Homework assignments

Due 2023-08-25: 1.8, 1.11, 1.13, 1.14, 1.17. (Scan to pdf and upload to CANVAS.)

Due 2023-09-01: 2.2, 2.4, 2.5, 2.7, 2.14. (Scan to pdf and upload to CANVAS.)

Due 2023-09-11: 2.11; 3.6; 3.12; 4.5; 4.8

Contents

Lecture 1

Definitions

The first synthetic description of curvature is due to Abraham Wald [17] published in 1936; it was his student work, written under the supervision of Karl Menger. This publication was not noticed for about 50 years [6]. In 1941, similar definitions were rediscovered by Alexandr Alexandrov [5].

A Notations

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

We will denote by $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$ the n-dimensional sphere (with angle metric), Euclidean space, and Lobachevsky space respectively. More generally, $\mathbb{M}^n(\kappa)$ will denote the m o d e l n-s p a c e of curvature κ; that is,

- if $\kappa > 0$, then $\mathbb{M}^n(\kappa)$ is the n-sphere of radius $\frac{1}{\sqrt{\kappa}}$, so $\mathbb{S}^n = \mathbb{M}^n(1)$
- $\mathbb{M}^n(0) = \mathbb{E}^n$,
- if $\kappa < 0$, then $\mathbb{M}^n(\kappa)$ is the Lobachevsky n-space $\mathbb{H}^n$ rescaled by factor $\frac{1}{\sqrt{-\kappa}}$; in particular $\mathbb{M}^n(-1) = \mathbb{H}^n$.

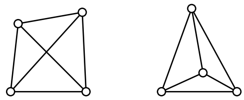
B Wald's approach

Wald noticed that a *typical* quadruple x_1, x_2, x_3, x_4 of points in a metric space admits model configurations in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \in \mathbb{M}^3(\kappa)$ with

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{M}^3(\kappa)} = |x_i - x_j|_{\mathcal{X}}$$

for κ in a closed interval, say

$$[\kappa_{\min}(x_1,x_2,x_3,x_4),\kappa_{\max}(x_1,x_2,x_3,x_4)]\subset\mathbb{R}.$$



In $\mathbb{M}^3(\kappa_{\min})$ and $\mathbb{M}^3(\kappa_{\max})$, the points $\tilde x_1,\tilde x_2,\tilde x_3,\tilde x_4$ form degenerate tetrahedrons shown on the diagram (for $\kappa_{\min}$ it is a convex quadrangle and for $\kappa_{\max}$ — a triangle with a point inside). In the interior of the interval, the tetrahedron is nondegenerate.

Moreover, one can use $[-\infty,\infty)$ instead of $\mathbb{R}$ and let

$$\kappa_{\min}(x_1,x_2,x_3,x_4)=-\infty$$

if there is *almost* model quadruple in $\mathbb{M}^3(\kappa)$ for $\kappa\to-\infty$; that is, for any $\varepsilon>0$ there is a quadruple $\tilde x_1,\tilde x_2,\tilde x_3,\tilde x_4\in\mathbb{M}^3(\kappa)$ such that $\kappa\leqslant-\frac{1}{\varepsilon}$, and

$$|\tilde x_i-\tilde x_j|_{\mathbb{M}^3(\kappa)}\lesseqgtr|x_i-x_j|_{\mathcal{X}}\pm\varepsilon$$

for all i and j. In this case the interval

$$[\kappa_{\min}(x_1,x_2,x_3,x_4),\kappa_{\max}(x_1,x_2,x_3,x_4)]\subset[-\infty,\infty)$$

is defined for *any* quadruple.

We will not use these statements further in the sequel, so we omit the proofs. The just wanted to describe the first step in the theory.

1.1. Exercise. *Let x_1,x_2,x_3,x_4 be a quadruple in a metric space such that $\kappa_{\min}(x_1,x_2,x_3,x_4)=-\infty$. Show that two maximal numbers from the following three are equal to each other.*

$$\begin{aligned}a&=|x_1-x_2|+|x_3-x_4|,\\ b&=|x_1-x_3|+|x_2-x_4|,\\ c&=|x_1-x_4|+|x_2-x_3|.\end{aligned}$$

1.2. Exercise. *Suppose that x_1,x_2,x_3,x_4 in a metric space such that*

$$\begin{aligned}|x_1-x_2|&=|x_1-x_3|=|x_1-x_4|=1,\\ |x_2-x_3|&=|x_3-x_4|=|x_4-x_1|=2.\end{aligned}$$

Show that

$$\kappa_{\min}(x_1,x_2,x_3,x_4)=\kappa_{\max}(x_1,x_2,x_3,x_4)=-\infty.$$

1.3. Exercise. *Let x_1, x_2, x_3, x_4 be a quadruple in $\mathbb{E}^2$. Suppose that triangle $[x_1x_2x_3]$ is degerate, but $[x_2x_3x_4]$ is not. Show that*

$$\kappa_{\min}(x_1, x_2, x_3, x_4) = \kappa_{\max}(x_1, x_2, x_3, x_4) = 0.$$

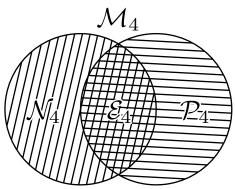
1.4. Wald-style definition. *Let $\kappa \in \mathbb{R}$. A metric space $\mathcal{X}$ has curvature $\geqslant \kappa$ (or $\leqslant \kappa$) if for any quadruple $x_1, x_2, x_3, x_4 \in \mathcal{X}$ we have $\kappa_{\max}(x_1, x_2, x_3, x_4) \geqslant \kappa$ (or $\kappa_{\min}(x_1, x_2, x_3, x_4) \leqslant \kappa$ respectively).*

C Substance

Consider the space $\mathcal{M}_4$ of all isometry classes of 4-point metric spaces. Each element in $\mathcal{M}_4$ can be described by 6 numbers — the distances between all 6 pairs of its points, say $\ell_{i,j}$ for $1 \leqslant i < j \leqslant 4$ modulo permutations of the index set $(1, 2, 3, 4)$. These 6 numbers are subject to 12 triangle inequalities; that is,

$$\ell_{i,j} + \ell_{j,k} \geqslant \ell_{i,k}$$

holds for all i, j and k, where we assume that $\ell_{j,i} = \ell_{i,j}$, and $\ell_{i,i} = 0$.



The space $\mathcal{M}_4$ comes with topology. It can be defined as a quotient topology of the cone in $\mathbb{R}^6$ by permutations of the 4 points of the space.

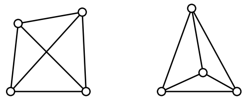
Consider the subset $\mathcal{E}_4 \subset \mathcal{M}_4$ of all isometry classes of 4-point metric spaces that admit isometric embeddings into Euclidean space.

1.5. Claim. *The complement $\mathcal{M}_4 \setminus \mathcal{E}_4$ has two connected components.*

1.6. Exercise. *Spend 10 minutes trying to prove the claim.*

The definition of Alexandrov spaces is based on the claim above. Let us denote one of the components by $\mathcal{P}_4$ and the other by $\mathcal{N}_4$. Here $\mathcal{P}$ and $\mathcal{N}$ stand for positive and negative curvature because spheres have no quadruples of type $\mathcal{N}_4$ and hyperbolic space has no quadruples of type $\mathcal{P}_4$.

A metric space that has no quadruples of points of type $\mathcal{P}_4$ or $\mathcal{N}_4$ respectively is called an Alexandrov space with non-positive (CAT(0)) or non-negative curvature (CBB(0)).



Let us describe the subdivision into $\mathcal{P}_4$, $\mathcal{E}_4$, and $\mathcal{N}_4$ intuitively. Imagine that you move out of $\mathcal{E}_4$ — your path is a one-parameter family of 4-point metric spaces. The last thing you see in $\mathcal{E}_4$ is one of the two plane configurations shown on the diagram. If you see the right configuration then you move into $\mathcal{N}_4$; if it is the one on the left, then you move into $\mathcal{P}_4$. More degenerate pictures can be avoided; for example, a triangle with a point on a side. From such a configuration one may move in $\mathcal{N}_4$ and $\mathcal{P}_4$ (as well as come back to $\mathcal{E}_4$).

Here is an exercise, solving which would force you to rebuild a considerable part of Alexandrov geometry. It is wise to spend some time thinking about this exercise before proceeding.

1.7. Advanced exercise. *Assume $\mathcal{X}$ is a complete metric space with length metric (see Section 1F), containing only quadruples of type $\mathcal{E}_4$. Show that $\mathcal{X}$ is isometric to a convex set in a Hilbert space.*

In the definition above, one can take $\mathbb{M}^3(\kappa)$ instead of $\mathbb{E}^3$. In this case, one obtains the definition of spaces with curvature bounded above or below by κ (CAT(κ) or CBB(κ)). The parameter κ has three interesting choices -1, 0, and 1; the rest can be obtained from these three applying rescaling.

D Geodesics, triangles, and angles

Geodesics. Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ a real interval. A distance-preserving map $\gamma\colon \mathbb{I}\to\mathcal{X}$ is called a geodesic[1]; in other words, $\gamma\colon \mathbb{I}\to\mathcal{X}$ is a geodesic if

$$|\gamma(s)-\gamma(t)|_{\mathcal{X}}=|s-t|$$

for any pair $s,t\in\mathbb{I}$.

If $\gamma\colon [a,b]\to\mathcal{X}$ is a geodesic such that $p=\gamma(a)$, $q=\gamma(b)$, then we say that γ is a geodesic from p to q. In this case, the image of γ is denoted by $[pq]$, and, with abuse of notations, we also call it a geodesic. We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$.

In general, a geodesic from p to q need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we assume that we have chosen such geodesic.

[1] Others call it differently: *shortest path*, *minimizing geodesic*. Also, note that the meaning of the term *geodesic* is different from what is used in Riemannian geometry, altho they are closely related.

A geodesic path is a geodesic with constant-speed parameterization by the unit interval $[0,1]$.

A metric space is called geodesic if any pair of its points can be joined by a geodesic.

Triangles. Given a triple of points p,q,r in a metric space $\mathcal{X}$, a choice of geodesics $([qr],[rp],[pq])$ will be called a triangle; we will use the short notation $[pqr]=[pqr]_{\mathcal{X}}=([qr],[rp],[pq])$.

Given a triple $p,q,r\in\mathcal{X}$ there may be no triangle $[pqr]$ simply because one of the pairs of these points cannot be joined by a geodesic. Also, many different triangles with these vertices may exist, any of which can be denoted by $[pqr]$. If we write $[pqr]$, it means that we have chosen such a triangle.

Model triangles. Given three points p,q,r in a metric space $\mathcal{X}$, let us define its model triangle $[\tilde p\tilde q\tilde r]$ (briefly, $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$) to be a triangle in the Euclidean plane $\mathbb{E}^2$ such that

$$|\tilde p-\tilde q|_{\mathbb{E}^2}=|p-q|_{\mathcal{X}},\qquad |\tilde q-\tilde r|_{\mathbb{E}^2}=|q-r|_{\mathcal{X}},\qquad |\tilde r-\tilde p|_{\mathbb{E}^2}=|r-p|_{\mathcal{X}}.$$

The same way we can define the hyperbolic and the spherical model triangles $\tilde\triangle(pqr)_{\mathbb{H}^2}$, $\tilde\triangle(pqr)_{\mathbb{S}^2}$ in the Lobachevsky plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2$. In the latter case, the model triangle is said to be defined if in addition

$$|p-q|+|q-r|+|r-p|<2{\cdot}\pi.$$

In this case, the model triangle again exists and is unique up to an isometry of $\mathbb{S}^2$.

Model angles. If $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$ and $|p-q|,|p-r|>0$, the angle measure of $[\tilde p\tilde q\tilde r]$ at $\tilde p$ will be called the model angle of the triple p, q, r and will be denoted by $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{E}^2}$.

The same way we define $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$; in particular, $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{H}^2}$ and $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{S}^2}$. We may use the notation $\tilde\measuredangle(p\,{}^q_r)$ if it is evident which of the model spaces is meant.

1.8. Exercise. *Show that for any triple of point p, q, and r, the function*

$$\kappa\mapsto\tilde\measuredangle(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$$

is nondecreasing in its domain of definition.

Hinges. Let $p,x,y\in\mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair of geodesics $([px],[py])$ will be called a hinge and will be denoted by $[p\,{}^x_y]=([px],[py])$.

E Definitions

In this section we write inequalities that describe the sets $\mathcal{E}_4 \cup \mathcal{P}_4$ and $\mathcal{E}_4 \cup \mathcal{N}_4$ from Section 1C.

Curvature bounded below. Let p, x, y, z be a quadruple of points in a metric space. If the inequality

$$\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^y_z)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^z_x)_{\mathbb{E}^2} \leqslant 2\cdot\pi \qquad ❶$$

holds, then we say that the quadruple meets CBB(0) *comparison*.

1.9. Definition. *A metric space $\mathcal{X}$ has nonnegative curvature in the sense of Alexandrov (briefly, $\mathcal{X} \in \mathrm{CBB}(0)$ if $\mathrm{CBB}(0)$ comparison holds for any quadruple in $\mathcal{X}$ such that each model angle in ❶ is defined.*

If instead of $\mathbb{E}^2$, we use $\mathbb{S}^2$ or $\mathbb{H}^2$, then we get the definition of CBB(1) and CBB(−1) comparisons. Note that $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2}$ and $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2}$ are defined if $p \ne x$, $p \ne y$, but for $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2}$ we need in addition, $|p-x| + |p-y| + |x-y| < 2\cdot\pi$.

More generally, one may apply this definition to $\mathbb{M}^2(\kappa)$. This way we define $\mathrm{CBB}(\kappa)$ comparison for any real κ.

1.10. Exercise. *Show that $\mathbb{E}^n$ is $\mathrm{CBB}(0)$.*

1.11. Exercise. *Show that a metric space $\mathcal{X}$ is $\mathrm{CBB}(0)$ if and only if for any quadruple of points $p, x_1, x_2, x_3 \in \mathcal{X}$ there is a quadruple of points $q, y_1, y_2, y_3 \in \mathbb{E}^3$ such that*

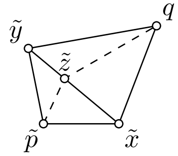
$$|p - x_i|_{\mathcal{X}} \geqslant |q - y_i|_{\mathbb{E}^2} \quad \textit{and} \quad |x_i - x_j|_{\mathcal{X}} \leqslant |y_i - y_j|_{\mathbb{E}^2}$$

for all i and j.

Curvature bounded above. Given a quadruple of points p, q, x, y in a metric space $\mathcal{X}$, consider two model triangles $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{E}^2}$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)_{\mathbb{E}^2}$ with common side $[\tilde x\tilde y]$.

If the inequality

$$|p - q|_{\mathcal{X}} \leqslant |\tilde p - \tilde z|_{\mathbb{E}^2} + |\tilde z - \tilde q|_{\mathbb{E}^2}$$



holds for any point $\tilde z \in [\tilde x\tilde y]$, then we say that the quadruple p, q, x, y satisfies CAT(0) *comparison*.

1.12. Definition. *A metric space $\mathcal{X}$ has nonpositive curvature in the sense of Alexandrov (briefly, $\mathcal{X} \in \mathrm{CAT}(0)$) if $\mathrm{CAT}(0)$ comparison holds for any quadruple in $\mathcal{X}$.*

If we do the same for spherical model triangles $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)_{\mathbb{S}^2}$ and $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\triangle}(qxy)_{\mathbb{S}^2}$, then we arrive at the definition of CAT(1) comparison. One of the spherical model triangles might undefined; it happens if

$$|p-x|+|p-y|+|x-y| \geqslant 2{\cdot}\pi \quad \text{or} \quad |q-x|+|q-y|+|x-y| \geqslant 2{\cdot}\pi.$$

In this case, it is assumed that CAT(1) comparison automatically holds for this quadruple.

We can do the same for $\mathbb{M}^2(\kappa)$. In this case, we arrive at the definition of CAT(κ) comparison. However, we will mostly consider CAT(0) comparison and occasionally CAT(1) comparison; so, if you see CAT(κ), then it is safe to assume that κ is 0 or 1.

Here CAT is an acronym for Cartan, Alexandrov, and Toponogov, but usually pronounced as “cat” in the sense of “miauw”. The term was coined by Mikhael Gromov in 1987. Originally, Alexandrov used $\mathfrak{R}_\kappa$ domain; this term is still in use.

1.13. Exercise. *Show that a metric space $\mathcal{U}$ is* CAT(0) *if and only if for any quadruple of points p, q, x, y in $\mathcal{U}$ there is a quadruple $\tilde{p}, \tilde{q}, \tilde{x}, \tilde{y}$ in $\mathbb{E}^2$ such that*

$$\begin{aligned} |\tilde{p}-\tilde{q}| &\geqslant |p-q|, & |\tilde{x}-\tilde{y}| &\geqslant |x-y|,\\ |\tilde{p}-\tilde{x}| &\leqslant |p-x|, & |\tilde{p}-\tilde{y}| &\leqslant |p-y|,\\ |\tilde{q}-\tilde{x}| &\leqslant |q-x|, & |\tilde{q}-\tilde{y}| &\leqslant |q-y|. \end{aligned}$$

1.14. Exercise. *Assume that a quadruple of points in a metric space satisfies* CBB(0) *and* CAT(0) *comparisons for all labelings. Show that it is isometric to a quadruple in* $\mathbb{E}^3$.

The definitions stated in the this section can be applied to any metric space. However, interesting things happen only for the so-called *geodesic* or at least *length spaces.*

F Length and length spaces

Length. A curve is defined as a continuous map from a real interval $\mathbb{I}$ to a metric space. If $\mathbb{I} = [0,1]$, then the curve is called a path.

1.15. Definition. *Let $\mathcal{X}$ be a metric space and $\alpha\colon \mathbb{I} \to \mathcal{X}$ be a curve. We define the length of α as*

$$\operatorname{length}\alpha := \sup_{t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n} \sum_i |\alpha(t_i) - \alpha(t_{i-1})|.$$

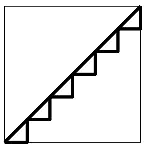
A curve α is called rectifiable *if* $\operatorname{length}\alpha < \infty$*.*

The following theorem is assumed to be known; see [7, 12].

1.16. Theorem. *Length is a lower semi-continuous with respect to the pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\gamma_n\colon \mathbb{I} \to \mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty\colon \mathbb{I} \to \mathcal{X}$; that is, for any fixed $t \in \mathbb{I}$ we have $\gamma_n(t) \to \gamma_\infty(t)$ as $n \to \infty$. Then

❶ $$\varliminf_{n\to\infty} \operatorname{length}\gamma_n \geqslant \operatorname{length}\gamma_\infty.$$



Note that the inequality ❶ might be strict. For example, the diagonal γ_∞ of the unit square can be approximated by stairs-like polygonal curves γ_n with sides parallel to the sides of the square (γ_6 is on the picture). In this case

$$\operatorname{length}\gamma_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length}\gamma_n = 2$$

for any n.

Length spaces. Let $\mathcal{X}$ be a metric space. If for any $\varepsilon > 0$ and any pair of points $x, y \in \mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length}\alpha < |x - y| + \varepsilon,$$

then $\mathcal{X}$ is called a length space and the metric on $\mathcal{X}$ is called a length metric.

Evidently, any geodesic space is a length space.

1.17. Exercise. *Show that any compact length space is geodesic.*

Induced length metric. Directly from the definition, it follows that if $\alpha\colon [0,1] \to \mathcal{X}$ is a path from x to y (that is, $\alpha(0) = x$ and $\alpha(1) = y$), then

$$\operatorname{length}\alpha \geqslant |x - y|.$$

Set

$$\|x - y\| = \inf\{\operatorname{length}\alpha\}$$

where the greatest lower bound is taken for all paths from x to y. It is straightforward to check that $(x, y) \mapsto \|x - y\|$ is an ∞-metric; that is, $(x, y) \mapsto \|x - y\|$ is a metric in the extended positive reals $[0, \infty]$. The metric $\|{*} - {*}\|$ is called the induced length metric.

1.18. Exercise. *Let $\mathcal{X}$ be a complete length space. Show that for any compact subset $K \subset \mathcal{X}$ there is a compact path-connected subset $K' \subset \mathcal{X}$ that contains K.*

1.19. Exercise. *Suppose $(\mathcal{X}, |{*}-{*}|)$ is a complete metric space. Show that $(\mathcal{X}, \|{*}-{*}\|)$ is complete.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x-y|_A = \inf_\alpha \{\,\mathrm{length}\,\alpha\,\},$$

where the greatest lower bound is taken for all paths α from x to y in A. In other words, $|{*}-{*}|_A$ denotes the induced length metric on the subspace A. (The notation $|{*}-{*}|_A$ conflicts with the previously defined notation for distance $|x-y|_{\mathcal{X}}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.)

G Embedding theorem

The following theorem is historically the first remarkable result in Alexandrov geometry. The main part of the following theorem is due to Alexandr Alexandrov [4]. The last part is very difficult; it was proved by Aleksei Pogorelov [13].

1.20. Theorem. *A metric space $\mathcal{X}$ is isometric to the surface of a convex body in the Euclidean space if and only if $\mathcal{X}$ is a geodesic* CBB(0) *space that is homeomorphic to $\mathbb{S}^2$.*

Moreover, $\mathcal{X}$ determines the convex body up to congruence.

The convex body above is a compact convex subset in $\mathbb{E}^3$; we assume that it does not lie in a line but might degenerate to a plane figure, say F. In the latter case, its surface is defined as two copies of F glued along the boundary. For nondegenerate convex body B, its surface is its boundary ∂B equipped with the induced length metric.

The only-if part of the theorem is the simplest; we will give a complete proof of it eventually. The if part will be sketched. We will not touch the last part.

Lecture 2

Angles

A Definition

The angle measure of a hinge $[p\,^x_y]$ is defined as the following limit

$$\measuredangle[p\,^x_y] = \lim_{\bar{x},\bar{y}\to p} \tilde{\measuredangle}(p\,^{\bar{x}}_{\bar{y}}),$$

where $\bar{x} \in\ ]px]$ and $\bar{y} \in\ ]py]$.

Note that if $\measuredangle[p\,^x_y]$ is defined, then

$$0 \leqslant \measuredangle[p\,^x_y] \leqslant \pi.$$

2.1. Exercise. *Suppose that in the above definition, one uses spherical or hyperbolic model angles instead of Euclidean. Show that it does not change the value* $\measuredangle[p\,^x_y]$.

2.2. Exercise. *Give an example of a hinge* $[p\,^x_y]$ *in a metric space with an undefined angle measure* $\measuredangle[p\,^x_y]$.

B Triangle inequality

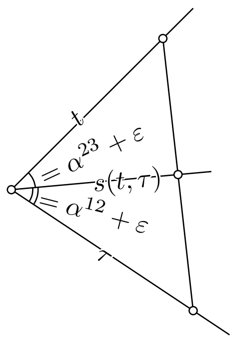
2.3. Proposition. *Let* $[px_1]$, $[px_2]$, *and* $[px_3]$ *be three geodesics in a metric space. Suppose all the angle measures* $\alpha_{ij} = \measuredangle[p\,^{x_i}_{x_j}]$ *are defined. Then*

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23}.$$

Proof. Since $\alpha_{13} \leqslant \pi$, we can assume that $\alpha_{12} + \alpha_{23} < \pi$. Denote by γ_i the unit-speed parametrization of $[px_i]$ from p to x_i. Given any

$\varepsilon > 0$, for all sufficiently small $t, \tau, s \in \mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned}|\gamma_1(t) - \gamma_3(\tau)| \leqslant |\gamma_1(t) - \gamma_2(s)| + |\gamma_2(s) - \gamma_3(\tau)| < \\ < \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha_{12} + \varepsilon)} + \\ + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\alpha_{23} + \varepsilon)} \leqslant\end{aligned}$$



Below we define $s(t, \tau)$ so that for $s = s(t, \tau)$, this chain of inequalities can be continued as follows:

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon.$$

Hence the result follows.

To define $s(t, \tau)$, consider three half-lines $\tilde\gamma_1$, $\tilde\gamma_2$, $\tilde\gamma_3$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde\gamma_1, \tilde\gamma_2) = \alpha_{12} + \varepsilon$, $\measuredangle(\tilde\gamma_2, \tilde\gamma_3) = \alpha_{23} + \varepsilon$, and $\measuredangle(\tilde\gamma_1, \tilde\gamma_3) = \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon$. We parametrize each half-line by the distance from the starting point. Given two positive numbers $t, \tau \in \mathbb{R}_{\geqslant 0}$, let $s = s(t, \tau)$ be the number such that $\tilde\gamma_2(s) \in [\tilde\gamma_1(t)\ \tilde\gamma_3(\tau)]$. Clearly, $s \leqslant \max\{t, \tau\}$, so t, τ, s may be taken sufficiently small. $\square$

2.4. Exercise. *Prove that the sum of adjacent angles is at least π.*

More precisely: suppose two hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ are adjacent; that is, they share side $[pz]$, and the union of two sides $[px]$ and $[py]$ form a geodesic $[xy]$. Show that

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \geqslant \pi$$

whenever each angle on the left-hand side is defined.

Give an example showing that the inequality can be strict.

2.5. Exercise. *Assume that the angle measure of $[q\,{}^p_x]$ is defined. Let γ be the unit speed parametrization of $[qx]$ from q to x. Show that*

$$|p - \gamma(t)| \leqslant |q - p| - t{\cdot}\cos(\measuredangle[q\,{}^p_x]) + o(t).$$

C Alexandrov's lemma

Recall that $[xy]$ denotes a geodesic from x to y; set

$$]xy] = [xy] \setminus \{x\}, \quad [xy[= [xy] \setminus \{y\}, \quad]xy[= [xy] \setminus \{x, y\}.$$

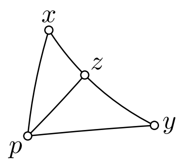
2.6. Lemma. *Let p,x,y,z be distinct points in a metric space such that $z\in\,]xy[$. Then the following expressions for the Euclidean model angles have the same sign:*

(a) $\tilde\measuredangle(x\,{}^{p}_{y})-\tilde\measuredangle(x\,{}^{p}_{z})$,

(b) $\tilde\measuredangle(z\,{}^{p}_{x})+\tilde\measuredangle(z\,{}^{p}_{y})-\pi$.

The same holds for the hyperbolic and spherical model angles, but in the latter case, one has to assume in addition that

$$|p-z|+|p-y|+|x-y|<2{\cdot}\pi.$$



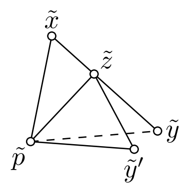
Proof. Consider the model triangle $[\tilde x\tilde p\tilde z]=\tilde\triangle(xpz)$. Take a point $\tilde y$ on the extension of $[\tilde x\tilde z]$ beyond $\tilde z$ so that $|\tilde x-\tilde y|=|x-y|$ (and therefore $|\tilde x-\tilde z|=|x-z|$).

Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:

(i) $\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}]-\tilde\measuredangle(x\,{}^{p}_{y})$,

(ii) $|\tilde p-\tilde y|-|p-y|$,

(iii) $\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}]-\tilde\measuredangle(z\,{}^{p}_{y})$.



Since

$$\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}]=\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde z}]=\tilde\measuredangle(x\,{}^{p}_{z})$$

and

$$\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}]=\pi-\measuredangle[\tilde z\,{}^{\tilde x}_{\tilde p}]=\pi-\tilde\measuredangle(z\,{}^{x}_{p}),$$

the statement follows.

The spherical and hyperbolic cases can be proved the same way. □

2.7. Exercise. *Assume p,x,y,z are as in Alexandrov's lemma. Show that*

$$\tilde\measuredangle(p\,{}^{x}_{y})\geqslant\tilde\measuredangle(p\,{}^{x}_{z})+\tilde\measuredangle(p\,{}^{z}_{y}),$$

with equality if and only if the expressions in (a) and (b) vanish.

D CBB comparison

Note that

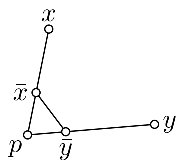
$$p\in\,]xy[\quad\Longrightarrow\quad\tilde\measuredangle(p\,{}^{x}_{y})=\pi.$$

Applying it with Alexandrov's lemma and CBB(0) comparison, we get the following claim and its corollary.

2.8. Claim. *If p,x,y,z are points in a* CBB(0) *such that $p\in\,]xy[$, then*

$$\tilde\measuredangle(x\,{}^{y}_{z})\leqslant\tilde\measuredangle(x\,{}^{p}_{z}).$$

2.9. Exercise. *Let $[p\,{}^{x}_{y}]$ be a hinge in a* CBB(0) *space. Consider the function*



$$f\colon(|p-\bar x|,|p-\bar y|)\mapsto\tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x\in\,]px]$ and $\bar y\in\,]py]$. Show that f is nonincreasing in each argument.

Note that 2.9 implies the following.

2.10. Claim. *For any hinge $[p\,{}^{x}_{y}]$ in a* CBB(0) *space, the angle measure $\measuredangle[p\,{}^{x}_{y}]$ is defined, and*

$$\measuredangle[p\,{}^{x}_{y}]\geqslant\tilde\measuredangle(p\,{}^{x}_{y}).$$

2.11. Exercise. *Let $[p\,{}^{x}_{y}]$ be a hinge in a* CBB(0) *space. Suppose $\measuredangle[p\,{}^{x}_{y}]=0$; show that $[px]\subset[py]$ or $[py]\subset[px]$.*

2.12. Exercise. *Let $[xy]$ be a geodesic in a* CBB(0) *space. Suppose $z\in\,]xy[$ show that there is a unique geodesic $[xz]$ and $[xz]\subset[xy]$.*

2.13. Exercise. *Let $[p\,{}^{x}_{z}]$ and $[p\,{}^{y}_{z}]$ be adjacent hinges in a* CBB(0) *space. Show that*

$$\measuredangle[p\,{}^{x}_{z}]+\measuredangle[p\,{}^{y}_{z}]=\pi.$$

2.14. Exercise. *Let p,x,y in a* CBB(0) *space and $v,w\in\,]xy[$. Show that*

$$\tilde\measuredangle(x\,{}^{y}_{p})=\tilde\measuredangle(x\,{}^{v}_{p})\quad\iff\quad\tilde\measuredangle(x\,{}^{y}_{p})=\tilde\measuredangle(x\,{}^{w}_{p}).$$

E Hinge comparison

Let $[p\,{}^{x}_{y}]$ be a hinge in a CBB(0) space. By 2.11, the angle measure $\measuredangle[p\,{}^{x}_{y}]$ is defined and

$$\measuredangle[p\,{}^{x}_{y}]\geqslant\tilde\measuredangle(p\,{}^{x}_{y}).$$

Further, according to 2.13, we have

$$\measuredangle[p\,{}^{x}_{z}]+\measuredangle[p\,{}^{y}_{z}]=\pi$$

for adjacent hinges $[p\,{}^{x}_{z}]$ and $[p\,{}^{y}_{z}]$ in a CBB(0) space.

The following theorem implies that a geodesic space is CBB(0) if the above conditions hold for all its hinges.

2.15. Theorem. *A geodesic space $\mathcal{L}$ is* CBB(0) *if the following conditions hold.*

(a) For any hinge $[x\,{}^{p}_{y}]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^{p}_{y}]$ is defined and

$$\measuredangle[x\,{}^{p}_{y}] \geqslant \tilde\measuredangle(x\,{}^{p}_{y}).$$

(b) For any two adjacent hinges $[p\,{}^{x}_{z}]$ and $[p\,{}^{y}_{z}]$ in $\mathcal{L}$, we have

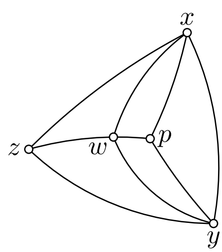
$$\measuredangle[p\,{}^{x}_{z}] + \measuredangle[p\,{}^{y}_{z}] \leqslant \pi.$$

Proof. Consider a point $w \in \,]pz[$ close to p. From *(b)*, it follows that

$$\measuredangle[w\,{}^{x}_{z}] + \measuredangle[w\,{}^{x}_{p}] \leqslant \pi \quad \text{and} \quad \measuredangle[w\,{}^{y}_{z}] + \measuredangle[w\,{}^{y}_{p}] \leqslant \pi.$$

Since $\measuredangle[w\,{}^{x}_{y}] \leqslant \measuredangle[w\,{}^{x}_{p}] + \measuredangle[w\,{}^{y}_{p}]$ (see 2.3), we get

$$\measuredangle[w\,{}^{x}_{z}] + \measuredangle[w\,{}^{y}_{z}] + \measuredangle[w\,{}^{x}_{y}] \leqslant 2\cdot\pi.$$



Applying *(a)*,

$$\tilde\measuredangle(w\,{}^{x}_{z}) + \tilde\measuredangle(w\,{}^{y}_{z}) + \tilde\measuredangle(w\,{}^{x}_{y}) \leqslant 2\cdot\pi.$$

Passing to the limits $w \to p$, we have

$$\tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{y}_{z}) + \tilde\measuredangle(p\,{}^{x}_{y}) \leqslant 2\cdot\pi.$$

□

F Equivalent conditions

The following theorem summarizes 2.8, 2.10, 2.13, 2.15.

2.16. Theorem. *Let $\mathcal{L}$ be a geodesic space. Then the following conditions are equivalent.*

(a) $\mathcal{L}$ is CBB(0).

(b) (adjacent angle comparison) for any geodesic $[xy]$ and point $z \in \,]xy[$, $z \neq p$ in $\mathcal{L}$, we have

$$\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) \leqslant \pi.$$

(c) (point-on-side comparison) for any geodesic $[xy]$ and $z \in \,]xy[$ in $\mathcal{L}$, we have

$$\tilde\measuredangle(x\,{}^{p}_{y}) \leqslant \tilde\measuredangle(x\,{}^{p}_{z}).$$

(d) *(hinge comparison) for any hinge $[x\,{}^{p}_{y}]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^{p}_{y}]$ is defined and*

$$\measuredangle[x\,{}^{p}_{y}] \geqslant \tilde\measuredangle(x\,{}^{p}_{y}).$$

Moreover,

$$\measuredangle[z\,{}^{p}_{y}] + \measuredangle[z\,{}^{p}_{x}] \leqslant \pi$$

for any adjacent hinges $[z\,{}^{p}_{y}]$ and $[z\,{}^{p}_{x}]$.

Moreover, the implications (a)⇒(b)⇒(c)⇒(d) hold in any space, not necessarily geodesic.

2.17. Advanced Exercise. *Construct a geodesic space $\mathcal{X} \notin \mathrm{CBB}(0)$ that meets the following condition: for any 3 points $p, x, y \in \mathcal{X}$ there is a geodesic $[xy]$ such that for any $z \in \,]xy[$*

$$\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) \leqslant \pi.$$

G Function comparison

Real-to-real functions. Choose $\lambda \in \mathbb{R}$. Let $s\colon \mathbb{I} \to \mathbb{R}$ be a locally Lipschitz function defined on an interval $\mathbb{I}$. We say that s is λ-concave if $s'' \leqslant \lambda$, where the second derivative s'' is understood in the sense of distributions.

Equivalently, *s is λ-concave if the function $h\colon t \mapsto s(t) - \lambda\cdot\frac{t^2}{2}$ is concave.* Concavity can be defined via Jensen inequality; that is,

$$h(s\cdot t_0 + (1-s)\cdot t_1) \geqslant s\cdot h(t_0) + (1-s)\cdot h(t_1)$$

for any $t_0, t_1 \in \mathbb{I}$ and $s \in [0,1]$. It could be also defined via the existence of (local) upper support at any point: *for any $t_0 \in \mathbb{I}$ there is a linear function ℓ that (locally) supports h at t_0 from above; that is, $\ell(t_0) = h(t_0)$ and $\ell(t) \geqslant h(t)$ for any t (in a neighborhood of t_0).*

The equivalence of these definitions is assumed to be known. We will also use that *λ-concave functions are one-side differentiable.*

Functions on metric space. A function on a metric space $\mathcal{L}$ will usually mean a *locally Lipschitz real-valued function defined in an open subset of $\mathcal{L}$.* The domain of definition of a function f will be denoted by $\operatorname{Dom} f$.

Let f be a function on a metric space $\mathcal{L}$. We say that f is λ-concave (briefly $f'' \leqslant \lambda$) if for any unit-speed geodesic $\gamma\colon \mathbb{I} \to \operatorname{Dom} f$ the real-to-real function $t \mapsto f\circ\gamma(t)$ is λ-concave.

The following proposition is conceptual — it reformulates a global geometric condition into an infinitesimal condition on distance functions.

2.18. Proposition. *A geodesic space $\mathcal{L}$ is* CBB(0) *if and only if $f'' \leqslant 1$ for any function f of the following type*

$$f\colon x \mapsto \tfrac{1}{2}\cdot|p-x|^2.$$

Proof. Choose a unit-speed geodesic γ in $\mathcal{L}$ and two points $x=\gamma(t_0)$, $y=\gamma(t_1)$ for some $t_0<t_1$. Consider the model triangle $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)$. Let $\tilde\gamma\colon[t_0,t_1]\to\mathbb{E}^2$ be the unit-speed parametrization of $[\tilde x\tilde y]$ from $\tilde x$ to $\tilde y$.

Set

$$\tilde r(t):=|\tilde p-\tilde\gamma(t)|, \qquad\qquad r(t):=|p-\gamma(t)|.$$

Clearly, $\tilde r(t_0)=r(t_0)$ and $\tilde r(t_1)=r(t_1)$. Note that the point-on-side comparison (2.16*c*) is equivalent to

$$\text{❶}\qquad t_0\leqslant t\leqslant t_1 \qquad\Longrightarrow\qquad \tilde r(t)\leqslant r(t)$$

for any γ and $t_0<t_1$.

Set

$$\tilde h(t)=\tfrac12\cdot\tilde r^2(t)-\tfrac12\cdot t^2, \qquad\qquad h=\tfrac12\cdot r^2(t)-\tfrac12\cdot t^2.$$

Note that $\tilde h$ is linear, $\tilde h(t_0)=h(t_0)$ and $\tilde h(t_1)=h(t_1)$. Observe that the Jensen inequality for the function h is equivalent to ❶. Hence the proposition follows. ☐

H Comments

All the discussed statements admit natural generalizations to CBB(κ) spaces. Most of the time the proof is the same with uglier formulas.

For example, the function comparison of CBB(-1) states that $f''\leqslant f$ for any function of the type $f=\cosh\circ\operatorname{dist}_p$. Similarly, the function comparison of CBB(1) states that for any point p, we have $f''\leqslant -f$ for the function $f=-\cos\circ\operatorname{dist}_p$ defined in $\mathrm{B}(p,\pi)$. The meaning of these inequalities is the same — distance functions in CBB(κ) are more concave than distance functions in $\mathbb{M}(\kappa)$. The inequality $f''\leqslant\varphi$ means that for any point p in the domain of definition and any $\varepsilon>0$, there is a neighborhood $U\ni p$ such that $f''\leqslant\varphi(p)+\varepsilon$ in U. Here we assume that f and φ are continuous and defined in open set.

Lecture 3

Surface of convex body

Recall that (for us) a convex body is a compact convex subset in $\mathbb{E}^3$; we assume that it does not lie in a line but it might degenerate to a plane figure.

Suppose B is a nondegenerate convex body; that is, it has nonempty interior. Then the surface of B is defined as its boundary ∂B equipped with the induced length metric.

If a convex body degenerates to a plane convex figure, say F, then its surface is defined as a doubling of F along its boundary; that is, two copies of F glued along the boundary ∂F. Intuitively, one can regard these copies as different sides of F — we live on its surface and to get from one side to the other one has to cross the boundary.

3.1. Exercise. *Show that surface of a convex body is homeomorphic to $\mathbb{S}^2$.*

In this lecture, we will prove that *surface of a convex body is* CBB(0). The latter, together with the exercise, gives the only-if part in the main part of the embedding theorem (1.20).

A Convex polyhedra

Recall that a convex polyhedron is a convex hull of a finite set of points. Extremal points of a convex polyhedron are called its vertices. As for convex bodies, our convex polyhedra might degenerate to a plane polygon, but we assume that it does not belong to a line.

Observe that a surface, say Σ, of a convex polyhedron P admits a triangulation such that each triangle is isometric to a plane triangle. In other words, Σ is a polyhedral surfaces; that is, it is a

2-dimensional manifold with length metric that admits a triangulation such that each triangle is isometric to a solid plane triangle. A triangulation of polyhedral surface will be assumed to satisfy this condition.

The total angle around a vertex p in Σ is defined as the sum of angles at p of all triangles in the triangulation that contain p.

Note that if a point p is not a vertex of P, then

◇ p lies in the interior of a face of P, and its neighborhood in Σ is a piece of plane, or

◇ p lies on an edge, and its neighborhood is two half-planes glued along the boundary.

In both cases, a neighborhood of p in Σ (with the induced length metric) is isometric to an open domain of the plane.

3.2. Claim. *Let Σ be the surface of a convex polyhedron P. Then, the total angle around a vertex in Σ cannot exceed $2{\cdot}\pi$.*

In the proof, we will use the following exercise which is the triangle inequality for angles (or the spherical triangle inequality); it easily follows from 2.3.

3.3. Exercise. *Let w_1, w_2, w_3 be unit vectors in $\mathbb{E}^3$. Denote by $\theta_{i,j}$ the angle between the vectors v_i and v_j. Show that*

$$\theta_{1,3} \leqslant \theta_{1,2} + \theta_{2,3}$$

and in case of equality, the vectors w_1, w_2, w_3 lie in a plane.

Proof. Consider the intersection of P with a small sphere centered at p; it is a convex spherical polygon, say F. Applying rescaling we may assume that the sphere has unit radius. We need to show that the perimeter of F does not exceed $2{\cdot}\pi$.

Note that F lies in a hemisphere, say H. Moreover, there is a decreasing sequence

$$H = H_0 \supset H_1 \supset \dots \supset H_n = F,$$

such that H_{i+1} is obtained from H_i by cutting along a chord.

By 3.3, we have

$$2{\cdot}\pi = \operatorname{perim} H = \operatorname{perim} H_0 \geqslant \operatorname{perim} H_1 \geqslant \ldots \geqslant \operatorname{perim} H_n = \operatorname{perim} F$$

— hence the result. □

A vertex of a triangulation of a polyhedral surface is called essential if the total angle around it is not $2{\cdot}\pi$.

3.4. Exercise. *Show that any vertex of a polyhedron is an essential vertex of its surface; that is, the inequality in the claim is strict.*

3.5. Exercise. *Show that geodesics on a surface of convex polyhedron do not pass thru its essential vertices.*

B Surface of convex polyhedron

Let p be a vertex of a polyhedron. If θ_p is the total angle around p, then the value $2{\cdot}\pi - \theta_p$ is called the curvature of the polyhedral surface at p; if p is not a vertex, then its curvature is defined to be zero.

3.6. Exercise. *Assume that the surface of a nondegenerate tetrahedron T has curvature π at each of its vertices. Show that*

(a) all faces of T are congruent;

(b) the line passing thru midpoints of opposite edges of T intersects these edges at right angles.

Note that the claim above says that *surface of a convex polyhedron has nondegenerate curvature.* However this definition works only for polyhedral surfaces. Now we show that it agrees with the CBB(0) definition.

3.7. Proposition. *A polyhedral surface with nonnegative curvature at each vertex is* CBB(0)*.*

Proof. Denote the surface by Σ. By 2.18, it is sufficient to check that $\operatorname{dist}_p^2 \circ \gamma$ is 1-concave for any geodesic γ and a point p in Σ.

We can assume that p is not a vertex; the vertex case can be done by approximation. Further, by 3.5, we may assume that γ does not contain vertices.

Given a point $x = \gamma(t_0)$, choose a geodesic $[px]$. Again, by 3.5, $[px]$ does not contain vertices. Therefore a small neighborhood of $U \supset [px]$ can be unfolded on a plane; denote this map by $z \mapsto \tilde{z}$. Note that this way we map part of γ in U to a line segment. Let

$$\tilde{f}(t) := \tfrac{1}{2}{\cdot}\operatorname{dist}_{\tilde{p}}^2 \circ \tilde{\gamma}(t).$$

Note that $\tilde{f}(t_0) \geqslant f(t_0)$. Further, since the unfolding $z \mapsto \tilde{z}$ preserves lengths of curves, we get $\tilde{f}(t) \geqslant f(t)$ if t is sufficiently close to t_0. That is, $\tilde{f}$ is a local upper support of f at t_0. Evidently, $\tilde{f}'' \equiv 1$; therefore $f'' \leqslant 1$. It remains to apply 2.18. □

3.8. Exercise. *Prove the converse to the proposition; that is, show that if a poyhedral surface is* CBB(0)*, then it has nonnegative curvature at each vertex.*

C Surface of convex body

3.9. Lemma. *Let $K_1, K_2, \dots$ be a sequence of convex bodies that converges to K_∞ in the sense of Hausdorff. Assume K_∞ is nondegenerate. Then the surface of K_n converges to the surface of K_∞ in the sense of Gromov–Hausdorff.*

In the following proof we use that the closest-point projection form the Euclidean space to a convex body is short; that is, distance-nonexpanding [11, 12.3].

Proof. Without loss of generality, we may assume that

$$\overline{\mathrm{B}}(0,r) \subset K_\infty \subset \overline{\mathrm{B}}(0,1)$$

for some $r > 0$. Note that there is a sequence $\varepsilon_n \to 0$ such that

$$K_n \subset (1+\varepsilon_n)\cdot K_\infty \quad \text{and} \quad K_\infty \subset (1+\varepsilon_n)\cdot K_n$$

for each n.

Given $x \in K_n$, denote by $g_n(x)$ the closest-point projection of $(1+\varepsilon_n)\cdot x$ to K_∞. Similarly, given $x \in K_\infty$, denote by $h_n(x)$ the closest point projection of $(1+\varepsilon_n)\cdot x$ to K_n. Note that

$$|g_n(x) - g_n(y)| \leqslant (1+\varepsilon_n)\cdot|x-y|$$

and

$$|h_n(x) - h_n(y)| \leqslant (1+\varepsilon_n)\cdot|x-y|.$$

Denote by Σ_∞ and Σ_n the surface of K_∞ and K_n respectively. The above inequlities imply

$$|g_n(x) - g_n(y)|_{\Sigma_\infty} \leqslant (1+\varepsilon_n)\cdot|x-y|_{\Sigma_n}$$

for any $x, y \in \Sigma_n$, and

$$|h_n(x) - h_n(y)|_{\Sigma_n} \leqslant (1+\varepsilon_n)\cdot|x-y|_{\Sigma_\infty}.$$

for any $x, y \in \Sigma_\infty$. Therefore, g_n is a δ_n-isometry $\Sigma_n \to \Sigma_\infty$ for a sequence $\delta_n \to 0$. □

3.10. Proposition. *The surface of a nondegenerate convex body is* CBB(0)*.*

Note that any convex body is a Hausdorff limit of a sequence of convex polyhedra. Therefore, the proposition follows from 3.7, 3.9, and the following claim.

3.11. Claim. *A Gromov–Hausdorff limit of* CBB(0) *spaces is* CBB(0).

Despite its simplicity, this claim is the main source of applications of Alexandrov geometry.

Proof. Let $\mathcal{L}_\infty$ be Gromov–Hausdorff limit of CBB(0) spaces $\mathcal{L}_1, \mathcal{L}_2, \dots$

Choose a quadruple of points p, x, y, z in $\mathcal{L}_\infty$. From convergence we may choose a sequence of quadruples p_n, x_n, y_n, z_n in $\mathcal{L}_n$ that converge to p, x, y, z; in particular, each of six distances between pairs of p_n, x_n, y_n, z_n converges to the corresponding distance between the pair of p, x, y, z. By CBB(0) comparison in $\mathcal{L}_n$,

$$\tilde\measuredangle(p_n\,{}^{x_n}_{y_n}) + \tilde\measuredangle(p_n\,{}^{y_n}_{z_n}) + \tilde\measuredangle(p_n\,{}^{z_n}_{x_n}) \leqslant 2\cdot\pi.$$

Passing to the limit we get

$$\tilde\measuredangle(p\,{}^{x}_{y}) + \tilde\measuredangle(p\,{}^{y}_{z}) + \tilde\measuredangle(p\,{}^{z}_{x}) \leqslant 2\cdot\pi.$$

□

Recall that surface of a degenerate convex body is defined as its doubling. More precisely, suppose F is a convex plane figure. Consider product space $F \times \{0,1\}$ with semimetric defined by

$$|(x,i)-(y,j)| = \begin{cases} |x-y| & \text{if } i=j \\ \inf\{\,|x-z|+|y-z| \ :\ z\in\partial F\,\} & \text{if } i\neq j \end{cases}$$

Then the corresponding metric space is the doubling of F along its boundary.

3.12. Exercise. *Suppose $F_1, F_2, \dots$ is a sequence of convex plane figures that converges to F_∞ in the sense of Hausdorff. Show that doublings of F_n converge to the doubling of F_∞ in the sense of Gromov–Hausdorff.*

Conclude that surfaces of degenerate convex bodies are CAT(0).

Note that 3.10 and 3.12 imply that *surface of a convex body is* CBB(0)*;* so the only-if part in the main part of the embedding theorem (1.20) is proved.

Lecture 4

Alexandrov embedding theorem

We will prove the Cauchy theorem, and then modify it to prove the Alexandrov uniqueness theorem. Further, we sketch a proof of the Alexandrov embedding theorem.

A Cauchy theorem

Recall that *surfaces* of convex polyhedrons are considered with the induced length metric..

4.1. Theorem. *Let K and K' be two non-degenerate convex polyhedrons in $\mathbb{E}^3$; denote their surfaces by P and P'. Suppose there is an isometry $P \to P'$ that sends each face of K to a face of K'. Then K is congruent to K'; moreover the isometry $P \to P'$ can be extended to a motion of $\mathbb{E}^3$ that maps K to K'.*

Proof. Consider the graph Γ formed by the edges of K; the edges of K' form the same graph.

For an edge e in Γ, denote by α_e and α'_e the corresponding dihedral angles in K and K' respectively. Mark e by plus if $\alpha_e < \alpha'_e$ and by minus if $\alpha_e > \alpha'_e$.

Now remove from Γ everything that was not marked; that is, leave only the edges marked by $(+)$ or $(-)$ and their endpoints.

Note that the theorem follows if Γ is an empty graph; assume the contrary.

The graph Γ is embedded into P, which is homeomorphic to the sphere. In particular, the edges coming from one vertex have a natural cyclic order. Given a vertex v of Γ, count the *number of sign changes* around v; that is, the number of consequent pairs edges with different signs.

4.2. Local lemma. *For any vertex of Γ the number of sign changes is at least 4.*

In other words, at each vertex of Γ, one can choose 4 edges marked by $(+)$, $(-)$, $(+)$, $(-)$ in the same cyclical order. Note that the local lemma contradicts the following.

4.3. Global lemma. *Let Γ be a nonempty subgraph of the graph formed by the edges of a convex polyhedron. Then it is impossible to mark all of the edges of Γ by $(+)$ or $(-)$ such that the number of sign changes around each vertex of Γ is at least 4.*

It remains to prove these two lemmas. $\square$

B Local lemma

Next lemma is the main ingredient in our proof of the local lemma.

4.4. Arm lemma. *Assume that $A = [a_0a_1 \dots a_n]$ is a convex polygon in $\mathbb{E}^2$ and $A' = [a'_0a'_1 \dots a'_n]$ be a polygonal line in $\mathbb{E}^3$ such that*

$$|a_i - a_{i+1}| = |a'_i - a'_{i+1}|$$

for any $i \in \{0, \dots, n-1\}$ and

$$\measuredangle a_i \leqslant \measuredangle a'_i$$

for each $i \in \{1, \dots, n-1\}$. Then

$$|a_0 - a_n| \leqslant |a'_0 - a'_n|$$

and equality holds if and only if A is congruent to A'.

One may view the polygonal lines $[a_0a_1 \dots a_n]$ and $[a'_0a'_1 \dots a'_n]$ as a robot's arm in two positions. The arm lemma states that when the arm opens, the distance between the shoulder and tip of a finger increases.

4.5. Exercise. *Show that the arm lemma does not hold if instead of the convexity, one only the local convexity; that is, if you go along the polygonal line $a_0a_1 \dots a_n$, then you only turn left.*

4.6. Exercise. *Suppose $A = [a_1 \dots a_n]$ and $A' = [a'_1 \dots a'_n]$ be non-congruent convex plane polygons with equal corresponding sides. Mark* *each vertex a_i with plus (minus) if the interior angle of A at a_i is smaller (respectively bigger) than the interior angle of A' at a_i'. Show that there are at least 4 sign changes around A.*

Give an example showing the statement does not hold without assuming convexity.

Proof. We will view $\mathbb{E}^2$ as the xy-plane in $\mathbb{E}^3$; so both A and A' lie in $\mathbb{E}^3$. Let a_m be the vertex of A that lies on the maximal distance to the line (a_0a_n).

Let us shift indexes of a_i and a_i' down by m, so that

$$\begin{array}{lllll} a_{-m} := a_0, & \dots & a_0 := a_m, & \dots & a_k := a_n, \\ a'_{-m} := a'_0, & \dots & a'_0 := a'_m, & \dots & a'_k := a'_n, \end{array}$$

where $k = n - m$. (Here the symbol ":=" means an assignment as in programming.)

Without loss of generality, we may assume that

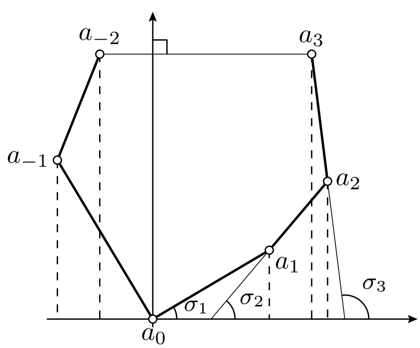
- ◇ $a_0 = a_0'$ and they both coincide with the origin $(0,0,0) \in \mathbb{E}^3$;
- ◇ all a_i lie in the xy-plane and the x-axis is parallel to the line $(a_{-m}a_k)$;
- ◇ the angle $\measuredangle a_0'$ lies in xy-plane and contains the angle $\measuredangle a_0$ inside and the directions to a'_{-1},a_{-1}, a_1 and a_1' from a_0 appear in the same cyclic order.

Denote by x_i and x_i' the projections of a_i and a_i' to the x-axis. We can assume in addition that $x_k \geqslant x_{-m}$. In this case,

$$|a_k - a_{-m}| = x_k - x_{-m}.$$

Since the projection is a distance non-expanding, we also have

$$|a_k' - a_{-m}'| \geqslant x_k' - x_{-m}'.$$



Therefore it is sufficient to show that

$$x_k' - x_{-m}' \geqslant x_k - x_{-m}.$$

The latter holds if

❶ $\quad x_i' - x_{i-1}' \geqslant x_i - x_{i-1}.$

for each i. It remains to prove ❶.

Let us assume that $i > 0$; the case $i \leqslant 0$ is similar. Denote by σ_i (σ'_i) the angle between the vector $w_i = a_i - a_{i-1}$ (respectively $w'_i = a'_i - a'_{i-1}$) and the x-axis. Note that

❷
$$\begin{aligned} x_i - x_{i-1} &= |a_i - a_{i-1}|\cdot \cos\sigma_i, \\ x'_i - x'_{i-1} &= |a_i - a_{i-1}|\cdot \cos\sigma'_i \end{aligned}$$

for each $i > 0$. By construction $\sigma_1 \geqslant \sigma'_1$. Note that $\measuredangle(w_{i-1}, w_i) = = \pi - \measuredangle a_i$. From convexity of $[a_1 a_1 \dots a_i]$, we have

$$\sigma_i = \sigma_1 + (\pi - \measuredangle a_1) + \dots + (\pi - \measuredangle a_i)$$

for any $i > 0$. Since $\measuredangle(w'_{i-1}, w'_i) = \pi - \measuredangle a'_i$, applying 3.3 several times, we get

$$\sigma'_i \leqslant \sigma'_1 + (\pi - \measuredangle a'_1) + \dots + (\pi - \measuredangle a'_i).$$

Since $\measuredangle a'_j \geqslant \measuredangle a_j$ for each j, we get $\sigma'_i \leqslant \sigma_i$, and therefore

$$\cos\sigma'_i \geqslant \cos\sigma_i$$

Applying ❷, we get ❶.

In the case of equality, we have $\sigma_i = \sigma'_i$, which implies $\measuredangle a_i = \measuredangle a'_i$ for each i. This also implies that all a'_i lie in xy-plane. The latter easily follows from the equality case in 3.3. □

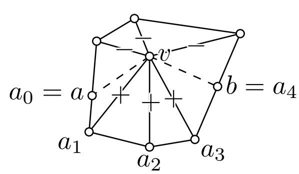
Proof of the local lemma (4.2). Assume that the local lemma does not hold at the vertex v of Γ. Cut from P a small pyramid Δ with the vertex v. One can choose two points a and b on the base of Δ so that on one side of the segments $[va]$ and $[vb]$ we have only pluses and on the other side only minuses.

The base of Δ has two polygonal lines with ends at a and b. Choose the one that has only pluses; denote it by $a_0 a_1 \dots a_n$; so $a = a_0$ and $b = a_n$. Denote by $a'_0 a'_1 \dots a'_n$ the corresponding line in P'; let $a' = a'_0$ and $b' = a'_n$.

Since each marked edge passing thru a_i has a (+) on it or nothing, we have

$$\measuredangle a_i \leqslant \measuredangle a'_i$$

for each i.



4.7. Exercise. *Prove the last statement.*

By the construction we have $|a_i - a_{i-1}| = |a'_i - a'_{i-1}|$ for all i. By the arm lemma (4.4), we get

❸ $|a - b| \leqslant |a' - b'|$.

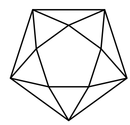
Swap K and K' and repeat the same construction for a plane passing thru a' and b'. We get

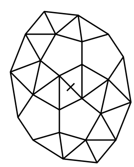
❹ $|a - b| \geqslant |a' - b'|$.

The claims ❸ and ❹ together imply $|a-b| = |a'-b'|$. The equality case in the arm lemma implies that no edge at v is marked; that is, v is not a vertex of Γ — a contradiction. □

From the proof, it follows that the local lemma is indeed local — it works for two noncongruent convex polyhedral angles with equal corresponding faces. Use this observation to solve the following exercise.

4.8. Exercise. *Consider two polyhedral discs in $\mathbb{E}^3$ glued from regular polygons by the rule on the diagrams. Assume that each disc is part of a surface of a convex polyhedron.*





(a) *The first configuration is rigid; that is, one can not fix the position of the pentagon and continuously move the remaining 5 vertices in a new position so that each triangle moves by a one-parameter family of isometries of $\mathbb{E}^3$.*

(b) *Show that the second configuration has a rotational symmetry with the axis passing thru the midpoint of the marked edge.*

C Global lemma

The proof of the global lemma is based on counting the sign changes in two ways; first while moving around each vertex of Γ and second while moving around each of the regions separated by Γ on the surface P. If two edges are adjacent at a vertex, then they are also adjacent in a region. The converse is true as well. Therefore, both countings give the same number.

It is instructive to do the next exercise before diving into the proof.

4.9. Exercise. *Try to mark the edges of an octahedron by pluses and minuses such that there would be 4 sign changes at each vertex.*

Show that this is impossible.

Proof of 4.3. We can assume that Γ is connected; that is, one can get from any vertex to any other vertex by walking along edges. (If not, pass to a connected component of Γ.)

Denote by k and l the number of vertices and edges in Γ. Denote by m the number of *regions* that Γ cuts from P. Since Γ is connected, each region is homeomorphic to an open disc.

4.10. Exercise. *Prove the last statement.*

Now we can apply Euler's formula

❶ $$k - l + m = 2.$$

Denote by s the total number of sign changes in Γ for all vertices. By the local lemma (4.2), we have

❷ $$4{\cdot}k \leqslant s.$$

Let us get an upper bound on s by counting the number of sign changes when you go around each region. Denote by m_n the number of regions bounded by n edges; if an edge appears twice when it is counted twice. Note that each region is bounded by at least 3 edges; therefore

❸ $$m = m_3 + m_4 + m_5 + \ldots$$

Counting edges and using the fact that each edge belongs to exactly two regions, we get

$$2{\cdot}l = 3{\cdot}m_3 + 4{\cdot}m_4 + 5{\cdot}m_5 + \ldots$$

Combining this with Euler's formula (❶), we get

❹ $$4{\cdot}k = 8 + 2{\cdot}m_3 + 4{\cdot}m_4 + 6{\cdot}m_5 + 8{\cdot}m_6 + \ldots$$

Observe that the number of sign changes in n-gon regions has to be even and $\leqslant n$. Therefore

❺ $$s \leqslant 2{\cdot}m_3 + 4{\cdot}m_4 + 4{\cdot}m_5 + 6{\cdot}m_6 + \ldots$$

Clearly, ❷ and ❺ contradict ❹. □

D Uniqueness

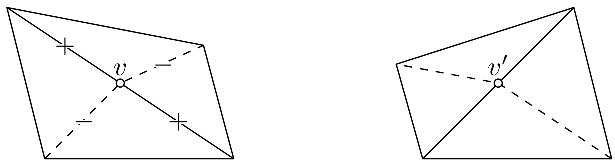
Alexandrov's uniqueness theorem states that the conclusion of the Cauchy theorem (4.1) still holds without the face-to-face assumption.

4.11. Theorem. *Any two convex polyhedrons in $\mathbb{E}^3$ with isometric surfaces are congruent.*

Moreover, any isometry between surfaces of convex polyhedrons can be extended to an isometry of the whole $\mathbb{E}^3$*.*

Needed modifications in the proof of 4.1. Suppose $\iota\colon P \to P'$ be an isometry between surfaces of K and K'. Mark in P all the edges of K and all the inverse images of edges in K'; further, these will be called fake edges. The marked lines divide P into convex polygons, and the restriction of ι to each polygon is a rigid motion. These polygons play the role of faces in the proof above.

A vertex of the obtained graph can be a vertex of K, or it can be a fake vertex; that is, it might be an intersection of an edge and a fake edge.



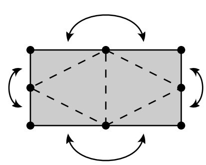
For the first type of vertex, the local lemma can be proved the same way. For a fake vertex v, it is easy to see that both parts of the edge coming thru v are marked with minus while both of the fake edges at v are marked with plus. Therefore, the local lemma holds for the fake vertices as well.

What remains in the proof needs no modifications. □

4.12. Exercise. *Let K be a convex polyhedron in $\mathbb{E}^3$; denote by P its surface. Show that each isometry $\iota\colon P \to P$, can be extended to an isometry of* $\mathbb{E}^3$*.*

E Existence

4.13. Theorem. *A polyhedral metric on the sphere is isometric to the surface of a convex polyhedron (possibly degenerate to a flat polygon) if and only if it has nonnegative curvature at each point.*



By 4.11, a convex polyhedron is completely defined by the intrinsic metric of its surface. By 4.13, it follows that knowing the metric we could find the position of the edges. However, in practice, it is not easy to do.

For example, the surface glued from a rectangle as shown on the diagram defines a tetrahedron. Some of the glued lines appear inside facets of the tetrahedron and some edges (dashed lines) do not follow the sides of the rectangle.

Space of polyhedrons. Let us denote by $\mathbf{K}$ the space of all convex polyhedrons in the Euclidean space, including polyhedrons that degenerate to a plane polygon. Polyhedra in $\mathbf{K}$ will be considered up to a motion of the space, and the whole space $\mathbf{K}$ will be considered with Hausdorff distance up to a motion of the space; that is, the distance between K and K' is the exact lower bound on Hausdorff distance from $\iota(K)$ to K', where ι is arbitrary motion of $\mathbb{E}^3$.

Further, denote by $\mathbf{K}_n$ the polyhedrons in $\mathbf{K}$ with exactly n vertices. Since any polyhedron has at least 3 vertices, the space $\mathbf{K}$ admits a subdivision into a countable number of subsets $\mathbf{K}_3, \mathbf{K}_4, \dots$

Space of polyhedral metrics. The space of polyhedral metrics on the sphere with nonnegative curvature will be denoted by $\mathbf{P}$. The metrics in $\mathbf{P}$ will be considered up to an isometry, and the whole space $\mathbf{P}$ will be equipped with the topology induced by the Gromov–Hausdorff metric.

The subset of $\mathbf{P}$ of all metrics with exactly n essential vertices will be denoted by $\mathbf{P}_n$. It is easy to see that any metric in $\mathbf{P}$ has at least 3 essential vertices. Therefore $\mathbf{P}$ is subdivided into countably many subsets $\mathbf{P}_3, \mathbf{P}_4, \dots$

From a polyhedron to its surface. By 3.7, passing from a polyhedron to its surface defines a map

$$\iota\colon \mathbf{K} \to \mathbf{P}.$$

By 3.4, the number of vertices of a polyhedron is equal to the number of essential vertices on its surface. In other words, $\iota(\mathbf{K}_n) \subset \mathbf{P}_n$ for any $n \geqslant 3$.

Using the introduced notation, we can unite 4.11 and 4.13 in the following more exact statement.

4.14. Reformulation. *For any integer $n \geqslant 3$, the map ι induces a bijection between $\mathbf{K}_n$ and $\mathbf{P}_n$.*

The proof is based on a construction of a one-parameter family of polyhedrons that starts at an arbitrary polyhedron and ends at a polyhedron with its surface isometric to the given one. This type of argument is called the *continuity method*; it is often used in the theory of differential equations.

Sketch. By 4.11, the map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is injective. Let us prove that it is surjective.

4.15. Lemma. *For any integer $n \geqslant 3$, the space $\mathbf{P}_n$ is connected.*

The proof of this lemma is not complicated, but it requires ingenuity; it can be done by the direct construction of a one-parameter family of metrics in $\mathbf{P}_n$ that connects two given metrics. Such a family can be obtained by a sequential application of the following construction and its inverse.

Let $P \in \mathbf{P}_n$. Suppose v and w are essential vertices in P. Let us cut P along a geodesic from v to w. Note that the geodesic cannot pass thru an essential vertex of P. Further, note that there is a three-parameter family of patches that can be used to patch the cut so that the obtained metric remains in $\mathbf{P}_n$; in particular, the obtained metric has exactly n essential vertices (after the patching, the vertices v and w may become inessential).

4.16. Lemma. *The map $\iota\colon \mathbf{K}_n \to \mathbf{P}_n$ is open, that is, it maps any open set in $\mathbf{K}_n$ to an open set in $\mathbf{P}_n$.*

In particular, for any $n \geqslant 3$, the image $\iota(\mathbf{K}_n)$ is open in $\mathbf{P}_n$.

This statement is very close to the so-called *invariance of domain theorem*; the latter states that a continuous injective map between manifolds of the same dimension is open.

Recall that ι is injective. The proof of the invariance of domain theorem can be adapted to our case since both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n - 6)$-dimensional and both look like manifolds, altho, formally speaking, they are *not* manifolds. In a more technical language, $\mathbf{K}_n$ and $\mathbf{P}_n$ have the natural structure of $(3{\cdot}n - 6)$-dimensional *orbifolds*, and the map ι respects the *orbifold structure*.

We will only show that both spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ are $(3{\cdot}n - 6)$-dimensional.

Choose $K \in \mathbf{K}_n$. Note that K is uniquely determined by the $3{\cdot}n$ coordinates of its n vertices. We can assume that the first vertex is the origin, the second has two vanishing coordinates and the third has one vanishing coordinate; therefore, all polyhedrons in $\mathbf{K}_n$ that lie sufficiently close to K can be described by $3{\cdot}n - 6$ parameters. If K has no symmetries, then this description can be made one-to-one; in this case, a neighborhood of K in $\mathbf{K}_n$ is a $(3{\cdot}n - 6)$-dimensional manifold. If K has a nontrivial symmetry group, then this description is not one-to-one but it does not have an impact on the dimension of $\mathbf{K}_n$.

The case of polyhedral metrics is analogous. We need to construct a subdivision of the sphere into plane triangles using only essential vertices. By Euler's formula, there are exactly $3{\cdot}n-6$ edges in this subdivision. Note that the lengths of edges completely describe the metric, and slight changes in these lengths produce a metric with the same property. Again, if P has no symmetries, then this description is one-to-one.

4.17. Lemma. *The map $\iota\colon \mathbf{K}_n\to\mathbf{P}_n$ is closed; that is, the image of a closed set in $\mathbf{K}_n$ is closed in $\mathbf{P}_n$.*

In particular, for any $n\geqslant 3$, the set $\iota(\mathbf{K}_n)$ is closed in $\mathbf{P}_n$.

Choose a closed set Z in $\mathbf{K}_n$. Denote by $\bar Z$ the closure of Z in $\mathbf{K}$; note that $Z=\mathbf{K}_n\cap\bar Z$. Assume $K_1,K_2,\dots\in Z$ is a sequence of polyhedrons that converges to a polyhedron $K_\infty\in\bar Z$. By 3.9, $\iota(K_n)$ converges to $\iota(K_\infty)$ in $\mathbf{P}$. In particular, $\iota(\bar Z)$ is closed in $\mathbf{P}$.

Since $\iota(\mathbf{K}_n)\subset\mathbf{P}_n$ for any $n\geqslant 3$, we have $\iota(Z)=\iota(\bar Z)\cap\mathbf{P}_n$; that is, $\iota(Z)$ is closed in $\mathbf{P}_n$.

Summarizing, $\iota(\mathbf{K}_n)$ is a nonempty closed and open set in $\mathbf{P}_n$, and $\mathbf{P}_n$ is connected for any $n\geqslant 3$. Therefore, $\iota(\mathbf{K}_n)=\mathbf{P}_n$; that is, $\iota\colon\mathbf{K}_n\to\mathbf{P}_n$ is surjective. □

F Approximation

By now, the embedding theorem is proved for polyhedral metrics on the sphere. The general case is done by approximation, using the following statement.

4.18. Proposition. *Let $K_1,K_2,\dots$ be a sequence of convex bodies that converge to K_∞ in the sense of Hausdorff. Then the surface of K_n converges to the surface of K_∞ in the sense of Gromov–Hausdorff.*

If K_∞ is nondegenerate, then the statement follows from 3.9. The degenerate case is left as an exercise.

Let $\mathcal{X}_\infty$ be a geodesic CBB(0) space that is homeomorphic to $\mathbb{S}^2$. Suppose that $\mathcal{X}_\infty$ is a Gromov–Hausdorff limit of a sequence of spheres with polyhedral metrics $\mathcal{X}_1,\mathcal{X}_2,\dots$ By 4.13, there is a sequence of convex polyhedra $K_1,K_2,\dots$ with surfaces isometric to $\mathcal{X}_1,\mathcal{X}_2,\dots$, respectively. Note that $\operatorname{diam}K_n\leqslant\operatorname{diam}\mathcal{X}_n$ for any n. Therefore we can assume that all polyhedra $K_1,K_2,\dots$ lie in a closed ball of sufficiently large radius.

Applying Blaschke selection theorem, we can pass to a subsequence of $K_1,K_2,\dots$ that converges in the sense of Hausdorff; denote its limit by K_∞. By 4.18 the surface of K_∞ is isometric to $\mathcal{X}_\infty$.

Therefore it remains to prove the following lemma.

4.19. Lemma. *Let $\mathcal{X}$ be a geodesic* CBB(0) *space that is homeomorphic to $\mathbb{S}^2$. Then there is a sphere with polyhedral metrics $\mathcal{X}'$ that is arbitrarily close to $\mathcal{X}$ in the sense of Gromov–Hausdorff.*

Idea behind the proof. Suppose we can triangulate $\mathcal{X}_\infty$ by small geodesic triangles; that is, we can choose a finite set of points $p_1, \dots, p_n \in \in \mathcal{X}_\infty$ and some geodesics $[p_i p_j]$ that cut $\mathcal{X}_\infty$ into regions of small diameter bounded by geodesic triangles $[p_i p_j p_k]$. (The actual proof constructs a triangulation with a weaker property.)

Observe that total angle around each p_i cannot exceed $2{\cdot}\pi$. That is, suppose $p_{j_1}, \dots, p_{j_k}$ are points connected to p_i by geodesics. Assume that they are ordered in the natural cyclic order. Then

$$\measuredangle[p_i\,{}^{p_{j_1}}_{p_{j_2}}] + \dots + \measuredangle[p_i\,{}^{p_{j_{k-1}}}_{p_{j_k}}] + \measuredangle[p_i\,{}^{p_{j_k}}_{p_{j_1}}] \leqslant 2{\cdot}\pi.$$

By comparison, we get

❶ $$\tilde\measuredangle(p_i\,{}^{p_{j_1}}_{p_{j_2}}) + \dots + \tilde\measuredangle(p_i\,{}^{p_{j_{k-1}}}_{p_{j_k}}) + \tilde\measuredangle(p_i\,{}^{p_{j_k}}_{p_{j_1}}) \leqslant 2{\cdot}\pi.$$

Now let us exchange each triangle by its model triangle. That is, consider a model triangle for each region in the subdivision of $\mathcal{X}$ and glue them together by the same rule. By ❶, the obtained polyhedral surface $\mathcal{X}'$ has nonnegative curvature. It remains to show that this way we can produce $\mathcal{X}'$ arbitrarily close to $\mathcal{X}$.

Denote by $p_i \to p_i'$ the natural map; it takes p_i in $\mathcal{X}$ and returns the corresponding point in $\mathcal{X}'$. Observe that

❷ $$|p_i' - p_j'|_{\mathcal{X}'} \leqslant |p_i - p_j|_{\mathcal{X}}.$$

Indeed, choose a geodesic γ from p_i to p_j. Let $p_i = x_0, x_1, \dots, x_n = p_j$ be the points of intersections of γ with the edges of the triangulation listed as they appear on γ. For each i, denote by x_i' the corresponding point in $\mathcal{X}'$. By comparison, we get

$$|x_k' - x_{k-1}'|_{\mathcal{X}'} \leqslant |x_k - x_{k-1}|_{\mathcal{X}}.$$

for each k. Therefore, ❷ follows.

Suppose $\varepsilon > 0$ is small, the points $p_1, \dots, p_n$ form an ε-net in $\mathcal{X}$, all edges of the triangulation are smaller than ε and

❸ $$|p_i' - p_j'|_{\mathcal{X}'} \geqslant |p_i - p_j|_{\mathcal{X}} - 100{\cdot}\varepsilon.$$

Then, together with the inequality above it proves that the lemma.

Note that the sides of the model triangles are local geodesics in $\mathcal{X}'$, but not necessarily geodesic; that is they do not have to be length-minimizing. Now, let us make another unjustified assumption: *Suppose that the sides of model triangles in $\mathcal{X}'$ are geodesics.* (The actual proof does not use this assumption.)

Choose a geodesic γ' from p_i' to p_j' in $\mathcal{X}'$. Note that γ' visits each triangle in the triangulation of $\mathcal{X}'$ at most once.

Let $p_i' = x_0', x_1', \dots, x_n' = p_j'$ be the points of intersections of γ' with the edges of the triangulation listed from p_i' to p_j'. For each i, denote by x_i the corresponding point in $\mathcal{X}$. Let Δ_k' be the triangle that contains arc $[x_{k-1}'x_k']$ of γ' and Δ_k the corresponding triangle in $\mathcal{X}$. Note that

$$|x_k' - x_{k-1}'|_{\mathcal{X}'} \geqslant |x_k - x_{k-1}|_{\mathcal{X}} - \varepsilon{\cdot}K(\Delta_k), \quad \text{❹}$$

where $K(\Delta_k)$ denotes the access of Δ_k; that is, the sum of its internal angles minus π.

Euler's formula and ❶ imply that the sum of all accesses is at most $4{\cdot}\pi$. Therefore, summing up ❹, we get

$$|p_i' - p_j'|_{\mathcal{X}'} \geqslant |p_i - p_j|_{\mathcal{X}} - 4{\cdot}\pi{\cdot}\varepsilon.$$

Whence ❸ follows. □

G Comments

This lecture contains selected material from Alexandrov's book [2].

In Euclid's Elements, solids were called equal if the same holds for their faces, but no proof was given. Adrien-Marie Legendre became interested in this problem towards the end of the 18th century. He discussed it with his colleague Joseph-Louis Lagrange, who suggested this problem to Augustin-Louis Cauchy in 1813; soon he proved it [8]. This theorem is included in many popular books [1, 9, 15].

The observation that the face-to-face condition can be removed was made by Alexandr Alexandrov [3].

Arm lemma. Original Cauchy's proof [8] also used a version of the arm lemma, but its proof contained a small mistake (corrected in one century).

Our proof of the arm lemma is due to Stanisław Zaremba. This and a couple of other proofs can be found in the letters between him and Isaac Schoenberg [14].

The following variation of the arm lemma makes sense for nonconvex spherical polygons. It is due to Viktor Zalgaller [18]. It can be used instead of the standard arm lemma.

4.20. Another arm lemma. *Let $A = [a_1 \dots a_n]$ and $A' = [a_1' \dots \dots a_n']$ be two spherical n-gons (not necessarily convex). Assume that A lies in a half-sphere, the corresponding sides of A and A' are equal* *and each angle of A is at least the corresponding angle in A'. Then A is congruent to A'.*

Global lemma. A more visual proof of the global lemma is given in [2, II §1.3].

Existence theorem. This theorem was proved by Alexandr Alexandrov [3]. Our sketch is taken from [10]; a complete proof is nicely written in [2]. In the original proof, the spaces $\mathbf{K}_n$ and $\mathbf{P}_n$ were modified so the they become $(3{\cdot}n-6)$-dimensional manifolds. It was done by introducing extra structure (for $\mathbf{K}_n$ it is orientation + a marked vertex and an edge) that *brakes symmetries* of the spaces. After that one could apply the domain invariance theorem directly. Alternatively, one may first remove from $\mathbf{K}_n$ and $\mathbf{P}_n$ elements (polyhedron or surface)with nontrivial symmetries (after that the spaces become manifolds) and show that any symmetric polyhedron (or surface) can be approximated by a non-symmetric polyhedron (or surface).

A very different proof was found by Yuri Volkov in his thesis [16]; it uses a deformation of three-dimensional polyhedral space.

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