \leqslant \end{aligned}$$



Below we define $s(t, \tau)$ so that for $s = s(t, \tau)$, this chain of inequalities can be continued as follows:

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon.$$

Hence the result follows.

To define $s(t, \tau)$, consider three half-lines $\tilde\gamma_1$, $\tilde\gamma_2$, $\tilde\gamma_3$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde\gamma_1, \tilde\gamma_2) = \alpha_{12} + \varepsilon$, $\measuredangle(\tilde\gamma_2, \tilde\gamma_3) = \alpha_{23} + \varepsilon$, and $\measuredangle(\tilde\gamma_1, \tilde\gamma_3) = \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon$. We parametrize each half-line by the distance from the starting point. Given two positive numbers $t, \tau \in \mathbb{R}_{\geqslant 0}$, let $s = s(t, \tau)$ be the number such that $\tilde\gamma_2(s) \in [\tilde\gamma_1(t)\ \tilde\gamma_3(\tau)]$. Clearly, $s \leqslant \max\{t, \tau\}$, so t, τ, s may be taken sufficiently small. $\square$

3.7. Exercise. *Prove that the sum of adjacent angles is at least π.*

More precisely: suppose two hinges $[p\,{}^{x}_{z}]$ and $[p\,{}^{y}_{z}]$ are adjacent; that is, they share side $[pz]$, and the union of two sides $[px]$ and $[py]$ form a geodesic $[xy]$. Show that

$$\measuredangle[p\,{}^{x}_{z}] + \measuredangle[p\,{}^{y}_{z}] \geqslant \pi$$

whenever each angle on the left-hand side is defined.

The above inequality can be strict. For example in a metric tree angles between any two different edges coming out of the same vertex are all equal to π.

3.8. Exercise. *Assume that a hinge $[q\,{}^p_x]$ with defined angle measure. Let γ be the unit speed parametrization of $[qx]$ from q to x. Show that*

$$|p-\gamma(t)| \leqslant |q-p| - t\cdot\cos(\measuredangle[q\,{}^p_x]) + o(t).$$

C Alexandrov's lemma

Recall that $[xy]$ denotes a geodesic from x to y; set

$$]xy] = [xy]\setminus\{x\}, \quad [xy[= [xy]\setminus\{y\}, \quad]xy[= [xy]\setminus\{x,y\}.$$

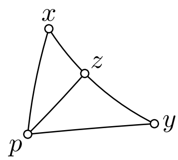
3.9. Lemma. *Let p,x,y,z be distinct points in a metric space such that $z\in\,]xy[$. Then the following expressions for the Euclidean model angles have the same sign:*

(a) $\tilde\measuredangle(x\,{}^p_y) - \tilde\measuredangle(x\,{}^p_z)$,

(b) $\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) - \pi$.

The same holds for the hyperbolic and spherical model angles, but in the latter case, one has to assume in addition that

$$|p-z|+|p-y|+|x-y| < 2\cdot\pi.$$



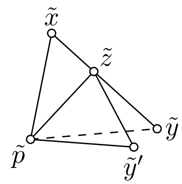
Proof. Consider the model triangle $[\tilde x\tilde p\tilde z] = \tilde\triangle(xpz)$. Take a point $\tilde y$ on the extension of $[\tilde x\tilde z]$ beyond $\tilde z$ so that $|\tilde x-\tilde y| = |x-y|$ (and therefore $|\tilde x-\tilde z| = |x-z|$).

Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:

(i) $\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(x\,{}^p_y)$,

(ii) $|\tilde p-\tilde y| - |p-y|$,

(iii) $\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(z\,{}^p_y)$.



Since

$$\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] = \measuredangle[\tilde x\,{}^{\tilde p}_{\tilde z}] = \tilde\measuredangle(x\,{}^p_z)$$

and

$$\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] = \pi - \measuredangle[\tilde z\,{}^{\tilde x}_{\tilde p}] = \pi - \tilde\measuredangle(z\,{}^x_p),$$

the first statement follows. □

3.10. Exercise. *Assume p,x,y,z are as in Alexandrov's lemma. Show that*

$$\tilde\measuredangle(p\,{}^x_y) \geqslant \tilde\measuredangle(p\,{}^x_z) + \tilde\measuredangle(p\,{}^z_y),$$

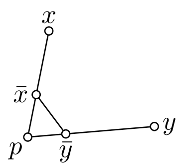
with equality if and only if the expressions in (a) and (b) vanish.

Note that if $p \in\ ]xy[$, then $\tilde\measuredangle(p\,{}^x_y) = \pi$. Applying Alexandrov's lemma and CBB(0) comparison, we get the following claim and its corollary.

3.11. Claim. *If p, x, y, z are points in a* CBB(0) *such that $p \in\ ]xy[$, then*

$$\tilde\measuredangle(x\,{}^y_z) \leqslant \tilde\measuredangle(x\,{}^p_z).$$

3.12. Corollary. *Let $[p\,{}^x_y]$ be a hinge in a* CBB(0) *space. Consider the function*



$$f\colon (|p-\bar x|, |p-\bar y|) \mapsto \tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x \in\ ]px]$ and $\bar y \in\ ]py]$. Show that f is nonincreasing in each argument.

Note that 3.12 implies the following generalization of 2.2.

3.13. Claim. *For any hinge $[p\,{}^x_y]$ in a* CBB(0) *space, the angle measure $\measuredangle[p\,{}^x_y]$ is defined, and*

$$\measuredangle[p\,{}^x_y] \geqslant \tilde\measuredangle(p\,{}^x_y).$$

3.14. Exercise. *Let $[p\,{}^x_y]$ be a hinge in a* CBB(0) *space. Suppose $\measuredangle[p\,{}^x_y] = 0$ show that $[px] \subset [py]$ or $[py] \subset [px]$.*

3.15. Exercise. *Let $[xy]$ be a geodesic in a* CBB(0) *space. Suppose $z \in\ ]xy[$ show that there is a unique geodesic $[xz]$ and $[xz] \subset [xy]$.*

3.16. Exercise. *Let $[p\,{}^x_z]$ and $[p\,{}^y_z]$ be adjacent hinges in a* CBB(0) *space. Show that*

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] = \pi.$$

3.17. Exercise. *Let p, x, y in a* CBB(0) *space and $v, w \in\ ]xy[$. Show that*

$$\tilde\measuredangle(x\,{}^y_p) = \tilde\measuredangle(x\,{}^v_p) \quad\Longleftrightarrow\quad \tilde\measuredangle(x\,{}^y_p) = \tilde\measuredangle(x\,{}^w_p).$$

Recall that a triangle $[xyz]$ in a space $\mathcal{X}$ is a triple of minimizing geodesics $[xy]$, $[yz]$, and $[zx]$. Consider the model triangle $[\tilde x\tilde y\tilde z] = \tilde\triangle(xyz)_{\mathbb{E}^2}$ in the Euclidean plane. The natural map $[\tilde x\tilde y\tilde z] \to [xyz]$ sends a point $\tilde p \in [\tilde x\tilde y] \cup [\tilde y\tilde z] \cup [\tilde z\tilde x]$ to the corresponding point $p \in [xy] \cup [yz] \cup [zx]$; that is, if $\tilde p$ lies on $[\tilde y\tilde z]$, then $p \in [yz]$ and $|\tilde y - \tilde p| = |y-p|$ (and therefore $|\tilde z - \tilde p| = |z-p|$).

3.18. Definition. *A triangle* $[xyz]$ *in the metric space* $\mathcal{X}$ *is called* thin *(or* fat*) if the natural map* $\tilde{\triangle}(xyz)_{\mathbb{E}^2} \to [xyz]$ *is distance nonincreasing (or respectively distance nondecreasing).*

3.19. Exercise. *Show that any triangle in a* CBB(0) *space is fat.*

D Comments

All the discussed statements admit natural generalizations to CBB(κ) spaces. Most of the time the proof is the same with uglier formulas. However, for the CBB(1) case in 3.1 one needs to assume in addition that space has intrinsic metric and the proof requires the globalization theorem which will be discussed later.

Lecture 4

CBB: globalization

A Hinge comparison

Let $[p\,{}^{x}_{y}]$ be a hinge in a CBB(0) space. By 3.14, the angle measure $\measuredangle[p\,{}^{x}_{y}]$ is defined and

$$\measuredangle[p\,{}^{x}_{y}] \geqslant \tilde\measuredangle(p\,{}^{x}_{y}).$$

Further, according to 3.16, we have

$$\measuredangle[p\,{}^{x}_{z}] + \measuredangle[p\,{}^{y}_{z}] = \pi$$

for adjacent hinges $[p\,{}^{x}_{z}]$ and $[p\,{}^{y}_{z}]$ in a CBB(0) space.

The following theorem implies that a geodesic space is CBB(0) if the above conditions hold for all its hinges.

4.1. Theorem. *A geodesic space $\mathcal{L}$ is* CBB(0) *if the following conditions hold.*

(a) For any hinge $[x\,{}^{p}_{y}]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^{p}_{y}]$ is defined and

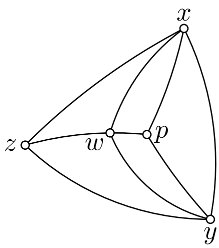
$$\measuredangle[x\,{}^{p}_{y}] \geqslant \tilde\measuredangle(x\,{}^{p}_{y}).$$

(b) For any two adjacent hinges $[p\,{}^{x}_{z}]$ and $[p\,{}^{y}_{z}]$ in $\mathcal{L}$, we have

$$\measuredangle[p\,{}^{x}_{z}] + \measuredangle[p\,{}^{y}_{z}] \leqslant \pi.$$

Proof. Consider a point $w \in\,]pz[$ close to p. From *(b)*, it follows that

$$\measuredangle[w\,{}^{x}_{z}] + \measuredangle[w\,{}^{x}_{p}] \leqslant \pi \quad\text{and}\quad \measuredangle[w\,{}^{y}_{z}] + \measuredangle[w\,{}^{y}_{p}] \leqslant \pi.$$



Since $\measuredangle[w\,{}^x_y] \leqslant \measuredangle[w\,{}^x_p] + \measuredangle[w\,{}^y_p]$ (see 3.6), we get

$$\measuredangle[w\,{}^x_z] + \measuredangle[w\,{}^y_z] + \measuredangle[w\,{}^x_y] \leqslant 2\cdot\pi.$$

Applying *(a)*,

$$\tilde\measuredangle(w\,{}^x_z) + \tilde\measuredangle(w\,{}^y_z) + \tilde\measuredangle(w\,{}^x_y) \leqslant 2\cdot\pi.$$

Passing to the limits $w \to p$, we have

$$\tilde\measuredangle(p\,{}^x_z) + \tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^x_y) \leqslant 2\cdot\pi.$$

□

B Equivalent conditions

The following theorem summarizes 3.11, 3.13, 3.16, 4.1.

4.2. Theorem. *Let $\mathcal{L}$ be a geodesic space. Then the following conditions are equivalent.*

(a) $\mathcal{L}$ *is* CBB(0).

(b) (adjacent angle comparison) for any geodesic $[xy]$ and point $z \in\,]xy[$, $z \ne p$ in $\mathcal{L}$, we have

$$\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) \leqslant \pi.$$

(c) (point-on-side comparison) for any geodesic $[xy]$ and $z \in\,]xy[$ in $\mathcal{L}$, we have

$$\tilde\measuredangle(x\,{}^p_y) \leqslant \tilde\measuredangle(x\,{}^p_z).$$

(d) (hinge comparison) for any hinge $[x\,{}^p_y]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and

$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle(x\,{}^p_y).$$

Moreover,

$$\measuredangle[z\,{}^p_y] + \measuredangle[z\,{}^p_x] \leqslant \pi$$

for any adjacent hinges $[z\,{}^p_y]$ and $[z\,{}^p_x]$.

Moreover, the implications (a)⇒(b)⇒(c)⇒(d) hold in any space, not necessarily geodesic.

4.3. Advanced Exercise. *Construct a geodesic space $\mathcal{X} \notin$ CBB(0) that meets the following condition: for any 3 points $p, x, y \in \mathcal{X}$ there is a geodesic $[xy]$ such that for any $z \in\,]xy[$*

$$\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) \leqslant \pi.$$

C Globalization

A metric space $\mathcal{L}$ is l o c a l l y CBB(0) if any point $p \in \mathcal{L}$ admits a neighborhood $U \ni p$ such that the CBB(0) comparison holds for any quadruple of points in U.

4.4. Globalization theorem. *Any locally* CBB(0) *compact geodesic space is* CBB(0).

Proof modulo the key lemma. Let $\mathcal{L}$ be a locally CBB(0) compact geodesic space. Note that condition 4.1*b* holds in $\mathcal{L}$ (the proof is the same as for CBB(0) space). It remains to prove that 4.1*a* holds in $\mathcal{L}$; that is,

❶ $$\measuredangle[x\,{}^{p}_{y}] \geqslant \tilde\measuredangle(x\,{}^{p}_{y}).$$

First note that ❶ holds for hinges in a small neighborhood of any point; this can be proved the same way as 3.13 and 3.16, applying the local version of CBB(0) comparison. Since $\mathcal{L}$ is compact, there is $\varepsilon > 0$ such that ❶ holds if $|x-p| + |p-y| < \varepsilon$. Applying the key lemma several times we get that ❶ holds for any given hinge. □

4.5. Key lemma. *Let $\mathcal{L}$ be a locally* CBB(0) *geodesic space. Assume that the comparison*

$$\measuredangle[x\,{}^{p}_{q}] \geqslant \tilde\measuredangle(x\,{}^{p}_{q})$$

holds for any hinge $[x\,{}^{p}_{q}]$ with $|x-y|+|x-q| < \tfrac{2}{3}\cdot\ell$. Then comparison

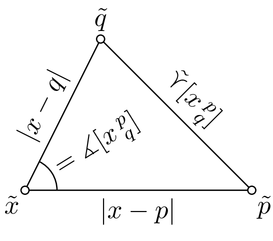
$$\measuredangle[x\,{}^{p}_{q}] \geqslant \tilde\measuredangle(x\,{}^{p}_{q})$$

holds for any hinge $[x\,{}^{p}_{q}]$ with $|x-p| + |x-q| < \ell$.

Let $[x\,{}^{p}_{q}]$ be a hinge in a metric space $\mathcal{L}$ with defined angle measure. Denote by $\tilde\Upsilon[x\,{}^{p}_{q}]$ its m o d e l s i d e; this is the opposite side in a flat triangle with the same angle and two adjacent sides as in $[x\,{}^{p}_{q}]$.

More precisely, consider the model hinge $[\tilde x\,{}^{\tilde p}_{\tilde q}]$ in $\mathbb{E}^2$ that is defined by

$$\begin{aligned}\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde q}]_{\mathbb{E}^2} &= \measuredangle[x\,{}^{p}_{q}]_{\mathcal{L}},\\ |\tilde x-\tilde p|_{\mathbb{E}^2} &= |x-p|_{\mathcal{L}},\\ |\tilde x-\tilde q|_{\mathbb{E}^2} &= |x-q|_{\mathcal{L}};\end{aligned}$$



then

$$\tilde\Upsilon[x\,{}^{p}_{q}]_{\mathcal{L}} := |\tilde p - \tilde q|_{\mathbb{E}^2}.$$

Note that

$$\tilde\Upsilon[x\,{}^{p}_{q}] \geqslant |p-q| \quad\iff\quad \measuredangle[x\,{}^{p}_{q}] \geqslant \tilde\measuredangle(x\,{}^{p}_{q}).$$

We will use it in the following proof.

Proof. It is sufficient to prove the inequality

❷ $$\tilde\Upsilon[x\,{}^{p}_{q}] \geqslant |p-q|$$

for any hinge $[x\,{}^{p}_{q}]$ with $|x-p|+|x-q|<\ell$.

Consider a hinge $[x\,{}^{p}_{q}]$ such that

$$\tfrac{2}{3}\cdot\ell \leqslant |p-x|+|x-q| < \ell.$$

First, let us construct a new smaller hinge $[x'\,{}^{p}_{q}]$ with

❸ $$|p-x|+|x-q| \geqslant |p-x'|+|x'-q|,$$

such that

❹ $$\tilde\Upsilon[x\,{}^{p}_{q}] \geqslant \tilde\Upsilon[x'\,{}^{p}_{q}].$$

Construction. Assume $|x-q|\geqslant|x-p|$; otherwise switch the roles of p and q in the following construction. Take $x'\in[xq]$ such that

❺ $$|p-x|+3\cdot|x-x'| = \tfrac{2}{3}\cdot\ell.$$

Choose a geodesic $[x'p]$ and consider the hinge $[x'\,{}^{p}_{q}]$ formed by $[x'p]$ and $[x'q]\subset[xq]$. Then ❸ follows from the triangle inequality.

Further, note that

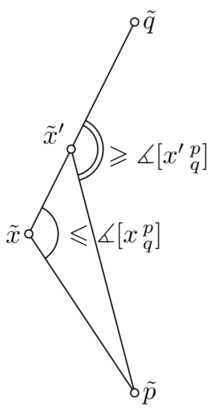
$$|p-x|+|x-x'| < \tfrac{2}{3}\cdot\ell, \qquad |p-x'|+|x'-x| < \tfrac{2}{3}\cdot\ell.$$

In particular,

❻ $$\measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde\measuredangle(x\,{}^{p}_{x'}) \quad\text{and}\quad \measuredangle[x'\,{}^{p}_{x}] \geqslant \tilde\measuredangle(x'\,{}^{p}_{x}).$$

Now, let $[\tilde x\tilde x'\tilde p] = \tilde\triangle(xx'p)$. Take $\tilde q$ on the extension of $[\tilde x\tilde x']$ beyond x' such that $|\tilde x-\tilde q| = |x-q|$ (and therefore $|\tilde x'-\tilde q| = |x'-q|$). By ❻,

$$\measuredangle[x\,{}^{p}_{q}] = \measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde\measuredangle(x\,{}^{p}_{x'}) \quad\Rightarrow\quad \tilde\Upsilon[x\,{}^{q}_{p}] \geqslant |\tilde p-\tilde q|.$$

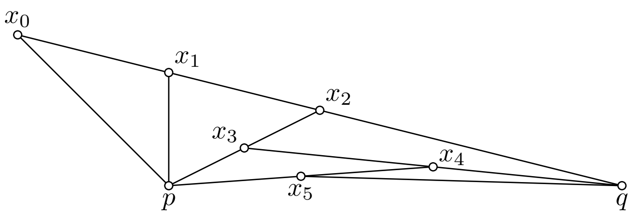


Hence

$$\begin{aligned}\measuredangle[\tilde x'\,{}^{\tilde p}_{\tilde q}] &= \pi - \tilde\measuredangle(x'\,{}^{p}_{x}) \geqslant \\ &\geqslant \pi - \measuredangle[x'\,{}^{p}_{x}] = \\ &= \measuredangle[x'\,{}^{p}_{q}],\end{aligned}$$

and ❹ follows.

Let us continue the proof. Set $x_0=x$. Let us apply inductively the above construction to get a sequence of hinges $[x_n\,{}^p_q]$ with $x_{n+1}=x'_n$. From ❹, we have that the sequence $s_n=\tilde\Upsilon[x_n\,{}^p_q]$ is nonincreasing.



The sequence might terminate at some n only if $|p-x_n|+|x_n-q|<$ $<\tfrac23\cdot\ell$. In this case, by the assumptions of the lemma, $\tilde\Upsilon[x_n\,{}^p_q]\geqslant|p-q|$. Since the sequence s_n is nonincreasing, inequality ❷ follows.

Otherwise, the sequence $r_n=|p-x_n|+|x_n-q|$ is nonincreasing, and $r_n\geqslant\tfrac23\cdot\ell$ for all n. Note that by construction, the distances $|x_n-$ $-x_{n+1}|$, $|x_n-p|$, and $|x_n-q|$ are bounded away from zero for all large n. Indeed, since on each step, we move x_n toward to the point p or q that is further away, the distances $|x_n-p|$ and $|x_n-q|$ become about the same. Namely, by ❺, we have that $|p-x_n|-|x_n-q|\leqslant\tfrac29\cdot\ell$ for all large n. Since $|p-x_n|+|x_n-q|\geqslant\tfrac23\cdot\ell$, we have $|x_n-p|\geqslant\tfrac{\ell}{100}$ and $|x_n-q|\geqslant\tfrac{\ell}{100}$. Further, since $r_n\geqslant\tfrac23\cdot\ell$, ❺ implies that $|x_n-$ $-x_{n+1}|>\tfrac{\ell}{100}$.

Since the sequence r_n is nonincreasing, it converges. In particular, $r_n-r_{n+1}\to0$ as $n\to\infty$. It follows that $\tilde\measuredangle(x_n\,{}^{p_n}_{x_{n+1}})\to\pi$, where $p_n=p$ if $x_{n+1}\in[x_nq]$, and otherwise $p_n=q$. Since $\measuredangle[x_n\,{}^{p_n}_{x_{n+1}}]\geqslant\tilde\measuredangle(x_n\,{}^{p_n}_{x_{n+1}})$, we have $\measuredangle[x_n\,{}^{p_n}_{x_{n+1}}]\to\pi$ as $n\to\infty$.

It follows that

$$r_n-s_n=|p-x_n|+|x_n-q|-\tilde\Upsilon[x_n\,{}^p_q]\to0.$$

Together with the triangle inequality

$$|p-x_n|+|x_n-q|\geqslant|p-q|$$

this yields

$$\lim_{n\to\infty}\tilde\Upsilon[x_n\,{}^p_q]\geqslant|p-q|.$$

Applying monotonicity of the sequence $s_n=\tilde\Upsilon[x_n\,{}^p_q]$, we obtain ❷. $\square$

D On general curvature bound

The globalization theorem can be generalized to CBB(κ) spaces for any real κ. The case $\kappa \leqslant 0$ is proved the same way, but the case $\kappa > 0$ requires minor modifications.

4.6. Exercise. *Suppose* $\kappa \leqslant \mathrm{K}$. *Show that*

$$\tilde{\measuredangle}(x\,{}^{y}_{z})_{\mathbb{M}(\kappa)} \leqslant \tilde{\measuredangle}(x\,{}^{y}_{z})_{\mathbb{M}(\mathrm{K})}$$

if the right-hand side is defined.

Conclude that any CBB(K) *space is locally* CBB(κ).

The exercise and the globalization theorem (here we need a more general version 4.10) imply that *any geodesic* CBB(K) *space is* CBB(κ). Recall that CBB(κ) stands for *curvature bounded below by* κ; so, for geodesic spaces it makes sense. However, as you can see from the following exercise, it does not make much sense in general.

4.7. Exercise. *Let* $\mathcal{X}$ *be the set* $\{p, x_1, x_2, x_3\}$ *with the metric defined by*

$$|p - x_i| = \pi, \quad |x_i - x_j| = 2{\cdot}\pi$$

for all $i \neq j$. *Show that* $\mathcal{X}$ *is* CBB(1), *but not* CBB(0).

E Remarks

The globalization theorem is also known as the *generalized Toponogov theorem.*

Recall that a metric space $\mathcal{X}$ is called complete if any Cauchy sequence of points in $\mathcal{X}$ converges. The compactness condition in our version of the theorem can be traded to completeness by using the following exercise.

4.8. Exercise. *Let* $\mathcal{X}$ *be a complete metric space. Suppose* $r\colon \mathcal{X} \to \mathbb{R}$ *is a positive continuous function. Show that for any* $\varepsilon > 0$ *there is a point* $p \in \mathcal{X}$ *such that*

$$r(x) > (1 - \varepsilon){\cdot}r(p)$$

for any $x \in \overline{\mathrm{B}}[p, \tfrac{1}{\varepsilon}{\cdot}r(p)]$.

Let us mention two more general versions of the globalization theorem.

Recall that length space is a metric space such that any two points p and q can be connected by a path with length arbitrarily close to $|p - q|$. Note that any geodesic space is length, but not the other way around. The following theorem was proved already in the paper of Michael Gromov, Yuriy Burago, and Grigory Perelman [11].

4.9. Theorem. *Any complete locally* CBB(κ) *length space is* CBB(κ).

The next result is mine [16].

4.10. Theorem. *Any locally* CBB(κ) *geodesic space is* CBB(κ).

In the two-dimensional case, the globalization theorem was proved by Paolo Pizzetti [17]; later it was reproved independently by Alexandr Alexandrov [9]. Victor Toponogov [23] proved it for Riemannian manifolds of all dimensions.

I took the proof from our book [2] (with generality reduction). It uses simplifications obtained by Conrad Plaut [18] and Dmitry Burago, Yuriy Burago, and Sergei Ivanov [10]. The same proof was rediscovered independently by Urs Lang and Viktor Schroeder [14]. Another simplified version was obtained by Katsuhiro Shiohama [21].

The question whether 4.1a suffices to conclude that $\mathcal{L}$ is CBB(κ) is a long-standing open problem (possibly dating back to Alexandrov); in print, it was first stated in [10, footnote in 4.1.5].

4.11. Open question. *Let $\mathcal{L}$ be a complete geodesic space (you can also assume that $\mathcal{L}$ is homeomorphic to $\mathbb{S}^2$ or $\mathbb{R}^2$) such that for any hinge $[x\,{}^p_y]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and*

$$\measuredangle[x\,{}^p_y] \geqslant \tilde{\measuredangle}(x\,{}^p_y).$$

Is it true that $\mathcal{L}$ is CBB(0)*?*

Index

Bibliography

[1] М. Айгнер и Г. Циглер. *Доказательства из Книги.* Любое издание.

[2] S. Alexander, V. Kapovitch, and A. Petrunin. *Alexandrov geometry: foundations.* 2019.

[3] А. Д. Александров. *Выпуклые многогранники.* 1950. [English translation: Alexandrov, A. D., *Convex polyhedra,* 2005].

[4] А. Д. Александров. «Существование выпуклого многогранника и выпуклой поверхности с заданной метрикой». *Матем. сб.* 11(53).1-2 (1942), 15—65.

[5] А. Д. Александров. «Внутренняя геометрия произвольной выпуклой поверхности». 32.7 (1941), 467—470.

[6] А. Д. Александров. «О склеивании выпуклых поверхностей». 54.2 (1946), 99—102.

[7] А. Д. Александров. *Внутренняя геометрия выпуклых поверхностей.* 1948.

[8] A. Alexandroff. "The inner geometry of an arbitrary convex surface". *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 32 (1941), 467–470.

[9] A. D. Alexandrow. "Über eine Verallgemeinerung der Riemannschen Geometrie". *Schr. Forschungsinst. Math.* 1 (1957), 33–84.

[10] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry.* Vol. 33. Graduate Studies in Mathematics. 2001. [Русский перевод: Бураго Д.Ю., Бураго Ю.Д., Иванов С.В. «Курс метрической геометрии»].

[11] Г. Я. Перельман Ю. Д. Бураго М. Л. Громов. «Пространства А. Д. Александрова с ограниченными снизу кривизнами». *УМН* 47.2(284) (1992), 3—51. [English translation: Yu. D. Burago, M. L. Gromov, G. Ya. Perel'man, "A. D. Alexandrov spaces with curvature bounded below", Russian Math. Surveys, 47:2 (1992), 1–58].

[12] A. Cauchy. "Sur les polygones et les polyèdres: second Mémoire". *Journal de l'Ecole Polytechnique* 16.9 (1813), 26–38.

[13] Н. П. Долбилин. *Жемчужины теории многогранников.* 2000.

[14] U. Lang and V. Schroeder. "On Toponogov's comparison theorem for Alexandrov spaces". *Enseign. Math.* 59.3-4 (2013), 325–336.

[15] Н. Лебедева и А. Петрунин. *Теорема Александрова о вложении многогранников.* 2022. [English translation: N. Lebedeva, A. Petrunin, *Alexandrov's embedding theorem.*]

[16] A. Petrunin. “A globalization for non-complete but geodesic spaces”. *Math. Ann.* 366.1-2 (2016), 387–393.

[17] P. Pizzetti. “Paragone fra due triangoli a lati uguali”. *Atti della Reale Accademia dei Lincei, Rendiconti (5). Classe di Scienze Fisiche, Matematiche e Naturali* 16.1 (1907), 6–11.

[18] C. Plaut. “Spaces of Wald curvature bounded below”. *J. Geom. Anal.* 6.1 (1996), 113–134.

[19] А. В. Погорелов. *Однозначная определённость общих выпуклых поверхностей.* Монографии института математики, вып. II. 1952.

[20] I. J. Schoenberg and S. C. Zaremba. “On Cauchy’s lemma concerning convex polygons”. *Canadian J. Math.* 19 (1967), 1062–1071.

[21] K. Shiohama. *An introduction to the geometry of Alexandrov spaces.* Vol. 8. Lecture Notes Series. 1993.

[22] С. Л. Табачников и Д. Б. Фукс. *Математический дивертисмент.* 2016.

[23] В. А. Топоногов. «Римановы пространства кривизны, ограниченной снизу». *Успехи математических наук* 14.1 (85) (1959), 87—130.

[24] Ю. А. Волков. «Существование многогранника с данной разверткой». *Зап. научн. сем. ПОМИ* (2018), 50—78.

[25] A. Wald. „Begründung eiiner Koordinatenlosen Differentialgeometrie der Flächen“. *Ergebnisse eines mathematischen Kolloquium* 6 (1935), 24–46.

[26] В. А. Залгаллер. «О деформациях многоугольника на сфере». *УМН* 11.5(71) (1956), 177—178.