Homework assignments

Due 2023-08-25: 1.8, 1.11, 1.13, 1.14, 1.17. (Scan to pdf and upload to CANVAS.)

Due 2023-09-01: 2.2, 2.4, 2.5, 2.7, 2.14. (Scan to pdf and upload to CANVAS.)

Contents

Lecture 1

Definitions

The first synthetic description of curvature is due to Abraham Wald [7] published in 1936; it was his student work, written under the supervision of Karl Menger. This publication was not noticed for about 50 years [3]. In 1941, similar definitions were rediscovered by Alexandr Alexandrov [2].

A Notations

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

We will denote by $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$ the n-dimensional sphere (with angle metric), Euclidean space, and Lobachevsky space respectively. More generally, $\mathbb{M}^n(\kappa)$ will denote the m o d e l n-s p a c e of curvature κ; that is,

- ◇ if $\kappa > 0$, then $\mathbb{M}^n(\kappa)$ is the n-sphere of radius $\frac{1}{\sqrt{\kappa}}$, so $\mathbb{S}^n = \mathbb{M}^n(1)$
- ◇ $\mathbb{M}^n(0) = \mathbb{E}^n$,
- ◇ if $\kappa < 0$, then $\mathbb{M}^n(\kappa)$ is the Lobachevsky n-space $\mathbb{H}^n$ rescaled by factor $\frac{1}{\sqrt{-\kappa}}$; in particular $\mathbb{M}^n(-1) = \mathbb{H}^n$.

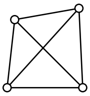
B Wald's approach

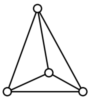
Wald noticed that a *typical* quadruple x_1, x_2, x_3, x_4 of points in a metric space admits model configurations in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \in \mathbb{M}^3(\kappa)$ with

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{M}^3(\kappa)} = |x_i - x_j|_{\mathcal{X}}$$

for κ in a closed interval, say

$$[\kappa_{\min}(x_1,x_2,x_3,x_4),\kappa_{\max}(x_1,x_2,x_3,x_4)]\subset\mathbb{R}.$$





In $\mathbb{M}^3(\kappa_{\min})$ and $\mathbb{M}^3(\kappa_{\max})$, the points $\tilde x_1,\tilde x_2,\tilde x_3,\tilde x_4$ form degenerate tetrahedrons shown on the diagram (for $\kappa_{\min}$ it is a convex quadrangle and for $\kappa_{\max}$ — a triangle with a point inside). In the interior of the interval, the tetrahedron is nondegenerate.

Moreover, one can use $[-\infty,\infty)$ instead of $\mathbb{R}$ and let

$$\kappa_{\min}(x_1,x_2,x_3,x_4)=-\infty$$

if there is *almost* model quadruple in $\mathbb{M}^3(\kappa)$ for $\kappa\to-\infty$; that is, for any $\varepsilon>0$ there is a quadruple $\tilde x_1,\tilde x_2,\tilde x_3,\tilde x_4\in\mathbb{M}^3(\kappa)$ such that $\kappa\leqslant-\frac{1}{\varepsilon}$, and

$$|\tilde x_i-\tilde x_j|_{\mathbb{M}^3(\kappa)}\lesseqgtr|x_i-x_j|_{\mathcal{X}}\pm\varepsilon$$

for all i and j. In this case the interval

$$[\kappa_{\min}(x_1,x_2,x_3,x_4),\kappa_{\max}(x_1,x_2,x_3,x_4)]\subset[-\infty,\infty)$$

is defined for *any* quadruple.

We will not use these statements further in the sequel, so we omit the proofs. The just wanted to describe the first step in the theory.

1.1. Exercise. *Let x_1,x_2,x_3,x_4 be a quadruple in a metric space such that $\kappa_{\min}(x_1,x_2,x_3,x_4)=-\infty$. Show that two maximal numbers from the following three are equal to each other.*

$$\begin{aligned}a&=|x_1-x_2|+|x_3-x_4|,\\ b&=|x_1-x_3|+|x_2-x_4|,\\ c&=|x_1-x_4|+|x_2-x_3|.\end{aligned}$$

1.2. Exercise. *Suppose that x_1,x_2,x_3,x_4 in a metric space such that*

$$\begin{aligned}|x_1-x_2|&=|x_1-x_3|=|x_1-x_4|=1,\\ |x_2-x_3|&=|x_3-x_4|=|x_4-x_1|=2.\end{aligned}$$

Show that

$$\kappa_{\min}(x_1,x_2,x_3,x_4)=\kappa_{\max}(x_1,x_2,x_3,x_4)=-\infty.$$

1.3. Exercise. *Let x_1, x_2, x_3, x_4 be a quadruple in $\mathbb{E}^2$. Suppose that triangle $[x_1x_2x_3]$ is degerate, but $[x_2x_3x_4]$ is not. Show that*

$$\kappa_{\min}(x_1, x_2, x_3, x_4) = \kappa_{\max}(x_1, x_2, x_3, x_4) = 0.$$

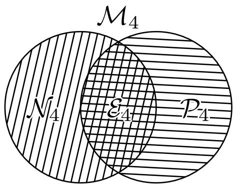
1.4. Wald-style definition. *Let $\kappa \in \mathbb{R}$. A metric space $\mathcal{X}$ has curvature $\geqslant \kappa$ (or $\leqslant \kappa$) if for any quadruple $x_1, x_2, x_3, x_4 \in \mathcal{X}$ we have $\kappa_{\max}(x_1, x_2, x_3, x_4) \geqslant \kappa$ (or $\kappa_{\min}(x_1, x_2, x_3, x_4) \leqslant \kappa$ respectively).*

C Substance

Consider the space $\mathcal{M}_4$ of all isometry classes of 4-point metric spaces. Each element in $\mathcal{M}_4$ can be described by 6 numbers — the distances between all 6 pairs of its points, say $\ell_{i,j}$ for $1 \leqslant i < j \leqslant 4$ modulo permutations of the index set $(1, 2, 3, 4)$. These 6 numbers are subject to 12 triangle inequalities; that is,

$$\ell_{i,j} + \ell_{j,k} \geqslant \ell_{i,k}$$

holds for all i, j and k, where we assume that $\ell_{j,i} = \ell_{i,j}$, and $\ell_{i,i} = 0$.



The space $\mathcal{M}_4$ comes with topology. It can be defined as a quotient topology of the cone in $\mathbb{R}^6$ by permutations of the 4 points of the space.

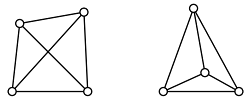
Consider the subset $\mathcal{E}_4 \subset \mathcal{M}_4$ of all isometry classes of 4-point metric spaces that admit isometric embeddings into Euclidean space.

1.5. Claim. *The complement $\mathcal{M}_4 \setminus \mathcal{E}_4$ has two connected components.*

1.6. Exercise. *Spend 10 minutes trying to prove the claim.*

The definition of Alexandrov spaces is based on the claim above. Let us denote one of the components by $\mathcal{P}_4$ and the other by $\mathcal{N}_4$. Here $\mathcal{P}$ and $\mathcal{N}$ stand for positive and negative curvature because spheres have no quadruples of type $\mathcal{N}_4$ and hyperbolic space has no quadruples of type $\mathcal{P}_4$.

A metric space that has no quadruples of points of type $\mathcal{P}_4$ or $\mathcal{N}_4$ respectively is called an Alexandrov space with non-positive (CAT(0)) or non-negative curvature (CBB(0)).



Let us describe the subdivision into $\mathcal{P}_4$, $\mathcal{E}_4$, and $\mathcal{N}_4$ intuitively. Imagine that you move out of $\mathcal{E}_4$ — your path is a one-parameter family of 4-point metric spaces. The last thing you see in $\mathcal{E}_4$ is one of the two plane configurations shown on the diagram. If you see the right configuration then you move into $\mathcal{N}_4$; if it is the one on the left, then you move into $\mathcal{P}_4$. More degenerate pictures can be avoided; for example, a triangle with a point on a side. From such a configuration one may move in $\mathcal{N}_4$ and $\mathcal{P}_4$ (as well as come back to $\mathcal{E}_4$).

Here is an exercise, solving which would force you to rebuild a considerable part of Alexandrov geometry. It is wise to spend some time thinking about this exercise before proceeding.

1.7. Advanced exercise. *Assume $\mathcal{X}$ is a complete metric space with length metric (see Section 1F), containing only quadruples of type $\mathcal{E}_4$. Show that $\mathcal{X}$ is isometric to a convex set in a Hilbert space.*

In the definition above, one can take $\mathbb{M}^3(\kappa)$ instead of $\mathbb{E}^3$. In this case, one obtains the definition of spaces with curvature bounded above or below by κ (CAT(κ) or CBB(κ)). The parameter κ has three interesting choices -1, 0, and 1; the rest can be obtained from these three applying rescaling.

D Geodesics, triangles, and angles

Geodesics. Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ a real interval. A distance-preserving map $\gamma\colon \mathbb{I}\to\mathcal{X}$ is called a g e o d e s i c[1]; in other words, $\gamma\colon \mathbb{I}\to\mathcal{X}$ is a geodesic if

$$|\gamma(s)-\gamma(t)|_{\mathcal{X}}=|s-t|$$

for any pair $s,t\in\mathbb{I}$.

If $\gamma\colon [a,b]\to\mathcal{X}$ is a geodesic such that $p=\gamma(a)$, $q=\gamma(b)$, then we say that γ is a geodesic from p to q. In this case, the image of γ is denoted by $[pq]$, and, with abuse of notations, we also call it a g e o d e s i c. We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$.

In general, a geodesic from p to q need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we assume that we have chosen such geodesic.

[1]Others call it differently: *shortest path*, *minimizing geodesic*. Also, note that the meaning of the term *geodesic* is different from what is used in Riemannian geometry, altho they are closely related.

A *geodesic path* is a geodesic with constant-speed parameterization by the unit interval $[0,1]$.

A metric space is called *geodesic* if any pair of its points can be joined by a geodesic.

Triangles. Given a triple of points p,q,r in a metric space $\mathcal{X}$, a choice of geodesics $([qr],[rp],[pq])$ will be called a *triangle*; we will use the short notation $[pqr]=[pqr]_{\mathcal{X}}=([qr],[rp],[pq])$.

Given a triple $p,q,r\in\mathcal{X}$ there may be no triangle $[pqr]$ simply because one of the pairs of these points cannot be joined by a geodesic. Also, many different triangles with these vertices may exist, any of which can be denoted by $[pqr]$. If we write $[pqr]$, it means that we have chosen such a triangle.

Model triangles. Given three points p,q,r in a metric space $\mathcal{X}$, let us define its *model triangle* $[\tilde p\tilde q\tilde r]$ (briefly, $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$) to be a triangle in the Euclidean plane $\mathbb{E}^2$ such that

$$|\tilde p-\tilde q|_{\mathbb{E}^2}=|p-q|_{\mathcal{X}},\qquad |\tilde q-\tilde r|_{\mathbb{E}^2}=|q-r|_{\mathcal{X}},\qquad |\tilde r-\tilde p|_{\mathbb{E}^2}=|r-p|_{\mathcal{X}}.$$

The same way we can define the *hyperbolic* and the *spherical model triangles* $\tilde\triangle(pqr)_{\mathbb{H}^2}$, $\tilde\triangle(pqr)_{\mathbb{S}^2}$ in the Lobachevsky plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2$. In the latter case, the model triangle is said to be defined if in addition

$$|p-q|+|q-r|+|r-p|<2{\cdot}\pi.$$

In this case, the model triangle again exists and is unique up to an isometry of $\mathbb{S}^2$.

Model angles. If $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$ and $|p-q|,|p-r|>0$, the angle measure of $[\tilde p\tilde q\tilde r]$ at $\tilde p$ will be called the *model angle* of the triple p, q, r and will be denoted by $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{E}^2}$.

The same way we define $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$; in particular, $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{H}^2}$ and $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{S}^2}$. We may use the notation $\tilde\measuredangle(p\,{}^q_r)$ if it is evident which of the model spaces is meant.

1.8. Exercise. *Show that for any triple of point p, q, and r, the function*

$$\kappa\mapsto\tilde\measuredangle(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$$

is nondecreasing in its domain of definition.

Hinges. Let $p,x,y\in\mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair of geodesics $([px],[py])$ will be called a *hinge* and will be denoted by $[p\,{}^x_y]=([px],[py])$.

E Definitions

In this section we write inequalities that describe the sets $\mathcal{E}_4 \cup \mathcal{P}_4$ and $\mathcal{E}_4 \cup \mathcal{N}_4$ from Section 1C.

Curvature bounded below. Let p, x, y, z be a quadruple of points in a metric space. If the inequality

❶ $$\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^y_z)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^z_x)_{\mathbb{E}^2} \leqslant 2\cdot\pi$$

holds, then we say that the quadruple meets CBB(0) comparison.

1.9. Definition. *A metric space $\mathcal{X}$ has nonnegative curvature in the sense of Alexandrov (briefly, $\mathcal{X} \in$ CBB(0) if* CBB(0) *comparison holds for any quadruple in $\mathcal{X}$ such that each model angle in* ❶ *is defined.*

If instead of $\mathbb{E}^2$, we use $\mathbb{S}^2$ or $\mathbb{H}^2$, then we get the definition of CBB(1) and CBB(−1) comparisons. Note that $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2}$ and $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2}$ are defined if $p \ne x$, $p \ne y$, but for $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2}$ we need in addition, $|p-x| + |p-y| + |x-y| < 2\cdot\pi$.

More generally, one may apply this definition to $\mathbb{M}^2(\kappa)$. This way we define CBB(κ) comparison for any real κ.

1.10. Exercise. *Show that $\mathbb{E}^n$ is* CBB(0).

1.11. Exercise. *Show that a metric space $\mathcal{X}$ is* CBB(0) *if and only if for any quadruple of points $p, x_1, x_2, x_3 \in \mathcal{X}$ there is a quadruple of points $q, y_1, y_2, y_3 \in \mathbb{E}^3$ such that*

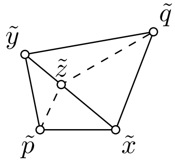
$$|p - x_i|_{\mathcal{X}} \geqslant |q - y_i|_{\mathbb{E}^2} \quad \textit{and} \quad |x_i - x_j|_{\mathcal{X}} \leqslant |y_i - y_j|_{\mathbb{E}^2}$$

for all i and j.

Curvature bounded above. Given a quadruple of points p, q, x, y in a metric space $\mathcal{X}$, consider two model triangles $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{E}^2}$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)_{\mathbb{E}^2}$ with common side $[\tilde x\tilde y]$.

If the inequality

$$|p - q|_{\mathcal{X}} \leqslant |\tilde p - \tilde z|_{\mathbb{E}^2} + |\tilde z - \tilde q|_{\mathbb{E}^2}$$



holds for any point $\tilde z \in [\tilde x\tilde y]$, then we say that the quadruple p, q, x, y satisfies CAT(0) comparison.

1.12. Definition. *A metric space $\mathcal{X}$ has nonpositive curvature in the sense of Alexandrov (briefly, $\mathcal{X} \in$ CAT(0)) if* CAT(0) *comparison holds for any quadruple in $\mathcal{X}$.*

If we do the same for spherical model triangles $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)_{\mathbb{S}^2}$ and $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\triangle}(qxy)_{\mathbb{S}^2}$, then we arrive at the definition of CAT(1) comparison. One of the spherical model triangles might undefined; it happens if

$$|p-x|+|p-y|+|x-y|\geqslant 2\cdot\pi \quad \text{or} \quad |q-x|+|q-y|+|x-y|\geqslant 2\cdot\pi.$$

In this case, it is assumed that CAT(1) comparison automatically holds for this quadruple.

We can do the same for $\mathbb{M}^2(\kappa)$. In this case, we arrive at the definition of CAT(κ) comparison. However, we will mostly consider CAT(0) comparison and occasionally CAT(1) comparison; so, if you see CAT(κ), then it is safe to assume that κ is 0 or 1.

Here CAT is an acronym for Cartan, Alexandrov, and Toponogov, but usually pronounced as "cat" in the sense of "miauw". The term was coined by Mikhael Gromov in 1987. Originally, Alexandrov used $\mathfrak{R}_\kappa$ domain; this term is still in use.

1.13. Exercise. *Show that a metric space $\mathcal{U}$ is* CAT(0) *if and only if for any quadruple of points p,q,x,y in $\mathcal{U}$ there is a quadruple $\tilde{p},\tilde{q},\tilde{x},\tilde{y}$ in $\mathbb{E}^2$ such that*

$$\begin{aligned} |\tilde{p}-\tilde{q}| &\geqslant |p-q|, & |\tilde{x}-\tilde{y}| &\geqslant |x-y|, \\ |\tilde{p}-\tilde{x}| &\leqslant |p-x|, & |\tilde{p}-\tilde{y}| &\leqslant |p-y|, \\ |\tilde{q}-\tilde{x}| &\leqslant |q-x|, & |\tilde{q}-\tilde{y}| &\leqslant |q-y|. \end{aligned}$$

1.14. Exercise. *Assume that a quadruple of points in a metric space satisfies* CBB(0) *and* CAT(0) *comparisons for all labelings. Show that it is isometric to a quadruple in $\mathbb{E}^3$.*

The definitions stated in the this section can be applied to any metric space. However, interesting things happen only for the so-called *geodesic* or at least *length spaces*.

F Length and length spaces

Length. A curve is defined as a continuous map from a real interval $\mathbb{I}$ to a metric space. If $\mathbb{I}=[0,1]$, then the curve is called a path.

1.15. Definition. *Let $\mathcal{X}$ be a metric space and $\alpha\colon\mathbb{I}\to\mathcal{X}$ be a curve. We define the length of α as*

$$\operatorname{length}\alpha := \sup_{t_0\leqslant t_1\leqslant\ldots\leqslant t_n}\sum_i |\alpha(t_i)-\alpha(t_{i-1})|.$$

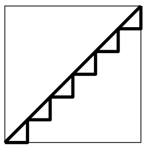
A curve α is called rectifiable *if* $\operatorname{length}\alpha < \infty$.

The following theorem is assumed to be known; see [4, 5].

1.16. Theorem. *Length is a lower semi-continuous with respect to the pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\gamma_n\colon \mathbb{I}\to\mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty\colon \mathbb{I}\to\mathcal{X}$; that is, for any fixed $t\in\mathbb{I}$ we have $\gamma_n(t)\to\gamma_\infty(t)$ as $n\to\infty$. Then

$$\underline{\lim}_{n\to\infty} \operatorname{length}\gamma_n \geqslant \operatorname{length}\gamma_\infty. \tag{❶}$$



Note that the inequality ❶ might be strict. For example, the diagonal γ_∞ of the unit square can be approximated by stairs-like polygonal curves γ_n with sides parallel to the sides of the square (γ_6 is on the picture). In this case

$$\operatorname{length}\gamma_\infty = \sqrt{2} \quad\text{and}\quad \operatorname{length}\gamma_n = 2$$

for any n.

Length spaces. Let $\mathcal{X}$ be a metric space. If for any $\varepsilon > 0$ and any pair of points $x, y \in \mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length}\alpha < |x-y| + \varepsilon,$$

then $\mathcal{X}$ is called a length space and the metric on $\mathcal{X}$ is called a length metric.

Evidently, any geodesic space is a length space.

1.17. Exercise. *Show that any compact length space is geodesic.*

Induced length metric. Directly from the definition, it follows that if $\alpha\colon[0,1]\to\mathcal{X}$ is a path from x to y (that is, $\alpha(0)=x$ and $\alpha(1)=y$), then

$$\operatorname{length}\alpha \geqslant |x-y|.$$

Set

$$\|x-y\| = \inf\{\operatorname{length}\alpha\}$$

where the greatest lower bound is taken for all paths from x to y. It is straightforward to check that $(x,y)\mapsto\|x-y\|$ is an ∞-metric; that is, $(x,y)\mapsto\|x-y\|$ is a metric in the extended positive reals $[0,\infty]$. The metric $\|{*}-{*}\|$ is called the induced length metric.

1.18. Exercise. *Let $\mathcal{X}$ be a complete length space. Show that for any compact subset $K \subset \mathcal{X}$ there is a compact path-connected subset $K' \subset \mathcal{X}$ that contains K.*

1.19. Exercise. *Suppose $(\mathcal{X}, |{*}-{*}|)$ is a complete metric space. Show that $(\mathcal{X}, \|{*}-{*}\|)$ is complete.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x-y|_A = \inf_{\alpha}\{\,\mathrm{length}\,\alpha\,\},$$

where the greatest lower bound is taken for all paths α from x to y in A. In other words, $|{*}-{*}|_A$ denotes the induced length metric on the subspace A. (The notation $|{*}-{*}|_A$ conflicts with the previously defined notation for distance $|x-y|_{\mathcal{X}}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.)

G Embedding theorem

The following theorem is historically the first remarkable result in Alexandrov geometry. The main part of the following theorem is due to Alexandr Alexandrov [1]. The last part is very difficult; it was proved by Aleksei Pogorelov [6].

1.20. Theorem. *A metric space $\mathcal{X}$ is isometric to the surface of a convex body in the Euclidean space if and only if $\mathcal{X}$ is a geodesic* CBB(0) *space that is homeomorphic to $\mathbb{S}^2$.*

Moreover, $\mathcal{X}$ determines the convex body up to congruence.

The convex body above is a compact convex subset in $\mathbb{E}^3$; we assume that it does not lie in a line but might degenerate to a plane figure, say F. In the latter case, its surface is defined as two copies of F glued along the boundary. For nondegenerate convex body B, its surface is its boundary ∂B equipped with the induced length metric.

The only-if part of the theorem is the simplest; we will give a complete proof of it eventually. The if part will be sketched. We will not touch the last part.

Lecture 2

Angles

A Definition

The angle measure of a hinge $[p\,{}^x_y]$ is defined as the following limit

$$\measuredangle[p\,{}^x_y] = \lim_{\bar x,\bar y\to p} \tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x \in\,]px]$ and $\bar y \in\,]py]$.

Note that if $\measuredangle[p\,{}^x_y]$ is defined, then

$$0 \leqslant \measuredangle[p\,{}^x_y] \leqslant \pi.$$

2.1. Exercise. *Suppose that in the above definition, one uses spherical or hyperbolic model angles instead of Euclidean. Show that it does not change the value* $\measuredangle[p\,{}^x_y]$.

2.2. Exercise. *Give an example of a hinge* $[p\,{}^x_y]$ *in a metric space with an undefined angle measure* $\measuredangle[p\,{}^x_y]$.

B Triangle inequality

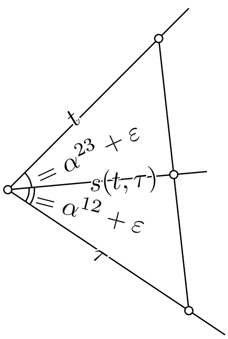
2.3. Proposition. *Let* $[px_1]$, $[px_2]$, *and* $[px_3]$ *be three geodesics in a metric space. Suppose all the angle measure* $\alpha_{ij} = \measuredangle[p\,{}^{x_i}_{x_j}]$ *are defined. Then*

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23}.$$

Proof. Since $\alpha_{13} \leqslant \pi$, we can assume that $\alpha_{12} + \alpha_{23} < \pi$. Denote by γ_i the unit-speed parametrization of $[px_i]$ from p to x_i. Given any

$\varepsilon > 0$, for all sufficiently small $t, \tau, s \in \mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned}|\gamma_1(t) - \gamma_3(\tau)| \leqslant |\gamma_1(t) - \gamma_2(s)| + |\gamma_2(s) - \gamma_3(\tau)| <\\ < \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha_{12} + \varepsilon)} +\\ + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\alpha_{23} + \varepsilon)} \leqslant\end{aligned}$$



Below we define $s(t,\tau)$ so that for $s = s(t,\tau)$, this chain of inequalities can be continued as follows:

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon.$$

Hence the result follows.

To define $s(t,\tau)$, consider three half-lines $\tilde\gamma_1$, $\tilde\gamma_2$, $\tilde\gamma_3$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde\gamma_1, \tilde\gamma_2) = \alpha_{12} + \varepsilon$, $\measuredangle(\tilde\gamma_2, \tilde\gamma_3) = \alpha_{23} + \varepsilon$, and $\measuredangle(\tilde\gamma_1, \tilde\gamma_3) = \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon$. We parametrize each half-line by the distance from the starting point. Given two positive numbers $t, \tau \in \mathbb{R}_{\geqslant 0}$, let $s = s(t,\tau)$ be the number such that $\tilde\gamma_2(s) \in [\tilde\gamma_1(t)\ \tilde\gamma_3(\tau)]$. Clearly, $s \leqslant \max\{t, \tau\}$, so t, τ, s may be taken sufficiently small. $\square$

2.4. Exercise. *Prove that the sum of adjacent angles is at least π.*

More precisely: suppose two hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ are adjacent; that is, they share side $[pz]$, and the union of two sides $[px]$ and $[py]$ form a geodesic $[xy]$. Show that

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \geqslant \pi$$

whenever each angle on the left-hand side is defined.

Give an example showing that the inequality can be strict.

2.5. Exercise. *Assume that a hinge $[q\,{}^p_x]$ with defined angle measure. Let γ be the unit speed parametrization of $[qx]$ from q to x. Show that*

$$|p - \gamma(t)| \leqslant |q - p| - t{\cdot}\cos(\measuredangle[q\,{}^p_x]) + o(t).$$

C Alexandrov's lemma

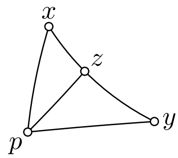
Recall that $[xy]$ denotes a geodesic from x to y; set

$$]xy] = [xy] \setminus \{x\}, \quad [xy[= [xy] \setminus \{y\}, \quad]xy[= [xy] \setminus \{x, y\}.$$

2.6. Lemma. *Let p, x, y, z be distinct points in a metric space such that $z \in \,]xy[$. Then the following expressions for the Euclidean model angles have the same sign:*

(a) $\tilde\measuredangle(x\,{}^{p}_{y}) - \tilde\measuredangle(x\,{}^{p}_{z})$,

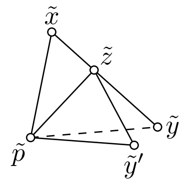
(b) $\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) - \pi$.



The same holds for the hyperbolic and spherical model angles, but in the latter case, one has to assume in addition that

$$|p-z| + |p-y| + |x-y| < 2{\cdot}\pi.$$

Proof. Consider the model triangle $[\tilde x\tilde p\tilde z] = \tilde\triangle(xpz)$. Take a point $\tilde y$ on the extension of $[\tilde x\tilde z]$ beyond $\tilde z$ so that $|\tilde x - \tilde y| = |x-y|$ (and therefore $|\tilde x - \tilde z| = |x - z|$).



Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:

(i) $\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(x\,{}^{p}_{y})$,

(ii) $|\tilde p - \tilde y| - |p - y|$,

(iii) $\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(z\,{}^{p}_{y})$.

Since

$$\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] = \measuredangle[\tilde x\,{}^{\tilde p}_{\tilde z}] = \tilde\measuredangle(x\,{}^{p}_{z})$$

and

$$\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] = \pi - \measuredangle[\tilde z\,{}^{\tilde x}_{\tilde p}] = \pi - \tilde\measuredangle(z\,{}^{x}_{p}),$$

the statement follows.

The spherical and hyperbolic cases can be proved the same way. □

2.7. Exercise. *Assume p, x, y, z are as in Alexandrov's lemma. Show that*

$$\tilde\measuredangle(p\,{}^{x}_{y}) \geqslant \tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{z}_{y}),$$

with equality if and only if the expressions in (a) and (b) vanish.

D CBB comparison

Note that

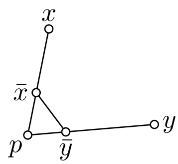
$$p \in \,]xy[\quad\Longrightarrow\quad \tilde\measuredangle(p\,{}^{x}_{y}) = \pi.$$

Applying it with Alexandrov's lemma and CBB(0) comparison, we get the following claim and its corollary.

2.8. Claim. *If p,x,y,z are points in a* CBB(0) *such that $p\in\,]xy[$, then*

$$\tilde\measuredangle(x\,{}^{y}_{z})\leqslant\tilde\measuredangle(x\,{}^{p}_{z}).$$

2.9. Exercise. *Let $[p\,{}^{x}_{y}]$ be a hinge in a* CBB(0) *space. Consider the function*



$$f\colon(|p-\bar x|,|p-\bar y|)\mapsto\tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x\in\,]px]$ and $\bar y\in\,]py]$. Show that f is nonincreasing in each argument.

Note that 2.9 implies the following.

2.10. Claim. *For any hinge $[p\,{}^{x}_{y}]$ in a* CBB(0) *space, the angle measure $\measuredangle[p\,{}^{x}_{y}]$ is defined, and*

$$\measuredangle[p\,{}^{x}_{y}]\geqslant\tilde\measuredangle(p\,{}^{x}_{y}).$$

2.11. Exercise. *Let $[p\,{}^{x}_{y}]$ be a hinge in a* CBB(0) *space. Suppose $\measuredangle[p\,{}^{x}_{y}]=0$ show that $[px]\subset[py]$ or $[py]\subset[px]$.*

2.12. Exercise. *Let $[xy]$ be a geodesic in a* CBB(0) *space. Suppose $z\in\,]xy[$ show that there is a unique geodesic $[xz]$ and $[xz]\subset[xy]$.*

2.13. Exercise. *Let $[p\,{}^{x}_{z}]$ and $[p\,{}^{y}_{z}]$ be adjacent hinges in a* CBB(0) *space. Show that*

$$\measuredangle[p\,{}^{x}_{z}]+\measuredangle[p\,{}^{y}_{z}]=\pi.$$

2.14. Exercise. *Let p,x,y in a* CBB(0) *space and $v,w\in\,]xy[$. Show that*

$$\tilde\measuredangle(x\,{}^{y}_{p})=\tilde\measuredangle(x\,{}^{v}_{p})\quad\Longleftrightarrow\quad\tilde\measuredangle(x\,{}^{y}_{p})=\tilde\measuredangle(x\,{}^{w}_{p}).$$

E Hinge comparison

Let $[p\,{}^{x}_{y}]$ be a hinge in a CBB(0) space. By 2.11, the angle measure $\measuredangle[p\,{}^{x}_{y}]$ is defined and

$$\measuredangle[p\,{}^{x}_{y}]\geqslant\tilde\measuredangle(p\,{}^{x}_{y}).$$

Further, according to 2.13, we have

$$\measuredangle[p\,{}^{x}_{z}]+\measuredangle[p\,{}^{y}_{z}]=\pi$$

for adjacent hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ in a CBB(0) space.

The following theorem implies that a geodesic space is CBB(0) if the above conditions hold for all its hinges.

2.15. Theorem. *A geodesic space $\mathcal{L}$ is* CBB(0) *if the following conditions hold.*

(a) For any hinge $[x\,{}^p_y]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and

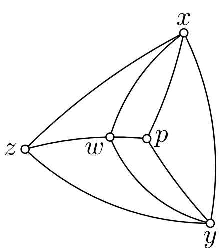
$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle(x\,{}^p_y).$$

(b) For any two adjacent hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ in $\mathcal{L}$, we have

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \leqslant \pi.$$

Proof. Consider a point $w \in \,]pz[$ close to p. From *(b)*, it follows that

$$\measuredangle[w\,{}^x_z] + \measuredangle[w\,{}^x_p] \leqslant \pi \quad \text{and} \quad \measuredangle[w\,{}^y_z] + \measuredangle[w\,{}^y_p] \leqslant \pi.$$



Since $\measuredangle[w\,{}^x_y] \leqslant \measuredangle[w\,{}^x_p] + \measuredangle[w\,{}^y_p]$ (see 2.3), we get

$$\measuredangle[w\,{}^x_z] + \measuredangle[w\,{}^y_z] + \measuredangle[w\,{}^x_y] \leqslant 2{\cdot}\pi.$$

Applying *(a)*,

$$\tilde\measuredangle(w\,{}^x_z) + \tilde\measuredangle(w\,{}^y_z) + \tilde\measuredangle(w\,{}^x_y) \leqslant 2{\cdot}\pi.$$

Passing to the limits $w \to p$, we have

$$\tilde\measuredangle(p\,{}^x_z) + \tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^x_y) \leqslant 2{\cdot}\pi.$$

□

F Equivalent conditions

The following theorem summarizes 2.8, 2.10, 2.13, 2.15.

2.16. Theorem. *Let $\mathcal{L}$ be a geodesic space. Then the following conditions are equivalent.*

(a) $\mathcal{L}$ is CBB(0).

(b) (adjacent angle comparison) for any geodesic $[xy]$ and point $z \in \,]xy[$, $z \ne p$ in $\mathcal{L}$, we have

$$\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) \leqslant \pi.$$

(c) (point-on-side comparison) for any geodesic $[xy]$ and $z \in \,]xy[$ in $\mathcal{L}$, we have

$$\tilde\measuredangle(x\,{}^p_y) \leqslant \tilde\measuredangle(x\,{}^p_z).$$

(d) (hinge comparison) for any hinge $[x\,{}^{p}_{y}]$ *in* $\mathcal{L}$, *the angle* $\measuredangle[x\,{}^{p}_{y}]$ *is defined and*

$$\measuredangle[x\,{}^{p}_{y}] \geqslant \tilde{\measuredangle}(x\,{}^{p}_{y}).$$

Moreover,

$$\measuredangle[z\,{}^{p}_{y}] + \measuredangle[z\,{}^{p}_{x}] \leqslant \pi$$

for any adjacent hinges $[z\,{}^{p}_{y}]$ *and* $[z\,{}^{p}_{x}]$.

Moreover, the implications (a)⇒(b)⇒(c)⇒(d) hold in any space, not necessarily geodesic.

2.17. Advanced Exercise. *Construct a geodesic space* $\mathcal{X} \notin \mathrm{CBB}(0)$ *that meets the following condition: for any 3 points* $p, x, y \in \mathcal{X}$ *there is a geodesic* $[xy]$ *such that for any* $z \in\,]xy[$

$$\tilde{\measuredangle}(z\,{}^{p}_{x}) + \tilde{\measuredangle}(z\,{}^{p}_{y}) \leqslant \pi.$$

G Function comparison

Real-to-real functions. Choose $\lambda \in \mathbb{R}$. Let $s\colon \mathbb{I} \to \mathbb{R}$ be a locally Lipschitz function defined on an interval $\mathbb{I}$. We say that s is λ-concave if $s'' \leqslant \lambda$, where the second derivative s'' is understood in the sense of distributions.

Equivalently, *s is λ-concave if the function $h\colon t \mapsto s(t) - \lambda\cdot\frac{t^2}{2}$ is concave.* Concavity can be defined via Jensen inequality; that is,

$$h(s\cdot t_0 + (1-s)\cdot t_1) \geqslant s\cdot h(t_0) + (1-s)\cdot h(t_1)$$

for any $t_0, t_1 \in \mathbb{I}$ and $s \in [0,1]$. It could be also defined via existence of (local) upper support at any point: *for any $t_0 \in \mathbb{I}$ there is a linear function ℓ that (locally) supports h at t_0 from above; that is, $\ell(t_0) = h(t_0)$ and $\ell(t) \geqslant h(t)$ for any t (in a neighborhood of t_0).*

The equivalence of these definitions is assumed to be known. We will also use that *λ-concave functions are one-side differentiable.*

A function on a metric space $\mathcal{L}$ will usually mean a *locally Lipschitz real-valued function defined in an open subset of $\mathcal{L}$.* The domain of definition of a function f will be denoted by $\operatorname{Dom} f$.

Functions on metric space. Let f be a function on a metric space $\mathcal{L}$. We say that f is λ-concave (briefly $f'' \leqslant \lambda$) if for any unit-speed geodesic $\gamma\colon \mathbb{I} \to \operatorname{Dom} f$ the real-to-real function $t \mapsto f\circ\gamma(t)$ is λ-concave.

The following proposition is conceptual — it reformulates a global geometric condition into an infinitesimal condition on distance functions.

2.18. Proposition. *A geodesic space $\mathcal{L}$ is* CBB(0) *if and only if $f'' \leqslant 1$ for any function f of the following type*

$$f\colon x \mapsto \tfrac{1}{2}\cdot|p-x|^2.$$

Proof. Choose a unit-speed geodesic γ in $\mathcal{L}$ and two points $x=\gamma(t_0)$, $y=\gamma(t_1)$ for some $t_0<t_1$. Consider the model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$. Let $\tilde{\gamma}\colon [t_0,t_1]\to\mathbb{E}^2$ be the unit-speed parametrization of $[\tilde{x}\tilde{y}]$ from $\tilde{x}$ to $\tilde{y}$.

Set

$$\tilde{r}(t) := |\tilde{p}-\tilde{\gamma}(t)|, \qquad\qquad r(t) := |p-\gamma(t)|.$$

Clearly, $\tilde{r}(t_0)=r(t_0)$ and $\tilde{r}(t_1)=r(t_1)$. Note that the point-on-side comparison (2.16c) is equivalent to

❶ $$t_0\leqslant t\leqslant t_1 \qquad \Longrightarrow \qquad \tilde{r}(t)\leqslant r(t)$$

for any γ and $t_0<t_1$.

Set

$$\tilde{h}(t)=\tfrac{1}{2}\cdot\tilde{r}^2(t)-\tfrac{1}{2}\cdot t^2, \qquad\qquad h=\tfrac{1}{2}\cdot r^2(t)-\tfrac{1}{2}\cdot t^2.$$

Note that $\tilde{h}$ is linear, $\tilde{h}(t_0)=h(t_0)$ and $\tilde{h}(t_1)=h(t_1)$. Observe that the Jensen inequality for the function h is equivalent to ❶. Hence the proposition follows. □

H Comments

All the discussed statements admit natural generalizations to CBB(κ) spaces. Most of the time the proof is the same with uglier formulas.

For example, the function comparison of CBB(-1) states that $f''\leqslant f$ for any function of the type $f=\cosh\circ\mathrm{dist}_p$. Similarly, the function comparison of CBB(1) states that for any point p we have $f''\leqslant -f$ for the function $f=-\cos\circ\mathrm{dist}_p$ defined in $\mathrm{B}(p,\pi)$. The meaning of these inequalities is the same — distance functions in CBB(κ) are more concave than distance functions in $\mathbb{M}(\kappa)$. The inequality $f''\leqslant\varphi$ can means that for any point p in the domain of definition and any $\varepsilon>0$ there is a neighborhood $U\ni p$ such that $f''\leqslant\varphi(p)+\varepsilon$ in U. Here we assume that f and φ are continuous and defined in open set.

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