Алгебра и анализ
Том 5 (1993), вып. 1

St. Petersburg Math. J.
Vol. 5 (1994), No. 1

EXTREMAL SUBSETS IN ALEKSANDROV SPACES AND THE GENERALIZED LIBERMAN THEOREM

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Dedicated to To A. D. Aleksandrov on the occasion of his 80th birthday

ABSTRACT. On the basis of the notion of extremal subset, we construct a natural stratification of Aleksandrov space that takes into account both its topological and metric singularities, and establish a number of its properties. The most important of these is the quasigeodesicity of the strata, which generalizes a classical theorem of Liberman on the shortest curves on convex hypersurfaces in $\mathbb{R}^n$.

INTRODUCTION

The present paper continues the study of finite dimensional Aleksandrov spaces (with curvature bounded from below) started in [3]. It is also closely connected with [6]. We will freely use the notions, results and standard notation from the papers mentioned.

As shown in [6], an Aleksandrov space possesses a canonical topological stratification whose strata are topological manifolds (Theorem II). At the same time, metric singularities may be arranged chaotically, constituting no strata. Nevertheless "essential" metric singularities often constitute strata, for example, when they arise as the result of factorization by a group of isometries.

In this paper we describe a stratification of an Aleksandrov space that takes into account both its topological and metric singularities: the closures of its strata are all possible primitive extremal subsets of the space (see Subsection 3.8). We also establish a number of natural properties of this stratification, the most important of them being the "total quasigeodesicity" (in a natural sense) of the strata, see Theorems 5.2 and 5.3. The latter property is a natural generalization of a classical theorem of Liberman.

Historical remarks. Liberman's theorems [4] describe external geometric properties of the shortest curves on convex hypersurfaces in $\mathbb{R}^n$. They are based on an important observation: if a cylinder whose directrix is such a shortest curve is transversal to the hypersurface, then the shortest curve is a convex curve with respect to the (flat) inner metric of the cylinder. The convexity will be preserved if we replace the cylinder by a cone with vertex inside the convex body and with the same directrix. It is precisely the last statement that we call here the Liberman theorem.

Independently of Liberman's theorems, in the fifties A. D. Aleksandrov defined and studied quasigeodesic curves on a convex surface in $\mathbb{R}^3$ ([1], see also [2]). The quasigeodesic curves constitute exactly the closure of the class of geodesics (that is, locally shortest curves). Aleksandrov's definitions and constructions admit no

1991 *Mathematics Subject Classification.* Primary 54E40, 58E20.
Key words and phrases. Aleksandrov space, space with stratification, extremal subset, quasigeodesic curve.

direct generalizations to higher dimensions. For polyhedra such a generalization was attempted by Milka [5].

In the second author's MS thesis (Quasigeodesics in multidimensional Aleksandrov spaces with curvature bounded from below, Leningrad State University, 1991, hereafter referred to as [Pet]) an entirely different definition of a quasigeodesic in an Aleksandrov space (with curvature bounded from below) of arbitrary dimension was given, and some of the basic properties of quasigeodesics were studied. From the point of view of this definition (see Subsection 5.1 below), the Liberman Theorem can be restated in the following way: each geodesic on a convex hypersurface in $\mathbb{R}^n$ is a quasigeodesic for the corresponding convex body regarded as an Aleksandrov space of nonnegative curvature. The present paper gives a generalization of this form of the Liberman theorem. The proof uses only the contents of §1 and of Subsections 3.1, 3.2(2).

Convention. In the course of the paper M will denote a compact n-dimensional Aleksandrov space with curvature $\geqslant 0$, $n \geqslant 2$. (All the assertions proved remain valid for any other lower bound of the curvature; the proofs either carry over *verbatim* or need only a slight modification. As a rule compactness is inessential; in the cases when an assertion is formally wrong for a noncompact space, it still holds for any of its relatively compact domains.) Σ will always denote a complete space of curvature $\geqslant 1$, and to simplify the base step of the inductive arguments we do not exclude spaces of dimension 1 (a segment of length $\leqslant \pi$ and a circle of length $\leqslant 2\pi$) and even spaces of dimension 0 (a point and a pair of points at distance π), the segment and the point being regarded as spaces with boundary, while the circle and the pair of points are regarded as spaces without boundary.

§1. Definition of extremal subsets, their elementary properties, and examples

1.1. **Definition.** A closed subset $F \subset M$ is said to be *extremal* if for any distance function $f = \operatorname{dist}(q)$, $q \in M$, $f(p) = |pq|$, the following condition is fulfilled:

If f has a local minimum on F at a point $p \neq q$, then p is a critical point of maximum type for f on M, i.e.,

$$\varlimsup_{p_i \in M,\ p_i \to p} \frac{(f(p_i) - f(p))}{|pp_i|} \leqslant 0.$$

In the case where $M = \Sigma$, that is, in the case of curvatures $\geqslant 1$, one more condition is imposed. If $F \subset \Sigma$ is empty or consists of one point $p \in \Sigma$, then we require in addition that $\operatorname{diam} \Sigma \leqslant \pi/2$ or $\Sigma \subset \overline{B}_p(\pi/2)$, respectively. It is obvious that the space itself is its own extremal subset.

1.2. **Examples.** Let $F \subset M$ be a closed subset satisfying the following condition: with each its point p, F contains all points q from some neighborhood $U_p \subset M$ whose conic neighborhoods are homeomorphic to a conic neighborhood of p (the homeomorphism carrying q to p). Then F is an extremal subset. In particular, the boundary ∂M (cf. Subsection 4.6 of the first author's preprint "Aleksandrov spaces with curvature bounded from below II, 1991") and the closures of the strata of the canonical stratification are extremal subsets.

Indeed, if a point $p \in F$ is not a critical point of maximum type for the function $f = \operatorname{dist}(q)$ on M, then it is a regular point of f. Therefore, by assertion (A) of Main Theorem 1.4 of [6], a small neighborhood of p can be topologically trivialized with respect to f. So we can find a sequence of points $p_i \to p$ with $f(p_i) < f(p)$

such that a conic neighborhood of each p_i is homeomorphic to a conic neighborhood of p. But then $p_i \in F$, and p is not a point of local minimum for f on F, as desired.

1.3. **Definition.** By the tangent space $\Sigma_p F$ to a closed set F at a point $p \in F$ we mean the set of limit points in Σ_p of the directions of shortest curves pp_i, where $p_i \in F$, $p_i \to p$. (Evidently, $\Sigma_p M = \Sigma_p$.) We will see below that for extremal sets F this definition is equivalent to the standard definition of the tangent space as the set of directions of all curves that start at p and lie in F (see 3.3).

1.4. **Proposition.** *Let $F \subset M$ be an extremal subset. Then for every $p \in F$ the set $\Sigma_p F \subset \Sigma_p$ is also extremal. The converse is true if F contains at least two points.*

Proof. Leaving aside the trivial cases in which $\Sigma_p F$ is empty or consists of one point, assume that the extremality condition for $\Sigma_p F$ is violated at a point ξ. Then obviously there are $\eta, \zeta \in \Sigma_p$ such that the function $\operatorname{dist}(\eta)$ has at ξ a strict global minimum on $\Sigma_p F$, $|\xi\zeta|$ and $|\xi\eta| < \pi/2$ and $\tilde{\angle}\eta\xi\zeta > \pi/2$. We send out some curves $\eta(t)$ and $\zeta(t)$ in the directions η and ζ and consider a sequence $p_i \to p$, $p_i \in F$, with the directions of the shortest curves pp_i converging to ξ. For each p_i we find a point q_i on $\eta(t)$ and a point r_i on $\zeta(t)$ with

$$|pq_i| \cdot \cos|\eta\xi| = |pp_i| = |pr_i| \cdot \cos|\zeta\xi|.$$

Let s_i be a minimum point of $\operatorname{dist}(q_i)$ on F. Note that since the directions of pq_i tend to η and those of ps_i cannot have limit points in $\overline{B}_\eta(\eta\xi) \setminus \{\xi\}$,

$$\overline{\lim} \frac{|pq_i| \cdot \sin|\xi\eta|}{|q_i s_i|} \leqslant 1.$$

On the other hand, $|q_i s_i| \leqslant |q_i p_i|$ and

$$\overline{\lim} \frac{|pq_i| \cdot \sin|\eta\xi|}{|q_i p_i|} = 1.$$

So $|q_i s_i|/|q_i p_i| \to 1$ and the limit set of directions of ps_i lies in $\overline{B}_\eta(|\eta\xi|)$. It follows that these directions tend to ξ. But then $\tilde{\angle}q_i s_i r_i \to \tilde{\angle}\eta\xi\zeta$, that is, for i sufficiently large we have $\tilde{\angle}q_i s_i r_i > \pi/2$, which contradicts the extremality of F. The first part of Proposition 1.4 is proved.

To prove the converse statement note that the condition of extremality of F at p means that the situation in which for some $\xi \in \Sigma_p$ both $|\xi\Sigma_p F| \geqslant \pi/2$ and $\max_{\eta \in \Sigma_p} |\xi\eta| > \pi/2$, is forbidden. Suppose to the contrary that such a situation occurs. Let η be a point of Σ_p farthest from ξ and let ζ be a point of $\Sigma_p F$ nearest to ξ. If $\Sigma_p F$ has at least two points, we can assume $\zeta \neq \eta$, and then $\tilde{\angle}\xi\zeta\eta > \pi/2$, which contradicts the extremality of $\Sigma_p F$ at ζ. Also if $\Sigma_p F = \varnothing$ or $\Sigma_p F$ is a singleton and $\eta = \zeta$, then according to our definition the relation $|\xi\eta| \leqslant \pi/2$ must hold, a contradiction. □

1.4.1. The arguments proving the second part of 1.4 also allow us to prove the following.

Proposition. *Let $F \subsetneqq \Sigma$ be a proper extremal subset of Σ and let p be the point of Σ farthest from F. Then the distance of each point of Σ to p is at most $\pi/2$.*

Proof. Let q be any point of Σ. Since p is the point farthest from F, we can find a point $r \in F$ with $|pr| = |pF|$ and $\angle rpq \leqslant \pi/2$. On the other hand, $\angle prq \leqslant \pi/2$, since F is extremal. The second part of the proof of 1.4 implies $|pF| \leqslant \pi/2$. Therefore $|pq| \leqslant \pi/2$. □

1.5. The following assertion is a modification of Lemma 2.3 of [6]. The definition of the class $\mathrm{DER}(\Sigma)$ can also be found there (Subsection 2.1).

Proposition. *Let $F \subset \Sigma$ be an extremal subset, let $f_i \in \mathrm{DER}(\Sigma)$, $i = 0, \dots, k+1$, $k \geqslant 0$, and assume that $\varepsilon = \min_{i \neq j}(-\langle f_i, f_j \rangle) > 0$. Then*

(1) $\exists q \in F : f_i(q) \geqslant \varepsilon$ *for* $0 \leqslant i \leqslant k$;

(2) $\exists p \in F : f_i(p) = 0$ *for* $1 \leqslant i \leqslant k$, $f_0(p) \geqslant \varepsilon$, $f_{k+1}(p) \leqslant -\varepsilon$.

Proof. First of all note that $F \neq \varnothing$, since the condition $\operatorname{diam} \Sigma \leqslant \pi/2$ would contradict the assumption $\langle f_0, f_{k+1} \rangle < 0$. To prove (1), we first find a point $\bar{q} \in \Sigma$ satisfying the inequalities in (1) (this is possible by [6, Lemma 2.3]), and then let $q \in F$ be the point of F nearest to $\bar{q}$. Clearly, $|q\bar{q}| \leqslant \pi/2$ (see 1.4.1). Also, because of the extremality of F, we have $\langle f_i', \chi_{\bar{q}}' \rangle_q \geqslant 0$, whence by the inequality $\langle f', g' \rangle_q \leqslant \langle f, g \rangle - f(q)g(q)$ (see [6, (2.3)]) we see that for $i = 0, \dots, k$ $f_i(q) \cdot (-\cos|q\bar{q}|) \leqslant -\varepsilon$, or $f_i(q) \geqslant \varepsilon$.

Assertion (2) is proved in the same way as assertion (c) of [6, Lemma 2.3], by using Proposition 1.4, only for X we should take the set $\{\xi \in F : f_i(x) \geqslant 0$ for $0 \leqslant i \leqslant k\}$. $\square$

1.6. **Corollary.** *Let $F \subset M$ be an extremal subset, let $p \in F$, and let f be a Lipschitz function on M with $f_p' \in \mathrm{DER}(\Sigma_p)$. Assume that $\varepsilon = \max_{\xi \in \Sigma_p} f'(\xi) > 0$. Then*

$$\max_{\xi \in \Sigma_p F} f'(\xi) = \varepsilon \quad \textit{and} \quad \min_{\xi \in \Sigma_p F} f'(\xi) \leqslant -\varepsilon.$$

It suffices to apply 1.5 to the case $\Sigma = \Sigma_p$. $\square$

§2. Topological structure of extremal subsets

The extremal subsets, as well as the space M itself, possess a canonical stratification whose strata are topological manifolds. To construct this stratification we can use an analog of elementary Morse theory described in [6].

Let $F \subset M$ be an extremal subset. Admissible mappings are defined as the restrictions to F of admissible mappings of M to $\mathbb{R}^k$ (it must be pointed out that we make a distinction between the restrictions of different mappings even if their values on F coincide). A point $p \in F$ is said to be regular for such a mapping if it is regular for the corresponding mapping of M. Then Properties 1.0–1.3 of [6] hold for F. (Indeed, Properties 1.0 and 1.1 are trivial; Property 1.2 is proved for F on the basis of the above Proposition 1.5 in the same way as the corresponding property for M was proved in [6] on the basis of Lemma 2.3. Finally, Property 1.3 for F is a formal consequence of this property for M, with the exception of the nonemptiness condition $K_\rho \cap g_{k+1}^{-1}(0) \cap g^{-1}(v) \neq \varnothing$, which could be omitted in the formulation of 1.3(c) since it follows from 1.3(d) and Property 1.2.)

However, the proof of Main Theorem 1.4 of [6] does not hold for F, since $K_\rho \setminus g_{k+1}^{-1}(0)$ can turn out to be empty for various values of k. Nevertheless it can be rescued as follows. By analogy with MCS-spaces of dimension n (see [6, Introduction]) we can also define $\widetilde{\mathrm{MCS}}$-spaces of dimension $\leqslant n$, requiring that a neighborhood of a point in such a space be homeomorphic to the open cone over a compact $\widetilde{\mathrm{MCS}}$-space of dimension $\leqslant n-1$. In the same way as in the case of MCS-spaces we define a canonical stratification into topological manifolds, the sole difference being that for $\widetilde{\mathrm{MCS}}$-spaces the strata of lower dimension can be open in the space, that is, the space can have different dimension at different points. Now if we modify assertion (C) of Main Theorem 1.4 of [6] replacing MCS-spaces of dimension $n-k$ with $\widetilde{MCS}$-spaces of dimension $\leqslant n-k$, then the proof presented in [6] will cover also the case of extremal subsets. Thus assertions (A) and (B) of Main Theorem 1.4 hold for extremal subsets, and the same is true for Local Theorem I and finally for Stratification Theorem II of [6]. The stratification constructed is called canonical.

§3. Various properties of extremal subsets. Decomposition into primitive components. Stratification of an Aleksandrov space

Our first aim is to prove that the inner metric of an extremal subset is locally finite and locally equivalent to the restriction of the metric of the ambient space.

3.1. **Lemma.** *There exists $\varepsilon > 0$ (depending on M only) such that:*
(1) if $p, q \in M$, $0 < |pq| < \varepsilon^2$, then either

$$\max_{\xi \in \Sigma_q} (\operatorname{dist}(p))'_{(q)}(\xi) > \varepsilon,$$

or

$$\max_{\xi \in \Sigma_p} (\operatorname{dist}(q))'_{(p)}(\xi) > \varepsilon;$$

(2) if $p \in M$ and $F \subset M$ is an extremal subset with $0 < |pF| < \varepsilon^2$, then

$$\max_{\xi \in \Sigma_p} (\operatorname{dist}(F))'_{(p)}(\xi) > \varepsilon.$$

Proof. To prove (1) it suffices to notice that for small $\varepsilon > 0$ and $0 < |pq| < \varepsilon^2$ we can use volume estimates (cf. [3], 8.6) and find a point $x \in M$ with $|px| > 3\varepsilon^{-1}|pq|$ (and so $\pi - \varepsilon < \tilde{\angle} xpq + \tilde{\angle} xqp$) such that for the direction $\xi \in \Sigma_p$ of some shortest curve px we have $\left|\min_{\eta \in q' \subset \Sigma_p} (|\xi\eta| - \pi/2)\right| > 3\varepsilon$.

Assertion (2) is proved similarly: one of the alternatives of (1) is excluded, since if $q \in F$ is a point of F nearest to p, then $\max_{\xi \in \Sigma_q} (\operatorname{dist}(p))'_{(q)}(\xi) \leqslant 0$, because of the extremality of F. $\square$

3.2. **Corollaries.** (1) *Under the notation of Lemma 3.1, let the point p lie in an extremal subset G. Then the conclusion $\max_{\xi \in \Sigma_p} (\operatorname{dist})' > \varepsilon$ can be replaced by $\max_{\xi \in \Sigma_p G} (\operatorname{dist})' > \varepsilon$ and $\min_{\xi \in \Sigma_p G} (\operatorname{dist})' < -\varepsilon$.*

Indeed, this follows from 1.6.

(2) *On each extremal subset G an inner metric is locally induced and, moreover, it is dominated by the external metric times ε^{-1}.*

Indeed, if $p, q \in G$, we can apply Corollary 3.2(1), and then the proof can be concluded in a standard way.

(3) *If under the assumptions of 3.1(1) we have $p \in F$, $q \in G$, where F and G are extremal subsets, then there is a point $r \in F \cap G$ with $\max\{|pr|, |qr|\} \leqslant \varepsilon^{-1}|pq|$.*

This follows from 3.2(1).

(4) *If under the assumptions of 3.1(2) we have $|pF| \leqslant \varepsilon^2/2$, G is an extremal subset containing p, and $0 < R < \varepsilon^2/2$, then we can construct a point $\widehat{p} \in G$ with $|p\widehat{p}| = R$ and $|\widehat{p}F| \geqslant \varepsilon R$.*

This follows from 3.2(1).

3.3. Now we are able to prove the equivalence of the two natural definitions of tangent space for extremal subsets; see Subsection 1.3.

Proposition. *Let $F \subset M$ be an extremal subset, $p \in F$, $\xi \in \Sigma_p F$. Then there exists a curve starting from p in the direction ξ and running over F.*

Proof. It is sufficient to find for each small $\delta > 0$ a number $R(\delta) > 0$ (not depending on ξ) such that for any $R \in (0, R(\delta))$ there is a point $z \in F$ with $|pz| = R$ and $z' \subset B_\xi(\delta) \subset \Sigma_p$. After that the proof will be easily concluded by using Corollary 3.2(2).

Let $q \neq p$ be a point near p such that $q' \subset B_\xi(\delta/4) \subset \Sigma_p$ and $\measuredangle xpy - \widetilde{\measuredangle} xpy < \delta/4$ for any $x, y \in B_p(|pq|)$. Set $R(\delta) = |pq|/2$. To justify our choice, we find for arbitrarily small $\nu > 0$ a sequence $z_0, \dots, z_N \in F$ with $z_0 = p$, $|pz_N| > |pq|/2$, $|z_i z_{i+1}| < \nu$ $\forall i < N$ and

$$\widetilde{\measuredangle} qpz_i \leqslant \frac{\delta}{4}(2 - |qz_i|/|pq|) \quad \text{for all} \quad i > 0. \tag{3.1}$$

We can easily make z_1 satisfy these conditions, and if they are satisfied for z_i, then we can use Corollary 1.6 to construct the next point z_{i+1}. Indeed, for arbitrarily small $\mu > 0$ 1.6 allows us to find z_{i+1} near z_i with

$$\widetilde{\measuredangle} qz_i z_{i+1} \leqslant \pi - \widetilde{\measuredangle} qz_i p + \mu. \tag{3.2}$$

In particular,

$$|qz_{i+1}| \leqslant |qz_i| - (1 - \delta)|z_i z_{i+1}|. \tag{3.3}$$

If μ equals zero, then (3.2) immediately implies that $\widetilde{\measuredangle} qpz_{i+1} \leqslant \widetilde{\measuredangle} qpz_i$. It is easily seen that μ can be chosen so small that (3.1) will hold for z_{i+1} if it holds for z_i. It remains to ensure $|pz_N| > |pq|/2$ for a suitable N. If $|z_i z_{i+1}|$ can be separated from zero, the inequality is evident by (3.3), and if not, we can use a standard trick (see, e.g., Subsection 6.3 of the preprint mentioned in 1.2). $\square$

3.4. Our next aim is to show that if F and G are extremal subsets, then $F \cap G$ and $\overline{G \setminus F}$ are also extremal subsets (if they are nonempty) (see 3.5).

Lemma. *Let F and G be extremal subsets in M, $p \in F \cap G$. Then*

(1) $\Sigma_p(F \cap G) = \Sigma_p F \cap \Sigma_p G$,

(2) $\Sigma_p(\overline{G \setminus F}) = \overline{\Sigma_p G \setminus \Sigma_p F}$.

Proof. (1) The inclusion $\Sigma_p(F \cap G) \subset \Sigma_p F \cap \Sigma_p G$ is trivial. To prove the reverse inclusion we fix $\xi \in \Sigma_p F \cap \Sigma_p G$ and, using Proposition 3.3, construct some sequences $q_i \in F$, $q_i \to p$, and $r_i \in G$, $r_i \to p$, with the directions of pq_i, pr_i tending to ξ and $pq_i = pr_i$. Then $|q_i r_i| = o(|pq_i|)$, and applying Corollary 3.2(3) we can find some points $s_i \in F \cap G$ with $|q_i s_i| = o(|pq_i|)$. Hence for $i \to \infty$ the directions of the shortest curves ps_i tend to ξ, as required.

(2) Here the inclusion $\overline{\Sigma_p G \setminus \Sigma_p F} \subset \Sigma_p(\overline{G \setminus F})$ is obvious, since $\Sigma_p(\overline{G \setminus F})$ is closed.

To prove the reverse inclusion, we fix $\xi \in \Sigma_p(\overline{G \setminus F})$ and a sequence $p_i \to p$ of points $p_i \in G \setminus F$ with the directions of pp_i converging to ξ. Fix $\delta > 0$. Then, using Corollary 3.2(4) and choosing suitable $R_i = \delta^2 |pp_i|$, we can assume $\delta < \ldots$ and construct a sequence of points $\widehat{p}_i \in G \setminus F$, $\widehat{p}_i \to p$, such that any direction η that is a limit point for the directions of shortest curves $p\widehat{p}_i$ satisfies $|\xi\eta| < \delta$ and $|\eta \Sigma_p F| > \delta^3$. (Here we have used Proposition 3.3 for F.) So, $\xi \in \overline{\Sigma_p G \setminus \Sigma_p F}$, as required. $\square$

3.5. **Proposition.** *Let F and G be extremal subsets of M. Then:*

(1) *$F \cap G$ is also an extremal subset of M,*

(2) $\overline{G\setminus F}$ *is also an extremal subset of* M *if it is nonempty.*

Proof. (1) In view of Proposition 1.4 it suffices to consider separately the case where $M=\Sigma$ and $F\cap G$ is either empty or a singleton, and then to use induction on dimension (with an evident base step), referring to 3.4(1). So, let $M=\Sigma$ and $F\cap G\subset\{p\}$, and let q be the point of Σ farthest from p. We have to prove that $|pq|\leqslant\pi/2$. Suppose the contrary. Then the function $\operatorname{dist}(q)$ has no critical points in $B_q(|pq|)$ except the point q itself. Since $q\notin F\cap G$, we can assume for the sake of definiteness that $q\notin F$, and hence $|qF|>\pi/2$. But this contradicts the extremality of F (see 1.4.1).

(2) In this case also, Proposition 1.4 and Lemma 3.4(2) permit us to apply induction on the dimension (with the evident base step). (Note that if $p\in\overline{G\setminus F}\cap F$, then $\Sigma_pG\setminus\Sigma_pF\neq\varnothing$ by Lemma 3.4(2).) The case $M=\Sigma$, $G\setminus F=\{p\}$ must be considered separately. It should be verified that $\Sigma\subset B_p(\pi/2)$. Suppose the contrary, and let q be the point of Σ farthest from p. Then $F\neq\varnothing$, and $F\neq\{q\}$, hence there exists a point $r\in F\setminus\{p,q\}$ nearest to p in F. Because of extremality of F we have $\tilde{\angle}prq\leqslant\pi/2$; on the other hand, $\tilde{\angle}rpq\leqslant\pi/2$ since p is an isolated point of the extremal subset G. But these two inequalities are incompatible with $|pr|\leqslant|pq|$ and $|pq|>\pi/2$. □

3.6. **Proposition.** *The number of extremal subsets of a given space* M *is finite.*

Proof. Indeed, since the function $\operatorname{dist}(p)$ has no critical points in a small punctured neighborhood of p, any extremal subset touching such a neighborhood passes through p. On the other hand, Lemma 3.4(2) implies that two extremal subsets with common tangent space at p do coincide near p. Therefore our proposition follows by induction on dimension. □

3.7. Now we are able to decompose an arbitrary extremal subset into primitive components.

Definition. An extremal subset is said to be *primitive* if it contains no proper extremal subset with nonempty relative interior.

Proposition. *Any extremal subset can be represented in a unique way as a union of primitive extremal subsets with nonempty relative interior.*

This easily follows from 3.5 and 3.6. □

3.8. **The stratification of an Aleksandrov space.** Let F be a primitive extremal subset of M. The *main part* $\mathring{F}$ of F is defined as the set of all points of F that do not lie in other primitive extremal subsets. Clearly, $\mathring{F}$ is open and everywhere dense in F. Also, it is easily seen that the collection of all main parts of primitive extremal subsets of M constitutes a disjoint covering of M.

Finally, we can show that for any F its main part $\mathring{F}$ is a topological manifold. (Indeed, since assertion (A) of Main Theorem 1.4 of [6] is true for extremal subsets (see §2), it follows that the closure of each stratum of the canonical stratification of F is an extremal subset (cf. 1.2). Hence, F being primitive, there is only one stratum whose closure has nonempty interior in F, and all other strata are covered by smaller primitive subsets. Therefore $\mathring{F}$ is contained in a single stratum, as required.)

The covering of M by main parts of primitive extremal subsets gives the stratification refined in comparison with the canonical topological one mentioned in Introduction.

§4. Factorization and extremal subsets

4.1. **Proposition.** *Let Γ be a compact group acting on M by isometries, let $\pi: M \to M/\Gamma$ be the natural projection and let $F \subset M$ be an extremal subset. Then $\pi(F)$ is an extremal subset of M/Γ (the latter being an Aleksandrov space, see [3, 4.6]).*

The proof is simple, and we leave it to the reader. □

4.2. **Proposition.** *Let Δ be a compact group acting on M by isometries, let $\Gamma \subset \Delta$ be a closed subgroup, let F be the set of fixed points of Γ and let $\pi: M \to M/\Delta$ be the natural projection. Then $\pi(F)$ is an extremal subset of M/Δ.*

Proof. We start with the additional condition specific for the case $M = \Sigma$ (see Definition 1.1), which can be checked easily. Indeed, if $\overline{B}_p(\pi/2) \neq \Sigma/\Delta$ for some $p \in \Sigma/\Delta$, then the function $\operatorname{dist}(\pi^{-1}(p)) - \pi/2)$ is strictly concave where positive, so its maximum point is unique and belongs to F, whence $\pi(F) \setminus \{p\}$ is nonempty.

Now we turn to the general setting. Suppose that there are points $\overline{p} \in \pi(F)$, $\overline{q}, \overline{r} \in M/\Delta$ with $|\overline{p}\overline{q}| = |\overline{q}\pi(F)| > 0$ but $\widetilde{\angle} qpr > \pi/2$. Choose some points $p \in F$, $q \in \pi^{-1}(\overline{q})$, $r \in \pi^{-1}(\overline{r})$ with $|pq| = |\overline{p}\overline{q}|$, $|pr| = |\overline{p}\overline{r}|$. Obviously, $\widetilde{\angle} spq > \pi/2$ for any $s \in \Gamma \cdot q$. Let $\Omega = \bigcup_{s \in \Gamma \cdot q} s' \subset \Sigma_p$. Then $|\Omega q'| > \pi/2$. Let ω be the point of Σ_p farthest from Ω. It follows that ω is unique and $|\omega q'| < \pi/2$ (cf. the proof of 1.4.1). Obviously, Ω is invariant with respect to the induced action of Γ on Σ_p, and so ω is a fixed point of this action. If a shortest curve comes out in the direction ω, then it is also fixed under the action of Γ, and we obtain a contradiction with $|qF| = |qp|$. If there is no such shortest curve, we can apply Lemma 4.3 on strictly convex hulls stated below. □

4.3. Fix $\lambda > 0$ and for an arbitrary compact set K consider the family of all 1-Lipschitz functions f such that the set $f^{-1}([0, +\infty))$ is absolutely convex, contains K, and f is λ-concave on this set. Let f_K denote the minimum of all the functions from this family. (It is clear that f_K itself belongs to it.) Then the set $f_K^{-1}([0, +\infty))$ is called the *λ-convex hull* of K and the maximum point of f_K (which is evidently unique) is called the *soul* of this hull.

Lemma on strictly convex hulls. *There exists a constant $\varepsilon > 0$ depending only on λ and M (more exactly, on the volume and the lower bound of curvature of M) such that each compact K of diameter $d < \varepsilon$ possesses a λ-convex hull of diameter $\varepsilon^{-1}d$.*

(For the proof see 4.5.)

4.4. Now the proof of Proposition 4.2 can be concluded in the following way. Take a point x near p with the direction of px very close to ω and notice that $\operatorname{diam}(\Gamma \cdot x) = o(|px|)$. Hence the direction of the shortest curve $p\overline{x}$, where $\overline{x}$ is the soul of the λ-convex hull of $\Gamma \cdot x$, also will be close to ω, and so $|q\overline{x}| < |qp|$. But on the other hand, $\overline{x}$ is clearly a fixed point of Γ. This contradicts the fact that $|qF| = |qp|$. The proof of Proposition 4.2 is completed. □

4.5. *The proof of Lemma 4.3.* Suppose that such $\varepsilon > 0$ does not exist. Then there is a sequence $\varepsilon_i \to 0$ and a sequence of compact sets K_i with $\operatorname{diam} K_i = d_i < \varepsilon_i$ such that K_i has no λ-convex hull of diameter $\leqslant \varepsilon_i^{-1} d_i$. In other words, each compact $\widetilde{K}_i = d_i^{-1} K_i$ of diameter 1 in $\widetilde{M}_i = d_i^{-1} M$ has no λd_i-convex hull of diameter $\leqslant \varepsilon_i^{-1}$. Choose in each $\widetilde{K}_i$ a point $\widetilde{p}_i$ and consider the Gromov–Hausdorff limit

$(\overline{M}, p)$ of the spaces $(\widetilde{M}_i, \widetilde{p}_i)$ along some subsequence of values of i. Clearly, $\overline{M}$ is an n-dimensional complete noncompact Aleksandrov space of nonnegative curvature and its limit set $L\overline{M}$ is of dimension $(n-1)$.

Fix a small $\delta > 0$ and choose on the limit set a maximal net of points l_α with pairwise distances $\geqslant \delta$. These points give rise to certain rays r_α with origin at p, and Busemann functions b_α. Since the normalized spheres $R^{-1}S_p(R)$ converge to $L\overline{M}$ in the Gromov–Hausdorff sense, there is $c > 0$ and $\overline{R} > 100$ such that with each point $x \in S_p(\overline{R})$ we can associate a set $\mathscr{A}_x$ of at least $c\delta^{1-n}$ values of α with $b_\alpha(p) \geqslant b_\alpha(x) + \overline{R}/2$. Now we can find $\widehat{R} > 100\overline{R}$ such that for the corresponding distance functions of points $q_\alpha = r_\alpha \cap S_p(\widehat{R})$ similar inequalities hold: $|q_\alpha p| \geqslant |q_\alpha x| + \overline{R}/3$. Define a function h_x setting

$$h_x = (\#\mathscr{A}_x)^{-1} \sum_{\alpha \in \mathscr{A}_x} \varphi(\operatorname{dist}(q_\alpha)),$$

where φ is determined by the conditions

$$\begin{aligned}
\varphi'(t) &= 1 && \text{for } t \leqslant \widehat{R} - 2\overline{R},\\
\varphi'(t) &= 1/2 && \text{for } t \geqslant \widehat{R} + 2\overline{R},\\
\varphi''(t) &= -\frac{1}{8\overline{R}} && \text{for } \widehat{R} - 2\overline{R} < t < \widehat{R} + 2\overline{R}.
\end{aligned}$$

Then the function h_x is 1-Lipschitz and $\overline{\lambda}$-concave in $B_p(2\overline{R})$ for some $\overline{\lambda} > 0$ depending on c, $\overline{R}$, $\widehat{R}$, but not on x (the $\overline{\lambda}$-concavity of h_x is proved in the same way as Lemma 3.6 of [6]). Furthermore, obviously $h_x(p) \geqslant h_x(x) + \overline{R}/6$. Now if we transfer the construction of the functions h_x to $\widetilde{M}_i$ (that is, choose there some points corresponding to the points q_α by the Hausdorff approximation and use the same representation in terms of distance functions as for h_x), the functions obtained will possess the same properties (the $\overline{\lambda}$-concavity is to be verified anew, but the verification is quite similar to the proof of Lemma 3.6 of [6]). Hence for i sufficiently large the compact $\widetilde{K}_i$ can be "separated" from any point $x \in S_p(\overline{R})$ by a function which is $\overline{\lambda}$-concave in $B_{\widetilde{p}_i}(2\overline{R})$ and 1-Lipschitz. Thus for sufficiently large i the compact $\widetilde{K}_i$ possesses a λd_i-convex hull lying in $B_{\widetilde{p}_i}(\overline{R})$. This contradicts our assumptions. □

§5. Generalized Liberman Theorem

5.1. Recall that if three points a, b, c lie on a shortest curve in the order indicated, then for any point p we have $|pb| \geqslant |\widetilde{p}\widetilde{b}|$, where $\widetilde{b}$ is the point on the base of the comparison triangle $\widetilde{p}\widetilde{a}\widetilde{c}$ corresponding to b. This inequality can be expressed analytically in terms of the function $f = \operatorname{dist}(p)$. For nonnegative curvature this relation looks as follows:

$$|ab|(f^2(c) - f^2(b)) - |bc|(f^2(b) - f^2(a)) \leqslant (|ab| + |bc|)|ab||bc|. \tag{5.1}$$

The same relation holds for any three points a, b, c lying on a geodesic (that is, a locally shortest curve), $|ab|$ and $|bc|$ being replaced with $|\breve{ab}|$ and $|\breve{bc}|$, the lengths of the corresponding arcs of the geodesic. This makes the following definition natural.

Definition. A rectifiable curve $\gamma \subset M$ is said to be *quasigeodesic*, if for any a, b, c lying on γ in the order indicated and any point $p \in M$ inequality (5.1) holds with distances $|ab|$ and $|bc|$ replaced with the γ-arc lengths $|\breve{ab}|$ and $|\breve{bc}|$.

An equivalent definition expressed in geometric terms is given in [Pet] (see Introduction).

The following Theorem 5.2 is a crucial step in the proof of the generalized Liberman Theorem 5.3.

5.2. **Theorem.** *Let $F \subset M$ be an extremal subset, and let $p, q \in F$ be sufficiently close to each other. Then there is at least one curve $\breve{pq}$ which is a shortest curve in the inner metric of F and is a quasigeodesic in M.*

Proof. For any $m \in \mathbb{N}$ consider all the possible polygonal geodesics $a_0 a_1 \dots a_m$ with $a_i \in F$, $a_0 = p$, $a_m = q$, and among them choose those with minimal value of

$$S = m \sum_{i=0}^{m-1} |a_i a_{i+1}|^2.$$

Note that the limit curves, $m \to \infty$, for these extremal polygonal geodesics are the shortest curves $\breve{pq}$ in F, and S tends to $|\breve{pq}|^2$. (Indeed, on the one hand, we consider the polygonal geodesic with vertices a_i dividing $\breve{pq}$ into equal parts and see that for an extremal polygonal geodesic we have $S \leqslant |\breve{pq}|^2$. In particular, for extremal polygonal geodesics, $\max_{0 \leqslant i \leqslant m-1} |a_i a_{i+1}| \to 0$ for $m \to \infty$, so the limit curves lie on F. On the other hand, by the Schwarz inequality, $\sum_{i=0}^{m-1} |a_i a_{i+1}| \leqslant \sqrt{S}$, so the length of the limit curve is not greater than $\lim_{m\to\infty} \sqrt{S} \leqslant |\breve{pq}|$, i.e., it is the shortest curve). Note also that the link lengths of an extremal polygonal geodesic approach their mean value. More precisely, let $|a_i a_{i+1}| = \alpha_i + |\breve{pq}|/m$. Then, since $\sum_{i=0}^{m-1} |a_i a_{i+1}| \to |\breve{pq}|$, we have $\sum_{i=0}^{m-1} \alpha_i \to 0$, and since

$$S = |\breve{pq}|^2 + 2|\breve{pq}| \sum_{i=0}^{m-1} \alpha_i + \sum_{i=0}^{m-1} m\alpha_i^2 \longrightarrow |\breve{pq}|^2,$$

we obtain

$$\sum_{i=0}^{m-1} m\alpha_i^2 \longrightarrow 0, \tag{5.2}$$

and, by the Schwarz inequality,

$$\sum_{i=0}^{m-1} |\alpha_i| \longrightarrow 0. \tag{5.3}$$

To verify the quasigeodesicity condition for the limit shortest curve $\breve{pq}$ we will ascertain a similar condition for the approximating polygonal geodesic $a_0 a_1 \dots a_m$ and then pass to the limit. Let $r \in M \setminus F$ be an arbitrary point. The extremality of $a_0 a_1 \dots a_m$ and of F, and Corollary 1.6 (applied to $f = S$) imply that for any direction $\xi \in \Sigma_{a_i}$

$$|a_{i-1} a_i| \cdot \cos |a'_{i-1} \xi| + |a_i a_{i+1}| \cdot \cos |a'_{i+1} \xi| \geqslant 0.$$

In particular, this is true for $\xi \in r' \subset \Sigma_{a_i}$, whence by the comparison theorem we get

$$\Delta_{i+1} - \Delta_i \leqslant |a_{i-1} a_i|^2 + |a_i a_{i+1}|^2, \quad \text{where } \Delta_j = |r a_j|^2 - |r a_{j-1}|^2. \tag{5.4}$$

Now we fix three indices $i_1 < i_2 < i_3$ and sum up the inequalities (5.4) in a special way,

(5.5)
$$\begin{aligned}
&(i_2 - i_1)(|ra_{i_3}|^2 - |ra_{i_2}|^2) - (i_3 - i_2)(|ra_{i_2}|^2 - |ra_{i_1}|^2) \\
&\quad = (i_2 - i_1)\Big(\sum_{i=i_2+1}^{i_3} \Delta_i\Big) - (i_3 - i_2)\sum_{i=i_1+1}^{i_2} \Delta_i \\
&\quad = \sum_{i=i_2+1}^{i_3}\sum_{j=i_1+1}^{i_2} (\Delta_i - \Delta_j) \\
&\quad = \sum_{i=i_1+1}^{i_2-1} (\Delta_{i+1} - \Delta_i)(i_3 - i_2)(i - i_1) + \sum_{i=i_2}^{i_2-1} (\Delta_{i+1} - \Delta_i)(i_2 - i_1)(i_3 - i) \\
&\quad \leqslant (i_3 - i_2)\sum_{i=i_1}^{i_2-1} |a_i a_{i+1}|^2 (2i + 1 - 2i_1) + (i_2 - i_1)\sum_{i=i_2}^{i_3-1} |a_i a_{i+1}|^2 (2i_3 - 2i - 1).
\end{aligned}$$

Now let $x_1, x_2, x_3 \in \breve{pq}$ and assume that $a_{i_1(m)}$, $a_{i_2(m)}$, and $a_{i_3(m)}$ tend to x_1, x_2, and x_3, respectively. Then

$$\begin{aligned}
|\breve{x_1 x_2}| &= \lim_{m\to\infty} \sum_{i=i_1(m)}^{i_2(m)-1} |a_i a_{i+1}| \\
&= \lim_{m\to\infty} \Big(\frac{i_2(m) - i_1(m)}{m} \breve{pq} + \sum_{i=i_1(m)}^{i_2(m)-1} \alpha_i \Big) \\
&\overset{(5.3)}{=} \lim_{m\to\infty} \frac{i_2(m) - i_1(m)}{m} |\breve{pq}|.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\lim_{m\to\infty} \sum_{i=i_1(m)}^{i_2(m)-1} |a_i a_{i+1}|^2 (2i + 1 - 2i_1(m)) \\
&\quad = \lim_{m\to\infty} \sum_{i=i_1(m)}^{i_2(m)-1} \alpha_i^2 (2i + 1 - 2i_1(m)) \\
&\qquad + 2\sum_{i=i_1(m)}^{i_2(m)-1} \alpha_i \frac{2i + 1 - 2i_1(m)}{m} |\breve{pq}| + \frac{|\breve{pq}|^2}{m^2} \sum_{i=i_1(m)}^{i_2(m)-1} (2i + 1 - 2i_1(m)) \\
&\overset{(5.2,\ 5.3)}{=} |\breve{pq}|^2 \cdot \frac{(i_2 - i_1)^2}{m^2}.
\end{aligned}$$

Therefore, passing to the limit in (5.5) (after division by m and multiplying by $|\breve{pq}|$), we obtain

$$|\breve{x_1 x_2}|(|rx_3|^2 - |rx_2|^2) - |\breve{x_2 x_3}|(|rx_2|^2 - |rx_1|^2) \leqslant |\breve{x_1 x_2}||\breve{x_2 x_3}|^2 + |\breve{x_2 x_3}||\breve{x_1 x_2}|^2,$$

which is none other than the quasigeodesicity condition (for the case of nonnegative curvature) expressed analytically. Theorem 5.2 is proved. □

5.3. **Generalized Liberman Theorem.** *Any shortest curve $\breve{pq}$ in the inner metric of an extremal subset $F \subset M$ is a quasigeodesic for M.*

We outline two proofs. In the first of them we use some basic properties of quasigeodesics established in [Pet].

First proof. Let us split $\breve{pq}$ into N equal parts by $p = p_0, p_1, p_2, \dots, p_N = q$. By Theorem 5.2 there exist some shortest curves $\breve{p_ip_{i+1}}$ in the inner metric of F which are quasigeodesics for M. The curve $(\breve{pq})_N$ composed of $\breve{p_ip_{i+1}}$ is also a shortest curve in the inner metric of F, and to prove its quasigeodesicity it suffices to verify for the points p_i that the entrance direction at p_i is polar to the exit one, see [Pet]. (Two directions $\xi, \eta \in \Sigma$ are said to be polar if $|\xi\zeta| + |\eta\zeta| \leqslant \pi$ for all $\zeta \in \Sigma$.) For some $\varepsilon > 0$ take some points r_ε on $\breve{p_ip_{i+1}}$ and s_ε on $\breve{p_ip_{i-1}}$ with $|\breve{p_ir_\varepsilon}| = |\breve{p_is_\varepsilon}| = \varepsilon$. ¿From Theorem 5.2 it is clear that there exists a shortest line $\breve{r_\varepsilon s_\varepsilon}$ in F which is quasigeodesic in M. Let $\varepsilon \to 0$ along some sequence ε_i and consider a cone K_{p_i} which is the Hausdorff limit of $(\varepsilon^{-1}M, p_i)$, K_{p_i} is the cone over the space of directions Σ_{p_i}. At the same time, $\breve{p_ip_{i\pm1}}$ converge to rays in K_{p_i}, since a quasigeodesic has both right and left directions at each point, and the $\breve{r_\varepsilon s_\varepsilon}$ converge to a quasigeodesic $\breve{rs}$, since $\dim K_{p_i} = \dim M$; see [Pet].

For a cone K, $|x|$ will denote the distance from its vertex to a point x. Then for $K = K_{p_i}$ we have $|\breve{rs}| = |r| + |s|$, $|r| = |s|$.

Lemma. *Let $\gamma(t) \subset K$ be a quasigeodesic in a nonnegatively curved cone. Then $|\gamma(t)|^2 = t^2 + 2kt + C$ for some k and C.*

Proof. The assertion is equivalent to $\frac{d^2}{dt^2}|\gamma(t)|^2 = 2$. The inequality $(|\gamma(t)|^2)'' \leqslant 2$ follows from the definition of a quasigeodesic, if we take as p the vertex of K. Further, taking as p the point at infinity in the direction $\gamma(t)$, we obtain $(|\gamma(t)|^2)'' \geqslant 2$, as required. □

For $K = K_{p_i}$ the lemma implies that

$$|s|^2 = |\breve{rs}|^2 + k|\breve{rs}| + |r|^2,$$

and so, since $|s|^2 = |r|^2$, we obtain

$$0 = |\breve{rs}|^2 + k|\breve{rs}|,$$

i.e., $k = -|\breve{rs}|$. Hence the distance from the midpoint of $\breve{rs}$ to the vertex of the cone K_{p_i} equals

$$|\breve{rs}|^2/4 - |\breve{rs}|^2/2 + |\breve{rs}|^2/4 = 0,$$

that is, these points coincide. Hence, clearly, the direction of entrance of $(\breve{pq})_N$ to p_i is polar to that of exit from p_i, and so $(\breve{pq})_N$ is a quasigeodesic. For $N \to \infty$ we have $(\breve{pq})_N \to \breve{pq}$ and so $\breve{pq}$ is also a quasigeodesic. Theorem 5.3 is proved. □

Second proof. Repeat the proof of Theorem 5.2 replacing S by the expression

$$m\sum_{i=0}^{m-1} |a_ia_{i+1}|^2 + \frac{1}{m}\sum_{i=0}^{m-1} |a_i\breve{pq}|^2,$$

where $\breve{pq}$ is the shortest curve we are interested in. □

§6. Open questions

6.1. Is it true that the inner metric of a primitive extremal subset has a curvature bounded from below?

At present it has not been proved even for the inner metric of the boundary.