METRIC MINIMZING SURFACES REVISITED

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Abstract. A surface which does not admit a length nonincreasing deformation is called *metric minimizing*. We show that metric minimizing surfaces in CAT[0] spaces are CAT[0].

1. Introduction

This paper substantially extends results in [7]; it also fills a gap in the proof.

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2. Definitions

Let X be a set. A *pseudometric* $|{*}-{*}|$ on a set X is a function $X\times X\to[0,\infty]$ such that

- ⋄ $|x-x|=0$, for any $x\in X$;
- ⋄ $|x-y|=|y-x|$, for any $x,y\in X$;
- ⋄ $|x-y|+|y-z|\geqslant|x-z|$ for any $x,y,z\in X$.

If in addition $|x-y|=0$ implies $x=y$ then the pseudometric $|{*}-{*}|$ is called *metric*.

A set with a pseudometric or metric will be called *pseudometric* or correspondingly *metric space*.

The value $|x-y|$ will be also called *distance* form x to y. Note that by our definition, the distance between points might be infinite.[1]

For any pseudometric on a set there is an equivalence relation "$\sim$" such that $x\sim y$ if and only if $|x-y|=0$. The pseudometric $|{*}-{*}|$ defines a genuine metric on the set of equivalence classes of $\sim$. We will say that the obtained metric space is *defined* by the original pseudometric space.

Induced pseudometrics. Let X and Y be metric spaces. Given a continuous map $f\colon X\to Y$, define a *connecting pseudometric* $|{*}-{*}|_f$ on X in the following way

$$|x-y|_f=\inf\{\operatorname{diam}f(K)\},$$

where the infimum is taken for all connected sets $K\subset X$ which contain x and y; if there is no such set we set $|x-y|_f=\infty$.

The intrinsic metric induced by $|{*}-{*}|_f$ will be denoted as $\|{*}-{*}\|_f$. That is,

$$\|x-y\|_f=\liminf_{\varepsilon\to0+}\left\{\sum_{i=1}^{n}|x_i-x_{i-1}|_f\right\},$$

[1] Let us mention the following construction to remove a possible psychological barrier with infinity — it will not be used further.

Consider an equivalence relation "$\approx$" on a metric space defined as

$$x\approx y\iff|x-y|<\infty.$$

Note that its equivalence classes form *usual metric spaces*; that is $|x-y|<\infty$ for any pair of points x and y in the class. It implies that metric spaces in our definition can be thought of disjoint union of some collection *usual metric spaces*.

where the infimum is taken for all arrays of points $x = x_0, x_1, \dots, x_n = y$ such that $|x_i - x_{i-1}|_f < \varepsilon$ for any i.

The metric spaces defined by the pseudometrics $|* - *|_f$ and $\|* - *\|_f$ on X will be denoted as $|X|_f$ and $\|X\|_f$ correspondingly.

Remarks.

- ◇ If f is injective, and the image $f(X)$ is compact then the space $\|X\|_f$ is isometric to the image $f(X)$ in Y equipped with induced intrinsic metric.
- ◇ Both pseudometrics $|* - *|_f$ and $\|* - *\|_f$ define the same equivalence classes on X; that is,

$$|x - y|_f = 0 \quad \iff \quad \|x - y\|_f = 0.$$

The equivalence class of $x \in X$ will be denoted as $[x]_f$; it can be regarded as a point in $|X|_f$ and in $\|X\|_f$.

Given two maps $f, h\colon X \to Y$ we will write $f \succcurlyeq h$ if

$$\|x - y\|_f \geqslant \|x - y\|_h$$

for any pair of points $x, y \in X$. We will write $f \succ h$ if in addition the inequality is strict for at least one pair of points then.

Metric minimizing map. Let X and Y be metric spaces and $A \subset X$ be a closed subset.

The map $f\colon X \to Y$ is called *metric minimizing relative to A* if there is no map $h\colon X \to Y$ such that $f \succ h$ and h agrees with f on A; that is, $h|_A = f|_A$.

We say that $f\colon X \to Y$ is *strict metric minimizing relative to A* if there is no map $h\colon X \to Y$ distinct from f such that $f \succcurlyeq h$, and $h|_A = f|_A$.

2.1. Proposition. *Let X and Y be metric spaces and $A \subset X$ be a closed subset. Assume X is connected and $f\colon X \to Y$ is a metric minimizing map relative to A. Then for any point $x \in X$ any connected component of $X \backslash [x]_f$ intersects A.*

Proof. Assume contrary. Denote by W the connected component of $X \backslash [x]_f$ such that $A \cap W = \varnothing$. Let us define the new map $h\colon X \to Y$ by setting $h(z) = f(x)$ for any $z \in W$ and $h(z) = f(z)$ for any $z \notin W$.

By construction f and h agree on A and $f \succcurlyeq h$.

Note that $\|x - y\|_f > 0 = \|x - y\|_h$ for any $y \in W$. Therefore $f \succ h$, a contradiction. □

The following two propositions follow directly from the definition of metric minimizing maps.

2.2. Proposition. *Let X and Y be metric spaces and $A \subset X$ be a closed subset. Assume $f\colon X \to Y$ is a metric minimizing map relative to A.*

Given a closed subset $W \subset X$, set

$$A_W = \partial_X W \cup (A \cap W),$$

where $\partial_X W$ denotes the boundary of W relative to X. Then the restriction $f|_W$ is metric minimizing relative to A_W.

2.3. Proposition. *Let X, X' and Y be metric spaces, $A \subset X$ be a closed subset. Assume $f\colon X \to Y$ is a metric minimizing map relative to A which factors through a continuous surjective map $\varphi\colon X \to X'$; that is $f = f' \circ \varphi$ for a map $f'\colon X' \to Y$. Then $f'\colon X' \to Y$ is metric minimizing relative to $\varphi(A)$.*

3. Metric minimizing graphs

3.1. Proposition. *Assume Γ is a finite graph and A is the collection of its vertexes. Let $Y \in \mathrm{CAT}[0]$ and $f\colon \Gamma \to Y$ be an arbitrary map. Then there is a metric minimizing map $f'\colon \Gamma \to Y$ relative to A such that $f'|_A = f|_A$ and $f \succcurlyeq f'$.*

Proof. Given a map $h\colon \Gamma \to Y$, denote its total length by $\operatorname{length} h$. That is, if $e_1, \dots, e_k$ be the edges of Γ then each restriction $h|_{e_i}$ is a curve and

$$\operatorname{length} h \stackrel{\text{def}}{=\!=} \sum \operatorname{length}(h|_{e_i}).$$

Note that there is a sequence of maps $f_n\colon \Gamma \to Y$ such that (1) $f = f_0 \succcurlyeq f_1 \succcurlyeq \dots$, (2) $f_n|_A = f|_A$ for each n and (3) if for some $h\colon \Gamma \to Y$ we have $h|_A = f|_A$ and $f_n \succ h$ for each n then

$$\operatorname{length} h = \lim_{n\to\infty} \operatorname{length} f_n.$$

Indeed, the sequence f_n can be chosen so that

$$\operatorname{length} f_n - \tfrac{1}{n} < \inf \{\, \operatorname{length} h \mid f_{n-1} \succcurlyeq h,\ h|_A = f|_A \,\},$$

where $\operatorname{length} f_n$ denotes the sum of lengths of all edges of Γ. Since $f_n \succ h$ implies $\operatorname{length} f_n > \operatorname{length} h$, the conditions above hold.

Fix an ultra-filter ω on the set of natural numbers. Let Y^ω be the ultra-power of Y; recall that Y can be (and will be) considered as a subspace of Y^ω. Pass to the ultra-limit $f_\omega\colon \Gamma \to Y^\omega$ of $f_n\colon \Gamma \to Y$. Note that f_ω is a metric minimizing map.

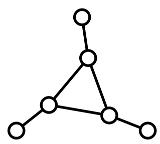
It remains to show that $f_\omega(\Gamma) \subset Y$. Assume contrary, then there is a subsequence of f_n which ω-converges to a metric minimizing map, say $v_\omega\colon \Gamma \to Y^\omega$ distinct from f_ω. Denote by $g_\omega(x)$ the midpoint of $[v_\omega(x) f_\omega(x)]$. Note that $f_\omega \succ g_\omega$ and $g_\omega|_A = f|_A$, a contradiction. □

3.2. Proposition. *Let Y be a $\mathrm{CAT}[0]$ space, Γ a finite graph and A a subset of its vertexes. Assume $f\colon \Gamma \to Y$ is metric minimizing relative to A. Then*

- ⋄ *each edge of Γ maps to a geodesic*
- ⋄ *for any vertex $v \notin A$ and any $x \neq f(v)$ there is an edge $[vw]$ in Γ such that $\measuredangle[f(v)\,{}^{f(w)}_{x}] \geqslant \frac{\pi}{2}$.*

Moreover, f is strictly metric minimizing relative to A.

As one may see from the diagram, the two conditions in the proposition do not guarantee that the map f is metric minimizing.



Proof. By Proposition 2.2, the restriction of f to any edge $[vw]$ of Γ is metric minimizing relative to $\{v, w\}$. Hence the first condition follows.

Assume the second condition does not hold at a vertex $v \notin A$; that is, there is a point $x \in Y$ such that $\measuredangle[f(v)\,{}^{f(w)}_{x}] < \frac{\pi}{2}$ for any adjacent vertex w. In this case moving $f(v)$ toward x along $[f(v)x]$ decrease the lengths of all edges adjacent to v, a contradiction.

To prove the last statement, assume there is a map f' distinct from f such that $f|_A = f'|_A$ and $f \succcurlyeq f'$. Denote by $g(x)$ the midpoint of $f(x)$ and $f'(x)$ for any $x \in \Gamma$. By comparison $f \succcurlyeq g$. It follows that the tautological map $\|\Gamma\|_f \to \|\Gamma\|_g$ is an isometry. The later implies that the distance $|f(v) - g(v)|$ is the same for all the vertices v in Γ. Since we have $|f(v) - g(v)| = 0$ for any $v \in A$, we get $f(v) = g(v)$ for any vertex v in Γ. Hence $f = f'$, a contradiction. □

Assume Γ is a finite graph embedded in the plane $\mathbb{R}^2$; in particular Γ is planar.

The complement to the unbounded connected component of $\mathbb{R}^2 \backslash \Gamma$ will be called filling of Γ; it will be denoted as $\operatorname{Fill} \Gamma$.

The vertex of Γ will be called *boundary vertex* if it lies in the boundary $\partial_{\mathbb{R}^2}[\operatorname{Fill}\Gamma]$, otherwise it will be called *interior vertex.*

3.3. Corollary. *Let Y be a* CAT[0] *space and Γ an embedded graph in $\mathbb{R}^2$. Assume $f\colon \Gamma\to Y$ is a metric minimizing map relative to the boundary vertices. Then one can equip* Fill Γ *with a* CAT[0] *pseudometric and extend f to a short map $\bar f\colon \operatorname{Fill}\Gamma\to Y$ which is length preserving on Γ.*

Proof. Fix a cycle γ in Γ which bounds one of the discs in the complement $\mathbb{R}^2\backslash\Gamma$. Set $\ell=\operatorname{length}\gamma$.

By Reshetnyak's majorization theorem, there is a convex polygon P (possibly degenerate) with perimeter ℓ which admits a short map to Y in such a way that γ is formed by the image of the boundary. Note that each angle of P is at least as big as the angle between the corresponding edges.

Prepare a polygon as above for each disc in the complement of Γ and glue these polygons into $|\Gamma|_f$ along the natural map. The obtained space D is simply connected. Therefore in order to show that D is CAT[0], we need to check that the sum of the angles around each interior vertex in Γ is at least $2{\cdot}\pi$.

Assume contrary, that is, the sum of the angles around a fixed interior vertex v is less than $2{\cdot}\pi$. The space of directions $\Sigma_{f(v)}$ is a CAT[1] space. The directions of the edges from v have a natural cyclic order say $\xi_1,\dots,\xi_k$ such that

$$\measuredangle(\xi_1,\xi_2)+\dots+\measuredangle(\xi_k,\xi_1)<2{\cdot}\pi.$$

By Reshetnyak's majorization theorem, the closed broken line $\xi_1,\dots,\xi_k$ is majorized by a convex spherical polygon P. Note that P lies in an open hemisphere with the pole at some point in P. Choose $x\in Y$ so that the direction form $f(v)$ to x coinsides with the image of the pole in $\Sigma_{f(v)}$. This choice of x contradicts the condition in Proposition 3.2. □

4. Metric minimizing discs

Let us denote by $\mathbb{D}$ the closed unit disc in the plane, its boundary $\partial\mathbb{D}$ is a unit circle.

Let X be a Hausdorff space and $f\colon\mathbb{D}\to X$ be a continuous map. We say that f is a *no-bubble map* if for any point $p\in X$ every connected component of the complement $\mathbb{D}\backslash f^{-1}\{p\}$ contains a point from $\partial\mathbb{D}$.

The following topological notion will be important throughout the text.

4.1. Definition. *A compact simply connected topological space which admits an embedding into the plane is called a* disc retract*.*

We will need the following disc version of Moore's theorem proved in [1].

4.2. Proposition. *Let X be a Hausdorff space and $f\colon\mathbb{D}\to X$ be a no-bubble map. Then $|\mathbb{D}|_f$ is homeomorphic to a disc retract.*

Proof. Note that $|\mathbb{D}|_f$ is a Hausdorff space and the forgetful map $h\colon\mathbb{D}\to|\mathbb{D}|_f$ is continuous. Since and $\mathbb{D}$ is compact, the space $|\mathbb{D}|_f$ comes with the quotient topology for h.

Consider the disc $\mathbb{D}$ as a subset in the sphere $\mathbb{S}^2$. For any $x\in\mathbb{D}$, set

$$[x]=h^{-1}\{h(x)\}$$

and if $x\in\mathbb{S}^2\backslash\mathbb{D}$, set $[x]=\{x\}$. Denote by $\mathbb{S}^2_f$ the set of classes $[x]$ with the quotient topology induced by the map $\iota_f\colon x\mapsto[x]$.

Note that for any point x, the set $[x]$ is connected and compact, with connected complement. Therefore $\mathbb{S}^2_f$ is homeomorphic to $\mathbb{S}^2$ by Moore's theorem, see [1].

It remains to note that $|\mathbb{D}|_f$ is embedded into $\mathbb{S}^2_f$ and its complement is homeomorphic to an open disc. □

Given a disc retract X, define its interior as a maximal open set which is homeomorphic to an open set in the plane. By Domain Invariance Theorem, interior is well defined. The complement to the interior of X will be called boundary and denoted as ∂X.

Applying 2.1 and 4.2, we get the following.

4.3. Proposition. *Let $f\colon \mathbb{D} \to Y$ be a metric minimizing map relative to $\partial\mathbb{D}$. Then f is a no-bubble map.*

In particular, $|\mathbb{D}|_f$ is homeomorphic to a disc retract. Moreover $\partial|\mathbb{D}|_f$ is the image of $\partial\mathbb{D}$ under the map $x \mapsto [x]_f$.

Note that the metric minimizing map $f\colon \mathbb{D} \to Y$ factors through a map

$$f'\colon |\mathbb{D}|_f \to Y.$$

By 2.3 and 4.3 the map f' is metric minimizing relative to $\partial|\mathbb{D}|_f$.

5. Key Lemma

5.1. Key Lemma. *Let Y be a* CAT[0] *space and $s\colon \mathbb{D} \to Y$ be a metric minimizing disc relative to the boundary $\partial\mathbb{D}$. Given a finite set $F \subset \mathbb{D}$ there is (1) a* CAT[0] *space W, which is a disc retract, and (2) maps $p\colon F \to W$ and $q\colon W \to Y$ such that*

$$s(x) = q \circ p(x)$$

for any $x \in F \cap \partial\mathbb{D}$ and

$$\|p(x) - p(y)\|_q \leqslant \|x - y\|_s$$

for any $x, y \in F$.

Proof. Let us connect each pair x, y of points in F by geodesics if $\|x - y\|_s < \infty$.

We can assume that every pair of the constructed geodesics are either disjoint, or their intersection is formed by finite collections of arcs and points.

Indeed, assuming some number of geodesics $\gamma_1, \dots, \gamma_n$ which meets the above property is already chosen and we need to choose one more geodesic connecting x to y. Choose a minimizing geodesic γ_{n+1} which maximize the time it spends in $\gamma_1, \dots, \gamma_n$ in the order of importance. Namely,

- ⋄ among all minimizing geodesics connecting x to y choose one which spends maximal time in γ_1 — in this case γ_{n+1} intesects γ_1 along the empty set, one-point set or a closed arc.
- ⋄ among all minimizing geodesics as above choose one which spends maximal time in γ_2 — in this case γ_{n+1} intesects γ_2 along at most two arcs and points.
- ⋄ and so on.

In particular the set of all these geodesics forms a finite graph, say Γ, embedded in $|\mathbb{D}|_s$.

According to Proposition 4.3, $|\mathbb{D}|_s$ admits an embedding into the plane. Therefore Γ can be considered as a graph embedded into the plane.

It remains to apply Corollary 3.3. □

6. Compactness of planar CAT[0] spaces

Let $\mathcal{K}_\ell$ be the set of isometry classes of CAT[0] metrics on a disc retract with rectifiable boundary curves of length at most ℓ.

Here is the main statement in this section.

6.1. Compactness lemma. *$\mathcal{K}_\ell$ is compact in the Gromov–Hausdorff topology.*

It follows immediately lemmas 6.2 and 6.3 proved below.

6.2. Lemma. *$\mathcal{K}_\ell$ is precompact in the Gromov–Hausdorff topology.*

Further $\operatorname{area} K$ denotes the two-dimensional Hausdorff measure of a metric space K.

Proof. Let K be a metric space with isometry class in $\mathcal{K}_\ell$. By Reshetnyak's theorem there is a short map from a convex plane figure F with perimeter at most ℓ onto K. In particular, $\operatorname{area} K \leqslant \operatorname{area} F \leqslant \ell^2$.

Fix $\varepsilon > 0$. Set $m = \lceil 10\cdot\frac{\ell}{\varepsilon}\rceil$. Choose m points $y_1, \dots, y_m$ on ∂D which divide ∂D into arcs of equal length.

Consider the maximal set of points $\{x_1, \dots, x_n\}$ such that $d(x_i, x_j) > \varepsilon$ and $d(x_i, y_j) > \varepsilon$.

Note that the set $\{x_1, \dots, x_n, y_1, \dots, y_m\}$ forms an ε-net in $(\mathbb{D}, d)$.

Further note that the balls $B_i = B_{\varepsilon/2}(x_i)$ do not overlap. By comparison,

$$\operatorname{area} B_i \geqslant \tfrac{\pi\cdot\varepsilon^2}{4}.$$

It follows that $n \leqslant 2\cdot\left(\frac{\ell}{\varepsilon}\right)^2$. That is, there is a function $N(\varepsilon)$, which returns a positive integer for any $\varepsilon > 0$ such that for any $(\mathbb{D}, d)$ contains an ε-net with at most $N(\varepsilon)$ points.

In other words, $\mathcal{K}_\ell$ is uniformly totally bounded. Any class of metrics with such property is precompact in Gromov–Hausdorff topology; see for example [3, 7.4.15]. □

6.3. Lemma. *$\mathcal{K}_\ell$ is closed in the Gromov–Hausdorff topology.*

Proof. For each n, consider a boundary curve $\gamma_n\colon \mathbb{S}^1 \to X_n$ which is composition of the boundary curve in the disc with strong deformational retract $\mathbb{D} \to X_n$.

Let $f_n\colon F_n \to X_n$ be the majorizor of γ_n; that is, F_n is a convex figure in the plane, f_n is a short map which is length preserving on the boundary and γ_n forms the boundary curve.

In addition we can assume that for each n the map f_n is minimal in the following sense: if $f_n'\colon F_n \to X_n$ is an other majorization then inequality

$$|f_n'(x) - f_n'(y)|_{X_n} > |f_n(x) - f_n(y)|_{X_n} \qquad ❶$$

holds for at least one pair of points $x, y \in F_n$. In this case we have

$$\operatorname{diam}[f_n(\Omega)] = \operatorname{diam}[f_n(\partial\Omega)] \qquad ❷$$

for any open set $\Omega \subset F_n$. Indeed if this is not the case for some Ω, one can redefine the map f_n so that any $x \in \Omega$ maps to the closest-point projection to $\operatorname{Conv}[\partial\Omega]$. For obtained map f_n' violates ❶.

Consider the limit figure F_∞ and the limit map $f_\infty\colon F_\infty \to X_\infty$. Note that f_∞ is onto and

$$\operatorname{diam}[f_\infty(\Omega)] = \operatorname{diam}[f_\infty(\partial\Omega)] \qquad ❸$$

for any open set $\Omega \subset F_\infty$. In particular f_∞ has no bubbles.

It remains to apply Proposition 4.2 □

7. Two-dimensional case

Let W be a metric space. A continuous map $s\colon \mathbb{D}\to W$ is called *saddle* if for any closed convex subset $K\subset W$ holds that the boundary $\partial\mathbb{D}$ intersects any connected component of $\mathbb{D}\backslash s^{-1}(K)$.

Applying this definition to one-point sets, we get that any saddle map has no bubbles.

7.1. Proposition. *Let W be a* CAT[0] *space then any metric minimizing map $s\colon \mathbb{D}\to W$ is saddle.*

Proof. Assume contrary, let K be a convex set in W and Ω be a violating component of the complement $\mathbb{D}\backslash s^{-1}(K)$. Redefine s for each $x\in\Omega$ by moving $s(x)$ to its closest point projection on K. Denote by s' the new map.

Since K is convex the closest point projection is short, therefore $s\succcurlyeq s'$. That is s is not strictly metric minimizing. It remains to apply Proposition 10.3. □

7.2. Shefel's theorem. *Let W be a* CAT[0] *space which is a disc retract. Assume $s\colon \mathbb{D}\to W$ is a saddle map. Then $\|\mathbb{D}\|_s$ is* CAT[0].

The statement above is a slight generalization of theorem proved by Shefel in [10]; originally it was proved if the ambient space is the plane and we need it for a disc retract. The proof below roughly the same as the original. However, Shefel's proof was written very tight and we have decided to provide all details.

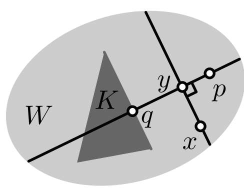
Note that since W is a disc retract, its boundary ∂W is compact. Therefore the space $\mathrm{Geod}(W)$ of geodesic segments with endpoints on ∂W is compact as well. Further, according to Lemma **??**, any geodesic in W can be extended to a geodesic in $\mathrm{Geod}(W)$.

7.3. Lemma. *Let W be a* CAT[0] *disc retract. If $K\subset W$ is a closed convex subset and p is a point in $W\backslash K$, then there is a (possibly degenerated) geodesic $\sigma\in\mathrm{Geod}(W)$ which separates p and K; that is, K and p lie in the different connected components of $W\backslash\sigma$.*

Moreover one can assume that σ lies in the given countable dense subset $\mathcal{G}$ of $\mathrm{Geod}(W)$.

This lemma analogous to the following statement in the plane geometry: *any closed convex set in the plane is an intersection of open half planes.* A Euclidean proof admits a straightforward generalization to the CAT[0] disc retracts.

Proof. Let q be the nearest point projection of p onto K.



Assume there is point $x\notin[pq]$ which lies sufficiently close to the midpoint of $[pq]$. Then the nearest point projection y of x onto $[pq]$ lies in the interior of $[pq]$. Extend the geodesic $[xy]$ to a geodesic σ in $\mathrm{Geod}(W)$. Note that σ separates K from p.

If there is no point $x\notin[pq]$ which lies sufficiently close to the midpoint of $[pq]$ then the midpoint m of $[pq]$ separates K from p. This point forms a degenerate geodesic from $\mathrm{Geod}\,W$.

If σ is not degenerate then any geodesic from $\mathcal{G}$ which is sufficiently close σ also separates K from p. In the degenerate case all the, the point m separates K from p and the same holds for any geodesic from $\mathrm{Geod}(W)$ sufficiently close to m which goes from one of connected component of $W\backslash\{m\}$ to an other one. The latter set is open in $\mathrm{Geod}(W)$. Hence the result follows. □

7.4. Cutting hat lemma. *Let W be a* CAT[0] *space, $K\subset W$ be a closed convex set and $\varepsilon>0$. Assume $s\colon \mathbb{D}\to W$ is a saddle map and $u\colon \mathbb{D}\to W$ is a continuous* *map such that*

$$|s(x) - u(x)| < \varepsilon$$

and $u(x) \in K$ *for any point in the boundary of some open set* $\Omega \subset \mathbb{D}$.

Then there is a continuous map $v\colon \mathbb{D} \to K$ *such that*

(i) $|s(x) - v(x)| < \varepsilon$ *holds for any* $x \in \mathbb{D}$.

(ii) $u(x) = v(x)$ *for any* $x \notin \Omega$ *and* $v(x) \in K$ *for any* $x \in \Omega$.

Moreover,

(iii) we can assume that

$$v(\Omega) \subset \operatorname{Conv} u(\partial\Omega),$$

where $\operatorname{Conv} S$ *denotes the minimal closed convex set containing* S.

(iv) if u *has no bubbles, we can assume that the constructed map* v *has no bubbles.*

Proof. Let $\pi\colon W \to K$ denote the closest point projection of W onto K. Note that π is a short map. Choose an open neighborhood U of $\partial\Omega$ such that $|s(x) - \pi\circ u(x)| < \varepsilon$ for $x \in U$. The assumptions imply that $s(\Omega)$ is contained in the ε-neighborhood of K; that is, $|s(x) - \pi\circ s(x)| < \varepsilon$ for $x \in \Omega$. It follows that for points in U the geodesic segment $\pi\circ s(x)\pi\circ u(x)$ is contained in the ε-ball around $s(x)$. Fix a continuous function $\lambda\colon \Omega \to [0,1]$ which is equal to 1 on $\partial\Omega$ and has support in U. For $x \notin \Omega$ set $v(x) = u(x)$ and if $x \in \Omega$, set $v(x)$ to be the point which divides the geodesic $\pi\circ s(x)\pi\circ u(x)$ at proportion $\lambda(x)$. By construction v satisfies both conditions.

Finally note that $\operatorname{Conv} u(\partial\Omega) \subset K$. Therefore we obtain statement (a) by applying the lemma for the convex set $K' = \operatorname{Conv} u(\partial\Omega)$.

To prove (iv), we need to remove bubbles form the constructed map v. Namely for any x we can choose a maximal open set Υ_x such that for some point $p \in W$ the complement $\mathbb{D}\backslash v^{-1}\{p\}$ has Υ_x as a connected component with no points from $\partial\mathbb{D}$. Redefine v by setting $v(x) = p$. □

7.5. Lemma. *Let* W *be a* $\mathrm{CAT}[0]$ *disc retract. Let* Δ *be a triangle in the euclidean plane and* Δ' *a triangle in* W. *If* $f\colon \Delta \to \Delta'$ *is a map which restricts to constant speed parametrizations on each side, then it extends to a map* $F\colon \operatorname{Conv}(\Delta) \to \operatorname{Conv}(\Delta')$ *between convex hulls which induces an isometry* $\bar F\colon \|\operatorname{Conv}(\Delta)\|_F \to \operatorname{Conv}(\Delta')$.

Proof. There are obvious cases, namely if Δ' is either embedded or contained in a single geodesic. For the general case we observe that if we cut Δ into two triangles $\Delta^\pm$ by a geodesic which joins a vertex to its opposite side, and if the claim is true for both triangles Δ^+ and Δ^-, then it is also true for Δ. Now let us assume that Δ' is neither embedded nor contained in a single geodesic. Then there is a vertex of Δ' such that the two adjacent sides have a closed nontrivial interval in common. Choose a point p in the interior of a side of Δ which maps to an endpoint of this interval. Cut Δ into two triangles $\Delta^\pm$ by joining p to its opposite vertex. We obtain natural maps $f^\pm : \Delta^\pm \to \Delta'$ where one of them say f^+ is degenerated, in the sense that its image is contained in a side of Δ', and f^- has a nondegenerated vertex, in the sense that the image of adjacent sides only intersect in the image of the vertex. Repeating this modification at most twice, reduces the general case to the above obvious ones. □

7.6. Lemma. *Let* W *be a* $\mathrm{CAT}[0]$ *disc retract and* N *is a positive integer. Consider the class of metric spaces* $\mathcal{S}_N$ *which are homeomorphic to a disc retract and can be covered by at most* N *isometric copies of closed convex subsets of* W. *Then* $\mathcal{S}_N$ *is compact in Gromov–Hausdorff topology.*

Proof. $\mathcal{S}_N$ is precompact by [3, 7.4.15]. So let Z be a limit space. Clearly, Z is covered by at most N convex subsets of W and by Lemma 6.3 it is homeomorphic to a disc retract. □

7.7. Lemma. *Let X be a metric space and $u\colon \mathbb{D}\to X$ a continuous map. Denote by $\pi\colon \mathbb{D}\to \|\mathbb{D}\|_u$ the natrual projection and by $\bar u\colon \|\mathbb{D}\|_u\to X$ the induced map. Assume that $\mathbb{D}$ has a decomposition $\mathbb{D}=\bigcup_{n=1}^N C_n$ into a finite number of sets such that $\bar u$ embeds each $\pi(C_n)$ isometrically as a closed subset into X. Then π is continuous.*

Proof. Let $x_k\to x$ be a converging sequence in $\mathbb{D}$. After passing to a subsequence we may assume that all x_k are contained in a single set C_n. By assumptions $u(x_k)\to u(x)\in u(C_n)=\bar u\circ\pi(C_n)$. Since $\bar u$ restricts to an isometry on $\pi(C_n)$, the claim follows. □

Proof of Shefel's theorem 7.2. Fix a fine triangulation τ of $\mathbb{D}$. Map the vertices of τ by s, and extend it to the 1-skeleton by mapping edges with constant speed to corresponding geodesics. Then use Lemma 7.5 to extend the map to the whole disc.

The obtained map $u\colon \mathbb{D}\to W$ can be made arbitrary close to s assuming that the triangulation is fine. Say given $\varepsilon>0$ we can assume that

$$|s(x)-u(x)|_X<\varepsilon \tag{4}$$

for any $x\in\mathbb{D}$.

Note that the image of each solid triangle $\triangle$ is the closed region bounded by the geodesic triangle in W with the corresponding vertices. Moreover, $\|\triangle\|_u$ is isometric to the image $u(\triangle)$ by Lemma 7.5. Denote by N the number of triangles in τ. Then, $\|\mathbb{D}\|_u$ decomposes into N convex sets which are mapped isometrically by the induced map $\bar u$, in particular, $\|\mathbb{D}\|_u$ belongs to $\mathcal{S}_N$.

Recall that the space $\mathrm{Geod}(W)$ of all geodesics in W with endpoints on ∂W is compact. Fix a dense sequence of geodesics $\mathcal{G}=\{\gamma_1,\gamma_2,\dots\}$ in $\mathrm{Geod}(W)$. The geodesics form convex sets in W, so we can apply the Cutting hat lemma 7.4 recursively. We obtain a sequence of maps, say $u=u_0,u_1,u_2,\dots$ such that

$$|u_n(x)-s(x)|<\varepsilon$$

for any x.

Furthermore, each space $\|\mathbb{D}\|_{u_n}$ belongs to $\mathcal{S}_N$ and has a decomposition into at most N convex sets which are mapped isometrically by the induced map

$$\bar u_n\colon \|\mathbb{D}\|_{u_n}\to W.$$

There is a more precise description, namely each of the convex sets is obtained from one of the triangles in τ by applying cuts along the geodesics γ_k. In particular, the decompositions are nested.

Since $\mathcal{S}_N$ is compact, we can pass to a partial limit of $\|\mathbb{D}\|_{u_n}$ as $n\to\infty$, say Q_ε. Denote by $w_\varepsilon\colon Q_\varepsilon\to W$ the limit of the short maps $\bar u_n$.

7.8. Sublemma. *The map w_ε is saddle.*

Proof. Assume that w_ε is not saddle. Then there is a geodesic $\gamma\in\mathrm{Geod}(W)$ such that $\mathbb{D}-w_\varepsilon^{-1}(\gamma)$ has a component U which is disjoint from $\partial\mathbb{D}$. Choose a point p in U. By Lemma 7.3, there is a geodesic $\gamma'\in\mathrm{Geod}(W)$ which separates $w_\varepsilon(p)$ and γ. Then there exists a disc D in U such that $w_\varepsilon(\partial D)$ and $w_\varepsilon(p)$ lie in different components of $W-\gamma'$. Choose $\delta>0$ such that the distance from γ' of $w_\varepsilon(\partial D)$ and $w_\varepsilon(p)$ is larger than δ. Then there is a natural number M such that the distance from γ' of $\bar u_n(\partial D_n)$ and $\bar u_n(p_n)$ is larger than $\frac{\delta}{2}$ where D_n and p_n are lifts of D

respectively p. By density of $\mathcal{G}$, there is a geodesic $\gamma_k \in \mathcal{G}$ with $k > M$ which has distance less than $\frac{\delta}{2}$ from γ'. Hence $\bar{u}_k^{-1}(\gamma')$ separates p_k from the boundary. By Lemma 7.7, the projection $\pi_k \colon \mathbb{D} \to \|\mathbb{D}\|_{u_k}$ is continuous and therefore $u_k^{-1}(\gamma')$ separates some lift $\hat{p}_k$ of p_k from the boundary. This is a contradiction because, by construction, γ_k cannot cut a hat from u_k. □

7.9. Sublemma. *The space Q_ε is* CAT[0].

Proof. It is enough to show that any point in the interior of Q_ε has a CAT[0] neighborhood. We first show that the total angle around any such point, say $z \in Q_\varepsilon$ is at least $2\cdot\pi$. Since the decompositions of $\|\mathbb{D}\|_{u_n}$ into convex subset of W are nested for varying n, they induce such unique such limit decomposition of Q_ε. Moreover, w_ε restricts to an isometry on each of the convex sets occuring in this decomposition. Therefore, w_ε is a local radial isometry. If the total angle at z is too small, then $w_\varepsilon(z) \notin \mathrm{Conv}[w_\varepsilon(\partial\Omega)]$ for a small neighborhood $\Omega \ni w$ in Q_ε. By Lemma 7.3 we could cut a hat from w_ε contradicting that w_ε is saddle. To conclude that z has a CAT[0] neighborhood, we first cut a small ball around z into to halves using a geodesic through z. Next observe that for a small enough radius we can cut each of this halves into sectors which are contained in a single convex set from our decomposition. The claim follows from Reshetnyak's gluing theorem. □

Now, assume that the boundary of $\|\mathbb{D}\|_s$ is rectifiable. By the Lemma on compactness 6.1, we can pass to a partial limit, say Q_0, of Q_ε as $\varepsilon \to 0$ and denote $w_0 \colon Q_0 \to W$ the limit map. By construction, Q_0 is a CAT[0] disc retract which majorizes $\|\mathbb{D}\|_s$, that is the limit map lifts to a short map $Q_0 \to \|\mathbb{D}\|_s$ which maps ∂Q_0 onto $\partial\|\mathbb{D}\|_s$ in a length-preserving way.

Since Q_0 is a CAT[0] space, its boundary can be majorized by a convex figure in the plane. Hence the same holds for $\|\mathbb{D}\|_s$.

The same argument can be repeated for all disc retracts in $\|\mathbb{D}\|_s$ with rectifiable boundary, in particular for all solid triangles in $\|\mathbb{D}\|_s$. Existence of majorization implies that all triangles in $\|\mathbb{D}\|_s$ are thin, that is $\|\mathbb{D}\|_s$ is CAT[0]. □

8. Main theorem

8.1. Main Theorem. *Let Y be a* CAT[0] *space and $s \colon \mathbb{D} \to Y$ be a metric minimizing map relative to the boundary $\partial\mathbb{D}$. Assume $\|\mathbb{D}\|_s$ is separable. Then $\|\mathbb{D}\|_s$ is a* CAT[0] *space.*

Proof of Main theorem. First let us show that $\|\mathbb{D}\|_s$ is geodesic.

Indeed, given two points p and q in $|\mathbb{D}|_s$ choose a sequence of constant-speed paths γ_n from p to q such that

$$\mathrm{length}\,\gamma_n \to \|p - q\|_s$$

as $n \to \infty$.

Note that γ_n are Lipschitz maps in $|\mathbb{D}|_s$. Since $|\mathbb{D}|_s$ is compact, we can pass to a partial limit γ of γ_n. Clearly

$$\mathrm{length}\,\gamma = \|p - q\|_s;$$

that is γ forms a geodesic in $\|\mathbb{D}\|_s$

Note that it is sufficient to prove the theorem in the case that $\partial\|\mathbb{D}\|_s$ formed by a rectifiable simple closed curve.

Indeed, fix a triangle $\triangle$ in $\|\mathbb{D}\|_s$. If the above case has been proven, then closure of each connected open component bounded by $\triangle$ is CAT[0]. In particular $\triangle$ is thin. Since $\triangle$ is arbitrary, the statement follows.

Given a finite set $F \subset \mathbb{D}$, denote by $\mathcal{W}_F$ the set of isometry classes of spaces W which meet the conditions of the Key Lemma 5.1 for F; according to lemma $\mathcal{W}_F \neq \varnothing$. Note that for two finite sets $F \subset F'$ in $\mathbb{D}$, we have $\mathcal{W}_F \supset \mathcal{W}_{F'}$.

According to Lemma on compactness (6.1) $\mathcal{W}_F$ is compact. Therefore

$$\mathcal{W} = \bigcap_F \mathcal{W}_F \neq \varnothing$$

where the intersection is taken over all finite subsets F in $\mathbb{D}$.

Fix a space W from $\mathcal{W}$ and a dense sequence of points $\{x_1, x_2, \dots\}$ in $\|\mathbb{D}\|_s$, such that its subsequence of points in $\partial\|\mathbb{D}\|_s$ also dense in $\partial\|\mathbb{D}\|_s$.

Set $F_n = \{x_1, \dots, x_n\}$. Denote by $p_n \colon F_n \to W$ a map satisfying the conditions in the Key Lemma 5.1.

Passing to a subsequence by n we can ensure that the sequence $p_n(x_k)$ converges as $n \to \infty$ for every fixed k. Set

$$p(x_k) = \lim_{n\to\infty} p_n(x_k).$$

Note that p is short. Since $\{x_k\}$ is dense in $\|\mathbb{D}\|_s$, the map p can be extended to whole disc $\|\mathbb{D}\|_s$ as a short map. The obtained map will be still denoted as p.

Pass to an ultralimit $\mathfrak{q}$ of the maps q_n provided by Key Lemma 5.1 the target of $\mathfrak{q}$ is an ultrapower $\mathfrak{Y}$ of Y, which is CAT[0] space containing Y as a convex subspace. The closest point projection of $\mathfrak{q}$ to Y will be denoted by q. By construction, $\mathfrak{q}$ is short and closest point projection is short, therefore q is also short.

Summarizing, the space W is CAT[0] disc retract. By construction, there are two short maps $\|\mathbb{D}\|_s \xrightarrow{p} W \xrightarrow{q} Y$ such that

$$q \circ p|_{\partial\mathbb{D}} = s|_{\partial\mathbb{D}}.$$

Since s is metric minimizing, both maps p and q are length preserving and metric minimizing.

It remains to apply Shefel's theorem 7.2. □

Remark. An argument similar to the the problem "Saddle surface" in [8] can be used to show that the constructed map p is injective. In particular $\|\mathbb{D}\|_s$ is isometric to W. Unfortunately the proof requres some regularity of $\|\mathbb{D}\|_s$ which only follows from Shefel's theorem.

Here is *not a proof* of the statement above. We cheat in the use of order; it is hard (if at all possible) to fix its precise meaning.

Not a proof. Assume $w = p(x) = p(y)$ for distinct points $x, y \in \|\mathbb{D}\|_s$ Note that w lies in the interior of W. Choose a geodesic γ which pass through w and goes from boundary to boundary of W. The inverse image $p^{-1}(\gamma)$ is a contractabe set with two ends at $\partial\|\mathbb{D}\|_s$, say a and b. In particular the there is well defined order of points on $p^{-1}(\gamma)$. We can assume that the points a, x, y, b appear in the same order on $p^{-1}(\gamma)$.

Note that there is a continuous one parameter family of geodesics γ_t passing through w with the ends at ∂W such that $\gamma = \gamma_0$ and γ_1 is γ with reversed parametrization. Note that the order of x and y on $p^{-1}(\gamma_t)$ does not change in t. On the other hand the order on γ_0 and on γ_1 are opposite, a contradiction. □

9. Harmonic maps

In this section we show that harmonic maps from disc to CAT[0] space are metric minimizing, see Proposition 9.2.

The following theorem was proved by Korevaar and Schoen in [5].

9.1. Dirichlet problem. *Let γ be a continuous circle in a* CAT[0] *space X. Then there is a unique energy minimizing disc u which spans γ; that is, u has the least energy among all discs spanning γ. The energy minimizer u is locally Lipschitz continuous in the interior of $\mathbb{D}$ and continuous on all of $\mathbb{D}$.*

Moreover, the local Lipschitz constant of u depends only on the energy of u and the distance to the boundary.

For a Sobolev map u from the disc $\mathbb{D}$ to the Euclidean space, the energy of a map $u\colon \mathbb{D}\to X$ is defined as

$$E(u)=\int\limits_{\mathbb{D}}|du|^2\cdot d\,\mathrm{area}\,.$$

In [9], Reshetnyak gives an appropriate generalization of Sobolev maps and energy with target in general metric space. This generalization is used in the formulation above.

9.2. Proposition. *Let X be a* CAT[0] *space and $u\colon \mathbb{D}\to X$ is an energy minimizing disc with fixed boundary curve. Then u is metric minimizing.*

In the proof we will use that if $u\colon \mathbb{D}\to X$ is Soblev and $u\succcurlyeq s$ then s is Sobolev and $E(u)\geqslant E(s)$.

Proof. By Proposition 10.6 there is a metric minimizing disc s with $u\succcurlyeq s$ and $u|_{\partial\mathbb{D}}=s|_{\partial\mathbb{D}}$. Then $E(u)=E(s)$, that is s is also energy minimizing. By uniqueness in 9.1, we get $u=s$. □

10. Existence and uniqueness

10A. Intrinsic metric determines metric minimizing discs.

Let $\gamma_{\pm1}\colon I\to X$ be two rectifiable paths, parametrized by arc length. We say that γ_{-1} is *parallel* to γ_{+1}, if $d(\gamma_{-1},\gamma_{+1})$ is constant on I.

10.1. Lemma. *Let $\gamma_{\pm1}\colon I\to X$ be two rectifiable paths of equal length $L>0$. Set $\gamma_t:=h(\cdot,t)$ where $h\colon[-1,1]\times[0,L]\to X$ denotes the geodesic homotopy between their arclength parametrizations. If γ_0 has length L, then γ_{-1} and γ_{+1} are parallel. Moreover, $h([-1,1]\times[0,L])$ is intrinsically flat.*

Proof. Convexity of d implies $|\dot\gamma_0(t)|\le 1$. By assumtion, length of γ_0 equals L, hence $|\dot\gamma_0(t)|=1$ for almost all $t\in[0,L]$. So γ_0 is parametrized by arc length. Since the energy of a unit speed path equals its length, we conclude from energy convexity

$$\int\limits_0^L|\nabla d(\gamma_{-1},\gamma_{+1})|^2d\mathcal{H}^1=0.$$

Therefore, γ_{-1} is parallel to γ_{+1}.

For the second claim, denote H the surface $h([-1,1]\times[0,L])$ equipped with the pull-back metric. By 8.1, H is CAT[0]. Now let $\varepsilon>0$ and choose an array of points $x_0^0,x_1^0,\dots,x_k^0$ on γ_0 such that $\sum_{i=0}^k d(x_i^0,x_{i+1}^0)\ge L-\varepsilon$. For $t\in[-1,1]$ denote by x_i^t the point γ_t corresponding to x_i^0. Now use the comparison triangles $\triangle(x_j^{-1},x_{j+1}^{-1},x_{j+1}^{+1})$ and $\triangle(x_j^{+1},x_{j+1}^{+1},x_{j+1}^{-1})$, $j=0,\dots k{-}1$, to glue a flat comparison surface S_ε for H. From Reshetnyak's majorization theorem, we obtain a short map $f_\varepsilon\colon S_\varepsilon\to H$. Denote by by $\hat x_i^t$ the point on on S_ε corresponding to x_i^t. Then

$$L-\varepsilon\le\sum_{i=0}^k d(x_i^0,x_{i+1}^0)\le\sum_{i=0}^k d(\hat x_i^0,\hat x_{i+1}^0)\le\frac12\sum_{i=0}^k d(x_i^{-1},x_{i+1}^{-1})+\frac12\sum_{i=0}^k d(x_i^{+1},x_{i+1}^{+1})\le L.$$

Hence we can choose a sequence $\varepsilon_j \to 0$ such that S_{ε_j} converges to a flat surface S and f_{ε_j} converges to a short map $f\colon S \to H$. Note that f is surjective. Moreover, S has two transversal foliations, one by parallel paths of constant length L and one by geodesic segments of constant length. It follows that the Jacbian of f is almost everywhere equal to one. Thus f is area preserving and therefore an isometry by **??**. □

10.2. Uniqueness Theorem. *Let X be a* CAT[0] *space and $s_0, s_1\colon \mathbb{D} \to X$ be metric minimizing discs. If $\|\mathbb{D}\|_{s_0}$ is isometric to $\|\mathbb{D}\|_{s_1}$, then $s_0 = s_1$.*

Proof. For $t \in [0,1]$ denote $s_t\colon \mathbb{D} \to X$ the map obtained by geodesic interpolation. Then $\|\mathbb{D}\|_{s_t}$ is isometric to $\|\mathbb{D}\|_{s_0}$ for all $t \in [0,1]$. In particular, every s_t is metric minimizing. Let γ be a path $\mathbb{D}$ connecting two boundary points and such that $s_0 \circ \gamma$ is rectifiable. Moreover, assume that the parametrization is such that $s_0 \circ \gamma$ is parametrized by arc length. Then each $s_t \circ \gamma$ is parametrized by arclength and Lemma 10.1 we obtain $s_t \circ \gamma \cong s_0 \circ \gamma$. The claim follows. □

10.3. Corollary. *Any metric minimizing disc in a* CAT[0] *space is strictly metric minimizing.*

10B. Metric minimizing discs below given continuous ones.

10.4. Intrinsic continuity. *A map $f\colon Y \to Z$ from a metric space Y to a metric space Z is called* intrinsically continuous*, if for every $y \in Y$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that any $y' \in Y$ with $|y, y'| < \delta$ can be connected to y by a path γ with* $\mathrm{length}(f \circ \gamma) < \varepsilon$.

10.5. Remark.

(1) *The main example of intrinsically continuous maps are Lipschitz maps from geodesic spaces to arbitrary metric spaces.*
(2) *Restrictions of intrinsically continuous maps to compact sets are uniformly intrinsically continuous.*
(3) *If a map $f\colon Y \to Z$ is intrinsically continuous, then the natural projection $\pi_f\colon Y \to \|Y\|_f$ is continuous. If moreover Y is compact, then $\|Y\|_f$ equipped with the induced metric is a geodesic space.*

10B.1. *Proper metric spaces.*

Let X be a proper metric space. Denote the set of continuous maps $\mathbb{D} \to X$ equipped with the compact-open topology by $\mathcal{C}(\mathbb{D}, X)$. In Section 2 we defined a partial order on $\mathcal{C}(D, X)$ by $g \succcurlyeq f$.

We address the problem of finding a metric minimizing disc below a given continuous one.

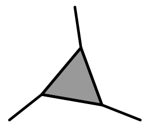
For a map $f \in \mathcal{C}(D, X)$ we define its relative sublevel $\mathcal{F}(f)$ consisting of all maps $g \in \mathcal{C}(D, X)$ with $g|_{\partial\mathbb{D}} = f|_{\partial\mathbb{D}}$ and $f \succcurlyeq g$. If f is intrinsically continuous as a map from $\mathbb{D}$, then so is every element in $\mathcal{F}(f)$. In this case, the family $\mathcal{F}(f)$ is equicontinuous. By Arzelà-Ascoli, it is precompact, since X is proper and it is closed by the semi-continuity of length structures. Hence, we conclude that $\mathcal{F}(f)$ contains a minimal element. This element is clearly metric minimizing, since the induced metric determines the length structure. This explains

10.6. Proposition. *If X is a proper metric space, then for every intrinsically continuous map $f\colon \mathbb{D} \to X$, there is a metric minimizing map $s\colon \mathbb{D} \to X$ such that $\|* - *\|_s \le \|* - *\|_f$.*

11. Smooth surfaces

In this section we show that any strictly saddle surface is locally metric minimizing. So in particular, the metric minimizing surfaces do generalize strictly saddle surfaces.

11.1. Proposition. *Any smooth strictly saddle surface in $\mathbb{R}^3$ is locally metric minimizing.*



In general smooth saddle surface may not be globally metric minimizing. An example homeomorphic to a pair of pants can be found among surfaces sorrounding the triangle with three segments as shown on the picture. We could not find examples like that among strictly saddle surfaces or discs, but it is very likely that such examples do exist.

Let $s\colon \mathbb{D}\to\mathbb{R}^3$ be a smooth map.

Fix an array of vector fields $\boldsymbol{v}=(v_1,\dots,v_k)$ in $\mathbb{D}$. Consider the energy functional

$$E_{\boldsymbol{v}}s \stackrel{\text{def}}{=\!=} \sum_i \int\limits_{\mathbb{D}} |v_is|^2\cdot d_x\,\mathrm{area}\,.$$

Set

$$\Delta_{\boldsymbol{v}}s=\sum_i v_i(v_is).$$

It is convenient to think of operator $s\mapsto\Delta_{\boldsymbol{v}}s$ as an analog of the Laplasian.

Note that

(i) $E_{\boldsymbol{v}}$ is well defined for any lipscitz map s.

(ii) $E_{\boldsymbol{v}}$ is convex, that is

$$E_{\boldsymbol{v}}[t\cdot s_1+(1-t)\cdot s_2]\leqslant t\cdot E_{\boldsymbol{v}}s_1+(1-t)\cdot E_{\boldsymbol{v}}s_2.$$

(iii) If s is smooth $E_{\boldsymbol{v}}$-minimizing map in the class of Lipschitz maps with given boundary data then s is metric minimizing.

(iv) A smooth map $s\colon\mathbb{D}\to\mathbb{R}^3$ is $E_{\boldsymbol{v}}$-minimizing map among the class of Lipschitz maps with given boundary if and only if

$$\Delta_{\boldsymbol{v}}s=0.$$

The discussion above reduces the Proposition above to the following.

11.2. Claim. *Assume $s\colon\mathbb{D}\to\mathbb{R}^3$ is a smooth strictly saddle surface. Then for any interior point $p\in\mathbb{D}$ there is an array of 4 vector fields $\boldsymbol{v}=(v_1,v_2,v_3,v_4)$ such that the equation*

$$\Delta_{\boldsymbol{v}}s=0 \qquad ❺$$

holds in an open neighborhood of p.

Proof. Denote by κ_1,κ_2 the principle curvatures, and by e_1,e_2 the corresponding unit principle vectors. Further, denote by by a_1,a_2 a pair of asymptotic vectors; we can assume that a_1,a_2 form coordinate vector fields in a neighborhood of x.

Set $v_1=\frac{1}{\sqrt{|\kappa_1|}}\cdot e_1$ and $v_2=\frac{1}{\sqrt{|\kappa_2|}}\cdot e_2$. It remains to show that one can choose smooth functions λ_1 and λ_2 so that ❺ holds in a neighborhood of x for $v_3=\lambda_1\cdot a_1$ and $v_4=\lambda_1\cdot a_1$.

Note that the sum $v_1(v_1s)+v_2(v_2s)$ has vanishing normal part. That is

$$v_1(v_1s)+v_2(v_2s)$$

is a tangent vector to the surface.

Since a_i are asymptotic, the vectors $a_1(a_1 s)$ and $a_2(a_2 s)$ have vanishing normal part. Therefore, for any choice of λ_i, the following tow vectors are also tangent

$$v_3(v_3 s) = \lambda_1^2 \cdot a_1(a_1 s) + \tfrac{1}{2} \cdot a_1 \lambda_1^2 \cdot a_1 s$$
$$v_4(v_4 s) = \lambda_2^2 \cdot a_2(a_2 s) + \tfrac{1}{2} \cdot a_2 \lambda_2^2 \cdot a_2 s.$$

Set $w = (\lambda_1^2, \lambda_2^2)$. Note that the system ❺ can be rewritten as

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) w_x + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) w_y = h(x, y, w),$$

where $h \colon \mathbb{R}^3 \to \mathbb{R}^2$ is a smooth function.

Change coordinate system, by setting $x = t + z$ and and $y = t - z$. Then the system takes form

$$w_t + \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) w_z = h(t + z, t - z, w)$$

According by [2, Theorem 3.6], it can be solved locally for smooth initial data at $t = 0$.

It remains to choose v_3 and v_4 for solution so that $\lambda_1, \lambda_2 > 0$ in a small neighborhood of p. □

12. Final remarks

Separability condition. Note that since disc $\mathbb{D}$ is compact, for any continuous $s \colon \mathbb{D} \to Y$ into a metric space, the space $|\mathbb{D}|_s$ is compact. On the other hand the space $\|\mathbb{D}\|_s$ might not be even separable, say if s is given as a cone over Koch snowflake then $|\mathbb{D}|_s$ is homeomorphic to the disc while $\|\mathbb{D}\|_s$ is homeomorphic to uncountable set of segments glued at one end.

The separablity condition in the main theorem 8.1 might be not necesary. While we tried to remove it, we found the following statement which we want to share.

12.1. Proposition. *Let $Y \in \mathrm{CAT}[0]$ and $f \colon \mathbb{D} \to Y$ be a metric minimizing disc. Then $\|\mathbb{D}\|_f$ is a complete metric space.*

Proof. Denote by W be completion of $\|\mathbb{D}\|_f$.

By Proposition 4.3, the space $|\mathbb{D}|_f$ is compact. Therefore the tautological map $f \colon \|\mathbb{D}\|_f \to |\mathbb{D}|_f$ extends to a continuous map $\bar{f} \colon W \to |\mathbb{D}|_f$.

If $\|\mathbb{D}\|_f$ is a proper subset of W then from above we get that then $\bar{f}$ is not injective.

In this case there is a rectifiable path $\gamma \colon [0, 1] \to W$, such that $\gamma(t) \in \|\mathbb{D}\|_f$ for all $t < 1$ and $\gamma(1) \notin \|\mathbb{D}\|_f$. Note that the path $\bar{f} \circ \gamma$ is also rectifiable and in particular continuous, so it can be lifted to $\|\mathbb{D}\|_f$. That is, $\gamma(1) \in \|\mathbb{D}\|_f$, a contradiction. □

Smooth metric minimizing surfaces. At the moment we do not see any way to show that given surface is metric minimizing except constructing energy for which it is energy minimizing, as we did in Section 11. Calculations show that generic smooth saddle surfaces in $\mathbb{R}^4$ is not energy minimizing for any energy.

We expect that generic smooth saddle surfaces in $\mathbb{R}^4$ are not metric minimizing; that is arbitrary small neighborhood of any point admit a deformation which shrinks the intrinsic metric and keeps the boundary fixed.

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