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PARALLEL TRANSPORTATION FOR ALEXANDROV SPACE WITH CURVATURE BOUNDED BELOW

A. Petrunin

In this paper we construct a "synthetic" parallel transportation along a geodesic in Alexandrov space with curvature bounded below, and prove an analog of the second variation formula for this case. A closely related construction has been made for Alexandrov space with bilaterally bounded curvature by Igor Nikolaev (see [N]).

Naturally, as we have a more general situation, the constructed transportation does not have such good properties as in the case of bilaterally bounded curvature. In particular, we cannot prove the uniqueness in any good sense. Nevertheless the constructed transportation is enough for the most important applications such as Synge's lemma and Frankel's theorem. Recently by using this parallel transportation together with techniques of harmonic functions on Alexandrov space, we have proved an isoperimetric inequality of Gromov's type.

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1 Parallel Transportation and Second Variation

1.0 Notation and conventions. The general reference for background on Alexandrov spaces is [BGP], nevertheless we recall some notation here.

• M will always denote a complete Alexandrov space with curvature $\geq k \ \ (\geq 1)$, Σ, usually a complete Alexandrov space with curvature ≥ 1 and X, a length-metric space.

• $C = C(\Sigma)$ will denote a cone over Σ, usually with curvature ≥ 0, and o will denote its center. If $x, y \in C$ then

$$|x| \stackrel{\text{def}}{=} |ox|$$

$$\langle x, y\rangle \stackrel{\text{def}}{=} \frac{|x|^2 + |y|^2 - |xy|^2}{2} = |x| \cdot |y| \cos \angle xoy\,.$$

An element of a cone C we will call a "vector".

• We also use C for a constant and it is ok to change its value even in the same formula.

• Σ_p is the space of directions of M at a point p. For $q \neq p$ we denote by q' or more specifically q'_p a direction in Σ_p of some shortest path pq (see [BGP, §7]). If H is a subset of M then

$$\Sigma_p(H) = \left\{ \theta \in \Sigma_p;\ \exists q_i \in H, \theta = \lim_{|pq_i| \to 0} (q_i)'_p \right\}.$$

• C_p is the tangent cone at point p, where $C_p = C(\Sigma_p)$. C_p can be also defined as a limit $C_p = \lim_{\lambda\to\infty}(\lambda M, p)$ (see [BGP, 7.7, 7.8.1]). We set $C_p(H) = C(\Sigma_p(H)) \subset C_p$.

• For $a \in C_p$ we say $a = \log_p q$ if there is a shortest path between p and q which starts from p tangent to a and has length $|a| = |pq|$. Note that in a space of curvature ≥ 0, $\log_p$ is a noncontracting map.

• $\exp_p \colon C_p \to M$ maps every element $x \in C_p$ to the end point of some quasi-geodesic which leaves p in the direction of x and has length $|x|$. (About existence of quasi-geodesics see [PPe2, §5])

Remark. The mappings $\exp_p$ and $\log_p$ are defined in a nonunique way. The only properties of $\exp_p$ we will use are

$$\exp_p \circ \log_p = \mathrm{id}\,, \quad \log_p \circ \exp_p(v) = v + o_p(v)$$

and

$$\left| x \log_p(q) \right| \geq \left| \exp_p(x) q \right| + O(|x|^2)\,,$$

(for curvature ≥ 0 we can ignore the O-term).

Remark on orientability. With respect to the Riemannian case we have an additional difficulty with the definition of orientability. It is easy to define orientability in the standard way using the atlas of distance coordinates. This atlas in general does not cover all our space, therefore for the nonorientable case we will distinguish two different cases: locally orientable and locally nonorientable. Locally orientable is if every point has an orientable neighborhood, and locally nonorientable otherwise. Every point has a ball neighborhood which is homeomorphic to the tangent cone at this point (see [P1] or [P2]), therefore local orientability is equivalent to orientability of all tangent spaces (or spaces of directions). There is an equivalent topological classification: M is locally orientable if at every point p we have $H^n(M, M\backslash p) = \mathbb{Z}$, where $n = \dim M$. Indeed from the same result of Perelman, we have $H^n(M, M\backslash p) = H^{n-1}(\Sigma_p)$, hence if Σ_p is orientable then $H^n(M, M\backslash p) = \mathbb{Z}$.

Also from Perelman's result one has natural orientation on the tangent cones from orientation on Alexandrov space.

Let us define the normal cone to a geodesic.

From [BGP, 7.15] if p is not an end point of a shortest path γ then

$$C_p = L_p \times \mathbb{R} \quad (\text{where} \quad L_p = \{x \in C_p; x \perp \gamma\})\,.$$

Theorem 1.1. *Let p and q be points of a shortest path γ in M, that are not end points. Then*

A. (Yu. Burago's conjecture) *L_p is isometric to L_q.*

B. (Second variation) *For any sequence $\varepsilon_n \to 0$, there is a subsequence $\{\epsilon_n\} \subset \{\varepsilon_n\}$ and an isometry $T : L_p \to L_q$ which preserves orientation of a small neighborhood of the shortest path pq (if it is orientable), such that*

$$\left| \exp_p(\epsilon_n x) \exp_q(\epsilon_n T(y))\right| \leq |pq| + \frac{|xy|^2 \epsilon_n^2}{2|pq|} - \frac{k|pq|}{6}\sigma\epsilon_n^2 + o(\epsilon_n^2)\,,$$

where $\sigma = \sigma(x,y) = |x|^2 + \langle x, y\rangle + |y|^2$.

Remark. To simplify calculations we will consider only the case of nonnegative curvature. Therefore the formula in (B) reduces to

$$\left| \exp_p(\epsilon_n x) \exp_q(\epsilon_n T(y))\right| \leq |pq| + \frac{|xy|^2 \epsilon_n^2}{2|pq|} + o(\epsilon_n^2)\,.$$

1.2. Part (A) and part (B) will be proved separately. Sections 1.2-1.10 will deal with part (A) and sections 1.11-1.16 with part (B).

For completeness we prove the following well known folklore lemma.

Let K_1 and K_2 be two compact metric spaces. We say (as in [BGP, 7.13]) $K_1 \geq K_2$ if there is a noncontracting map $K_2 \to K_1$.

Lemma. *Let K_1 and K_2 be two compact metric spaces, such that $K_1 \geq K_2$ and $K_2 \geq K_1$. Then K_1 is isometric to K_2.*

Proof. Let $f_1 : K_1 \to K_2$ and $f_2 : K_2 \to K_1$ be non contracting mappings. It is sufficient to prove that $F = f_2 \circ f_1$ is an isometry. Let $N_\varepsilon(K_1)$ be the maximal number such that there exist $\{r_i\} \subset K_1$, $i = \{1, 2, \ldots, N_\varepsilon\}$, such that $|r_i r_j| \geq \varepsilon$ ($N_\varepsilon < \infty$ because K_1 is compact). Then the finite subsets $F^n(\{r_i\})$ are also ε-nets (because F is a noncontracting map, and if there existed a point $p \in K_1$ such that $|p\,F^n(r_i)| \geq \varepsilon$ for all i then there exist $N_\varepsilon + 1$ points with the same condition, but N_ε is maximal, a contradiction).

On the other hand, the sequence

$$a_n(i,j) = \left|F^n(r_i)\, F^n(r_j)\right|$$

is nondecreasing and bounded (K_1 is compact and hence $a_n(i,j) \leq \mathrm{Diam}(K_1) < \infty$), therefore

$$\lim_{n\to\infty}\big(|F^n(r_i)\,F^n(r_j)| - |F^{n-1}(r_i)\,F^{n-1}(r_j)|\big) = 0$$

and hence

$$\lim_{n\to\infty}\max_{i,j}\big\{|F^n(r_i)F^n(r_j)| - |F^{n-1}(r_i)F^{n-1}(r_j)|\big\} = 0\,.$$

Let $x, y \in K_1$. For every n we can find i and j, such that

$$\big|F(x)\,F^n(r_i)\big| < \varepsilon\,, \quad \big|F(y)\,F^n(r_j)\big| < \varepsilon\,.$$

Since F is noncontracting, we obtain

$$\big|x\,F^{n-1}(r_i)\big| < \varepsilon\,, \quad \big|y\,F^{n-1}(r_j)\big| < \varepsilon\,.$$

Therefore

$$\big|F(x)\,F(y)\big| - |xy| \leq \big|F^n(r_i)\,F^n(r_j)\big| - \big|F^{n-1}(r_i)\,F^{n-1}(r_j)\big| + 2\varepsilon\,.$$

Passing to the limit as $n \to \infty$ and $\varepsilon \to 0$ we obtain the lemma. □

1.3. Let us define the projections:

$$\begin{aligned} &\mathrm{pr}_L\colon C_p \to L_p\,, && \mathrm{pr}_L(x,y) = x\,,\\ &\mathrm{pr}_{\mathbb{R}}\colon C_p \to \mathbb{R}\,, && \mathrm{pr}_{\mathbb{R}}(x,y) = y \end{aligned}$$

where

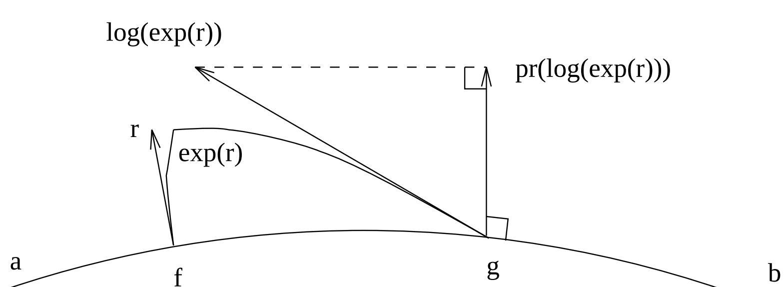
$$C_p = L_p \times \mathbb{R} \;\; \big(L_p = \{x \in C_p; x \perp \gamma\} \text{ and } y \in \mathbb{R}\big)\,.$$

1.4. Let f and g be two points inside a shortest path. Define

$$\pi\colon C_f \to L_g$$

by

$$\pi = \mathrm{pr}_L \circ \log_g \circ \exp_f$$



LEMMA. *There is a constant C such that*

$$\pi(r\varepsilon)/\varepsilon \in B_{C|r|}(o)$$

for any $\varepsilon > 0$ and $r \in C_f$. (Here o is the vertex of the tangent cone at g).

Proof. The idea of the proof is due to [BGP, 7.17].

From the definition of quasi-geodesic we obtain

$$\begin{aligned}|\log_g \exp_f r| = |g \exp_f r| &\le \sqrt{|fg|^2 + |r|^2 - 2|r| \cdot |fg| \cos\angle(r, f'_g)} \\ &= |fg| - |r|\cos\angle(r, f'_g) + O(|r|^2)\,;\end{aligned}$$

and if a is the left endpoint of γ and b is the right endpoint, then

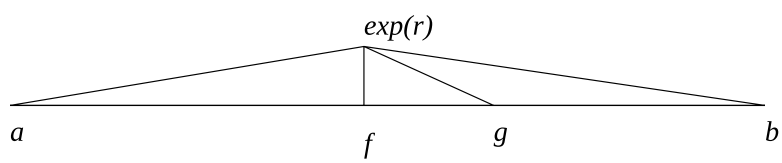
$$\begin{aligned}|\log_g \circ \exp_f r| = |g \exp_f r| &> |ga| - |a \exp_f r| \\ &\ge |ga| - \sqrt{|af|^2 + |r|^2 + 2|r| \cdot |af| \cos\angle(r, f'_g)} \\ &= |fg| - |r|\cos\angle(r, f'_g) + O(|r|^2)\,.\end{aligned}$$

Therefore

$$|g \exp_f r| = |\log_g \exp_f r| = |fg| - |r|\cos\angle(r, f'_g) + O(|r|^2)\,. \qquad (*)$$

Analogously

$$|b \exp_g r| = |bf| - |r|\cos\angle(r, f'_g) + O(|r|^2)$$



Using the last two equations together with the law of cosines we obtain

$$\cos(\tilde{\angle} b\, g \exp_f r) = -1 + O(|r|^2)\,.$$

Therefore

$$\tilde{\angle} b\, g \exp_f r > \pi - C|r|\,.$$

Hence

$$\angle(\log_g f, \log_g \exp_f r) = \angle f g \exp_f r \le \pi - \tilde{\angle} b\, g \exp_f r < C|r| \qquad (**)$$

(compare [BGP, 7.17]). The last C depends only on the distance from f and g to the ends of γ.

Now using $(*)$ and $(**)$ we obtain

$$|\pi(r)| \le C|r|\big(|fg| + O(|r|^2)\big) \le C|r|$$

and therefore

$$\pi(r\varepsilon)/\varepsilon \in B_{C|r|}(o)\,. \qquad \square$$

1.5. Let $Q_f = \{r_i\}$ be a countable everywhere dense subset of L_f and $\{\varepsilon_n\} \to 0$ be a sequence of positive numbers. Then using the lemma from 1.4 and compactness of $B_{C|r|}(o)$ (see [BGP, 7.3]) we can pass to a subsequence, so that the following limit exists

$$\lim_{n\to\infty} \pi(r_1\varepsilon_n)/\varepsilon_n\,.$$

After that pass to a subsequence again, such that there exists

$$\lim_{n\to\infty} \pi(r_2\varepsilon_n)/\varepsilon_n$$

and so on. By choosing a diagonal subsequence we obtain a sequence such that the following limit exists for every r_i

$$\lim_{n\to\infty} \pi(r_i\varepsilon_n)/\varepsilon_n\,.$$

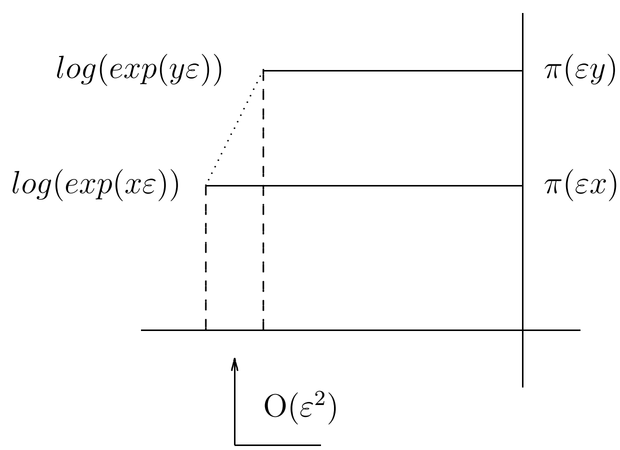
Thus we define the map

$$\Pi\colon Q_f \subset L_f \to L_g$$
$$\Pi(r_i) \stackrel{\text{def}}{=} \lim_{n\to\infty} \pi(r_i\varepsilon_n)/\varepsilon_n\,.$$

LEMMA. Π *is a noncontracting mapping.*

Proof. Let $x, y \in Q_f$. Using $(*)$ and $(**)$ (see 1.4) we can estimate

$$\left|\mathrm{pr}_{\mathbb{R}} \log_g \exp_f(x\varepsilon_n) - \mathrm{pr}_{\mathbb{R}} \log_g \exp_f(y\varepsilon_n)\right| = O(\varepsilon_n^2)\,.$$



By the noncontracting property of $\log_g$ (curvature ≥ 0) and fact that $C_p = \lim(M/\epsilon, p)$ we have

$$\left|\log_g \exp_f(x\varepsilon_n)\ \log_g \exp_f(y\varepsilon_n)\right| \geq \left|\exp_f(x\varepsilon_n)\exp_f(y\varepsilon_n)\right| \geq |xy|\varepsilon_n + o(\varepsilon_n)\,.$$

Therefore

$$
\begin{aligned}
&\big|\pi(x\varepsilon_n)\,\pi(y\varepsilon_n)\big| \\
&\quad = \sqrt{|\exp_f(x\varepsilon_n)\exp_f(y\varepsilon_n)|^2 - |\mathrm{pr}_{\mathbb{R}}\log_g\exp_f(x\varepsilon_n) - \mathrm{pr}_{\mathbb{R}}\log_g\exp_f(y\varepsilon_n)|^2} \\
&\quad = |xy|\varepsilon_n + o(\varepsilon_n)
\end{aligned}
$$

passing to the limit $\varepsilon_n \to 0$

$$\big|\Pi(x)\Pi(y)\big| \geq |xy|\,. \qquad \square$$

Construction of $\widetilde{\Pi}$. For any $v \in L_f \backslash Q_f$ take a sequence of vectors $x_i \in Q_f$ such that $x_i \to v$ and there is a limit of the sequence $\{\Pi(x_i)\}$ (we can find such a sequence using the lemma in 1.4). Now extend Π to $\widetilde{\Pi} : L_f \to L_g$, such that

$$\widetilde{\Pi}(v) = \lim_{i\to\infty} \Pi(x_i) \qquad (\widetilde{\Pi}|_{Q_f} = \Pi)\,.$$

Remark. It is easy to see that $\widetilde{\Pi}$ is a noncontracting mapping as is Π.

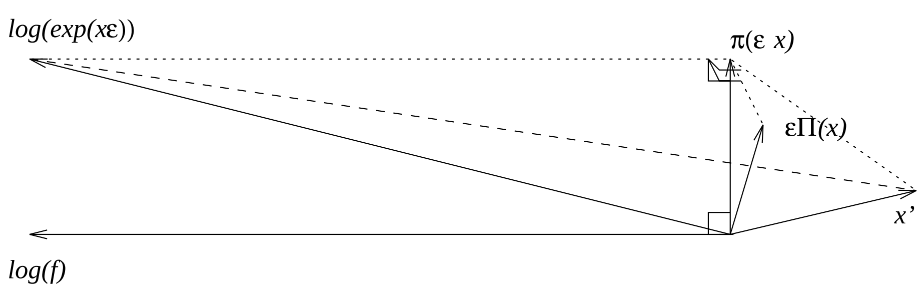
1.6 Key Lemma for A. *For any $x \in Q_f \subset L_f$ and $x' \in L_g$*

$$\big|\exp_f(\epsilon_n x)\exp_g(\epsilon_n x')\big|^2 \leq |fg|^2 + \big(|x|^2 - |\widetilde{\Pi}(x)|^2 + |x'\widetilde{\Pi}(x)|^2\big)\epsilon_n^2 + o(\epsilon_n^2)$$

or equivalently

$$\big|\exp_f(\epsilon_n x)\exp_g(\epsilon_n x')\big| \leq |fg| + \frac{1}{2|fg|}\big(|x|^2 - |\widetilde{\Pi}(x)|^2 + |x'\widetilde{\Pi}(x)|^2\big)\epsilon_n^2 + o(\epsilon_n^2)\,.$$

Proof. To understand the proof of this fact, it is much more useful to look at the following picture of C_g than to look at the formulas:



In the proof we use only the definition of quasi-geodesic and the Pythagorean theorem.

It is easy to see that

$$\big|\exp_f(\epsilon_n x)\, g\big|^2 \leq |fg|^2 + |x|^2\epsilon_n^2\,.$$

Further

$$\begin{aligned}
&\big|\exp_f(\epsilon_n x)\ \exp_g(\epsilon_n x')\big|^2\\
&\quad\le \big|\log_g\exp_f(\epsilon_n x)\ \epsilon_n x'\big|^2\\
&\quad= \big|\log_g\exp_f(\epsilon_n x)\,\pi(\epsilon_n x)\big|^2+\big|\pi(\epsilon_n x)\,\epsilon_n x'\big|^2\\
&\quad= \big|\log_g\exp_f(\epsilon_n x)\big|^2-\big|\pi(\epsilon_n x)\big|^2+\big|\pi(\epsilon_n x)\,\epsilon_n x'\big|^2 = {???}
\end{aligned}$$

By the definition of Π and $\widetilde{\Pi}$ we have

$$\begin{aligned}
\big|\widetilde{\Pi}(x)\big|^2\epsilon_n^2-\big|\pi(x\epsilon_n)\big|^2 &\le \big(\widetilde{\Pi}(x)\epsilon_n+\pi(x\epsilon_n)\big)\big|\widetilde{\Pi}(x)\epsilon_n\pi(x\epsilon_n)\big|\\
&= O(\epsilon_n)o(\epsilon_n)=o(\epsilon_n^2)\,;
\end{aligned}$$

and also

$$\begin{aligned}
\big|\widetilde{\Pi}(x)x'\big|^2\epsilon_n^2-\big|\pi(x\epsilon_n)x'\epsilon_n\big|^2 &\le \big(|\widetilde{\Pi}(x)x'|\epsilon_n+|\pi(x\epsilon_n)x'|\epsilon_n\big)\big|\widetilde{\Pi}(x)\epsilon_n\pi(x\epsilon_n)\big|\\
&= o(\epsilon_n^2)\,.
\end{aligned}$$

Therefore we can continue:

$$\begin{aligned}
{???} &= \big|\log_g\exp_f(\epsilon_n x)\big|^2-\big|\widetilde{\Pi}(x)\big|^2\epsilon_n^2+\big|\widetilde{\Pi}(x)\,x'\big|^2\epsilon_n^2+o(\epsilon_n^2)\\
&= \big|\exp_f(\epsilon_n x)\,g\big|^2-\big|\widetilde{\Pi}(x)\big|^2\epsilon_n^2+\big|\widetilde{\Pi}(x)\,x'\big|^2\epsilon_n^2+o(\epsilon_n^2)\\
&\le |fg|^2+\big(|x|^2-|\widetilde{\Pi}(x)|^2+|\widetilde{\Pi}(x)\,x'|^2\big)\epsilon_n^2+o(\epsilon_n^2)\,. \qquad \square
\end{aligned}$$

1.7. We carry out the proof in sections 1.7-1.9.

Suppose the shortest path pq is divided into N equal parts by the points

$$p=p_0,p_1,\dots,p_N=q\,.$$

Let us take some countable everywhere dense subsets $Q_{p_n}\subset L_{p_n}$ for any $n\in\{0,...,N\}$ and a sequence $\varepsilon_n\to 0$. Construct $\Pi_N\colon Q_{p_0}\subset L_{p_0}\to L_{p_1}$ for some subsequence $\{\varepsilon_n^1\}$ of $\{\varepsilon_n\}$. After that find a new subsequence $\{\varepsilon_n^2\}\subset\{\varepsilon_n^1\}$ and build a new $\Pi_N\colon Q_{p_1}\subset L_{p_1}\to L_{p_2}$ and so on to $\{\varepsilon_n^N\}\subset\{\varepsilon_n^{N-1}\}\subset ...\subset\{\varepsilon_n\}$ and $\Pi_N\colon Q_{p_{n-1}}\subset L_{p_{n-1}}\to L_{p_n}$ for any n. Assume $\epsilon_n=\varepsilon_n^N$. Now we can extend every Π to respective $\widetilde{\Pi}$.

Set $\widetilde{\Pi}_N^k=\overbrace{\widetilde{\Pi}_N\circ\widetilde{\Pi}_N\circ ...\circ\widetilde{\Pi}_N}^{k\text{ times}}\colon L_{p_{n-k}}\to L_{p_n}$.

1.8 LEMMA. *There is a constant C, such that*

$$\big|\widetilde{\Pi}_N^N(v)\big|\le\big(1+\tfrac{C}{N}\big)\,|v|$$

for any $v\in L_p$.

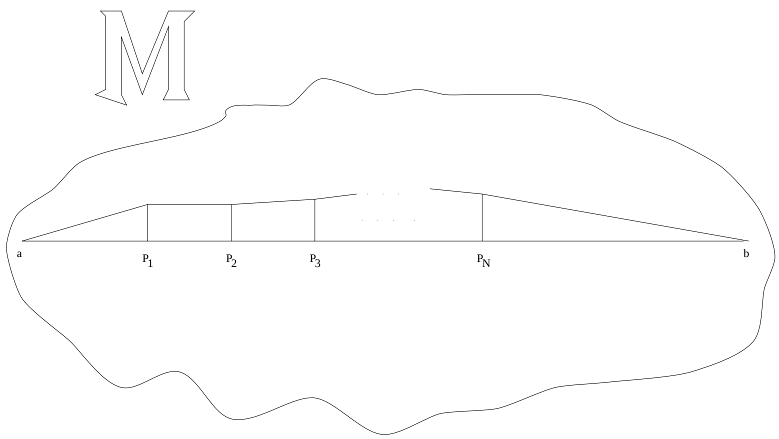
Proof. The idea of this estimate was prompted by G. Perelman.

Let $x \in Q_p$, then

$$\left|a\ \exp_{p_0}(\epsilon_n x)\right| \le \sqrt{|ap_0|^2 + |x|^2\epsilon_n^2} \le |ap_0| + C|x|^2\epsilon_n^2$$

and

$$\left|\exp_{p_N}(\epsilon_n \widetilde{\Pi}_N^N(x))\, b\right| \le \sqrt{|p_N b|^2 + |\widetilde{\Pi}_N^N(x)|^2} \le |p_N b| + C\left|\widetilde{\Pi}_N^N(x)\right|^2\epsilon_n^2 .$$



From Lemma 1.6 and the triangle inequality it is not hard to get

$$\begin{aligned}|ab| &\le |a\exp_{p_0}(\epsilon_n x)| + \big(|\exp_{p_0}(\epsilon_n x)\exp_{p_1}(\epsilon_n\widetilde{\Pi}_N^1(x))| \\ &\quad + |\exp_{p_1}(\epsilon_n\widetilde{\Pi}_N^1(x))\exp_{p_2}(\epsilon_n\widetilde{\Pi}_N^2(x))| + \ldots \\ &\quad + |\exp_{p_{N-1}}(\epsilon_n\widetilde{\Pi}_N^{N-1}(x))\ \exp_{p_N}(\epsilon\widetilde{\Pi}_N^N(x))|\big) + |\exp_{p_N}(\epsilon_n\widetilde{\pi}_N^N(x))b| \\ &\le |ap_0| + |p_0p_1| + |p_1p_2| + \ldots + |p_{N-1}p_N| + |p_N b| \\ &\quad + \bigg[C|x|^2 + \frac{N}{2|pq|}\big\{(|(x)|^2 - |\widetilde{\Pi}_N^1(x)|^2) + (|\widetilde{\Pi}_N^1(x)|^2 - |\widetilde{\Pi}_N^2(x)|^2) + \ldots \\ &\quad + (|\widetilde{\Pi}_N^{N-1}(x)|^2 - |\widetilde{\Pi}_N^N(x)|^2)\big\} + C|\widetilde{\Pi}_N^N(x)|^2\bigg]\epsilon_n^2 + o(\epsilon_n^2) .\end{aligned}$$

Hence

$$\left(\frac{N}{2|pq|} - C\right)\left|\widetilde{\Pi}_N^N(x)\right|^2 \le \left(\frac{N}{2|pq|} + C\right)|x|^2 .$$

Therefore for $x \in Q_p$

$$\left|\widetilde{\Pi}_N^N(x)\right| \le \left(1 + \tfrac{C}{N}\right)|x|$$

and as a corollary

$$\left|\widetilde{\Pi}_N^N(v)\right| \le \left(1 + \tfrac{C}{N}\right)|v|$$

for any $v \in L_p$.

1.9. Take a sequence $N_i \to \infty$ such that there is a pointwise limit

$$f_1 = \lim_{i\to\infty} \widetilde{\Pi}_{N_i}^{N_i},$$

on some countable everywhere dense subset $Q_p \subset L_p$. One can repeat the same "Construction" as in 1.5 to extend f_1 to $\tilde{f}_1 : L_p \to L_q$. It is a noncontracting mapping and from Lemma 1.8 $|\tilde{f}_1(v)| \le |v|$. In the same way we can construct such a mapping $\tilde{f}_2 : L_q \to L_p$. Using Lemma 1.3 for $K_1 = B_R(o) \subset L_p$, $K_2 = B_R(o) \subset L_q$ we obtain part (A) of the theorem. □

1.10. By *Regular point* of an Alexandrov space we understand a point with Euclidean space as a tangent cone.

COROLLARY OF THEOREM 1.2(A). *Let x lie on a shortest path pq. Then $\Sigma_x \ge \Sigma_p$ and $\Sigma_x \ge \Sigma_q$ (see 1.3). In particular if p or q are regular points of M then all points on a shortest path pq are regular. (In other words, the set of regular points is convex.)*

Proof. It follows immediately from the semicontinuity property of the space of directions [BGP, 7.14] and Theorem 1.2(A).

REMARK. It is true also that $\Sigma_x \ge S(\Sigma_{q'}(\Sigma_p))$ and $\Sigma_x \ge S(\Sigma_{p'}(\Sigma_q))$. The proof uses a stronger version of the semicontinuity property, namely if $M_i \to M$ is a sequence of Alexandrov spaces with curvature $\ge C$ and $p_i \in M_i$ such that $p_i \to p \in M$ then

$$\liminf \Sigma_{p_i} \ge \Sigma_p .$$

1.11. Starting from this point we are preparing for part (B) of the Theorem 1.1.

Lyrical digression. One can see that the construction of parallel transport almost contains the second variation formula, but when we pass to the limit we lose everything. The rest of this section is very technical dealing with this obstacle. There is nothing good in the proof below except that it is, hopefully, right.

LEMMA (Isoperimetric inequality). *Let U be a metric space which is a $e^{\pm\delta}$-bi-Lipschitz homeomorph to an open Euclidean n-dimensional domain. Then for every compact $K \subset U$ we have*

$$\mathrm{Vol}_n(K)^{1/n} \le e^{2\delta}\omega_n \mathrm{Vol}_{n-1}^{1/(n-1)}(\partial K)$$

where ω_n is the constant in the standard Euclidean isoperimetric inequality.

Proof. It is easy to see that for a $e^{\pm\delta}$-bi-Lipschitz homomorphism $f\colon U \to \mathbb{R}^n$ we have

$$\mathrm{Vol}_n^{1/n}(K) \le e^{\delta} \cdot \mathrm{Vol}_n^{1/n}(f(K))$$

and

$$\mathrm{Vol}_{n-1}^{1/(n-1)}(\partial K) \ge e^{-\delta}\mathrm{Vol}_{n-1}^{1/(n-1)}(\partial f(K))\,.$$

Further $f(K) \subset \mathbb{R}^n$ and hence we have

$$\mathrm{Vol}_n(f(K))^{1/n} \le \omega_n \mathrm{Vol}_{n-1}^{1/(n-1)}(\partial f(K))\,.$$

Therefore we obtain our version of the isoperimetric inequality. $\square$

Let $B_t(K)$ be the t-neighborhood of a subset K. Using the inequality above it is easy to obtain the following

Corollary (Coarea formula). *Under the assumption of the last lemma, let $\overline{B}_t(K)$ be compact subsets of U for $0 \le t < \epsilon$. Then*

$$\tfrac{d}{dt}\mathrm{Vol}_n(B_t(K)) \ge \mathrm{Vol}_n(B_t K)^{n-1/n} \cdot e^{-2(n-1)\delta}\omega_n^{1-n}$$

and therefore

$$\mathrm{Vol}_n^{1/n}(B_t(K)) - \mathrm{Vol}_n^{1/n}(K) \ge t c_n e^{-2(n-1)\delta}\,,$$

where $c_n = \omega_n^{1-n}/n$ is the constant in the standard Euclidean coarea formula.

1.12. Sections 1.12-1.15 will be devoted to proving the following

Key lemma for B. *Given $\varepsilon_n \to 0$, and $Q_p \subset L_p$, $Q_q \subset L_q$ are countable everywhere dense subsets of regular points. There is a subsequence $\{\epsilon_n\} \subset \{\varepsilon_n\}$ and an isometry $T : L_p \to L_q$ which preserves orientation of a small neighborhood of pq, such that for any $x \in Q_p$ and $y \in T^{-1}(Q_q)$*

$$\left|\exp_p(\epsilon_n x)\exp_q(\epsilon_n T(y))\right| \le |pq| + \frac{|xy|^2\epsilon_n^2}{2|pq|} + o(\epsilon_n^2)\,.$$

Proof. Let r be the midpoint of segment pq. Let us consider the same construction as in the proof of (A) for pairs p, r and q, r and subsets $Q_p \subset L_p$ and $Q_q \subset L_q$. We obtain a series of mappings $\widetilde{\Pi}_N^N : L_p \to L_r$ and $\widetilde{\Pi}_N^N : L_q \to L_r$. As in 1.9 we take some sequence N_i such that the following limits exist

$$f = \lim_{i\to\infty} \widetilde{\Pi}_{N_i}^{N_i} : L_q \to L_r \quad \text{and} \quad g = \lim_{i\to\infty} \widetilde{\Pi}_{N_i}^{N_i} : L_q \to L_r\,.$$

From 1.9 these mappings f and g are isometries. We can assume that all of $\widetilde{\Pi}_N^k$ are defined for one subsequence of $\{\epsilon_n\}$.

We claim that $T = f \circ g^{-1}$ is the mapping we need.

It is easy to verify that T preserves orientation of a small neighborhood of pq.

We have to prove that for any $x \in Q_p$ and $y \in T^{-1}(Q_q)$

$$\left|\exp_p(\epsilon_n x)\exp_q(\epsilon_n T(y))\right| \le |pq| + \frac{|xy|^2\epsilon_n^2}{2|pq|} + o(\epsilon_n^2)\,,$$

or equivalently for any $\lambda > |xy|/|pq|$

$$\left|\exp_p(\epsilon_n x)\exp_q(\epsilon_n T(y))\right| \le |pq|\left(1 + \frac{\lambda^2\epsilon_n^2}{2}\right) + o(\epsilon_n^2)\,.$$

Starting from this point we fix $x, y \in Q_p$ and some $\lambda > |xy|/|pq|$ and always assume that $N \in \{N_i\}$.

1.13 Warning. In the next few sections I'm trying to avoid unnecessary indexes (not to confuse you but to make it easier to understand), do not be surprised.

Let $\mathcal{B}(X)$ be the set of all subsets of X. Recall that $B_t(Y)$ is the t-neighborhood of a subset $Y \subset X$. Let us consider the following map

$$\Phi : \mathcal{B}(L_f) \to \mathcal{B}(L_g)\,, \qquad \Phi = B_{\lambda|fg|} \circ \widetilde{\Pi}\,.$$

Now using Lemma 1.6 we obtain that if $x' \in \Phi(x)$ and $x \in Q_f \subset L_f$ then

$$\left|\exp_f(\epsilon_n x)\exp_g(\epsilon_n x')\right| \le |fg| + \frac{1}{2|fg|}\big(|x|^2 - |\widetilde{\Pi}(x)|^2 + \lambda^2|fg|^2\big)\epsilon_n^2 + o(\epsilon_n^2)\,. \tag{\#}$$

Therefore using that $\widetilde{\Pi}$ is a noncontracting mapping (see Lemma 1.5)

$$\left|\exp_f(\epsilon_n x)\exp_g(\epsilon_n x')\right| \le |fg|\left(1 + \frac{\lambda^2\epsilon_n^2}{2}\right) + o(\epsilon_n^2)\,. \tag{\#\#}$$

Consider a series of mappings Φ_N corresponding to $\widetilde{\Pi}_N$.

Set

$$\Phi_N^k : \mathcal{B}(L_{p_{n-k}}) \to \mathcal{B}(L_{p_n})\,, \qquad \Phi_N^k(x) = \overbrace{\Phi_N \circ \Phi_N \circ \ldots \Phi_N}^{k \text{ times}}\,.$$

Putting together inequality (##) for different $\widetilde{\Pi}$-s we obtain that if $x' \in \Phi_N^k(x)$ and $x \in Q_{p_{n-k}}$ then

$$\left|\exp_{p_{n-k}}(\epsilon_n x)\exp_{p_n}(\epsilon_n x')\right| \le |p_{n-k}p_n|\left(1 + \frac{\lambda^2\epsilon_n^2}{2}\right) + o(\epsilon_n^2)\,. \tag{\#\#\#}$$

1.13a. We can consider a series of mappings:

$$\Upsilon_N : L_{p_{n-1}} \to L_{p_n}$$

and

$$\Upsilon_N^k = \overbrace{\Upsilon_N \circ \Upsilon_N \circ \ldots \Upsilon_N}^{k \text{ times}} : L_{p_{n-k}} \to L_{p_n}$$

such that $\Upsilon_N^k : L_{p_0} \to L_{p_k}$ is an isometry with $\Upsilon_N^k(o) = o$ that minimizes

$$\max_{v \in B_1(o) \subset L_{p_0}} \left\{ |\Upsilon_N^k(v) \widetilde{\Pi}_N^k(v)| \right\}.$$

for $k < N$.

Proposition. *Let $k = k(N)$, $n = n(N)$ be any sequences such that $1 \le k \le n \le N$, and $x = x_N \in L_{p_{n-k}}$ be a sequence such that $|x| \le R$ for some fixed R. Then*

$$\lim_{N\to\infty} \left| \widetilde{\Pi}_N^k(x)\, \Upsilon_N^k(x) \right| = 0\,. \tag{####}$$

In particular $\lim_{N\to\infty} \Upsilon_N^N = \tilde{f}$ where $\tilde{f}$ is from 1.9.

Proof. This is true because $\widetilde{\Pi}_N^k$ is close to an isometry for large N (a direct corollary of 1.9 and 1.5).

Lemma. *Let $k = k(N)$, $n = n(N)$ be any sequences such that $1 \le k \le n \le N$, and $x = x_N \in L_{p_{n-k}}$ be a sequence such that $|x| \le R$ for some R. Then for any fixed $\varepsilon > 0$ we have*

$$\Phi_N^k(x) \subset \Upsilon_N^k\big(B_{\lambda|p_{n-k}p_n|+\varepsilon}(x)\big)$$

for sufficiently large N.

Proof. Let us prove first that there is R' such that for sufficiently large N, $|\Phi_N^k(x)| \le R'$. The following proof is very similar to the one in 1.8.

Take a sequence $x_0 = x, x_1 \in \Phi_N(x_0),\ldots,x_k \in \Phi_N(x_{k-1})$. We must prove that there is R' such that for any such a sequence we have $|x_k| \le R'$. We can assume that $\max_i |x_i| = |x_k|$.

First of all

$$\left| a \exp_{p_{n-k}}(x_0 \epsilon_n) \right| \le |a p_{n-k}| + C|x_0|^2 \epsilon_n^2\,,$$

and

$$\left| \exp_{p_{n-k}}(x_k \epsilon_n)\, b \right| \le |p_n b| + C|x_k|^2 \epsilon_n^2\,.$$

Now

$$|ab| \le \left| a\ \exp_{p_{n-k}}(\epsilon_n x_0) \right|$$

$$+\big(|\exp_{p_{n-k}}(\epsilon_n x_0)\ \exp_{p_{n-k+1}}(\epsilon_n x_1)| + |\exp_{p_{n-k+1}}(\epsilon_n x_1)\ \exp_{p_2}(\epsilon_n x_2)| + \ldots$$

$$+|\exp_{p_{N-1}}(\epsilon_n x_{k-1})\ \exp_{p_N}(\epsilon_n x_k)|\big) + \big|\exp_{p_N}(\epsilon_n x_k)\, b\big| \le ???$$

By inequality

$$\begin{aligned} ???\ &\le |ap_0| + |p_{n-k}p_{n-k+1}| + |p_{n-k+1}p_{n-k+2}| + \ldots + |p_{n-1}p_n| + |p_n b| \\ &+\Big[C|x|^2 + \frac{N}{2|pq|}\big\{(|(x_0)|^2 - |\widetilde{\Pi}_N(x_0)|^2) + (|x_1|^2 - |\widetilde{\Pi}_N(x_1)|^2) + \ldots \\ &\ldots + (|x_{k-1}|^2 - |\widetilde{\Pi}_N(x_{k-1})|^2)\big\} + \lambda^2|p_{n-k}p_n|/2 + C|x_k|^2\Big]\epsilon_n^2 + o(\epsilon_n^2) \le ??? \end{aligned}$$

Using $|\widetilde{\Pi}_N(x_i)x_{i+1}| \le \lambda|p_i p_{i+1}|$, we have

$$|x_{i+1}|^2 - |\widetilde{\Pi}_N(x_i)|^2 \le 2\lambda|p_i p_{i+1}| \max\big\{|x_i|, |\widetilde{\Pi}_N(x_i)|\big\}\,.$$

Therefore we can continue

$$\begin{aligned} ??? \le |ab| + \Big[C|x|^2 + \frac{N}{2|pq|}\big\{|(x_0)|^2 - |(x_k)|^2 + 2\lambda|p_{n-k}p_n| \max_i\{|x_i|, |\widetilde{\Pi}_N(x_i)|\}\big\} \\ +\lambda^2|p_{n-k}p_n|/2 + C|x_k|^2\Big]\epsilon_n^2 \\ +o(\epsilon_n^2)\,. \end{aligned}$$

Therefore for all sufficiently large N we have

$$|x_k| \le |x_0| + 2\lambda|p_{n-k}p_n| \le R + 2\lambda|pq| = R'\,.$$

Reasoning by contradiction, we assume that the lemma is false. Then there is a sequence of points $y = y_N \in L_{p_{n-k}}$ such that $\Upsilon_N^k(y) \in \Phi_N^k(x)$ and $|yx| > k\lambda|pq|/N + \varepsilon$. Using that $\widetilde{\Pi}_N$ are noncontracting mappings we obtain

$$\begin{aligned} \big|\widetilde{\Pi}_N(y)\,\Phi_N(x)\big| &= \big|\widetilde{\Pi}_N(y)\,B_{\lambda|pq|/N} \circ \widetilde{\Pi}_N(x)\big|e \\ &\ge \big|\widetilde{\Pi}_N(y)\,\widetilde{\Pi}_N(x)\big| - \lambda|pq|/N \ge |yx| - \lambda|pq|/N\,. \end{aligned}$$

In the next step

$$\begin{aligned} \big|\widetilde{\Pi}_N^2(y)\,\Phi_N^2(x)\big| &= \big|\widetilde{\Pi}_N \circ \widetilde{\Pi}_N(y)\,B_{\lambda|pq|/N} \circ \widetilde{\Pi}_N \circ \Phi_N(x)\big| \\ &\ge \big|\widetilde{\Pi}_N(y)\,\Phi_N(x)\big| - \lambda|pq|/N \ge |yx| - 2\lambda|pq|/N\,, \end{aligned}$$

after k-th step we have

$$|\widetilde{\Pi}_N^k(y)\,\Phi_N^k(x)| \ge |yx| - k\lambda|pq|/N\,.$$

From above $|y| \le R'$ for some R', and therefore the properties Υ_N^k insure that if N is sufficiently large we obtain $|\widetilde{\Pi}_N^k(y)\,\Upsilon_N^k(y)| \le \epsilon$. Therefore, for large N

$$0 = \big|\Upsilon_N^k(y)\,\Phi_N^k(x)\big| \ge \big|\widetilde{\Pi}_N^k(y)\,\Phi_N^k(x)\big| - \big|\widetilde{\Pi}_N^k(y)\,\Upsilon_N^k(y)\big|$$

$$> |yx| - k\lambda|pq|/N - \varepsilon > 0\,,$$

a contradiction. □

1.14 LEMMA. *If U is a neighborhood of x which is $e^{\pm\delta}$-bi-Lipschitz equivalent to a domain in Euclidean space and for $0 \le i \le k$*

$$\Phi_N^i(x) \subset \Upsilon_N^i(U)\,,$$

then

$$\mathrm{Vol}_{n-1}(\Phi_N^k(x)) \ge \big(1 - l_{1.14}(\delta)\big)\mathrm{Vol}(B_1^{n-1})\left(\frac{k\lambda|pq|}{N}\right)^{n-1}$$

where $l_{1.14}(\delta) = 2n^2\delta$ and B_1^{n-1} is a unit ball in Euclidean space.

Proof. This is an immediate corollary of the noncontracting property of $\widetilde{\Pi}$ (see Lemma 1.5) and the coarea formula (see Corollary 1.11). □

COROLLARY. *If U is an open neighborhood of x which is $e^{\pm\delta}$-bi-Lipschitz equivalent to a domain in Euclidean space and*

$$\Phi_N^k(x) \subset \Upsilon_N^k(U)\,,$$

then

$$\limsup_{N\to\infty} d_H(\Phi_N^k(x)), B_{\lambda|p_{n-k}p_n|}(\Upsilon_N^k(x)) \le \lambda c_{1.14}(\delta) \limsup_{N\to\infty} |p_{n-k}p_n|$$

where $c_{1.14}(\delta) = 2(2n^2\delta)^{1/n}$ and d_H is the Hausdorff distance between subsets of L_{p_n}.

Proof. Since L_{p_n} is an Alexandrov space with curvature ≥ 0 (see [BGP, 10.2]), this follows immediately from Lemmas 1.13a and 1.14 and the fact that the volume of a ball in L_{p_n} does not exceed the volume of a ball of the same radius in Euclidean space. □

1.15. Now we are ready to complete the proof of the Key Lemma for B (1.12).

Let us consider a shortest path β between $x \in Q_p$ and $y \in T^{-1}(Q_q)$ in L_p. Assume z is the midpoint of β.

CLAIM. *There is $\rho > 0$ such that for all sufficiently large N*

$$\mathrm{Vol}\big(B_\rho(\Upsilon_N^N(z)) \cap \Phi_N^N(x)\big) > \tfrac{1}{2}\mathrm{Vol}(B_\rho(\Upsilon_N^N(z))\,.$$

REMARK. It is easy to see that if all points of the shortest path γ are regular then this claim is a direct corollary of Lemma 1.14 and its corollary. The following trick to treat the general case was suggested by Perelman.

Proof. The points x and y are regular, therefore by the corollary in 1.10 all points on β are regular. It is easy to see that for any $\delta > 0$ there is a neighborhood U_δ of β which is $e^{\pm\delta}$-bi-Lipschitz equivalent to a domain in Euclidean space.

Using the corollary in 1.14 step by step we will prove that for $\varepsilon > 0$ such that $(1-\varepsilon)|pr| > |xz|$ we have

$$\lim_{N\to\infty} \left|\Upsilon_N^{N-[N\varepsilon]}(z)\Phi_N^{N-[N\varepsilon]}(x)\right| = 0\,.$$

Indeed, take $\delta > 0$ such that $(1-2c_{1.14}(\delta))(1-\varepsilon)|pr| > |xz|$. Construct an open neighborhood U_δ of β which is $e^{\pm\delta}$-bi-Lipschitz equivalent to a domain in Euclidean space. Take a compact neighborhood U'_δ of β, such that $U'_\delta \subset U_\delta$. By Lemma 1.13a for sufficiently large $m \in \mathbb{N}$ we have that if $x' \in \Upsilon_N^{k_1}(U'_\delta)$, then for any $k_2 \le N/m$ and sufficiently large N

$$\Phi_N^{k_2}(x') \subset \Upsilon_N^{k_1+k_2}(U_\delta)\,.$$

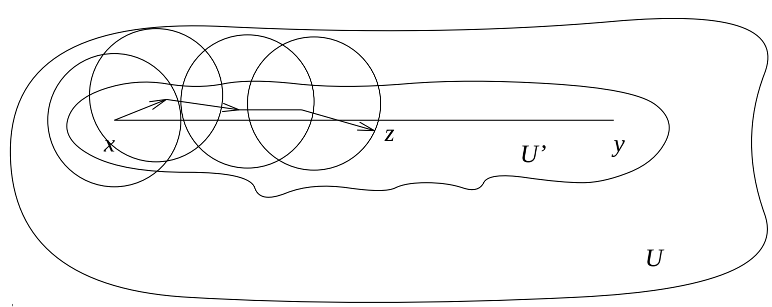
(This will allow us to use Corollary 1.14 for such a point x'.) Also one can assume that $2x_{1.14}|pr|/m$-neighborhood of β is in U'_δ.

Note that if $x' \in \Phi_N^{k_1}(x)$ then $\Phi_N^{k_2}(x') \subset \Phi_N^{k_1+k_2}(x)$.

Now fix m, U_δ and U'_δ.

Let us divide the shortest path xz in m equal parts by points $x = x_0, x_1, \ldots, x_m = z$. Let $k = [N(1-\varepsilon)/m]$. It is easy to see that if N is sufficiently large $[N\varepsilon] + k \ge N - km \ge [N\varepsilon]$.

From Corollary 1.14, for sufficiently large N there is a point $x'_1 \in \Phi_N^k(x)$ such that $|x'_1\Upsilon_N^k(x_1)| < 2c_{1.14}|pr|/m$. Using the same corollary for x'_1 we obtain existence of $x'_2 \in \Phi_N^k(x'_1) \subset \Phi_N^{2k}(x)$ such that $|x'_2\Upsilon_N^{2k}(x_2)| < 2c_{1.14}|pr|/m$; and so on to obtain $x'_m \in \Phi_N^{mk}(x)$ such that $|x'_m\Upsilon_N^{mk}(z)| < 2c_{1.14}|pr|/m$. Using the same corollary again we obtain existence of $x''_m \in \Phi_N^{N-mk-[N\varepsilon]}(x'_m) \subset \Phi_N^{N-[N\varepsilon]}(x)$ such that $|x''_m\Upsilon_N^{N-[N\varepsilon]}(z)| < 2c_{1.14}|pr|/m$.



Now $2c_{1.14}|pr|/m \to 0$ when $m \to \infty$.

Therefore we can find $\varepsilon > 0$ such that

$$\lim_{N\to\infty} \left|\Upsilon_N^{N-[N\varepsilon]}(z)\Phi_N^{N-[N\varepsilon]}(x)\right| = 0\,.$$

Let $x' \in \Phi_N^{N-[N\varepsilon]}(x)$ be the closest point to $\Upsilon_N^N(z)$. From Lemmas 1.13a and 1.14 we obtain that for sufficiently small δ and ε and large N we have

$$\begin{aligned}\operatorname{Vol}\bigl(B_{\lambda\varepsilon|pr|}(\Upsilon_N^N(z)) \cap \Phi_N^N(x)\bigr) &\geq \operatorname{Vol}\bigl(B_{\lambda\varepsilon|pr|}(\Upsilon_N^N(z)) \cap \Phi^{[N\epsilon]}(x')\bigr)\\ &\leq \tfrac{1}{2}\operatorname{Vol}\bigl(B_{\lambda\varepsilon|pr|}(\Upsilon_N^N(z))\bigr)\,.\end{aligned}$$

Therefore the Claim is true for $\rho = \lambda\varepsilon|pr|$. □

Using this Claim we obtain that $\Phi_N^N(x) \cap \Phi_N^N(T(y)) \neq \emptyset$.

Let $z' \in \Phi_N^N(x) \cap \Phi_N^N(T(y)) \subset L_r$. Then by (###)

$$\begin{aligned}\left|\exp_p(\epsilon_n x)\exp_q(\epsilon_n T(y))\right| &\leq \left|\exp_p(\epsilon_n x)\exp_r(\epsilon_n z')\right|\\ &\quad + \left|\exp_r(\epsilon_n z')\exp_q(\epsilon_n T(y))\right|\\ &\leq |pq|\left(1+\frac{\lambda^2\epsilon_n^2}{2}\right) + o(\epsilon_n^2)\,.\end{aligned}$$

This completes the proof of the Key Lemma. □

1.16 *Proof of the Theorem, part B.* Take a sequence of points p_n, q_n which lie in the shortest path pq, with $|p_n p| = |q_n q| \to 0$. Assume r is the middle of segment pq. Let $T_{p_n} : L_{p_n} \to L_r$ and $T_{q_n} : L_{q_n} \to L_r$ be the mappings from the Key Lemma for B (1.12). We can assume that all of these mappings are constructed for some subsequence $\{\epsilon_n\} \subset \{\varepsilon_n\}$.

Take countable everywhere dense subsets $Q_p \subset L_p$ and $Q_q \subset L_q$, pass to a subsequence $\{\epsilon_n\}$ and construct mappings $\widetilde{\Pi}_{p_n} : L_p \to L_{p_n}$ and $\widetilde{\Pi}_{q_n} : L_q \to L_{q_n}$ (see 1.5). We can easily find sequences $\{p_n\}$ and $\{q_n\}$ such that the limits $f = \lim_{n\to\infty} T_{p_n} \circ \widetilde{\Pi}_{p_n} : L_p \to L_r$ and $g = \lim_{i\to\infty} T_{q_n} \circ \widetilde{\Pi}_{q_n} : L_q \to L_r$ exist. Exactly as in 1.9 we can see that f and g are isometries.

We claim that $T = f \circ g^{-1}$ is the mapping which we need.

Assume the statement B of the theorem is false for the constructed mapping T. Then there are vectors $x, y \in L_p$ and a subsequence $\{\epsilon_n\}$ such that

$$\left|\exp_p(\epsilon_n x)\exp_q(\epsilon_n T(y))\right| \geq |pq| + \frac{|xy|^2\epsilon_n^2}{2|pq|} + c\epsilon_n^2$$

for some $c > 0$.

Let $Q_p' = Q_p \cup x$ and $Q_q' = Q_q \cup T(y)$. We can construct $\widetilde{\Pi}'_{q_n}$ and $\widetilde{\Pi}'_{p_n}$ for these subsets and a subsequence of $\{\epsilon_n'\}$. It is easy to see that $\widetilde{\Pi}'_{q_n}$ and $\widetilde{\Pi}'_{p_n}$ can be taken such that $\widetilde{\Pi}'_{p_n}|_{L_p\setminus x} = \widetilde{\Pi}_{p_n}|_{L_p\setminus x}$ and $\widetilde{\Pi}'_{q_n}|_{L_q\setminus T(y)} = \widetilde{\Pi}_{q_n}|_{L_q\setminus T(y)}$.

Note that the construction of T is invariant under changing Π-s to Π'-s. Construct Φ'_{p_n} and Φ'_{q_n} from these Π'-s as in 1.13.

Now $\Phi'_{p_n}(x) \subset L_{p_n}$ and $\Phi'_{q_n}(T(y)) \subset L_{q_n}$ are open subsets and therefore for any $x, y \in L_p$ there are elements $x_n, \in Q_{p_n} \subset L_{p_n}$ and $y \in T_{p_n}^{-1} \circ T_{q_n}(Q_{q_n})$ such that $x_n \in \Phi'(x)$ and $T_n(y_n) \in \Phi'(T(y))$. Using the Key Lemma we obtain

$$\begin{aligned} \big|\exp_p(\epsilon_n x)\exp_q(\epsilon_n T(y))\big| &\le \big|\exp_p(\epsilon_n x)\exp_{p_n}(\epsilon_n x_n)\big| \\ +\big|\exp_p(\epsilon_n x_n)\exp_q(\epsilon_n T_n(y_n))\big| &+ \big|\exp_p(\epsilon_n T_n(y_n))\exp_q(\epsilon_n T(y))\big| \\ &\le |pq|\left(1+\frac{\lambda^2\epsilon_n^2}{2}\right)+o(\epsilon_n^2) \end{aligned}$$

if $\lambda > |x_n y_n|/|p_n q_n|$. But it is easy to see that $\lim_{i\to\infty}|x_n y_n|/|p_n q_n| = |xy|/|pq|$ and therefore we obtain that

$$\big|\exp_p(\epsilon_n x)\exp_q(\epsilon_n T(y))\big| \le |pq|\left(1+\frac{\lambda^2\epsilon_n^2}{2}\right)+o(\epsilon_n^2)$$

for any $\lambda > |xy|/|pq|$, or

$$\big|\exp_p(\epsilon_n x)\exp_q(\epsilon_n T(y))\big| \le |pq|\left(1+\frac{|xy|^2\epsilon_n^2}{2|pq|^2}\right)+o(\epsilon_n^2)\,. \qquad \square$$

2 Synge's Theorem

2.0. The following theorem was proved by J.L. Synge in 1936 for Riemannian manifolds. Here we generalize this theorem to the case of Alexandrov spaces. Our proof is only a small modification of the original one; it is an example of the application of our second variation formula.

Generalized Synge's Theorem 2.1. a) *If M is an even dimensional orientable Alexandrov space with curvature ≥ 1, then M is simply connected.*

b) *If M is an odd dimension locally orientable (see 1.0) Alexandrov space with curvature ≥ 1, then M is orientable.*

In order to formulate the following Generalized Synge's Lemma we need a formal generalization of Alexandrov space.

Definition. A metric space will be called an *Alexandrov domain* if for every point there exists a compact neighborhood which is an Alexandrov space (with the same curvature bound).

In particular, every open subset of Alexandrov space is an Alexandrov domain because every point has a convex neighborhood (see [PPe1, 4.3]).

2.2 Generalized Synge's Lemma. A. *Let Γ be an Alexandrov domain with curvature ≥ 1, $\gamma \subset \Gamma$ be a closed path and one of the following be true:*

a) *Γ is orientable and even dimensional;*

or

b) *arbitrarily small neighborhoods of γ in Γ are nonorientable and Γ has odd dimension.*

Then for every $\varepsilon > 0$ there is an ε-close path γ_ε (ε-close means that for some parameterizations $|\gamma_\varepsilon(t)\gamma(t)| < \varepsilon$) such that

$$length(\gamma_\varepsilon) < length(\gamma)\,.$$

B. *Let Σ be an orientable Alexandrov space with curvature ≥ 1, $T : \Sigma \to \Sigma$*

a) *Σ is even dimensional and T preserves orientation;*

or

b) *Σ is odd dimensional and T reverses orientation.*

Then T has a fixed point.

Proof. We will prove this using the induction scheme $\mathrm{B}_n \to \mathrm{A}_{n+2} \to \mathrm{B}_{n+2} \dots$ and take as a base the trivial cases B_0 and B_1 (lower indexes indicate the dimension).

($\mathrm{A}_n \to \mathrm{B}_n$) Let us consider the function

$$f\colon \Sigma \to \mathbb{R}_+\,, \qquad f(x) = |xT(x)|$$

and let p be a minimum point of f on Σ. If $f(p) = 0$ then p is a fixed point of T. Therefore assume $f(p) > 0$.

Let us consider projection to the quotient space of an ϵ-neighborhood of a shortest path γ between p and $T(p)$ by isometry T. The resulting metric space Γ is easily an Alexandrov domain and γ is glued in a closed path. Therefore using (A) we obtain that there is an ε-close path γ_ε such that

$$length(\gamma_\varepsilon) < length(\gamma)\,.$$

Let us consider any point on γ_ε. It is easy to see that for the preimage $q \in \Sigma$ of this point we obtain

$$f(q) \leq \mathrm{length}(\gamma_\varepsilon) < \mathrm{length}(\gamma) = f(p)\,,$$

a contradiction.

($\mathrm{B}_n \to \mathrm{A}_{n+2}$) We may assume γ is a closed geodesic (locally shortest path).

Let us divide our closed geodesic into shortest paths p_ip_{i+1} such that each of them lies in a compact convex subset which is an Alexandrov space. For every such a path p_ip_{i+1} ($p_0 = p_N$) consider the map $T_i\colon L_{p_{i-1}} \to L_{p_i}$ from Theorem 1.2(B). Using this theorem we can choose these maps such that for some sequence $\epsilon_n \to 0$ and every i and x we have

$$\left| \exp_{p_{i-1}}(\epsilon_n x) \exp_{p_i}(\epsilon_n T_i(x)) \right| = |p_{i-1}p_i| \left(1 - \tfrac{k}{2}\epsilon_n^2 x^2\right) + o(\epsilon_n^2)\,,$$

and moreover T_i preserve orientation of a small neighborhood of the segment $p_{i-1}p_i$.

As a result we obtain a map $T = T_N \circ ... \circ T_2 \circ T_1 : L_{p_0} \to L_{p_0}$ which preserves orientation in even dimensional case and reverses orientation in the odd dimensional case. T is an isometry, and we can consider the restriction of this map to Λ_{p_0}, where $L_{p_0} = C(\Lambda_{p_0})$. Therefore using ($\mathrm{B}_n$) we obtain a fixed point $x_0 \in L_{p_0}$ of this mapping. Assume $x_N = T_N \circ ... \circ T_2 \circ T_1(x_0)$ ($x_N = x_0$). Then using the second variation formula we obtain

$$\begin{aligned} \mathit{length}(\gamma) > \left| \exp_{p_0}(\epsilon_n x_0) \exp_{p_1}(\epsilon_n x_1) \right| + \left| \exp_{p_1}(\epsilon_n x_1) \exp_{p_2}(\epsilon_n x_2) \right| + ... \\ + \left| \exp_{p_{N-1}}(\epsilon_n x_{N-1}) \exp_{p_0}(\epsilon_n x_0) \right|. \end{aligned}$$

Therefore the broken geodesic

$$\exp_{p_0}(\epsilon_n x_0) \exp_{p_1}(\epsilon_n x_1) ... \exp_{p_{N-1}}(\epsilon_n x_{N-1}) \exp_{p_0}(\epsilon_n x_0)$$

can be taken as γ_ϵ for $\epsilon \gg \epsilon_n$. □

3 Frankel's Theorem

3.0. The following theorem was proved by Theodore Frankel [F, Th.1] in 1961 for the Riemannian case. Unfortunately totally geodesic submanifolds in a Riemannian manifold with positive curvature are very rare phenomena. Thus Frankel's theorem turns out to be particularly important for the Alexandrov case because extremal subsets are always totally quasi-geodesic (see the corollary below). Again the ideology of our proof is the same as in the original; it is another illustration of the work of our second variation formula.

3.1. Let H be a subset of M, and denote by $|**|_H$ the intrinsic metric in H.

Definition. A connected closed subset H of an Alexandrov space M is called totally quasi-geodesic if

1. For any compact subset $K \subset H$ there is $\varepsilon > 0$ such that

$$|pq|_H < \varepsilon^{-1}|pq|$$

for any $p, q \in K$.

2. Any shortest path in the intrinsic metric of H is a quasi-geodesic and moreover it can be prolonged infinitely on both sides in H as a quasi-geodesic in M.

If H is totally quasi-geodesic then for any point $p \in H$ we know $C_p(H)$ is a totally quasi-geodesic subset of C_p and $\Sigma_p(H)$ is a totally quasi-geodesic subset of Σ_p (here $C_p(H) = C(\Sigma_p(H))$ (see [PPe2, 2.3(3)] or [Pe, I 2.4]). In particular from any point $p \in H$ and $\xi \in \Sigma_p(H)$, there is a quasi-geodesic in H with this initial date, which is infinitely long on both sides.

Theorem 3.2. *Let F and G be two totally quasi-geodesic subsets of an Alexandrov space Σ with curvature ≥ 1, and suppose $\dim F + \dim G \geq \dim \Sigma$. Then $F \cap G \neq \emptyset$.*

COROLLARY 3.3. *Let F and G be two extremal subsets of an Alexandrov space Σ with curvature ≥ 1, where $\dim F + \dim G \geq \dim \Sigma$. Then $F \cap G \neq \emptyset$.*

Proof of the corollary. We only need to verify that any extremal subset is a totally quasi-geodesic subset.

Part 1 of the definition is easily seen from [PPe1, 3.2(2)]. Part 2 of the definition is a direct corollary of the generalized Lieberman lemma ([PPe1, 5.3] or [Pe, II 1.1]) and construction of quasi-geodesic (see [PPe2, 6.3(b)] or [Pe, I 4.1(A′)]). Now we only need to apply the theorem.

3.4. Now we prepare for the proof of the theorem.

LEMMA. *For a totally quasi-geodesic subset F of a compact Alexandrov space M and for arbitrary $\varepsilon > 0$ there is a function $f : F \to \mathbb{R}$ such that*

a) *For any quasi-geodesic $\gamma \in F$ we have $(f \circ \gamma)'' \leq \varepsilon$, $|(f \circ \gamma)'| < \varepsilon$, $|f| < \varepsilon$.*

b) *If p is a singular point of F, i.e. $C_p(F) \neq \mathbb{R}^k$ where k is the dimension of F, then there is a pair of polar directions $(\xi, \xi^*) \in C_p(F) \subset C_p$ such that*

$$\frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \xi^*} < 0 \,.$$

Proof. We claim that there is $\delta(\varepsilon)$ such that the function

$$f(p) = \delta(\varepsilon) \oint_{x \in M} \mathrm{dist}_x^2(p) dh_n$$

is the function we need. From the definition of quasi-geodesic it is easy to see that a) is true for $\delta(\varepsilon) \leq \varepsilon \min\{1, \mathrm{Diam}^{-2}(M)\}$. It is easy to see that for all pairs of polar vectors $(\xi, \xi^*) \in C_p(F)$

$$\frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \xi^*} \le 0\,.$$

Assume the inequality is exact. It means that for all such pairs (ξ, ξ^*) we have

$$\big\langle \lambda \log_p(x), \xi \big\rangle + \big\langle \lambda \log_p(x), \xi^* \big\rangle = 0$$

for almost all $x \in M$ and $\lambda \ge 0$. It follows that all pairs (ξ, ξ^*) are opposite, i.e. there is a representation $C_p = \mathbb{R} \times C_p'$ such that $\xi = (|\xi|, o)$ and $\xi^* = (-|\xi|, o)$. Thus there is a representation $C_p = \mathbb{R}^{k'} \times C_p'$ such that $C_p(F) \subset \mathbb{R}^{k'} \times o$. F is totally quasi-geodesic, therefore $C_p(F)$ is a linear subspace of $\mathbb{R}^{k'} \times o$. Hence p is a regular point of F. □

3.5 *Proof of the Theorem.* The proof is carried in sections 3.5-3.8.

Let $h = |FG| = \min_{p \in F,\, q \in G} |pq|$. If $h = 0$ then $F \cap G \neq \emptyset$, therefore assume $h > 0$.

Take some $\varepsilon > 0$. Consider functions f on F and g as in Lemma 3.4 and set

$$\Psi(p,q) = \big(|pq| + f(p) + g(q)\big)$$

for $p \in F$, $q \in G$. Let (p,q) be a minimum pair for Ψ. Lemma 3.3(b) guarantees that p (q) is a regular point of F (G).

Therefore for any vector $v \in C_p(F)$ there is an opposite vector $v^* \in C_p(F)$. Using that (p,q) is a minimum pair of Ψ we obtain

$$\langle v, q_p' \rangle \le d_p f(v)\,, \qquad \langle v^*, q_p' \rangle \le d_p f(v^*)\,.$$

v is opposite to v^* and f is ε-concave (see Lemma 3.4(a)), therefore

$$\langle v, q_p' \rangle + \langle v^*, q_p' \rangle \ge 0\,, \qquad d_p f(v) + d_p f(v^*) \le 0\,,$$

hence

$$\langle v, q_p' \rangle = d_p f \qquad (\langle v, p_q' \rangle = d_q g)\,,$$

for any $v \in C_p(F)$, $(C_q(G))$. In particular $d_p f$ $(d_q g)$ is a linear function on $C_p(F)$ $(C_q(G))$.

Let us consider a shortest path pq, let r be the center of this shortest path and $\{p_n\}, \{q_n\}$ be two sequences of points on pq such that $|p_n p| = |q_n q| \to 0$. Let us consider two isometries $T_{p_n} : L_{p_n} \to L_r$, $T_{q_n} : L_{q_n} \to L_r$ as in Theorem 1.2(B). Then $T_n = T_{q_n}^{-1} \circ T_{p_n}$ satisfy the inequality 1.2(B).

3.6. Now we start to construct mappings $\Pi_n : C_p(F) \to L_{p_n}$ $(C_q(G) \to L_{q_n})$. This construction is almost the same as the construction which we made in 1.4, 1.5, 1.6 and we write only the necessary changes in these sections.

(Changes to 1.4) First of all F is totally quasi-geodesic, therefore one can construct an exponential map $\exp_p$ such that $\exp_p(C_p(F)) \subset F$. Using this exponential map we can construct a map $\pi : C_p \to L_{p_n}$. For the restriction $\pi|_{C_p(F)}$ we can obtain the statement of Lemma 1.4, using the same proof but with our function $f(x)$ instead of $|ax|$.

(Changes to 1.5) Take a countable, everywhere dense subset $Q_p \subset C_p(F)$ and pass to a subsequence of $\{\epsilon_n\}$ such that the following limit exists for any $v \in Q_p$:

$$\lim_{n\to\infty} \pi(v\varepsilon_n)/\varepsilon_n\,.$$

This defines a map

$$\Pi\colon Q_p \subset C_p(F) \to L_{p_n}$$
$$\Pi(v) \stackrel{\text{def}}{=} \lim_{n\to\infty} \pi(v\varepsilon_n)/\varepsilon_n\,.$$

The same argument as in Lemma 1.5 shows that for any $v, u \in Q_p$ we have

$$\big|\Pi(v)\Pi(u)\big|^2 \ge |uv|^2 - \big|df(u) - df(v)\big|^2\,.$$

Set

$$\|uv\| \stackrel{\text{def}}{=} \sqrt{|uv|^2 - |df(u) - df(v)|^2}\,.$$

Therefore we obtain that our mapping Π is noncontracting with respect to $\|{*}{*}\|$.

Note that $(C_p(F), \|{*}{*}\|)$ is flat and we have

$$(1-\varepsilon^2)|{*}{*}| \le \|{*}{*}\| \le |{*}{*}|\,.$$

(Changes to 1.6) The same argument as in the proof of Lemma 1.6 shows that for any $v \in Q_f \subset C_p$

$$\begin{aligned}
&f(\exp_p(\epsilon_n v)) + \big|\exp_p(\epsilon_n v)\exp_{p_n}(\epsilon_n \Pi(v))\big| \\
&\quad\le f(p) + |pp_n| + \frac{\varepsilon|v|^2\epsilon_n^2}{2} + \frac{1}{2|pp_n|}\big(\|v\|^2 - |\Pi(v)|^2\big)\epsilon_n^2 + o(\epsilon_n^2) \\
&\quad\le f(p) + |pp_n| + \varepsilon\|v\|^2\epsilon_n^2 + \frac{1}{2|pp_n|}\big(\|v\|^2 - |\Pi(v)|^2\big)\epsilon_n^2 + o(\epsilon_n^2)\,,
\end{aligned}$$

because if ε is sufficiently small then $|v| \le 2\|v\|$. Using that (p,q) is a minimum pair for Ψ we obtain

$$\begin{aligned}
g(q) + |qq_n| + \big|q_n \exp_{p_n}(\epsilon_n\Pi(v))\big| + \big|\exp_{p_n}(\epsilon_n\Pi(v))\exp_p(\epsilon_n v)\big| + f(\exp_p(\epsilon_n v)) \\
\ge \Psi(\exp_p(\epsilon_n v), q) \ge \Psi(q,p)\,.
\end{aligned}$$

Hence

$$\left(\varepsilon + \frac{1}{2|pp_n|}\right)\|v\|^2 \ge \left(\frac{1}{2|q_np_n|} + \frac{1}{2|pp_n|}\right)|\Pi(v)|^2\,.$$

3.7. Therefore for some $C>0$

$$|\Pi(v)| \le \bigl(1+C|pp_n|\bigr)\|v\|\,.$$

Pass to subsequences of $\{p_n\},\{q_n\}$ such that there are the following limits:

$$T_p=\lim_{n\to\infty} T_{p_n}\circ\Pi_n$$

and

$$T_q=\lim_{n\to\infty} T_{q_n}\circ\Pi_n\,.$$

From above (see 3.6, Changes to 1.5) T_p and T_q are noncontracting mappings which preserve the norm. Therefore for any $u,v\in(C_p(F),\|**\|)$ we have

$$\widetilde{\angle} uov \le \widetilde{\angle} T_p(u)oT_p(v)$$

and this inequality is exact if and only if $|T_p(u)\,T_p(v)|=\|uv\|$.

Using that $(C_p(F),\|**\|)$ is flat we obtain that for any two vectors $u,v\in(C_p(F),\|**\|)$ there is a vector $w\in(C_p(F),\|**\|)$ such that

$$\widetilde{\angle} uov+\widetilde{\angle} vow+\widetilde{\angle} wou=2\pi\,.$$

Therefore

$$\widetilde{\angle} T_p(u)oT_p(v)+\widetilde{\angle} T_p(v)oT_p(w)+\widetilde{\angle} T_p(w)oT_p(u)\ge\widetilde{\angle} uov+\widetilde{\angle} vow+\widetilde{\angle} wou=2\pi.$$

Since L_r is an Alexandrov space with curvature ≥ 0, the inequality is exact and therefore T_p and T_q are isometries with respect to $\|**\|$.

3.8. Hence $\operatorname{clos}(\operatorname{Im} f)$ and $\operatorname{clos}(\operatorname{Im} g)\ne\emptyset$ are linear subspaces of the cone L_r. $\dim F+\dim G\ge\dim\Sigma$, therefore,

$$\operatorname{clos}(\operatorname{Im} f)\cap\operatorname{clos}(\operatorname{Im} g)\ne\emptyset\,.$$

Hence for any $\nu>0$ there is a pair of vectors $u\in C_p(F)$ and $v\in C_q(G)$ and sufficiently large n such that $|T_{q_n}\circ\Pi_n(v)\;T_{p_n}\circ\Pi_n(u)|\le\nu\|v\|$, or equivalently $|\Pi_n(v)\;T^{-1}\circ\Pi_n(u)|\le\nu\|v\|$. Set $v'=\Pi_n(v)$ and $u'=T^{-1}\circ\Pi_n(u)$. We can assume that all mappings Π_n and T_n are constructed for one sequence $\{\epsilon_n\}$.

Now let us put together inequalities:

$$\begin{aligned}
\Psi(p,q) &\le \Psi\bigl(\exp_q(\epsilon_n u),\exp_p(\epsilon_n v)\bigr)\\
&\le g(\exp_q(\epsilon_n u))+\bigl|\exp_q(\epsilon_n u)\exp_{q_n}(\epsilon_n T(u')\bigr|\\
&\quad+\bigl|\exp_{q_n}(\epsilon_n T(u'))\exp_{p_n}(\epsilon_n v')\bigr|\\
&\quad+\bigl|\exp_{p_n}(\epsilon_n v')\exp_p(\epsilon_n v)\bigr|+f(\exp_p(\epsilon_n v))\\
&\le \Psi(pq)+\varepsilon\bigl(\|v\|^2+\|u\|^2\bigr)\epsilon_n^2
\end{aligned}$$

$$+ \frac{1}{2|pp_n|}\big(\|v\|^2 - |\Pi(v)|^2 + \|u\|^2 - |\Pi(u)|^2\big)\epsilon_n^2$$
$$+ |v'u'|^2\epsilon_n^2 - \frac{|p_nq_n|}{6}\big(|v'|^2 + \langle v', u'\rangle + |u'|^2\big)\epsilon_n^2 + o(\epsilon_n^2)$$
$$\leq \Psi(pq)+\varepsilon\|v\|^2(1+\nu)^2\epsilon_n^2+\|v\|^2\nu^2\epsilon_n^2-\frac{|p_nq_n|}{2}(\|v\|^2)(1-\nu)^2\epsilon_n^2+o(\epsilon_n^2)\,.$$

Therefore

$$\varepsilon(1+\nu)^2 + \nu^2 > \frac{|p_nq_n|}{2}(1-\nu)^2$$

and this is impossible for sufficiently small ε and ν. □

Comments

We have seen that the constructed parallel transportation has almost the same properties as parallel transportation in the Riemannian case. But this construction has a lot of choices: we need to fix a sequence $\{\epsilon_n\}$, exponential mappings and direction of transportation. We do not have any example when it does really depend on these choices and it would be interesting to avoid this, in particular it would give us a uniquely defined integral curvature for Alexandrov spaces. Furthermore it would make the second variation formula much easier to use.

It would also be interesting to find some connections between our parallel transportation and other natural parallel transportations. For example: the parallel transportation which can be constructed along a curve on a surface in $\mathbb{R}^n$; also there is a "direct" construction of connection on Alexandrov space (see [P3, 4.3]).

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Anton Petrunin
Current address:
Department of Mathematics
SUNY
Stony Brook, NY 11794-3651
USA

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