Homework assignments

Due Feb. 23: 1.4, 1.7, 1.8, 1.10, 1.13.

Contents

Lecture 1

Alexandrov embedding theorem

This lecture contains selected material from Alexandrov's book [1].

We give a proof of the Cauchy theorem and then modify it to prove the Alexandrov uniqueness theorem. Further, we sketch a proof of the Alexandrov embedding theorem.

A Cauchy theorem

Further, surfaces of convex polyhedrons will be considered with intrinsic metric; it is defined as the length of a shortest path on the surface between points. Shortest paths parametrized by arclength will be called geodesics; this term has a slightly different meaning in Riemannian geometry.

1.1. Theorem. *Let K and K' be two non-degenerate convex polyhedrons in $\mathbb{R}^3$; denote their surfaces by P and P'. Suppose there is an isometry $P \to P'$ that sends each face of K to a face of K'. Then K is congruent to K'.*

Proof. Consider the graph Γ formed by the edges of K; the edges of K' form the same graph.

For an edge e in Γ, denote by α_e and α'_e the corresponding dihedral angles in K and K' respectively. Mark e by plus if $\alpha_e < \alpha'_e$ and by minus if $\alpha_e > \alpha'_e$.

Now remove from Γ everything which was not marked; that is, leave only the edges marked by $(+)$ or $(-)$ and their endpoints.

Note that the theorem follows if Γ is an empty graph; assume the contrary.

The graph Γ is embedded into P, which is homeomorphic to the sphere. In particular, the edges coming from one vertex have a natural cyclic order. Given a vertex v of Γ, count the number of sign changes around v; that is, the number of consequent pairs edges with different signs.

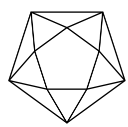
1.2. Local lemma. *For any vertex of Γ the number of sign changes is at least 4.*

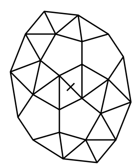
In other words, at each vertex of Γ, one can choose 4 edges marked by $(+)$, $(-)$, $(+)$, $(-)$ in the same cyclical order. Note that the local lemma contradicts the following.

1.3. Global lemma. *Let Γ be a nonempty subgraph of the graph formed by the edges of a convex polyhedron. Then it is impossible to mark all of the edges of Γ by $(+)$ or $(-)$ such that the number of sign changes around each vertex of Γ is at least 4.*

It remains to prove these two lemmas. □

1.4. Exercise. *Consider two polyhedral discs in $\mathbb{R}^3$ glued by the rule on the diagrams from regular polygons. Assume that each disc is a part of a surface of a convex polyhedron. Use the local lemma to show the following.*





(a) The first configuration is rigid; that is, one can not fix the position of the pentagon and continuously move the remaining 5 vertices in a new position so that each triangle moves by a one-parameter family of isometries of $\mathbb{R}^3$.

(b) Show that the second configuration has a rotational symmetry with the axis passing through the midpoint of the marked edge.

B Local lemma

Next lemma is the main ingredient in our proof of the local lemma.

1.5. Arm lemma. *Assume that $A = [a_0 a_1 \dots a_n]$ is a convex polygon in $\mathbb{R}^2$ and $A' = [a_0' a_1' \dots a_n']$ be a polygonal line in $\mathbb{R}^3$ such that*

$$|a_i - a_{i+1}| = |a_i' - a_{i+1}'|$$

for any $i \in \{0, \dots, n-1\}$ and

$$\measuredangle a_i \leqslant \measuredangle a_i'$$

for each $i \in \{1, \dots, n-1\}$*. Then*

$$|a_0 - a_n| \leqslant |a'_0 - a'_n|$$

and equality holds if and only if A *is congruent to* A'*.*

One may view the polygonal lines $[a_0 a_1 \dots a_n]$ and $[a'_0 a'_1 \dots a'_n]$ as a robot's arm in two positions. The arm lemma states that when the arm opens, the distance between the shoulder and tips of the fingers increases.

1.6. Exercise. *Show that the arm lemma does not hold if instead of the convexity, one only the local convexity; that is, if you go along the polygonal line* $a_0 a_1 \dots a_n$*, then you only turn left.*

1.7. Exercise. *Suppose* $A = [a_1 \dots a_n]$ *and* $A' = [a'_1 \dots a'_n]$ *be non-congruent convex plane polygons with equal corresponding sides. Mark each vertex* a_i *with plus (minus) if the interior angle of* A *at* a_i *is smaller (respectively bigger) than the interior angle of* A' *at* a'_i*. Show that there are at least 4 sign changes around* A*.*

Give an example showing the statement does not hold without assuming convexity.

In the proof, we will use the following exercise which is the triangle inequality angles (or the spherical triangle inequality).

1.8. Exercise. *Let* w_1, w_2, w_3 *be unit vectors in* $\mathbb{R}^3$*. Denote by* $\theta_{i,j}$ *the angle between the vectors* v_i *and* v_j*. Show that*

$$\theta_{1,3} \leqslant \theta_{1,2} + \theta_{2,3}$$

and in case of equality, the vectors w_1, w_2, w_3 *lie in a plane.*

Proof. We will view $\mathbb{R}^2$ as the xy-plane in $\mathbb{R}^3$; so both A and A' lie in $\mathbb{R}^3$. Let a_m be the vertex of A that lies on the maximal distance to the line $(a_0 a_n)$.

Let us shift indexes of a_i and a'_i down by m, so that

$$\begin{array}{lllll} a_{-m} := a_0, & \dots & a_0 := a_m, & \dots & a_k := a_n, \\ a'_{-m} := a'_0, & \dots & a'_0 := a'_m, & \dots & a'_k := a'_n, \end{array}$$

where $k = n - m$. (Here the symbol ":=" means an assignment as in programming.)

Without loss of generality, we may assume that

- ⋄ $a_0 = a'_0$ and they both coincide with the origin $(0,0,0) \in \mathbb{R}^3$;
- ⋄ all a_i lie in the xy-plane and the x-axis is parallel to the line $(a_{-m} a_k)$;

◇ the angle $\measuredangle a_0'$ lies in xy-plane and contains the angle $\measuredangle a_0$ inside and the directions to a_{-1}',a_{-1}, a_1 and a_1' from a_0 appear in the same cyclic order.

Denote by x_i and x_i' the projections of a_i and a_i' to the x-axis. We can assume in addition that $x_k \geqslant x_{-m}$. In this case

$$|a_k - a_{-m}| = x_k - x_{-m}.$$

Since the projection is a distance non-expanding, we also have

$$|a_k' - a_{-m}'| \geqslant x_k' - x_{-m}'.$$

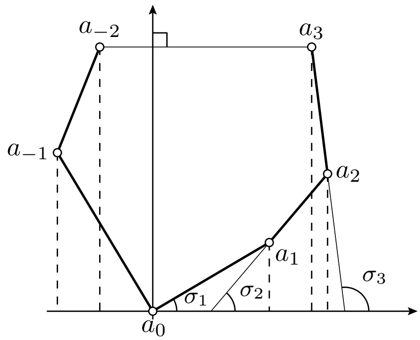
Therefore it is sufficient to show that

$$x_k' - x_{-m}' \geqslant x_k - x_{-m}.$$

The latter holds if

❶ $x_i' - x_{i-1}' \geqslant x_i - x_{i-1}.$

for each i. It remains to prove ❶.



Let us assume that $i > 0$; the case $i \leqslant 0$ is similar. Denote by σ_i (σ_i') the angle between the vector $w_i = a_i - a_{i-1}$ (respectively $w_i' = a_i' - a_{i-1}'$) and the x-axis. Note that

❷
$$\begin{aligned} x_i - x_{i-1} &= |a_i - a_{i-1}|\cdot \cos\sigma_i, \\ x_i' - x_{i-1}' &= |a_i - a_{i-1}|\cdot \cos\sigma_i' \end{aligned}$$

for each $i > 0$. By construction $\sigma_1 \geqslant \sigma_1'$. Note that $\measuredangle(w_{i-1}, w_i) = = \pi - \measuredangle a_i$. From convexity of $[a_1 a_1 \dots a_i]$, we have

$$\sigma_i = \sigma_1 + (\pi - \measuredangle a_1) + \dots + (\pi - \measuredangle a_i)$$

for any $i > 0$. Since $\measuredangle(w_{i-1}', w_i') = \pi - \measuredangle a_i'$, applying 1.8 several times, we get

$$\sigma_i' \leqslant \sigma_1' + (\pi - \measuredangle a_1') + \dots + (\pi - \measuredangle a_i').$$

Since $\measuredangle a_j' \geqslant \measuredangle a_j$ for each j, we get $\sigma_i' \leqslant \sigma_i$, and therefore

$$\cos\sigma_i' \geqslant \cos\sigma_i$$

Applying ❷, we get ❶.

In the case of equality, we have $\sigma_i = \sigma_i'$, which implies $\measuredangle a_i = \measuredangle a_i'$ for each i. This also implies that all a_i' lie in xy-plane. The latter easily follows from the equality case in 1.8. □

Proof of the local lemma (1.2). Assume that the local lemma does not hold at the vertex v of Γ. Cut from P a small pyramid Δ with the vertex v. One can choose two points a and b on the base of Δ so that on one side of the segments $[va]$ and $[vb]$ we have only pluses and on the other side only minuses.

The base of Δ has two polygonal lines with ends at a and b. Choose the one that has only pluses; denote it by $a_0a_1\dots a_n$; so $a=a_0$ and $b=a_n$. Denote by $a_0'a_1'\dots a_n'$ the corresponding line in P'; let $a'=a_0'$ and $b'=a_n'$.

Since each marked edge passing through a_i has a $(+)$ on it or nothing, we have

$$\measuredangle a_i \leqslant \measuredangle a_i'$$

for each i.

1.9. Exercise. *Prove the last statement.*

By the construction we have $|a_i-a_{i-1}|=|a_i'-a_{i-1}'|$ for all i. By the arm lemma (1.5), we get

❸ $|a-b|\leqslant|a'-b'|$.

Swap K and K' and repeat the same construction. We get

❹ $|a-b|\geqslant|a'-b'|$.

The claims ❸ and ❹ together imply $|a-b|=|a'-b'|$. The equality case in arm lemma implies that no edge at v is marked; that is, v is not a vertex of Γ — a contradiction. □

C Global lemma

The proof of the global lemma is based on counting the sign changes in two ways; first while moving around each vertex of Γ and second while moving around each of the regions separated by Γ on the surface P. If two edges are adjacent at a vertex, then they are also adjacent in a region. The converse is true as well. Therefore, both countings give the same number.

It is instructive to do the next exercise before diving into the proof.

1.10. Exercise. *Try to mark the edges of an octahedron by pluses and minuses such that there would be 4 sign changes at each vertex.*

Show that this is impossible.

Proof of 1.3. We can assume that Γ is connected; that is, one can get from any vertex to any other vertex by walking along edges. (If not, pass to a connected component of Γ.)

Denote by k and l the number of vertices and edges in Γ. Denote by m the number of regions that Γ cuts from P. Since Γ is connected, each region is homeomorphic to an open disc.

1.11. Exercise. *Prove the last statement.*

Now we can apply Euler's formula

❶ $$k - l + m = 2.$$

Denote by s the total number of sign changes in Γ for all vertices. By the local lemma (1.2), we have

❷ $$4{\cdot}k \leqslant s.$$

Let us get an upper bound on s by counting the number of sign changes when you go around each region. Denote by m_n the number of regions bounded by n edges; if an edge appears twice when it is counted twice. Note that each region is bounded by at least 3 edges; therefore

❸ $$m = m_3 + m_4 + m_5 + \dots$$

Counting edges and using the fact that each edge belongs to exactly two regions, we get

$$2{\cdot}l = 3{\cdot}m_3 + 4{\cdot}m_4 + 5{\cdot}m_5 + \dots$$

Combining this with Euler's formula (❶), we get

❹ $$4{\cdot}k = 8 + 2{\cdot}m_3 + 4{\cdot}m_4 + 6{\cdot}m_5 + 8{\cdot}m_6 + \dots$$

Observe that the number of sign changes in n-gon regions has to be even and $\leqslant n$. Therefore

❺ $$s \leqslant 2{\cdot}m_3 + 4{\cdot}m_4 + 4{\cdot}m_5 + 6{\cdot}m_6 + \dots$$

Clearly, ❷ and ❺ contradict ❹. □

D Uniqueness

Alexandrov's uniqueness theorem states that the conclusion of the Cauchy theorem (1.1) still holds without the face-to-face assumption.

1.12. Theorem. *Any two convex polyhedrons in $\mathbb{R}^3$ with isometric surfaces are congruent.*

Needed modifications in the proof of 1.1. Suppose $\iota\colon P \to P'$ be an isometry between surfaces of K and K'. Mark in P all the edges of K and all the inverse images of edges in K'; further, these will be called fake edges. The marked lines divide P into convex polygons, and the restriction of ι to each polygon is a rigid motion. These polygons play the role of faces in the proof above.

A vertex of the obtained graph can be a vertex of K or it can be a fake vertex; that is, it might be an intersection of an edge and a fake edge.

For the first type of vertex, the local lemma can be proved the same way. For a fake vertex v, it is easy to see that both parts of the edge coming through v are marked with minus while both of the fake edges at v are marked with plus. Therefore, the local lemma holds for the fake vertices as well.

What remains in the proof needs no modifications. □

E Existence

Let P be a surface with a polyhedral metric. The curvature of a point $p \in P$ is defined as $2\cdot\pi - \theta$, where θ is the total angle around p.

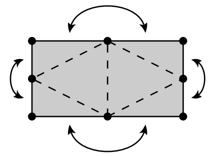
1.13. Exercise. *Suppose P is the surface of a convex polyhedron. Show that P is homeomorphic to the sphere, and it has nonnegative curvature at every point.*

Alexandrov's theorem states that the converse holds if one includes in the consideration twice covered polygons. In other words, we assume that a polyhedron can degenerate to a plane polygon; in this case, its surface is defined as two copies of the polygon glued along the boundaries.

1.14. Theorem. *A polyhedral metric on the sphere is isometric to the surface of a convex polyhedron (possibly degenerate) if and only if it has nonnegative curvature at each point.*

By 1.12, a convex polyhedron is completely defined by the intrinsic metric of its surface. By 1.14, it follows that knowing the metric we could find the position of the edges. However, in practice, it is not easy to do.

For example, the surface glued from a rectangle as shown on the diagram defines a tetrahedron. Some of the glued lines appear inside facets of the tetrahedron and some edges (dashed lines) do not follow the sides of the rectangle.



Space of polyhedrons. Let us denote by $\mathbf{K}$ the space of all convex polyhedrons in the Euclidean space, including polyhedrons that degenerate to a plane polygon. Polyhedra in $\mathbf{K}$ will be considered up to a motion of the space, and the whole space $\mathbf{K}$ will be considered with the natural topology (so far an intuitive meaning of closeness of two polyhedrons should be sufficient).

Further, denote by $\mathbf{K}_n$ the polyhedrons in $\mathbf{K}$ with exactly n vertices. Since any polyhedron has at least 3 vertices, the space $\mathbf{K}$ admits a subdivision into a countable number of subsets $\mathbf{K}_3, \mathbf{K}_4, \dots$

Space of polyhedral metrics. The space of polyhedral metrics on the sphere with nonnegative curvature will be denoted by $\mathbf{P}$. The metrics in $\mathbf{P}$ will be considered up to an isometry, and the whole space $\mathbf{P}$ will be equipped with the natural topology (again, an intuitive meaning of closeness of two metrics is sufficient).

A point on the sphere with positive curvature will be called an essential vertex. The subset of $\mathbf{P}$ of all metrics with exactly n essential vertices will be denoted by $\mathbf{P}_n$. It is easy to see that any metric in $\mathbf{P}$ has at least 3 essential vertices. Therefore $\mathbf{P}$ is subdivided into countably many subsets $\mathbf{P}_3, \mathbf{P}_4, \dots$

From a polyhedron to its surface. By 1.13, passing from a polyhedron to its surface defines a map

$$\iota\colon \mathbf{K} \to \mathbf{P}.$$

Note that the number of vertices of a polyhedron is equal to the number of essential vertices of its surface. In other words, $\iota(\mathbf{K}_n) \subset \mathbf{P}_n$ for any $n \geqslant 3$.

Using the introduced notation, we can unite 1.12 and 1.14 in the following statement.

1.15. Reformulation. *For any integer $n \geqslant 3$, the map ι induces a bijection between $\mathbf{K}_n$ and $\mathbf{P}_n$.*

The proof is based on a construction of a one-parameter family of polyhedrons that starts at an arbitrary polyhedron and ends at a polyhedron with its surface isometric to the given one. This type of argument is called the continuity method; it is often used in the theory of differential equations.

Sketch. By 1.12, the map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is injective. Let us prove that it is surjective.

1.16. Lemma. *For any integer $n \geqslant 3$, the space $\mathbf{P}_n$ is connected.*

The proof of this lemma is not complicated, but it requires ingenuity; it can be done by the direct construction of a one-parameter family of metrics in $\mathbf{P}_n$ that connects two given metrics. Such a family can be obtained by a sequential application of the following construction and its inverse.

Let $P \in \mathbf{P}_n$. Suppose v and w are essential vertices in P. Let us cut P along a geodesic from v to w. Note that the geodesic cannot pass thru an essential vertex of P. Further, note that there is a three-parameter family of patches that can be used to patch the cut so that the obtained metric remains in $\mathbf{P}_n$; in particular, the obtained metric has exactly n essential vertices (after the patching, the vertices v and w may become inessential).

1.17. Lemma. *The map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is open, that is, it maps any open set in $\mathbf{K}_n$ to an open set in $\mathbf{P}_n$.*

In particular, for any $n \geqslant 3$, the image $\iota(\mathbf{K}_n)$ is open in $\mathbf{P}_n$.

This statement is very close to the so-called invariance of domain theorem; the latter states that a continuous injective map between manifolds of the same dimension is open.

Recall that ι is injective. The proof of the invariance of domain theorem can be adapted to our case since both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n - 6)$-dimensional and both look like manifolds, altho, formally speaking, they are *not* manifolds. In a more technical language, $\mathbf{K}_n$ and $\mathbf{P}_n$ have the natural structure of $(3{\cdot}n-6)$-dimensional orbifolds, and the map ι respects the orbifold structure.

We will only show that both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n - 6)$-dimensional.

Choose $K \in \mathbf{K}_n$. Note that K is uniquely determined by the $3{\cdot}n$ coordinates of its n vertices. We can assume that the first vertex is the origin, the second has two vanishing coordinates and the third has one vanishing coordinate; therefore, all polyhedrons in $\mathbf{K}_n$ that lie sufficiently close to K can be described by $3{\cdot}n - 6$ parameters. If K has no symmetries, then this description can be made one-to-one; in this case, a neighborhood of K in $\mathbf{K}_n$ is a $(3{\cdot}n - 6)$-dimensional manifold. If K has a nontrivial symmetry group, then this description is not one-to-one but it does not have an impact on the dimension of $\mathbf{K}_n$.

The case of polyhedral metrics is analogous. We need to construct a subdivision of the sphere into plane triangles using only essential vertices. By Euler's formula, there are exactly $3{\cdot}n-6$ edges in this subdivision. Note that the lengths of edges completely describe the metric, and slight changes in these lengths produce a metric with the same property. Again, if P has no symmetries, then this description is one-to-one.

1.18. Lemma. *The map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is closed; that is, the image of a closed set in $\mathbf{K}_n$ is closed in $\mathbf{P}_n$.*

In particular, for any $n \geqslant 3$, the set $\iota(\mathbf{K}_n)$ is closed in $\mathbf{P}_n$.

Choose a closed set Z in $\mathbf{K}_n$. Denote by $\bar{Z}$ the closure of Z in $\mathbf{K}$; note that $Z = \mathbf{K}_n \cap \bar{Z}$. Assume $K_1, K_2, \dots \in Z$ is a sequence of polyhedrons that converges to a polyhedron $K_\infty \in \bar{Z}$. Note that $\iota(K_n)$ converges to $\iota(K_\infty)$ in $\mathbf{P}$. In particular, $\iota(\bar{Z})$ is closed in $\mathbf{P}$.

Since $\iota(\mathbf{K}_n) \subset \mathbf{P}_n$ for any $n \geqslant 3$, we have $\iota(Z) = \iota(\bar{Z}) \cap \mathbf{P}_n$; that is, $\iota(Z)$ is closed in $\mathbf{P}_n$.

Summarizing, $\iota(\mathbf{K}_n)$ is a nonempty closed and open set in $\mathbf{P}_n$, and $\mathbf{P}_n$ is connected for any $n \geqslant 3$. Therefore, $\iota(\mathbf{K}_n) = \mathbf{P}_n$; that is, $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is surjective. □

F Comments

In Euclid's Elements, solids were called equal if the same holds for their faces, but no proof was given. Adrien-Marie Legendre became interested in this problem towards the end of the 18th century. He discussed it with his colleague Joseph-Louis Lagrange, who suggested this problem to Augustin-Louis Cauchy in 1813; soon he proved it. In 1950, Alexandrov understood that the condition of the equality of faces can be weakened.

Arm lemma. Original Cauchy's proof [3] also used a version of the arm lemma, but its proof contained a small mistake (corrected in one century).

Our proof of arm lemma is due to Stanisław Zaremba. This and a couple of other proofs can be found in the letters between Isaac Schoenber and Stanisław Zaremba [4].

The following variation of the arm lemma makes sense for nonconvex spherical polygons. It is due to Viktor Zalgaller [6]. It can be used instead of the standard arm lemma.

1.19. Another arm lemma. *Let $A = [a_1 \dots a_n]$ and $A' = [a'_1 \dots$ $\dots a'_n]$ be two spherical n-gons (not necessarily convex). Assume that A lies in a half-sphere, the corresponding sides of A and A' are equal and each angle of A is at least the corresponding angle in A'. Then A is congruent to A'.*

Global lemma. A more visual proof of the global lemma is given in [1, II §1.3].

Existance theorem. This theorem was proved by A. D. Alexandrov [2]; our sketch has only minor differences. A complete proof is nicely written in [1]. A very different proof was found by Yu. A. Volkov in his thesis [5]; it uses a deformation of three-dimensional polyhedral space.

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