Homework assignments

Scan to pdf and upload to CANVAS.

Due 2023-08-25: 1.8, 1.11, 1.13, 1.14, 1.17.

Due 2023-09-01: 2.2, 2.4, 2.5, 2.7, 2.14.

Due 2023-09-11: 2.11, 3.6, 3.13, 4.5, 4.8.

Due 2023-09-18: 5.3, 5.9, 5.10, 5.16, 5.29.

Due 2023-09-25: 6.11, 6.15, 7.9, 8.13, 8.17.

Due 2023-10-02: 6.13, 8.19, 8.21, 8.28, 9.18.

Due 2023-10-09: 5.11, 7.10, ~~9.27,~~ 10.6, 10.7.

Due 2023-10-23: 5.4, 5.12, 10.3, 10.16, 10.19.

Due 2023-10-30: ~~2.9,~~ 2.12, 2.13, 8.2, 11.3 11.7.

Due 2023-11-06: 11.13, 12.3, 12.7, 12.8, 12.9.

Due 2023-11-13: 13.1, 13.4, 13.9, 13.10, 13.11.

Due 2023-11-27: 14.7, 14.8, 14.9, A.6, A.7+A.2.

MT1

- 2♠ Geodesics, hinges, triangles, model triangles, model angles, angle measure, triangle inequality for angles 2.3
- 3♠ Alexandrov's lemma (2.6), CBB and CAT comparisons, existance of angles in CBB and CAT spaces (2.9 and 5.2). Uniqueness of geodesics in CAT spaces 5.1.
- 4♠ CBB: triangle comparisons 2.16. Function comparison 2.18.
- 5♠ Surface of convex polyhedra 3.7 and convergence of convex surfaces (3.9).
- 6♠ Cauchy theorem: Arm lemma 4.4 and Local lemma 4.2.
- 2♣ Cauchy theorem: Global lemma 4.3.
- 3♣ Thin triangles, the inheritance lemma 5.13, Reshetnyak's gluing theorem 5.14.
- 4♣ Majorization theorem 6.4.
- 5♣ Patchwork along a geodesic 7.2.
- 6♣ Patchwork globalization theorem 7.7.
- 2♢ Hadamard–Cartan globalization theorem 7.6.
- 3♢ Product of CAT(0) spaces 8.1.
- 4♢ Flag complexes, no-triangle condition 8.14, Gromov's theorem 8.16, cubical complexes.
- 5♢ Two-convexity, sets with smooth boundary (9.10).
- 6♢ Simply-connected subsets of the plane (9.11).
- 2♡ Barycentric simplex 10.1.
- 3♡ Convexity of up-set 10.5.
- 4♡ Bi-Hölder embedding 10.8.
- 5♡ Dimension theorem 10.14.

Contents

Lecture 1

Definitions

The first synthetic description of curvature is due to Abraham Wald [99] published in 1936; it was his student work, written under the supervision of Karl Menger. This publication was not noticed for about 50 years [18]. In 1941, similar definitions were rediscovered by Alexandr Alexandrov [12].

A Notations

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

We will denote by $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$ the n-dimensional sphere (with angle metric), Euclidean space, and Lobachevsky space respectively. More generally, $\mathbb{M}^n(\kappa)$ will denote the m o d e l n-s p a c e of curvature κ; that is,

- ◇ if $\kappa > 0$, then $\mathbb{M}^n(\kappa)$ is the n-sphere of radius $\frac{1}{\sqrt{\kappa}}$, so $\mathbb{S}^n = \mathbb{M}^n(1)$
- ◇ $\mathbb{M}^n(0) = \mathbb{E}^n$,
- ◇ if $\kappa < 0$, then $\mathbb{M}^n(\kappa)$ is the Lobachevsky n-space $\mathbb{H}^n$ rescaled by factor $\frac{1}{\sqrt{-\kappa}}$; in particular $\mathbb{M}^n(-1) = \mathbb{H}^n$.

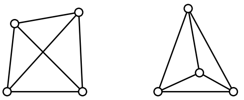
B Wald's approach

Wald noticed that a *typical* quadruple x_1, x_2, x_3, x_4 of points in a metric space admits model configurations in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \in \mathbb{M}^3(\kappa)$ with

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{M}^3(\kappa)} = |x_i - x_j|_{\mathcal{X}}$$

for κ in a closed interval, say

$$[\kappa_{\min}(x_1,x_2,x_3,x_4),\kappa_{\max}(x_1,x_2,x_3,x_4)]\subset\mathbb{R}.$$



In $\mathbb{M}^3(\kappa_{\min})$ and $\mathbb{M}^3(\kappa_{\max})$, the points $\tilde x_1,\tilde x_2,\tilde x_3,\tilde x_4$ form degenerate tetrahedrons shown on the diagram (for $\kappa_{\min}$ it is a convex quadrangle and for $\kappa_{\max}$ — a triangle with a point inside). In the interior of the interval, the tetrahedron is nondegenerate.

Moreover, one can use $[-\infty,\infty)$ instead of $\mathbb{R}$ and let

$$\kappa_{\min}(x_1,x_2,x_3,x_4)=-\infty$$

if there is *almost* model quadruple in $\mathbb{M}^3(\kappa)$ for $\kappa\to-\infty$; that is, for any $\varepsilon>0$ there is a quadruple $\tilde x_1,\tilde x_2,\tilde x_3,\tilde x_4\in\mathbb{M}^3(\kappa)$ such that $\kappa\leqslant-\frac{1}{\varepsilon}$, and

$$|\tilde x_i-\tilde x_j|_{\mathbb{M}^3(\kappa)}\lesseqgtr|x_i-x_j|_{\mathcal{X}}\pm\varepsilon$$

for all i and j. In this case the interval

$$[\kappa_{\min}(x_1,x_2,x_3,x_4),\kappa_{\max}(x_1,x_2,x_3,x_4)]\subset[-\infty,\infty)$$

is defined for *any* quadruple.

We will not use these statements further in the sequel, so we omit the proofs. The just wanted to describe the first step in the theory.

1.1. Exercise. *Let x_1,x_2,x_3,x_4 be a quadruple in a metric space such that $\kappa_{\min}(x_1,x_2,x_3,x_4)=-\infty$. Show that two maximal numbers from the following three are equal to each other.*

$$\begin{aligned}a&=|x_1-x_2|+|x_3-x_4|,\\ b&=|x_1-x_3|+|x_2-x_4|,\\ c&=|x_1-x_4|+|x_2-x_3|.\end{aligned}$$

1.2. Exercise. *Suppose that x_1,x_2,x_3,x_4 in a metric space such that*

$$\begin{aligned}|x_1-x_2|&=|x_1-x_3|=|x_1-x_4|=1,\\ |x_2-x_3|&=|x_3-x_4|=|x_4-x_1|=2.\end{aligned}$$

Show that

$$\kappa_{\min}(x_1,x_2,x_3,x_4)=\kappa_{\max}(x_1,x_2,x_3,x_4)=-\infty.$$

1.3. Exercise. *Let x_1, x_2, x_3, x_4 be a quadruple in $\mathbb{E}^2$. Suppose that triangle $[x_1 x_2 x_3]$ is degerate, but $[x_2 x_3 x_4]$ is not. Show that*

$$\kappa_{\min}(x_1, x_2, x_3, x_4) = \kappa_{\max}(x_1, x_2, x_3, x_4) = 0.$$

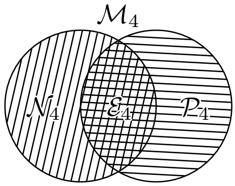
1.4. Wald-style definition. *Let $\kappa \in \mathbb{R}$. A metric space $\mathcal{X}$ has curvature $\geqslant \kappa$ (or $\leqslant \kappa$) if for any quadruple $x_1, x_2, x_3, x_4 \in \mathcal{X}$ we have $\kappa_{\max}(x_1, x_2, x_3, x_4) \geqslant \kappa$ (or $\kappa_{\min}(x_1, x_2, x_3, x_4) \leqslant \kappa$ respectively).*

C Substance

Consider the space $\mathcal{M}_4$ of all isometry classes of 4-point metric spaces. Each element in $\mathcal{M}_4$ can be described by 6 numbers — the distances between all 6 pairs of its points, say $\ell_{i,j}$ for $1 \leqslant i < j \leqslant 4$ modulo permutations of the index set $(1, 2, 3, 4)$. These 6 numbers are subject to 12 triangle inequalities; that is,

$$\ell_{i,j} + \ell_{j,k} \geqslant \ell_{i,k}$$

holds for all i, j and k, where we assume that $\ell_{j,i} = \ell_{i,j}$, and $\ell_{i,i} = 0$.



The space $\mathcal{M}_4$ comes with topology. It can be defined as a quotient topology of the cone in $\mathbb{R}^6$ by permutations of the 4 points of the space.

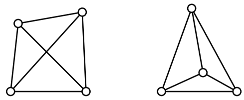
Consider the subset $\mathcal{E}_4 \subset \mathcal{M}_4$ of all isometry classes of 4-point metric spaces that admit isometric embeddings into Euclidean space.

1.5. Claim. *The complement $\mathcal{M}_4 \setminus \mathcal{E}_4$ has two connected components.*

1.6. Exercise. *Spend 10 minutes trying to prove the claim.*

The definition of Alexandrov spaces is based on the claim above. Let us denote one of the components by $\mathcal{P}_4$ and the other by $\mathcal{N}_4$. Here $\mathcal{P}$ and $\mathcal{N}$ stand for positive and negative curvature because spheres have no quadruples of type $\mathcal{N}_4$ and hyperbolic space has no quadruples of type $\mathcal{P}_4$.

A metric space that has no quadruples of points of type $\mathcal{P}_4$ or $\mathcal{N}_4$ respectively is called an Alexandrov space with non-positive (CAT(0)) or non-negative curvature (CBB(0)).



Let us describe the subdivision into $\mathcal{P}_4$, $\mathcal{E}_4$, and $\mathcal{N}_4$ intuitively. Imagine that you move out of $\mathcal{E}_4$ — your path is a one-parameter family of 4-point metric spaces. The last thing you see in $\mathcal{E}_4$ is one of the two plane configurations shown on the diagram. If you see the right configuration then you move into $\mathcal{N}_4$; if it is the one on the left, then you move into $\mathcal{P}_4$. More degenerate pictures can be avoided; for example, a triangle with a point on a side. From such a configuration one may move in $\mathcal{N}_4$ and $\mathcal{P}_4$ (as well as come back to $\mathcal{E}_4$).

Here is an exercise, solving which would force you to rebuild a considerable part of Alexandrov geometry. It is wise to spend some time thinking about this exercise before proceeding.

1.7. Advanced exercise. *Assume $\mathcal{X}$ is a complete metric space with length metric (see Section 1F), containing only quadruples of type $\mathcal{E}_4$. Show that $\mathcal{X}$ is isometric to a convex set in a Hilbert space.*

In the definition above, one can take $\mathbb{M}^3(\kappa)$ instead of $\mathbb{E}^3$. In this case, one obtains the definition of spaces with curvature bounded above or below by κ (CAT(κ) or CBB(κ)). The parameter κ has three interesting choices -1, 0, and 1; the rest can be obtained from these three applying rescaling.

D Geodesics, triangles, and angles

Geodesics. Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ a real interval. A distance-preserving map $\gamma\colon \mathbb{I}\to\mathcal{X}$ is called a g e o d e s i c[1]; in other words, $\gamma\colon \mathbb{I}\to\mathcal{X}$ is a geodesic if

$$|\gamma(s)-\gamma(t)|_{\mathcal{X}}=|s-t|$$

for any pair $s,t\in\mathbb{I}$.

If $\gamma\colon [a,b]\to\mathcal{X}$ is a geodesic such that $p=\gamma(a)$, $q=\gamma(b)$, then we say that γ is a geodesic from p to q. In this case, the image of γ is denoted by $[pq]$, and, with abuse of notations, we also call it a g e o d e s i c. We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$.

In general, a geodesic from p to q need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we assume that we have chosen such geodesic.

[1]Others call it differently: *shortest path*, *minimizing geodesic*. Also, note that the meaning of the term *geodesic* is different from what is used in Riemannian geometry, altho they are closely related.

A geodesic path is a geodesic with constant-speed parameterization by the unit interval $[0,1]$.

A metric space is called geodesic if any pair of its points can be joined by a geodesic.

Triangles. Given a triple of points p,q,r in a metric space $\mathcal{X}$, a choice of geodesics $([qr],[rp],[pq])$ will be called a triangle; we will use the short notation $[pqr]=[pqr]_{\mathcal{X}}=([qr],[rp],[pq])$.

Given a triple $p,q,r\in\mathcal{X}$ there may be no triangle $[pqr]$ simply because one of the pairs of these points cannot be joined by a geodesic. Also, many different triangles with these vertices may exist, any of which can be denoted by $[pqr]$. If we write $[pqr]$, it means that we have chosen such a triangle.

Model triangles. Given three points p,q,r in a metric space $\mathcal{X}$, let us define its model triangle $[\tilde p\tilde q\tilde r]$ (briefly, $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$) to be a triangle in the Euclidean plane $\mathbb{E}^2$ such that

$$|\tilde p-\tilde q|_{\mathbb{E}^2}=|p-q|_{\mathcal{X}},\qquad |\tilde q-\tilde r|_{\mathbb{E}^2}=|q-r|_{\mathcal{X}},\qquad |\tilde r-\tilde p|_{\mathbb{E}^2}=|r-p|_{\mathcal{X}}.$$

The same way we can define the hyperbolic and the spherical model triangles $\tilde\triangle(pqr)_{\mathbb{H}^2}$, $\tilde\triangle(pqr)_{\mathbb{S}^2}$ in the Lobachevsky plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2$. In the latter case, the model triangle is said to be defined if in addition

$$|p-q|+|q-r|+|r-p|<2\cdot\pi.$$

In this case, the model triangle again exists and is unique up to an isometry of $\mathbb{S}^2$.

Model angles. If $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$ and $|p-q|,|p-r|>0$, the angle measure of $[\tilde p\tilde q\tilde r]$ at $\tilde p$ will be called the model angle of the triple p, q, r and will be denoted by $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{E}^2}$.

The same way we define $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$; in particular, $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{H}^2}$ and $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{S}^2}$. We may use the notation $\tilde\measuredangle(p\,{}^q_r)$ if it is evident which of the model spaces is meant.

1.8. Exercise. *Show that for any triple of point p, q, and r, the function*

$$\kappa\mapsto\tilde\measuredangle(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$$

is nondecreasing in its domain of definition.

Hinges. Let $p,x,y\in\mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair of geodesics $([px],[py])$ will be called a hinge and will be denoted by $[p\,{}^x_y]=([px],[py])$.

E Definitions

In this section we write inequalities that describe the sets $\mathcal{E}_4 \cup \mathcal{P}_4$ and $\mathcal{E}_4 \cup \mathcal{N}_4$ from Section 1C.

Curvature bounded below. Let p, x, y, z be a quadruple of points in a metric space. If the inequality

$$\text{❶} \qquad \tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^y_z)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^z_x)_{\mathbb{E}^2} \leqslant 2\cdot\pi$$

holds, then we say that the quadruple meets CBB(0) comparison.

1.9. Definition. *A metric space $\mathcal{X}$ has nonnegative curvature in the sense of Alexandrov (briefly, $\mathcal{X} \in \mathrm{CBB}(0)$ if* CBB(0) *comparison holds for any quadruple in $\mathcal{X}$ such that each model angle in* ❶ *is defined.*

If instead of $\mathbb{E}^2$, we use $\mathbb{S}^2$ or $\mathbb{H}^2$, then we get the definition of CBB(1) and CBB(−1) comparisons. Note that $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2}$ and $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2}$ are defined if $p \ne x$, $p \ne y$, but for $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2}$ we need in addition, $|p-x| + |p-y| + |x-y| < 2\cdot\pi$.

More generally, one may apply this definition to $\mathbb{M}^2(\kappa)$. This way we define CBB(κ) comparison for any real κ.

1.10. Exercise. *Show that $\mathbb{E}^n$ is* CBB(0).

1.11. Exercise. *Show that a metric space $\mathcal{X}$ is* CBB(0) *if and only if for any quadruple of points $p, x_1, x_2, x_3 \in \mathcal{X}$ there is a quadruple of points $q, y_1, y_2, y_3 \in \mathbb{E}^3$ such that*

$$|p - x_i|_{\mathcal{X}} \geqslant |q - y_i|_{\mathbb{E}^2} \quad \textit{and} \quad |x_i - x_j|_{\mathcal{X}} \leqslant |y_i - y_j|_{\mathbb{E}^2}$$

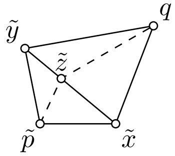
for all i and j.

Curvature bounded above. Given a quadruple of points p, q, x, y in a metric space $\mathcal{X}$, consider two model triangles $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{E}^2}$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)_{\mathbb{E}^2}$ with common side $[\tilde x\tilde y]$.

If the inequality

$$|p - q|_{\mathcal{X}} \leqslant |\tilde p - \tilde z|_{\mathbb{E}^2} + |\tilde z - \tilde q|_{\mathbb{E}^2}$$

holds for any point $\tilde z \in [\tilde x\tilde y]$, then we say that the quadruple p, q, x, y satisfies CAT(0) comparison.



1.12. Definition. *A metric space $\mathcal{X}$ has nonpositive curvature in the sense of Alexandrov (briefly, $\mathcal{X} \in \mathrm{CAT}(0)$) if* CAT(0) *comparison holds for any quadruple in $\mathcal{X}$.*

If we do the same for spherical model triangles $[\tilde p\tilde x\tilde y]=\tilde\triangle(pxy)_{\mathbb{S}^2}$ and $[\tilde q\tilde x\tilde y]=\tilde\triangle(qxy)_{\mathbb{S}^2}$, then we arrive at the definition of CAT(1) comparison. One of the spherical model triangles might undefined; it happens if

$$|p-x|+|p-y|+|x-y|\geqslant 2\cdot\pi\quad\text{or}\quad|q-x|+|q-y|+|x-y|\geqslant 2\cdot\pi.$$

In this case, it is assumed that CAT(1) comparison automatically holds for this quadruple.

We can do the same for $\mathbb{M}^2(\kappa)$. In this case, we arrive at the definition of CAT(κ) comparison. However, we will mostly consider CAT(0) comparison and occasionally CAT(1) comparison; so, if you see CAT(κ), then it is safe to assume that κ is 0 or 1.

Here CAT is an acronym for Cartan, Alexandrov, and Toponogov, but usually pronounced as "cat" in the sense of "miauw". The term was coined by Mikhael Gromov in 1987. Originally, Alexandrov used $\mathfrak{R}_\kappa$ domain; this term is still in use.

1.13. Exercise. *Show that a metric space $\mathcal{U}$ is* CAT(0) *if and only if for any quadruple of points p,q,x,y in $\mathcal{U}$ there is a quadruple $\tilde p,\tilde q,\tilde x,\tilde y$ in $\mathbb{E}^2$ such that*

$$\begin{aligned}
|\tilde p-\tilde q|&\geqslant|p-q|, & |\tilde x-\tilde y|&\geqslant|x-y|,\\
|\tilde p-\tilde x|&\leqslant|p-x|, & |\tilde p-\tilde y|&\leqslant|p-y|,\\
|\tilde q-\tilde x|&\leqslant|q-x|, & |\tilde q-\tilde y|&\leqslant|q-y|.
\end{aligned}$$

1.14. Exercise. *Assume that a quadruple of points in a metric space satisfies* CBB(0) *and* CAT(0) *comparisons for all labelings. Show that it is isometric to a quadruple in $\mathbb{E}^3$.*

The definitions stated in the this section can be applied to any metric space. However, interesting things happen only for the so-called *geodesic* or at least *length spaces*.

F Length and length spaces

Length. A curve is defined as a continuous map from a real interval $\mathbb{I}$ to a metric space. If $\mathbb{I}=[0,1]$, then the curve is called a path.

1.15. Definition. *Let $\mathcal{X}$ be a metric space and $\alpha\colon\mathbb{I}\to\mathcal{X}$ be a curve. We define the length of α as*

$$\operatorname{length}\alpha:=\sup_{t_0\leqslant t_1\leqslant\ldots\leqslant t_n}\sum_i|\alpha(t_i)-\alpha(t_{i-1})|.$$

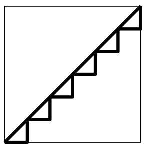
A curve α is called rectifiable *if* $\operatorname{length}\alpha < \infty$.

The following theorem is assumed to be known; see [25, 78].

1.16. Theorem. *Length is a lower semi-continuous with respect to the pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\gamma_n\colon \mathbb{I} \to \mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty\colon \mathbb{I} \to \mathcal{X}$; that is, for any fixed $t \in \mathbb{I}$ we have $\gamma_n(t) \to \gamma_\infty(t)$ as $n \to \infty$. Then

$$\underline{\lim}_{n\to\infty} \operatorname{length}\gamma_n \geqslant \operatorname{length}\gamma_\infty. \qquad ❶$$



Note that the inequality ❶ might be strict. For example, the diagonal γ_∞ of the unit square can be approximated by stairs-like polygonal curves γ_n with sides parallel to the sides of the square (γ_6 is on the picture). In this case

$$\operatorname{length}\gamma_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length}\gamma_n = 2$$

for any n.

Length spaces. Let $\mathcal{X}$ be a metric space. If for any $\varepsilon > 0$ and any pair of points $x, y \in \mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length}\alpha < |x - y| + \varepsilon,$$

then $\mathcal{X}$ is called a length space and the metric on $\mathcal{X}$ is called a length metric.

Evidently, any geodesic space is a length space.

1.17. Exercise. *Show that any compact length space is geodesic.*

Induced length metric. Directly from the definition, it follows that if $\alpha\colon [0,1] \to \mathcal{X}$ is a path from x to y (that is, $\alpha(0) = x$ and $\alpha(1) = y$), then

$$\operatorname{length}\alpha \geqslant |x - y|.$$

Set

$$\|x - y\| = \inf\{\operatorname{length}\alpha\}$$

where the greatest lower bound is taken for all paths from x to y. It is straightforward to check that $(x, y) \mapsto \|x - y\|$ is an ∞-metric; that is, $(x, y) \mapsto \|x - y\|$ is a metric in the extended positive reals $[0, \infty]$. The metric $\|{*} - {*}\|$ is called the induced length metric.

1.18. Exercise. *Let $\mathcal{X}$ be a complete length space. Show that for any compact subset $K \subset \mathcal{X}$ there is a compact path-connected subset $K' \subset \mathcal{X}$ that contains K.*

1.19. Exercise. *Suppose $(\mathcal{X}, |{*}-{*}|)$ is a complete metric space. Show that $(\mathcal{X}, \|{*}-{*}\|)$ is complete.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x-y|_A = \inf_{\alpha}\{\,\operatorname{length}\alpha\,\},$$

where the greatest lower bound is taken for all paths α from x to y in A. In other words, $|{*}-{*}|_A$ denotes the induced length metric on the subspace A. (The notation $|{*}-{*}|_A$ conflicts with the previously defined notation for distance $|x-y|_{\mathcal{X}}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.)

G Embedding theorem

The following theorem is historically the first remarkable result in Alexandrov geometry. The main part of the following theorem is due to Alexandr Alexandrov [9]. The last part is very difficult; it was proved by Aleksei Pogorelov [81].

1.20. Theorem. *A metric space $\mathcal{X}$ is isometric to the surface of a convex body in the Euclidean space if and only if $\mathcal{X}$ is a geodesic* $\operatorname{CBB}(0)$ *space that is homeomorphic to $\mathbb{S}^2$.*

Moreover, $\mathcal{X}$ determines the convex body up to congruence.

The convex body above is a compact convex subset in $\mathbb{E}^3$; we assume that it does not lie in a line but might degenerate to a plane figure, say F. In the latter case, its surface is defined as two copies of F glued along the boundary. For nondegenerate convex body B, its surface is its boundary ∂B equipped with the induced length metric.

The only-if part of the theorem is the simplest; we will give a complete proof of it eventually. The if part will be sketched. We will not touch the last part.

Lecture 2

Angles

A Definition

The angle measure of a hinge $[p\,{}^x_y]$ is defined as the following limit

$$\measuredangle[p\,{}^x_y] = \lim_{\bar{x},\bar{y}\to p} \tilde{\measuredangle}(p\,{}^{\bar{x}}_{\bar{y}}),$$

where $\bar{x} \in\,]px]$ and $\bar{y} \in\,]py]$.

Note that if $\measuredangle[p\,{}^x_y]$ is defined, then

$$0 \leqslant \measuredangle[p\,{}^x_y] \leqslant \pi.$$

2.1. Exercise. *Suppose that in the above definition, one uses spherical or hyperbolic model angles instead of Euclidean. Show that it does not change the value $\measuredangle[p\,{}^x_y]$.*

2.2. Exercise. *Give an example of a hinge $[p\,{}^x_y]$ in a metric space with an undefined angle measure $\measuredangle[p\,{}^x_y]$.*

B Triangle inequality

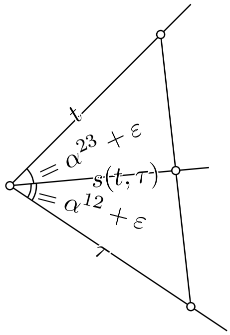
2.3. Proposition. *Let $[px_1]$, $[px_2]$, and $[px_3]$ be three geodesics in a metric space. Suppose all the angle measures $\alpha_{ij} = \measuredangle[p\,{}^{x_i}_{x_j}]$ are defined. Then*

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23}.$$

Proof. Since $\alpha_{13} \leqslant \pi$, we can assume that $\alpha_{12} + \alpha_{23} < \pi$. Denote by γ_i the unit-speed parametrization of $[px_i]$ from p to x_i. Given any

$\varepsilon > 0$, for all sufficiently small $t, \tau, s \in \mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned}|\gamma_1(t) - \gamma_3(\tau)| \leqslant |\gamma_1(t) - \gamma_2(s)| + |\gamma_2(s) - \gamma_3(\tau)| < \\ < \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha_{12} + \varepsilon)} + \\ + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\alpha_{23} + \varepsilon)} \leqslant\end{aligned}$$



Below we define $s(t, \tau)$ so that for $s = s(t, \tau)$, this chain of inequalities can be continued as follows:

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon.$$

Hence the result follows.

To define $s(t, \tau)$, consider three half-lines $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, $\tilde{\gamma}_3$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde{\gamma}_1, \tilde{\gamma}_2) = \alpha_{12} + \varepsilon$, $\measuredangle(\tilde{\gamma}_2, \tilde{\gamma}_3) = \alpha_{23} + \varepsilon$, and $\measuredangle(\tilde{\gamma}_1, \tilde{\gamma}_3) = \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon$. We parametrize each half-line by the distance from the starting point. Given two positive numbers $t, \tau \in \mathbb{R}_{\geqslant 0}$, let $s = s(t, \tau)$ be the number such that $\tilde{\gamma}_2(s) \in [\tilde{\gamma}_1(t)\ \tilde{\gamma}_3(\tau)]$. Clearly, $s \leqslant \max\{t, \tau\}$, so t, τ, s may be taken sufficiently small. $\square$

2.4. Exercise. *Prove that the sum of adjacent angles is at least π.*

More precisely: suppose two hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ are adjacent*; that is, they share side $[pz]$, and the union of two sides $[px]$ and $[py]$ form a geodesic $[xy]$. Show that*

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \geqslant \pi$$

whenever each angle on the left-hand side is defined.

Give an example showing that the inequality can be strict.

2.5. Exercise. *Assume that the angle measure of $[q\,{}^p_x]$ is defined. Let γ be the unit speed parametrization of $[qx]$ from q to x. Show that*

$$|p - \gamma(t)| \leqslant |q - p| - t{\cdot}\cos(\measuredangle[q\,{}^p_x]) + o(t).$$

C Alexandrov's lemma

Recall that $[xy]$ denotes a geodesic from x to y; set

$$]xy] = [xy] \setminus \{x\}, \quad [xy[= [xy] \setminus \{y\}, \quad]xy[= [xy] \setminus \{x, y\}.$$

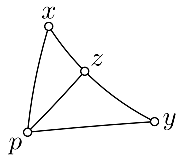
2.6. Lemma. *Let p, x, y, z be distinct points in a metric space such that $z \in\,]xy[$. Then the following expressions for the Euclidean model angles have the same sign:*

(a) $\tilde\measuredangle(x\,{}^{p}_{y}) - \tilde\measuredangle(x\,{}^{p}_{z})$,

(b) $\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) - \pi$.

The same holds for the hyperbolic and spherical model angles, but in the latter case, one has to assume in addition that

$$|p-z| + |p-y| + |x-y| < 2{\cdot}\pi.$$



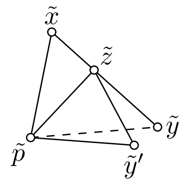
Proof. Consider the model triangle $[\tilde x\tilde p\tilde z] = \tilde\triangle(xpz)$. Take a point $\tilde y$ on the extension of $[\tilde x\tilde z]$ beyond $\tilde z$ so that $|\tilde x - \tilde y| = |x-y|$ (and therefore $|\tilde x - \tilde z| = |x - z|$).

Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:

(i) $\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(x\,{}^{p}_{y})$,

(ii) $|\tilde p - \tilde y| - |p - y|$,

(iii) $\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(z\,{}^{p}_{y})$.



Since

$$\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] = \measuredangle[\tilde x\,{}^{\tilde p}_{\tilde z}] = \tilde\measuredangle(x\,{}^{p}_{z})$$

and

$$\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] = \pi - \measuredangle[\tilde z\,{}^{\tilde x}_{\tilde p}] = \pi - \tilde\measuredangle(z\,{}^{x}_{p}),$$

the statement follows.

The spherical and hyperbolic cases can be proved the same way. $\square$

2.7. Exercise. *Assume p, x, y, z are as in Alexandrov's lemma. Show that*

$$\tilde\measuredangle(p\,{}^{x}_{y}) \geqslant \tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{z}_{y}),$$

with equality if and only if the expressions in (a) and (b) vanish.

D CBB comparison

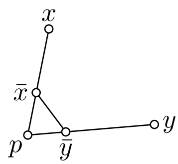
Note that

$$p \in\,]xy[\quad\Longrightarrow\quad \tilde\measuredangle(p\,{}^{x}_{y}) = \pi.$$

Applying it with Alexandrov's lemma and CBB(0) comparison, we get the following claim and its corollary.

2.8. Claim. *If p,x,y,z are points in a* CBB(0) *such that $p\in\,]xy[$, then*
$$\tilde\measuredangle(x\,{}^y_z)\leqslant\tilde\measuredangle(x\,{}^p_z).$$

2.9. Exercise. *Let $[p\,{}^x_y]$ be a hinge in a* CBB(0) *space. Consider the function*



$$f\colon (|p-\bar x|,|p-\bar y|)\mapsto\tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$
where $\bar x\in\,]px]$ and $\bar y\in\,]py]$. Show that f is nonincreasing in each argument.

Note that 2.9 implies the following.

2.10. Claim. *For any hinge $[p\,{}^x_y]$ in a* CBB(0) *space, the angle measure $\measuredangle[p\,{}^x_y]$ is defined, and*
$$\measuredangle[p\,{}^x_y]\geqslant\tilde\measuredangle(p\,{}^x_y).$$

2.11. Exercise. *Let $[p\,{}^x_y]$ be a hinge in a* CBB(0) *space. Suppose $\measuredangle[p\,{}^x_y]=0$; show that $[px]\subset[py]$ or $[py]\subset[px]$.*

2.12. Exercise. *Let $[xy]$ be a geodesic in a* CBB(0) *space. Suppose $z\in\,]xy[$ show that there is a unique geodesic $[xz]$ and $[xz]\subset[xy]$.*

2.13. Exercise. *Let $[p\,{}^x_z]$ and $[p\,{}^y_z]$ be adjacent hinges in a* CBB(0) *space. Show that*
$$\measuredangle[p\,{}^x_z]+\measuredangle[p\,{}^y_z]=\pi.$$

2.14. Exercise. *Let p,x,y in a* CBB(0) *space and $v,w\in\,]xy[$. Show that*
$$\tilde\measuredangle(x\,{}^y_p)=\tilde\measuredangle(x\,{}^v_p)\quad\Longleftrightarrow\quad\tilde\measuredangle(x\,{}^y_p)=\tilde\measuredangle(x\,{}^w_p).$$

E Hinge comparison

Let $[p\,{}^x_y]$ be a hinge in a CBB(0) space. By 2.11, the angle measure $\measuredangle[p\,{}^x_y]$ is defined and
$$\measuredangle[p\,{}^x_y]\geqslant\tilde\measuredangle(p\,{}^x_y).$$

Further, according to 2.13, we have
$$\measuredangle[p\,{}^x_z]+\measuredangle[p\,{}^y_z]=\pi$$

for adjacent hinges $[p\,^{x}_{z}]$ and $[p\,^{y}_{z}]$ in a CBB(0) space.

The following theorem implies that a geodesic space is CBB(0) if the above conditions hold for all its hinges.

2.15. Theorem. *A geodesic space $\mathcal{L}$ is* CBB(0) *if the following conditions hold.*

(a) For any hinge $[x\,^{p}_{y}]$ in $\mathcal{L}$, the angle $\measuredangle[x\,^{p}_{y}]$ is defined and

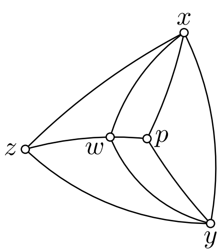
$$\measuredangle[x\,^{p}_{y}] \geqslant \tilde\measuredangle(x\,^{p}_{y}).$$

(b) For any two adjacent hinges $[p\,^{x}_{z}]$ and $[p\,^{y}_{z}]$ in $\mathcal{L}$, we have

$$\measuredangle[p\,^{x}_{z}] + \measuredangle[p\,^{y}_{z}] \leqslant \pi.$$

Proof. Consider a point $w \in\,]pz[$ close to p. From *(b)*, it follows that

$$\measuredangle[w\,^{x}_{z}] + \measuredangle[w\,^{x}_{p}] \leqslant \pi \quad \text{and} \quad \measuredangle[w\,^{y}_{z}] + \measuredangle[w\,^{y}_{p}] \leqslant \pi.$$



Since $\measuredangle[w\,^{x}_{y}] \leqslant \measuredangle[w\,^{x}_{p}] + \measuredangle[w\,^{y}_{p}]$ (see 2.3), we get

$$\measuredangle[w\,^{x}_{z}] + \measuredangle[w\,^{y}_{z}] + \measuredangle[w\,^{x}_{y}] \leqslant 2\cdot\pi.$$

Applying *(a)*,

$$\tilde\measuredangle(w\,^{x}_{z}) + \tilde\measuredangle(w\,^{y}_{z}) + \tilde\measuredangle(w\,^{x}_{y}) \leqslant 2\cdot\pi.$$

Passing to the limits $w \to p$, we have

$$\tilde\measuredangle(p\,^{x}_{z}) + \tilde\measuredangle(p\,^{y}_{z}) + \tilde\measuredangle(p\,^{x}_{y}) \leqslant 2\cdot\pi.$$

□

F Equivalent conditions

The following theorem summarizes 2.8, 2.10, 2.13, 2.15.

2.16. Theorem. *Let $\mathcal{L}$ be a geodesic space. Then the following conditions are equivalent.*

(a) $\mathcal{L}$ is CBB(0).

(b) (adjacent angle comparison) for any geodesic $[xy]$ and point $z \in\,]xy[$, $z \ne p$ in $\mathcal{L}$, we have

$$\tilde\measuredangle(z\,^{p}_{x}) + \tilde\measuredangle(z\,^{p}_{y}) \leqslant \pi.$$

(c) (point-on-side comparison) for any geodesic $[xy]$ and $z \in\,]xy[$ in $\mathcal{L}$, we have

$$\tilde\measuredangle(x\,^{p}_{y}) \leqslant \tilde\measuredangle(x\,^{p}_{z}).$$

(d) *(hinge comparison) for any hinge $[x\,{}^{p}_{y}]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^{p}_{y}]$ is defined and*

$$\measuredangle[x\,{}^{p}_{y}] \geqslant \tilde\measuredangle(x\,{}^{p}_{y}).$$

Moreover,

$$\measuredangle[z\,{}^{p}_{y}] + \measuredangle[z\,{}^{p}_{x}] \leqslant \pi$$

for any adjacent hinges $[z\,{}^{p}_{y}]$ and $[z\,{}^{p}_{x}]$.

Moreover, the implications (a)⇒(b)⇒(c)⇒(d) hold in any space, not necessarily geodesic.

2.17. Advanced Exercise. *Construct a geodesic space $\mathcal{X} \notin \mathrm{CBB}(0)$ that meets the following condition: for any 3 points $p, x, y \in \mathcal{X}$ there is a geodesic $[xy]$ such that for any $z \in \,]xy[$*

$$\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) \leqslant \pi.$$

G Function comparison

Real-to-real functions. Choose $\lambda \in \mathbb{R}$. Let $s\colon \mathbb{I} \to \mathbb{R}$ be a locally Lipschitz function defined on an interval $\mathbb{I}$. We say that s is λ-concave if $s'' \leqslant \lambda$, where the second derivative s'' is understood in the sense of distributions.

Equivalently, *s is λ-concave if the function $h\colon t \mapsto s(t) - \lambda\cdot\frac{t^2}{2}$ is concave.* Concavity can be defined via Jensen inequality; that is,

$$h(s\cdot t_0 + (1-s)\cdot t_1) \geqslant s\cdot h(t_0) + (1-s)\cdot h(t_1)$$

for any $t_0, t_1 \in \mathbb{I}$ and $s \in [0,1]$. It could be also defined via the existence of (local) upper support at any point: *for any $t_0 \in \mathbb{I}$ there is a linear function ℓ that (locally) supports h at t_0 from above; that is, $\ell(t_0) = h(t_0)$ and $\ell(t) \geqslant h(t)$ for any t (in a neighborhood of t_0).*

The equivalence of these definitions is assumed to be known. We will also use that *λ-concave functions are one-side differentiable.*

Functions on metric space. A function on a metric space $\mathcal{L}$ will usually mean a *locally Lipschitz real-valued function defined in an open subset of $\mathcal{L}$.* The domain of definition of a function f will be denoted by $\operatorname{Dom} f$.

Let f be a function on a metric space $\mathcal{L}$. We say that f is λ-concave (briefly $f'' \leqslant \lambda$) if for any unit-speed geodesic $\gamma\colon \mathbb{I} \to \operatorname{Dom} f$ the real-to-real function $t \mapsto f\circ\gamma(t)$ is λ-concave.

The following proposition is conceptual — it reformulates a global geometric condition into an infinitesimal condition on distance functions.

2.18. Proposition. *A geodesic space $\mathcal{L}$ is* CBB(0) *if and only if $f'' \leqslant 1$ for any function f of the following type*

$$f\colon x \mapsto \tfrac{1}{2}{\cdot}|p - x|^2.$$

Proof. Choose a unit-speed geodesic γ in $\mathcal{L}$ and two points $x = \gamma(t_0)$, $y = \gamma(t_1)$ for some $t_0 < t_1$. Consider the model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$. Let $\tilde{\gamma}\colon [t_0, t_1] \to \mathbb{E}^2$ be the unit-speed parametrization of $[\tilde{x}\tilde{y}]$ from $\tilde{x}$ to $\tilde{y}$.

Set

$$\tilde{r}(t) := |\tilde{p} - \tilde{\gamma}(t)|, \qquad\qquad r(t) := |p - \gamma(t)|.$$

Clearly, $\tilde{r}(t_0) = r(t_0)$ and $\tilde{r}(t_1) = r(t_1)$. Note that the point-on-side comparison (2.16c) is equivalent to

$$❶ \qquad t_0 \leqslant t \leqslant t_1 \qquad \Longrightarrow \qquad \tilde{r}(t) \leqslant r(t)$$

for any γ and $t_0 < t_1$.

Set

$$\tilde{h}(t) = \tfrac{1}{2}{\cdot}\tilde{r}^2(t) - \tfrac{1}{2}{\cdot}t^2, \qquad\qquad h = \tfrac{1}{2}{\cdot}r^2(t) - \tfrac{1}{2}{\cdot}t^2.$$

Note that $\tilde{h}$ is linear, $\tilde{h}(t_0) = h(t_0)$ and $\tilde{h}(t_1) = h(t_1)$. Observe that the Jensen inequality for the function h is equivalent to ❶. Hence the proposition follows. □

H Comments

All the discussed statements admit natural generalizations to CBB(κ) spaces. Most of the time the proof is the same with uglier formulas.

For example, the function comparison of CBB(-1) states that $f'' \leqslant f$ for any function of the type $f = \cosh\circ\operatorname{dist}_p$. Similarly, the function comparison of CBB(1) states that for any point p, we have $f'' \leqslant -f$ for the function $f = -\cos\circ\operatorname{dist}_p$ defined in $\mathrm{B}(p, \pi)$. The meaning of these inequalities is the same — distance functions in CBB(κ) are more concave than distance functions in $\mathbb{M}(\kappa)$. The inequality $f'' \leqslant \varphi$ means that for any point p in the domain of definition and any $\varepsilon > 0$, there is a neighborhood $U \ni p$ such that $f'' \leqslant \varphi(p) + \varepsilon$ in U. Here we assume that f and φ are continuous and defined in open set.

Lecture 3

Surface of convex body

Recall that (for us) a convex body is a compact convex subset in $\mathbb{E}^3$; we assume that it does not lie in a line but it might degenerate to a plane figure.

Suppose B is a nondegenerate convex body; that is, it has nonempty interior. Then the surface of B is defined as its boundary ∂B equipped with the induced length metric.

If a convex body degenerates to a plane convex figure, say F, then its surface is defined as a doubling of F along its boundary; that is, two copies of F glued along the boundary ∂F. Intuitively, one can regard these copies as different sides of F — we live on its surface and to get from one side to the other one has to cross the boundary.

3.1. Exercise. *Show that surface of a convex body is homeomorphic to $\mathbb{S}^2$.*

In this lecture, we will prove that *surface of a convex body is* CBB(0). The latter, together with the exercise, gives the only-if part in the main part of the embedding theorem (1.20).

A Convex polyhedra

Recall that a convex polyhedron is a convex hull of a finite set of points. Extremal points of a convex polyhedron are called its vertices. As for convex bodies, our convex polyhedra might degenerate to a plane polygon, but we assume that it does not belong to a line.

Observe that a surface, say Σ, of a convex polyhedron P admits a triangulation such that each triangle is isometric to a plane triangle. In other words, Σ is a polyhedral surfaces; that is, it is a

2-dimensional manifold with length metric that admits a triangulation such that each triangle is isometric to a solid plane triangle. A triangulation of polyhedral surface will be assumed to satisfy this condition.

The total angle around a vertex p in Σ is defined as the sum of angles at p of all triangles in the triangulation that contain p.

Note that if a point p is not a vertex of P, then

- ◇ p lies in the interior of a face of P, and its neighborhood in Σ is a piece of plane, or
- ◇ p lies on an edge, and its neighborhood is two half-planes glued along the boundary.

In both cases, a neighborhood of p in Σ (with the induced length metric) is isometric to an open domain of the plane.

3.2. Claim. *Let Σ be the surface of a convex polyhedron P. Then, the total angle around a vertex in Σ cannot exceed $2{\cdot}\pi$.*

In the proof, we will use the following exercise which is the triangle inequality for angles (or the spherical triangle inequality); it easily follows from 2.3.

3.3. Exercise. *Let w_1, w_2, w_3 be unit vectors in $\mathbb{E}^3$. Denote by $\theta_{i,j}$ the angle between the vectors v_i and v_j. Show that*

$$\theta_{1,3} \leqslant \theta_{1,2} + \theta_{2,3}$$

and in case of equality, the vectors w_1, w_2, w_3 lie in a plane.

Proof. Consider the intersection of P with a small sphere centered at p; it is a convex spherical polygon, say F. Applying rescaling we may assume that the sphere has unit radius. We need to show that the perimeter of F does not exceed $2{\cdot}\pi$.

Note that F lies in a hemisphere, say H. Moreover, there is a decreasing sequence

$$H = H_0 \supset H_1 \supset \dots \supset H_n = F,$$

such that H_{i+1} is obtained from H_i by cutting along a chord.

By 3.3, we have

$$2{\cdot}\pi = \operatorname{perim} H = \operatorname{perim} H_0 \geqslant \operatorname{perim} H_1 \geqslant \ldots \geqslant \operatorname{perim} H_n = \operatorname{perim} F$$

— hence the result. □

A vertex of a triangulation of a polyhedral surface is called essential if the total angle around it is not $2{\cdot}\pi$.

3.4. Exercise. *Show that any vertex of a polyhedron is an essential vertex of its surface; that is, the inequality in the claim is strict.*

3.5. Exercise. *Show that geodesics on a surface of convex polyhedron do not pass thru its essential vertices.*

B Surface of convex polyhedron

Let p be a vertex of a polyhedron. If θ_p is the total angle around p, then the value $2\cdot\pi - \theta_p$ is called the curvature of the polyhedral surface at p; if p is not a vertex, then its curvature is defined to be zero.

3.6. Exercise. *Assume that the surface of a nondegenerate tetrahedron T has curvature π at each of its vertices. Show that*

(a) all faces of T are congruent;

(b) the line passing thru midpoints of opposite edges of T intersects these edges at right angles.

Note that the claim above says that *surface of a convex polyhedron has nondegenerate curvature.* However this definition works only for polyhedral surfaces. Now we show that it agrees with the CBB(0) definition.

3.7. Proposition. *A polyhedral surface with nonnegative curvature at each vertex is* CBB(0).

Proof. Denote the surface by Σ. By 2.18, it is sufficient to check that $\mathrm{dist}_p^2 \circ \gamma$ is 1-concave for any geodesic γ and a point p in Σ.

We can assume that p is not a vertex; the vertex case can be done by approximation. Further, by 3.5, we may assume that γ does not contain vertices.

Given a point $x = \gamma(t_0)$, choose a geodesic $[px]$. Again, by 3.5, $[px]$ does not contain vertices. Therefore a small neighborhood of $U \supset [px]$ can be unfolded on a plane; denote this map by $z \mapsto \tilde{z}$. Note that this way we map part of γ in U to a line segment. Let

$$\tilde{f}(t) := \tfrac{1}{2}\cdot\mathrm{dist}_{\tilde{p}}^2 \circ \tilde{\gamma}(t).$$

Note that $\tilde{f}(t_0) \geqslant f(t_0)$. Further, since the unfolding $z \mapsto \tilde{z}$ preserves lengths of curves, we get $\tilde{f}(t) \geqslant f(t)$ if t is sufficiently close to t_0. That is, $\tilde{f}$ is a local upper support of f at t_0. Evidently, $\tilde{f}'' \equiv 1$; therefore $f'' \leqslant 1$. It remains to apply 2.18. □

3.8. Exercise. *Prove the converse to the proposition; that is, show that if a poyhedral surface is* CBB(0)*, then it has nonnegative curvature at each vertex.*

C Surface of convex body

3.9. Lemma. *Let $K_1, K_2, \dots$ be a sequence of convex bodies that converges to K_∞ in the sense of Hausdorff. Assume K_∞ is nondegenerate. Then the surface of K_n converges to the surface of K_∞ in the sense of Gromov–Hausdorff.*

In the following proof we use that the closest-point projection form the Euclidean space to a convex body is short; that is, distance-nonexpanding [76, 12.3].

Proof. Without loss of generality, we may assume that

$$\overline{\mathrm{B}}(0,r)\subset K_\infty\subset\overline{\mathrm{B}}(0,1)$$

for some $r>0$. Note that there is a sequence $\varepsilon_n\to 0$ such that

$$K_n\subset(1+\varepsilon_n)\cdot K_\infty\quad\text{and}\quad K_\infty\subset(1+\varepsilon_n)\cdot K_n$$

for each n.

Given $x\in K_n$, denote by $g_n(x)$ the closest-point projection of $(1+\varepsilon_n)\cdot x$ to K_∞. Similarly, given $x\in K_\infty$, denote by $h_n(x)$ the closest point projection of $(1+\varepsilon_n)\cdot x$ to K_n. Note that

$$|g_n(x)-g_n(y)|\leqslant(1+\varepsilon_n)\cdot|x-y|$$

and

$$|h_n(x)-h_n(y)|\leqslant(1+\varepsilon_n)\cdot|x-y|.$$

Denote by Σ_∞ and Σ_n the surface of K_∞ and K_n respectively. The above inequlities imply

$$|g_n(x)-g_n(y)|_{\Sigma_\infty}\leqslant(1+\varepsilon_n)\cdot|x-y|_{\Sigma_n}$$

for any $x,y\in\Sigma_n$, and

$$|h_n(x)-h_n(y)|_{\Sigma_n}\leqslant(1+\varepsilon_n)\cdot|x-y|_{\Sigma_\infty}.$$

for any $x,y\in\Sigma_\infty$. Therefore, g_n is a δ_n-isometry $\Sigma_n\to\Sigma_\infty$ for a sequence $\delta_n\to 0$. $\square$

3.10. Proposition. *The surface of a nondegenerate convex body is* CBB(0).

Note that any convex body is a Hausdorff limit of a sequence of convex polyhedra. Therefore, the proposition follows from 3.7, 3.9, and the following claim.

3.11. Claim. *A Gromov–Hausdorff limit of* CBB(0) *spaces is* CBB(0).

Despite its simplicity, this claim is the main source of applications of Alexandrov geometry.

Proof. Let $\mathcal{L}_\infty$ be Gromov–Hausdorff limit of CBB(0) spaces $\mathcal{L}_1, \mathcal{L}_2, \dots$

Choose a quadruple of points p, x, y, z in $\mathcal{L}_\infty$. From convergence we may choose a sequence of quadruples p_n, x_n, y_n, z_n in $\mathcal{L}_n$ that converge to p, x, y, z; in particular, each of six distances between pairs of p_n, x_n, y_n, z_n converges to the corresponding distance between the pair of p, x, y, z. By CBB(0) comparison in $\mathcal{L}_n$,

$$\tilde\measuredangle(p_n\,{}^{x_n}_{y_n}) + \tilde\measuredangle(p_n\,{}^{y_n}_{z_n}) + \tilde\measuredangle(p_n\,{}^{z_n}_{x_n}) \leqslant 2\cdot\pi.$$

Passing to the limit we get

$$\tilde\measuredangle(p\,{}^{x}_{y}) + \tilde\measuredangle(p\,{}^{y}_{z}) + \tilde\measuredangle(p\,{}^{z}_{x}) \leqslant 2\cdot\pi.$$

□

The following exercise can be solved along the same lines.

3.12. Exercise. *Show that a Gromov–Hausdorff limit of* CAT(0) *spaces is* CAT(0).

Recall that surface of a degenerate convex body is defined as its doubling. More precisely, suppose F is a convex plane figure. Consider product space $F \times \{0, 1\}$ with semimetric defined by

$$|(x, i) - (y, j)| = \begin{cases} |x - y| & \text{if } i = j \\ \inf\{\, |x - z| + |y - z| \;:\; z \in \partial F \,\} & \text{if } i \neq j \end{cases}$$

Then the corresponding metric space is the doubling of F along its boundary.

3.13. Exercise. *Suppose $F_1, F_2, \dots$ is a sequence of convex plane figures that converges to F_∞ in the sense of Hausdorff. Show that doublings of F_n converge to the doubling of F_∞ in the sense of Gromov–Hausdorff.*

Conclude that surfaces of degenerate convex bodies are CAT(0).

Note that 3.10 and 3.13 imply that *surface of a convex body is* CBB(0)*;* so the only-if part in the main part of the embedding theorem (1.20) is proved.

Lecture 4

Alexandrov embedding theorem

We will prove the Cauchy theorem, and then modify it to prove the Alexandrov uniqueness theorem. Further, we sketch a proof of the Alexandrov embedding theorem.

A Cauchy theorem

Recall that *surfaces* of convex polyhedrons are considered with the induced length metric..

4.1. Theorem. *Let K and K' be two non-degenerate convex polyhedrons in $\mathbb{E}^3$; denote their surfaces by P and P'. Suppose there is an isometry $P \to P'$ that sends each face of K to a face of K'. Then K is congruent to K'; moreover the isometry $P \to P'$ can be extended to a motion of $\mathbb{E}^3$ that maps K to K'.*

Proof. Consider the graph Γ formed by the edges of K; the edges of K' form the same graph.

For an edge e in Γ, denote by α_e and α'_e the corresponding dihedral angles in K and K' respectively. Mark e by plus if $\alpha_e < \alpha'_e$ and by minus if $\alpha_e > \alpha'_e$.

Now remove from Γ everything that was not marked; that is, leave only the edges marked by $(+)$ or $(-)$ and their endpoints.

Note that the theorem follows if Γ is an empty graph; assume the contrary.

The graph Γ is embedded into P, which is homeomorphic to the sphere. In particular, the edges coming from one vertex have a natural cyclic order. Given a vertex v of Γ, count the *number of sign changes* around v; that is, the number of consequent pairs edges with different signs.

4.2. Local lemma. *For any vertex of* Γ *the number of sign changes is at least* 4.

In other words, at each vertex of Γ, one can choose 4 edges marked by $(+)$, $(-)$, $(+)$, $(-)$ in the same cyclical order. Note that the local lemma contradicts the following.

4.3. Global lemma. *Let* Γ *be a nonempty subgraph of the graph formed by the edges of a convex polyhedron. Then it is impossible to mark all of the edges of* Γ *by* $(+)$ *or* $(-)$ *such that the number of sign changes around each vertex of* Γ *is at least* 4.

It remains to prove these two lemmas. □

B Local lemma

Next lemma is the main ingredient in our proof of the local lemma.

4.4. Arm lemma. *Assume that* $A = [a_0 a_1 \dots a_n]$ *is a convex polygon in* $\mathbb{E}^2$ *and* $A' = [a_0' a_1' \dots a_n']$ *be a polygonal line in* $\mathbb{E}^3$ *such that*

$$|a_i - a_{i+1}| = |a_i' - a_{i+1}'|$$

for any $i \in \{0, \dots, n-1\}$ *and*

$$\measuredangle a_i \leqslant \measuredangle a_i'$$

for each $i \in \{1, \dots, n-1\}$. *Then*

$$|a_0 - a_n| \leqslant |a_0' - a_n'|$$

and equality holds if and only if A *is congruent to* A'.

One may view the polygonal lines $[a_0 a_1 \dots a_n]$ and $[a_0' a_1' \dots a_n']$ as a robot's arm in two positions. The arm lemma states that when the arm opens, the distance between the shoulder and tip of a finger increases.

4.5. Exercise. *Show that the arm lemma does not hold if instead of the convexity, one only the local convexity; that is, if you go along the polygonal line* $a_0 a_1 \dots a_n$, *then you only turn left.*

4.6. Exercise. *Suppose* $A = [a_1 \dots a_n]$ *and* $A' = [a_1' \dots a_n']$ *be non-congruent convex plane polygons with equal corresponding sides. Mark* *each vertex a_i with plus (minus) if the interior angle of A at a_i is smaller (respectively bigger) than the interior angle of A' at a_i'. Show that there are at least 4 sign changes around A.*

Give an example showing the statement does not hold without assuming convexity.

Proof. We will view $\mathbb{E}^2$ as the xy-plane in $\mathbb{E}^3$; so both A and A' lie in $\mathbb{E}^3$. Let a_m be the vertex of A that lies on the maximal distance to the line $(a_0 a_n)$.

Let us shift indexes of a_i and a_i' down by m, so that

$$\begin{array}{lllll} a_{-m} := a_0, & \dots & a_0 := a_m, & \dots & a_k := a_n, \\ a'_{-m} := a'_0, & \dots & a'_0 := a'_m, & \dots & a'_k := a'_n, \end{array}$$

where $k = n - m$. (Here the symbol ":=" means an assignment as in programming.)

Without loss of generality, we may assume that

- ◇ $a_0 = a_0'$ and they both coincide with the origin $(0,0,0) \in \mathbb{E}^3$;
- ◇ all a_i lie in the xy-plane and the x-axis is parallel to the line $(a_{-m}a_k)$;
- ◇ the angle $\measuredangle a_0'$ lies in xy-plane and contains the angle $\measuredangle a_0$ inside and the directions to a'_{-1},a_{-1}, a_1 and a_1' from a_0 appear in the same cyclic order.

Denote by x_i and x_i' the projections of a_i and a_i' to the x-axis. We can assume in addition that $x_k \geqslant x_{-m}$. In this case,

$$|a_k - a_{-m}| = x_k - x_{-m}.$$

Since the projection is a distance non-expanding, we also have

$$|a_k' - a_{-m}'| \geqslant x_k' - x_{-m}'.$$

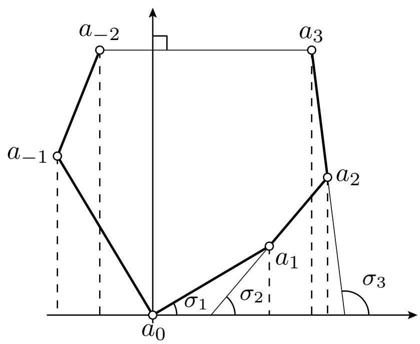
Therefore it is sufficient to show that

$$x_k' - x_{-m}' \geqslant x_k - x_{-m}.$$

The latter holds if

❶ $\quad x_i' - x_{i-1}' \geqslant x_i - x_{i-1}.$

for each i. It remains to prove ❶.



Let us assume that $i > 0$; the case $i \leqslant 0$ is similar. Denote by σ_i (σ_i') the angle between the vector $w_i = a_i - a_{i-1}$ (respectively $w_i' = a_i' - a_{i-1}'$) and the x-axis. Note that

❷
$$\begin{aligned} x_i - x_{i-1} &= |a_i - a_{i-1}| \cdot \cos\sigma_i, \\ x_i' - x_{i-1}' &= |a_i - a_{i-1}| \cdot \cos\sigma_i' \end{aligned}$$

for each $i > 0$. By construction $\sigma_1 \geqslant \sigma_1'$. Note that $\measuredangle(w_{i-1}, w_i) = \pi - \measuredangle a_i$. From convexity of $[a_1 a_1 \dots a_i]$, we have

$$\sigma_i = \sigma_1 + (\pi - \measuredangle a_1) + \dots + (\pi - \measuredangle a_i)$$

for any $i > 0$. Since $\measuredangle(w_{i-1}', w_i') = \pi - \measuredangle a_i'$, applying 3.3 several times, we get

$$\sigma_i' \leqslant \sigma_1' + (\pi - \measuredangle a_1') + \dots + (\pi - \measuredangle a_i').$$

Since $\measuredangle a_j' \geqslant \measuredangle a_j$ for each j, we get $\sigma_i' \leqslant \sigma_i$, and therefore

$$\cos\sigma_i' \geqslant \cos\sigma_i$$

Applying ❷, we get ❶.

In the case of equality, we have $\sigma_i = \sigma_i'$, which implies $\measuredangle a_i = \measuredangle a_i'$ for each i. This also implies that all a_i' lie in xy-plane. The latter easily follows from the equality case in 3.3. □

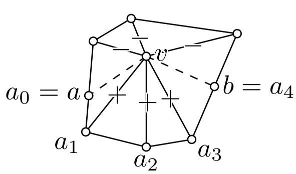
Proof of the local lemma (4.2). Assume that the local lemma does not hold at the vertex v of Γ. Cut from P a small pyramid Δ with the vertex v. One can choose two points a and b on the base of Δ so that on one side of the segments $[va]$ and $[vb]$ we have only pluses and on the other side only minuses.

The base of Δ has two polygonal lines with ends at a and b. Choose the one that has only pluses; denote it by $a_0 a_1 \dots a_n$; so $a = a_0$ and $b = a_n$. Denote by $a_0' a_1' \dots a_n'$ the corresponding line in P'; let $a' = a_0'$ and $b' = a_n'$.

Since each marked edge passing thru a_i has a (+) on it or nothing, we have

$$\measuredangle a_i \leqslant \measuredangle a_i'$$

for each i.



4.7. Exercise. *Prove the last statement.*

By the construction we have $|a_i - a_{i-1}| = |a_i' - a_{i-1}'|$ for all i. By the arm lemma (4.4), we get

❸ $|a - b| \leqslant |a' - b'|$.

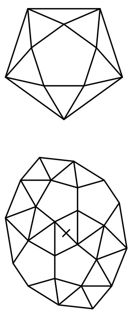
Swap K and K' and repeat the same construction for a plane passing thru a' and b'. We get

❹ $|a - b| \geqslant |a' - b'|$.

The claims ❸ and ❹ together imply $|a-b| = |a'-b'|$. The equality case in the arm lemma implies that no edge at v is marked; that is, v is not a vertex of Γ — a contradiction. □

From the proof, it follows that the local lemma is indeed local — it works for two noncongruent convex polyhedral angles with equal corresponding faces. Use this observation to solve the following exercise.

4.8. Exercise. *Consider two polyhedral discs in $\mathbb{E}^3$ glued from regular polygons by the rule on the diagrams. Assume that each disc is part of a surface of a convex polyhedron.*



(a) *The first configuration is rigid; that is, one can not fix the position of the pentagon and continuously move the remaining 5 vertices in a new position so that each triangle moves by a one-parameter family of isometries of $\mathbb{E}^3$.*
(b) *Show that the second configuration has a rotational symmetry with the axis passing thru the midpoint of the marked edge.*

C Global lemma

The proof of the global lemma is based on counting the sign changes in two ways; first while moving around each vertex of Γ and second while moving around each of the regions separated by Γ on the surface P. If two edges are adjacent at a vertex, then they are also adjacent in a region. The converse is true as well. Therefore, both countings give the same number.

It is instructive to do the next exercise before diving into the proof.

4.9. Exercise. *Try to mark the edges of an octahedron by pluses and minuses such that there would be 4 sign changes at each vertex.*

Show that this is impossible.

Proof of 4.3. We can assume that Γ is connected; that is, one can get from any vertex to any other vertex by walking along edges. (If not, pass to a connected component of Γ.)

Denote by k and l the number of vertices and edges in Γ. Denote by m the number of *regions* that Γ cuts from P. Since Γ is connected, each region is homeomorphic to an open disc.

4.10. Exercise. *Prove the last statement.*

Now we can apply Euler's formula

❶ $$k - l + m = 2.$$

Denote by s the total number of sign changes in Γ for all vertices. By the local lemma (4.2), we have

❷ $$4{\cdot}k \leqslant s.$$

Let us get an upper bound on s by counting the number of sign changes when you go around each region. Denote by m_n the number of regions bounded by n edges; if an edge appears twice when it is counted twice. Note that each region is bounded by at least 3 edges; therefore

❸ $$m = m_3 + m_4 + m_5 + \dots$$

Counting edges and using the fact that each edge belongs to exactly two regions, we get

$$2{\cdot}l = 3{\cdot}m_3 + 4{\cdot}m_4 + 5{\cdot}m_5 + \dots$$

Combining this with Euler's formula (❶), we get

❹ $$4{\cdot}k = 8 + 2{\cdot}m_3 + 4{\cdot}m_4 + 6{\cdot}m_5 + 8{\cdot}m_6 + \dots$$

Observe that the number of sign changes in n-gon regions has to be even and $\leqslant n$. Therefore

❺ $$s \leqslant 2{\cdot}m_3 + 4{\cdot}m_4 + 4{\cdot}m_5 + 6{\cdot}m_6 + \dots$$

Clearly, ❷ and ❺ contradict ❹. □

D Uniqueness

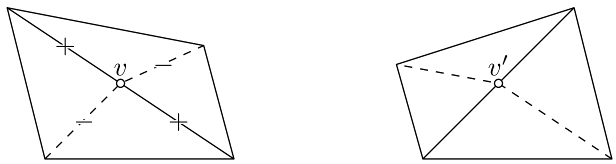
Alexandrov's uniqueness theorem states that the conclusion of the Cauchy theorem (4.1) still holds without the face-to-face assumption.

4.11. Theorem. *Any two convex polyhedrons in $\mathbb{E}^3$ with isometric surfaces are congruent.*

Moreover, any isometry between surfaces of convex polyhedrons can be extended to an isometry of the whole $\mathbb{E}^3$.

Needed modifications in the proof of 4.1. Suppose $\iota\colon P \to P'$ be an isometry between surfaces of K and K'. Mark in P all the edges of K and all the inverse images of edges in K'; further, these will be called f a k e edges. The marked lines divide P into convex polygons, and the restriction of ι to each polygon is a rigid motion. These polygons play the role of faces in the proof above.

A vertex of the obtained graph can be a vertex of K, or it can be a fake vertex; that is, it might be an intersection of an edge and a fake edge.



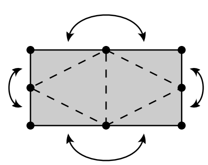
For the first type of vertex, the local lemma can be proved the same way. For a fake vertex v, it is easy to see that both parts of the edge coming thru v are marked with minus while both of the fake edges at v are marked with plus. Therefore, the local lemma holds for the fake vertices as well.

What remains in the proof needs no modifications. □

4.12. Exercise. *Let K be a convex polyhedron in $\mathbb{E}^3$; denote by P its surface. Show that each isometry $\iota\colon P \to P$, can be extended to an isometry of $\mathbb{E}^3$.*

E Existence

4.13. Theorem. *A polyhedral metric on the sphere is isometric to the surface of a convex polyhedron (possibly degenerate to a flat polygon) if and only if it has nonnegative curvature at each point.*



By 4.11, a convex polyhedron is completely defined by the intrinsic metric of its surface. By 4.13, it follows that knowing the metric we could find the position of the edges. However, in practice, it is not easy to do.

For example, the surface glued from a rectangle as shown on the diagram defines a tetrahedron. Some of the glued lines appear inside facets of the tetrahedron and some edges (dashed lines) do not follow the sides of the rectangle.

Space of polyhedrons. Let us denote by $\mathbf{K}$ the space of all convex polyhedrons in the Euclidean space, including polyhedrons that degenerate to a plane polygon. Polyhedra in $\mathbf{K}$ will be considered up to a motion of the space, and the whole space $\mathbf{K}$ will be considered with Hausdorff distance up to a motion of the space; that is, the distance between K and K' is the exact lower bound on Hausdorff distance from $\iota(K)$ to K', where ι is arbitrary motion of $\mathbb{E}^3$.

Further, denote by $\mathbf{K}_n$ the polyhedrons in $\mathbf{K}$ with exactly n vertices. Since any polyhedron has at least 3 vertices, the space $\mathbf{K}$ admits a subdivision into a countable number of subsets $\mathbf{K}_3, \mathbf{K}_4, \dots$

Space of polyhedral metrics. The space of polyhedral metrics on the sphere with nonnegative curvature will be denoted by $\mathbf{P}$. The metrics in $\mathbf{P}$ will be considered up to an isometry, and the whole space $\mathbf{P}$ will be equipped with the topology induced by the Gromov–Hausdorff metric.

The subset of $\mathbf{P}$ of all metrics with exactly n essential vertices will be denoted by $\mathbf{P}_n$. It is easy to see that any metric in $\mathbf{P}$ has at least 3 essential vertices. Therefore $\mathbf{P}$ is subdivided into countably many subsets $\mathbf{P}_3, \mathbf{P}_4, \dots$

From a polyhedron to its surface. By 3.7, passing from a polyhedron to its surface defines a map

$$\iota\colon \mathbf{K} \to \mathbf{P}.$$

By 3.4, the number of vertices of a polyhedron is equal to the number of essential vertices on its surface. In other words, $\iota(\mathbf{K}_n) \subset \mathbf{P}_n$ for any $n \geqslant 3$.

Using the introduced notation, we can unite 4.11 and 4.13 in the following more exact statement.

4.14. Reformulation. *For any integer $n \geqslant 3$, the map ι induces a bijection between $\mathbf{K}_n$ and $\mathbf{P}_n$.*

The proof is based on a construction of a one-parameter family of polyhedrons that starts at an arbitrary polyhedron and ends at a polyhedron with its surface isometric to the given one. This type of argument is called the *continuity method*; it is often used in the theory of differential equations.

Sketch. By 4.11, the map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is injective. Let us prove that it is surjective.

4.15. Lemma. *For any integer $n \geqslant 3$, the space $\mathbf{P}_n$ is connected.*

The proof of this lemma is not complicated, but it requires ingenuity; it can be done by the direct construction of a one-parameter family of metrics in $\mathbf{P}_n$ that connects two given metrics. Such a family can be obtained by a sequential application of the following construction and its inverse.

Let $P \in \mathbf{P}_n$. Suppose v and w are essential vertices in P. Let us cut P along a geodesic from v to w. Note that the geodesic cannot pass thru an essential vertex of P. Further, note that there is a three-parameter family of patches that can be used to patch the cut so that the obtained metric remains in $\mathbf{P}_n$; in particular, the obtained metric has exactly n essential vertices (after the patching, the vertices v and w may become inessential).

4.16. Lemma. *The map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is open, that is, it maps any open set in $\mathbf{K}_n$ to an open set in $\mathbf{P}_n$.*

In particular, for any $n \geqslant 3$, the image $\iota(\mathbf{K}_n)$ is open in $\mathbf{P}_n$.

This statement is very close to the so-called *invariance of domain theorem*; the latter states that a continuous injective map between manifolds of the same dimension is open.

Recall that ι is injective. The proof of the invariance of domain theorem can be adapted to our case since both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n - 6)$-dimensional and both look like manifolds, altho, formally speaking, they are *not* manifolds. In a more technical language, $\mathbf{K}_n$ and $\mathbf{P}_n$ have the natural structure of $(3{\cdot}n - 6)$-dimensional *orbifolds*, and the map ι respects the *orbifold structure.*

We will only show that both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n - 6)$-dimensional.

Choose $K \in \mathbf{K}_n$. Note that K is uniquely determined by the $3{\cdot}n$ coordinates of its n vertices. We can assume that the first vertex is the origin, the second has two vanishing coordinates and the third has one vanishing coordinate; therefore, all polyhedrons in $\mathbf{K}_n$ that lie sufficiently close to K can be described by $3{\cdot}n - 6$ parameters. If K has no symmetries, then this description can be made one-to-one; in this case, a neighborhood of K in $\mathbf{K}_n$ is a $(3{\cdot}n - 6)$-dimensional manifold. If K has a nontrivial symmetry group, then this description is not one-to-one but it does not have an impact on the dimension of $\mathbf{K}_n$.

The case of polyhedral metrics is analogous. We need to construct a subdivision of the sphere into plane triangles using only essential vertices. By Euler's formula, there are exactly $3{\cdot}n-6$ edges in this subdivision. Note that the lengths of edges completely describe the metric, and slight changes in these lengths produce a metric with the same property. Again, if P has no symmetries, then this description is one-to-one.

4.17. Lemma. *The map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is closed; that is, the image of a closed set in $\mathbf{K}_n$ is closed in $\mathbf{P}_n$.*

In particular, for any $n \geqslant 3$, the set $\iota(\mathbf{K}_n)$ is closed in $\mathbf{P}_n$.

Choose a closed set Z in $\mathbf{K}_n$. Denote by $\bar{Z}$ the closure of Z in $\mathbf{K}$; note that $Z = \mathbf{K}_n \cap \bar{Z}$. Assume $K_1, K_2, \dots \in Z$ is a sequence of polyhedrons that converges to a polyhedron $K_\infty \in \bar{Z}$. By 3.9, $\iota(K_n)$ converges to $\iota(K_\infty)$ in $\mathbf{P}$. In particular, $\iota(\bar{Z})$ is closed in $\mathbf{P}$.

Since $\iota(\mathbf{K}_n) \subset \mathbf{P}_n$ for any $n \geqslant 3$, we have $\iota(Z) = \iota(\bar{Z}) \cap \mathbf{P}_n$; that is, $\iota(Z)$ is closed in $\mathbf{P}_n$.

Summarizing, $\iota(\mathbf{K}_n)$ is a nonempty closed and open set in $\mathbf{P}_n$, and $\mathbf{P}_n$ is connected for any $n \geqslant 3$. Therefore, $\iota(\mathbf{K}_n) = \mathbf{P}_n$; that is, $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is surjective. □

F Approximation

By now, the embedding theorem is proved for polyhedral metrics on the sphere. The general case is done by approximation, using the following statement.

4.18. Proposition. *Let $K_1, K_2, \dots$ be a sequence of convex bodies that converge to K_∞ in the sense of Hausdorff. Then the surface of K_n converges to the surface of K_∞ in the sense of Gromov–Hausdorff.*

If K_∞ is nondegenerate, then the statement follows from 3.9. The degenerate case is left as an exercise.

Let $\mathcal{X}_\infty$ be a geodesic CBB(0) space that is homeomorphic to $\mathbb{S}^2$. Suppose that $\mathcal{X}_\infty$ is a Gromov–Hausdorff limit of a sequence of spheres with polyhedral metrics $\mathcal{X}_1, \mathcal{X}_2, \dots$ By 4.13, there is a sequence of convex polyhedra $K_1, K_2, \dots$ with surfaces isometric to $\mathcal{X}_1, \mathcal{X}_2, \dots$, respectively. Note that $\operatorname{diam} K_n \leqslant \operatorname{diam} \mathcal{X}_n$ for any n. Therefore we can assume that all polyhedra $K_1, K_2, \dots$ lie in a closed ball of sufficiently large radius.

Applying Blaschke selection theorem, we can pass to a subsequence of $K_1, K_2, \dots$ that converges in the sense of Hausdorff; denote its limit by K_∞. By 4.18 the surface of K_∞ is isometric to $\mathcal{X}_\infty$.

Therefore it remains to prove the following lemma.

4.19. Lemma. *Let $\mathcal{X}$ be a geodesic* CBB(0) *space that is homeomorphic to $\mathbb{S}^2$. Then there is a sphere with polyhedral metrics $\mathcal{X}'$ that is arbitrarily close to $\mathcal{X}$ in the sense of Gromov–Hausdorff.*

Idea behind the proof. Suppose we can triangulate $\mathcal{X}_\infty$ by small geodesic triangles; that is, we can choose a finite set of points $p_1, \dots, p_n \in \in \mathcal{X}_\infty$ and some geodesics $[p_ip_j]$ that cut $\mathcal{X}_\infty$ into regions of small diameter bounded by geodesic triangles $[p_ip_jp_k]$. (The actual proof constructs a triangulation with a weaker property.)

Observe that total angle around each p_i cannot exceed $2{\cdot}\pi$. That is, suppose $p_{j_1}, \dots, p_{j_k}$ are points connected to p_i by geodesics. Assume that they are ordered in the natural cyclic order. Then

$$\measuredangle[p_i\,{}^{p_{j_1}}_{p_{j_2}}] + \dots + \measuredangle[p_i\,{}^{p_{j_{k-1}}}_{p_{j_k}}] + \measuredangle[p_i\,{}^{p_{j_k}}_{p_{j_1}}] \leqslant 2{\cdot}\pi.$$

By comparison, we get

❶ $$\tilde\measuredangle(p_i\,{}^{p_{j_1}}_{p_{j_2}}) + \dots + \tilde\measuredangle(p_i\,{}^{p_{j_{k-1}}}_{p_{j_k}}) + \tilde\measuredangle(p_i\,{}^{p_{j_k}}_{p_{j_1}}) \leqslant 2{\cdot}\pi.$$

Now let us exchange each triangle by its model triangle. That is, consider a model triangle for each region in the subdivision of $\mathcal{X}$ and glue them together by the same rule. By ❶, the obtained polyhedral surface $\mathcal{X}'$ has nonnegative curvature. It remains to show that this way we can produce $\mathcal{X}'$ arbitrarily close to $\mathcal{X}$.

Denote by $p_i \to p_i'$ the natural map; it takes p_i in $\mathcal{X}$ and returns the corresponding point in $\mathcal{X}'$. Observe that

❷ $$|p_i' - p_j'|_{\mathcal{X}'} \leqslant |p_i - p_j|_{\mathcal{X}}.$$

Indeed, choose a geodesic γ from p_i to p_j. Let $p_i = x_0, x_1, \dots, x_n = p_j$ be the points of intersections of γ with the edges of the triangulation listed as they appear on γ. For each i, denote by x_i' the corresponding point in $\mathcal{X}'$. By comparison, we get

$$|x_k' - x_{k-1}'|_{\mathcal{X}'} \leqslant |x_k - x_{k-1}|_{\mathcal{X}}.$$

for each k. Therefore, ❷ follows.

Suppose $\varepsilon > 0$ is small, the points $p_1, \dots, p_n$ form an ε-net in $\mathcal{X}$, all edges of the triangulation are smaller than ε and

❸ $$|p_i' - p_j'|_{\mathcal{X}'} \geqslant |p_i - p_j|_{\mathcal{X}} - 100{\cdot}\varepsilon.$$

Then, together with the inequality above it proves that the lemma.

Note that the sides of the model triangles are local geodesics in $\mathcal{X}'$, but not necessarily geodesic; that is they do not have to be length-minimizing. Now, let us make another unjustified assumption: *Suppose that the sides of model triangles in $\mathcal{X}'$ are geodesics.* (The actual proof does not use this assumption.)

Choose a geodesic γ' from p'_i to p'_j in $\mathcal{X}'$. Note that γ' visits each triangle in the triangulation of $\mathcal{X}'$ at most once.

Let $p'_i = x'_0, x'_1, \dots, x'_n = p'_j$ be the points of intersections of γ' with the edges of the triangulation listed from p'_i to p'_j. For each i, denote by x_i the corresponding point in $\mathcal{X}$. Let Δ'_k be the triangle that contains arc $[x'_{k-1}x'_k]$ of γ' and Δ_k the corresponding triangle in $\mathcal{X}$. Note that

❹ $$|x'_k - x'_{k-1}|_{\mathcal{X}'} \geqslant |x_k - x_{k-1}|_{\mathcal{X}} - \varepsilon{\cdot}K(\Delta_k),$$

where $K(\Delta_k)$ denotes the access of Δ_k; that is, the sum of its internal angles minus π.

Euler's formula and ❶ imply that the sum of all accesses is at most $4{\cdot}\pi$. Therefore, summing up ❹, we get

$$|p'_i - p'_j|_{\mathcal{X}'} \geqslant |p_i - p_j|_{\mathcal{X}} - 4{\cdot}\pi{\cdot}\varepsilon.$$

Whence ❸ follows. □

G Comments

This lecture contains selected material from Alexandrov's book [7].

In Euclid's Elements, solids were called equal if the same holds for their faces, but no proof was given. Adrien-Marie Legendre became interested in this problem towards the end of the 18th century. He discussed it with his colleague Joseph-Louis Lagrange, who suggested this problem to Augustin-Louis Cauchy in 1813; soon he proved it [34]. This theorem is included in many popular books [1, 45, 94].

The observation that the face-to-face condition can be removed was made by Alexandr Alexandrov [8].

Arm lemma. Original Cauchy's proof [34] also used a version of the arm lemma, but its proof contained a small mistake (corrected in one century).

Our proof of the arm lemma is due to Stanisław Zaremba. This and a couple of other proofs can be found in the letters between him and Isaac Schoenberg [88].

The following variation of the arm lemma makes sense for nonconvex spherical polygons. It is due to Viktor Zalgaller [100]. It can be used instead of the standard arm lemma.

4.20. Another arm lemma. *Let $A = [a_1 \dots a_n]$ and $A' = [a'_1 \dots \dots a'_n]$ be two spherical n-gons (not necessarily convex). Assume that A lies in a half-sphere, the corresponding sides of A and A' are equal* *and each angle of A is at least the corresponding angle in A'. Then A is congruent to A'.*

Global lemma. A more visual proof of the global lemma is given in [7, II §1.3]. This argument was reused by Anton Klyachko [61] in his car-crash lemma.

Existence theorem. This theorem was proved by Alexandr Alexandrov [8]. Our sketch is taken from [64]; a complete proof is nicely written in [7]. In the original proof, the spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ were modified so the they become $(3{\cdot}n-6)$-dimensional manifolds. It was done by introducing extra structure (for $\mathbf{K}_n$ it is orientation + a marked vertex and an edge) that *brakes symmetries* of the spaces. After that one could apply the domain invariance theorem directly. Alternatively, one may first remove from $\mathbf{K}_n$ and $\mathbf{P}_n$ elements (polyhedron or surface)with nontrivial symmetries (after that the spaces become manifolds) and show that any symmetric polyhedron (or surface) can be approximated by a non-symmetric polyhedron (or surface).

A very different proof was found by Yuri Volkov in his thesis [98]; it uses a deformation of three-dimensional polyhedral space.

Lecture 5

Gluing and billiards

This lecture is nearly a copy of [5, Chapter 2]; here we define upper curvature bound in the sense of Alexandrov, prove Reshetnyak's gluing theorem, and apply it to a problem in billiards.

A Geodesics

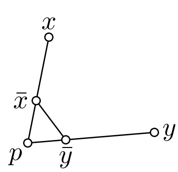
The CAT comparison can be applied to any metric space, but it is usually applied to geodesic spaces (or complete length spaces). To simplify the presentation we will assume in addition that the space is proper. The latter means that any closed ball is compact.

Recall that function is proper if inverse image of any compact set is compact. Note that *a metric space is proper if and only if the distance function from one (and therefore any) point is proper.*

5.1. Proposition. *Let $\mathcal{U}$ be a complete geodesic* CAT(0) *space. Then any two points in $\mathcal{U}$ are joined by a unique geodesic.*

Proof. Suppose there are two geodesics between x and y. Then we can choose two points $p \neq q$ on these geodesics such that $|x-p| = |x-q|$ and therefore $|y-p| = |y-q|$.

Observe that the model triangles $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$ and $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\triangle}(qxy)$ are degenerate and moreover $\tilde{p} = \tilde{q}$. Applying CAT(0) comparison with $\tilde{z} = \tilde{p} = \tilde{q}$, we get that $|p-q| = 0$, a contradiction. $\square$



5.2. Exercise. *Given $[p\,{}^{x}_{y}]$ in a* CAT(0) *space $\mathcal{U}$, consider the function*

$$f\colon (|p-\bar{x}|, |p-\bar{y}|) \mapsto \tilde{\measuredangle}(p\,{}^{\bar{x}}_{\bar{y}}),$$

where $\bar{x} \in\,]px]$ and $\bar{y} \in\,]py]$. Show that f is nondecreasing in each argument.

Conclude that any hinge in a CAT(0) *space has defined angle.*

5.3. Exercise. *Fix a point p in a be a complete geodesic* CAT(0) *space $\mathcal{U}$. Given a point $x \in \mathcal{U}$, denote by $\gamma_x\colon [0,1] \to \mathcal{U}$ a (necessarily unique) geodesic path from p to x.*

Show that the family of maps $h_t\colon \mathcal{U} \to \mathcal{U}$ defined by

$$h_t(x) = \gamma_x(t)$$

is a homotopy; it is called geodesic homotopy. Conclude that $\mathcal{U}$ is contractible.

The geodesic homotopy introduced in the previous exercise should help to solve the next one.

5.4. Exercise. *Let $\mathcal{U}$ be a complete geodesic* CAT(0) *space. Assume $\mathcal{U}$ is a topological manifold. Show that any geodesic in $\mathcal{U}$ can be extended as a two-side infinite geodesic.*

B Thin triangles

Let us recall the definition of thin triangles.

5.5. Definition. *A triangle $[xyz]$ in the metric space $\mathcal{U}$ is called thin if the natural map $\tilde{\triangle}(xyz)_{\mathbb{E}^2} \to [xyz]$ is distance nonincreasing.*

Analogously, a triangle $[xyz]$ is called spherically thin if the natural map from the spherical model triangle $\tilde{\triangle}(xyz)_{\mathbb{S}^2}$ to $[xyz]$ is distance nonincreasing.

5.6. Proposition. *A geodesic space is* CAT(0) *(*CAT(1)*) if and only if all its triangles are thin (respectively, all its triangles of perimeter $< 2\cdot\pi$ are spherically thin).*

Proof; if part. Apply the triangle inequality and thinness of triangles $[pxy]$ and $[qxy]$, where p, q, x, and y are as in the definition of the $\mathrm{CAT}(\kappa)$ comparison.

Only-if part. Applying CAT(0) comparison to a quadruple p, q, x, y with $q \in [xy]$ shows that any triangle satisfies point-side comparison, that is, the distance from a vertex to a point on the opposite side is no greater than the corresponding distance in the Euclidean model triangle.

Now consider a triangle $[xyz]$ and let $p \in [xy]$ and $q \in [xz]$. Let $\tilde{p}$, $\tilde{q}$ be the corresponding points on the sides of the model triangle $\tilde{\triangle}(xyz)_{\mathbb{E}^2}$. Applying 5.2, we get that

$$\tilde{\measuredangle}(x\,{}^{y}_{z})_{\mathbb{E}^2} \geqslant \tilde{\measuredangle}(x\,{}^{p}_{q})_{\mathbb{E}^2}.$$

Therefore $|\tilde{p} - \tilde{q}|_{\mathbb{E}^2} \geqslant |p - q|$.

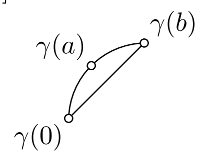
The CAT(1) argument is the same. □

A curve $\gamma\colon \mathbb{I} \to \mathcal{U}$ is called a *local geodesic* if for any $t \in \mathbb{I}$ there is a neighborhood U of t in $\mathbb{I}$ such that the restriction $\gamma|_U$ is a geodesic.

5.7. Proposition. *Suppose $\mathcal{U}$ is a proper geodesic* CAT(0) *space. Then any local geodesic in $\mathcal{U}$ is a geodesic.*

Analogously, if $\mathcal{U}$ is a proper geodesic CAT(1) *space, then any local geodesic in $\mathcal{U}$ which is shorter than π is a geodesic.*

Proof. Suppose $\gamma\colon [0,\ell] \to \mathcal{U}$ is a local geodesic that is not a geodesic. Choose a to be the maximal value such that γ is a geodesic on $[0,a]$. Further, choose $b > a$ so that γ is a geodesic on $[a,b]$.



Since the triangle $[\gamma(0)\gamma(a)\gamma(b)]$ is thin (see the next section) and $|\gamma(0) - \gamma(b)| < b$ we have

$$|\gamma(a-\varepsilon) - \gamma(a+\varepsilon)| < 2{\cdot}\varepsilon$$

for all small $\varepsilon > 0$. That is, γ is not length-minimizing on the interval $[a-\varepsilon, a+\varepsilon]$ for any $\varepsilon > 0$, a contradiction.

The spherical case is done in the same way. □

5.8. Exercise. *Let $\mathcal{U}$ be a complete geodesic space. Show that $\mathcal{U}$ is* CAT(0) *if and only if the function $f = \frac{1}{2}{\cdot}\mathrm{dist}_p^2$ is 1-convex for any $p \in \mathcal{U}$.*

5.9. Exercise. *Suppose $\gamma_1, \gamma_2\colon [0,1] \to \mathcal{U}$ are two geodesic paths in a complete geodesic* CAT(0) *space $\mathcal{U}$. Show that*

$$t \mapsto |\gamma_1(t) - \gamma_2(t)|_{\mathcal{U}}$$

is a convex function.

5.10. Exercise. *Let A be a convex closed set in a proper geodesic* CAT(0) *space $\mathcal{U}$; that is, if $x, y \in A$, then $[xy] \subset A$. Show that for any $r > 0$ the closed r-neighborhood of A is convex; that is, the set*

$$A_r = \{\, x \in \mathcal{U} \;:\; \mathrm{dist}_A x \leqslant r \,\}$$

is convex.

5.11. Exercise. *Let $\mathcal{U}$ be a proper geodesic* CAT(0) *space and $K \subset \mathcal{U}$ be a closed convex set. Show that:*

(a) *For each point $p \in \mathcal{U}$ there is a unique point $p^* \in K$ that minimizes the distance $|p - p^*|$.*
(b) *The closest-point projection $p \mapsto p^*$ defined by (a) is short.*

Recall that a set A in a metric space $\mathcal{U}$ is called locally convex if for any point $p \in A$ there is an open neighborhood $\mathcal{U} \ni p$ such that any geodesic in $\mathcal{U}$ with ends in A lies in A.

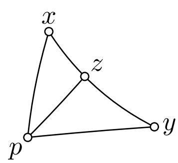
5.12. Exercise. *Let $\mathcal{U}$ be a proper geodesic* CAT(0) *space. Show that any closed, connected, locally convex set in $\mathcal{U}$ is convex.*

C Inheritance lemma

5.13. Inheritance lemma. *Assume that a triangle $[pxy]$ in a metric space is decomposed into two triangles $[pxz]$ and $[pyz]$; that is, $[pxz]$ and $[pyz]$ have a common side $[pz]$, and the sides $[xz]$ and $[zy]$ together form the side $[xy]$ of $[pxy]$.*

If both triangles $[pxz]$ and $[pyz]$ are thin, then the triangle $[pxy]$ is also thin.

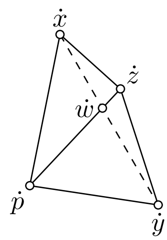
Analogously, if $[pxy]$ has perimeter $< 2\cdot\pi$ and both triangles $[pxz]$ and $[pyz]$ are spherically thin, then triangle $[pxy]$ is spherically thin.



Proof. Construct the model triangles $[\dot p\dot x\dot z] = \tilde\triangle(pxz)_{\mathbb{E}^2}$ and $[\dot p\dot y\dot z] = \tilde\triangle(pyz)_{\mathbb{E}^2}$ so that $\dot x$ and $\dot y$ lie on opposite sides of $[\dot p\dot z]$.

Let us show that

❶ $$\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) \geqslant \pi.$$



If not, then for some point $\dot w \in [\dot p\dot z]$, we have

$$|\dot x - \dot w| + |\dot w - \dot y| < |\dot x - \dot z| + |\dot z - \dot y| = |x - y|.$$

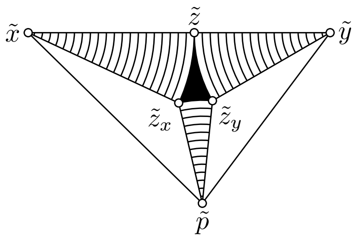
Let $w \in [pz]$ correspond to $\dot w$; that is, $|z - w| = |\dot z - \dot w|$. Since $[pxz]$ and $[pyz]$ are thin, we have

$$|x - w| + |w - y| < |x - y|,$$

contradicting the triangle inequality.

Denote by $\dot D$ the union of two solid triangles $[\dot p\dot x\dot z]$ and $[\dot p\dot y\dot z]$. Further, denote by $\tilde D$ the solid triangle $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{E}^2}$. By ❶, there is a short map $F\colon \tilde D \to \dot D$ that sends

$$\tilde p \mapsto \dot p, \qquad \tilde x \mapsto \dot x, \qquad \tilde z \mapsto \dot z, \qquad \tilde y \mapsto \dot y.$$



Indeed, by Alexandrov's lemma (2.6), there are nonoverlapping triangles

$$[\tilde{p}\tilde{x}\tilde{z}_x] \xlongequal{iso} [\dot{p}\dot{x}\dot{z}]$$

and

$$[\tilde{p}\tilde{y}\tilde{z}_y] \xlongequal{iso} [\dot{p}\dot{y}\dot{z}]$$

inside the triangle $[\tilde{p}\tilde{x}\tilde{y}]$.

Connect the points in each pair $(\tilde{z}, \tilde{z}_x)$, $(\tilde{z}_x, \tilde{z}_y)$ and $(\tilde{z}_y, \tilde{z})$ with arcs of circles centered at $\tilde{y}$, $\tilde{p}$, and $\tilde{x}$ respectively. Define F as follows:

- ◇ Map $\mathrm{Conv}[\tilde{p}\tilde{x}\tilde{z}_x]$ isometrically onto $\mathrm{Conv}[\dot{p}\dot{x}\dot{z}]$; similarly map $\mathrm{Conv}[\tilde{p}\tilde{y}\tilde{z}_y]$ onto $\mathrm{Conv}[\dot{p}\dot{y}\dot{z}]$.
- ◇ If x is in one of the three circular sectors, say at distance r from its center, set $F(x)$ to be the point on the corresponding segment $[pz]$, $[xz]$ or $[yz]$ whose distance from the left-hand endpoint of the segment is r.
- ◇ Finally, if x lies in the remaining curvilinear triangle $\tilde{z}\tilde{z}_x\tilde{z}_y$, set $F(x) = z$.

By construction, F satisfies the conditions.

By assumption, the natural maps $[\dot{p}\dot{x}\dot{z}] \to [pxz]$ and $[\dot{p}\dot{y}\dot{z}] \to [pyz]$ are short. By composition, the natural map from $[\tilde{p}\tilde{x}\tilde{y}]$ to $[pyz]$ is short, as claimed.

The spherical case is done along the same lines. □

D Reshetnyak's gluing

Suppose $\mathcal{U}^1$ and $\mathcal{U}^2$ are proper geodesic spaces with isometric closed convex sets $A^i \subset \mathcal{U}^i$ and let $\iota\colon A^1 \to A^2$ be an isometry. Consider the space $\mathcal{W}$ of all equivalence classes in $\mathcal{U}^1 \sqcup \mathcal{U}^2$ with the equivalence relation given by $a \sim \iota(a)$ for any $a \in A^1$.

It is straightforward to see that $\mathcal{W}$ is a proper geodesic space when equipped with the following metric

$$\begin{aligned}
|x-y|_{\mathcal{W}} &:= |x-y|_{\mathcal{U}^i} \\
&\qquad \text{if} \quad x, y \in \mathcal{U}^i, \quad \text{and} \\
|x-y|_{\mathcal{W}} &:= \min\left\{\, |x-a|_{\mathcal{U}^1} + |y-\iota(a)|_{\mathcal{U}^2} \;:\; a \in A^1 \,\right\} \\
&\qquad \text{if} \quad x \in \mathcal{U}^1 \quad \text{and} \quad y \in \mathcal{U}^2.
\end{aligned}$$

Abusing notation, we denote by x and y the points in $\mathcal{U}^1 \sqcup \mathcal{U}^2$ and their equivalence classes in $\mathcal{U}^1 \sqcup \mathcal{U}^2/{\sim}$.

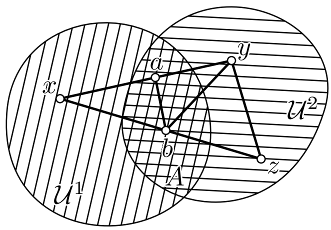
The space $\mathcal{W}$ is called the gluing of $\mathcal{U}^1$ and $\mathcal{U}^2$ along ι. If one applies this construction to two copies of one space $\mathcal{U}$ with a set $A \subset \mathcal{U}$ and the identity map $\iota\colon A \to A$, then the obtained space is called the double of $\mathcal{U}$ along A.

We can (and will) identify $\mathcal{U}^i$ with its image in $\mathcal{W}$; this way both subsets $A^i \subset \mathcal{U}^i$ will be identified and denoted further by A. Note that $A = \mathcal{U}^1 \cap \mathcal{U}^2 \subset \mathcal{W}$, therefore A is also a convex set in $\mathcal{W}$.

5.14. Reshetnyak gluing. *Suppose $\mathcal{U}^1$ and $\mathcal{U}^2$ are proper geodesic* CAT(0) *spaces with isometric closed convex sets $A^i \subset \mathcal{U}^i$, and $\iota\colon A^1 \to A^2$ is an isometry. Then the gluing of $\mathcal{U}^1$ and $\mathcal{U}^2$ along ι is a* CAT(0) *proper geodesic space.*

Proof. By construction of the gluing space, the statement can be reformulated in the following way:

5.15. Reformulation of 5.14. *Let $\mathcal{W}$ be a proper geodesic space with two closed convex sets $\mathcal{U}^1, \mathcal{U}^2 \subset \mathcal{W}$ such that $\mathcal{U}^1 \cup \mathcal{U}^2 = \mathcal{W}$ and $\mathcal{U}^1$, $\mathcal{U}^2$ are* CAT(0)*. Then $\mathcal{W}$ is* CAT(0)*.*



It suffices to show that any triangle $[xyz]$ in $\mathcal{W}$ is thin. This is obviously true if all three points x, y, z lie in one of $\mathcal{U}^i$. Thus, without loss of generality, we may assume that $x \in \mathcal{U}^1$ and $y, z \in \mathcal{U}^2$.

Choose points $a, b \in A = \mathcal{U}^1 \cap \mathcal{U}^2$ that lie respectively on the sides $[xy], [xz]$. Note that

- ⋄ the triangle $[xab]$ lies in $\mathcal{U}^1$,
- ⋄ both triangles $[yab]$ and $[ybz]$ lie in $\mathcal{U}^2$.

In particular, each triangle $[xab]$, $[yab]$, and $[ybz]$ is thin.

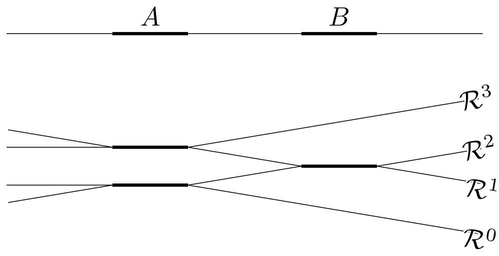
Applying the inheritance lemma (5.13) twice, we get that $[xyb]$ and consequently $[xyz]$ is thin. □

5.16. Exercise. *Suppose $\mathcal{U}$ is a geodesic space and $A \subset \mathcal{U}$ is a closed subset. Assume that the doubling of $\mathcal{U}$ in A is* CAT(0)*. Show that A is a convex set of $\mathcal{U}$.*

E Puff pastry

In this section, we introduce the notion of Reshetnyak puff pastry. This construction will be used in the next section to prove the collision theorem (5.27).

Let $\boldsymbol{A} = (A^1, \dots, A^N)$ be an array of convex closed sets in the Euclidean space $\mathbb{E}^m$. Consider an array of $N+1$ copies of $\mathbb{E}^m$. Assume that the space $\mathcal{R}$ is obtained by gluing successive pairs of spaces along $A^1, \dots, A^N$ respectively.



Puff pastry for (A, B, A).

The resulting space $\mathcal{R}$ will be called the Reshetnyak puff pastry for array $\boldsymbol{A}$. The copies of $\mathbb{E}^m$ in the puff pastry $\mathcal{R}$ will be called levels; they will be denoted by $\mathcal{R}^0, \dots, \mathcal{R}^N$. The point in the k-th level $\mathcal{R}^k$ that corresponds to $x \in \mathbb{E}^m$ will be denoted by x^k.

Given $x \in \mathbb{E}^m$, any point $x^k \in \mathcal{R}$ is called a lifting of x. The map $x \mapsto x^k$ defines an isometry $\mathbb{E}^m \to \mathcal{R}^k$; in particular, we can talk about liftings of subsets in $\mathbb{E}^m$.

Note that:

◇ The intersection $A^1 \cap \dots \cap A^N$ admits a unique lifting in $\mathcal{R}$.

◇ Moreover, $x^i = x^j$ for some $i < j$ if and only if

$$x \in A^{i+1} \cap \dots \cap A^j.$$

◇ The restriction $\mathcal{R}^k \to \mathbb{E}^m$ of the natural projection $x^k \mapsto x$ is an isometry.

5.17. Observation. *Any Reshetnyak puff pastry is a proper geodesic* CAT(0) *space.*

Proof. Apply Reshetnyak gluing theorem (5.14) recursively for the convex sets in the array. □

5.18. Proposition. *Assume $(A^1, \dots, A^N)$ and $(\check{A}^1, \dots, \check{A}^N)$ are two arrays of convex closed sets in $\mathbb{E}^m$ such that $A^k \subset \check{A}^k$ for each k. Let $\mathcal{R}$ and $\check{\mathcal{R}}$ be the corresponding Reshetnyak puff pastries. Then the map $\mathcal{R} \to \check{\mathcal{R}}$ defined by $x^k \mapsto \check{x}^k$ is short.*

Moreover, if

❶ $$|x^i - y^j|_{\mathcal{R}} = |\check{x}^i - \check{y}^j|_{\check{\mathcal{R}}}$$

for some $x, y \in \mathbb{E}^m$ and $i, j \in \{0, \dots, n\}$, then the unique geodesic $[\check{x}^i\check{y}^j]_{\check{\mathcal{R}}}$ is the image of the unique geodesic $[x^i y^j]_{\mathcal{R}}$ under the map $x^i \mapsto \check{x}^i$.

Proof. The first statement in the proposition follows from the construction of Reshetnyak puff pastries.

By Observation 5.17, $\mathcal{R}$ and $\check{\mathcal{R}}$ are proper geodesic CAT(0) spaces; hence $[x^i y^j]_{\mathcal{R}}$ and $[\check{x}^i\check{y}^j]_{\check{\mathcal{R}}}$ are unique. By ❶, since the map $\mathcal{R} \to \check{\mathcal{R}}$ is short, the image of $[x^i y^j]_{\mathcal{R}}$ is a geodesic of $\check{\mathcal{R}}$ joining $\check{x}^i$ to $\check{y}^j$. Hence the second statement follows. □

5.19. Definition. *Consider a Reshetnyak puff pastry $\mathcal{R}$ with the levels $\mathcal{R}^0, \dots, \mathcal{R}^N$. We say that $\mathcal{R}$ is end-to-end convex if $\mathcal{R}^0 \cup \mathcal{R}^N$, the union of its lower and upper levels, forms a convex set in $\mathcal{R}$; that is, if $x, y \in \mathcal{R}^0 \cup \mathcal{R}^N$, then $[xy]_{\mathcal{R}} \subset \mathcal{R}^0 \cup \mathcal{R}^N$.*

Note that if $\mathcal{R}$ is the Reshetnyak puff pastry for an array of convex sets $\boldsymbol{A} = (A^1, \dots, A^N)$, then $\mathcal{R}$ is end-to-end convex if and only if the union of the lower and the upper levels $\mathcal{R}^0 \cup \mathcal{R}^N$ is isometric to the double of $\mathbb{E}^m$ along the nonempty intersection $A^1 \cap \dots \cap A^N$.

5.20. Observation. *Let $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ be arrays of convex bodies in $\mathbb{E}^m$. Assume that array $\boldsymbol{A}$ is obtained by inserting in $\check{\boldsymbol{A}}$ several copies of the bodies which were already listed in $\check{\boldsymbol{A}}$.*

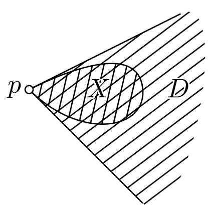
For example, if $\check{\boldsymbol{A}} = (A, C, B, C, A)$, by placing B in the second place and A in the fourth place, we obtain $\boldsymbol{A} = (A, B, C, A, B, C, A)$.

Denote by $\check{\mathcal{R}}$ and $\mathcal{R}$ the Reshetnyak puff pastries for $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ respectively.

If $\check{\mathcal{R}}$ is end-to-end convex, then so is $\mathcal{R}$.

Proof. Without loss of generality, we may assume that $\boldsymbol{A}$ is obtained by inserting one element in $\check{\boldsymbol{A}}$, say at the place number k.

Note that $\check{\mathcal{R}}$ is isometric to the puff pastry for $\boldsymbol{A}$ with A^k replaced by $\mathbb{E}^m$. It remains to apply Proposition 5.18. □



Let X be a convex set in a Euclidean space. By a dihedral angle, we understand an intersection of two half-spaces; the intersection of corresponding hyperplanes is called the edge of the angle. We say that a dihedral angle D supports X at a point $p \in X$ if D contains X and the edge of D contains p.

5.21. Lemma. *Let A and B be two convex sets in $\mathbb{E}^m$. Assume that any dihedral angle supporting $A \cap B$ has angle measure at least α. Then* *the Reshetnyak puff pastry for the array*

$$(\underbrace{A, B, A, \dots}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \ \textit{times}}).$$

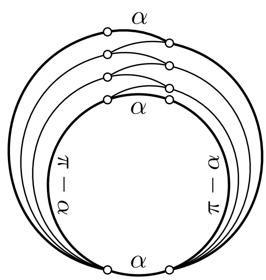
is end-to-end convex.

The proof of the lemma is based on a partial case, which we formulate as a sublemma.

5.22. Sublemma. *Let $\ddot{A}$ and $\ddot{B}$ be two half-planes in $\mathbb{E}^2$, where $\ddot{A} \cap \ddot{B}$ is an angle with measure α. Then the Reshetnyak puff pastry for the array*

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \dots}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \ \textit{times}})$$

is end-to-end convex.

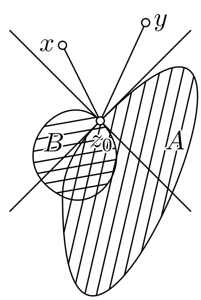


Proof. Note that the puff pastry $\ddot{\mathcal{R}}$ is isometric to the cone over the space glued from the unit circles as shown on the diagram.

All the short arcs on the diagram have length α; the long arcs have length $\pi - \alpha$, so making a circuit along any path will take $2 \cdot \pi$.

The end-to-end convexity of $\ddot{\mathcal{R}}$ is equivalent to the fact that any geodesic shorter than π with the ends on the inner and the outer circles lies completely in the union of these two circles.

The latter holds if the zigzag line in the picture has length at least π. This line is formed by $\lceil \frac{\pi}{\alpha} \rceil$ arcs with length α each. Hence the sublemma. $\square$



In the proof of 5.21, we will use the following exercise in convex geometry:

5.23. Exercise. *Let A and B be two closed convex sets in $\mathbb{E}^m$ and $A \cap B \neq \varnothing$. Given two points $x, y \in \mathbb{E}^m$ let $f(z) = |x - z| + |y - z|$.*

Let $z_0 \in A \cap B$ be a point of minimum of $f|_{A \cap B}$.

Show that there are half-spaces $\dot{A}$ and $\dot{B}$ such that $\dot{A} \supset A$ and $\dot{B} \supset B$ and z_0 is also a point of minimum of the restriction $f|_{\dot{A} \cap \dot{B}}$.

Proof of 5.21. Fix arbitrary $x, y \in \mathbb{E}^m$. Choose a point $z \in A \cap B$ for which the sum

$$|x - z| + |y - z|$$

is minimal. To show the end-to-end convexity of $\mathcal{R}$, it is sufficient to prove the following:

❷ *The geodesic* $[x^0y^N]_{\mathcal{R}}$ *contains* $z^0 = z^N \in \mathcal{R}$.

Without loss of generality, we may assume that $z \in \partial A \cap \partial B$. Indeed, since the puff pastry for the 1-array (B) is end-to-end convex, Proposition 5.18 together with 5.20 imply ❷ in case z lies in the interior of A. The same way we can treat the case when z lies in the interior of B.

Note that $\mathbb{E}^m$ admits an isometric splitting $\mathbb{E}^{m-2} \times \mathbb{E}^2$ such that

$$\dot{A} = \mathbb{E}^{m-2} \times \ddot{A}$$
$$\dot{B} = \mathbb{E}^{m-2} \times \ddot{B}$$

where $\ddot{A}$ and $\ddot{B}$ are half-planes in $\mathbb{E}^2$.

Using Exercise 5.23, let us replace each A by $\dot{A}$ and each B by $\dot{B}$ in the array, to get the array

$$(\underbrace{\dot{A}, \dot{B}, \dot{A}, \ldots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \text{ times}}).$$

The corresponding puff pastry $\dot{\mathcal{R}}$ splits as a product of $\mathbb{E}^{m-2}$ and a puff pastry, call it $\ddot{\mathcal{R}}$, glued from the copies of the plane $\mathbb{E}^2$ for the array

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \ldots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \text{ times}}).$$

Note that the dihedral angle $\dot{A} \cap \dot{B}$ is at least α. Therefore the angle measure of $\ddot{A} \cap \ddot{B}$ is also at least α. According to Sublemma 5.22 and Observation 5.20, $\ddot{\mathcal{R}}$ is end-to-end convex.

Since $\dot{\mathcal{R}} \stackrel{iso}{=\!=} \mathbb{E}^{m-2} \times \ddot{\mathcal{R}}$, the puff pastry $\dot{\mathcal{R}}$ is also end-to-end convex.

It follows that the geodesic $[\dot{x}^0\dot{y}^N]_{\dot{\mathcal{R}}}$ contains $\dot{z}^0 = \dot{z}^N \in \dot{\mathcal{R}}$. By Proposition 5.18, the image of $[\dot{x}^0\dot{y}^N]_{\dot{\mathcal{R}}}$ under the map $\dot{x}^k \mapsto x^k$ is the geodesic $[x^0y^N]_{\mathcal{R}}$. Hence ❷ and the lemma follow. $\square$

F Wide corners

We say that a closed convex set $A \subset \mathbb{E}^m$ has ε-wide corners for given $\varepsilon > 0$ if together with each point p, the set A contains a small right circular cone with the tip at p and aperture ε; that is, ε is the maximum angle between two generating lines of the cone.

For example, a plane polygon has ε-wide corners if all its interior angles are at least ε.

We will consider finite collections of closed convex sets $A^1,\dots,A^n\subset$ $\subset\mathbb{E}^m$ such that for any subset $F\subset\{1,\dots,n\}$, the intersection $\bigcap_{i\in F}A^i$ has ε-wide corners. In this case, we may say briefly *all intersections of A^i have ε-wide corners.*

5.24. Exercise. *Assume $A^1,\dots,A^n\subset\mathbb{E}^m$ are compact, convex sets with a common interior point. Show that all intersections of A^i have ε-wide corners for some $\varepsilon>0$.*

5.25. Exercise. *Assume $A^1,\dots,A^n\subset\mathbb{E}^m$ are convex sets with nonempty interiors that have a common center of symmetry. Show that all intersections of A^i have ε-wide corners for some $\varepsilon>0$.*

The proof of the following proposition is based on 5.21; this lemma is essentially the case $n=2$ in the proposition.

5.26. Proposition. *Given $\varepsilon>0$ and a positive integer n, there is an array of integers $\boldsymbol{j}_\varepsilon(n)=(j_1,\dots,j_N)$ such that:*

(a) For each k we have $1\leqslant j_k\leqslant n$, and each number $1,\dots,n$ appears in $\boldsymbol{j}_\varepsilon$ at least once.

(b) If $A^1,\dots,A^n$ is a collection of closed convex sets in $\mathbb{E}^m$ with a common point and all their intersections have ε-wide corners, then the puff pastry for the array $(A^{j_1},\dots,A^{j_N})$ is end-to-end convex.

Moreover, we can assume that $N\leqslant(\lceil\frac{\pi}{\varepsilon}\rceil+1)^n$.

Proof. The array $\boldsymbol{j}_\varepsilon(n)=(j_1,\dots,j_N)$ is constructed recursively. For $n=1$, we can take $\boldsymbol{j}_\varepsilon(1)=(1)$.

Assume that $\boldsymbol{j}_\varepsilon(n)$ is constructed. Let us replace each occurrence of n in $\boldsymbol{j}_\varepsilon(n)$ by the alternating string

$$\underbrace{n,n+1,n,\dots}_{\lceil\frac{\pi}{\varepsilon}\rceil+1\text{ times}}.$$

Denote the obtained array by $\boldsymbol{j}_\varepsilon(n+1)$.

By Lemma 5.21, the end-to-end convexity of the puff pastry for $\boldsymbol{j}_\varepsilon(n+1)$ follows from the end-to-end convexity of the puff pastry for the array where each string

$$\underbrace{A^n,A^{n+1},A^n,\dots}_{\lceil\frac{\pi}{\varepsilon}\rceil+1\text{ times}}$$

is replaced by $Q=A^n\cap A^{n+1}$. End-to-end convexity of the latter follows by the assumption on $\boldsymbol{j}_\varepsilon(n)$, since all the intersections of $A^1,\dots,A^{n-1},Q$ have ε-wide corners.

The upper bound on N follows directly from the construction. $\square$

G Billiards

Let $A^1, A^2, \dots A^n$ be a finite collection of closed convex sets in $\mathbb{E}^m$. Assume that for each i the boundary ∂A^i is a smooth hypersurface.

Consider the billiard table formed by the closure of the complement

$$T = \overline{\mathbb{E}^m \setminus \bigcup_i A^i}.$$

The sets A^i will be called walls of the table and the billiards described above will be called billiards with convex walls.

A billiard trajectory on the table is a unit-speed broken line γ that follows the standard law of billiards at the breakpoints on ∂A^i — in particular, the angle of reflection is equal to the angle of incidence. The breakpoints of the trajectory will be called collisions. We assume the trajectory meets only one wall at a time.

Recall that the definition of sets with ε-wide corners is given in 5F.

5.27. Collision theorem. *Assume $T \subset \mathbb{E}^m$ is a billiard table with n convex walls. Assume that the walls of T have a common interior point and all their intersections have ε-wide corners. Then the number of collisions of any trajectory in T is bounded by a number N which depends only on n and ε.*

As we will see from the proof, the value N can be found explicitly; $N = (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^{n^2}$ will do.

5.28. Corollary. *Consider n homogeneous hard balls moving freely and colliding elastically in $\mathbb{R}^3$. Every ball moves along a straight line with constant speed until two balls collide, and then the new velocities of the two balls are determined by the laws of classical mechanics. We assume that only two balls can collide at the same time.*

Then the total number of collisions cannot exceed some number N that depends on the radii and masses of the balls. If the balls are identical, then N depends only on n.

5.29. Exercise. *Show that in the case of identical balls in the one-dimensional space (in $\mathbb{R}$) the total number of collisions cannot exceed $N = \frac{n \cdot (n-1)}{2}$.*

The proof below admits a straightforward generalization to all dimensions.

Proof of 5.28 modulo 5.27. Denote by $a_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ the center of the i-th ball. Consider the corresponding point in $\mathbb{R}^{3 \cdot N}$

$$\begin{aligned} \boldsymbol{a} &= (a_1, a_2, \dots, a_n) = \\ &= (x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_n, y_n, z_n). \end{aligned}$$

The i-th and j-th balls intersect if

$$|a_i - a_j| \leqslant R_i + R_j,$$

where R_i denotes the radius of the i-th ball. These inequalities define $\frac{n\cdot(n-1)}{2}$ cylinders

$$C_{i,j} = \left\{ (a_1, a_2, \dots, a_n) \in \mathbb{R}^{3\cdot n} \ : \ |a_i - a_j| \leqslant R_i + R_j \right\}.$$

The closure of the complement

$$T = \overline{\mathbb{R}^{3\cdot n} \setminus \bigcup_{i<j} C_{i,j}}$$

is the configuration space of our system. Its points correspond to valid positions of the system of balls.

The evolution of the system of balls is described by the motion of the point $\boldsymbol{a} \in \mathbb{R}^{3\cdot n}$. It moves along a straight line at a constant speed until it hits one of the cylinders $C_{i,j}$; this event corresponds to a collision in the system of balls.

Consider the norm of $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{R}^{3\cdot n}$ defined by

$$\|\boldsymbol{a}\| = \sqrt{M_1 \cdot |a_1|^2 + \dots + M_n \cdot |a_n|^2},$$

where $|a_i| = \sqrt{x_i^2 + y_i^2 + z_i^2}$ and M_i denotes the mass of the i-th ball. In the metric defined by $\|{*}\|$, the collisions follow the standard law of billiards.

By construction, the number of collisions of hard balls that we need to estimate is the same as the number of collisions of the corresponding billiard trajectory on the table with $C_{i,j}$ as the walls.

Note that each cylinder $C_{i,j}$ is a convex set; it has smooth boundary, and it is centrally symmetric around the origin. By 5.25, all the intersections of the walls have ε-wide corners for some $\varepsilon > 0$ that depend on the radiuses R_i and the masses M_i. It remains to apply the collision theorem (5.27). $\square$

Now we present the proof of the collision theorem (5.27) based on the results developed in the previous section.

Proof of 5.27. Let us apply induction on n.

Base: $n = 1$. The number of collisions cannot exceed 1. Indeed, by the convexity of A^1, if the trajectory is reflected once in ∂A^1, then it cannot return to A^1.

Step. Assume γ is a trajectory that meets the walls in the order $A^{i_1}, \dots, A^{i_N}$ for a large integer N.

Consider the array

$$\boldsymbol{A}_\gamma = (A^{i_1}, \dots, A^{i_N}).$$

The induction hypothesis implies:

❶ *There is a positive integer M such that any M consecutive elements of $\boldsymbol{A}_\gamma$ contain each A^i at least once.*

Let $\mathcal{R}_\gamma$ be the Reshetnyak puff pastry for $\boldsymbol{A}_\gamma$.

Consider the lift of γ to $\mathcal{R}_\gamma$, defined by $\bar\gamma(t) = \gamma^k(t) \in \mathcal{R}_\gamma$ for any moment of time t between the k-th and $(k+1)$-th collisions. Since γ follows the standard law of billiards at breakpoints, the lift $\bar\gamma$ is locally a geodesic in $\mathcal{R}_\gamma$. By 5.17, the puff pastry $\mathcal{R}_\gamma$ is a proper geodesic CAT(0) space. Therefore $\bar\gamma$ is a geodesic.

Since γ does not meet $A^1 \cap \dots \cap A^n$, the lift $\bar\gamma$ does not lie in $\mathcal{R}^0_\gamma \cup \mathcal{R}^N_\gamma$. In particular, $\mathcal{R}_\gamma$ is not end-to-end convex.

Let

$$\boldsymbol{B} = (A^{j_1}, \dots, A^{j_K})$$

be the array provided by Proposition 5.26; so $\boldsymbol{B}$ contains each A^i at least once and the puff pastry $\mathcal{R}_{\boldsymbol{B}}$ for $\boldsymbol{B}$ is end-to-end convex. If N is sufficiently large, namely $N \geqslant K{\cdot}M$, then ❶ implies that $\boldsymbol{A}_\gamma$ can be obtained by inserting a finite number of A^i's in $\boldsymbol{B}$.

By 5.20, $\mathcal{R}_\gamma$ is end-to-end convex — a contradiction. □

H Comments

The gluing theorem (5.14) was proved by Yuri Reshetnyak [83]. It can be extended to all geodesic CAT(0) spaces. It also admits a natural generalization to geodesic CAT(κ) spaces; see the book of Martin Bridson and André Haefliger [23] and our book [6] for details.

The collision theorem (5.27) was proved by Dmitri Burago, Serge Ferleger and Alexey Kononenko [26]. Its corollary (5.28) answers a question posed by Yakov Sinai [48]. Puff pastry is used to bound topological entropy of the billiard flow and to approximate the shortest billiard path that touches given lines in a given order; see the papers of Dmitri Burago with Serge Ferleger, and Alexey Kononenko [27], and with Dimitri Grigoriev and Anatol Slissenko [28]. The lecture of Dmitri Burago [24] gives a short survey on the subject.

Note that the interior points of the walls play a key role in the proof despite that the trajectories never go inside the walls. In a similar fashion, puff pastry was used by Stephanie Alexander and Richard Bishop [3] to find the upper curvature bound for warped products.

Joel Hass [54] constructed an example of a Riemannian metric on the 3-ball with negative curvature and concave boundary. This example might decrease your appetite for generalizing the collision theorem — while locally such a 3-ball looks as good as the billiards table in the theorem, the number of collisions is obviously infinite.

It was shown by Dmitri Burago and Sergei Ivanov [30] that the number of collisions that may occur between n identical balls in $\mathbb{R}^3$ grows at least exponentially in n; the two-dimensional remains open.

Lecture 6

Majorization

A Formulation

6.1. Definition. *Let $\mathcal{X}$ be a metric space, $\tilde{\alpha}$ be a simple closed curve of finite length in $\mathbb{E}^2$, and $D \subset \mathbb{E}^2$ be a closed region bounded by $\tilde{\alpha}$. A length-nonincreasing map $F\colon D \to \mathcal{X}$ is called m a j o r i z i n g if it is length-preserving on $\tilde{\alpha}$.*

In this case, we say that D m a j o r i z e s the curve $\alpha = F \circ \tilde{\alpha}$ under the map F.

The following proposition is a consequence of the definition.

6.2. Proposition. *Let α be a closed curve in a metric space $\mathcal{X}$. Suppose $D \subset \mathbb{E}^2$ majorizes α under $F\colon D \to \mathcal{X}$. Then any geodesic subarc of α is the image under F of a subarc of $\partial_{\mathbb{E}^2} D$ that is geodesic in the length metric of D.*

In particular, if D is convex, then the corresponding subarc is a geodesic in $\mathbb{E}^2$.

Proof. For a geodesic subarc $\gamma\colon [a,b] \to \mathcal{X}$ of $\alpha = F \circ \tilde{\alpha}$, set

$$\begin{aligned} \tilde{r} &= |\tilde{\gamma}(a) - \tilde{\gamma}(b)|_D, & \tilde{\gamma} &= (F|_{\partial D})^{-1} \circ \gamma, \\ s &= \operatorname{length}\gamma, & \tilde{s} &= \operatorname{length}\tilde{\gamma}. \end{aligned}$$

Then

$$\tilde{r} \geqslant r = s = \tilde{s} \geqslant \tilde{r}.$$

Therefore $\tilde{s} = \tilde{r}$. □

6.3. Corollary. *Assume a convex region $D \subset \mathbb{E}^2$ majorizes $[pxy]$. Then D is a s o l i d m o d e l t r i a n g l e of $[pxy]$; that is, $D = \operatorname{Conv}[\tilde{p}\tilde{x}\tilde{y}]$*

for a model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$. *Moreover, the majorizing map sends* $\tilde{p}$, $\tilde{x}$ *and* $\tilde{y}$ *respectively to* p, x *and* y.

Now we come to the main theorem of this section.

6.4. Majorization theorem. *Any closed rectifiable curve* α *in a geodesic* CAT(0) *space is majorized by a convex plane figure.*

B Triangles

The case when α is a triangle, say $[pxy]$, is the base in the following proof, and it is nontrivial. In this case, by Corollary 6.3, the majorizing convex region the solid model triangle.

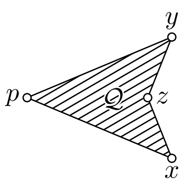
6.5. Line-of-sight map. *Let* p *be a point and* α *be a curve of finite length in a geodesic space* $\mathcal{X}$. *Let* $\mathring{\alpha}\colon [0,1] \to \mathcal{U}$ *be the constant-speed parametrization of* α. *If* $\gamma_t\colon [0,1] \to \mathcal{U}$ *is a geodesic path from* p *to* $\mathring{\alpha}(t)$, *we say*

$$[0,1] \times [0,1] \to \mathcal{U}\colon (t,s) \mapsto \gamma_t(s)$$

is a line-of-sight map *from* p *to* α.

We will show that there is a majorizing map for $[pxy]$ whose image W is the image of the line-of-sight map for $[xy]$ from p, but as one can see from the following example, the line-of-sight map is not majorizing in general.

Example. Let $\mathcal{Q}$ be a solid quadrangle $[pxzy]$ in $\mathbb{E}^2$ formed by two congruent triangles, which is non-convex at z (as in the picture). Equip $\mathcal{Q}$ with the length metric. Then $\mathcal{Q}$ is CAT(0) by Reshetnyak gluing (5.14). For triangle $[pxy]_{\mathcal{Q}}$ in $\mathcal{Q}$ and its model triangle $[\tilde{p}\tilde{x}\tilde{y}]$ in $\mathbb{E}^2$, we have



$$|\tilde{x} - \tilde{y}| = |x - y|_{\mathcal{Q}} = |x - z| + |z - y|.$$

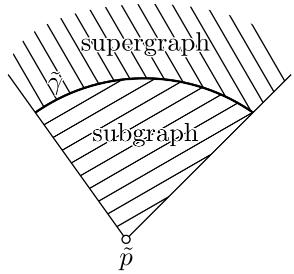
Then the map F defined by matching line-of-sight parameters satisfies $F(\tilde{x}) = x$ and $|x - F(\tilde{w})| > |\tilde{x} - \tilde{w}|$ if $\tilde{w}$ is near the midpoint $\tilde{z}$ of $[\tilde{x}\tilde{y}]$ and lies on $[\tilde{p}\tilde{z}]$. Indeed, for $\varepsilon = 1 - s$ we have

$$|\tilde{x} - \tilde{w}| = |\tilde{x} - \tilde{\gamma}_{\frac{1}{2}}(s)| = |x - z| + o(\varepsilon)$$

and

$$|x - F(\tilde{w})| = |x - \gamma_{\frac{1}{2}}(s)| = |x - z| - \varepsilon \cdot \cos \measuredangle[z\,{}^{p}_{x}] + o(\varepsilon).$$

Thus F is not majorizing.



6.6. Definition. *Let $\tilde{\gamma}\colon \mathbb{I} \to \mathbb{E}^2$ be a curve and $\tilde{p} \in \mathbb{E}^2$ be such that the direction of $[\tilde{p}\,\tilde{\gamma}(t)]$ turns monotonically as t grows.*

The set formed by all geodesics from $\tilde{p}$ to the points on $\tilde{\gamma}$ is called the subgraph of $\tilde{\gamma}$ with respect to $\tilde{p}$.

The set of all points $\tilde{x} \in \mathbb{E}^2$ such that a geodesic $[\tilde{p}\tilde{x}]$ intersects $\tilde{\gamma}$ is called the supergraph of $\tilde{\gamma}$ with respect to $\tilde{p}$.

The curve $\tilde{\gamma}$ is called convex (concave) with respect to $\tilde{p}$ if the subgraph (supergraph) of $\tilde{\gamma}$ with respect to $\tilde{p}$ is convex.

The curve $\tilde{\gamma}$ is called locally convex (concave) with respect to $\tilde{p}$ if for any interior value t_0 in $\mathbb{I}$ there is a subsegment $(a,b) \subset \mathbb{I}$, $(a,b) \ni t_0$, such that the restriction $\tilde{\gamma}|_{(a,b)}$ is convex (concave) with respect to $\tilde{p}$.

Our first lemma gives a model space construction based on repeated application of the argument in the proof of the inheritance lemma (5.13).

6.7. Lemma. *In $\mathbb{E}^2$, let β be a curve from x to y that is concave with respect to p. Let D be the subgraph of β with respect to p.*

(a) Then β forms a geodesic $[xy]_D$ in D and therefore β, $[px]$ and $[py]$ form a triangle $[pxy]_D$ in the length metric of D.

(b) Let $[\tilde{p}\tilde{x}\tilde{y}]$ be the model triangle for $[pxy]_D$. Then there is a short map

$$G\colon \operatorname{Conv}[\tilde{p}\tilde{x}\tilde{y}] \to D$$

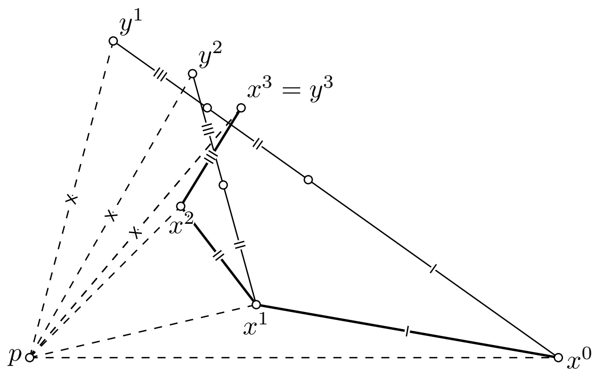
such that $\tilde{p} \mapsto p$, $\tilde{x} \mapsto x$, $\tilde{y} \mapsto y$, and G is length-preserving on each side of $[\tilde{p}\tilde{x}\tilde{y}]$. In particular, Conv$[\tilde{p}\tilde{x}\tilde{y}]$ *majorizes triangle $[pxy]_D$ in D under G.*

Proof. We prove the lemma for a polygonal line β; the general case then follows by approximation. Namely, since β is concave it can be approximated by polygonal lines that are concave with respect to p, with their lengths converging to length β. Passing to a partial limit we will obtain the needed map G.

Suppose $\beta = x^0x^1\dots x^n$ is a polygonal line with $x^0 = x$ and $x^n = y$. Consider a sequence of polygonal lines $\beta_i = x^0x^1\dots x^{i-1}y_i$ such that $|p - y_i| = |p - y|$ and β_i has same length as β; that is,

$$|x^{i-1} - y_i| = |x^{i-1} - x^i| + |x^i - x^{i+1}| + \dots + |x^{n-1} - x^n|.$$

Clearly $\beta_n = \beta$. Sequentially applying Alexandrov's lemma (2.6) shows that each of the polygonal lines $\beta_{n-1}, \beta_{n-2}, \dots, \beta_1$ is concave



with respect to p. Let D_i be the subgraph of β_i with respect to p. Applying the argument in the inheritance lemma (5.13) gives a short map $G_i\colon D_i \to D_{i+1}$ that maps $y_i \mapsto y_{i+1}$ and does not move p and x (in fact, G_i is the identity everywhere except on $\mathrm{Conv}[px^{i-1}y_i]$). Thus the composition

$$G_{n-1}\circ\dots\circ G_1\colon D_1 \to D_n$$

is short. The result follows since $D_1 \stackrel{iso}{=\!=} \mathrm{Conv}[\tilde p\tilde x\tilde y]$. □

6.8. Lemma. *Let $\mathcal{X}$ be a metric space, $\gamma\colon \mathbb{I}\to\mathcal{X}$ be a 1-Lipschitz curve, $p\in\mathcal{X}$, and $\tilde p\in\mathbb{E}^2$. Then there exists a unique up to rotation curve $\tilde\gamma\colon\mathbb{I}\to\mathbb{E}^2$, parametrized by arc-length, such that $|\tilde p-\tilde\gamma(t)| = |p-\gamma(t)|$ for all t and the direction of $[\tilde p\tilde\gamma(t)]$ monotonically turns around $\tilde p$ counterclockwise as t increases.*

If p, $\tilde p$, γ, and $\tilde\gamma$ are as above, then $\tilde\gamma$ is called the development of γ with respect to p; the point $\tilde p$ is called the basepoint of the development.

Proof. Consider the functions $\rho,\theta\colon\mathbb{I}\to\mathbb{R}$ defined as

$$\rho(t) = |p-\gamma(t)|, \qquad\qquad \theta(t) = \int\limits_{t_0}^{t}\frac{\sqrt{1-(\rho')^2}}{\rho},$$

where $t_0\in\mathbb{I}$ is a fixed number and $\int$ denotes Lebesgue integral. Since γ is 1-Lipshitz, so is $\rho(t)$, and thus the function θ is defined and non-decreasing.

It is straightforward to check that (ρ,θ) uniquely describe $\tilde\gamma$ in polar coordinates on $\mathbb{E}^2$ with center at $\tilde p$. □

6.9. Exercise. *A geodesic space $\mathcal{U}$ is* CAT(0) *if and only if development of any geodesic with respect to any point is concave.*

6.10. Lemma. *Let $[pxy]$ be a triangle in a geodesic* CAT(0) *space $\mathcal{U}$. In $\mathbb{E}^2$, let $\tilde\gamma$ be the κ-development of $[xy]$ with respect to p, where $\tilde\gamma$ has basepoint $\tilde p$ and subgraph D. Consider the map $H\colon D\to\mathcal{U}$ that sends the point with parameter (t,s) under the line-of-sight map for $\tilde\gamma$ with respect to $\tilde p$, to the point with the same parameter under the line-of-sight map f for $[xy]$ with respect to p. Then H is length-nonincreasing. In particular, D majorizes triangle $[pxy]$.*

Proof. Let $\gamma\colon[0,T]\to\mathcal{U}$ be a unit-speed paremetrization of $[xy]$; so, $T=|x-y|$. Choose a partition

$$0=t^0<t^1<\dots<t^n=T,$$

and set $x^i=\gamma(t^i)$. Construct a chain of model triangles $[\tilde p\tilde x^{i-1}\tilde x^i]= =\tilde\triangle(px^{i-1}x^i)$, with $\tilde x^0=\tilde x$ and the direction of $[\tilde p\tilde x^i]$ turning counterclockwise as i grows. Let D_n be the subgraph with respect to $\tilde p$ of the polygonal line $\tilde x^0\dots\tilde x^n$.

Let δ_n be the maximum radius of a circle inscribed in any of the triangles $[\tilde p\tilde x^{i-1}\tilde x^i]$.

Now we construct a map $H_n\colon D_n\to\mathcal{U}$ that increases distances by at most $2\cdot\delta_n$. Suppose $w\in D_n$. Then w lies on or inside some triangle $[\tilde p\tilde x^{i-1}\tilde x^i]$. Define $H_n(w)$ by first mapping w to a nearest point on $[\tilde p\tilde x^{i-1}\tilde x^i]$ (choosing one if there are several), followed by the natural map to the triangle $[px^{i-1}x^i]$.

Since triangles in $\mathcal{U}$ are thin, the restriction of H_n to each triangle $[\tilde p\tilde x^{i-1}\tilde x^i]$ is short. Then the triangle inequality implies that the restriction of H_n to

$$U_n=\bigcup_{1\leqslant i\leqslant n}[\tilde p\tilde x^{i-1}\tilde x^i]$$

is short with respect to the length metric on D_n. Since nearest-point projection from D_n to U_n increases the D_n-distance between two points by at most $2\cdot\delta_n$, the map H_n also increases the D_n-distance by at most $2\cdot\delta_n$.

Consider converging sequences $v_n\to v$ and $w_n\to w$ such that $v_n,w_n\in D_n$ and therefore $v,w\in D$. Note that

$$❶\qquad |H_n(v_n)-H_n(w_n)|\leqslant|v_n-w_n|_{D_n}+2\cdot\delta_n,$$

for each n. Since $\delta_n\to0$ and geodesics in $\mathcal{U}$ vary continuously with their endpoints (7.7), we have $H_n(v_n)\to H(v)$ and $H_n(w_n)\to H(w)$. Therefore the left-hand side in ❶ converges to $|H(v)-H(w)|$ and the right-hand side converges to $|v-w|_D$, it follows that H is short. ☐

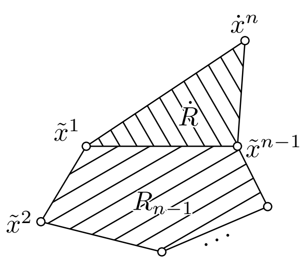
Proof of 6.4 for triangles. Suppose α is a triangle, say $[pxy]$.

Let $\tilde{\gamma}$ be the development of $[xy]$ with respect to p, where $\tilde{\gamma}$ has basepoint $\tilde{p}$ and subgraph D. By 6.9, $\tilde{\gamma}$ is concave. By 6.7, there is a short map $G\colon \operatorname{Conv}\tilde{\triangle}(pxy) \to D$. Further, by 6.10, D majorizes $[pxy]$ under a majorizing map $H\colon D \to \mathcal{U}$. Clearly $H \circ G$ is a majorizing map for $[pxy]$. $\square$

C Polygons

In the following proofs, $x^1 \dots x^n$ $(n \geqslant 3)$ denotes a polygonal line $x^1, \dots, x^n$, and $[x^1 \dots x^n]$ denotes the corresponding (closed) polygon. For a subset R of the ambient metric space, we denote by $[x^1 \dots x^n]_R$ a polygon in the length metric of R.

Proof of 6.4 for polygons. We begin by proving the theorem in case α is polygonal.



Now we claim that any closed n-gon $[x^1x^2 \dots x^n]$ in a CAT(0) space is majorized by a convex polygonal region

$$R_n = \operatorname{Conv}[\tilde{x}^1\tilde{x}^2 \dots \tilde{x}^n]$$

under a map F_n such that $F_n\colon \tilde{x}^i \mapsto x^i$ for each i.

The base case $n = 3$ is proved above. Assume the statement is true for $(n-1)$-gons, $n \geqslant 4$. Then $[x^1x^2 \dots x^{n-1}]$ is majorized by a convex polygonal region

$$R_{n-1} = \operatorname{Conv}[\tilde{x}^1\tilde{x}^2, \dots \tilde{x}^{n-1}],$$

in $\mathbb{E}^2$ under a map F_{n-1} satisfying $F_{n-1}(\tilde{x}^i) = x^i$ for all i. Take $\dot{x}^n \in \mathbb{E}^2$ such that $[\tilde{x}^1\tilde{x}^{n-1}\dot{x}^n] = \tilde{\triangle}(x^1x^{n-1}x^n)$ and this triangle lies on the other side of $[\tilde{x}^1\tilde{x}^{n-1}]$ from R_{n-1}. Let $\dot{R} = \operatorname{Conv}[\tilde{x}^1\tilde{x}^{n-1}\dot{x}^n]$, and $\dot{F}\colon \dot{R} \to \mathcal{U}$ be a majorizing map for $[x^1x^{n-1}x^n]$ as provided above.

Set $R = R_{n-1} \cup \dot{R}$, where R carries its length metric. Since F_n and F agree on $[\tilde{x}^1\tilde{x}^{n-1}]$, we may define $F\colon R \to \mathcal{U}$ by

$$F(x) = \begin{cases} F_{n-1}(x), & x \in R_{n-1}, \\ \dot{F}(x), & x \in \dot{R}. \end{cases}$$

Then F is length-nonincreasing and is a majorizing map for $[x^1x^2 \dots x^n]$ (as in Definition 6.1).

If R is a convex subset of $\mathbb{E}^2$, we are done.

If R is not convex, the total internal angle of R at $\tilde{x}^1$ or $\tilde{x}^{n-1}$ or both is $> \pi$. By relabeling we may suppose this holds for $\tilde{x}^{n-1}$.

The region R is obtained by gluing R_{n-1} to $\dot{R}$ by $[x^1x^{n-1}]$. Thus, by Reshetnyak gluing (5.14), R carrying its length metric is a CAT(0)-space. Moreover $[\tilde{x}^{n-2}\tilde{x}^{n-1}] \cup [\tilde{x}^{n-1}\dot{x}^n]$ is a geodesic of R. Thus $[\tilde{x}^1\tilde{x}^2\dots\tilde{x}^{n-2}\dot{x}^n]_R$ is a closed $(n-1)$-gon in R, to which the induction hypothesis applies. The resulting short map from a convex region in $\mathbb{E}^2$ to R, followed by F, is the desired majorizing map. $\square$

If $p_1\dots p_n$ is a polygon, then values $\theta_i = \pi - \measuredangle[p_i\,{}^{p_{i-1}}_{p_{i+1}}]$ for all i (mod n) are called external angles of the polygon. The following exercise is a generalization of Fenchel's theorem.

6.11. Exercise. *Show that the sum of external angles of any polygon in a complete length* CAT(0) *space cannot be smaller than* $2{\cdot}\pi$.

The following exercise is a version of the Fáry–Milnor theorem for CAT(0) spaces.

6.12. Very advanced exercise. *Suppose that a simple polygon β in a complete length* CAT(0) *space does not bound an embedded disc. Show that the sum of external angles of β cannot be smaller than* $4{\cdot}\pi$.

Give an example of such a polygon β with the sum of external angles exactly $4{\cdot}\pi$.

6.13. Exercise. *Prove the following generalization of the arm lemma (4.4).*

6.14. Arm lemma. *Let $P = [x^0x^1\dots x^{n+1}]$ be a polygon in a geodesic* CAT(0) *space $\mathcal{U}$. Suppose $\tilde{P} = [\tilde{x}^0\tilde{x}^1\dots\tilde{x}^{n+1}]$ is a convex polygon in $\mathbb{E}^2$ such that*

❶ $$|\tilde{x}^i - \tilde{x}^{i-1}|_{\mathbb{E}^2} = |x^i - x^{i-1}|_{\mathcal{U}} \quad \text{and} \quad \measuredangle[x^i\,{}^{x^{i-1}}_{x^{i+1}}] \geqslant \measuredangle[\tilde{x}^i\,{}^{\tilde{x}^{i-1}}_{\tilde{x}^{i+1}}]$$

for all i. Then $|\tilde{x}^0 - \tilde{x}^{n+1}|_{\mathbb{E}^2} \leqslant |x^0 - x^{n+1}|_{\mathcal{U}}$.

D General case

If the space is proper, then the general case follows applying polygonal case to inscribed polygonal lines and passing to the limit. The statement holds for any geodesic CAT(0) space but one need to be more careful [6].

The following exercise is the rigidity case of the majorization theorem.

6.15. Exercise. *Let $\mathcal{U}$ be a geodesic* CAT(0) *space and $\alpha\colon [0,\ell] \to \mathcal{U}$ be a closed curve with arclength parametrization. Assume there is a closed convex curve $\tilde{\alpha}\colon [0,\ell] \to \mathbb{E}^2$ such that*

$$|\alpha(t_0) - \alpha(t_1)|_{\mathcal{U}} = |\tilde{\alpha}(t_0) - \tilde{\alpha}(t_1)|_{\mathbb{E}^2}$$

for any t_0 and t_1. Show that there is a distance-preserving map $F\colon \operatorname{Conv}\tilde\alpha \to \mathcal{U}$ such that $F\colon \tilde\alpha(t) \mapsto \alpha(t)$ for any t.

6.16. Exercise. *Two majorizations $F\colon D \to \mathcal{U}$ and $F'\colon D' \to \mathcal{U}$ will be called equivalent if $F' = F \circ \iota$ for an isometry $\iota\colon D \to D'$.*

Show that a closed rectifiable curve in a CAT(0) *space has an isometric majorization map if and only if the majorization map is unique up to equivalence.*

E Comments

The statements in this section can be generalized to CAT(±1) spaces; in the CAT(1) case one has to assume that the closed curve has length at most $2{\cdot}\pi$.

The majorization theorem was proved by Yuriy Reshetnyak [84]; our proof uses a trick that we learned from the lectures of Werner Ballmann [17]. Another proof can be built on generalized Kirszbraun's theorem, but it works only for complete spaces.

The definition of develpoment appears in [13] and an earlier form of it can be found in [65].

6.17. Open problem. *Let α be a closed rectifiable curve in a* CAT(0) *space $\mathcal{U}$. Note that if α is a geodesic triangle or it bounds an isometric copy of convex plane figure in $\mathcal{U}$, then α has a unique (up to congruence) majorizing convex figure.*

What about the converse?

Lecture 7

Globalization for CATs

This lecture is nearly a copy of [5, Sections 3.1–3.3]; here we introduce locally CAT(0) spaces and prove the globalization theorem that provides a sufficient condition for locally CAT(0) spaces to be globally CAT(0).

A Locally CAT spaces

We say that a space $\mathcal{U}$ is *locally CAT(0)* (or *locally CAT(1)*) if a small closed ball centered at any point p in $\mathcal{U}$ is CAT(0) (or CAT(1), respectively).

For example, the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is locally isometric to $\mathbb{R}$, and so $\mathbb{S}^1$ is locally CAT(0). On the other hand, $\mathbb{S}^1$ is not CAT(0), since closed local geodesics in $\mathbb{S}^1$ are not geodesics, so $\mathbb{S}^1$ does not meet 5.7.

If $\mathcal{U}$ is a proper geodesic space, then it is locally CAT(0) (or locally CAT(1)) if and only if each point $p \in \mathcal{U}$ admits an open neighborhood Ω that is geodesic and such that any triangle in Ω is thin (or spherically thin, respectively).

B Space of local geodesic paths

A constant-speed parameterization of a local geodesic by the unit interval $[0, 1]$ is called a *local geodesic path*.

In this section, we will study the behavior of local geodesics in locally CAT(κ) spaces. The results will be used in the proof of the globalization theorem (7.6).

Recall that a *path* is a curve parametrized by $[0, 1]$. The space of paths in a metric space $\mathcal{U}$ comes with the natural metric

❶ $$|\alpha - \beta| = \sup\{\, |\alpha(t) - \beta(t)|_{\mathcal{U}} \,:\, t \in [0,1]\,\}\,.$$

7.1. Proposition. *Let $\mathcal{U}$ be a proper geodesic, locally* CAT(κ) *space.*

Assume $\gamma_n\colon [0,1] \to \mathcal{U}$ is a sequence of local geodesic paths converging to a path $\gamma_\infty\colon [0,1] \to \mathcal{U}$. Then γ_∞ is a local geodesic path. Moreover

$$\operatorname{length}\gamma_n \to \operatorname{length}\gamma_\infty$$

as $n \to \infty$.

Proof; CAT(0) *case.* Fix $t \in [0,1]$. Let $R > 0$ be sufficiently small, so that $\overline{\mathrm{B}}[\gamma_\infty(t), R]$ forms a proper geodesic CAT(0) space.

Assume that a local geodesic σ is shorter than $R/2$ and intersects the ball $\mathrm{B}(\gamma_\infty(t), R/2)$. Then σ cannot leave the ball $\overline{\mathrm{B}}[\gamma_\infty(t), R]$. By 5.7, σ is a geodesic. In particular, for all sufficiently large n, any arc of γ_n of length $R/2$ or less containing $\gamma_n(t)$ is a geodesic.

Since $\mathcal{B} = \overline{\mathrm{B}}[\gamma_\infty(t), R]$ is a proper geodesic CAT(0) space, by 5.1, geodesic segments in $\mathcal{B}$ depend uniquely on their endpoint pairs. Thus there is a subinterval $\mathbb{I}$ of $[0,1]$, that contains a neighborhood of t in $[0,1]$ and such that the arc $\gamma_n|_{\mathbb{I}}$ is minimizing for all large n. It follows that $\gamma_\infty|_{\mathbb{I}}$ is a geodesic, and therefore γ_∞ is a local geodesic.

The CAT(1) case is done in the same way, but one has to assume in addition that $R < \pi$. □

The following lemma allows a local geodesic path to be moved continuously so that its endpoints follow given trajectories.

7.2. Patchwork along a geodesic. *Let $\mathcal{U}$ be a proper geodesic, locally* CAT(0) *space, and $\gamma\colon [0,1] \to \mathcal{U}$ be a locally geodesic path.*

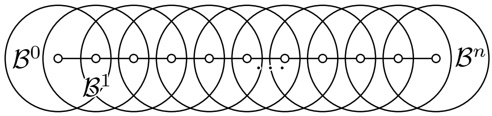
Then there is a proper geodesic CAT(0) *space $\mathcal{N}$, an open set $\hat{\Omega} \subset \mathcal{N}$, and a geodesic path $\hat{\gamma}\colon [0,1] \to \hat{\Omega}$, such that there is an open locally distance-preserving map $\Phi\colon \hat{\Omega} \looparrowright \mathcal{U}$ satisfying $\Phi \circ \hat{\gamma} = \gamma$.*

If $\operatorname{length}\gamma < \pi$, *then the same holds in the* CAT(1) *case. Namely, we assume that $\mathcal{U}$ is a proper geodesic, locally* CAT(1) *space and construct a proper geodesic* CAT(1) *space $\mathcal{N}$ with the same property as above.*

Proof. Fix $r > 0$ so that for each $t \in [0,1]$, the closed ball $\overline{\mathrm{B}}[\gamma(t), r]$ forms a proper geodesic CAT(0) space.

Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that

$$\mathrm{B}(\gamma(t_i), r) \supset \gamma([t_{i-1}, t_i])$$



for all $n > i > 0$. Set $\mathcal{B}_i = \overline{\mathrm{B}}[\gamma(t_i), r]$. We can assume in addition that $\mathcal{B}_{i-1} \cap \mathcal{B}_{i+1} \subset \mathcal{B}_i$ if $0 < i < n$.

Consider the disjoint union $\bigsqcup_i \mathcal{B}_i = \{ (i, x) : x \in \mathcal{B}_i \}$ with the minimal equivalence relation $\sim$ such that $(i, x) \sim (i-1, x)$ for all i. Let $\mathcal{N}$ be the space obtained by gluing the $\mathcal{B}_i$ along $\sim$.

Note that $A_i = \mathcal{B}_i \cap \mathcal{B}_{i-1}$ is convex in $\mathcal{B}_i$ and in $\mathcal{B}_{i-1}$. Applying the Reshetnyak gluing theorem (5.14) n times, we conclude that $\mathcal{N}$ is a proper geodesic CAT(0) space.

For $t \in [t_{i-1}, t_i]$, define $\hat{\gamma}(t)$ as the equivalence class of $(i, \gamma(t))$ in $\mathcal{N}$. Let $\hat{\Omega}$ be the ε-neighborhood of $\hat{\gamma}$ in $\mathcal{N}$, where $\varepsilon > 0$ is chosen so that $\mathrm{B}(\gamma(t), \varepsilon) \subset \mathcal{B}_i$ for all $t \in [t_{i-1}, t_i]$.

Define $\Phi\colon \hat{\Omega} \to \mathcal{U}$ by sending the equivalence class of (i, x) to x. It is straightforward to check that Φ, $\hat{\gamma}$, and $\hat{\Omega} \subset \mathcal{N}$ satisfy the conclusion of the lemma.

The CAT(1) case is proved in the same way. □

Recall that local geodesics are geodesics in any CAT(0) space; see 5.7. Using it with 7.2 and the uniqueness of geodesics (5.7), we get the following.

7.3. Corollary. *If $\mathcal{U}$ is a proper geodesic, locally* CAT(0) *space, then for any pair of points $p, q \in \mathcal{U}$, the space of all local geodesic paths from p to q is discrete; that is, for any local geodesic path γ connecting p to q, there is $\varepsilon > 0$ such that for any other local geodesic path δ from p to q we have $|\gamma(t) - \delta(t)|_{\mathcal{U}} > \varepsilon$ for some $t \in [0, 1]$.*

Analogously, if $\mathcal{U}$ is a proper geodesic, locally CAT(1) *space, then for any pair of points $p, q \in \mathcal{U}$, the space of all local geodesic paths shorter than π from p to q is discrete.*

7.4. Corollary. *If $\mathcal{U}$ is a proper geodesic, locally* CAT(0) *space, then for any path α there is a choice of local geodesic path γ_α connecting the ends of α such that the map $\alpha \mapsto \gamma_\alpha$ is continuous, and if α is a local geodesic path then $\gamma_\alpha = \alpha$.*

Analogously, if $\mathcal{U}$ is a proper geodesic, locally CAT(1) *space, then for any path α shorter than π, there is a choice of local geodesic path γ_α shorter than π connecting the ends of α such that the map $\alpha \mapsto \gamma_\alpha$ is continuous, and if α is a local geodesic path then $\gamma_\alpha = \alpha$.*

Proof of 7.4. We do the CAT(0) case; the CAT(1) case is analogous.

Consider the maximal interval $\mathbb{I} \subset [0,1]$ containing 0 such that there is a continuous one-parameter family of local geodesic paths γ_t for $t \in \mathbb{I}$ connecting $\alpha(0)$ to $\alpha(t)$, with $\gamma_t(0) = \gamma_0(t) = \alpha(0)$ for any t.

By 7.1, $\mathbb{I}$ is closed, so we may assume $\mathbb{I} = [0,s]$ for some $s \in [0,1]$.

Applying patchwork (7.2) to γ_s, we find that $\mathbb{I}$ is also open in $[0,1]$. Hence $\mathbb{I} = [0,1]$. Set $\gamma_\alpha = \gamma_1$.

By construction, if α is a local geodesic path, then $\gamma_\alpha = \alpha$.

Moreover, from 7.3, the construction $\alpha \mapsto \gamma_\alpha$ produces close results for sufficiently close paths in the metric defined by ❶; that is, the map $\alpha \mapsto \gamma_\alpha$ is continuous. ☐

Given a path $\alpha\colon [0,1] \to \mathcal{U}$, we denote by $\bar{\alpha}$ the same path traveled in the opposite direction; that is,

$$\bar{\alpha}(t) = \alpha(1-t).$$

The product of two paths will be denoted with "$*$"; if two paths α and β connect the same pair of points, then the product $\bar{\alpha} * \beta$ is a closed curve.

7.5. Exercise. *Assume $\mathcal{U}$ is a proper geodesic, locally* CAT(1) *space. Consider the construction $\alpha \mapsto \gamma_\alpha$ provided by Corollary 7.4.*

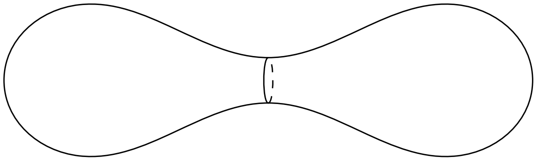
Assume that α and β are two paths connecting the same pair of points in $\mathcal{U}$, where each is shorter than π and the product $\bar{\alpha} * \beta$ is null-homotopic in the class of closed curves shorter than $2\cdot\pi$. Show that $\gamma_\alpha = \gamma_\beta$.

C Globalization

7.6. Globalization theorem. *If a proper geodesic, locally* CAT(0) *space is simply connected, then it is* CAT(0).

Analogously, if $\mathcal{U}$ is a proper geodesic, locally CAT(1) *space such that any closed curve $\gamma\colon \mathbb{S}^1 \to \mathcal{U}$ shorter than $2\cdot\pi$ is null-homotopic in the class of closed curves shorter than $2\cdot\pi$. Then $\mathcal{U}$ is* CAT(1).

The surface on the diagram is an example of a simply connected space that is locally CAT(1) but not CAT(1). To contract the marked



curve one has to increase its length to $2\cdot\pi$ or more; in particular, the surface does not satisfy the assumption of the globalization theorem.

The proof of the globalization theorem relies on the following theorem, which is essentially [10, Satz 9].

7.7. Patchwork globalization theorem. *A proper geodesic, locally* CAT(0) *space* $\mathcal{U}$ *is* CAT(0) *if and only if all pairs of points in* $\mathcal{U}$ *are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

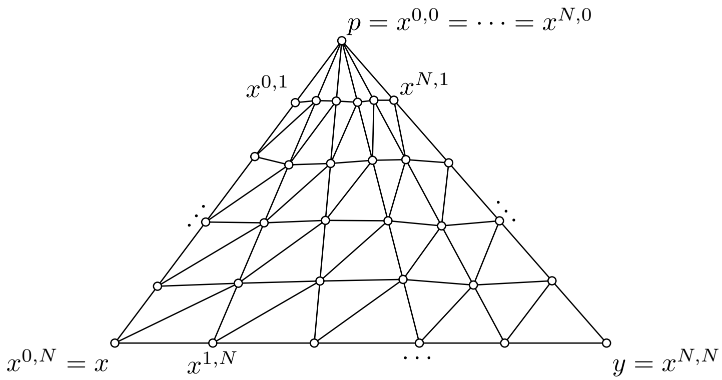
Analogously, a proper geodesic, locally CAT(1) *space* $\mathcal{U}$ *is* CAT(1) *if and only if all pairs of points in* $\mathcal{U}$ *at distance less than* π *are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

The proof uses a thin-triangle decomposition with the inheritance lemma (5.13) and the line-of-sight map (6.5).

Proof of the patchwork globalization theorem (7.7). Note that the implication "only if" follows from 5.1 and 5.9; it remains to prove the "if" part.

Fix a triangle $[pxy]$ in $\mathcal{U}$. We need to show that $[pxy]$ is thin.

By the assumptions, the line-of-sight map $(t,s)\mapsto\gamma_t(s)$ from p to $[xy]$ is uniquely defined and continuous.



Fix a partition

$$0 = t^0 < t^1 < \ldots < t^N = 1,$$

and set $x^{i,j} = \gamma_{t^i}(t^j)$. Since the line-of-sight map is continuous and $\mathcal{U}$ is locally CAT(0), we may assume that the triangles

$$[x^{i,j}x^{i,j+1}x^{i+1,j+1}] \quad \text{and} \quad [x^{i,j}x^{i+1,j}x^{i+1,j+1}]$$

are thin for each pair i, j.

Now we show that the thin property propagates to $[pxy]$ by repeated application of the inheritance lemma (5.13):

- ◇ For fixed i, sequentially applying the lemma shows that the triangles $[px^{i,1}x^{i+1,2}]$, $[px^{i,2}x^{i+1,2}]$, $[px^{i,2}x^{i+1,3}]$, and so on are thin.

In particular, for each i, the long triangle $[px^{i,N}x^{i+1,N}]$ is thin.

- ◇ By the same lemma the triangles $[px^{0,N}x^{2,N}]$, $[px^{0,N}x^{3,N}]$, and so on, are thin.

In particular, $[pxy] = [px^{0,N}x^{N,N}]$ is thin. □

Proof of the globalization theorem; CAT(0) *case.* Let $\mathcal{U}$ be a proper geodesic, locally CAT(0) space that is simply connected. Given a path α in $\mathcal{U}$, denote by γ_α the local geodesic path provided by 7.4. Since the map $\alpha \mapsto \gamma_\alpha$ is continuous, by 7.3 we have $\gamma_\alpha = \gamma_\beta$ for any pair of paths α and β homotopic relative to the ends.

Since $\mathcal{U}$ is simply connected, any pair of paths with common ends are homotopic. In particular, if α and β are local geodesics from p to q, then $\alpha = \gamma_\alpha = \gamma_\beta = \beta$ by Corollary 7.4. It follows that any two points $p, q \in \mathcal{U}$ are joined by a unique local geodesic that depends continuously on (p, q).

Since $\mathcal{U}$ is geodesic, it remains to apply the patchwork globalization theorem (7.7).

CAT(1) *case.* The proof goes along the same lines, but one needs to use Exercise 7.5. □

7.8. Corollary. *Any compact geodesic, locally* CAT(0) *space that contains no closed local geodesics is* CAT(0).

Analogously, any compact geodesic, locally CAT(1) *space that contains no closed local geodesics shorter than* $2{\cdot}\pi$ *is* CAT(1).

Proof. By the globalization theorem (7.6), we need to show that the space is simply connected. Assume the contrary. Fix a nontrivial homotopy class of closed curves.

Denote by ℓ the exact lower bound for the lengths of curves in the class. Note that $\ell > 0$; otherwise, there would be a closed noncontractible curve in a CAT(0) neighborhood of some point, contradicting 5.3.

Since the space is compact, the class contains a length-minimizing curve, which must be a closed local geodesic.

The CAT(1) case is analogous, one only has to consider a homotopy class of closed curves shorter than $2{\cdot}\pi$. □

7.9. Exercise. *Prove that any compact geodesic, locally* CAT(0) *space* $\mathcal{X}$ *that is not* CAT(0) *contains a* g e o d e s i c c i r c l e; *that is, a simple* *closed curve γ such that for any two points $p, q \in \gamma$, one of the arcs of γ with endpoints p and q is a geodesic.*

Formulate and prove the analogous statement for CAT(1) *spaces.*

7.10. Advanced exercise. *Let $\mathcal{U}$ be a proper geodesic* CAT(0) *space. Assume $\tilde{\mathcal{U}} \to \mathcal{U}$ is a metric double cover branching along a geodesic γ. (Formally speaking, $\tilde{\mathcal{U}}$ is completion of a double cover of the complement $\mathcal{U} \setminus \gamma$. For example, 3-dimensional Euclidean space admits a double cover branching along a line.)*

Show that $\tilde{\mathcal{U}}$ is CAT(0)*.*

D Remarks

The lemma about patchwork along a geodesic and its proof were suggested to us by Alexander Lytchak. This statement was originally proved by Stephanie Alexander and Richard Bishop [4] using a different method.

As was mentioned earlier, the motivation for the notion of CAT(κ) spaces comes from the fact that a Riemannian manifold is locally CAT(κ) if and only if it has sectional curvature at most κ. This easily follows from Rauch comparison for Jacobi fields and Proposition 5.6.

In the globalization theorem (7.6), properness can be weakened to completeness [see 6, and the references therein]. The original formulation of the globalization theorem, or Hadamard–Cartan theorem, states that *if M is a complete Riemannian manifold with sectional curvature at most* 0*, then the exponential map at any point $p \in M$ is a covering*; in particular, it implies that *the universal cover of M is diffeomorphic to the Euclidean space of the same dimension.*

In this generality, this theorem appeared in the lectures of Elie Cartan [33]. This theorem was proved for surfaces in Euclidean 3-space by Hans von Mangoldt [68] and a few years later independently for two-dimensional Riemannian manifolds by Jacques Hadamard [52].

Formulations for metric spaces of different generality were proved by Herbert Busemann [31], Willi Rinow [85], Mikhael Gromov [49, p. 119]. A detailed proof of Gromov's statement was given by Werner Ballmann [16] when $\mathcal{U}$ is proper, and by Stephanie Alexander and Richard Bishop [4] in more generality.

For proper CAT(1) spaces, the globalization theorem was proved by Brian Bowditch [21].

The globalization theorem holds for complete length spaces (not necessarily proper spaces) [6].

The patchwork globalization (7.7) is proved by Alexandrov [10, Satz 9]. For proper spaces one can remove the continuous dependence from the formulation; it follows from uniqueness. For complete spaces, the latter is not true [23, Chapter I, Exercise 3.14].

Lecture 8

Polyhedral spaces

This lecture is nearly a copy of [5, Sections 3.4–3.8]; here we describe a set of rules for gluing Euclidean cubes that produce a locally CAT(0) space and use these rules to construct exotic examples of aspherical manifolds.

A Products, cones, and suspension

Given two metric spaces $\mathcal{U}$ and $\mathcal{V}$, the product space $\mathcal{U}\times\mathcal{V}$ is defined as the set of all pairs (u,v) where $u\in\mathcal{U}$ and $v\in\mathcal{V}$ with the metric defined by Pythagorean theorem

$$|(u_1,v_1)-(u_2,v_2)|_{\mathcal{U}\times\mathcal{V}}=\sqrt{|u_1-u_2|^2_{\mathcal{U}}+|v_1-v_2|^2_{\mathcal{V}}}.$$

8.1. Proposition. *Let $\mathcal{U}$ and $\mathcal{V}$ be* CAT(0) *spaces. Then the product space $\mathcal{U}\times\mathcal{V}$ is* CAT(0).

Proof. Fix a quadruple in $\mathcal{U}\times\mathcal{V}$:

$$p=(p_1,p_2),\qquad q=(q_1,q_2),\qquad x=(x_1,x_2),\qquad y=(y_1,y_2).$$

For the quadruple p_1,q_1,x_1,y_1 in $\mathcal{U}$, construct two model triangles $[\tilde p_1\tilde x_1\tilde y_1]=\tilde\triangle(p_1x_1y_1)_{\mathbb{E}^2}$ and $[\tilde q_1\tilde x_1\tilde y_1]=\tilde\triangle(q_1x_1y_1)_{\mathbb{E}^2}$. Similarly, for the quadruple p_2,q_2,x_2,y_2 in $\mathcal{V}$ construct two model triangles $[\tilde p_2\tilde x_2\tilde y_2]$ and $[\tilde q_2\tilde x_2\tilde y_2]$.

Consider four points in $\mathbb{E}^4=\mathbb{E}^2\times\mathbb{E}^2$

$$\tilde p=(\tilde p_1,\tilde p_2),\qquad \tilde q=(\tilde q_1,\tilde q_2),\qquad \tilde x=(\tilde x_1,\tilde x_2),\qquad \tilde y=(\tilde y_1,\tilde y_2).$$

Note that the triangles $[\tilde{p}\tilde{x}\tilde{y}]$ and $[\tilde{q}\tilde{x}\tilde{y}]$ in $\mathbb{E}^4$ are isometric to the model triangles $\tilde{\triangle}(pxy)_{\mathbb{E}^2}$ and $\tilde{\triangle}(qxy)_{\mathbb{E}^2}$.

If $\tilde{z} = (\tilde{z}_1, \tilde{z}_2) \in [\tilde{x}\tilde{y}]$, then $\tilde{z}_1 \in [\tilde{x}_1\tilde{y}_1]$ and $\tilde{z}_2 \in [\tilde{x}_2\tilde{y}_2]$ and

$$\begin{aligned}
|\tilde{z} - \tilde{p}|^2_{\mathbb{E}^4} &= |\tilde{z}_1 - \tilde{p}_1|^2_{\mathbb{E}^2} + |\tilde{z}_2 - \tilde{p}_2|^2_{\mathbb{E}^2},\\
|\tilde{z} - \tilde{q}|^2_{\mathbb{E}^4} &= |\tilde{z}_1 - \tilde{q}_1|^2_{\mathbb{E}^2} + |\tilde{z}_2 - \tilde{q}_2|^2_{\mathbb{E}^2},\\
|p - q|^2_{\mathcal{U}\times\mathcal{V}} &= |p_1 - q_1|^2_{\mathcal{U}} + |p_2 - q_2|^2_{\mathcal{V}}.
\end{aligned}$$

Therefore CAT(0) comparison for the quadruples p_1, q_1, x_1, y_1 in $\mathcal{U}$ and p_2, q_2, x_2, y_2 in $\mathcal{V}$ implies CAT(0) comparison for the quadruples p, q, x, y in $\mathcal{U} \times \mathcal{V}$. $\square$

8.2. Exercise. *Assume $\mathcal{U}$ and $\mathcal{V}$ are* CBB(0) *spaces. Show that the product space $\mathcal{U} \times \mathcal{V}$ is* CBB(0).

Recall that cone $\mathcal{V} = \operatorname{Cone}\mathcal{U}$ over a metric space $\mathcal{U}$ is defined as the metric space whose underlying set consists of equivalence classes in $[0,\infty) \times \mathcal{U}$ with the equivalence relation "$\sim$" given by $(0,p) \sim (0,q)$ for any points $p, q \in \mathcal{U}$, and whose metric is given by the cosine rule

$$|(p,s) - (q,t)|_{\mathcal{V}} = \sqrt{s^2 + t^2 - 2{\cdot}s{\cdot}t{\cdot}\cos\alpha},$$

where $\alpha = \min\{\pi, |p-q|_{\mathcal{U}}\}$. Points in $\mathcal{V}$ might be called vectors, they come with the norm, scalar product, and multiplication by nonnegative reals. The space $\mathcal{U}$ can be identified with the subset $x \in \mathcal{V}$ such that $|x| = 1$.

8.3. Proposition. *Let $\mathcal{U}$ be a metric space. Then* $\operatorname{Cone}\mathcal{U}$ *is* CAT(0) *if and only if $\mathcal{U}$ is* CAT(1).

Proof; if part. Given a point $x \in \operatorname{Cone}\mathcal{U}$, denote by x' its projection to $\mathcal{U}$ and by $|x|$ the distance from x to the tip of the cone; if x is the tip, then $|x| = 0$ and we can take any point of $\mathcal{U}$ as x'.

Let p, q, x, y be a quadruple in $\operatorname{Cone}\mathcal{U}$. Assume that the spherical model triangles $[\tilde{p}'\tilde{x}'\tilde{y}']_{\mathbb{S}^2} = \tilde{\triangle}(p'x'y')_{\mathbb{S}^2}$ and $[\tilde{q}'\tilde{x}'\tilde{y}']_{\mathbb{S}^2} = \tilde{\triangle}(q'x'y')_{\mathbb{S}^2}$ are defined. Consider the following points in $\mathbb{E}^3 = \operatorname{Cone}\mathbb{S}^2$:

$$\tilde{p} = |p|{\cdot}\tilde{p}', \qquad \tilde{q} = |q|{\cdot}\tilde{q}', \qquad \tilde{x} = |x|{\cdot}\tilde{x}', \qquad \tilde{y} = |y|{\cdot}\tilde{y}'.$$

Note that $[\tilde{p}\tilde{x}\tilde{y}]_{\mathbb{E}^3} \overset{iso}{=\!=} \tilde{\triangle}(pxy)_{\mathbb{E}^2}$ and $[\tilde{q}\tilde{x}\tilde{y}]_{\mathbb{E}^3} \overset{iso}{=\!=} \tilde{\triangle}(qxy)_{\mathbb{E}^2}$. Further, note that if $\tilde{z} \in [\tilde{x}\tilde{y}]_{\mathbb{E}^3}$, then $\tilde{z}' = \tilde{z}/|\tilde{z}|$ lies on the geodesic $[\tilde{x}'\tilde{y}']_{\mathbb{S}^2}$. Therefore the CAT(1) comparison for $|p'-q'|$ with $\tilde{z}' \in [\tilde{x}'\tilde{y}']_{\mathbb{S}^2}$ implies the CAT(0) comparison for $|p - q|$ with $\tilde{z} \in [\tilde{x}\tilde{y}]_{\mathbb{E}^3}$.

If at least one of the model triangles $\tilde\triangle(p'x'y')_{\mathbb{S}^2}$ and $\tilde\triangle(q'x'y')_{\mathbb{S}^2}$ is undefined, then the statement follows from the triangle inequalities

$$
\begin{aligned}
|p'-x'|_{\mathcal{U}}+|q'-x'|_{\mathcal{U}} &\geqslant |p'-q'|_{\mathcal{U}}\\
|p'-y'|_{\mathcal{U}}+|q'-y'|_{\mathcal{U}} &\geqslant |p'-q'|_{\mathcal{U}}
\end{aligned}
$$

This case is left as an exercise.

Only-if part. Suppose that $\tilde p'$, $\tilde q'$, $\tilde x'$, $\tilde y'$ are defined as above. Assume all these points lie in a half-space of $\mathbb{E}^3=\operatorname{Cone}\mathbb{S}^2$ with origin at its boundary. Then we can choose positive values a, b, c, and d such that the points $a{\cdot}\tilde p'$, $b{\cdot}\tilde q'$, $c{\cdot}\tilde x'$, $d{\cdot}\tilde y'$ lie in one plane. Consider the corresponding points $a{\cdot}p'$, $b{\cdot}q'$, $c{\cdot}x'$, $d{\cdot}y'$ in $\operatorname{Cone}\mathcal{U}$. Applying the CAT(0) comparison for these points leads to CAT(1) comparison for the quadruple p', q', x', y' in $\mathcal{U}$.

It remains to consider the case when $\tilde p'$, $\tilde q'$, $\tilde x'$, $\tilde y'$ do not in a half-space. Fix $\tilde z'\in[\tilde x'\tilde y']_{\mathbb{S}^2}$. Observe that

$$
|\tilde p'-\tilde x'|_{\mathbb{S}^2}+|\tilde q'-\tilde x'|_{\mathbb{S}^2}\leqslant|\tilde p'-\tilde z'|_{\mathbb{S}^2}+|\tilde q'-\tilde z'|_{\mathbb{S}^2}
$$

or

$$
|\tilde p'-\tilde y'|_{\mathbb{S}^2}+|\tilde q'-\tilde y'|_{\mathbb{S}^2}\leqslant|\tilde p'-\tilde z'|_{\mathbb{S}^2}+|\tilde q'-\tilde z'|_{\mathbb{S}^2}.
$$

That is, in this case, the CAT(1) comparison follows from the triangle inequality. □

A similar argument gives the following, but the proof requires the globalization theorem that will be proved much latter.

8.4. Proposition. *Let $\mathcal{L}$ be a geodesic space. Then* $\operatorname{Cone}\mathcal{L}$ *is* CBB(0) *if and only if $\mathcal{L}$ is* CBB(1).

Suspension is a spherical analog of cone construction.

The suspension $\mathcal{V}=\operatorname{Susp}\mathcal{U}$ over a metric space $\mathcal{U}$ is defined as the metric space whose underlying set consists of equivalence classes in $[0,\pi]\times\mathcal{U}$ with the equivalence relation "$\sim$" given by $(0,p)\sim(0,q)$ and $(\pi,p)\sim(\pi,q)$ for any points $p,q\in\mathcal{U}$, and whose metric is given by the spherical cosine rule

$$
\cos|(p,s)-(q,t)|_{\operatorname{Susp}\mathcal{U}}=\cos s\cdot\cos t-\sin s\cdot\sin t\cdot\cos\alpha,
$$

where $\alpha=\min\{\pi,|p-q|_{\mathcal{U}}\}$.

The points in $\mathcal{V}$ formed by the equivalence classes of $0\times\mathcal{U}$ and $\pi\times\mathcal{U}$ are called the north and the south poles of the suspension.

8.5. Exercise. *Let $\mathcal{U}$ be a metric space. Show that the spaces*

$$\mathbb{R} \times \mathrm{Cone}\,\mathcal{U} \quad \textit{and} \quad \mathrm{Cone}[\mathrm{Susp}\,\mathcal{U}]$$

are isometric.

The following statement is a direct analog of 8.3 and it can be proved along the same lines.

8.6. Proposition. *Let $\mathcal{U}$ be a metric space. Then a neighborhood $\mathcal{N}$ of the north in* $\mathrm{Susp}\,\mathcal{U}$ *is* CAT(1) *if and only if $\mathcal{U}$ is* CAT(1).

B Polyhedral spaces

8.7. Definition. *A geodesic space $\mathcal{P}$ is called a (spherical) polyhedral space if it admits a finite triangulation τ such that every simplex in τ is isometric to a simplex in a Euclidean space (or respectively a unit sphere) of appropriate dimension.*

By triangulation of a polyhedral space, we will always understand a triangulation as above.

Note that according to the above definition, all polyhedral spaces are compact.

The dimension of a polyhedral space $\mathcal{P}$ is defined as the maximal dimension of the simplices in one (and therefore any) triangulation of $\mathcal{P}$.

Links. Let $\mathcal{P}$ be a polyhedral space and σ be a simplex in a triangulation τ of $\mathcal{P}$.

The simplices that contain σ form an abstract simplicial complex called the link of σ, denoted by Link_σ. If m is the dimension of σ, then the set of vertices of Link_σ is formed by the $(m+1)$-simplices that contain σ; the set of its edges is formed by the $(m+2)$-simplices that contain σ; and so on.

The link Link_σ can be identified with the subcomplex of τ formed by all the simplices σ' such that $\sigma \cap \sigma' = \varnothing$ but both σ and σ' are faces of a simplex of τ.

The points in Link_σ can be identified with the normal directions to σ at a point in its interior. The angle metric between directions makes Link_σ into a spherical polyhedral space. We will always consider the link with this metric.

Tangent space and space of directions. Let $\mathcal{P}$ be a polyhedral space (Euclidean or spherical) and τ be its triangulation. If a point

$p \in \mathcal{P}$ lies in the interior of a k-simplex σ of τ then the tangent space $\mathrm{T}_p = \mathrm{T}_p\mathcal{P}$ is naturally isometric to

$$\mathbb{E}^k \times (\mathrm{Cone}\,\mathrm{Link}_\sigma).$$

If $\mathcal{P}$ is an m-dimensional polyhedral space, then for any $p \in \mathcal{P}$ the space of directions Σ_p is a spherical polyhedral space of dimension at most $m - 1$.

In particular, for any point p in σ, the isometry class of Link_σ together with $k = \dim \sigma$ determines the isometry class of Σ_p, and the other way around — Σ_p and k determines the isometry class of Link_σ.

A small neighborhood of p is isometric to a neighborhood of the tip of $\mathrm{Cone}\,\Sigma_p$. In fact, if this property holds at any point of a compact length space $\mathcal{P}$, then $\mathcal{P}$ is a polyhedral space [64].

C CAT test

The following theorem provides a combinatorial description of polyhedral spaces with curvature bounded above.

8.8. Theorem. *Let $\mathcal{P}$ be a polyhedral space and τ be its triangulation. Then $\mathcal{P}$ is locally* CAT(0) *if and only if the link of each simplex in τ has no closed local geodesic shorter than $2{\cdot}\pi$.*

Analogously, let $\mathcal{P}$ be a spherical polyhedral space and τ be its triangulation. Then $\mathcal{P}$ is CAT(1) *if and only if neither $\mathcal{P}$ nor the link of any simplex in τ has a closed local geodesic shorter than $2{\cdot}\pi$.*

Proof. The "only if" part follows from 5.7, 8.6, and 8.3.

To prove the "if" part, we apply induction on $\dim \mathcal{P}$. The base case $\dim \mathcal{P} = 0$ is evident. Let us start with the CAT(1) case.

Step. Assume that the theorem is proved in the case $\dim \mathcal{P} < m$. Suppose $\dim \mathcal{P} = m$.

Fix a point $p \in \mathcal{P}$. A neighborhood of p is isometric to a neighborhood of the north pole in the suspension over the space of directions Σ_p.

Note that Σ_p is a spherical polyhedral space, and its links are isometric to links of $\mathcal{P}$. By the induction hypothesis, Σ_p is CAT(1). Thus, by the second part of Exercise 8.3, $\mathcal{P}$ is locally CAT(1).

Applying the second part of Corollary 7.8, we get the statement.

The CAT(0) case is done in exactly the same way except we need to use the first part of Exercise 8.3 and the first part of Corollary 7.8 on the last step. $\square$

8.9. Exercise. *Let $\mathcal{P}$ be a polyhedral space such that any two points can be connected by a unique geodesic. Show that $\mathcal{P}$ is* CAT(0).

8.10. Advanced exercise. *Construct a Euclidean polyhedral metric on $\mathbb{S}^3$ such that the total angle around each edge in its triangulation is at least $2{\cdot}\pi$.*

Let us formulate an analogous test for spaces with lower curvature bound.

8.11. Theorem. *Let $\mathcal{P}$ be a polyhedral space and τ be a triangulation of $\mathcal{P}$. Then $\mathcal{P}$ is* CBB(0) *if and only if the following conditions hold.*

(a) *τ is pure; that is, any simplex in τ is a face of some simplex of dimension exactly m.*

(b) *The link of any simplex of dimension $m-1$ is formed by single point or two points.*

(c) *The link of any simplex of dimension $\leqslant m-2$ is connected.*

(d) *Any link of any simplex of dimension $m-2$ has diameter at most π.*

The condition *(c)* can be reformulated in the following way:

(c)′ Any path γ in $\mathcal{P}$ is a limit of paths that cross only simplexes of dimension m and $m-1$.

Further, modulo the other conditions, the condition *(d)* is equivalent to the following:

(d)′ The link of any simplex of dimension $m-2$ is isometric to a circle of length $\leqslant 2{\cdot}\pi$ or a closed real interval of length $\leqslant \pi$.

D Flag complexes

8.12. Definition. *A simplicial complex $\mathcal{S}$ is called flag if whenever $\{v^0, \dots, v^k\}$ is a set of distinct vertices of $\mathcal{S}$ that are pairwise joined by edges, then the vertices $v^0, \dots, v^k$ span a k-simplex in $\mathcal{S}$.*

If the above condition is satisfied for $k = 2$, then we say that $\mathcal{S}$ satisfies the no-triangle condition.

Note that every flag complex is determined by its one-skeleton. Moreover, for any graph, its cliques (that is, complete subgraphs) define a flag complex. For that reason, flag complexes are also called clique complexes.

8.13. Exercise. *Show that the barycentric subdivision of any simplicial complex is a flag complex.*

Use the flag condition (see 8.16 below) to conclude that any finite simplicial complex is homeomorphic to a proper length CAT(1) *space.*

8.14. Proposition. *A simplicial complex $\mathcal{S}$ is flag if and only if $\mathcal{S}$ as well as all the links of all its simplices satisfy the no-triangle condition.*

From the definition of flag complex, we get the following.

8.15. Observation. *Any link of any simplex in a flag complex is flag.*

Proof of 8.14. By Observation 8.15, the no-triangle condition holds for any flag complex and the links of all its simplices.

Now assume that a complex $\mathcal{S}$ and all its links satisfy the no-triangle condition. It follows that $\mathcal{S}$ includes a 2-simplex for each triangle. Applying the same observation for each edge we get that $\mathcal{S}$ includes a 3-simplex for any complete graph with 4 vertices. Repeating this observation for triangles, 4-simplices, 5-simplices, and so on, we get that $\mathcal{S}$ is flag. □

All-right triangulation. A triangulation of a spherical polyhedral space is called an all-right triangulation if each simplex of the triangulation is isometric to a spherical simplex all of whose angles are right. Similarly, we say that a simplicial complex is equipped with an all-right spherical metric if it is a length metric and each simplex is isometric to a spherical simplex all of whose angles are right.

Spherical polyhedral CAT(1) spaces glued from right-angled simplices admit the following characterization discovered by Mikhael Gromov [49, p. 122].

8.16. Flag condition. *Assume that a spherical polyhedral space $\mathcal{P}$ admits an all-right triangulation τ. Then $\mathcal{P}$ is* CAT(1) *if and only if τ is flag.*

Proof; only-if part. Assume there are three vertices v^1, v^2, and v^3 of τ that are pairwise joined by edges but do not span a triangle. Note that in this case

$$\measuredangle[v^1\,{}^{v^2}_{v^3}] = \measuredangle[v^2\,{}^{v^3}_{v^1}] = \measuredangle[v^3\,{}^{v^1}_{v^2}] = \pi.$$

Equivalently,

❶ *The product of the geodesics $[v^1v^2]$, $[v^2v^3]$, and $[v^3v^1]$ forms a locally geodesic loop in $\mathcal{P}$ of length $\tfrac{3}{2}\cdot\pi$.*

Now assume that $\mathcal{P}$ is CAT(1). Then by 8.6, $\mathrm{Link}_\sigma\mathcal{P}$ is CAT(1) for every simplex σ in τ.

Each of these links is an all-right spherical complex and by 7.8, none of these links can contain a geodesic circle shorter than $2\cdot\pi$.

Therefore Proposition 8.14 and ❶ imply the "only if" part.

If part. By 8.15 and 7.8, it is sufficient to show that any closed local geodesic γ in a flag complex $\mathcal{S}$ with all-right metric has length at least $2{\cdot}\pi$.

Recall that the closed star of a vertex v (briefly $\overline{\mathrm{Star}}_v$) is formed by all the simplices containing v. Similarly, Star_v, the open star of v, is the union of all simplices containing v with faces opposite v removed.

Choose a vertex v such that Star_v contains a point $\gamma(t_0)$ of γ. Consider the maximal arc γ_v of γ that contains the point $\gamma(t_0)$ and runs in Star_v. Note that the distance $|v-\gamma_v(t)|_{\mathcal{P}}$ behaves in exactly the same way as the distance from the north pole in $\mathbb{S}^2$ to a geodesic in the northern hemisphere; that is, there is a geodesic $\tilde\gamma_v$ in the northern hemisphere of $\mathbb{S}^2$ such that for any t we have

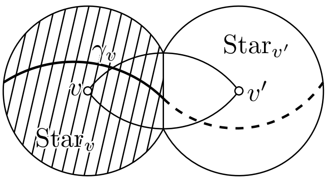
$$|v-\gamma_v(t)|_{\mathcal{P}} = |n-\tilde\gamma_v(t)|_{\mathbb{S}^2},$$

where n denotes the north pole of $\mathbb{S}^2$. In particular,

$$\operatorname{length}\gamma_v = \pi;$$

that is, γ spends time π on every visit to Star_v.

After leaving Star_v, the local geodesic γ has to enter another simplex, say σ'. Since τ is flag, the simplex σ' has a vertex v' not joined to v by an edge; that is,



$$\mathrm{Star}_v \cap \mathrm{Star}_{v'} = \varnothing$$

The same argument as above shows that γ spends time π on every visit to $\mathrm{Star}_{v'}$. Therefore the total length of γ is at least $2{\cdot}\pi$. □

8.17. Exercise. *Assume that a spherical polyhedral space $\mathcal{P}$ admits a triangulation τ such that all edge lengths of all simplices are at least $\frac{\pi}{2}$. Show that $\mathcal{P}$ is* CAT(1) *if τ is flag.*

8.18. Exercise. *Let P be a convex polyhedron in $\mathbb{E}^3$ with n faces $F_1,\dots,F_n$. Suppose that each face of P has only obtuse or right angles. Let us take 2^n copies of P indexed by an n-bit array. Glue two copies of P along F_i if their arrays differ only in the i-th bit. Show that the obtained space is a locally* CAT(0) *topological manifold.*

The space of trees. The following construction is given by Louis Billera, Susan Holmes, and Karen Vogtmann [19].

Let $\mathcal{T}_n$ be the set of all metric trees with n end vertices labeled by $a^1, \dots, a^n$. To describe one tree in $\mathcal{T}_n$ we may fix a topological tree t with end vertices $a^1, \dots,$ a^n, and all other vertices of degree 3, and prescribe the lengths of $2{\cdot}n - 3$ edges. If the length of an edge vanishes, we assume that this edge degenerates; such a tree can be also described using a different topological tree t'. The subset of $\mathcal{T}_n$ corresponding to the given topological tree t can be identified with the octant

$$\left\{\, (x_1, \dots, x_{2\cdot n-3}) \in \mathbb{R}^{2\cdot n-3} \;:\; x_i \geqslant 0 \,\right\}.$$

Equip each such subset with the metric induced from $\mathbb{R}^{2\cdot n-3}$ and consider the length metric on $\mathcal{T}_n$ induced by these metrics.

8.19. Exercise. *Show that $\mathcal{T}_n$ with the described metric is* CAT(0).

E Cubical complexes

The definition of a cubical complex mostly repeats the definition of a simplicial complex, with simplices replaced by cubes.

Formally, a cubical complex is defined as a subcomplex of the unit cube in the Euclidean space $\mathbb{R}^N$ of large dimension; that is, a collection of faces of the cube such that together with each face it contains all its sub-faces. Each cube face in this collection will be called a cube of the cubical complex.

Note that according to this definition, any cubical complex is finite.

The union of all the cubes in a cubical complex $\mathcal{Q}$ will be called its underlying space. A homeomorphism from the underlying space of $\mathcal{Q}$ to a topological space $\mathcal{X}$ is called a cubulation of $\mathcal{X}$.

The underlying space of a cubical complex $\mathcal{Q}$ will be always considered with the length metric induced from $\mathbb{R}^N$. In particular, with this metric, each cube of $\mathcal{Q}$ is isometric to the unit cube of the corresponding dimension.

It is straightforward to construct a triangulation of the underlying space of $\mathcal{Q}$ such that each simplex is isometric to a Euclidean simplex. In particular, the underlying space of $\mathcal{Q}$ is a Euclidean polyhedral space.

The link of a cube in a cubical complex is defined similarly to the link of a simplex in a simplicial complex. It is a simplicial complex that admits a natural all-right triangulation — each simplex corresponds to an adjusted cube.

Cubical analog of a simplicial complex. Let $\mathcal{S}$ be a finite simplicial complex and $\{v_1, \dots, v_N\}$ be the set of its vertices.

Consider $\mathbb{R}^N$ with the standard basis $\{e_1, \dots, e_N\}$. Denote by $\square^N$ the standard unit cube in $\mathbb{R}^N$; that is,

$$\square^N = \left\{ (x_1, \dots, x_N) \in \mathbb{R}^N \ : \ 0 \leqslant x_i \leqslant 1 \text{ for each } i \right\}.$$

Given a k-dimensional simplex $\langle v_{i_0}, \dots, v_{i_k} \rangle$ in $\mathcal{S}$, mark the $(k+1)$-dimensional faces in $\square^N$ (there are 2^{N-k} of them) which are parallel to the coordinate $(k+1)$-plane spanned by $e_{i_0}, \dots, e_{i_k}$.

Note that the set of all marked faces of $\square^N$ forms a cubical complex; it will be called the cubical analog of $\mathcal{S}$ and will be denoted as $\square_{\mathcal{S}}$.

8.20. Proposition. *Let $\mathcal{S}$ be a finite connected simplicial complex and $\mathcal{Q} = \square_{\mathcal{S}}$ be its cubical analog. Then the underlying space of $\mathcal{Q}$ is connected and the link of any vertex of $\mathcal{Q}$ is isometric to $\mathcal{S}$ equipped with the all-right spherical metric.*

In particular, if $\mathcal{S}$ is a flag complex, then $\mathcal{Q}$ is a locally $\mathrm{CAT}(0)$*, and therefore its universal cover $\tilde{\mathcal{Q}}$ is* $\mathrm{CAT}(0)$.

Proof. The first part of the proposition follows from the construction of $\square_{\mathcal{S}}$.

If $\mathcal{S}$ is flag, then by the flag condition (8.16) the link of any cube in $\mathcal{Q}$ is $\mathrm{CAT}(1)$. Therefore, by the cone construction (8.3) $\mathcal{Q}$ is locally $\mathrm{CAT}(0)$. It remains to apply the globalization theorem (7.6). $\square$

From Proposition 8.20, it follows that the cubical analog of any flag complex is aspherical. The following exercise states that the converse also holds; see [42, 5.4].

8.21. Exercise. *Show that a finite simplicial complex is flag if and only if its cubical analog is aspherical.*

F Construction

By 5.3, any complete length $\mathrm{CAT}(0)$ space is contractible. Therefore, by the globalization theorem (7.6), all proper length, locally $\mathrm{CAT}(0)$ spaces are aspherical; that is, they have contractible universal covers. This observation will be used to construct examples of aspherical spaces.

Let $\mathcal{X}$ be a proper topological space. Recall that $\mathcal{X}$ is called simply connected at infinity if for any compact set $K \subset \mathcal{X}$ there is a bigger compact set $K' \supset K$ such that $\mathcal{X} \setminus K'$ is path-connected and any loop which lies in $\mathcal{X} \setminus K'$ is null-homotopic in $\mathcal{X} \setminus K$.

Recall that path-connected spaces are not empty by definition. Therefore compact spaces are not simply connected at infinity.

The following example was constructed by Michael Davis [41].

8.22. Proposition. *For any $m \geqslant 4$, there is a closed aspherical m-dimensional manifold whose universal cover is not simply connected at infinity.*

In particular, the universal cover of this manifold is not homeomorphic to the m-dimensional Euclidean space.

The proof requires the following lemma.

8.23. Lemma. *Let $\mathcal{S}$ be a finite flag complex, $\mathcal{Q} = \square_{\mathcal{S}}$ be its cubical analog and $\tilde{\mathcal{Q}}$ be the universal cover of $\mathcal{Q}$.*

Assume $\tilde{\mathcal{Q}}$ is simply connected at infinity. Then $\mathcal{S}$ is simply connected.

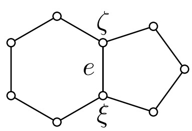
Proof. Assume $\mathcal{S}$ is not simply connected. Equip $\mathcal{S}$ with an all-right spherical metric. Choose a shortest noncontractible circle $\gamma\colon \mathbb{S}^1 \to \mathcal{S}$ formed by the edges of $\mathcal{S}$.

Note that γ forms a one-dimensional subcomplex of $\mathcal{S}$ which is a closed local geodesic. Denote by G the subcomplex of $\mathcal{Q}$ which corresponds to γ.

Fix a vertex $v \in G$; let G_v be the connected component of v in G. Let $\tilde{G}$ be a connected component of the inverse image of G_v in $\tilde{\mathcal{Q}}$ for the universal cover $\tilde{\mathcal{Q}} \to \mathcal{Q}$. Fix a point $\tilde{v} \in \tilde{G}$ in the inverse image of v.

Note that

❶ *$\tilde{G}$ is a convex set in $\tilde{\mathcal{Q}}$.*



Indeed, according to Proposition 8.20, $\tilde{\mathcal{Q}}$ is CAT(0). By Exercise 5.12, it is sufficient to show that $\tilde{G}$ is locally convex in $\tilde{\mathcal{Q}}$, or equivalently, G is locally convex in $\mathcal{Q}$.

Note that the latter can only fail if γ contains two vertices, say ξ and ζ in $\mathcal{S}$, which are joined by an edge not in γ; denote this edge by e.

Each edge of $\mathcal{S}$ has length $\frac{\pi}{2}$. Therefore each of the two circles formed by e and an arc of γ from ξ to ζ is shorter than γ. Moreover, at least one of them is noncontractible since γ is noncontractible. That is, γ is not a shortest noncontractible circle, a contradiction. △

Further, note that $\tilde{G}$ is homeomorphic to the plane since $\tilde{G}$ is a two-dimensional manifold without boundary which by the above is CAT(0) and hence is contractible.

Denote by C_R the circle of radius R in $\tilde{G}$ centered at $\tilde{v}$. All C_R are homotopic to each other in $\tilde{G} \setminus \{\tilde{v}\}$ and therefore in $\tilde{\mathcal{Q}} \setminus \{\tilde{v}\}$.

Note that the map $\tilde{\mathcal{Q}} \setminus \{\tilde{v}\} \to \mathcal{S}$ which returns the direction of $[\tilde{v}x]$ for any $x \neq \tilde{v}$, maps C_R to a circle homotopic to γ. Therefore C_R is not contractible in $\tilde{\mathcal{Q}} \setminus \{\tilde{v}\}$.

If R is large, the circle C_R lies outside of any fixed compact set K' in $\tilde{\mathcal{Q}}$. From above C_R is not contractible in $\tilde{\mathcal{Q}} \setminus K$ if $K \supset \tilde{v}$. It follows that $\tilde{\mathcal{Q}}$ is not simply connected at infinity, a contradiction. $\square$

The proof of the following exercise is analogous. It will be used later in the proof of Proposition 8.25 — a more geometric version of Proposition 8.22.

8.24. Exercise. *Under the assumptions of Lemma 8.23, for any vertex v in $\mathcal{S}$ the complement $\mathcal{S} \setminus \{v\}$ is simply connected.*

Proof of 8.22. Let Σ^{m-1} be an $(m-1)$-dimensional smooth homology sphere that is not simply connected, and bounds a contractible smooth compact m-dimensional manifold $\mathcal{W}$.

For $m \geqslant 5$, the existence of such $(\mathcal{W}, \Sigma)$ is proved by Michel Kervaire [59]. For $m = 4$, it follows from the construction of Barry Mazur [69].

Pick any triangulation τ of W and let $\mathcal{S}$ be the resulting subcomplex that triangulates Σ.

We can assume that $\mathcal{S}$ is flag; otherwise, pass to the barycentric subdivision of τ and apply Exercise 8.13.

Let $\mathcal{Q} = \square_{\mathcal{S}}$ be the cubical analog of $\mathcal{S}$.

By Proposition 8.20, $\mathcal{Q}$ is a homology manifold. It follows that $\mathcal{Q}$ is a piecewise linear manifold with a finite number of singularities at its vertices.

Removing a small contractible neighborhood V_v of each vertex v in $\mathcal{Q}$, we can obtain a piecewise linear manifold $\mathcal{N}$ whose boundary is formed by several copies of Σ.

Let us glue a copy of $\mathcal{W}$ along its boundary to each copy of Σ in the boundary of $\mathcal{N}$. This results in a closed manifold $\mathcal{M}$ with polyhedral metric which is homotopically equivalent to $\mathcal{Q}$.

Indeed, since both V_v and $\mathcal{W}$ are contractible, the identity map of their common boundary Σ can be extended to a homotopy equivalence $V_v \to \mathcal{W}$ relative to the boundary. Therefore the identity map on $\mathcal{N}$ extends to homotopy equivalences $f\colon \mathcal{Q} \to \mathcal{M}$ and $g\colon \mathcal{M} \to \mathcal{Q}$.

Finally, by Lemma 8.23, the universal cover $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ is not simply connected at infinity.

The same holds for the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$. The latter follows since the constructed homotopy equivalences $f\colon \mathcal{Q} \to \mathcal{M}$ and $g\colon \mathcal{M} \to$ $\to \mathcal{Q}$ lift to proper maps $\tilde{f}\colon \tilde{\mathcal{Q}} \to \tilde{\mathcal{M}}$ and $\tilde{g}\colon \tilde{\mathcal{M}} \to \tilde{\mathcal{Q}}$; that is, for any compact sets $A \subset \tilde{\mathcal{Q}}$ and $B \subset \tilde{\mathcal{M}}$, the inverse images $\tilde{g}^{-1}(A)$ and $\tilde{f}^{-1}(B)$ are compact. $\square$

The following proposition was proved by Fredric Ancel, Michael Davis, and Craig Guilbault [14]; it could be considered as a more geometric version of Proposition 8.22.

8.25. Proposition. *Given $m \geqslant 5$, there is a Euclidean polyhedral space $\mathcal{P}$ such that:*

(a) $\mathcal{P}$ is homeomorphic to a closed m-dimensional manifold.

(b) $\mathcal{P}$ is locally $\mathrm{CAT}(0)$.

(c) The universal cover of $\mathcal{P}$ is not simply connected at infinity.

Dale Rolfsen [86] has shown that there are no three-dimensional examples of that type. Paul Thurston [95] conjectured that the same holds in the four-dimensional case.

Proof. Apply Exercise 8.24 to the barycentric subdivision of the simplicial complex $\mathcal{S}$ provided by Exercise 8.26. □

8.26. Exercise. *Given an integer $m \geqslant 5$, construct a finite $(m-1)$-dimensional simplicial complex $\mathcal{S}$ such that* $\mathrm{Cone}\,\mathcal{S}$ *is homeomorphic to $\mathbb{E}^m$ and $\pi_1(\mathcal{S}\setminus\{v\}) \neq 0$ for some vertex v in $\mathcal{S}$.*

G Remarks

There is a good-looking description of polyhedral $\mathrm{CAT}(\kappa)$ and $\mathrm{CBB}(\kappa)$ spaces [6, 12.2 and 12.5], but in fact, it is hard to check even in very simple cases. For example, the description of those coverings of $\mathbb{S}^3$ branching at three great circles which are $\mathrm{CAT}(1)$ requires quite a bit of work [35] — try to guess the answer before reading.

Another example is the braid space $\mathcal{B}_n$ that is the universal cover of $\mathbb{C}^n$ infinitely branching in complex hyperplanes $z_i = z_j$ with the induced length metric. So far it is not known if $\mathcal{B}_n$ is $\mathrm{CAT}(0)$ for any $n \geqslant 4$ [73]. Understanding this space could help to study the braid group. This circle of questions is closely related to the generalization of the flag condition (8.16) to spherical simplices with few acute dihedral angles.

The construction used in the proof of Proposition 8.22 admits a number of modifications, several of which are discussed in a survey by Michael Davis [42].

A similar argument was used by Michael Davis, Tadeusz Januszkiewicz, and Jean-François Lafont [44]. They constructed a closed smooth four-dimensional manifold M with universal cover $\tilde{M}$ diffeomorphic to $\mathbb{R}^4$, such that M admits a polyhedral metric which is locally $\mathrm{CAT}(0)$, but does not admit a Riemannian metric with nonpositive sectional curvature. Another example of that type was constructed by Stephan Stadler [92]. There are no lower-dimensional examples of this type —

the two-dimensional case follows from the classification of surfaces, and the three-dimensional case follows from the geometrization conjecture.

It is noteworthy that any complete, simply connected Riemannian manifold with nonpositive curvature is homeomorphic to the Euclidean space of the same dimension. In fact, by the globalization theorem (7.6), the exponential map at a point of such a manifold is a homeomorphism. In particular, there is no Riemannian analog of Proposition 8.25.

Recall that a triangulation of an m-dimensional manifold defines a piecewise linear structure if the link of every simplex Δ is homeomorphic to the sphere of dimension $m-1-\dim\Delta$. According to Stone's theorem [43, 93], the triangulation of $\mathcal{P}$ in Proposition 8.25 cannot be made piecewise linear — despite the fact that $\mathcal{P}$ is a manifold, its triangulation does not induce a piecewise linear structure.

The flag condition also leads to the so-called hyperbolization procedure, a flexible tool for constructing aspherical spaces; a good survey on the subject is given by Ruth Charney and Michael Davis [36].

The CAT(0) property of a cube complex admits interesting (and useful) geometric descriptions if one exchanged the ℓ^2-metric to a natural ℓ^1 or ℓ^∞ on each cube.

8.27. Theorem. *The following three conditions are equivalent.*

- *(a) A cube complex Q equipped with ℓ^2-metric is* CAT(0)*.*
- *(b) A cube complex Q equipped with ℓ^∞-metric is injective.*
- *(c) A cube complex Q equipped with ℓ^1-metric is median. The latter means that for any three points x, y, z there is a unique point m (it is called the median of x, y, and z) and a choice of geodesics such that $[xy] \ni m$, $[xz] \ni m$ and $[yz] \ni m$.*

A very readable paper on the subject was written by Brian Bowditch [22].

8.28. Exercise. *Prove the implication (c)⇒(a) in the theorem.*

All the topics discussed in this lecture link Alexandrov geometry with the fundamental group. The theory of hyperbolic groups, a branch of geometric group theory, introduced by Mikhael Gromov [49], could be considered as a further step in this direction.

Lecture 9

Subsets

This lecture is nearly a copy of [5, Chapter 4]; here we give a partial answer to the following question: *Which subsets of Euclidean space, equipped with their induced length-metrics, are* CAT(0)*?*

A Motivating examples

Consider three subgraphs of different quadric surfaces:

$$\begin{aligned}
A &= \{\, (x,y,z) \in \mathbb{E}^3 \;:\; z \leqslant x^2 + y^2 \,\},\\
B &= \{\, (x,y,z) \in \mathbb{E}^3 \;:\; z \leqslant -x^2 - y^2 \,\},\\
C &= \{\, (x,y,z) \in \mathbb{E}^3 \;:\; z \leqslant x^2 - y^2 \,\}.
\end{aligned}$$

9.1. Question. *Which of the sets A, B and C, if equipped with the induced length metric, are* CAT(0) *and why?*

The answers are given below, but it is instructive to think about these questions before reading further.

A. No, A is not CAT(0).

The boundary ∂A is the paraboloid described by $z = x^2 + y^2$; in particular it bounds an open convex set in $\mathbb{E}^3$ whose complement is A. The closest point projection of $A \to \partial A$ is short (Exercise 5.11). It follows that ∂A is a convex set in A equipped with its induced length metric.

Therefore if A is CAT(0), then so is ∂A. The latter is not true: ∂A is a smooth convex surface, and has strictly positive curvature by the Gauss formula.

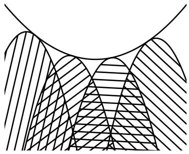
B. Yes, B is CAT(0).

Evidently B is a convex closed set in $\mathbb{E}^3$. Therefore the length metric on B coincides with the Euclidean metric and CAT(0) comparison holds.

C. Yes, C is CAT(0), but the proof is not as easy as before.

Set $f_t(x,y) = x^2 - y^2 - 2{\cdot}(x-t)^2$. Consider the one-parameter family of sets

$$V_t = \left\{\, (x,y,z) \in \mathbb{E}^3 \;:\; z \leqslant f_t(x,y) \,\right\}.$$



Each set V_t is a solid paraboloid tangent to ∂C along the parabola $y \mapsto (t, y, t^2 - y^2)$. The set V_t is closed and convex for any t, and

$$C = \bigcup_t V_t.$$

Further note that the function $t \mapsto f_t(x,y)$ is concave for any fixed x, y. Therefore

❶ *if $a < b < c$, then $V_b \supset V_a \cap V_c$.*

Consider the finite union

$$C' = V_{t_1} \cup \dots \cup V_{t_n}.$$

The inclusion ❶ makes it possible to apply Reshetnyak gluing theorem 5.14 recursively and show that C' is CAT(0).

By approximation, the CAT(0) comparison holds for any 4 points in C; hence C is CAT(0). More precisely, choose $x_1, x_2, x_3, x_4 \in C$ and 6 geodesics $[x_i x_j]_C$ between them. Choose $\varepsilon > 0$, shift each $[x_i x_j]_C$ down by ε, and reconnect it to x_i and x_j by vertical ε-segments. Denote the obtained curve by $\gamma_{i,j}$; note that

$$\operatorname{length} \gamma_{i,j} = |x_i - x_j|_C + 2{\cdot}\varepsilon.$$

We may assume that C' contains all $\gamma_{i,j}$. It follows that

$$|x_i - x_j|_C \leqslant |x_i - x_j|_{C'} \leqslant |x_i - x_j|_C + 2{\cdot}\varepsilon$$

Since and C' is CAT(0), $\varepsilon > 0$ is arbitrary, so is C.

Remark. The set C is not convex, but it is *two-convex* as defined in the next section. As you will see, two-convexity is closely related to the inheritance of an upper curvature bound by a subset.

B Two-convexity

9.2. Definition. *We say that a subset $K \subset \mathbb{E}^m$ is* two-convex *if the following condition holds for any plane $W \subset \mathbb{E}^m$: If γ is a simple closed curve in $W \cap K$ that is null-homotopic in K, then it is null-homotopic in $W \cap K$, and in particular the disc in W bounded by γ lies in K.*

Note that two-convex sets do not have to be connected or simply connected. The following two propositions follow immediately from the definition.

9.3. Proposition. *Any subset in $\mathbb{E}^2$ is two-convex.*

9.4. Proposition. *The intersection of an arbitrary collection of two-convex sets in $\mathbb{E}^m$ is two-convex.*

9.5. Proposition. *The interior of any two-convex set in $\mathbb{E}^m$ is a two-convex set.*

Proof. Fix a two-convex set $K \subset \mathbb{E}^m$ and a 2-plane W; denote by $\operatorname{Int} K$ the interior of K. Let γ be a closed simple curve in $W \cap \operatorname{Int} K$ that is contractible in the interior of K.

Since K is two-convex, the plane disc D bounded by γ lies in K. The same holds for the translations of D by small vectors. Therefore D lies in $\operatorname{Int} K$; that is, $\operatorname{Int} K$ is two-convex. □

9.6. Definition. *Given a subset $K \subset \mathbb{E}^m$, define its two-convex hull (briefly, $\operatorname{Conv}_2 K$) as the intersection of all two-convex subsets containing K.*

Note that by Proposition 9.4, the two-convex hull of any set is two-convex. Further, by 9.5, the two-convex hull of an open set is open.

The next proposition describes closed two-convex sets with smooth boundary.

9.7. Proposition. *Let $K \subset \mathbb{E}^m$ be a closed subset.*

Assume that the boundary of K is a smooth hypersurface S. Consider the unit normal vector field ν on S that points outside of K. Denote by $k_1 \leqslant \ldots \leqslant k_{m-1}$ the principal curvature functions of S with respect to ν (note that if K is convex, then $k_1 \geqslant 0$).

Then K is two-convex if and only if $k_2(p) \geqslant 0$ for any point $p \in S$. Moreover, if $k_2(p) < 0$ at some point p, then Definition 9.2 fails for some curve γ forming a triangle in an arbitrary small neighborhood of p.

The following proof was given by Mikhael Gromov [50, §$\tfrac12$], but we added a few details. The proof uses a straightforward modification of the Morse theory for manifolds with boundary; the paper of Sergei Vakhrameev [97] contains all the necessary lemmas.

Proof; only-if part. If $k_2(p) < 0$ for some $p \in S$, consider the plane W containing p and spanned by the first two principal directions at p. Choose a small triangle in W which surrounds p and move it slightly in the direction of $\nu(p)$. We get a triangle $[xyz]$ which is null-homotopic in K, but the solid triangle $\Delta = \operatorname{Conv}\{x, y, x\}$ bounded by $[xyz]$ does not lie in K completely. Therefore K is not two-convex. (See figure in the "only if" part of the smooth two-convexity theorem (9.10).)

If part. Recall that a smooth function $f\colon \mathbb{E}^m \to \mathbb{R}$ is called strongly convex if its Hessian is positive definite at each point.

Suppose $f\colon \mathbb{E}^m \to \mathbb{R}$ is a smooth strongly convex function such that the restriction $f|_S$ is a Morse function. Note that a generic smooth strongly convex function $f\colon \mathbb{E}^m \to \mathbb{R}$ has this property.

For a critical point p of $f|_S$, the outer normal vector $\nu(p)$ is parallel to the gradient $\nabla_p f$; we say that p is a positive critical point if $\nu(p)$ and $\nabla_p f$ point in the same direction, and negative otherwise. If f is generic, then we can assume that the sign is defined for all critical points; that is, $\nabla_p f \neq 0$ for any critical point p of $f|_S$.

Since $k_2 \geqslant 0$ and the function f is strongly convex, the negative critical points of $f|_S$ have index at most 1.

Given a real value s, set

$$K_s = \{\, x \in K \,:\, f(x) < s \,\}.$$

Assume $\varphi_0\colon \mathbb{D} \to K$ is a continuous map of the disc $\mathbb{D}$ such that $\varphi_0(\partial\mathbb{D}) \subset K_s$.

Note that by the Morse lemma, there is a homotopy $\varphi_t\colon \mathbb{D} \to K$ rel $\partial\mathbb{D}$ such that $\varphi_1(\mathbb{D}) \subset K_s$.

Indeed, we can construct a homotopy $\varphi_t\colon \mathbb{D} \to K$ that decreases the maximum of $f \circ \varphi$ on $\mathbb{D}$ until the maximum occurs at a critical point p of $f|_S$. This point cannot be negative; otherwise, its index would be at least 2. If this critical point is positive, then it is easy to decrease the maximum a little by pushing the disc from S into K in the direction of $-\nabla f_p$.

Consider a closed curve $\gamma\colon \mathbb{S}^1 \to K$ that is null-homotopic in K. Note that the distance function

$$f_0(x) = |\operatorname{Conv}\gamma - x|_{\mathbb{E}^m}$$

is convex. Therefore f_0 can be approximated by smooth strongly convex functions f in general position. From above, there is a disc in

K with boundary γ that lies arbitrarily close to $\operatorname{Conv}\gamma$. Since K is closed, the statement follows. □

Note that the "if" part proves a somewhat stronger statement. Namely, any plane curve γ (not necessary simple) which is contractible in K is also contractible in the intersection of K with the plane of γ. The latter condition does not hold for the complement of two planes in $\mathbb{E}^4$, which is two-convex by Proposition 9.4; see also Exercise 9.18 below. The following proposition shows that there are no such examples in $\mathbb{E}^3$.

9.8. Proposition. *Let $\Omega\subset\mathbb{E}^3$ be an open two-convex subset. Then for any plane $W\subset\mathbb{E}^3$, any closed curve in $W\cap\Omega$ that is null-homotopic in Ω is also null-homotopic in $W\cap\Omega$.*

This statement is intuitively obvious, but the proof is not trivial; it use the following classical result. An alternative definition of two-convexity using homology instead of homotopy is mentioned in the last section. For this definition the proof is simpler.

9.9. Loop theorem. *Let M be a three-dimensional manifold with nonempty boundary ∂M. Assume $f\colon(\mathbb{D},\partial\mathbb{D})\to(M,\partial M)$ is a continuous map from the disc $\mathbb{D}$ such that the boundary curve $f|_{\partial\mathbb{D}}$ is not null-homotopic in ∂M. Then there is an embedding $h\colon(\mathbb{D},\partial\mathbb{D})\to(M,\partial M)$ with the same property.*

The theorem is due to Christos Papakyriakopoulos [a proof can be found in 55].

Proof of 9.8. Fix a closed plane curve γ in $W\cap\Omega$ that is null-homotopic in Ω. Suppose γ is not contractible in $W\cap\Omega$.

Let $\varphi\colon\mathbb{D}\to\Omega$ be a map of the disc with the boundary curve γ.

Since Ω is open we can first change φ slightly so that $\varphi(x)\notin W$ for $1-\varepsilon<|x|<1$ for some small $\varepsilon>0$. By further changing φ slightly we can assume that it is transversal to W on $\operatorname{Int}\mathbb{D}$ and agrees with the previous map near $\partial\mathbb{D}$.

This means that $\varphi^{-1}(W)\cap\operatorname{Int}\mathbb{D}$ consists of finitely many simple closed curves which cut $\mathbb{D}$ into several components. Consider one of the "innermost" components c'; that is, c' is a boundary curve of a disc $\mathbb{D}'\subset\mathbb{D}$, $\varphi(c')$ is a closed curve in W and $\varphi(\mathbb{D}')$ completely lies in one of the two half-spaces with boundary W. Denote this half-space by H.

If $\varphi(c')$ is not contractible in $W\cap\Omega$, then applying the loop theorem to $M^3=H\cap\Omega$ we conclude that there exists a simple closed curve $\gamma'\subset\Omega\cap W$ which is not contractible in $\Omega\cap W$ but is contractible in $\Omega\cap H$. This contradicts two-convexity of Ω.

Hence $\varphi(c')$ is contractible in $W \cap \Omega$. Therefore φ can be changed in a small neighborhood U of $\mathbb{D}'$ so that the new map $\hat{\varphi}$ maps U to one side of W. In particular, the set $\hat{\varphi}^{-1}(W)$ consists of the same curves as $\varphi^{-1}(W)$ with the exception of c'.

Repeating this process several times we reduce the problem to the case where $\varphi^{-1}(W) \cap \operatorname{Int} \mathbb{D} = \varnothing$. This means that $\varphi(\mathbb{D})$ lies entirely in one of the half-spaces bounded by W.

Again applying the loop theorem, we obtain a simple closed curve in $W \cap \Omega$ which is not contractible in $W \cap \Omega$ but is contractible in Ω. This again contradicts two-convexity of Ω. Hence γ is contractible in $W \cap \Omega$ as claimed. □

C Sets with smooth boundary

In this section, we characterize the subsets with smooth boundary in $\mathbb{E}^m$ that form CAT(0) spaces.

9.10. Smooth two-convexity theorem. *Let K be a closed, simply connected subset in $\mathbb{E}^m$ equipped with the induced length metric. Assume K is bounded by a smooth hypersurface. Then K is* CAT(0) *if and only if K is two-convex.*

This theorem is a baby case of a result of Stephanie Alexander, David Berg, and Richard Bishop [2], which is briefly discussed at the end of the lecture. The proof below is based on the argument in Section 9A.

Proof. Denote by S and by Ω the boundary and the interior of K respectively. Since K is connected and S is smooth, Ω is also connected.

Denote by $k_1(p) \leqslant \ldots \leqslant k_{m-1}(p)$ the principal curvatures of S at $p \in S$ with respect to the normal vector $\nu(p)$ pointing out of K. By Proposition 9.7, K is two-convex if and only if $k_2(p) \geqslant 0$ for any $p \in S$.

Only-if part. Assume K is not two-convex. Then by Proposition 9.7, there is a triangle $[xyz]$ in K which is null-homotopic in K, but the solid triangle $\Delta = \operatorname{Conv}\{x, y, z\}$ does not lie in K completely. Evidently the triangle $[xyz]$ is not thin in K. Hence K is not CAT(0).

If part. Since K is simply connected, by the globalization theorem (7.6) it suffices to show that any point $p \in K$ admits a CAT(0) neighborhood.

If $p \in \operatorname{Int} K$, then it admits a neighborhood isometric to a CAT(0) subset of $\mathbb{E}^m$. Fix $p \in S$. Assume that $k_2(p) > 0$. Fix a sufficiently small $\varepsilon > 0$ and set $K' = K \cap \overline{\mathrm{B}}[p, \varepsilon]$. Let us show that

❶ *K' is* CAT(0).

Consider the coordinate system with the origin at p and the principal directions and $\nu(p)$ as the coordinate directions. For small $\varepsilon > 0$, the set K' can be described as a subgraph

$$K' = \{\, (x_1, \dots, x_m) \in \overline{\mathrm{B}}[p, \varepsilon] \ : \ x_m \leqslant f(x_1, \dots, x_{m-1} \,\} .$$

Fix $s \in [-\varepsilon, \varepsilon]$. Since ε is small and $k_2(p) > 0$, the restriction $f|_{x_1 = s}$ is concave in the $(m-2)$-dimensional cube defined by the inequalities $|x_i| < 2{\cdot}\varepsilon$ for $2 \leqslant i \leqslant m - 1$.

Fix a negative real value $\lambda < k_1(p)$. Given $s \in (-\varepsilon, \varepsilon)$, consider the set

$$V_s = \{\, (x_1, \dots, x_m) \in K' \ : \ x_m \leqslant f(x_1, \dots, x_{m-1}) + \lambda{\cdot}(x_1 - s)^2 \,\} .$$

Note that the function

$$(x_1, \dots, x_{m-1}) \mapsto f(x_1, \dots, x_{m-1}) + \lambda{\cdot}(x_1 - s)^2$$

is concave near the origin. Since ε is small, we can assume that the V_s are convex subsets of $\mathbb{E}^m$.

Further note that

$$K' = \bigcup_{s \in [-\varepsilon, \varepsilon]} V_s.$$

Also, the same argument as in 9.1 shows that

❷ *If $a < b < c$, then $V_b \supset V_a \cap V_c$.*

Given an array of values $s^1 < \dots < s^k$ in $[-\varepsilon, \varepsilon]$, set $V^i = V_{s^i}$ and consider the unions

$$W^i = V^1 \cup \dots \cup V^i$$

equipped with the induced length metric.

Note that the array (s^n) can be chosen in such a way that W^k is arbitrarily close to K' in the sense of Hausdorff.

Arguing as in 9A, we get that the following claim implies ❶.

❸ *All W^i are* CAT(0).

This claim is proved by induction. Base: $W^1 = V^1$ is CAT(0) as a convex subset in $\mathbb{E}^m$.

Step: Assume that W^i is CAT(0). According to ❷,

$$V^{i+1} \cap W^i = V^{i+1} \cap V^i.$$

Moreover, this is a convex set in $\mathbb{E}^m$ and therefore it is a convex set in W^i and in V^{i+1}. By the Reshetnyak gluing theorem, W^{i+1} is CAT(0). Hence the claim follows. △

Note that we have proved the following:

❹ *K' is* CAT(0) *if K is strongly two-convex, that is, $k_2(p) > 0$ at any point $p \in S$.*

It remains to show that p admits a CAT(0) neighborhood in the case $k_2(p) = 0$.

Choose a coordinate system $(x_1, \dots, x_m)$ as above, so that the $(x_1, \dots, x_{m-1})$-coordinate hyperplane is the tangent subspace to S at p.

Fix $\varepsilon > 0$ so that a neighborhood of p in S is the graph

$$x_m = f(x_1, \dots, x_{m-1})$$

of a function f defined on the open ball B of radius ε centered at the origin in the $(x_1, \dots, x_{m-1})$-hyperplane. Fix a smooth positive strongly convex function $\varphi\colon B \to \mathbb{R}_+$ such that $\varphi(x) \to \infty$ as x approaches the boundary of B. Note that for $\delta > 0$, the subgraph K_δ defined by the inequality

$$x_m \leqslant f(x_1, \dots, x_{m-1}) - \delta{\cdot}\varphi(x_1, \dots, x_{m-1})$$

is strongly two-convex. By ❹, K_δ is CAT(0).

Finally as $\delta \to 0$, the closed ε-neighborhoods of p in K_δ converge to the closed ε-neighborhood of p in K. By Exercise 3.12, the ε-neighborhood of p is CAT(0). □

D Open plane sets

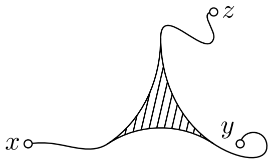
In this section, we consider inheritance of upper curvature bounds by subsets of the Euclidean plane.

9.11. Theorem. *Let Ω be an open simply connected subset of $\mathbb{E}^2$. Equip Ω with its induced length metric and denote its completion by K. Then K is* CAT(0).

The assumption that the set Ω is open is not critical; instead one can assume that the induced length metric takes finite values at all points of Ω. We sketch the proof given by Richard Bishop [20] and leave the details to be finished as an exercise. A generalization of this result is proved by Alexander Lytchak and Stefan Wenger [66, Proposition 12.1]; this paper also contains a far-reaching application.

Sketch of proof. It is sufficient to show that any triangle in K is thin, as defined in 5.5.

Note that K admits a length-preserving map to $\mathbb{E}^2$ that extends the embedding $\Omega \hookrightarrow \mathbb{E}^2$. Therefore each triangle $[xyz]$ in K can be mapped to the plane in a length-preserving way. Since Ω is simply connected, any open region, say Δ, that is surrounded by the image of $[xyz]$ lies completely in Ω.



Note that in each triangle $[xyz]$ in K, the sides $[xy]$, $[yz]$ and $[zx]$ intersect each other along a geodesic starting at a common vertex, possibly a one-point geodesic. In other words, every triangle in K looks like the one in the diagram.

Indeed, assuming the contrary, there will be a *lune* in K bounded by two minimizing geodesics with common ends but no other common points. The image of this lune in the plane must have concave sides, since otherwise one could shorten the sides by pushing them into the interior. Evidently, there is no plane lune with concave sides, a contradiction.

Note that it is sufficient to consider only simple triangles $[xyz]$, that is, triangles whose sides $[xy]$, $[yz]$ and $[zx]$ intersect each other only at the common vertices. If this is not the case, chopping the overlapping part of sides reduces to the injective case (this is formally stated in Exercise 9.12).

Again, the open region, say Δ, bounded by the image of $[xyz]$ has concave sides in the plane, since otherwise one could shorten the sides by pushing them into Ω. It remains to solve Exercise 9.13. $\square$

9.12. Exercise. *Assume that $[pq]$ is a common part of the two sides $[px]$ and $[py]$ of the triangle $[pxy]$. Consider the triangle $[qxy]$ whose sides are formed by arcs of the sides of $[pxy]$. Show that if $[qxy]$ is thin, then so is $[pxy]$.*

9.13. Exercise. *Assume S is a closed plane region whose boundary is a plane triangle T with concave sides. Equip S with the induced length metric. Show that the triangle T is thin in S.*

Here is a spherical analog of Theorem 9.11, which can be proved along the same lines. It will be used in the next section.

9.14. Proposition. *Let Θ be an open connected subset of the unit sphere $\mathbb{S}^2$ that does not contain a closed hemisphere. Equip Θ with the induced length metric. Let $\tilde\Theta$ be a metric cover of Θ such that any closed curve in $\tilde\Theta$ shorter than $2\cdot\pi$ is contractible.*

Show that the completion of $\tilde\Theta$ is CAT(1).

9.15. Exercise. *Prove the following partial case of the proposition:*

Let K be closed subset of the unit sphere $\mathbb{S}^2$ that does not contain a closed hemisphere. Suppose K is simply connected and bounded by *a simple Lipschitz curve. Show that K with induced length metric is* CAT(1).

E Shefel's theorem

In this section, we will formulate our version of a theorem of Samuel Shefel (9.17) and prove a couple of its corollaries.

It seems that Shefel was very intrigued by the survival of metric properties under affine transformation. To describe an instance of such phenomena, note that two-convexity survives under affine transformations of a Euclidean space. Therefore, as a consequence of the smooth two-convexity theorem (9.10), the following holds.

9.16. Corollary. *Let K be closed connected subset of Euclidean space equipped with the induced length metric. Assume K is* CAT(0) *and bounded by a smooth hypersurface. Then any affine transformation of K is also* CAT(0).

By Corollary 9.19, an analogous statement holds for sets bounded by Lipschitz surfaces in the three-dimensional Euclidean space. In higher dimensions this is no longer true.

9.17. Two-convexity theorem. *Let Ω be a connected open set in $\mathbb{E}^3$. Equip Ω with the induced length metric and denote by $\tilde{K}$ the completion of the universal metric cover of Ω. Then $\tilde{K}$ is* CAT(0) *if and only if Ω is two-convex.*

The proof of this statement will be given in the following three sections. First we prove its polyhedral analog, then we prove some properties of two-convex hulls in three-dimensional Euclidean space and only then do we prove the general statement.

The following exercise shows that the analogous statement does not hold in higher dimensions.

9.18. Exercise. *Let Π_1, Π_2 be two planes in $\mathbb{E}^4$ intersecting at a single point. Let $\tilde{K}$ be the completion of the universal metric cover of $\mathbb{E}^4 \setminus (\Pi_1 \cup \Pi_2)$.*

Show that $\tilde{K}$ is CAT(0) *if and only if $\Pi_1 \perp \Pi_2$.*

Before coming to the proof of the two-convexity theorem, let us formulate a few corollaries. The following corollary is a generalization of the smooth two-convexity theorem (9.10) for three-dimensional Euclidean space.

9.19. Corollary. *Let K be a closed subset in $\mathbb{E}^3$ bounded by a Lipschitz hypersurface. Then K with the induced length metric is* CAT(0) *if and only if the interior of K is two-convex and simply connected.*

Proof. Set $\Omega = \operatorname{Int} K$. Since K is simply connected and bounded by a surface, Ω is also simply connected.

Apply the two-convexity theorem to Ω. Note that the completion of Ω equipped with the induced length metric is isometric to K with the induced length metric. Hence the result. $\square$

Note that the Lipschitz condition is used just once to show that the completion of Ω is isometric to K with the induced length metric. This property holds for a wider class of hypersurfaces; for instance Alexander horned ball might have $\mathrm{CAT}(0)$ induced length metric.

Let U be an open set in $\mathbb{R}^2$. A continuous function $f\colon U \to \mathbb{R}$ is called saddle if for any linear function $\ell\colon \mathbb{R}^2 \to \mathbb{R}$, the difference $f - \ell$ does not have local maxima or local minima in U. Equivalently, the open subgraph and epigraph of f

$$\begin{aligned}&\left\{\,(x,y,z)\in\mathbb{E}^3 \;:\; z < f(x,y),\;\; (x,y)\in U\,\right\},\\ &\left\{\,(x,y,z)\in\mathbb{E}^3 \;:\; z > f(x,y),\;\; (x,y)\in U\,\right\}\end{aligned}$$

are two-convex.

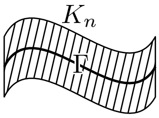
9.20. Theorem. *Let $f\colon \mathbb{D} \to \mathbb{R}$ be a Lipschitz function which is saddle in the interior of the closed unit disc $\mathbb{D}$. Then the graph*

$$\Gamma = \left\{\,(x,y,z)\in\mathbb{E}^3 \;:\; z = f(x,y)\,\right\},$$

equipped with induced length metric is $\mathrm{CAT}(0)$.

Proof. Since the function f is Lipschitz, its graph Γ with the induced length metric is bi-Lipschitz equivalent to $\mathbb{D}$ with the Euclidean metric.

Consider the sequence of sets



$$K_n = \left\{\,(x,y,z)\in\mathbb{E}^3 \;:\; z \lesseqgtr f(x,y) \pm \tfrac{1}{n},\;\; (x,y)\in\mathbb{D}\,\right\}.$$

Note that each K_n is closed and simply connected. By definition K is also two-convex. Moreover the boundary of K_n is a Lipschitz surface.

Equip K_n with the induced length metric. By Corollary 9.19, K_n is $\mathrm{CAT}(0)$. It remains to note that $K_n \to \Gamma$ in the sense of Gromov–Hausdorff, and apply 3.12. $\square$

F Polyhedral case

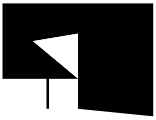
Now we are back to the proof of the two-convexity theorem (9.17).

Recall that a subset P of $\mathbb{E}^m$ is called a polytope if it can be presented as a union of a finite number of simplices. Similarly, a spherical polytope is a union of a finite number of simplices in $\mathbb{S}^m$.

Note that any polytope admits a finite triangulation. Therefore any polytope equipped with the induced length metric forms a Euclidean polyhedral space as defined in 8.7.

9.21. Lemma. *The two-convexity theorem (9.17) holds if the set Ω is the interior of a polytope.*

The statement might look obvious, but there is a hidden obstacle in the proof that is related to the following. Let P be a polytope and Ω its interior, both considered with the induced length metrics. Typically, the completion K of Ω is isometric to P — in this case, the lemma follows easily from 8.8.



However in general we only have a locally distance-preserving map $K \to P$; it does not have to be onto and it may not be injective. An example can be guessed from the picture. Nevertheless, is easy to see that K is always a polyhedral space.

The proof uses the following two exercises.

9.22. Exercise. *Show that any closed path of length $< 2{\cdot}\pi$ in the units sphere $\mathbb{S}^2$ lies in an open hemisphere.*

9.23. Exercise. *Assume Ω is an open subset in $\mathbb{E}^3$ that is not two-convex. Show that there is a plane W such that the complement $W\setminus\Omega$ contains an isolated point and a small circle around this point in W is contractible in Ω.*

Proof of 9.21. The "only if" part can be proved in the same way as in the smooth two-convexity theorem (9.10) with additional use of Exercise 9.23.

If part. Assume that Ω is two-convex. Denote by $\tilde\Omega$ the universal metric cover of Ω. Let $\tilde K$ and K be the corresponding completions of $\tilde\Omega$ and Ω.

The main step is to show that $\tilde K$ is CAT(0).

Note that K is a polyhedral space and the covering $\tilde\Omega \to \Omega$ extends to a covering map $\tilde K \to K$ which might be branching at some vertices.[1]

Fix a point $\tilde p \in \tilde K \setminus \tilde\Omega$; denote by p the image of $\tilde p$ in K. Note that $\tilde K$ is a ramified cover of K and hence is locally contractible. Thus, any loop in $\tilde K$ is homotopic to a loop in $\tilde\Omega$ which is simply connected. Therefore $\tilde K$ is simply connected too.

[1]For example, if $K = \{\,(x,y,z)\in\mathbb{E}^3 : |z| \leqslant |x|+|y| \leqslant 1\,\}$ and p is the origin, then Σ_p, the space of directions at p, is not simply connected and $\tilde K \to K$ branches at p.

Thus, by the globalization theorem (7.6), it is sufficient to show that

❶ *a small neighborhood of $\tilde p$ in $\tilde K$ is* CAT(0).

Recall that $\Sigma_{\tilde p} = \Sigma_{\tilde p}\tilde K$ denotes the space of directions at $\tilde p$. Note that a small neighborhood of $\tilde p$ in $\tilde K$ is isometric to an open set in the cone over $\Sigma_{\tilde p}\tilde K$. By Exercise 8.3, ❶ follows once we can show that

❷ $\Sigma_{\tilde p}$ *is* CAT(1).

By rescaling, we can assume that every face of K which does not contain p lies at distance at least 2 from p. Denote by $\mathbb{S}^2$ the unit sphere centered at p, and set $\Theta = \mathbb{S}^2 \cap \Omega$. Note that $\Sigma_p K$ is isometric to the completion of Θ and $\Sigma_{\tilde p}\tilde K$ is the completion of the regular metric covering $\tilde\Theta$ of Θ induced by the universal metric cover $\tilde\Omega \to \Omega$.

By 9.14, it remains to show the following:

❸ *Any closed curve in $\tilde\Theta$ shorter than $2{\cdot}\pi$ is contractible.*

Fix a closed curve $\tilde\gamma$ of length $< 2{\cdot}\pi$ in $\tilde\Theta$. Its projection γ in $\Theta \subset \mathbb{S}^2$ has the same length. Therefore, by Exercise 9.22, γ lies in an open hemisphere. Then for a plane Π passing close to p, the central projection γ' of γ to Π is defined and lies in Ω. By construction of $\tilde\Theta$, the curve γ and therefore γ' are contractible in Ω. From two-convexity of Ω and Proposition 9.8, the curve γ' is contractible in $\Pi \cap \Omega$.

It follows that γ is contractible in Θ and therefore $\tilde\gamma$ is contractible in $\tilde\Theta$. □

G Two-convex hulls

The following proposition describes a construction which produces the two-convex hull $\mathrm{Conv}_2\,\Omega$ of an open set $\Omega \subset \mathbb{E}^3$. This construction is very close to the one given by Samuel Shefel [89].

9.24. Proposition. *Let $\Pi_1, \Pi_2 \dots$ be an everywhere dense sequence of planes in $\mathbb{E}^3$. Given an open set Ω, consider the recursively defined sequence of open sets $\Omega = \Omega_0 \subset \Omega_1 \subset \dots$ such that Ω_n is the union of Ω_{n-1} and all the bounded components of $\mathbb{E}^3 \setminus (\Pi_n \cup \Omega_{n-1})$. Then*

$$\mathrm{Conv}_2\,\Omega = \bigcup_n \Omega_n.$$

Proof. Set

❶ $$\Omega' = \bigcup_n \Omega_n.$$

Note that Ω' is a union of open sets; in particular, Ω' is open.

Let us show that

❷ $$\operatorname{Conv}_2 \Omega \supset \Omega'.$$

Suppose we already know that $\operatorname{Conv}_2 \Omega \supset \Omega_{n-1}$. Fix a bounded component $\mathfrak{C}$ of $\mathbb{E}^3 \setminus (\Pi_n \cup \Omega_{n-1})$. It is sufficient to show that $\mathfrak{C} \subset \operatorname{Conv}_2 \Omega$.

By 9.5, $\operatorname{Conv}_2 \Omega$ is open. Therefore, if $\mathfrak{C} \not\subset \operatorname{Conv}_2 \Omega$, then there is a point $p \in \mathfrak{C} \setminus \operatorname{Conv}_2 \Omega$ lying at maximal distance from Π_n. Denote by W_p the plane containing p which is parallel to Π_n.

Note that p lies in a bounded component of $W_p \setminus \operatorname{Conv}_2 \Omega$. In particular p can be surrounded by a simple closed curve γ in $W_p \cap \cap \operatorname{Conv}_2 \Omega$. Since p lies at maximal distance from Π_n, the curve γ is null-homotopic in $\operatorname{Conv}_2 \Omega$. Therefore $p \in \operatorname{Conv}_2 \Omega$, a contradiction.

By induction, $\operatorname{Conv}_2 \Omega \supset \Omega_n$ for each n. Therefore ❶ implies ❷.

It remains to show that Ω' is two-convex. Assume the contrary; that is, there is a plane Π and a simple closed curve $\gamma\colon \mathbb{S}^1 \to \Pi \cap \Omega'$ which is null-homotopic in Ω', but not null-homotopic in $\Pi \cap \Omega'$.

By approximation we can assume that $\Pi = \Pi_n$ for a large n, and that γ lies in Ω_{n-1}. By the same argument as in the proof of Proposition 9.8 using the loop theorem, we can assume that there is an embedding $\varphi\colon \mathbb{D} \to \Omega'$ such that $\varphi|_{\partial \mathbb{D}} = \gamma$ and $\varphi(D)$ lies entirely in one of the half-spaces bounded by Π. By the n-step of the construction, the entire bounded domain U bounded by Π_n and $\varphi(D)$ is contained in Ω' and hence γ is contractible in $\Pi \cap \Omega'$, a contradiction. □

9.25. Key lemma. *The two-convex hull of the interior of a polytope in $\mathbb{E}^3$ is also the interior of a polytope.*

Proof. Fix a polytope P in $\mathbb{E}^3$. Set $\Omega = \operatorname{Int} P$. We may assume that Ω is dense in P (if not, redefine P as the closure of Ω). Denote by $F_1, \dots, F_m$ the facets of P. By subdividing F_i if necessary, we may assume that all F_i are convex polygons.

Set $\Omega' = \operatorname{Conv}_2 \Omega$ and let P' be the closure of Ω'. Further, for each i, set $F_i' = F_i \setminus \Omega'$. In other words, F_i' is the subset of the facet F_i which remains on the boundary of P'.

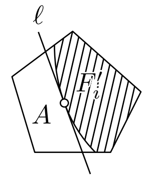
From the construction of the two-convex hull (9.24) we have:

❸ *F_i' is a convex subset of F_i.*

Further, since Ω' is two-convex we obtain the following:

❹ *Each connected component of the complement $F_i \setminus F_i'$ is convex.*

Indeed, assume a connected component A of $F_i \setminus F_i'$ fails to be convex. Then there is a supporting line ℓ to F_i' touching F_i' at a single point in the interior of F_i. Then one could rotate the plane of F_i slightly around ℓ and move it parallelly to cut a "cap" from the complement of Ω. The latter means that Ω is not two-convex, a contradiction. $\triangle$



From ❸ and ❹, we conclude

❺ *F_i' is a convex polygon for each i.*

Consider the complement $\mathbb{E}^3 \setminus \Omega$ equipped with the length metric. By construction of the two-convex hull (9.24), the complement $L = \mathbb{E}^3 \setminus (\Omega' \cup P)$ is locally convex; that is, any point of L admits a convex neighborhood.

Summarizing: (1) Ω' is a two-convex open set, (2) the boundary $\partial\Omega'$ contains a finite number of polygons F_i' and the remaining part S of the boundary is locally concave. It remains to show that (1) and (2) imply that S and therefore $\partial\Omega'$ are piecewise linear.

9.26. Exercise. *Prove the last statement.* $\square$

H Proof of Shefel's theorem

Proof of 9.17. The "only if" part can be proved in the same way as in the smooth two-convexity theorem (9.10) with the additional use of Exercise 9.23.

If part. Suppose Ω is two-convex. We need to show that $\tilde{K}$ is CAT(0).

Fix a quadruple of points $x^1, x^2, x^3, x^4 \in \tilde{\Omega}$. Let us show that CAT(0) comparison holds for this quadruple.

Fix $\varepsilon > 0$. Choose six broken lines in $\tilde{\Omega}$ connecting all pairs of points x^1, x^2, x^3, x^4, where the length of each broken line is at most ε bigger than the distance between its ends in the length metric on $\tilde{\Omega}$. Denote by X the union of these broken lines. Choose a polytope P in Ω such that its interior $\operatorname{Int} P$ contains the projections of all six broken lines and discs which contract all the loops created by them (it is sufficient to take 3 discs).

Denote by Ω' the two-convex hull of the interior of P. According to the key lemma (9.25), Ω' is the interior of a polytope.

Equip Ω' with the induced length metric. Consider the universal metric cover $\tilde{\Omega}'$ of Ω'. (The covering $\tilde{\Omega}' \to \Omega'$ might be nontrivial — even if $\operatorname{Int} P$ is simply connected, its two-convex hull Ω' might not be simply connected.) Denote by $\tilde{K}'$ the completion of $\tilde{\Omega}'$.

By Lemma 9.21, $\tilde{K}'$ is CAT(0).

By construction of $\operatorname{Int} P$, the embedding $\operatorname{Int} P \hookrightarrow \Omega'$ admits a lift $\iota\colon X \hookrightarrow \tilde{K}'$. By construction, ι almost preserves the distances between the points x^1, x^2, x^3, x^4; namely

$$|\iota(x^i) - \iota(x^j)|_L \lesseqgtr |x^i - x^j|_{\operatorname{Int} P} \pm \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and CAT(0) comparison holds in $\tilde{K}'$, we get that CAT(0) comparison holds in Ω for x^1, x^2, x^3, x^4.

The statement follows since the quadruple $x^1, x^2, x^3, x^4 \in \tilde{\Omega}$ is arbitrary. $\square$

9.27. Exercise. *Assume $K \subset \mathbb{E}^m$ is a closed set bounded by a Lipschitz hypersurface. Equip K with the induced length metric. Show that if K is* CAT(0)*, then K is two-convex.*

I Remarks

Under the name $(n-2)$-convex sets, two-convex sets in $\mathbb{E}^n$ were introduced by Mikhael Gromov [50]. In addition to the inheritance of upper curvature bounds by two-convex sets discussed in this lecture, these sets appear as the maximal open sets with vanishing curvature in Riemannian manifolds with non-negative or non-positive sectional curvature [see Lemma 5.8 in 32, 15 and 72].

Two-convex sets could be defined using homology instead of homotopy, as in Gromov's formulation of the Leftschetz theorem [50, §$\tfrac{1}{2}$]. Namely, we can say that K is two-convex if the following condition holds: if a one-dimensional cycle z has support in the intersection of K with a plane W and bounds in K, then it bounds in $K \cap W$.

The resulting definition is equivalent to the one used above. But unlike our definition it can be generalized to define k-convex sets in $\mathbb{E}^m$ for $k > 2$. With this homological definition one can also avoid the use of the loop theorem, whose proof is quite involved. Nevertheless, we chose the definition using homotopies since it is easier to visualize.

Both definitions work well for open sets; for general sets one should be able to give a similar definition using an appropriate homotopy/homology theory.

In [2], Stephanie Alexander, David Berg and Richard Bishop gave the exact upper bound on Alexandrov's curvature for the Riemannian manifolds with boundary. This theorem includes the smooth two-convexity theorem (9.10) as a partial case. Namely they show the following.

9.28. Theorem. *Let M be a Riemannian manifold with boundary ∂M. A direction tangent to the boundary will be called concave if there is a short geodesic in this direction which leaves the boundary and goes into the interior of M. A sectional direction (that is, a 2-plane) tangent to the boundary will be called* concave *if all the directions in it are concave.*

Denote by κ an upper bound of sectional curvatures of M and sectional curvatures of ∂M in the concave sectional directions. Then M is locally $\mathrm{CAT}(\kappa)$.

9.29. Corollary. *Let M be a Riemannian manifold with boundary ∂M. Assume that all the sectional curvatures of M and ∂M are bounded above by κ. Then M is locally* $\mathrm{CAT}(\kappa)$.

Theorem 9.20 is from Shefel's original paper [90]. It is related to Alexandrov's theorem about ruled surfaces [11].

Let D be an embedded closed disc in $\mathbb{E}^3$. We say that D is saddle if each connected component which any plane cuts from D contains a point on the boundary ∂D. If D is locally described by a Lipschitz embedding, then this condition is equivalent to saying that D is two-convex.

9.30. Shefel's conjecture. *Any saddle surface in $\mathbb{E}^3$ equipped with the length-metric is locally* $\mathrm{CAT}(0)$.

The conjecture is open even for the surfaces described by a bi-Lipschitz embedding of a disc. From another result of Samuel Shefel [90], it follows that a saddle surface satisfies the isoperimetric inequality $a \leqslant C\cdot\ell^2$ where a is the area of a disc bounded by a curve of length ℓ and $C = \frac{1}{3\cdot\pi}$. By a result of Alexander Lytchak and Stefan Wenger [66], Shefel's conjecture is equivalent to the isoperimetric inequality with the optimal constant $C = \frac{1}{4\cdot\pi}$. [For more on the subject, see 75, and the references therein.]

Lecture 10

Barycenters

A Definition

Let us denote by $\Delta^k \subset \mathbb{R}^{k+1}$ the standard k-simplex; that is, $\boldsymbol{m} = (m_0, \dots, m_k) \in \Delta^k$ if $m_0 + \dots + m_k = 1$ and $m_i \geqslant 0$ for all i.

Consider a point array $\boldsymbol{p} = (p_0, \dots, p_k)$ in a Euclidean space $\mathbb{E}^n$. Recall that

$$z = m_0{\cdot}p_0 + \dots + m_k{\cdot}p_k$$

is called barycenter of point array $\boldsymbol{p} = p_0, \dots, p_k$ with masses $\boldsymbol{m} = (m_0, \dots, m_k) \in \Delta^k$. Equivalently,

❶ $$z := \operatorname{MinPoint}(m_0{\cdot}f_0 + \dots + m_k{\cdot}f_k),$$

where $f_i = \frac{1}{2}{\cdot}\operatorname{dist}_{p_i}^2$ for each i, and $\operatorname{MinPoint} f$ denotes a point of minimum of function f.

The map $\mathfrak{S}\colon \Delta^k \mapsto \mathbb{E}^n$ defined by $\mathfrak{S}\colon \boldsymbol{m} \mapsto z$ is called barycentric simplex of the array $\boldsymbol{p}$. If needed we may denote this map by $\mathfrak{S}_{\boldsymbol{p}}$ or, more generally, $\mathfrak{S}_{\boldsymbol{f}}$. The latter means that we define the map using ❶ for an array of functions $\boldsymbol{f} = (f_0, f_1, \dots, f_k)$. Note that the definition ❶ makes sense for any array of functions in a metric space; altho, in this case, the map might be undefined or nonuniquely defined.

Further, we will work with this definition in CAT(0) spaces instead of $\mathbb{E}^n$. It will be used to define and study dimension of CAT spaces. We will use that on a geodesic CAT(0) space, functions of the type $f = \frac{1}{2}{\cdot}\operatorname{dist}_p^2$ are 1-convex; see 5.8. Besides that, we will not use CAT(0) condition for a while.

B Barycentric simplex

10.1. Theorem. *Let $\mathcal{X}$ be a complete geodesic space and $\boldsymbol{f} = (f_0, \dots \dots, f_k)\colon \mathcal{X} \to \mathbb{R}^{k+1}$ be an array of nonnegative 1-convex locally Lipschitz functions. Then the barycentric simplex $\mathfrak{S}_{\boldsymbol{f}}\colon \Delta^k \to \mathcal{X}$ is a uniquely defined Lipshitz map.*

In particular, we have that the barycentric simplex $\mathfrak{S}_{\boldsymbol{p}}$ any point array $\boldsymbol{p} = (p_0, \dots, p_k)$ in a complete geodesic $\mathrm{CAT}(0)$ *space is a uniquely defined Lipshitz map.*

10.2. Lemma. *Suppose $\mathcal{X}$ is a complete geodesic space and $f\colon \mathcal{X} \to \to \mathbb{R}$ is a locally Lipschitz,* 1*-convex function. Then* $\operatorname{MinPoint} f$ *is uniquely defined.*

Proof. Note that

❶ *if z is a midpoint of the geodesic $[xy]$, then*

$$s \leqslant f(z) \leqslant \tfrac{1}{2}\cdot f(x) + \tfrac{1}{2}\cdot f(y) - \tfrac{1}{8}\cdot|x-y|^2,$$

where s is the infimum of f.

Uniqueness. Assume that x and y are distinct minimum points of f. From ❶ we have

$$f(z) < f(x) = f(y)$$

— a contradiction.

Existence. Fix a point $p \in \mathcal{X}$, and let $L \in \mathbb{R}$ be a Lipschitz constant of f in a neighborhood of p.

Choose a sequence of points $p_n \in \mathcal{X}$ such that $f(p_n) \to s$. Applying ❶ for $x = p_n$, $y = p_m$, we see that p_n is a Cauchy sequence. Thus the sequence p_n converges to a minimum point of f. □

Proof of 10.1. Since each f_i is 1-convex, for any $\boldsymbol{x} = (x_0, x_1, \dots, x_k) \in \in \Delta^k$ the convex combination

$$\left(\sum_i x_i\cdot f_i\right)\colon \mathcal{X} \to \mathbb{R}$$

is also 1-convex. Therefore, according to 10.2, the barycentric simplex $\mathfrak{S}_{\boldsymbol{f}}$ is uniquely defined on Δ^k.

For $\boldsymbol{x}, \boldsymbol{y} \in \Delta^k$, let

$$\begin{aligned} f_{\boldsymbol{x}} &= \sum_i x_i\cdot f_i, & f_{\boldsymbol{y}} &= \sum_i y_i\cdot f_i,\\ p &= \mathfrak{S}_{\boldsymbol{f}}(\boldsymbol{x}), & q &= \mathfrak{S}_{\boldsymbol{f}}(\boldsymbol{y}), \end{aligned}$$

Choose a geodesic γ from p to q; suppose $s = |p - q|$ and so $\gamma(0) = p$ and $\gamma(s) = q$. Observe the following:

- ⋄ The function $\varphi(t) = f_{\boldsymbol{x}} \circ \gamma(t)$ has a minimum at 0. Therefore $\varphi^+(0) \geqslant 0$.
- ⋄ The function $\psi(t) = f_{\boldsymbol{y}} \circ \gamma(t)$ has a minimum at s. Therefore $\psi^-(s) \geqslant 0$.

From 1-convexity of $f_{\boldsymbol{y}}$, we have

$$\psi^+(0) + \psi^-(s) + s \leqslant 0.$$

Let L be a Lipschitz constant for all f_i in a neighborhood $\Omega \ni p$. Then

$$\psi^+(0) \leqslant \varphi^+(0) + L \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_1,$$

where $\|\boldsymbol{x} - \boldsymbol{y}\|_1 = \sum_{i=0}^{k} |x_i - y_i|$. It follows that given $\boldsymbol{x} \in \Delta^k$, there is a constant L such that

$$\begin{aligned} |\mathfrak{S}_{\boldsymbol{f}}(\boldsymbol{x}) - \mathfrak{S}_{\boldsymbol{f}}(\boldsymbol{y})| = s \leqslant \\ \leqslant L \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_1 \end{aligned}$$

for any $\boldsymbol{y} \in \Delta^k$. In particular, there is $\varepsilon > 0$ such that if $\|\boldsymbol{x} - \boldsymbol{y}\|_1 < \varepsilon$, $\|\boldsymbol{x} - \boldsymbol{z}\|_1 < \varepsilon$, then $\mathfrak{S}_{\boldsymbol{f}}(\boldsymbol{y})$, $\mathfrak{S}_{\boldsymbol{f}}(\boldsymbol{z}) \in \Omega$. Thus the same argument as above implies

$$|\mathfrak{S}_{\boldsymbol{f}}(\boldsymbol{y}) - \mathfrak{S}_{\boldsymbol{f}}(\boldsymbol{z})| \leqslant L \cdot \|\boldsymbol{y} - \boldsymbol{z}\|_1$$

for any $\boldsymbol{y}$ and $\boldsymbol{z}$ sufficiently close to $\boldsymbol{x}$; that is, $\mathfrak{S}_{\boldsymbol{f}}$ is locally Lipschitz. Since Δ^k is compact, $\mathfrak{S}_{\boldsymbol{f}}$ is Lipschitz. □

10.3. Exercise. *Let G be a subgroup of the group of isometries of a proper geodesic* CAT(0) *space. Assume that*

(a) G is finite, or

(b) G is compact.

Show that the action of G has a fixed point.

C Convexity of up-set

10.4. Definition. *For two real arrays $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbb{R}^{k+1}$, $\boldsymbol{v} = (v_0, \dots, v_k)$ and $\boldsymbol{w} = (w_0, \dots, w_k)$, we will write $\boldsymbol{v} \succcurlyeq \boldsymbol{w}$ if $v_i \geqslant w_i$ for each i.*

Given a subset $Q \subset \mathbb{R}^{k+1}$, denote by $\operatorname{Up} Q$ the smallest upper set containing Q; that is,

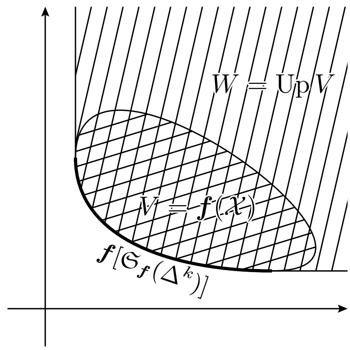
$$\operatorname{Up} Q = \left\{\, \boldsymbol{v} \in \mathbb{R}^{k+1} \ : \ \exists\, \boldsymbol{w} \in Q \text{ such that } \boldsymbol{v} \succcurlyeq \boldsymbol{w} \,\right\},$$

10.5. Proposition. *Let $\mathcal{X}$ be a complete geodesic space and $\boldsymbol{f} = (f_0, \dots, f_k) \colon \mathcal{X} \to \mathbb{R}^{k+1}$ be an array of nonnegative 1-convex locally Lipschitz functions. Consider the set $W = \operatorname{Up}[\boldsymbol{f}(\mathcal{X})] \subset \mathbb{R}^{k+1}$. Then*

(a) *The set W is convex.*
(b) $\boldsymbol{f}[\mathfrak{S}_{\boldsymbol{f}}(\Delta^k)] \subset \partial W$. *Moreover,* $\boldsymbol{f}[\mathfrak{S}_{\boldsymbol{f}}(\Delta^k) \setminus \mathfrak{S}_{\boldsymbol{f}}(\partial\Delta^k)]$ *is an open set in* ∂W.
(c) $W = \operatorname{Up}(\boldsymbol{f}[\mathfrak{S}_{\boldsymbol{f}}(\Delta^k)])$; *in other words,* $\operatorname{Up}(\boldsymbol{f}[\mathfrak{S}_{\boldsymbol{f}}(\Delta^k)]) \supset \boldsymbol{f}(\mathcal{X})$.

Note that since Δ^k is compact, we also get that W is closed.

Proof. Let $V = \boldsymbol{f}(\mathcal{X}) \subset \mathbb{R}^{k+1}$; so $W = \operatorname{Up} V$. Denote by $\bar{V}$ the closure of V.



(a). Convexity of all f_i implies that for any two points $p, q \in \mathcal{X}$ and $t \in [0,1]$ we have

$$(1-t)\cdot\boldsymbol{f}(p) + t\cdot\boldsymbol{f}(q) \succcurlyeq \boldsymbol{f} \circ \gamma(t),$$

where γ denotes a geodesic path from p to q. Therefore, W is convex.

(b)+(c). Choose $p \in \mathfrak{S}_{\boldsymbol{f}}(\Delta^k)$. Note that if $\boldsymbol{f}(p) \succcurlyeq \boldsymbol{w}$ for some $\boldsymbol{w} \in W$, then $\boldsymbol{f}(p) = \boldsymbol{w}$. It follows that $\boldsymbol{f}(p) \in \partial W$; therefore $\boldsymbol{f}[\mathfrak{S}_{\boldsymbol{f}}(\Delta^k)]$ lies in a convex hypersurface ∂W.

Choose $\boldsymbol{w} \in W$. Observe that $\boldsymbol{w} \succcurlyeq \boldsymbol{v}$ for some $\boldsymbol{v} \in \bar{V} \cap \partial W$. Note that W is supported at $\boldsymbol{v}$ by a hyperplane

$$\Pi = \left\{\, (x_1, \dots, x_k) \in \mathbb{R}^k \;:\; m_0\cdot x_0 + \dots + m_k\cdot x_k = \text{const} \,\right\}$$

for some $\boldsymbol{m} = (m_0, \dots, m_k) \in \Delta^k$. Let $p = \mathfrak{S}_{\boldsymbol{f}}(\boldsymbol{m})$. By 10.2, $\boldsymbol{f}(p) = \boldsymbol{v}$; in particular $\boldsymbol{v} \in V$.

Note that $p \in \mathfrak{S}_{\boldsymbol{f}}(\Delta^k) \setminus \mathfrak{S}_{\boldsymbol{f}}(\partial\Delta^k)$ if and only if $\boldsymbol{f}(p)$ is supported by a plane as above for some $\boldsymbol{m} \in \Delta^k$, but it is not supported by a plane for some $\boldsymbol{m} \in \partial\Delta^k$. This condition is open, therefore $\mathfrak{S}_{\boldsymbol{f}}(\Delta^k) \setminus \mathfrak{S}_{\boldsymbol{f}}(\partial\Delta^k)$ is an open set. □

D Nondegenerate simplex

Given an array $\boldsymbol{f} = (f_0, \dots, f_k)$, we denote by $\boldsymbol{f}^{-i}$ the subarray of $\boldsymbol{f}$ with f_i removed; that is,

$$\boldsymbol{f}^{-i} := (f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_k).$$

It should be clear from the definition that $\mathfrak{S}_{\boldsymbol{f}^{-i}}$ coincides with the restriction of $\mathfrak{S}_{\boldsymbol{f}}$ to the corresponding facet of Δ^k.

If $\operatorname{Im}\mathfrak{S}_{\boldsymbol{f}}$ is not covered by $\operatorname{Im}\mathfrak{S}_{\boldsymbol{f}^{-i}}$ for all i, then we say that $\mathfrak{S}_{\boldsymbol{f}}$ is nondegenerate. In other words, $\mathfrak{S}_{\boldsymbol{f}}$ is nondegenerate if

$$\mathfrak{S}_{\boldsymbol{f}}(\Delta^k) \setminus \mathfrak{S}_{\boldsymbol{f}}(\partial\Delta^k) \neq \varnothing.$$

10.6. Exercise. *Let $\mathcal{U}$ be a complete geodesic* CAT(0) *space.*

Show that the image 1-dimensional barycentric simplex for a pair of points $p_0, p_1 \in \mathcal{U}$ is the geodesic $[p_0p_1]$.

Construct a CAT(0) *space with a three-point array (p_0, p_1, p_2) such that its barycentric simplex is nondegenerate and noninjective.*

10.7. Exercise. *Let $\boldsymbol{p} = (p_0, \dots p_k)$ be a point array in a complete length* CAT(0) *space $\mathcal{U}$, and $B_i = \overline{\mathrm{B}}[p_i, r_i]$ for some array of positive reals $(r_0, r_1, \dots, r_k)$.*

(a) Suppose $\bigcap_i B_i \neq \varnothing$. Show that

$$\operatorname{Im}\mathfrak{S}_{\boldsymbol{p}} \subset \bigcup_i B_i.$$

(b) Suppose $\bigcap_i B_i = \varnothing$, but $\bigcap_{i\neq j} B_i \neq \varnothing$ for any j. Show that $\mathfrak{S}_{\boldsymbol{p}}$ is nondegenerate.

(c) Suppose $\mathfrak{S}_{\boldsymbol{p}}$ is nondegenerate. Show that the condition in (b) hold for some array of positive reals $(r_0, \dots, r_k)$.

E bi-Hölder embedding

10.8. Theorem. *Let $\mathcal{X}$ be a complete geodesic space and $\boldsymbol{f} = (f_0, \dots$ $\dots, f_k)\colon \mathcal{X} \to \mathbb{R}^{k+1}$ be an array of 1-convex locally Lipschitz functions. Then the set*

$$Z = \mathfrak{S}_{\boldsymbol{f}}(\Delta^k) \setminus \mathfrak{S}_{\boldsymbol{f}}(\partial\Delta^k)$$

is $C^{\frac12}$-bi-Hölder to an open domain in $\mathbb{R}^k$.

Proof. Let $\operatorname{proj}\colon \mathbb{R}^{k+1} \to \Pi$ be orthogonal projection to the hyperplane $x_0 + \dots + x_k = 0$. Let us show that the restriction $\operatorname{proj}\circ\boldsymbol{f}|_Z$ is a bi-Hölder embedding.

The map $\operatorname{proj}\circ\boldsymbol{f}$ is Lipschitz; it remains to construct its right inverse and show that it is $C^{\frac12}$-continuous.

Given $\boldsymbol{v} = (v_0, v_1, \dots, v_k) \in \Pi$, consider the function $h_{\boldsymbol{v}}\colon \mathcal{X} \to \mathbb{R}$ defined by

$$h_{\boldsymbol{v}}(p) = \max_i\{f_i(p) - v_i\}.$$

Note that $h_{\boldsymbol{v}}$ is 1-convex. Let

$$\Phi(\boldsymbol{v}) := \operatorname{MinPoint} h_{\boldsymbol{v}}.$$

According to Lemma 10.2, $\Phi(\boldsymbol{v})$ is uniquely defined.

If $\boldsymbol{v} = \operatorname{proj} \boldsymbol{f}(p)$, then

$$f_i \circ \Phi(\boldsymbol{v}) \leqslant f_i(p)$$

for any i. In particular, if $p \in \mathfrak{S}_{\boldsymbol{f}}(\Delta^k)$, then $p = \Phi(\boldsymbol{v})$. That is, Φ is a right inverse of the restriction $\boldsymbol{f}|_{\mathfrak{S}_{\boldsymbol{f}}(\Delta^k)}$.

Given $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^{k+1}$, set $p = \Phi(\boldsymbol{v})$ and $q = \Phi(\boldsymbol{w})$. Since $h_{\boldsymbol{v}}$ and $h_{\boldsymbol{w}}$ are 1-convex, we have

$$h_{\boldsymbol{v}}(q) \geqslant h_{\boldsymbol{v}}(p) + \tfrac{1}{2}\cdot|p-q|^2, \qquad h_{\boldsymbol{w}}(p) \geqslant h_{\boldsymbol{w}}(q) + \tfrac{1}{2}\cdot|p-q|^2.$$

Therefore,

$$\begin{aligned}|p-q|^2 &\leqslant 2\cdot \sup_{x\in\mathcal{X}}\{|h_{\boldsymbol{v}}(x) - h_{\boldsymbol{w}}(x)|\} \leqslant \\ &\leqslant 2\cdot \max_i\{|v_i - w_i|\}.\end{aligned}$$

In particular, Φ is $C^{\frac{1}{2}}$-continuous.

Finally, by 10.5b, $\boldsymbol{f}(Z)$ is a k-dimensional manifold, hence the result. □

F Topological dimension

Let $\mathcal{X}$ be a metric space and $\{V_\beta\}_{\beta\in\mathcal{B}}$ be an open cover of $\mathcal{X}$. Let us recall two notions in general topology:

- ◇ The order of $\{V_\beta\}$ is the supremum of all integers n such that there is a collection of $n+1$ elements of $\{V_\beta\}$ with nonempty intersection.
- ◇ An open cover $\{W_\alpha\}_{\alpha\in\mathcal{A}}$ of $\mathcal{X}$ is called a refinement of $\{V_\beta\}_{\beta\in\mathcal{B}}$ if for any $\alpha\in\mathcal{A}$ there is $\beta\in\mathcal{B}$ such that $W_\alpha \subset V_\beta$.

10.9. Definition. *Let $\mathcal{X}$ be a metric space. The topological dimension of $\mathcal{X}$ is defined to be the minimum of nonnegative integers n such that for any open cover of $\mathcal{X}$ there is a finite open refinement with order n.*

If no such n exists, the topological dimension of $\mathcal{X}$ is infinite.

The topological dimension of $\mathcal{X}$ will be denoted by $\operatorname{TopDim}\mathcal{X}$.

The invariants satisfying the following two statements 10.10 and 10.11 are commonly called “dimension”; for that reason, we call these statements axioms.

10.10. Normalization axiom. *For any $m \in \mathbb{Z}_{\geqslant 0}$,*

$$\operatorname{TopDim} \mathbb{E}^m = m.$$

10.11. Cover axiom. *If $\{A_n\}_{n=1}^{\infty}$ is a countable closed cover of $\mathcal{X}$, then*

$$\operatorname{TopDim} \mathcal{X} = \sup_n \{\operatorname{TopDim} A_n\}.$$

On product spaces. The following inequality holds for arbitrary metric spaces

$$\operatorname{TopDim}(\mathcal{X} \times \mathcal{Y}) \leqslant \operatorname{TopDim} \mathcal{X} + \operatorname{TopDim} \mathcal{Y}.$$

It is strict for a pair of Pontryagin surfaces [82].

10.12. Definition. *Let $\mathcal{X}$ be a metric space and $F\colon \mathcal{X} \to \mathbb{R}^m$ be a continuous map. A point $\boldsymbol{z} \in \operatorname{Im} F$ is called a stable value of F if there is $\varepsilon > 0$ such that $\boldsymbol{z} \in \operatorname{Im} F'$ for any ε-close to F continuous map $F'\colon \mathcal{X} \to \mathbb{R}^m$, that is, $|F'(x) - F(x)| < \varepsilon$ for all $x \in \mathcal{X}$.*

The next theorem follows from [57, theorems VI 1&2]. (This theorem also holds for non-separable metric spaces [71], [46, 3.2.10]).

10.13. Stable value theorem. *Let $\mathcal{X}$ be a separable metric space. Then $\operatorname{TopDim} \mathcal{X} \geqslant m$ if and only if there is a map $F\colon \mathcal{X} \to \mathbb{R}^m$ with a stable value.*

G Dimension theorem

10.14. Theorem. *For any proper geodesic $\mathrm{CAT}(0)$ space $\mathcal{U}$, the following statements are equivalent:*

(a)

$$\operatorname{TopDim} \mathcal{U} \geqslant m.$$

(b) For some $z \in \mathcal{U}$ there is an array of $m+1$ balls $B_i = \mathrm{B}(a_i, r_i)$ such that

$$\bigcap_i B_i = \varnothing \quad \text{and} \quad \bigcap_{i \neq j} B_i \neq \varnothing \quad \text{for each } j.$$

(c) There is a $C^{\frac{1}{2}}$-embedding of an open set in $\mathbb{R}^m$ to $\mathcal{U}$; that is, Φ is bi-Hölder with exponent $\frac{1}{2}$.

10.15. Lemma. *Let $\mathcal{U}$ be a proper geodesic* CAT(0) *space and $\rho\colon \mathcal{U} \to \mathbb{R}$ be a continuous positive function. Then there is a locally finite countable simplicial complex $\mathcal{N}$, a locally Lipschitz map $\Phi\colon \mathcal{U} \to \mathcal{N}$, and a Lipschitz map $\Psi\colon \mathcal{N} \to \mathcal{U}$ such that:*

(a) The displacement of the composition $\Psi\circ\Phi\colon \mathcal{U} \to \mathcal{U}$ is bounded by ρ; that is,

$$|x - \Psi\circ\Phi(x)| < \rho(x)$$

for any $x \in \mathcal{U}$.

(b) If $\operatorname{TopDim}\mathcal{U} \leqslant m$*, then the Ψ-image of $\mathcal{N}$ coincides with the image of its m-skeleton.*

Proof. Choose a locally finite countable covering $\{\,\Omega_\alpha : \alpha\in\mathcal{A}\,\}$ of $\mathcal{U}$ such that $\Omega_\alpha \subset \mathrm{B}(x, \tfrac13\cdot\rho(x))$ for any $x\in\Omega_\alpha$.

Denote by $\mathcal{N}$ the nerve of the covering $\{\Omega_\alpha\}$; that is, $\mathcal{N}$ is an abstract simplicial complex with vertex set $\mathcal{A}$, such that a finite subset $\{\alpha_0,\dots,\alpha_n\}\subset\mathcal{A}$ forms a simplex if and only if

$$\Omega_{\alpha_0}\cap\dots\cap\Omega_{\alpha_n}\ne\varnothing.$$

Choose a Lipschitz partition of unity $\varphi_\alpha\colon \mathcal{U}\to[0,1]$ subordinate to $\{\Omega_\alpha\}$. Consider the map $\Phi\colon\mathcal{U}\to\mathcal{N}$ such that the barycentric coordinate of $\Phi(p)$ is $\varphi_\alpha(p)$. Note that Φ is locally Lipschitz. Clearly, the Φ-preimage of any open simplex in $\mathcal{N}$ lies in Ω_α for some $\alpha\in\mathcal{A}$.

For each $\alpha\in\mathcal{A}$, choose $x_\alpha\in\Omega_\alpha$. Let us extend the map $\alpha\mapsto x_\alpha$ to a map $\Psi\colon\mathcal{N}\to\mathcal{U}$ that is barycentric on each simplex. According to 10.1, this extension exists, and Ψ is locally Lipschitz.

(a). Fix $x\in\mathcal{U}$. Denote by Δ the minimal simplex that contains $\Phi(x)$, and let $\alpha_0,\alpha_1,\dots,\alpha_n$ be the vertexes of Δ. Note that α is a vertex of Δ if and only if $\varphi_\alpha(x)>0$. Thus

$$|x - x_{\alpha_i}| < \tfrac13\cdot\rho(x)$$

for any i. Therefore

$$\operatorname{diam}\Psi(\Delta)\leqslant\max_{i,j}\{|x_{\alpha_i}-x_{\alpha_j}|\}<\tfrac23\cdot\rho(x).$$

In particular,

$$|x-\Psi\circ\Phi(x)|\leqslant|x-x_{\alpha_0}|+\operatorname{diam}\Psi(\Delta)<\rho(x).$$

(b). Assume the contrary; that is, $\Psi(\mathcal{N})$ is not included in the Ψ-image of the m-skeleton of $\mathcal{N}$. Then for some $k>m$, there is a k-simplex Δ^k

in $\mathcal{N}$ such that the barycentric simplex $\sigma = \Psi|_{\Delta^k}$ is nondegenerate; that is,

$$W = \Psi(\Delta^k) \setminus \Psi(\partial\Delta^k) \neq \varnothing.$$

By 10.8, $\operatorname{TopDim}\mathcal{U} \geqslant k$ — a contradiction. $\square$

Proof of 10.14; (b)⇒(c)⇒(a). The implication *(b)⇒(c)* follows from Lemma 10.7 and Theorem 10.8, and *(c)⇒(a)* is trivial.

(a)⇒(b). According to 10.13, there is a continuous map $F\colon \mathcal{U} \to \mathbb{R}^m$ with a stable value.

Fix $\varepsilon > 0$. Since F is continuous, there is a continuous positive function ρ defined on $\mathcal{U}$ such that

$$|x - y| < \rho(x) \quad \Rightarrow \quad |F(x) - F(y)| < \tfrac{1}{3}\cdot\varepsilon.$$

Apply 10.15 to ρ. For the resulting simplicial complex $\mathcal{N}$ and the maps $\Phi\colon \mathcal{U} \to \mathcal{N}$, $\Psi\colon \mathcal{N} \to \mathcal{U}$, we have

$$|F \circ \Psi \circ \Phi(x) - F(x)| < \tfrac{1}{3}\cdot\varepsilon$$

for any $x \in \mathcal{U}$.

Arguing by contradiction, assume $\operatorname{TopDim}\mathcal{U} < m$. By 10.15*b*, the image $F_\varepsilon \circ \Psi \circ \Phi(K)$ lies in the F_ε-image of the $(m-1)$-skeleton of $\mathcal{N}$; In particular, it can be covered by a countable collection of Lipschitz images of $(m-1)$-simplexes. Hence $\mathbf{0} \in \mathbb{R}^m$ is not a stable value of $F_\varepsilon \circ \Psi \circ \Phi$. Since $\varepsilon > 0$ is arbitrary, we get the result. $\square$

The following exercise is a generalization of Helly's theorem; for closely related statements see [60, Prop. 5.3] and [58].

10.16. Exercise. *Let $K_1, \dots, K_n$ be closed convex subsets in a proper length* CAT(0) *space $\mathcal{U}$. Suppose that* $\operatorname{TopDim}\mathcal{U} = m$ *and any $m + 1$ subsets from $\{K_1, \dots K_n\}$ have a common point. Show that all subsets $K_1, \dots, K_n$ have a common point.*

H Hausdorff dimension

10.17. Definition. *Let $\mathcal{X}$ be a metric space. Its Hausdorff dimension is defined as*

$$\operatorname{HausDim}\mathcal{X} = \sup\{\, \alpha \in \mathbb{R} \,:\, \operatorname{HausMes}_\alpha(\mathcal{X}) > 0 \,\},$$

where $\operatorname{HausMes}_\alpha$ *denotes the α-dimensional Hausdorff measure.*

The following theorem follows from [57, theorems V 8 and VII 2].

10.18. Szpilrajn's theorem. *Let $\mathcal{X}$ be a separable metric space. Assume* $\operatorname{TopDim}\mathcal{X} \geqslant m$. *Then* $\operatorname{HausMes}_m \mathcal{X} > 0$.

In particular, $\operatorname{TopDim}\mathcal{X} \leqslant \operatorname{HausDim}\mathcal{X}$.

Except for Szpilrajn's theorem, there are no other relations between topological and Husdorff dimension of separable spaces. Moreover, the following exercise implies that the same holds for compact geodesic CAT(0) spaces of topological dimension at least 1.

10.19. Exercise. *Construct a metric on the binary tree such that it has compact completion of arbitrary Huasdorff dimension $\alpha \geqslant 1$.*

Concclude that for any integer $m \geqslant 1$ and real $\alpha \geqslant m$ there is a compact CAT(0) *space with topological dimension m and Hausdorff dimension α.*

I Remarks

The barycenters in CAT(κ) spaces were introduced by Bruce Kleiner [60]. He also proved the dimension theorem; an improvement was made by Alexander Lytchak [67].

It is not known if the dimension theorem holds for arbitrary complete geodesic CAT(κ) spaces. It was conjectured by Bruce Kleiner [60], see also [51, p. 133]. For separable spaces, the answer is "yes", and it follows from Kleiner's argument [6, Corollary 14.13].

One may wonder if bi-Hölder condition 10.14c can be improved to bi-Lipschitz; it seem to be unknown even for compact spaces. However if a compact geodesic CAT(0) space $\mathcal{U}$ has finite topological dimension m, then a slight modification of Kleiner's technique can be used to show that there is a bi-Lipschitz embedding of an m-cube into $\mathcal{U}$ [6, Theorem 14.15]. In particular, there is a bi-Lipschitz embedding of an n-cube for any $n \leqslant m$. If $\operatorname{TopDim}\mathcal{U} = \infty$, then we expect existence of a bi-Lipschitz embedding of an n-cube for any integer $n \geqslant 1$. The statement is trivial for $n = 1$; in this case any geodesic gives an isometric embedding. For $n = 2$, one can get it from the the fact that minimal (or metric minimizing) surfaces in $\mathcal{U}$ are CAT(0) (any such surface is locally bi-Lipschitz to the Euclidean plane). For $n \geqslant 3$ the question remains open.

Lecture 11

Quotients

A Quotient space

Suppose that a group G acts isometrically on a metric space $\mathcal{X}$. Note that

$$|G\cdot x - G\cdot y|_{\mathcal{X}/G} := \inf\{\, |x - g\cdot y|_{\mathcal{X}} \,:\, g \in G \,\}$$

defines a semimetric on the orbit space $\mathcal{X}/G$. Moreover, it is a genuine metric if the orbits of the action are closed.

11.1. Theorem. *Assume that group G acts isometrically on a proper* CBB(0) *space $\mathcal{L}$ and has closed orbits. Then the quotient space $\mathcal{L}/G$ is* CBB(0).

Proof. Denote by $\sigma\colon \mathcal{L} \to \mathcal{L}/G$ the quotient map.

Fix a quadruple of points $p, x_1, x_2, x_3 \in \mathcal{L}/G$. Choose an arbitrary $\hat{p} \in \mathcal{L}$ such that $\sigma(\hat{p}) = p$. Since $\mathcal{L}$ is proper, we can choose the points $\hat{x}_i \in \mathcal{L}$ such that $\sigma(\hat{x}_i) = x_i$ and

$$|p - x_i|_{\mathcal{L}/G} = |\hat{p} - \hat{x}_i|_{\mathcal{L}}$$

for all i.

Note that

$$|x_i - x_j|_{\mathcal{L}/G} \leqslant |\hat{x}_i - \hat{x}_j|_{\mathcal{L}}$$

for all i and j. Therefore

❶ $$\tilde{\measuredangle}(p\,{}^{x_i}_{x_j}) \leqslant \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_i}_{\hat{x}_j})$$

holds for all i and j.

By CBB(0) comparison in $\mathcal{L}$, we have

$$\tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_1}_{\hat{x}_2}) + \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_2}_{\hat{x}_3}) + \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_3}_{\hat{x}_1}) \leqslant 2\cdot\pi.$$

Applying ❶, we get

$$\tilde\measuredangle(p^{x_1}_{x_2}) + \tilde\measuredangle(p^{x_2}_{x_3}) + \tilde\measuredangle(p^{x_3}_{x_1}) < 2{\cdot}\pi;$$

that is, the CBB(0) comparison holds for any quadruple in $\mathcal{L}/G$. □

11.2. Advanced exercise. *Let G be a compact Lie group with a bi-invariant Riemannian metric. Show that G is isometric to a quotient of the Hilbert space by an isometric group action.*

Conclude that $G \in$ CBB(0).

B Generalizations

A map $\sigma\colon \mathcal{X} \to \mathcal{Y}$ between the metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is called a submetry if

$$\sigma(\mathrm{B}(p,r)_{\mathcal{X}}) = \mathrm{B}(\sigma(p),r)_{\mathcal{Y}}$$

for any $p \in \mathcal{X}$ and $r \geqslant 0$.

Suppose G and $\mathcal{L}$ are as in 11.1. Observe that the quotient map $\sigma\colon \mathcal{L} \to \mathcal{L}/G$ is a submetry. The following two exercises show that this is not the only source of submetries.

11.3. Exercise. *Construct submetries*

(a) $\sigma_1\colon \mathbb{S}^2 \to [0,\pi]$
(b) $\sigma_2\colon \mathbb{S}^2 \to [0,\tfrac{\pi}{2}]$
(c) $\sigma_n\colon \mathbb{S}^2 \to [0,\tfrac{\pi}{n}]$ *(for integer $n \geqslant 1$)*

such that the fibers $\sigma_n^{-1}\{x\}$ are connected and have an empty interior for any x.

11.4. Exercise. *Let $\sigma\colon \mathbb{E}^2 \to [0,\infty)$ be a submetry. Show that $K = \sigma^{-1}\{0\}$ is a closed convex set in $\mathbb{E}^2$ and $\sigma(x) = \mathrm{dist}_K x$.*

The proof of 11.1 works for submetries. Therefore we get the following.

11.5. Generalization. *Let $\sigma\colon \mathcal{L} \to \mathcal{M}$ be a submetry. Suppose $\mathcal{L}$ is a* CBB(0) *space, then so is $\mathcal{M}$.*

Theorems 11.1 and 11.5 admit straightforward generalizations to CBB(−1) spaces. In the CBB(1) case, the proof produces a slightly weaker statement — *any open $\tfrac{\pi}{2}$-ball in the quotient of* CBB(1) *is* CBB(1); in particular, the quotient space is *locally* CBB(1). If the space is geodesic, then the globalization theorem implies that it is globally CBB(1). The same holds for the targets of submetry from a CBB(1) space. In other words, if $\mathcal{L}$ is a geodesic space, then analogs of 11.1 and 11.5 hold for CBB(κ) spaces with arbitrary κ.

C Hopf's conjecture

Recall that Hopf's conjecture states that $\mathbb{S}^2 \times \mathbb{S}^2$ *does not admit a Riemannian metric with positive sectional curvature.* The following partial result was obtained by Wu-Yi Hsiang and Bruce Kleiner [56].

11.6. Theorem. *There is no Riemannian metric on $\mathbb{S}^2 \times \mathbb{S}^2$ with sectional curvature $\geqslant 1$ and a nontrivial isometric $\mathbb{S}^1$-action.*

We will give a ruf sketch that will use many statements and notions that are not yet introduced. Nevertheless, it should be possible to follow the proof with an intuitive understanding of the notions (boundary and dimension of a quotient space). The statements include

- The Toponogov comparison theorem and the globalization theorem. The former says that *a complete Riemannian manifold has sectional curvature $\geqslant \kappa$ if and only if the corresponding metric space is* $\mathrm{CBB}(\kappa)$. The later says that *a complete geodesic locally* $\mathrm{CBB}(\kappa)$ *space is* $\mathrm{CBB}(\kappa)$.
- Doubling theorem: *Doubling of a finite-dimensional geodesic* $\mathrm{CBB}(\kappa)$ *space is a geodesic* $\mathrm{CBB}(\kappa)$ *space*
- Splitting theorem: *A geodesic* $\mathrm{CBB}(0)$ *space that contains a line splits isometrically as* $\mathbb{R} \times \mathcal{X}$; here the line is a both-sided infinite geodesic.

In addition, we will use the following exercise that will be proved latter:

11.7. Exercise. *Suppose $\mathbb{S}^1 \curvearrowright \mathbb{S}^3$ is an isometric action without fixed points and $\Sigma = \mathbb{S}^3/\mathbb{S}^1$ is its quotient space. Then there is a distance noncontracting map $\Sigma \to \tfrac{1}{2}\cdot\mathbb{S}^2$, where $\tfrac{1}{2}\cdot\mathbb{S}^2$ is the standard 2-sphere rescaled with a factor $\tfrac{1}{2}$.*

Sketch of 11.6. Assume $\mathcal{M} = (\mathbb{S}^2 \times \mathbb{S}^2, g)$ is a counterexample. By the Toponogov theorem, $\mathcal{M}$ is $\mathrm{CBB}(1)$. By 11.1, the quotient space $\mathcal{L} = \mathcal{M}/\mathbb{S}^1$ is $\mathrm{CBB}(1)$; evidently, $\mathcal{L}$ is 3-dimensional.

Denote by $F \subset \mathcal{M}$ the fixed point set of the $\mathbb{S}^1$-action. Each connected component of F is either an isolated point or a 2-dimensional geodesic submanifold in $\mathcal{M}$; the latter has to have positive curvature and therefore it is either $\mathbb{S}^2$ or $\mathbb{RP}^2$. Notice that

- each isolated point contributes 1 to the Euler characteristic of $\mathcal{M}$,
- each sphere contributes 2 to the Euler characteristic of $\mathcal{M}$, and
- each projective plane contributes 1 to the Euler characteristic of $\mathcal{M}$.

Since $\chi(\mathcal{M}) = 4$, we are in one of the following three cases:

- F has exactly 4 isolated points,

◇ F has one 2-dimensional submanifold and at least 2 isolated points,

◇ F has at least two 2-dimensional submanifolds.

Each case is covered separately.

Case 1. Suppose F has exactly 4 isolated points x_1, x_2, x_3, and x_4. Denote by y_1, y_2, y_3, and y_4 the corresponding points in $\mathcal{L}$. Note that $\Sigma_{y_i}\mathcal{L}$ is isometric to a quotient of $\mathbb{S}^3$ by an isometric $\mathbb{S}^1$-action without fixed points.

By 11.7, each angle $\measuredangle[y_i\,{}^{y_j}_{y_k}] \leqslant \frac{\pi}{2}$ for any three distinct points y_i, y_j, y_k. In particular, all four triangles $[y_1y_2y_3]$, $[y_1y_2y_4]$, $[y_1y_3y_4]$, and $[y_2y_3y_4]$ are nondegenerate. By the comparison, the sum of angles in each triangle is strictly greater than π.

Denote by σ the sum of all 12 angles in 4 triangles $[y_1y_2y_3]$, $[y_1y_2y_4]$, $[y_1y_3y_4]$, and $[y_2y_3y_4]$. From above,

$$\sigma > 4{\cdot}\pi.$$

On the other hand, by 11.7 any triangle in $\Sigma_{y_1}\mathcal{L}$ has perimeter at most π. In particular,

$$\measuredangle[y_1\,{}^{y_2}_{y_3}] + \measuredangle[y_1\,{}^{y_3}_{y_4}] + \measuredangle[y_1\,{}^{y_4}_{y_2}] \leqslant \pi.$$

Apply the same argument in $\Sigma_{y_2}\mathcal{L}$, $\Sigma_{y_3}\mathcal{L}$, and $\Sigma_{y_4}\mathcal{L}$. Adding the results we get

$$\sigma \leqslant 4{\cdot}\pi$$

— a contradiction.

Case 2. Let F contain one surface S. Note that the projection of S to $\mathcal{L}$ forms its boundary $\partial\mathcal{L}$. Note that doubling $\hat{\mathcal{L}}$ of $\mathcal{L}$ across its boundary has 4 singular points — each singular point of $\mathcal{L}$ corresponds to two singular points of $\hat{\mathcal{L}}$.

By the Doubling theorem, $\hat{\mathcal{L}}$ is a geodesic CBB(1) space. Therefore we arrive at a contradiction the same way as in the first case.

Case 3. Suppose F contains at least two surfaces. Then $\partial\mathcal{L}$ has at least two connected components; choose two of them A and B. Denote by γ a geodesic that minimizes the distance from A to B.

Let

$$\dots,\mathcal{L}_{-1},\mathcal{L}_0,\mathcal{L}_1,\dots$$

be a two-sided infinite sequence of copies on $\partial\mathcal{L}$. Let us glue $\mathcal{L}_i$ to $\mathcal{L}_{i+1}$ along A if i is even and along B if i is odd.

By the Doubling theorem, every point in the obtained space $\mathcal{N}$ has a neighborhood that is isometric to a neighborhood of the corresponding point in $\mathcal{L}$ or its doubling. By the globalization theorem, $\mathcal{N}$ is CBB(1).

Note that the copies of γ in $\mathcal{L}_i$ form a line in $\mathcal{N}$. By the splitting theorem, $\mathcal{N}$ is isometric to a product $\mathcal{N}'\oplus\mathbb{R}$. The latter is impossible for a CBB(1) space — a contradiction. (Here we used that the dimension of $\mathcal{N}$ is bigger than 1. According to our definitions, $\mathbb{R}$ is CBB(1); it splits trivially, but such examples exist only in dimension 1.) $\square$

D Erdős' problem rediscovered

A point p in a CBB(κ) space is called extremal if $\measuredangle[p\,{}^x_y]\leqslant\frac{\pi}{2}$ for any hinge $[p\,{}^x_y]$ with the vertex at p.

11.8. Theorem. *Let $\mathcal{L}$ be a compact m-dimensional* CBB(0) *space. Then it has at most 2^m one-point extremal sets.*

The proof is a translation of the proof of a classical problem in discrete geometry to Alexandrov's language.

11.9. Erdős' problem. *Let F be a set of points in $\mathbb{E}^m$ such that any triangle formed by three distinct points in F has no obtuse angles. Then $|F|\leqslant 2^m$. Moreover, if $|F|=2^m$ then F consists of the vertexes of an m-dimensional rectangle.*

This problem was posed by Paul Erdős [47] and solved by Ludwig Danzer and Branko Grünbaum [40]. Grigori Perelman noticed that after proper definitions, the same proof works in Alexandrov spaces [74]; so it proves 11.8. We will use the notion of volume; it is not defined so far and should be understood intuitively.

Proof of 11.8. Let $\{p_1,\dots,p_N\}$ be extremal points in $\mathcal{L}$. For each p_i consider its open Voronoi domain V_i; that is,

$$V_i=\{\,x\in\mathcal{L}\,:\,|p_i-x|<|p_j-x|\text{ for any }j\neq i\,\}.$$

Clearly $V_i\cap V_j=\varnothing$ if $i\neq j$.

Suppose $0<\alpha\leqslant 1$. Given a point $x\in\mathcal{L}$, choose a geodesic $[p_ix]$ and denote by x_i the point on $[p_ix]$ such that $|p_i-x_i|=\alpha{\cdot}|p_i-x|$; let $\Phi_i\colon x\to x_i$ be the corresponding map. By the comparison,

$$|x_i-y_i|\geqslant\alpha{\cdot}|x-y|$$

for any x, y, and i. Therefore

$$\operatorname{vol}(\Phi_i\mathcal{L})\geqslant\alpha^m\cdot\operatorname{vol}\mathcal{L}.$$

Suppose $\alpha<\frac12$. Then $x_i\in V_i$ for any $x\in\mathcal{L}$. Indeed, assume $x_i\notin V_i$, then threre is p_j such that $|p_i-x_i|\geqslant|p_j-x_i|$. Then from comparison, we have $\tilde\measuredangle(p_j\,{}^{p_i}_{x})_{\mathbb{E}^2} > \frac{\pi}{2}$; that is, p_j does not form a one-point extremal set. It follows that $\operatorname{vol} V_i \geqslant \alpha^m \cdot \operatorname{vol}\mathcal{L}$ for any $0 < \alpha < \frac{1}{2}$ and hence

$$\operatorname{vol} V_i \geqslant \tfrac{1}{2^m} \cdot \operatorname{vol}\mathcal{L}$$

and hence

$$N \leqslant 2^m.$$

□

E Crystallographic actions

An isometric action $\Gamma \curvearrowright \mathbb{E}^m$ is called crystallographic if it is properly discontinuous (that is, for any compact set $K \subset \mathbb{E}^m$ and $x \in \mathbb{E}^m$ there only finitely many $g \in \Gamma$ such that $g\cdot x \in K$) and cocompact (that is, the quotient space $\mathcal{L} = \mathbb{E}^m/\Gamma$ is compact).

Let F be a maximal finite subgroup of Γ; that is, if H is a finite group H such that $F < H < \Gamma$, then $F = H$. Denote by $\#(\Gamma)$ the number of maximal finite subgroups of Γ up to conjugation.

11.10. Open question. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Is it true that $\#(\Gamma) \leqslant 2^m$?*

Note that any finite subgroup F of Γ fixes an affine subspace A_F in $\mathbb{E}^m$. If F is maximal, then A_F completely describes F. Denote by $\#_k(\Gamma)$ the number of maximal finite subgroups $F < \Gamma$ (up to conjugation) such that $\dim A_F = k$.

Choose a finite subgroup $F < \Gamma$; consider a conjugate subgroup $F' = g\cdot F\cdot g^{-1}$. Note that $A_{F'} = g\cdot A_F$. In particular, the subspaces A_F and $A_{F'}$ have the same image in the quotient space $\mathcal{L} = \mathbb{E}^m/\Gamma$. It follows that to count subgroups up to conjugation, we need to count the images of their fixed set. Therefore, by the lemma below, $\#_0(\Gamma)$ cannot exceed the number of extremal points in $\mathcal{L} = \mathbb{E}^m/\Gamma$. Combining this observation with 11.8, we get the following.

11.11. Proposition. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Then $\#_0(\Gamma) \leqslant 2^m$.*

11.12. Lemma. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action and F be a maximal finite subgroup of Γ that fixes an isolated point p. Then the image of p in the quotient space $\mathcal{L} = \mathbb{E}^m/\Gamma$ is an extremal point.*

Proof. Let q be the image of p. Suppose q is not extremal; that is, $\measuredangle[q\,{}^{y_1}_{y_2}] > \frac{\pi}{2}$ for some hinge $[q\,{}^{y_1}_{y_2}]$ in $\mathcal{L}$.

Choose the inverse images $x_1, x_2 \in \mathbb{E}^m$ of $y_1, y_2 \in \mathcal{L}$ such that $|p-x_i|_{\mathbb{E}^m} = |q-y_i|_{\mathcal{L}}$. Note that $\measuredangle[p\,{}^{x_1}_{x_2}] \geqslant \measuredangle[q\,{}^{y_1}_{y_2}] > \frac{\pi}{2}$. Moreover, since p is fixed by F, we have

$$\text{❶} \qquad \measuredangle[p\,{}^{x_1}_{g\cdot x_2}] > \tfrac{\pi}{2}$$

for any $g \in F$.

Denote by z the barycenter of the orbit $G\cdot x_2$. Note that z is a fixed point of F. By ❶, $z \neq p$; so F must fix the line pz. But p is an isolated fixed point of F — a contradiction. $\square$

11.13. Exercise. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Apply the theorems used in 11.6 to show that*

(a) $\#_{m-1}(\Gamma) \leqslant 2$, and

(b) if $\#_{m-1}(\Gamma) = 1$, then $\#_0(\Gamma) \leqslant 2^{m-1}$.

Construct crystallographic actions with equalities in (a) and (b).

F Remarks

It is expected that *no geodesic* CBB(1) *space with a nontrivial isometric $\mathbb{S}^1$-action can be homeomorphic to* $\mathbb{S}^2\times\mathbb{S}^2$; so 11.6 holds for general CBB(1) space. The proof of 11.6 would work if we had the following generalization of 11.7.

11.14. Conjecture. *Let Σ be a geodesic* CBB(1) *space homeomorphic to $\mathbb{S}^3$. Suppose $\mathbb{S}^1$ acts on Σ isometrically. Then any triangle in $\Sigma/\mathbb{S}^1$ has perimeter at most π.*

Compact geodesic m-dimensional CBB(0) spaces with the maximal number of extremal points include m-dimensional rectangles and the quotients of flat tori by reflections across a point. (This action has 2^m isolated fixed points; each corresponds to an extremal point in the quotient space $\mathcal{L} = \mathbb{T}^m/\mathbb{Z}_2$.) Nina Lebedeva has proved [63] that *every m-dimensional geodesic* CBB(0) *space with 2^m extremal points is a quotient of Euclidean space by a crystallographic action.*

Counting maximal finite subgroups in a crystallographic group Γ is equivalent to counting the so-called primitive extremal subsets in the quotient space $\mathcal{L} = \mathbb{E}^m/\Gamma$. So, 11.11 would follow from the next conjecture.

11.15. Conjecture. *Any m-dimensional compact geodesic* CBB(0) *space has at most 2^m primitive extremal subset.*

A closed subset E in a CBB(κ) space is called e x t r e m a l if $\measuredangle[p\,{}^{x}_{y}] \leqslant \frac{\pi}{2}$ for any $x \notin E$ and $p \in E$ such that $|x-p|$ takes minimal value. An extremal subset is called primitive if it contains no proper extremal subsets. For example, the whole space and empty set are aslo extremal in any space. Also every vertex, edge, or face (as well as their union) of the cube is an extremal subset of the cube. Vertices of the cube are the only its primitive extremal subsets.

Lecture 12

CBB: globalization

Recall that a *triangle* $[xyz]$ in a space $\mathcal{X}$ is a triple of minimizing geodesics $[xy]$, $[yz]$, and $[zx]$. Consider the model triangle $[\tilde{x}\tilde{y}\tilde{z}] = \tilde{\triangle}(xyz)_{\mathbb{E}^2}$ in the Euclidean plane. The *natural map* $[\tilde{x}\tilde{y}\tilde{z}] \to [xyz]$ sends a point $\tilde{p} \in [\tilde{x}\tilde{y}] \cup [\tilde{y}\tilde{z}] \cup [\tilde{z}\tilde{x}]$ to the corresponding point $p \in [xy] \cup [yz] \cup [zx]$; that is, if $\tilde{p}$ lies on $[\tilde{y}\tilde{z}]$, then $p \in [yz]$ and $|\tilde{y} - \tilde{p}| = |y - p|$ (and therefore $|\tilde{z} - \tilde{p}| = |z - p|$).

12.1. Definition. *A triangle $[xyz]$ in the metric space $\mathcal{X}$ is called thin (or fat) if the natural map $\tilde{\triangle}(xyz)_{\mathbb{E}^2} \to [xyz]$ is distance nonincreasing (or respectively distance nondecreasing).*

12.2. Exercise. *Show that any triangle in a* CBB(0) *space is fat.*

12.3. Exercise. *Let $\mathcal{W}$ be $\mathbb{R}^2$ with the metric induced by a norm. Suppose that*

(a) $\mathcal{W}$ is CBB(0), *or*

(b) $\mathcal{W}$ is CAT(0).

Show that $\mathcal{W}$ is isometric to the Euclidean plane $\mathbb{E}^2$.

A Globalization

A metric space $\mathcal{L}$ is *locally* CBB(0) if any point $p \in \mathcal{L}$ admits a neighborhood $U \ni p$ such that the CBB(0) comparison holds for any quadruple of points in U.

12.4. Globalization theorem. *Any locally* CBB(0) *compact geodesic space is* CBB(0).

Proof modulo the key lemma. Let $\mathcal{L}$ be a locally CBB(0) compact geodesic space. Note that condition 2.15b holds in $\mathcal{L}$ (the proof is the same as for CBB(0) spaces). It remains to prove that 2.15a holds in $\mathcal{L}$; that is,

❶ $$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle(x\,{}^p_y).$$

First note that ❶ holds for hinges in a small neighborhood of any point; this can be proved the same way as 2.10 and 2.13, applying the local version of the CBB(0) comparison. Since $\mathcal{L}$ is compact, there is $\varepsilon > 0$ such that ❶ holds if $|x-p| + |p-y| < \varepsilon$. Applying the key lemma several times we get that ❶ holds for any given hinge. □

12.5. Key lemma. *Let $\mathcal{L}$ be a locally* CBB(0) *geodesic space. Assume that the comparison*

$$\measuredangle[x\,{}^p_q] \geqslant \tilde\measuredangle(x\,{}^p_q)$$

holds for any hinge $[x\,{}^p_q]$ with $|x-y| + |x-q| < \tfrac{2}{3}\cdot\ell$. Then the comparison

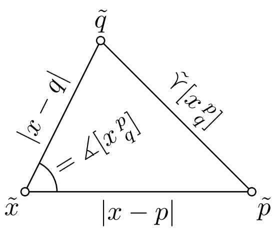
$$\measuredangle[x\,{}^p_q] \geqslant \tilde\measuredangle(x\,{}^p_q)$$

holds for any hinge $[x\,{}^p_q]$ with $|x-p| + |x-q| < \ell$.

Let $[x\,{}^p_q]$ be a hinge in a metric space $\mathcal{L}$ with defined angle measure. Denote by $\tilde\Upsilon[x\,{}^p_q]$ its model side; this is the opposite side in a flat triangle with the same angle and two adjacent sides as in $[x\,{}^p_q]$.

More precisely, consider the model hinge $[\tilde x\,{}^{\tilde p}_{\tilde q}]$ in $\mathbb{E}^2$ that is defined by

$$\begin{aligned}
\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde q}]_{\mathbb{E}^2} &= \measuredangle[x\,{}^p_q]_{\mathcal{L}},\\
|\tilde x - \tilde p|_{\mathbb{E}^2} &= |x-p|_{\mathcal{L}},\\
|\tilde x - \tilde q|_{\mathbb{E}^2} &= |x-q|_{\mathcal{L}};
\end{aligned}$$



then

$$\tilde\Upsilon[x\,{}^p_q]_{\mathcal{L}} := |\tilde p - \tilde q|_{\mathbb{E}^2}.$$

Note that

$$\tilde\Upsilon[x\,{}^p_q] \geqslant |p-q| \quad\iff\quad \measuredangle[x\,{}^p_q] \geqslant \tilde\measuredangle(x\,{}^p_q).$$

We will use it in the following proof.

Proof. It is sufficient to prove the inequality

❷ $$\tilde\Upsilon[x\,{}^p_q] \geqslant |p-q|$$

for any hinge $[x\,^p_q]$ with $|x-p|+|x-q|<\ell$.

Consider a hinge $[x\,^p_q]$ such that

$$\tfrac{2}{3}\cdot\ell \leqslant |p-x|+|x-q|<\ell.$$

First, let us construct a new smaller hinge $[x'\,^p_q]$ with

❸ $$|p-x|+|x-q| \geqslant |p-x'|+|x'-q|,$$

such that

❹ $$\tilde\Upsilon[x\,^p_q] \geqslant \tilde\Upsilon[x'\,^p_q].$$

Construction. Assume $|x-q|\geqslant|x-p|$; otherwise, switch the roles of p and q in the following construction. Take $x'\in[xq]$ such that

❺ $$|p-x|+3\cdot|x-x'| = \tfrac{2}{3}\cdot\ell.$$

Choose a geodesic $[x'p]$ and consider the hinge $[x'\,^p_q]$ formed by $[x'p]$ and $[x'q]\subset[xq]$. Then ❸ follows from the triangle inequality.

Further, note that

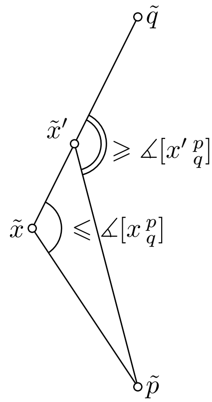
$$|p-x|+|x-x'|<\tfrac{2}{3}\cdot\ell, \qquad |p-x'|+|x'-x|<\tfrac{2}{3}\cdot\ell.$$

In particular,

❻ $$\measuredangle[x\,^p_{x'}] \geqslant \tilde\measuredangle(x\,^p_{x'}) \quad\text{and}\quad \measuredangle[x'\,^p_x] \geqslant \tilde\measuredangle(x'\,^p_x).$$

Now, let $[\tilde x\tilde x'\tilde p] = \tilde\triangle(xx'p)$. Take $\tilde q$ on the extension of $[\tilde x\tilde x']$ beyond x' such that $|\tilde x-\tilde q| = |x-q|$ (and therefore $|\tilde x'-\tilde q| = |x'-q|$). By ❻,

$$\measuredangle[x\,^p_q] = \measuredangle[x\,^p_{x'}] \geqslant \tilde\measuredangle(x\,^p_{x'}) \quad\Rightarrow\quad \tilde\Upsilon[x\,^q_p] \geqslant |\tilde p-\tilde q|.$$

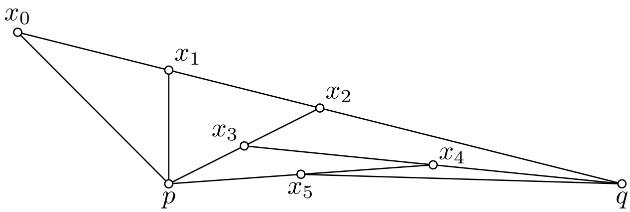


Hence

$$\begin{aligned}\measuredangle[\tilde x'\,^{\tilde p}_{\tilde q}] &= \pi - \tilde\measuredangle(x'\,^p_x) \geqslant\\ &\geqslant \pi - \measuredangle[x'\,^p_x] =\\ &= \measuredangle[x'\,^p_q],\end{aligned}$$

and ❹ follows.

Let us continue the proof. Set $x_0=x$. Let us apply inductively the above construction to get a sequence of hinges $[x_n\,^p_q]$ with $x_{n+1}=x_n'$. From ❹, we have that the sequence $s_n = \tilde\Upsilon[x_n\,^p_q]$ is nonincreasing.



The sequence might terminate at some n only if $|p-x_n|+|x_n-q| < \frac{2}{3}\cdot\ell$. In this case, by the assumptions of the lemma, $\tilde\Upsilon[x_n{}^p_q] \geqslant |p-q|$. Since the sequence s_n is nonincreasing, inequality ❷ follows.

Otherwise, the sequence $r_n = |p-x_n| + |x_n - q|$ is nonincreasing, and $r_n \geqslant \frac{2}{3}\cdot\ell$ for all n. Note that by construction, the distances $|x_n - x_{n+1}|$, $|x_n - p|$, and $|x_n - q|$ are bounded away from zero for all large n. Indeed, since on each step, we move x_n toward to the point p or q that is further away, the distances $|x_n - p|$ and $|x_n - q|$ become about the same. Namely, by ❺, we have that $|p-x_n| - |x_n - q| \leqslant \frac{2}{9}\cdot\ell$ for all large n. Since $|p - x_n| + |x_n - q| \geqslant \frac{2}{3}\cdot\ell$, we have $|x_n - p| \geqslant \frac{\ell}{100}$ and $|x_n - q| \geqslant \frac{\ell}{100}$. Further, since $r_n \geqslant \frac{2}{3}\cdot\ell$, ❺ implies that $|x_n - x_{n+1}| > \frac{\ell}{100}$.

Since the sequence r_n is nonincreasing, it converges. In particular, $r_n - r_{n+1} \to 0$ as $n \to \infty$. It follows that $\tilde\measuredangle(x_n{}^{p_n}_{x_{n+1}}) \to \pi$, where $p_n = p$ if $x_{n+1} \in [x_n q]$, and otherwise $p_n = q$. Since $\measuredangle[x_n{}^{p_n}_{x_{n+1}}] \geqslant \tilde\measuredangle(x_n{}^{p_n}_{x_{n+1}})$, we have $\measuredangle[x_n{}^{p_n}_{x_{n+1}}] \to \pi$ as $n \to \infty$.

It follows that

$$r_n - s_n = |p - x_n| + |x_n - q| - \tilde\Upsilon[x_n{}^p_q] \to 0.$$

Together with the triangle inequality

$$|p - x_n| + |x_n - q| \geqslant |p - q|$$

this yields

$$\lim_{n\to\infty} \tilde\Upsilon[x_n{}^p_q] \geqslant |p - q|.$$

Applying the monotonicity of the sequence $s_n = \tilde\Upsilon[x_n{}^p_q]$, we obtain ❷. □

B On general curvature bound

The globalization theorem can be generalized to CBB(κ) spaces for any real κ. The case $\kappa \leqslant 0$ is proved the same way, but the case $\kappa > 0$ requires minor modifications.

12.6. Exercise. *Suppose* $\kappa \leqslant \mathrm{K}$. *Show that*

$$\tilde{\measuredangle}(x\,{}^y_z)_{\mathbb{M}(\kappa)} \leqslant \tilde{\measuredangle}(x\,{}^y_z)_{\mathbb{M}(\mathrm{K})}$$

if the right-hand side is defined.

Conclude that any CBB(K) *space is locally* CBB(κ).

The exercise and the globalization theorem (here we need a more general version 12.11) imply that *any geodesic* CBB(K) *space is* CBB(κ). Recall that CBB(κ) stands for *curvature bounded below by* κ; so, for geodesic spaces it makes sense. However, as you can see from the following exercise, it does not make much sense in general.

12.7. Exercise. *Let* $\mathcal{X}$ *be the set* $\{p, x_1, x_2, x_3\}$ *with the metric defined by*

$$|p - x_i| = \pi, \quad |x_i - x_j| = 2{\cdot}\pi$$

for all $i \neq j$. *Show that* $\mathcal{X}$ *is* CBB(1), *but not* CBB(0).

12.8. Exercise. *Let* p *and* q *be points in a* CBB(1) *geodesic space* $\mathcal{L}$. *Suppose* $|p - q| > \pi$. *Denote by* m *the midpoint of* $[pq]$. *Show that for any hinge* $[m\,{}^x_p]$ *we have either* $\measuredangle[m\,{}^x_p] = 0$ *or* $\measuredangle[m\,{}^x_p] = \pi$. *Conclude that* $\mathcal{L}$ *is isometric to a real interval or a circle.*

C Remarks

The globalization theorem is also known as the *generalized Toponogov theorem.*

Recall that a metric space $\mathcal{X}$ is called complete if any Cauchy sequence of points in $\mathcal{X}$ converges. The compactness condition in our version of the theorem can be traded for completeness by using the following exercise.

12.9. Exercise. *Let* $\mathcal{X}$ *be a complete metric space. Suppose* $r\colon \mathcal{X} \to \mathbb{R}$ *is a positive continuous function. Show that for any* $\varepsilon > 0$ *there is a point* $p \in \mathcal{X}$ *such that*

$$r(x) > (1 - \varepsilon){\cdot}r(p)$$

for any $x \in \overline{\mathrm{B}}[p, \tfrac{1}{\varepsilon}{\cdot}r(p)]$.

Let us mention two more general versions of the globalization theorem.

Recall that a length space is a metric space such that any two points p and q can be connected by a path with a length arbitrarily close to $|p - q|$. Note that any geodesic space is length, but not the other way around. The following theorem was already proved in the paper of Michael Gromov, Yuriy Burago, and Grigory Perelman [29].

12.10. Theorem. *Any complete locally* CBB(κ) *length space is* CBB(κ)*.*

The next result is mine [77].

12.11. Theorem. *Any locally* CBB(κ) *geodesic space is* CBB(κ)*.*

In the two-dimensional case, the globalization theorem was proved by Paolo Pizzetti [79]; later it was reproved independently by Alexandr Alexandrov [13]. Victor Toponogov [96] proved it for Riemannian manifolds of all dimensions.

I took the proof from our book [6] (with generality reduction). It uses simplifications obtained by Conrad Plaut [80] and Dmitry Burago, Yuriy Burago, and Sergei Ivanov [25]. The same proof was rediscovered independently by Urs Lang and Viktor Schroeder [62]. Another simplified version was obtained by Katsuhiro Shiohama [91].

The question of whether 2.15a suffices to conclude that $\mathcal{L}$ is CBB(κ) is a long-standing open problem (possibly dating back to Alexandrov); in print, it was first stated in [25, footnote in 4.1.5].

12.12. Open question. *Let $\mathcal{L}$ be a complete geodesic space (you can also assume that $\mathcal{L}$ is homeomorphic to $\mathbb{S}^2$ or $\mathbb{R}^2$) such that for any hinge $[x\,{}^p_y]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and*

$$\measuredangle[x\,{}^p_y] \geqslant \tilde{\measuredangle}(x\,{}^p_y).$$

Is it true that $\mathcal{L}$ is CBB(0)*?*

Lecture 13

Semiconcave functions

A Semiconcave functions

We will write $f'' \leqslant \varphi$ if for any point $x \in \operatorname{Dom} f$ and any $\varepsilon > 0$ there is a neighborhood $U \ni x$ such that the restriction $f|_U$ is $(\varphi(x)+\varepsilon)$-concave. Here we assume that φ is continuous and defined in $\operatorname{Dom} f$.

If $f'' \leqslant \varphi$ for some continuous function φ, then f is called semiconcave.

13.1. Exercise. *Let f be a distance function on a geodesic* CBB(0) *space $\mathcal{L}$; that is, $f(x) \equiv |p - x|$ for some $p \in \mathcal{L}$. Show that $f'' \leqslant \frac{1}{f}$. In particular, f is semiconcave in $\mathcal{L} \setminus \{p\}$.*

B Completion

Given a metric space $\mathcal{X}$, consider the set $\mathcal{C}$ of all Cauchy sequences in $\mathcal{X}$. Note that for any two Cauchy sequences (x_n) and (y_n) the right-hand side in ❶ is defined; moreover, it defines a semimetric on $\mathcal{C}$

❶ $$|(x_n) - (y_n)|_{\mathcal{C}} := \lim_{n\to\infty} |x_n - y_n|_{\mathcal{X}}.$$

The corresponding metric space is called the completion of $\mathcal{X}$; it will be denoted by $\bar{\mathcal{X}}$.

It is straightforward to check that *completion is complete.*

For each point $x \in \mathcal{X}$, one can consider a constant sequence $x_n = x$ which is Cauchy. It defines a natural inclusion map $\mathcal{X} \hookrightarrow \bar{\mathcal{X}}$. It is easy to check that this map is distance-preserving. In particular, we can (and will) consider $\mathcal{X}$ as a subset of $\bar{\mathcal{X}}$. Note that $\mathcal{X}$ *is a dense subset in its completion* $\bar{\mathcal{X}}$.

C Space of directions

Let $\mathcal{X}$ be a space with defined angles. Given $p \in \mathcal{X}$, consider the set $\mathfrak{S}_p$ of all nontrivial unit-speed geodesics starting at p. By 2.3, the triangle inequality holds for $\measuredangle$ on $\mathfrak{S}_p$, that is, $(\mathfrak{S}_p, \measuredangle)$ forms a semimetric space.

The metric space corresponding to $(\mathfrak{S}_p, \measuredangle)$ is called the space of geodesic directions at p, denoted by Σ'_p or $\Sigma'_p\mathcal{X}$. The elements of Σ'_p are called geodesic directions at p. Each geodesic direction is formed by an equivalence class of geodesics starting from p for the equivalence relation

$$[px] \sim [py] \qquad \Longleftrightarrow \qquad \measuredangle[p\,{}^x_y] = 0;$$

the direction of $[px]$ is denoted by $\uparrow_{[px]}$. (If $\mathcal{X}$ is CBB, then by 2.11, $[px] \sim [py]$ if and only if $[px] \subset [py]$ or $[px] \supset [py]$.)

The completion of Σ'_p is called the space of directions at p and is denoted by Σ_p or $\Sigma_p\mathcal{X}$. The elements of Σ_p are called directions at p.

D Tangent space

Recall that Euclidean cone $\mathcal{V} = \operatorname{Cone}\mathcal{X}$ over a metric space $\mathcal{X}$ is defined as the metric space whose underlying set consists of equivalence classes in $[0, \infty) \times \mathcal{X}$ with the equivalence relation "$\sim$" given by $(0, p) \sim (0, q)$ for any points $p, q \in \mathcal{X}$, and whose metric is given by the cosine rule

$$|(s, p) - (t, q)|_{\mathcal{V}} = \sqrt{s^2 + t^2 - 2{\cdot}s{\cdot}t{\cdot}\cos\theta},$$

where $\theta = \min\{\pi, |p - q|_{\mathcal{X}}\}$.

Note that $\operatorname{Cone}\mathbb{S}^n$ *is isometric to* $\mathbb{E}^{n+1}$. This is a leading example; further, we generalize several notions of Euclidean space to the Euclidean cones.

The point in $\mathcal{V}$ that corresponds $(t, x) \in [0, \infty) \times \mathcal{X}$ will be denoted by $t{\cdot}x$. The point in $\mathcal{V}$ formed by the equivalence class of $\{0\} \times \mathcal{X}$ is called the origin of the cone and is denoted by 0 or $0_{\mathcal{V}}$. For $v \in \mathcal{V}$ the distance $|0 - v|_{\mathcal{V}}$ is called the norm of v and is denoted by $|v|$ or $|v|_{\mathcal{V}}$. The scalar product $\langle v, w\rangle$ of $v = s{\cdot}p$ and $w = t{\cdot}q$ is defined by

$$\langle v, w\rangle := |v|{\cdot}|w|{\cdot}\cos\theta$$

where $\theta = \min\{\pi, |p - q|_{\mathcal{X}}\}$; we set $\langle v, w\rangle := 0$ if $v = 0$ or $w = 0$.

Tangent space. The Euclidean cone $\operatorname{Cone}\Sigma_p$ over the space of directions Σ_p is called the tangent space at p and denoted by T_p or

$\mathrm{T}_p\mathcal{X}$. The elements of $\mathrm{T}_p\mathcal{X}$ will be called tangent vectors at p (despite the fact that T_p is only a cone — not a vector space). The space of directions Σ_p can be (and will be) identified with the unit sphere in T_p.

13.2. Proposition. *Tangent spaces of* $\mathrm{CBB}(\kappa)$ *space are* $\mathrm{CBB}(0)$.

Proof. Consider the tangent space $\mathrm{T}_p = \mathrm{Cone}\,\Sigma_p$ of a $\mathrm{CBB}(\kappa)$ space $\mathcal{L}$ at a point p. We need to show that the $\mathrm{CBB}(0)$ comparison holds for a given quadruple $v_0, v_1, v_2, v_3 \in \mathrm{T}_p$.

Recall that the space of geodesic directions Σ'_p is dense in Σ_p. It follows that the subcone $\mathrm{T}'_p = \mathrm{Cone}\,\Sigma'_p$ is dense in T_p. Therefore, it is sufficient to consider the case $v_0, v_1, v_2, v_3 \in \mathrm{T}'_p$.

For each i, choose a geodesic γ_i from p in the direction of v_i; assume γ_i has speed $|v_i|$ for each i. Since the angles are defined, we have

❶
$$|\gamma_i(\varepsilon) - \gamma_j(\varepsilon)|_{\mathcal{L}} = \varepsilon\cdot|v_i - v_j|_{\mathrm{T}_p} + o(\varepsilon)$$

for $\varepsilon > 0$. The quadruple $\gamma_0(\varepsilon), \gamma_1(\varepsilon), \gamma_2(\varepsilon), \gamma_3(\varepsilon)$ meets the $\mathrm{CBB}(\kappa)$ comparison. After rescaling all the distances by $\frac{1}{\varepsilon}$, it becomes the $\mathrm{CBB}(\varepsilon^2\cdot\kappa)$ comparison. Passing to the limit as $\varepsilon \to 0$ and applying ❶, we get the $\mathrm{CBB}(0)$ comparison for v_0, v_1, v_2, v_3. □

13.3. Exercise. *Show that tangent spaces of* $\mathrm{CAT}(\kappa)$ *space are* $\mathrm{CAT}(0)$.

E Differential

Let $\mathcal{X}$ be a space with defined angles. Let f be a semiconcave function on $\mathcal{X}$ and $p \in \mathrm{Dom}\, f$. Choose a unit-speed geodesic γ that starts at p; let $\xi \in \Sigma_p$ be its direction. Define

$$(\boldsymbol{d}_p f)(\xi) := (f\circ\gamma)^+(0),$$

here $(f\circ\gamma)^+$ denotes the right derivative of $(f\circ\gamma)$; it is defined since f is semiconcave.

By the following exercise, the value $(\boldsymbol{d}_p f)(\xi)$ is defined; that is, it does not depend on the choice of γ. Moreover, $\boldsymbol{d}_p f$ is a Lipschitz function on Σ'_p. It follows that the function $\boldsymbol{d}_p f\colon \Sigma'_p \to \mathbb{R}$ can be extended to a Lipschitz function $\boldsymbol{d}_p f\colon \Sigma_p \to \mathbb{R}$. Further, we can extend it to the tangent space by setting

$$(\boldsymbol{d}_p f)(r\cdot\xi) := r\cdot(\boldsymbol{d}_p f)(\xi)$$

for any $r \geqslant 0$ and $\xi \in \Sigma_p$. The obtained function $\boldsymbol{d}_p f\colon \mathrm{T}_p \to \mathbb{R}$ is Lipschitz; it is called the differential of f at p.

13.4. Exercise. *Let f be a semiconcave function on a geodesic space $\mathcal{X}$ with defined angles. Suppose γ_1 and γ_2 are unit-speed geodesics that start at $p \in \operatorname{Dom} f$; denote by θ the angle between γ_1 and γ_2 at p. Show that*

$$|(f\circ\gamma_1)^+(0) - (f\circ\gamma_2)^+(0)| \leqslant L{\cdot}\theta,$$

where L is the Lipschitz constant of f in a neighborhood of p.

13.5. Exercise. *Let p and q be distinct points in a* CBB(0) *space. Denote by ξ the direction of a geodesic $[pq]$ at p. Show that*

$$\boldsymbol{d}_p\mathrm{dist}_q(v) \leqslant -\langle \xi, v\rangle$$

for any $v \in \mathrm{T}_p$.

F Gradient

13.6. Definition. *Let f be a semiconcave function on a geodesic space $\mathcal{X}$ with defined angles. A tangent vector $g \in \mathrm{T}_p$ is called a* g r a d i e n t *of f at p (briefly, $g = \nabla_p f$) if*

(a) $(\boldsymbol{d}_p f)(w) \leqslant \langle g, w\rangle$ for any $w \in \mathrm{T}_p$, and
(b) $(\boldsymbol{d}_p f)(g) = \langle g, g\rangle$.

13.7. Proposition. *Suppose that a semiconcave function f is defined in a neighborhood of a point p in a* CBB(κ) *space. Then the gradient $\nabla_p f$ is uniquely defined.*

13.8. Key lemma. *Let f be a λ-concave function that is defined in a neighborhood of a point p in a geodesic* CBB(κ) *space $\mathcal{L}$. Then for any $u, v \in \mathrm{T}_p$, we have*

$$s{\cdot}\sqrt{|u|^2 + 2{\cdot}\langle u, v\rangle + |v|^2} \geqslant (\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v),$$

where

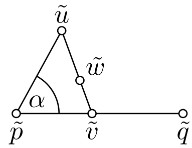
$$s = \sup\left\{\, (\boldsymbol{d}_p f)(\xi) \;:\; \xi \in \Sigma_p \,\right\}.$$

Note that in Euclidean space we have

$$|u+v| = \sqrt{|u|^2 + 2{\cdot}\langle u, v\rangle + |v|^2}.$$

The right-hand side makes sense in any Euclidean cone, but the sum in the left-hand side does not.

Proof. We will assume $\kappa = 0$; the general case requires only minor modifications. We can assume that $v \neq 0$, $w \neq 0$, and $\alpha = \measuredangle(u, v) > 0$; otherwise, the statement is trivial.



Prepare a model configuration of five points: $\tilde p$, $\tilde u$, $\tilde v$, $\tilde q$, $\tilde w \in \mathbb{E}^2$ such that

◇ $\measuredangle[\tilde p\,{}^{\tilde u}_{\tilde v}] = \alpha$,
◇ $|\tilde p - \tilde u| = |u|$,
◇ $|\tilde p - \tilde v| = |v|$,
◇ $\tilde q$ lies on an extension of $[\tilde p\tilde v]$ so that $\tilde v$ is the midpoint of $[\tilde p\tilde q]$,
◇ $\tilde w$ is the midpoint between $\tilde u$ and $\tilde v$.

Note that

$$❶ \qquad |\tilde p - \tilde w| = \tfrac12\cdot\sqrt{|u|^2 + 2\cdot\langle u, v\rangle + |v|^2}.$$

We can assume that there are geodesics in the directions of u and v; the latter follows since the geodesic space of directions Σ'_p is dense in Σ_p. Choose geodesics γ_u and γ_v in the direction in the directions of u and v; let us assume that they are parametrized with speed $|u|$ and $|v|$ respectively. For all small $t > 0$, construct points $u_t, v_t, q_t, w_t \in \mathcal{L}$ as follows.

◇ $v_t = \gamma_v(t), \quad q_t = \gamma_v(2\cdot t)$
◇ $u_t = \gamma_u(t)$.
◇ w_t is the midpoint of $[u_t v_t]$.

Clearly

$$|p - u_t| = t\cdot|u|, \qquad |p - v_t| = t\cdot|v|, \qquad |p - q_t| = 2\cdot t\cdot|v|.$$

Since $\measuredangle(u, v)$ is defined, we have

$$|u_t - v_t| = t\cdot|\tilde u - \tilde v| + o(t), \qquad |u_t - q_t| = t\cdot|\tilde u - \tilde q| + o(t).$$

From the point-on-side and hinge comparisons ($2.16c$+$2.16d$), we have

$$\tilde\measuredangle(v_t\,{}^{p}_{w_t}) \geqslant \tilde\measuredangle(v_t\,{}^{p}_{u_t}) \geqslant \measuredangle[\tilde v\,{}^{\tilde p}_{\tilde u}] + \tfrac{o(t)}{t}$$

and

$$\tilde\measuredangle(v_t\,{}^{q_t}_{w_t}) \geqslant \tilde\measuredangle(v_t\,{}^{q_t}_{u_t}) \geqslant \measuredangle[\tilde v\,{}^{\tilde q}_{\tilde u}] + \tfrac{o(t)}{t}.$$

Clearly, $\measuredangle[\tilde v\,{}^{\tilde p}_{\tilde u}] + \measuredangle[\tilde v\,{}^{\tilde q}_{\tilde u}] = \pi$. From the adjacent angle comparison ($2.16b$), $\tilde\measuredangle(v_t\,{}^{p}_{u_t}) + \tilde\measuredangle(v_t\,{}^{u_t}_{q_t}) \leqslant \pi$. Hence $\tilde\measuredangle(v_t\,{}^{p}_{w_t}) \to \measuredangle[\tilde v\,{}^{\tilde p}_{\tilde w}]$ as $t \to 0+$ and thus

$$|p - w_t| = t\cdot|\tilde p - \tilde w| + o(t).$$

Without loss of generality, we can assume that $f(p) = 0$. Since f is λ-concave, we have

$$\begin{aligned} 2\cdot f(w_t) &\geqslant f(u_t) + f(v_t) + \tfrac{\lambda}{4}\cdot|u_t - v_t|^2 = \\ &= t\cdot[(\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v)] + o(t). \end{aligned}$$

Applying λ-concavity of f, we have

$$\begin{aligned}(\boldsymbol{d}_p f)(\uparrow_{[pw_t]}) &\geqslant \frac{f(w_t) - \frac{\lambda}{2}\cdot|p - w_t|^2}{|p - w_t|} \geqslant \\ &\geqslant \frac{t\cdot[(\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v)] + o(t)}{2\cdot t\cdot|\tilde{p} - \tilde{w}| + o(t)}.\end{aligned}$$

By ❶, the key lemma follows. □

13.9. Exercise. *Let* $[q\,{}^p_x]$ *be a hinge in a* CBB(0) *space and* $y \in\,]qp[$. *Suppose that* γ *is the unit speed parametrization of* $[qx]$ *from* q *to* x. *Show that*

$$|y - \gamma(t)| = |y - q| - t\cdot\cos(\measuredangle[q\,{}^p_x]) + o(t).$$

Conclude that

$$(\boldsymbol{d}_q \mathrm{dist}_y)(w) = -\langle\uparrow_{[qp]}, w\rangle$$

for any $w \in \mathrm{T}_x$

Proof of 13.7; uniqueness. If $g, g' \in \mathrm{T}_p$ are two gradients of f, then

$$\langle g, g\rangle = (\boldsymbol{d}_p f)(g) \leqslant \langle g, g'\rangle, \qquad \langle g', g'\rangle = (\boldsymbol{d}_p f)(g') \leqslant \langle g, g'\rangle.$$

Therefore,

$$|g - g'|^2 = \langle g, g\rangle - 2\cdot\langle g, g'\rangle + \langle g', g'\rangle \leqslant 0.$$

It follows that $g = g'$.

Existence. Note first that if $\boldsymbol{d}_p f \leqslant 0$, then one can take $\nabla_p f = 0$.

Otherwise, if $s = \sup\{\,(\boldsymbol{d}_p f)(\xi) : \xi \in \Sigma_p\,\} > 0$, it is sufficient to show that there is $\bar{\xi} \in \Sigma_p$ such that

❷ $$(\boldsymbol{d}_p f)\left(\bar{\xi}\right) = s.$$

Indeed, suppose $\bar{\xi}$ exists. Applying 13.8 for $u = \bar{\xi}$, $v = \varepsilon\cdot w$ with $\varepsilon \to 0+$, we get

$$(\boldsymbol{d}_p f)(w) \leqslant \langle w, s\cdot\bar{\xi}\rangle$$

for any $w \in \mathrm{T}_p$; that is, $s\cdot\bar{\xi}$ is the gradient at p.

Take a sequence of directions $\xi_n \in \Sigma_p$, such that $(\boldsymbol{d}_p f)(\xi_n) \to s$. Applying 13.8 for $u = \xi_n$ and $v = \xi_m$, we get

$$s \geqslant \frac{(\boldsymbol{d}_p f)(\xi_n) + (\boldsymbol{d}_p f)(\xi_m)}{\sqrt{2 + 2\cdot\cos\measuredangle(\xi_n, \xi_m)}}.$$

Therefore $\measuredangle(\xi_n, \xi_m) \to 0$ as $n, m \to \infty$; that is, the sequence ξ_n is Cauchy. Clearly, $\overline{\xi} = \lim_n \xi_n$ meets ❷. □

13.10. Exercise. *Let f and g be locally Lipschitz semiconcave functions defined in a neighborhood of a point p in a* CBB *space. Show that*

$$|\nabla_p f - \nabla_p g|^2_{\mathrm{T}_p} \leqslant s\cdot(|\nabla_p f| + |\nabla_p g|),$$

where

$$s = \sup \{\, |(\boldsymbol{d}_p f)(\xi) - (\boldsymbol{d}_p g)(\xi)| \;:\; \xi \in \Sigma_p \,\}\,.$$

Conclude that if the sequence of restrictions $\boldsymbol{d}_p f_n|_{\Sigma_p}$ converges uniformly, then $\nabla_p f_n$ converges as $n \to \infty$. Here we assume that all functions f_1, $f_2, \dots$ are semiconcave and locally Lipschitz.

13.11. Exercise. *Let f be a locally Lipschitz λ-concave function on a complete geodesic* CBB(κ) *space $\mathcal{L}$.*

(a) Suppose $s \geqslant 0$. Show that $|\nabla_x f| > s$ if and only if for some point y we have

$$f(y) - f(x) > s\cdot\ell + \lambda\cdot\tfrac{\ell^2}{2},$$

were $\ell = |x - y|$.

(b) Show that $x \mapsto |\nabla_x f|$ is lower semicontinuous; that is, if $x_n \to \to x_\infty$, then

$$|\nabla_{x_\infty} f| \leqslant \varliminf_{n\to\infty} |\nabla_{x_n} f|.$$

G Remarks

The following example was constructed by Stephanie Halbeisen [53]. It is better to keep this example in your head in Section 13F.

13.12. Example. *There is a complete length* CBB *space $\check{\mathcal{L}}$ with a point $p \in \check{\mathcal{L}}$ such that the space of directions $\Sigma_p\check{\mathcal{L}}$ is not a π-length space, and therefore the tangent space $\mathrm{T}_p\check{\mathcal{L}}$ is not a length space.*

If the dimension is finite, such examples do not exist; for proper spaces the question is open, see 13.15.

Construction. Let $\mathbb{H}$ be a Hilbert space formed by infinite sequences of real numbers $\boldsymbol{x} = (x_0, x_1, \dots)$ with the ℓ^2-norm $|\boldsymbol{x}|^2 = \sum_i (x_i)^2$. Fix $\varepsilon = 0.001$ and consider two functions $f, \check{f} : \mathbb{H} \to \mathbb{R}$:

$$f(\boldsymbol{x}) = |\boldsymbol{x}|,$$

$$\check f(\boldsymbol{x}) = \max\left\{|\boldsymbol{x}|, \max_{n\geqslant 1}\{(1+\varepsilon)\cdot x_n - \tfrac{1}{n}\}\right\}.$$

Both of these functions are convex and Lipschitz, therefore their graphs in $\mathbb{H}\times\mathbb{R}$ equipped with its length metric form infinite-dimensional CBB(0) spaces, say $\mathcal{L}$ and $\check{\mathcal{L}}$ (this can be proved similarly to 3.10).

Let p be the origin of $\mathbb{H}\times\mathbb{R}$. Note that $\check{\mathcal{L}}\cap\mathcal{L}$ is a star-shaped subset of $\mathbb{H}$ with center at p. Further, $\check{\mathcal{L}}\setminus\mathcal{L}$ consists of a countable number of disjoint sets

$$\Omega_n = \left\{\,(\boldsymbol{x}, \check f(\boldsymbol{x}))\in\check{\mathcal{L}} \,:\, (1+\varepsilon)\cdot x_n - \tfrac{1}{n} > |\boldsymbol{x}|\,\right\}.$$

Note that $|\Omega_n - p| > \frac{1}{n}$ for each n. It follows that for any geodesic $[pq]$ in $\check{\mathcal{L}}$, a small subinterval $[p\bar q]\subset[pq]$ is a straight line segment in $\mathbb{H}\times\mathbb{R}$, and also a geodesic in $\mathcal{L}$. Thus we can treat $\Sigma_p\mathcal{L}$ and $\Sigma_p\check{\mathcal{L}}$ as one set, with two angle metrics $\measuredangle$ and $\check\measuredangle$. Let us denote by $\measuredangle_{\mathbb{H}\times\mathbb{R}}$ the angle in $\mathbb{H}\times\mathbb{R}$.

The space $\mathcal{L}$ is isometric to the Euclidean cone over $\Sigma_p\mathcal{L}$ with vertex at p; $\Sigma_p\mathcal{L}$ is isometric to a sphere in Hilbert space with radius $\frac{1}{\sqrt2}$. In particular, $\measuredangle$ is the length metric of $\measuredangle_{\mathbb{H}\times\mathbb{R}}$ on $\Sigma_p\mathcal{L}$.

Therefore in order to show that $\check\measuredangle$ does not define a length metric on $\Sigma_p\mathcal{L}$, it is sufficient to construct a pair of directions (ξ_+,ξ_-) such that

$$\check\measuredangle(\xi_+,\xi_-) < \measuredangle(\xi_+,\xi_-).$$

Set $\boldsymbol{e}_0 = (1,0,0,\dots)$, $\boldsymbol{e}_1 = (0,1,0,\dots),\dots\in\mathbb{H}$. Consider the following two half-lines in $\mathbb{H}\times\mathbb{R}$:

$$\gamma_+(t) = \tfrac{t}{\sqrt2}\cdot(\boldsymbol{e}_0, 1) \quad\text{and}\quad \gamma_-(t) = \tfrac{t}{\sqrt2}\cdot(-\boldsymbol{e}_0, 1),\qquad t\in[0,+\infty).$$

They form unit-speed geodesics in both $\mathcal{L}$ and $\check{\mathcal{L}}$. Let $\xi_\pm$ be the directions of $\gamma_\pm$ at p. Denote by σ_n the half-planes in $\mathbb{H}$ spanned by $\boldsymbol{e}_0$ and $\boldsymbol{e}_n$; that is, $\sigma_n = \{\,x\cdot\boldsymbol{e}_0 + y\cdot\boldsymbol{e}_n \,:\, y\geqslant 0\,\}$. Consider a sequence of 2-dimensional sectors $Q_n = \check{\mathcal{L}}\cap(\sigma_n\times\mathbb{R})$. For each n, the sector Q_n intersects Ω_n and is bounded by two geodesic half-lines $\gamma_\pm$. Note that $Q_n \xrightarrow{\text{GH}} Q$, where Q is a solid Euclidean angle in $\mathbb{E}^2$ with angle measure $\beta < \measuredangle(\xi_+,\xi_-) = \frac{\pi}{\sqrt2}$. Indeed, Q_n is path-isometric to the subset of $\mathbb{E}^3$ described by

$$y\geqslant 0 \quad\text{and}\quad z = \max\left\{\sqrt{x^2+y^2}, (1+\varepsilon)\cdot y - \tfrac{1}{n}\right\}$$

with length metric. Thus its limit Q is path-isometric to the subset of $\mathbb{E}^3$ described by

$$y\geqslant 0 \quad\text{and}\quad z = \max\left\{\sqrt{x^2+y^2}, (1+\varepsilon)\cdot y\right\}$$

with length metric. In particular, for any $t,\tau\geqslant 0$,

$$\begin{aligned}|\gamma_+(t)-\gamma_-(\tau)|_{\check{\mathcal{L}}}\leqslant \lim_{n\to\infty}|\gamma_+(t)-\gamma_-(\tau)|_{Q_n}=\\=\tilde{\Upsilon}\{\beta;t,\tau\},\end{aligned}$$

where $\tilde{\Upsilon}\{\beta;t,\tau\}:=\sqrt{t^2+\tau^2-2{\cdot}t{\cdot}\tau{\cdot}\cos\beta}$. That is, $\check{\measuredangle}(\xi_+,\xi_-)\leqslant\beta<\measuredangle(\xi_+,\xi_-)$. ☐

13.13. Exercise. *Show that tangent space of any geodesic* CAT(0) *space is geodesic.*

13.14. Exercise. *Construct a non-compact complete geodesic* CBB(0) *space that contains no half-lines.*

13.15. Open question. *Let $\mathcal{L}$ be a proper length* CBB(κ) *space. Is it true that for any $p\in\mathcal{L}$, the tangent space* T_p *is a length space?*

Lecture 14

Gradient flow

A Velocity of curve

Let α be a curve in a geodesic CBB(κ) space $\mathcal{L}$; choose $p=\alpha(t_0)$. If for any choice of geodesics $[p\,\alpha(t_0+\varepsilon)]$ the vectors

$$\tfrac{1}{\varepsilon}\cdot|p-\alpha(t_0+\varepsilon)|\cdot{\uparrow}_{[p\alpha(t_0+\varepsilon)]}$$

converge as $\varepsilon\to 0+$, then their limit in T_p is called the right derivative of α at t_0; it will be denoted by $\alpha^+(t_0)$. In addition, $\alpha^+(t_0):=0$ if $\frac{1}{\varepsilon}\cdot|p-\alpha(t_0+\varepsilon)|\to 0$ as $\varepsilon\to 0+$.

14.1. Claim. *Let α be a curve in a geodesic* CBB(κ) *space $\mathcal{L}$. Suppose f a semiconcave Lipschitz function defined in a neighborhood of $p=\alpha(0)$, and $\alpha^+(0)$ is defined. Then*

$$(f\circ\alpha)^+(0)=(\boldsymbol{d}_pf)(\alpha^+(0)).$$

Proof. Without loss of generality, we can assume that $f(p)=0$. Suppose f and therefore $\boldsymbol{d}_pf$ are L-Lipschitz. Further, we will always assume that $\varepsilon>0$.

Choose a constant-speed geodesic γ that starts from p, such that the distance $s=|\alpha^+(0)-\gamma^+(0)|_{\mathrm{T}_p}$ is small. Observe that by the definition of differential,

$$(f\circ\gamma)^+(0)=\boldsymbol{d}_pf(\gamma^+(0)).$$

By comparison and the definition of α^+,

$$|\alpha(\varepsilon)-\gamma(\varepsilon)|_{\mathcal{L}}\leqslant s\cdot\varepsilon+o(\varepsilon)$$

Therefore

$$|f\circ\alpha(\varepsilon)-f\circ\gamma(\varepsilon)|\leqslant L{\cdot}s{\cdot}\varepsilon+o(\varepsilon).$$

Suppose $(f\circ\alpha)^+(0)$ is defined. Then

$$|(f\circ\alpha)^+(0)-(f\circ\gamma)^+(0)|\leqslant L{\cdot}s.$$

Since $\boldsymbol{d}_pf$ is L-Lipschitz, we also get

$$|\boldsymbol{d}_pf(\alpha^+(0))-\boldsymbol{d}_pf(\gamma^+(0))|\leqslant L{\cdot}s.$$

It follows that the needed identity holds up to error $2{\cdot}L{\cdot}s$. The statement follows since $s>0$ can be chosen arbitrarily.

Finally, even if $(f\circ\alpha)^+(0)$ is undefined, we can arrive to the same conclusion using all partial limits $\frac{1}{\varepsilon_n}{\cdot}[f\circ\alpha(\varepsilon_n)-f(p)]$ for $\varepsilon_n\to 0+$ in the place of $(f\circ\alpha)^+(0)$. □

B Gradient curves

14.2. Definition. *Let $f\colon\mathcal{L}\to\mathbb{R}$ be a locally Lipschitz and semiconcave function on a complete geodesic* CBB(κ) *space $\mathcal{L}$.*

A locally Lipschitz curve $\alpha\colon[t_{\min},t_{\max})\to\operatorname{Dom}f$ will be called an f-gradient curve if

$$\alpha^+=\nabla_\alpha f;$$

that is, for any $t\in[t_{\min},t_{\max})$, $\alpha^+(t)$ is defined and $\alpha^+(t)=\nabla_{\alpha(t)}f$.

A complete proof of the following theorem takes about 5 pages [6]; it mimics the standard Picard theorem on the existence and uniqueness of solutions of ordinary differential equations. We omit the proof of existence; the uniqueness is proved in the next section.

14.3. Picard theorem. *Let $f\colon\mathcal{L}\to\mathbb{R}$ be a locally Lipschitz and λ-concave function on a complete geodesic* CBB(κ) *space $\mathcal{L}$. Then for any $p\in\operatorname{Dom}f$, there are unique $t_{\max}\in(0,\infty]$ and f-gradient curve $\alpha\colon[0,t_{\max})\to\mathcal{L}$ with $\alpha(0)=p$ such that any sequence $t_n\to t_{\max}-$, the sequence $\alpha(t_n)$ does not have a limit point in* $\operatorname{Dom}f$.

Note that the theorem only says that the future of a gradient curve is determined by its present, but it says nothing about its past.

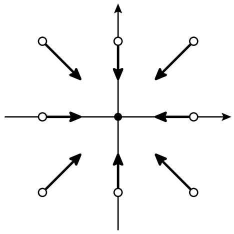
Here is an example showing that the past is not determined by the present. Consider the function $f\colon x\mapsto-|x|$ on the real line $\mathbb{R}$. The tangent space $\mathrm{T}_x\mathbb{R}$ can be identified with $\mathbb{R}$. Note that

$$\nabla_xf=\begin{cases}1&\text{if}\quad x<0,\\0&\text{if}\quad x=0,\\-1&\text{if}\quad x>0.\end{cases}$$

So, the f-gradient curves go to the origin with unit speed and then stand there forever. In particular, if α is an f-gradient curve that starts at x, then $\alpha(t)=0$ for any $t\geqslant|x|$.

Here is a slightly more interesting example; it shows that gradient curves can merge even in the region where $|\nabla f|\neq 0$. Hence their *past* cannot be uniquely determined from their *present*.

14.4. Example. *Consider the function $f\colon(x,y)\mapsto -|x|-|y|$ on the (x,y)-plane. Note that f is concave; its gradient field is sketched on the figure.*



Let α be an f-gradient curve that starts at (x,y) for $x>y>0$. Then

$$\alpha(t)=\begin{cases}(x-t,y-t) & \textit{for}\quad 0\leqslant t\leqslant x-y,\\ (x-t,0) & \textit{for}\quad x-y\leqslant t\leqslant x,\\ (0,0) & \textit{for}\quad x\leqslant t.\end{cases}$$

C Distance estimates

14.5. Observation. *Let α is a gradient curve of a λ-concave function f defined on a complete geodesic* CBB *space. Choose point p; let $\ell(t):=\operatorname{dist}_p\circ\alpha(t)$ and $q=\alpha(t_0)$. Then*

$$\ell^+(t_0)\leqslant -\left(f(p)-f(q)-\tfrac{\lambda}{2}\cdot\ell^2(t_0)\right)/\ell(t_0)$$

Proof. Let γ be the unit-speed parametrization of $[qp]$ from q to p, so $q=\gamma(0)$. Then

$$\begin{aligned}\ell^+(t_0)&=(\boldsymbol{d}_q\operatorname{dist}_p)(\nabla_q f)\leqslant\\ &\leqslant-\langle\uparrow_{[qp]},\nabla_q f\rangle\leqslant\\ &\leqslant-\boldsymbol{d}_q f(\uparrow_{[qp]})=\\ &=-(f\circ\gamma)^+(0)\leqslant\\ &\leqslant-\left(f(p)-f(q)-\tfrac{\lambda}{2}\cdot\ell^2(t_0)\right)/\ell(t_0)\end{aligned}$$

In the above calculations we consequently applied 14.1, 13.5, the definition of gradient, the definition of differential, and concavity of $t\mapsto$ $\mapsto f\circ\gamma(t)-\frac{\lambda}{2}\cdot t^2$. □

Note that the following estimate implies uniqueness in the Picard theorem (14.3).

14.6. First distance estimate. *Let f be a λ-concave locally Lipschitz function on a complete geodesic* CBB(κ) *space $\mathcal{L}$. Then*

$$|\alpha(t)-\beta(t)| \leqslant e^{\lambda\cdot t}\cdot|\alpha(0)-\beta(0)|$$

for any $t \geqslant 0$ and any two f-gradient curves α and β.

Moreover, the statement holds for a locally Lipschitz λ-concave function defined in an open domain if there is a geodesic $[\alpha(t)\,\beta(t)]$ in Dom f *for any t.*

Proof. Fix a choice of geodesic $[\alpha(t)\,\beta(t)]$ for each t. Let $\ell(t) = |\alpha(t)-$ $-\beta(t)|$. Note that

$$\ell^+(t) \leqslant -\langle\uparrow_{[\alpha(t)\beta(t)]}, \nabla_{\alpha(t)}f\rangle - \langle\uparrow_{[\beta(t)\alpha(t)]}, \nabla_{\beta(t)}f\rangle \leqslant \lambda\cdot\ell(t).$$

Here one has to apply 14.5 for distance to the midpoint m of $[\alpha(t)\,\beta(t)]$, and then apply the triangle inequality. Hence the result. $\square$

The following exercise describes a global geometric property of a gradient curve without direct reference to its function. It uses the notion of *self-contracting curves* introduced by Aris Daniilidis, Olivier Ley, and Stéphane Sabourau [39].

14.7. Exercise. *Let $f\colon \mathcal{L} \to \mathbb{R}$ be a locally Lipschitz and concave function on a complete geodesic* CBB(κ) *space $\mathcal{L}$. Then*

$$t_1 \leqslant t_2 \leqslant t_3 \quad \Longrightarrow \quad |\alpha(t_1)-\alpha(t_3)|_{\mathcal{L}} \geqslant |\alpha(t_2)-\alpha(t_3)|_{\mathcal{L}}.$$

for any f-gradient curve α.

14.8. Exercise. *Let f be a locally Lipschitz concave function defined on a* CBB(κ) *space. Suppose $\hat\alpha\colon [0,\ell]$ is an arc-length reparametrization of an f-gradient curve. Show that $(f\circ\hat\alpha)$ is concave.*

The following exercise implies that gradient curves for a uniformly converging sequence of λ-concave functions converge to the gradient curves of the limit function.

14.9. Exercise. *Let f and g be λ-concave locally Lipschitz functions on a complete geodesic* CBB(κ) *space $\mathcal{L}$. Suppose $\alpha,\beta\colon [0,t_{\max}) \to \mathcal{L}$ are respectively f- and g-gradient curves. Assume $|f-g| < \varepsilon$; let $\ell\colon t \mapsto |\alpha(t)-\beta(t)|$. Show that*

$$\ell^+ \leqslant \lambda\cdot\ell + \tfrac{2\cdot\varepsilon}{\ell}.$$

Conclude that $\alpha(0)=\beta(0)$ and $t_{\max}<\infty$ then

$$|\alpha(t)-\beta(t)| \leqslant c\cdot\sqrt{\varepsilon\cdot t}$$

for some constant $c = c(t_{\max},\lambda)$.

D Gradient flow

Let $\mathcal{L}$ be a complete geodesic CBB(κ) space and f be a locally Lipschitz semiconcave function defined on an open set of $\mathcal{L}$. If there is an f-gradient curve α such that $\alpha(0)=x$ and $\alpha(t)=y$, then we will write

$$\mathrm{Flow}_f^t(x)=y.$$

The partially defined map Flow_f^t from $\mathcal{L}$ to itself is called the *f-gradient flow* for time t. Note that

$$\mathrm{Flow}_f^{t_1+t_2}=\mathrm{Flow}_f^{t_1}\circ\mathrm{Flow}_f^{t_2};$$

in other words, gradient flow is given by an action of the semigroup $(\mathbb{R}_{\geqslant 0},+)$.

From the first distance estimate 14.6, it follows that for any $t\geqslant 0$, the domain of definition of Flow_f^t is an open subset of $\mathcal{L}$. In some cases, it is globally defined. For example, if f is a λ-concave function defined on the whole space $\mathcal{L}$, then $\mathrm{Flow}_f^t(x)$ is defined for all $x\in\mathcal{L}$ and $t\geqslant 0$; see [6, 16.19].

Now let us reformulate statements obtained earlier using this new terminology. Again, from the first distance estimate, we have the following.

14.10. Proposition. *Let $\mathcal{L}$ be a complete geodesic* CBB(κ) *space and $f\colon\mathcal{L}\to\mathbb{R}$ be a semiconcave function. Then the map $x\mapsto\mathrm{Flow}_f^t(x)$ is locally Lipschitz.*

Moreover, if f is λ-concave, then Flow_f^t is $e^{\lambda\cdot t}$-Lipschitz.

The next proposition follows from 14.9.

14.11. Proposition. *Let $\mathcal{L}$ be a complete geodesic* CBB(κ) *space. Suppose $f_n\colon\mathcal{L}\to\mathbb{R}$ is a sequence of λ-concave functions that converges to $f_\infty\colon\mathcal{L}_\infty\to\mathbb{R}$. Then for any $x\in\mathcal{L}$ and $t\geqslant 0$, we have*

$$\mathrm{Flow}_{f_n}^t(x)\to\mathrm{Flow}_{f_\infty}^t(x)$$

as $n\to\infty$

Lecture 15

Line splitting

A Busemann function

A half-line is a distance-preserving map from $\mathbb{R}_{\geqslant 0} = [0, \infty)$ to a metric space. In other words, a half-line is a geodesic defined on the real half-line $\mathbb{R}_{\geqslant 0}$. If $\gamma\colon [0, \infty) \to \mathcal{X}$ is a half-line, then the limit

❶ $$\mathrm{bus}_\gamma(x) = \lim_{t\to\infty} |\gamma(t) - x| - t$$

is called the Busemann function of γ.

(The Busemann function bus_γ mimics the distance function from the ideal point of γ.)

15.1. Proposition. *For any half-line γ in a metric space $\mathcal{X}$, its Busemann function $\mathrm{bus}_\gamma\colon \mathcal{X} \to \mathbb{R}$ is defined. Moreover, bus_γ is 1-Lipschitz and $\mathrm{bus}_\gamma \circ\gamma(t) + t = 0$ for any t.*

Proof. By the triangle inequality, the function

$$t \mapsto |\gamma(t) - x| - t$$

is nonincreasing for any fixed x.

Since $t = |\gamma(0) - \gamma(t)|$, the triangle inequality implies that

$$|\gamma(t) - x| - t \geqslant -|\gamma(0) - x|.$$

Thus the limit in ❶ is defined, and it is 1-Lipschitz as a limit of 1-Lipschitz functions. The last statement follows since $|\gamma(t) - \gamma(t_0)| = t - t_0$ for all large t. $\square$

Note that 13.1 implies the following.

15.2. Observation. *Any Busemann function on a geodesic* CBB(0) *space is concave.*

B Splitting theorem

A line is a distance-preserving map from $\mathbb{R}$ to a metric space. In other words, a line is a geodesic defined on the real line $\mathbb{R}$.

Let $\mathcal{X}$ be a metric space and $A, B \subset \mathcal{X}$. We will write

$$\mathcal{X} = A \oplus B$$

if there are projections $\mathrm{proj}_A\colon \mathcal{X} \to A$ and $\mathrm{proj}_B\colon \mathcal{X} \to B$ such that

$$|x-y|^2 = |\mathrm{proj}_A(x) - \mathrm{proj}_A(y)|^2 + |\mathrm{proj}_B(x) - \mathrm{proj}_B(y)|^2$$

for any two points $x, y \in \mathcal{X}$.

Note that if

$$\mathcal{X} = A \oplus B$$

then

- ◇ A intersects B at a single point,
- ◇ both sets A and B are convex sets in $\mathcal{X}$; the latter means that any geodesic with the ends in A (or B) lies in A (or B).

15.3. Line splitting theorem. *Let $\mathcal{L}$ be a complete geodesic* CBB(0) *space and γ be a line in $\mathcal{L}$. Then*

$$\mathcal{L} = \mathcal{L}' \oplus \gamma(\mathbb{R})$$

for some subset $\mathcal{L}' \subset \mathcal{L}$.

Before going into the proof, let us state a corollary of the theorem.

15.4. Corollary. *Let $\mathcal{L}$ be a complete geodesic* CBB(0) *space. Then there is an isometric splitting*

$$\mathcal{L} = \mathcal{L}' \oplus H$$

where $H \subset \mathcal{L}$ is a subset isometric to a Hilbert space, and $\mathcal{L}' \subset \mathcal{L}$ is a convex subset that contains no line.

The following lemma is closely relevant to the first distance estimate (14.6); its proof goes along the same lines.

15.5. Lemma. *Suppose $f\colon \mathcal{L} \to \mathbb{R}$ be a concave 1-Lipschitz function on a geodesic* CBB(0) *space $\mathcal{L}$. Consider two f-gradient curves α and β. Then for any $t, s \geqslant 0$ we have*

$$|\alpha(s) - \beta(t)|^2 \leqslant |p-q|^2 + 2\cdot(f(p) - f(q))\cdot(s-t) + (s-t)^2,$$

where $p = \alpha(0)$ and $q = \beta(0)$.

Proof. Since f is 1-Lipschitz, $|\nabla f| \leqslant 1$. Therefore

$$f \circ \beta(t) \leqslant f(q) + t$$

for any $t \geqslant 0$.

Set $\ell(t) = |p - \beta(t)|$. Applying 14.5, we get

$$\begin{aligned}(\ell^2)^+(t) &\leqslant 2\cdot (f \circ \beta(t) - f(p)) \leqslant \\ &\leqslant 2\cdot (f(q) + t - f(p)).\end{aligned}$$

Therefore

$$\ell^2(t) - \ell^2(0) \leqslant 2\cdot (f(q) - f(p)) \cdot t + t^2.$$

It proves the needed inequality in case $s = 0$. Combining it with the first distance estimate (14.6), we get the result in case $s \leqslant t$. The case $s \geqslant t$ follows by switching the roles of s and t. □

Proof of 15.3. Consider two Busemann functions, bus_+ and bus_-, asociated with half-lines $\gamma : [0, \infty) \to \mathcal{L}$ and $\gamma : (-\infty, 0] \to \mathcal{L}$ respectively; that is,

$$\mathrm{bus}_\pm(x) := \lim_{t\to\infty} |\gamma(\pm t) - x| - t.$$

According to 15.2, both functions $\mathrm{bus}_\pm$ are concave.

Fix $x \in \mathcal{L}$. Since γ is a line, we have $\mathrm{bus}_+(x) + \mathrm{bus}_-(x) \geqslant 0$. On the other hand, by 2.18, $f(t) = \mathrm{dist}_x^2 \circ \gamma(t)$ is 2-concave. In particular, $f(t) \leqslant t^2 + at + b$ for some constants $a, b \in \mathbb{R}$. Passing to the limit as $t \to \pm\infty$, we have $\mathrm{bus}_+(x) + \mathrm{bus}_-(x) \leqslant 0$. Hence

$$\mathrm{bus}_+(x) + \mathrm{bus}_-(x) = 0$$

for any $x \in \mathcal{L}$. In particular, the functions $\mathrm{bus}_\pm$ are affine; that is, they are convex and concave at the same time.

Note that for any x,

$$\begin{aligned}|\nabla_x \mathrm{bus}_\pm| &= \sup \{ \boldsymbol{d}_x \mathrm{bus}_\pm(\xi) \ : \ \xi \in \Sigma_x \} = \\ &= \sup \{ -\boldsymbol{d}_x \mathrm{bus}_\mp(\xi) \ : \ \xi \in \Sigma_x \} \equiv \\ &\equiv 1.\end{aligned}$$

Observe that α is a $\mathrm{bus}_\pm$-gradient curve if and only if α is a geodesic such that $(\mathrm{bus}_\pm \circ\alpha)^+ = 1$. Indeed, if α is a geodesic, then $(\mathrm{bus}_\pm \circ\alpha)^+ \leqslant 1$ and the equality holds only if $\nabla_\alpha \mathrm{bus}_\pm = \alpha^+$. Now suppose $\nabla_\alpha \mathrm{bus}_\pm = \alpha^+$. Then $|\alpha^+| \leqslant 1$ and $(\mathrm{bus}_\pm \circ\alpha)^+ = 1$; therefore

$$\begin{aligned}|t_0 - t_1| &\geqslant |\alpha(t_0) - \alpha(t_1)| \geqslant \\ &\geqslant |\,\mathrm{bus}_\pm \circ\alpha(t_0) - \mathrm{bus}_\pm \circ\alpha(t_1) = \\ &= |t_0 - t_1|.\end{aligned}$$

It follows that for any $t>0$, the $\mathrm{bus}_\pm$-gradient flows commute; that is,

$$\mathrm{Flow}^t_{\mathrm{bus}_+}\circ\mathrm{Flow}^t_{\mathrm{bus}_-}=\mathrm{id}_{\mathcal{L}}\,.$$

Setting

$$\mathrm{Flow}^t=\begin{cases}\mathrm{Flow}^t_{\mathrm{bus}_+} & \text{if } t\geqslant 0\\ \mathrm{Flow}^t_{\mathrm{bus}_-} & \text{if } t\leqslant 0\end{cases}$$

defines an $\mathbb{R}$-action on $\mathcal{L}$.

Consider the level set $\mathcal{L}'=\mathrm{bus}_+^{-1}(0)=\mathrm{bus}_-^{-1}(0)$; it is a closed convex subset of $\mathcal{L}$, and therefore forms an Alexandrov space. Consider the map $h\colon\mathcal{L}'\times\mathbb{R}\to\mathcal{L}$ defined by $h\colon(x,t)\mapsto\mathrm{Flow}^t(x)$. Note that h is onto. Applying Lemma 15.5 for $\mathrm{Flow}^t_{\mathrm{bus}_+}$ and $\mathrm{Flow}^t_{\mathrm{bus}_-}$ shows that h is short and non-contracting at the same time; that is, h is an isometry. □

C Polar vectors

Here we give a corollary of 13.10. It will be used to prove basic properties of the tangent space.

15.6. Anti-sum lemma. *Let $\mathcal{L}$ be a complete geodesic* CBB *space and $p\in\mathcal{L}$.*

Given two vectors $u,v\in\mathrm{T}_p$, there is a unique vector $w\in\mathrm{T}_p$ such that

$$\langle u,x\rangle+\langle v,x\rangle+\langle w,x\rangle\geqslant 0$$

for any $x\in\mathrm{T}_p$, and

$$\langle u,w\rangle+\langle v,w\rangle+\langle w,w\rangle=0.$$

15.7. Exercise. *Suppose $u,v,w\in\mathrm{T}_p$ are as in 15.6. Show that*

$$|w|^2\leqslant|u|^2+|v|^2+2\cdot\langle u,v\rangle.$$

If T_p were geodesic, then the lemma would follow from the existence of the gradient, applied to the function $\mathrm{T}_p\to\mathbb{R}$ defined by $x\mapsto-(\langle u,x\rangle+\langle v,x\rangle)$ which is concave. However, the tangent space T_p might fail to be geodesic; see Halbeisen's example [6].

Applying the above lemma for $u=v$, we have the following statement.

15.8. Existence of polar vector. *Let $\mathcal{L}$ be a complete geodesic* CBB *space and $p\in\mathcal{L}$. Given a vector $u\in\mathrm{T}_p$, there is a unique vector*

$u^* \in \mathrm{T}_p$ such that $\langle u^*, u^*\rangle + \langle u, u^*\rangle = 0$ and u^* is polar to u; that is, $\langle u^*, x\rangle + \langle u, x\rangle \geqslant 0$ for any $x \in \mathrm{T}_p$.

In particular, for any vector $u \in \mathrm{T}_p$ there is a polar vector $u^* \in \mathrm{T}_p$ such that $|u^*| \leqslant |u|$.

15.9. Example. *Let $\mathcal{L}$ be the upper half plane in $\mathbb{E}^2$; that is, $\mathcal{L} = \{(x, y) \in \mathbb{E}^2 \mid y \geqslant 0\}$. It is a complete geodesic* CBB(0) *space. For $p = 0$, the tangent space T_p can be canonically identified with $\mathcal{L}$. If $y > 0$, then $u = (x, y) \in \mathrm{T}_p$ has many polar vectors; it includes $u^* = (-x, 0)$ which is provided by 15.8, but the vector $w = (-x, y)$ is polar as well.*

In this case, w is the only polar vector with the same magnitude. If the dimension is finite, then Milka's lemma guarantees the existence of such a polar vector.

Proof of 15.6. By 13.9, we can choose two sequences of points a_n, b_n such that

$$\begin{aligned}\boldsymbol{d}_p\mathrm{dist}_{a_n}(w) &= -\langle \uparrow_{[pa_n]}, w\rangle\\ \boldsymbol{d}_p\mathrm{dist}_{b_n}(w) &= -\langle \uparrow_{[pb_n]}, w\rangle\end{aligned}$$

for any $w \in \mathrm{T}_p$ and $\uparrow_{[pa_n]} \to u/|u|$, $\uparrow_{[pb_n]} \to v/|v|$ as $n \to \infty$

Consider a sequence of functions

$$f_n = |u|\cdot\mathrm{dist}_{a_n} + |v|\cdot\mathrm{dist}_{b_n}.$$

Note that

$$(\boldsymbol{d}_p f_n)(x) = -|u|\cdot\langle \uparrow_{[pa_n]}, x\rangle - |v|\cdot\langle \uparrow_{[pb_n]}, x\rangle.$$

Thus we have the following uniform convergence for $x \in \Sigma_p$:

$$(\boldsymbol{d}_p f_n)(x) \to -\langle u, x\rangle - \langle v, x\rangle$$

as $n \to \infty$, According to 13.10, the sequence $\nabla_p f_n$ converges. Let

$$w = \lim_{n\to\infty} \nabla_p f_n.$$

By the definition of gradient,

$$\begin{aligned}\langle w, w\rangle &= \lim_{n\to\infty} \langle \nabla_p f_n, \nabla_p f_n\rangle = \\ &= \lim_{n\to\infty} (\boldsymbol{d}_p f_n)(\nabla_p f_n) = \\ &= -\langle u, w\rangle - \langle v, w\rangle,\end{aligned} \qquad \begin{aligned}\langle w, x\rangle &= \lim_{n\to\infty} \langle \nabla_p f_n, x\rangle \geqslant \\ &\geqslant \lim_{n\to\infty} (\boldsymbol{d}_p f_n)(x) = \\ &= -\langle u, x\rangle - \langle v, x\rangle.\end{aligned}$$

□

D Linear subspace of tangent space

15.10. Definition. *Let $\mathcal{L}$ be a complete geodesic* CBB(κ) *space, $p \in \mathcal{L}$ and $u, v \in \mathrm{T}_p$. We say that vectors u and v are opposite to each other, (briefly, $u + v = 0$) if $|u| = |v| = 0$ or $\measuredangle(u, v) = \pi$ and $|u| = |v|$.*

The subcone

$$\mathrm{Lin}_p = \{\, v \in \mathrm{T}_p \,:\, \exists\, w \in \mathrm{T}_p \quad \textit{such that} \quad w + v = 0 \,\}$$

will be called the linear subcone of T_p.

15.11. Proposition. *Let $\mathcal{L}$ be a complete geodesic* CBB *space and $p \in \mathcal{L}$. Given two vectors $u, v \in \mathrm{T}_p$, the following statements are equivalent:*

(a) $u + v = 0$;
(b) $\langle u, x\rangle + \langle v, x\rangle = 0$ for any $x \in \mathrm{T}_p$;
(c) $\langle u, \xi\rangle + \langle v, \xi\rangle = 0$ for any $\xi \in \Sigma_p$.

Proof. The equivalence *(b)*⇔*(c)* is trivial.

The condition $u + v = 0$ is equivalent to

$$\langle u, u\rangle = -\langle u, v\rangle = \langle v, v\rangle;$$

thus *(b)*⇒*(a)*.

Recall that T_p is CBB(0). Note that the hinges $[0\,{}^{u}_{x}]$ and $[0\,{}^{v}_{x}]$ are adjacent. By 2.13, $\measuredangle[0\,{}^{u}_{x}] + \measuredangle[0\,{}^{v}_{x}] = 0$; hence *(a)*⇒*(b)*. □

15.12. Exercise. *Let $\mathcal{L}$ be a complete geodesic* CBB *space and $p \in \mathcal{L}$. Then for any three vectors $u, v, w \in \mathrm{T}_p$, if $u + v = 0$ and $u + w = 0$ then $v = w$.*

Let $u \in \mathrm{Lin}_p$; that is, $u + v = 0$ for some $v \in \mathrm{T}_p$. Given $s < 0$, let

$$s{\cdot}u := (-s){\cdot}v.$$

So we can multiply any vector in Lin_p by any real number (positive and negative). By 15.12, this multiplication is uniquely defined; by 15.11; we have identity

$$\langle -v, x\rangle = -\langle v, x\rangle;$$

later we will see that it extends to a linear structure on Lin_p.

15.13. Exercise. *Suppose $u, v, w \in \mathrm{T}_p$ are as in 15.6. Show that*

$$\langle u, x\rangle + \langle v, x\rangle + \langle w, x\rangle = 0$$

for any $x \in \mathrm{Lin}_p$.

15.14. Exercise. *Let* $\mathcal{L}$ *be a complete geodesic* CBB *space,* $p \in \mathcal{L}$ *and* $u \in \mathrm{T}_p$. *Suppose* $u^* \in \mathrm{T}_p$ *is provided by 15.8; that is,*

$$\langle u^*, u^*\rangle + \langle u, u^*\rangle = 0 \quad \textit{and} \quad \langle u^*, x\rangle + \langle u, x\rangle \geqslant 0$$

for any $x \in \mathrm{T}_p$. *Show that*

$$u = -u^* \quad \iff \quad |u| = |u^*|.$$

15.15. Theorem. *Let* p *be a point in a complete geodesic* CBB(κ). *Then* Lin_p *is isometric to a Hilbert space.*

Proof. Note that Lin_p is a closed subset of T_p; in particular, it is complete.

If any two vectors in Lin_p can be connected by a geodesic in Lin_p, then the statement follows from the splitting theorem (15.3). By Menger's lemma (A.11), it is sufficient to show that for any two vectors $x, y \in \mathrm{Lin}_p$ there is a midpoint $w \in \mathrm{Lin}_p$.

Choose $w \in \mathrm{T}_p$ to be the anti-sum of $u = -\tfrac{1}{2}\cdot x$ and $v = -\tfrac{1}{2}\cdot y$; see 15.6. By 15.7 and 15.13,

$$\begin{aligned}
|w|^2 &\leqslant \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle,\\
\langle w, x\rangle &= \tfrac{1}{2}\cdot|x|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle,\\
\langle w, y\rangle &= \tfrac{1}{2}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle,
\end{aligned}$$

It follows that

$$\begin{aligned}
|x - w|^2 &= |x|^2 + |w|^2 - 2\cdot\langle w, x\rangle \leqslant\\
&\leqslant \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 - \tfrac{1}{2}\cdot\langle x, y\rangle =\\
&= \tfrac{1}{4}\cdot|x - y|^2.
\end{aligned}$$

That is, $|x-w| \leqslant \tfrac{1}{2}\cdot|x-y|$, and similarly $|y-w| \leqslant \tfrac{1}{2}\cdot|x-y|$. Therefore w is a midpoint of x and y. In addition we get equality

$$|w|^2 = \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle.$$

It remains to show that $w \in \mathrm{Lin}_p$. Let w^* be the polar vector provided by 15.8. Note that

$$|w^*| \leqslant |w|, \quad \langle w^*, x\rangle + \langle w, x\rangle = 0, \quad \langle w^*, y\rangle + \langle w, y\rangle = 0.$$

The same calculation as above shows that w^* is a midpoint of $-x$ and $-y$ and

$$|w^*|^2 = \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle = |w|^2.$$

By 15.14, $w = -w^*$; hence $w \in \mathrm{Lin}_p$. $\square$

15.16. Exercise. *Let p be a point in a complete geodesic* $\mathrm{CBB}(\kappa)$ *space $\mathcal{L}$ and $f = \mathrm{dist}_p$. Denote by S the subset of points $x \in \mathcal{L}$ such that $|\nabla_x f| = 1$.*

(a) Show that S is a dense G-delta set.

(b) Show that

$$\nabla_x f + \uparrow_{[xp]} = 0$$

for any $x \in S$; in particular, $\uparrow_{[xp]} \in \mathrm{Lin}_x$.

(c) Show that if $|\nabla_x f| = 1$, then $\boldsymbol{d}_x f(w) = \langle \nabla_x f, w\rangle$ for any $w \in \mathrm{T}_x$.

Note that 15.16*b* implies the following.

15.17. Corollary. *Given a countable set of points X in a complete geodesic* $\mathrm{CBB}(\kappa)$ *space $\mathcal{L}$ there is a G-delta dense set $S \subset \mathcal{L}$ such that $\uparrow_{[sx]} \in \mathrm{Lin}_s$ for any $s \in S$ and $x \in X$.*

E Menger's lemma

15.18. Lemma. *Let $\mathcal{X}$ be a complete metric space. Assume that for any pair of points $x, y \in \mathcal{X}$, there is a midpoint z. Then $\mathcal{X}$ is a geodesic space.*

This lemma is due to Karl Menger [70, Section 6].

Proof. Choose $x, y \in \mathcal{X}$; set $\alpha(0) = x$, and $\alpha(1) = y$.

Let $\alpha(\frac{1}{2})$ be a midpoint between $\alpha(0)$ and $\alpha(1)$. Further, let $\alpha(\frac{1}{4})$ and $\alpha(\frac{3}{4})$ be midpoints between the pairs $(\alpha(0), \alpha(\frac{1}{2}))$ and $(\alpha(\frac{1}{2}), \alpha(1))$ respectively. Applying the above procedure recursively, on the n-th step we define $\alpha(\frac{k}{2^n})$, for every odd integer k such that $0 < \frac{k}{2^n} < 1$, as a midpoint of the already defined $\alpha(\frac{k-1}{2^n})$ and $\alpha(\frac{k+1}{2^n})$.

This way we define $\alpha(t)$ for all dyadic rationals t in $[0, 1]$. Moreover, α has Lipschitz constant $|x - y|$. Since $\mathcal{X}$ is complete, the map α can be extended continuously to $[0, 1]$. Moreover,

➊ $$\operatorname{length}\alpha \leqslant |x - y|.$$

$\square$

F Comments

The splitting theorem has an interesting history that starts with Stefan Cohn-Vossen [38]. Our proof is based on the idea of Jeff Cheeger and Detlef Gromoll [37].

Corollary 15.17 is the key to the following result: *all reasonable definitions of dimension give the same result on complete geodesic* CBB *spaces.* We might come back to it after studying the opposite curvature bound.

Appendix A

Ultralimits

Ultralimits provide a very general way to pass to a limit. This procedure works for *any* sequence of metric spaces, its result reminds limit in the sense of Gromov–Hausdorff, but has some strange features; for example, the limit of a constant sequence of spaces $\mathcal{X}_n = \mathcal{X}$ is *not* $\mathcal{X}$ in general (see A.14*b*).

In geometry, ultralimits are used mostly as a canonical way to pass to a convergent subsequence. It is very useful in the proofs where one needs to repeat "pass to convergent subsequence" too many times.

This appendix is taken from [78].

A Faces of ultrafilters

Measure-theoretic definition. Recall that $\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers.

A.1. Definition. *A finitely additive measure ω on $\mathbb{N}$ is called an* u l t r a f i l t e r *if it meets the following condition:*

(a) $\omega(\mathbb{N}) = 1$ and $\omega(S) = 0$ or 1 for any subset $S \subset \mathbb{N}$.

An ultrafilter ω is called n o n p r i n c i p a l *if in addition*

(b) $\omega(F) = 0$ for any finite subset $F \subset \mathbb{N}$.

If $\omega(S) = 0$ for some subset $S \subset \mathbb{N}$, we say that S is ω-s m a l l. If $\omega(S) = 1$, we say that S contains ω-a l m o s t a l l elements of $\mathbb{N}$.

A.2. Advanced exercise. *Let ω be an ultrafilter on $\mathbb{N}$ and $f\colon \mathbb{N} \to \to \mathbb{N}$. Suppose that $\omega(S) \leqslant \omega(f^{-1}(S))$ for any set $S \subset \mathbb{N}$. Show that $f(n) = n$ for ω-almost all $n \in \mathbb{N}$.*

Classical definition. More commonly, a nonprincipal ultrafilter is defined as a collection, say $\mathfrak{F}$, of subsets in $\mathbb{N}$ such that

1. if $P \in \mathfrak{F}$ and $Q \supset P$, then $Q \in \mathfrak{F}$,
2. if $P, Q \in \mathfrak{F}$, then $P \cap Q \in \mathfrak{F}$,
3. for any subset $P \subset \mathbb{N}$, either P or its complement is an element of $\mathfrak{F}$.
4. if $F \subset \mathbb{N}$ is finite, then $F \notin \mathfrak{F}$.

Setting $P \in \mathfrak{F} \Leftrightarrow \omega(P) = 1$ makes these two definitions equivalent.

A nonempty collection of sets $\mathfrak{F}$ that does not include the empty set and satisfies only conditions 1 and 2 is called a filter; if, in addition, $\mathfrak{F}$ satisfies condition 3, it is called an ultrafilter. From Zorn's lemma, it follows that every filter contains an ultrafilter. Thus there is an ultrafilter $\mathfrak{F}$ contained in the filter of all complements of finite sets. Clearly, this ultrafilter $\mathfrak{F}$ is nonprincipal.

Stone–Čech compactification. Given a set $S \subset \mathbb{N}$, consider subset Ω_S of all ultrafilters ω such that $\omega(S) = 1$. It is straightforward to check that the sets Ω_S for all subsets $S \subset \mathbb{N}$ form a topology on the set of ultrafilters on $\mathbb{N}$. The obtained space was first considered by Andrey Tikhonov and called Stone–Čech compactification of $\mathbb{N}$; it is usually denoted as $\beta\mathbb{N}$.

Let ω_n be a principal ultrafilter such that $\omega_n(\{n\}) = 1$; that is, $\omega_n(S) = 1$ if and only if $n \in S$. Note that $n \mapsto \omega_n$ defines an embedding $\mathbb{N} \hookrightarrow \beta\mathbb{N}$; so, we can (and will) consider $\mathbb{N}$ as a subset of $\beta\mathbb{N}$.

The space $\beta\mathbb{N}$ is the maximal compact Hausdorff space that contains $\mathbb{N}$ as an everywhere dense subset. More precisely, the inclusion $\mathbb{N} \hookrightarrow \beta\mathbb{N}$ has the following universal property: *for any compact Hausdorff space $\mathcal{X}$ and a map $f\colon \mathbb{N} \to \mathcal{X}$ there is a unique continuous map $\bar f\colon \beta\mathbb{N} \to X$ such that the restriction $\bar f|_{\mathbb{N}}$ coincides with f.*

B Ultralimits of points

Let us fix a nonprincipal ultrafilter ω once and for all.

Assume x_n is a sequence of points in a metric space $\mathcal{X}$. Let us define the ω-limit of a sequence $x_1, x_2, \dots$ as the point $x_\omega \in \mathcal{X}$ such that for any $\varepsilon > 0$, point x_n lies in $\mathrm{B}(x_\omega, \varepsilon)$ for ω-almost all n; that is, if

$$S_\varepsilon = \{\, n \in \mathbb{N} \; : \; |x_\omega - x_n| < \varepsilon \,\},$$

then $\omega(S_\varepsilon) = 1$ for any $\varepsilon > 0$. In this case, we will write

$$x_\omega = \lim_{n \to \omega} x_n \quad \text{or} \quad x_n \to x_\omega \text{ as } n \to \omega.$$

For example, if ω_n is the *principal* ultrafilter defined by $\omega_n\{n\} = 1$ for some $n \in \mathbb{N}$, then $x_{\omega_n} = x_n$.

The sequence x_n can be regarded as a map $\mathbb{N} \to \mathcal{X}$ defined by $n \mapsto x_n$. If $\mathcal{X}$ is compact, then the map $\mathbb{N} \to \mathcal{X}$ can be extended to a continuous map $\beta\mathbb{N} \to \mathcal{X}$ from the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Then the ω-limit x_ω is the image of ω.

Note that the ω-limits of a sequence and its subsequence may differ. For example, sequence $y_n = -(-1)^n$ is a subsequence of $x_n = (-1)^n$, but for any ultrafilter ω, we have

$$\lim_{n\to\omega} x_n \neq \lim_{n\to\omega} y_n.$$

A.3. Proposition. *Let x_n be a sequence of points in a metric space $\mathcal{X}$. Assume $x_n \to x_\omega$ as $n \to \omega$. Then x_ω is a partial limit of x_n; that is, there is a subsequence of x_n that converges to x_ω in the usual sense.*

Proof. Given $\varepsilon > 0$, let $S_\varepsilon = \{\, n \in \mathbb{N} : |x_n - x_\omega| < \varepsilon \,\}$. Recall that $\omega(S_\varepsilon) = 1$ for any $\varepsilon > 0$.

Since ω is nonprincipal, the set S_ε is infinite for any $\varepsilon > 0$. Therefore, we can choose an increasing sequence n_k such that $n_k \in S_{\frac{1}{k}}$ for each $k \in \mathbb{N}$. Clearly, $x_{n_k} \to x_\omega$ as $k \to \infty$. □

A.4. Proposition. *Any sequence x_n of points in a compact metric space $\mathcal{X}$ has a unique ω-limit x_ω.*

In particular, a bounded sequence of real numbers has a unique ω-limit.

The proposition is analogous to the Bolzano–Weierstrass theorem, and it can be proved the same way. The following lemma is an ultralimit analog of the Cauchy convergence test.

A.5. Lemma. *A sequence of points in a metric space converges if and only if all its subsequences have the same ω-limit.*

Proof. The only-if part is evident; it remains to prove the if part. Suppose z is a ω-limit of all subsequences of $x_1, x_2, \dots$ By A.3, z is a partial limit of x_n. If $x_1, x_2, \dots$ is Cauchy, then $x_n \to z$, and the lemma is proved.

Assume $x_1, x_2, \dots$ is not Cauchy. Then for some $\varepsilon > 0$, there is a subsequence y_n of x_n such that $|x_n - y_n| \geqslant \varepsilon$ for all n. Therefore $|x_\omega - y_\omega| \geqslant \varepsilon$ — a contradiction. □

Recall that ℓ^∞ denotes the space of bounded sequences of real numbers equipped with the sup-norm.

A.6. Exercise. *Construct a linear functional $L\colon \ell^\infty \to \mathbb{R}$ such that for any sequence $\boldsymbol{s} = (s_1, s_2, \dots) \in \ell^\infty$ the image $L(\boldsymbol{s})$ is a partial limit of $s_1, s_2, \dots$*

A.7. Exercise. *Suppose that $f\colon \mathbb{N}\to\mathbb{N}$ is a map such that*

$$\lim_{n\to\omega} x_n = \lim_{n\to\omega} x_{f(n)}$$

for any bounded sequence x_n of real numbers. Show that $f(n)=n$ for ω-almost all $n\in\mathbb{N}$.

C An illustration

In this section, we illustrate the power of ultralimits by proving the following simple claim.

A.8. Claim. *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact spaces. Suppose that for every $n\in\mathbb{N}$ there is a $\tfrac1n$-isometry $f_n\colon\mathcal{X}\to\mathcal{Y}$. Then there is an isometry $\mathcal{X}\to\mathcal{Y}$.*

Proof. Consider the ω-limit f_ω of f_n; according to A.4, f_ω is defined. Since

$$|f_n(x)-f_n(x')| \lessgtr |x-x'|\pm\tfrac1n$$

we get that

$$|f_\omega(x)-f_\omega(x')| = |x-x'|$$

for any $x,x'\in\mathcal{X}$; that is, f_ω is distance-preserving.

Further, since f_n is a $\tfrac1n$-isometry, for any $y\in\mathcal{Y}$ there is a sequence $x_1,x_2,\dots\in\mathcal{X}$ such that $|f_n(x_n)-y|\leqslant\tfrac1n$ for any n. Therefore,

$$f_\omega(x_\omega)=y,$$

where x_ω is the ω-limit of x_n; that is, f_ω is onto.

It follows that $f_\omega\colon\mathcal{X}\to\mathcal{Y}$ is an isometry. □

D Ultralimits of spaces

Recall that ω is a fixed nonprincipal ultrafilter on $\mathbb{N}$.

Let $\mathcal{X}_n$ be a sequence of metric spaces. Consider all sequences of points $x_n\in\mathcal{X}_n$. On the set of all such sequences, define an ∞-semimetric by

❶ $$|(x_n)-(y_n)| := \lim_{n\to\omega} |x_n-y_n|_{\mathcal{X}_n}.$$

Note that the ω-limit on the right-hand side is always defined and takes a value in $[0,\infty]$. (The ω-convergence to ∞ is defined analogously to the usual convergence to ∞; that is, $\lim x_n=\infty \iff \lim\tfrac{1}{x_n}=0$).

Let $\mathcal{X}_\omega$ be the corresponding metric space; that is, the underlying set of $\mathcal{X}_\omega$ is formed by classes of equivalence of sequences of points $x_n \in \mathcal{X}_n$ defined by

$$(x_n) \sim (y_n) \ \Leftrightarrow\ \lim_{n\to\omega} |x_n - y_n| = 0$$

and the distance is defined by ❶.

The space $\mathcal{X}_\omega$ is called the ω-limit of $\mathcal{X}_n$. (It is also called ω-product; this term is motivated by the fact that $\mathcal{X}_\omega$ is a quotient of the product $\prod \mathcal{X}_n$) Typically $\mathcal{X}_\omega$ will denote the ω-limit of sequence $\mathcal{X}_n$; we may also write

$$\mathcal{X}_n \to \mathcal{X}_\omega \text{ as } n \to \omega, \quad \text{or} \quad \mathcal{X}_\omega = \lim_{n\to\omega} \mathcal{X}_n.$$

Given a sequence $x_n \in \mathcal{X}_n$, we will denote by x_ω its equivalence class which is a point in $\mathcal{X}_\omega$; it can be written as

$$x_n \to x_\omega \text{ as } n \to \omega, \quad \text{or} \quad x_\omega = \lim_{n\to\omega} x_n.$$

A.9. Observation. *The ω-limit of any sequence of metric spaces is complete.*

Proof. Let $\mathcal{X}_n$ be a sequence of metric spaces and $\mathcal{X}_n \to \mathcal{X}_\omega$ as $n \to \omega$. Choose a Cauchy sequence $x_1, x_2, \ldots \in \mathcal{X}_\omega$. Passing to a subsequence, we can assume that $|x_k - x_m|_{\mathcal{X}_\omega} < \frac{1}{k}$ if $k < m$.

Choose a double sequence $x_{n,m} \in \mathcal{X}_n$ such that for any fixed m we have $x_{n,m} \to x_m$ as $n \to \omega$. Note that for any $k < m$ the inequality $|x_{n,k} - x_{n,m}| < \frac{1}{k}$ holds for ω-almost all n.

Given $m \in \mathbb{N}$, consider the subset $S_m \subset \mathbb{N}$ defined by

$$S_m = \left\{\, n \geqslant m \,:\, |x_{n,k} - x_{n,l}| < \tfrac{1}{k} \quad \text{for all} \quad k < l \leqslant m \,\right\}.$$

Note that

- ⋄ $\mathbb{N} = S_1 \supset S_2 \supset \ldots$
- ⋄ $\omega(S_m) = 1$ for each m, and
- ⋄ $\min S_m \geqslant m$.

Consider the sequence $y_n = x_{n,m(n)}$, where $m(n)$ is the largest value such that $n \in S_{m(n)}$; from above, $m(n) \leqslant n$. Denote by $y_\omega \in \mathcal{X}_\omega$ the ω-limit of y_n.

Observe that $|y_m - x_{n,m}| < \frac{1}{m}$ for ω-almost all n. It follows that $|x_m - y_\omega| \leqslant \frac{1}{m}$ for any m. Therefore, $x_n \to y_\omega$ as $n \to \infty$. That is, any Cauchy sequence in $\mathcal{X}_\omega$ converges. □

A.10. Observation. *The ω-limit of any sequence of length spaces is geodesic.*

Proof. If $\mathcal{X}_n$ is a sequence of length spaces, then for any sequence of pairs $x_n, y_n \in X_n$ there is a sequence of $\frac{1}{n}$-midpoints z_n.

Let $x_n \to x_\omega$, $y_n \to y_\omega$ and $z_n \to z_\omega$ as $n \to \omega$. Note that z_ω is a midpoint of x_ω and y_ω in $\mathcal{X}_\omega$.

By A.9, $\mathcal{X}_\omega$ is complete. Applying Menger's lemma (A.11), we get the statement. □

A.11. Exercise. *Show that an ultralimit of metric trees is a metric tree.*

A.12. Exercise. *Suppose that $\mathcal{X}_\infty$ and $\mathcal{X}_1, \mathcal{X}_2, \dots$ are compact metric spaces. Assume $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$. Show that $\mathcal{X}_\omega \stackrel{iso}{=\!=} \mathcal{X}_\infty$.*

Pointed limit. If $\operatorname{diam} \mathcal{X}_n \to \infty$ as $n \to \omega$, then the metric on $\mathcal{X}_\omega$ takes value ∞; so $\mathcal{X}_\omega$ has at least two metric components.

To specify a metric component in $\mathcal{X}_\omega$, we may choose a sequence of marked points p_n in $\mathcal{X}_n$, and pass to the metric component of its ω-limit p_ω in $\mathcal{X}_\omega$. The obtained metric component $\mathcal{Z} = (\mathcal{X}_\omega)_{p_\omega}$ with marked p_ω is called the pointed ω-limit of $(\mathcal{X}_n, p_n)$. Note that $\mathcal{Z}$ is a genuine metric space.

If, in the definition of ultralimit, we consider only sequences $x_n \in \in \mathcal{X}_n$ with such that $|p_n - x_n|$ is bounded, then arrive at $\mathcal{Z}$.

For proper metric spaces, there is a relation between the pointed ultralimit and the pointed Gromov–Hausdorff limit. Namely, *if $(\mathcal{X}_\infty, p_\infty)$ and $(\mathcal{X}_1, p_1), (\mathcal{X}_2, p_2), \dots$ are proper pointed metric spaces such that $(\mathcal{X}_n, p_n) \xrightarrow{\text{GH}} (\mathcal{X}_\infty, p_\infty)$, then $(\mathcal{X}_\infty, p_\infty)$ is isometric to the pointed ω-limit of $(\mathcal{X}_n, p_n)$.* The proof is the same as for A.13.

E Ultrapower

If all the metric spaces in the sequence are identical $\mathcal{X}_n = \mathcal{X}$, its ω-limit $\lim_{n\to\omega} \mathcal{X}_n$ is denoted by $\mathcal{X}^\omega$ and called ω-power of $\mathcal{X}$ (also known as ω-completion).

A.13. Exercise. *For any point $x \in \mathcal{X}$, consider the constant sequence $x_n = x$ and set $\iota(x) = \lim_{n\to\omega} x_n \in \mathcal{X}^\omega$.*

(a) Show that $\iota \colon \mathcal{X} \to \mathcal{X}^\omega$ is distance-preserving embedding. (So we can and will consider $\mathcal{X}$ as a subset of $\mathcal{X}^\omega$.)

(b) Show that ι is onto if and only if $\mathcal{X}$ is compact.

(c) Show that if $\mathcal{X}$ is proper, then $\iota(\mathcal{X})$ forms a metric component of $\mathcal{X}^\omega$; that is, a subset of $\mathcal{X}^\omega$ that lies at a finite distance from a given point.

If $\mathcal{X}$ is a genuine metric space, then the metric component of $\mathcal{X}$ in $\mathcal{X}^\omega$ is also called the ultrapower of $\mathcal{X}$; if needed, we may call it the small ultrapower, and the whole space $\mathcal{X}^\omega$ could be called the big ultrapower of $\mathcal{X}$. Note that the small ultrapower of genuine metric space is a genuine metric space. Further, according to *(c)*, *proper metric space is isometric to its small ultrapower.*

Note that *(b)* implies that the inclusion $\mathcal{X} \hookrightarrow \mathcal{X}^\omega$ is not onto if the space $\mathcal{X}$ is not compact. However, the spaces $\mathcal{X}$ and $\mathcal{X}^\omega$ might be isometric; here is an example:

A.14. Exercise. *Let $\mathcal{X}$ be an infinite countable set with discrete metric; that is $|x-y|_{\mathcal{X}} = 1$ if $x \neq y$. Show that*

(a) $\mathcal{X}^\omega$ is not isometric to $\mathcal{X}$, but

(b) $\mathcal{X}^\omega$ is isometric to $(\mathcal{X}^\omega)^\omega$.

A.15. Exercise. *Given a nonprincipal ultrafilter ω, construct an ultrafilter ω_1 such that*

$$\mathcal{X}^{\omega_1} \stackrel{iso}{=\!=} (\mathcal{X}^\omega)^\omega$$

for any metric space $\mathcal{X}$.

A.16. Observation. *Let $\mathcal{X}$ be a complete metric space. Then $\mathcal{X}^\omega$ is geodesic space if and only if $\mathcal{X}$ is a length space.*

Proof. The if part follows from A.10; it remains to prove the only-if part

Assume $\mathcal{X}^\omega$ is geodesic. Then any pair of points $x, y \in \mathcal{X}$ has a midpoint $z_\omega \in \mathcal{X}^\omega$. Fix a sequence of points $z_n \in \mathcal{X}$ such that $z_n \to z_\omega$ as $n \to \omega$.

Note that $|x - z_n|_{\mathcal{X}} \to \frac{1}{2}\cdot|x-y|_{\mathcal{X}}$ and $|y - z_n|_{\mathcal{X}} \to \frac{1}{2}\cdot|x-y|_{\mathcal{X}}$ as $n \to \omega$. In particular, for any $\varepsilon > 0$, the point z_n is an ε-midpoint of x and y for ω-almost all n. It remains to apply Menger's lemma (A.11). □

A.17. Exercise. *Assume $\mathcal{X}$ is a complete length space and $p, q \in \mathcal{X}$ cannot be joined by a geodesic in $\mathcal{X}$. Show that there is at least a continuum of distinct geodesics between p and q in the ultrapower $\mathcal{X}^\omega$.*

A.18. Exercise. *Construct a proper metric space $\mathcal{X}$ such that its big ultrapower $\mathcal{X}^\omega$ is not locally compact.*

F Tangent and asymptotic spaces

Choose a space $\mathcal{X}$ and a sequence λ_n of positive numbers. Consider the sequence of rescalings $\mathcal{X}_n = \lambda_n{\cdot}\mathcal{X} = (\mathcal{X}, \lambda_n{\cdot}|*-*|_{\mathcal{X}})$.

Choose a point $p \in \mathcal{X}$ and denote by p_n the corresponding point in $\mathcal{X}_n$. Consider the ω-limit $\mathcal{X}_\omega$ of $\mathcal{X}_n$ (one may denote it by $\lambda_\omega{\cdot}\mathcal{X}$); set p_ω to be the ω-limit of p_n.

If $\lambda_n \to \infty$ as $n \to \omega$, then the metric component of p_ω in $\mathcal{X}_\omega$ is called λ_ω-tangent space at p and denoted by $\mathrm{T}_p^{\lambda_\omega}\mathcal{X}$ (or $\mathrm{T}_p^{\omega}\mathcal{X}$ if $\lambda_n = n$).

If $\lambda_n \to 0$ as $n \to \omega$, then the metric component of p_ω is called λ_ω-asymptotic space[1] and denoted by $\operatorname{Asym}\mathcal{X}$ or $\operatorname{Asym}^{\lambda_\omega}\mathcal{X}$. Note that the space $\operatorname{Asym}\mathcal{X}$ and its point p_ω do not depend on the choice of $p \in \mathcal{X}$.

The following exercise states that the constructions above depend on the sequence λ_n and a nonprincipal ultrafilter ω.

A.19. Exercise. *Construct a metric space $\mathcal{X}$ with a point p such that the tangent space $\mathrm{T}_p^{\lambda_\omega}\mathcal{X}$ (or its asymptotic cone $\operatorname{Asym}^{\lambda_\omega}\mathcal{X}$) depends on the sequence λ_n and/or ultrafilter ω.*

For nice spaces, different choices of the sequence of coefficients and ultrafilter may give the same space; some examples are given in the following exercise.

A.20. Exercise. *Let $\mathcal{T} = \operatorname{Asym}\mathcal{L}$, where $\mathcal{L}$ is one of the following spaces*

- *(i) Lobachevsky plane,*
- *(ii) Lobachevsky space, or*
- *(iii) 3-regular metric tree; that is, the degree of any vertex. Assume that each edge has unit length.*

- *(a) Show that $\mathcal{T}$ is a complete metric tree.*
- *(b) Show that $\mathcal{T}$ is one-point-homogeneous; that is, given two points $s, t \in \mathcal{T}$ there is an isometry of $\mathcal{T}$ that maps s to t.*
- *(c) Show that $\mathcal{T}$ has continuum degree at any point; that is, for any point $t \in \mathcal{T}$ the set of connected components of the complement $\mathcal{T}\setminus\{t\}$ has cardinality continuum.*

A.21. Exercise. *Consider the cylinder $\mathbb{S}^1\times[0,1]$ with the standard metric. Let $\mathcal{X}$ be the quotient space $\mathbb{S}^1\times[0,1]/\mathbb{S}^1\times\{0\}$; that is,*

$$|(u_1,t_1)-(u_2,t_2)|_{\mathcal{X}} := \min\{\,|(u_1,t_1)-(u_2,t_2)|_{\mathbb{S}^1\times[0,1]}, t_1+t_2\,\}.$$

Describe the ultratangent space $\mathrm{T}_o^{\omega}\mathcal{X}$, where $o \in \mathcal{X}$ is the point that corresponds to $\mathbb{S}^1\times\{0\}$.

[1] Often it is called an *asymptotic cone* despite that it is not a cone in general; this name is used since in good cases it has a cone structure.

G Remarks

A nonprincipal ultrafilter ω is called selective if for any partition of $\mathbb{N}$ into sets $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ such that $\omega(C_\alpha) = 0$ for each α, there is a set $S \subset \mathbb{N}$ such that $\omega(S) = 1$ and $S \cap C_\alpha$ is a one-point set for each $\alpha \in \mathcal{A}$.

The existence of a selective ultrafilter follows from the continuum hypothesis [87].

If needed, we may assume that the chosen ultrafilter ω is selective. In this case, *the subsequence $(x_n)_{n\in S}$ in A.3 can be chosen so that $\omega(S) = 1$.*

Index

Bibliography

[1] М. Айгнер и Г. Циглер. *Доказательства из Книги.* Любое издание.

[2] S. Alexander, D. Berg, and R. Bishop. “Geometric curvature bounds in Riemannian manifolds with boundary”. *Trans. Amer. Math. Soc.* 339.2 (1993), 703–716.

[3] S. Alexander and R. Bishop. “Warped products of Hadamard spaces”. *Manuscripta Math.* 96.4 (1998), 487–505.

[4] S. Alexander and R. Bishop. “The Hadamard–Cartan theorem in locally convex metric spaces”. *Enseign. Math. (2)* 36.3-4 (1990), 309–320.

[5] S. Alexander, V. Kapovitch, and A. Petrunin. *An invitation to Alexandrov geometry: CAT(0) spaces.* 2019.

[6] S. Alexander, V. Kapovitch, and A. Petrunin. *Alexandrov geometry: foundations.* 2022. arXiv: `1903.08539 [math.DG]`.

[7] А. Д. Александров. *Выпуклые многогранники.* 1950. [English translation: Alexandrov, A. D., *Convex polyhedra,* 2005].

[8] А. Д. Александров. «Существование выпуклого многогранника и выпуклой поверхности с заданной метрикой». *Матем. сб.* 11(53).1-2 (1942), 15—65.

[9] А. Д. Александров. *Внутренняя геометрия выпуклых поверхностей.* 1948.

[10] A. D. Alexandrow. “Über eine Verallgemeinerung der Riemannschen Geometrie”. *Schr. Forschungsinst. Math.* 1 (1957), 33–84.

[11] А. Д. Александров. «Линейчатые поверхности в метрических пространствах». *Вестник ЛГУ* 2 (1957), 15—44.

[12] A. Alexandroff. “The inner geometry of an arbitrary convex surface”. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 32 (1941), 467–470.

[13] A. D. Alexandrow. “Über eine Verallgemeinerung der Riemannschen Geometrie”. *Schr. Forschungsinst. Math.* 1 (1957), 33–84.

[14] F. D. Ancel, M. W. Davis, and C. R. Guilbault. “CAT(0) reflection manifolds”. *Geometric topology (Athens, GA, 1993).* Vol. 2. AMS/IP Stud. Adv. Math. Amer. Math. Soc., Providence, RI, 1997, 441–445.

[15] L. Andersson and R. Howard. “Comparison and rigidity theorems in semi-Riemannian geometry”. *Comm. Anal. Geom.* 6.4 (1998), 819–877.

[16] W. Ballmann. *Lectures on spaces of nonpositive curvature.* Vol. 25. DMV Seminar. With an appendix by Misha Brin. 1995.

[17] W. Ballmann. *Lectures on spaces of nonpositive curvature.* Vol. 25. DMV Seminar. With an appendix by Misha Brin. 1995.

[18] V. N. Berestovskii. "Spaces with bounded curvature and distance geometry". *Sibirsk. Mat. Zh.* 27.1 (1986), 11–25, 197.

[19] L. Billera, S. Holmes, and K. Vogtmann. "Geometry of the space of phylogenetic trees". *Adv. in Appl. Math.* 27.4 (2001), 733–767.

[20] R. Bishop. "The intrinsic geometry of a Jordan domain". *Int. Electron. J. Geom.* 1.2 (2008), 33–39.

[21] B. H. Bowditch. "Notes on locally CAT(1) spaces". *Geometric group theory (Columbus, OH, 1992).* Vol. 3. Ohio State Univ. Math. Res. Inst. Publ. de Gruyter, Berlin, 1995, 1–48.

[22] B. Bowditch. "Median and injective metric spaces". *Math. Proc. Cambridge Philos. Soc.* 168.1 (2020), 43–55.

[23] M. Bridson and A. Haefliger. *Metric spaces of non-positive curvature.* Vol. 319. Grundlehren der Mathematischen Wissenschaften. 1999.

[24] D. Burago. "Hard balls gas and Alexandrov spaces of curvature bounded above". *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998).* Extra Vol. II. 1998, 289–298.

[25] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry.* Vol. 33. Graduate Studies in Mathematics. 2001. [Русский перевод: Бураго Д. Ю., Бураго Ю. Д., Иванов С. В. «Курс метрической геометрии», 2004.]

[26] D. Burago, S. Ferleger, and A. Kononenko. "Uniform estimates on the number of collisions in semi-dispersing billiards". *Ann. of Math. (2)* 147.3 (1998), 695–708.

[27] D. Burago, S. Ferleger, and A. Kononenko. "Topological entropy of semi-dispersing billiards". *Ergodic Theory Dynam. Systems* 18.4 (1998), 791–805.

[28] D. Burago, D. Grigoriev, and A. Slissenko. "Approximating shortest path for the skew lines problem in time doubly logarithmic in 1/epsilon". *Theoret. Comput. Sci.* 315.2-3 (2004), 371–404.

[29] Г. Я. Перельман Ю. Д. Бураго М. Л. Громов. «Пространства А. Д. Александрова с ограниченными снизу кривизнами». *УМН* 47.2(284) (1992), 3—51. [English translation: Yu. D. Burago, M. L. Gromov, G. Ya. Perel'man, "A. D. Alexandrov spaces with curvature bounded below", Russian Math. Surveys, 47:2 (1992), 1–58].

[30] D. Burago and S. Ivanov. "Examples of exponentially many collisions in a hard ball system". *Ergodic Theory Dynam. Systems* 41.9 (2021), 2754–2769.

[31] H. Busemann. "Spaces with non-positive curvature". *Acta Math.* 80 (1948), 259–310.

[32] S. V. Buyalo. "Volume and fundamental group of a manifold of nonpositive curvature". *Mat. Sb. (N.S.)* 122(164).2 (1983), 142–156.

[33] É. Cartan. *Leçons sur la Géométrie des Espaces de Riemann.* 1928.

[34] A. Cauchy. "Sur les polygones et les polyèdres: second Mémoire". *Journal de l'Ecole Polytechnique* 16.9 (1813), 26–38.

[35] R. Charney and M. Davis. "Singular metrics of nonpositive curvature on branched covers of Riemannian manifolds". *Amer. J. Math.* 115.5 (1993), 929–1009.

[36] R. Charney and M. Davis. “Strict hyperbolization”. *Topology* 34.2 (1995), 329–350.

[37] J. Cheeger and D. Gromoll. “The splitting theorem for manifolds of non-negative Ricci curvature”. *J. Differential Geometry* 6 (1971/72), 119–128.

[38] S. Cohn-Vossen. „Totalkrümmung und geodätische Linien auf einfachzusammenhängenden offenen vollständigen Flächenstücken“. *Mat. Sb.* 1(43).2 (1936).

[39] A. Daniilidis, O. Ley, and S. Sabourau. “Asymptotic behaviour of self-contracted planar curves and gradient orbits of convex functions”. *J. Math. Pures Appl. (9)* 94.2 (2010), 183–199.

[40] L. Danzer und B. Grünbaum. „Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee“. *Math. Z.* 79 (1962), 95–99.

[41] M. Davis. “Groups generated by reflections and aspherical manifolds not covered by Euclidean space”. *Ann. of Math. (2)* 117.2 (1983), 293–324.

[42] M. Davis. “Exotic aspherical manifolds”. *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001).* Vol. 9. ICTP Lect. Notes. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002, 371–404.

[43] M. Davis and T. Januszkiewicz. “Hyperbolization of polyhedra”. *J. Differential Geom.* 34.2 (1991), 347–388.

[44] M. Davis, T. Januszkiewicz, and J.-F. Lafont. “4-dimensional locally CAT(0)-manifolds with no Riemannian smoothings”. *Duke Math. J.* 161.1 (2012), 1–28.

[45] Н. П. Долбилин. *Жемчужины теории многогранников.* 2000.

[46] R. Engelking. *Dimension theory.* 1978.

[47] P. Erdős. “Some unsolved problems”. *Michigan Math. J.* 4 (1957), 291–300.

[48] Г. А. Гальперин. «О системах локально взаимодействующих и отталкивающихся частиц, движущихся в пространстве». *Труды Московского математического общества* 43.0 (1981), 142—196.

[49] M. Gromov. “Hyperbolic groups”. *Essays in group theory.* Vol. 8. Math. Sci. Res. Inst. Publ. Springer, New York, 1987, 75–263.

[50] M. Gromov. “Sign and geometric meaning of curvature”. *Rend. Sem. Mat. Fis. Milano* 61 (1991), 9–123 (1994).

[51] M. Gromov. “Asymptotic invariants of infinite groups”. *Geometric group theory, Vol. 2 (Sussex, 1991).* Vol. 182. London Math. Soc. Lecture Note Ser. 1993, 1–295.

[52] J. Hadamard. “Sur la forme des lignes géodésiques à l’infini et sur les géodésiques des surfaces réglées du second ordre”. *Bull. Soc. Math. France* 26 (1898), 195–216.

[53] S. Halbeisen. “On tangent cones of Alexandrov spaces with curvature bounded below”. *Manuscripta Math.* 103.2 (2000), 169–182.

[54] J. Hass. “Bounded 3-manifolds admit negatively curved metrics with concave boundary”. *J. Differential Geom.* 40.3 (1994), 449–459.

[55] A. Hatcher. *Algebraic topology.* 2002. [Русский перевод: Хатчер А. «Алгебраическая топология», 2011.]

[56] W.-Y. Hsiang and B. Kleiner. “On the topology of positively curved 4-manifolds with symmetry”. *J. Differential Geom.* 29.3 (1989), 615–621.

[57] W. Hurewicz and H. Wallman. *Dimension Theory.* Princeton Mathematical Series, v. 4. 1941.

[58] S. Ivanov. "On Helly's theorem in geodesic spaces". *Electron. Res. Announc. Math. Sci.* 21 (2014), 109–112.

[59] M. Kervaire. "Smooth homology spheres and their fundamental groups". *Trans. Amer. Math. Soc.* 144 (1969), 67–72.

[60] B. Kleiner. "The local structure of length spaces with curvature bounded above". *Math. Z.* 231.3 (1999), 409–456.

[61] A. Klyachko. "A funny property of sphere and equations over groups". *Comm. Algebra* 21.7 (1993), 2555–2575.

[62] U. Lang and V. Schroeder. "On Toponogov's comparison theorem for Alexandrov spaces". *Enseign. Math.* 59.3-4 (2013), 325–336.

[63] N. Lebedeva. "Alexandrov spaces with maximal number of extremal points". *Geom. Topol.* 19.3 (2015), 1493–1521.

[64] Н. Лебедева и А. Петрунин. *Теорема Александрова о вложении многогранников.* 2022. [English translation: N. Lebedeva, A. Petrunin, *Alexandrov's embedding theorem.*]

[65] И. М. Либерман. «Геодезические линии на выпуклых поверхностях». *Докл. АН СССР* 32.2 (1941), 310—312.

[66] A. Lytchak and S. Wenger. "Isoperimetric characterization of upper curvature bounds". *Acta Math.* 221.1 (2018), 159–202.

[67] A. Lytchak. "Differentiation in metric spaces". *St. Petersburg Math. J.* 16.6 (2005), 1017–1041.

[68] H. von Mangoldt. "Ueber diejenigen Punkte auf positiv gekrümmten Flächen, welche die Eigenschaft haben, dass die von ihnen ausgehenden geodätischen Linien nie aufhören, kürzeste Linien zu sein". *J. Reine Angew. Math.* 91 (1881), 23–53.

[69] B. Mazur. "A note on some contractible 4-manifolds". *Ann. of Math. (2)* 73 (1961), 221–228.

[70] K. Menger. „Untersuchungen über allgemeine Metrik". *Math. Ann.* 100.1 (1928), 75–163.

[71] J. Nagata. *Modern dimension theory.* Revised. Vol. 2. Sigma Series in Pure Mathematics. 1983.

[72] D. Panov and A. Petrunin. "Sweeping out sectional curvature". *Geom. Topol.* 18.2 (2014), 617–631.

[73] D. Panov and A. Petrunin. "Ramification conjecture and Hirzebruch's property of line arrangements". *Compos. Math.* 152.12 (2016), 2443–2460.

[74] G. Perelman. "Spaces with curvature bounded below". *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994).* 1995, 517–525.

[75] A. Petrunin and S. Stadler. "Metric-minimizing surfaces revisited". *Geom. Topol.* 23.6 (2019), 3111–3139.

[76] A. Petrunin and S. Zamora Barrera. *What is differential geometry: curves and surfaces.* 2021. arXiv: `2012.11814 [math.HO]`.

[77] A. Petrunin. "A globalization for non-complete but geodesic spaces". *Math. Ann.* 366.1-2 (2016), 387–393.

[78] A. Petrunin. *Pure metric geometry: introductory lectures.* 2023. arXiv: 2007.09846 [math.MG].

[79] P. Pizzetti. “Paragone fra due triangoli a lati uguali”. *Atti della Reale Accademia dei Lincei, Rendiconti (5). Classe di Scienze Fisiche, Matematiche e Naturali* 16.1 (1907), 6–11.

[80] C. Plaut. “Spaces of Wald curvature bounded below”. *J. Geom. Anal.* 6.1 (1996), 113–134.

[81] А. В. Погорелов. *Однозначная определённость общих выпуклых поверхностей.* Монографии института математики, вып. II. 1952.

[82] L. Pontrjagin. “Sur une hypothèse fondamentale de la théorie de la dimension.” *C. R. Acad. Sci., Paris* 190 (1930), 1105–1107.

[83] Yu. G. Reshetnyak. “On the theory of spaces with curvature no greater than K”. *Mat. Sb. (N.S.)* 52 (94) (1960), 789–798.

[84] Yu. G. Reshetnyak. “Inextensible mappings in a space of curvature no greater than K”. *Siberian Mathematical Journal* 9.4 (1968), 683–689.

[85] W. Rinow. *Die innere Geometrie der metrischen Räume.* Die Grundlehren der mathematischen Wissenschaften, Bd. 105. 1961.

[86] D. Rolfsen. “Strongly convex metrics in cells”. *Bull. Amer. Math. Soc.* 74 (1968), 171–175.

[87] W. Rudin. “Homogeneity problems in the theory of Čech compactifications”. *Duke Math. J.* 23 (1956), 409–419.

[88] I. J. Schoenberg and S. C. Zaremba. “On Cauchy’s lemma concerning convex polygons”. *Canadian J. Math.* 19 (1967), 1062–1071.

[89] С. З. Шефель. “О внутренней геометрии седловых поверхностей”. *Сибирский математический журнал* 5.6 (1964), 1382–1396.

[90] С. З. Шефель. “О седловых поверхностях, ограниченных спрямляемой кривой”. *Доклады Академии наук* 162.2 (1965), 294–296. English translation "On saddle surfaces bounded by a rectifiable curve", Soviet Math. Dokl. 6 (1965), 684–687.

[91] K. Shiohama. *An introduction to the geometry of Alexandrov spaces.* Vol. 8. Lecture Notes Series. 1993.

[92] S. Stadler. “An obstruction to the smoothability of singular nonpositively curved metrics on 4-manifolds by patterns of incompressible tori”. *Geom. Funct. Anal.* 25.5 (2015), 1575–1587.

[93] D. Stone. “Geodesics in piecewise linear manifolds”. *Trans. Amer. Math. Soc.* 215 (1976), 1–44.

[94] С. Л. Табачников и Д. Б. Фукс. *Математический дивертисмент.* 2016.

[95] P. Thurston. “CAT(0) 4-manifolds possessing a single tame point are Euclidean”. *J. Geom. Anal.* 6.3 (1996), 475–494 (1997).

[96] В. А. Топоногов. «Римановы пространства кривизны, ограниченной снизу». *Успехи математических наук* 14.1 (85) (1959), 87—130.

[97] S. A. Vakhrameev. “Morse lemmas for smooth functions on manifolds with corners”. Vol. 100. 4. Dynamical systems, 8. 2000, 2428–2445.

[98] Ю. А. Волков. «Существование многогранника с данной разверткой». *Зап. научн. сем. ПОМИ* (2018), 50—78.

[99] A. Wald. „Begründung eiiner Koordinatenlosen Differentialgeometrie der Flächen“. *Ergebnisse eines mathematischen Kolloquium* 6 (1935), 24–46.

[100] В. А. Залгаллер. «О деформациях многоугольника на сфере». *УМН* 11.5(71) (1956), 177—178.