Homework assignments

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Due 2023-08-25: 1.8, 1.11, 1.13, 1.14, 1.17.

Due 2023-09-01: 2.2, 2.4, 2.5, 2.7, 2.14.

Due 2023-09-11: 2.11, 3.6, 3.12, 4.5, 4.8.

Due 2023-09-18: 5.3, 5.9, 5.10, 5.16, 5.29.

Contents

Lecture 1

Definitions

The first synthetic description of curvature is due to Abraham Wald [43] published in 1936; it was his student work, written under the supervision of Karl Menger. This publication was not noticed for about 50 years [14]. In 1941, similar definitions were rediscovered by Alexandr Alexandrov [10].

A Notations

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

We will denote by $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$ the n-dimensional sphere (with angle metric), Euclidean space, and Lobachevsky space respectively. More generally, $\mathbb{M}^n(\kappa)$ will denote the model n-space of curvature κ; that is,

- ◇ if $\kappa > 0$, then $\mathbb{M}^n(\kappa)$ is the n-sphere of radius $\frac{1}{\sqrt{\kappa}}$, so $\mathbb{S}^n = \mathbb{M}^n(1)$
- ◇ $\mathbb{M}^n(0) = \mathbb{E}^n$,
- ◇ if $\kappa < 0$, then $\mathbb{M}^n(\kappa)$ is the Lobachevsky n-space $\mathbb{H}^n$ rescaled by factor $\frac{1}{\sqrt{-\kappa}}$; in particular $\mathbb{M}^n(-1) = \mathbb{H}^n$.

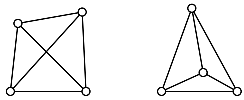
B Wald's approach

Wald noticed that a *typical* quadruple x_1, x_2, x_3, x_4 of points in a metric space admits model configurations in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \in \mathbb{M}^3(\kappa)$ with

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{M}^3(\kappa)} = |x_i - x_j|_{\mathcal{X}}$$

for κ in a closed interval, say

$$[\kappa_{\min}(x_1, x_2, x_3, x_4), \kappa_{\max}(x_1, x_2, x_3, x_4)] \subset \mathbb{R}.$$



In $\mathbb{M}^3(\kappa_{\min})$ and $\mathbb{M}^3(\kappa_{\max})$, the points $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$ form degenerate tetrahedrons shown on the diagram (for $\kappa_{\min}$ it is a convex quadrangle and for $\kappa_{\max}$ — a triangle with a point inside). In the interior of the interval, the tetrahedron is nondegenerate.

Moreover, one can use $[-\infty, \infty)$ instead of $\mathbb{R}$ and let

$$\kappa_{\min}(x_1, x_2, x_3, x_4) = -\infty$$

if there is *almost* model quadruple in $\mathbb{M}^3(\kappa)$ for $\kappa \to -\infty$; that is, for any $\varepsilon > 0$ there is a quadruple $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \in \mathbb{M}^3(\kappa)$ such that $\kappa \leqslant -\frac{1}{\varepsilon}$, and

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{M}^3(\kappa)} \lessgtr |x_i - x_j|_{\mathcal{X}} \pm \varepsilon$$

for all i and j. In this case the interval

$$[\kappa_{\min}(x_1, x_2, x_3, x_4), \kappa_{\max}(x_1, x_2, x_3, x_4)] \subset [-\infty, \infty)$$

is defined for *any* quadruple.

We will not use these statements further in the sequel, so we omit the proofs. The just wanted to describe the first step in the theory.

1.1. Exercise. *Let x_1, x_2, x_3, x_4 be a quadruple in a metric space such that $\kappa_{\min}(x_1, x_2, x_3, x_4) = -\infty$. Show that two maximal numbers from the following three are equal to each other.*

$$\begin{aligned} a &= |x_1 - x_2| + |x_3 - x_4|, \\ b &= |x_1 - x_3| + |x_2 - x_4|, \\ c &= |x_1 - x_4| + |x_2 - x_3|. \end{aligned}$$

1.2. Exercise. *Suppose that x_1, x_2, x_3, x_4 in a metric space such that*

$$\begin{aligned} |x_1 - x_2| &= |x_1 - x_3| = |x_1 - x_4| = 1, \\ |x_2 - x_3| &= |x_3 - x_4| = |x_4 - x_1| = 2. \end{aligned}$$

Show that

$$\kappa_{\min}(x_1, x_2, x_3, x_4) = \kappa_{\max}(x_1, x_2, x_3, x_4) = -\infty.$$

1.3. Exercise. *Let x_1, x_2, x_3, x_4 be a quadruple in $\mathbb{E}^2$. Suppose that triangle $[x_1x_2x_3]$ is degerate, but $[x_2x_3x_4]$ is not. Show that*

$$\kappa_{\min}(x_1, x_2, x_3, x_4) = \kappa_{\max}(x_1, x_2, x_3, x_4) = 0.$$

1.4. Wald-style definition. *Let $\kappa \in \mathbb{R}$. A metric space $\mathcal{X}$ has curvature $\geqslant \kappa$ (or $\leqslant \kappa$) if for any quadruple $x_1, x_2, x_3, x_4 \in \mathcal{X}$ we have $\kappa_{\max}(x_1, x_2, x_3, x_4) \geqslant \kappa$ (or $\kappa_{\min}(x_1, x_2, x_3, x_4) \leqslant \kappa$ respectively).*

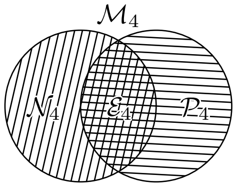
C Substance

Consider the space $\mathcal{M}_4$ of all isometry classes of 4-point metric spaces. Each element in $\mathcal{M}_4$ can be described by 6 numbers — the distances between all 6 pairs of its points, say $\ell_{i,j}$ for $1 \leqslant i < j \leqslant 4$ modulo permutations of the index set $(1, 2, 3, 4)$. These 6 numbers are subject to 12 triangle inequalities; that is,

$$\ell_{i,j} + \ell_{j,k} \geqslant \ell_{i,k}$$

holds for all i, j and k, where we assume that $\ell_{j,i} = \ell_{i,j}$, and $\ell_{i,i} = 0$.

The space $\mathcal{M}_4$ comes with topology. It can be defined as a quotient topology of the cone in $\mathbb{R}^6$ by permutations of the 4 points of the space.



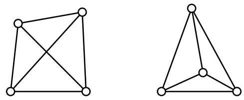
Consider the subset $\mathcal{E}_4 \subset \mathcal{M}_4$ of all isometry classes of 4-point metric spaces that admit isometric embeddings into Euclidean space.

1.5. Claim. *The complement $\mathcal{M}_4 \setminus \mathcal{E}_4$ has two connected components.*

1.6. Exercise. *Spend 10 minutes trying to prove the claim.*

The definition of Alexandrov spaces is based on the claim above. Let us denote one of the components by $\mathcal{P}_4$ and the other by $\mathcal{N}_4$. Here $\mathcal{P}$ and $\mathcal{N}$ stand for p o s i t i v e and n e g a t i v e c u r v a t u r e because spheres have no quadruples of type $\mathcal{N}_4$ and hyperbolic space has no quadruples of type $\mathcal{P}_4$.

A metric space that has no quadruples of points of type $\mathcal{P}_4$ or $\mathcal{N}_4$ respectively is called an Alexandrov space with non-positive (CAT(0)) or non-negative curvature (CBB(0)).



Let us describe the subdivision into $\mathcal{P}_4$, $\mathcal{E}_4$, and $\mathcal{N}_4$ intuitively. Imagine that you move out of $\mathcal{E}_4$ — your path is a one-parameter family of 4-point metric spaces. The last thing you see in $\mathcal{E}_4$ is one of the two plane configurations shown on the diagram. If you see the right configuration then you move into $\mathcal{N}_4$; if it is the one on the left, then you move into $\mathcal{P}_4$. More degenerate pictures can be avoided; for example, a triangle with a point on a side. From such a configuration one may move in $\mathcal{N}_4$ and $\mathcal{P}_4$ (as well as come back to $\mathcal{E}_4$).

Here is an exercise, solving which would force you to rebuild a considerable part of Alexandrov geometry. It is wise to spend some time thinking about this exercise before proceeding.

1.7. Advanced exercise. *Assume $\mathcal{X}$ is a complete metric space with length metric (see Section 1F), containing only quadruples of type $\mathcal{E}_4$. Show that $\mathcal{X}$ is isometric to a convex set in a Hilbert space.*

In the definition above, one can take $\mathbb{M}^3(\kappa)$ instead of $\mathbb{E}^3$. In this case, one obtains the definition of spaces with curvature bounded above or below by κ (CAT(κ) or CBB(κ)). The parameter κ has three interesting choices -1, 0, and 1; the rest can be obtained from these three applying rescaling.

D Geodesics, triangles, and angles

Geodesics. Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ a real interval. A distance-preserving map $\gamma\colon \mathbb{I}\to\mathcal{X}$ is called a geodesic[1]; in other words, $\gamma\colon \mathbb{I}\to\mathcal{X}$ is a geodesic if

$$|\gamma(s)-\gamma(t)|_{\mathcal{X}}=|s-t|$$

for any pair $s,t\in\mathbb{I}$.

If $\gamma\colon [a,b]\to\mathcal{X}$ is a geodesic such that $p=\gamma(a)$, $q=\gamma(b)$, then we say that γ is a geodesic from p to q. In this case, the image of γ is denoted by $[pq]$, and, with abuse of notations, we also call it a geodesic. We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$.

In general, a geodesic from p to q need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we assume that we have chosen such geodesic.

[1]Others call it differently: *shortest path*, *minimizing geodesic*. Also, note that the meaning of the term *geodesic* is different from what is used in Riemannian geometry, altho they are closely related.

A *geodesic path* is a geodesic with constant-speed parameterization by the unit interval $[0,1]$.

A metric space is called *geodesic* if any pair of its points can be joined by a geodesic.

Triangles. Given a triple of points p,q,r in a metric space $\mathcal{X}$, a choice of geodesics $([qr],[rp],[pq])$ will be called a *triangle*; we will use the short notation $[pqr]=[pqr]_{\mathcal{X}}=([qr],[rp],[pq])$.

Given a triple $p,q,r\in\mathcal{X}$ there may be no triangle $[pqr]$ simply because one of the pairs of these points cannot be joined by a geodesic. Also, many different triangles with these vertices may exist, any of which can be denoted by $[pqr]$. If we write $[pqr]$, it means that we have chosen such a triangle.

Model triangles. Given three points p,q,r in a metric space $\mathcal{X}$, let us define its *model triangle* $[\tilde p\tilde q\tilde r]$ (briefly, $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$) to be a triangle in the Euclidean plane $\mathbb{E}^2$ such that

$$|\tilde p-\tilde q|_{\mathbb{E}^2}=|p-q|_{\mathcal{X}},\qquad|\tilde q-\tilde r|_{\mathbb{E}^2}=|q-r|_{\mathcal{X}},\qquad|\tilde r-\tilde p|_{\mathbb{E}^2}=|r-p|_{\mathcal{X}}.$$

The same way we can define the *hyperbolic* and the *spherical model triangles* $\tilde\triangle(pqr)_{\mathbb{H}^2}$, $\tilde\triangle(pqr)_{\mathbb{S}^2}$ in the Lobachevsky plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2$. In the latter case, the model triangle is said to be defined if in addition

$$|p-q|+|q-r|+|r-p|<2{\cdot}\pi.$$

In this case, the model triangle again exists and is unique up to an isometry of $\mathbb{S}^2$.

Model angles. If $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$ and $|p-q|,|p-r|>0$, the angle measure of $[\tilde p\tilde q\tilde r]$ at $\tilde p$ will be called the *model angle* of the triple p, q, r and will be denoted by $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{E}^2}$.

The same way we define $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$; in particular, $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{H}^2}$ and $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{S}^2}$. We may use the notation $\tilde\measuredangle(p\,{}^q_r)$ if it is evident which of the model spaces is meant.

1.8. Exercise. *Show that for any triple of point p, q, and r, the function*

$$\kappa\mapsto\tilde\measuredangle(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$$

is nondecreasing in its domain of definition.

Hinges. Let $p,x,y\in\mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair of geodesics $([px],[py])$ will be called a *hinge* and will be denoted by $[p\,{}^x_y]=([px],[py])$.

E Definitions

In this section we write inequalities that describe the sets $\mathcal{E}_4 \cup \mathcal{P}_4$ and $\mathcal{E}_4 \cup \mathcal{N}_4$ from Section 1C.

Curvature bounded below. Let p, x, y, z be a quadruple of points in a metric space. If the inequality

❶ $$\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^y_z)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^z_x)_{\mathbb{E}^2} \leqslant 2\cdot\pi$$

holds, then we say that the quadruple meets CBB(0) comparison.

1.9. Definition. *A metric space $\mathcal{X}$ has nonnegative curvature in the sense of Alexandrov (briefly, $\mathcal{X} \in \mathrm{CBB}(0)$ if $\mathrm{CBB}(0)$ comparison holds for any quadruple in $\mathcal{X}$ such that each model angle in ❶ is defined.*

If instead of $\mathbb{E}^2$, we use $\mathbb{S}^2$ or $\mathbb{H}^2$, then we get the definition of CBB(1) and CBB(−1) comparisons. Note that $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2}$ and $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2}$ are defined if $p \ne x$, $p \ne y$, but for $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2}$ we need in addition, $|p-x| + |p-y| + |x-y| < 2\cdot\pi$.

More generally, one may apply this definition to $\mathbb{M}^2(\kappa)$. This way we define $\mathrm{CBB}(\kappa)$ comparison for any real κ.

1.10. Exercise. *Show that $\mathbb{E}^n$ is $\mathrm{CBB}(0)$.*

1.11. Exercise. *Show that a metric space $\mathcal{X}$ is $\mathrm{CBB}(0)$ if and only if for any quadruple of points $p, x_1, x_2, x_3 \in \mathcal{X}$ there is a quadruple of points $q, y_1, y_2, y_3 \in \mathbb{E}^3$ such that*

$$|p - x_i|_{\mathcal{X}} \geqslant |q - y_i|_{\mathbb{E}^2} \quad \textit{and} \quad |x_i - x_j|_{\mathcal{X}} \leqslant |y_i - y_j|_{\mathbb{E}^2}$$

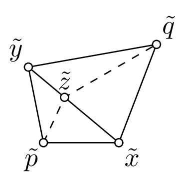
for all i and j.

Curvature bounded above. Given a quadruple of points p, q, x, y in a metric space $\mathcal{X}$, consider two model triangles $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{E}^2}$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)_{\mathbb{E}^2}$ with common side $[\tilde x\tilde y]$.

If the inequality

$$|p - q|_{\mathcal{X}} \leqslant |\tilde p - \tilde z|_{\mathbb{E}^2} + |\tilde z - \tilde q|_{\mathbb{E}^2}$$

holds for any point $\tilde z \in [\tilde x\tilde y]$, then we say that the quadruple p, q, x, y satisfies CAT(0) comparison.



1.12. Definition. *A metric space $\mathcal{X}$ has nonpositive curvature in the sense of Alexandrov (briefly, $\mathcal{X} \in \mathrm{CAT}(0)$) if $\mathrm{CAT}(0)$ comparison holds for any quadruple in $\mathcal{X}$.*

If we do the same for spherical model triangles $[\tilde p\tilde x\tilde y]=\tilde\triangle(pxy)_{\mathbb{S}^2}$ and $[\tilde q\tilde x\tilde y]=\tilde\triangle(qxy)_{\mathbb{S}^2}$, then we arrive at the definition of CAT(1) comparison. One of the spherical model triangles might undefined; it happens if

$$|p-x|+|p-y|+|x-y|\geqslant 2{\cdot}\pi \quad \text{or} \quad |q-x|+|q-y|+|x-y|\geqslant 2{\cdot}\pi.$$

In this case, it is assumed that CAT(1) comparison automatically holds for this quadruple.

We can do the same for $\mathbb{M}^2(\kappa)$. In this case, we arrive at the definition of CAT(κ) comparison. However, we will mostly consider CAT(0) comparison and occasionally CAT(1) comparison; so, if you see CAT(κ), then it is safe to assume that κ is 0 or 1.

Here CAT is an acronym for Cartan, Alexandrov, and Toponogov, but usually pronounced as "cat" in the sense of "miauw". The term was coined by Mikhael Gromov in 1987. Originally, Alexandrov used $\mathfrak{R}_\kappa$ domain; this term is still in use.

1.13. Exercise. *Show that a metric space $\mathcal{U}$ is* CAT(0) *if and only if for any quadruple of points p,q,x,y in $\mathcal{U}$ there is a quadruple $\tilde p,\tilde q,\tilde x,\tilde y$ in $\mathbb{E}^2$ such that*

$$\begin{aligned}
|\tilde p-\tilde q|&\geqslant|p-q|, & |\tilde x-\tilde y|&\geqslant|x-y|,\\
|\tilde p-\tilde x|&\leqslant|p-x|, & |\tilde p-\tilde y|&\leqslant|p-y|,\\
|\tilde q-\tilde x|&\leqslant|q-x|, & |\tilde q-\tilde y|&\leqslant|q-y|.
\end{aligned}$$

1.14. Exercise. *Assume that a quadruple of points in a metric space satisfies* CBB(0) *and* CAT(0) *comparisons for all labelings. Show that it is isometric to a quadruple in $\mathbb{E}^3$.*

The definitions stated in the this section can be applied to any metric space. However, interesting things happen only for the so-called *geodesic* or at least *length spaces*.

F Length and length spaces

Length. A curve is defined as a continuous map from a real interval $\mathbb{I}$ to a metric space. If $\mathbb{I}=[0,1]$, then the curve is called a path.

1.15. Definition. *Let $\mathcal{X}$ be a metric space and $\alpha\colon\mathbb{I}\to\mathcal{X}$ be a curve. We define the length of α as*

$$\operatorname{length}\alpha:=\sup_{t_0\leqslant t_1\leqslant\ldots\leqslant t_n}\sum_i|\alpha(t_i)-\alpha(t_{i-1})|.$$

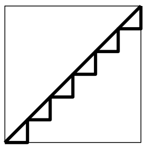
A curve α is called r e c t i f i a b l e *if* $\operatorname{length}\alpha < \infty$.

The following theorem is assumed to be known; see [18, 35].

1.16. Theorem. *Length is a lower semi-continuous with respect to the pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\gamma_n\colon \mathbb{I} \to \mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty\colon \mathbb{I} \to \mathcal{X}$; that is, for any fixed $t \in \mathbb{I}$ we have $\gamma_n(t) \to \gamma_\infty(t)$ as $n \to \infty$. Then

$$❶ \qquad \varliminf_{n\to\infty} \operatorname{length}\gamma_n \geqslant \operatorname{length}\gamma_\infty.$$



Note that the inequality ❶ might be strict. For example, the diagonal γ_∞ of the unit square can be approximated by stairs-like polygonal curves γ_n with sides parallel to the sides of the square (γ_6 is on the picture). In this case

$$\operatorname{length}\gamma_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length}\gamma_n = 2$$

for any n.

Length spaces. Let $\mathcal{X}$ be a metric space. If for any $\varepsilon > 0$ and any pair of points $x, y \in \mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length}\alpha < |x - y| + \varepsilon,$$

then $\mathcal{X}$ is called a l e n g t h s p a c e and the metric on $\mathcal{X}$ is called a l e n g t h m e t r i c.

Evidently, any geodesic space is a length space.

1.17. Exercise. *Show that any compact length space is geodesic.*

Induced length metric. Directly from the definition, it follows that if $\alpha\colon [0,1] \to \mathcal{X}$ is a path from x to y (that is, $\alpha(0) = x$ and $\alpha(1) = y$), then

$$\operatorname{length}\alpha \geqslant |x - y|.$$

Set

$$\|x - y\| = \inf\{\operatorname{length}\alpha\}$$

where the greatest lower bound is taken for all paths from x to y. It is straightforward to check that $(x, y) \mapsto \|x - y\|$ is an ∞-m e t r i c; that is, $(x, y) \mapsto \|x - y\|$ is a metric in the extended positive reals $[0, \infty]$. The metric $\|{*} - {*}\|$ is called the i n d u c e d l e n g t h m e t r i c.

1.18. Exercise. *Let $\mathcal{X}$ be a complete length space. Show that for any compact subset $K \subset \mathcal{X}$ there is a compact path-connected subset $K' \subset \mathcal{X}$ that contains K.*

1.19. Exercise. *Suppose $(\mathcal{X}, |{*}-{*}|)$ is a complete metric space. Show that $(\mathcal{X}, \|{*}-{*}\|)$ is complete.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x - y|_A = \inf_\alpha \{\, \mathrm{length}\, \alpha \,\},$$

where the greatest lower bound is taken for all paths α from x to y in A. In other words, $|{*}-{*}|_A$ denotes the induced length metric on the subspace A. (The notation $|{*}-{*}|_A$ conflicts with the previously defined notation for distance $|x-y|_\mathcal{X}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.)

G Embedding theorem

The following theorem is historically the first remarkable result in Alexandrov geometry. The main part of the following theorem is due to Alexandr Alexandrov [8]. The last part is very difficult; it was proved by Aleksei Pogorelov [36].

1.20. Theorem. *A metric space $\mathcal{X}$ is isometric to the surface of a convex body in the Euclidean space if and only if $\mathcal{X}$ is a geodesic* $\mathrm{CBB}(0)$ *space that is homeomorphic to $\mathbb{S}^2$.*

Moreover, $\mathcal{X}$ determines the convex body up to congruence.

The convex body above is a compact convex subset in $\mathbb{E}^3$; we assume that it does not lie in a line but might degenerate to a plane figure, say F. In the latter case, its surface is defined as two copies of F glued along the boundary. For nondegenerate convex body B, its surface is its boundary ∂B equipped with the induced length metric.

The only-if part of the theorem is the simplest; we will give a complete proof of it eventually. The if part will be sketched. We will not touch the last part.

Lecture 2

Angles

A Definition

The angle measure of a hinge $[p\,{}^x_y]$ is defined as the following limit

$$\measuredangle[p\,{}^x_y] = \lim_{\bar{x},\bar{y}\to p} \tilde{\measuredangle}(p\,{}^{\bar{x}}_{\bar{y}}),$$

where $\bar{x} \in\,]px]$ and $\bar{y} \in\,]py]$.

Note that if $\measuredangle[p\,{}^x_y]$ is defined, then

$$0 \leqslant \measuredangle[p\,{}^x_y] \leqslant \pi.$$

2.1. Exercise. *Suppose that in the above definition, one uses spherical or hyperbolic model angles instead of Euclidean. Show that it does not change the value $\measuredangle[p\,{}^x_y]$.*

2.2. Exercise. *Give an example of a hinge $[p\,{}^x_y]$ in a metric space with an undefined angle measure $\measuredangle[p\,{}^x_y]$.*

B Triangle inequality

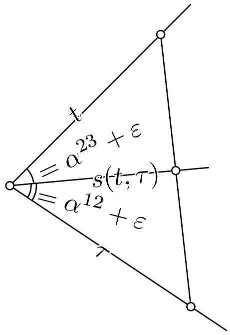
2.3. Proposition. *Let $[px_1]$, $[px_2]$, and $[px_3]$ be three geodesics in a metric space. Suppose all the angle measures $\alpha_{ij} = \measuredangle[p\,{}^{x_i}_{x_j}]$ are defined. Then*

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23}.$$

Proof. Since $\alpha_{13} \leqslant \pi$, we can assume that $\alpha_{12} + \alpha_{23} < \pi$. Denote by γ_i the unit-speed parametrization of $[px_i]$ from p to x_i. Given any

$\varepsilon > 0$, for all sufficiently small $t, \tau, s \in \mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned}|\gamma_1(t) - \gamma_3(\tau)| \leqslant |\gamma_1(t) - \gamma_2(s)| + |\gamma_2(s) - \gamma_3(\tau)| < \\ < \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha_{12} + \varepsilon)} + \\ + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\alpha_{23} + \varepsilon)} \leqslant\end{aligned}$$



Below we define $s(t, \tau)$ so that for $s = s(t, \tau)$, this chain of inequalities can be continued as follows:

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon.$$

Hence the result follows.

To define $s(t, \tau)$, consider three half-lines $\tilde\gamma_1$, $\tilde\gamma_2$, $\tilde\gamma_3$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde\gamma_1, \tilde\gamma_2) = \alpha_{12} + \varepsilon$, $\measuredangle(\tilde\gamma_2, \tilde\gamma_3) = \alpha_{23} + \varepsilon$, and $\measuredangle(\tilde\gamma_1, \tilde\gamma_3) = \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon$. We parametrize each half-line by the distance from the starting point. Given two positive numbers $t, \tau \in \mathbb{R}_{\geqslant 0}$, let $s = s(t, \tau)$ be the number such that $\tilde\gamma_2(s) \in [\tilde\gamma_1(t)\ \tilde\gamma_3(\tau)]$. Clearly, $s \leqslant \max\{t, \tau\}$, so t, τ, s may be taken sufficiently small. $\square$

2.4. Exercise. *Prove that the sum of adjacent angles is at least π.*

More precisely: suppose two hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ are adjacent; that is, they share side $[pz]$, and the union of two sides $[px]$ and $[py]$ form a geodesic $[xy]$. Show that

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \geqslant \pi$$

whenever each angle on the left-hand side is defined.

Give an example showing that the inequality can be strict.

2.5. Exercise. *Assume that the angle measure of $[q\,{}^p_x]$ is defined. Let γ be the unit speed parametrization of $[qx]$ from q to x. Show that*

$$|p - \gamma(t)| \leqslant |q - p| - t{\cdot}\cos(\measuredangle[q\,{}^p_x]) + o(t).$$

C Alexandrov's lemma

Recall that $[xy]$ denotes a geodesic from x to y; set

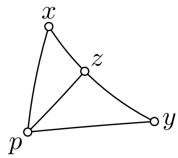
$$]xy] = [xy] \setminus \{x\}, \quad [xy[= [xy] \setminus \{y\}, \quad]xy[= [xy] \setminus \{x, y\}.$$

2.6. Lemma. *Let p, x, y, z be distinct points in a metric space such that $z \in \,]xy[$. Then the following expressions for the Euclidean model angles have the same sign:*

(a) $\tilde\measuredangle(x\,{}^{p}_{y}) - \tilde\measuredangle(x\,{}^{p}_{z})$,

(b) $\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) - \pi$.

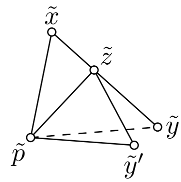
The same holds for the hyperbolic and spherical model angles, but in the latter case, one has to assume in addition that



$$|p-z| + |p-y| + |x-y| < 2{\cdot}\pi.$$

Proof. Consider the model triangle $[\tilde x\tilde p\tilde z] = \tilde\triangle(xpz)$. Take a point $\tilde y$ on the extension of $[\tilde x\tilde z]$ beyond $\tilde z$ so that $|\tilde x-\tilde y| = |x-y|$ (and therefore $|\tilde x-\tilde z| = |x-z|$).

Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:



(i) $\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(x\,{}^{p}_{y})$,

(ii) $|\tilde p-\tilde y| - |p-y|$,

(iii) $\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(z\,{}^{p}_{y})$.

Since

$$\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] = \measuredangle[\tilde x\,{}^{\tilde p}_{\tilde z}] = \tilde\measuredangle(x\,{}^{p}_{z})$$

and

$$\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] = \pi - \measuredangle[\tilde z\,{}^{\tilde x}_{\tilde p}] = \pi - \tilde\measuredangle(z\,{}^{x}_{p}),$$

the statement follows.

The spherical and hyperbolic cases can be proved the same way. $\square$

2.7. Exercise. *Assume p, x, y, z are as in Alexandrov's lemma. Show that*

$$\tilde\measuredangle(p\,{}^{x}_{y}) \geqslant \tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{z}_{y}),$$

with equality if and only if the expressions in (a) and (b) vanish.

D CBB comparison

Note that

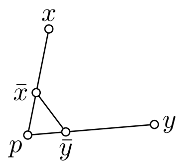
$$p \in \,]xy[\quad\Longrightarrow\quad \tilde\measuredangle(p\,{}^{x}_{y}) = \pi.$$

Applying it with Alexandrov's lemma and CBB(0) comparison, we get the following claim and its corollary.

2.8. Claim. *If p, x, y, z are points in a* CBB(0) *such that $p \in\,]xy[$, then*

$$\tilde\measuredangle(x\,{}^y_z) \leqslant \tilde\measuredangle(x\,{}^p_z).$$

2.9. Exercise. *Let $[p\,{}^x_y]$ be a hinge in a* CBB(0) *space. Consider the function*



$$f\colon (|p-\bar x|, |p-\bar y|) \mapsto \tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x \in\,]px]$ and $\bar y \in\,]py]$. Show that f is nonincreasing in each argument.

Note that 2.9 implies the following.

2.10. Claim. *For any hinge $[p\,{}^x_y]$ in a* CBB(0) *space, the angle measure $\measuredangle[p\,{}^x_y]$ is defined, and*

$$\measuredangle[p\,{}^x_y] \geqslant \tilde\measuredangle(p\,{}^x_y).$$

2.11. Exercise. *Let $[p\,{}^x_y]$ be a hinge in a* CBB(0) *space. Suppose $\measuredangle[p\,{}^x_y] = 0$; show that $[px] \subset [py]$ or $[py] \subset [px]$.*

2.12. Exercise. *Let $[xy]$ be a geodesic in a* CBB(0) *space. Suppose $z \in\,]xy[$ show that there is a unique geodesic $[xz]$ and $[xz] \subset [xy]$.*

2.13. Exercise. *Let $[p\,{}^x_z]$ and $[p\,{}^y_z]$ be adjacent hinges in a* CBB(0) *space. Show that*

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] = \pi.$$

2.14. Exercise. *Let p, x, y in a* CBB(0) *space and $v, w \in\,]xy[$. Show that*

$$\tilde\measuredangle(x\,{}^y_p) = \tilde\measuredangle(x\,{}^v_p) \quad\iff\quad \tilde\measuredangle(x\,{}^y_p) = \tilde\measuredangle(x\,{}^w_p).$$

E Hinge comparison

Let $[p\,{}^x_y]$ be a hinge in a CBB(0) space. By 2.11, the angle measure $\measuredangle[p\,{}^x_y]$ is defined and

$$\measuredangle[p\,{}^x_y] \geqslant \tilde\measuredangle(p\,{}^x_y).$$

Further, according to 2.13, we have

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] = \pi$$

for adjacent hinges $[p\,{}^{x}_{z}]$ and $[p\,{}^{y}_{z}]$ in a CBB(0) space.

The following theorem implies that a geodesic space is CBB(0) if the above conditions hold for all its hinges.

2.15. Theorem. *A geodesic space $\mathcal{L}$ is* CBB(0) *if the following conditions hold.*

(a) For any hinge $[x\,{}^{p}_{y}]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^{p}_{y}]$ is defined and

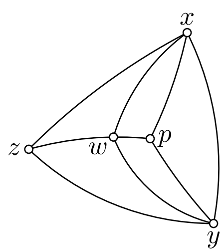
$$\measuredangle[x\,{}^{p}_{y}] \geqslant \tilde\measuredangle(x\,{}^{p}_{y}).$$

(b) For any two adjacent hinges $[p\,{}^{x}_{z}]$ and $[p\,{}^{y}_{z}]$ in $\mathcal{L}$, we have

$$\measuredangle[p\,{}^{x}_{z}] + \measuredangle[p\,{}^{y}_{z}] \leqslant \pi.$$

Proof. Consider a point $w \in\,]pz[$ close to p. From *(b)*, it follows that

$$\measuredangle[w\,{}^{x}_{z}] + \measuredangle[w\,{}^{x}_{p}] \leqslant \pi \quad \text{and} \quad \measuredangle[w\,{}^{y}_{z}] + \measuredangle[w\,{}^{y}_{p}] \leqslant \pi.$$



Since $\measuredangle[w\,{}^{x}_{y}] \leqslant \measuredangle[w\,{}^{x}_{p}] + \measuredangle[w\,{}^{y}_{p}]$ (see 2.3), we get

$$\measuredangle[w\,{}^{x}_{z}] + \measuredangle[w\,{}^{y}_{z}] + \measuredangle[w\,{}^{x}_{y}] \leqslant 2{\cdot}\pi.$$

Applying *(a)*,

$$\tilde\measuredangle(w\,{}^{x}_{z}) + \tilde\measuredangle(w\,{}^{y}_{z}) + \tilde\measuredangle(w\,{}^{x}_{y}) \leqslant 2{\cdot}\pi.$$

Passing to the limits $w \to p$, we have

$$\tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{y}_{z}) + \tilde\measuredangle(p\,{}^{x}_{y}) \leqslant 2{\cdot}\pi.$$

□

F Equivalent conditions

The following theorem summarizes 2.8, 2.10, 2.13, 2.15.

2.16. Theorem. *Let $\mathcal{L}$ be a geodesic space. Then the following conditions are equivalent.*

(a) $\mathcal{L}$ is CBB(0).

(b) (adjacent angle comparison) for any geodesic $[xy]$ and point $z \in\,]xy[$, $z \neq p$ in $\mathcal{L}$, we have

$$\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) \leqslant \pi.$$

(c) (point-on-side comparison) for any geodesic $[xy]$ and $z \in\,]xy[$ in $\mathcal{L}$, we have

$$\tilde\measuredangle(x\,{}^{p}_{y}) \leqslant \tilde\measuredangle(x\,{}^{p}_{z}).$$

(d) *(hinge comparison) for any hinge* $[x\,{}^{p}_{y}]$ *in* $\mathcal{L}$, *the angle* $\measuredangle[x\,{}^{p}_{y}]$ *is defined and*

$$\measuredangle[x\,{}^{p}_{y}] \geqslant \tilde\measuredangle(x\,{}^{p}_{y}).$$

Moreover,

$$\measuredangle[z\,{}^{p}_{y}] + \measuredangle[z\,{}^{p}_{x}] \leqslant \pi$$

for any adjacent hinges $[z\,{}^{p}_{y}]$ *and* $[z\,{}^{p}_{x}]$.

Moreover, the implications (a)⇒(b)⇒(c)⇒(d) hold in any space, not necessarily geodesic.

2.17. Advanced Exercise. *Construct a geodesic space* $\mathcal{X} \notin \mathrm{CBB}(0)$ *that meets the following condition: for any 3 points* $p, x, y \in \mathcal{X}$ *there is a geodesic* $[xy]$ *such that for any* $z \in\,]xy[$

$$\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) \leqslant \pi.$$

G Function comparison

Real-to-real functions. Choose $\lambda \in \mathbb{R}$. Let $s\colon \mathbb{I} \to \mathbb{R}$ be a locally Lipschitz function defined on an interval $\mathbb{I}$. We say that s is λ-concave if $s'' \leqslant \lambda$, where the second derivative s'' is understood in the sense of distributions.

Equivalently, *s is λ-concave if the function $h\colon t \mapsto s(t) - \lambda\cdot\frac{t^2}{2}$ is concave.* Concavity can be defined via Jensen inequality; that is,

$$h(s\cdot t_0 + (1-s)\cdot t_1) \geqslant s\cdot h(t_0) + (1-s)\cdot h(t_1)$$

for any $t_0, t_1 \in \mathbb{I}$ and $s \in [0,1]$. It could be also defined via the existence of (local) upper support at any point: *for any $t_0 \in \mathbb{I}$ there is a linear function ℓ that (locally) supports h at t_0 from above; that is, $\ell(t_0) = h(t_0)$ and $\ell(t) \geqslant h(t)$ for any t (in a neighborhood of t_0).*

The equivalence of these definitions is assumed to be known. We will also use that *λ-concave functions are one-side differentiable.*

Functions on metric space. A function on a metric space $\mathcal{L}$ will usually mean a *locally Lipschitz real-valued function defined in an open subset of $\mathcal{L}$.* The domain of definition of a function f will be denoted by $\operatorname{Dom} f$.

Let f be a function on a metric space $\mathcal{L}$. We say that f is λ-concave (briefly $f'' \leqslant \lambda$) if for any unit-speed geodesic $\gamma\colon \mathbb{I} \to \operatorname{Dom} f$ the real-to-real function $t \mapsto f\circ\gamma(t)$ is λ-concave.

The following proposition is conceptual — it reformulates a global geometric condition into an infinitesimal condition on distance functions.

2.18. Proposition. *A geodesic space $\mathcal{L}$ is* CBB(0) *if and only if $f'' \leqslant 1$ for any function f of the following type*

$$f\colon x \mapsto \tfrac{1}{2}\cdot|p-x|^2.$$

Proof. Choose a unit-speed geodesic γ in $\mathcal{L}$ and two points $x=\gamma(t_0)$, $y=\gamma(t_1)$ for some $t_0<t_1$. Consider the model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$. Let $\tilde{\gamma}\colon[t_0,t_1]\to\mathbb{E}^2$ be the unit-speed parametrization of $[\tilde{x}\tilde{y}]$ from $\tilde{x}$ to $\tilde{y}$.

Set

$$\tilde{r}(t) := |\tilde{p}-\tilde{\gamma}(t)|, \qquad\qquad r(t) := |p-\gamma(t)|.$$

Clearly, $\tilde{r}(t_0)=r(t_0)$ and $\tilde{r}(t_1)=r(t_1)$. Note that the point-on-side comparison (2.16c) is equivalent to

$$❶ \qquad t_0\leqslant t\leqslant t_1 \qquad\Longrightarrow\qquad \tilde{r}(t)\leqslant r(t)$$

for any γ and $t_0<t_1$.

Set

$$\tilde{h}(t)=\tfrac{1}{2}\cdot\tilde{r}^2(t)-\tfrac{1}{2}\cdot t^2, \qquad\qquad h=\tfrac{1}{2}\cdot r^2(t)-\tfrac{1}{2}\cdot t^2.$$

Note that $\tilde{h}$ is linear, $\tilde{h}(t_0)=h(t_0)$ and $\tilde{h}(t_1)=h(t_1)$. Observe that the Jensen inequality for the function h is equivalent to ❶. Hence the proposition follows. □

H Comments

All the discussed statements admit natural generalizations to CBB(κ) spaces. Most of the time the proof is the same with uglier formulas.

For example, the function comparison of CBB(-1) states that $f''\leqslant f$ for any function of the type $f=\cosh\circ\operatorname{dist}_p$. Similarly, the function comparison of CBB(1) states that for any point p, we have $f''\leqslant -f$ for the function $f=-\cos\circ\operatorname{dist}_p$ defined in $\mathrm{B}(p,\pi)$. The meaning of these inequalities is the same — distance functions in CBB(κ) are more concave than distance functions in $\mathbb{M}(\kappa)$. The inequality $f''\leqslant\varphi$ means that for any point p in the domain of definition and any $\varepsilon>0$, there is a neighborhood $U\ni p$ such that $f''\leqslant\varphi(p)+\varepsilon$ in U. Here we assume that f and φ are continuous and defined in open set.

Lecture 3

Surface of convex body

Recall that (for us) a convex body is a compact convex subset in $\mathbb{E}^3$; we assume that it does not lie in a line but it might degenerate to a plane figure.

Suppose B is a nondegenerate convex body; that is, it has nonempty interior. Then the surface of B is defined as its boundary ∂B equipped with the induced length metric.

If a convex body degenerates to a plane convex figure, say F, then its surface is defined as a doubling of F along its boundary; that is, two copies of F glued along the boundary ∂F. Intuitively, one can regard these copies as different sides of F — we live on its surface and to get from one side to the other one has to cross the boundary.

3.1. Exercise. *Show that surface of a convex body is homeomorphic to* $\mathbb{S}^2$.

In this lecture, we will prove that *surface of a convex body is* CBB(0). The latter, together with the exercise, gives the only-if part in the main part of the embedding theorem (1.20).

A Convex polyhedra

Recall that a convex polyhedron is a convex hull of a finite set of points. Extremal points of a convex polyhedron are called its vertices. As for convex bodies, our convex polyhedra might degenerate to a plane polygon, but we assume that it does not belong to a line.

Observe that a surface, say Σ, of a convex polyhedron P admits a triangulation such that each triangle is isometric to a plane triangle. In other words, Σ is a polyhedral surfaces; that is, it is a

2-dimensional manifold with length metric that admits a triangulation such that each triangle is isometric to a solid plane triangle. A triangulation of polyhedral surface will be assumed to satisfy this condition.

The total angle around a vertex p in Σ is defined as the sum of angles at p of all triangles in the triangulation that contain p.

Note that if a point p is not a vertex of P, then

- ◇ p lies in the interior of a face of P, and its neighborhood in Σ is a piece of plane, or
- ◇ p lies on an edge, and its neighborhood is two half-planes glued along the boundary.

In both cases, a neighborhood of p in Σ (with the induced length metric) is isometric to an open domain of the plane.

3.2. Claim. *Let Σ be the surface of a convex polyhedron P. Then, the total angle around a vertex in Σ cannot exceed $2\cdot\pi$.*

In the proof, we will use the following exercise which is the triangle inequality for angles (or the spherical triangle inequality); it easily follows from 2.3.

3.3. Exercise. *Let w_1, w_2, w_3 be unit vectors in $\mathbb{E}^3$. Denote by $\theta_{i,j}$ the angle between the vectors v_i and v_j. Show that*

$$\theta_{1,3} \leqslant \theta_{1,2} + \theta_{2,3}$$

and in case of equality, the vectors w_1, w_2, w_3 lie in a plane.

Proof. Consider the intersection of P with a small sphere centered at p; it is a convex spherical polygon, say F. Applying rescaling we may assume that the sphere has unit radius. We need to show that the perimeter of F does not exceed $2\cdot\pi$.

Note that F lies in a hemisphere, say H. Moreover, there is a decreasing sequence

$$H = H_0 \supset H_1 \supset \dots \supset H_n = F,$$

such that H_{i+1} is obtained from H_i by cutting along a chord.

By 3.3, we have

$$2\cdot\pi = \operatorname{perim} H = \operatorname{perim} H_0 \geqslant \operatorname{perim} H_1 \geqslant \dots \geqslant \operatorname{perim} H_n = \operatorname{perim} F$$

— hence the result. □

A vertex of a triangulation of a polyhedral surface is called essential if the total angle around it is not $2\cdot\pi$.

3.4. Exercise. *Show that any vertex of a polyhedron is an essential vertex of its surface; that is, the inequality in the claim is strict.*

3.5. Exercise. *Show that geodesics on a surface of convex polyhedron do not pass thru its essential vertices.*

B Surface of convex polyhedron

Let p be a vertex of a polyhedron. If θ_p is the total angle around p, then the value $2\cdot\pi - \theta_p$ is called the curvature of the polyhedral surface at p; if p is not a vertex, then its curvature is defined to be zero.

3.6. Exercise. *Assume that the surface of a nondegenerate tetrahedron T has curvature π at each of its vertices. Show that*

(a) all faces of T are congruent;

(b) the line passing thru midpoints of opposite edges of T intersects these edges at right angles.

Note that the claim above says that *surface of a convex polyhedron has nondegenerate curvature*. However this definition works only for polyhedral surfaces. Now we show that it agrees with the CBB(0) definition.

3.7. Proposition. *A polyhedral surface with nonnegative curvature at each vertex is* CBB(0).

Proof. Denote the surface by Σ. By 2.18, it is sufficient to check that $\operatorname{dist}_p^2\circ\gamma$ is 1-concave for any geodesic γ and a point p in Σ.

We can assume that p is not a vertex; the vertex case can be done by approximation. Further, by 3.5, we may assume that γ does not contain vertices.

Given a point $x = \gamma(t_0)$, choose a geodesic $[px]$. Again, by 3.5, $[px]$ does not contain vertices. Therefore a small neighborhood of $U \supset [px]$ can be unfolded on a plane; denote this map by $z \mapsto \tilde z$. Note that this way we map part of γ in U to a line segment. Let

$$\tilde f(t) := \tfrac12\cdot\operatorname{dist}_{\tilde p}^2\circ\tilde\gamma(t).$$

Note that $\tilde f(t_0) \geqslant f(t_0)$. Further, since the unfolding $z \mapsto \tilde z$ preserves lengths of curves, we get $\tilde f(t) \geqslant f(t)$ if t is sufficiently close to t_0. That is, $\tilde f$ is a local upper support of f at t_0. Evidently, $\tilde f'' \equiv 1$; therefore $f'' \leqslant 1$. It remains to apply 2.18. □

3.8. Exercise. *Prove the converse to the proposition; that is, show that if a poyhedral surface is* CBB(0)*, then it has nonnegative curvature at each vertex.*

C Surface of convex body

3.9. Lemma. *Let $K_1, K_2, \dots$ be a sequence of convex bodies that converges to K_∞ in the sense of Hausdorff. Assume K_∞ is nondegenerate. Then the surface of K_n converges to the surface of K_∞ in the sense of Gromov–Hausdorff.*

In the following proof we use that the closest-point projection form the Euclidean space to a convex body is short; that is, distance-nonexpanding [34, 12.3].

Proof. Without loss of generality, we may assume that

$$\overline{\mathrm{B}}(0,r)\subset K_\infty\subset\overline{\mathrm{B}}(0,1)$$

for some $r>0$. Note that there is a sequence $\varepsilon_n\to 0$ such that

$$K_n\subset(1+\varepsilon_n)\cdot K_\infty\quad\text{and}\quad K_\infty\subset(1+\varepsilon_n)\cdot K_n$$

for each n.

Given $x\in K_n$, denote by $g_n(x)$ the closest-point projection of $(1+\varepsilon_n)\cdot x$ to K_∞. Similarly, given $x\in K_\infty$, denote by $h_n(x)$ the closest point projection of $(1+\varepsilon_n)\cdot x$ to K_n. Note that

$$|g_n(x)-g_n(y)|\leqslant(1+\varepsilon_n)\cdot|x-y|$$

and

$$|h_n(x)-h_n(y)|\leqslant(1+\varepsilon_n)\cdot|x-y|.$$

Denote by Σ_∞ and Σ_n the surface of K_∞ and K_n respectively. The above inequlities imply

$$|g_n(x)-g_n(y)|_{\Sigma_\infty}\leqslant(1+\varepsilon_n)\cdot|x-y|_{\Sigma_n}$$

for any $x,y\in\Sigma_n$, and

$$|h_n(x)-h_n(y)|_{\Sigma_n}\leqslant(1+\varepsilon_n)\cdot|x-y|_{\Sigma_\infty}.$$

for any $x,y\in\Sigma_\infty$. Therefore, g_n is a δ_n-isometry $\Sigma_n\to\Sigma_\infty$ for a sequence $\delta_n\to 0$. □

3.10. Proposition. *The surface of a nondegenerate convex body is* CBB(0).

Note that any convex body is a Hausdorff limit of a sequence of convex polyhedra. Therefore, the proposition follows from 3.7, 3.9, and the following claim.

3.11. Claim. *A Gromov–Hausdorff limit of* CBB(0) *spaces is* CBB(0).

Despite its simplicity, this claim is the main source of applications of Alexandrov geometry.

Proof. Let $\mathcal{L}_\infty$ be Gromov–Hausdorff limit of CBB(0) spaces $\mathcal{L}_1, \mathcal{L}_2, \dots$

Choose a quadruple of points p, x, y, z in $\mathcal{L}_\infty$. From convergence we may choose a sequence of quadruples p_n, x_n, y_n, z_n in $\mathcal{L}_n$ that converge to p, x, y, z; in particular, each of six distances between pairs of p_n, x_n, y_n, z_n converges to the corresponding distance between the pair of p, x, y, z. By CBB(0) comparison in $\mathcal{L}_n$,

$$\tilde{\measuredangle}(p_n\,{}^{x_n}_{y_n}) + \tilde{\measuredangle}(p_n\,{}^{y_n}_{z_n}) + \tilde{\measuredangle}(p_n\,{}^{z_n}_{x_n}) \leqslant 2{\cdot}\pi.$$

Passing to the limit we get

$$\tilde{\measuredangle}(p\,{}^{x}_{y}) + \tilde{\measuredangle}(p\,{}^{y}_{z}) + \tilde{\measuredangle}(p\,{}^{z}_{x}) \leqslant 2{\cdot}\pi.$$

□

Recall that surface of a degenerate convex body is defined as its doubling. More precisely, suppose F is a convex plane figure. Consider product space $F \times \{0, 1\}$ with semimetric defined by

$$|(x,i) - (y,j)| = \begin{cases} |x-y| & \text{if } i = j \\ \inf\{\, |x-z| + |y-z| \;:\; z \in \partial F \,\} & \text{if } i \neq j \end{cases}$$

Then the corresponding metric space is the doubling of F along its boundary.

3.12. Exercise. *Suppose $F_1, F_2, \dots$ is a sequence of convex plane figures that converges to F_∞ in the sense of Hausdorff. Show that doublings of F_n converge to the doubling of F_∞ in the sense of Gromov–Hausdorff.*

Conclude that surfaces of degenerate convex bodies are CAT(0).

Note that 3.10 and 3.12 imply that *surface of a convex body is* CBB(0); so the only-if part in the main part of the embedding theorem (1.20) is proved.

Lecture 4

Alexandrov embedding theorem

We will prove the Cauchy theorem, and then modify it to prove the Alexandrov uniqueness theorem. Further, we sketch a proof of the Alexandrov embedding theorem.

A Cauchy theorem

Recall that *surfaces* of convex polyhedrons are considered with the induced length metric..

4.1. Theorem. *Let K and K' be two non-degenerate convex polyhedrons in $\mathbb{E}^3$; denote their surfaces by P and P'. Suppose there is an isometry $P \to P'$ that sends each face of K to a face of K'. Then K is congruent to K'; moreover the isometry $P \to P'$ can be extended to a motion of $\mathbb{E}^3$ that maps K to K'.*

Proof. Consider the graph Γ formed by the edges of K; the edges of K' form the same graph.

For an edge e in Γ, denote by α_e and α'_e the corresponding dihedral angles in K and K' respectively. Mark e by plus if $\alpha_e < \alpha'_e$ and by minus if $\alpha_e > \alpha'_e$.

Now remove from Γ everything that was not marked; that is, leave only the edges marked by $(+)$ or $(-)$ and their endpoints.

Note that the theorem follows if Γ is an empty graph; assume the contrary.

The graph Γ is embedded into P, which is homeomorphic to the sphere. In particular, the edges coming from one vertex have a natural cyclic order. Given a vertex v of Γ, count the *number of sign changes* around v; that is, the number of consequent pairs edges with different signs.

4.2. Local lemma. *For any vertex of* Γ *the number of sign changes is at least* 4.

In other words, at each vertex of Γ, one can choose 4 edges marked by $(+)$, $(-)$, $(+)$, $(-)$ in the same cyclical order. Note that the local lemma contradicts the following.

4.3. Global lemma. *Let* Γ *be a nonempty subgraph of the graph formed by the edges of a convex polyhedron. Then it is impossible to mark all of the edges of* Γ *by* $(+)$ *or* $(-)$ *such that the number of sign changes around each vertex of* Γ *is at least* 4.

It remains to prove these two lemmas. $\square$

B Local lemma

Next lemma is the main ingredient in our proof of the local lemma.

4.4. Arm lemma. *Assume that* $A = [a_0 a_1 \dots a_n]$ *is a convex polygon in* $\mathbb{E}^2$ *and* $A' = [a_0' a_1' \dots a_n']$ *be a polygonal line in* $\mathbb{E}^3$ *such that*

$$|a_i - a_{i+1}| = |a_i' - a_{i+1}'|$$

for any $i \in \{0, \dots, n-1\}$ *and*

$$\measuredangle a_i \leqslant \measuredangle a_i'$$

for each $i \in \{1, \dots, n-1\}$. *Then*

$$|a_0 - a_n| \leqslant |a_0' - a_n'|$$

and equality holds if and only if A *is congruent to* A'.

One may view the polygonal lines $[a_0 a_1 \dots a_n]$ and $[a_0' a_1' \dots a_n']$ as a robot's arm in two positions. The arm lemma states that when the arm opens, the distance between the shoulder and tip of a finger increases.

4.5. Exercise. *Show that the arm lemma does not hold if instead of the convexity, one only the local convexity; that is, if you go along the polygonal line* $a_0 a_1 \dots a_n$, *then you only turn left.*

4.6. Exercise. *Suppose* $A = [a_1 \dots a_n]$ *and* $A' = [a_1' \dots a_n']$ *be non-congruent convex plane polygons with equal corresponding sides. Mark* *each vertex a_i with plus (minus) if the interior angle of A at a_i is smaller (respectively bigger) than the interior angle of A' at a_i'. Show that there are at least 4 sign changes around A.*

Give an example showing the statement does not hold without assuming convexity.

Proof. We will view $\mathbb{E}^2$ as the xy-plane in $\mathbb{E}^3$; so both A and A' lie in $\mathbb{E}^3$. Let a_m be the vertex of A that lies on the maximal distance to the line $(a_0 a_n)$.

Let us shift indexes of a_i and a_i' down by m, so that

$$\begin{array}{lllll} a_{-m} := a_0, & \dots & a_0 := a_m, & \dots & a_k := a_n, \\ a'_{-m} := a'_0, & \dots & a'_0 := a'_m, & \dots & a'_k := a'_n, \end{array}$$

where $k = n - m$. (Here the symbol ":=" means an assignment as in programming.)

Without loss of generality, we may assume that

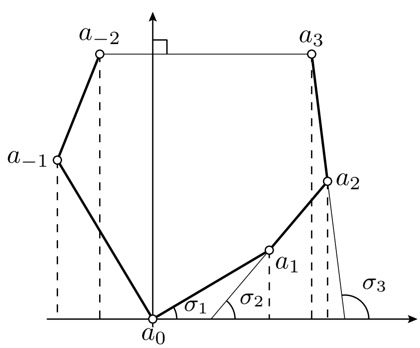
- ◇ $a_0 = a_0'$ and they both coincide with the origin $(0,0,0) \in \mathbb{E}^3$;
- ◇ all a_i lie in the xy-plane and the x-axis is parallel to the line $(a_{-m} a_k)$;
- ◇ the angle $\measuredangle a_0'$ lies in xy-plane and contains the angle $\measuredangle a_0$ inside and the directions to a'_{-1},a_{-1}, a_1 and a_1' from a_0 appear in the same cyclic order.

Denote by x_i and x_i' the projections of a_i and a_i' to the x-axis. We can assume in addition that $x_k \geqslant x_{-m}$. In this case,

$$|a_k - a_{-m}| = x_k - x_{-m}.$$

Since the projection is a distance non-expanding, we also have

$$|a_k' - a_{-m}'| \geqslant x_k' - x_{-m}'.$$



Therefore it is sufficient to show that

$$x_k' - x_{-m}' \geqslant x_k - x_{-m}.$$

The latter holds if

❶ $$x_i' - x_{i-1}' \geqslant x_i - x_{i-1}.$$

for each i. It remains to prove ❶.

Let us assume that $i > 0$; the case $i \leqslant 0$ is similar. Denote by σ_i (σ'_i) the angle between the vector $w_i = a_i - a_{i-1}$ (respectively $w'_i = a'_i - a'_{i-1}$) and the x-axis. Note that

❷
$$\begin{aligned} x_i - x_{i-1} &= |a_i - a_{i-1}|\cdot \cos\sigma_i, \\ x'_i - x'_{i-1} &= |a_i - a_{i-1}|\cdot \cos\sigma'_i \end{aligned}$$

for each $i > 0$. By construction $\sigma_1 \geqslant \sigma'_1$. Note that $\measuredangle(w_{i-1}, w_i) = \pi - \measuredangle a_i$. From convexity of $[a_1 a_1 \dots a_i]$, we have

$$\sigma_i = \sigma_1 + (\pi - \measuredangle a_1) + \dots + (\pi - \measuredangle a_i)$$

for any $i > 0$. Since $\measuredangle(w'_{i-1}, w'_i) = \pi - \measuredangle a'_i$, applying 3.3 several times, we get

$$\sigma'_i \leqslant \sigma'_1 + (\pi - \measuredangle a'_1) + \dots + (\pi - \measuredangle a'_i).$$

Since $\measuredangle a'_j \geqslant \measuredangle a_j$ for each j, we get $\sigma'_i \leqslant \sigma_i$, and therefore

$$\cos\sigma'_i \geqslant \cos\sigma_i$$

Applying ❷, we get ❶.

In the case of equality, we have $\sigma_i = \sigma'_i$, which implies $\measuredangle a_i = \measuredangle a'_i$ for each i. This also implies that all a'_i lie in xy-plane. The latter easily follows from the equality case in 3.3. □

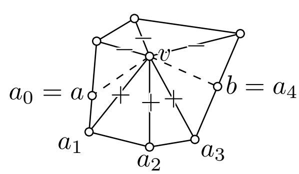
Proof of the local lemma (4.2). Assume that the local lemma does not hold at the vertex v of Γ. Cut from P a small pyramid Δ with the vertex v. One can choose two points a and b on the base of Δ so that on one side of the segments $[va]$ and $[vb]$ we have only pluses and on the other side only minuses.

The base of Δ has two polygonal lines with ends at a and b. Choose the one that has only pluses; denote it by $a_0 a_1 \dots a_n$; so $a = a_0$ and $b = a_n$. Denote by $a'_0 a'_1 \dots a'_n$ the corresponding line in P'; let $a' = a'_0$ and $b' = a'_n$.

Since each marked edge passing thru a_i has a (+) on it or nothing, we have

$$\measuredangle a_i \leqslant \measuredangle a'_i$$

for each i.



4.7. Exercise. *Prove the last statement.*

By the construction we have $|a_i - a_{i-1}| = |a'_i - a'_{i-1}|$ for all i. By the arm lemma (6.14), we get

❸ $|a-b| \leqslant |a'-b'|$.

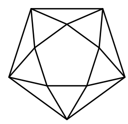
Swap K and K' and repeat the same construction for a plane passing thru a' and b'. We get

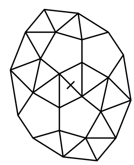
❹ $|a-b| \geqslant |a'-b'|$.

The claims ❸ and ❹ together imply $|a-b| = |a'-b'|$. The equality case in the arm lemma implies that no edge at v is marked; that is, v is not a vertex of Γ — a contradiction. □

From the proof, it follows that the local lemma is indeed local — it works for two noncongruent convex polyhedral angles with equal corresponding faces. Use this observation to solve the following exercise.

4.8. Exercise. *Consider two polyhedral discs in $\mathbb{E}^3$ glued from regular polygons by the rule on the diagrams. Assume that each disc is part of a surface of a convex polyhedron.*





(a) The first configuration is rigid; that is, one can not fix the position of the pentagon and continuously move the remaining 5 vertices in a new position so that each triangle moves by a one-parameter family of isometries of $\mathbb{E}^3$.

(b) Show that the second configuration has a rotational symmetry with the axis passing thru the midpoint of the marked edge.

C Global lemma

The proof of the global lemma is based on counting the sign changes in two ways; first while moving around each vertex of Γ and second while moving around each of the regions separated by Γ on the surface P. If two edges are adjacent at a vertex, then they are also adjacent in a region. The converse is true as well. Therefore, both countings give the same number.

It is instructive to do the next exercise before diving into the proof.

4.9. Exercise. *Try to mark the edges of an octahedron by pluses and minuses such that there would be 4 sign changes at each vertex.*

Show that this is impossible.

Proof of 4.3. We can assume that Γ is connected; that is, one can get from any vertex to any other vertex by walking along edges. (If not, pass to a connected component of Γ.)

Denote by k and l the number of vertices and edges in Γ. Denote by m the number of *regions* that Γ cuts from P. Since Γ is connected, each region is homeomorphic to an open disc.

4.10. Exercise. *Prove the last statement.*

Now we can apply Euler's formula

❶ $$k - l + m = 2.$$

Denote by s the total number of sign changes in Γ for all vertices. By the local lemma (4.2), we have

❷ $$4{\cdot}k \leqslant s.$$

Let us get an upper bound on s by counting the number of sign changes when you go around each region. Denote by m_n the number of regions bounded by n edges; if an edge appears twice when it is counted twice. Note that each region is bounded by at least 3 edges; therefore

❸ $$m = m_3 + m_4 + m_5 + \dots$$

Counting edges and using the fact that each edge belongs to exactly two regions, we get

$$2{\cdot}l = 3{\cdot}m_3 + 4{\cdot}m_4 + 5{\cdot}m_5 + \dots$$

Combining this with Euler's formula (❶), we get

❹ $$4{\cdot}k = 8 + 2{\cdot}m_3 + 4{\cdot}m_4 + 6{\cdot}m_5 + 8{\cdot}m_6 + \dots$$

Observe that the number of sign changes in n-gon regions has to be even and $\leqslant n$. Therefore

❺ $$s \leqslant 2{\cdot}m_3 + 4{\cdot}m_4 + 4{\cdot}m_5 + 6{\cdot}m_6 + \dots$$

Clearly, ❷ and ❺ contradict ❹. □

D Uniqueness

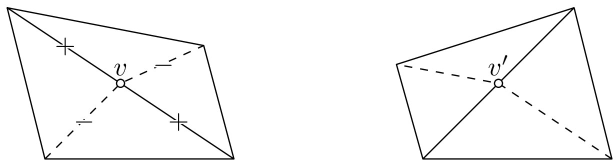
Alexandrov's uniqueness theorem states that the conclusion of the Cauchy theorem (4.1) still holds without the face-to-face assumption.

4.11. Theorem. *Any two convex polyhedrons in $\mathbb{E}^3$ with isometric surfaces are congruent.*

Moreover, any isometry between surfaces of convex polyhedrons can be extended to an isometry of the whole $\mathbb{E}^3$.

Needed modifications in the proof of 4.1. Suppose $\iota\colon P \to P'$ be an isometry between surfaces of K and K'. Mark in P all the edges of K and all the inverse images of edges in K'; further, these will be called f a k e edges. The marked lines divide P into convex polygons, and the restriction of ι to each polygon is a rigid motion. These polygons play the role of faces in the proof above.

A vertex of the obtained graph can be a vertex of K, or it can be a fake vertex; that is, it might be an intersection of an edge and a fake edge.



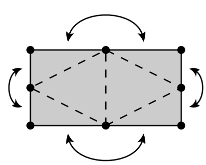
For the first type of vertex, the local lemma can be proved the same way. For a fake vertex v, it is easy to see that both parts of the edge coming thru v are marked with minus while both of the fake edges at v are marked with plus. Therefore, the local lemma holds for the fake vertices as well.

What remains in the proof needs no modifications. □

4.12. Exercise. *Let K be a convex polyhedron in $\mathbb{E}^3$; denote by P its surface. Show that each isometry $\iota\colon P \to P$, can be extended to an isometry of $\mathbb{E}^3$.*

E Existence

4.13. Theorem. *A polyhedral metric on the sphere is isometric to the surface of a convex polyhedron (possibly degenerate to a flat polygon) if and only if it has nonnegative curvature at each point.*



By 4.11, a convex polyhedron is completely defined by the intrinsic metric of its surface. By 4.13, it follows that knowing the metric we could find the position of the edges. However, in practice, it is not easy to do.

For example, the surface glued from a rectangle as shown on the diagram defines a tetrahedron. Some of the glued lines appear inside facets of the tetrahedron and some edges (dashed lines) do not follow the sides of the rectangle.

Space of polyhedrons. Let us denote by $\mathbf{K}$ the space of all convex polyhedrons in the Euclidean space, including polyhedrons that degenerate to a plane polygon. Polyhedra in $\mathbf{K}$ will be considered up to a motion of the space, and the whole space $\mathbf{K}$ will be considered with Hausdorff distance up to a motion of the space; that is, the distance between K and K' is the exact lower bound on Hausdorff distance from $\iota(K)$ to K', where ι is arbitrary motion of $\mathbb{E}^3$.

Further, denote by $\mathbf{K}_n$ the polyhedrons in $\mathbf{K}$ with exactly n vertices. Since any polyhedron has at least 3 vertices, the space $\mathbf{K}$ admits a subdivision into a countable number of subsets $\mathbf{K}_3, \mathbf{K}_4, \dots$

Space of polyhedral metrics. The space of polyhedral metrics on the sphere with nonnegative curvature will be denoted by $\mathbf{P}$. The metrics in $\mathbf{P}$ will be considered up to an isometry, and the whole space $\mathbf{P}$ will be equipped with the topology induced by the Gromov–Hausdorff metric.

The subset of $\mathbf{P}$ of all metrics with exactly n essential vertices will be denoted by $\mathbf{P}_n$. It is easy to see that any metric in $\mathbf{P}$ has at least 3 essential vertices. Therefore $\mathbf{P}$ is subdivided into countably many subsets $\mathbf{P}_3, \mathbf{P}_4, \dots$

From a polyhedron to its surface. By 3.7, passing from a polyhedron to its surface defines a map

$$\iota\colon \mathbf{K} \to \mathbf{P}.$$

By 3.4, the number of vertices of a polyhedron is equal to the number of essential vertices on its surface. In other words, $\iota(\mathbf{K}_n) \subset \mathbf{P}_n$ for any $n \geqslant 3$.

Using the introduced notation, we can unite 4.11 and 4.13 in the following more exact statement.

4.14. Reformulation. *For any integer $n \geqslant 3$, the map ι induces a bijection between $\mathbf{K}_n$ and $\mathbf{P}_n$.*

The proof is based on a construction of a one-parameter family of polyhedrons that starts at an arbitrary polyhedron and ends at a polyhedron with its surface isometric to the given one. This type of argument is called the *continuity method*; it is often used in the theory of differential equations.

Sketch. By 4.11, the map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is injective. Let us prove that it is surjective.

4.15. Lemma. *For any integer $n \geqslant 3$, the space $\mathbf{P}_n$ is connected.*

The proof of this lemma is not complicated, but it requires ingenuity; it can be done by the direct construction of a one-parameter family of metrics in $\mathbf{P}_n$ that connects two given metrics. Such a family can be obtained by a sequential application of the following construction and its inverse.

Let $P \in \mathbf{P}_n$. Suppose v and w are essential vertices in P. Let us cut P along a geodesic from v to w. Note that the geodesic cannot pass thru an essential vertex of P. Further, note that there is a three-parameter family of patches that can be used to patch the cut so that the obtained metric remains in $\mathbf{P}_n$; in particular, the obtained metric has exactly n essential vertices (after the patching, the vertices v and w may become inessential).

4.16. Lemma. *The map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is open, that is, it maps any open set in $\mathbf{K}_n$ to an open set in $\mathbf{P}_n$.*

In particular, for any $n \geqslant 3$, the image $\iota(\mathbf{K}_n)$ is open in $\mathbf{P}_n$.

This statement is very close to the so-called *invariance of domain theorem*; the latter states that a continuous injective map between manifolds of the same dimension is open.

Recall that ι is injective. The proof of the invariance of domain theorem can be adapted to our case since both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n-6)$-dimensional and both look like manifolds, altho, formally speaking, they are *not* manifolds. In a more technical language, $\mathbf{K}_n$ and $\mathbf{P}_n$ have the natural structure of $(3{\cdot}n-6)$-dimensional *orbifolds*, and the map ι respects the *orbifold structure*.

We will only show that both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n-6)$-dimensional.

Choose $K \in \mathbf{K}_n$. Note that K is uniquely determined by the $3{\cdot}n$ coordinates of its n vertices. We can assume that the first vertex is the origin, the second has two vanishing coordinates and the third has one vanishing coordinate; therefore, all polyhedrons in $\mathbf{K}_n$ that lie sufficiently close to K can be described by $3{\cdot}n-6$ parameters. If K has no symmetries, then this description can be made one-to-one; in this case, a neighborhood of K in $\mathbf{K}_n$ is a $(3{\cdot}n-6)$-dimensional manifold. If K has a nontrivial symmetry group, then this description is not one-to-one but it does not have an impact on the dimension of $\mathbf{K}_n$.

The case of polyhedral metrics is analogous. We need to construct a subdivision of the sphere into plane triangles using only essential vertices. By Euler's formula, there are exactly $3{\cdot}n-6$ edges in this subdivision. Note that the lengths of edges completely describe the metric, and slight changes in these lengths produce a metric with the same property. Again, if P has no symmetries, then this description is one-to-one.

4.17. Lemma. *The map $\iota\colon \mathbf{K}_n\to\mathbf{P}_n$ is closed; that is, the image of a closed set in $\mathbf{K}_n$ is closed in $\mathbf{P}_n$.*

In particular, for any $n\geqslant 3$, the set $\iota(\mathbf{K}_n)$ is closed in $\mathbf{P}_n$.

Choose a closed set Z in $\mathbf{K}_n$. Denote by $\bar Z$ the closure of Z in $\mathbf{K}$; note that $Z=\mathbf{K}_n\cap\bar Z$. Assume $K_1,K_2,\dots\in Z$ is a sequence of polyhedrons that converges to a polyhedron $K_\infty\in\bar Z$. By 3.9, $\iota(K_n)$ converges to $\iota(K_\infty)$ in $\mathbf{P}$. In particular, $\iota(\bar Z)$ is closed in $\mathbf{P}$.

Since $\iota(\mathbf{K}_n)\subset\mathbf{P}_n$ for any $n\geqslant 3$, we have $\iota(Z)=\iota(\bar Z)\cap\mathbf{P}_n$; that is, $\iota(Z)$ is closed in $\mathbf{P}_n$.

Summarizing, $\iota(\mathbf{K}_n)$ is a nonempty closed and open set in $\mathbf{P}_n$, and $\mathbf{P}_n$ is connected for any $n\geqslant 3$. Therefore, $\iota(\mathbf{K}_n)=\mathbf{P}_n$; that is, $\iota\colon\mathbf{K}_n\to\mathbf{P}_n$ is surjective. □

F Approximation

By now, the embedding theorem is proved for polyhedral metrics on the sphere. The general case is done by approximation, using the following statement.

4.18. Proposition. *Let $K_1,K_2,\dots$ be a sequence of convex bodies that converge to K_∞ in the sense of Hausdorff. Then the surface of K_n converges to the surface of K_∞ in the sense of Gromov–Hausdorff.*

If K_∞ is nondegenerate, then the statement follows from 3.9. The degenerate case is left as an exercise.

Let $\mathcal{X}_\infty$ be a geodesic CBB(0) space that is homeomorphic to $\mathbb{S}^2$. Suppose that $\mathcal{X}_\infty$ is a Gromov–Hausdorff limit of a sequence of spheres with polyhedral metrics $\mathcal{X}_1,\mathcal{X}_2,\dots$ By 4.13, there is a sequence of convex polyhedra $K_1,K_2,\dots$ with surfaces isometric to $\mathcal{X}_1,\mathcal{X}_2,\dots$, respectively. Note that $\operatorname{diam}K_n\leqslant\operatorname{diam}\mathcal{X}_n$ for any n. Therefore we can assume that all polyhedra $K_1,K_2,\dots$ lie in a closed ball of sufficiently large radius.

Applying Blaschke selection theorem, we can pass to a subsequence of $K_1,K_2,\dots$ that converges in the sense of Hausdorff; denote its limit by K_∞. By 4.18 the surface of K_∞ is isometric to $\mathcal{X}_\infty$.

Therefore it remains to prove the following lemma.

4.19. Lemma. *Let $\mathcal{X}$ be a geodesic* CBB(0) *space that is homeomorphic to $\mathbb{S}^2$. Then there is a sphere with polyhedral metrics $\mathcal{X}'$ that is arbitrarily close to $\mathcal{X}$ in the sense of Gromov–Hausdorff.*

Idea behind the proof. Suppose we can triangulate $\mathcal{X}_\infty$ by small geodesic triangles; that is, we can choose a finite set of points $p_1, \dots, p_n \in \in \mathcal{X}_\infty$ and some geodesics $[p_i p_j]$ that cut $\mathcal{X}_\infty$ into regions of small diameter bounded by geodesic triangles $[p_i p_j p_k]$. (The actual proof constructs a triangulation with a weaker property.)

Observe that total angle around each p_i cannot exceed $2{\cdot}\pi$. That is, suppose $p_{j_1}, \dots, p_{j_k}$ are points connected to p_i by geodesics. Assume that they are ordered in the natural cyclic order. Then

$$\measuredangle[p_i\,{}^{p_{j_1}}_{p_{j_2}}] + \dots + \measuredangle[p_i\,{}^{p_{j_{k-1}}}_{p_{j_k}}] + \measuredangle[p_i\,{}^{p_{j_k}}_{p_{j_1}}] \leqslant 2{\cdot}\pi.$$

By comparison, we get

❶ $$\tilde\measuredangle(p_i\,{}^{p_{j_1}}_{p_{j_2}}) + \dots + \tilde\measuredangle(p_i\,{}^{p_{j_{k-1}}}_{p_{j_k}}) + \tilde\measuredangle(p_i\,{}^{p_{j_k}}_{p_{j_1}}) \leqslant 2{\cdot}\pi.$$

Now let us exchange each triangle by its model triangle. That is, consider a model triangle for each region in the subdivision of $\mathcal{X}$ and glue them together by the same rule. By ❶, the obtained polyhedral surface $\mathcal{X}'$ has nonnegative curvature. It remains to show that this way we can produce $\mathcal{X}'$ arbitrarily close to $\mathcal{X}$.

Denote by $p_i \to p_i'$ the natural map; it takes p_i in $\mathcal{X}$ and returns the corresponding point in $\mathcal{X}'$. Observe that

❷ $$|p_i' - p_j'|_{\mathcal{X}'} \leqslant |p_i - p_j|_{\mathcal{X}}.$$

Indeed, choose a geodesic γ from p_i to p_j. Let $p_i = x_0, x_1, \dots, x_n = p_j$ be the points of intersections of γ with the edges of the triangulation listed as they appear on γ. For each i, denote by x_i' the corresponding point in $\mathcal{X}'$. By comparison, we get

$$|x_k' - x_{k-1}'|_{\mathcal{X}'} \leqslant |x_k - x_{k-1}|_{\mathcal{X}}.$$

for each k. Therefore, ❷ follows.

Suppose $\varepsilon > 0$ is small, the points $p_1, \dots, p_n$ form an ε-net in $\mathcal{X}$, all edges of the triangulation are smaller than ε and

❸ $$|p_i' - p_j'|_{\mathcal{X}'} \geqslant |p_i - p_j|_{\mathcal{X}} - 100{\cdot}\varepsilon.$$

Then, together with the inequality above it proves that the lemma.

Note that the sides of the model triangles are local geodesics in $\mathcal{X}'$, but not necessarily geodesic; that is they do not have to be length-minimizing. Now, let us make another unjustified assumption: *Suppose that the sides of model triangles in $\mathcal{X}'$ are geodesics.* (The actual proof does not use this assumption.)

Choose a geodesic γ' from p'_i to p'_j in $\mathcal{X}'$. Note that γ' visits each triangle in the triangulation of $\mathcal{X}'$ at most once.

Let $p'_i = x'_0, x'_1, \dots, x'_n = p'_j$ be the points of intersections of γ' with the edges of the triangulation listed from p'_i to p'_j. For each i, denote by x_i the corresponding point in $\mathcal{X}$. Let Δ'_k be the triangle that contains arc $[x'_{k-1}x'_k]$ of γ' and Δ_k the corresponding triangle in $\mathcal{X}$. Note that

$$\text{❹} \qquad |x'_k - x'_{k-1}|_{\mathcal{X}'} \geqslant |x_k - x_{k-1}|_{\mathcal{X}} - \varepsilon{\cdot}K(\Delta_k),$$

where $K(\Delta_k)$ denotes the access of Δ_k; that is, the sum of its internal angles minus π.

Euler's formula and ❶ imply that the sum of all accesses is at most $4{\cdot}\pi$. Therefore, summing up ❹, we get

$$|p'_i - p'_j|_{\mathcal{X}'} \geqslant |p_i - p_j|_{\mathcal{X}} - 4{\cdot}\pi{\cdot}\varepsilon.$$

Whence ❸ follows. □

G Comments

This lecture contains selected material from Alexandrov's book [6].

In Euclid's Elements, solids were called equal if the same holds for their faces, but no proof was given. Adrien-Marie Legendre became interested in this problem towards the end of the 18th century. He discussed it with his colleague Joseph-Louis Lagrange, who suggested this problem to Augustin-Louis Cauchy in 1813; soon he proved it [25]. This theorem is included in many popular books [1, 26, 41].

The observation that the face-to-face condition can be removed was made by Alexandr Alexandrov [7].

Arm lemma. Original Cauchy's proof [25] also used a version of the arm lemma, but its proof contained a small mistake (corrected in one century).

Our proof of the arm lemma is due to Stanisław Zaremba. This and a couple of other proofs can be found in the letters between him and Isaac Schoenberg [40].

The following variation of the arm lemma makes sense for nonconvex spherical polygons. It is due to Viktor Zalgaller [44]. It can be used instead of the standard arm lemma.

4.20. Another arm lemma. *Let $A = [a_1 \dots a_n]$ and $A' = [a'_1 \dots \dots a'_n]$ be two spherical n-gons (not necessarily convex). Assume that A lies in a half-sphere, the corresponding sides of A and A' are equal* *and each angle of A is at least the corresponding angle in A'. Then A is congruent to A'.*

Global lemma. A more visual proof of the global lemma is given in [6, II §1.3].

Existence theorem. This theorem was proved by Alexandr Alexandrov [7]. Our sketch is taken from [31]; a complete proof is nicely written in [6]. In the original proof, the spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ were modified so the they become $(3\cdot n-6)$-dimensional manifolds. It was done by introducing extra structure (for $\mathbf{K}_n$ it is orientation + a marked vertex and an edge) that *brakes symmetries* of the spaces. After that one could apply the domain invariance theorem directly. Alternatively, one may first remove from $\mathbf{K}_n$ and $\mathbf{P}_n$ elements (polyhedron or surface)with nontrivial symmetries (after that the spaces become manifolds) and show that any symmetric polyhedron (or surface) can be approximated by a non-symmetric polyhedron (or surface).

A very different proof was found by Yuri Volkov in his thesis [42]; it uses a deformation of three-dimensional polyhedral space.

Lecture 5

Gluing and billiards

This lecture is nearly a copy of [4, Chapter 2]; here we define upper curvature bound in the sense of Alexandrov, prove Reshetnyak's gluing theorem, and apply it to a problem in billiards.

A Geodesics

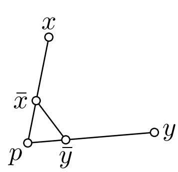
The CAT comparison can be applied to any metric space, but it is usually applied to geodesic spaces (or complete length spaces). To simplify the presentation we will assume in addition that the space is proper. The latter means that any closed ball is compact.

Recall that function is proper if inverse image of any compact set is compact. Note that *a metric space is proper if and only if the distance function from one (and therefore any) point is proper.*

5.1. Proposition. *Let* $\mathcal{U}$ *be a complete geodesic* CAT(0) *space. Then any two points in* $\mathcal{U}$ *are joined by a unique geodesic.*

Proof. Suppose there are two geodesics between x and y. Then we can choose two points $p \neq q$ on these geodesics such that $|x-p| = |x-q|$ and therefore $|y-p| = |y-q|$.

Observe that the model triangles $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)$ are degenerate and moreover $\tilde p = \tilde q$. Applying CAT(0) comparison with $\tilde z = \tilde p = \tilde q$, we get that $|p-q| = 0$, a contradiction. $\square$



5.2. Exercise. *Given* $[p\,{}^{x}_{y}]$ *in a* CAT(0) *space* $\mathcal{U}$, *consider the function*

$$f\colon (|p-\bar x|, |p-\bar y|) \mapsto \tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar{x} \in\,]px]$ *and* $\bar{y} \in\,]py]$. *Show that* f *is nondecreasing in each argument.*

Conclude that any hinge in a CAT(0) *space has defined angle.*

5.3. Exercise. *Fix a point* p *in a be a complete geodesic* CAT(0) *space* $\mathcal{U}$. *Given a point* $x \in \mathcal{U}$, *denote by* $\gamma_x\colon [0,1] \to \mathcal{U}$ *a (necessarily unique) geodesic path from* p *to* x.

Show that the family of maps $h_t\colon \mathcal{U} \to \mathcal{U}$ *defined by*

$$h_t(x) = \gamma_x(t)$$

is a homotopy; it is called geodesic homotopy. Conclude that $\mathcal{U}$ *is contractible.*

The geodesic homotopy introduced in the previous exercise should help to solve the next one.

5.4. Exercise. *Let* $\mathcal{U}$ *be a complete geodesic* CAT(0) *space. Assume* $\mathcal{U}$ *is a topological manifold. Show that any geodesic in* $\mathcal{U}$ *can be extended as a two-side infinite geodesic.*

B Thin triangles

Let us recall the definition of thin triangles.

5.5. Definition. *A triangle* $[xyz]$ *in the metric space* $\mathcal{U}$ *is called thin if the natural map* $\tilde{\triangle}(xyz)_{\mathbb{E}^2} \to [xyz]$ *is distance nonincreasing.*

Analogously, a triangle $[xyz]$ *is called spherically thin if the natural map from the spherical model triangle* $\tilde{\triangle}(xyz)_{\mathbb{S}^2}$ *to* $[xyz]$ *is distance nonincreasing.*

5.6. Proposition. *A geodesic space is* CAT(0) *(*CAT(1)*) if and only if all its triangles are thin (respectively, all its triangles of perimeter* $< 2\cdot\pi$ *are spherically thin).*

Proof; if part. Apply the triangle inequality and thinness of triangles $[pxy]$ and $[qxy]$, where p, q, x, and y are as in the definition of the CAT(κ) comparison.

Only-if part. Applying CAT(0) comparison to a quadruple p, q, x, y with $q \in [xy]$ shows that any triangle satisfies point-side comparison, that is, the distance from a vertex to a point on the opposite side is no greater than the corresponding distance in the Euclidean model triangle.

Now consider a triangle $[xyz]$ and let $p \in [xy]$ and $q \in [xz]$. Let $\tilde{p}$, $\tilde{q}$ be the corresponding points on the sides of the model triangle $\tilde{\triangle}(xyz)_{\mathbb{E}^2}$. Applying 5.2, we get that

$$\tilde{\measuredangle}(x\,{}^{y}_{z})_{\mathbb{E}^2} \geqslant \tilde{\measuredangle}(x\,{}^{p}_{q})_{\mathbb{E}^2}.$$

Therefore $|\tilde{p} - \tilde{q}|_{\mathbb{E}^2} \geqslant |p - q|$.

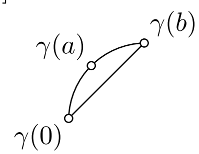
The CAT(1) argument is the same. □

A curve $\gamma\colon \mathbb{I} \to \mathcal{U}$ is called a *local geodesic* if for any $t \in \mathbb{I}$ there is a neighborhood U of t in $\mathbb{I}$ such that the restriction $\gamma|_U$ is a geodesic.

5.7. Proposition. *Suppose $\mathcal{U}$ is a proper geodesic* CAT(0) *space. Then any local geodesic in $\mathcal{U}$ is a geodesic.*

Analogously, if $\mathcal{U}$ is a proper geodesic CAT(1) *space, then any local geodesic in $\mathcal{U}$ which is shorter than π is a geodesic.*

Proof. Suppose $\gamma\colon [0, \ell] \to \mathcal{U}$ is a local geodesic that is not a geodesic. Choose a to be the maximal value such that γ is a geodesic on $[0, a]$. Further, choose $b > a$ so that γ is a geodesic on $[a, b]$.



Since the triangle $[\gamma(0)\gamma(a)\gamma(b)]$ is thin (see the next section) and $|\gamma(0) - \gamma(b)| < b$ we have

$$|\gamma(a - \varepsilon) - \gamma(a + \varepsilon)| < 2{\cdot}\varepsilon$$

for all small $\varepsilon > 0$. That is, γ is not length-minimizing on the interval $[a - \varepsilon, a + \varepsilon]$ for any $\varepsilon > 0$, a contradiction.

The spherical case is done in the same way. □

5.8. Exercise. *Let $\mathcal{U}$ be a complete geodesic space. Show that $\mathcal{U}$ is* CAT(0) *if and only if the function $f = \frac{1}{2}{\cdot}\mathrm{dist}_p^2$ is 1-convex for any $p \in \mathcal{U}$.*

5.9. Exercise. *Suppose $\gamma_1, \gamma_2\colon [0, 1] \to \mathcal{U}$ are two geodesic paths in a complete geodesic* CAT(0) *space $\mathcal{U}$. Show that*

$$t \mapsto |\gamma_1(t) - \gamma_2(t)|_{\mathcal{U}}$$

is a convex function.

5.10. Exercise. *Let A be a convex closed set in a proper geodesic* CAT(0) *space $\mathcal{U}$; that is, if $x, y \in A$, then $[xy] \subset A$. Show that for any $r > 0$ the closed r-neighborhood of A is convex; that is, the set*

$$A_r = \{\, x \in \mathcal{U} \,:\, \mathrm{dist}_A x \leqslant r \,\}$$

is convex.

5.11. Exercise. *Let $\mathcal{U}$ be a proper geodesic* CAT(0) *space and $K \subset \mathcal{U}$ be a closed convex set. Show that:*

(a) *For each point $p\in\mathcal{U}$ there is a unique point $p^*\in K$ that minimizes the distance $|p-p^*|$.*
(b) *The closest-point projection $p\mapsto p^*$ defined by (a) is short.*

Recall that a set A in a metric space $\mathcal{U}$ is called locally convex if for any point $p\in A$ there is an open neighborhood $\mathcal{U}\ni p$ such that any geodesic in $\mathcal{U}$ with ends in A lies in A.

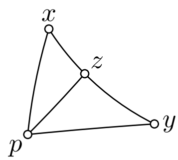
5.12. Exercise. *Let $\mathcal{U}$ be a proper geodesic* CAT(0) *space. Show that any closed, connected, locally convex set in $\mathcal{U}$ is convex.*

C Inheritance lemma

5.13. Inheritance lemma. *Assume that a triangle $[pxy]$ in a metric space is decomposed into two triangles $[pxz]$ and $[pyz]$; that is, $[pxz]$ and $[pyz]$ have a common side $[pz]$, and the sides $[xz]$ and $[zy]$ together form the side $[xy]$ of $[pxy]$.*

If both triangles $[pxz]$ and $[pyz]$ are thin, then the triangle $[pxy]$ is also thin.

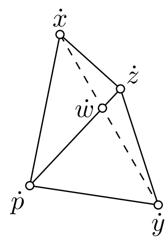
Analogously, if $[pxy]$ has perimeter $<2\cdot\pi$ and both triangles $[pxz]$ and $[pyz]$ are spherically thin, then triangle $[pxy]$ is spherically thin.



Proof. Construct the model triangles $[\dot p\dot x\dot z]=\tilde\triangle(pxz)_{\mathbb{E}^2}$ and $[\dot p\dot y\dot z]=$ $=\tilde\triangle(pyz)_{\mathbb{E}^2}$ so that $\dot x$ and $\dot y$ lie on opposite sides of $[\dot p\dot z]$.

Let us show that

❶ $$\tilde\measuredangle(z\,{}^p_x)+\tilde\measuredangle(z\,{}^p_y)\geqslant\pi.$$



If not, then for some point $\dot w\in[\dot p\dot z]$, we have

$$|\dot x-\dot w|+|\dot w-\dot y|<|\dot x-\dot z|+|\dot z-\dot y|=|x-y|.$$

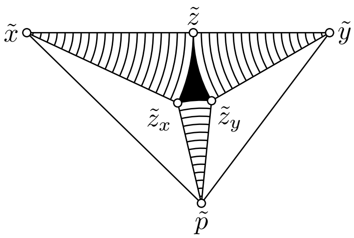
Let $w\in[pz]$ correspond to $\dot w$; that is, $|z-$ $-w|=|\dot z-\dot w|$. Since $[pxz]$ and $[pyz]$ are thin, we have

$$|x-w|+|w-y|<|x-y|,$$

contradicting the triangle inequality.

Denote by $\dot D$ the union of two solid triangles $[\dot p\dot x\dot z]$ and $[\dot p\dot y\dot z]$. Further, denote by $\tilde D$ the solid triangle $[\tilde p\tilde x\tilde y]=\tilde\triangle(pxy)_{\mathbb{E}^2}$. By ❶, there is a short map $F\colon\tilde D\to\dot D$ that sends

$$\tilde p\mapsto\dot p,\qquad\tilde x\mapsto\dot x,\qquad\tilde z\mapsto\dot z,\qquad\tilde y\mapsto\dot y.$$



Indeed, by Alexandrov's lemma (2.6), there are nonoverlapping triangles

$$[\tilde{p}\tilde{x}\tilde{z}_x] \stackrel{iso}{=\!=} [\dot{p}\dot{x}\dot{z}]$$

and

$$[\tilde{p}\tilde{y}\tilde{z}_y] \stackrel{iso}{=\!=} [\dot{p}\dot{y}\dot{z}]$$

inside the triangle $[\tilde{p}\tilde{x}\tilde{y}]$.

Connect the points in each pair $(\tilde{z}, \tilde{z}_x)$, $(\tilde{z}_x, \tilde{z}_y)$ and $(\tilde{z}_y, \tilde{z})$ with arcs of circles centered at $\tilde{y}$, $\tilde{p}$, and $\tilde{x}$ respectively. Define F as follows:

- ◇ Map $\mathrm{Conv}[\tilde{p}\tilde{x}\tilde{z}_x]$ isometrically onto $\mathrm{Conv}[\dot{p}\dot{x}\dot{z}]$; similarly map $\mathrm{Conv}[\tilde{p}\tilde{y}\tilde{z}_y]$ onto $\mathrm{Conv}[\dot{p}\dot{y}\dot{z}]$.
- ◇ If x is in one of the three circular sectors, say at distance r from its center, set $F(x)$ to be the point on the corresponding segment $[pz]$, $[xz]$ or $[yz]$ whose distance from the left-hand endpoint of the segment is r.
- ◇ Finally, if x lies in the remaining curvilinear triangle $\tilde{z}\tilde{z}_x\tilde{z}_y$, set $F(x) = z$.

By construction, F satisfies the conditions.

By assumption, the natural maps $[\dot{p}\dot{x}\dot{z}] \to [pxz]$ and $[\dot{p}\dot{y}\dot{z}] \to [pyz]$ are short. By composition, the natural map from $[\tilde{p}\tilde{x}\tilde{y}]$ to $[pyz]$ is short, as claimed.

The spherical case is done along the same lines. □

D Reshetnyak's gluing

Suppose $\mathcal{U}^1$ and $\mathcal{U}^2$ are proper geodesic spaces with isometric closed convex sets $A^i \subset \mathcal{U}^i$ and let $\iota\colon A^1 \to A^2$ be an isometry. Consider the space $\mathcal{W}$ of all equivalence classes in $\mathcal{U}^1 \sqcup \mathcal{U}^2$ with the equivalence relation given by $a \sim \iota(a)$ for any $a \in A^1$.

It is straightforward to see that $\mathcal{W}$ is a proper geodesic space when equipped with the following metric

$$\begin{aligned} |x-y|_{\mathcal{W}} &:= |x-y|_{\mathcal{U}^i} \\ &\qquad \text{if} \quad x,y \in \mathcal{U}^i, \quad \text{and} \\ |x-y|_{\mathcal{W}} &:= \min\left\{\, |x-a|_{\mathcal{U}^1} + |y-\iota(a)|_{\mathcal{U}^2} \;:\; a \in A^1 \,\right\} \\ &\qquad \text{if} \quad x \in \mathcal{U}^1 \quad \text{and} \quad y \in \mathcal{U}^2. \end{aligned}$$

Abusing notation, we denote by x and y the points in $\mathcal{U}^1 \sqcup \mathcal{U}^2$ and their equivalence classes in $\mathcal{U}^1 \sqcup \mathcal{U}^2/\!\sim$.

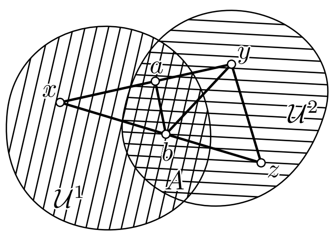
The space $\mathcal{W}$ is called the gluing of $\mathcal{U}^1$ and $\mathcal{U}^2$ along ι. If one applies this construction to two copies of one space $\mathcal{U}$ with a set $A \subset \mathcal{U}$ and the identity map $\iota\colon A \to A$, then the obtained space is called the double of $\mathcal{U}$ along A.

We can (and will) identify $\mathcal{U}^i$ with its image in $\mathcal{W}$; this way both subsets $A^i \subset \mathcal{U}^i$ will be identified and denoted further by A. Note that $A = \mathcal{U}^1 \cap \mathcal{U}^2 \subset \mathcal{W}$, therefore A is also a convex set in $\mathcal{W}$.

5.14. Reshetnyak gluing. *Suppose $\mathcal{U}^1$ and $\mathcal{U}^2$ are proper geodesic* $\mathrm{CAT}(0)$ *spaces with isometric closed convex sets $A^i \subset \mathcal{U}^i$, and $\iota\colon A^1 \to A^2$ is an isometry. Then the gluing of $\mathcal{U}^1$ and $\mathcal{U}^2$ along ι is a* $\mathrm{CAT}(0)$ *proper geodesic space.*

Proof. By construction of the gluing space, the statement can be reformulated in the following way:

5.15. Reformulation of 5.14. *Let $\mathcal{W}$ be a proper geodesic space with two closed convex sets $\mathcal{U}^1, \mathcal{U}^2 \subset \mathcal{W}$ such that $\mathcal{U}^1 \cup \mathcal{U}^2 = \mathcal{W}$ and $\mathcal{U}^1$, $\mathcal{U}^2$ are* $\mathrm{CAT}(0)$*. Then $\mathcal{W}$ is* $\mathrm{CAT}(0)$*.*



It suffices to show that any triangle $[xyz]$ in $\mathcal{W}$ is thin. This is obviously true if all three points x, y, z lie in one of $\mathcal{U}^i$. Thus, without loss of generality, we may assume that $x \in \mathcal{U}^1$ and $y, z \in \mathcal{U}^2$.

Choose points $a, b \in A = \mathcal{U}^1 \cap \mathcal{U}^2$ that lie respectively on the sides $[xy]$, $[xz]$. Note that

- ◇ the triangle $[xab]$ lies in $\mathcal{U}^1$,
- ◇ both triangles $[yab]$ and $[ybz]$ lie in $\mathcal{U}^2$.

In particular, each triangle $[xab]$, $[yab]$, and $[ybz]$ is thin.

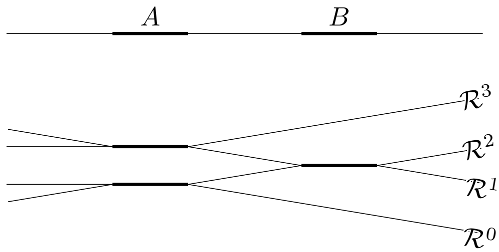
Applying the inheritance lemma (5.13) twice, we get that $[xyb]$ and consequently $[xyz]$ is thin. $\square$

5.16. Exercise. *Suppose $\mathcal{U}$ is a geodesic space and $A \subset \mathcal{U}$ is a closed subset. Assume that the doubling of $\mathcal{U}$ in A is* $\mathrm{CAT}(0)$*. Show that A is a convex set of $\mathcal{U}$.*

E Puff pastry

In this section, we introduce the notion of Reshetnyak puff pastry. This construction will be used in the next section to prove the collision theorem (5.27).

Let $\boldsymbol{A} = (A^1, \dots, A^N)$ be an array of convex closed sets in the Euclidean space $\mathbb{E}^m$. Consider an array of $N+1$ copies of $\mathbb{E}^m$. Assume that the space $\mathcal{R}$ is obtained by gluing successive pairs of spaces along $A^1, \dots, A^N$ respectively.



Puff pastry for (A, B, A).

The resulting space $\mathcal{R}$ will be called the Reshetnyak puff pastry for array $\boldsymbol{A}$. The copies of $\mathbb{E}^m$ in the puff pastry $\mathcal{R}$ will be called levels; they will be denoted by $\mathcal{R}^0, \dots, \mathcal{R}^N$. The point in the k-th level $\mathcal{R}^k$ that corresponds to $x \in \mathbb{E}^m$ will be denoted by x^k.

Given $x \in \mathbb{E}^m$, any point $x^k \in \mathcal{R}$ is called a lifting of x. The map $x \mapsto x^k$ defines an isometry $\mathbb{E}^m \to \mathcal{R}^k$; in particular, we can talk about liftings of subsets in $\mathbb{E}^m$.

Note that:

◇ The intersection $A^1 \cap \dots \cap A^N$ admits a unique lifting in $\mathcal{R}$.

◇ Moreover, $x^i = x^j$ for some $i < j$ if and only if

$$x \in A^{i+1} \cap \dots \cap A^j.$$

◇ The restriction $\mathcal{R}^k \to \mathbb{E}^m$ of the natural projection $x^k \mapsto x$ is an isometry.

5.17. Observation. *Any Reshetnyak puff pastry is a proper geodesic* CAT(0) *space.*

Proof. Apply Reshetnyak gluing theorem (5.14) recursively for the convex sets in the array. □

5.18. Proposition. *Assume $(A^1, \dots, A^N)$ and $(\check{A}^1, \dots, \check{A}^N)$ are two arrays of convex closed sets in $\mathbb{E}^m$ such that $A^k \subset \check{A}^k$ for each k. Let $\mathcal{R}$ and $\check{\mathcal{R}}$ be the corresponding Reshetnyak puff pastries. Then the map $\mathcal{R} \to \check{\mathcal{R}}$ defined by $x^k \mapsto \check{x}^k$ is short.*

Moreover, if

❶ $$|x^i - y^j|_{\mathcal{R}} = |\check{x}^i - \check{y}^j|_{\check{\mathcal{R}}}$$

for some $x, y \in \mathbb{E}^m$ and $i, j \in \{0, \dots, n\}$, then the unique geodesic $[\check{x}^i \check{y}^j]_{\check{\mathcal{R}}}$ is the image of the unique geodesic $[x^i y^j]_{\mathcal{R}}$ under the map $x^i \mapsto \check{x}^i$.

Proof. The first statement in the proposition follows from the construction of Reshetnyak puff pastries.

By Observation 5.17, $\mathcal{R}$ and $\check{\mathcal{R}}$ are proper geodesic CAT(0) spaces; hence $[x^i y^j]_{\mathcal{R}}$ and $[\check{x}^i \check{y}^j]_{\check{\mathcal{R}}}$ are unique. By ❶, since the map $\mathcal{R} \to \check{\mathcal{R}}$ is short, the image of $[x^i y^j]_{\mathcal{R}}$ is a geodesic of $\check{\mathcal{R}}$ joining $\check{x}^i$ to $\check{y}^j$. Hence the second statement follows. $\square$

5.19. Definition. *Consider a Reshetnyak puff pastry $\mathcal{R}$ with the levels $\mathcal{R}^0, \dots, \mathcal{R}^N$. We say that $\mathcal{R}$ is end-to-end convex if $\mathcal{R}^0 \cup \mathcal{R}^N$, the union of its lower and upper levels, forms a convex set in $\mathcal{R}$; that is, if $x, y \in \mathcal{R}^0 \cup \mathcal{R}^N$, then $[xy]_{\mathcal{R}} \subset \mathcal{R}^0 \cup \mathcal{R}^N$.*

Note that if $\mathcal{R}$ is the Reshetnyak puff pastry for an array of convex sets $\boldsymbol{A} = (A^1, \dots, A^N)$, then $\mathcal{R}$ is end-to-end convex if and only if the union of the lower and the upper levels $\mathcal{R}^0 \cup \mathcal{R}^N$ is isometric to the double of $\mathbb{E}^m$ along the nonempty intersection $A^1 \cap \dots \cap A^N$.

5.20. Observation. *Let $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ be arrays of convex bodies in $\mathbb{E}^m$. Assume that array $\boldsymbol{A}$ is obtained by inserting in $\check{\boldsymbol{A}}$ several copies of the bodies which were already listed in $\check{\boldsymbol{A}}$.*

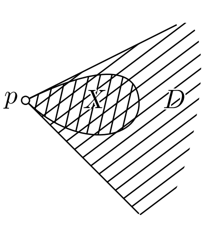
For example, if $\check{\boldsymbol{A}} = (A, C, B, C, A)$, by placing B in the second place and A in the fourth place, we obtain $\boldsymbol{A} = (A, B, C, A, B, C, A)$.

Denote by $\check{\mathcal{R}}$ and $\mathcal{R}$ the Reshetnyak puff pastries for $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ respectively.

If $\check{\mathcal{R}}$ is end-to-end convex, then so is $\mathcal{R}$.

Proof. Without loss of generality, we may assume that $\boldsymbol{A}$ is obtained by inserting one element in $\check{\boldsymbol{A}}$, say at the place number k.

Note that $\check{\mathcal{R}}$ is isometric to the puff pastry for $\boldsymbol{A}$ with A^k replaced by $\mathbb{E}^m$. It remains to apply Proposition 5.18. $\square$



Let X be a convex set in a Euclidean space. By a dihedral angle, we understand an intersection of two half-spaces; the intersection of corresponding hyperplanes is called the edge of the angle. We say that a dihedral angle D supports X at a point $p \in X$ if D contains X and the edge of D contains p.

5.21. Lemma. *Let A and B be two convex sets in $\mathbb{E}^m$. Assume that any dihedral angle supporting $A \cap B$ has angle measure at least α. Then* *the Reshetnyak puff pastry for the array*

$$(\underbrace{A, B, A, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \ times}).$$

is end-to-end convex.

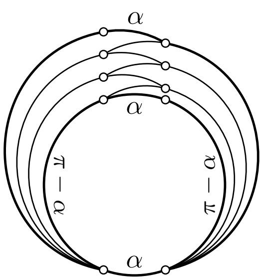
The proof of the lemma is based on a partial case, which we formulate as a sublemma.

5.22. Sublemma. *Let $\ddot{A}$ and $\ddot{B}$ be two half-planes in $\mathbb{E}^2$, where $\ddot{A}\cap\ddot{B}$ is an angle with measure α. Then the Reshetnyak puff pastry for the array*

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \ times})$$

is end-to-end convex.

Proof. Note that the puff pastry $\ddot{\mathcal{R}}$ is isometric to the cone over the space glued from the unit circles as shown on the diagram.

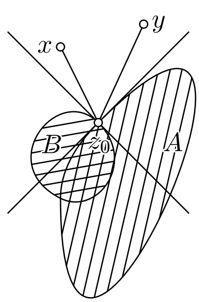


All the short arcs on the diagram have length α; the long arcs have length $\pi - \alpha$, so making a circuit along any path will take $2{\cdot}\pi$.

The end-to-end convexity of $\ddot{\mathcal{R}}$ is equivalent to the fact that any geodesic shorter than π with the ends on the inner and the outer circles lies completely in the union of these two circles.

The latter holds if the zigzag line in the picture has length at least π. This line is formed by $\lceil\frac{\pi}{\alpha}\rceil$ arcs with length α each. Hence the sublemma. □

In the proof of 5.21, we will use the following exercise in convex geometry:



5.23. Exercise. *Let A and B be two closed convex sets in $\mathbb{E}^m$ and $A \cap B \neq \varnothing$. Given two points $x, y \in \mathbb{E}^m$ let $f(z) = |x - z| + |y - z|$.*

Let $z_0 \in A\cap B$ be a point of minimum of $f|_{A\cap B}$.

Show that there are half-spaces $\dot{A}$ and $\dot{B}$ such that $\dot{A} \supset A$ and $\dot{B} \supset B$ and z_0 is also a point of minimum of the restriction $f|_{\dot{A}\cap\dot{B}}$.

Proof of 5.21. Fix arbitrary $x, y \in \mathbb{E}^m$. Choose a point $z \in A \cap B$ for which the sum

$$|x - z| + |y - z|$$

is minimal. To show the end-to-end convexity of $\mathcal{R}$, it is sufficient to prove the following:

❷ *The geodesic* $[x^0y^N]_\mathcal{R}$ *contains* $z^0 = z^N \in \mathcal{R}$.

Without loss of generality, we may assume that $z \in \partial A \cap \partial B$. Indeed, since the puff pastry for the 1-array (B) is end-to-end convex, Proposition 5.18 together with 5.20 imply ❷ in case z lies in the interior of A. The same way we can treat the case when z lies in the interior of B.

Note that $\mathbb{E}^m$ admits an isometric splitting $\mathbb{E}^{m-2} \times \mathbb{E}^2$ such that

$$\dot{A} = \mathbb{E}^{m-2} \times \ddot{A}$$
$$\dot{B} = \mathbb{E}^{m-2} \times \ddot{B}$$

where $\ddot{A}$ and $\ddot{B}$ are half-planes in $\mathbb{E}^2$.

Using Exercise 5.23, let us replace each A by $\dot{A}$ and each B by $\dot{B}$ in the array, to get the array

$$(\underbrace{\dot{A}, \dot{B}, \dot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \text{ times}}).$$

The corresponding puff pastry $\dot{\mathcal{R}}$ splits as a product of $\mathbb{E}^{m-2}$ and a puff pastry, call it $\ddot{\mathcal{R}}$, glued from the copies of the plane $\mathbb{E}^2$ for the array

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \text{ times}}).$$

Note that the dihedral angle $\dot{A} \cap \dot{B}$ is at least α. Therefore the angle measure of $\ddot{A} \cap \ddot{B}$ is also at least α. According to Sublemma 5.22 and Observation 5.20, $\ddot{\mathcal{R}}$ is end-to-end convex.

Since $\dot{\mathcal{R}} \stackrel{iso}{=\!=} \mathbb{E}^{m-2} \times \ddot{\mathcal{R}}$, the puff pastry $\dot{\mathcal{R}}$ is also end-to-end convex.

It follows that the geodesic $[\dot{x}^0\dot{y}^N]_{\dot{\mathcal{R}}}$ contains $\dot{z}^0 = \dot{z}^N \in \dot{\mathcal{R}}$. By Proposition 5.18, the image of $[\dot{x}^0\dot{y}^N]_{\dot{\mathcal{R}}}$ under the map $\dot{x}^k \mapsto x^k$ is the geodesic $[x^0y^N]_\mathcal{R}$. Hence ❷ and the lemma follow. $\square$

F Wide corners

We say that a closed convex set $A \subset \mathbb{E}^m$ has ε-wide corners for given $\varepsilon > 0$ if together with each point p, the set A contains a small right circular cone with the tip at p and aperture ε; that is, ε is the maximum angle between two generating lines of the cone.

For example, a plane polygon has ε-wide corners if all its interior angles are at least ε.

We will consider finite collections of closed convex sets $A^1, \dots, A^n \subset \subset \mathbb{E}^m$ such that for any subset $F \subset \{1, \dots, n\}$, the intersection $\bigcap_{i \in F} A^i$ has ε-wide corners. In this case, we may say briefly *all intersections of A^i have ε-wide corners.*

5.24. Exercise. *Assume $A^1, \dots, A^n \subset \mathbb{E}^m$ are compact, convex sets with a common interior point. Show that all intersections of A^i have ε-wide corners for some $\varepsilon > 0$.*

5.25. Exercise. *Assume $A^1, \dots, A^n \subset \mathbb{E}^m$ are convex sets with nonempty interiors that have a common center of symmetry. Show that all intersections of A^i have ε-wide corners for some $\varepsilon > 0$.*

The proof of the following proposition is based on 5.21; this lemma is essentially the case $n = 2$ in the proposition.

5.26. Proposition. *Given $\varepsilon > 0$ and a positive integer n, there is an array of integers $\boldsymbol{j}_\varepsilon(n) = (j_1, \dots, j_N)$ such that:*

(a) For each k we have $1 \leqslant j_k \leqslant n$, and each number $1, \dots, n$ appears in $\boldsymbol{j}_\varepsilon$ at least once.

(b) If $A^1, \dots, A^n$ is a collection of closed convex sets in $\mathbb{E}^m$ with a common point and all their intersections have ε-wide corners, then the puff pastry for the array $(A^{j_1}, \dots, A^{j_N})$ is end-to-end convex.

Moreover, we can assume that $N \leqslant (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^n$.

Proof. The array $\boldsymbol{j}_\varepsilon(n) = (j_1, \dots, j_N)$ is constructed recursively. For $n = 1$, we can take $\boldsymbol{j}_\varepsilon(1) = (1)$.

Assume that $\boldsymbol{j}_\varepsilon(n)$ is constructed. Let us replace each occurrence of n in $\boldsymbol{j}_\varepsilon(n)$ by the alternating string

$$\underbrace{n, n+1, n, \dots}_{\lceil \frac{\pi}{\varepsilon} \rceil + 1 \text{ times}}.$$

Denote the obtained array by $\boldsymbol{j}_\varepsilon(n+1)$.

By Lemma 5.21, the end-to-end convexity of the puff pastry for $\boldsymbol{j}_\varepsilon(n+1)$ follows from the end-to-end convexity of the puff pastry for the array where each string

$$\underbrace{A^n, A^{n+1}, A^n, \dots}_{\lceil \frac{\pi}{\varepsilon} \rceil + 1 \text{ times}}$$

is replaced by $Q = A^n \cap A^{n+1}$. End-to-end convexity of the latter follows by the assumption on $\boldsymbol{j}_\varepsilon(n)$, since all the intersections of $A^1, \dots, A^{n-1}, Q$ have ε-wide corners.

The upper bound on N follows directly from the construction. $\square$

G Billiards

Let $A^1, A^2, \dots A^n$ be a finite collection of closed convex sets in $\mathbb{E}^m$. Assume that for each i the boundary ∂A^i is a smooth hypersurface.

Consider the billiard table formed by the closure of the complement

$$T = \overline{\mathbb{E}^m \setminus \bigcup_i A^i}.$$

The sets A^i will be called walls of the table and the billiards described above will be called billiards with convex walls.

A billiard trajectory on the table is a unit-speed broken line γ that follows the standard law of billiards at the breakpoints on ∂A^i — in particular, the angle of reflection is equal to the angle of incidence. The breakpoints of the trajectory will be called collisions. We assume the trajectory meets only one wall at a time.

Recall that the definition of sets with ε-wide corners is given in 5F.

5.27. Collision theorem. *Assume $T \subset \mathbb{E}^m$ is a billiard table with n convex walls. Assume that the walls of T have a common interior point and all their intersections have ε-wide corners. Then the number of collisions of any trajectory in T is bounded by a number N which depends only on n and ε.*

As we will see from the proof, the value N can be found explicitly; $N = (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^{n^2}$ will do.

5.28. Corollary. *Consider n homogeneous hard balls moving freely and colliding elastically in $\mathbb{R}^3$. Every ball moves along a straight line with constant speed until two balls collide, and then the new velocities of the two balls are determined by the laws of classical mechanics. We assume that only two balls can collide at the same time.*

Then the total number of collisions cannot exceed some number N that depends on the radii and masses of the balls. If the balls are identical, then N depends only on n.

5.29. Exercise. *Show that in the case of identical balls in the one-dimensional space (in $\mathbb{R}$) the total number of collisions cannot exceed $N = \frac{n\cdot(n-1)}{2}$.*

The proof below admits a straightforward generalization to all dimensions.

Proof of 5.28 modulo 5.27. Denote by $a_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ the center of the i-th ball. Consider the corresponding point in $\mathbb{R}^{3\cdot N}$

$$\begin{aligned}\boldsymbol{a} &= (a_1, a_2, \dots, a_n) = \\ &= (x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_n, y_n, z_n).\end{aligned}$$

The i-th and j-th balls intersect if

$$|a_i - a_j| \leqslant R_i + R_j,$$

where R_i denotes the radius of the i-th ball. These inequalities define $\frac{n\cdot(n-1)}{2}$ cylinders

$$C_{i,j} = \left\{ (a_1, a_2, \dots, a_n) \in \mathbb{R}^{3\cdot n} \ : \ |a_i - a_j| \leqslant R_i + R_j \right\}.$$

The closure of the complement

$$T = \overline{\mathbb{R}^{3\cdot n} \setminus \bigcup_{i<j} C_{i,j}}$$

is the configuration space of our system. Its points correspond to valid positions of the system of balls.

The evolution of the system of balls is described by the motion of the point $\boldsymbol{a} \in \mathbb{R}^{3\cdot n}$. It moves along a straight line at a constant speed until it hits one of the cylinders $C_{i,j}$; this event corresponds to a collision in the system of balls.

Consider the norm of $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{R}^{3\cdot n}$ defined by

$$\|\boldsymbol{a}\| = \sqrt{M_1 \cdot |a_1|^2 + \dots + M_n \cdot |a_n|^2},$$

where $|a_i| = \sqrt{x_i^2 + y_i^2 + z_i^2}$ and M_i denotes the mass of the i-th ball. In the metric defined by $\|*\|$, the collisions follow the standard law of billiards.

By construction, the number of collisions of hard balls that we need to estimate is the same as the number of collisions of the corresponding billiard trajectory on the table with $C_{i,j}$ as the walls.

Note that each cylinder $C_{i,j}$ is a convex set; it has smooth boundary, and it is centrally symmetric around the origin. By 5.25, all the intersections of the walls have ε-wide corners for some $\varepsilon > 0$ that depend on the radiuses R_i and the masses M_i. It remains to apply the collision theorem (5.27). $\square$

Now we present the proof of the collision theorem (5.27) based on the results developed in the previous section.

Proof of 5.27. Let us apply induction on n.

Base: $n = 1$. The number of collisions cannot exceed 1. Indeed, by the convexity of A^1, if the trajectory is reflected once in ∂A^1, then it cannot return to A^1.

Step. Assume γ is a trajectory that meets the walls in the order $A^{i_1}, \dots, A^{i_N}$ for a large integer N.

Consider the array

$$\boldsymbol{A}_\gamma = (A^{i_1}, \dots, A^{i_N}).$$

The induction hypothesis implies:

❶ *There is a positive integer M such that any M consecutive elements of $\boldsymbol{A}_\gamma$ contain each A^i at least once.*

Let $\mathcal{R}_\gamma$ be the Reshetnyak puff pastry for $\boldsymbol{A}_\gamma$.

Consider the lift of γ to $\mathcal{R}_\gamma$, defined by $\bar\gamma(t) = \gamma^k(t) \in \mathcal{R}_\gamma$ for any moment of time t between the k-th and $(k+1)$-th collisions. Since γ follows the standard law of billiards at breakpoints, the lift $\bar\gamma$ is locally a geodesic in $\mathcal{R}_\gamma$. By 5.17, the puff pastry $\mathcal{R}_\gamma$ is a proper geodesic CAT(0) space. Therefore $\bar\gamma$ is a geodesic.

Since γ does not meet $A^1 \cap \dots \cap A^n$, the lift $\bar\gamma$ does not lie in $\mathcal{R}^0_\gamma \cup \mathcal{R}^N_\gamma$. In particular, $\mathcal{R}_\gamma$ is not end-to-end convex.

Let

$$\boldsymbol{B} = (A^{j_1}, \dots, A^{j_K})$$

be the array provided by Proposition 5.26; so $\boldsymbol{B}$ contains each A^i at least once and the puff pastry $\mathcal{R}_{\boldsymbol{B}}$ for $\boldsymbol{B}$ is end-to-end convex. If N is sufficiently large, namely $N \geqslant K \cdot M$, then ❶ implies that $\boldsymbol{A}_\gamma$ can be obtained by inserting a finite number of A^i's in $\boldsymbol{B}$.

By 5.20, $\mathcal{R}_\gamma$ is end-to-end convex — a contradiction. □

H Comments

The gluing theorem (5.14) was proved by Yuri Reshetnyak [37]. It can be extended to all geodesic CAT(0) spaces. It also admits a natural generalization to geodesic CAT(κ) spaces; see the book of Martin Bridson and André Haefliger [16] and our book [5] for details.

The collision theorem (5.27) was proved by Dmitri Burago, Serge Ferleger and Alexey Kononenko [19]. Its corollary (5.28) answers a question posed by Yakov Sinai [27]. Puff pastry is used to bound topological entropy of the billiard flow and to approximate the shortest billiard path that touches given lines in a given order; see the papers of Dmitri Burago with Serge Ferleger, and Alexey Kononenko [20], and with Dimitri Grigoriev and Anatol Slissenko [21]. The lecture of Dmitri Burago [17] gives a short survey on the subject.

Note that the interior points of the walls play a key role in the proof despite that the trajectories never go inside the walls. In a similar fashion, puff pastry was used by Stephanie Alexander and Richard Bishop [2] to find the upper curvature bound for warped products.

Joel Hass [30] constructed an example of a Riemannian metric on the 3-ball with negative curvature and concave boundary. This example might decrease your appetite for generalizing the collision theorem — while locally such a 3-ball looks as good as the billiards table in the theorem, the number of collisions is obviously infinite.

It was shown by Dmitri Burago and Sergei Ivanov [22] that the number of collisions that may occur between n identical balls in $\mathbb{R}^3$ grows at least exponentially in n; the two-dimensional remains open.

Lecture 6

Majorization

A Formulation

6.1. Definition. *Let $\mathcal{X}$ be a metric space, $\tilde{\alpha}$ be a simple closed curve of finite length in $\mathbb{E}^2$, and $D \subset \mathbb{E}^2$ be a closed region bounded by $\tilde{\alpha}$. A length-nonincreasing map $F\colon D \to \mathcal{X}$ is called m a j o r i z i n g if it is length-preserving on $\tilde{\alpha}$.*

In this case, we say that D m a j o r i z e s the curve $\alpha = F \circ \tilde{\alpha}$ under the map F.

The following proposition is a consequence of the definition.

6.2. Proposition. *Let α be a closed curve in a metric space $\mathcal{X}$. Suppose $D \subset \mathbb{E}^2$ majorizes α under $F\colon D \to \mathcal{X}$. Then any geodesic subarc of α is the image under F of a subarc of $\partial_{\mathbb{E}^2} D$ that is geodesic in the length metric of D.*

In particular, if D is convex, then the corresponding subarc is a geodesic in $\mathbb{E}^2$.

Proof. For a geodesic subarc $\gamma\colon [a,b] \to \mathcal{X}$ of $\alpha = F \circ \tilde{\alpha}$, set

$$\begin{aligned} \tilde{r} &= |\tilde{\gamma}(a) - \tilde{\gamma}(b)|_D, & \tilde{\gamma} &= (F|_{\partial D})^{-1} \circ \gamma, \\ s &= \operatorname{length} \gamma, & \tilde{s} &= \operatorname{length} \tilde{\gamma}. \end{aligned}$$

Then

$$\tilde{r} \geqslant r = s = \tilde{s} \geqslant \tilde{r}.$$

Therefore $\tilde{s} = \tilde{r}$. ☐

6.3. Corollary. *Assume a convex region $D \subset \mathbb{E}^2$ majorizes $[pxy]$. Then D is a s o l i d m o d e l t r i a n g l e of $[pxy]$; that is, $D = \operatorname{Conv}[\tilde{p}\tilde{x}\tilde{y}]$*

for a model triangle $[\tilde p\tilde x\tilde y]=\tilde\triangle(pxy)$*. Moreover, the majorizing map sends* $\tilde p$*,* $\tilde x$ *and* $\tilde y$ *respectively to* p*,* x *and* y*.*

Now we come to the main theorem of this section.

6.4. Majorization theorem. *Any closed rectifiable curve* α *in a geodesic* CAT(0) *space is majorized by a convex plane figure.*

B Triangles

The case when α is a triangle, say $[pxy]$, is the base in the following proof, and it is nontrivial. In this case, by Corollary 6.3, the majorizing convex region the solid model triangle.

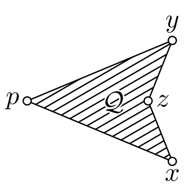
6.5. Line-of-sight map. *Let* p *be a point and* α *be a curve of finite length in a geodesic space* $\mathcal{X}$*. Let* $\mathring\alpha:[0,1]\to\mathcal{U}$ *be the constant-speed parametrization of* α*. If* $\gamma_t\colon[0,1]\to\mathcal{U}$ *is a geodesic path from* p *to* $\mathring\alpha(t)$*, we say*

$$[0,1]\times[0,1]\to\mathcal{U}\colon (t,s)\mapsto\gamma_t(s)$$

is a line-of-sight map from p *to* α*.*

We will show that there is a majorizing map for $[pxy]$ whose image W is the image of the line-of-sight map for $[xy]$ from p, but as one can see from the following example, the line-of-sight map is not majorizing in general.

Example. Let $\mathcal{Q}$ be a solid quadrangle $[pxzy]$ in $\mathbb{E}^2$ formed by two congruent triangles, which is non-convex at z (as in the picture). Equip $\mathcal{Q}$ with the length metric. Then $\mathcal{Q}$ is CAT(0) by Reshetnyak gluing (5.14). For triangle $[pxy]_{\mathcal{Q}}$ in $\mathcal{Q}$ and its model triangle $[\tilde p\tilde x\tilde y]$ in $\mathbb{E}^2$, we have



$$|\tilde x-\tilde y|=|x-y|_{\mathcal{Q}}=|x-z|+|z-y|.$$

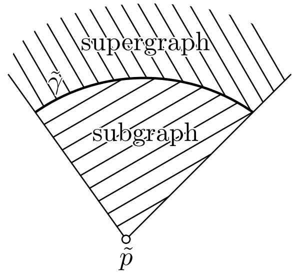
Then the map F defined by matching line-of-sight parameters satisfies $F(\tilde x)=x$ and $|x-F(\tilde w)|>|\tilde x-\tilde w|$ if $\tilde w$ is near the midpoint $\tilde z$ of $[\tilde x\tilde y]$ and lies on $[\tilde p\tilde z]$. Indeed, for $\varepsilon=1-s$ we have

$$|\tilde x-\tilde w|=|\tilde x-\tilde\gamma_{\frac12}(s)|=|x-z|+o(\varepsilon)$$

and

$$|x-F(\tilde w)|=|x-\gamma_{\frac12}(s)|=|x-z|-\varepsilon\cdot\cos\measuredangle[z\,{}^p_x]+o(\varepsilon).$$

Thus F is not majorizing.



6.6. Definition. *Let $\tilde\gamma\colon \mathbb{I}\to\mathbb{E}^2$ be a curve and $\tilde p\in\mathbb{E}^2$ be such that the direction of $[\tilde p\,\tilde\gamma(t)]$ turns monotonically as t grows.*

The set formed by all geodesics from $\tilde p$ to the points on $\tilde\gamma$ is called the subgraph of $\tilde\gamma$ with respect to $\tilde p$.

The set of all points $\tilde x\in\mathbb{E}^2$ such that a geodesic $[\tilde p\tilde x]$ intersects $\tilde\gamma$ is called the supergraph of $\tilde\gamma$ with respect to $\tilde p$.

The curve $\tilde\gamma$ is called convex (concave) with respect to $\tilde p$ if the subgraph (supergraph) of $\tilde\gamma$ with respect to $\tilde p$ is convex.

The curve $\tilde\gamma$ is called locally convex (concave) with respect to $\tilde p$ if for any interior value t_0 in $\mathbb{I}$ there is a subsegment $(a,b)\subset\mathbb{I}$, $(a,b)\ni t_0$, such that the restriction $\tilde\gamma|_{(a,b)}$ is convex (concave) with respect to $\tilde p$.

Our first lemma gives a model space construction based on repeated application of the argument in the proof of the inheritance lemma (5.13).

6.7. Lemma. *In $\mathbb{E}^2$, let β be a curve from x to y that is concave with respect to p. Let D be the subgraph of β with respect to p.*

(a) Then β forms a geodesic $[xy]_D$ in D and therefore β, $[px]$ and $[py]$ form a triangle $[pxy]_D$ in the length metric of D.

(b) Let $[\tilde p\tilde x\tilde y]$ be the model triangle for $[pxy]_D$. Then there is a short map

$$G\colon \operatorname{Conv}[\tilde p\tilde x\tilde y]\to D$$

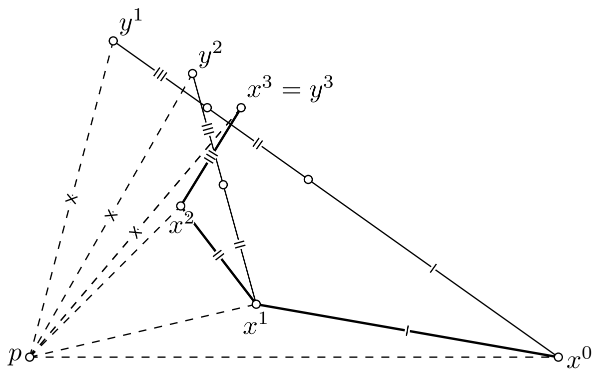
such that $\tilde p\mapsto p$, $\tilde x\mapsto x$, $\tilde y\mapsto y$, and G is length-preserving on each side of $[\tilde p\tilde x\tilde y]$. In particular, Conv$[\tilde p\tilde x\tilde y]$ *majorizes triangle $[pxy]_D$ in D under G.*

Proof. We prove the lemma for a polygonal line β; the general case then follows by approximation. Namely, since β is concave it can be approximated by polygonal lines that are concave with respect to p, with their lengths converging to length β. Passing to a partial limit we will obtain the needed map G.

Suppose $\beta = x^0x^1\dots x^n$ is a polygonal line with $x^0=x$ and $x^n=y$. Consider a sequence of polygonal lines $\beta_i = x^0x^1\dots x^{i-1}y_i$ such that $|p-y_i|=|p-y|$ and β_i has same length as β; that is,

$$|x^{i-1}-y_i| = |x^{i-1}-x^i| + |x^i-x^{i+1}| + \dots + |x^{n-1}-x^n|.$$

Clearly $\beta_n=\beta$. Sequentially applying Alexandrov's lemma (2.6) shows that each of the polygonal lines $\beta_{n-1},\beta_{n-2},\dots,\beta_1$ is concave



with respect to p. Let D_i be the subgraph of β_i with respect to p. Applying the argument in the inheritance lemma (5.13) gives a short map $G_i\colon D_i \to D_{i+1}$ that maps $y_i \mapsto y_{i+1}$ and does not move p and x (in fact, G_i is the identity everywhere except on $\mathrm{Conv}[px^{i-1}y_i]$). Thus the composition

$$G_{n-1} \circ \dots \circ G_1 \colon D_1 \to D_n$$

is short. The result follows since $D_1 \overset{iso}{=\!=} \mathrm{Conv}[\tilde p \tilde x \tilde y]$. $\square$

6.8. Lemma. *Let $\mathcal{X}$ be a metric space, $\gamma\colon \mathbb{I} \to \mathcal{X}$ be a 1-Lipschitz curve, $p \in \mathcal{X}$, and $\tilde p \in \mathbb{E}^2$. Then there exists a unique up to rotation curve $\tilde\gamma\colon \mathbb{I} \to \mathbb{E}^2$, parametrized by arc-length, such that $|\tilde p - \tilde\gamma(t)| = |p - \gamma(t)|$ for all t and the direction of $[\tilde p\tilde\gamma(t)]$ monotonically turns around $\tilde p$ counterclockwise as t increases.*

If p, $\tilde p$, γ, and $\tilde\gamma$ are as above, then $\tilde\gamma$ is called the development of γ with respect to p; the point $\tilde p$ is called the basepoint of the development.

Proof. Consider the functions ρ, $\theta\colon \mathbb{I} \to \mathbb{R}$ defined as

$$\rho(t) = |p - \gamma(t)|, \qquad\qquad \theta(t) = \int\limits_{t_0}^{t} \frac{\sqrt{1 - (\rho')^2}}{\rho},$$

where $t_0 \in \mathbb{I}$ is a fixed number and $\int$ denotes Lebesgue integral. Since γ is 1-Lipshitz, so is $\rho(t)$, and thus the function θ is defined and non-decreasing.

It is straightforward to check that (ρ, θ) uniquely describe $\tilde\gamma$ in polar coordinates on $\mathbb{E}^2$ with center at $\tilde p$. $\square$

6.9. Exercise. *A geodesic space $\mathcal{U}$ is* CAT(0) *if and only if development of any geodesic with respect to any point is concave.*

6.10. Lemma. *Let $[pxy]$ be a triangle in a geodesic* $\mathrm{CAT}(0)$ *space* $\mathcal{U}$*. In* $\mathbb{E}^2$*, let* $\tilde\gamma$ *be the* κ*-development of* $[xy]$ *with respect to* p*, where* $\tilde\gamma$ *has basepoint* $\tilde p$ *and subgraph* D*. Consider the map* $H\colon D\to\mathcal{U}$ *that sends the point with parameter* (t,s) *under the line-of-sight map for* $\tilde\gamma$ *with respect to* $\tilde p$*, to the point with the same parameter under the line-of-sight map* f *for* $[xy]$ *with respect to* p*. Then* H *is length-nonincreasing. In particular,* D *majorizes triangle* $[pxy]$*.*

Proof. Let $\gamma\colon[0,T]\to\mathcal{U}$ be a unit-speed paremetrization of $[xy]$; so, $T=|x-y|$. Choose a partition

$$0=t^0<t^1<\dots<t^n=T,$$

and set $x^i=\gamma(t^i)$. Construct a chain of model triangles $[\tilde p\tilde x^{i-1}\tilde x^i]=$ $=\tilde\triangle(px^{i-1}x^i)$, with $\tilde x^0=\tilde x$ and the direction of $[\tilde p\tilde x^i]$ turning counterclockwise as i grows. Let D_n be the subgraph with respect to $\tilde p$ of the polygonal line $\tilde x^0\dots\tilde x^n$.

Let δ_n be the maximum radius of a circle inscribed in any of the triangles $[\tilde p\tilde x^{i-1}\tilde x^i]$.

Now we construct a map $H_n\colon D_n\to\mathcal{U}$ that increases distances by at most $2\cdot\delta_n$. Suppose $w\in D_n$. Then w lies on or inside some triangle $[\tilde p\tilde x^{i-1}\tilde x^i]$. Define $H_n(w)$ by first mapping w to a nearest point on $[\tilde p\tilde x^{i-1}\tilde x^i]$ (choosing one if there are several), followed by the natural map to the triangle $[px^{i-1}x^i]$.

Since triangles in $\mathcal{U}$ are thin, the restriction of H_n to each triangle $[\tilde p\tilde x^{i-1}\tilde x^i]$ is short. Then the triangle inequality implies that the restriction of H_n to

$$U_n=\bigcup_{1\leqslant i\leqslant n}[\tilde p\tilde x^{i-1}\tilde x^i]$$

is short with respect to the length metric on D_n. Since nearest-point projection from D_n to U_n increases the D_n-distance between two points by at most $2\cdot\delta_n$, the map H_n also increases the D_n-distance by at most $2\cdot\delta_n$.

Consider converging sequences $v_n\to v$ and $w_n\to w$ such that $v_n,w_n\in D_n$ and therefore $v,w\in D$. Note that

❶ $$|H_n(v_n)-H_n(w_n)|\leqslant|v_n-w_n|_{D_n}+2\cdot\delta_n,$$

for each n. Since $\delta_n\to0$ and geodesics in $\mathcal{U}$ vary continuously with their endpoints (7.7), we have $H_n(v_n)\to H(v)$ and $H_n(w_n)\to H(w)$. Therefore the left-hand side in ❶ converges to $|H(v)-H(w)|$ and the right-hand side converges to $|v-w|_D$, it follows that H is short. $\square$

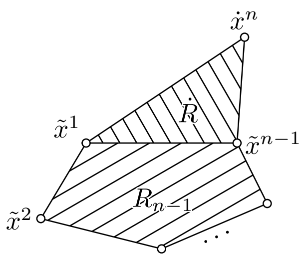
Proof of 6.4 for triangles. Suppose α is a triangle, say $[pxy]$.

Let $\tilde{\gamma}$ be the development of $[xy]$ with respect to p, where $\tilde{\gamma}$ has basepoint $\tilde{p}$ and subgraph D. By 6.9, $\tilde{\gamma}$ is concave. By 6.7, there is a short map $G\colon \operatorname{Conv}\tilde{\triangle}(pxy) \to D$. Further, by 6.10, D majorizes $[pxy]$ under a majorizing map $H\colon D \to \mathcal{U}$. Clearly $H \circ G$ is a majorizing map for $[pxy]$. $\square$

C Polygons

In the following proofs, $x^1 \dots x^n$ $(n \geqslant 3)$ denotes a polygonal line $x^1, \dots, x^n$, and $[x^1 \dots x^n]$ denotes the corresponding (closed) polygon. For a subset R of the ambient metric space, we denote by $[x^1 \dots x^n]_R$ a polygon in the length metric of R.

Proof of 6.4 for polygons. We begin by proving the theorem in case α is polygonal.



Now we claim that any closed n-gon $[x^1x^2 \dots x^n]$ in a CAT(0) space is majorized by a convex polygonal region

$$R_n = \operatorname{Conv}[\tilde{x}^1\tilde{x}^2 \dots \tilde{x}^n]$$

under a map F_n such that $F_n\colon \tilde{x}^i \mapsto x^i$ for each i.

The base case $n = 3$ is proved above. Assume the statement is true for $(n-1)$-gons, $n \geqslant 4$. Then $[x^1x^2 \dots x^{n-1}]$ is majorized by a convex polygonal region

$$R_{n-1} = \operatorname{Conv}[\tilde{x}^1\tilde{x}^2, \dots \tilde{x}^{n-1}],$$

in $\mathbb{E}^2$ under a map F_{n-1} satisfying $F_{n-1}(\tilde{x}^i) = x^i$ for all i. Take $\dot{x}^n \in \mathbb{E}^2$ such that $[\tilde{x}^1\tilde{x}^{n-1}\dot{x}^n] = \tilde{\triangle}(x^1x^{n-1}x^n)$ and this triangle lies on the other side of $[\tilde{x}^1\tilde{x}^{n-1}]$ from R_{n-1}. Let $\dot{R} = \operatorname{Conv}[\tilde{x}^1\tilde{x}^{n-1}\dot{x}^n]$, and $\dot{F}\colon \dot{R} \to \mathcal{U}$ be a majorizing map for $[x^1x^{n-1}x^n]$ as provided above.

Set $R = R_{n-1} \cup \dot{R}$, where R carries its length metric. Since F_n and F agree on $[\tilde{x}^1\tilde{x}^{n-1}]$, we may define $F\colon R \to \mathcal{U}$ by

$$F(x) = \begin{cases} F_{n-1}(x), & x \in R_{n-1}, \\ \dot{F}(x), & x \in \dot{R}. \end{cases}$$

Then F is length-nonincreasing and is a majorizing map for $[x^1x^2 \dots x^n]$ (as in Definition 6.1).

If R is a convex subset of $\mathbb{E}^2$, we are done.

If R is not convex, the total internal angle of R at $\tilde{x}^1$ or $\tilde{x}^{n-1}$ or both is $> \pi$. By relabeling we may suppose this holds for $\tilde{x}^{n-1}$.

The region R is obtained by gluing R_{n-1} to $\dot{R}$ by $[x^1x^{n-1}]$. Thus, by Reshetnyak gluing (5.14), R carrying its length metric is a CAT(0)-space. Moreover $[\tilde{x}^{n-2}\tilde{x}^{n-1}] \cup [\tilde{x}^{n-1}\dot{x}^n]$ is a geodesic of R. Thus $[\tilde{x}^1\tilde{x}^2\dots\tilde{x}^{n-2}\dot{x}^n]_R$ is a closed $(n-1)$-gon in R, to which the induction hypothesis applies. The resulting short map from a convex region in $\mathbb{E}^2$ to R, followed by F, is the desired majorizing map. $\square$

If $p_1\dots p_n$ is a polygon, then values $\theta_i = \pi - \measuredangle[p_i\,{}^{p_{i-1}}_{p_{i+1}}]$ for all i $(\mathrm{mod}\ n)$ are called external angles of the polygon. The following exercise is a generalization of Fenchel's theorem.

6.11. Exercise. *Show that the sum of external angles of any polygon in a complete length* CAT(0) *space cannot be smaller than* $2{\cdot}\pi$.

The following exercise is a version of the Fáry–Milnor theorem for CAT(0) spaces.

6.12. Very advanced exercise. *Suppose that a simple polygon β in a complete length* CAT(0) *space does not bound an embedded disc. Show that the sum of external angles of β cannot be smaller than* $4{\cdot}\pi$.

Give an example of such a polygon β with the sum of external angles exactly $4{\cdot}\pi$.

6.13. Exercise. *Prove the following generalization of the arm lemma (6.14).*

6.14. Arm lemma. *Let $P = [x^0x^1\dots x^{n+1}]$ be a polygon in a geodesic* CAT(0) *space $\mathcal{U}$. Suppose $\tilde{P} = [\tilde{x}^0\tilde{x}^1\dots\tilde{x}^{n+1}]$ is a convex polygon in $\mathbb{E}^2$ such that*

$$❶\qquad |\tilde{x}^i - \tilde{x}^{i-1}|_{\mathbb{E}^2} = |x^i - x^{i-1}|_{\mathcal{U}} \quad \text{and} \quad \measuredangle[x^i\,{}^{x^{i-1}}_{x^{i+1}}] \geqslant \measuredangle[\tilde{x}^i\,{}^{\tilde{x}^{i-1}}_{\tilde{x}^{i+1}}]$$

for all i. Then $|\tilde{x}^0 - \tilde{x}^{n+1}|_{\mathbb{E}^2} \leqslant |x^0 - x^{n+1}|_{\mathcal{U}}$.

D General case

If the space is proper, then the general case follows applying polygonal case to inscribed polygonal lines and passing to the limit. The statement holds for any geodesic CAT(0) space but one need to be more careful [5].

The following exercise is the rigidity case of the majorization theorem.

6.15. Exercise. *Let $\mathcal{U}$ be a geodesic* CAT(0) *space and $\alpha\colon [0,\ell] \to \mathcal{U}$ be a closed curve with arclength parametrization. Assume there is a closed convex curve $\tilde{\alpha}\colon [0,\ell] \to \mathbb{E}^2$ such that*

$$|\alpha(t_0) - \alpha(t_1)|_{\mathcal{U}} = |\tilde{\alpha}(t_0) - \tilde{\alpha}(t_1)|_{\mathbb{E}^2}$$

for any t_0 *and* t_1. *Then there is a distance-preserving map* $F\colon \operatorname{Conv}\tilde\alpha \to \mathcal{U}$ *such that* $F\colon \tilde\alpha(t)\mapsto \alpha(t)$ *for any* t.

6.16. Exercise. *Two majorizations* $F\colon D\to\mathcal{U}$ *and* $F'\colon D'\to\mathcal{U}$ *will be called equivalent if* $F' = F\circ\iota$ *for an isometry* $\iota\colon D\to D'$.

Show that a closed rectifiable curve in a CAT(0) *space has an isometric majorization map if and only if the majorization map is unique up to equivalence.*

E Comments

The statements in this section can be generalized to CAT(±1) spaces; in the CAT(1) case one has to assume that the closed curve has length at most $2{\cdot}\pi$.

The majorization theorem was proved by Yuriy Reshetnyak [38]; our proof uses a trick that we learned from the lectures of Werner Ballmann [13]. Another proof can be built on generalized Kirszbraun's theorem, but it works only for complete spaces.

The definition of develpoment appears in [11] and an earlier form of it can be found in [32].

6.17. Open problem. *Let* α *be a closed rectifiable curve in a* CAT(0) *space* $\mathcal{U}$. *Note that if* α *is a geodesic triangle or it bounds an isometric copy of convex plane figure in* $\mathcal{U}$, *then* α *has a unique (up to congruence) majorizing convex figure.*

What about the converse?

Lecture 7

Globalization for CATs

This lecture is nearly a copy of [4, Sections 3.1–3.3]; here we introduce locally CAT(0) spaces and prove the globalization theorem that provides a sufficient condition for locally CAT(0) spaces to be globally CAT(0).

A Locally CAT spaces

We say that a space $\mathcal{U}$ is *locally* CAT(0) (or *locally* CAT(1)) if a small closed ball centered at any point p in $\mathcal{U}$ is CAT(0) (or CAT(1), respectively).

For example, the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is locally isometric to $\mathbb{R}$, and so $\mathbb{S}^1$ is locally CAT(0). On the other hand, $\mathbb{S}^1$ is not CAT(0), since closed local geodesics in $\mathbb{S}^1$ are not geodesics, so $\mathbb{S}^1$ does not meet 5.7.

If $\mathcal{U}$ is a proper geodesic space, then it is locally CAT(0) (or locally CAT(1)) if and only if each point $p \in \mathcal{U}$ admits an open neighborhood Ω that is geodesic and such that any triangle in Ω is thin (or spherically thin, respectively).

B Space of local geodesic paths

A constant-speed parameterization of a local geodesic by the unit interval $[0,1]$ is called a *local geodesic path*.

In this section, we will study the behavior of local geodesics in locally CAT(κ) spaces. The results will be used in the proof of the globalization theorem (7.6).

Recall that a *path* is a curve parametrized by $[0,1]$. The space of paths in a metric space $\mathcal{U}$ comes with the natural metric

❶ $$|\alpha - \beta| = \sup\{\,|\alpha(t) - \beta(t)|_{\mathcal{U}} \,:\, t \in [0,1]\,\}.$$

7.1. Proposition. *Let $\mathcal{U}$ be a proper geodesic, locally* $\mathrm{CAT}(\kappa)$ *space.*

Assume $\gamma_n\colon [0,1] \to \mathcal{U}$ is a sequence of local geodesic paths converging to a path $\gamma_\infty\colon [0,1] \to \mathcal{U}$. Then γ_∞ is a local geodesic path. Moreover

$$\operatorname{length}\gamma_n \to \operatorname{length}\gamma_\infty$$

as $n \to \infty$.

Proof; $\mathrm{CAT}(0)$ *case.* Fix $t \in [0,1]$. Let $R > 0$ be sufficiently small, so that $\overline{\mathrm{B}}[\gamma_\infty(t), R]$ forms a proper geodesic $\mathrm{CAT}(0)$ space.

Assume that a local geodesic σ is shorter than $R/2$ and intersects the ball $\mathrm{B}(\gamma_\infty(t), R/2)$. Then σ cannot leave the ball $\overline{\mathrm{B}}[\gamma_\infty(t), R]$. By 5.7, σ is a geodesic. In particular, for all sufficiently large n, any arc of γ_n of length $R/2$ or less containing $\gamma_n(t)$ is a geodesic.

Since $\mathcal{B} = \overline{\mathrm{B}}[\gamma_\infty(t), R]$ is a proper geodesic $\mathrm{CAT}(0)$ space, by 5.1, geodesic segments in $\mathcal{B}$ depend uniquely on their endpoint pairs. Thus there is a subinterval $\mathbb{I}$ of $[0,1]$, that contains a neighborhood of t in $[0,1]$ and such that the arc $\gamma_n|_{\mathbb{I}}$ is minimizing for all large n. It follows that $\gamma_\infty|_{\mathbb{I}}$ is a geodesic, and therefore γ_∞ is a local geodesic.

The $\mathrm{CAT}(1)$ case is done in the same way, but one has to assume in addition that $R < \pi$. $\square$

The following lemma allows a local geodesic path to be moved continuously so that its endpoints follow given trajectories.

7.2. Patchwork along a geodesic. *Let $\mathcal{U}$ be a proper geodesic, locally* $\mathrm{CAT}(0)$ *space, and $\gamma\colon [0,1] \to \mathcal{U}$ be a locally geodesic path.*

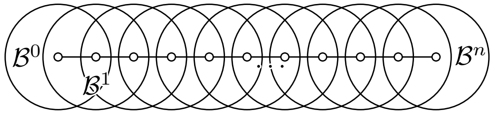
Then there is a proper geodesic $\mathrm{CAT}(0)$ *space $\mathcal{N}$, an open set $\hat{\Omega} \subset \mathcal{N}$, and a geodesic path $\hat{\gamma}\colon [0,1] \to \hat{\Omega}$, such that there is an open locally distance-preserving map $\Phi\colon \hat{\Omega} \looparrowright \mathcal{U}$ satisfying $\Phi \circ \hat{\gamma} = \gamma$.*

If $\operatorname{length}\gamma < \pi$, *then the same holds in the* $\mathrm{CAT}(1)$ *case. Namely, we assume that $\mathcal{U}$ is a proper geodesic, locally* $\mathrm{CAT}(1)$ *space and construct a proper geodesic* $\mathrm{CAT}(1)$ *space $\mathcal{N}$ with the same property as above.*

Proof. Fix $r > 0$ so that for each $t \in [0,1]$, the closed ball $\overline{\mathrm{B}}[\gamma(t), r]$ forms a proper geodesic $\mathrm{CAT}(0)$ space.

Choose a partition $0 = t_0 < t_1 < \dots < t_n = 1$ such that

$$\mathrm{B}(\gamma(t_i), r) \supset \gamma([t_{i-1}, t_i])$$



for all $n > i > 0$. Set $\mathcal{B}_i = \overline{\mathrm{B}}[\gamma(t_i), r]$. We can assume in addition that $\mathcal{B}_{i-1} \cap \mathcal{B}_{i+1} \subset \mathcal{B}_i$ if $0 < i < n$.

Consider the disjoint union $\bigsqcup_i \mathcal{B}_i = \{ (i, x) : x \in \mathcal{B}_i \}$ with the minimal equivalence relation $\sim$ such that $(i, x) \sim (i-1, x)$ for all i. Let $\mathcal{N}$ be the space obtained by gluing the $\mathcal{B}_i$ along $\sim$.

Note that $A_i = \mathcal{B}_i \cap \mathcal{B}_{i-1}$ is convex in $\mathcal{B}_i$ and in $\mathcal{B}_{i-1}$. Applying the Reshetnyak gluing theorem (5.14) n times, we conclude that $\mathcal{N}$ is a proper geodesic CAT(0) space.

For $t \in [t_{i-1}, t_i]$, define $\hat{\gamma}(t)$ as the equivalence class of $(i, \gamma(t))$ in $\mathcal{N}$. Let $\hat{\Omega}$ be the ε-neighborhood of $\hat{\gamma}$ in $\mathcal{N}$, where $\varepsilon > 0$ is chosen so that $\mathrm{B}(\gamma(t), \varepsilon) \subset \mathcal{B}_i$ for all $t \in [t_{i-1}, t_i]$.

Define $\Phi \colon \hat{\Omega} \to \mathcal{U}$ by sending the equivalence class of (i, x) to x. It is straightforward to check that Φ, $\hat{\gamma}$, and $\hat{\Omega} \subset \mathcal{N}$ satisfy the conclusion of the lemma.

The CAT(1) case is proved in the same way. $\square$

Recall that local geodesics are geodesics in any CAT(0) space; see 5.7. Using it with 7.2 and the uniqueness of geodesics (5.7), we get the following.

7.3. Corollary. *If $\mathcal{U}$ is a proper geodesic, locally* CAT(0) *space, then for any pair of points $p, q \in \mathcal{U}$, the space of all local geodesic paths from p to q is discrete; that is, for any local geodesic path γ connecting p to q, there is $\varepsilon > 0$ such that for any other local geodesic path δ from p to q we have $|\gamma(t) - \delta(t)|_{\mathcal{U}} > \varepsilon$ for some $t \in [0, 1]$.*

Analogously, if $\mathcal{U}$ is a proper geodesic, locally CAT(1) *space, then for any pair of points $p, q \in \mathcal{U}$, the space of all local geodesic paths shorter than π from p to q is discrete.*

7.4. Corollary. *If $\mathcal{U}$ is a proper geodesic, locally* CAT(0) *space, then for any path α there is a choice of local geodesic path γ_α connecting the ends of α such that the map $\alpha \mapsto \gamma_\alpha$ is continuous, and if α is a local geodesic path then $\gamma_\alpha = \alpha$.*

Analogously, if $\mathcal{U}$ is a proper geodesic, locally CAT(1) *space, then for any path α shorter than π, there is a choice of local geodesic path γ_α shorter than π connecting the ends of α such that the map $\alpha \mapsto \gamma_\alpha$ is continuous, and if α is a local geodesic path then $\gamma_\alpha = \alpha$.*

Proof of 7.4. We do the CAT(0) case; the CAT(1) case is analogous.

Consider the maximal interval $\mathbb{I} \subset [0,1]$ containing 0 such that there is a continuous one-parameter family of local geodesic paths γ_t for $t \in \mathbb{I}$ connecting $\alpha(0)$ to $\alpha(t)$, with $\gamma_t(0) = \gamma_0(t) = \alpha(0)$ for any t.

By 7.1, $\mathbb{I}$ is closed, so we may assume $\mathbb{I} = [0,s]$ for some $s \in [0,1]$.

Applying patchwork (7.2) to γ_s, we find that $\mathbb{I}$ is also open in $[0,1]$. Hence $\mathbb{I} = [0,1]$. Set $\gamma_\alpha = \gamma_1$.

By construction, if α is a local geodesic path, then $\gamma_\alpha = \alpha$.

Moreover, from 7.3, the construction $\alpha \mapsto \gamma_\alpha$ produces close results for sufficiently close paths in the metric defined by ❶; that is, the map $\alpha \mapsto \gamma_\alpha$ is continuous. $\square$

Given a path $\alpha\colon [0,1] \to \mathcal{U}$, we denote by $\bar\alpha$ the same path traveled in the opposite direction; that is,

$$\bar\alpha(t) = \alpha(1-t).$$

The product of two paths will be denoted with "$*$"; if two paths α and β connect the same pair of points, then the product $\bar\alpha * \beta$ is a closed curve.

7.5. Exercise. *Assume $\mathcal{U}$ is a proper geodesic, locally* CAT(1) *space. Consider the construction $\alpha \mapsto \gamma_\alpha$ provided by Corollary 7.4.*

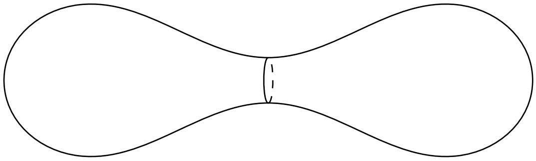
Assume that α and β are two paths connecting the same pair of points in $\mathcal{U}$, where each is shorter than π and the product $\bar\alpha * \beta$ is null-homotopic in the class of closed curves shorter than $2{\cdot}\pi$. Show that $\gamma_\alpha = \gamma_\beta$.

C Globalization

7.6. Globalization theorem. *If a proper geodesic, locally* CAT(0) *space is simply connected, then it is* CAT(0).

Analogously, if $\mathcal{U}$ is a proper geodesic, locally CAT(1) *space such that any closed curve $\gamma\colon \mathbb{S}^1 \to \mathcal{U}$ shorter than $2{\cdot}\pi$ is null-homotopic in the class of closed curves shorter than $2{\cdot}\pi$. Then $\mathcal{U}$ is* CAT(1).

The surface on the diagram is an example of a simply connected space that is locally CAT(1) but not CAT(1). To contract the marked



curve one has to increase its length to $2{\cdot}\pi$ or more; in particular, the surface does not satisfy the assumption of the globalization theorem.

The proof of the globalization theorem relies on the following theorem, which is essentially [9, Satz 9].

7.7. Patchwork globalization theorem. *A proper geodesic, locally* $\mathrm{CAT}(0)$ *space* $\mathcal{U}$ *is* $\mathrm{CAT}(0)$ *if and only if all pairs of points in* $\mathcal{U}$ *are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

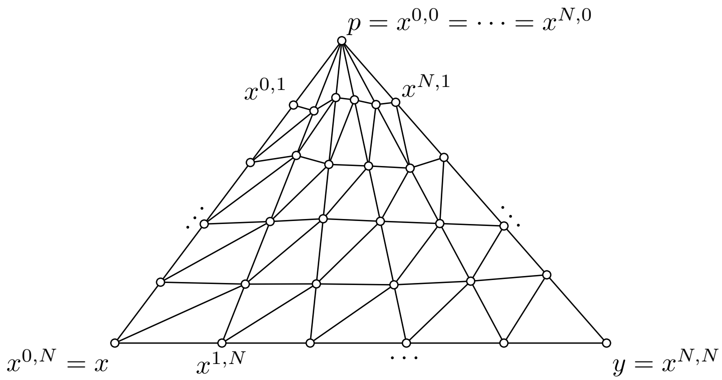
Analogously, a proper geodesic, locally $\mathrm{CAT}(1)$ *space* $\mathcal{U}$ *is* $\mathrm{CAT}(1)$ *if and only if all pairs of points in* $\mathcal{U}$ *at distance less than* π *are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

The proof uses a thin-triangle decomposition with the inheritance lemma (5.13) and the line-of-sight map (6.5).

Proof of the patchwork globalization theorem (7.7). Note that the implication "only if" follows from 5.1 and 5.9; it remains to prove the "if" part.

Fix a triangle $[pxy]$ in $\mathcal{U}$. We need to show that $[pxy]$ is thin.

By the assumptions, the line-of-sight map $(t,s)\mapsto\gamma_t(s)$ from p to $[xy]$ is uniquely defined and continuous.



Fix a partition

$$0 = t^0 < t^1 < \ldots < t^N = 1,$$

and set $x^{i,j} = \gamma_{t^i}(t^j)$. Since the line-of-sight map is continuous and $\mathcal{U}$ is locally $\mathrm{CAT}(0)$, we may assume that the triangles

$$[x^{i,j}x^{i,j+1}x^{i+1,j+1}] \quad \text{and} \quad [x^{i,j}x^{i+1,j}x^{i+1,j+1}]$$

are thin for each pair i, j.

Now we show that the thin property propagates to $[pxy]$ by repeated application of the inheritance lemma (5.13):

- ◇ For fixed i, sequentially applying the lemma shows that the triangles $[px^{i,1}x^{i+1,2}]$, $[px^{i,2}x^{i+1,2}]$, $[px^{i,2}x^{i+1,3}]$, and so on are thin.

In particular, for each i, the long triangle $[px^{i,N}x^{i+1,N}]$ is thin.

- ◇ By the same lemma the triangles $[px^{0,N}x^{2,N}]$, $[px^{0,N}x^{3,N}]$, and so on, are thin.

In particular, $[pxy] = [px^{0,N}x^{N,N}]$ is thin. □

Proof of the globalization theorem; CAT(0) *case.* Let $\mathcal{U}$ be a proper geodesic, locally CAT(0) space that is simply connected. Given a path α in $\mathcal{U}$, denote by γ_α the local geodesic path provided by 7.4. Since the map $\alpha \mapsto \gamma_\alpha$ is continuous, by 7.3 we have $\gamma_\alpha = \gamma_\beta$ for any pair of paths α and β homotopic relative to the ends.

Since $\mathcal{U}$ is simply connected, any pair of paths with common ends are homotopic. In particular, if α and β are local geodesics from p to q, then $\alpha = \gamma_\alpha = \gamma_\beta = \beta$ by Corollary 7.4. It follows that any two points $p, q \in \mathcal{U}$ are joined by a unique local geodesic that depends continuously on (p, q).

Since $\mathcal{U}$ is geodesic, it remains to apply the patchwork globalization theorem (7.7).

CAT(1) *case.* The proof goes along the same lines, but one needs to use Exercise 7.5. □

7.8. Corollary. *Any compact geodesic, locally* CAT(0) *space that contains no closed local geodesics is* CAT(0).

Analogously, any compact geodesic, locally CAT(1) *space that contains no closed local geodesics shorter than* $2{\cdot}\pi$ *is* CAT(1).

Proof. By the globalization theorem (7.6), we need to show that the space is simply connected. Assume the contrary. Fix a nontrivial homotopy class of closed curves.

Denote by ℓ the exact lower bound for the lengths of curves in the class. Note that $\ell > 0$; otherwise, there would be a closed noncontractible curve in a CAT(0) neighborhood of some point, contradicting 5.3.

Since the space is compact, the class contains a length-minimizing curve, which must be a closed local geodesic.

The CAT(1) case is analogous, one only has to consider a homotopy class of closed curves shorter than $2{\cdot}\pi$. □

7.9. Exercise. *Prove that any compact geodesic, locally* CAT(0) *space* $\mathcal{X}$ *that is not* CAT(0) *contains a* geodesic circle*; that is, a simple* *closed curve γ such that for any two points $p, q \in \gamma$, one of the arcs of γ with endpoints p and q is a geodesic.*

Formulate and prove the analogous statement for CAT(1) *spaces.*

7.10. Advanced exercise. *Let $\mathcal{U}$ be a proper geodesic* CAT(0) *space. Assume $\tilde{\mathcal{U}} \to \mathcal{U}$ is a metric double cover branching along a geodesic. (For example, 3-dimensional Euclidean space admits a double cover branching along a line.)*

Show that $\tilde{\mathcal{U}}$ is CAT(0).

D Remarks

The lemma about patchwork along a geodesic and its proof were suggested to us by Alexander Lytchak. This statement was originally proved by Stephanie Alexander and Richard Bishop [3] using a different method.

As was mentioned earlier, the motivation for the notion of CAT(κ) spaces comes from the fact that a Riemannian manifold is locally CAT(κ) if and only if it has sectional curvature at most κ. This easily follows from Rauch comparison for Jacobi fields and Proposition 5.6.

In the globalization theorem (7.6), properness can be weakened to completeness [see 5, and the references therein]. The original formulation of the globalization theorem, or Hadamard–Cartan theorem, states that *if M is a complete Riemannian manifold with sectional curvature at most* 0*, then the exponential map at any point $p \in M$ is a covering*; in particular, it implies that *the universal cover of M is diffeomorphic to the Euclidean space of the same dimension.*

In this generality, this theorem appeared in the lectures of Elie Cartan [24]. This theorem was proved for surfaces in Euclidean 3-space by Hans von Mangoldt [33] and a few years later independently for two-dimensional Riemannian manifolds by Jacques Hadamard [29].

Formulations for metric spaces of different generality were proved by Herbert Busemann [23], Willi Rinow [39], Mikhael Gromov [28, p. 119]. A detailed proof of Gromov's statement was given by Werner Ballmann [12] when $\mathcal{U}$ is proper, and by Stephanie Alexander and Richard Bishop [3] in more generality.

For proper CAT(1) spaces, the globalization theorem was proved by Brian Bowditch [15].

The globalization theorem holds for complete length spaces (not necessarily proper spaces) [5].

The patchwork globalization (7.7) is proved by Alexandrov [9, Satz 9]. For proper spaces one can remove the continuous dependence from the formulation; it follows from uniqueness. For complete spaces, the latter is not true [16, Chapter I, Exercise 3.14].

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