Homework assignments

Due 2023-03-02: 1.6, 1.7, 1.9, 1.10, 1.14.

Due 2023-03-09: 1.13, 2.3, 2.6, 2.8, 2.9.

Due 2023-03-16: 3.5, 3.7, 3.8, 3.10, 3.14.

Due 2023-03-23: 3.16, 3.17, 3.19, 4.7, 4.9.

Due 2023-03-30: 4.8, 5.2, 5.3, 5.4, 5.9.

Due 2023-04-06: 5.10, 5.11, 6.7, 6.8, 6.9.

Due 2023-04-13: 7.7, 7.12, 7.13, 7.14, 7.16(a+b).

Due 2023-04-20: 8.1, 8.4, 8.9, 8.25, 8.29.

Contents

Lecture 1

Alexandrov embedding theorem

This lecture contains selected material from Alexandrov's book [6].

We give a proof of the Cauchy theorem and then modify it to prove the Alexandrov uniqueness theorem. Further, we sketch a proof of the Alexandrov embedding theorem.

A Cauchy theorem

Further, *surfaces* of convex polyhedrons will be considered with intrinsic metric; it is defined as the length of a shortest path on the surface between points. Shortest paths parametrized by arclength will be called geodesics; this term has a slightly different meaning in Riemannian geometry.

1.1. Theorem. *Let K and K' be two non-degenerate convex polyhedrons in $\mathbb{E}^3$; denote their surfaces by P and P'. Suppose there is an isometry $P \to P'$ that sends each face of K to a face of K'. Then K is congruent to K'.*

Proof. Consider the graph Γ formed by the edges of K; the edges of K' form the same graph.

For an edge e in Γ, denote by α_e and α'_e the corresponding dihedral angles in K and K' respectively. Mark e by plus if $\alpha_e < \alpha'_e$ and by minus if $\alpha_e > \alpha'_e$.

Now remove from Γ everything which was not marked; that is, leave only the edges marked by $(+)$ or $(-)$ and their endpoints.

Note that the theorem follows if Γ is an empty graph; assume the contrary.

The graph Γ is embedded into P, which is homeomorphic to the sphere. In particular, the edges coming from one vertex have a natural cyclic order. Given a vertex v of Γ, count the *number of sign changes* around v; that is, the number of consequent pairs edges with different signs.

1.2. Local lemma. *For any vertex of* Γ *the number of sign changes is at least* 4.

In other words, at each vertex of Γ, one can choose 4 edges marked by $(+)$, $(-)$, $(+)$, $(-)$ in the same cyclical order. Note that the local lemma contradicts the following.

1.3. Global lemma. *Let* Γ *be a nonempty subgraph of the graph formed by the edges of a convex polyhedron. Then it is impossible to mark all of the edges of* Γ *by* $(+)$ *or* $(-)$ *such that the number of sign changes around each vertex of* Γ *is at least* 4.

It remains to prove these two lemmas. □

B Local lemma

Next lemma is the main ingredient in our proof of the local lemma.

1.4. Arm lemma. *Assume that* $A = [a_0a_1 \dots a_n]$ *is a convex polygon in* $\mathbb{E}^2$ *and* $A' = [a'_0a'_1 \dots a'_n]$ *be a polygonal line in* $\mathbb{E}^3$ *such that*

$$|a_i - a_{i+1}| = |a'_i - a'_{i+1}|$$

for any $i \in \{0, \dots, n-1\}$ *and*

$$\measuredangle a_i \leqslant \measuredangle a'_i$$

for each $i \in \{1, \dots, n-1\}$. *Then*

$$|a_0 - a_n| \leqslant |a'_0 - a'_n|$$

and equality holds if and only if A *is congruent to* A'.

One may view the polygonal lines $[a_0a_1 \dots a_n]$ and $[a'_0a'_1 \dots a'_n]$ as a robot's arm in two positions. The arm lemma states that when the arm opens, the distance between the shoulder and tips of the fingers increases.

1.5. Exercise. *Show that the arm lemma does not hold if instead of the convexity, one only the local convexity; that is, if you go along the polygonal line* $a_0a_1 \dots a_n$, *then you only turn left.*

1.6. Exercise. *Suppose $A = [a_1 \dots a_n]$ and $A' = [a'_1 \dots a'_n]$ be non-congruent convex plane polygons with equal corresponding sides. Mark each vertex a_i with plus (minus) if the interior angle of A at a_i is smaller (respectively bigger) than the interior angle of A' at a'_i. Show that there are at least 4 sign changes around A.*

Give an example showing the statement does not hold without assuming convexity.

In the proof, we will use the following exercise which is the triangle inequality angles (or the spherical triangle inequality).

1.7. Exercise. *Let w_1, w_2, w_3 be unit vectors in $\mathbb{E}^3$. Denote by $\theta_{i,j}$ the angle between the vectors v_i and v_j. Show that*

$$\theta_{1,3} \leqslant \theta_{1,2} + \theta_{2,3}$$

and in case of equality, the vectors w_1, w_2, w_3 lie in a plane.

Proof. We will view $\mathbb{E}^2$ as the xy-plane in $\mathbb{E}^3$; so both A and A' lie in $\mathbb{E}^3$. Let a_m be the vertex of A that lies on the maximal distance to the line $(a_0 a_n)$.

Let us shift indexes of a_i and a'_i down by m, so that

$$\begin{array}{ccccc} a_{-m} := a_0, & \dots & a_0 := a_m, & \dots & a_k := a_n, \\ a'_{-m} := a'_0, & \dots & a'_0 := a'_m, & \dots & a'_k := a'_n, \end{array}$$

where $k = n - m$. (Here the symbol ":=" means an assignment as in programming.)

Without loss of generality, we may assume that

- ◇ $a_0 = a'_0$ and they both coincide with the origin $(0, 0, 0) \in \mathbb{E}^3$;
- ◇ all a_i lie in the xy-plane and the x-axis is parallel to the line $(a_{-m} a_k)$;
- ◇ the angle $\measuredangle a'_0$ lies in xy-plane and contains the angle $\measuredangle a_0$ inside and the directions to a'_{-1},a_{-1}, a_1 and a'_1 from a_0 appear in the same cyclic order.

Denote by x_i and x'_i the projections of a_i and a'_i to the x-axis. We can assume in addition that $x_k \geqslant x_{-m}$. In this case,

$$|a_k - a_{-m}| = x_k - x_{-m}.$$

Since the projection is a distance non-expanding, we also have

$$|a'_k - a'_{-m}| \geqslant x'_k - x'_{-m}.$$

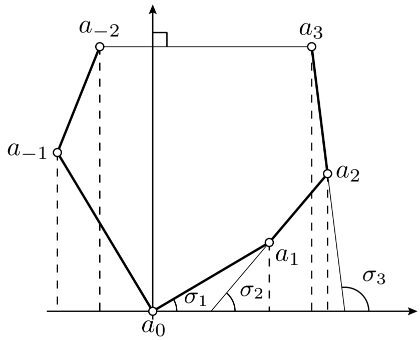
Therefore it is sufficient to show that

$$x'_k - x'_{-m} \geqslant x_k - x_{-m}.$$

The latter holds if

❶ $x'_i - x'_{i-1} \geqslant x_i - x_{i-1}.$

for each i. It remains to prove ❶.



Let us assume that $i > 0$; the case $i \leqslant 0$ is similar. Denote by σ_i (σ'_i) the angle between the vector $w_i = a_i - a_{i-1}$ (respectively $w'_i = a'_i - a'_{i-1}$) and the x-axis. Note that

$$❷ \qquad \begin{aligned} x_i - x_{i-1} &= |a_i - a_{i-1}|\cdot \cos\sigma_i, \\ x'_i - x'_{i-1} &= |a_i - a_{i-1}|\cdot \cos\sigma'_i \end{aligned}$$

for each $i > 0$. By construction $\sigma_1 \geqslant \sigma'_1$. Note that $\measuredangle(w_{i-1}, w_i) = \pi - \measuredangle a_i$. From convexity of $[a_1 a_1 \dots a_i]$, we have

$$\sigma_i = \sigma_1 + (\pi - \measuredangle a_1) + \dots + (\pi - \measuredangle a_i)$$

for any $i > 0$. Since $\measuredangle(w'_{i-1}, w'_i) = \pi - \measuredangle a'_i$, applying 1.7 several times, we get

$$\sigma'_i \leqslant \sigma'_1 + (\pi - \measuredangle a'_1) + \dots + (\pi - \measuredangle a'_i).$$

Since $\measuredangle a'_j \geqslant \measuredangle a_j$ for each j, we get $\sigma'_i \leqslant \sigma_i$, and therefore

$$\cos\sigma'_i \geqslant \cos\sigma_i$$

Applying ❷, we get ❶.

In the case of equality, we have $\sigma_i = \sigma'_i$, which implies $\measuredangle a_i = \measuredangle a'_i$ for each i. This also implies that all a'_i lie in xy-plane. The latter easily follows from the equality case in 1.7. □

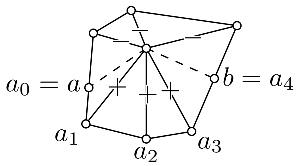
Proof of the local lemma (1.2). Assume that the local lemma does not hold at the vertex v of Γ. Cut from P a small pyramid Δ with the vertex v. One can choose two points a and b on the base of Δ so that on one side of the segments $[va]$ and $[vb]$ we have only pluses and on the other side only minuses.

The base of Δ has two polygonal lines with ends at a and b. Choose the one that has only pluses; denote it by $a_0 a_1 \dots a_n$; so $a = a_0$ and $b = a_n$. Denote by $a'_0 a'_1 \dots a'_n$ the corresponding line in P'; let $a' = a'_0$ and $b' = a'_n$.

Since each marked edge passing thru a_i has a $(+)$ on it or nothing, we have

$$\measuredangle a_i \leqslant \measuredangle a'_i$$

for each i.



1.8. Exercise. *Prove the last statement.*

By the construction we have $|a_i - a_{i-1}| = |a'_i - a'_{i-1}|$ for all i. By the arm lemma (1.4), we get

❸ $|a - b| \leqslant |a' - b'|$.

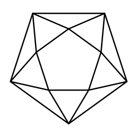
Swap K and K' and repeat the same construction. We get

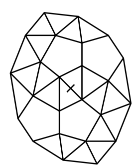
❹ $|a - b| \geqslant |a' - b'|$.

The claims ❸ and ❹ together imply $|a-b| = |a'-b'|$. The equality case in the arm lemma implies that no edge at v is marked; that is, v is not a vertex of Γ — a contradiction. □

From the proof, it follows that the local lemma is indeed local — it works for two noncongruent convex polyhedral angles with equal corresponding faces. Use this observation to solve the following exercise.

1.9. Exercise. *Consider two polyhedral discs in $\mathbb{E}^3$ glued from regular polygons by the rule on the diagrams. Assume that each disc is part of a surface of a convex polyhedron.*





(a) The first configuration is rigid; that is, one can not fix the position of the pentagon and continuously move the remaining 5 vertices in a new position so that each triangle moves by a one-parameter family of isometries of $\mathbb{E}^3$.

(b) Show that the second configuration has a rotational symmetry with the axis passing thru the midpoint of the marked edge.

C Global lemma

The proof of the global lemma is based on counting the sign changes in two ways; first while moving around each vertex of Γ and second while moving around each of the regions separated by Γ on the surface P. If two edges are adjacent at a vertex, then they are also adjacent in a region. The converse is true as well. Therefore, both countings give the same number.

It is instructive to do the next exercise before diving into the proof.

1.10. Exercise. *Try to mark the edges of an octahedron by pluses and minuses such that there would be 4 sign changes at each vertex.*

Show that this is impossible.

Proof of 1.3. We can assume that Γ is connected; that is, one can get from any vertex to any other vertex by walking along edges. (If not, pass to a connected component of Γ.)

Denote by k and l the number of vertices and edges in Γ. Denote by m the number of *regions* that Γ cuts from P. Since Γ is connected, each region is homeomorphic to an open disc.

1.11. Exercise. *Prove the last statement.*

Now we can apply Euler's formula

❶ $$k - l + m = 2.$$

Denote by s the total number of sign changes in Γ for all vertices. By the local lemma (1.2), we have

❷ $$4{\cdot}k \leqslant s.$$

Let us get an upper bound on s by counting the number of sign changes when you go around each region. Denote by m_n the number of regions bounded by n edges; if an edge appears twice when it is counted twice. Note that each region is bounded by at least 3 edges; therefore

❸ $$m = m_3 + m_4 + m_5 + \dots$$

Counting edges and using the fact that each edge belongs to exactly two regions, we get

$$2{\cdot}l = 3{\cdot}m_3 + 4{\cdot}m_4 + 5{\cdot}m_5 + \dots$$

Combining this with Euler's formula (❶), we get

❹ $$4{\cdot}k = 8 + 2{\cdot}m_3 + 4{\cdot}m_4 + 6{\cdot}m_5 + 8{\cdot}m_6 + \dots$$

Observe that the number of sign changes in n-gon regions has to be even and $\leqslant n$. Therefore

❺ $$s \leqslant 2{\cdot}m_3 + 4{\cdot}m_4 + 4{\cdot}m_5 + 6{\cdot}m_6 + \dots$$

Clearly, ❷ and ❺ contradict ❹. $\square$

D Uniqueness

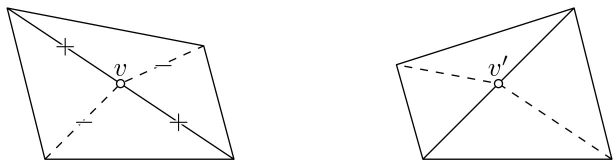
Alexandrov's uniqueness theorem states that the conclusion of the Cauchy theorem (1.1) still holds without the face-to-face assumption.

1.12. Theorem. *Any two convex polyhedrons in $\mathbb{E}^3$ with isometric surfaces are congruent.*

Moreover any isometry between surfaces of convex polyhedrons can be extended to an isometry of the whole $\mathbb{E}^3$.

Needed modifications in the proof of 1.1. Suppose $\iota\colon P \to P'$ be an isometry between surfaces of K and K'. Mark in P all the edges of K and all the inverse images of edges in K'; further, these will be called fake edges. The marked lines divide P into convex polygons, and the restriction of ι to each polygon is a rigid motion. These polygons play the role of faces in the proof above.

A vertex of the obtained graph can be a vertex of K or it can be a fake vertex; that is, it might be an intersection of an edge and a fake edge.



For the first type of vertex, the local lemma can be proved the same way. For a fake vertex v, it is easy to see that both parts of the edge coming thru v are marked with minus while both of the fake edges at v are marked with plus. Therefore, the local lemma holds for the fake vertices as well.

What remains in the proof needs no modifications. □

1.13. Exercise. *Let K be a convex polyhedron in $\mathbb{E}^3$; denote by P its surface. Show that each isometry $\iota\colon P \to P$, can be extended to an isometry of $\mathbb{E}^3$.*

E Existence

Let P be a surface with a polyhedral metric. The curvature of a point $p \in P$ is defined as $2\cdot\pi - \theta$, where θ is the total angle around p.

1.14. Exercise. *Suppose P is the surface of a convex polyhedron. Show that P is homeomorphic to the sphere, and it has nonnegative curvature at every point.*

1.15. Exercise. *Assume that the surface of a nonregular tetrahedron T has curvature π at each of its vertices. Show that*

(a) all faces of T are congruent;

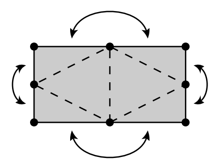
(b) the line passing thru midpoints of opposite edges of T intersects these edges at right angles.

Alexandrov's theorem states that the converse holds if one includes in the consideration *twice covered polygons*. In other words, we assume that a polyhedron can degenerate to a plane polygon; in this case, its surface is defined as two copies of the polygon glued along the boundaries. Intuitively, one can regard these copies as different sides of the polygon. To get from one side to the other one has to cross the boundary of the polygon.

1.16. Theorem. *A polyhedral metric on the sphere is isometric to the surface of a convex polyhedron (possibly degenerate) if and only if it has nonnegative curvature at each point.*

By 1.12, a convex polyhedron is completely defined by the intrinsic metric of its surface. By 1.16, it follows that knowing the metric we could find the position of the edges. However, in practice, it is not easy to do.

For example, the surface glued from a rectangle as shown on the diagram defines a tetrahedron. Some of the glued lines appear inside facets of the tetrahedron and some edges (dashed lines) do not follow the sides of the rectangle.



Space of polyhedrons. Let us denote by $\mathbf{K}$ the space of all convex polyhedrons in the Euclidean space, including polyhedrons that degenerate to a plane polygon. Polyhedra in $\mathbf{K}$ will be considered up to a motion of the space, and the whole space $\mathbf{K}$ will be considered with the natural topology (so far an intuitive meaning of closeness of two polyhedrons should be sufficient).

Further, denote by $\mathbf{K}_n$ the polyhedrons in $\mathbf{K}$ with exactly n vertices. Since any polyhedron has at least 3 vertices, the space $\mathbf{K}$ admits a subdivision into a countable number of subsets $\mathbf{K}_3, \mathbf{K}_4, \ldots$

Space of polyhedral metrics. The space of polyhedral metrics on the sphere with nonnegative curvature will be denoted by $\mathbf{P}$. The metrics in $\mathbf{P}$ will be considered up to an isometry, and the whole space $\mathbf{P}$ will be equipped with the natural topology (again, an intuitive meaning of closeness of two metrics is sufficient).

A point on the sphere with positive curvature will be called an e s s e n t i a l v e r t e x. The subset of $\mathbf{P}$ of all metrics with exactly n essential vertices will be denoted by $\mathbf{P}_n$. It is easy to see that any metric in $\mathbf{P}$ has at least 3 essential vertices. Therefore $\mathbf{P}$ is subdivided into countably many subsets $\mathbf{P}_3, \mathbf{P}_4, \dots$

From a polyhedron to its surface. By 1.14, passing from a polyhedron to its surface defines a map

$$\iota\colon \mathbf{K} \to \mathbf{P}.$$

Note that the number of vertices of a polyhedron is equal to the number of essential vertices on its surface. In other words, $\iota(\mathbf{K}_n) \subset \mathbf{P}_n$ for any $n \geqslant 3$.

Using the introduced notation, we can unite 1.12 and 1.16 in the following statement.

1.17. Reformulation. *For any integer $n \geqslant 3$, the map ι induces a bijection between $\mathbf{K}_n$ and $\mathbf{P}_n$.*

The proof is based on a construction of a one-parameter family of polyhedrons that starts at an arbitrary polyhedron and ends at a polyhedron with its surface isometric to the given one. This type of argument is called the *continuity method*; it is often used in the theory of differential equations.

Sketch. By 1.12, the map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is injective. Let us prove that it is surjective.

1.18. Lemma. *For any integer $n \geqslant 3$, the space $\mathbf{P}_n$ is connected.*

The proof of this lemma is not complicated, but it requires ingenuity; it can be done by the direct construction of a one-parameter family of metrics in $\mathbf{P}_n$ that connects two given metrics. Such a family can be obtained by a sequential application of the following construction and its inverse.

Let $P \in \mathbf{P}_n$. Suppose v and w are essential vertices in P. Let us cut P along a geodesic from v to w. Note that the geodesic cannot pass thru an essential vertex of P. Further, note that there is a three-parameter family of patches that can be used to patch the cut so that the obtained metric remains in $\mathbf{P}_n$; in particular, the obtained metric has exactly n essential vertices (after the patching, the vertices v and w may become inessential).

1.19. Lemma. *The map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is open, that is, it maps any open set in $\mathbf{K}_n$ to an open set in $\mathbf{P}_n$.*

In particular, for any $n \geqslant 3$, the image $\iota(\mathbf{K}_n)$ is open in $\mathbf{P}_n$.

This statement is very close to the so-called *invariance of domain theorem*; the latter states that a continuous injective map between manifolds of the same dimension is open.

Recall that ι is injective. The proof of the invariance of domain theorem can be adapted to our case since both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n-6)$-dimensional and both look like manifolds, altho, formally speaking, they are *not* manifolds. In a more technical language, $\mathbf{K}_n$ and $\mathbf{P}_n$ have the natural structure of $(3{\cdot}n-6)$-dimensional *orbifolds*, and the map ι respects the *orbifold structure*.

We will only show that both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n-6)$-dimensional.

Choose $K\in\mathbf{K}_n$. Note that K is uniquely determined by the $3{\cdot}n$ coordinates of its n vertices. We can assume that the first vertex is the origin, the second has two vanishing coordinates and the third has one vanishing coordinate; therefore, all polyhedrons in $\mathbf{K}_n$ that lie sufficiently close to K can be described by $3{\cdot}n-6$ parameters. If K has no symmetries, then this description can be made one-to-one; in this case, a neighborhood of K in $\mathbf{K}_n$ is a $(3{\cdot}n-6)$-dimensional manifold. If K has a nontrivial symmetry group, then this description is not one-to-one but it does not have an impact on the dimension of $\mathbf{K}_n$.

The case of polyhedral metrics is analogous. We need to construct a subdivision of the sphere into plane triangles using only essential vertices. By Euler's formula, there are exactly $3{\cdot}n-6$ edges in this subdivision. Note that the lengths of edges completely describe the metric, and slight changes in these lengths produce a metric with the same property. Again, if P has no symmetries, then this description is one-to-one.

1.20. Lemma. *The map $\iota\colon\mathbf{K}_n\to\mathbf{P}_n$ is closed; that is, the image of a closed set in $\mathbf{K}_n$ is closed in $\mathbf{P}_n$.*

In particular, for any $n\geqslant 3$, the set $\iota(\mathbf{K}_n)$ is closed in $\mathbf{P}_n$.

Choose a closed set Z in $\mathbf{K}_n$. Denote by $\bar Z$ the closure of Z in $\mathbf{K}$; note that $Z=\mathbf{K}_n\cap\bar Z$. Assume $K_1,K_2,\dots\in Z$ is a sequence of polyhedrons that converges to a polyhedron $K_\infty\in\bar Z$. Note that $\iota(K_n)$ converges to $\iota(K_\infty)$ in $\mathbf{P}$. In particular, $\iota(\bar Z)$ is closed in $\mathbf{P}$.

Since $\iota(\mathbf{K}_n)\subset\mathbf{P}_n$ for any $n\geqslant 3$, we have $\iota(Z)=\iota(\bar Z)\cap\mathbf{P}_n$; that is, $\iota(Z)$ is closed in $\mathbf{P}_n$.

Summarizing, $\iota(\mathbf{K}_n)$ is a nonempty closed and open set in $\mathbf{P}_n$, and $\mathbf{P}_n$ is connected for any $n\geqslant 3$. Therefore, $\iota(\mathbf{K}_n)=\mathbf{P}_n$; that is, $\iota\colon\mathbf{K}_n\to\mathbf{P}_n$ is surjective. $\square$

F Comments

In Euclid's Elements, solids were called equal if the same holds for their faces, but no proof was given. Adrien-Marie Legendre became interested in this problem towards the end of the 18th century. He discussed it with his colleague Joseph-Louis Lagrange, who suggested this problem to Augustin-Louis Cauchy in 1813; soon he proved it [21]. This theorem is included in many popular books [1, 25, 39].

The observation that the face-to-face condition can be removed was made by Alexandr Alexandrov [7].

Arm lemma. Original Cauchy's proof [21] also used a version of the arm lemma, but its proof contained a small mistake (corrected in one century).

Our proof of the arm lemma is due to Stanisław Zaremba. This and a couple of other proofs can be found in the letters between him and Isaac Schoenberg [37].

The following variation of the arm lemma makes sense for nonconvex spherical polygons. It is due to Viktor Zalgaller [43]. It can be used instead of the standard arm lemma.

1.21. Another arm lemma. *Let $A = [a_1 \dots a_n]$ and $A' = [a'_1 \dots$ $\dots a'_n]$ be two spherical n-gons (not necessarily convex). Assume that A lies in a half-sphere, the corresponding sides of A and A' are equal and each angle of A is at least the corresponding angle in A'. Then A is congruent to A'.*

Global lemma. A more visual proof of the global lemma is given in [6, II §1.3].

Existence theorem. This theorem was proved by Alexandr Alexandrov [7]. Our sketch is taken from [29]; a complete proof is nicely written in [6]. A very different proof was found by Yuri Volkov in his thesis [41]; it uses a deformation of three-dimensional polyhedral space.

Lecture 2

CBB: definition

A Distances and geodesics

Distances. The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$. The function $(x,y)\mapsto|x-y|_{\mathcal{X}}$ is called metric; it has to meet the following conditions for any three points $x,y,z\in\mathcal{X}$:

(a) $|x-y|_{\mathcal{X}}\geqslant 0$,

(b) $|x-y|_{\mathcal{X}}=0 \iff x=y$,

(c) $|x-y|_{\mathcal{X}}=|y-x|_{\mathcal{X}}$,

(d) $|x-y|_{\mathcal{X}}+|y-z|_{\mathcal{X}}\geqslant|x-z|_{\mathcal{X}}$.

Geodesics. Let $\mathbb{I}$ be a real interval. A distance-preserving map γ from $\mathbb{I}$ to a metric space $\mathcal{X}$ is called a geodesic[1]; in other words, $\gamma\colon\mathbb{I}\to\mathcal{X}$ is a geodesic if

$$|\gamma(s)-\gamma(t)|_{\mathcal{X}}=|s-t|$$

for any pair $s,t\in\mathbb{I}$.

If $\gamma\colon[a,b]\to\mathcal{X}$ is a geodesic such that $p=\gamma(a)$, $q=\gamma(b)$, then we say that γ is a geodesic from p to q. In this case, the image of γ is denoted by $[pq]$, and, with abuse of notations, we also call it a geodesic. We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$.

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[1] Others call it differently: *shortest path*, *minimizing geodesic*. Also, note that the meaning of the term *geodesic* is different from what is used in Riemannian geometry, altho they are closely related.

In general, a geodesic from p to q need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we assume that we have chosen such geodesic.

Geodesic path. A geodesic path is a geodesic with constant-speed parameterization by the unit interval $[0,1]$.

Geodesic space. A metric space is called geodesic if any pair of its points can be joined by a geodesic.

B Triangles, hinges, and angles

Triangles. Given a triple of points p, q, r in a metric space $\mathcal{X}$, a choice of geodesics $([qr],[rp],[pq])$ will be called a triangle; we will use the short notation $[pqr] = [pqr]_{\mathcal{X}} = ([qr],[rp],[pq])$.

Given a triple $p, q, r \in \mathcal{X}$ there may be no triangle $[pqr]$ simply because one of the pairs of these points cannot be joined by a geodesic. Also, many different triangles with these vertices may exist, any of which can be denoted by $[pqr]$. If we write $[pqr]$, it means that we have chosen such a triangle.

Model triangles. Given three points p, q, r in a metric space $\mathcal{X}$, let us define its model triangle $[\tilde{p}\tilde{q}\tilde{r}]$ (briefly, $[\tilde{p}\tilde{q}\tilde{r}] = \tilde{\triangle}(pqr)_{\mathbb{E}^2}$) to be a triangle in the Euclidean plane $\mathbb{E}^2$ such that

$$|\tilde{p}-\tilde{q}|_{\mathbb{E}^2} = |p-q|_{\mathcal{X}}, \qquad |\tilde{q}-\tilde{r}|_{\mathbb{E}^2} = |q-r|_{\mathcal{X}}, \qquad |\tilde{r}-\tilde{p}|_{\mathbb{E}^2} = |r-p|_{\mathcal{X}}.$$

The same way we can define the hyperbolic and the spherical model triangles $\tilde{\triangle}(pqr)_{\mathbb{H}^2}$, $\tilde{\triangle}(pqr)_{\mathbb{S}^2}$ in the Lobachevsky plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2$. In the latter case, the model triangle is said to be defined if in addition

$$|p-q| + |q-r| + |r-p| < 2{\cdot}\pi.$$

In this case, the model triangle again exists and is unique up to an isometry of $\mathbb{S}^2$.

Model angles. If $[\tilde{p}\tilde{q}\tilde{r}] = \tilde{\triangle}(pqr)_{\mathbb{E}^2}$ and $|p-q|, |p-r| > 0$, the angle measure of $[\tilde{p}\tilde{q}\tilde{r}]$ at $\tilde{p}$ will be called the model angle of the triple p, q, r and will be denoted by $\tilde{\measuredangle}(p\,{}^q_r)_{\mathbb{E}^2}$. The same way we define $\tilde{\measuredangle}(p\,{}^q_r)_{\mathbb{H}^2}$ and $\tilde{\measuredangle}(p\,{}^q_r)_{\mathbb{S}^2}$; in the latter case, we assume in addition that the model triangle $\tilde{\triangle}(pqr)_{\mathbb{S}^2}$ is defined.

We may use the notation $\tilde{\measuredangle}(p\,{}^q_r)$ if it is evident which of the model spaces $\mathbb{H}^2$, $\mathbb{E}^2$ or $\mathbb{S}^2$ is meant.

Hinges. Let $p, x, y \in \mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair of geodesics $([px],[py])$ will be called a hinge and will be denoted by $[p\,{}^x_y] = ([px],[py])$.

C Baby Toponogov

Recall that polyhedral space is a geodesic space that admits a finite triangulation such that each simplex is isometric to a simplex in a Euclidean space. If, in addition, it is homeomorphic to a surface (without boundary), then it is called a polyhedral surface. A point on a polyhedral surface with nonzero curvature is called an essential vertex. Any other point on the surface will be called regular. Note that *any regular point has a neighborhood that is isometric to an open set in the Euclidean plane.*

2.1. Exercise. *Let P be a non-negatively curved polyhedral surface.*

(a) Show that a geodesic in P cannot pass thru an essential vertex.

(b) Show that if two geodesics in P intersect at two points, then these are the endpoints for both geodesics.

The next theorem gives a global geometric property of non-negatively curved polyhedral surfaces.

Given a hinge $[p\,^x_y]$ in a non-negatively curved polyhedral surface P, denote by $\measuredangle[p\,^x_y]$ the minimal angle that the hinge cuts from P at p. (Soon we will give a more general definition of $\measuredangle[p\,^x_y]$; see 3B.)

2.2. Theorem. *Let P be a polyhedral surface. Assume P has non-negative curvature at each point (see 1E). Then*

$$\measuredangle[p\,^x_y] \geqslant \tilde\measuredangle(p\,^x_y)$$

for any hinge $[p\,^x_y]$ in P.

The following exercise will be used in the proof.

2.3. Exercise. *Let $f\colon[0,\ell]\to\mathbb{R}$ be a continuous function such that for any $t\in\,]0,\ell[$ there is a linear function h that locally supports f from above; that is, $h(t_0)=f(t_0)$, and there is $\varepsilon>0$ such that $h(t)\geqslant f(t)$ if $|t-t_0|<\varepsilon$. Show that f is concave.*

Proof. Let $[pxy]$ be a triangle in P and let $[\tilde p\tilde x\tilde y]$ be the model triangle of $[pxy]$. Set $\ell=|x-y|_P=|\tilde x-\tilde y|_{\mathbb{E}^2}$.

Denote by $\gamma(t)$ and $\tilde\gamma(t)$ the geodesics $[xy]$ and $[\tilde x\tilde y]$ parametrized by length starting from x and $\tilde x$, respectively. Observe that it is sufficient to show that

$$\text{❶}\qquad |p-\gamma(t)|\leqslant|\tilde p-\tilde\gamma(t)|$$

for any t in $[0,\ell]$.

We may assume that p is a regular point; otherwise, move it slightly and apply approximation.

From the cosine law, we get that the function

$$\tilde f(t) = |\tilde p - \tilde\gamma(t)|^2 - t^2$$

is linear. Consider the function

$$f(t) = |p - \gamma(t)|^2 - t^2.$$

Note that $f(0) = \tilde f(0)$, $f(\ell) = \tilde f(\ell)$, and the inequality ❶ is equivalent to

❷ $$f(t) \geqslant \tilde f(t).$$

By Jensen's inequality, ❷ holds if f is concave.

By 2.1, $\gamma(t_0)$ is regular. Since p is regular, a geodesic $[p\gamma(t)]$ contains only regular points. Therefore for small $\varepsilon > 0$, the ε-neighborhood of $[p\gamma(t)]$, say Ω, contains only regular points. We may assume that Ω is homeomorphic to a disc; in this case, there is a locally distance-preserving embedding $\iota\colon \Omega \to \mathbb{E}^2$. Note the image $\iota[p\gamma(t)]$ is a line segment that and $\iota(\Omega)$ is the ε-neighborhood of $\iota[p\gamma(t)]$ in $\mathbb{E}^2$; in particular, $\iota(\Omega)$ is convex. Thus $\iota(\Omega)$ contains a triangle with base $\iota[\gamma(t_0 - \varepsilon)\ \gamma(t_0 + \varepsilon)]$ and vertex $\iota(p)$.

Clearly, for any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ we have

$$|\iota(p) - \iota(\gamma(t))| \geqslant |p - \gamma(t)|.$$

Note that the function

$$h(t) = |\iota(p) - \iota(\gamma(t))|^2 - t^2$$

is linear. From above, h supports f locally at t_0. It remains to apply 2.3. □

D Definition

2.4. Definition. *A metric space $\mathcal{X}$ has* n o n n e g a t i v e c u r v a t u r e *in the sense of Alexandrov (briefly, $\mathcal{X} \in \mathrm{CBB}(0)$) if the inequality*

❶ $$\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^y_z)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^z_x)_{\mathbb{E}^2} \leqslant 2\cdot\pi$$

holds for any quadruple $p, x, y, z \in \mathcal{X}$ such that each model angle in ❶ is defined.

The inequality ❶ is called CBB(0) c o m p a r i s o n *for the quadruple p, x, y, z. If instead of $\mathbb{E}^2$, we use $\mathbb{S}^2$ or $\mathbb{H}^2$, then we get the definition of* CBB(1) *and* CBB(−1) *comparisons. (Note that $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2}$ and $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2}$*

are defined if $p\ne x$, $p\ne y$, but for $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2}$ we need in addition, $|p-x|+|p-y|+|x-y|<2\cdot\pi$.)

More generally, one may apply this definition to $\mathbb{M}^2(\kappa)$ — the model plane of curvature κ, defined as follows: $\mathbb{M}(0)=\mathbb{E}^2$, if $\kappa>0$, then $\mathbb{M}(\kappa)$ is the sphere of radius $\frac{1}{\sqrt{\kappa}}$ and if $\kappa<0$, then it is Lobachevsky plane rescaled by factor $\frac{1}{\sqrt{-\kappa}}$. This way we define $\mathrm{CBB}(\kappa)$ *comparison for any real κ.*

While this definition can be applied to any metric space, it is usually applied to geodesic spaces (or, at least, length spaces that will be defined later).

2.5. Exercise. *Show that Euclidean space $\mathbb{E}^n$ is* $\mathrm{CBB}(0)$.

2.6. Exercise. *Show that a polyhedral surface is* $\mathrm{CBB}(0)$ *if and only if it has nonnegative curvature in the sense of 1E.*

The following theorem generalizes 1.12 and 1.16.

2.7. Theorem. *A metric space $\mathcal{X}$ is isometric to the surface of a convex body in the Euclidean space if and only if $\mathcal{X}$ is a geodesic* $\mathrm{CBB}(0)$ *space that is homeomorphic to $\mathbb{S}^2$.*

Moreover, $\mathcal{X}$ determines the convex body up to congruence.

As before, a convex body can degenerate to a plane figure F; in this case, its surface is defined as two copies of F glued along the boundary.

The main part is due to Alexadr Alexandrov [10]; its proof is an application of 1.16 together with approximation. The last part is very difficult; it was proved by Aleksei Pogorelov [35].

Eventually, we will prove the only-if part of the theorem, which is the simplest part of the theorem; it requires only 1.14 which is the only-if part of 1.16. To do this we will need to introduce the convergence of subsets in Euclidean space (Hausdorff convergence) and convergence of metric spaces (Gromov–Hausdorff convergence); it will be done in the next lecture.

2.8. Exercise. *Show that a metric space $\mathcal{X}$ is* $\mathrm{CBB}(0)$ *if and only if for any quadruple of points $p,x_1,x_2,x_3\in\mathcal{X}$ there is a quadruple of points $q,y_1,y_2,y_3\in\mathbb{E}^3$ such that*

$$|p-x_i|_{\mathcal{X}}\geqslant|q-y_i|_{\mathbb{E}^2}\quad\textit{and}\quad|x_i-x_j|_{\mathcal{X}}\leqslant|y_i-y_j|_{\mathbb{E}^2}$$

for all i and j.

2.9. Exercise. *Show that $\mathbb{R}^2$ with metric induced by a norm is* $\mathrm{CBB}(0)$ *if and only if it is isometric to the Euclidean plane $\mathbb{E}^2$.*

E Four-point metric spaces

Let us give a more conceptual way to think about the comparison inequality in 2.4 and an analogous inequality for upper-curvature bound that will appear later.

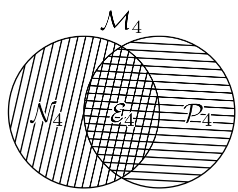
Consider the space $\mathcal{M}_4$ of all isometry classes of 4-point metric spaces. Each element in $\mathcal{M}_4$ can be described by 6 numbers — the distances between all 6 pairs of its points, say $\ell_{i,j}$ for $1 \leqslant i < j \leqslant 4$ modulo permutations of the index set $(1, 2, 3, 4)$. These 6 numbers are subject to 12 triangle inequalities; that is,

$$\ell_{i,j} + \ell_{j,k} \geqslant \ell_{i,k}$$

holds for all i, j and k, where we assume that $\ell_{j,i} = \ell_{i,j}$ and $\ell_{i,i} = 0$.

The space $\mathcal{M}_4$ comes with topology. It can be defined as a quotient of the cone in $\mathbb{R}^6$ by permutations of the 4 points of the space.

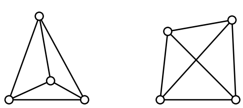
Consider the subset $\mathcal{E}_4 \subset \mathcal{M}_4$ of all isometry classes of 4-point metric spaces that admit isometric embeddings into Euclidean space.



2.10. Advanced exercise. *The complement $\mathcal{M}_4 \backslash \mathcal{E}_4$ has two connected components.*

Let us denote one of the components by $\mathcal{P}_4$ and the other by $\mathcal{N}_4$. Here $\mathcal{P}$ and $\mathcal{N}$ stand for positive and negative curvature because spheres have no quadruples of type $\mathcal{N}_4$ and Lobachevsky space has no quadruples of type $\mathcal{P}_4$.

A metric space that has no quadruples of points of type $\mathcal{P}_4$ or $\mathcal{N}_4$ respectively is called CAT(0) and CBB(0).



Let us describe the subdivision into $\mathcal{P}_4$, $\mathcal{E}_4$, and $\mathcal{N}_4$ intuitively. Imagine that you move out of $\mathcal{E}_4$ — your path is a one-parameter family of 4-point metric spaces. The last thing you see in $\mathcal{E}_4$ is one of the two plane configurations shown on the diagram. If you see the left configuration then you move into $\mathcal{N}_4$; if it is the one on the right, then you move into $\mathcal{P}_4$. More degenerate pictures can be avoided; for example, a triangle with a point on a side. From such a configuration one may move in $\mathcal{N}_4$ and $\mathcal{P}_4$ (as well as come back to $\mathcal{E}_4$).

Here is an exercise, solving which would force you to rebuild a considerable part of Alexandrov geometry. It might be helpful to spend some time thinking about this exercise before proceeding.

2.11. Advanced exercise. *Assume $\mathcal{X}$ is a geodesic space, containing only quadruples of type $\mathcal{E}_4$. Show that $\mathcal{X}$ is isometric to a convex set in a Hilbert space.*

In the definition above, instead of Euclidean space, one can take Lobachevsky space of curvature -1. In this case, one obtains the definition of spaces with curvature bounded above or below by -1 (CAT(-1) or CBB(-1)).

To define spaces with curvature bounded above or below by 1 (CAT(1) or CBB(1)), one has to take the unit 3-sphere and specify that only the quadruples of points such that each of the four triangles has perimeter less than $2\cdot\pi$ are checked.

F Comments

The first synthetic description of curvature is due to Abraham Wald [42]; it was given in a lone publication on a "coordinateless description of Gauss surfaces" published in 1936. In 1941, similar definitions were rediscovered by Alexandr Alexandrov [11].

In Alexandrov's work, the first applications of this approach were given. Mainly: the main part of 2.7 [7, 8] and the gluing theorem [9], which gave a flexible tool to modify non-negatively curved metrics on a sphere. These two results together formed the foundation of the branch of geometry now called Alexandrov geometry; they gave a very intuitive geometric tool to study embeddings and bending of surfaces in Euclidean space and changed the subject dramatically.

In particular, the existence of bending of a large spherical dome (sphere with a small disc removed) easily follows from these two theorems; moreover, it provides an intuitive description of such bending that can be extended to a closed convex surface.

Lecture 3

CBB: first steps

In this lecture, we start to study metric spaces that satisfy CBB comparison [see 2.4]. Most of the covered material will not be used further, it served as a motivation for CBB comparison.

A Quotients and submetries

3.1. Theorem. *Assume that group G acts isometrically on a* CBB(0) *space $\mathcal{L}$ and has closed orbits. Then the quotient space $\mathcal{L}/G$ is* CBB(0)*.*

Proof. Denote by $\sigma\colon \mathcal{L} \to \mathcal{L}/G$ the quotient map.

Fix a quadruple of points $p, x_1, x_2, x_3 \in \mathcal{L}/G$. Choose an arbitrary $\hat{p} \in \mathcal{L}$ such that $\sigma(\hat{p}) = p$. Note that we can choose the points $\hat{x}_1, \hat{x}_2, \hat{x}_3 \in \mathcal{L}$ such that $\sigma(\hat{x}_i) = x_i$ and

$$|p - x_i|_{\mathcal{L}/G} \lesseqgtr |\hat{p} - \hat{x}_i|_{\mathcal{L}} \pm \delta$$

for all i and any fixed $\delta > 0$.

Given $\varepsilon > 0$, the value δ can be chosen in such a way that the inequality

$$\text{❶} \qquad \tilde{\measuredangle}(p\,{}^{x_i}_{x_j}) < \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_i}_{\hat{x}_j}) + \varepsilon$$

holds for all i and j.

By CBB(0) comparison in $\mathcal{L}$, we have

$$\tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_1}_{\hat{x}_2}) + \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_2}_{\hat{x}_3}) + \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_3}_{\hat{x}_1}) \leqslant 2{\cdot}\pi.$$

Applying ❶, we get

$$\tilde{\measuredangle}(p\,{}^{x_1}_{x_2}) + \tilde{\measuredangle}(p\,{}^{x_2}_{x_3}) + \tilde{\measuredangle}(p\,{}^{x_3}_{x_1}) < 2\cdot\pi + 3\cdot\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we have

$$\tilde{\measuredangle}(p\,{}^{x_1}_{x_2}) + \tilde{\measuredangle}(p\,{}^{x_2}_{x_3}) + \tilde{\measuredangle}(p\,{}^{x_3}_{x_1}) \leqslant 2\cdot\pi;$$

that is, the CBB(0) comparison holds for this quadruple in $\mathcal{L}/G$. $\square$

A map $\sigma\colon \mathcal{X} \to \mathcal{Y}$ between the metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is called a submetry if

$$\sigma(\mathrm{B}(p,r)_{\mathcal{X}}) = \mathrm{B}(\sigma(p),r)_{\mathcal{Y}}$$

for any $p \in \mathcal{X}$ and $r \geqslant 0$.

Suppose G and $\mathcal{L}$ are as in 3.1. Observe that the quotient map $\sigma\colon \mathcal{L} \to \mathcal{L}/G$ is a submetry. Moreover, the proof above works for any submetry. Therefore we get the following.

3.2. Generalization. *Let $\sigma\colon \mathcal{L} \to \mathcal{M}$ be a submetry. Suppose $\mathcal{L}$ is a* CBB(0) *space, then so is $\mathcal{M}$.*

3.3. Advanced exercise. *Let G be a compact Lie group with a bi-invariant Riemannian metric. Show that G is isometric to a quotient of the Hilbert space by isometric group action.*

Conclude that $G \in$ CBB(0)*.*

B Angles

The angle measure of a hinge $[p\,{}^x_y]$ is defined as the following limit

$$\measuredangle[p\,{}^x_y] = \lim_{\bar{x},\bar{y}\to p} \tilde{\measuredangle}(p\,{}^{\bar{x}}_{\bar{y}}),$$

where $\bar{x} \in\,]px]$ and $\bar{y} \in\,]py]$.

Note that if $\measuredangle[p\,{}^x_y]$ is defined, then

$$0 \leqslant \measuredangle[p\,{}^x_y] \leqslant \pi.$$

3.4. Exercise. *Suppose that in the above definition, one uses spherical or hyperbolic model angles instead of Euclidean. Show that it does not change the value $\measuredangle[p\,{}^x_y]$.*

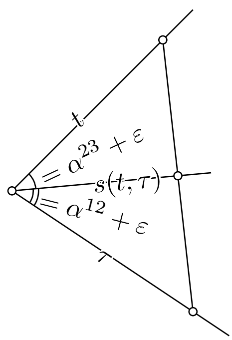
3.5. Exercise. *Give an example of a hinge $[p\,{}^x_y]$ in a metric space with an undefined angle $\measuredangle[p\,{}^x_y]$.*

3.6. Triangle inequality for angles. *Let $[px_1]$, $[px_2]$, and $[px_3]$ be three geodesics in a metric space. If all of the angles $\alpha_{ij} = \measuredangle[p\,{}^{x_i}_{x_j}]$ are defined then they satisfy the triangle inequality:*

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23}.$$

Proof. Since $\alpha_{13} \leqslant \pi$, we can assume that $\alpha_{12} + \alpha_{23} < \pi$. Denote by γ_i the unit-speed parametrization of $[px_i]$ from p to x_i. Given any $\varepsilon > 0$, for all sufficiently small $t, \tau, s \in \mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned}|\gamma_1(t) - \gamma_3(\tau)| &\leqslant |\gamma_1(t) - \gamma_2(s)| + |\gamma_2(s) - \gamma_3(\tau)| < \\ &< \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha_{12} + \varepsilon)} + \\ &\quad + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\alpha_{23} + \varepsilon)} \leqslant\end{aligned}$$



Below we define $s(t,\tau)$ so that for $s = s(t,\tau)$, this chain of inequalities can be continued as follows:

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon.$$

Hence the result follows.

To define $s(t,\tau)$, consider three half-lines $\tilde\gamma_1$, $\tilde\gamma_2$, $\tilde\gamma_3$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde\gamma_1, \tilde\gamma_2) = \alpha_{12} + \varepsilon$, $\measuredangle(\tilde\gamma_2, \tilde\gamma_3) = \alpha_{23} + \varepsilon$, and $\measuredangle(\tilde\gamma_1, \tilde\gamma_3) = \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon$. We parametrize each half-line by the distance from the starting point. Given two positive numbers $t, \tau \in \mathbb{R}_{\geqslant 0}$, let $s = s(t,\tau)$ be the number such that $\tilde\gamma_2(s) \in [\tilde\gamma_1(t)\ \tilde\gamma_3(\tau)]$. Clearly, $s \leqslant \max\{t, \tau\}$, so t, τ, s may be taken sufficiently small. $\square$

3.7. Exercise. *Prove that the sum of adjacent angles is at least π.*

More precisely: suppose two hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ are adjacent; that is, they share side $[pz]$, and the union of two sides $[px]$ and $[py]$ form a geodesic $[xy]$. Show that

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \geqslant \pi$$

whenever each angle on the left-hand side is defined.

The above inequality can be strict. For example in a metric tree angles between any two different edges coming out of the same vertex are all equal to π.

3.8. Exercise. *Assume that a hinge $[q\,{}^p_x]$ with defined angle measure. Let γ be the unit speed parametrization of $[qx]$ from q to x. Show that*

$$|p-\gamma(t)| \leqslant |q-p| - t\cdot\cos(\measuredangle[q\,{}^p_x]) + o(t).$$

C Alexandrov's lemma

Recall that $[xy]$ denotes a geodesic from x to y; set

$$]xy] = [xy]\setminus\{x\}, \quad [xy[= [xy]\setminus\{y\}, \quad]xy[= [xy]\setminus\{x,y\}.$$

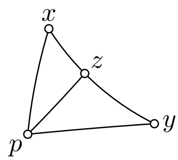
3.9. Lemma. *Let p, x, y, z be distinct points in a metric space such that $z \in\,]xy[$. Then the following expressions for the Euclidean model angles have the same sign:*

(a) $\tilde\measuredangle(x\,{}^p_y) - \tilde\measuredangle(x\,{}^p_z)$,

(b) $\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) - \pi$.

The same holds for the hyperbolic and spherical model angles, but in the latter case, one has to assume in addition that

$$|p-z| + |p-y| + |x-y| < 2\cdot\pi.$$



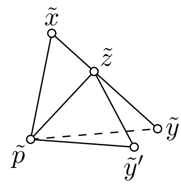
Proof. Consider the model triangle $[\tilde x\tilde p\tilde z] = \tilde\triangle(xpz)$. Take a point $\tilde y$ on the extension of $[\tilde x\tilde z]$ beyond $\tilde z$ so that $|\tilde x-\tilde y| = |x-y|$ (and therefore $|\tilde x-\tilde z| = |x-z|$).

Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:

(i) $\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(x\,{}^p_y)$,

(ii) $|\tilde p-\tilde y| - |p-y|$,

(iii) $\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(z\,{}^p_y)$.



Since

$$\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] = \measuredangle[\tilde x\,{}^{\tilde p}_{\tilde z}] = \tilde\measuredangle(x\,{}^p_z)$$

and

$$\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] = \pi - \measuredangle[\tilde z\,{}^{\tilde x}_{\tilde p}] = \pi - \tilde\measuredangle(z\,{}^x_p),$$

the first statement follows. □

3.10. Exercise. *Assume p, x, y, z are as in Alexandrov's lemma. Show that*

$$\tilde\measuredangle(p\,{}^x_y) \geqslant \tilde\measuredangle(p\,{}^x_z) + \tilde\measuredangle(p\,{}^z_y),$$

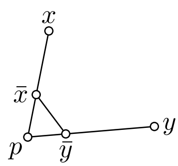
with equality if and only if the expressions in (a) and (b) vanish.

Note that if $p \in\,]xy[$, then $\tilde\measuredangle(p\,{}^{x}_{y}) = \pi$. Applying Alexandrov's lemma and CBB(0) comparison, we get the following claim and its corollary.

3.11. Claim. *If p, x, y, z are points in a* CBB(0) *such that $p \in\,]xy[$, then*

$$\tilde\measuredangle(x\,{}^{y}_{z}) \leqslant \tilde\measuredangle(x\,{}^{p}_{z}).$$

3.12. Corollary. *Let $[p\,{}^{x}_{y}]$ be a hinge in a* CBB(0) *space. Consider the function*



$$f\colon (|p-\bar x|, |p-\bar y|) \mapsto \tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x \in\,]px]$ and $\bar y \in\,]py]$. Show that f is nonincreasing in each argument.

Note that 3.12 implies the following generalization of 2.2.

3.13. Claim. *For any hinge $[p\,{}^{x}_{y}]$ in a* CBB(0) *space, the angle measure $\measuredangle[p\,{}^{x}_{y}]$ is defined, and*

$$\measuredangle[p\,{}^{x}_{y}] \geqslant \tilde\measuredangle(p\,{}^{x}_{y}).$$

3.14. Exercise. *Let $[p\,{}^{x}_{y}]$ be a hinge in a* CBB(0) *space. Suppose $\measuredangle[p\,{}^{x}_{y}] = 0$ show that $[px] \subset [py]$ or $[py] \subset [px]$.*

3.15. Exercise. *Let $[xy]$ be a geodesic in a* CBB(0) *space. Suppose $z \in\,]xy[$ show that there is a unique geodesic $[xz]$ and $[xz] \subset [xy]$.*

3.16. Exercise. *Let $[p\,{}^{x}_{z}]$ and $[p\,{}^{y}_{z}]$ be adjacent hinges in a* CBB(0) *space. Show that*

$$\measuredangle[p\,{}^{x}_{z}] + \measuredangle[p\,{}^{y}_{z}] = \pi.$$

3.17. Exercise. *Let p, x, y in a* CBB(0) *space and $v, w \in\,]xy[$. Show that*

$$\tilde\measuredangle(x\,{}^{y}_{p}) = \tilde\measuredangle(x\,{}^{v}_{p}) \quad\iff\quad \tilde\measuredangle(x\,{}^{y}_{p}) = \tilde\measuredangle(x\,{}^{w}_{p}).$$

Recall that a t r i a n g l e $[xyz]$ in a space $\mathcal{X}$ is a triple of minimizing geodesics $[xy]$, $[yz]$, and $[zx]$. Consider the model triangle $[\tilde x\tilde y\tilde z] = \tilde\triangle(xyz)_{\mathbb{E}^2}$ in the Euclidean plane. The n a t u r a l m a p $[\tilde x\tilde y\tilde z] \to [xyz]$ sends a point $\tilde p \in [\tilde x\tilde y] \cup [\tilde y\tilde z] \cup [\tilde z\tilde x]$ to the corresponding point $p \in [xy] \cup [yz] \cup [zx]$; that is, if $\tilde p$ lies on $[\tilde y\tilde z]$, then $p \in [yz]$ and $|\tilde y - \tilde p| = |y-p|$ (and therefore $|\tilde z - \tilde p| = |z-p|$).

3.18. Definition. *A triangle $[xyz]$ in the metric space $\mathcal{X}$ is called $thin$ (or fat) if the natural map $\tilde{\triangle}(xyz)_{\mathbb{E}^2} \to [xyz]$ is distance nonincreasing (or respectively distance nondecreasing).*

3.19. Exercise. *Show that any triangle in a* CBB(0) *space is fat.*

D Comments

All the discussed statements admit natural generalizations to CBB(κ) spaces. Most of the time the proof is the same with uglier formulas. However, for the CBB(1) case in 3.1 one needs to assume in addition that space has intrinsic metric and the proof requires the globalization theorem which will be discussed later.

Lecture 4

CBB: globalization

A Hinge comparison

Let $[p\,^x_y]$ be a hinge in a CBB(0) space. By 3.14, the angle measure $\measuredangle[p\,^x_y]$ is defined and

$$\measuredangle[p\,^x_y] \geqslant \tilde{\measuredangle}(p\,^x_y).$$

Further, according to 3.16, we have

$$\measuredangle[p\,^x_z] + \measuredangle[p\,^y_z] = \pi$$

for adjacent hinges $[p\,^x_z]$ and $[p\,^y_z]$ in a CBB(0) space.

The following theorem implies that a geodesic space is CBB(0) if the above conditions hold for all its hinges.

4.1. Theorem. *A geodesic space $\mathcal{L}$ is* CBB(0) *if the following conditions hold.*

(a) For any hinge $[x\,^p_y]$ in $\mathcal{L}$, the angle $\measuredangle[x\,^p_y]$ is defined and

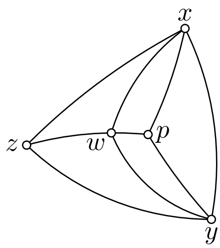
$$\measuredangle[x\,^p_y] \geqslant \tilde{\measuredangle}(x\,^p_y).$$

(b) For any two adjacent hinges $[p\,^x_z]$ and $[p\,^y_z]$ in $\mathcal{L}$, we have

$$\measuredangle[p\,^x_z] + \measuredangle[p\,^y_z] \leqslant \pi.$$

Proof. Consider a point $w \in\ ]pz[$ close to p. From *(b)*, it follows that

$$\measuredangle[w\,^x_z] + \measuredangle[w\,^x_p] \leqslant \pi \quad \text{and} \quad \measuredangle[w\,^y_z] + \measuredangle[w\,^y_p] \leqslant \pi.$$



Since $\measuredangle[w\,{}^x_y] \leqslant \measuredangle[w\,{}^x_p] + \measuredangle[w\,{}^y_p]$ (see 3.6), we get

$$\measuredangle[w\,{}^x_z] + \measuredangle[w\,{}^y_z] + \measuredangle[w\,{}^x_y] \leqslant 2{\cdot}\pi.$$

Applying *(a)*,

$$\tilde\measuredangle(w\,{}^x_z) + \tilde\measuredangle(w\,{}^y_z) + \tilde\measuredangle(w\,{}^x_y) \leqslant 2{\cdot}\pi.$$

Passing to the limits $w \to p$, we have

$$\tilde\measuredangle(p\,{}^x_z) + \tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^x_y) \leqslant 2{\cdot}\pi.$$

□

B Equivalent conditions

The following theorem summarizes 3.11, 3.13, 3.16, 4.1.

4.2. Theorem. *Let $\mathcal{L}$ be a geodesic space. Then the following conditions are equivalent.*

(a) *$\mathcal{L}$ is* CBB(0).

(b) *(adjacent angle comparison) for any geodesic $[xy]$ and point $z \in]xy[$, $z \ne p$ in $\mathcal{L}$, we have*

$$\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) \leqslant \pi.$$

(c) *(point-on-side comparison) for any geodesic $[xy]$ and $z \in]xy[$ in $\mathcal{L}$, we have*

$$\tilde\measuredangle(x\,{}^p_y) \leqslant \tilde\measuredangle(x\,{}^p_z).$$

(d) *(hinge comparison) for any hinge $[x\,{}^p_y]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and*

$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle(x\,{}^p_y).$$

Moreover,

$$\measuredangle[z\,{}^p_y] + \measuredangle[z\,{}^p_x] \leqslant \pi$$

for any adjacent hinges $[z\,{}^p_y]$ and $[z\,{}^p_x]$.

Moreover, the implications (a)⇒(b)⇒(c)⇒(d) hold in any space, not necessarily geodesic.

4.3. Advanced Exercise. *Construct a geodesic space $\mathcal{X} \notin$* CBB(0) *that meets the following condition: for any 3 points $p, x, y \in \mathcal{X}$ there is a geodesic $[xy]$ such that for any $z \in]xy[$*

$$\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) \leqslant \pi.$$

C Globalization

A metric space $\mathcal{L}$ is locally CBB(0) if any point $p \in \mathcal{L}$ admits a neighborhood $U \ni p$ such that the CBB(0) comparison holds for any quadruple of points in U.

4.4. Globalization theorem. *Any locally* CBB(0) *compact geodesic space is* CBB(0).

Proof modulo the key lemma. Let $\mathcal{L}$ be a locally CBB(0) compact geodesic space. Note that condition 4.1b holds in $\mathcal{L}$ (the proof is the same as for CBB(0) space). It remains to prove that 4.1a holds in $\mathcal{L}$; that is,

❶ $$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle(x\,{}^p_y).$$

First note that ❶ holds for hinges in a small neighborhood of any point; this can be proved the same way as 3.13 and 3.16, applying the local version of CBB(0) comparison. Since $\mathcal{L}$ is compact, there is $\varepsilon > 0$ such that ❶ holds if $|x-p| + |p-y| < \varepsilon$. Applying the key lemma several times we get that ❶ holds for any given hinge. $\square$

4.5. Key lemma. *Let $\mathcal{L}$ be a locally* CBB(0) *geodesic space. Assume that the comparison*

$$\measuredangle[x\,{}^p_q] \geqslant \tilde\measuredangle(x\,{}^p_q)$$

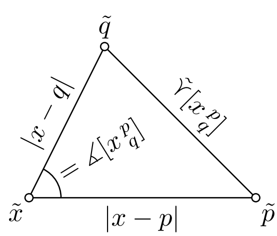
holds for any hinge $[x\,{}^p_q]$ with $|x-y| + |x-q| < \tfrac{2}{3}\cdot\ell$. Then comparison

$$\measuredangle[x\,{}^p_q] \geqslant \tilde\measuredangle(x\,{}^p_q)$$

holds for any hinge $[x\,{}^p_q]$ with $|x-p| + |x-q| < \ell$.

Let $[x\,{}^p_q]$ be a hinge in a metric space $\mathcal{L}$ with defined angle measure. Denote by $\tilde{\Upsilon}[x\,{}^p_q]$ its model side; this is the opposite side in a flat triangle with the same angle and two adjacent sides as in $[x\,{}^p_q]$.

More precisely, consider the model hinge $[\tilde x\,{}^{\tilde p}_{\tilde q}]$ in $\mathbb{E}^2$ that is defined by



$$\begin{aligned}
\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde q}]_{\mathbb{E}^2} &= \measuredangle[x\,{}^p_q]_{\mathcal{L}},\\
|\tilde x - \tilde p|_{\mathbb{E}^2} &= |x-p|_{\mathcal{L}},\\
|\tilde x - \tilde q|_{\mathbb{E}^2} &= |x-q|_{\mathcal{L}};
\end{aligned}$$

then

$$\tilde{\Upsilon}[x\,{}^p_q]_{\mathcal{L}} := |\tilde p - \tilde q|_{\mathbb{E}^2}.$$

Note that

$$\tilde\Upsilon[x\,{}^{p}_{q}] \geqslant |p-q| \quad\iff\quad \measuredangle[x\,{}^{p}_{q}] \geqslant \tilde\measuredangle(x\,{}^{p}_{q}).$$

We will use it in the following proof.

Proof. It is sufficient to prove the inequality

❷ $$\tilde\Upsilon[x\,{}^{p}_{q}] \geqslant |p-q|$$

for any hinge $[x\,{}^{p}_{q}]$ with $|x-p|+|x-q|<\ell$.

Consider a hinge $[x\,{}^{p}_{q}]$ such that

$$\tfrac{2}{3}\cdot\ell \leqslant |p-x|+|x-q| < \ell.$$

First, let us construct a new smaller hinge $[x'\,{}^{p}_{q}]$ with

❸ $$|p-x|+|x-q| \geqslant |p-x'|+|x'-q|,$$

such that

❹ $$\tilde\Upsilon[x\,{}^{p}_{q}] \geqslant \tilde\Upsilon[x'\,{}^{p}_{q}].$$

Construction. Assume $|x-q|\geqslant|x-p|$; otherwise switch the roles of p and q in the following construction. Take $x'\in[xq]$ such that

❺ $$|p-x|+3\cdot|x-x'| = \tfrac{2}{3}\cdot\ell.$$

Choose a geodesic $[x'p]$ and consider the hinge $[x'\,{}^{p}_{q}]$ formed by $[x'p]$ and $[x'q]\subset[xq]$. Then ❸ follows from the triangle inequality.

Further, note that

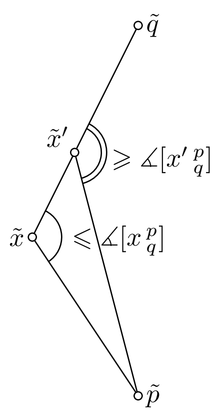
$$|p-x|+|x-x'| < \tfrac{2}{3}\cdot\ell, \qquad |p-x'|+|x'-x| < \tfrac{2}{3}\cdot\ell.$$

In particular,

❻ $$\measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde\measuredangle(x\,{}^{p}_{x'}) \quad\text{and}\quad \measuredangle[x'\,{}^{p}_{x}] \geqslant \tilde\measuredangle(x'\,{}^{p}_{x}).$$

Now, let $[\tilde x\tilde x'\tilde p] = \tilde\triangle(xx'p)$. Take $\tilde q$ on the extension of $[\tilde x\tilde x']$ beyond x' such that $|\tilde x-\tilde q| = |x-q|$ (and therefore $|\tilde x'-\tilde q| = |x'-q|$). By ❻,

$$\measuredangle[x\,{}^{p}_{q}] = \measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde\measuredangle(x\,{}^{p}_{x'}) \quad\Rightarrow\quad \tilde\Upsilon[x\,{}^{q}_{p}] \geqslant |\tilde p-\tilde q|.$$

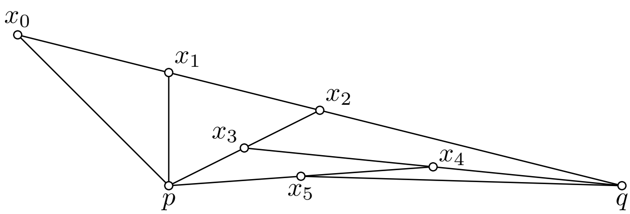


Hence

$$\begin{aligned}\measuredangle[\tilde x'\,{}^{\tilde p}_{\tilde q}] &= \pi - \tilde\measuredangle(x'\,{}^{p}_{x}) \geqslant\\ &\geqslant \pi - \measuredangle[x'\,{}^{p}_{x}] =\\ &= \measuredangle[x'\,{}^{p}_{q}],\end{aligned}$$

and ❹ follows.

Let us continue the proof. Set $x_0 = x$. Let us apply inductively the above construction to get a sequence of hinges $[x_n\,{}^{p}_{q}]$ with $x_{n+1} = x'_n$. From ❹, we have that the sequence $s_n = \tilde\Upsilon[x_n\,{}^{p}_{q}]$ is nonincreasing.



The sequence might terminate at some n only if $|p-x_n|+|x_n-q| < \frac{2}{3}\cdot\ell$. In this case, by the assumptions of the lemma, $\tilde\Upsilon[x_n\,{}^{p}_{q}] \geqslant |p-q|$. Since the sequence s_n is nonincreasing, inequality ❷ follows.

Otherwise, the sequence $r_n = |p - x_n| + |x_n - q|$ is nonincreasing, and $r_n \geqslant \frac{2}{3}\cdot\ell$ for all n. Note that by construction, the distances $|x_n - x_{n+1}|$, $|x_n - p|$, and $|x_n - q|$ are bounded away from zero for all large n. Indeed, since on each step, we move x_n toward to the point p or q that is further away, the distances $|x_n - p|$ and $|x_n - q|$ become about the same. Namely, by ❺, we have that $|p - x_n| - |x_n - q| \leqslant \frac{2}{9}\cdot\ell$ for all large n. Since $|p - x_n| + |x_n - q| \geqslant \frac{2}{3}\cdot\ell$, we have $|x_n - p| \geqslant \frac{\ell}{100}$ and $|x_n - q| \geqslant \frac{\ell}{100}$. Further, since $r_n \geqslant \frac{2}{3}\cdot\ell$, ❺ implies that $|x_n - x_{n+1}| > \frac{\ell}{100}$.

Since the sequence r_n is nonincreasing, it converges. In particular, $r_n - r_{n+1} \to 0$ as $n \to \infty$. It follows that $\tilde\measuredangle(x_n\,{}^{p_n}_{x_{n+1}}) \to \pi$, where $p_n = p$ if $x_{n+1} \in [x_n q]$, and otherwise $p_n = q$. Since $\measuredangle[x_n\,{}^{p_n}_{x_{n+1}}] \geqslant \tilde\measuredangle(x_n\,{}^{p_n}_{x_{n+1}})$, we have $\measuredangle[x_n\,{}^{p_n}_{x_{n+1}}] \to \pi$ as $n \to \infty$.

It follows that

$$r_n - s_n = |p - x_n| + |x_n - q| - \tilde\Upsilon[x_n\,{}^{p}_{q}] \to 0.$$

Together with the triangle inequality

$$|p - x_n| + |x_n - q| \geqslant |p - q|$$

this yields

$$\lim_{n\to\infty} \tilde\Upsilon[x_n\,{}^{p}_{q}] \geqslant |p - q|.$$

Applying monotonicity of the sequence $s_n = \tilde\Upsilon[x_n\,{}^{p}_{q}]$, we obtain ❷. $\square$

D On general curvature bound

The globalization theorem can be generalized to CBB(κ) spaces for any real κ. The case $\kappa \leqslant 0$ is proved the same way, but the case $\kappa > 0$ requires minor modifications.

4.6. Exercise. *Suppose $\kappa \leqslant \mathrm{K}$. Show that*

$$\tilde\measuredangle(x\,{}^y_z)_{\mathbb{M}(\kappa)} \leqslant \tilde\measuredangle(x\,{}^y_z)_{\mathbb{M}(\mathrm{K})}$$

if the right-hand side is defined.

Conclude that any CBB(K) *space is locally* CBB(κ).

The exercise and the globalization theorem (here we need a more general version 4.11) imply that *any geodesic* CBB(K) *space is* CBB(κ). Recall that CBB(κ) stands for *curvature bounded below by κ*; so, for geodesic spaces it makes sense. However, as you can see from the following exercise, it does not make much sense in general.

4.7. Exercise. *Let $\mathcal{X}$ be the set $\{p, x_1, x_2, x_3\}$ with the metric defined by*

$$|p - x_i| = \pi, \quad |x_i - x_j| = 2{\cdot}\pi$$

for all $i \neq j$. Show that $\mathcal{X}$ is CBB(1), *but not* CBB(0).

4.8. Exercise. *Let p and q be points in a* CBB(1) *geodesic space $\mathcal{L}$. Suppose $|p - q| > \pi$. Denote by m the midpoint of $[pq]$. Show that for any hinge $[m\,{}^x_p]$ we have either $\measuredangle[m\,{}^x_p] = 0$ or $\measuredangle[m\,{}^x_p] = \pi$. Conclude that $\mathcal{L}$ is isometric to a real interval or a circle.*

E Remarks

The globalization theorem is also known as the *generalized Toponogov theorem.*

Recall that a metric space $\mathcal{X}$ is called c o m p l e t e if any Cauchy sequence of points in $\mathcal{X}$ converges. The compactness condition in our version of the theorem can be traded to completeness by using the following exercise.

4.9. Exercise. *Let $\mathcal{X}$ be a complete metric space. Suppose $r\colon \mathcal{X} \to \mathbb{R}$ is a positive continuous function. Show that for any $\varepsilon > 0$ there is a point $p \in \mathcal{X}$ such that*

$$r(x) > (1 - \varepsilon){\cdot}r(p)$$

for any $x \in \overline{\mathrm{B}}[p, \tfrac{1}{\varepsilon}{\cdot}r(p)]$.

Let us mention two more general versions of the globalization theorem.

Recall that length space is a metric space such that any two points p and q can be connected by a path with length arbitrarily close to $|p-q|$. Note that any geodesic space is length, but not the other way around. The following theorem was proved already in the paper of Michael Gromov, Yuriy Burago, and Grigory Perelman [19].

4.10. Theorem. *Any complete locally* $\mathrm{CBB}(\kappa)$ *length space is* $\mathrm{CBB}(\kappa)$.

The next result is mine [31].

4.11. Theorem. *Any locally* $\mathrm{CBB}(\kappa)$ *geodesic space is* $\mathrm{CBB}(\kappa)$.

In the two-dimensional case, the globalization theorem was proved by Paolo Pizzetti [33]; later it was reproved independently by Alexandr Alexandrov [12]. Victor Toponogov [40] proved it for Riemannian manifolds of all dimensions.

I took the proof from our book [5] (with generality reduction). It uses simplifications obtained by Conrad Plaut [34] and Dmitry Burago, Yuriy Burago, and Sergei Ivanov [15]. The same proof was rediscovered independently by Urs Lang and Viktor Schroeder [28]. Another simplified version was obtained by Katsuhiro Shiohama [38].

The question whether 4.1a suffices to conclude that $\mathcal{L}$ is $\mathrm{CBB}(\kappa)$ is a long-standing open problem (possibly dating back to Alexandrov); in print, it was first stated in [15, footnote in 4.1.5].

4.12. Open question. *Let $\mathcal{L}$ be a complete geodesic space (you can also assume that $\mathcal{L}$ is homeomorphic to $\mathbb{S}^2$ or $\mathbb{R}^2$) such that for any hinge $[x\,{}^p_y]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and*

$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle(x\,{}^p_y).$$

Is it true that $\mathcal{L}$ is $\mathrm{CBB}(0)$*?*

Lecture 5

Semiconcave functions

A Real-to-real functions

Choose $\lambda \in \mathbb{R}$. Let $s\colon \mathbb{I} \to \mathbb{R}$ be a locally Lipschitz function defined on an interval $\mathbb{I}$. We say that s is λ-concave if $s'' \leqslant \lambda$, where the second derivative s'' is understood in the sense of distributions.

Equivalently, *s is λ-concave if the function $h\colon t \mapsto s(t) - \lambda\cdot\frac{t^2}{2}$ is concave.* Concavity can be defined via Jensen inequality; that is,

$$h(s\cdot t_0 + (1-s)\cdot t_1) \geqslant s\cdot h(t_0) + (1-s)\cdot h(t_1)$$

for any $t_0, t_1 \in \mathbb{I}$ and $s \in [0,1]$. It could be also defined via existence of (local) upper support at any point: *for any $t_0 \in \mathbb{I}$ there is a linear function ℓ that (locally) supports h at t_0 from above; that is, $\ell(t_0) = h(t_0)$ and $\ell(t) \geqslant h(t)$ for any t (in a neighborhood of t_0).*

The equivalence of these definitions is assumed to be known. We will also use that *λ-concave functions are one-side differentiable.*

B Function comparison

A function on a metric space $\mathcal{L}$ will usually mean a *locally Lipschitz real-valued function defined in an open subset of $\mathcal{L}$.* The domain of definition of a function f will be denoted by $\operatorname{Dom} f$.

Let f be a function on a metric space $\mathcal{L}$. We say that f is λ-concave (briefly $f'' \leqslant \lambda$) if for any unit-speed geodesic $\gamma\colon \mathbb{I} \to \operatorname{Dom} f$ the real-to-real function $t \mapsto f\circ\gamma(t)$ is λ-concave.

The following proposition is conceptual — it reformulates a global geometric condition into an infinitesimal condition on distance functions.

5.1. Proposition. *A geodesic space $\mathcal{L}$ is* CBB(0) *if and only if $f'' \leqslant 1$ for any function f of the following type*

$$f\colon x \mapsto \tfrac{1}{2}\cdot|p - x|^2.$$

Proof. Choose a unit-speed geodesic γ in $\mathcal{L}$ and two points $x = \gamma(t_0)$, $y = \gamma(t_1)$ for some $t_0 < t_1$. Consider the model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$. Let $\tilde{\gamma}\colon [t_0, t_1] \to \mathbb{E}^2$ be the unit-speed parametrization of $[\tilde{x}\tilde{y}]$ from $\tilde{x}$ to $\tilde{y}$.

Set

$$\tilde{r}(t) := |\tilde{p} - \tilde{\gamma}(t)|, \qquad r(t) := |p - \gamma(t)|.$$

Clearly, $\tilde{r}(t_0) = r(t_0)$ and $\tilde{r}(t_1) = r(t_1)$. Note that the point-on-side comparison (4.2c) is equivalent to

❶ $$t_0 \leqslant t \leqslant t_1 \qquad \Longrightarrow \qquad \tilde{r}(t) \leqslant r(t)$$

for any γ and $t_0 < t_1$.

Set

$$\tilde{h}(t) = \tfrac{1}{2}\cdot\tilde{r}^2(t) - \tfrac{1}{2}\cdot t^2, \qquad h = \tfrac{1}{2}\cdot r^2(t) - \tfrac{1}{2}\cdot t^2.$$

Note that $\tilde{h}$ is linear, $\tilde{h}(t_0) = h(t_0)$ and $\tilde{h}(t_1) = h(t_1)$. Observe that the Jensen inequality for the function h is equivalent to ❶. Hence the proposition follows. □

C Semiconcave functions

We will write $f'' \leqslant \varphi$ if for any point $x \in \operatorname{Dom} f$ and any $\varepsilon > 0$ there is a neighborhood $U \ni x$ such that the restriction $f|_U$ is $(\varphi(x) + \varepsilon)$-concave. Here we assume that φ is continuous and defined in $\operatorname{Dom} f$.

If $f'' \leqslant \varphi$ for some continuous function φ, then f is called semiconcave.

5.2. Exercise. *Let f be a distance function on a geodesic* CBB(0) *space $\mathcal{L}$; that is, $f(x) \equiv |p - x|$ for some $p \in \mathcal{L}$. Show that $f'' \leqslant \frac{1}{f}$. In particular, f is semiconcave in $\mathcal{L} \setminus \{p\}$.*

D Completion

Given a metric space $\mathcal{X}$, consider the set $\mathcal{C}$ of all Cauchy sequences in $\mathcal{X}$. Note that for any two Cauchy sequences (x_n) and (y_n) the right-hand side in ❶ is defined; moreover, it defines a semimetric on $\mathcal{C}$

❶ $$|(x_n) - (y_n)|_{\mathcal{C}} := \lim_{n\to\infty} |x_n - y_n|_{\mathcal{X}}.$$

The corresponding metric space is called the completion of $\mathcal{X}$; it will be denoted by $\bar{\mathcal{X}}$.

It is straightforward to check that *completion is complete.*

For each point $x \in \mathcal{X}$, one can consider a constant sequence $x_n = x$ which is Cauchy. It defines a natural inclusion map $\mathcal{X} \hookrightarrow \bar{\mathcal{X}}$. It is easy to check that this map is distance-preserving. In particular, we can (and will) consider $\mathcal{X}$ as a subset of $\bar{\mathcal{X}}$. Note that $\mathcal{X}$ *is a dense subset in its completion* $\bar{\mathcal{X}}$.

E Space of directions

Let $\mathcal{X}$ be a space with defined angles. Given $p \in \mathcal{X}$, consider the set $\mathfrak{S}_p$ of all nontrivial unit-speed geodesics starting at p. By 3.6, the triangle inequality holds for $\measuredangle$ on $\mathfrak{S}_p$, that is, $(\mathfrak{S}_p, \measuredangle)$ forms a semimetric space.

The metric space corresponding to $(\mathfrak{S}_p, \measuredangle)$ is called the space of geodesic directions at p, denoted by Σ'_p or $\Sigma'_p\mathcal{X}$. The elements of Σ'_p are called geodesic directions at p. Each geodesic direction is formed by an equivalence class of geodesics starting from p for the equivalence relation

$$[px] \sim [py] \qquad \Longleftrightarrow \qquad \measuredangle[p\,{}^x_y] = 0;$$

the direction of $[px]$ is denoted by $\uparrow_{[px]}$. (If $\mathcal{X}$ is CBB, then by 3.14, $[px] \sim [py]$ if and only if $[px] \subset [py]$ or $[px] \supset [py]$.)

The completion of Σ'_p is called the space of directions at p and is denoted by Σ_p or $\Sigma_p\mathcal{X}$. The elements of Σ_p are called directions at p.

F Tangent space

Cone construction. The Euclidean cone $\mathcal{V} = \operatorname{Cone}\mathcal{X}$ over a metric space $\mathcal{X}$ is defined as the metric space whose underlying set consists of equivalence classes in $[0, \infty) \times \mathcal{X}$ with the equivalence relation "$\sim$" given by $(0, p) \sim (0, q)$ for any points $p, q \in \mathcal{X}$, and whose metric is given by the cosine rule

$$|(s, p) - (t, q)|_{\mathcal{V}} = \sqrt{s^2 + t^2 - 2{\cdot}s{\cdot}t{\cdot}\cos\theta},$$

where $\theta = \min\{\pi, |p - q|_{\mathcal{X}}\}$.

Note that $\operatorname{Cone}\mathbb{S}^n$ *is isometric to* $\mathbb{E}^{n+1}$. This is a leading example; further, we generalize several notions of Euclidean space to the Euclidean cones.

The point in $\mathcal{V}$ that corresponds $(t,x) \in [0,\infty)\times\mathcal{X}$ will be denoted by $t{\cdot}x$. The point in $\mathcal{V}$ formed by the equivalence class of $\{0\}\times\mathcal{X}$ is called the origin of the cone and is denoted by 0 or $0_{\mathcal{V}}$. For $v\in\mathcal{V}$ the distance $|0-v|_{\mathcal{V}}$ is called the norm of v and is denoted by $|v|$ or $|v|_{\mathcal{V}}$. The scalar product $\langle v,w\rangle$ of $v=s{\cdot}p$ and $w=t{\cdot}q$ is defined by

$$\langle v,w\rangle := |v|{\cdot}|w|{\cdot}\cos\theta$$

where $\theta=\min\{\pi,|p-q|_{\mathcal{X}}\}$; we set $\langle v,w\rangle:=0$ if $v=0$ or $w=0$.

Tangent space. The Euclidean cone $\operatorname{Cone}\Sigma_p$ over the space of directions Σ_p is called the tangent space at p and denoted by T_p or $\mathrm{T}_p\mathcal{X}$. The elements of $\mathrm{T}_p\mathcal{X}$ will be called tangent vectors at p (despite the fact that T_p is only a cone — not a vector space). The space of directions Σ_p can be (and will be) identified with the unit sphere in T_p.

5.3. Exercise. *Show that tangent spaces of* $\mathrm{CBB}(\kappa)$ *space are* $\mathrm{CBB}(0)$.

G Differential

Let $\mathcal{X}$ be a space with defined angles. Let f be a semiconcave function on $\mathcal{X}$ and $p\in\operatorname{Dom}f$. Choose a unit-speed geodesic γ that starts at p; let $\xi\in\Sigma_p$ be its direction. Define

$$(\boldsymbol{d}_pf)(\xi):=(f\circ\gamma)^+(0),$$

here $(f\circ\gamma)^+$ denotes the right derivative of $(f\circ\gamma)$; it is defined since f is semiconcave.

By the following exercise, the value $(\boldsymbol{d}_pf)(\xi)$ is defined; that is, it does not depend on the choice of γ. Moreover, $\boldsymbol{d}_pf$ is a Lipschitz function on Σ'_p. It follows that the function $\boldsymbol{d}_pf\colon\Sigma'_p\to\mathbb{R}$ can be extended to a Lipschitz function $\boldsymbol{d}_pf\colon\Sigma_p\to\mathbb{R}$. Further, we can extend it to the tangent space by setting

$$(\boldsymbol{d}_pf)(r{\cdot}\xi):=r{\cdot}(\boldsymbol{d}_pf)(\xi)$$

for any $r\geqslant 0$ and $\xi\in\Sigma_p$. The obtained function $\boldsymbol{d}_pf\colon\mathrm{T}_p\to\mathbb{R}$ is Lipschitz; it is called the differential of f at p.

5.4. Exercise. *Let f be a semiconcave function on a geodesic space $\mathcal{X}$ with defined angles. Suppose γ_1 and γ_2 are unit-speed geodesics that start at $p\in\operatorname{Dom}f$; denote by θ the angle between γ_1 and γ_2 at p. Show that*

$$|(f\circ\gamma_1)^+(0)-(f\circ\gamma_2)^+(0)|\leqslant L{\cdot}\theta,$$

where L is the Lipschitz constant of f in a neighborhood of p.

5.5. Exercise. *Let p and q be distinct points in a* CBB(0) *space. Denote by ξ the direction of a geodesic $[pq]$ at p. Show that*

$$\boldsymbol{d}_p \mathrm{dist}_q(v) \leqslant -\langle \xi, v \rangle$$

for any $v \in \mathrm{T}_p$.

H Gradient

5.6. Definition. *Let f be a semiconcave function on a geodesic space $\mathcal{X}$ with defined angles. A tangent vector $g \in \mathrm{T}_p$ is called a gradient of f at p (briefly, $g = \nabla_p f$) if*

(a) $(\boldsymbol{d}_p f)(w) \leqslant \langle g, w \rangle$ for any $w \in \mathrm{T}_p$, and

(b) $(\boldsymbol{d}_p f)(g) = \langle g, g \rangle$.

5.7. Proposition. *Suppose that a semiconcave function f is defined in a neighborhood of a point p in a* CBB(κ) *space. Then the gradient $\nabla_p f$ is uniquely defined.*

5.8. Key lemma. *Let f be a λ-concave function that is defined in a neighborhood of a point p in a geodesic* CBB(κ) *space $\mathcal{L}$. Then for any $u, v \in \mathrm{T}_p$, we have*

$$s \cdot \sqrt{|u|^2 + 2 \cdot \langle u, v \rangle + |v|^2} \geqslant (\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v),$$

where

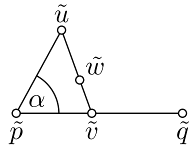
$$s = \sup \{ (\boldsymbol{d}_p f)(\xi) \ : \ \xi \in \Sigma_p \} .$$

Note that in Euclidean space we have

$$|u + v| = \sqrt{|u|^2 + 2 \cdot \langle u, v \rangle + |v|^2}.$$

The right-hand side makes sense in any Euclidean cone, but the sum in the left-hand side does not.

Proof. We will assume $\kappa = 0$; the general case requires only minor modifications. We can assume that $v \neq 0$, $w \neq 0$, and $\alpha = \measuredangle(u, v) > 0$; otherwise, the statement is trivial.



Prepare a model configuration of five points: $\tilde p$, $\tilde u$, $\tilde v$, $\tilde q$, $\tilde w \in \mathbb{E}^2$ such that

◇ $\measuredangle[\tilde p\,{}^{\tilde u}_{\tilde v}] = \alpha$,
◇ $|\tilde p - \tilde u| = |u|$,
◇ $|\tilde p - \tilde v| = |v|$,
◇ $\tilde q$ lies on an extension of $[\tilde p\tilde v]$ so that $\tilde v$ is the midpoint of $[\tilde p\tilde q]$,
◇ $\tilde w$ is the midpoint between $\tilde u$ and $\tilde v$.

Note that

❶ $$|\tilde p - \tilde w| = \tfrac12\cdot\sqrt{|u|^2 + 2\cdot\langle u, v\rangle + |v|^2}.$$

We can assume that there are geodesics in the directions of u and v; the latter follows since the geodesic space of directions Σ_p' is dense in Σ_p. Choose geodesics γ_u and γ_v in the direction in the directions of u and v; let us assume that they are parametrized with speed $|u|$ and $|v|$ respectively. For all small $t > 0$, construct points $u_t, v_t, q_t, w_t \in \mathcal{L}$ as follows.

◇ $v_t = \gamma_v(t)$, $\quad q_t = \gamma_v(2\cdot t)$
◇ $u_t = \gamma_u(t)$.
◇ w_t is the midpoint of $[u_t v_t]$.

Clearly

$$|p - u_t| = t\cdot|u|, \qquad |p - v_t| = t\cdot|v|, \qquad |p - q_t| = 2\cdot t\cdot|v|.$$

Since $\measuredangle(u, v)$ is defined, we have

$$|u_t - v_t| = t\cdot|\tilde u - \tilde v| + o(t), \qquad |u_t - q_t| = t\cdot|\tilde u - \tilde q| + o(t).$$

From the point-on-side and hinge comparisons (4.2c+4.2d), we have

$$\tilde\measuredangle(v_t\,{}^{p}_{w_t}) \geqslant \tilde\measuredangle(v_t\,{}^{p}_{u_t}) \geqslant \measuredangle[\tilde v\,{}^{\tilde p}_{\tilde u}] + \tfrac{o(t)}{t}$$

and

$$\tilde\measuredangle(v_t\,{}^{q_t}_{w_t}) \geqslant \tilde\measuredangle(v_t\,{}^{q_t}_{u_t}) \geqslant \measuredangle[\tilde v\,{}^{\tilde q}_{\tilde u}] + \tfrac{o(t)}{t}.$$

Clearly, $\measuredangle[\tilde v\,{}^{\tilde p}_{\tilde u}] + \measuredangle[\tilde v\,{}^{\tilde q}_{\tilde u}] = \pi$. From the adjacent angle comparison (4.2b), $\tilde\measuredangle(v_t\,{}^{p}_{u_t}) + \tilde\measuredangle(v_t\,{}^{u_t}_{q_t}) \leqslant \pi$. Hence $\tilde\measuredangle(v_t\,{}^{p}_{w_t}) \to \measuredangle[\tilde v\,{}^{\tilde p}_{\tilde w}]$ as $t \to 0+$ and thus

$$|p - w_t| = t\cdot|\tilde p - \tilde w| + o(t).$$

Without loss of generality, we can assume that $f(p) = 0$. Since f is λ-concave, we have

$$\begin{aligned}2\cdot f(w_t) &\geqslant f(u_t) + f(v_t) + \tfrac{\lambda}{4}\cdot|u_t - v_t|^2 = \\ &= t\cdot[(\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v)] + o(t).\end{aligned}$$

Applying λ-concavity of f, we have

$$\begin{aligned}(\boldsymbol{d}_p f)(\uparrow_{[pw_t]}) &\geqslant \frac{f(w_t) - \frac{\lambda}{2}\cdot|p - w_t|^2}{|p - w_t|} \geqslant \\ &\geqslant \frac{t\cdot[(\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v)] + o(t)}{2\cdot t\cdot|\tilde p - \tilde w| + o(t)}.\end{aligned}$$

By ❶, the key lemma follows. □

5.9. Exercise. *Let $[q\,{}^p_x]$ be a hinge in a* CBB(0) *space and $y \in\,]qp[$. Suppose that γ is the unit speed parametrization of $[qx]$ from q to x. Show that*

$$|y - \gamma(t)| = |y - q| - t\cdot \cos(\measuredangle[q\,{}^p_x]) + o(t).$$

Conclude that

$$(\boldsymbol{d}_q \mathrm{dist}_y)(w) = -\langle \uparrow_{[qp]}, w\rangle$$

for any $w \in \mathrm{T}_x$

Proof of 5.7; uniqueness. If $g, g' \in \mathrm{T}_p$ are two gradients of f, then

$$\langle g, g\rangle = (\boldsymbol{d}_p f)(g) \leqslant \langle g, g'\rangle, \qquad \langle g', g'\rangle = (\boldsymbol{d}_p f)(g') \leqslant \langle g, g'\rangle.$$

Therefore,

$$|g - g'|^2 = \langle g, g\rangle - 2\cdot\langle g, g'\rangle + \langle g', g'\rangle \leqslant 0.$$

It follows that $g = g'$.

Existence. Note first that if $\boldsymbol{d}_p f \leqslant 0$, then one can take $\nabla_p f = 0$.

Otherwise, if $s = \sup\{\, (\boldsymbol{d}_p f)(\xi) : \xi \in \Sigma_p \,\} > 0$, it is sufficient to show that there is $\bar\xi \in \Sigma_p$ such that

❷ $$(\boldsymbol{d}_p f)\left(\bar\xi\right) = s.$$

Indeed, suppose $\bar\xi$ exists. Applying 5.8 for $u = \bar\xi$, $v = \varepsilon\cdot w$ with $\varepsilon \to 0+$, we get

$$(\boldsymbol{d}_p f)(w) \leqslant \langle w, s\cdot\bar\xi\rangle$$

for any $w \in \mathrm{T}_p$; that is, $s\cdot\bar\xi$ is the gradient at p.

Take a sequence of directions $\xi_n \in \Sigma_p$, such that $(\boldsymbol{d}_p f)(\xi_n) \to s$. Applying 5.8 for $u = \xi_n$ and $v = \xi_m$, we get

$$s \geqslant \frac{(\boldsymbol{d}_p f)(\xi_n) + (\boldsymbol{d}_p f)(\xi_m)}{\sqrt{2 + 2\cdot \cos\measuredangle(\xi_n, \xi_m)}}.$$

Therefore $\measuredangle(\xi_n, \xi_m) \to 0$ as $n, m \to \infty$; that is, the sequence ξ_n is Cauchy. Clearly $\overline{\xi} = \lim_n \xi_n$ meets ❷. □

5.10. Exercise. *Let f and g be locally Lipschitz semiconcave functions defined in a neighborhood of a point p in a* CBB *space. Show that*

$$|\nabla_p f - \nabla_p g|^2_{\mathrm{T}_p} \leqslant s\cdot(|\nabla_p f| + |\nabla_p g|),$$

where

$$s = \sup\left\{\, |(\boldsymbol{d}_p f)(\xi) - (\boldsymbol{d}_p g)(\xi)| \;:\; \xi \in \Sigma_p \,\right\}.$$

Conclude that if the sequence of restrictions $\boldsymbol{d}_p f_n|_{\Sigma_p}$ converges uniformly, then $\nabla_p f_n$ converges as $n \to \infty$. Here we assume that all functions f_1, $f_2, \dots$ are semiconcave and locally Lipschitz.

5.11. Exercise. *Let f be a locally Lipschitz semiconcave function on a complete geodesic* $\mathrm{CBB}(\kappa)$ *space $\mathcal{L}$.*

(a) Suppose $s \geqslant 0$. Show that $|\nabla_x f| > s$ if and only if

$$f(y) - f(x) > s\cdot\ell + \lambda\cdot\tfrac{\ell^2}{2}$$

for some point y; here $\ell = |x - y|$.

(b) Show that $x \mapsto |\nabla_x f|$ is lower semicontinuous; that is, if $x_n \to \to x_\infty$, then

$$|\nabla_{x_\infty} f| \leqslant \varliminf_{n\to\infty} |\nabla_{x_n} f|.$$

I Comments

The function comparison of $\mathrm{CBB}(-1)$ states that $f'' \leqslant f$ for any function of the type $f = \cosh\circ\mathrm{dist}_p$. Similarly, the function comparison of $\mathrm{CBB}(1)$ states that for any point p we have $f'' \leqslant -f$ for the function $f = -\cos\circ\mathrm{dist}_p$ defined in $\mathrm{B}(p, \pi)$. The meaning of these inequalities is the same — distance functions in $\mathrm{CBB}(\kappa)$ are more concave than distance functions in $\mathbb{M}(\kappa)$.

Lecture 6

Gradient flow

A Velocity of curve

Let α be a curve in a geodesic CBB(κ) space $\mathcal{L}$; choose $p = \alpha(t_0)$. If for any choice of geodesics $[p\,\alpha(t_0+\varepsilon)]$ the vectors

$$\tfrac{1}{\varepsilon}\cdot|p-\alpha(t_0+\varepsilon)|\cdot{\uparrow}_{[p\alpha(t_0+\varepsilon)]}$$

converge as $\varepsilon \to 0+$, then their limit in T_p is called the right derivative of α at t_0; it will be denoted by $\alpha^+(t_0)$. In addition, $\alpha^+(t_0) := 0$ if $\tfrac{1}{\varepsilon}\cdot|p-\alpha(t_0+\varepsilon)| \to 0$ as $\varepsilon \to 0+$.

6.1. Claim. *Let α be a curve in a geodesic* CBB(κ) *space $\mathcal{L}$. Suppose f a semiconcave Lipschitz function defined in a neighborhood of $p = \alpha(0)$, and $\alpha^+(0)$ is defined. Then*

$$(f\circ\alpha)^+(0) = (\boldsymbol{d}_p f)(\alpha^+(0)).$$

Proof. Without loss of generality, we can assume that $f(p) = 0$. Suppose f and therefore $\boldsymbol{d}_p f$ are L-Lipschitz. Further, we will always assume that $\varepsilon > 0$.

Choose a constant-speed geodesic γ that starts from p, such that the distance $s = |\alpha^+(0) - \gamma^+(0)|_{\mathrm{T}_p}$ is small. Observe that by the definition of differential,

$$(f\circ\gamma)^+(0) = \boldsymbol{d}_p f(\gamma^+(0)).$$

By comparison and the definition of α^+,

$$|\alpha(\varepsilon) - \gamma(\varepsilon)|_{\mathcal{L}} \leqslant s\cdot\varepsilon + o(\varepsilon)$$

Therefore

$$|f\circ\alpha(\varepsilon)-f\circ\gamma(\varepsilon)|\leqslant L{\cdot}s{\cdot}\varepsilon+o(\varepsilon).$$

Suppose $(f\circ\alpha)^+(0)$ is defined. Then

$$|(f\circ\alpha)^+(0)-(f\circ\gamma)^+(0)|\leqslant L{\cdot}s.$$

Since $\boldsymbol{d}_pf$ is L-Lipschitz, we also get

$$|\boldsymbol{d}_pf(\alpha^+(0))-\boldsymbol{d}_pf(\gamma^+(0))|\leqslant L{\cdot}s.$$

It follows that the needed identity holds up to error $2{\cdot}L{\cdot}s$. The statement follows since $s>0$ can be chosen arbitrarily.

Finally, even if $(f\circ\alpha)^+(0)$ is undefined, we can arrive to the same conclusion using all partial limits $\frac{1}{\varepsilon_n}{\cdot}[f\circ\alpha(\varepsilon_n)-f(p)]$ for $\varepsilon_n\to 0+$ in the place of $(f\circ\alpha)^+(0)$. □

B Gradient curves

6.2. Definition. *Let $f\colon\mathcal{L}\to\mathbb{R}$ be a locally Lipschitz and semiconcave function on a complete geodesic* CBB(κ) *space $\mathcal{L}$.*

A locally Lipschitz curve $\alpha\colon[t_{\min},t_{\max})\to\operatorname{Dom}f$ will be called an f-gradient curve if

$$\alpha^+=\nabla_\alpha f;$$

that is, for any $t\in[t_{\min},t_{\max})$, $\alpha^+(t)$ is defined and $\alpha^+(t)=\nabla_{\alpha(t)}f$.

A complete proof of the following theorem takes about 5 pages [5]; it mimics the standard Picard theorem on the existence and uniqueness of solutions of ordinary differential equations. We omit the proof of existence; the uniqueness is proved in the next section.

6.3. Picard theorem. *Let $f\colon\mathcal{L}\to\mathbb{R}$ be a locally Lipschitz and λ-concave function on a complete geodesic* CBB(κ) *space $\mathcal{L}$. Then for any $p\in\operatorname{Dom}f$, there are unique $t_{\max}\in(0,\infty]$ and f-gradient curve $\alpha\colon[0,t_{\max})\to\mathcal{L}$ with $\alpha(0)=p$ such that any sequence $t_n\to t_{\max}-$, the sequence $\alpha(t_n)$ does not have a limit point in* $\operatorname{Dom}f$.

Note that the theorem only says that the future of a gradient curve is determined by its present, but it says nothing about its past.

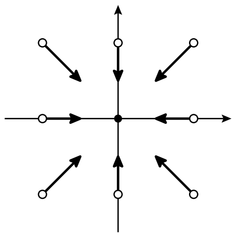
Here is an example showing that the past is not determined by the present. Consider the function $f\colon x\mapsto-|x|$ on the real line $\mathbb{R}$. The tangent space $\mathrm{T}_x\mathbb{R}$ can be identified with $\mathbb{R}$. Note that

$$\nabla_xf=\begin{cases}1&\text{if}\quad x<0,\\0&\text{if}\quad x=0,\\-1&\text{if}\quad x>0.\end{cases}$$

So, the f-gradient curves go to the origin with unit speed and then stand there forever. In particular, if α is an f-gradient curve that starts at x, then $\alpha(t)=0$ for any $t\geqslant|x|$.

Here is a slightly more interesting example; it shows that gradient curves can merge even in the region where $|\nabla f|\neq 0$. Hence their *past* cannot be uniquely determined from their *present*.

6.4. Example. *Consider the function $f\colon(x,y)\mapsto -|x|-|y|$ on the (x,y)-plane. Note that f is concave; its gradient field is sketched on the figure.*



Let α be an f-gradient curve that starts at (x,y) for $x>y>0$. Then

$$\alpha(t)=\begin{cases}(x-t,y-t) & \textit{for}\quad 0\leqslant t\leqslant x-y,\\ (x-t,0) & \textit{for}\quad x-y\leqslant t\leqslant x,\\ (0,0) & \textit{for}\quad x\leqslant t.\end{cases}$$

C Distance estimates

6.5. Observation. *Let α is a gradient curve of a λ-concave function f defined on a complete geodesic* CBB *space. Choose point p; let $\ell(t):=\operatorname{dist}_p\circ\alpha(t)$ and $q=\alpha(t_0)$. Then*

$$\ell^+(t_0)\leqslant -\left(f(p)-f(q)-\tfrac{\lambda}{2}\cdot\ell^2(t_0)\right)/\ell(t_0)$$

Proof. Let γ be the unit-speed parametrization of $[qp]$ from q to p, so $q=\gamma(0)$. Then

$$\begin{aligned}\ell^+(t_0)&=(\boldsymbol{d}_q\operatorname{dist}_p)(\nabla_q f)\leqslant\\ &\leqslant -\langle\uparrow_{[qp]},\nabla_q f\rangle\leqslant\\ &\leqslant -\boldsymbol{d}_q f(\uparrow_{[qp]})=\\ &=-(f\circ\gamma)^+(0)\leqslant\\ &\leqslant -\left(f(p)-f(q)-\tfrac{\lambda}{2}\cdot\ell^2(t_0)\right)/\ell(t_0)\end{aligned}$$

In the above calculations we consequently applied 6.1, 5.5, the definition of gradient, the definition of differential, and concavity of $t\mapsto\mapsto f\circ\gamma(t)-\tfrac{\lambda}{2}\cdot t^2$. $\square$

Note that the following estimate implies uniqueness in the Picard theorem (6.3).

6.6. First distance estimate. *Let f be a λ-concave locally Lipschitz function on a complete geodesic* CBB(κ) *space $\mathcal{L}$. Then*

$$|\alpha(t)-\beta(t)|\leqslant e^{\lambda\cdot t}\cdot|\alpha(0)-\beta(0)|$$

for any $t\geqslant 0$ and any two f-gradient curves α and β.

Moreover, the statement holds for a locally Lipschitz λ-concave function defined in an open domain if there is a geodesic $[\alpha(t)\,\beta(t)]$ in $\operatorname{Dom} f$ *for any t.*

Proof. Fix a choice of geodesic $[\alpha(t)\,\beta(t)]$ for each t. Let $\ell(t)=|\alpha(t)-\beta(t)|$. Note that

$$\ell^{+}(t)\leqslant-\langle\uparrow_{[\alpha(t)\beta(t)]},\nabla_{\alpha(t)}f\rangle-\langle\uparrow_{[\beta(t)\alpha(t)]},\nabla_{\beta(t)}f\rangle\leqslant\lambda\cdot\ell(t).$$

Here one has to apply 6.5 for distance to the midpoint m of $[\alpha(t)\,\beta(t)]$, and then apply the triangle inequality. Hence the result. □

The following exercise describes a global geometric property of a gradient curve without direct reference to its function. It uses the notion of *self-contracting curves* introduced by Aris Daniilidis, Olivier Ley, and Stéphane Sabourau [24].

6.7. Exercise. *Let $f\colon\mathcal{L}\to\mathbb{R}$ be a locally Lipschitz and concave function on a complete geodesic* CBB(κ) *space $\mathcal{L}$. Then*

$$t_1\leqslant t_2\leqslant t_3\quad\Longrightarrow\quad|\alpha(t_1)-\alpha(t_3)|_{\mathcal{L}}\geqslant|\alpha(t_2)-\alpha(t_3)|_{\mathcal{L}}.$$

for any f-gradient curve α.

6.8. Exercise. *Let f be a locally Lipschitz concave function defined on a* CBB(κ) *space. Suppose $\hat\alpha\colon[0,\ell]$ is an arc-length reparametrization of an f-gradient curve. Show that $(f\circ\hat\alpha)$ is concave.*

The following exercise implies that gradient curves for a uniformly converging sequence of λ-concave functions converge to the gradient curves of the limit function.

6.9. Exercise. *Let f and g be λ-concave locally Lipschitz functions on a complete geodesic* CBB(κ) *space $\mathcal{L}$. Suppose $\alpha,\beta\colon[0,t_{\max})\to\mathcal{L}$ are respectively f- and g-gradient curves. Assume $|f-g|<\varepsilon$; let $\ell\colon t\mapsto|\alpha(t)-\beta(t)|$. Show that*

$$\ell^{+}\leqslant\lambda\cdot\ell+\tfrac{2\cdot\varepsilon}{\ell}.$$

Conclude that $\alpha(0)=\beta(0)$ and $t_{\max}<\infty$ then

$$|\alpha(t)-\beta(t)|\leqslant c\cdot\sqrt{\varepsilon\cdot t}$$

for some constant $c=c(t_{\max},\lambda)$.

D Gradient flow

Let $\mathcal{L}$ be a complete geodesic $\mathrm{CBB}(\kappa)$ space and f be a locally Lipschitz semiconcave function defined on an open set of $\mathcal{L}$. If there is an f-gradient curve α such that $\alpha(0)=x$ and $\alpha(t)=y$, then we will write

$$\mathrm{Flow}_f^t(x)=y.$$

The partially defined map Flow_f^t from $\mathcal{L}$ to itself is called the f-gradient flow for time t. Note that

$$\mathrm{Flow}_f^{t_1+t_2}=\mathrm{Flow}_f^{t_1}\circ\mathrm{Flow}_f^{t_2};$$

in other words, gradient flow is given by an action of the semigroup $(\mathbb{R}_{\geqslant 0},+)$.

From the first distance estimate 6.6, it follows that for any $t\geqslant 0$, the domain of definition of Flow_f^t is an open subset of $\mathcal{L}$. In some cases, it is globally defined. For example, if f is a λ-concave function defined on the whole space $\mathcal{L}$, then $\mathrm{Flow}_f^t(x)$ is defined for all $x\in\mathcal{L}$ and $t\geqslant 0$; see [5, 16.19].

Now let us reformulate statements obtained earlier using this new terminology. Again, from the first distance estimate, we have the following.

6.10. Proposition. *Let $\mathcal{L}$ be a complete geodesic* $\mathrm{CBB}(\kappa)$ *space and $f\colon\mathcal{L}\to\mathbb{R}$ be a semiconcave function. Then the map $x\mapsto\mathrm{Flow}_f^t(x)$ is locally Lipschitz.*

Moreover, if f is λ-concave, then Flow_f^t is $e^{\lambda\cdot t}$-Lipschitz.

The next proposition follows from 6.9.

6.11. Proposition. *Let $\mathcal{L}$ be a complete geodesic* $\mathrm{CBB}(\kappa)$ *space. Suppose $f_n\colon\mathcal{L}\to\mathbb{R}$ is a sequence of λ-concave functions that converges to $f_\infty\colon\mathcal{L}_\infty\to\mathbb{R}$. Then for any $x\in\mathcal{L}$ and $t\geqslant 0$, we have*

$$\mathrm{Flow}_{f_n}^t(x)\to\mathrm{Flow}_{f_\infty}^t(x)$$

as $n\to\infty$

Lecture 7

Line splitting

A Busemann function

A half-line is a distance-preserving map from $\mathbb{R}_{\geqslant 0} = [0, \infty)$ to a metric space. In other words, a half-line is a geodesic defined on the real half-line $\mathbb{R}_{\geqslant 0}$. If $\gamma\colon [0, \infty) \to \mathcal{X}$ is a half-line, then the limit

❶ $$\mathrm{bus}_\gamma(x) = \lim_{t\to\infty} |\gamma(t) - x| - t$$

is called the Busemann function of γ.

(The Busemann function bus_γ mimics the distance function from the ideal point of γ.)

7.1. Proposition. *For any half-line γ in a metric space $\mathcal{X}$, its Busemann function $\mathrm{bus}_\gamma\colon \mathcal{X} \to \mathbb{R}$ is defined. Moreover, bus_γ is 1-Lipschitz and $\mathrm{bus}_\gamma \circ\gamma(t) + t = 0$ for any t.*

Proof. By the triangle inequality, the function

$$t \mapsto |\gamma(t) - x| - t$$

is nonincreasing for any fixed x.

Since $t = |\gamma(0) - \gamma(t)|$, the triangle inequality implies that

$$|\gamma(t) - x| - t \geqslant -|\gamma(0) - x|.$$

Thus the limit in ❶ is defined, and it is 1-Lipschitz as a limit of 1-Lipschitz functions. The last statement follows since $|\gamma(t) - \gamma(t_0)| = t - t_0$ for all large t. $\square$

Note that 5.2 implies the following.

7.2. Observation. *Any Busemann function on a geodesic* CBB(0) *space is concave.*

B Splitting theorem

A line is a distance-preserving map from $\mathbb{R}$ to a metric space. In other words, a line is a geodesic defined on the real line $\mathbb{R}$.

Let $\mathcal{X}$ be a metric space and $A, B \subset \mathcal{X}$. We will write

$$\mathcal{X} = A \oplus B$$

if there are projections $\operatorname{proj}_A\colon \mathcal{X} \to A$ and $\operatorname{proj}_B\colon \mathcal{X} \to B$ such that

$$|x - y|^2 = |\operatorname{proj}_A(x) - \operatorname{proj}_A(y)|^2 + |\operatorname{proj}_B(x) - \operatorname{proj}_B(y)|^2$$

for any two points $x, y \in \mathcal{X}$.

Note that if

$$\mathcal{X} = A \oplus B$$

then

- A intersects B at a single point,
- both sets A and B are convex sets in $\mathcal{X}$; the latter means that any geodesic with the ends in A (or B) lies in A (or B).

7.3. Line splitting theorem. *Let $\mathcal{L}$ be a complete geodesic* CBB(0) *space and γ be a line in $\mathcal{L}$. Then*

$$\mathcal{L} = \mathcal{L}' \oplus \gamma(\mathbb{R})$$

for some subset $\mathcal{L}' \subset \mathcal{L}$.

Before going into the proof, let us state a corollary of the theorem.

7.4. Corollary. *Let $\mathcal{L}$ be a complete geodesic* CBB(0) *space. Then there is an isometric splitting*

$$\mathcal{L} = \mathcal{L}' \oplus H$$

where $H \subset \mathcal{L}$ is a subset isometric to a Hilbert space, and $\mathcal{L}' \subset \mathcal{L}$ is a convex subset that contains no line.

The following lemma is closely relevant to the first distance estimate (6.6); its proof goes along the same lines.

7.5. Lemma. *Suppose $f\colon \mathcal{L} \to \mathbb{R}$ be a concave 1-Lipschitz function on a geodesic* CBB(0) *space $\mathcal{L}$. Consider two f-gradient curves α and β. Then for any $t, s \geqslant 0$ we have*

$$|\alpha(s) - \beta(t)|^2 \leqslant |p - q|^2 + 2{\cdot}(f(p) - f(q)){\cdot}(s - t) + (s - t)^2,$$

where $p = \alpha(0)$ and $q = \beta(0)$.

Proof. Since f is 1-Lipschitz, $|\nabla f| \leqslant 1$. Therefore

$$f\circ\beta(t) \leqslant f(q) + t$$

for any $t \geqslant 0$.

Set $\ell(t) = |p-\beta(t)|$. Applying 6.5, we get

$$\begin{aligned}(\ell^2)^+(t) &\leqslant 2\cdot\left(f\circ\beta(t) - f(p)\right) \leqslant \\ &\leqslant 2\cdot\left(f(q) + t - f(p)\right).\end{aligned}$$

Therefore

$$\ell^2(t) - \ell^2(0) \leqslant 2\cdot\left(f(q)-f(p)\right)\cdot t + t^2.$$

It proves the needed inequality in case $s = 0$. Combining it with the first distance estimate (6.6), we get the result in case $s \leqslant t$. The case $s \geqslant t$ follows by switching the roles of s and t. $\square$

Proof of 7.3. Consider two Busemann functions, bus_+ and bus_-, asociated with half-lines $\gamma : [0,\infty) \to \mathcal{L}$ and $\gamma : (-\infty, 0] \to \mathcal{L}$ respectively; that is,

$$\operatorname{bus}_\pm(x) := \lim_{t\to\infty} |\gamma(\pm t) - x| - t.$$

According to 7.2, both functions $\operatorname{bus}_\pm$ are concave.

Fix $x \in \mathcal{L}$. Since γ is a line, we have $\operatorname{bus}_+(x) + \operatorname{bus}_-(x) \geqslant 0$. On the other hand, by 5.1, $f(t) = \operatorname{dist}_x^2\circ\gamma(t)$ is 2-concave. In particular, $f(t) \leqslant t^2 + at + b$ for some constants $a, b \in \mathbb{R}$. Passing to the limit as $t \to \pm\infty$, we have $\operatorname{bus}_+(x) + \operatorname{bus}_-(x) \leqslant 0$. Hence

$$\operatorname{bus}_+(x) + \operatorname{bus}_-(x) = 0$$

for any $x \in \mathcal{L}$. In particular, the functions $\operatorname{bus}_\pm$ are affine; that is, they are convex and concave at the same time.

Note that for any x,

$$\begin{aligned}|\nabla_x \operatorname{bus}_\pm| &= \sup\{\, \boldsymbol{d}_x \operatorname{bus}_\pm(\xi) \;:\; \xi\in\Sigma_x \,\} = \\ &= \sup\{\, -\boldsymbol{d}_x \operatorname{bus}_\mp(\xi) \;:\; \xi\in\Sigma_x \,\} \equiv \\ &\equiv 1.\end{aligned}$$

Observe that α is a $\operatorname{bus}_\pm$-gradient curve if and only if α is a geodesic such that $(\operatorname{bus}_\pm\circ\alpha)^+ = 1$. Indeed, if α is a geodesic, then $(\operatorname{bus}_\pm\circ\alpha)^+ \leqslant 1$ and the equality holds only if $\nabla_\alpha \operatorname{bus}_\pm = \alpha^+$. Now suppose $\nabla_\alpha \operatorname{bus}_\pm = \alpha^+$. Then $|\alpha^+| \leqslant 1$ and $(\operatorname{bus}_\pm\circ\alpha)^+ = 1$; therefore

$$\begin{aligned}|t_0 - t_1| &\geqslant |\alpha(t_0) - \alpha(t_1)| \geqslant \\ &\geqslant |\operatorname{bus}_\pm\circ\alpha(t_0) - \operatorname{bus}_\pm\circ\alpha(t_1) = \\ &= |t_0 - t_1|.\end{aligned}$$

It follows that for any $t>0$, the $\mathrm{bus}_\pm$-gradient flows commute; that is,

$$\mathrm{Flow}^t_{\mathrm{bus}_+}\circ\mathrm{Flow}^t_{\mathrm{bus}_-}=\mathrm{id}_{\mathcal{L}}.$$

Setting

$$\mathrm{Flow}^t=\begin{cases}\mathrm{Flow}^t_{\mathrm{bus}_+} & \text{if } t\geqslant 0\\ \mathrm{Flow}^t_{\mathrm{bus}_-} & \text{if } t\leqslant 0\end{cases}$$

defines an $\mathbb{R}$-action on $\mathcal{L}$.

Consider the level set $\mathcal{L}'=\mathrm{bus}_+^{-1}(0)=\mathrm{bus}_-^{-1}(0)$; it is a closed convex subset of $\mathcal{L}$, and therefore forms an Alexandrov space. Consider the map $h\colon\mathcal{L}'\times\mathbb{R}\to\mathcal{L}$ defined by $h\colon(x,t)\mapsto\mathrm{Flow}^t(x)$. Note that h is onto. Applying Lemma 7.5 for $\mathrm{Flow}^t_{\mathrm{bus}_+}$ and $\mathrm{Flow}^t_{\mathrm{bus}_-}$ shows that h is short and non-contracting at the same time; that is, h is an isometry. □

C Polar vectors

Here we give a corollary of 5.10. It will be used to prove basic properties of the tangent space.

7.6. Anti-sum lemma. *Let $\mathcal{L}$ be a complete geodesic* CBB *space and $p\in\mathcal{L}$.*

Given two vectors $u,v\in\mathrm{T}_p$, there is a unique vector $w\in\mathrm{T}_p$ such that

$$\langle u,x\rangle+\langle v,x\rangle+\langle w,x\rangle\geqslant 0$$

for any $x\in\mathrm{T}_p$, and

$$\langle u,w\rangle+\langle v,w\rangle+\langle w,w\rangle=0.$$

7.7. Exercise. *Suppose $u,v,w\in\mathrm{T}_p$ are as in 7.6. Show that*

$$|w|^2\leqslant|u|^2+|v|^2+2\cdot\langle u,v\rangle.$$

If T_p were geodesic, then the lemma would follow from the existence of the gradient, applied to the function $\mathrm{T}_p\to\mathbb{R}$ defined by $x\mapsto-(\langle u,x\rangle+\langle v,x\rangle)$ which is concave. However, the tangent space T_p might fail to be geodesic; see Halbeisen's example [5].

Applying the above lemma for $u=v$, we have the following statement.

7.8. Existence of polar vector. *Let $\mathcal{L}$ be a complete geodesic* CBB *space and $p\in\mathcal{L}$. Given a vector $u\in\mathrm{T}_p$, there is a unique vector*

$u^* \in \mathrm{T}_p$ such that $\langle u^*, u^*\rangle + \langle u, u^*\rangle = 0$ and u^* is polar to u; that is, $\langle u^*, x\rangle + \langle u, x\rangle \geqslant 0$ for any $x \in \mathrm{T}_p$.

In particular, for any vector $u \in \mathrm{T}_p$ there is a polar vector $u^* \in \mathrm{T}_p$ such that $|u^*| \leqslant |u|$.

7.9. Example. *Let $\mathcal{L}$ be the upper half plane in $\mathbb{E}^2$; that is, $\mathcal{L} = \{(x, y) \in \mathbb{E}^2 \mid y \geqslant 0\}$. It is a complete geodesic* CBB(0) *space. For $p = 0$, the tangent space T_p can be canonically identified with $\mathcal{L}$. If $y > 0$, then $u = (x, y) \in \mathrm{T}_p$ has many polar vectors; it includes $u^* = (-x, 0)$ which is provided by 7.8, but the vector $w = (-x, y)$ is polar as well.*

In this case, w is the only polar vector with the same magnitude. If the dimension is finite, then Milka's lemma guarantees the existence of such a polar vector.

Proof of 7.6. By 5.9, we can choose two sequences of points a_n, b_n such that

$$\begin{aligned} \boldsymbol{d}_p \mathrm{dist}_{a_n}(w) &= -\langle \uparrow_{[pa_n]}, w\rangle \\ \boldsymbol{d}_p \mathrm{dist}_{b_n}(w) &= -\langle \uparrow_{[pb_n]}, w\rangle \end{aligned}$$

for any $w \in \mathrm{T}_p$ and $\uparrow_{[pa_n]} \to u/|u|$, $\uparrow_{[pb_n]} \to v/|v|$ as $n \to \infty$

Consider a sequence of functions

$$f_n = |u|\cdot \mathrm{dist}_{a_n} + |v|\cdot \mathrm{dist}_{b_n}.$$

Note that

$$(\boldsymbol{d}_p f_n)(x) = -|u|\cdot\langle \uparrow_{[pa_n]}, x\rangle - |v|\cdot\langle \uparrow_{[pb_n]}, x\rangle.$$

Thus we have the following uniform convergence for $x \in \Sigma_p$:

$$(\boldsymbol{d}_p f_n)(x) \to -\langle u, x\rangle - \langle v, x\rangle$$

as $n \to \infty$, According to 5.10, the sequence $\nabla_p f_n$ converges. Let

$$w = \lim_{n\to\infty} \nabla_p f_n.$$

By the definition of gradient,

$$\begin{aligned} \langle w, w\rangle &= \lim_{n\to\infty} \langle \nabla_p f_n, \nabla_p f_n\rangle = \\ &= \lim_{n\to\infty} (\boldsymbol{d}_p f_n)(\nabla_p f_n) = \\ &= -\langle u, w\rangle - \langle v, w\rangle, \end{aligned} \qquad \begin{aligned} \langle w, x\rangle &= \lim_{n\to\infty} \langle \nabla_p f_n, x\rangle \geqslant \\ &\geqslant \lim_{n\to\infty} (\boldsymbol{d}_p f_n)(x) = \\ &= -\langle u, x\rangle - \langle v, x\rangle. \end{aligned}$$

□

D Linear subspace of tangent space

7.10. Definition. *Let $\mathcal{L}$ be a complete geodesic* $\mathrm{CBB}(\kappa)$ *space, $p \in \mathcal{L}$ and $u, v \in \mathrm{T}_p$. We say that vectors u and v are o p p o s i t e to each other, (briefly, $u + v = 0$) if $|u| = |v| = 0$ or $\measuredangle(u, v) = \pi$ and $|u| = |v|$.*

The subcone

$$\mathrm{Lin}_p = \{\, v \in \mathrm{T}_p \;:\; \exists\; w \in \mathrm{T}_p \quad \textit{such that} \quad w + v = 0 \,\}$$

will be called the l i n e a r s u b c o n e of T_p.

7.11. Proposition. *Let $\mathcal{L}$ be a complete geodesic* CBB *space and $p \in \mathcal{L}$. Given two vectors $u, v \in \mathrm{T}_p$, the following statements are equivalent:*

(a) $u + v = 0$;
(b) $\langle u, x\rangle + \langle v, x\rangle = 0$ for any $x \in \mathrm{T}_p$;
(c) $\langle u, \xi\rangle + \langle v, \xi\rangle = 0$ for any $\xi \in \Sigma_p$.

Proof. The equivalence *(b)*$\Leftrightarrow$*(c)* is trivial.

The condition $u + v = 0$ is equivalent to

$$\langle u, u\rangle = -\langle u, v\rangle = \langle v, v\rangle;$$

thus *(b)*$\Rightarrow$*(a)*.

Recall that T_p is $\mathrm{CBB}(0)$. Note that the hinges $[0\,{}^{u}_{x}]$ and $[0\,{}^{v}_{x}]$ are adjacent. By 3.16, $\measuredangle[0\,{}^{u}_{x}] + \measuredangle[0\,{}^{v}_{x}] = 0$; hence *(a)*$\Rightarrow$*(b)*. $\square$

7.12. Exercise. *Let $\mathcal{L}$ be a complete geodesic* CBB *space and $p \in \mathcal{L}$. Then for any three vectors $u, v, w \in \mathrm{T}_p$, if $u + v = 0$ and $u + w = 0$ then $v = w$.*

Let $u \in \mathrm{Lin}_p$; that is, $u + v = 0$ for some $v \in \mathrm{T}_p$. Given $s < 0$, let

$$s{\cdot}u := (-s){\cdot}v.$$

So we can multiply any vector in Lin_p by any real number (positive and negative). By 7.12, this multiplication is uniquely defined; by 7.11; we have identity

$$\langle -v, x\rangle = -\langle v, x\rangle;$$

later we will see that it extends to a linear structure on Lin_p.

7.13. Exercise. *Suppose $u, v, w \in \mathrm{T}_p$ are as in 7.6. Show that*

$$\langle u, x\rangle + \langle v, x\rangle + \langle w, x\rangle = 0$$

for any $x \in \mathrm{Lin}_p$.

7.14. Exercise. *Let* $\mathcal{L}$ *be a complete geodesic* CBB *space,* $p \in \mathcal{L}$ *and* $u \in \mathrm{T}_p$. *Suppose* $u^* \in \mathrm{T}_p$ *is provided by 7.8; that is,*

$$\langle u^*, u^*\rangle + \langle u, u^*\rangle = 0 \quad \textit{and} \quad \langle u^*, x\rangle + \langle u, x\rangle \geqslant 0$$

for any $x \in \mathrm{T}_p$. *Show that*

$$u = -u^* \quad \iff \quad |u| = |u^*|.$$

7.15. Theorem. *Let* p *be a point in a complete geodesic* CBB(κ). *Then* Lin_p *is isometric to a Hilbert space.*

Proof of 7.15. Note that Lin_p is a closed subset of T_p; in particular, it is complete.

If any two vectors in Lin_p can be connected by a geodesic in Lin_p, then the statement follows from the splitting theorem (7.3). By Menger's lemma (7.18), it is sufficient to show that for any two vectors $x, y \in \mathrm{Lin}_p$ there is a midpoint $w \in \mathrm{Lin}_p$.

Choose $w \in \mathrm{T}_p$ to be the anti-sum of $u = -\tfrac{1}{2}\cdot x$ and $v = -\tfrac{1}{2}\cdot y$; see 7.6. By 7.7 and 7.13,

$$\begin{aligned}
|w|^2 &\leqslant \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle,\\
\langle w, x\rangle &= \tfrac{1}{2}\cdot|x|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle,\\
\langle w, y\rangle &= \tfrac{1}{2}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle,
\end{aligned}$$

It follows that

$$\begin{aligned}
|x - w|^2 &= |x|^2 + |w|^2 - 2\cdot\langle w, x\rangle \leqslant\\
&\leqslant \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 - \tfrac{1}{2}\cdot\langle x, y\rangle =\\
&= \tfrac{1}{4}\cdot|x - y|^2.
\end{aligned}$$

That is, $|x-w| \leqslant \tfrac{1}{2}\cdot|x-y|$, and similarly $|y-w| \leqslant \tfrac{1}{2}\cdot|x-y|$. Therefore w is a midpoint of x and y. In addition we get equality

$$|w|^2 = \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle.$$

It remains to show that $w \in \mathrm{Lin}_p$. Let w^* be the polar vector provided by 7.8. Note that

$$|w^*| \leqslant |w|, \quad \langle w^*, x\rangle + \langle w, x\rangle = 0, \quad \langle w^*, y\rangle + \langle w, y\rangle = 0.$$

The same calculation as above shows that w^* is a midpoint of $-x$ and $-y$ and

$$|w^*|^2 = \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle = |w|^2.$$

By 7.14, $w = -w^*$; hence $w \in \mathrm{Lin}_p$. $\square$

7.16. Exercise. *Let p be a point in a complete geodesic* $\mathrm{CBB}(\kappa)$ *space $\mathcal{L}$ and $f = \mathrm{dist}_p$. Denote by S the subset of points $x \in \mathcal{L}$ such that $|\nabla_x f| = 1$.*

(a) Show that S is a dense G-delta set.

(b) Show that

$$\nabla_x f + {\uparrow}_{[xp]} = 0$$

for any $x \in S$; in particular, ${\uparrow}_{[xp]} \in \mathrm{Lin}_x$.

(c) Show that if $|\nabla_x f| = 1$, then $\boldsymbol{d}_x f(w) = \langle \nabla_x f, w\rangle$ for any $w \in \mathrm{T}_x$.

Note that 7.16*b* implies the following.

7.17. Corollary. *Given a countable set of points X in a complete geodesic* $\mathrm{CBB}(\kappa)$ *space $\mathcal{L}$ there is a G-delta dense set $S \subset \mathcal{L}$ such that ${\uparrow}_{[sx]} \in \mathrm{Lin}_s$ for any $s \in S$ and $x \in X$.*

E Menger's lemma

7.18. Lemma. *Let $\mathcal{X}$ be a complete metric space. Assume that for any pair of points $x, y \in \mathcal{X}$, there is a midpoint z. Then $\mathcal{X}$ is a geodesic space.*

This lemma is due to Karl Menger [30, Section 6].

Proof. Choose $x, y \in \mathcal{X}$; set $\alpha(0) = x$, and $\alpha(1) = y$.

Let $\alpha(\tfrac{1}{2})$ be a midpoint between $\alpha(0)$ and $\alpha(1)$. Further, let $\alpha(\tfrac{1}{4})$ and $\alpha(\tfrac{3}{4})$ be midpoints between the pairs $(\alpha(0), \alpha(\tfrac{1}{2}))$ and $(\alpha(\tfrac{1}{2}), \alpha(1))$ respectively. Applying the above procedure recursively, on the n-th step we define $\alpha(\tfrac{k}{2^n})$, for every odd integer k such that $0 < \tfrac{k}{2^n} < 1$, as a midpoint of the already defined $\alpha(\tfrac{k-1}{2^n})$ and $\alpha(\tfrac{k+1}{2^n})$.

This way we define $\alpha(t)$ for all dyadic rationals t in $[0, 1]$. Moreover, α has Lipschitz constant $\tfrac{1}{|x-y|}$. Since $\mathcal{X}$ is complete, the map α can be extended continuously to $[0, 1]$. Moreover,

❶ $$\mathrm{length}\,\alpha \leqslant |x - y|.$$

$\square$

F Comments

The splitting theorem has an interesting history that starts with Stefan Cohn-Vossen [23]. Our proof is based on the idea of Jeff Cheeger and Detlef Gromoll [22].

Corollary 7.17 is the key to the following result: *all reasonable definitions of dimension give the same result on complete geodesic* CBB *spaces.* We might come back to it after studying the opposite curvature bound.

Lecture 8

Gluing and billiards

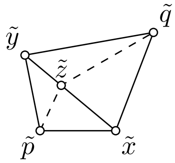
This lecture is nearly a copy of [3, Chapter 2]; here we define upper curvature bound in the sense of Alexandrov, prove Reshetnyak's gluing theorem, and apply it to a problem in billiards.

A Curvature bounded above

Given a quadruple of points p, q, x, y in a metric space $\mathcal{X}$, consider two model triangles $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)_{\mathbb{E}^2}$ and $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\triangle}(qxy)_{\mathbb{E}^2}$ with common side $[\tilde{x}\tilde{y}]$.

If the inequality

$$|p - q|_{\mathcal{X}} \leqslant |\tilde{p} - \tilde{z}|_{\mathbb{E}^2} + |\tilde{z} - \tilde{q}|_{\mathbb{E}^2}$$



holds for any point $\tilde{z} \in [\tilde{x}\tilde{y}]$, then we say that the quadruple p, q, x, y satisfies CAT(0) *comparison*.

If we do the same for spherical model triangles $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)_{\mathbb{S}^2}$ and $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\triangle}(qxy)_{\mathbb{S}^2}$, then we arrive at the definition of CAT(1) comparison. One of the spherical model triangles might undefined; it happens if

$$|p - x| + |p - y| + |x - y| \geqslant 2{\cdot}\pi \quad \text{or} \quad |q - x| + |q - y| + |x - y| \geqslant 2{\cdot}\pi.$$

In this case, it is assumed that CAT(1) comparison automatically holds for this quadruple.

We can do the same for the *model plane* of curvature κ; that is, a sphere if $\kappa > 0$, Euclidean plane if $\kappa = 0$ and Lobachevsky plane if $\kappa < 0$. In this case, we arrive at the definition of CAT(κ) comparison. However, we will mostly consider CAT(0) comparison and occasionally

CAT(1) comparison; so, if you see CAT(κ), then it is safe assuming that κ is 0 or 1.

If all quadruples in a metric space $\mathcal{X}$ satisfy CAT(κ) comparison, then we say that the space $\mathcal{X}$ is CAT(κ) (we use CAT(κ) as an adjective).

Here CAT is an acronym for Cartan, Alexandrov, and Toponogov, but usually pronounced as "cat" in the sense of "miauw". The term was coined by Mikhael Gromov in 1987. Originally, Alexandrov used $\mathfrak{R}_\kappa$ domain; this term is still in use.

8.1. Exercise. *Show that a metric space* $\mathcal{U}$ *is* CAT(0) *if and only if for any quadruple of points* p, q, x, y *in* $\mathcal{U}$ *there is a quadruple* $\tilde{p}, \tilde{q}, \tilde{x}, \tilde{y}$ *in* $\mathbb{E}^2$ *such that*

$$\begin{aligned} |\tilde{p}-\tilde{q}| &\geqslant |p-q|, & |\tilde{x}-\tilde{y}| &\geqslant |x-y|,\\ |\tilde{p}-\tilde{x}| &\leqslant |p-x|, & |\tilde{p}-\tilde{y}| &\leqslant |p-y|,\\ |\tilde{q}-\tilde{x}| &\leqslant |q-x|, & |\tilde{q}-\tilde{y}| &\leqslant |q-y|. \end{aligned}$$

B Geodesics

The CAT comparison can be applied to any metric space, but it is usually applied to geodesic spaces (or complete length spaces). To simplify the presentation we will assume in addition that the space is proper. The latter means that any closed ball is compact; equivalently, the distance function from one (and therefore any) point is proper.

8.2. Proposition. *Let* $\mathcal{U}$ *be a complete geodesic* CAT(0) *space. Then any two points in* $\mathcal{U}$ *are joined by a unique geodesic.*

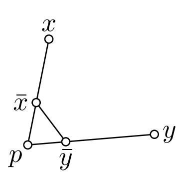
Proof. Suppose there are two geodesics between x and y. Then we can choose two points $p \neq q$ on these geodesics such that $|x-p| = |x-q|$ and therefore $|y-p| = |y-q|$.

Observe that the model triangles $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$ and $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\triangle}(qxy)$ are degenerate and moreover $\tilde{p} = \tilde{q}$. Applying CAT(0) comparison with $\tilde{z} = \tilde{p} = \tilde{q}$, we get that $|p-q| = 0$, a contradiction. $\square$

8.3. Exercise. *Given* $[p\,{}^x_y]$ *in a* CAT(0) *space* $\mathcal{U}$, *consider the function*

$$f\colon (|p-\bar{x}|, |p-\bar{y}|) \mapsto \tilde{\measuredangle}(p\,{}^{\bar{x}}_{\bar{y}}),$$

where $\bar{x} \in\,]px]$ *and* $\bar{y} \in\,]py]$. *Show that* f *is nondecreasing in each argument.*



Conclude that any hinge in a CAT(0) *space has defined angle.*

8.4. Exercise. *Fix a point p in a be a complete geodesic* CAT(0) *space $\mathcal{U}$. Given a point $x\in\mathcal{U}$, denote by $\gamma_x\colon[0,1]\to\mathcal{U}$ a (necessarily unique) geodesic path from p to x.*

Show that the family of maps $h_t\colon\mathcal{U}\to\mathcal{U}$ defined by

$$h_t(x)=\gamma_x(t)$$

is a homotopy; it is called geodesic homotopy. Conclude that $\mathcal{U}$ is contractible.

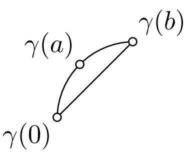
The geodesic homotopy introduced in the previous exercise should help to solve the next one.

8.5. Exercise. *Let $\mathcal{U}$ be a complete geodesic* CAT(0) *space. Assume $\mathcal{U}$ is a topological manifold. Show that any geodesic in $\mathcal{U}$ can be extended as a two-side infinite geodesic.*

A curve $\gamma\colon\mathbb{I}\to\mathcal{U}$ is called a local geodesic if for any $t\in\mathbb{I}$ there is a neighborhood U of t in $\mathbb{I}$ such that the restriction $\gamma|_U$ is a geodesic.

8.6. Proposition. *Suppose $\mathcal{U}$ is a proper geodesic* CAT(0) *space. Then any local geodesic in $\mathcal{U}$ is a geodesic.*

Analogously, if $\mathcal{U}$ is a proper geodesic CAT(1) *space, then any local geodesic in $\mathcal{U}$ which is shorter than π is a geodesic.*



Proof. Suppose $\gamma\colon[0,\ell]\to\mathcal{U}$ is a local geodesic that is not a geodesic. Choose a to be the maximal value such that γ is a geodesic on $[0,a]$. Further choose $b>a$ so that γ is a geodesic on $[a,b]$.

Since the triangle $[\gamma(0)\gamma(a)\gamma(b)]$ is thin and $|\gamma(0)-\gamma(b)|<b$ we have

$$|\gamma(a-\varepsilon)-\gamma(a+\varepsilon)|<2\cdot\varepsilon$$

for all small $\varepsilon>0$. That is, γ is not length-minimizing on the interval $[a-\varepsilon,a+\varepsilon]$ for any $\varepsilon>0$, a contradiction.

The spherical case is done in the same way. □

C Thin triangles

Recall that a triangle $[xyz]$ in a space $\mathcal{U}$ is a triple of minimizing geodesics $[xy]$, $[yz]$ and $[zx]$. Consider the model triangle $[\tilde x\tilde y\tilde z]=\tilde\triangle(xyz)_{\mathbb{E}^2}$ in the Euclidean plane. The natural map $[\tilde x\tilde y\tilde z]\to[xyz]$ sends a point $\tilde p\in[\tilde x\tilde y]\cup[\tilde y\tilde z]\cup[\tilde z\tilde x]$ to the corresponding point

$p \in [xy] \cup [yz] \cup [zx]$; that is, if $\tilde p$ lies on $[\tilde y\tilde z]$, then $p \in [yz]$ and $|\tilde y - \tilde p| = |y - p|$ (and therefore $|\tilde z - \tilde p| = |z - p|$).

The same way, the natural map can be defined for the spherical model triangle $\tilde\triangle(xyz)_{\mathbb{S}^2}$.

8.7. Definition. *A triangle $[xyz]$ in the metric space $\mathcal{U}$ is called* thin *if the natural map $\tilde\triangle(xyz)_{\mathbb{E}^2} \to [xyz]$ is distance nonincreasing.*

Analogously, a triangle $[xyz]$ is called spherically thin *if the natural map from the spherical model triangle $\tilde\triangle(xyz)_{\mathbb{S}^2}$ to $[xyz]$ is distance nonincreasing.*

8.8. Proposition. *A geodesic space is* $\mathrm{CAT}(0)$ *(*$\mathrm{CAT}(1)$*) if and only if all its triangles are thin (respectively, all its triangles of perimeter $< 2{\cdot}\pi$ are spherically thin).*

Proof; if part. Apply the triangle inequality and thinness of triangles $[pxy]$ and $[qxy]$, where p, q, x, and y are as in the definition of the $\mathrm{CAT}(\kappa)$ comparison.

Only-if part. Applying $\mathrm{CAT}(0)$ comparison to a quadruple p, q, x, y with $q \in [xy]$ shows that any triangle satisfies point-side comparison, that is, the distance from a vertex to a point on the opposite side is no greater than the corresponding distance in the Euclidean model triangle.

Now consider a triangle $[xyz]$ and let $p \in [xy]$ and $q \in [xz]$. Let $\tilde p$, $\tilde q$ be the corresponding points on the sides of the model triangle $\tilde\triangle(xyz)_{\mathbb{E}^2}$. Applying 8.3, we get that

$$\tilde\measuredangle(x\,{}^{y}_{z})_{\mathbb{E}^2} \geqslant \tilde\measuredangle(x\,{}^{p}_{q})_{\mathbb{E}^2}.$$

Therefore $|\tilde p - \tilde q|_{\mathbb{E}^2} \geqslant |p - q|$.

The $\mathrm{CAT}(1)$ argument is the same. □

8.9. Exercise. *Suppose $\gamma_1, \gamma_2\colon [0,1] \to \mathcal{U}$ be two geodesic paths in a complete geodesic* $\mathrm{CAT}(0)$ *space $\mathcal{U}$. Show that*

$$t \mapsto |\gamma_1(t) - \gamma_2(t)|_{\mathcal{U}}$$

is a convex function.

8.10. Exercise. *Let A be a convex closed set in a proper geodesic* $\mathrm{CAT}(0)$ *space $\mathcal{U}$; that is, if $x, y \in A$, then $[xy] \subset A$. Show that for any $r > 0$ the closed r-neighborhood of A is convex; that is, the set*

$$A_r = \{\, x \in \mathcal{U} \;:\; \mathrm{dist}_A x \leqslant r \,\}$$

is convex.

8.11. Exercise. *Let* $\mathcal{U}$ *be a proper geodesic* CAT(0) *space and* $K \subset \mathcal{U}$ *be a closed convex set. Show that:*

(a) *For each point* $p \in \mathcal{U}$ *there is a unique point* $p^* \in K$ *that minimizes the distance* $|p - p^*|$.

(b) *The closest-point projection* $p \mapsto p^*$ *defined by (a) is short.*

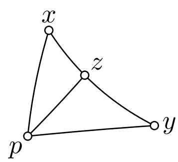
Recall that a set A in a metric space $\mathcal{U}$ is called locally convex if for any point $p \in A$ there is an open neighborhood $\mathcal{U} \ni p$ such that any geodesic in $\mathcal{U}$ with ends in A lies in A.

8.12. Exercise. *Let* $\mathcal{U}$ *be a proper geodesic* CAT(0) *space. Show that any closed, connected, locally convex set in* $\mathcal{U}$ *is convex.*

D Inheritance lemma

8.13. Inheritance lemma. *Assume that a triangle* $[pxy]$ *in a metric space is decomposed into two triangles* $[pxz]$ *and* $[pyz]$*; that is,* $[pxz]$ *and* $[pyz]$ *have a common side* $[pz]$*, and the sides* $[xz]$ *and* $[zy]$ *together form the side* $[xy]$ *of* $[pxy]$.

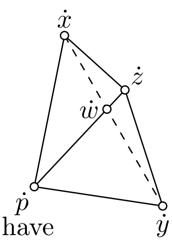
If both triangles $[pxz]$ *and* $[pyz]$ *are thin, then the triangle* $[pxy]$ *is also thin.*



Analogously, if $[pxy]$ *has perimeter* $< 2\cdot\pi$ *and both triangles* $[pxz]$ *and* $[pyz]$ *are spherically thin, then triangle* $[pxy]$ *is spherically thin.*

Proof. Construct the model triangles $[\dot p\dot x\dot z] = \tilde\triangle(pxz)_{\mathbb{E}^2}$ and $[\dot p\dot y\dot z] = \tilde\triangle(pyz)_{\mathbb{E}^2}$ so that $\dot x$ and $\dot y$ lie on opposite sides of $[\dot p\dot z]$.

Let us show that



$$❶ \qquad \tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) \geqslant \pi.$$

If not, then for some point $\dot w \in [\dot p\dot z]$, we have

$$|\dot x - \dot w| + |\dot w - \dot y| < |\dot x - \dot z| + |\dot z - \dot y| = |x - y|.$$

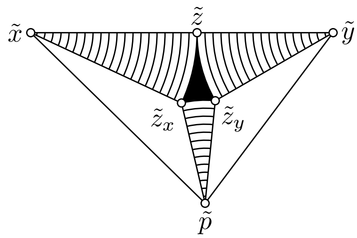
Let $w \in [pz]$ correspond to $\dot w$; that is, $|z - w| = |\dot z - \dot w|$. Since $[pxz]$ and $[pyz]$ are thin, we have

$$|x - w| + |w - y| < |x - y|,$$

contradicting the triangle inequality.

Denote by $\dot D$ the union of two solid triangles $[\dot p\dot x\dot z]$ and $[\dot p\dot y\dot z]$. Further, denote by $\tilde D$ the solid triangle $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{E}^2}$. By ❶, there is a short map $F\colon \tilde D \to \dot D$ that sends

$$\tilde p \mapsto \dot p, \qquad \tilde x \mapsto \dot x, \qquad \tilde z \mapsto \dot z, \qquad \tilde y \mapsto \dot y.$$



Indeed, by Alexandrov's lemma (3.9), there are nonoverlapping triangles

$$[\tilde{p}\tilde{x}\tilde{z}_x] \overset{iso}{=\!=} [\dot{p}\dot{x}\dot{z}]$$

and

$$[\tilde{p}\tilde{y}\tilde{z}_y] \overset{iso}{=\!=} [\dot{p}\dot{y}\dot{z}]$$

inside the triangle $[\tilde{p}\tilde{x}\tilde{y}]$.

Connect the points in each pair $(\tilde{z}, \tilde{z}_x)$, $(\tilde{z}_x, \tilde{z}_y)$ and $(\tilde{z}_y, \tilde{z})$ with arcs of circles centered at $\tilde{y}$, $\tilde{p}$, and $\tilde{x}$ respectively. Define F as follows:

- ◇ Map $\mathrm{Conv}[\tilde{p}\tilde{x}\tilde{z}_x]$ isometrically onto $\mathrm{Conv}[\dot{p}\dot{x}\dot{z}]$; similarly map $\mathrm{Conv}[\tilde{p}\tilde{y}\tilde{z}_y]$ onto $\mathrm{Conv}[\dot{p}\dot{y}\dot{z}]$.
- ◇ If x is in one of the three circular sectors, say at distance r from its center, set $F(x)$ to be the point on the corresponding segment $[pz]$, $[xz]$ or $[yz]$ whose distance from the left-hand endpoint of the segment is r.
- ◇ Finally, if x lies in the remaining curvilinear triangle $\tilde{z}\tilde{z}_x\tilde{z}_y$, set $F(x) = z$.

By construction, F satisfies the conditions.

By assumption, the natural maps $[\dot{p}\dot{x}\dot{z}] \to [pxz]$ and $[\dot{p}\dot{y}\dot{z}] \to [pyz]$ are short. By composition, the natural map from $[\tilde{p}\tilde{x}\tilde{y}]$ to $[pyz]$ is short, as claimed.

The spherical case is done along the same lines. □

E Reshetnyak's gluing

Suppose $\mathcal{U}^1$ and $\mathcal{U}^2$ are proper geodesic spaces with isometric closed convex sets $A^i \subset \mathcal{U}^i$ and let $\iota\colon A^1 \to A^2$ be an isometry. Consider the space $\mathcal{W}$ of all equivalence classes in $\mathcal{U}^1 \sqcup \mathcal{U}^2$ with the equivalence relation given by $a \sim \iota(a)$ for any $a \in A^1$.

It is straightforward to see that $\mathcal{W}$ is a proper geodesic space when equipped with the following metric

$$\begin{aligned}|x - y|_{\mathcal{W}} &:= |x - y|_{\mathcal{U}^i}\\ &\qquad \text{if} \quad x, y \in \mathcal{U}^i, \quad \text{and}\\ |x - y|_{\mathcal{W}} &:= \min\left\{\, |x - a|_{\mathcal{U}^1} + |y - \iota(a)|_{\mathcal{U}^2} \;:\; a \in A^1 \,\right\}\\ &\qquad \text{if} \quad x \in \mathcal{U}^1 \quad \text{and} \quad y \in \mathcal{U}^2.\end{aligned}$$

Abusing notation, we denote by x and y the points in $\mathcal{U}^1 \sqcup \mathcal{U}^2$ and their equivalence classes in $\mathcal{U}^1 \sqcup \mathcal{U}^2/{\sim}$.

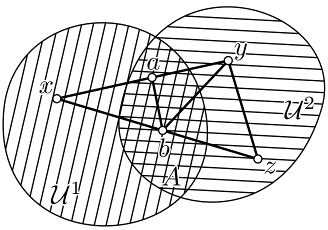
The space $\mathcal{W}$ is called the gluing of $\mathcal{U}^1$ and $\mathcal{U}^2$ along ι. If one applies this construction to two copies of one space $\mathcal{U}$ with a set $A \subset \mathcal{U}$ and the identity map $\iota\colon A \to A$, then the obtained space is called the double of $\mathcal{U}$ along A.

We can (and will) identify $\mathcal{U}^i$ with its image in $\mathcal{W}$; this way both subsets $A^i \subset \mathcal{U}^i$ will be identified and denoted further by A. Note that $A = \mathcal{U}^1 \cap \mathcal{U}^2 \subset \mathcal{W}$, therefore A is also a convex set in $\mathcal{W}$.

8.14. Reshetnyak gluing. *Suppose $\mathcal{U}^1$ and $\mathcal{U}^2$ are proper geodesic* CAT(0) *spaces with isometric closed convex sets $A^i \subset \mathcal{U}^i$, and $\iota\colon A^1 \to A^2$ is an isometry. Then the gluing of $\mathcal{U}^1$ and $\mathcal{U}^2$ along ι is a* CAT(0) *proper geodesic space.*

Proof. By construction of the gluing space, the statement can be reformulated in the following way:

8.15. Reformulation of 8.14. *Let $\mathcal{W}$ be a proper geodesic space which has two closed convex sets $\mathcal{U}^1, \mathcal{U}^2 \subset \mathcal{W}$ such that $\mathcal{U}^1 \cup \mathcal{U}^2 = \mathcal{W}$ and $\mathcal{U}^1$, $\mathcal{U}^2$ are* CAT(0)*. Then $\mathcal{W}$ is* CAT(0)*.*



It suffices to show that any triangle $[xyz]$ in $\mathcal{W}$ is thin. This is obviously true if all three points x, y, z lie in one of $\mathcal{U}^i$. Thus, without loss of generality, we may assume that $x \in \mathcal{U}^1$ and $y, z \in \mathcal{U}^2$.

Choose points $a, b \in A = \mathcal{U}^1 \cap \mathcal{U}^2$ that lie respectively on the sides $[xy]$, $[xz]$. Note that

- ⋄ the triangle $[xab]$ lies in $\mathcal{U}^1$,
- ⋄ both triangles $[yab]$ and $[ybz]$ lie in $\mathcal{U}^2$.

In particular, each triangle $[xab]$, $[yab]$ and $[ybz]$ is thin.

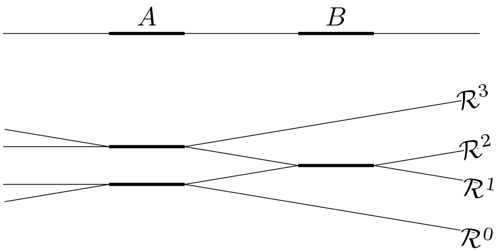
Applying the inheritance lemma (8.13) twice, we get that $[xyb]$ and consequently $[xyz]$ is thin. □

8.16. Exercise. *Suppose $\mathcal{U}$ is a geodesic space and $A \subset \mathcal{U}$ is a closed subset. Assume that the doubling of $\mathcal{U}$ in A is* CAT(0)*. Show that A is a convex set of $\mathcal{U}$.*

F Puff pastry

In this section, we introduce the notion of Reshetnyak puff pastry. This construction will be used in the next section to prove the collision theorem (8.27).

Let $\boldsymbol{A} = (A^1, \dots, A^N)$ be an array of convex closed sets in the Euclidean space $\mathbb{E}^m$. Consider an array of $N+1$ copies of $\mathbb{E}^m$. Assume that the space $\mathcal{R}$ is obtained by gluing successive pairs of spaces along $A^1, \dots, A^N$ respectively.



Puff pastry for (A, B, A).

The resulting space $\mathcal{R}$ will be called the Reshetnyak puff pastry for array $\boldsymbol{A}$. The copies of $\mathbb{E}^m$ in the puff pastry $\mathcal{R}$ will be called levels; they will be denoted by $\mathcal{R}^0, \dots, \mathcal{R}^N$. The point in the k-th level $\mathcal{R}^k$ that corresponds to $x \in \mathbb{E}^m$ will be denoted by x^k.

Given $x \in \mathbb{E}^m$, any point $x^k \in \mathcal{R}$ is called a lifting of x. The map $x \mapsto x^k$ defines an isometry $\mathbb{E}^m \to \mathcal{R}^k$; in particular, we can talk about liftings of subsets in $\mathbb{E}^m$.

Note that:

- ◇ The intersection $A^1 \cap \dots \cap A^N$ admits a unique lifting in $\mathcal{R}$.
- ◇ Moreover, $x^i = x^j$ for some $i < j$ if and only if
$$x \in A^{i+1} \cap \dots \cap A^j.$$
- ◇ The restriction $\mathcal{R}^k \to \mathbb{E}^m$ of the natural projection $x^k \mapsto x$ is an isometry.

8.17. Observation. *Any Reshetnyak puff pastry is a proper geodesic* CAT(0) *space.*

Proof. Apply Reshetnyak gluing theorem (8.14) recursively for the convex sets in the array. □

8.18. Proposition. *Assume* $(A^1, \dots, A^N)$ *and* $(\check{A}^1, \dots, \check{A}^N)$ *are two arrays of convex closed sets in* $\mathbb{E}^m$ *such that* $A^k \subset \check{A}^k$ *for each* k*. Let* $\mathcal{R}$ *and* $\check{\mathcal{R}}$ *be the corresponding Reshetnyak puff pastries. Then the map* $\mathcal{R} \to \check{\mathcal{R}}$ *defined by* $x^k \mapsto \check{x}^k$ *is short.*

Moreover, if

❶ $$|x^i - y^j|_{\mathcal{R}} = |\check{x}^i - \check{y}^j|_{\check{\mathcal{R}}}$$

for some $x, y \in \mathbb{E}^m$ and $i, j \in \{0, \dots, n\}$, then the unique geodesic $[\check{x}^i \check{y}^j]_{\check{\mathcal{R}}}$ is the image of the unique geodesic $[x^i y^j]_{\mathcal{R}}$ under the map $x^i \mapsto \check{x}^i$.

Proof. The first statement in the proposition follows from the construction of Reshetnyak puff pastries.

By Observation 8.17, $\mathcal{R}$ and $\check{\mathcal{R}}$ are proper geodesic CAT(0) spaces; hence $[x^i y^j]_{\mathcal{R}}$ and $[\check{x}^i \check{y}^j]_{\check{\mathcal{R}}}$ are unique. By ❶, since the map $\mathcal{R} \to \check{\mathcal{R}}$ is short, the image of $[x^i y^j]_{\mathcal{R}}$ is a geodesic of $\check{\mathcal{R}}$ joining $\check{x}^i$ to $\check{y}^j$. Hence the second statement follows. □

8.19. Definition. *Consider a Reshetnyak puff pastry $\mathcal{R}$ with the levels $\mathcal{R}^0, \dots, \mathcal{R}^N$. We say that $\mathcal{R}$ is end-to-end convex if $\mathcal{R}^0 \cup \mathcal{R}^N$, the union of its lower and upper levels, forms a convex set in $\mathcal{R}$; that is, if $x, y \in \mathcal{R}^0 \cup \mathcal{R}^N$, then $[xy]_{\mathcal{R}} \subset \mathcal{R}^0 \cup \mathcal{R}^N$.*

Note that if $\mathcal{R}$ is the Reshetnyak puff pastry for an array of convex sets $\boldsymbol{A} = (A^1, \dots, A^N)$, then $\mathcal{R}$ is end-to-end convex if and only if the union of the lower and the upper levels $\mathcal{R}^0 \cup \mathcal{R}^N$ is isometric to the double of $\mathbb{E}^m$ along the nonempty intersection $A^1 \cap \dots \cap A^N$.

8.20. Observation. *Let $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ be arrays of convex bodies in $\mathbb{E}^m$. Assume that array $\boldsymbol{A}$ is obtained by inserting in $\check{\boldsymbol{A}}$ several copies of the bodies which were already listed in $\check{\boldsymbol{A}}$.*

For example, if $\check{\boldsymbol{A}} = (A, C, B, C, A)$, by placing B in the second place and A in the fourth place, we obtain $\boldsymbol{A} = (A, B, C, A, B, C, A)$.

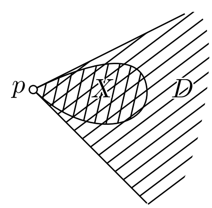
Denote by $\check{\mathcal{R}}$ and $\mathcal{R}$ the Reshetnyak puff pastries for $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ respectively.

If $\check{\mathcal{R}}$ is end-to-end convex, then so is $\mathcal{R}$.

Proof. Without loss of generality, we may assume that $\boldsymbol{A}$ is obtained by inserting one element in $\check{\boldsymbol{A}}$, say at the place number k.

Note that $\check{\mathcal{R}}$ is isometric to the puff pastry for $\boldsymbol{A}$ with A^k replaced by $\mathbb{E}^m$. It remains to apply Proposition 8.18. □

Let X be a convex set in a Euclidean space. By a dihedral angle, we understand an intersection of two half-spaces; the intersection of corresponding hyperplanes is called the edge of the angle. We say that a dihedral angle D supports X at a point $p \in X$ if D contains X and the edge of D contains p.



8.21. Lemma. *Let A and B be two convex sets in $\mathbb{E}^m$. Assume that any dihedral angle supporting $A \cap B$ has angle measure at least α. Then* *the Reshetnyak puff pastry for the array*

$$(\underbrace{A, B, A, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \ times}).$$

is end-to-end convex.

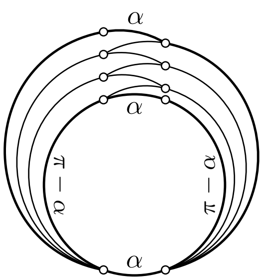
The proof of the lemma is based on a partial case, which we formulate as a sublemma.

8.22. Sublemma. *Let $\ddot{A}$ and $\ddot{B}$ be two half-planes in $\mathbb{E}^2$, where $\ddot{A}\cap\ddot{B}$ is an angle with measure α. Then the Reshetnyak puff pastry for the array*

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \ times})$$

is end-to-end convex.

Proof. Note that the puff pastry $\ddot{\mathcal{R}}$ is isometric to the cone over the space glued from the unit circles as shown on the diagram.

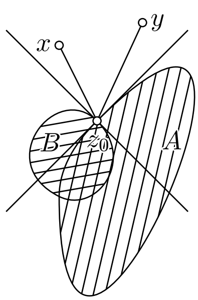


All the short arcs on the diagram have length α; the long arcs have length $\pi - \alpha$, so making a circuit along any path will take $2{\cdot}\pi$.

The end-to-end convexity of $\ddot{\mathcal{R}}$ is equivalent to the fact that any geodesic shorter than π with the ends on the inner and the outer circles lies completely in the union of these two circles.

The latter holds if the zigzag line in the picture has length at least π. This line is formed by $\lceil\frac{\pi}{\alpha}\rceil$ arcs with length α each. Hence the sublemma. □

In the proof of 8.21, we will use the following exercise in convex geometry:



8.23. Exercise. *Let A and B be two closed convex sets in $\mathbb{E}^m$ and $A \cap B \neq \varnothing$. Given two points $x, y \in \mathbb{E}^m$ let $f(z) = |x - z| + |y - z|$.*

Let $z_0 \in A\cap B$ be a point of minimum of $f|_{A\cap B}$.

Show that there are half-spaces $\dot{A}$ and $\dot{B}$ such that $\dot{A} \supset A$ and $\dot{B} \supset B$ and z_0 is also a point of minimum of the restriction $f|_{\dot{A}\cap\dot{B}}$.

Proof of 8.21. Fix arbitrary $x, y \in \mathbb{E}^m$. Choose a point $z \in A \cap B$ for which the sum

$$|x - z| + |y - z|$$

is minimal. To show the end-to-end convexity of $\mathcal{R}$, it is sufficient to prove the following:

❷ *The geodesic* $[x^0y^N]_{\mathcal{R}}$ *contains* $z^0 = z^N \in \mathcal{R}$.

Without loss of generality, we may assume that $z \in \partial A \cap \partial B$. Indeed, since the puff pastry for the 1-array (B) is end-to-end convex, Proposition 8.18 together with 8.20 imply ❷ in case z lies in the interior of A. The same way we can treat the case when z lies in the interior of B.

Note that $\mathbb{E}^m$ admits an isometric splitting $\mathbb{E}^{m-2} \times \mathbb{E}^2$ such that

$$\dot{A} = \mathbb{E}^{m-2} \times \ddot{A}$$
$$\dot{B} = \mathbb{E}^{m-2} \times \ddot{B}$$

where $\ddot{A}$ and $\ddot{B}$ are half-planes in $\mathbb{E}^2$.

Using Exercise 8.23, let us replace each A by $\dot{A}$ and each B by $\dot{B}$ in the array, to get the array

$$(\underbrace{\dot{A}, \dot{B}, \dot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \text{ times}}).$$

The corresponding puff pastry $\dot{\mathcal{R}}$ splits as a product of $\mathbb{E}^{m-2}$ and a puff pastry, call it $\ddot{\mathcal{R}}$, glued from the copies of the plane $\mathbb{E}^2$ for the array

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \text{ times}}).$$

Note that the dihedral angle $\dot{A} \cap \dot{B}$ is at least α. Therefore the angle measure of $\ddot{A} \cap \ddot{B}$ is also at least α. According to Sublemma 8.22 and Observation 8.20, $\ddot{\mathcal{R}}$ is end-to-end convex.

Since $\dot{\mathcal{R}} \overset{iso}{=\!=} \mathbb{E}^{m-2} \times \ddot{\mathcal{R}}$, the puff pastry $\dot{\mathcal{R}}$ is also end-to-end convex.

It follows that the geodesic $[\dot{x}^0\dot{y}^N]_{\dot{\mathcal{R}}}$ contains $\dot{z}^0 = \dot{z}^N \in \dot{\mathcal{R}}$. By Proposition 8.18, the image of $[\dot{x}^0\dot{y}^N]_{\dot{\mathcal{R}}}$ under the map $\dot{x}^k \mapsto x^k$ is the geodesic $[x^0y^N]_{\mathcal{R}}$. Hence ❷ and the lemma follow. □

G Wide corners

We say that a closed convex set $A \subset \mathbb{E}^m$ has ε-wide corners for given $\varepsilon > 0$ if together with each point p, the set A contains a small right circular cone with the tip at p and aperture ε; that is, ε is the maximum angle between two generating lines of the cone.

For example, a plane polygon has ε-wide corners if all its interior angles are at least ε.

We will consider finite collections of closed convex sets $A^1, \dots, A^n \subset \subset \mathbb{E}^m$ such that for any subset $F \subset \{1, \dots, n\}$, the intersection $\bigcap_{i \in F} A^i$ has ε-wide corners. In this case, we may say briefly *all intersections of A^i have ε-wide corners.*

8.24. Exercise. *Assume $A^1, \dots, A^n \subset \mathbb{E}^m$ are compact, convex sets with a common interior point. Show that all intersections of A^i have ε-wide corners for some $\varepsilon > 0$.*

8.25. Exercise. *Assume $A^1, \dots, A^n \subset \mathbb{E}^m$ are convex sets with nonempty interiors that have a common center of symmetry. Show that all intersections of A^i have ε-wide corners for some $\varepsilon > 0$.*

The proof of the following proposition is based on 8.21; this lemma is essentially the case $n = 2$ in the proposition.

8.26. Proposition. *Given $\varepsilon > 0$ and a positive integer n, there is an array of integers $\boldsymbol{j}_\varepsilon(n) = (j_1, \dots, j_N)$ such that:*

- *(a) For each k we have $1 \leqslant j_k \leqslant n$, and each number $1, \dots, n$ appears in $\boldsymbol{j}_\varepsilon$ at least once.*
- *(b) If $A^1, \dots, A^n$ is a collection of closed convex sets in $\mathbb{E}^m$ with a common point and all their intersections have ε-wide corners, then the puff pastry for the array $(A^{j_1}, \dots, A^{j_N})$ is end-to-end convex.*

Moreover we can assume that $N \leqslant (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^n$.

Proof. The array $\boldsymbol{j}_\varepsilon(n) = (j_1, \dots, j_N)$ is constructed recursively. For $n = 1$, we can take $\boldsymbol{j}_\varepsilon(1) = (1)$.

Assume that $\boldsymbol{j}_\varepsilon(n)$ is constructed. Let us replace each occurrence of n in $\boldsymbol{j}_\varepsilon(n)$ by the alternating string

$$\underbrace{n, n+1, n, \dots}_{\lceil \frac{\pi}{\varepsilon} \rceil + 1 \text{ times}}.$$

Denote the obtained array by $\boldsymbol{j}_\varepsilon(n+1)$.

By Lemma 8.21, end-to-end convexity of the puff pastry for $\boldsymbol{j}_\varepsilon(n+ +1)$ follows from end-to-end convexity of the puff pastry for the array where each string

$$\underbrace{A^n, A^{n+1}, A^n, \dots}_{\lceil \frac{\pi}{\varepsilon} \rceil + 1 \text{ times}}$$

is replaced by $Q = A^n \cap A^{n+1}$. End-to-end convexity of the latter follows by the assumption on $\boldsymbol{j}_\varepsilon(n)$, since all the intersections of $A^1, \dots, A^{n-1}, Q$ have ε-wide corners.

The upper bound on N follows directly from the construction. □

H Billiards

Let $A^1, A^2, \dots A^n$ be a finite collection of closed convex sets in $\mathbb{E}^m$. Assume that for each i the boundary ∂A^i is a smooth hypersurface.

Consider the billiard table formed by the closure of the complement

$$T = \overline{\mathbb{E}^m \setminus \bigcup_i A^i}.$$

The sets A^i will be called walls of the table and the billiards described above will be called billiards with convex walls.

A billiard trajectory on the table is a unit-speed broken line γ that follows the standard law of billiards at the breakpoints on ∂A^i — in particular, the angle of reflection is equal to the angle of incidence. The breakpoints of the trajectory will be called collisions. We assume the trajectory meets only one wall at a time.

Recall that the definition of sets with ε-wide corners is given in 8G.

8.27. Collision theorem. *Assume $T \subset \mathbb{E}^m$ is a billiard table with n convex walls. Assume that the walls of T have a common interior point and all their intersections have ε-wide corners. Then the number of collisions of any trajectory in T is bounded by a number N which depends only on n and ε.*

As we will see from the proof, the value N can be found explicitly; $N = (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^{n^2}$ will do.

8.28. Corollary. *Consider n homogeneous hard balls moving freely and colliding elastically in $\mathbb{R}^3$. Every ball moves along a straight line with constant speed until two balls collide, and then the new velocities of the two balls are determined by the laws of classical mechanics. We assume that only two balls can collide at the same time.*

Then the total number of collisions cannot exceed some number N that depends on the radii and masses of the balls. If the balls are identical, then N depends only on n.

8.29. Exercise. *Show that in the case of identical balls in the one-dimensional space (in $\mathbb{R}$) the total number of collisions cannot exceed $N = \frac{n\cdot(n-1)}{2}$.*

The proof below admits a straightforward generalization to all dimensions.

Proof of 8.28 modulo 8.27. Denote by $a_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ the center of the i-th ball. Consider the corresponding point in $\mathbb{R}^{3\cdot N}$

$$\begin{aligned}\boldsymbol{a} &= (a_1, a_2, \dots, a_n) = \\ &= (x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_n, y_n, z_n).\end{aligned}$$

The i-th and j-th ball intersect if

$$|a_i - a_j| \leqslant R_i + R_j,$$

where R_i denotes the radius of the i-th ball. These inequalities define $\frac{n\cdot(n-1)}{2}$ cylinders

$$C_{i,j} = \left\{ (a_1, a_2, \ldots, a_n) \in \mathbb{R}^{3\cdot n} \ : \ |a_i - a_j| \leqslant R_i + R_j \right\}.$$

The closure of the complement

$$T = \overline{\mathbb{R}^{3\cdot n} \setminus \bigcup_{i<j} C_{i,j}}$$

is the configuration space of our system. Its points correspond to valid positions of the system of balls.

The evolution of the system of balls is described by the motion of the point $\boldsymbol{a} \in \mathbb{R}^{3\cdot n}$. It moves along a straight line at a constant speed until it hits one of the cylinders $C_{i,j}$; this event corresponds to a collision in the system of balls.

Consider the norm of $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^{3\cdot n}$ defined by

$$\|\boldsymbol{a}\| = \sqrt{M_1\cdot|a_1|^2 + \cdots + M_n\cdot|a_n|^2},$$

where $|a_i| = \sqrt{x_i^2 + y_i^2 + z_i^2}$ and M_i denotes the mass of the i-th ball. In the metric defined by $\|{*}\|$, the collisions follow the standard law of billiards.

By construction, the number of collisions of hard balls that we need to estimate is the same as the number of collisions of the corresponding billiard trajectory on the table with $C_{i,j}$ as the walls.

Note that each cylinder $C_{i,j}$ is a convex set; it has smooth boundary, and it is centrally symmetric around the origin. By 8.25, all the intersections of the walls have ε-wide corners for some $\varepsilon > 0$ that depend on the radiuses R_i and the masses M_i. It remains to apply the collision theorem (8.27). $\square$

Now we present the proof of the collision theorem (8.27) based on the results developed in the previous section.

Proof of 8.27. Let us apply induction on n.

Base: $n = 1$. The number of collisions cannot exceed 1. Indeed, by the convexity of A^1, if the trajectory is reflected once in ∂A^1, then it cannot return to A^1.

Step. Assume γ is a trajectory that meets the walls in the order $A^{i_1}, \ldots, A^{i_N}$ for a large integer N.

Consider the array

$$\boldsymbol{A}_\gamma = (A^{i_1}, \dots, A^{i_N}).$$

The induction hypothesis implies:

❶ *There is a positive integer M such that any M consecutive elements of $\boldsymbol{A}_\gamma$ contain each A^i at least once.*

Let $\mathcal{R}_\gamma$ be the Reshetnyak puff pastry for $\boldsymbol{A}_\gamma$.

Consider the lift of γ to $\mathcal{R}_\gamma$, defined by $\bar\gamma(t) = \gamma^k(t) \in \mathcal{R}_\gamma$ for any moment of time t between the k-th and $(k+1)$-th collisions. Since γ follows the standard law of billiards at breakpoints, the lift $\bar\gamma$ is locally a geodesic in $\mathcal{R}_\gamma$. By 8.17, the puff pastry $\mathcal{R}_\gamma$ is a proper geodesic CAT(0) space. Therefore $\bar\gamma$ is a geodesic.

Since γ does not meet $A^1 \cap \dots \cap A^n$, the lift $\bar\gamma$ does not lie in $\mathcal{R}^0_\gamma \cup \mathcal{R}^N_\gamma$. In particular, $\mathcal{R}_\gamma$ is not end-to-end convex.

Let

$$\boldsymbol{B} = (A^{j_1}, \dots, A^{j_K})$$

be the array provided by Proposition 8.26; so $\boldsymbol{B}$ contains each A^i at least once and the puff pastry $\mathcal{R}_{\boldsymbol{B}}$ for $\boldsymbol{B}$ is end-to-end convex. If N is sufficiently large, namely $N \geqslant K\cdot M$, then ❶ implies that $\boldsymbol{A}_\gamma$ can be obtained by inserting a finite number of A^i's in $\boldsymbol{B}$.

By 8.20, $\mathcal{R}_\gamma$ is end-to-end convex — a contradiction. □

I Comments

The gluing theorem (8.14) was proved by Yuri Reshetnyak [36]. It can be extended to all geodesic CAT(0) spaces. It also admits a natural generalization to geodesic CAT(κ) spaces; see the book of Martin Bridson and André Haefliger [13] and our book [4] for details.

The collision theorem (8.27) was proved by Dmitri Burago, Serge Ferleger and Alexey Kononenko [16]. Its corollary (8.28) answers a question posed by Yakov Sinai [26]. Puff pastry is used to bound topological entropy of the billiard flow and to approximate the shortest billiard path that touches given lines in a given order; see the papers of Dmitri Burago with Serge Ferleger and Alexey Kononenko [17], and with Dimitri Grigoriev and Anatol Slissenko [18]. The lecture of Dmitri Burago [14] gives a short survey on the subject.

Note that the interior points of the walls play a key role in the proof despite that the trajectories never go inside the walls. In a similar fashion, puff pastry was used by Stephanie Alexander and Richard Bishop [2] to find the upper curvature bound for warped products.

Joel Hass [27] constructed an example of a Riemannian metric on the 3-ball with negative curvature and concave boundary. This example might decrease your appetite for generalizing the collision theorem — while locally such a 3-ball looks as good as the billiards table in the theorem, the number of collisions is obviously infinite.

It was shown by Dmitri Burago and Sergei Ivanov [20] that the number of collisions that may occur between n identical balls in $\mathbb{R}^3$ grows at least exponentially in n; the two-dimensional remains open.

Appendix A

Solutions

5.9; Let $\theta = \measuredangle[q\,{}^p_x]$. We need to prove two inequalities

$$|y-\gamma(t)| \leqslant |y-q| - t\cdot\cos\theta + o(t),$$
$$|y-\gamma(t)| \geqslant |y-q| - t\cdot\cos\theta + o(t).$$

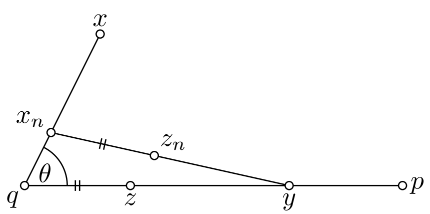
The first one follows from 3.8; it remains to prove the second one. Arguing by contradiction, assume there is a sequence $t_n \to 0+$ such that for some fixed $\varepsilon > 0$

$$\tilde\measuredangle(q\,{}^{x_n}_{y}) < \theta - \varepsilon$$

for any $x_n = \gamma(t_n)$. Since $|q-x_n| \to 0$, we get

❶ $$\measuredangle[x_n\,{}^q_y] > \pi - \theta + \tfrac{\varepsilon}{2}, \quad \text{and therefore} \quad \measuredangle[x_n\,{}^x_y] < \theta - \tfrac{\varepsilon}{2}$$

for all large n.[1]



Without loss of generality, we may assume that

❷ $$\tilde\measuredangle(q\,{}^x_z) > \theta - \tfrac{\varepsilon}{10}$$

[1] If the space is compact, then a subsequence of $[x_n y]$ should converge to a geodesic from q to y that makes angle $< \theta$ to $[qx]$. It follows that there is a pair of distinct geodesics from q to y which contradicts 3.15. With the use of ultralimits (see for example [32]), this argument works in the general case.

for some point $z\in\,]qy]$. (If it does not hold, shift x to q.)

For each n, choose $z_n\in\,]x_ny]$ such that $|x_n-z_n|=|q-z|$. Applying the triangle inequality, ❶, ❷, and the CBB(0) comparison, we get

$$|z-z_n|\geqslant|a-z|-|a-z_n|>\delta_0$$

for some $\delta_0>0$ and all large n. Hence

$$\measuredangle[y\,{}^{x_n}_{q}]=\measuredangle[y\,{}^{z_n}_{z}]\geqslant\tilde\measuredangle(y\,{}^{z_n}_{z})>\delta_1,\quad\text{and therefore}\quad\measuredangle[y\,{}^{x_n}_{p}]<\pi-\delta_1$$

for some $\delta_1>0$ and all large n. By CBB(0) comparison,

$$|q-x_n|<|p-q|-\delta_2$$

for some $\delta_2>0$ and all large n. Since $|q-x_n|\to0$, we arrive at a contradiction with the triangle inequality.

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