From Euclid to Alexandrov; a guided tour

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Chapter 0

Prerequisite

We assume that thereader familiar with the following topics:

1. Set theory
   - ◇ Operations on sets: union $A \cup B$, intersection $A \cap B$, set-theoretic difference $A \backslash B$, disjoint union $A \sqcup B$.
   - ◇ Equivalence relations, equivalence classes, quotient set.
2. Calculus
   - ◇ Converging sequences.
   - ◇ Cauchy sequence.
   - ◇ Continuous functions on the real line and in Euclidean space.
   - ◇ Open and closed sets in Euclidean space.
   - ◇ Compactness of subsets in Euclidean space.
   - ◇ Convex functions, Jensen's inequality.
3. Geometry
   - ◇ Motions of plane and space, conruent figures.
   - ◇ Convexity: definition, convex hull.
   - ◇ Convex polyhedra: vertices, faces and edges.
   - ◇ Platonic solids: cube, tetrahedron and octahedron.
   - ◇ Area and volume.
4. Combinatorics

- ◇ Basic definitions in graph theory: connectedness, degree of vertex, planar graph.

5. Linear algebra

   - ◇ Linear independence

Part I

The rules of the game

Chapter 1

Metric spaces

1.1 Definitions

1.1.1. Definition. *A metric space is a pair* (X, dist) *where* X *is a set and*

$$\operatorname{dist}\colon X \times X \to [0, \infty)$$

is a function such that

a) $\operatorname{dist}(x, y) = 0$ *if and only if* $x = y$*;*
b) $\operatorname{dist}(x, y) = \operatorname{dist}(y, x)$ *for any* $x, y \in X$*;*
c) $\operatorname{dist}(x, z) \leqslant \operatorname{dist}(x, y) + \operatorname{dist}(y, z)$ *for any* $x, y, z \in X$*.*

The condition (c) above is called *triangle inequality.* The set X is called *underlying set* of the metric space and its elements are called *points* of the metric space. The function $\operatorname{dist}\colon X \times X \to [0, \infty)$ is called a *metric*, the value $\operatorname{dist}(x, y) \geqslant 0$ is called the *distance* from x to y.

Examples:

⋄ Discrete metric. For any set X, the discrete metric on X is defined by

$$\operatorname{dist}(x, y) = \begin{bmatrix} 1 & \text{if} & x \neq y \\ 0 & \text{if} & x = y \end{bmatrix}$$

⋄ Euclidean space. The set is formed by arrays of n real numbers

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

and the distance function defined as

$$\operatorname{dist}(\boldsymbol{x},\boldsymbol{y}) \stackrel{def}{=\!=} \|\boldsymbol{x}-\boldsymbol{y}\|_2,$$

where

$$\|\boldsymbol{x}\|_2 \stackrel{def}{=\!=} \sqrt{x_1^2+x_2^2+\dots+x_n^2}.$$

1.1.2. Exercise. *Show that Euclidean space is a metric space.*

◇ Manhattan metric on the plane. The set is formed by all pairs $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ and the metric is defined as

$$\operatorname{dist}(\boldsymbol{x},\boldsymbol{y}) \stackrel{def}{=\!=} \|\boldsymbol{x}-\boldsymbol{y}\|_1,$$

where

$$\|\boldsymbol{x}\|_1 \stackrel{def}{=\!=} |x_1|+|x_2|.$$

◇ Space of functions with sup-norm. Given a set X, consider the set $\mathcal{F}(X)$ of all bounded functions $f\colon X\to\mathbb{R}$ with the metric defined as

$$\operatorname{dist}(f,g) \stackrel{def}{=\!=} \|f-g\|_\infty,$$

where

$$\|f\|_\infty \stackrel{def}{=\!=} \sup_{x\in X}|f(x)|.$$

◇ Subspaces. Given an arbitrary subset $A\subset X$ of a metric space (X,dist), one can give A the metric defined as the restriction of dist to $A\times A\subset X\times X$. In this situation, $(A, \operatorname{dist}|_{A\times A})$ is called a *subspace* of (X,dist).

Notation for distance

The distance $\operatorname{dist}(x,y)$ between x and y in a metric space X will be further also denoted as

$$|x-y| = |x-y|_X = \operatorname{dist}_x y = \operatorname{dist}_y x = \operatorname{dist}(x,y).$$

We will write $|x-y|_X$ to emphasize that the points x and y belong to the metric space X and the notation dist_x is used when we need to consider the distance to the point x as a function $\operatorname{dist}_x\colon X\to\mathbb{R}$.

It should be noted that the expression $x-y$ for two points in a metric space makes no sense and $|x-y|$ should be read as *distance from x to y*.

Isometries

1.1.3. Definition. *Let X and Y be metric spaces.*

a) A map $f\colon X \to Y$ is distance preserving *if*

$$|f(x) - f(x')|_Y = |x - x'|_X$$

for any $x, x' \in X$.

b) A distance preserving bijection $f\colon X \to Y$ is called an isometry*.*

c) The spaces X and Y are called isometric *(briefly $X \stackrel{iso}{=\!=} Y$) if there is an isometry $f\colon X \to Y$.*

Note that

◇ "$\stackrel{iso}{=\!=}$" is an equivalence relation on the class of metric spaces.

◇ Distance preserving map is necessarily injective.

◇ The existence of a distance preserving map $X \to Y$ is equivalent to the existence of subset of Y which is isometric to X.

1.2 Calculus in metric spaces

Convergence and continuity

1.2.1. Definition. *Let X be a metric space. A sequence of points $x_1, x_2, \ldots$ in X is called* convergent *if there is $x_\infty \in X$ such that $|x_\infty - x_n| \to 0$ as $n \to \infty$. That is, for every $\varepsilon > 0$, there is a natural number N such that for all $n \geq N$, we have*

$$|x_\infty - x_n| < \varepsilon.$$

In this case we say that the sequence (x_n) converges to x_∞ or x_∞ is the limit of the sequence (x_n) and write $x_\infty = \lim_{n\to\infty} x_n$ or $x_n \to x_\infty$ as $n \to \infty$.

Note that any converging sequence has unique limit.

1.2.2. Definition. *Let X and Y be metric spaces. A map $f\colon X \to Y$ is called continuous if for any convergent sequence $x_n \to x_\infty$ in X, the sequence $y_n = f(x_n)$ converges to $y_\infty = f(x_\infty)$ in Y.*

Equivalently, $f\colon X \to Y$ is continuous if for any $x \in X$ and any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x - x'|_X < \delta \;\Rightarrow\; |f(x) - f(x')|_Y < \varepsilon.$$

It is not hard to see that two definitions of continuity given above are equivalent; try to prove it.

Open and closed sets

1.2.3. Definition. *A subset A of a metric space X is called* closed *if whenever a sequence (x_n) of points from A converges in X, we have that $\lim_{n\to\infty} x_n \in A$.*

A set $\Omega \subset X$ is called open *if the complement $X \setminus \Omega$ is a closed set. Equivalently, $\Omega \subset X$ is open if for any $z \in \Omega$ there is $\varepsilon > 0$ such that the* ball

$$B_\varepsilon(z) = \{\, x \in X \mid |x - z| < \varepsilon \,\}$$

is contained in Ω.

The proof of equivalence of the two definitions of an open set given above is left to the reader.

Note that whole space as well as the empty set are both open and closed at the same time. Also, the half-closed interval $[0, 1)$ is neither open nor closed as a subset of the real line.

Closure, interior and boundary

Note that the intersection of an arbitrary number of closed sets is closed. It follows that for any set Q in a metric space X, there is a minimal closed set which contains Q; this set is called the *closure of* Q and is denoted as $\operatorname{Closure} Q$. The closure of Q can be obtained as the intersection of all closed sets $A \supset Q$. It is also equal to the set of all limit points for all sequences in Q.

Similarly, the union of an arbitrary number of open sets is open. It follows that for any set Q in a metric space, there is a maximal open set which is contained in Q; this set is called the *interior of* Q and is denoted as $\operatorname{Interior} Q$. The interior of Q can be obtained as the union of all open sets in Q. It is also equal to the set of all point $p \in Q$ such that $B_\varepsilon(p) \subset Q$ for some $\varepsilon > 0$.

Further the set-theoretic difference

$$\operatorname{Closure} Q \setminus \operatorname{Interior} Q$$

is called the *boundary* of Q and is denoted as ∂Q or $\partial_X Q$ if we want to emphasise that Q is a subset of X. Clearly, ∂Q is a closed set. A point p lies in the boundary of Q if and only if for any $\varepsilon > 0$, there are points $q \in Q$ and $q' \notin Q$ such that $|p - q|, |p - q'| < \varepsilon$.

1.3 Completeness

1.3.1. Definition. *Let X be a metric space. A sequence of points $x_1, x_2, x_3, \ldots$ in X is called* Cauchy, *if for every $\varepsilon > 0$, there is an integer N such that*

$$|x_m - x_n| < \varepsilon,$$

for any $m, n > N$.

1.3.2. Definition. *A metric space X is called* complete *if any Cauchy sequence in X is convergent.*

Examples:

◇ The real line as well as Euclidean space are complete.

◇ The subspace of real line formed by the rational numbers is not complete.

1.3.3. Exercise. *Let X be a complete metric space and $A \subset X$, then the subspace formed by A is complete if and only if A is closed.*

Completion

For any metric space X, there is a canonical construction of a complete metric space $\bar{X}$, which contains X. The space $\bar{X}$ is called the *completion* of X and it is constructed as a set of equivalence classes of Cauchy sequences in X.

For any two Cauchy sequences $\boldsymbol{x} = (x_n)$ and $\boldsymbol{y} = (y_n)$ in X, we may define their distance as

$$|\boldsymbol{x} - \boldsymbol{y}| = \lim_{n \to \infty} |x_n - y_n|.$$

This limit exists because the space of real numbers is complete.

The defined function $|\boldsymbol{x} - \boldsymbol{y}|$ satisfies all axioms of metric but 1.1.1a; i.e., two different Cauchy sequences may have the distance 0 (the functions of that type are called *pseudometrics*). But "having distance 0" is an equivalence relation on the set of all Cauchy sequences, and the set of equivalence classes, which we denote $\bar{X}$, is a metric space called the *completion* of X.

The original space is embedded in this space via the identification of an element x of X with the equivalence class of the sequence with constant value x. This defines a distance preserving map $X \to \bar{X}$, as required.

It is staightforward to verify all claims made about $\bar{X}$ and prove that the distance function given is well-defined and is indeed a metric on $\bar{X}$.

1.4 Compactness

1.4.1. Definition. *A metric space X is* compact *if any sequence of points in X contains a convergent subsequence.*

If a subset A in a metric space forms a compact subspace then we say that A is a compact subset *of X.*

The proofs of the following properties are left to the reader; these proofs can be also found in many books, for example in [20] (which is an excellent book).

Properties:

- ⋄ (Heine–Borel theorem.) A subset of Euclidean space is compact if and only if it is both closed and bounded.
- ⋄ Any closed subset of a compact space is compact.
- ⋄ Any compact subset of a metric space is closed.
- ⋄ Any compact metric space is complete.
- ⋄ The Cartesian product of two compact spaces $X \times Y$ equipped with the metric
$$|(x_0, y_0) - (x_1, y_1)| \stackrel{def}{=\!=} \max\{|x_0 - x_1|, |y_0 - y_1|\}$$
is compact.
- ⋄ If $f\colon X \to Y$ is continuous map between metric spaces and X is compact, then the image $f(X)$ is a compact subset of Y.
- ⋄ If $f\colon X \to Y$ is a continuous bijection between metric spaces and X is compact, then the inverse map $f^{-1}\colon Y \to X$ is continuous.
- ⋄ (Extreme Value Theorem) Prove that if X is a compact metric space, then any continuous function $f\colon X \to \mathbb{R}$ attains a global maximum value at some point of X. That is, there exists $x \in X$ such that $f(y) \leqslant f(x)$ for all $y \in X$. (Here, it is understood that the metric we are considering on $\mathbb{R}$ is the usual Euclidean metric.)

1.4.2. Definition. *A subset A of a metric space X is called* bounded *if for one (and therefore any) point x there is a constant $D < \infty$ such that*
$$|x - a|_X \leqslant D$$
for any $a \in A$.

1.4.3. Definition. *A metric space X is called* proper *if any bounded, closed set in X is compact.*

Nets and packings

1.4.4. Definition. *Let X be a metric space and let $\varepsilon > 0$. A set $A \subset X$ is called an* ε-net *of X if for any point $x \in X$ there is a point $a \in A$ such that $|x - a| \leqslant \varepsilon$.*

1.4.5. Definition. *Let X be a metric space and let $\varepsilon > 0$. A set of points $x_1, x_2, \dots, x_n \in X$ such that $|x_i - x_j| > \varepsilon$ for all $i \neq j$ is called an* ε-packing.

The supremum of all integers n for which there is a ε-packing $x_1, x_2, \dots, x_n \in \in X$ is denoted as $\operatorname{pack}_\varepsilon X$. *(*$\operatorname{pack}_\varepsilon$ *takes integer value or ∞.)*

If $n = \operatorname{pack}_\varepsilon X$ is finite, then an ε-packing $x_1, x_2, \dots, x_n \in X$ is called a maximal ε-packing.

1.4.6. Exercise. *Any maximal ε-packing is an ε-net.*

1.4.7. Exercise. *Let $\{a_1, a_2, \dots, a_n\}$ be a finite ε-net in a metric space X. Show that* $\operatorname{pack}_{2\cdot\varepsilon} X \leqslant n$.

1.4.8. Theorem. *Let X be a complete metric space. Then the following conditions are equivalent:*

a) X is compact;

b) $\operatorname{pack}_\varepsilon X$ *is finite for any $\varepsilon > 0$;*

c) X is totally bounded*; i.e., for any $\varepsilon > 0$ there is a finite ε-net in X.*

Proof; (a)⇒(c). Assume X has no finite ε-net. Then for any point array $z_1, z_2, \dots, z_{n-1}$ in X there is a $z_n \in X$ such that $|z_i - z_n| > \varepsilon$ for any $i < n$.

Applying the above statement inductively, we can construct an infinite sequence (z_n) in X such that $|z_i - z_j| > \varepsilon$ for all $i \neq j$. Therefore (z_n) has no convergent subsequence, a contradiction.

(c)⇔(b). Follows from exercises 1.4.7 and 1.4.6.

(c)⇒(a). Set $\varepsilon_k = \frac{1}{2^k}$. For each k, let $\{z_{1,k}, z_{2,k}, \dots, z_{n_k,k}\}$ be an ε_k-net. Note that for each k, the collection of balls

$$\bar{B}_{\varepsilon_k}(z_{i,k}) = \{\, x \in X \mid |x - z_i| \leqslant \varepsilon_k \,\}$$

cover all of X.

To show X is compact, given an infinite sequence $x_1, x_2, \dots$ in X, we must find a convergent subsequence. We shall apply a diagonal process to choose a subsequence $x_{n_1}, x_{n_2}, \dots$ with the following property: *for each k there is i_k such that $x_{n_m} \in \bar{B}_{\varepsilon_k}(z_{i_k,k})$ for all $m \geqslant k$.*

For the first step, note that since the finite collection of balls $\bar{B}_{\varepsilon_1}(z_{i,1})$ cover X, there must be an index i_1 such that $\bar{B}_{\varepsilon_1}(z_{i_1,1})$ contains infinitely many terms of our sequence (x_n). Let $F_1 = \bar{B}_{\varepsilon_1}(z_{i_1,1})$. Choose an integer n_1 such that $x_{n_1} \in F_1$.

For the second step, since the balls $\bar{B}_{\varepsilon_2}(z_{i,2})$ cover X, in particular they cover F_1. Since F_1 contains infinitely many points of our sequence (x_n), there must be an index i_2 such that $F_1 \cap \bar{B}_{\varepsilon_2}(z_{i_2,2})$ also contains infinitely many of the x_n. Let $F_2 = F_1 \cap \bar{B}_{\varepsilon_2}(z_{i_2,2})$ and choose $n_2 > n_1$ such that $x_{n_2} \in F_2$.

Proceeding inductively at the k-th step, we know that the balls $\bar{B}_{\varepsilon_k}(z_{i,k})$ cover X and hence F_{k-1}, which contains infinitely many terms of our sequence (x_n). So there must exist i_k so that $F_{k-1} \cap \bar{B}_{\varepsilon_k}(z_{i_k,k})$ also contains infinitely many of the x_n. Let $F_k = F_{k-1} \cap \bar{B}_{\varepsilon_k}(z_{i_k,k})$ and choose $n_k > n_{k-1}$ with $x_{n_k} \in F_k$.

Note that $F_{k+1} \subseteq F_k$, so that for all $m \geq k$, $x_{n_m} \in F_k \subseteq B_{\varepsilon_k}(z_{i_k,k})$ as desired. It follows that (x_{n_k}) is a Cauchy sequence. Since X is complete, (x_{n_k}) converges, and we have proven that X is compact. □

1.5 Topology

1.5.1. Definition. *Let X and Y be metric spaces. A map $f\colon X \to Y$ is called a* homeomorphism *if it is a continuous bijection and the inverse $f^{-1}\colon Y \to X$ is also continuous.*

Two metric spaces X and Y are called homeomorphic *(briefly $X \overset{hom}{=\!=} Y$) if there is homeomorphism $f\colon X \to Y$.*

It is straightforward to check that $\overset{hom}{=\!=}$ is an equivalence relation.

Every isometry is a homeomorphism, but not every homeomorphism is an isometry. The homeomorphism class of metric space is much larger than its isometry class.

1.5.2. Exercise. *Give an example of a continuous bijection between metric spaces that is not a homeomorphism.*

Let ϱ and ϱ' be two metrics on the same set X. We say that ϱ is *equivalent* to ϱ' if the identity map $(X, \varrho) \to (X, \varrho')$ is a homeomorphism. In other words, ϱ is equivalent to ϱ' if and only if for any sequence of points (x_n) and a point

x_∞ in X, we have

$$\varrho(x_n, x_\infty) \to 0 \quad \Leftrightarrow \quad \varrho'(x_n, x_\infty) \to 0.$$

The class of equivalent metrics on a fixed set X is described completely by all open subsets of X; i.e., a metric ϱ is equivalent to a metric ϱ' if and only if any set which is open with respect to the metric ϱ is open with respect to the metric ϱ' and visa versa.

1.5.3. Exercise. *Prove the last statement.*

The set of open sets in X is called the *topology* of X. A topology has to satisfy the following axioms:

1. the union of an arbitrary number of open sets is open,
2. the intersection of a finite number of open sets is open
3. the empty set and X are both open sets

A set with chosen topology is called *topological space.*

Many of concepts described above can be formulated in terms of toplogy. For example a map $f\colon X \to Y$ is continuous if and only if for any open set $U \subset Y$, the preimage $f^{-1}(U)$ is an open set in X. Compactness also can be described entirely in topological terms; see Theorem 20.1.2. On the other hand, completeness is not a topological property, for example the space $[0,1)$ is homeomorphic[1] to $[0,\infty)$, but $[0,1)$ is not complete whereas $[0,\infty)$ is.

Given a topological space X, we may say "consider a metric ϱ on X" meaning that d is a metric which describe the topology of X. In the case that X is a metric space, we mean that ϱ is equivalent to the original metric on X.

1.5.4. Definition. *A map $f\colon X \to Y$ is called an* embedding *if f gives a homeomorphism of X to its image $f(X)$ in Y.*

Note that any distance preserving map is an embedding, but the converse is not true.

Remarks. These definitions are a starting point of so called *point-set topology*; but this is as far as we need to go in this direction. We will only use the fact that any metric space is naturally a topological space.

A topological space which can appear this way is called *metrizable.* Not every topological space is metrizable; i.e., one can find a set X with a collection of *open sets* which satisfies above 3 axioms, but there is no metric on X for which these are the open sets; we will not consider such monsters in these lectures.

[1] say $x \mapsto \frac{x}{1-x}$ is a homeomorphism

Exercises

1.A. Give an example of a metric space X with a distance preserving map $f\colon X \to X$ which is not a bijection.

1.B. Let K be a compact metric space and $f\colon K \to K$ be a non-contracting map; i.e.,

$$|f(x) - f(y)|_K \geqslant |x - y|_K$$

for any $x, y \in K$. Prove that f is an isometry.

1.C. Let X be a metric space; fix $x \in X$. Show that the map

$$K_x\colon X \to \mathcal{F}(X)$$

defined by

$$K_x(z) = \operatorname{dist}_z - \operatorname{dist}_x$$

is distance preserving; i.e.

$$|K_x(y) - K_x(z)|_{\mathcal{F}(X)} = |y - z|_X$$

for any $y, z \in X$.

The map $K_x\colon X \to \mathcal{F}(X)$ is called the Kuratowski embedding with base x.

If X has bounded diameter, i.e., if there is a constant $D < \infty$, such that $|z - y| \leqslant D$ for any $z, y \in X$, then dist_x is bounded for any $x \in X$ and the map

$$K\colon X \to \mathcal{F}(X)$$

given by

$$K(x) = \operatorname{dist}_x$$

is also distance preserving. The latter map is also called the Kuratowski embedding.

1.D. Show that any compact space is isometric to a subset of $\mathcal{F}(\mathbb{N})$; i.e., the space of bounded sequences with the metric induced by sup-norm.

1.E. Show that completion of any metric space is isometric to the closure of its image under Kuratowski embedding.

1.F. Let X be a complete metric space and $\varrho\colon X \to \mathbb{R}$ be a continuous positive function.

Show that there is $x \in X$ such that

$$\varrho(y) > \tfrac{99}{100}\cdot\varrho(x)$$

for any $y \in B_{\varrho(x)}(x)$.

1.G. Show that a complete metric space X is compact if for any $\varepsilon > 0$ there is a compact ε-net in X.

Chapter 2

The space of sets

In this chapter we introduce Hausdorff convergence, a type of convergence for subsets of a metric space. This is a simple idea which seriously changed the shape of geometry sometime ago.

As a motivating example, I give a proof of isoperimetrical inequality. This proof was given by Blaschke in [5]; it use Hausdorff convergence together with Steiners's 4-joint method. Historically, this was the first essential application of Hausdorff convergence.

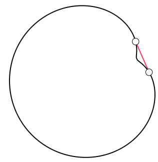
The material of this chapter I learned from the book of Wilhelm Blaschke [5]. This is a truly remarkable book which I recommend to read to everyone who can. (It is available in German and Russian; no English translations so far.)

2.1 Steiner's 4-joint method

2.1.1. Theorem. *Let F be a plane figure bounded by a curve of length 1 which has maximal area. Then F is a round disc.*

Note that if we would know that such F exists, then as a corollary we would obtain the isoperimetric inequality in the plane:

2.1.2. Theorem. *Any closed simple curve in the plane of length L bounds area at most $\frac{1}{4\cdot\pi}\cdot L^2$. Moreover, in case of equality, the curve is a circle.*



The existence of F will be proved later; in this proof we use Hausdorff convergence constructed later in this chapter.

The following argument was found by Jacob Steiner in 1842.

Proof of Theorem 2.1.1. Let F be a maximal such figure. We shall show that F is a disc. First note that F is convex (i.e. the straight light segment connecting any two points in F is also contained in F); otherwise one could make the perimeter of F smaller and the area larger.

Take any point P in the bounding curve of F (further denoted by ∂F). Consider a point $P' \in \partial F$ so that both arcs ∂F of from P to P' have the same length (which has to be $\frac{1}{2}$).

Divide F by the segment $[PP']$ into two parts F_1 and F_2. Without loss of generality we can assume that $\operatorname{area} F_1 \geqslant \operatorname{area} F_2$. Let F_1' be the reflection of F_1 in the line (PP'). Set $F' = F_1 \cup F_1'$

Note that

$$\operatorname{area} F' = 2 \cdot \operatorname{area} F_1 \geqslant \operatorname{area} F_1 + \operatorname{area} F_2 = \operatorname{area} F$$

and the perimeter of F' is the same as perimeter of F. Hence F' also has maximal possible area and $\operatorname{area} F_1 = \operatorname{area} F_2$.

Note that if F is not a round disc then points P and P' can be chosen so that F' is not a round disc. Chose arbitrary point $Q \in \partial F'$ and let Q' be the reflection of Q in (PP').

One can think of F' as quadrilateral $PQP'Q'$ with a lune attached at each side. Think about these lunes as being made of rigid material (say cut it from cardboard) and imagine that at each vertex P, Q, P', Q' we have a joint; so the quadrilateral $PQP'Q'$ can be moved continuously keeping its sides fixed.

Note that if $\angle PQP' \neq \frac{\pi}{2}$ then we can move this construction slightly and increase the area of the obtained figure.

Hence $\angle PQP' = \frac{\pi}{2}$ for any $Q \in \partial F'$. It follows that F' is a disc; hence F is also a disc. $\square$

As it was mentioned above, in order to prove isoperimetric inequality (Theorem 2.1.2) one only has to show existence of an extremal object (the figure F in Theorem 2.1.1) and then apply Steiner's argument. One possible approach is to cook up a compact metric space out of plane figures and show that volume and perimeter depend continuously (or semicontinuously) on the figure. Then existence of a maximal F would follow, as a continuous function on a compact metric space attains a maximum.

The first step is to define a metric on the set of figures in the plane; this is done in the next section.

2.2 Hausdorff metric

Let X be a metric space. Given a subset $A \subset X$, consider the distance function to A

$$\operatorname{dist}_A : X \to [0, \infty)$$

defined as

$$\operatorname{dist}_A x \stackrel{def}{=\!=} \inf_{a \in A} \{\operatorname{dist}_a x\}.$$

2.2.1. Definition. *Let A and B be two compact subsets of a metric space X. Then the Hausdorff distance between A and B is defined as*

$$|A - B|_{\mathcal{H}(X)} \stackrel{def}{=\!=} \sup_{x \in X} |\operatorname{dist}_A x - \operatorname{dist}_B x|.$$

2.2.2. Exercise. *Let A, B be two compact subsets of a metric space X. Show that $|A - B|_{\mathcal{H}(X)} \leqslant R$ if and only if $\operatorname{dist}_A b \leqslant R$ for any $b \in B$ and $\operatorname{dist}_B a \leqslant R$ for any $a \in A$.*

2.2.3. Exercise. *Show that the set of all nonempty compact subsets of a metric space X equipped with the Hausdorff metric forms a metric space.*

This new metric space will be denoted as $\mathcal{H}(X)$.

2.2.4. Exercise. *Let X be a metric space. Given a subset $A \subset X$ define its diameter as*

$$\operatorname{diam} A \stackrel{def}{=\!=} \sup_{a,b \in A} |a - b|.$$

Show that

$$\operatorname{diam} \colon \mathcal{H}(X) \to \mathbb{R}$$

is a continuous function.

2.2.5. Blaschke's compactness theorem. *Let X be a metric space. Then the space $\mathcal{H}(X)$ is compact if and only if X is compact.*

Proof; "only if" part. Note that the map $\iota \colon X \to \mathcal{H}(X)$, defined as $\iota \colon x \mapsto \{x\}$ (i.e., point x mapped to the one-point subset $\{x\}$ of X) is distance preserving. Thus X is isometric to the set $\iota(X)$ in $\mathcal{H}(X)$.

Note that for a nonempty subset $A \subset X$, we have $\operatorname{diam} A = 0$ if and only if A is a one-point set. Therefore, from Exercise 2.2.4, it follows that $\iota(X)$ is closed in $\mathcal{H}(X)$. Hence $\iota(X)$ is compact, as it is a closed subset of a compact space. Since X is isometric to $\iota(X)$, "only if" part follows. □

To prove "if" part we will need the following two lemmas.

2.2.6. Lemma. *Let $K_1 \supset K_2 \supset \dots$ be a sequence of nonempty compact sets in a metric space X then $K_\infty = \bigcap_n K_n$ is the Hausdorff limit of K_n; i.e., $|K_\infty - K_n|_{\mathcal{H}(X)} \to 0$ as $n \to \infty$.*

Proof. Note that K_∞ is compact and nonempty. If the assertion were false, then there is $\varepsilon > 0$ such that for each n one can choose $x_n \in K_n$ such that $\operatorname{dist}_{K_\infty} x_n \geqslant \varepsilon$. Note that $x_n \in K_1$ for each n. Since K_1 is compact, there is a *partial limit*[1] x_∞ of x_n. Clearly $\operatorname{dist}_{K_\infty} x_\infty \geqslant \varepsilon$.

On the other hand, since K_n is closed and $x_m \in K_n$ for $m \geqslant n$, we get $x_\infty \in K_n$ for each n. It follows that $x_\infty \in K_\infty$ and therefore $\operatorname{dist}_{K_\infty} x_\infty = 0$, a contradiction. □

The following statement was used in the previous proof.

2.2.7. Exercise. *Show that if $K_1 \supset K_2 \supset \dots$ is a nonempty compact subsets of a metric space, then*

$$K_\infty = \bigcap_{n=1}^{\infty} K_n$$

is nonempty and compact.

Give an example that shows that this statement is false if we replace "compact" with "closed".

2.2.8. Lemma. *If X is a compact metric space then $\mathcal{H}(X)$ is complete.*

Proof. Let (Q_n) be a Cauchy sequence in $\mathcal{H}(X)$. Passing to a subsequence of Q_n we may assume that

❶ $$|Q_n - Q_{n+1}|_{\mathcal{H}(X)} \leqslant \tfrac{1}{10^n}$$

for each n.

Set

$$K_n = \left\{\, x \in X \;\middle|\; \operatorname{dist}_{Q_n}(x) \leqslant \tfrac{1}{10^n} \,\right\}$$

[1] Partial limit is a limit of a subsequence.

Clearly, $|Q_n - K_n|_{\mathcal{H}(X)} \leqslant \frac{1}{10^n}$ and from ❶, we get $K_n \supset K_{n+1}$ for each n. Set

$$K_\infty = \bigcap_{n=1}^{\infty} K_n.$$

Applying Lemma 2.2.6, we get that $|K_n - K_\infty|_{\mathcal{H}(X)} \to 0$ as $n \to \infty$. Since $|Q_n - K_n|_{\mathcal{H}(X)} \leqslant \frac{1}{10^n}$, we get $|Q_n - K_\infty|_{\mathcal{H}(X)} \to 0$ as $n \to \infty$. Hence lemma follows. □

Proof of "if" part of Blaschke's compactness Theorem 2.2.5. According to Lemma 2.2.8, $\mathcal{H}(X)$ is complete. So to show that $\mathcal{H}(X)$ is compact, it only remains to show that $\mathcal{H}(X)$ is totally bounded; i.e., given $\varepsilon > 0$ there is a finite ε-net in $\mathcal{H}(X)$.

Choose a finite ε-net A in X. Denote by $\mathcal{A}$ the set of all subsets of A. Note that $\mathcal{A}$ is finite set in $\mathcal{H}(X)$. We shall show that $\mathcal{A}$ is an ε-net in $\mathcal{H}(X)$.

For each compact set $K \subset X$, consider the subset K' of all points $a \in A$ such that $\operatorname{dist}_K a \leqslant \varepsilon$. Then $K' \in \mathcal{A}$ and $|K - K'|_{\mathcal{H}(X)} \leqslant \varepsilon$. In other words $\mathcal{A}$ is a finite ε-net in $\mathcal{H}(X)$. □

2.3 Isoperimetric inequality: the end of proof

Let Q be a figure in the plane which is bounded by a closed curve. Note that if Q is not convex then the *convex hull*[2] of Q has bigger area and smaller perimeter. Therefore it is sufficient to prove the isoperimetric inequality only for convex figures. In other words, to prove Theorem 2.1.2, it is sufficient to prove the following.

2.3.1. Theorem. *Any convex plane figure wth area A is bounded by a curve of length at least $\sqrt{4{\cdot}\pi{\cdot}A}$. Moreover, in case of equality, the figure is a disc.*

Note that by Steiner's argument (2.1.1), Theorem 2.3.1 follows from the following.

2.3.2. Proposition. *There is a convex figure F in the plane, which maximize the area among all convex figures with perimeter* 1.

First note that any convex figure with perimeter 1 can be moved into a 1×1-square by parallel translation. Therefore it is sufficient to show existence

[2] i.e., the minimal convex set which contains the given set

of F which maximizes area among convex figures with perimeter 1 in a given 1×1-square, which will be denoted as $\square$.

Note that $\square$ is compact. According to Blaschke's compactness Theorem 2.2.5, $\mathcal{H}(\square)$ is compact. Let us denote by $\mathcal{C}$ the subset of $\mathcal{H}(\square)$ formed by all convex sets.

2.3.3. Exercise. *Show that $\mathcal{C}$ is a closed subset of $\mathcal{H}(\square)$.*

Since $\mathcal{H}(\square)$ is compact, the above exercise implies that $\mathcal{C}$ is also compact. The area and perimeter define two functions on $\mathcal{C}$.

$$\operatorname{area}\colon \mathcal{C} \to \mathbb{R} \quad \text{and} \quad \operatorname{perim}\colon \mathcal{C} \to \mathbb{R}.$$

In order to ensure existence of F it remains to prove the following claim:

2.3.4. Proposition. *Both functions*

$$\operatorname{area}\colon \mathcal{C} \to \mathbb{R} \quad \textit{and} \quad \operatorname{perim}\colon \mathcal{C} \to \mathbb{R}$$

are continuous.

(In other words, for any Hausdorff converging sequence $K_n \to K_\infty$ of convex compact sets in $\square$, we have

$$\operatorname{area} K_\infty = \lim_{n\to\infty} \operatorname{area} K_n \quad \textit{and} \quad \operatorname{perim} K_\infty = \lim_{n\to\infty} \operatorname{perim} K_n.)$$

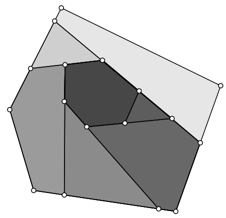
Indeed, since $\operatorname{perim}\colon \mathcal{C} \to \mathbb{R}$ is continuous, we have that the figures in $\mathcal{C}$ with perimeter 1 form a closed set, say $\mathcal{C}_1$. Since $\mathcal{C}$ is compact, so is $\mathcal{C}_1$. Then the restriction of $\operatorname{area}\colon \mathcal{C} \to \mathbb{R}$ to $\mathcal{C}_1$ admits a maximal value by Exercise 1.4; hence Proposition 2.3.2 follows.

In the proof of Proposition 2.3.4, we will use the following two lemmas.

2.3.5. Lemma. *Let P and Q be two convex compact figures in the plane. Then*

$$P \subset Q \quad \textit{and} \quad \operatorname{perim} P \leqslant \operatorname{perim} Q.$$

Proof. First note that it is sufficient to prove the lemma only for convex polygons P and Q. Note that if P is obtained from Q by intersecting Q with a half-plane then the inequality $\operatorname{perim} P \leqslant \operatorname{perim} Q$ follows from the triangle inequality.



Now note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \dots \subset P_n = Q$$

such that P_{i-1} is intersection of P_i with a half-plane. Therefore

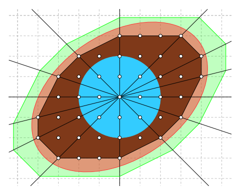
$$\begin{aligned}\operatorname{perim} P = \operatorname{perim} P_0 \leqslant \operatorname{perim} P_1 \leqslant \dots \\ \dots \leqslant \operatorname{perim} P_n = \operatorname{perim} Q\end{aligned}$$

and the lemma follows. □

2.3.6. Lemma. *Let Q be a compact convex plane figure and the interior of Q contains the origin $0 \in \mathbb{R}^2$. Then given $\varepsilon > 0$ there is a convex polygon $P \subset Q$ such that the rescaled polygon*

$$(1+\varepsilon)\cdot P = \left\{\, (1+\varepsilon)\cdot x \in \mathbb{R}^2 \;\middle|\; x \in P \,\right\}$$

contains Q.



Proof. Assume Q contains $\bar{B}_R(0)$ in its interior.

Fix small $\delta > 0$ and consider the set Z all points $(\delta\cdot m, \delta\cdot n) \in Q$ such that m and n are integers. Let P be the convex hull of Z. Since Q is convex, we have $P \subset Q$.

Note that for any point $q \in Q$ there is a point $p \in P$ such that

$$|p - q| \leqslant \sqrt{2}\cdot\delta.$$

Since δ is small, we can assume that $P \supset \bar{B}_R(0)$

In this case straightforward calculation show that the set $(1+\varepsilon)\cdot P$ contains all points on distance $\varepsilon\cdot R$ from P. In particular, lemma holds if in addition $\delta < \varepsilon\cdot\frac{R}{10}$. □

2.3.7. Exercise. *Perform the "straightforward calculation" in the above proof.*

Proof of Proposition 2.3.4. Assume K_∞ is nondegenerate; i.e. K_∞ contains a disc, say $B_R(z)$. Without loss of generality, we may assume that z is the origin $0 \in \mathbb{R}^2$; so we can apply Lemma 2.3.6 to K_∞.

Choose small $\varepsilon > 0$ and choose P as in the Lemma 2.3.6. Note that there is N such that

$$(1-2\cdot\varepsilon)\cdot P \subset K_n \subset (1+2\cdot\varepsilon)\cdot P$$

for all $n \geqslant N$. It follows that[3]

$$\begin{aligned}\operatorname{area} K_n &\lesseqgtr (1\pm 2\cdot\varepsilon)^2\cdot \operatorname{area} P \lesseqgtr \\ &\lesseqgtr (1\pm 2\cdot\varepsilon)^4\cdot \operatorname{area} K_\infty.\end{aligned}$$

From Lemma 2.3.5, we get

$$\begin{aligned}\operatorname{perim} K_n &\lesseqgtr (1\pm 2\cdot\varepsilon)\cdot \operatorname{perim} P \lesseqgtr \\ &\lesseqgtr (1\pm 2\cdot\varepsilon)^2\cdot \operatorname{perim} K_\infty.\end{aligned}$$

Hence the we get

$$\operatorname{area} K_\infty = \lim_{n\to\infty} \operatorname{area} K_n \quad\text{and}\quad \operatorname{perim} K_\infty = \lim_{n\to\infty} \operatorname{perim} K_n.$$

The following exercise finishes the proof. □

2.3.8. Exercise. *Prove that*

$$\operatorname{area} K_\infty = \lim_{n\to\infty} \operatorname{area} K_n \quad\textit{and}\quad \operatorname{perim} K_\infty = \lim_{n\to\infty} \operatorname{perim} K_n.$$

in case K_∞ is degenerate K_∞; i.e., if K_∞ is either one-point set or a segment.

(If ℓ is the length of a segment then its perimeter is defined to be $2\cdot\ell$.)

2.4 Comments

It seems that Hausdorff convergence was first introduced by Felix Hausdorff in his book [14] which appeared in print in 1914. Blaschke's book appears two years later and it introduce the same concept without a reference to Hausdorff. It it therefore likely that Hausdorff convergence was known in mathematical

[3] Here

$$a \lesseqgtr (1\pm\varepsilon)\cdot b$$

means two inequalities:

$$a \leqslant (1+\varepsilon)\cdot b \quad\text{and}\quad a \geqslant (1-\varepsilon)\cdot b.$$

The sign "$\lesseqgtr$" should be red as "more-or-less" or better as "less-or-more".

folklore before these books, but I would be very interested to know the real story.

The Hausdorff metric was only used to define convergence of compact subsets in a metric space and any equivalent metric would have done the job. In other words, the topology which the Hausdorff metric describes on the set of compact subsets is important, while the concrete metric is not.

One may wonder what will happen if we consider all subsets of the metric space X, instead of just the compact sets. In this case, distinct sets might have zero distance, say consider $X = \mathbb{R}$ and sets $A = [0, 1]$ and B is the set rational point in $[0, 1]$. It suggests that one should only consider closed sets. Further, the distance between closed subsets might be infinite. For example, consider $X = \mathbb{R}$ and the subsets $A = [0, \infty)$ and $B = \{0\}$. I.e., formally, the Hausdorff distance would not satisfy definition 1.1.1. Therefore, one would have to modify the definition of metric space to allow the distance between two points to be infinite, or one could restrict the set of subsets further to bounded closed sets. In the latter case, we get a nice metric space, say $\mathcal{H}'(X)$. Blashke's compactness theorem still holds for $\mathcal{H}'(X)$, but the proof becomes slightly more technical.

For unbounded closed sets, there is a better modification of Hausdorff convergence. Given a sequence of closed sets A_n, we say that it converges to a closed set A_∞ if the sequence of functions $\operatorname{dist}_{A_n}$ converge pointwise to $\operatorname{dist}_{A_\infty}$. This modification was introduced by Frolík in [12] and few years later by Wijsman in [23].

Note that in the above definition, if one changes *pointwise convergence* to *uniform convergence* then one obtains standard Hausdorff convergence. For example the sequence of sets $A_n = [0, n]$ in $\mathbb{R}$ diverges in the stadard Hausdorff sense, but converges to $[0, \infty)$ in the sense of Hausdorff–Wijsman.

Often one works with equivalence classes of subsets. For example, consider the set of compact sets in metric space X up to congruence. Two subsets $A, B \subset X$ are congruent (briefly, $A \cong B$) if there is an isometry $\iota\colon X \to X$ such that $\iota(A) = B$. In this case the Hausdorff metric can be used to define a metric on the congruemce classes. For example we may talk about distance between sets up to congruence; say denote by $\mathcal{H}(X)/\cong$ the set of congruence classes of compact subsets of X and equip $\mathcal{H}(X)/\cong$ with the metric defined as

$$❶ \qquad |[A] - [B]|_{\mathcal{H}(X)/\cong} = \inf_{\iota} |A - \iota(B)|_{\mathcal{H}(X)},$$

where the infimum is taken for all isometries $\iota\colon X \to X$.

2.4.1. Exercise. *Show that ❶ defines a metric on $\mathcal{H}(X)/\cong$.*

Exercises

2.A. Consider an n-gon A which maximize area among all n-gons with side-lengths $a_1, a_2, \dots, a_n$. Show that A is inscribed in a circle; i.e., all vetexes of A lie on a circle.

2.B. Let X and Y be two compact subsets in $\mathbb{R}^2$. Assume $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, is it true that $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, where ∂X denotes the boundary of X.

Does the converse holds; i.e. assume X and Y be two compact subsets in $\mathbb{R}^2$ and $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$. Is it true that $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$?

2.C. Let A and B be compact subsets of $\mathbb{R}^2$. Show that

$$|\operatorname{Conv} A - \operatorname{Conv} B|_{\mathcal{H}(\mathbb{R}^2)} \leqslant |A - B|_{\mathcal{H}(\mathbb{R}^2)},$$

where $\operatorname{Conv} A$ denotes the convex hull of A.

2.D. Let X be a compact metric space and $A, B \subset X$ be two finite ε-nets in X. Show that $|A - B|_{\mathcal{H}(X)} \leqslant \varepsilon$.

2.E. Recall that a figure of constant width a is a compact convex set in the plane such that the length of the orthogonal projection to any line is equal to a.

(i) Show that any set K in the plane with diameter 1 is a subset of a figure constant width 1.

(ii) Show that the area of any figure of constant width 1 is at least 1/100.

(iii) Let F be a compact set in the plane which can be presented as a union of figures of constant width 1. Assume $\operatorname{diam} F > 10000$. Show that either F can be presented as a union of two disjoint compact sets or $\operatorname{area} F > 1$.

(iv) Let $\mathcal{Q}$ be the set of all compact sets in the plane, which can be presented as a union of figures of constant width 1. Try to prove that area is a continuous function on $\mathcal{Q}$ (with respect to the Hausdorff metric).

(v) Use all above to show existance of a solution for the Lebesgue Minimal Problem; i.e., show that there is a plane figure F of least area which is capable of covering any plane figure of unit diameter.
Try to guess what is F.

2.F. Let X be a complete metric space and K_n be a sequence of compact sets which converges in the sence of Hausdorff. Show that closure Q of union $\bigcup_{n=1}^{\infty} K_n$ is compact.

Use this to show that in Lemma 2.2.8 compactness of X can be exchanged to completeness.

2.G. Let $K_1, K_2, \dots$ and K_∞ be compact convex sets in $\mathbb{R}^2$. Assume that K_∞ is nondegenerate; i.e., it contains interior points. Prove that $K_n \xrightarrow[\text{GH}]{} K_\infty$ if and only if

$$\operatorname{area}(K_n \backslash K_\infty) + \operatorname{area}(K_\infty \backslash K_n) \to 0.$$

Chapter 3

The space of spaces

Here we introduce so called Gromov–Hausdorff convergence for metric spaces. This convergence was introduced by Gromov around 1980, published in [13]. Very soon this notion began to be used in all branches of geometry. In fact today I have difficulty to understand how one could do geometry without this type of convergence.(Some types of convergences of metric spaces was considered before Gromov, but they had lack of generality; each type of convergence was desined to solve one particular problem.)

3.1 Gromov–Hausdorff metric

The goal of this section is to cook up a metric space out of metric spaces. More precisely, we want to define the so called Gromov–Hausdorff metric on the set of *isometry classes* of compact metric spaces. (Being isometric is an equivalence relation on the class of metric spaces, and an isometry class is an equivalence class with respect to this equivalence relation.)

The obtained metric space will be denoted as $\mathcal{M}$. Given two metric spaces X and Y, denote by $[X]$ and $[Y]$ their isometry classes; i.e., $X' \in [X]$ if and only if $X' \stackrel{iso}{=\!=} X$. Pedantically, the Gromov–Hausdorff distance from $[X]$ to $[Y]$ should be denoted as $|[X]-[Y]|_{\mathcal{M}}$; but we will often write it as $|X-Y|_{\mathcal{M}}$ and say (not quite correctly) "$|X-Y|_{\mathcal{M}}$ is the Gromov–Hausdorff distance from X to Y". In other words, from now on, if I say "metric space", you should guess from the context if I mean "metric space" or "isometry class of this metric space" (this is an abuse of notation).

Let us describe the idea behind the definition. First, we want to define the metric on $\mathcal{M}$ so that the distance between subspaces in the same metric space has to be no greater than the Hausdorff distance between them. In other words, if two subspaces of the same space are close to each other in the sense of Hausdorff distance in the ambient space, their isometry classes must be close to each other in $\mathcal{M}$. Second, we want the distance between isometric spaces to be zero. The Gromov–Hausdorff distance is in fact the maximum distance satisfying these two requirements.

3.1.1. Definition. *Let X and Y be compact metric spaces. The Gromov–Hausdorff distance $|X-Y|_{\mathcal{M}}$ between them is defined by the following relation.*

Given $r>0$, we have that $|X-Y|_{\mathcal{M}}<r$ if and only if there exist a metric space Z and subspaces X' and Y' in Z which are isometric to X and Y respectively and such that $|X'-Y'|_{\mathcal{H}(Z)}<r$. (Here $|X'-Y'|_{\mathcal{H}(Z)}$ denotes the Hausdorff distance between sets X' and Y' in Z.)

In other words, $|X-Y|_{\mathcal{M}}$ is the infimum of all $r>0$ for which the above Z, X' and Y' exist.

We say that a sequence of (isometry classes of) compact metric spaces X_n *converges in the sense of Gromov–Hausdorff* to the (isometry classes of) compact metric space X_∞ if $|X_n-X_\infty|_{\mathcal{M}}\to 0$ as $n\to\infty$; in this case we write $X_n \xrightarrow[\mathrm{GH}]{} X_\infty$.

3.1.2. Theorem. *The set of isometry classes of compact metric spaces equipped with Gromov–Hausdorff metric forms a metric space.*

This metric space will be denoted further as $\mathcal{M}$; named for "metric space".

Before proving this theorem, we give couple of variations of the definition of Gromov–Hausdorff distance.

Metrics on disjoined union of X and Y

Definition 3.1.1 deals with a huge class of metric spaces, namely, all metric spaces Z that contain subspaces isometric to X and Y. It is possible to reduce this class to metrics on the disjoint unions of X and Y. More precisely,

3.1.3. Proposition. *The Gromov–Hausdorff distance between two compact metric spaces X and Y is the infimum of $r>0$ such that there exists a metric $|*-*|_W$ on the disjoint union $W=X\sqcup Y$ such that the restrictions of $|*-*|_W$ to X and Y coincide with $|*-*|_X$ and $|*-*|_Y$ and $|X-Y|_{\mathcal{H}(W)}<r$.*

Proof. Identify $X\sqcup Y$ with $X'\cup Y'\subset Z$ (the notation is from Definition 3.1.1).

More formally, fix isometries $f\colon X \to X'$ and $g\colon Y \to Y'$, then define the distance between $x \in X$ and $y \in Y$ by $|x - y|_W = |f(x) - g(y)|_Z + \varepsilon$ for small enough $\varepsilon > 0$.[1] This yields a metric on $W = X \sqcup Y$ for which $|X - Y|_{\mathcal{H}(W)} <$ $< r$. ☐

A definition with fixed Z

3.1.4. Proposition. *In the Definition 3.1.1, one can fix the space Z once for all, by taking $Z = \mathcal{F}(\mathbb{N})$[2]. That is,*

$$|X - Y|_{\mathcal{M}} = \inf\{|X' - Y'|_{\mathcal{H}(\mathcal{F}(\mathbb{N}))}\}$$

where the infimum is taken over all pairs of sets X' and Y' in $\mathcal{F}(\mathbb{N})$ which isometric to X and Y correspondingly.

Proof. It is clear that $|X - Y|_{\mathcal{M}} \leqslant \inf\{|X' - Y'|_{\mathcal{H}(\mathcal{F}(\mathbb{N}))}\}$. Let W be an arbitrary metric space with the underlying set $X \sqcup Y$ as in the proof of Proposition 3.1.3. Note W is compact since it is union of two compact subsets $X, Y \subset W$. According to Problem 1.D, W admits a distance preserving map to $\mathcal{F}(\mathbb{N})$. So $\inf\{|X - Y)|_{\mathcal{H}(\mathcal{F}(\mathbb{N}))}\} \leqslant |X - Y|_{\mathcal{H}(W)}$, and taking the infimum over all such W gives $\inf\{|X - Y|_{\mathcal{H}(\mathcal{F}(\mathbb{N}))}\} \leqslant |X - Y|_{\mathcal{M}}$. ☐

3.1.5. Exercise. *Let X, Y be two compact sets in the Euclidean plane $\mathbb{R}^2$. Show that X is isometric to Y if and only if there is an isometry $\iota\colon \mathbb{R}^2 \to \mathbb{R}^2$ which sends X to Y.*

3.1.6. Exercise. *Find two isometric subsets X, Y of $\mathcal{F}(\mathbb{N})$ such that there is no isometry $\iota\colon \mathcal{F}(\mathbb{N}) \to \mathcal{F}(\mathbb{N})$ which sends X to Y.*

3.2 Almost isometries

3.2.1. Definition. *Let X and Y be metric spaces and $\varepsilon > 0$. A map*[3] *$f\colon X \to$ $\to Y$ is called an ε-isometry if*

$$|f(x) - f(x')|_Y \lesseqgtr |x - x'|_X \pm \varepsilon$$

[1] We add ε to ensure that $d(x, y) > 0$ for any $x \in X$ and $y \in Y$; so $|x - y|_W$ is indeed a metric.

[2] i.e., the space of bounded infinite sequences.

[3] possibly noncontinuous

for any $x, x' \in X$ and if $f(X)$ is an ε-net in Y.

3.2.2. Exercise. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two ε-isometries. Show that $g \circ f\colon X \to Z$ is a $(3{\cdot}\varepsilon)$-isometry.*

3.2.3. Exercise. *Assume $f\colon X \to Y$ is an ε-isometry. Show that there is a $(3{\cdot}\varepsilon)$-isometry $g\colon Y \to X$.*

3.2.4. Exercise. *Assume $|X - Y|_{\mathcal{M}} < \varepsilon$, show that there is a $(2{\cdot}\varepsilon)$-isometry $f\colon X \to Y$.*

3.2.5. Proposition. *Let X and Y be metric spaces and let $f\colon X \to Y$ be an ε-isometry. Then $|X - Y|_{\mathcal{M}} \leqslant 2{\cdot}\varepsilon$.*

Proof. Let us equip $W = X \sqcup Y$ with the metric defined the following way:

1. For any $x, x' \in X$
$$|x - x'|_W = |x - x'|_X;$$
2. For any $y, y' \in Y$,
$$|y - y'|_W = |y - y'|_Y$$
3. For any $x \in X$ and $y \in Y$,
$$|x - y|_W = \varepsilon + \inf_{x' \in X}\{|x - x'|_X + |f(x') - y|_Y\}.$$

3.2.6. Exercise. *Show $|{*} - {*}|_W$ is indeed a metric on W.*

Since $f(X)$ is an ε-net in Y, for any $y \in Y$ there is $x \in X$ such that $|f(x) - y|_Y \leqslant \varepsilon$; therefore $|x - y|_W \leqslant 2{\cdot}\varepsilon$. On the other hand for any $x \in X$, we have $|x - y|_W \leqslant \varepsilon$ for $y = f(x) \in Y$.

It follows that
$$|X - Y|_{\mathcal{H}(W)} \leqslant 2{\cdot}\varepsilon.$$
Hence the result. □

3.3 Gromov–Hausdorff metric is a metric

In this section we prove Theorem 3.1.2.

Let X, Y and Z be arbitrary compact metric spaces. We need to check the following (see Definition 1.1.1).

(i) $|X - Y|_{\mathcal{M}} \geqslant 0$;

(ii) $|X - Y|_{\mathcal{M}} = 0$ if and only if X is isometric to Y;
(iii) $|X - Y|_{\mathcal{M}} = |Y - X|_{\mathcal{M}}$;
(iv) $|X - Y|_{\mathcal{M}} + |Y - Z|_{\mathcal{M}} \geqslant |X - Z|_{\mathcal{M}}$.

Note that *(i)*, *(iii)* and "if"-part of *(ii)* follow directly from the definition of Gromov–Hausdorff metric (3.1.1).

Proof of (iv). Choose arbitrary $a, b \in \mathbb{R}$ such that

$$a > |X - Y|_{\mathcal{M}} \quad \text{and} \quad b > |Y - Z|_{\mathcal{M}}.$$

Choose two metrics on $U = X \sqcup Y$ and $V = Y \sqcup Z$ so that $|X - Y|_{\mathcal{H}(U)} < a$ and $|Y - Z|_{\mathcal{H}(V)} < b$ and the inclusions $X \hookrightarrow U$, $Y \hookrightarrow U$, $Y \hookrightarrow V$ and $Z \hookrightarrow V$ are distance preserving.

Consider the metric on $W = X \sqcup Z$ so that inclusions $X \hookrightarrow W$ and $Z \hookrightarrow W$ are distance preserving and

$$|x - z|_W = \inf_{y \in Y} \{|x - y|_U + |y - z|_V\}.$$

Note that $|{*} - {*}|_W$ is indeed a metric and

$$d_H^W(X, Z) < a + b.$$

The last inequality holds for any $a > |X - Y|_{\mathcal{M}}$ and $b > |Y - Z|_{\mathcal{M}}$; hence *(iv)* follows. □

Proof of "only if"-part of (ii). According to Exercise 3.2.4, for any sequence $\varepsilon_n \to 0^+$ there is a sequence of ε_n-isometries $f_n \colon X \to Y$.

Since X is compact, we can choose a countable dense set S in X. Use a diagonal procedure if necessary, to pass to a subsequence of (f_n) such that for every $x \in S$ the sequence $(f_n(x))$ converges in Y. Consider the pointwise limit map $f_\infty \colon S \to Y$ defined by

$$f_\infty(x) = \lim_{n \to \infty} f_n(x)$$

for every $x \in S$. Since

$$|f_n(x) - f_n(x')|_Y \lesseqgtr |x - x'|_X \pm \varepsilon_n,$$

we have

$$|f_\infty(x) - f_\infty(x')|_Y = \lim_{n \to \infty} |f_n(x) - f_n(x')|_Y = |x - x'|_X$$

for all $x, x' \in S$; i.e., $f_\infty \colon S \to Y$ is a distance-preserving map. Then f_∞ can be extended to a distance-preserving map from all of X to Y. The later is done by setting

$$f_\infty(x) = \lim_{n\to\infty} f_\infty(x_n)$$

for some (and therefore any) sequence of points (x_n) in S which converges to x in X. (Note that if $x_n \to x$ then (x_n) is Cauchy. Since f_∞ is distance preserving, $y_n = f_\infty(x_n)$ is also a Cauchy sequence in Y; therefore it converges.)

This way we obtain a distance preserving map $f_\infty \colon X \to Y$. It remains to show that f_∞ is surjective; i.e. $f_\infty(X) = Y$.

Note that in the same way we can obtain a distance preserving map $g_\infty \colon Y \to \to X$. If f_∞ is not surjective, then neither is $f_\infty \circ g_\infty \colon Y \to Y$. So $f_\infty \circ g_\infty$ is a distance preserving map from a compact space to itself which is not an isometry. The later contradicts Problem 1.B. □

3.3.1. Exercise. *Let P be a one-point metric space. Prove that*

$$|X - P|_{\mathcal{M}} = \frac{\operatorname{diam} X}{2}$$

for any compact metric space X.

3.3.2. Exercise. *Let X and Y be two compact metric spaces. Prove that*

$$|\operatorname{diam} X - \operatorname{diam} Y| \leqslant 2\cdot|X - Y|_{\mathcal{M}}.$$

In other words, diam *is a 2-Lipschitz function on $\mathcal{M}$.*

3.3.3. Exercise. *Assume X_n be a sequence of compact metric spaces which converges to a compact metric space X_∞ in the sense of Gromov–Hausdorff. Show that for any $\varepsilon > 0$*

$$\operatorname{pack}_\varepsilon X_n \geqslant \operatorname{pack}_\varepsilon X_\infty$$

for all large enough n. In particular, $\operatorname{pack}_\varepsilon$ *is a lower semicontinuous function on $\mathcal{M}$.*

3.4 Gromov–Hausdorff convegence

The Gromov–Hausdorff metric defines Gromov–Hausdorff convegence and this is the only thing it is good for. In other words in all applications, we use only topology on $\mathcal{M}$ and we do not care about particular value of Gromov–Hausdorff distance between spaces.

In order to determine that a given sequence of metric spaces (X_n) converges in the Gromov–Hausdorff sense to X_∞, it is sufficient to estimate distances $|X_n - X_\infty|_{\mathcal{M}}$ and check if $|X_n - X_\infty|_{\mathcal{M}} \to 0$. This problem turns to be simpler than finding Gromov–Hausdorff distance between a particular pair of spaces. The proposition below gives one way to do this.

3.4.1. Proposition. *A sequence of compact metric spaces (X_n) converges to X_∞ in the sense of Gromov–Hausdorff if and only if there is a sequence $\varepsilon_n \to 0^+$ and an ε_n-isometry $f_n\colon X_n \to X_\infty$ for each n.*

Proof. Follows from Propsition 3.2.5 and Exercise 3.2.4 □

3.5 Gromov's compactness theorem

The following theorem is analogous to Blaschke's compactness theorems (2.2.5).

3.5.1. Gromov's compactness theorem. *Let $\mathcal{Q}$ be a closed subset of $\mathcal{M}$. Then $\mathcal{Q}$ is compact if and only if there is a sequence of positive numbers $\varepsilon_1, \varepsilon_2, \dots$ such that $\varepsilon_n \to 0$ and*

$$\text{❶} \qquad \operatorname{pack}_{\varepsilon_n} X \leqslant n$$

for any space X in $\mathcal{Q}$.

3.5.2. Exercise. *Show that the conclusion of the theorem does not hold if the inequality ❶ holds only for $n \geqslant 2$.*

3.5.3. Lemma. *$\mathcal{M}$ is complete.*

Proof. Let (X_n) be a Cauchy sequence in $\mathcal{M}$. Passing to a subsequence if necessary, we can assume that $|X_n - X_{n+1}|_{\mathcal{M}} < \frac{1}{2^n}$ for each n. In particular, for each n one can equip $W_n = X_n \sqcup X_{n+1}$ with a metric such that inclusions $X_n \hookrightarrow W_n$ and $X_{n+1} \hookrightarrow W_n$ are distance preserving and

$$|X_n - X_{n+1}|_{\mathcal{H}(W_n)} < \tfrac{1}{2^n}$$

for each n.

Set W to be the disjoint union of all X_n. Let us equip W with a metric defined the following way:

◇ for any fixed n and any two points $x_n, x'_n \in X_n$ set

$$|x_n - x'_n|_W = |x_n - x'_n|_{X_n}$$

◇ for any positive integers $m > n$ and any two points $x_n \in X_n$ and $x_m \in X_m$ set

$$|x_n - x_m|_W = \inf\left\{\sum_{i=n}^{m-1} |x_i - x_{i+1}|_{W_i}\right\},$$

where the infimum is taken for all sequences $x_i \in X_i$.

3.5.4. Exercise. *Check that this indeed defines a metric on W.*

Let $\bar{W}$ be the completion of W. Note that $|X_m - X_n| < \frac{1}{2^{n-1}}$ if $m > n$. Therefore the union of $X_1 \cup X_2 \cup \dots \cup X_n$ forms a $\frac{1}{2^{n-1}}$-net in $\bar{W}$. Since each X_i is compact, we get that $\bar{W}$ admits a compact ε-net for any $\varepsilon > 0$. According to Problem 1.G, $\bar{W}$ is compact.

According to Blaschke's compactness theorem (2.2.5), we can pass to a subsequence of (X_n) which converge in $\mathcal{H}(\bar{W})$ and therefore in $\mathcal{M}$. □

Proof of 3.5.1; "only if" part. If there is no sequence $\varepsilon_n \to 0$ as described in the problem, then for a fixed fixed $\delta > 0$ there is a sequence of spaces $X_n \in \mathcal{Q}$ such that

$$\operatorname{pack}_\delta X_n \to \infty \quad \text{as} \quad n \to \infty.$$

Since $\mathcal{Q}$ is compact, this sequence has a partial limit say $X_\infty \in \mathcal{Q}$. It is easy to see that $\operatorname{pack}_{\delta/10} X_\infty = \infty$; the later contradicts Theorem 1.4.8.

"If" part. Let us fix the sequence $\varepsilon_n \to 0$ as in the problem and consider the set $\hat{\mathcal{Q}}$ of all (isometry classes of all) metric spaces X such that $\operatorname{pack}_{\varepsilon_n} X \leqslant n$ for any n. According to Exercise 3.3.3, $\hat{\mathcal{Q}}$ is closed in $\mathcal{M}$. Clearly $\mathcal{Q} \subset \hat{\mathcal{Q}}$. Therefore it is sufficient to prove that $\hat{\mathcal{Q}}$ is compact.

Note that $\operatorname{diam} X \leqslant \varepsilon_1$ for any $X \in \hat{\mathcal{Q}}$. Given positive integer n consider set of all metric spaces $\mathcal{W}_n$ with number of points at most n and diameter $\leqslant \varepsilon_1$. Note that $\mathcal{W}_n$ is compact for each n. Further a maximal ε_n-packing of any $X \in \hat{\mathcal{Q}}$ forms a subspace from $\mathcal{W}_n$. Therefore $\mathcal{W}_n \cap \hat{\mathcal{Q}}$ is a comapct ε_n-net in $\hat{\mathcal{Q}}$. Problem 1.G implies that $\hat{\mathcal{Q}}$ is compact. □

3.6 Comments

Given two metric spaces X and Y, we will write $X \preccurlyeq Y$ if there is a noncontracting map $f\colon X \to Y$; i.e., if

$$|x - x'|_X \leqslant |f(x) - f(x')|_Y$$

for any $x, x' \in X$.

Further, given $\varepsilon > 0$, we will write $X \preccurlyeq Y + \varepsilon$ if there is a map $f\colon X \to Y$ such that

$$|x - x'|_X \leqslant |f(x) - f(x')|_Y + \varepsilon$$

for any $x, x' \in X$.

Define

$$d'_{GH}(X, Y) = \inf \{\, \varepsilon \mid X \preccurlyeq Y + \varepsilon \ \text{ and } \ Y \preccurlyeq X + \varepsilon \,\}$$

It turns out that d'_{GH} is a different metric on the set of isometry classes of compact metric spaces; i.e., in general $d'_{GH}(X, Y) \neq |X - Y|_{\mathcal{M}}$. However, these two metrics define the same topology on $\mathcal{M}$. More precicely:

3.6.1. Proposition. *For any sequence of compact metric spaces (X_n) and a compact metric space X_∞, we have*

$$|X_n - X_\infty|_{\mathcal{M}} \to 0 \quad \Leftrightarrow \quad d'_{GH}(X_n, X_\infty) \to 0$$

as $n \to \infty$.

We will not give a proof of this proposition. Likely, we will not use it further in the lectures, but it might help you to build intuition for Gromov–Hausdorff convergence. If you want to prove it yourself look in the proof of Theorem 3.1.2 and try to modify it using ideas from the proof of Problem 1.B.

The Gromov–Hausdorff distance can be defined for arbitrary pair of metric space. Therefore it is natural to ask why we only consider compact metric spaces. First note the Gromov–Hausdorff distance from amy metric space X to its completion $\bar{X}$ is zero. Therfore if you want to end up in a metric space, it is better to consider only complete metric spaces.

Further, the distance between one-point-space and a metric spce with infinite diameter is infinite. Therefore one has to either consider only bounded metric spaces (i.e., the spaces with finite diameter) or relux the definition of metric space which allow metric to take infinite value.

Finally, the class of isometry classes of all bounded complete metric spaces forms a class, but not a set. Hence again we have two choices: either relux the definition of metric space so its points will form a class, or restrict further the class of spaces for which the isometry classes will form a set.

3.6.2. Exercise. *Prove that isometry classes of compact metric spaces form a set.*

Exercises

3.A. Let $X = \{x, y, z\}$ be a three point subset of Euclidean plane with distances

$$|x - y| = |y - z| = |z - x| = 1.$$

(i) Find the minimal Hausdorff distance from X to a one-point subset of the plane.
(ii) Find the Gromov–Hausdorff distance from X to the one-point metric space.

3.B. Let X and Y be a compact metric spaces which have isometric ε-nets. Show that

$$|X - Y|_{\mathcal{M}} \leqslant 2{\cdot}\varepsilon.$$

Is it always true that

$$|X - Y|_{\mathcal{M}} \leqslant \varepsilon?$$

3.C. Define the *radius of a metric space* X as

$$\operatorname{rad} X = \inf_x \left\{ \sup_y \{|x - y|_X\} \right\}.$$

Equivalently,

$$\operatorname{rad} X = \inf \{\, R > 0 \mid \text{there is } x \in X \text{ such that } B_R(x) \supset X \,\}.$$

(i) Show that for any compact metric space X we have

$$\tfrac{1}{2}\cdot \operatorname{diam} X \leqslant \operatorname{rad} X \leqslant \operatorname{diam} X.$$

(ii) Show that for any compact metric spaces X, Y we have

$$|\operatorname{rad} X - \operatorname{rad} Y| \leqslant 2 \cdot |X - Y|_{\mathcal{M}}.$$

3.D. Let X be a metric space. If two compact sets A, B in X are isometric, we will write $A \stackrel{iso}{=\!=} B$. Set

$$d(A, B) = \inf \left\{ \, |A' - B'|_{\mathcal{H}(X)} \; \middle| \; A' \stackrel{iso}{=\!=} A \text{ and } B' \stackrel{iso}{=\!=} B \right\}.$$

Note that if $X = \mathcal{F}(\mathbb{N})$ then according to Proposition 3.1.4, d is a metric on $\mathcal{H}(X)/ \stackrel{iso}{=\!=}$ (i.e., on the "$\stackrel{iso}{=\!=}$"-equivalecne classes of $\mathcal{H}(X)$).

Show that it does not hold for arbitrary metric space X. Understand the reason why it holds for $X = \mathcal{F}(\mathbb{N})$.

3.E. Let X be a comapact metric space. Denote by $\operatorname{Under}(X)$ the set of all isometry classes of metric spaces Y which admit a distance non-contracting map $Y \to X$.

a) Show that $\operatorname{Under}(X)$ forms a compact set in $\mathcal{M}$.
b) Show that for any compact set K in $\mathcal{M}$ there is a compact space X such that $\operatorname{Under}(X) \supset K$.

3.F. Consider the pairs (X, A), where X is a compact metric space and A is a closed subset in X. Two such pairs, say (X, A) and (X', A') will be called equivalent (briefly $(X, A) \sim (X', A')$) if there is an isometry $\iota \colon X \to X'$ such that $\iota(A) = A'$.

Modify the definition of Gromov–Hausdorff metric to construct a natural metric on the set of $\sim$-equivalence classes of the pairs (X, A).

Chapter 4

Geodesics

4.1 Length of curves

Recall that *real interval* is an arbitrary convex set of real line; i.e. a set $\mathbb{I}$ such that $a, b \in \mathbb{I}$ and $a < x < b$ implies $x \in \mathbb{I}$. For example, given two real numbers $a < b$ we may consider the following intervals

$$[a,b] \stackrel{def}{=\!=} \{\, x \in \mathbb{R} \mid a \leqslant x \leqslant b \,\}, \qquad [a,b) \stackrel{def}{=\!=} \{\, x \in \mathbb{R} \mid a \leqslant x < b \,\},$$
$$(a,b] \stackrel{def}{=\!=} \{\, x \in \mathbb{R} \mid a < x \leqslant b \,\}, \qquad (a,b) \stackrel{def}{=\!=} \{\, x \in \mathbb{R} \mid a < x < b \,\}.$$

In addition to the bounded intervals described above, there are unbounded intervals

$$[a,\infty) \stackrel{def}{=\!=} \{\, x \in \mathbb{R} \mid a \leqslant x \,\}, \qquad (a,\infty) \stackrel{def}{=\!=} \{\, x \in \mathbb{R} \mid a < x \,\},$$
$$(-\infty,a] \stackrel{def}{=\!=} \{\, x \in \mathbb{R} \mid a \geqslant x \,\}, \qquad (-\infty,a) \stackrel{def}{=\!=} \{\, x \in \mathbb{R} \mid a > x \,\}.$$

Finally the whole real line is also an interval

$$(-\infty,\infty) = \mathbb{R}.$$

4.1.1. Definition. *A* curve *is a continuous mapping $\alpha\colon \mathbb{I} \to X$, where $\mathbb{I}$ is a real interval and X is a metric space.*

If $\mathbb{I} = [a,b]$ and

$$\alpha(a) = p, \quad \alpha(b) = q,$$

we say that α *is a* curve from p to q.

4.1.2. Definition. *Let* $\alpha\colon \mathbb{I}\to X$ *be a curve. Define* length *of* α *as*

$$\begin{aligned}\operatorname{length}\alpha = \sup\{&|\alpha(t_0)-\alpha(t_1)|+|\alpha(t_1)-\alpha(t_2)|+\dots\\ &\dots+|\alpha(t_{k-1})-\alpha(t_k)|\}.\end{aligned}$$

where the supremum is taken over all k *and all sequences* $t_0<t_1<\dots<t_k$ *in* $\mathbb{I}$.

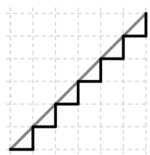
A curve is called rectifiable *if its length is finite.*

4.1.3. Exercise. *Construct a curve* $\alpha\colon[0,1]\to\mathbb{R}^2$ *which is not rectifiable.*

4.1.4. Semicontinuity of length. *Length is a lower semi-continuous functional on the space of curves* $\alpha\colon\mathbb{I}\to X$ *with respect to point-wise convergence.*

In other words: assume that a sequence of curves $\alpha_n\colon\mathbb{I}\to X$ *converges pointwise to a curve* $\alpha_\infty\colon\mathbb{I}\to X$*; i.e., for any fixed* $t\in\mathbb{I}$*, we have* $\alpha_n(t)\to\alpha_\infty(t)$ *as* $n\to\infty$*. Then*

➊ $$\liminf_{n\to\infty}\operatorname{length}\alpha_n\geqslant\operatorname{length}\alpha_\infty.$$



Note that the inequality ➊ might be strict. For example the diagonal of unit square say α_∞ (red on the picture) can be approximated by a sequence of stairs-like polygonal curves α_n with sides parallel to the sides of the squre, α_6 is black the picture. In this case $\operatorname{length}\alpha_\infty=\sqrt2$ and $\operatorname{length}\alpha_n=2$ for all n.

Proof. Fix $\varepsilon>0$ and choose a sequence $t_0<t_1<\dots<t_k$ in $\mathbb{I}$ such that

$$\begin{aligned}\operatorname{length}\alpha_\infty-(&|\alpha_\infty(t_0)-\alpha_\infty(t_1)|+|\alpha_\infty(t_1)-\alpha_\infty(t_2)|+\dots\\ &\dots+|\alpha_\infty(t_{k-1})-\alpha_\infty(t_k)|)<\varepsilon\end{aligned}$$

Set

$$\begin{aligned}\Sigma_n &\stackrel{def}{=\!=} |\alpha_n(t_0)-\alpha_n(t_1)|+|\alpha_n(t_1)-\alpha_n(t_2)|+\dots\\ &\qquad\dots+|\alpha_n(t_{k-1})-\alpha_n(t_k)|.\\ \Sigma_\infty &\stackrel{def}{=\!=} |\alpha_\infty(t_0)-\alpha_\infty(t_1)|+|\alpha_\infty(t_1)-\alpha_\infty(t_2)|+\dots\\ &\qquad\dots+|\alpha_\infty(t_{k-1})-\alpha_\infty(t_k)|.\end{aligned}$$

Note that $\Sigma_n \to \Sigma_\infty$ as $n \to \infty$ and $\Sigma_n \leqslant \operatorname{length} \alpha_n$ for each n. Hence

$$\liminf_{n\to\infty} \operatorname{length} \alpha_n \geqslant \operatorname{length} \alpha_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get ❶. □

4.2 Length spaces

4.2.1. Definition. *A metric space X is called* length space *if for any two points $x, y \in X$ and any $\varepsilon > 0$, there is a curve γ from x to y such that*

$$\operatorname{length} \gamma < |x - y|_X + \varepsilon.$$

Examples. The real line as well as higher dimensional Euclidean spaces are length spaces. A discrete space (with at least two points) is not a length space. Also, a circle in the plane forms a subspace of a length space which is not a length space.

4.2.2. Exercise. *Show that the set of (isometry classes of) length spaces in $\mathcal{M}$ is closed.*

Given a complete metric space (X, d), consider the function $\hat{d}\colon X \times X \to \mathbb{R}$ defined as

$$\hat{d}(x, y) \stackrel{def}{=\!=} \inf_\alpha \{\operatorname{length} \alpha\}$$

where the infimum is taken along all the curves α from x to y.

It is straightforward to see that $\hat{d}\colon X \times X \to \mathbb{R}$ satisfies all conditions of the metric if $\hat{d}(x, y) < \infty$ for all $x, y \in X$. In this case, the metric $\hat{d}$ will be called the *induced length metric* of d.

4.2.3. Exercise. *Construct a metric space (X, d) such that the length metric $\hat{d}$ is finite but (X, d) is not homeomorphic to $(X, \hat{d})$.*

4.3 Hopf–Rinow theorem

4.3.1. Definition. *A curve $\alpha\colon \mathbb{I}\to X$ is called a* geodesic[1] *if it is a distance preserving map; i.e., if*

$$|\alpha(t_0)-\alpha(t_1)|_X=|t_0-t_1|$$

for any $t_0,t_1\in\mathbb{I}$.

The metric space X is called geodesic *if any two points in X can be joined by a geodesic.*

4.3.2. Exercise. *Show that any proper length space is geodesic.*

4.3.3. Definition. *A metric space X is called* locally compact *if for any $x\in X$ there is an $\varepsilon>0$ such that the closed ball*

$$\bar B_\varepsilon(x)=\{\,y\in X\mid |x-y|_X\leqslant\varepsilon\,\}$$

is compact.

Note that any proper metric space is locally compact (see definition 1.4.3 on page 9). The converse does not hold in general. For example, any infinite set equipped with the discrete metric is locally compact, but not proper.

4.3.4. Exercise. *Give an example of metric space which is locally compact, but not complete.*

4.3.5. Hopf–Rinow theorem. *Any complete, locally compact length space is proper.*

Proof. Let X be a locally compact length space. Given $x\in X$, denote by $\varrho(x)$ the supremum of all $R>0$ such that the closed ball $\bar B_R(x)$ is compact. Since X is locally compact

➊ $$\varrho(x)>0\ \ \text{for any}\ \ x\in X.$$

It is sufficient to show that $\varrho(x)=\infty$ for some (and therefore any) point $x\in X$.

[1] formally our "geodesic" should be called "unit-speed minimizing geodesic", and the term "geodesic" is reserved for curves which *locally* satisfy the identity

$$|\alpha(t_0)-\alpha(t_1)|_X=\mathrm{Const}\cdot|t_0-t_1|$$

for some Const $\geqslant 0$.

Assume contrary; i.e. $\varrho(x) < \infty$. Let us show that the closed ball $W = \bar{B}_{\varrho(x)}(x)$ is compact. To prove this claim, notice that the closed ball $\bar{B}_{\varrho(x)}(x)$ is a closed set, therefore it forms a complete subspace of X (see Exercise 1.3.3). Further, since X is a length space, for any $\varepsilon > 0$, the set $\bar{B}_{\varrho(x)-\varepsilon}(x)$ is a compact ε-net in $\bar{B}_{\varrho(x)}(x)$; it remains to apply Problem 1.G.

Next we will reapeat the argument in the Lebesgue number lemma (20.1.1).

Note that if $\varrho(x)+|x-y| < \varrho(y)$, then $\bar{B}_{\varrho(x)+\varepsilon}(x)$ is a closed subset of $\bar{B}_{\varrho(y)}(y)$ for some $\varepsilon > 0$. In this case compactness of $\bar{B}_{\varrho(y)}(y)$ implies compactness of $\bar{B}_{\varrho(x)+\varepsilon}(x)$, a contradiction. Applying the same observation again switching x and y, we get

$$|\varrho(x) - \varrho(y)| \leqslant |x - y|_X;$$

in particular ϱ is a continuous function. Set $\varepsilon = \min_{y \in W}\{\varrho(y)\}$; the minimum is defined since W is compact. From ❶, we have $\varepsilon > 0$.

Choose a finite $\frac{\varepsilon}{10}$-net $\{a_1, a_2, \ldots, a_n\}$ in W. The union U of the closed balls $\bar{B}_\varepsilon(a_i)$ is compact. Clearly $\bar{B}_{\varrho(x)+\frac{\varepsilon}{10}}(x) \subset U$. Therefore $\bar{B}_{\varrho(x)+\frac{\varepsilon}{10}}(x)$ is compact or $\varrho(x) > \varrho(x)$, a contradiction. □

4.4 Geodesics, triangles and hinges

Let X be a metric space.

Geodesics. Given a pair of points $x, y \in X$, we will denote by $[xy]$ the image of a geodesic from x to y.

In general, a geodesic between x and y need not exist and if it exists, it need not be unique. However, once we write $[xy]$ we mean that we made a choice of a geodesic between x and y.

Also we will use the following notational short-cuts:

$$]xy[= [xy]\backslash\{x, y\}, \qquad]xy] = [xy]\backslash\{x\}, \qquad [xy[= [xy]\backslash\{y\}.$$

Triangles. For a triple of points $x, y, z \in X$, a choice of a triple of geodesics $([xy], [yz], [zx])$ will be called a *triangle* and we will use the short notation $[xyz] = ([xy], [yz], [zx])$.

Given a triple $x, y, z \in X$ there may be no triangle $[xyz]$ simply because one of the pairs of these points cannot be joined by a geodesic, and also there may be many different triangles with these vertices, any of which can be denoted by $[xyz]$. Once we write $[xyz]$, it means that we made a choice of such a triangle, i.e. a choice of each $[xy], [yz]$ and $[zx]$.

4.5 Model triangles and angles

Model triangles. Given $x, y, z \in X$. Let us define its *model triangle* $[\tilde{x}\tilde{y}\tilde{z}]$ (briefly, $[\tilde{x}\tilde{y}\tilde{z}] = \tilde{\triangle}(xyz)$) to be a triangle in the Euclidean plane such that

$$|\tilde{x} - \tilde{y}|_{\mathbb{R}^2} = |x - y|_X, \quad |\tilde{y} - \tilde{z}|_{\mathbb{R}^2} = |y - z|_X, \quad |\tilde{z} - \tilde{x}|_{\mathbb{R}^2} = |z - x|_X.$$

Note that a model triangle exists and is unique up to congruence.

In this case, a point $\tilde{p} \in [\tilde{x}\tilde{y}]$ is said to be *corresponding* to the point $p \in [xy]$ if $\tilde{p}$ divides $[\tilde{x}\tilde{y}]$ in the same ratio as p divides $[xy]$. (Equivalently, $|\tilde{x} - \tilde{p}|_{\mathbb{R}^2} = |x - p|_X$ or $|\tilde{y} - \tilde{p}|_{\mathbb{R}^2} = |y - p|_X$.)

Model angles. Given $x, y, z \in X$ such that

$$|x - y|, |x - z| > 0,$$

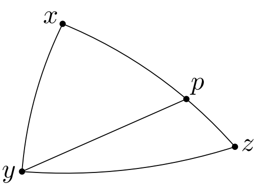
the angle measure of the model triangle $[\tilde{x}\tilde{y}\tilde{z}] = \tilde{\triangle}(xyz)$ at $\tilde{x}$ will be called the *model angle* of the triple x, y, z and it will be denoted by $\tilde{\measuredangle}(x\,{}^{y}_{z})$.

Fat and thin triangles. Let $[x_1x_2x_3]$ be a triangle in a metric space and $[\tilde{x}_1\tilde{x}_2\tilde{x}_3] = \tilde{\triangle}(x_1x_2x_3)$ be its model triangle. For any point $\tilde{z}$ $[\tilde{x}_1\tilde{x}_2\tilde{x}_3]$, one can consider the corresponding point z on the sides of the original $[x_1x_2x_3]$; i.e., if $\tilde{z} \in [\tilde{x}_i\tilde{x}_j]$ then $z \in [x_ix_j]$ and it divides $[x_ix_j]$ in the same ratio as $\tilde{z}$ divides $[\tilde{x}_i\tilde{x}_j]$. This construction defines so called *natural map* $\tilde{z} \mapsto z$ from the model triangle $[\tilde{x}_1\tilde{x}_2\tilde{x}_3] = [\tilde{x}_1\tilde{x}_2] \cup [\tilde{x}_2\tilde{x}_3] \cup [\tilde{x}_3\tilde{x}_1]$ to the original triangle $[x_1x_2x_3] = [x_1x_2] \cup [x_2x_3] \cup [x_3x_1]$.

4.5.1. Definition. *We say the triangle $[x_1x_2x_3]$ in a metric space is* fat *(respectively* thin*) if the natural map $[\tilde{x}_1\tilde{x}_2\tilde{x}_3] \to [x_1x_2x_3]$ is distance non-contracting (respectively distance non-expanding).*

For example, in Euclidean space any triangle is both fat and thin. In fact, it is true that any proper length space with this property is isometric to a convex subset of a Hilbert space.

The following is a statement in plane geometry, although its formulation uses fancier terms.



4.5.2. Alexandrov's lemma. *Let x, y, z, p be distinct points in a metric space such that $p \in \,]xz[$. Then the following expressions have the same sign:*

a) $\tilde{\measuredangle}(x\,{}^{y}_{p}) - \tilde{\measuredangle}(x\,{}^{y}_{z})$,

b) $\pi - \tilde{\measuredangle}(p\,{}^{y}_{x}) - \tilde{\measuredangle}(p\,{}^{y}_{z})$.

Proof. Consider the model triangle $[\tilde{x}\tilde{y}\tilde{p}] = \tilde{\triangle}xyp$. Take a point $\tilde{z}$ on the extension of $[\tilde{x}\tilde{p}]$ beyond $\tilde{p}$ so that $|\tilde{x}-\tilde{z}| = |x-z|$ (and therefore $|\tilde{p}-\tilde{z}| = |p-z|$).

Since increasing a side in a planar triangle increases the opposite angle, the following expressions have the same sign:

(i) $\measuredangle[\tilde{x}\,{}^{\tilde{y}}_{\tilde{z}}] - \tilde{\measuredangle}(x\,{}^{y}_{z})$;

(ii) $|\tilde{y} - \tilde{z}| - |y - z|$;

(iii) $\measuredangle[\tilde{p}\,{}^{\tilde{y}}_{\tilde{z}}] - \tilde{\measuredangle}(p\,{}^{y}_{z})$.

Since

$$\measuredangle[\tilde{x}\,{}^{\tilde{y}}_{\tilde{z}}] = \measuredangle[\tilde{x}\,{}^{\tilde{y}}_{\tilde{p}}] = \tilde{\measuredangle}(x\,{}^{y}_{p})$$

and

$$\measuredangle[\tilde{p}\,{}^{\tilde{y}}_{\tilde{z}}] = \pi - \measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{y}}] = \pi - \tilde{\measuredangle}(p\,{}^{x}_{y}),$$

the statement follows. □

4.6 Hinges and angles

Hinges. Let $p, x, y \in X$ be a triple of points such that p is distinct from x and y. A pair geodesics $([px], [py])$ will be called a *hinge* and it will be denoted by $[p\,{}^{x}_{y}] = ([px], [py])$.

Angles. Given a hinge $[p\,{}^{x}_{y}]$, we define its *angle* as

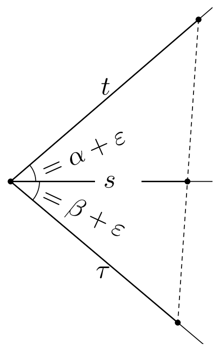
$$\measuredangle[p\,{}^{x}_{y}] \xlongequal{def} \lim_{\bar{x},\bar{y}\to p} \tilde{\measuredangle}(p\,{}^{\bar{x}}_{\bar{y}}),$$

where $\bar{x} \in\,]px]$ and $\bar{y} \in\,]py]$, if the limit exists.

4.6.1. Exercise. *Given an example of a hinge in a proper length space for which the angle is not defined, i.e. the above limit does not exist.*

4.6.2. Triangle inequality for angles. *Let $[px]$, $[py]$ and $[pz]$ be three geodesics in a metric space. If all of the angles $\alpha = \measuredangle[p\,{}^{x}_{y}]$, $\beta = \measuredangle[p\,{}^{y}_{z}]$ and $\gamma = \measuredangle[p\,{}^{x}_{z}]$ are defined, then they satisfy the triangle inequality:*

$$\gamma \leqslant \alpha + \beta.$$



Proof. Since $\gamma \leqslant \pi$, we can assume that $\alpha + \beta < \pi$. Parametrize $[px]$, $[py]$, and $[pz]$ by arc-length starting from p and denote the obtained curves by σ_x, σ_y and σ_z. Given any $\varepsilon > 0$, for all sufficiently small $t, \tau, s \in \mathbb{R}_+$ we have

$$\begin{aligned}|\sigma_x(t) - \sigma_z(\tau)| &\leqslant |\sigma_x(t) - \sigma_y(s)| + |\sigma_y(s) - \sigma_z(\tau)| \\ &< \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha+\varepsilon)} + \\ &\quad + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\beta+\varepsilon)}\end{aligned}$$

Below we define $s(t, \tau)$ so that for $s = s(t, \tau)$, this chain continues

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha+\beta+2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\gamma \leqslant \alpha + \beta + 2{\cdot}\varepsilon.$$

Hence the result.

To define $s(t, \tau)$, consider three rays $\tilde\sigma_x$, $\tilde\sigma_y$, $\tilde\sigma_z$ in the Euclidean plane starting at one point, such that $\measuredangle(\tilde\sigma_x, \tilde\sigma_y) = \alpha + \varepsilon$, $\measuredangle(\tilde\sigma_y, \tilde\sigma_z) = \beta + \varepsilon$ and $\measuredangle(\tilde\sigma_x, \tilde\sigma_z) = \alpha + \beta + 2{\cdot}\varepsilon$. We parametrize each ray by length from the common end. Given two positive numbers $t, \tau \in \mathbb{R}_+$, let $s = s(t, \tau)$ be the number such that $\tilde\sigma_y(s) \in [\tilde\sigma_x(t)\ \tilde\sigma_z(\tau)]$. Clearly $s \leqslant \max\{t, \tau\}$, so t, τ, s may be taken sufficiently small. ☐

Exercises

4.A. Show that $\mathcal{H}(\mathbb{R}^2)$ is a length space.

4.B. Show that $\mathcal{M}$ is a length space.

4.C. Let $K_n \to K_\infty$ be a sequence of compact convex bodies in $\mathbb{R}^3$ which converges in the sense of Hausdorff. Assume K_∞ is nondegenerate (i.e., its interior is not empty). Denote by ∂K_n the surface of K_n; i.e. the boundary[2] of K_n equipped with induced length metric. Show that $\partial K_n \xrightarrow[\text{GH}]{} \partial K_\infty$.

[2] A point x belongs to the boundary of a subset S of a metric space if for any $\varepsilon > 0$, the ball $B_\varepsilon(x)$ contains a point in S as well as a point in the complement of S.

What happens if K_∞ degenerates to a plane figure or an interval?

4.D. Assume that for any three points x, y and z of a compact length metric space X we have

$$|x-z|_X \geqslant |x-y|_X, |y-z|_X \implies |x-z|_X = |x-y|_X + |y-z|_X$$

Show that X is isometric to a closed real interval.

4.E.[3] Let X a metric space with hinge $[x\,{}^y_z]$. Assume that the angle $\alpha = \measuredangle[x\,{}^y_z]$ is defined. Show that

$$|z-\bar{y}| \leqslant |z-x| - |x-\bar{y}|\cdot\cos\alpha + o(|x-\bar{y}|)$$

for $\bar{y} \in \,]xy]$.

4.F. Prove that the sum of adjacent angles is at least π.

More precisely: let X be a geodesic space and $p, x, y, z \in X$. If $p \in \,]xy[$, then

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \geqslant \pi$$

whenever each angle on the left-hand side is defined.

[3]Hint: Apply the definition of angle and triangle inequality

$$|z-\bar{y}| \geqslant |\bar{z}-\bar{y}| + |\bar{z}-z|$$

for $\bar{z} \in \,]xz]$.

Chapter 5

Polyhedral spaces

5.1 Simplexes

Simplex. Let $\{v_0, v_1, \dots, v_m\}$ be a set of points in $\mathbb{R}^N$ for $N \geqslant m$ such that the m vectors

$$v_1 - v_0, v_2 - v_0, \dots, v_m - v_0$$

are linearly independent. The convex hull $\Delta^m = \mathrm{Conv}(v_0, v_1, \dots, v_m)$ is called an *m-dimensional simplex*.

So, a 0-dimensional simplex is a one-point set; a 1-dimensional simplex is a line segment; a 2-dimensional simplex is a triangle; a 3-dimensional simplex is a tetrahedron.

If Δ^m as above then the convex hull of any $(k{+}1)$-point subset of $\{v_0, v_1, \dots, v_m\}$ also forms a k-dimensional simplex which will be called *face* of Δ^m.

Barycentric coordinates. Let $\Delta^m = \mathrm{Conv}(v_0, v_1, \dots, v_m)$ be an m-dimensional simplex. Note that $x \in \Delta^m$ if and only if

$$x = \lambda_0 {\cdot} v_0 + \lambda_1 {\cdot} v_1 + \dots + \lambda_n {\cdot} v_m$$

for some nonnegative real numbers $\lambda_0, \lambda_1, \dots, \lambda_m$ such that

$$\sum_{i=0}^{m} \lambda_i = 1.$$

In this case, the real array $(\lambda_0, \lambda_1, \dots, \lambda_m)$ will be called the *barycentric coordinates* of the point x.

5.1.1. Exercise. *Verify the claim that*

$$\operatorname{Conv}(v_0, v_1, \dots, v_m) = \left\{ \sum_{i=0}^{m} \lambda_i{\cdot}v_i \;\middle|\; \lambda_i \geqslant 0 \textit{ and } \sum_{i=0}^{m} \lambda_i = 1 \right\}.$$

5.2 Simplicial complexes

A *simplicial complex* is defined as a finite collection $\mathcal{K}$ of simplices in $\mathbb{R}^n$ that satisfies the following conditions:

- ◇ Any face of a simplex from $\mathcal{K}$ is also in $\mathcal{K}$.
- ◇ The intersection of any two simplices Δ_1 and $\Delta_2 \in \mathcal{K}$ is either an empty set or it is a face of both Δ_1 and Δ_2.

The dimension ($\dim \mathcal{K}$) of simplicial complex $\mathcal{K}$ is defined as the maximal dimension of all of its simplices.

Note that to describe a simplicial complex $\mathcal{K}$ in $\mathbb{R}^n$ it is sufficient to list the vertices $v_1, \dots, v_k$ of all simplices in $\mathcal{K}$ and the set of subsets $\mathcal{S}$ of $F = \{1, \dots, k\}$ such that $X \in \mathcal{S}$ if and only if there is a simplex of $\mathcal{K}$ with vertices in X. The information encoded in $\mathcal{S}$ is called *abstract simplicial complex.*

More formally, an *abstract simplicial complex* is a finite set F with a family $\mathcal{S}$ of subsets of F such that for every set X in $\mathcal{S}$, and every subset $Y \subset X$, Y also belongs to $\mathcal{S}$; in particular, $\varnothing \in \mathcal{S}$.

As it was noted above, given a simplicial complex $\mathcal{K}$ defines an abstract simplicial complex. On the other hand, given an abstract simplicial complex $\mathcal{S}$ on finite set $F = \{1, \dots, n\}$ one can construct geometric simplicial complex in $\mathbb{R}^n$, by taking its vertices in a basis of $\mathbb{R}^n$.

We say that a point x belongs to simplicial complex $\mathcal{K}$ if it belongs to one of its simplices. The set of all points of $\mathcal{K}$ is called *underlying set* of $\mathcal{K}$; it will be denoted as $|\mathcal{K}|$.

If $\mathcal{K}$ is a simplicial complex and $\{v_1, v_2, \dots, v_k\}$ is the set of all its vertices then any point x in $\mathcal{K}$ can be described through barycentric coordinates $(\lambda_1, \lambda_2, \dots, \lambda_n)$, so $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$, $\lambda_i \geqslant 0$ for any i and for any x the set $X = \{\, i \mid \lambda_i > 0 \,\}$ belong to the corresponding abstract complex.

5.3 Polytopes

The union of all simplices in $\mathcal{K}$ is called *underlying set* of $\mathcal{K}$; it will be denoted as $|\mathcal{K}|$.

A subset P of $\mathbb{R}^n$ is called *polytope* if it can be presented as underlying set of some geometric simplicial complex. Given a polytope P, any simplicial complex $\mathcal{K}$ with the underlying set P is called *triangulation*[1] of P. In general, a polytope may have many distinct triangulations.[2]

5.3.1. Exercise. *Let P be a polytope. Show that dimension of any triangulation of P has the same dimension.*

The exercise above makes possible to define the dimension of polytope is defined as the dimension of its triangulation.

With a slight abuse of notation we may say that a simplicial complex $\mathcal{K}$ is homeomorphic to simplicial complex $\mathcal{K}'$ if their underlying spaces are homeomorphic. A metric (or topological) space Q is called *topological polytope* if it admits a homeomorphsm from a polytope say P. Such a homeomorphism together with triangulation of P is called *topological triangulation* of Q.

5.3.2. Exercise. *Find a topological triangulation of sphere*[3] $\mathbb{S}^2$ *with minimal number of triangles.*

You will need at least 14 triangles to do topological triangulation the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Finding such triangulation might be interesting, but proving its minimality is not fun.

The dimension of topological polytope is also defined as the maximal dimension of the simplices in its triangulation. This value also the same for any triangulation, but the proof requires Domain Invariance Theorem (15.5.2).

5.4 Polyhedral spaces

5.4.1. Definition. *A complete length space P is called a* polyhedral space *if it admits a triangulation such that each simplex in P is isometric to a simplex in Euclidean space.*

Further by a *triangulation of a polyhedral space* we will understand the triangulation as in the definition. We will also assume that the triangulation is linear; i.e., the metric is induced by a linear map into Euclidean space which is linear with respect to barycentric coordinates. In particular, if

$$(\lambda_1, \lambda_2, \dots, \lambda_n) \quad \text{and} \quad (\mu_1, \mu_2, \dots, \mu_n)$$

[1] The term is a bit misleading; the triangulation may contain simplexes of arbitrary dimension

[2] In fact if the triangulation is unique then polytope is formed by a finite set of points.

[3] The unit n-sphere is $\mathbb{S}^n = \{\, (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1 \,\}$.

are baricentric coordinates of two points x and y in one simplex of the triangulation then the point with coordinates

$$\left(\tfrac{\lambda_1+\mu_1}{2}, \tfrac{\lambda_2+\mu_2}{2}, \dots, \tfrac{\lambda_n+\mu_n}{2}\right)$$

is a midpoint of x and y.

The supremum of the dimensions of all simplices in such a triangulation is called *dimension* of P and denoted as $\dim P$.

1-dimensional simplicial complexes are also called *graphs*. If any two vertices are connected by exactly one simple path then graph is called *tree*. By that reason 1-dimesional polyhedral spaces are also called *metric graphs* and if the polyhedral spaces is build on a tree, it is called *metric tree*.

5.4.2. Exercise. *Show that any convex nondegenerate polyhedron*[4] *in* $\mathbb{R}^m$ *is an* m*-dimensional polyhedral space.*

5.4.3. Exercise. *Show that boundary any convex nondegenerate polyhedron*[4] *in* $\mathbb{R}^m$ *equipped with its length metric is an* $(m-1)$*-dimensional polyhedral space.*

5.4.4. Exercise. *The dimension of a polyhedral space does not depend on the choice of triangulation.*

5.5 Euler formula

5.5.1. Theorem. *Let* Γ *be a connected graph embedded in* $\mathbb{S}^2$*. If* k *is the number of vertices and* l *is the number of edges of* Γ *and* m *is the number faces; i.e., the domainswhich* Γ *cuts from the* $\mathbb{S}^2$*. Then*

$$k - l + m = 2.$$

Proof. If the connected planar graph Γ has no edges, it is an isolated vertex and

$$k - l + m = 1 - 0 + 1 = 2.$$

Otherwise, choose any edge e. If e connects two distinct vertices, contract it, reducing k and l by one.

Otherwise, e is a loop; i.e., it connects a vertex to itself. By ???, e separates two faces. Remove e; it reduces l and m by one.

[4] i.e., a convex hull of finite number of points which has nonempty interior.

In either case the result follows by induction on $k + l + m$. □

5.5.2. Theorem. *Let P be a convex polytope in $\mathbb{R}^3$ and k, l and m denotes the number of its vertices edges and faces. Then*

$$k - l + m = 2.$$

5.5.3. Theorem. *Let $\mathcal{S}$ be a simplicial complex which is homeomorphic to $\mathbb{S}^2$. If k is the number of vertices in $\mathcal{S}$ then $\mathcal{S}$ has $l = 3{\cdot}k - 6$ edges and $m = 2{\cdot}k - 4$ triangles.*

Proof. Each edge appears as a side in exactly two triangles and each triangle has three sides; i.e., we have

$$3{\cdot}m = 2{\cdot}l.$$

Applying Euler's formula, we get the result. □

5.5.4. Theorem. *Let P be a polyhedral surface which is homeomorphic to $\mathbb{S}^2$. Then sum of the curvatures at all vertices of P is equal to $4{\cdot}\pi$.*

Proof. Choose a triangulation $\mathcal{T}$ of P. If k is the number of vertices of $\mathcal{T}$, According to Theorem 5.5.3, the number of triangles is $m = 2{\cdot}k - 4$. Therefore the total sum of all angles of all the triangles in $\mathcal{T}$ is

$$\pi{\cdot}(2{\cdot}k - 4).$$

Let $v_1, v_2, \dots, v_k$ be the vertices of $\mathcal{T}$, denote by α_i the total sum of angles and ω_i the curvature at v_i; so $\omega_i = 2{\cdot}\pi - \alpha_i$. Then

$$\alpha_1 + \dots + \alpha_k = \pi{\cdot}(2{\cdot}k - 4).$$

Therefore

$$\begin{aligned}\omega_1 + \dots + \omega_k &= 2{\cdot}\pi{\cdot}k - \pi{\cdot}(2{\cdot}k - 4) = \\ &= 4{\cdot}\pi.\end{aligned}$$

□

5.5.5. Theorem. *Let P be a polyhedral disc. Denote by Ω the total curvature of all interior vertices of P and T be total boundary turn. Then*

$$\mathrm{T} + \Omega = 2{\cdot}\pi.$$

Proof. Consider doubling W of P; i.e., two copies of P glued along the corresponding points of their boundaries.

Note that W is a polyhedral surface homeomorphic to $\mathbb{S}^2$. The total curvature of W is $2{\cdot}\Omega + 2{\cdot}\mathrm{T}$. Applying 5.5.4, we get the result. □

5.6 Polyhedral surfaces

A 2-dimensional polyhedral space P is called *polyhedral surface* if each point in P admits a neighborhood homeomorphic to an open set in a half-plane.

This condition can be reformulated in a more combinatorial fashion. Namely, every edge in a triangulation of P appears as a side of exactly one or two triangles and all the triangles with common vertex admit a cycle or linear order such that two triangles share a side if and only if they are neighbors in the cycle order.

5.6.1. Exercise. *Prove the equivalence of these two conditions.*

5.6.2. Definition. *Let P be a polyhedral surface. Given $p \in P$, consider a triangulation of P for which p is a vertex. Denote by α_p the sum of the angles around p. The value α_p will be called the* total angle around p *and the value*

$$\omega_p = 2{\cdot}\pi - \alpha_p$$

will be called the curvature of P at p.

Note that for any $p \in P$, there is a triangulation as described in the above definition, and that the curvature of P at p does not depend on the choice of such a triangulation.

5.6.3. Exercise. *Assume P is a polyhedral surface and $\mathcal{T}$ is a triangulation of P. Show that if the curvature of P at p is nonzero then p is a vertex of $\mathcal{T}$.*

5.6.4. Exercise. *Assume P is a polyhedral surface which is is homeomorphic to $\mathbb{S}^2$ and $\mathcal{T}$ is a triangulation of P. Show that the sum of the curvatures of P at all vertices of $\mathcal{T}$ is equal to $4{\cdot}\pi$.*

5.7 Comments

Nerves and partition of unity

Here we describe one source of examples of simplicial complexes which appear in many branches of mathematics. We will not need it further, but understanding these constructions might help you to understand idea behind the notion of simplicial complex, and for sure it will help you in the future (assuming you will do mathematics).

Nerve. Let $\mathcal{V}$ be a collection of subsets of some set. Consider the abstract simplicial complex, where $\mathcal{V}$ is the set of vertices and $\mathcal{S}$ is all collections of subsets in $\mathcal{V}$ which have non-empty intersection. We obtain a simplicial complex called the *nerve of* $\mathcal{V}$.

If $\mathcal{V}$ is finite then so is its nerve. If any set in $\mathcal{V}$ intersects only finitely many other sets in $\mathcal{V}$, then its nerve is locally finite.

5.7.1. Definition. *Given $L \geq 0$, a map $f\colon X \to Y$ between metric spaces is* L-Lipschitz *if*

$$|f(x) - f(x')|_Y \leqslant L{\cdot}|x - x'|_X$$

for all $x, x' \in X$. Note that this implies f is continuous.

A map $f\colon X \to Y$ is called Lipschitz *if it is L-Lipschitz for some real L.*

A map $f\colon X \to Y$ is called locally Lipschitz *if for any point $x \in X$ there is $\varepsilon > 0$ such that the restriction $f|_{B_\varepsilon(x)}$ is Lipschitz.*

5.7.2. Partition of unity. *Let $\mathcal{V} = \{V_1, V_2, \dots, V_n\}$ is a finite open covering of a metric space X. Then there are locally Lipschitz functions $\psi_i\colon X \to [0,1]$ such that if $\psi_i(x) > 0$ then $x \in V_i$ and*

$$\sum_i \psi_i(x) = 1$$

for any $x \in X$.

A collection of functions ψ_i with above properies is called a *partition of unity subordinate to the open cover* $\{V_1, V_2, \dots, V_n\}$.

Proof. Consider the functions $\varphi_i\colon X \to \mathbb{R}$ defined as following:

$$\varphi_i(x) = \operatorname{dist}_{X\setminus V_i} x.$$

Note φ_i is 1-Lipschitz for any i (see the definition below) and $\varphi_i(x) > 0$ if and only if $x \in V_i$. In particular,

$$\sum_i \varphi_i(x) > 0 \quad \text{for any} \quad x \in X.$$

Set

$$\psi_k(x) = \frac{\varphi_k(x)}{\sum_i \varphi_i(x)}.$$

Note that ψ_k are locally Lipschitz; $\varphi_i(x) \geqslant 0$ and if $\psi_i(x) > 0$ then $x \in V_i$; further

$$\sum_i \psi_i(x) = 1 \quad \text{for any} \quad x \in X.$$

□

Note that in the above proof for any point $x \in X$, the set

$$\{ V_i \mid \psi_i(x) > 0 \}$$

corresponds to one of the simplices in the nerve. Therefore

$$\boldsymbol{\psi} \colon x \mapsto (\psi_1(x), \psi_2(x), \dots, \psi_n(x))$$

can be thought of as a Lipschitz map from X to the nerve of $\{V_1, V_2, \dots, V_n\}$; where the point x is mapped to the point with barycentric coordinates $\psi_i(x)$.

Exercise

5.A. Show that there is a triangulation of $\mathbb{S}^3$ with 1000 vertices such that each pair of vertices is connected by an edge.

5.B. Let P be a (possibly nonconvex) polygon equipped with the induced length metric. Show that P admits a triangulation[5] such that the set of vertices of the triangulation is the set of vertices of P.

5.C.[6] Show that the analogous statement for a polyhedron 3-dimensional space does not hold.

5.D. Let $\mathcal{A}$ and $\mathcal{B}$ be two triangulations of a polyhedral space[5]. Show that there is a triangulation $\mathcal{C}$ such that each triangle of $\mathcal{A}$ and $\mathcal{B}$ is a union of triangles in $\mathcal{C}$.

5.E. Let F be a metric space with finite number of points. Show that F is isometric to a subset of metric tree if and only if for any four points x_0, x_1, y_0, y_1 in F we have

$$|x_0 - x_1|_F + |y_0 - y_1|_F \leqslant |x_0 - y_0|_F + |x_0 - y_1|_F + |x_1 - x_0|_F + |x_1 - y_1|_F.$$

5.F. Let P be a convex hull of the finite set of points $\{x_1, x_2, \dots, x_n\}$ in $\mathbb{R}^2$. Assume that positive reals $r_1, r_2, \dots, r_n$ are chosen in such a way that the balls

[5]Recall that triangulations of polyhedral space are always assumed to be linear; see Definition 5.4.1 and the discussion right after that.

[6]Hint: look at the Figure 42.6, page 387 *Lectures on Discrete and Polyhedral Geometry* by Igor Pak

$B_i = B_{r_i}(x_i)$ cover P and moreover each side of P is covered by the balls centered on this side.

Show that P admits a triangulation with vertices at x_i such that each triangle is covered by the three balls centered at its vertices.

5.G. Let P be a compact subset of Euclidean space. Show that P is a polytope for every point $x \in P$ there is a *cone*[7] K_x *with tip at* x and $\varepsilon > 0$ such that

$$B_\varepsilon(x) \cap P = B_\varepsilon(x) \cap K_x.$$

[7] A cone with tip x is a set formed by union of a set of rays starting at x.

Part II

No curvature

Chapter 6

Zalgaller's folding theorem

Let P be a polyhedral space. A map $f\colon P \to \mathbb{R}^n$ is called *piecewise distance preserving* if there is a triangulation of P such that for any simplex Δ in the triangulation, the restriction $f|_\Delta$ is distance preserving.

6.0.1. Exercise. *Show that any piecewise distance preserving map f is continuous and length-preserving. (It then follows from Exercise* **??** *that f is also distance non-expanding.)*

The following statement might look obvious, but try to prove it rigorously.

6.0.2. Exercise. *Suppose that an m-dimensional polyhedral space admits a piecewise distance preserving map into $\mathbb{R}^n$. Show that $n \geqslant m$.*

In fact, the converse of the statement in the exercise is true. In other words, dimension is the only obstruction to the existence of a piecewise distance preserving map into Euclidean space. The following theorem asserts this for the case where $m = 2$.

6.0.3. Zalgaller's theorem. *Any 2-dimensional polyhedral space admits a piecewise distance preserving map into the Euclidean plane.*

Imagine that you have a paper model of a 2-dimensional polyhedral space P in your hands, and you fold this model so that it lays flat on a table.[1] This is an intuitive way to think of a piecewise distance preserving map $f\colon P \to \mathbb{R}^2$. To make it closer to the actual definition, one has to imagine that the layers of

[1] We recommend that you create such a paper model, say the surface of a cube, and then try to fold it on the table.

paper can go through each other. Zalgaller's theorem says that such a "folding" is always possible; see also Exercise 10.0.3.

The following exercise shows that in this process, new folds may need to be introduced across the triangles of the given triangulation of P.

6.0.4. Exercise. *Let Δ be a non-degenerate 3-dimensional simplex in $\mathbb{R}^3$ and let $\partial\Delta$ be its boundary, equipped with the induced length metric. It is a polyhedral space glued from 4 triangles — the faces of Δ.*

Show that $\partial\Delta$ does not admit a map to $\mathbb{R}^2$ which is distance preserving on each of the 4 faces of Δ.

Describe explicitly a piecewise distance preserving map

$$f\colon \partial\Delta \to \mathbb{R}^2$$

which is distance preserving on 2 out of the 4 faces of Δ. (You will have to subdivide the other 2 faces into smaller triangles.)

Below we give two similar proofs of Zalgaller's theorem: the first with cheating and the second without. In the first proof, we use the following claim without proof.

❶ *Any 2-dimensional polyhedral space admits an* acute triangulation*, that is, a triangulation such that all of its triangles are acute.*

6.0.5. Exercise. *Show that any triangle admits an acute triangulation.*

Notice that there is more to proving ❶ than simply subdividing each triangle into acute triangles. One must be careful that the subdivisions of two triangles that share an edge are compatible on that common edge. See [21] for a proof of ❶.

Proof using ❶. Fix an acute triangulation $\mathcal{T}_0$ of P provided by ❶. Mark all its vertices in white and denote them by $\{w_1, \dots, w_k\}$.

For each w_i, consider its *Voronoi domain* V_i, which is the subset

$$V_i = \{\, x \in P \mid |x - w_i| \leqslant |x - w_j| \text{ for any } j \,\}.$$

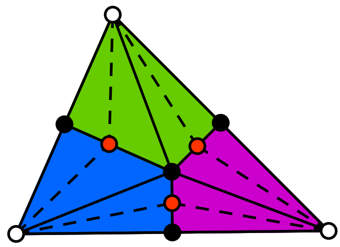
Denote by $S(w_i)$ the *star* of w_i, which is the union of all simplices of the triangulation $\mathcal{T}_0$ which contain w_i.

Since an acute triangle contains its own circumcenter, it is impossible for the Voronoi domain of a vertex of a triangle to cross the opposite edge. From this it follows that

$$V_i \subset S(w_i)$$

for all i. In particular, for any point $x \in V_i$, there is a unique geodesic $[w_i, x]$, which is a line segment in a single triangle or an edge of $\mathcal{T}_0$.

Note that in each triangle of $\mathcal{T}_0$, we have one point where three Voronoi domains meet and three points on the sides of triangle where pairs of Voronoi domains meet.Let us bisect each edge of $\mathcal{T}_0$ and subdivide each triangle into 6 triangles as it is done on the picture (solid lines only).



The Voronoi domains within one triangle.

In this way we obtain a new triangulation $\mathcal{T}_1$. We mark all the new vertices of $\mathcal{T}_1$ in black.

Note that

1. Each V_i is a union of all triangles and edges of $\mathcal{T}_1$ which have w_i as a vertex.
2. Each triangle in $\mathcal{T}_1$ has one white and two black vertices.
3. The triangles in $\mathcal{T}_1$ come in pairs of congruent triangles, they share two black vertices and have different white vertices.

Given a point $x \in P$, set

$$\varrho(x) = \min_i \{|w_i - x|\}.$$

Notice that if $x \in V_i$, then $\varrho(x) = |w_i - x|$. Given $x \in V_i$, we denote by $\vartheta_i(x)$ the minimum angle between $[w_i, x]$ and any edge of $\mathcal{T}_1$ coming from w_i.

By Property 3, if $x \in V_i \cap V_j$ then $\vartheta_i(x) = \vartheta_j(x)$. In other words, the function ϑ given by

$$\vartheta(x) = \vartheta_i(x), \qquad x \in V_i$$

is well-defined on the set $P \backslash \{w_1, \dots, w_n\}$. Moreover, ϑ is a continuous function.

We now describe the map $f \colon P \to \mathbb{R}^2$ using polar coordinates on $\mathbb{R}^2$. We define $f(w_i) = 0$ and $f(x) = (\varrho(x), \vartheta(x))$ if $x \in P \backslash \{w_1, \dots, w_n\}$.

Subdividing each triangle by the angle bisector at the white vertex (see the dashed lines on the picture) gives a new triangulation $\mathcal{T}_2$ which satisfies the conditions of the theorem for the constructed map f. $\square$

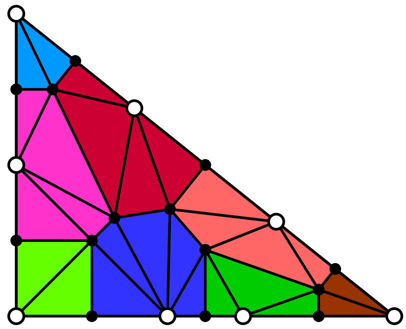
Now we modify the above proof so it does not use Claim ❶. The only property we really need for the triangulation is that $V_i \subset S(w_i)$. On one hand, this inclusion does not hold for a general triangulation. A simple example of this is obtained by gluing an equilateral triangle to the longer side of obtuse triangle. On the other hand, by increasing the number of Voronoi domains, we can arrange that this inclusion holds without making the triangulation acute.

Proof without using ❶. Fix a triangulation $\mathcal{T}_0$ of P that is not necessarily acute. We will construct new triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of P by subdividing the triangles of $\mathcal{T}_0$, and we will define a map $f\colon P \to \mathbb{R}^2$ which is distance preserving on each triangle of $\mathcal{T}_2$. The vertices of $\mathcal{T}_1$ will be colored either white or black in such a way that each triangle of $\mathcal{T}_1$ will have two black vertices and one white vertex.

We shall first describe the set of white vertices.

Fix a small number $\varepsilon > 0$. We mark in white all of the vertices of $\mathcal{T}_0$, as well as the points on the sides of triangles of $\mathcal{T}_0$ with the property that the distance to the closest vertex of the edge is an integer multiple of ε. In this way we mark a finite number of points white. Label the white points by $w_1, \dots, w_k$.

As in the previous proof, let V_i be the Voronoi domain of w_i, so that



A triangle Δ of $\mathcal{T}_0$ with marked white points and the intersections of their Voronoi domains with Δ.

$$V_i = \{\, x \in P \mid |x - w_i| \leqslant |x - w_j| \text{ for any } j \,\}.$$

We also let $S(w_i)$ be the star of w_i in $\mathcal{T}_0$, which is the union of all simplices of $\mathcal{T}_0$ which contain w_i. By the following exercise, we can assume that $V_i \subset S(w_i)$ for each i, by taking a suitably small ε.

6.0.6. Exercise. *Let ℓ be the minimal length of the edges in the triangulation, and let α be the minimal angle of all the triangles in $\mathcal{T}_0$. Show that if $\varepsilon < \frac{\ell\cdot\alpha}{100}$, then $V_i \subset S(w_i)$ for each i.*

Fix a triangle Δ of $\mathcal{T}_0$. Note that for any $w_i \in \Delta$, the intersection $V_i \cap \Delta$ is a convex polygon. This follows because for any two points $w_i, w_j \in \Delta$, the inequality

$$|x - w_i| \leqslant |x - w_j|$$

describes the set of all points $x \in \Delta$ which lies on one side of the bisecting perpendicular to w_i and w_j. Let us color the vertices of all of the polygons $V_i \cap \Delta$ in black, if they are not already white.

If $\mathcal{T}_0$ contains an edge E which is not a side of a triangle then color the midpoint of E in black.

We'll now describe the triangulation $\mathcal{T}_1$. The vertice of $\mathcal{T}_1$ are the black and white vertices. A white point w_i is connected by an edge to each black point in V_i. A pair of black vertices b and b' are connected by an edge if it forms a side of some $V_i \cap \Delta$ (for some Δ and $w_i \in \Delta$). In this case w_i, b and b' also form a triangle of $\mathcal{T}_1$. Notice that each black-black edge is a side of two congruent triangles of $\mathcal{T}_1$ with different white vertices.

The remaining part of the proof is the same as before. We define

$$\varrho(x) = \min_i\{|w_i - x|_P\}$$

and $\vartheta(x)$ for $x \in V_i$ as the minimal angle between $[w_i, x]$ and any edges in $\mathcal{T}_1$ coming from w_i. Then define the map $f\colon P \to \mathbb{R}^2$ so that $f(w_i) = 0$ for each i and $f(x) = (\varrho(x), \vartheta(x))$ in polar coordinates.

Further subdividing each triangle of $\mathcal{T}_1$ in two along the angle bisector from the white vertex produces a new triangulation $\mathcal{T}_2$. It is straightforward to see that the constructed map f is distance preserving on each triangle of $\mathcal{T}_2$. $\square$

Use Zalgaller's theorem to show the following.

6.0.7. Advanced exercise. *Any 2-dimensional polyhedral space is isometric to the underlying set of a simplicial complex in $\mathbb{R}^n$, equipped with its induced length metric.*

We end this section with an entertaining exercise.

6.0.8. Exercise. *Let $\mathcal{T}$ be a triangulation of a convex polygon Q in $\mathbb{R}^2$ such that each triangle is colored either black or white. Show that the following two conditions are equivalent.*

a) There is a piecewise distance preserving map $Q \to \mathbb{R}^2$ for this triangulation which preserves the orientation[2] of each white triangle and reverses the orientation of each black triangle.

b) The sum of black angles around any vertex of $\mathcal{T}$ which lies in the interior of Q is either 0, π or $2{\cdot}\pi$.

[2]We say that the motion preserves/reverses the orientation if it is a composition of even/odd number of reflections.

Chapter 7

Brehm's extension theorem

7.0.1. Brehm's extension theorem. *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be two collections of points in $\mathbb{R}^2$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j, and let A be a convex polygon which contains $a_1, \dots, a_n$. Then there is a piecewise distance preserving map $f\colon A \to \mathbb{R}^2$ such that $f(a_i) = b_i$ for all i.

In other words, if $F = \{a_1, \dots, a_n\}$ is a finite subset of a convex polygon A, then any distance non-expanding map $\varphi\colon F \to \mathbb{R}^2$ extends to a piecewise distance preserving map $f\colon A \to \mathbb{R}^2$.

Proof. The proof is by induction on n.

The base case $n = 1$ is trivial: we can take

$$f(x) = x + (b_1 - a_1),$$

which is distance preserving on all of A.

Applying the induction hypothesis to the last $n - 1$ pair of points, we get a piecewise distance preserving map $h\colon A \to \mathbb{R}^2$ such that $h(a_i) = b_i$ for all $i > 1$. We will use h to construct the desired map $f\colon A \to \mathbb{R}^2$.

Consider the set

$$\Omega = \{\, x \in A \mid |a_1 - x| < |b_1 - h(x)| \,\}.$$

We can assume that $a_1 \in \Omega$; otherwise $h(a_1) = b_1$ and we can take $f = h$. We make the following claim.

❶ *The set Ω is star-shaped with respect to a_1. That is, if $x \in \Omega$ then the line segment $[a_1, x]$ lies in Ω.*

Indeed, if $y \in [a_1, x]$ then

$$|a_1 - y| + |y - x| = |a_1 - x|.$$

Since $x \in \Omega$, we have

$$|a_1 - x| < |b_1 - h(x)|.$$

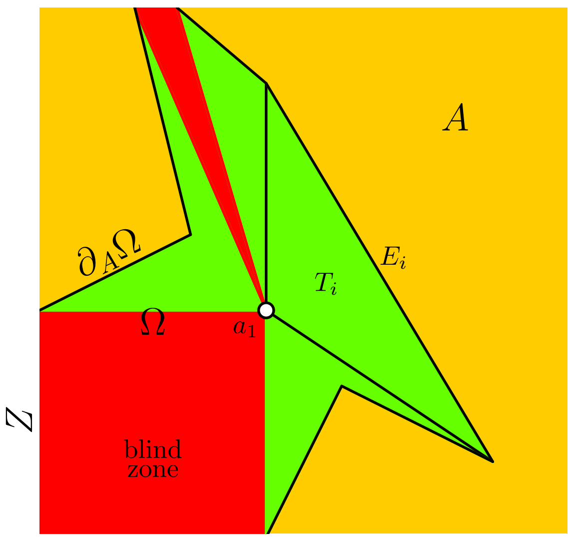
Since h is distance non-expanding (see Exercise 6.0.1), we have

$$|h(x) - h(y)| \leqslant |x - y|.$$

Combining the above with the triangle inequality, we see

$$\begin{aligned}
|a_1 - y| &= |a_1 - x| - |x - y| < \\
&< |b_1 - h(x)| - |h(x) - h(y)| \leqslant \\
&\leqslant |b_1 - h(y)|.
\end{aligned}$$

This proves $y \in \Omega$, which establishes Claim ❶.



Recall that $\partial_A\Omega$ denotes the boundary of Ω considered as a subset of the space A. This may be different a different set than $\partial_{\mathbb{R}^2}\Omega$. Note that

❷ $$|a_1 - x| = |b_1 - h(x)|$$

for any $x \in \partial_A\Omega$. To see this, consider a sequence of points in Ω that converges to x and another sequence of points in $A\backslash\Omega$ that converges to x, and then use the fact that h is continuous (Exercise 6.0.1).

Further, note the following.

❸ *The boundary $\partial_A\Omega$ is the union of a finite collection of line segments $E_1, \ldots, E_k$ which intersect each other only at the common endpoints. Moreover, h is distance preserving on each of these segments.*

Indeed, fix a triangulation of A so that h is distance preserving on each triangle. Note that for any point $x \in \partial_A\Omega$, this triangulation has a triangle $\Delta \ni x$ such that $\Delta \cap \Omega \neq \varnothing$. Fix such a triangle Δ. Since h is distance preserving on Δ, the restriction $h|_\Delta$ can be extended uniquely to an isometry $\iota\colon \mathbb{R}^2 \to \mathbb{R}^2$.

Set $b_1' = \iota^{-1}(b_1)$. Note that

$$|b_1' - x| = |b_1 - h(x)|$$

for any $x \in \Delta$, because ι is an isometry and $\iota|_\Delta = h|_\Delta$.

Observe that $a_1 \neq b_1'$. Assuming otherwise, we see

$$|a_1 - x| = |b_1' - x| = |b_1 - h(x)|$$

for any $x \in \Delta$, which gives the contradiction $\Delta \cap \Omega = \varnothing$.

Denote by ℓ_Δ the perpendicular bisector to $[a_1, b_1']$, which coincides with the set of all points equidistant from a_1 and b_1'. By the definition of Ω, for any $x \in \Delta$ we have that $x \in \Omega$ if and only if x and a_1 lie on the same side from ℓ_Δ. Therefore $\partial_A\Omega$ is the union of the intersections $\Delta \cap \ell_\Delta$ for all Δ as above. Hence ❸ follows, as there are only finitely many such Δ.

For each edge E_i in $\partial_A\Omega$, consider the triangle T_i with vertex a_1 and base E_i. Condition ❷ implies that there is an isometry ι_i of $\mathbb{R}^2$ such that $\iota_i(a_1) = b_1$ and $\iota_i(x) = h(x)$ for any $x \in E_i$.

Let us define $f(x) = h(x)$ for any $x \notin \Omega$ and $f(x) = \iota_i(x)$ for any $x \in T_i$. This defines f on $A\backslash\Omega$ and on all line segments from a_1 to $\partial_A\Omega$.

This completely defines f on A in the case where $\partial_A\Omega = \partial_{\mathbb{R}^2}\Omega$. If $Z = \partial_{\mathbb{R}^2}\Omega\backslash\partial_A\Omega$ is nonempty, then the points which lie on the lines between a_1 and the points in Z form a "blind zone" — this is the subset of A where f yet has to be defined.

Note that the closure of the blind zone is a union of a finite number of convex polygons $Q_1, \dots, Q_m$ which intersect only at the common vertex a_1. Each Q_i is bounded by a broken line in the closure of Z and two line segments from a_1 to the ends of this broken line.

So far the distance non-expanding map f is defined only on the two sides of each Q_i coming from a_1, and by construction it is distance preserving on each of these two sides. From the exercise below, it follows that one can extend f to each of Q_i while keeping it piecewise distance preserving. □

7.0.2. Exercise. *Let $Q = [a_1x_1 \dots x_k]$ be a convex polygon and b_1, y_1, y_k be points in the plane. Assume that*

$$|b_1 - y_1| = |a_1 - x_1|, \qquad |b_1 - y_k| = |a_1 - x_k|, \qquad |y_1 - y_k| \leqslant |x_1 - x_k|.$$

Then there is a piecewise distance preserving map $f\colon Q \to \mathbb{R}^2$ such that $f(x_1) = y_1$, $f(x_k) = y_k$ and $f(a_1) = b_1$.

Let us finish this lecture with some additional exercises.

7.0.3. Exercise. *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be two collections of points in $\mathbb{R}^2$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j. Let $A = \operatorname{Conv}\{a_1, \dots, a_n\}$ and $B = \operatorname{Conv}\{b_1, \dots, b_n\}$ be their convex hulls. Show that

$$\operatorname{perim} A \geqslant \operatorname{perim} B,$$

where $\operatorname{perim} A$ *denotes the perimeter of A.*

Is it true that

$$\operatorname{area} A \geqslant \operatorname{area} B?$$

The following exercise is a 2-dimensional case of Alexander's theorem [2]. It has quite a simple solution, but it plays an important role in discrete geometry, check for example the paper [4] by Bezdek and Connelly.

7.0.4. Advanced exercise. *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be two collections of points in $\mathbb{R}^2$. Let us consider $\mathbb{R}^2$ as a coordinate plane $\mathbb{R}^2 \times \{0\}$ in $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$.*

Construct a collection of curves $\alpha_i\colon [0,1] \to \mathbb{R}^4$ such that $\alpha_i(0) = a_i = (a_i, 0)$, $\alpha_i(1) = b_i = (b_i, 0)$ and the function $\ell_{i,j}(t) = |\alpha_i(t) - \alpha_j(t)|$ is monotonic (i.e., increasing, decreasing or constant) for each i and j.

7.0.5. Exercise. *Use Brehm's extension theorem to prove Kirszbraun's theorem, stated below, in the special case where Q is a finite set.*

7.0.6. Kirszbraun's theorem. *Let $Q \subset \mathbb{R}^2$ be arbitrary subset and $f\colon Q \to \mathbb{R}^2$ be a distance non-expanding map. Then f admits a distance non-expanding extension to all of $\mathbb{R}^2$. In other words there is a distance non-expanding map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the restriction $F|_Q$ coincides with f.*

Chapter 8

Akopyan's approximation theorem

Let P be a polyhedral space. A map $h\colon P \to \mathbb{R}^n$ is called *piecewise linear* if there is a triangulation of P such that restriction of h to any simplex Δ is a linear map. This means that if $v_0, \ldots, v_k$ are the vertices of Δ, then for any $x \in \Delta$ we have

$$h(x) = \lambda_0 \cdot h(v_0) + \cdots + \lambda_k \cdot h(v_k),$$

where $(\lambda_0, \ldots, \lambda_k)$ are the barycentric coordinates of x.

8.0.1. Exercise. *Show that if P is a 2-dimensional polyhedral space, then any piecewise distance preserving map $f\colon P \to \mathbb{R}^2$ is piecewise linear.*

In general, piecewise linear maps may not be injective, and they may either expand or contract distances. We shall be interested in approximating piecewise linear maps by piecewise distance preserving maps. Since all piecewise distance preserving maps are distance non-expanding, it only makes sense to try this approximation for distance non-expanding maps.

8.0.2. Akopyan's theorem. *Assume P is a 2-dimensional polyhedral space. Then any distance non-expanding piecewise linear map $h\colon P \to \mathbb{R}^2$ can be approximated by piecewise distance preserving maps.*

More precisely, for any $\varepsilon > 0$ there is a piecewise distance preserving map $f\colon P \to \mathbb{R}^2$ such that

$$|f(x) - h(x)| < \varepsilon$$

for all $x \in P$.

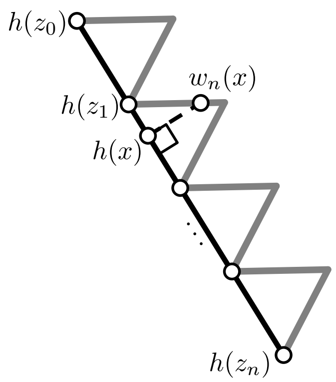
Akopyan's theorem implies the existence of many piecewise distance preserving maps from P into $\mathbb{R}^2$. In particular, it implies Zalgaller's theorem (6.0.3). To see this, consider the constant map $h\colon P \to \mathbb{R}^2$; i.e., the map which sends the whole space P to a single point. Since h is piecewise linear, we can apply Akopyan's theorem to produce piecewise distancing preserving map f arbitrarily close to h.

As you will see below, the proof of Akopyan's theorem will not use Zalgaller's theorem. We still consider the proof of Zalgaller's theorem to be important because it gives a very clear geometric description of the piecewise distance preserving map. In constrast, the maps produced by Akopyan's theorem will rely on the recursive construction of Brehm's theorem, which is harder to understand.

8.0.3. Exercise. *Show that if P is a convex polygon in $\mathbb{R}^2$, then the above theorem follows from Brehm's extension theorem (7.0.1).*

The main idea in the proof of Akopyan's theorem is to triangulate P and use Brehm's extension theorem on each triangle, as in the previous exercise. Unfortunately, it is not that simple. The big technical issue that arises is that if two triangles share a common edge, we need to ensure that the maps produced using Brehm's theorem agree on that common edge.

To address this issue, we will use the following *zigzag construction*. It produces a piecewise distance preserving map which is close to a given a distance non-expanding linear map defined on a line segment. For the construction, we fix a unit vector e in $\mathbb{R}^2$. The choice of e does not matter, but the same e must be used uniformly in all zigzag constructions that follow.



Zigzag construction. Let E be a line segment and $h\colon E \to \mathbb{R}^2$ be a distance non-expanding linear map. Let $\ell = \operatorname{length} E$ and $\ell' = \operatorname{length} h(E)$. Since h is distance non-expanding, we have $\ell' \leqslant \ell$.

Fix a positive integer n, and subdivide E into n equal intervals. Denote by $z_0, \dots, z_n$ the endpoints of these intervals.

Note that the image $h(E)$ is either a line segment or a point. In the first case, let u be a unit normal vector to $h(E)$; otherwise, let $u = e$.

Given $x\in E$, set

$$s_n(x)=\min_i\{|z_i-x|\},$$
$$w_n(x)=k{\cdot}s_n(x){\cdot}u+h(x),$$

where $k=\sqrt{1-(\ell'/\ell)^2}$. If we subdivide E further by adding the midpoints between any two consecutive endpoints, then w_n is distance preserving on each of the resulting subintervals. This shows that w_n is piecewise distance preserving. Moreover

$$|w_n(x)-h(x)|\leqslant\tfrac{\ell}{2\cdot n}$$

for any $x\in E$, because $k\leqslant 1$ and $s_n(x)\leqslant\frac{\ell}{2\cdot n}$.

The piecewise distance preserving map w_n is the result of the *n-step zigzag construction* applied to h.

Given a triangulation $\mathcal{T}$ of a polyhedral space P, let $\mathcal{T}^1$ denote the 1-skeleton of $\mathcal{T}$. This is the 1-dimensional subcomplex of $\mathcal{T}$ formed by all the vertices and edges in $\mathcal{T}$. Notice that $\mathcal{T}^1$ is a 1-dimensional polyhedral space when equipped with its induced length metric, which is different from the subspace metric it inherits from P.

The following proposition is the main technical step in the proof of Akopyan's theorem.

8.0.4. Proposition. *Let $\mathcal{T}^1$ be the 1-skeleton of a triangulation of a 2-dimensional polyhedral space P, and let $h\colon\mathcal{T}^1\to\mathbb{R}^2$ be a piecewise linear map such that*

$$|h(x)-h(y)|_{\mathbb{R}^2}\leqslant|x-y|_P$$

for any $x,y\in\mathcal{T}^1$. Then for any $\varepsilon>0$, there is a piecewise distance preserving map $w\colon\mathcal{T}^1\to\mathbb{R}^2$ such that

$$|w(x)-w(y)|_{\mathbb{R}^2}\leqslant|x-y|_P$$

for any $x,y\in\mathcal{T}^1$ and

$$|w(x)-h(x)|<\varepsilon$$

for all $x\in\mathcal{T}^1$.

Proof. First we prove the statement under the following additional assumption on h:

❶ *For some fixed $\delta > 0$, we have*

$$|h(x) - h(y)|_{\mathbb{R}^2} \leqslant (1-\delta)\cdot|x-y|_P$$

for any $x, y \in \mathcal{T}^1$ and

$$h(v) = h(x)$$

for any vertex v of $\mathcal{T}^1$ and point $x \in \mathcal{T}^1$ such that $|v - x|_P \leqslant \delta$.

Let $\mathcal{S}$ denote the subdivision of $\mathcal{T}^1$ such that h is linear on each edge of $\mathcal{S}$. Subdividing $\mathcal{S}$ further if necessary, we may assume without loss of generality that each edge of $\mathcal{S}$ which comes from a vertex of $\mathcal{T}^1$ has length δ. (To perform this subdivision, we have to assume that δ in ❶ is sufficiently small.)

Denote by ℓ the maximal length of the edges in $\mathcal{T}^1$. Let us apply the n-step zigzag construction to each edge of $\mathcal{S}$. Since the maps from the zigzag construction agree at the common vertices of different edges, we obtain a piecewise distance preserving map $w_n\colon \mathcal{T}^1 \to \mathbb{R}^2$ such that

❷
$$|w_n(x) - h(x)| \leqslant \tfrac{\ell}{2\cdot n}$$

for all $x \in \mathcal{T}^1$.

We shall show that the inequality

$$|w_n(x) - w_n(y)|_{\mathbb{R}^2} \leqslant |x-y|_P$$

holds for all $x, y \in \mathcal{T}^1$, provided n is sufficiently large. Notice that

❸ $|w_n(x) - w_n(y)| \leqslant |x - y|_P$ *if x and y lie on the same edge.*

This follows since $|x-y|_P = |x-y|_E$ whenever x and y lie on the same edge E of $\mathcal{T}^1$, as well as the fact that w_n is distance non-expanding on E.

From ❶ and ❷, we see that

$$\begin{aligned}
&|w_n(x) - w_n(y)|_{\mathbb{R}^2} \leqslant \\
&\qquad \leqslant |w_n(x) - h(x)|_{\mathbb{R}^2} \;+\; |h(x) - h(y)|_{\mathbb{R}^2} \;+\; |h(y) - w_n(y)|_{\mathbb{R}^2} \leqslant \\
&\qquad \leqslant |x-y|_P + \left(\tfrac{\ell}{n} - \delta\cdot|x-y|_P\right)
\end{aligned}$$

for any x and y in $\mathcal{T}^1$.

Now suppose $|w_n(x) - w_n(y)|_{\mathbb{R}^2} > |x-y|_P$ for some $x, y \in \mathcal{T}^1$. Then from above, we have $|x - y|_P < \frac{\ell}{n\cdot\delta}$, which shows that

❹
$$|x-y|_P < \tfrac{C}{n}.$$

for a constant C which does not depend on x or y. Thus, ❸ and ❹ imply the following.

❺ *For sufficiently large*[1] n, *if* $|w_n(x) - w_n(y)| > |x - y|_P$ *then both* x *and* y *lie on different edges which come from one vertex, say* v *of* $\mathcal{T}^1$, *and*

$$|x - v|_P,\ |y - v|_P \leqslant \delta.$$

Let x, y and v be as in ❺. Take the point x' on the same edge as y such that $|v-x'|_P = |v-x|_P$. It follows from the construction of w_n that $w_n(x') = w_n(x)$. (Notice that by ❶, w_n is produced by the zigzag construction in the case where the image of h is a point.) Therefore

$$\begin{aligned} |w_n(x) - w_n(y)|_{\mathbb{R}^2} &= |w_n(x') - w_n(y)|_{\mathbb{R}^2} \leqslant \\ &\leqslant |x' - y|_P = \\ &= \big||x - v|_P - |y - v|_P\big| \leqslant \\ &\leqslant |x - y|_P. \end{aligned}$$

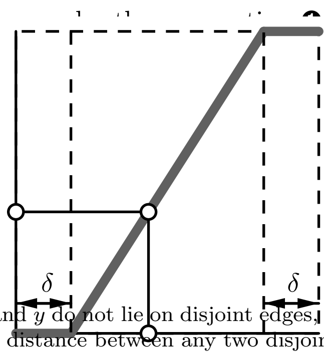
Thus we have shown that if n is sufficiently large, then the inequality

$$|w_n(x) - w_n(y)|_{\mathbb{R}^2} \leqslant |x - y|_P$$

holds for any pair $x, y \in \mathcal{T}^1$. Let $w = w_n$ for such an n which additionally satisfies $\frac{\ell}{2\cdot n} < \varepsilon$. Then from ❷, it follows that

$$|w(x) - h(x)| < \varepsilon$$

for all $x \in \mathcal{T}^1$. Thus we have proved the propositi



The graph of q_δ on one edge

It remains to be shown that the general case can be reduced to the case where ❶ holds. We shall achieve this by approximating h by a map that satisfies ❶.

For small $\delta > 0$ (say less than half the smallest edge length), consider the map

$$q_\delta \colon \mathcal{T}^1 \to \mathcal{T}^1$$

[1]The size of n does not depend on x or y. To ensure x and y do not lie on disjoint edges, n must be large enough so that C/n is less than the minimal distance between any two disjoint edges. To ensure that both x and y are within δ of v, we must choose n large enough in a way which will depend on the minimal angle in any triangle. For both issues, we are using the fact that $\mathcal{T}$ has a finite number of triangles.

which smashes the δ-neighborhood of each vertex of $\mathcal{T}^1$ to the vertex and linearly stretches the remaining part of the edge, as in the figure.

Let L_δ be the optimal Lipschitz constant of q_δ; i.e., the minimal number such that

$$|q_\delta(x) - q_\delta(y)|_P \leqslant L_\delta \cdot |x - y|_P$$

for all $x, y \in \mathcal{T}^1$. Notice that $L_\delta \to 1$ as $\delta \to 0^+$. Then the map

$$h_\delta \xlongequal{def} \tfrac{1-\delta}{L_\delta} \cdot (h \circ q_\delta)$$

is piecewise linear and satisfies condition ❶. Moreover, we can choose δ small enough so that

$$|h_\delta(x) - h(x)| < \tfrac{\varepsilon}{2}$$

for all $x \in \mathcal{T}^1$.

By the previous part of the proof, there is a piecewise distance preserving map $w\colon \mathcal{T}^1 \to \mathbb{R}^2$ such that

$$|w(x) - h_\delta(x)| < \tfrac{\varepsilon}{2}, \qquad |w(x) - w(y)|_{\mathbb{R}^2} \leqslant |x - y|_P$$

for all $x, y \in \mathcal{T}^1$. By the triangle inequality,

$$|w(x) - h(x)| < \varepsilon$$

for all $x \in \mathcal{T}^1$. $\square$

Proof of 8.0.2. Fix a fine triangulation $\mathcal{T}$ of P, one for which the diameter of each triangle is smaller than $\frac{\varepsilon}{3}$. Let $\mathcal{T}^1$ denotes the 1-skeleton of $\mathcal{T}$. By Proposition 8.0.4, there is a piecewise distance preserving map $w\colon \mathcal{T}^1 \to \mathbb{R}^2$ such that

$$|w(x) - h(x)|_{\mathbb{R}^2} < \tfrac{\varepsilon}{3}$$

for any $x \in \mathcal{T}^1$ and

$$|w(x) - w(y)|_{\mathbb{R}^2} \leqslant |x - y|_P$$

for any x and $y \in \mathcal{T}^1$.

We shall use Brehm's extension theorem (7.0.1) to extend w to a piecewise distance preserving map on P. To do this, let $\mathcal{S}$ be a subdivision of $\mathcal{T}^1$ so that w is distance preserving on each edge of $\mathcal{S}$. Fix a triangle Δ of $\mathcal{T}$. Let $a_1, \dots, a_n$ be the vertices of $\mathcal{S}$ on the boundary of Δ, and let $b_i = w(a_i)$ for each

i. By applying Brehm's theorem, we obtain a piecewise distance preserving map $f_\triangle\colon \Delta \to \mathbb{R}^2$.

Since w is distance preserving on each edge of $\mathcal{S}$, the maps $f_\triangle$ and w coincide on the boundary of Δ. In particular, if Δ and Δ' share a common edge, then $f_\triangle$ and $f_{\triangle'}$ agree on that common edge. Therefore the collection of maps $\{f_\triangle\}$ determines a single piecewise distance preserving map $f\colon P \to \mathbb{R}^2$.

We'll show f satisfies the conclusion of the theorem. Let $x \in P$ be arbitrary and let y be a point on the edge of a triangle in $\mathcal{T}$ that contains x. Then $|x-y| < \frac{\varepsilon}{3}$ by our choice of $\mathcal{T}$. We see

$$\begin{aligned}|f(x)-h(x)| &\leqslant |f(x)-w(y)| + |w(y)-h(y)| + |h(y)-h(x)| = \\ &= |f(x)-f(y)| + |w(y)-h(y)| + |h(y)-h(x)| \leqslant \\ &\leqslant 2\cdot|x-y| + |w(y)-h(y)| < \\ &< \varepsilon,\end{aligned}$$

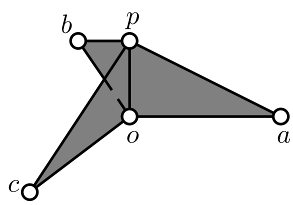
because $w(y) = f(y)$ and the maps f, h are distance non-expanding. □

We close this section with a counterexample explaining one way in which we cannot improve Akopyan's theorem. One might expect that a stronger statement holds, namely that the map f in Akopyan's theorem can be constructed so that it coincides with h on a given finite set of points. The following exercise shows that this cannot be done in general.

8.0.5. Exercise. *Consider the following 5 points in $\mathbb{R}^3$:*

$$o = (0,0,0),\ p = (0,0,1),\ a = (2,0,0),\ b = (-1,2,0),\ c = (-1,-2,0)$$

Let P be the "tripod" which is the polyhedral space consisting of the three triangles $\triangle opa$, $\triangle opb$ and $\triangle opc$ in $\mathbb{R}^3$, and equipped with the induced length metric.



Note that the restriction of the coordinate projection $\pi(x,y,z) = (x,y,0)$ to P is distance non-expanding and piecewise linear. We have that

$$\pi(o) = \pi(p) = o, \qquad \pi(a) = a, \qquad \pi(b) = b, \qquad \pi(c) = c.$$

Show that there is no piecewise distance preserving map $f\colon P \to \mathbb{R}^2 = \mathbb{R}^2 \times \times \{0\}$ such that $f(a) = a$, $f(b) = b$ and $f(c) = c$.

Chapter 9

Gromov's rumpling theorem

Recall that $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$, which we equip with its induced length metric. Here is our main theorem.

9.0.1. Theorem. *There is a length-preserving map $f\colon \mathbb{S}^2 \to \mathbb{R}^2$.*

Such a map f has to crease on an everywhere dense set in $\mathbb{S}^2$. More precisely, the restriction of f to any open subset of $\mathbb{S}^2$ cannot be injective.[1]

In the proof of the theorem we will use the following exercise.

9.0.2. Exercise. *Let K be a convex polyhedron in $\mathbb{R}^3$. Given a point x in $\mathbb{R}^3$, show that there is a unique point $\bar{x} \in K$ which minimizes the distance $|x - \bar{x}|$. Moreover show that the projection map*

$$\varphi\colon \mathbb{R}^3 \to K, \qquad \varphi(x) = \bar{x}$$

is distance non-expanding.

Proof of Theorem 9.0.1. Consider a nested sequence $K_0 \subset K_1 \subset \dots$ of convex polyhedra in $\mathbb{R}^3$ whose union is the open unit ball. Let $P_n = \partial K_n$ denotes

[1] To prove this, one can show that if f is injective and length-preserving on an open set $U \subset \mathbb{S}^2$, then f maps (sufficiently short) geodesics to straight lines (this requires the Domain Invariance Theorem, see for example Section 2.9 in [3].) It follows that the restriction of f to U is locally distance preserving, which is impossible.

the surface of K_n, equipped with the induced length metric. Note that P_n is a 2-dimensional polyhedral space for each n.

Let φ_n denote the projection onto K_n, as in Exercise 9.0.2, which is a distance non-expanding map. Since $K_n \subset K_{n+1}$, it follows that $\varphi_n(P_{n+1}) = P_n$. Note that one can triangulate P_n and P_{n+1} in such a way that the restriction of φ_n to any simplex of P_{n+1} is an orthogonal projection onto some simplex of P_n. In particular the restriction of φ_n to P_{n+1} is piecewise linear[2] and distance non-expanding with respect to the length metrics on P_{n+1} and P_n.

We claim that for any point $x \in \mathbb{S}^2$, there is unique sequence of points $x_n \in P_n$ such that $x_n \to x$ as $n \to \infty$ and $\varphi_n(x_{n+1}) = x_n$ for all n. The uniqueness follows since the maps φ_n are distance non-expanding. To show existence, fix any sequence $z_n \in P_n$ such that $z_n \to x$. Consider the double sequence $y_{n,m} \in P_n$, defined for $n \leqslant m$, such that $y_{n,n} = z_n$ and $y_{n,m} = \varphi_n(y_{n+1,m})$ if $0 \leqslant n < m$. Then set

$$x_n = \lim_{m\to\infty} y_{n,m}.$$

9.0.3. Exercise. *Show that the limit above exists and $\varphi_n(x_{n+1}) = x_n$ for any n. Then show that $x_n \to x$ as $n \to \infty$.*

Let $x_n \to x \in \mathbb{S}^2$ be the sequence as above. Define $\psi_n\colon \mathbb{S}^2 \to P_n$ by $\psi_n(x) = x_n$. We have that ψ_n is distance non-expanding, $\psi_n = \varphi_n \circ \psi_{n+1}$ for all n, and for any $p, q \in \mathbb{S}^2$,

$$❶ \qquad |p_n - q_n|_{P_n} \to |p - q|_{\mathbb{S}^2} \text{ as } n \to \infty,$$

where $p_n = \psi_n(p)$ and $q_n = \psi_n(q)$.

The desired length-preserving map $f\colon \mathbb{S}^2 \to \mathbb{R}^2$ will be a "limit" in some sense of a sequence of piecewise distance preserving maps $f_n\colon P_n \to \mathbb{R}^2$. The maps will be constructed recursively to satisfy

$$|f_{n+1}(x) - f_n(\varphi_n(x))| < \varepsilon_n$$

for a carefully chosen sequence (ε_n) of positive numbers that decays rapidly to 0.

Recursive construction of $f_n\colon P_n \to \mathbb{R}^2$ and ε_n. Assume we have a piecewise distance preserving map $f_n\colon P_n \to \mathbb{R}^2$ and a given ε_n. The composition $f_n \circ \varphi_n\colon P_{n+1} \to \mathbb{R}^2$ is piecewise linear and distance non-expanding. So we can

[2] See the definition of page 66.

apply Akopyan's Theorem 8.0.2 to construct a piecewise distance preserving map $f_{n+1}\colon P_{n+1} \to \mathbb{R}^2$ which is ε_n-close to $f_n \circ \varphi_n$.

Let $M(n+1)$ denote the number of triangles in a triangulation of P_{n+1} such that f_{n+1} is distance preserving on each triangle. Set

❷ $$\varepsilon_{n+1} = \frac{\varepsilon_n}{2\cdot M(n+1)}.$$

In this way, we recursively define the construction of f_n and ε_n. It goes as follows:

1. Choose an arbitrary $\varepsilon_0 > 0$ and take a piecewise distance preserving map $f_0\colon P_0 \to \mathbb{R}^2$, say the one provided by Zalgaller's folding theorem (6.0.3).
2. Use φ_0, f_0 and ε_0 to construct f_1.
3. Use f_1 to construct ε_1.
4. Use φ_1, f_1 and ε_1 to construct f_2.
5. Use f_2 to construct ε_2.
6. and so on.[3]

It remains to prove the following claim:

❸ *The sequence of maps $f_n \circ \psi_n\colon \mathbb{S}^2 \to \mathbb{R}^2$ converges to a length-preserving map $f\colon \mathbb{S}^2 \to \mathbb{R}^2$.*

Since ε_n decays faster than $\frac{\varepsilon_0}{2^n}$, the sequence $(f_n \circ \psi_n)(x) \in \mathbb{R}^2$ is Cauchy, hence convergent, for any fixed x. We define $f\colon \mathbb{S}^2 \to \mathbb{R}^2$ by $f(x) = \lim_{n\to\infty}(f_n \circ \psi_n)(x)$. By the recursive construction of f_n, we have that

$$|(f_n \circ \psi_n)(x) - f(x)| < \varepsilon_n$$

for any $x \in \mathbb{S}^2$ and any n. Since each $f_n \circ \psi_n$ is distance non-expanding, f is distance non-expanding as well.

It only remains to show that the constructed map $f\colon \mathbb{S}^2 \to \mathbb{R}^2$ is length-preserving. By Exercise **??**(**??**), it suffices to show that

❹ $$\operatorname{length}(f \circ \alpha) \geqslant |p - q|_{\mathbb{S}^2}$$

for any curve α between two points $p, q \in \mathbb{S}^2$. For the remainder of the proof, we will need the following definition; it should be considered as an analog of length of curve.

[3] The procedure is similar to walking on stairs: you take a right step, which makes it possible to take a left step, which in turn makes it possible to take a right step, and so on.

9.0.4. Definition. *Let X be a metric space and $\alpha\colon [a,b]\to X$ be a curve. Set*

$$\ell_k(\alpha) \stackrel{def}{=\!=} \sup\left\{\ \sum_{i=1}^{k} |\alpha(t_i)-\alpha(t_{i-1})|_X \ \middle|\ a=t_0<t_1<\dots<t_k=b \right\}.$$

Note that given a curve $\alpha\colon [a,b]\to X$, we have

$$\begin{aligned}&\ell_1(\alpha)\leqslant\ell_2(\alpha)\leqslant\ell_3(\alpha)\leqslant\dots,\\&\ell_k(\alpha)\to\operatorname{length}\alpha\quad\text{as}\quad k\to\infty,\\&\ell_k(\alpha)\leqslant\operatorname{length}\alpha\ \text{for any}\quad k.\end{aligned}$$

Moreover, if

$$\ell_k(\alpha)=\operatorname{length}\alpha$$

then α is a chain made from at most k geodesic segments.

The following exersise states that if two curves α and β are sufficiently close then $\ell_k(\alpha)\approx\ell_k(\beta)$. Note that according to the remark after 4.1.4, the value $|\operatorname{length}\alpha-\operatorname{length}\beta|$ might be large in this case.

9.0.5. Exercise. *Suppose that $\alpha,\beta:\mathbb{I}\to X$ are two curves which are close in the sense that*

$$|\alpha(t)-\beta(t)|_X<\varepsilon$$

for all $t\in\mathbb{I}$. Show that

$$|\ell_k(\alpha)-\ell_k(\beta)|\leqslant 2{\cdot}k{\cdot}\varepsilon.$$

Now we come back to the proof of ❹. Set $p_n=\psi_n(p)$ and $q_n=\psi_n(q)$. Let β be an arbitrary curve from p_n to q_n in P_n. Note that one can find a shorter curve γ from p_n to q_n whose image in any triangle of the triangulation of P_n is a line segment, and moreover the endpoints of these line segments lie on β. It follows that $f_n\circ\gamma$ is a broken line in $\mathbb{R}^2$ with at most $M(n)$ edges, whose vertices we denote, in order, by

$$f_n(p_n)=z_0,z_1,\dots,z_k=f_n(q_n).$$

Note that $k\leqslant M(n)$ and each z_i lies on the curve $f_n\circ\beta$. Therefore

❺
$$\begin{aligned}|p_n-q_n|_{P_n}&\leqslant\operatorname{length}\gamma=\\&=\ell_{M(n)}(f_n\circ\gamma)=\\&=|z_0-z_1|+\dots+|z_{k-1}-z_k|\leqslant\\&\leqslant\ell_{M(n)}(f_n\circ\beta);\end{aligned}$$

Fix a curve α from p to q in $\mathbb{S}^2$. By Exercise 9.0.5 and ❷, for all n we have

❻ $$|\ell_{M(n)}(f\circ\alpha)-\ell_{M(n)}(f_n\circ\psi_n\circ\alpha)|\leqslant 2{\cdot}M(n){\cdot}\varepsilon_n=\varepsilon_{n-1}.$$

Given $\varepsilon>0$, we can choose n large enough so that $\varepsilon_{n-1}\leqslant\frac{\varepsilon}{2}$ and

$$|p-q|_{\mathbb{S}^2}-|p_n-q_n|_{P_n}\leqslant\tfrac{\varepsilon}{2},$$

which can be arranged by ❶. Applying ❺ for $\beta=\psi_n\circ\alpha$ and ❻, we see

$$\begin{aligned}\operatorname{length}(f\circ\alpha)&\geqslant\ell_{M(n)}(f\circ\alpha)\geqslant\\&\geqslant\ell_{M(n)}(f_n\circ\psi_n\circ\alpha)-\varepsilon_{n-1}\geqslant\\&\geqslant|p_n-q_n|_{P_n}-\varepsilon_{n-1}\geqslant\\&\geqslant|p-q|_{\mathbb{S}^2}-\tfrac{\varepsilon}{2}-\varepsilon_{n-1}\geqslant\\&\geqslant|p-q|_{\mathbb{S}^2}-\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary,

$$\operatorname{length}(f\circ\alpha)\geqslant|p-q|_{\mathbb{S}^2}.$$

Hence ❹ follows. □

Chapter 10

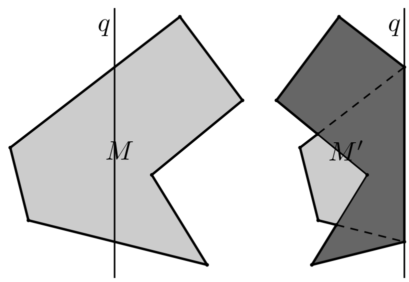
Arnold's problem on paper folding

This lecture is meant to be entertaining. Here we will discuss the following problem posted by V. Arnold in 1956 [**?**, Problem 1956-1].

10.0.1. Problem. *Is it possible to fold a square on the plane so that the obtained figure will have a longer perimeter?*

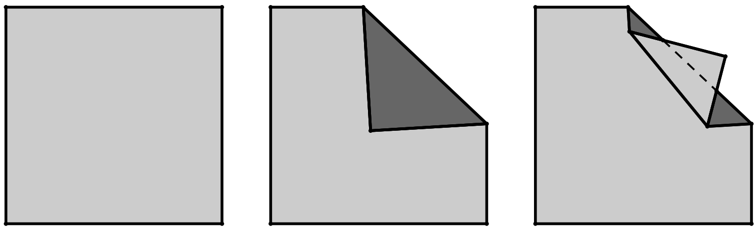
The answer to this problem depends on the meaning of word "fold".

For example, one can consider a sequence of *foldings* in which all layers are folded simultaneously along a line. By the following exercise, perimeter can never increase under a folding of this type.



10.0.2. Exercise. *Show that each fold described above indeed decreases perimeter. (Note that in general, the intersection of the line q with the polygon M in the picture may be a union of line segments.)*

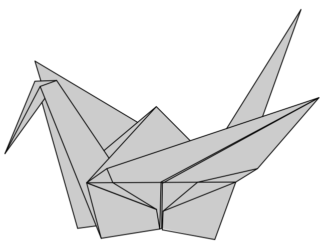
Using only the *foldings* described above makes it impossible to unfold a layer which lies on top of another layer, as shown in the following picture.

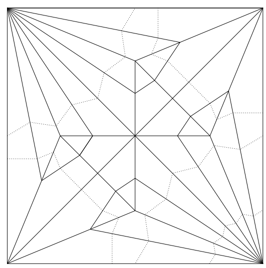


Note that the *unfold* increases the perimeter, although not beyond the perimeter of the original square. It is still unknown if it possible to increase perimeter by a sequence of such "folds" and "unfolds".

Japanese crane. Now let us consider a more general definition of folding. Imagine that we mark in advance the lines of folding and start to fold the paper in such a way that each domain between folds remains flat all the time.

If you understand "folding" this way, then the answer to the problem is "yes". In some sense, this problem was solved by origami practitioners well before it was even posed. The possibility to increase the perimeter slightly can be seen in the base for the crane. This was known by origami masters for centuries[1], but mathematicians learned this answer only in 1998[2].

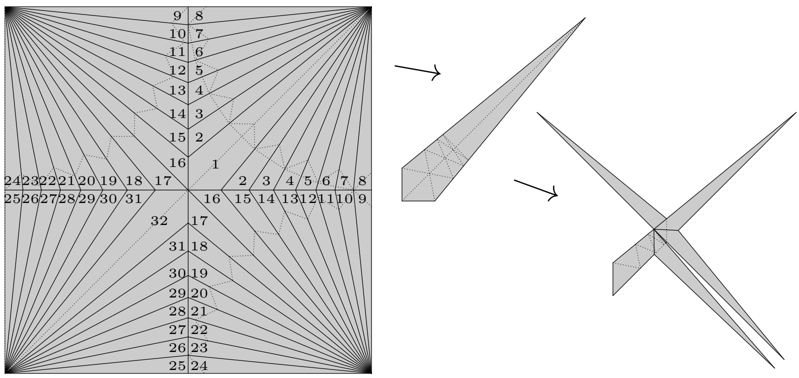




The base for the crane has four long ends and one short end. Two ends are used for wings and the other two have to be thinned, as one is used for the head and the other for the tail. Thinning twice each of the long ends makes it possible to produce a base which can then be folded into the plane to obtain a figure with larger perimeter.

[1] It appears in the oldest known book on origami, "Senbazuru Orikata," dated 1797; but for sure it was known much earlier.

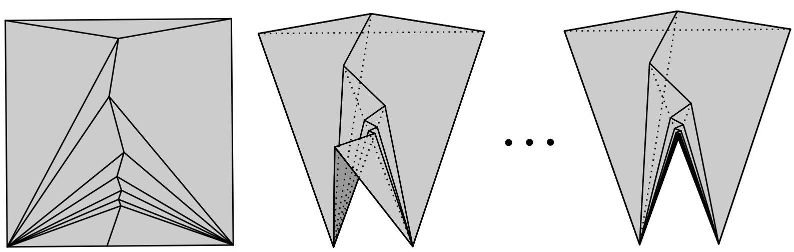
[2] Here is the html-file which tells how it happened.



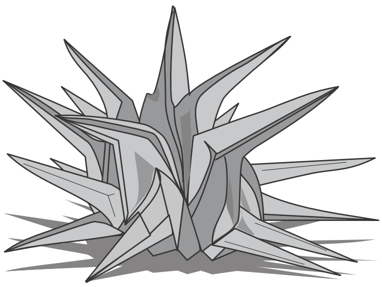
On the picture above, you can see the net of folds, the base, and the base with opened out ends. On the net of folds, you can see the number of the layer in the base. The dashed lines are the folds which appear at the opening out. The perimeter increases by about 0.5%, and there are 80 layers in the end. We do not know of a way to increase the perimeter with a smaller number of layers.

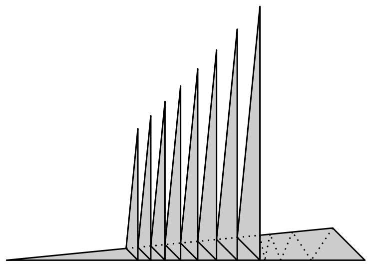
If a is the side of original square then it takes a bit less than a to go around each of four needles and it takes about $(\sqrt{2}-1)\cdot a$ to go around the short end, resulting in a longer perimeter. Thinning the ends many times makes possible to increase the perimeter by a value arbitrarily close to $(\sqrt{2}-1)\cdot a$.

The following picture describes another way to increase the perimeter, based on an idea of Yashenko [**?**]. It can be obtained by recursive application of one simple move. If one repeats this move sufficiently many times, we obtain a figure with a longer perimeter. Since each iteration adds two layers near the concave corner, the total number of layers in this model is much larger than in the crane base.



The sea urchin and the comb. It turns out the perimeter can be made arbitrarily large. This can be seen in the origami model for a sea urchin constructed by Robert Lang in 1987 [?]. In 2004, a complete solution was discovered independently by Alexei Tarasov [?]. Tarasov constructs a folding of a "comb", which is shown in the picture on the right. More importantly, he proves that the comb can be folded in a true way, in particular without starching and crooking the paper as is often done in origami.

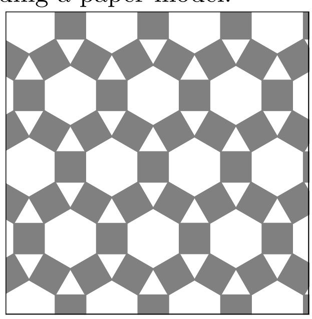




If you read the original pdf-file on a computer, you can extract the following three movies which describe Tarasov's solution,

- ⋄ The first movie shows the folding of whole comb.
- ⋄ The second movie shows the folding of one the needles separately.
- ⋄ The third movie shows the folding of two needles simultaneously.

Foldings in 4-dimensional space. One can define a "folding" as a piecewise distance preserving map from the square to the plane. These foldings are yet more general than those which appear above. The following exercise shows that it is not always possible to realize such a map by folding a paper model.



10.0.3. Exercise. *Consider the part of regular tessellation in the square $\square$ as on the picture.*

Show that there is a map $f\colon \square \to \mathbb{R}^2$ which is distance preserving on each polygon in the tessellation and such that it only reverses the orientation gray polygons.[3]

Show that it is not possible to make a paper folding model for f.[4]

The obstructions described in the above exercise disappear in $\mathbb{R}^4$; i.e., one can regard piecewise

[3]Less formally you need to "fold" along each segment in this tessellation.

[4]More formally, we need to think that the plane lies in the space $\mathbb{R}^3$, and we need to show that the map f cannot be approximated by injective continuous maps $\square \to \mathbb{R}^3$.

distance preserving maps as paper folding in 4-dimensional space. Moreover, one can actually fold the square in $\mathbb{R}^4$, as prescribed by a given piecewise distance preserving map. By this we mean one can construct a continuous one parameter family of piecewise distance preserving maps $f_t\colon \square \to \mathbb{R}^4$, $t \in [0,1]$ with fixed triangulation, say $\mathcal{T}$, such that

- ◇ f_0 is a distance preserving map from $\square$ to the coordinate plane $\mathbb{R}^2 \times \{0\}$ in $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$,
- ◇ the map f_1 is our given piecewise distance preserving map to the same coordinate plane,
- ◇ the map f_t is injective for any $t \neq 1$.

The proof of last statement is based on Exercise 7.0.4. Let $a_1, \dots, a_k$ be the vertices of $\mathcal{T}$ and $b_1, \dots, b_k$ be the corresponding images for the piecewise distance preserving map. Set $f_t(a_i) \in \mathbb{R}^2 \times \mathbb{R}^2$ to be

$$f_t(a_i) = \left(\frac{a_i + b_i}{2} + \cos(\pi\cdot t)\cdot\frac{a_i - b_i}{2},\ \sin(\pi\cdot t)\cdot\frac{a_i - b_i}{2}\right);$$

so $f_0(a_i) = (a_i, 0)$ and $f_1(a_i) = (b_i, 0)$ for any i. We can extend f_t linearly to each triangle of $\mathcal{T}$. Direct calculation show that $\ell_{i,j}(t) = |f_t(a_i) - f_t(a_j)|$ is monotonic in t; in particular if $|a_i - a_j| = |b_i - b_j|$ then $\ell_{ij}(t)$ is constant. This proves that f_t is piecewise distance preserving.

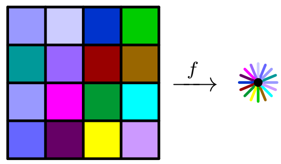
Further, direct calculations show that for any $x, y \in \square$,

$$|f_t(x) - f_t(y)|^2 = p - q\cdot\cos(\pi\cdot t)$$

for some constants p and q. Therefore if $x \neq y$ then $|f_t(x) - f_t(y)| > 0$ for any $t \neq 1$. In other words, f_t is injective for any $t \neq 1$.

It follows that for the paper folding in $\mathbb{R}^4$, the existence of perimeter increasing folds follows from Brehm's theorem. It is sufficient to construct a distance non-expanding map f from the square to the plane so that the perimeter of its image is sufficiently long. Then applying Brehm's theorem for a sufficiently dense finite set of points in the square, we get a piecewise distance preserving map h which is arbitrarily close to f. In particular we can arrange it so that the perimeter of the image $f(\square)$ is still sufficiently long.

The needed map can be constructed as follows: Fix a large n and divide the square $\square$ into n^2 squares with side length $\frac{a}{n}$. Let $d(x)$ denotes the distance from a point $x \in \square$ to the boundary of the small square which contains x. The function $d\colon \square \to \mathbb{R}$ takes values in $[0, \frac{a}{2\cdot n}]$. Further, let us enumerate the squares by integers from 1 to n^2. Given $x \in \square$ denote by $i(x)$ the (say minimal) number in this enumeration of a small square which contains x.



Now for each $i \in \{1, \dots, n^2\}$ choose a unit vector $u_i \in \mathbb{R}^2$ so that $u_i \neq u_j$ if $i \neq j$. Consider the map $f\colon \square \to \mathbb{R}^2$ defined by

$$f(x) = d(x) \cdot u_{i(x)}.$$

It is straightforward to check that the obtained map is distance non-expanding. The image $f(\square)$ consists of n^2 segments of length $\frac{a}{2\cdot n}$ which start at the origin. So the perimeter of $f(\square)$ is equal to $2\cdot n^2 \cdot \frac{a}{2\cdot n} = a\cdot n$. So by taking n large enough, one can make the perimeter of the image $f(\square)$ arbitrary large.

(The picture shows the case $n = 4$. In this case the perimeter of $f(\square)$ is $4\cdot a$, which is the same as perimeter of the original square. However for $n > 4$, it gets larger. When you calculate the perimeter of the degenerate figure, imagine going up and down each segment as you "traverse the boundary" and count the length of each segment twice.)

Final remarks

Zalgaller's folding theorem. Zalgaller's theorem holds in all dimensions: any m-dimensional polyhedral space P admits a piecewise distance preserving map to $\mathbb{R}^m$.

In [25], Zalgaller proved this statement for $m \leqslant 4$. The trick described in the "proof with no cheating" makes the proof work in all dimensions. This trick first appeared in Krat's thesis [16].

Brehm's extension theorem. This was proved by Brehm in [7] and rediscovered independently many years later by Akopyan and Tarasov in [**?**]. The proofs are based on the same idea.

Brehm's extension theorem holds in all dimensions and it can be proved along the same lines.

Kirszbraun theorem. This remarkable theorem was proved by Kirszbraun in his thesis, defended in 1930. A few years later he published the result in [**?**]. Independently the same result was reproved later by Valentine [**?**].

The paper of Danzer, Grünbaum and Klee [**?**], which is delightful to read, gives a very nice proof of this theorem is based on Helly's theorem on the intersection of convex sets.

Akopyan's approximation theorem. This theorem also admits direct generalizations to higher dimensions. Moreover the condition that f is piecewise linear is redundant. This is because any distance non-expanding map from a polyhedral space to $\mathbb{R}^m$ can be approximated by piecewise linear distance non-expanding maps.

The 2-dimensional case was proved by Krat in her thesis [16]. In [1], Akopyan noticed that Brehm's extension theorem simplifies the proof and also makes it possible to prove the higher dimensional case.

Much earlier, an analogous question was considered by Burago and Zalgaller. They proved that any piecewise linear embedding of a 2-dimensional polyhedral surface in $\mathbb{R}^3$ can be approximated by a piecewise distance preserving embedding; see [**?**] and [6].

Rumpling the sphere. Theorem 9.0.1 admits the following generalization, which can be proved along the same lines.

10.0.4. Gromov's rumpling theorem. *Let M be an m-dimensional Riemannian manifold. Then any distance non-expanding map $f\colon M \to \mathbb{R}^m$ can be approximated by length-preserving maps. More precisely, given $\varepsilon > 0$ there is a length-preserving map $f_\varepsilon\colon M \to \mathbb{R}^m$ such that*

$$|f_\varepsilon(x) - f(x)| < \varepsilon$$

for any $x \in M$.

This result is a partial case of Gromov's theorem in [**?**, Section 2.4.11]. The proof presented here is based on [19]; Gromov's original proof is different. The proof in [19] also makes it possible to construct surprising examples of spaces which admit length-preserving maps to $\mathbb{R}^m$, such as sub-Riemannian manifolds.

Gromov's theorem roughly states that length-preserving maps have no non-trivial global properties. This is a typical "local to global" problem. Here the length-preserving property is "local" and the only "global" consequence is trivial: it is the distance non-expanding property.

For such "local to global" problems the answer "no non-trivial global properties" is the most common, but it does not mean that it is easy to prove. There is machinery invented by Gromov which makes it possible to prove this for many local to global problems. This machinery is called the "h-principle" (homotopy principle). The h-principle is not a theorem, it is a property which often holds for different geometric structures. There are a few methods to prove the h-principle, including the one which is described in the proof of Theorem 9.0.1[5]. Gromov's rumpling theorem is one of the simplest examples. Other examples include

- ◇ The cone eversion theorem, which states that there is a continuously varying one-parameter family of smooth functions $f_t(x,y)$, $t \in [0,1]$, without critical points in the punctured plane $\mathbb{R}^2\backslash\{0\}$, such that $f_0(x,y) = -\sqrt{x^2+y^2}$ and $f_1(x,y) = \sqrt{x^2+y^2}$. See [11, Lecture 27] and read the whole book; it is nice.

[5] Usually, the h-principle is formulated in terms of partial differential equations, but one may think of "length-preserving maps" as weak solutions of a particular partial differential equation.

- ◇ The Nash–Kuiper theorem, which in particular implies the existence of C^1-smooth length-preserving maps $\mathbb{S}^2 \to \mathbb{R}^3$ whose image has arbitrarily small diameter.
- ◇ Smale's sphere eversion paradox, which states that there is a continuous one partameter family of smooth immersions $f_t\colon \mathbb{S}^2 \to \mathbb{R}^3$, $t \in [0,1]$ such that $f_0\colon \mathbb{S}^2 \to \mathbb{R}^3$ is the standard inclusion and $f_1(x) = -f_0(x)$ for all $x \in \mathbb{S}^2$.
- ◇ Combining the techniques of Smale and Nash–Kuiper, one can make sphere eversions f_t in the class of length-preserving C^1-smooth maps.

For further reading we suggest the comprehensive introduction to the h-principle by Eliashberg and Mishachev [15].

Paper folding. The aspects of paper folding related to geometric constructions are discussed in [**?**]; this paper is very entertaining. Interesting aspects of paper folding in the 3-dimensional space are covered in [11, Lecture 15].

Part III

Nonpositive curvature

Chapter 11

Saddle surfaces

11.1 Polyhedral saddle surfaces

A 2-dimensional polytope in $\mathbb{R}^3$ is called *polyhedral disc* if it is homeomorphic to a the unit disc

$$\mathbb{D} = \left\{\, (x,y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 \leqslant 1 \,\right\}.$$

A vertex of a triangulation of a polyhedral disc is called interior if it lies in the *interior vertex* of $\mathbb{D}$ and *boundary vertex* if it lies on the boundary of $\mathbb{D}$. The boundary and interior vertices can be defined in a more combinatorial fashion. The link of any vertex is formed by a broken line. For interior vertex the link is a closed line, and for boundary vertex it is open; i.e., it has the beginning and the end. This equivalence as well as the correctness of this definition folows from domain invariance; see 15.5.2.

A polyhedral disc P in $\mathbb{R}^3$ is called *saddle* if any interior vertex of P lies in the convex hull of its link.

11.1.1. Proposition. *Let P be a saddle polyhedral disc in $\mathbb{R}^3$ equipped with the induced length metric. Then P has non-positive curvature; that is the total angle around any point in the interior of P is at least $2{\cdot}\pi$.*

The proof relies on the following lemma

11.1.2. Hemisphere lemma. *Any closed curve of length $< 2{\cdot}\pi$ in $\mathbb{S}^2$ lies in an open hemisphere.*

Proof. Let α be a closed curve in $\mathbb{S}^2$ of length $2{\cdot}\ell$. Assume $\ell < \pi$.

Let us dived α in two subarcs, say α_1 and α_2 with length ℓ each. Denote by p and q their endpoints. Since $|p-q| \leqslant \ell < \pi$, there is a unique geodesic $[pq]$ in $\mathbb{S}^2$. Let z be the midpoint of $[pq]$.

We claim that α lies in the open hemisphere centered at z. If not, α intersects the boundary great circle, say at the point x. Without loss of generality we may assume that $x \in \check{\alpha}_1$.

Consider rotation of $\mathbb{S}^2$ by angle π with fixed point z. Arc α_1 together with its rotation form a closed curve of length $2\cdot\ell$ that passes through x and its antipodal point $x' = -x$. Thus

$$\begin{aligned}\ell &= \operatorname{length}\alpha_1 \geqslant \\ &\geqslant |x-x'|_{\mathbb{S}^2} = \\ &= \pi,\end{aligned}$$

where $|x-x'|_{\mathbb{S}^2}$ denotes the distance from x to x' in the length metric on $\mathbb{S}^2$, contradiction. □

Proof of Proposition 11.1.1. Let x be an interior vertex of the saddle surface Σ, denote by ϑ the total angle around x.

Choose sufficiently small $r > 0$; it should be smaller that the distance from x to any simplex of Σ which does not contain x. Note that the intersection of Σ with a sphere of radius r centered at x forms a closed curve, say α.

Since Σ is saddle, each equator in the sphere intersects α. Therefore by Hemisphere lemma, the $\operatorname{length}\alpha \geqslant 2\cdot\pi\cdot r$. On the other hand, $\operatorname{length}\alpha = \vartheta\cdot r$. Hence the result. □

11.2 Thin triangles

11.2.1. Definition of thin triangles . *Let $[x^1x^2x^3]$ be a triangle in a metric space. Consider its model triangle $[\tilde{x}^1\tilde{x}^2\tilde{x}^3] = \tilde{\triangle}(x^1x^2x^3)$ and the* natural map *$[\tilde{x}^1\tilde{x}^2\tilde{x}^3] \to [x^1x^2x^3]$ that sends a point $\tilde{z} \in [\tilde{x}^i\tilde{x}^j]$ to the corresponding point $z \in [x^ix^j]$ (i.e. such that $|\tilde{x}^i - \tilde{z}| = |x^i - z|$ and therefore $|\tilde{x}^j - \tilde{z}| = |x^j - z|$).*

We say the triangle $[x^1x^2x^3]$ is thin *if the natural map $[\tilde{x}^1\tilde{x}^2\tilde{x}^3] \to [x^1x^2x^3]$ is distance nonexpanding.*

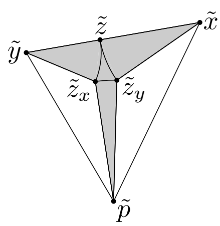
11.2.2. Inheritance lemma for thin triangles. *In a metric space, consider a triangle $[pxy]$ that* decomposes *into two triangles $[pxz]$ and $[pyz]$; that* *is, $[pxz]$ and $[pyz]$ have common side $[pz]$, and the sides $[xz]$ and $[zy]$ together form the side $[xy]$ of $[pxy]$.*

If both triangles $[pxz]$ and $[pyz]$ are thin, then triangle $[pxy]$ is thin.

We shall need the following model-space lemma.

11.2.3. Lemma. *Let $[\tilde p\tilde x\tilde y]$ be a triangle in $\mathbb{R}^2$ and $\tilde z \in [\tilde x\tilde y]$. Set $\tilde D = \operatorname{Conv}[\tilde p\tilde x\tilde y]$. Construct points $\dot p, \dot x, \dot y, \dot z \in \mathbb{R}^2$ such that $|\dot p - \dot x| = |\tilde p - \tilde x|$, $|\dot p - \dot y| = |\tilde p - \tilde y|$, $|\dot x - \dot z| = |\tilde x - \tilde z|$, $|\dot y - \dot z| = |\tilde y - \tilde z|$, $|\dot p - \dot z| \leqslant |\tilde p - \tilde z|$ and points $\dot x$ and $\dot y$ lie on either side of $[\dot p\dot z]$. Set $\dot D = \operatorname{Conv}[\dot p\dot x\dot z] \cup \operatorname{Conv}[\dot p\dot y\dot z]$.*

Then there is a distance nonexpanding map $F\colon \tilde D \to \dot D$ that maps $\tilde p$, $\tilde x$, $\tilde y$ and $\tilde z$ to $\dot p$, $\dot x$, $\dot y$ and $\dot z$ respectively.



Proof. By Alexandrov's lemma (4.5.2), there are nonoverlapping triangles $[\tilde p\tilde x\tilde z_y] \overset{iso}{=\!=} [\dot p\dot x\dot z]$ and $[\tilde p\tilde y\tilde z_x] \overset{iso}{=\!=} [\dot p\dot y\dot z]$ inside triangle $[\tilde p\tilde x\tilde y]$.

Connect points in each pair $(\tilde z, \tilde z_x)$, $(\tilde z_x, \tilde z_y)$ and $(\tilde z_y, \tilde z)$ with arcs of circles centered at $\tilde y$, $\tilde p$, and $\tilde x$ respectively. Define F as follows.

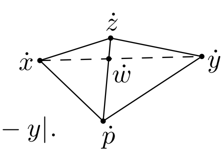
- ◇ Map $\operatorname{Conv}[\tilde p\tilde x\tilde z_y]$ isometrically onto $\operatorname{Conv}[\dot p\dot x\dot y]$; similarly map $\operatorname{Conv}[\tilde p\tilde y\tilde z_x]$ onto $\operatorname{Conv}[\dot p\dot y\dot z]$.
- ◇ If x is in one of the three circular sectors, say at distance r from center of the circle, let $F(x)$ be the point on the corresponding segment $[pz]$, $[xz]$ or $[yz]$ whose distance from the lefthand endpoint of the segment is r.
- ◇ Finally, if x lies in the remaining curvilinear triangle $\tilde z\tilde z_x\tilde z_y$, set $F(x) = z$.

By construction, F satisfies the conditions of the lemma. □

Proof of Inheritance for thin triangles (11.2.2). Construct model triangles $[\dot p\dot x\dot z] = \tilde\triangle(pxz)$ and $[\dot p\dot y\dot z] = \tilde\triangle(pyz)$ so that $\dot x$ and $\dot y$ lie on opposite sides of $[\dot p\dot z]$.

Suppose

$$\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) < \pi.$$



Then for some point $\dot w \in [\dot p\dot z]$, we have

$$|\dot x - \dot w| + |\dot w - \dot y| < |\dot x - \dot z| + |\dot z - \dot y| = |x - y|.$$

Let $w \in [pz]$ correspond to $\dot w$; i.e. $|z - w| = |\dot z - \dot w|$. Since $[pxz]$ and $[pyz]$ are thin, we have

$$|x - w| + |w - y| < |x - y|,$$

contradicting the triangle inequality.

Thus

$$\tilde{\measuredangle}(z\,{}^{p}_{x}) + \tilde{\measuredangle}(z\,{}^{p}_{y}) \geqslant \pi.$$

By Alexandrov's lemma (4.5.2), this is equivalent to

$$❶ \qquad \tilde{\measuredangle}(x\,{}^{p}_{z}) \leqslant \tilde{\measuredangle}(x\,{}^{p}_{y}).$$

Let $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$ and $\tilde{z} \in [\tilde{x}\tilde{y}]$ correspond to z; i.e. $|x - z| = |\tilde{x} - \tilde{z}|$. Inequality ❶ is equivalent to $|p - z| \leqslant |\tilde{p} - \tilde{z}|$. Hence Lemma 11.2.3 applies. Therefore there is a distance nonexpanding map F that sends $[\tilde{p}\tilde{x}\tilde{y}]$ to $\dot{D} = \mathrm{Conv}[\dot{p}\dot{x}\dot{z}] \cup \mathrm{Conv}[\dot{p}\dot{y}\dot{z}]$ in such a way that $\tilde{p} \mapsto \dot{p}$, $\tilde{x} \mapsto \dot{x}$, $\tilde{z} \mapsto \dot{z}$ and $\tilde{y} \mapsto \dot{y}$.

By assumption, the natural maps $[\dot{p}\dot{x}\dot{z}] \to [pxz]$ and $[\dot{p}\dot{y}\dot{z}] \to [pyz]$ are distance nonexpanding. By composition, the natural map from $[\tilde{p}\tilde{x}\tilde{y}]$ to $[pyz]$ is distance nonexpanding, as claimed. □

11.3 Polyhedral discs with non-positive curvature

11.3.1. Lemma. *In a polyhedral disc non-positive curvature any two points are joint by unique geodesic.*

Moreover the unique geodesic depends continuously from its end points.

Proof Let P be a polyhedral disc non-positive curvature and $x, y \in P$ and $|x - y| = \ell$. Since P is compact, there is a geodesic from x to y.

Assume that there are two geodesics $\alpha, \beta\colon [0, \ell] \to P$ from from x to y. Note that the intersection of α as well as β with any triangle in a triangulation of P is a line segment.

Without loss of generality, we may assume that $\alpha(t) = \beta(t)$ if and only if $t = 0$ or l. Otherwise choose a maximal with respect to inclusion open interval $(a, b) \subset [0, \ell]$ such that $\alpha(t) \neq \beta(t)$ for any $t \in (a, b)$ and set $x = \alpha(a)$ and $y = \alpha(b)$.

According to Jordan curve theorem (**??**), α and β cut from P a polyhedral disc P'. Denote by φ and ψ the angles of this disc at x and y. Note that turn of the boundary is $\pi - \varphi$ at x, $\pi - \psi$ at y and at the remaining boundary points it is nonpositive.

According to 5.5.5,

$$\mathrm{T} + \Omega = 2\cdot\pi.$$

where T and Ω are correspondingly the turn of boundary and the curvature of interior point of P'.

From above we have that $\mathrm{T} \leqslant 2\cdot\pi - \varphi - \psi$ and $\Omega \leqslant 0$. Hence $\varphi = \psi = 0$; in particular, the geodesics α and β coincide at the beginning edge, a contradiction.

It remains to prove the second part of lemma. Let $x_n \to x_\infty$ and $y_n \to y_\infty$ be two sequences of points in P. Let $\ell_n = |x_n - y_n|$, denote by $\alpha_n \colon [0, \ell_n] \to P$ the geodesic from x_n to y_n. Note that any converging subsequence of α_n converges to the unique geodesic from x to y. On the other hand, since P is compact, any sequence of geodesics has a converging subsequence (see ???). Hence the secont part of lemma follows. □

11.3.2. Proposition. *In a polyhedral disc with non-positive curvature any triangle is thin.*

11.4 Shefel's theorem

11.4.1. Shefel's theorem. *Let $P \subset \mathbb{R}^2$ be a convex polygon and $f \colon P \to \mathbb{R}$ a Lipschitz function. Assume that the graph*

$$\Gamma_f = \left\{\, (x,y,z) \in \mathbb{R}^3 \,\middle|\, z = f(x,y), \quad (x,y) \in P \,\right\}$$

equipped with length metric has nonpositive curvature.

Proof. Fix small $\varepsilon > 0$. Note that there is a piecewise linear function $h \colon P \to \mathbb{R}$ such that

$$h(x,y) \lesssim f(x,y) \pm \varepsilon$$

for any $(x,y) \in P$. In other words, the graph Γ_h of h lies between two graphs $\Gamma_{f+\varepsilon}$ and $\Gamma_{f-\varepsilon}$.

Now we start to cut hats from below of Γ_h. Cutting a hat is the following procedure...

Note that since Γ_f and therefore $\Gamma_{f+\varepsilon}$ has no hats. Hence by applying cutting a hat to any graph which lies below $\Gamma_{f+\varepsilon}$, we get a graph which is still below $\Gamma_{f+\varepsilon}$. In particular, after cutting all hats, we get a graph of a function which lies below $\Gamma_{f+\varepsilon}$.

❶ *After cutting all hats from Γ_h, we get a graph of a piecewise linear function.*

From ❶ it follows that for any n there is a saddle piecewise linear function $h_n \colon P \to \mathbb{R}$ such that $h(x) \lesssim f(x) \pm \frac{1}{n}$ for any $x \in P$.

Denote by $|x-y|_n$ the ??? and $|x-y|_\infty$ the ???. It is sufficient to show that

$$|x-y|_n \to |x-y|_\infty \text{ as } n \to \infty$$

for any $x, y \in P$,

$\square$

Chapter 12

??? Curvature bounded above

12.1 Definitions

12.1.1. Definition. *A proper length space X has non-negative curvature in the sense of Alexandrov (briefly $X \in \mathrm{CAT}[0]$*[1]*) if the following inequality*

❶ $$|z - p|_X \leqslant |\tilde{z} - \tilde{p}|_{\mathbb{R}^2}$$

holds for any triangle $[xyz]$ in X, its model triangle $[\tilde{x}\tilde{y}\tilde{z}] = \tilde{\triangle}(xyz)$, any point $p \in \,]xy[$ and the corresponding point $\tilde{p} \in \,]\tilde{x}\tilde{y}[$.

12.1.2. Unique geodesics. *In a $\mathrm{CAT}[0]$ space, pairs of points are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

Proof. Let $U \in \mathrm{CAT}[0]$, $p^1, p^2 \in U$. Suppose $p^1_n \to p^1$, $p^2_n \to p^2$ as $n \to \infty$. Let z_n be the midpoint of a geodesic $[p^1_n p^2_n]$ and z be the midpoint of a geodesic $[p^1 p^2]$.

[1] $\mathrm{CAT}[0]$ stays for "curvature bounded above by 0". If in the definition of model triangle, one exchanges the Euclidean plane with a sphere or Lobachevsky plane of constant curvature k, then one gets the definition of *spaces with curvature $\leqslant k$ in the sense of Alexandrov*, which denoted as $\mathrm{CAT}[k]$. We will only consider the case $k = 0$.

It suffices to show that

❷ $$|z_n - z| \to 0 \quad \text{as} \quad n \to \infty.$$

By the triangle inequality, the z_n are approximate midpoints of p^1 and p^2. Apply (2+2)-point comparison (**??**) to the quadruple p^1, p^2, z_n, z. For $p = p^1$ or $p = p^2$, we see that $\tilde\measuredangle(p\,{}^{z_n}_{z})$ is arbitrarily small when n is sufficiently large. By the law of cosines, ❷ follows. □

12.1.3. Theorem. *Let U be a geodesic space. Then*

a) $U \in \mathrm{CAT}[0]$

if and only if one of the following conditions holds for all $p, x, y \in U$:

b) (adjacent-angles comparison) for any geodesic $[xy]$ and $z \in \,]xy[$, we have

$$\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) \geqslant \pi.$$

c) (point-on-side comparison) for any geodesic $[xy]$ and $z \in \,]xy[$, we have

$$\tilde\measuredangle(x\,{}^{p}_{y}) \geqslant \tilde\measuredangle(x\,{}^{p}_{z}),$$

or equivalently,

$$|\tilde p - \tilde z| \geqslant |p - z|,$$

where $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)$, $\tilde z \in \,]\tilde x\tilde y[$, $|\tilde x - \tilde z| = |x - z|$.

d) (angle comparison) for any hinge $[x\,{}^{p}_{y}]$, the angle $\measuredangle[x\,{}^{p}_{y}]$ exists and

$$\measuredangle[x\,{}^{p}_{y}] \leqslant \tilde\measuredangle(x\,{}^{p}_{y}).$$

Remark. In the following proof, the part $(a)\Rightarrow(b)$ only requires that the (2+2)-point comparison (**??**) hold for any quadruple, and does not require the existence of geodesics. The same is true of the parts $(b)\Leftrightarrow(c)$ and $(c)\Rightarrow(d)$. Thus the conditions (b), (c) and (d) are valid for any metric space (not necessarily intrinsic) which satisfies (2+2)-point comparison (**??**). The converse does not hold; for example, all these conditions are vacuously true in a totally disconnected space, while (2+2)-point comparison is not.

Proof. (a) $\Rightarrow$ (b). By Alexandrov's lemma (4.5.2),

$$\tilde\measuredangle(p\,{}^{z}_{x}) + \tilde\measuredangle(p\,{}^{z}_{y}) < \tilde\measuredangle(p\,{}^{x}_{y}) \quad \text{or} \quad \tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) = \pi.$$

In the former case, (2+2)-point comparison (**??**) applied to the quadruple p, z, x, y implies

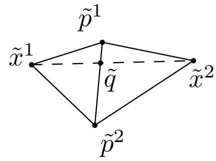
$$\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) \geqslant \tilde\measuredangle(z\,{}^{x}_{y}) = \pi.$$

(b) $\Leftrightarrow$ *(c).* Follows directly from Alexandrov's lemma (4.5.2).

(c) $\Rightarrow$ *(d).* By (c), for $\bar p \in\,]xp]$ and $\bar y \in\,]xy]$ the function $(|x-\bar p|, |x-\bar y|) \mapsto \tilde\measuredangle(x\,{}^{\bar p}_{\bar y})$ is nondecreasing in each argument. In particular, $\measuredangle[x\,{}^{p}_{y}] = \inf \tilde\measuredangle(x\,{}^{\bar p}_{\bar y})$. Thus $\measuredangle[x\,{}^{p}_{y}]$ exists and is at most $\tilde\measuredangle(x\,{}^{p}_{y})$.

(d) $\Rightarrow$ *(b).* By (d) and the triangle inequality for angles (4.6.2),

$$\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) \geqslant \measuredangle[z\,{}^{p}_{x}] + \measuredangle[z\,{}^{p}_{y}] \geqslant \pi.$$



(c) $\Rightarrow$ *(a).* Given a quadruple p^1, p^2, x^1, x^2, we must verify (2+2)-point comparison (**??**). Construct the model triangles $[\tilde p^1\tilde p^2\tilde x^1] = \tilde\triangle(p^1p^2x^1)$ and $[\tilde p^1\tilde p^2\tilde x^2] = \tilde\triangle(p^1p^2x^2)$, lying on either side of a common segment $[\tilde p^1\tilde p^2]$. We may suppose

$$\tilde\measuredangle(p^1\,{}^{p^2}_{x^1}) + \tilde\measuredangle(p^1\,{}^{p^2}_{x^2}) \leqslant \pi \quad\text{and}\quad \tilde\measuredangle(p^2\,{}^{p^1}_{x^1}) + \tilde\measuredangle(p^2\,{}^{p^1}_{x^2}) \leqslant \pi,$$

since otherwise (2+2)-point comparison holds trivially. Then $[\tilde p^1\tilde p^2]$ and $[\tilde x^1\tilde x^2]$ intersect, say at $\tilde q$.

By assumption, there is a geodesic $[p^1p^2]$. Choose $q \in [p^1p^2]$ corresponding to $\tilde q$; that is, $|p^1 - q| = |\tilde p^1 - \tilde q|$. Then

$$|x^1 - x^2| \leqslant |x^1 - q| + |q - x^2| \leqslant |\tilde x^1 - \tilde q| + |\tilde q - \tilde x^2| = |\tilde x^1 - \tilde x^2|,$$

where the second inequality follows from (c). Since increasing one side of a planar triangle increases the opposite angle,

$$\tilde\measuredangle(p^1\,{}^{x^1}_{x^2}) \leqslant \measuredangle[\tilde p^1\,{}^{\tilde x^1}_{\tilde x^2}] = \tilde\measuredangle(p^1\,{}^{p^2}_{x^1}) + \tilde\measuredangle(p^1\,{}^{p^2}_{x^2}).$$

□

12.2 Reshetnyak's gluing theorem

12.2.1. Reshetnyak's Gluing Theorem. *Let X be a proper geodesic space which contains two convex subsets X_1 and X_2 such that $X_i \in \mathrm{CAT}[0]$ and $X = X_1 \cup X_2$ then $X \in \mathrm{CAT}[0]$*

This theorem is mostly used to construct examples of CAT[0] spaces. One can take two CAT[0] spaces X_1 and X_2 with convex sets $C_i \subset X_i$ and an isometry $f\colon C_1 \to C_2$. Then attach these spaces together along the isometry f. It is easy to check that the induced map of X_i in the glued space, say X, is distance preserving and from ??? it follows that X is geodesic. Therefore, according to above theorem the resulting space X is a CAT[0] space.

For example, one can glue two copies of $\mathbb{R}^n$ along an isometry between closed unit balls in them, the resulting space will be CAT[0].

Proof. Set $C = X_1 \cap X_2$. Since both X_i are convex, we have that so is C.

We need to show that any given triangle $[xyz]$ in X is thin. The comparison trivially holds if all the vertices lie in one of X_i. Hence, without loss of generality, we may assume that $x \in X_1$ and $y, z \in X_2$.

Then there are points $\bar{y}, \bar{z} \in C$ such that $\bar{y} \in [xy]$ and $\bar{z} \in [xz]$. Decompose triangle $[xyz]$ into three triangles $[x\bar{y}\bar{z}]$, $[\bar{y}\bar{z}z]$ and $[\bar{y}zy]$. Applying Inheritance for thin triangles (11.2.2) twice, first for the triangle $[x\bar{y}z]$ decomposed into $[x\bar{y}\bar{z}]$, $[\bar{y}\bar{z}z]$ and then for the $[xyz]$ decomposed into $[x\bar{y}z]$, $[\bar{y}yz]$, we get that $[xyz]$ is thin. $\square$

Exercises

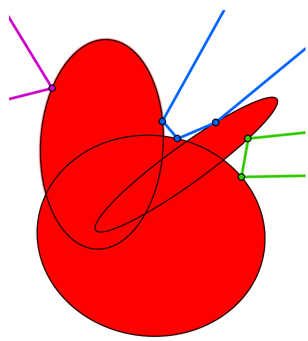
12.A. Let P be a two-dimensional polyhedral space. Show that $P \in \mathrm{CAT}[0]$ if and only if any two points in P are joined by unique geodesic.

Does the same hold for higher dimensional polyhedral spaces?

Chapter 13

An application to billiards

13.1 Piecewise concave table



In this section, we shall consider a billiard table formed by Euclidean space with a finite collection of convex sets removed.

Let $\{A_1, A_2, \dots, A_n\}$ be a finite collection of sets in $\mathbb{R}^m$. Consider the table

$$T = \mathrm{Closure}\left(\mathbb{R}^m \backslash \bigcup_{i=1}^{n} A_i\right).$$

Assume in addition that

(a) Each A_i is closed and convex.

(b) All A_i have nonempty intersection; i.e.,

$$\bigcap_{i=1}^{n} A_i \neq \varnothing.$$

(c) The boundaries $\partial A_1, \partial A_2, \dots, \partial A_n$ are *smooth hypersurfaces.*

(d) There is $\alpha < \pi$, such that for any i and j, the angle between outer normals to ∂A_i and ∂A_j at any common point $p \in \partial A_i \cap \partial A_j$ is at most α.

A billiard trajectory on such table is a unit-speed broken line with the break points at walls (i.e., at ∂A_i), such that at every break point the left and right velocities have equal projections onto the tangent plane to the wall. The number of break points of the trajectory will be called *number of collisions* of the trajectory. Further we will show the number of collisions of any trajectory has to be finite.

13.1.1. Exercise. *Assume that the collection of convex sets A_i in $\mathbb{R}^m$ satisfies conditions (b) and (c) above and in addition, all A_i are bounded. Show that condition (d) is also satisfied.*

13.1.2. Collision theorem. *Assume*

$$T = \mathrm{Closure}\left(\mathbb{R}^m \backslash \bigcup_{i=1}^{n} A_i\right)$$

is a billiard table which satisfies conditions (a)–(d) above. Then the number of collisions of any trajectory in T is bounded by a number N which depends only on n and α.

In the proof we will need the following lemma in convex geometry.

13.1.3. Lemma. *Let $\{A_1, A_2, \dots, A_n\}$ be a collection of sets in $\mathbb{R}^m$ which satisfies conditions (a)–(d). Set $K = \bigcap_{i=1}^{n} A_i$, then*

$$\mathrm{dist}_K \geqslant \sin\alpha \cdot \max_i \{\mathrm{dist}_{A_i}\}.$$

Moreover, given two points $x, y \in \mathbb{R}^m$, set $e_i(x,y) = \min_{a\in A_i}\{|x-a| + |a-y| - |x-y|\}$ and $E(x,y) = \min_{a\in K}\{|x-a| + |a-y| - |x-y|\}$ then

$$E(x,y) \geqslant \sin\alpha \cdot \max_i \{e_i(x,y)\}.$$

Proof. ??? □

Proof of Collision theorem. The proof is by induction on n.

The base case $n = 1$ is evident; the number of collisions cannot exceed 1. It follows from the convexity of A_1 that if the trajectory is reflected once in ∂A_1, then it cannot return to A_1.

The proof of the first step $n = 2$ is slightly simpler than the remaining steps. To simplify the presentation, we prove the case $n = 2$, and then describe the necessary modifications to prove the $n = 3$ case. Once this is done, the proof of the general case should be evident.

First step; $n = 2$. To reduce number of indices, set $A = A_1$ and $B = A_2$.

Note that any trajectory hits A and B in turn; i.e., the trajectory can not hit A (as well as B) twice in a raw.

Fix a large number N and assume a trajectory has at least $2 \cdot N$ collisuion. Then it should hit A, then B, then A and so on $2 \cdot N$ times and then finally B. Denote by $a_1 \in \partial A$, $b_1 \in \partial B, \dots b_N \in \partial B$ the brake points of this trajectory.

Prepare $2 \cdot N$ copies of $\mathbb{R}^m$, say $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_{2 \cdot N}$. Each $\mathcal{R}_i$ contains a copy of A and B, which will be denoted as A_i and B_i.

Let us glue a new space say $\mathcal{R}$ out of $\mathcal{R}_i$'s by gluing

1. $\mathcal{R}_1$ to $\mathcal{R}_2$ by identifying A_1 and A_2;
2. $\mathcal{R}_2$ to $\mathcal{R}_3$ by identifying B_2 and B_3;
3. $\mathcal{R}_3$ to $\mathcal{R}_4$ by identifying A_3 and A_4;
4. and so on;
5. $\mathcal{R}_{2 \cdot N-1}$ to $\mathcal{R}_{2 \cdot N}$ by identifying $A_{2 \cdot N-1}$ and $A_{2 \cdot N}$.

According to Reshetnyak's gluing theorem (12.2.1), $\mathcal{R} \in \mathrm{CAT}[0]$. Further we will view $\mathcal{R}_i$ as subsets of $\mathcal{R}$.

Denote by $f_i \colon \mathbb{R}^m \to \mathcal{R}$ the distance preserving map which identifies $\mathbb{R}^m$ and $\mathcal{R}_i$ and let $F \colon \mathcal{R} \to \mathbb{R}^m$ be the natural projection which identifies all $\mathcal{R}_i$ with $\mathbb{R}^m$. Consider broken geodesic γ connecting points $f_1(a_1), f_2(b_2), f_3(a_3), \dots, f_{2 \cdot N}$ in $\mathcal{R}$. Note that γ forms a local geodesic in $\mathcal{R}$. Therefore, according to ??? γ is a geodesic with end points in $A_1 \cup A_{2 \cdot N}$, but all interior points of γ do not belong to $A_1 \cup A_{2 \cdot N}$.

Hence to arrive to a contradiction, it is sufficient to prove the following claim.

➊ *The union $A_1 \cup A_{2 \cdot N}$ is a convex set in $\mathcal{R}$.*

Due to the convexity of A_1 and $A_{2 \cdot N}$, it is sufficient to show that for any points $p \in A_1 \backslash A_{2 \cdot N}$ and $q \in A_{2 \cdot N} \backslash A_1$, the geodesic $[pq]$ contain a point $x \in K =$ $= A_1 \cap A_{2 \cdot N}$.

Assume contrary. Mark points $p = p_1, q_2, p_3, \dots, p_{2 \cdot N} = q$ of $[pq]$ such that $p_i \in A_i$ and $q_j \in B_j$. Note that $[p_{2 \cdot i-1} q_{2 \cdot i}] \in \mathcal{R}_{2 \cdot i-1}$ and $[q_{2 \cdot i} p_{2 \cdot i+1}] \in \mathcal{R}_{2 \cdot i}$ for each i. We will construct a curve passing throug K and connecting p to q which is shorter than $[pq]$; this way we arrive to a contradiction.

Second step; $n = 3$.

□

The Baby collision theorem (13.1.2) admits a straightforward generalization to higher dimensions and to an arbitrary finite number of sets B_i. Namely, the following result holds:

13.1.4. Adult collision theorem. *Let $B_1, B_2, \dots, B_n$ be a finite collection of open convex sets in $\mathbb{R}^m$. Assume each B_i contains the origin $0 \in \mathbb{R}^m$ and for each i, the boundary $W_i = \partial B_i$ is a smooth hypersurface. Further, assume there is $\varepsilon > 0$ such that for any point $p \in W_i \cap W_j$ the angle between the tangent hyperplane of W_i and W_j at p is at least ε.*

Then there is a natural number N which depends only on ε such that the number of collisions of any trajectory in T is at most N.

The proof of the Collision theorem can be given along the same lines as its baby case. We glue the space in a similar way: Take a copy of $\mathbb{R}^m$ and glue to it n copies of $\mathbb{R}^m$, along each B_i. Further glue to each copy $n-1$ copies of $\mathbb{R}^m$...

Now we will show how to apply the above theorem to prove the following.

13.2 On number of collisions of balls

13.2.1. Theorem. *Consider n identical homogeneous hard balls moving freely and colliding elastically in empty space $\mathbb{R}^3$. Every ball moves along a straight line with constant speed until two balls collide, and then the new velocities of the two balls are determined by the laws of classical mechanics.*

Then the total number of collisions cannot exceed some number N which depends only on n.

Proof. A position of a collection of n balls can be represented by a point in $\mathbb{R}^{3\cdot n}$. If $a_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ is the center of the i-th ball then the corresponding point in $\mathbb{R}^{3\cdot N}$ is

$$(a_1, a_2, \dots, a_n) = (x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_n, y_n, z_n).$$

Not every point in $\mathbb{R}^{3\cdot n}$ represents a valid configuration of balls. We have to exclude positions where some of the balls overlap. The i-th and j-th ball intersect if

$$|a_i - a_j| < 2.$$

This inequality defines a cylinder $C_{ij} \subset \mathbb{R}^{3\cdot n}$. The complement

$$\mathbb{R}^{3\cdot n} \backslash \bigcup_{i \neq j} C_{ij}$$

is the configuration space of our system. Its points correspond to valid positions of the system of balls.

The evolution of the system of balls traces a path in the configuration space. It is easy to verify that the point representing the configuration of balls moves straight and at a constant speed until it hits one of the cylinders C_{ij} (this event corresponds to a collision in the system of balls), and then it continues following the standard law of billiard collision: the angle of reflection is equal to the angle of incidence.

Note that each cylinder C_{ij} is an open convex set with smooth boundaries which contains a unit ball around the origin. It is easy to check that the sets C_{ij} satisfy the conditions (a)–(d) on page 98; we can take $\alpha = \frac{\pi}{3}$ in (d). Therefore our theorem follows from Theorem 13.1.2. $\square$

Part IV

Nonnegative curvature

Chapter 14

Convex surfaces

14.1 Surface of convex polyhedron

Let K be a convex polyhedron in $\mathbb{R}^3$.

Assume K is *non-degenerate*; i.e., it has nonempty interior. In this case the boundary of K equipped with the induced length metric will be called the *surface of K*. According to Exercise 5.4.3, the surface of a polyhedron is a polyhedral space.

In the *degenerate* case, i.e., if K is a flat polygon, the surface of K is defined differently. You have to imagine that you are living in $\mathbb{R}^3$ and can walk on a flat polygon made from rigid material; so to get from one of its sides to the other, you have to travel over a boundary edge.

More formally, we define the surface of a convex polygon K as its *doubling*; i.e., two copies of K glued along the corresponding points of their boundaries.

We shall be interested in when an abstract polyhedral space P can be realized as the surface of a convex polyhedron in $\mathbb{R}^3$. The proposition below says that a necessary condition for this is that P must be homeomorphic to $\mathbb{S}^2$.

14.1.1. Proposition. *Let K be a non degenerate convex polyhedron or a convex polygon. Then the surface of K is homeomorphic to $\mathbb{S}^2$.*

Proof. Any convex flat polygon, is homeomorphic to disc. Therefore its doubling is homeomorphic to a sphere.

If K is nondegenerate polyhedron, we may assume that the origin $0 \in \mathbb{R}^3$ lies in the interior of K. Consider the map $f\colon P \to \mathbb{S}^2$ defined by $f(x) = x/|x|$;

in other words, a point $x \in P$ is mapped to the intersection of the ray $[0x)$ with the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

Since K is convex and the origin 0 is an interior point, for any $z \neq 0$, the ray $[0z)$ intersects P at exactly one point. It follows that, $f\colon P \to \mathbb{S}^2$ is a bijection.

Since $0 \notin P$, the map $f\colon P \to \mathbb{S}^2$ is continuous. Given $y \in \mathbb{S}^2$, denote by $h(y)$ the length of the intersection of the ray $[0y)$ with P. Then it is easy to see that $f^{-1}(y) = h(y)\cdot y$; since h is continuous function on $\mathbb{S}^2$, so is f^{-1}. (Alternatively, you may apply Exercise 1.4.) Therefore $f\colon P \to \mathbb{S}^2$ is a homeomorphism. □

14.2 Surface of convex body

14.2.1. Theorem. *Let K be a nondegenerate convex body then its surface has nonnegative curvature in the sense of Alexandrov.*

14.3 Curvature

By Proposition 14.1.1, if P which be realized as the surface of a nondegenerate convex polyhedron or polygon in $\mathbb{R}^3$ then it is homeomorphic to $\mathbb{S}^2$; in particular P is a polyhedral surface. Now we will formulate another necessary condition on P.

14.3.1. Exercise. *Let P be the surface of a convex polyhedron K. Show that the curvature of any point $p \in P$ is non-negative.*

Further in this section, we study geometry of polyhedral surfaces with non-negative curvature. We start with an exercise.

14.3.2. Exercise. *Let P be a non-negatively curved polyhedral surface.*

a) Show that if two geodesics in P intersect at two points, then these are the end points for both geodesics.

b) Show that a geodesic in P cannot pass through a vertex of P.

Here is the main theorem in the section; it gives a global geometric property of non-negatively curved polyhedral surface.

14.3.3. Theorem. *Let P be a polyhedral surface. Assume P has non-negative curvature at each point. Then*

$$\measuredangle[p\,{}^x_y] \geqslant \tilde{\measuredangle}(p\,{}^x_y)$$

for any hinge $[p\,{}^x_y]$ *in* P.

14.3.4. Corollary. *Let* P *be a polyhedral surface. Then* P *has non-negative curvature at each point if and only if*

$$\tilde\measuredangle(p\,{}^x_y) + \tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^z_x) \leqslant 2\cdot\pi$$

for any hinge quadruple of points p, x, y, z *in* P *such that* p *is distinct from each of* x, y *and* z.

Proof. Let $[pxy]$ be a triangle in P and let $[\tilde p\tilde x\tilde y]$ be the model triangle of $[pxy]$. Set $\ell = |x-y|_P = |\tilde x - \tilde y|_{\mathbb{R}^2}$.

Denote by $\gamma(t)$ the geodesic $[xy]$ parametrized by length starting from x and let $\tilde\gamma(t)$ be the geodesic $[\tilde x\tilde y]$ parametrized by length starting from $\tilde x$. It is sufficient to show that

$$❶ \qquad |p-\gamma(t)| \leqslant |\tilde p - \tilde\gamma(t)|$$

for any t in $[0,\ell]$.

A point p in P will be calles *regular* if p is not a vertex of P or equivalently, the curvature of P vanish at p. Since any vertex can be approximated by regular points, we may assume that p is a regular point.

From the cosine law, we get that the function

$$\tilde f(t) = |\tilde p - \tilde\gamma(t)|^2 - t^2$$

is linear. Consider the function

$$f(t) = |p-\gamma(t)|^2 - t^2.$$

Note that

$$\begin{aligned} f(0) &= \tilde f(0),\\ f(\ell) &= \tilde f(\ell). \end{aligned}$$

Note that the inequality ❶ is equivalent to

$$❷ \qquad f(t) \geqslant \tilde f(t).$$

According to Jensen's inequality, to prove ❷ it is sufficient to show that f is a concave function. The latter follows once we prove the following:

❸ *For any* $t_0 \in \,]0,\ell[$ *there is a* supporting linear function*; i.e., a function* $h(t)$ *such that*

$$h(t_0) = f(t_0) \quad \textit{and} \quad h(t) \geqslant f(t)$$

for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ *and some fixed* $\varepsilon > 0$.

Note that according to Exercise 14.3.2, $\gamma(t_0)$ is regular. Since p is regular, a geodesic $[p\gamma(t)]$ contains only regular points. Therefore for small enough $\varepsilon > 0$, the ε-neighborhood of $[p\gamma(t)]$, say Ω contains only regular points. We may assume that Ω is homeomorphic to a disc; in this case there is a locally distance preserving embedding $\iota\colon \Omega \to \mathbb{R}^2$. Note the image $\iota([p\gamma(t)])$ is a line segmentthat and $\iota(\Omega)$ is the ε-neighborhood of $\iota([p\gamma(t)])$ in $\mathbb{R}^2$; in particular $\iota(\Omega)$ is convex. Thus $\iota(\Omega)$ contains a triangle with base $\iota([\gamma(t_0 - \varepsilon)\ \gamma(t_0 + \varepsilon)])$ and vertex $\iota(p)$.

Clearly, for any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ we have

$$|\iota(p) - \iota(\gamma(t))| \geqslant |p - \gamma(t)|.$$

Note that the function

$$h(t) = |\iota(p) - \iota(\gamma(t))|^2 - t^2$$

is linear and it satisfies the condition ❸. □

If in the above proof the geodesic from p to $\gamma(t_0)$ is not unique, then the inequality $h(t) \geqslant f(t)$ might be strict for t arbitrary close to t_0.

14.4 Cauchy's theorem

The first step in the proof of Alexandrov's theorem (15.1.2) is Alexandrov's uniqueness theorem (14.7.1) which in turn generalizes Cauchy's theorem formulated below. We start with the proof of Cauchy's theorem and then modify it to prove Alexandrov's uniqueness theorem.

14.4.1. Cauchy's theorem. *Let K and K' be two non-degenerate convex polyhedra in $\mathbb{R}^3$; denote their surfaces*[1] *by P and P'. If there is an isometry $P \to P'$ which sends each face of K to a face of K', then K is congruent to K'.*

First I break the proof into two parts, "local" and "global", which will be proved in the following sections.

[1] Their boundaries equipped with the induced length metric.

Outline of the proof. Consider the graph Γ formed by the edges of K (the edges of K' form the same graph).

For an edge e in Γ,

◇ denote by α_e the corresponding dihedral angle of K;

◇ denote by α'_e the corresponding dihedral angle of K'.

Mark an edge e of Γ with $(+)$ if $\alpha_e < \alpha'_e$ and with $(-)$ if $\alpha_e > \alpha'_e$.

Now remove from Γ everything which was not marked; i.e., leave only the edges marked by $(+)$ or $(-)$ and their endpoints. The statement of Cauchy's theorem is equivalent to the fact that Γ is an empty graph. Let us assume the contrary and try to arrive at a contradiction.

Note that Γ is embedded into P, which is homeomorphic to $\mathbb{S}^2$ (see Proposition 14.1.1). In particular, the edges coming from one vertex have a natural cyclical order. Given a vertex v of Γ, we can count the *number of sign changes* around v; i.e., the number of pairs of adjacent edges which are marked by different signs.

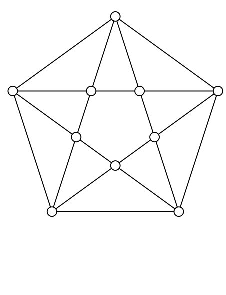
14.4.2. Local lemma. *For any vertex of* Γ *the number of sign changes is at least* 4.

In other words, the local lemma states that at each vertex of Γ, one can choose 4 edges marked by $(+)$, $(-)$, $(+)$ and $(-)$ which are in the same cyclical order.

Once the Local lemma is proved, we get a contradiction by applying the following lemma.

14.4.3. Global lemma. *Let* Γ *be a nonempty sub-graph of the graph formed by the edges of a convex polyhedron. Then it is impossible to mark all of the edges of* Γ *by* $(+)$ *or* $(-)$ *such that the number of sign changes around each vertex of* Γ *is at least* 4. □

14.4.4. Exercise. *Assume that we glue one pentagon and 10 triangles in* $\mathbb{R}^3$ *along the rule shown in the picture. Assume that it forms a part of a surface of a convex polyhedron and each vertex is a vertex of the polyhedron.*



Use the local lemma to show that this configuration is rigid; say one can not fix the position of the pentagon and continuously move the remaining 5 vertices in a new position so that each triangle moves by a one parameter family of isometries of $\mathbb{R}^3$.

14.5 Arm lemma and Local lemma

To prove the Local lemma, we will need the following.

14.5.1. Arm lemma. *Assume that $A = [a_0 a_1 \dots a_n]$ is a convex polygon in $\mathbb{R}^2$ and $A' = [a_0' a_1' \dots a_n']$ be a closed broken line in $\mathbb{R}^3$ such that*

$$|a_i - a_{i+1}| = |a_i' - a_{i+1}'|$$

for any $i \in \{0, \dots, n-1\}$ and

$$\angle a_i \leqslant \angle a_i'$$

for each $i \in \{1, \dots, n-1\}$. Then

$$|a_0 - a_n| \leqslant |a_0' - a_n'|$$

and equality holds if and only if A is congruent to A'.

One may view the broken lines $[a_0 a_1 \dots a_n]$ and $[a_0' a_1' \dots a_n']$ as a robot's arm in two positions. The arm lemma states that when the arm opens, the distance between the "shoulder" and "tips of the fingers" increases.

14.5.2. Exercise. *Show that the arm lemma does not hold if instead of the convexity, one only the local convexity; i.e., if you go along the broken line $a_0 a_1 \dots a_n$, then you only turn left.*

In the proof, we will use the following exercise. Equivalently, one can think of this as of triangle inequality on the unit sphere $\mathbb{S}^2$ with the induced length metric.

14.5.3. Exercise. *Let w_1, w_2, w_3 be unit vectors in $\mathbb{R}^3$. Denote by $\vartheta_{i,j}$ the angle between the vectors v_i and v_j. Then*

$$\vartheta_{1,3} \leqslant \vartheta_{1,2} + \vartheta_{2,3}$$

and in case of equality, the the vectors w_1, w_2, w_3 lie in a plane.

Proof. We will view $\mathbb{R}^2$ as the xy-plane in $\mathbb{R}^3$, so that both A and A' lie in $\mathbb{R}^3$. Let a_m be the vertex of A which has maximal distance to the line $(a_0 a_n)$.

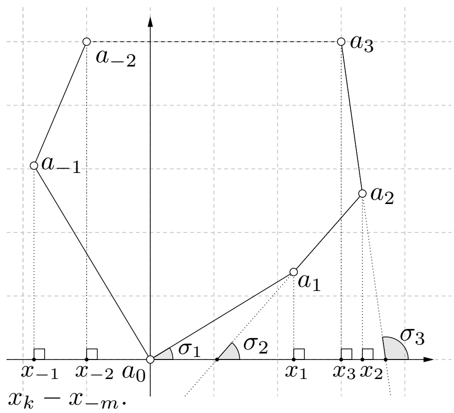
Let us shift indexes of a_i and a_i' down by m, so that

$$\begin{aligned} a_{-m} &:= a_0, & a'_{-m} &:= a'_0,\\ a_{-(m-1)} &:= a_1, & a'_{-(m-1)} &:= a'_1,\\ &\vdots & &\vdots\\ a_0 &:= a_m, & a'_0 &:= a'_m,\\ &\vdots & &\vdots\\ a_k &:= a_n & a'_k &:= a'_n, \end{aligned}$$

where $k = n - m$. The symbol ":=" means *an assignment* as in programming (the order of variables in an assignment statement is important: $a := b$ means that both a and b take the value b and $b := a$ means that both take the value a).

Without loss of generality, we may assume that

- ⋄ $a_0 = a_0'$ and they both coinside with the origin $(0,0,0) \in \mathbb{R}^3$;
- ⋄ all a_i lie in the xy-plane and the x-axis is parallel to the line $(a_{-m}a_k)$;
- ⋄ the angle $\angle a_0'$ lies in xy-plane and contains the angle $\angle a_0$ inside and the directions to a'_{-1},a_{-1}, a_1 and a_1' from a_0 appear in the same cyclic order.



Denote by x_i and x_i' the projections of a_i and a_i' to x-axis. We can assume in addition that $x_k \geqslant x_{-m}$. In this case

$$|a_k - a_{-m}| = x_k - x_{-m}$$

and

$$|a'_k - a'_{-m}| \geqslant x'_k - x'_{-m}.$$

The latter follows because projection is a distance non-expanding map.

Therefore it is sufficient to show that

$$x'_k - x'_{-m} \geqslant x_k - x_{-m}.$$

The later holds if

❶ $$x'_i - x'_{i-1} \geqslant x_i - x_{i-1}.$$

for each i.

It remains to prove ❶; in the proof we assume that $i > 0$, the case $i \leqslant 0$ is similar. Let us

◇ denote by σ_i the angle between the vector $w_i = a_i - a_{i-1}$ and the x-axis;

◇ denote by σ'_i the angle between the vector $w'_i = a'_i - a'_{i-1}$ and the x-axis.

Note that

❷
$$\begin{aligned} x_i - x_{i-1} &= |a_i - a_{i-1}| \cdot \cos \sigma_i, \\ x'_i - x'_{i-1} &= |a_i - a_{i-1}| \cdot \cos \sigma'_i \end{aligned}$$

for each $i > 0$. By construction $\sigma_1 \geqslant \sigma'_1$. Note that $\measuredangle(w_{i-1}, w_i) = \pi - \measuredangle a_i$. From convexity of $[a_1 a_1 \dots a_i]$, we have

$$\sigma_i = \sigma_1 + (\pi - \measuredangle a_1) + \dots + (\pi - \measuredangle a_i)$$

for any $i > 0$. Since $\measuredangle(w'_{i-1}, w'_i) = \pi - \measuredangle a'_i$. Applying Exercise 14.5.3 $i - 1$ times, we get

$$\sigma'_i \leqslant \sigma'_1 + (\pi - \measuredangle a'_1) + \dots + (\pi - \measuredangle a'_i).$$

Since $\measuredangle a_j \leqslant \measuredangle a'_j$ for each j, summing it we get

$$\sigma_i \geqslant \sigma'_i.$$

Applying ❷, we get ❶.

In the case of equality, we have $\sigma_i = \sigma'_i$ for each i, that implies $\measuredangle a_i = \measuredangle a'_i$ for each i. This also implies that all a'_i lie in xy-plane. The latter easily follows from the equality case in the Exercise 14.5.3. □

Proof of Local lemma (14.4.2). Assume that the Local lemma does not hold at the vertex v of Γ. Let cut from P a small pyramid Δ with vertex v. Then one can choose two points a and b on the base of Δ so that on one side of the segments $[va]$ and $[vb]$ we have only $(+)$'s, and on the other side only $(-)$'s.

The base polygon is formed by two broken lines with ends at a and b. Assume that

$$a = a_0,\ a_1,\ \dots,\ a_n = b$$

form the broken line along the side marked with $(+)$'s. Denote by

$$a' = a'_0,\ a'_1,\ \dots,\ a'_n = b'$$

the corresponding points in P'. Since each marked edge passing through a_i has a $(+)$ on it or nothing, we have

$$\measuredangle a_{i-1} a_i a_{i+1} \leqslant \measuredangle a'_{i-1} a'_i a'_{i+1}$$

for each i.

14.5.4. Exercise. *Prove the last statement.*

By construction we have $|a_i - a_{i-1}| = |a'_i - a'_{i-1}|$ for all i. By the Arm lemma (14.5.1), we get

❸ $|a - b| \leqslant |a' - b'|$ *and equality holds if no edge from v is marked with a* $(+)$.

Repeating the same construction exchanging the places of K and K' gives

❹ $|a - b| \geqslant |a' - b'|$ *and equality holds no edge from v is marked with a* $(-)$.

The claims ❸ and ❹ together imply $|a - b| = |a' - b'|$ and it follows that no edge at v is marked; i.e., v is not a vertex of Γ, a contradiction. □

14.6 Global lemma

Before going into the proof, we suggest to do the following.

14.6.1. Exercise. *Try to mark the edges of an octahedron by* $(+)$*'s and* $(-)$*'s such that there would be 4 sign changes at each vertex.*

Show that this is impossible.

The proof of the Global lemma is based on counting the sign changes in two ways; the first is as one moves around each vertex of Γ and the second is as one moves around each of the regions separated by Γ on the surface P. If two edges are adjacent at a vertex, then they are also adjacent in moving around the region to whose boundary they belong. The converse is true as well. Therefore, both of the ways of counting give the same number.

Proof of 14.4.3. We can assume that Γ is connected; that is, one can get from any vertex to any other vertex by walking along edges. (If not, pass to any connected component of Γ.) Denote by k and l the number of vertices and edges respectively in Γ. Denote by m the number of *regions* which Γ cuts from P. Since Γ is connected, each region is homeomorphic to an open disc.

14.6.2. Exercise. *Prove the last statement.*

Now we can apply Euler's formula

❶ $$k - l + m = 2.$$

Denote by s the total number of sign changes in Γ for all vertices. By the Local lemma (14.4.2), we have

❷ $$4{\cdot}k \leqslant s.$$

Now let us get an upper bound on s by counting the number of sign changes when you go around each region. Denote by m_n the number of regions which are bounded by n edges; if an edge appear twice when you go around the regions it is counted twice. Note that each region is bounded by at least 3 edges; therefore

❸ $$m = m_3 + m_4 + m_5 + \dots$$

Counting edges and using the fact that each edge belongs to exactly two regions, we get

$$2{\cdot}l = 3{\cdot}m_3 + 4{\cdot}m_4 + 5{\cdot}m_5 + \dots.$$

Combining this with Euler's formula (❶), we get

❹ $$4{\cdot}k = 8 + 2{\cdot}m_3 + 4{\cdot}m_4 + 6{\cdot}m_5 + \dots$$

Observe that the number of sign changes in n-gon regions has to be even number which is at most n. Therefore

❺ $$s \leqslant 2{\cdot}m_3 + 4{\cdot}m_4 + 4{\cdot}m_5 + 6{\cdot}m_6 + \dots$$

Clearly, ❷ and ❺ contradict ❹. □

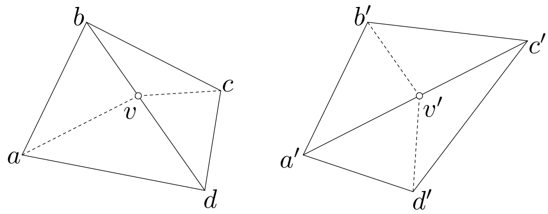
14.7 Alexandrov's uniqueness theorem

Alexandrov's uniqueness theorem states that the conclusion of Cauchy's theorem (14.4.1) still holds if one removes the phrase "which sends each face of K to a face of K'" from it. For your convenience we repeat the formulation here:

14.7.1. Alexandrov's uniqueness theorem. *Any two convex polyhedra in $\mathbb{R}^3$ with isometric surfaces are congruent.*

The proof is along the same lines as the proof of Cauchy's theorem. We will only describe the necessary modifications.

Let $\iota\colon P \to P'$ be an isometry. Mark in P all the edges of K and all the ι-preimages of edges of K', which will further be called fake edges. These lines divide P into convex polygons, say $\{Q_i\}$, and the restriction of ι to each Q_i is a rigid move. These polygons will play the role of faces in the proof of Cauchy's theorem.



A fake vertex $v \in K$ and the corresponding point $v' \in K'$.

A vertex of Q_i can be a vertex of K or it can be a fake vertex; i.e., lie on intersection of an edge and fake edge. For the first type of vertex, the Local lemma can be proved in exactly the same way. For a fake vertex v, it is easy to see that both parts of the edge coming through v are marked with $(+)$ while both of the fake edges at v are marked with $(-)$. Therefore, the Local lemma holds for the fake vertices as well.

The remainder of the proof needs no further modifications.

14.8 Comments

In the Euclid's Elements, solids were called equal if the same holds for their faces, but no proof was given. Legendre became interested in it towards the end of the 18th century and talked to his colleague Lagrange about it, who in turn suggested this problem to Cauchy in 1813 who soon proved it. In 1950, Alexandrov understood that the condition on the equality of faces can be exchanged by much weaker condition on the surface of polyhedra.

We present a proof which has only minor modifications of Alexandrov's original proof in [3]. Two alternative proofs due to Pogorelov and Senkin–Zalgaller are nicely discussed in Pak's lecture notes [**?**], both of these proofs are fun to read.

Arm lemma. The original Cauchy's proof (if you read French, see [9]) also used Arm lemma, but its proof contained an error which was corrected in 1928 by Steinitz.

The proof of Arm Lemma which we present is due to Zaremba. This and yet couple other beautiful proofs can be found in the letters between Schoenber and Zaremba published in [22].

The following variation of Arm Lemma which makes sense for nonconvex shperical polygons.

14.8.1. Zalgaller's Arm Lemma. *Let $A = a_1a_2 \dots a_n$ and $A' = a'_1a'_2 \dots a'_n$ be two spherical n-gons (not nesessury convex). Assume that A lies in a half-sphere, corresponding sides of A and A' are equal and each angle of A is bigger or equal to the corresponding angle in A'. Then A is congruent to A'.*

See the original paper in Russian [24] or its translation in ???. This lemma is used in the Senkin–Zalgaller proof of the uniqueness theorem mentioned above.

Global lemma. Alexandrov's book includes an other proof of Global lemma in addition to the one given here. This proof require no calculations, it is easier to explain but harder to write down.

Exercises

14.A. Show that the sum of the exterior angles of any closed broken line in $\mathbb{R}^3$ is at least $2{\cdot}\pi$.

Hint: Use induction on the number of edges.

Use the previous problem to solve the next one.

14.B. Consider a convex polygons $A = [a_1a_2 \dots a_n]$ in $\mathbb{R}^2$ and a closed broken line $A' = [a'_1a'_2 \dots a'_n]$ in $\mathbb{R}^3$. Let us enumerate the vertices in an n-periodic way; i.e., set $a_{n+k} = a_k$ and $a'_{n+k} = a'_k$ for any k.

Assume $|a_i - a_{i-1}| = |a'_i - a'_{i-1}|$ and $|a_i - a_{i-2}| \leqslant |a'_i - a'_{i-2}|$ for all i. Show that A is congruent to A'.

14.C. Give a complete proof of the Arm lemma (14.5.1) using the following plan.

Use induction on n. Prove the base case $n = 2$.

For a point b on the broken line $[a_0a_1 \dots a_n]$, denote by b' the corresponding point on $[a'_0a'_1 \dots a'_n]$; i.e., if $b \in [a_{i-1}a_i]$ then $b' \in [a'_{i-1}a'_i]$ and $|a_i - b| = |a'_i - b'|$.

Assume $|a_0 - a_n| \geqslant |a'_0 - a'_n|$. Applying the induction hypothesis, $|a_i - a_j| \leqslant |a'_i - a'_j|$ if $|i - j| < n$. Moreover for any point b on the broken line $[a_0a_1 \dots a_n]$, we have $|a_i - b| \leqslant |a'_i - b'|$ if $0 < i < n$.

Choose $b \in [a_{n-1}a_n]$ to be the closest point to a_n auch that $|a_0 - b| = |a'_0 - b'|$.

Show that the case $b = a_{n-1}$ can be reduced to the case $n = 2$.

In the remaining case, apply the previous problem to the broken lines $[a_0a_1 \dots \dots a_{n-1}b]$ and $[a'_0a'_1 \dots a'_{n-1}b']$.

14.D. Assume that the surface of a nonregular tetrahedron T has curvature π at each of it vertices. Show that

i) all faces of T are congruent;
ii) the line passing through midpoints of opposite edges of T itersects these edges at right angles.

14.E. Let K be a convex polyhedron in $\mathbb{R}^3$; denote by P its surface. Show that each isometry $\iota\colon P \to P$, can be extended to an isometry of $\mathbb{R}^3$.

The following problem is a quantitative version of Exercise 14.3.2

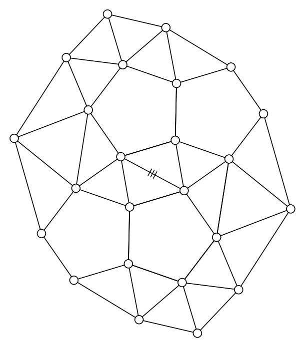
14.F. Let P be a non-negatively curved polyhedral space homeomorphic to a sphere.

Assume two vertices in P are jointed by two geodesics, denote by m_1 and m_2 their midpoints. Show that

$$|m_1 - m_2|_P > \tfrac{1}{100}\cdot\ell\cdot\varepsilon$$

where ε is the minimum of curvatures of the vertices of P and ℓ is the minimal distance between pairs of vertices of P.

14.G. Assume that we glue in $\mathbb{R}^3$ four regular pentagons and 22 equilateral triangles along the rule shown on the picture such that they form a part of surface of a convex polyhedron.



Use the Local lemma to show that this configuration has a rotational symmetry with axis passing through the midpoint of the marked edge.

Chapter 15

Alexandrov's existence theorem

15.1 The goal

In this chapter we prove the following theorem.

15.1.1. Alexandrov's existence theorem. *Let P be a polyhedral space which is homeomorphic to $\mathbb{S}^2$. If P has non-negative curvature at each point, then there is a convex polyhedron K whose surface is isometric to P.*

(The convex polyhedron in the theorem can degenerate to a convex polygon. In this case its surface is defined as its doubling; i.e., two copies of the polygon glued along the boundary. See Section 14.1.)

The proof of this theorem relies on Alexandrov's uniquencess theorem (14.7.1). Together with Alexandrov's uniquencess theorem, Alexandrov's existence theorem give an "if and only if" characterization of surfaces of convex polyhedra. Namely the following result holds.

15.1.2. Alexandrov's theorem. *A metric space P is isometric to the surface of a convex polyhedron in $\mathbb{R}^3$ if and only if P is a polyhedral space which is homeomorphic to $\mathbb{S}^2$, and the curvature of P is non-negative at each point.*

Moreover, the polyhedron associated to P is unique up to isometry of $\mathbb{R}^3$.

The plan. In Section 15.2 we consider the space of convex polyhedra in $\mathbb{R}^3$. In Section 15.3 we consider the space of polyhedal spaces. In these two sections we introduce a number of notations, and prove that if we fix the number k of vertices, then after removing neglectable sets, both of the spaces become $(3{\cdot}k-6)$-dimensional manifolds.

In Section 15.4, we use the introduced notation to give an outline of the proof of the existence theorem. After that, it remains to prove three lemmas: the Open lemma, the Closed lemma and the Connecting lemma. The Closed lemma and the Open lemma are proved in Section 15.5. The proof of the Connecting lemma (15.4.4) occupies two sections 15.6 and 15.7. In Section 15.6 we give a couple more definitions and introduce Alexandrov's patching construction, which is a key ingredient in the proof.

15.2 Space of convex polyhedra

In this section, we consider the space $\mathbf{K}_k$ formed by the congruence classes of convex polyhedra in $\mathbb{R}^3$ with exactly k vertices. The main result of this section is Lemma 15.2.2.

Assume K is a convex polyhedron in $\mathbb{R}^3$ with exactly k vertices. Let us denote by $[K]$ its congruence class; i.e., if K' is another convex polyhedron in $\mathbb{R}^3$ with exactly k vertices, then $[K]=[K']$ if and only if K is congruent to K'.

The set of all congruence class of convex polyhedra in $\mathbb{R}^3$ with exactly k vertices will be denoted by $\mathbf{K}_k$. This set is equipped with the metric

$$|[P]-[P']|_{\mathbf{K}_k} \stackrel{def}{=\!=} \inf_{\iota}\{|P-\iota(P')|_{\mathcal{H}(\mathbb{R}^3)}\},$$

where the infimum is over all isometries $\iota\colon \mathbb{R}^3\to\mathbb{R}^3$. (In other words, $\mathbf{K}_k$ is the subspace of $\mathcal{H}(\mathbb{R}^3)/_{\cong}$ formed by congruence classes of convex polyhedra with exactly k vertices, see Exercise 2.4.1.)

The statement K *is a convex polyhedron with exactly k vertices in* $\mathbb{R}^3$ can be written as $[K]\in\mathbf{K}_k$, but we often will write it as

$$K\in\,]\mathbf{K}_k[\,,$$

formally here $]\mathbf{K}_k[$ stands for the union of all congruence classes in $\mathbf{K}_k$.

We say that $K\in\,]\mathbf{K}_k[$ *has a symmetry* if there is an isometry $\iota\colon\mathbb{R}^3\to\mathbb{R}^3$, which is distinct from identity, such that $\iota(K)=K$. If there is no such isometry, we say that K has *no symmetry.*

Note that any segment or triangle in $\mathbb{R}^3$ has a symmetry. On the other hand, for any $k\geqslant 4$, a generic $K\in\,]\mathbf{K}_k[$ has no symmetry.

15.2.1. Exercise. *Prove that the set of congruence classes of polyhedra with no symmetry form an open dense subset of $\mathbf{K}_k$ for any $k \geqslant 4$.*

15.2.2. Lemma. *Fix an integer $k \geqslant 4$. Assume $K \in]\mathbf{K}_k[$ has no symmetry. Then $[K]$ admits an open neighborhood in $\mathbf{K}_k$ which is homeomorphic to an open domain in $\mathbb{R}^{3\cdot k-6}$.*

In the proof, we will use the following technical statement. Roughly it states that if two polyhedra are close in the sense of Hausdorff, then their vertex sets are also close. (Do not be scared by the long formulation; this statement is much easier to prove than it is to formulate.)

15.2.3. Exercise. *Let K be a convex polyhedron in $\mathbb{R}^3$ with k vertices, say w_1, $w_2, \dots, w_k$. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that if K' is another convex polyhedron with k vertices such that $|K - K'|_{\mathcal{H}(\mathbb{R}^3)} < \delta$ then the vertices of K' can be labeled by w_1', $w_2', \dots, w_k'$ in such a way that*

$$|w_i - w_i'| < \varepsilon$$

for each i.

Moreover for the unique orientation preserving isometries $\iota, \iota' \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\begin{aligned} \iota(w_1) &= (0,0,0) & \iota'(w_1') &= (0,0,0),\\ \iota(w_2) &= (x_2,0,0), & \iota'(w_2') &= (x_2',0,0),\\ \iota(w_3) &= (x_3,y_3,0), & \iota'(w_2') &= (x_3',y_3',0) \end{aligned}$$

with $x_2, x_2', y_3, y_3' > 0$ we have

$$|\iota(w_i) - \iota'(w_i')| < \varepsilon$$

for each i.

Proof of Lemma 15.2.2. Let K be a convex polyhedron with vertices $w_1, w_2, \dots, w_k$.

Let us choose orthogonal coordinates (x, y, z) in $\mathbb{R}^3$ so that

$$\begin{aligned} w_1 &= (0,0,0),\\ w_2 &= (x_2,0,0),\\ w_3 &= (x_3,y_3,0),\\ w_4 &= (x_4,y_4,z_4),\\ &\ \vdots\\ w_k &= (x_k,y_k,z_k) \end{aligned}$$

and $x_2, y_3 > 0$.

Note that one can choose small enough $\varepsilon > 0$ such that if

❶ $$x_i' \lesssim x_i \pm \varepsilon, \quad y_i' \lesssim y_i \pm \varepsilon, \quad z_i' \lesssim z_i \pm \varepsilon$$

for all i then the points

❷ $$\begin{aligned} w_1' &= (0,0,0), \\ w_2' &= (x_2',0,0), \\ w_3' &= (x_3',y_3',0), \\ w_4' &= (x_4',y_4',z_4'), \\ &\vdots \\ w_k' &= (x_k',y_k',z_k'). \end{aligned}$$

form vertices of a convex polyhedron, say K'.

Since K has no symmetry and $\varepsilon > 0$ is small, the polyhedron K' also has no symmetry.

15.2.4. Exercise. *Prove the last statement.*

Therefore
$$(x_2', x_3', y_3', x_4', y_4', z_4', \dots, x_k', y_k', z_k') \mapsto [K']$$
describes a continuous injective map $f\colon U \to \mathbf{K}_k$, where

$$U = (x_2 - \varepsilon, x_2 + \varepsilon) \times (x_3 - \varepsilon, x_3 + \varepsilon) \times (y_3 - \varepsilon, y_3 + \varepsilon) \times \dots$$
$$\dots \times (z_k - \varepsilon, z_k + \varepsilon) \subset \mathbb{R}^{3\cdot k - 6}.$$

It remains to show that $f(U)$ contains a small neghborhood of the congruence class $[K]$ in $\mathbf{K}_k$. In other words, given a convex polyhedron K' with exactly k vertices such that $|K - K'|_{\mathcal{H}(\mathbb{R}^3)}$ is sufficiently small, we need to find an isometry of $\mathbb{R}^3$ which moves K' into a new position, say K'', such that its vertices could be described as in ❷ and ❶ holds for all i. This follows from the Exercise 15.2.3. □

15.3 Space of polyhedral spaces

The main result of this section is Lemma 15.3.1.

Let P be a polyhedral space with nonnegative curvature that is homeomorphic to $\mathbb{S}^2$. A point $v \in P$ will be called a *vertex of P* if v has positive curvature.

Fix $k \geqslant 4$. Let us denote by $\mathbf{P}_k$ the subspace of $\mathcal{M}$ formed by the isometry classes of polyhedral spaces with nonnegative curvature, with exactly k vertices, and which are homeomorphic to $\mathbb{S}^2$.

For $\mathbf{P}_k$ we will use conventions similar to the ones made for $\mathbf{K}_k$. Namely given a polyhedral space P, we denote by $[P]$ its isometry class. The statement *P is a polyhedral space with nonnegative curvature, with exactly k vertices, and which is homeomorphic to* $\mathbb{S}^2$ can be written as $[P] \in \mathbf{P}_k$, but we often will write it as

$$P \in \,]\mathbf{P}_k[\,,$$

formally here $]\mathbf{P}_k[$ stays for the union of all isometry classes in $\mathbf{P}_k$.

A space $P \in \,]\mathbf{P}_k[$ is called *realizable* if there is $K \in \,]\mathbf{K}_k[$ such that P is isometric to the surface of K. So the Alexandrov's existence theorem can be stated as *any space in* $]\mathbf{P}_k[$ *is realizable.*

15.3.1. Lemma. *Let $P \in \,]\mathbf{P}_k[$ be a realizable polyhedral with no symmetry. Then $[P]$ has a neighborhood in $\mathbf{P}_k$ which is homeomorphic to an open domain in $\mathbb{R}^{3\cdot k-6}$.*

In the proof we will use the following technical statement.

15.3.2. Exercise. *Let $P, P' \in \,]\mathbf{P}_k[$ and $\varepsilon > 0$. Assume P admits a triangulation $\mathcal{T}$ with k vertices such that each edge is formed by the unique geodesic between its endpoints. Then there is $\delta > 0$ such that if $f\colon P' \to P$ is a δ-isometry then*

a) The vertices of P and P' can be labeled as $v_1, v_2, \dots, v_k$ and $v'_1, v'_2, \dots, v'_k$ correspondingly in such a way that $|f(v'_i) - v_i| < \varepsilon$ for any i.

b) P' admits a triangulation $\mathcal{T}'$ such that a pair of vertices (v'_i, v'_j) is connected by an edge if and only if the corresponding pair (v_i, v_j) is connected by an edge in $\mathcal{T}$ and each edge of $\mathcal{T}'$ is the necessarily unique geodesic.

Proof of Lemma 15.3.1. Since P is realizable, we can identify P with the surface of a convex polyhedron, say K. Since P has no symmetry, Exercise 14.E implies that K has no symmetry.

Note that one can triangulate each face of K in a such a way that only vertices of the face are the vertices of the triangulation. This follows from Exercise 5.B, but it is much easier since the face is convex; see the comment in the solution to this exercise.

These triangulations together give a triangulation, say $\mathcal{T}$, of P with vertices only at the vertices of P. In particular $\mathcal{T}$ has exactly k vertices; by Euler's formula (5.5.1), $\mathcal{T}$ has $3{\cdot}k-6$ edges. Note that each edge of $\mathcal{T}$ is a line segment in $\mathbb{R}^3$. Therefore each edge of $\mathcal{T}$ is the necessarily unique geodesic between its endpoints in P. (This property is not achieved if K degenerates to a plane polygon, but since P has no symmetry, it cannot be degenerate.)

Note that the triangulation $\mathcal{T}$ together with the lengths of its edges, say a_1, $a_2, \dots, a_l$, describe P up to isometry. Indeed, the side lengths describe each triangle in $\mathcal{T}$ and P can be obtained by gluing these triangles according the rule encoded in $\mathcal{T}$.

The same construction can be performed if we change the lengths of edges a bit. Say, fix a sufficiently small $\varepsilon > 0$ and make the lengths of edges to be a_1', $a_2', \dots, a_l'$, such that

$$a_i' \lessgtr a_i \pm \varepsilon$$

for each i. If ε is small enough, the sides of each triangle of $\mathcal{T}$ satisfy the strict triangle inequality. Therefore one can cut such triangles from the plane and glue them together according the rule encoded in $\mathcal{T}$ and obtain a polyhedral space P' which is homeomorphic to $\mathbb{S}^2$. Further note that the angle of a triangle depends continuously on its sides. Hence for small enough $\varepsilon > 0$, the curvature of any vertex in P' is still positive; i.e., $P' \in]\mathbf{P}_k[$.

Clearly if $\varepsilon > 0$ is small enough then so is $|P - P'|_{\mathcal{M}}$. I.e.,

$$f\colon (a_1', a_2', \dots, a_l') \mapsto [P']$$

give a continuous map $f\colon U \to \mathbf{P}_k$, where

$$U = (a_1 - \varepsilon, a_1 + \varepsilon) \times (a_2 - \varepsilon, a_2 + \varepsilon) \times \dots \times (a_l - \varepsilon, a_l + \varepsilon) \subset \mathbb{R}^l.$$

It remains to establish the following two claims.

❶ *f is injective.*

❷ *$f(U)$ contains a neighborhood of $[P]$ in $\mathbf{P}_k$.*

Assume f is not injective for all $\varepsilon > 0$. I.e., there is a collection of numbers $(a_1', a_2', \dots, a_l')$ arbitrarily close to $(a_1, a_2, \dots, a_l)$ such that $K' = f(a_1', a_2', \dots, a_l')$ has a symmetry. Note that a symmetry of K' has to give a nontrivial permutation of the vertexes of K'. Passing to the limit as $a_i' \to a_i$ for each i, we get that K has a symmetry, a contradiction.

The condition ❷ follows from Exercise 3.2.4. □

15.4 Outline of the proof

Let us denote by $\mathbf{R}_k$ the subset of $\mathbf{P}_k$ formed by isometry classes of realizable spaces. As was already mentioned, it is sufficient to prove that

❶ $\mathbf{R}_k = \mathbf{P}_k$ *for any integer* $k \geqslant 3$.

In the proof, we use the following lemmas.

15.4.1. Closed lemma. $\mathbf{R}_k$ *is a closed subset of* $\mathbf{P}_k$.

Let us denote by $\mathbf{P}'_k$ the subspace of $\mathbf{P}_k$ formed by the isometry classes of spaces without symmetry.

15.4.2. Exercise. $\mathbf{P}'_k$ *is an open dense*[1] *subspace of* $\mathbf{P}$.

Further, set $\mathbf{R}'_k = \mathbf{R}_k \cap \mathbf{P}'_k$; i.e. $P \in]\mathbf{R}'_k[$ if P is a realizable space without symmetry from $]\mathbf{P}_k[$.

15.4.3. Open lemma. $\mathbf{R}'_k$ *is an open subset of* $\mathbf{P}'_k$.

15.4.4. Connecting lemma. *???Any two points in the space* $\mathbf{P}_k$ *can be connected by a path. Moreover, any path in* $\mathbf{P}_k$ *with endpoints in* $\mathbf{P}'_k$ *can be deformed into a path in* $\mathbf{P}'_k$

??? $\mathbf{P}'_k$ *is connected; i.e.,* $\varnothing$ *and whole* $\mathbf{P}'_k$ *are the only subsets of* $\mathbf{P}'_k$ *which are open and closed at the same time.*

Proof of Existence theorem modulo the lemmas above. From the Closed and Open lemmas, it follows that $\mathbf{R}'_k = \mathbf{R}_k \cap \mathbf{P}'_k$ is a subset of $\mathbf{P}'_k$ which is closed and open at the same time. Clearly $\mathbf{R}'_k$ is nonempty. Therefore, by the Connecting lemma, $\mathbf{R}'_k = \mathbf{P}'_k$. Finally, applying Exercise 15.4.2 and the Closed lemma again, we get $\mathbf{R}_k = \mathbf{P}_k$. □

In the next section we prove the Closed and Open lemmas. The Connecting lemma is proved in sections 15.6 and 15.7. The proof of the connecting lemma goes by induction on k; it is the hardest part of the remaining part of the proof.

15.5 Closed and open lemmas

Proof of Closed lemma. We need to show that P_n is a sequence of realizable spaces from $]\mathbf{P}_k[$ which is converging in the sense of Gromov–Hausdorff to a space $P_\infty \in]\mathbf{P}_k[$ then P_∞ is realizable.

[1] i.e., the closure of $\mathbf{P}'_k$ is $\mathbf{P}_k$.

15.5.1. Exercise. *The diameter of convex polyhedron does not exceed the diameter of its surface.*

For each P_n, consider $K_n \in]\mathbf{K}_k[$ such that the surface of K_n is isometric to P_n. Note that diameter of all P_n is bounded above by some real constant R. According to the exercise, we may assume that all K_n lie in a fixed ball of radius R in $\mathbb{R}^3$. Therefore by Blaschke's compactness theorem (2.2.5) we may choose a subsequence of K_n which converge in the sense of Hausdorff to say K_∞. Clearly K_∞ is a polyhedron with at most k vertices. According to Problem 4.C, the surface of K_∞ is isometric to P_∞. In particular, by ???, K_∞ has exactly k vertices and hence P_∞ is realizable. $\square$

The proof of Open lemma relies on lemmas 15.2.2, 15.3.1 and the following theorem; its rigorous proof of it given in Section 20.2.

15.5.2. Domain invariance theorem. *If Ω is an open subset of $\mathbb{R}^n$ and $f\colon \Omega \to \mathbb{R}^n$ is an injective continuous map, then the image $f(\Omega)$ is open.*

Note that from Exercise 1.4 it follows that f is a homeomorphism from Ω to $f(\Omega)$. So this theorem might seem trivial, since a homeomorphic mapping of one space onto another always takes open sets into open sets by the definition of homeomorphism. However, what was said above implies only that the image of an open set is open as a subset of $f(\Omega)$, but not as a subset of $\mathbb{R}^n$. For example, under the identity mapping $[0, \infty) \to \mathbb{R}$ the image is not an open subset of $\mathbb{R}$, although it is open as a subset of $[0, \infty)$.

Proof of Open lemma. Consider map $\Phi_k\colon \mathbf{K}_k \to \mathbb{P}_k$ which sends congruence class $[K] \in \mathbf{K}_k$ to the isometry class of its surface.

According to Exercise ???, Φ_k is continuous and by Alexandrov uniqness theorem Φ_k is injective. If $[K] \in \mathbf{K}'_k$ then according to Problem ???, $\Phi_k[K] \in \mathbb{P}'_k$. By Lemma ???, we can choose small neighborhoods $U \ni [K]$ in $\mathbf{K}'_k$ and $V \ni [K]$ in $\mathbb{P}'_k$ such that both U and V are homeomorphic to open sets in $\mathbb{R}^{3\cdot n-6}$ and $\Phi(U) \subset V$. By Domain invariance theorem (15.5.2), $\Phi_k(U)$ is open subset of V. In particular, the image $\Phi_k(\mathbf{K}'_k)$ is open. $\square$

15.6 Deformation by patching

In this section we describe a construction which will be used in the next section to prove Connecting lemma (15.4.4).

Deformation of metric. Let P be a compact metric space space; as usual, denote by $|{*}-{*}|$ the metric on P.

A family of metrics $|{*}-{*}|_t$, $t \in [0,1]$ on P will be called *deformation of P* if $|{*}-{*}|_0 = |{*}-{*}|$ and

$$(t,x,y) \mapsto |x-y|_t$$

is a continuous function on $[0,1]\times P\times P$.

Set $P_t = (P, |{*}-{*}|_t)$; i.e., P_t is a metric space with underlying space P and the metric $|{*}-{*}|_t$, in particular $P_0 = P$.

Note that the family of metric spaces P_t is continuous in the sense of Gromov–Hausdorff; i.e., for any $t_0 \in [0,1]$ we have

$$t \to t_0 \quad \Rightarrow \quad P_t \xrightarrow[\text{GH}]{} P_{t_0}.$$

Deformations and continuous families. To make a deformation from a one parameter family of metric spaces P_t one has to produce a homeomorphism $h_t\colon P_0 \to P_t$ for each t, such that

$$(t,x,y) \mapsto |h_t(x)-h_t(y)|_{P_t}$$

is a continuous function on $[0,1]\times P_0\times P_0$.

Thus one may think of deformation as about a family of metric spaces P_t with a family of homeomorphisms $h_t\colon P_0 \to P_t$ as above.

Note that not any one parameter family of metric spaces which is continuous in the sense of Gromov–Hausdorff forms a deformation.

Now assume in addition that $P_t \in \,]\mathbf{P}_k[$ for any $t \in [0,1]$. Fix a homeomorphism $\mathbb{S}^2 \to P_0$; so we may think that $P_t = (\mathbb{S}^2, |{*}-{*}|_t)$ for all t. Note that in this case, one can label vertices of P_t, say as $v_t^1, v_t^2, \dots, v_t^k$ in such a way that $t \mapsto v_t^i$ forms a path in $\mathbb{S}^2$. Moreover this labeling is unique up to permutation of the upper indexes.

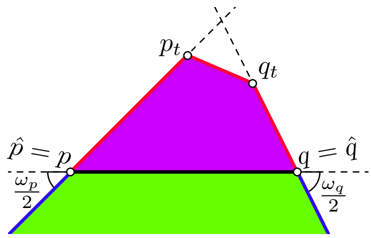
15.6.1. Exercise. *Prove the last two statements.*

This labeling gives a bijection between the vertices of P_0 and P_t, which will be referred further as the *vertex correspondence of the deformation.* We also will say that the vertex $v_t^i \in P_t$ *corresponds* to vertex $v^i \in P$.

Let us use the introduced notation to formulate a slightly stronger version of Connecting Lemma (15.4.4), which turns out to be easier to prove using induction by k.

Further we say that a *deformation* $(P, |{*}-{*}|_t)$ *is from P to P'* if $P \overset{iso}{=\!=} (P, |{*}-{*}|_0)$ and $P' \overset{iso}{=\!=} (P, |{*}-{*}|_1)$.

Patching construction. Choose any two vertices p and q in P denote their curvatures by ω_p and ω_q.



Cut P along an arbitrary geodesic $[pq]$. In the obtained hole, we will glue a one-parameter family of patches A_t; this way we obtain a deformation P_t of P.

Consider a one parameter family of convex quadrilaterals $[\hat{p}p_tq_t\hat{q}]$ in the plane such that

❶ $$\measuredangle p_t\hat{p}\hat{q} = \tfrac{\omega_q}{2}, \quad \measuredangle \hat{p}\hat{q}q_t = \tfrac{\omega_q}{2}$$

and $|\hat{p}-\hat{q}|_{\mathbb{R}^2} = |p-q|_P$. We assume that the points p_t and q_t depend continuously on the parameter $t \in [0,1]$ and $p_0 = \hat{p}$, $q_0 = \hat{q}$; so at $t = 0$ the quadrilateral degenerates to the segment $[\hat{p}\hat{q}]$.

To construct the patch A_t, take two copies of $[\hat{p}p_tq_t\hat{q}]$ and identify them along corresponding points on the sides $[\hat{p}p_t]$, $[p_tq_t]$, $[q_t\hat{q}]$. Now glue A_t in the hole of P so that $\hat{p}$ is glued to p and $\hat{q}$ to q. Denote by P_t the obtained space.

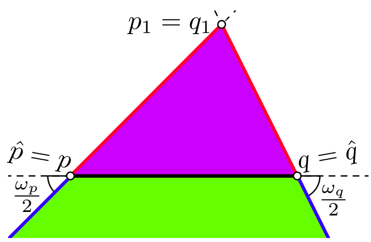
15.6.2. Exercise. *Prove that P_t is a deformation of P by constructing the needed one parameter family of homeomorphisms $P \to P_t$.*

The polyhedral space P_t has vertices at the points p_t and q_t, it also inherits every vertex of P except p and q. To see this, notice that a neighborhood of $[pq]$ in P is isometric to two copies of the green quadrilateral on the picture, with identified the corresponding points on the sides marked by blue and black. The patch A_t is isometric to two copies of the purple quadrilateral, with identified corresponding points on the sides marked by red. After cutting P along $[pq]$ (the black edge on the picture) and gluing in the patch A_t, it will look like two copies of the union of two quadrilaterals, green and purple, with the corresponding points on the sides marked by blue and red identified.

Case of small curvature. If in addition

$$\omega_p + \omega_q < 2{\cdot}\pi$$

then one can choose $|p_1 - q_1|$ arbitrary small and even equal zero.



In the later case the quadrilateral degenerates to a triangle; so P_1 has $k-1$ vertices, one less than P_0.

Now we will use the patching construction to prove the following Lemma. Briefly, it states that if $k \geqslant 4$, then any space P_0 with isometry class in $\mathbf{P}_k$

and two marked vertices p_0, q_0 can be deformed in $\mathbf{P}_k$ so that the the vertices corresponding to p_0 and q_0 get much closer than any other pair of vertices of and their curvatures getting small.

15.7 Connecting lemma; first part

In this section we use the patching construction described above to prove the following lemma.

15.7.1. Lemma. *Given $P, P' \in]\mathbf{P}_k[$, there is a deformation P_t from P to P' such that $P_t \in]\mathbf{P}_k[$ for all t.*

Moreover:

a) Any bijection between vertices of P and P' can be realized as the vertex correspondence of the deformation P_t.

b) ??? The space P_t has no symmetry for any $t \neq 0, 1$.

First we prove the following.

15.7.2. Lemma. *Let $k \geqslant 4$, $\varepsilon > 0$, $P_0 \in]\mathbf{P}_k[$ and p_0, q_0 be two vertices of P_0. Then there is a deformation $P_t \in]\mathbf{P}_k[$ from P_0 to P_1 such that*

a) The corresponding vertices p_1 and q_1 lie on the distance at least 10 times smaller than the distance between any other pair of vertices in P_1

b) The curvatures of P_1 at p_1 and q_1 are smaller than ε.

Proof. Note that the patching construction makes possible to a move any portion of curvature from one vertex to an other as far as at each vertex the curvature is in the range $(0, 2{\cdot}\pi)$. I.e., given two vertices x_0 and y_0 in P_0 we can construct a deformation P_t with corresponding vertices x_t and y_t such that the curvatures ω_{x_1} and ω_{y_1} take any values in $(0, 2{\cdot}\pi)$ such that

$$\omega_{x_1} + \omega_{y_1} = \omega_{x_0} + \omega_{y_0}.$$

I claim that almost all curvature of P_1 can be moved to a given two vertices. This can be easily arranged by applying the patching construction recursively; i.e.,

- ◇ Start with P_0; choose a pair of vertices and construct a deformation P_t.
- ◇ Choose an other pair of vertices in P_1 and apply the construction again to obtain a deformation from P_1 to P_2 and $t \in [1, 2]$.
- ◇ Repeat this as many times as necessary.
- ◇ After n times you get a "long" deformation P_t with $t \in [0, n]$, which can be linearly reparametrized by $[0, 1]$.

Therefore, if $k \geqslant 4$, then almost all curvature can be moved to the vertices different form p_1 and q_1, making the curvatures at p_1 and q_1 arbitrary small. Once it is done we may apply the case of small curvature of the patching construction to meet the condition on the distance $|p_1 - q_1|_{P_1}$. □

Proof of Lemma 15.7.1. I will apply induction on k.

Base case, $k = 3$. Let u, v and w be the vertices of P and u', v' and w' be the vertices of P'. Assume we want to construct a deformation P_t for which u corresponds to u', v corresponds to v' and w corresponds to w'.

Cut P by geodesics $[uv]$, $[vw]$ and $[wu]$. Note that according to Exercise 14.3.2, these geodesics intersect only at the common vertices. Therefore after the cutting, we obtain two congruent flat triangles. It follows that P is completely determined by three distances $a = |u - v|_P$, $b = |v - w|_P$ and $c = |w - u|_P$.

Repeat the same procedure for P', we obtain three numbers a', b' and c'.

Set

$$a_t = (1-t)\cdot a + t\cdot a' \qquad b_t = (1-t)\cdot b + t\cdot b' \qquad c_t = (1-t)\cdot c + t\cdot c'.$$

The both triples a, b, c and a', b', c' satisfy the strict triangle inequality. Therefore the same holds for the triple a_t, b_t, c_t for any $t \in [0, 1]$.

Now, for each $t \in [0, 1]$, glue the space P_t from two copies of triangle with sides a_t, b_t and c_t. denote by u_t, v_t and w_t the corresponding vertices of P_t.

To make a deformation from the family P_t, choose a homeomorphism $P \to P_t$ which sends linearly triangle to triangle and such that $u \mapsto u_t$, $v \mapsto v_t$ and $w \mapsto w_t$.

Induction step. Fix $k \geqslant 4$ and assume that the lemma is already proved if the number of vertices is strictly less then k.

Applying Lemma 15.7.2, we may assume that P has two vertices p and q with very small curvatures and on distance at least 10 times smaller than the distance between any other pair of vertices in P. Analogously, we may assume in P' the corresponding (by the given bijection) vertices p' and q' satisfy the same condition.

Let us apply the case of small curvature of the patching construction for P. I.e., let us cut P along $[pq]$ and glue in the hole a patch glued from two congruent triangles. This way we obtain a new polyhedral space, say Q, which isometry class belongs to $\mathbf{P}_{k-1}$. In the obtained space Q, the points p and q become regular, instead we get an other vertex which will be denoted by z. Note that there are unique geodesics $[zp]$ and $[zq]$ and they cut at z two equal angles.

Let $Q' \in]\mathbf{P}_{k-1}[$ be the result of the same construction for P'; we denote by z' the vertex which appear in Q' instead of p' and q'.

Applying the induction hypothesis, we can find a deformation Q_t from Q to Q' such that z corresponds to z' and and the vertex correspondence for the rest of the vertices taken the same as the correspondence for P and P'. Denote by z_t the vertex of Q_t which corresponds to $z \in Q$.

Now we will use the obtained deformation Q_t to construct the needed deformation P_t. Note that we can choose points $p_t, q_t \in Q_t$ such that

(i) $p_0 = p$, $p_1 = p'$, $q_0 = q$, $q_1 = q'$.

(ii) For any t, each distance $|z_t - p_t|_{Q_t}$ and $|z_t - q_t|_{Q_t}$ is positive but smaller then the distance between any pair of vertices of Q_t

(iii) The (necessary unique) geodesics $[z_t p_t]$ and $[z_t q_t]$ cut two equal angles at z_t.

(iv) p_t, q_t depend continuously on t; i.e., for the one parameter family of homeomorphisms $h_t\colon Q \to Q_t$ of the deformation, the maps $t \mapsto h_t^{-1}(p_t)$ and $t \mapsto h_t^{-1}(q_t)$ are both continuous paths in Q.

Note that these conditions imply that the existence of two geodesics, say γ_1 and γ_2 from p_t to q_t which form a digon in Q_t containing z_t inside. Cut Q_t along the digon, in the remaining part (the one without z_t) glue the corresponding points on γ_1 and γ_2. We obtain a polyhedral space P_t; the points p_t and q_t become vertices in P_t.

By construction, $P_0 = P$ and $P_1 = P'$. It remains to construct a homeomorphism $P \to P_t$ for each t which makes a deformation from the family P_t. The later is left to the reader. □

15.8 ???Braking the symmetry

It remains to show that ???. Note that any symmetry of P_t induce a nontrivial permutation of its vertices. In particular, if the curvatures of all vertices are different then the space has no symmetry. If both P and P' have no symmetry, it is easy to modify the above deformation in such a way that all curvatures are different for all values with exception for finite number of values $t_1, t_2, \dots, t_n$ and at each t_i there is exactly one pair of vertices with the same curvature. If P_{t_i} has a symmetry then the induced permutation only changes the places of these two vertices.

15.9 Remarks

The rest of the proof presented here is nearly the same as in original one.

There is an alternative, very interesting proof due to Volkov, ???.

Connecting lema. The patching construction described below was introduced by Alexandrov in ???; it was much later than his publiction of Existence theorem ???. It seems that I was the first to realize that this construction simplifies the the proof of Alexandrov existence theorem.

The Connecting lemma has a simple but not elementary proof using Teichmüller theory.

Here is the idea. Let P be a polyhedral space with exactly k vertices. Note that P with all its vertices removed admits natural complex structure. It describes a map, say Ψ_k from $\mathbf{P}_k$ to the Teichmüller space $\mathcal{T}_k$ which is the space of that the space of conformal structures on sphere with k points removed. (The map consists of forgetting the metric, and remembering only the conformal structure.)

The space $\mathcal{T}_k$ is homeomorphic to the space of configurations of k points in the $\mathbb{S}^2$ up to a conformal map $\mathbb{S}^2 \to \mathbb{S}^2$. It is easy to see that $\mathcal{T}_k$ is connected.

Further the preimage $\Psi_k^{-1}(T) \subset \mathbf{P}_k$ is also connected for any $T \in \mathcal{T}_k$. To prove the later statement, one has to observe that P can be uniquely recovered if one knows $T = \Psi_k(P)$, the curvatures at each vertex and the total area of P.

Exercises

15.A. Consider a regular octahedra H, with vertices a, a', b, b', c, c' and assume that the pairs (a, a'), (b, b') and (c, c') are opposite.

Cut from H a pyramid with vertex a by a plane Π parallel to the plane containing b, b', c and c'. Denote by P the remaining surface of H; it is bounded by a broken line $xyx'y'$ which bounds a square in Π.

Consider the patch R_α, which is a planar rhombus with the same side length as $xyx'y'$ and with angle α at one vertex. Glue R_α to P along $xyx'y'$ by a length-preserving map of its boundary vertex-to-vertex. Denote the obtained space as P_α.

1. For which α does the space P_α has non-negative curvature? For such α denote by K_α the convex polyhedron with surface isometric to P_α.
2. For which pairs of α, α' are the polyhedra K_α and $K_{\alpha'}$ congruent?
3. Construct another polygonal patch (not a rhombus) which gives a space with non-negative curvature.

4. Characterize all such patches.

15.B. Let P be a non-negatively curved polyhedral metric on $\mathbb{S}^2$. Cut a triangle Δ from P along geodesics and equip it with the induced length metric. Show that Δ is isometric to a planar triangle if and only if the sum of its angles is π.

Hint: Use Problem 15.C to prove the "only if" part. To prove the "if" part, construct a distance preserving map explicitly.

15.C. Let P be a non-negatively curved polyhedral space homeomorphic to a sphere and let Δ be a triangle in P bounded by 3 geodesics. Denote by α, β and γ the angles of Δ. Show that $\alpha + \beta + \gamma - \pi$ is equal to the sum of curvatures of all points in the interior of Δ.

In particular, the sum of the angles of any triangle in P is at least π.

Hint: Pass to the doubling of Δ and apply 5.6.4 together with Exercise 14.3.2.

15.D. Let P be the surface of a regular tetrahedron. Find a periodic local geodesic[2] in P. Show that any two distinct vertices can be joined by arbitrary long local geodesic.

15.E. Let P be a non-negatively curved polyhedral space homeomorphic to a sphere with exactly 4 vertices a, b, c and d. Let us draw on P a geodesic between each pair of vertices $[ab]$, $[bc]$, $[cd]$ and $[da]$. Show that either these geodesics intersect only at the common ends, or exactly two of them intersect at an interior point.

In the latter case, show that P is isometric to the doubling of a convex quadrilateral.

(You may use Alexandrov's theorem, doing this problem directly is harder.)

15.F. Show that $\mathbf{K}_k$ is connected.[3]

???

Base. $\mathbf{R}_3 = \mathbf{P}_3$.

Proof. Assume that $P \in]\mathbf{P}_3[$ has exactly three vertices, say u, v, and w. It is sufficient to show that

❶ *P is isometric to a doubling of a planar triangle. i.e., the surface of this triangle in $\mathbb{R}^3$.*

---

[2] see Definition **??**

[3] Note that Alexandrov's theorem implies that $\mathbf{P}_k$ is also connected.

Choose geodesics $[uv]$, $[vw]$ and $[wu]$ between each pair of points. According to Exercise 14.3.2 these geodesics do not intersect each other at the interior points.

Cut P along the geodesics $[uv]$, $[vw]$ and $[wu]$. As a result we get two congruent flat triangles Δ and Δ'. Hence ❶ follows. $\square$

Chapter 16

Curvature bounded below

16.1 Definitions

16.1.1. Definition. *A proper length space X has non-negative curvature in the sense of Alexandrov (briefly $X \in \mathrm{CBB}[0]$[1]) if any triangle in X is fat (see Definition 4.5.1).*

Note that Euclidean space, as well as any convex set in the Euclidean space form a CBB[0] space.

The following theorem gives a number of equivalent ways to define CBB[0] spaces.

16.1.2. Theorem. *Let X be a proper length space. Then the following are equivalent.*

a) $X \in \mathrm{CBB}[0]$.

b) ((1+3)-point comparison) if p is distinct from x, y, and z then

$$\tilde\measuredangle(p\,{}^x_y) + \tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^z_x) \leqslant 2\cdot\pi.$$

c) (adjacent angle comparison) for any geodesic $[xy]$ and $p \in\ ]xy[$, $z \neq p$ we have

$$\tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^z_x) \leqslant \pi.$$

[1] CBB[0] stays for "curvature bounded below by 0". If in the definition of model triangle, one exchanges the Euclidean plane with a sphere or Lobachevsky plane of constant curvature k, then one gets the definition of *spaces with curvature $\geqslant k$ in the sense of Alexandrov*, which denoted as $\mathrm{CBB}[k]$. We will only consider the case $k = 0$.

d) (hinge comparison) for any hinge $[x\,{}^y_z]$, *the angle* $\measuredangle[x\,{}^y_z]$ *is defined and*

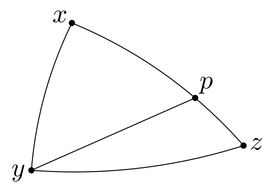
$$\measuredangle[x\,{}^y_z] \geqslant \tilde\measuredangle(x\,{}^y_z).$$

Moreover, if $p \in\,]xy[$, $z \neq p$ *then*

$$\measuredangle[p\,{}^y_z] + \measuredangle[p\,{}^z_x] \leqslant \pi$$

for any two hinges $[p\,{}^z_y]$ *and* $[p\,{}^z_x]$ *with common side* $[pz]$.[2]

Proof of Theorem 16.1.2. Note that X is a geodesic space[3], as it is proper and length (see Exercise 4.3.2).



(b) ⇒ *(c).* Since $p \in\,]xy[$, we have $\tilde\measuredangle(p\,{}^x_y) = \pi$. Thus, (1+3)-point comparison

$$\tilde\measuredangle(p\,{}^x_y) + \tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^z_x) \leqslant 2\cdot\pi$$

implies

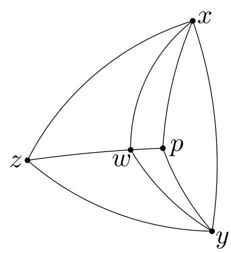
$$\tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^z_x) \leqslant \pi.$$

(c) ⇔ *(a).* Follows directly from Alexandrov's lemma (4.5.2).

(c) + (a) ⇒ *(d).* From Proposition **??**, we get that for $\bar y \in\,]xy]$ and $\bar z \in\,]xz]$ the function $(|x-\bar y|, |x-\bar z|) \mapsto \tilde\measuredangle(x\,{}^{\bar y}_{\bar z})$ is nonincreasing in each argument. In particular, $\measuredangle[x\,{}^y_z] = \sup\{\tilde\measuredangle(x\,{}^{\bar y}_{\bar z})\}$. Thus, $\measuredangle[x\,{}^y_z]$ is defined and it is at least $\tilde\measuredangle(x\,{}^y_z)$.

From above and (c), it follows that $\measuredangle[p\,{}^y_z] + \measuredangle[p\,{}^z_x] \leqslant \pi$.

(d) ⇒ *(b).* Consider a point $w \in\,]pz[$ close to p. From (d), it follows that



$$\measuredangle[w\,{}^x_z] + \measuredangle[w\,{}^x_p] \leqslant \pi \quad\text{and}\quad \measuredangle[w\,{}^y_z] + \measuredangle[w\,{}^y_p] \leqslant \pi.$$

By the triangle inequality for angles (see 4.6.2), we have $\measuredangle[w\,{}^x_y] \leqslant \measuredangle[w\,{}^x_p] + \measuredangle[w\,{}^y_p]$. Therefore we get

$$\measuredangle[w\,{}^x_z] + \measuredangle[w\,{}^y_z] + \measuredangle[w\,{}^x_y] \leqslant 2\cdot\pi.$$

Applying the first inequality in (d), we obtain

$$\tilde\measuredangle(w\,{}^x_z) + \tilde\measuredangle(w\,{}^y_z) + \tilde\measuredangle(w\,{}^x_y) \leqslant 2\cdot\pi.$$

[2] It is not known if the last inequality is necessary, even for spaces homeomorphic to $\mathbb{S}^2$.

[3] I.e., any two points in X can be joined by a geodesic.

Passing to the limits $w \to p$, we obtain

$$\tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{y}_{z}) + \tilde\measuredangle(p\,{}^{x}_{y}) \leqslant 2\cdot\pi. \qquad \square$$

16.1.3. Proposition. *Let (X_n) be a sequence of compact* CBB[0] *spaces which converge to a compact space X_∞ in the sense of Gromov–Hausdorff.*

Then $X_\infty \in$ CBB[0].

Proof. According to Exercise 4.2.2, X_∞ is a length space.

For each natural number n, choose $f_n\colon X_n \to X_\infty$ to be an ε_n-isometry for some sequence $\varepsilon_n \to 0$ as $n \to \infty$. Take arbitrary points $p_\infty, x_\infty, y_\infty, z_\infty \in X_\infty$ and choose points $p_n, x_n, y_n, z_n \in X_n$ such that $f_n(p_n) \to p_\infty$, $f_n(x_n) \to x_\infty$, $f_n(y_n) \to y_\infty$, $f_n(z_n) \to z_\infty$ as $n \to \infty$.

Since $X_n \in$ CBB[0], from 16.1.2*b* we have

$$\tilde\measuredangle(p_n\,{}^{x_n}_{y_n}) + \tilde\measuredangle(p_n\,{}^{y_n}_{z_n}) + \tilde\measuredangle(p_n\,{}^{z_n}_{x_n}) \leqslant 2\cdot\pi.$$

Clearly,

$$\tilde\measuredangle(p_n\,{}^{x_n}_{y_n}) \to \tilde\measuredangle(p_\infty\,{}^{x_\infty}_{y_\infty}),\ \tilde\measuredangle(p_n\,{}^{y_n}_{z_n}) \to \tilde\measuredangle(p_\infty\,{}^{y_\infty}_{z_\infty}),\ \tilde\measuredangle(p_n\,{}^{z_n}_{x_n}) \to \tilde\measuredangle(p_\infty\,{}^{z_\infty}_{x_\infty})$$

as $n \to \infty$. Therefore

$$\tilde\measuredangle(p_\infty\,{}^{x_\infty}_{y_\infty}) + \tilde\measuredangle(p_\infty\,{}^{y_\infty}_{z_\infty}) + \tilde\measuredangle(p_\infty\,{}^{z_\infty}_{x_\infty}) \leqslant 2\cdot\pi.$$

The proposition follows from 16.1.2*b*. $\square$

16.2 ???Surface of convex body

The Theorem 14.3.3 together with Exercise 14.3.1 implies that that surface of any convex polyhdron is a CBB[0] space. Further, applying Problem 4.C and Proposition 16.1.3, we get the following.

16.2.1. Proposition. *The surface of a convex body in $\mathbb{R}^3$ is a* CBB[0] *space.*

In this section we prove the following theorem, which is a convese of the above proposition.

16.2.2. Theorem. *A metric space X is isometric to the surface of a convex body*[4] *if and only if X is a* CBB[0]*-space which is homeomorphic to $\mathbb{S}^2$.*

[4] We allow a convex body to degenerate to a planar figure but not to a segment. As in the case of convex polyhedra, the surface of a planar figure is defined as its doubling.

The following proposition gives the main step in the proof of the theorem above.

16.2.3. Proposition. *Given a* CBB[0] *space X homeomorphic to a sphere, there is a non-negatively curved polyhedral space $\tilde X$ homeomorphic to $\mathbb{S}^2$ for which the Gromov–Hausdorff distance $|X-\tilde X|_{\mathcal{M}}$ arbitrarily small.*

Proof with cheating. In the proof we will use two claims without proof; the proofs are not hard but tedious.

❶ *Given $\varepsilon>0$, there is a triangulation $\mathcal{T}$ of X with edges formed by geodesics and the diameter of each triangle is less than ε.*

Fix small $\varepsilon>0$ and choose a triangulation $\mathcal{T}$ of X provided by the claim ❶.

16.2.4. Exercise. *The sum of the angles at one of the vertices of $\mathcal{T}$ is $\leqslant 2{\cdot}\pi$.*

For each triangle in $\mathcal{T}$, construct a model triangle and glue them together the same way as the corresponding triangles in X. Denote the obtained polyhedral space by $\tilde X$, and given a vertex v_i of $\mathcal{T}$, we will denote by $\tilde v_i$ the corresponding point in $\tilde X$. Applying the hinge comparison (16.1.2d) we have that the sum of the angles around a vertex $\tilde v_i$ of $\tilde X$ is at most as big as the sum of the angles around the corresponding vertex v_i in X. Hence $\tilde X$ is non-negatively curved.

Now to prove the proposition, it is sufficient to show that if $\varepsilon>0$ is small enough then $\tilde X$ is sufficiently close to X.

First note that the set of vertices $\{v_i\}$ forms an ε-net in X and the set of vertices $\{\tilde v_i\}$ forms an ε-net in $\tilde X$. Therefore it is sufficient to show that the inequalities

❷ $$|\tilde v_i-\tilde v_j|_{\tilde X}\leqslant|v_i-v_j|_X$$

❸ $$|v_i-v_j|_X-4{\cdot}\pi{\cdot}\varepsilon\leqslant|\tilde v_i-\tilde v_j|_{\tilde X}$$

hold for any i and j.

To prove ❷, consider a geodesic $[v_iv_j]$ in X. Let $v_i=x_0,x_1,\dots,x_n=v_j$ be the points of intersection of $[v_iv_j]$ with the edges of triangulation listed in the order from v_i to v_j.[5]

Fix $k\in\{1,\dots,n\}$. Let $[pqr]$ be the triangle in $\mathcal{T}$ which contains $[x_{k-1}x_k]$ inside. Without loss of generality, we can assume that $x_{k-1}\in[pq]$ and $x_k\in$

[5]Note that according problem 16.A and 16.C, if a geodesic $[v_iv_j]$ intersect an edge at two points then it contains this edge. In this case one can take as x any point on this edge. Taking this into account, we have that n is finite.

$[pr]$. Applying definition of CBB[0] spaces twice, first for the triangle $[pqr]$ and $x_k \in [pr]$ and then for the triangle $[pqx_k]$ and $x_{k-1} \in [pq]$ we get that

$$|\tilde{x}_k - \tilde{x}_{k-1}|_{\tilde{X}} \leqslant |x_k - x_{k-1}|_X$$

holds for each k. Summing up, we get ❷.

The proof of ❸ is similar, but in this part we use yet the following claim without proof.

❹ *There is a constant $\varrho = \varrho(X) > 0$ such that for all small $\varepsilon > 0$, the Gromov–Hausdorff distance from $\tilde{X}$ to any real segment is at least ϱ.*

Consider a geodesic $[\tilde{v}_i\tilde{v}_j]$ in $\tilde{X}$. Let $\tilde{v}_i = \tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_m = \tilde{v}_j$ be the points of intersection of $[\tilde{v}_i\tilde{v}_j]$ with the edges of triangulation listed in the order from $\tilde{v}_i$ to $\tilde{v}_j$. Assume $[\tilde{v}_i\tilde{v}_j]$ crosses twice one edge of $\mathcal{T}$. Say $[\tilde{v}_i\tilde{v}_j]$ crosses edge ℓ at y_k and y_m. Since the length of ℓ is smaller than ε and $[\tilde{v}_i\tilde{v}_j]$ is a geodesic, we get that $|y_k - y_m|_{\tilde{X}} < \varepsilon$. Hence the geodesic $[y_k y_m]$ together with the segment of ℓ from y_k to y_m forms a closed curve on $\tilde{X}$ of length $< 2{\cdot}\varepsilon$.

According to Claim **??**, $[\tilde{v}_i\tilde{v}_j]$ intersects each triangle at most once.

Fix $k \in \{1, \dots, n\}$. Let $[\tilde{p}\tilde{q}\tilde{r}]$ be the triangle in $\tilde{X}$ which contains $[\tilde{y}_{k-1}\tilde{y}_k]$ inside. Without loss of generality, we can assume that $\tilde{y}_{k-1} \in [\tilde{p}\tilde{q}]$ and $\tilde{y}_k \in [\tilde{p}\tilde{r}]$. Set

$$\begin{aligned}
\alpha &= \measuredangle[p\,{}^{q}_{r}], & \beta &= \measuredangle[q\,{}^{p}_{r}], & \gamma &= \measuredangle[r\,{}^{p}_{q}],\\
\tilde{\alpha} &= \tilde{\measuredangle}(p\,{}^{q}_{r}) & \tilde{\beta} &= \tilde{\measuredangle}(q\,{}^{p}_{r}) & \tilde{\gamma} &= \tilde{\measuredangle}(r\,{}^{p}_{q})
\end{aligned}$$

By hinge comparison, (16.1.2d) we have

$$\alpha \geqslant \tilde{\alpha}, \qquad \beta \geqslant \tilde{\beta}, \qquad \gamma \geqslant \tilde{\gamma}$$

For the triangle $\Delta = [pqr]$ we define its *curvature* as

$$\kappa(\Delta) = \alpha + \beta + \gamma - \pi = \alpha + \beta + \gamma - (\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}).$$

From the above $\kappa(\Delta) \geqslant 0$. Together with the rule of cosines, straightforward estimates give the following:

$$\begin{aligned}
|y_k - y_{k-1}|_X &\leqslant \sqrt{|p - y_{k-1}|^2 + |p - y_k|^2 - 2|p - y_{k-1}|{\cdot}|p - y_k|{\cdot}\cos\alpha}\\
&\leqslant |\tilde{y}_k - \tilde{y}_{k-1}|_{\tilde{X}} + \varepsilon{\cdot}(\alpha - \tilde{\alpha})\\
&\leqslant |\tilde{y}_k - \tilde{y}_{k-1}|_{\tilde{X}} + \varepsilon{\cdot}\kappa(\Delta).
\end{aligned}$$

Summing it up, we get

$$\begin{aligned}|v_i - v_j|_X &\leqslant \sum_{k=1}^{m} |y_k - y_{k-1}|_X \\ &\leqslant \sum_{k=1}^{m} |\tilde{y}_k - \tilde{y}_{k-1}|_{\tilde{X}} + \varepsilon\cdot \sum_{\Delta} \kappa(\Delta).\end{aligned}$$

where the last sum is taken over all triangles Δ in $\mathcal{T}$. Hence ❸ boils down to the inequality

$$\sum_{\Delta \text{ in } \mathcal{T}} \kappa(\Delta) \leqslant 4\cdot\pi.$$

16.2.5. Exercise. *Prove the last inequality.*[6] □

Theorem 16.2.2 and Proposition **??** will be proved next week.

Proof of Theorem 16.2.2

"Only if" part. Assume X is a surface of a convex body B (B might degenerate to a flat figure, but not a line segment). The same argument as in Proposition 14.1.1, shows that X is homeomorphic to $\mathbb{S}^2$. The convex body B can be approximated by a sequence of convex polyhedra K_n in the sense of Hausdorff. The proof of the last statement is the same as in Lemma 2.3.6.

Denote by P_n the surface of K_n. According to Problem 4.C, P_n converge to X in the sense of Gromov–Hausdorff. Applying Propositions 16.1.3 and **??**, we get that $X \in \mathrm{CBB}[0]$.

"If" part. Assume X is a $\mathrm{CBB}[0]$ space which is homeomorphic to a sphere. According to Proposition 16.2.3, there is a sequence of non-negatively curved polyhedral spaces P_n which converge to X in Gromov–Hausdorff sense and such that each P_n is homeomorphic to $\mathbb{S}^2$.

Applying Alexandrov's existence theorem (15.1.1), we obtain a sequence of convex polyhedra K_n such that the surface of K_n is isometric to P_n for each n. Note that for all n the diameter of K_n is bounded by the diameter of P_n. Since $P_n \to X$ in the sense of Gromov–Hausdorff we have that $\operatorname{diam} P_n \to \operatorname{diam} X$ as $n \to \infty$ because diam is a continuous function (see Exercise 3.3.2.) In particular, $\operatorname{diam} P_n \leqslant C$ for some fixed constant C and all n.

[6] Hint: Use Exercise 16.2.4 and Euler's formula the same way as Exercise 5.6.4.

Without loss of generality we may assume that each K_n contains the origin $0 \in \mathbb{R}^3$. Therefore K_n lies in a fixed bounded region for all large n. Applying Blaschke's compactness theorem (2.2.5), we can pass to a Hausdorff-converging subsequence of K_n. Denote by B its limit.

According to Problem 4.C, X is isometric to the surface of B. □

16.3 Comments

The Theorem 14.3.3 admits the following generalization to higher dimensions.

16.3.1. Theorem. *Let P be m-dimensional polyhedral space. Then $P \in$ CBB[0] if and only if each of the following conditions hold.*

a) Any simplex in P is a face in an m-dimensional simplex.
b) Any $(m-1)$-dimensional simplex in P is a face of one or two an m-dimensional simplecies.
c) The link of any simplex of dimension $\leqslant m-2$ is connected.
d) The angle around any simplex of dimension $m-2$ is $\leqslant 2{\cdot}\pi$.

Globalization theorem.

Applications in Riemannian geometry.

Exercises

16.A. Let $X \in$ CBB[0] and $[x\,{}^{y}_{z}]$ be a hinge in X. Assume $\measuredangle[x\,{}^{y}_{z}] = 0$. Show that either $[xy] \subset [xz]$ or $[xz] \subset [xy]$.

16.B. Let $X \in$ CBB[0]. Show that given three distinct points x, y and p in X, there is at most one geodesic from x to y which passes through p.

16.C. Let $X \in$ CBB[0] and $[z\,{}^{x}_{p}]$ and $[z\,{}^{y}_{p}]$ be two hinges with common side $[zp]$ in X. Assume that points p, x, y and z are distinct and $z \in [xy]$. Show that

$$\measuredangle[z\,{}^{p}_{y}] + \measuredangle[z\,{}^{p}_{x}] = \pi.$$

16.D. Let X be a CBB[0] space with is homeomorphic to $\mathbb{S}^2$. Assume that $\operatorname{diam} X = D$ and X contains a closed simple curve of length ε which cuts X into two domains of diameter at least R each. Set $\mathbb{I} = [0, D]$. Prove that

$$|X - \mathbb{I}|_{\mathcal{M}} \leqslant \tfrac{D}{R}{\cdot}\varepsilon.$$

Chapter 17

An application to crystallography

Here we apply spaces with non-negative curvature in the sence of Alexandrov to crystallography. The proof relies on the following problem in discrete geometry.

17.1 Erdős problem

17.1.1. Problem. *Assume $x_1, x_2, \dots, x_m$ is a collection of points in n-dimensional Euclidean space such that $\angle[x_i\,{}^{x_j}_{x_k}] \leqslant \frac{\pi}{2}$ for any distinct i, j and k. Show that $m \leqslant 2^n$ and moreover, if $m = 2^n$ then the x_i form the set of vertices of a right parallelepiped.*

This problem was posed by Erdős and solved by Danzer and Grünbaum.

Proof. Let K be the convex hull of $x_1, x_2, \dots, x_m$. Without loss of generality we may assume that K is a non-degenerate convex polyhedron. (Otherwise, instead of $\mathbb{R}^n$ take the minimal subspace which contain K; its dimension has to be $< n$.)

First let us prove the following

❶ *$\measuredangle[x_i\,{}^{v}_{w}] \leqslant \frac{\pi}{2}$ for each i and any $v, w \in K$.*

Indeed, assume contrary. For fixed x_i and v, the set of points H_v containing x_i and all $w \in \mathbb{R}^n$ such that $\measuredangle[x_i\,{}^{v}_{w}] \leqslant \frac{\pi}{2}$ is a half-space. Thus, if $\measuredangle[x_i\,{}^{v}_{w}] > \frac{\pi}{2}$ for some $w \in K$ then $K \not\subset H_v$ and so $x_j \notin H_v$ for some j; i.e., $\measuredangle[x_i\,{}^{v}_{x_j}] > \frac{\pi}{2}$ for some

j. Repeating the same argument for x_j instead of v, we get that $\measuredangle[x_i\,{}^{x_k}_{x_j}] > \frac{\pi}{2}$ for some j and k, a contradiction.

For each x_i denote by K_i the dilation of K with center x_i and coefficient $\frac{1}{2}$.

❷ *For any $i \neq j$, the polyhedra K_i and K_j have no common interior points. In particular,* $\mathrm{vol}(K_i \cap K_j) = 0$.

Assume there is an interior point v of $K_i \cap K_j$. Without loss of generality, we can assume that $|v - x_i| \neq |v - x_j|$. Then there are points $y_i, y_j \in K$ such that v is the midpoint of $[x_i y_i]$ and $[x_j y_j]$. Hence $x_i y_j y_i x_j$ is a parallelogram, therefore $\measuredangle[x_i\,{}^{x_j}_{y_j}] + \measuredangle[x_j\,{}^{x_i}_{y_i}] = \pi$. From ❶ we get that $\measuredangle[x_i\,{}^{x_j}_{y_j}] = \measuredangle[x_j\,{}^{x_i}_{y_i}] = \frac{\pi}{2}$. I.e., $x_i y_j y_i x_j$ is a rectangle and therefore $|v - x_i| = |v - x_j|$, a contradiction.

Clearly $\mathrm{vol}\, K_i = \frac{1}{2^n} \cdot \mathrm{vol}\, K$ and $K_i \subset K$ for each i. From ❷, we get

$$\sum_{i=1}^{m} \mathrm{vol}\, K_i \leqslant \mathrm{vol}\, K.$$

Hence the result. □

17.2 Isometric actions

Let X be a metric space. Denote by $\mathrm{Isom}\, X$ the set of all isometries of X. Let G be a nonempty subset of $\mathrm{Isom}\, X$ such that the following condition[1] holds:

◇ *Given two isometries $f, g \in G$ the composition $f \circ g$ as well as the inverse f^{-1} are in G.*

In this case we say that the *group G acts on X by isometries*. For example, one can take $G = \mathrm{Isom}\, X$ or $G = \{\mathrm{id}_X\}$.

In this case, given $g \in G$ and $x \in X$ the g-image of x will denoted by $g \cdot x$, that is, $g \cdot x = g(x)$. Consider the relation "$\sim$" on X such that $x \sim y$ if and only if there if there is $g \in G$ such that $g \cdot x = y$. Given $x \in X$, the G-orbit of x is defined as

$$G \cdot x = \{\, g \cdot x \mid g \in G. \}$$

Note that $G \cdot x = G \cdot y$ if and only if $x \sim y$; i.e., $\sim$ is an equivalence relation and G-orbits form the $\sim$-equivalence classes.

The set of G-orbits of X is denoted by X/G. Denote by $\pi \colon X \to X/G$ the natural surjective map, $\pi \colon x \mapsto G \cdot x$.

[1] In the language of group theory, $\mathrm{Isom}\, X$ is a group and G is a subgroup of $\mathrm{Isom}\, X$.

In the case that every G-orbit is a closed subset of X, then the set of orbits X/G can be equipped with the following metric

$$|\pi(x) - \pi(y)|_{X/G} \stackrel{def}{=\!=} \inf_{g \in G} \{|x - g{\cdot}y|_X\}.$$

17.2.1. Exercise. *Check that $|{*} - {*}|_{X/G}$ is a metric on X/G.*

The set X/G equipped with this metric is called the quotient space; for the quotient space, we keep the same notation X/G.

Note that

❶ $$|\pi(x) - \pi(y)| \leqslant |x - y|$$

for any $x, y \in X$; i.e. $\pi\colon X \to X/G$ is a distance non-expanding map.

Examples:

- ◇ $\mathbb{Z}$ acts on $\mathbb{R}$ by shifts $n{\cdot}x \stackrel{def}{=\!=} 2{\cdot}\pi{\cdot}n + x$; in this case $\mathbb{R}/\mathbb{Z}$ is isometric to $\mathbb{S}^1 \subset \mathbb{R}^2$ with the induced length metric.
- ◇ The set of all isometries of $\mathbb{R}$ which can be presented as a composition of a finite number of shifts $x \mapsto x + 1$ and reflections $x \mapsto -x$ forms a group, say G; in this case $\mathbb{R}/G$ is isometric to the interval $[0, \frac{1}{2}]$.
- ◇ $\mathbb{S}^1$ acts on $\mathbb{R}^2$ by rotations which fix the origin; in this case $\mathbb{R}^2/\mathbb{S}^1$ is isometric to the ray $[0, \infty)$.

17.2.2. Proposition. *Let X be a proper length space such that a group G acts on X by isometries and has closed orbits. Then X/G a proper length space.*

Proof.??? Given $\bar{x}$ and $\bar{y}$ in X/G, choose an arbitrary $x \in X$ such that $\pi(x) = \bar{x}$. Further, since X is proper and the orbits are closed, we can choose y such that $\pi(y) = \bar{y}$ and

$$|x - y| = |\bar{x} - \bar{y}|.$$

I.e. for any $\bar{y} \in \bar{B}_r(\bar{x})$ there is $y \in \bar{B}_r(x)$ such that $\pi(y) = \bar{y}$, or equivalently $\pi(\bar{B}_r(x)) = \bar{B}_r(\bar{x})$ for any $r > 0$. Since X is proper, $\bar{B}_r(x)$ is compact; hence $\bar{B}_r(\bar{x}) \subset X/G$ is compact for any $r > 0$ because it is the continuous image of a compact set, and therefore X/G is proper.

It remains to show that X/G is a length space. Let γ be a curve from x to y such that

$$\operatorname{length} \gamma < |x - y|_X + \varepsilon$$

Consider curve $\bar\gamma = \pi\circ\gamma$ in X/G. Note that $\bar\gamma$ is a curve from $\bar x$ to $\bar y$ and from ❶, we get

$$\operatorname{length}\bar\gamma \leqslant \operatorname{length}\gamma$$

Hence

$$\operatorname{length}\bar\gamma < |\bar x - \bar y|_{X/G} + \varepsilon \qquad \square$$

17.2.3. Proposition. *Let $X \in \mathrm{CBB}[0]$ such that a group G acts on X by isometries and has closed orbits. Then $X/G \in \mathrm{CBB}[0]$*

Proof. Applying Proposition 17.2.2, we get that X/G is a proper and length space. By Theorem 16.1.2*b*, it remains to show that

❷ $$\tilde\measuredangle(\bar p\,{}^{\bar x}_{\bar y}) + \tilde\measuredangle(\bar p\,{}^{\bar y}_{\bar z}) + \tilde\measuredangle(\bar p\,{}^{\bar z}_{\bar x}) \leqslant 2{\cdot}\pi.$$

holds for any $\bar p, \bar x, \bar y, \bar z \in X/G$.

Choose arbitrary $p \in X$ such that $\pi(p) = \bar p$. Since X is proper and orbits are closed, we can choose x, y and $z \in X$ such that

$$\begin{aligned}
\pi(x) &= \bar x, & |p-x|_X &= |\bar p - \bar x|_{X/G},\\
\pi(y) &= \bar y, & |p-y|_X &= |\bar p - \bar y|_{X/G},\\
\pi(z) &= \bar z, & |p-z|_X &= |\bar p - \bar z|_{X/G}.
\end{aligned}$$

By the definition of metric in X/G, we have

$$|x-y|_X \geqslant |\bar x - \bar y|_{X/G}, \quad |y-z|_X \geqslant |\bar y - \bar z|_{X/G}, \quad |z-x|_X \geqslant |\bar z - \bar x|_{X/G}.$$

Taking all this into account, we get

❸ $$\tilde\measuredangle(p\,{}^{x}_{y}) \geqslant \tilde\measuredangle(\bar p\,{}^{\bar x}_{\bar y}), \quad \tilde\measuredangle(p\,{}^{y}_{z}) \geqslant \tilde\measuredangle(\bar p\,{}^{\bar y}_{\bar z}), \quad \tilde\measuredangle(p\,{}^{z}_{x}) \geqslant \tilde\measuredangle(\bar p\,{}^{\bar z}_{\bar x}),$$

Since $X \in \mathrm{CBB}[0]$, we have

$$\tilde\measuredangle(p\,{}^{x}_{y}) + \tilde\measuredangle(p\,{}^{y}_{z}) + \tilde\measuredangle(p\,{}^{z}_{x}) \leqslant 2{\cdot}\pi.$$

This inequality plus ❸ implies ❷. $\square$

17.3 Number of isolated fixed-point orbits

Let $X \in \mathrm{CBB}[0]$. A point $p \in X$ is called *extremal* if $\measuredangle[p\,^x_y] \leqslant \frac{\pi}{2}$ for any hinge $[p\,^x_y]$ in X.

For example if $X \stackrel{iso}{=\!=} [0,1]$ then both ends 0 and 1 are extremal points and the remaining points are not extremal. Further in a n-dimensional cube, each of 2^n vertices is an extremal point and the remaining points are not extremal. In a regular triangle, the vertices are the only extremal points, In a regular pentagon there are no extremal points.

Let G act by isometries on $\mathbb{R}^n$. According to Proposition 17.2.3, $\mathbb{R}^n/G \in \mathrm{CBB}[0]$. Assume that the action of G on $\mathbb{R}^n$ is *properly discontinuous*; i.e., given a compact set $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, there are only finitely many elements $g \in G$ such that $g \cdot x \in K$.

In this case $\mathbb{R}^n/G$ is a polyhedral space; it is not hard to prove, but we will not give a proof here. In particular, we can talk about volume and dimension of $\mathbb{R}^n/G$.

It tuns out that a point $\bar{x} \in \mathbb{R}^n/G$ is extremal if and only if one (and therefore any) point $x \in \mathbb{R}^n$ such that $\pi(x) = \bar{x}$ is an isolated fixed point for some subgroup of G. More precisely, if G_x is the subset[2] of all isometries g in G such that $g \cdot x = x$, then for any $y \neq x$ there is $g \in G_x$ such that $g \cdot y \neq y$. Therefore counting the number of such G-orbits is equivalent to the counting extremal points in $\mathbb{R}^n/G$. All this means that the following theorem implies some nontrivial information about the G-action.

17.3.1. Theorem. *Suppose G acts by isometries on $\mathbb{R}^n$ and this action is properly discontinuous. Then $\mathbb{R}^n/G$ has at most 2^n extremal points.*

Before going into proofs, let us consider a couple of examples which show that $\mathbb{R}^n/G$ can have exactly 2^n extremal points.

Consider the set of all isometries which can be presented as a composition of parallel translations $x \mapsto x + v$ with a vector v with all integer coordinates, and all the reflections in the coordinate hyperplanes. This defines a group action, say G on $\mathbb{R}^n$ for which the quotient $\mathbb{R}^n/G$ is isometric to the cube $[0, \frac{1}{2}]^n$. Note that each vertex of the cube is an extremal point.

[2] The subset G_x is in fact a subgroup of G and it is called the *stabilizer* of x.

Part V

Details

Chapter 18

Digressions into discrete geometry

This complete chapter is beautiful and useless

18.1 Bezdek–Connelly theorem

This is yet another relaxing section.

18.1.1. Proposition. *Let $R > 0$ and $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two collections of points in $\mathbb{R}^m$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j. Then

$$\bigcap_{i=1}^{n} B_R(a_i) \neq \varnothing \quad \Longrightarrow \quad \bigcap_{i=1}^{n} B_R(b_i) \neq \varnothing.$$

Proof. Applying Kirszbraun's theorem (**??**), we get a distance non-contracting map $f\colon \mathbb{R}^m \to \mathbb{R}^m$ such that $f(a_i) = b_i$. Choose $x \in \bigcap_{i=1}^{n} B_R(a_i)$. Since f distance non-contracting,

$$|f(x) - b_i| \leqslant |x - a_i| \leqslant R$$

for each i. In particular, $f(x) \in B_R(b_i)$ for each i; hence the result. $\square$

Given a set $A \subset \mathbb{R}^m$, let $\mathrm{vol}_m(A)$ denote m-dimensional volume of A. For example vol_1 measures length in $\mathbb{R}^1$, vol_2 measures area in $\mathbb{R}^2$, and vol_3 measures the usual notion of volume in $\mathbb{R}^3$.

18.1.2. Conjecture. *Let $R > 0$ and $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two collections of points in $\mathbb{R}^m$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j. Then

❶ $$\mathrm{vol}_m\left(\bigcap_{i=1}^{n} B_R(a_i)\right) \leqslant \mathrm{vol}_m\left(\bigcap_{i=1}^{n} B_R(b_i)\right)$$

and

❷ $$\mathrm{vol}_m\left(\bigcup_{i=1}^{n} B_R(a_i)\right) \geqslant \mathrm{vol}_m\left(\bigcup_{i=1}^{n} B_R(b_i)\right).$$

18.1.3. Exercise. *Prove this conjecture in case $m = 1$.*

The inequality ❷ was conjectured by Poulsen in 1954 and Kneser in 1955 and Hadwiger in 1956. The inequality ❶ was conjectured much later, it appears in list of problems of Klee and Wagon published in 1991. Both inequalities are trivial in the case $m = 1$; both are open problems for $m > 2$. Both were proved in case $m = 2$ by Bezdek and Connelly in 2002. Here we will describe ideas in their proof without going into details.

Not quite working idea. Assume one can construct n smooth curves $\alpha_i\colon [0,1] \to \mathbb{R}^2$ such that

❸ $\alpha_i(0) = a_i$, $\alpha_i(1) = b_i$ and $\ell_{i,j}(t) = |\alpha_i(t) - \alpha_j(t)|$ is nonincreasing.

In this case one can consider functions

$$v(t) = \mathrm{area}\left(\bigcap_{i=1}^{n} B_R(\alpha(t))\right) \quad \text{and} \quad V(t) = \mathrm{area}\left(\bigcup_{i=1}^{n} B_R(\alpha(t))\right).$$

In order to prove ❶, it is sufficient to show that $v'(t) \geqslant 0$ for all t. Similarly, to prove ❷, it is sufficient to show that $V'(t) \leqslant 0$ for all t. The latter is a calculus problem; it is technical, but straightforward.

As one can see by the following exercise, this approach cannot lead to a solution directly.

18.1.4. Exercise. *Construct points $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R}^2$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j, but there are no curves $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \colon [0,1] \to \mathbb{R}^2$ which satisfy ❸.

In their proof, Bezdek and Connelly found a work around which uses the following theorems.

18.1.5. Alexander's theorem. *Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two collections of points in $\mathbb{R}^m$. Viewing $\mathbb{R}^{2\cdot m}$ as $\mathbb{R}^m \times \mathbb{R}^m$, we shall consider $\mathbb{R}^m = \mathbb{R}^m \times \{0\}$ as a coordinate subspace of $\mathbb{R}^{2\cdot m}$. Then there is a choice of curves $\alpha_i \colon [0,1] \to \mathbb{R}^{2\cdot m}$ such that $\alpha_i(0) = a_i = (a_i, 0)$, $\alpha_i(1) = b_i = (b_i, 0)$ and the function $\ell_{i,j}(t) = |\alpha_i(t) - \alpha_j(t)|$ is monotonic for each i and j.*

Proof. Straightforward calculations show that conclusion of the theorem hold for

$$\alpha_i(t) = \left(\frac{a_i + b_i}{2} + \cos(\pi\cdot t)\cdot\frac{a_i - b_i}{2},\ \sin(\pi\cdot t)\cdot\frac{a_i - b_i}{2}\right).$$

□

18.1.6. Archimedes' theorem. *Consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Denote by $\Pi \colon \mathbb{S}^2 \to \mathbb{R}$ a coordinate projection. Then for any subinterval interval $[a,b] \subset \subset [-1,1]$, we have*

$$\operatorname{area}\left[\Pi^{-1}([a,b])\right] = 2\cdot\pi\cdot(b-a).$$

In other words, the area of the unit sphere which lies between two cutting parallel planes of distance h is equal to $2\cdot\pi\cdot h$.

Proof. The set $\Pi^{-1}([a,b])$ is a surface of revolution of function

$$f(x) = \sqrt{1 - x^2}$$

restricted to the interval $[a,b]$. Therefore

$$\begin{aligned}\operatorname{area}\left[\Pi^{-1}([a,b])\right] &= 2\cdot\pi\cdot\int_a^b f(x)\cdot\sqrt{1 + (f'(x))^2}\cdot dx \\ &= 2\cdot\pi\cdot(b-a).\end{aligned}$$

□

Let us denote by $\mathbb{B}^m$ the unit ball in $\mathbb{R}^m$ and vol_m the m-dimensional volume. Note that when we write $\mathrm{vol}_m \mathbb{S}^m$, we are considering the m-dimensional analogue of surface area of the unit sphere, not the $(m+1)$-dimensional volume of the region it bounds in $\mathbb{R}^{m+1}$. Archimedes' theorem admits the following generalization in higher dimensions. The proof goes along the same lines. The case $m = 1$ coincides with the original Archimedes' theorem; we will use only the case $m = 2$.

18.1.7. Generalized Archimedes' theorem. *Let m be a positive integer. Consider the unit sphere $\mathbb{S}^{m+1} \subset \mathbb{R}^{m+2}$. Denote by $\Pi\colon \mathbb{S}^{m+1} \to \mathbb{R}^m$ a coordinate projection; clearly $\Pi(\mathbb{S}^{m+1}) = \mathbb{B}^m$. Then for any domain $\Omega \subset \mathbb{B}^m$, we have*

$$\mathrm{vol}_{m+1}\left[\Pi^{-1}(\Omega)\right] = 2{\cdot}\pi{\cdot}\,\mathrm{vol}_m \Omega.$$

In particular

$$\mathrm{vol}_{m+1} \mathbb{S}^{m+1} = 2{\cdot}\pi{\cdot}\,\mathrm{vol}_m \mathbb{B}^m.$$

The working idea of Conjecture 18.1.2 for $m = 2$. According to Alexander's theorem we can construct curves $\alpha_i\colon [0,1] \to \mathbb{R}^4$ which satisfy ❸. Since we are considering $\mathbb{R}^2$ as a subspace of $\mathbb{R}^4$, given $x \in \mathbb{R}^2 \subset \mathbb{R}^4$ we need to distinguish the notions of balls centered at x in $\mathbb{R}^2$ and balls centered at x in $\mathbb{R}^4$. We will denote them $B_R(x;\mathbb{R}^2)$ and $B_R(x;\mathbb{R}^4)$ respectively. From the Generalized Archimedes' theorem, we get that

❹
$$\begin{aligned}\mathrm{area}\left(\bigcap_{i=1}^n B_R(a_i;\mathbb{R}^2)\right) &= \tfrac{1}{2\cdot\pi\cdot R}\cdot \mathrm{vol}_3\left(\partial\bigcap_{i=1}^n B_R(a_i;\mathbb{R}^4)\right)\\ \mathrm{area}\left(\bigcap_{i=1}^n B_R(b_i;\mathbb{R}^2)\right) &= \tfrac{1}{2\cdot\pi\cdot R}\cdot \mathrm{vol}_3\left(\partial\bigcap_{i=1}^n B_R(b_i;\mathbb{R}^4)\right)\end{aligned}$$

Now consider the function

$$w(t) = \tfrac{1}{2\cdot\pi\cdot R}\cdot \mathrm{vol}_3\left(\partial\bigcap_{i=1}^n B_R(\alpha_i(t);\mathbb{R}^4)\right).$$

If $w(t)$ nonincreasing for all t, then together with ❹ it would imply ❶.

To show that $w(t)$ nonincreasing one needs to calculate its derivative and show that it is nonpositive. The latter follows since

$$w'(t) = -\sum_{i<j} \ell'_{i,j}(t){\cdot}\vartheta_{i,j}(t),$$

for some nonnegative values $\vartheta_{i,j}(t)$. In fact for any fixed t, the value $\vartheta_{i,j}(t)$ can be expressed using only the values $\ell_{i,j}(t)$ and R. (The inequality $\vartheta_{i,j} \geqslant 0$ follows from the geometric interpretation of this number as the area of certain sets in $\bigcap_{i=1}^n B_R(\alpha_i(t);\mathbb{R}^4)$, which we do not discuss here.)

The inequality ❷ is proved in a similar way. We have to define

$$W(t) = \tfrac{1}{2\cdot\pi\cdot R}\cdot \mathrm{vol}_3\left(\partial \bigcup_{i=1}^n B_R(\alpha_i(t);\mathbb{R}^4)\right)$$

and prove that

$$W'(t) = \sum_{i<j} \ell'_{i,j}(t)\cdot\Theta_{i,j}(t),$$

for some nonnegative values $\Theta_{i,j}(t)$. Then, apply the Generalized Archimedes theorem to get

$$\mathrm{area}\left(\bigcup_{i=1}^n B_R(a_i;\mathbb{R}^2)\right) = W(0),$$
$$\mathrm{area}\left(\bigcup_{i=1}^n B_R(b_i;\mathbb{R}^2)\right) = W(1).$$

Hence ❷.

18.1.8. Exercise. *Try to understand why the same idea does not work in the case $m = 3$.*

18.1.9. Exercise. *Construct points $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R}^2$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j, but

$$\mathrm{length}\left(\partial \bigcup_{i=1}^4 B_R(a_i)\right) < \mathrm{length}\left(\partial \bigcup_{i=1}^4 B_R(b_i)\right).$$

18.1.10. Exercise. *Apply inequality ❷ for $R \to \infty$ to show that if $a_1, a_2, \dots$ $\dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}^2$ such that*

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all i and j then

$$\mathrm{length}\left(\partial\,\mathrm{Conv}(a_1, a_2, \dots, a_n)\right) \geqslant \mathrm{length}\left(\partial\,\mathrm{Conv}(b_1, b_2, \dots, b_n)\right).$$

Chapter 19

Flexible polyhedra

In this chapter we construct few examples of flexible polyhedra (with self intersections and without). These examples are not used directly further in the lectures, but they might help to build right intuition in forcomming two chapters.

19.1 Flexible vs. rigid

Consider a simplicial complex $\mathcal{T}$ which is homeomorphic to the sphere $\mathbb{S}^2$. We are interested in maps $f\colon \mathcal{T} \to \mathbb{R}^3$ which are linear on each triangle. (We assume that the image of each triangle in $\mathcal{T}$ is not degenerate.

Let k, l and m be the number of vertices, edges, and triangles in $\mathcal{T}$. Label the vertices of $\mathcal{T}$ as $\{v_1, v_2, \dots, v_k\}$. Note that the map f is completely determined by values

$$w_i = f(v_i) = (x_i, y_i, z_i) \in \mathbb{R}^3.$$

Consider the following phisical model described by the map f. If a pair (v_i, v_j) is connected by an edge in $\mathcal{T}$, let us connect the vertices w_i and w_j by a rigid bar and connect all the bars coming from one vertex w_i with joint-hinge. If the obtained model admits a motion different from an isometry of Euclidean space, then f is said to be *flexible*; if not, then f is *rigid*.

One can reformulate the flexibility/rigidity the following way. Consider polyhedral metric on $\mathbb{S}^2$ which makes the restriction of f to each triangle to be distance preserving. This metric will be called *the metric induced by f*. Note that the induced metric is completely determined by the lengths of images of all edges of $\mathcal{T}$. Without loss of generality, we may assume that v_1, v_2 and v_3 are vertexes of one triangle in $\mathcal{T}$. The map is called flexible if there is a nontrivial continuous one-parameter family of maps f_t, such that $f_0 = f$ and $f_t(v_1)$, $f_t(v_2)$, $f_t(v_3)$ do not depend on t; if there is no such f_t then f is called rigid.

Counting constrains. Without loss of generality, we may assume that w_1 is the origin of $\mathbb{R}^3$; w_2 lies on x-axis and w_3 lies in the xy-plane. In other words,

$$w_1 = (0,0,0), \qquad w_2 = (x_2,0,0), \qquad w_3 = (x_3,y_3,0)$$

Thus, up to isometry of $\mathbb{R}^3$, the map f is completely described by $3{\cdot}k - 6$ numbers

❶ $$(x_2), (x_3, y_3), (x_4, y_4, z_4), \dots, (x_k, y_k, z_k).$$

Now let us count number of constraints. We want to preserve the length of each bar; i.e., if vertices v_i and v_j are connected by an edge in $\mathcal{T}$ then

❷ $$|w_i - w_j|_{\mathbb{R}^3} = a_{ij},$$

where a_{ij} is the length of this edge. (The equation ❷ is quadratic if written in the variables x_i, y_i, z_i, x_j, y_j and z_j.) All together f has to satisfy system of l equations of the form ❷.

According to Euler's formula, we get

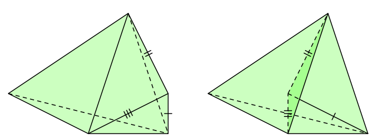
$$k - l + m = \chi(\mathbb{S}^2) = 2.$$

Further each edge appears as a side in exactly two triangles and each triangle has three sides; i.e., we have $3{\cdot}m = 2{\cdot}l$. Therefore

$$l = 3{\cdot}k - 6.$$

Thus, the number of equations (l) coincides with the number of parameters ($3{\cdot}k - 6$) in ❶.

Untill we understand our equasions better, this coincidence does not mean much. But assuming that the equasions are independed (in fact this is the case), it means that f is rigid for "generic" choice of points w_i.



Example. The system ❷ describes all the positions of k points with given distances between the chousen l pairs. For example on the picture you see two configurations of 5 points which have the same distances between 9 pairs marked by the edges. Obvoisly these configurations could not be obtained one from the other by rigid move of $\mathbb{R}^3$.

On the other hand each of these two maps is rigid. In particular, it is impossible to get from one these polyhedra to the other one by a continuous deformation which keeps each edge rigid.

19.1.1. Exercise. *Prove the rigidity of these two configurations.*

19.2 Bricard octhedra

In this section we construct the first nontrivial example of *flexible polyhedral map*. These so called Bricard's octahedra are the piecewise linear maps $f\colon \mathbb{S}^2 \to \to \mathbb{R}^3$ with triangulation as in the surface of octahedra. The map have self-intersections, i.e., f fails to be embedding. You may look at the animated image of Bricard's octahedron.

Construction. The polyhedral space P is glued out of 8 triangles in the same way as the usual octahedron. The triangulation, say $\mathcal{T}$, is the same as in an octahedron; it has 8 triangles, 12 edges and 6 vertices. We contruct Bricard's octahedra such that P isometric to the surface of convex centrally simmetric octahedron, say K which is not non-regular.

The flexible map $f\colon P \to \mathbb{R}^3$ is isometric on each face, but it does not map P to the surface of K. The 6 vertices of K will be denoted as x, y, z, x', y', z', any pair of these vertices is connected by an edge, with exception for the 3 pairs (x, x'), (y, y') and (z, z'). Since K is centrally simmetric, the midpoints of the line segments $[xx']$, $[yy']$ and $[zz']$ coincide. In particular, the points x, x', y and y' lie on one plane, say Π.

The flexible map is obtained by reflecting z' along with the 4 edges coming from z' through Π. The new map is distance preserving on each of the 8 triangles of $\mathcal{T}$.

Why it is flexible. Note first that the part glued out of 4 triangles with vertex z is flexible. This is true if each angle $\angle xzy$, $\angle xzy'$, $\angle x'zy$, $\angle x'zy'$ is strictly larger than the sum of the remaining three angles. In particular, if the spherical quadrilateral Q with vertices formed by unit vectors in the directions of rays $[zx)$, $[zy)$, $[zx')$ and $[zy')$ is not degenerate. Note that flexibility of Q in the sphere is equivalent to flexibility of our 4 triangles in $\mathbb{R}^3$. I.e., we can fix vertices x, y and z and move x' and y' along nontrivial curves $x'(t)$, $y'(t)$ such that $x'(0) = x'$, $y'(0) = y'$ and all of the distances

$$|z - x'(t)|, \qquad |z - y'(t)|, \qquad |y - x'(t)|, \qquad |x'(t) - y'(t)|, \qquad |y'(t) - x|$$

stay constant while t changes.

Set $z'(t)$ to be the rotation of z by angle π around the line passing through the midpoints of line segments $[x\,x'(t)]$ and $[y\,y'(t)]$. For $t = 0$, the midpoints of $[x\,x'(0)]$ and $[y\,y'(0)]$ coincide; in this case z' is the rotation of z by angle π around the line passing through the midpoint of $[x\,x'(0)]$ and perpendicular to the plane containing $x, y, x'(0), y'(0)$. In this case we have

$$\begin{aligned} |z'(t) - x| &= |z - x'(t)|, & |z'(t) - y| &= |z - y'(t)|, \\ |z'(t) - x'(t)| &= |z - x|, & |z'(t) - y'(t)| &= |z - y|. \end{aligned}$$

Therfore

$$|z'(t) - x|, \qquad |z'(t) - y|, \qquad |z'(t) - x'(t)|, \qquad |z'(t) - y'(t)|$$

stay constant while t changes.

19.3 Connelly sphere

Here we use Bricard's octahedra to produce a flexible polyhedral embedding which we call *Connelly sphere*. It uses 3 Bricard's octahedra and gives a flexible embedding of sphere for a triangulation with 24 vertices.

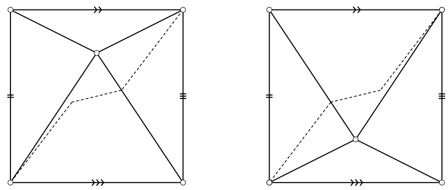
We suggest to do the following exexcise before going into construction.

19.3.1. Exercise. *Print 158, cut the figure and glue the marked sides. You obtain 6 out of 8 triangles in a Bricard's octahedra.*

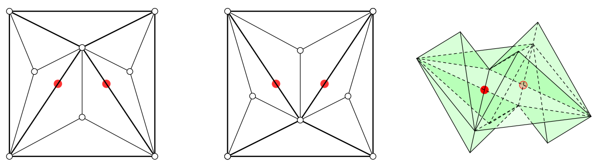
Move this models, folding only along marked lines to make another flat polygon (not the square which you started with). Imagine the missing edge and understand that its length does not change while you move the model.

Start with a Bricard's octahedron glued from two squares as in the exercise. The identified sides are decorated the same way. One the picture below, you see "upper" and "lower" sides[1], both are squares and the triangulation is marked by solid lines. The dashed line is a line of self-intersection after a small deformation.

[1]These sides on the same level, we call one upper and one lower to distiguish them.



Remove 3 big triangles from the upper side and exchange each by 3 triangles with common vertex quite a bit above of the corresponding triangle. This way you exchange 3 triangles to 9 and add 3 extra vertices. Do the same for the lower side, but choose the vertex quite a bit below the corresponding triangle. The next picture shows how it will look from above, form below and a side view. The new model is still flexible and for small deformations its self intersections appear only at the spots marked by red.



Consider one of the red spots on the upper side. Cover it by two small triangles say Δ_1 and Δ_2 joined along the edge. Construct a Bricard's octahedron as in the exercise with Δ_1 and Δ_2 as the missing faces. Remove Δ_1 and Δ_2 from the model and glue instead the 6 faces of the Bricard's octahedron. This new model is still flexible. For the right choice of the small triangles and the Bricard's octahedron, we can get rid of this self-intersection near this red spot.

If the same is done to the other red spot, we get a flexible embedding.

19.4 Comments

The Bricard's octahedra were discoverd by Raoul Bricard in 1897; in fact he classified all the flexible octahedra. This classification includes the flexible octahedra constructed in Problem 19.B.

The Connelly sphere was discovered by Robert Connelly in 1977, see [10]. The number of vertecies in this eaxample can be easely reduced to 20. An other construction due to Klaus Steffen reduces the number of vertices to 9. It is unknown if this number can be reduced further, but maybe no one wants to know.

In the late 1970s Connelly and Sullivan formulated so called *Bellows conjecture* stating that flexible polyhedron "can not be used as bellows". More precisely that the volume of a flexible polyhedron is invariant under flexing. This conjecture was proved for by Sabitov in 1996 [17].

There is also so called *Strong belloes conjecture* which states that if K and K' be the regions bounded by a flexible polyhedron before and after flexing then K and K' are scissors congruet; i.e., one can cut K into polhedons and rearange them into form of K'. This conjecture is still open.

The flexibility and rigidity were extensively studied in the smooth category. A C^∞-smooth embedding $f_0 \colon \mathbb{S}^2 \to \mathbb{R}^3$, is called *flexible* if there is one parameter family of C^∞-smooth embeddings f_t, $t \in [0,1]$, which is continuous in t such that the which give the same induced metric on $\mathbb{S}^2$, but such that f_t can not be presented as composition of isometry $\iota_t \colon \mathbb{R}^3 \to \mathbb{R}^3$ and f_0; otherwise the embedding f_0 is called *rigid*. The existence of flexible smooth embedding is unknown by now; the same applies for flexible smooth immersions which could be defined the same way.

Exercises

19.A. How many combinatorically different[2] triangulations of $\mathbb{S}^2$ with 5 vertices are there? Show that any map for corresponding simplicial complex(es) such that no 4 vertices lie in one plane is(are) rigid.

19.B. Let $A = \{x, y, x', y'\} \in \mathbb{R}^3$ be a set of distinct points such that $|x - y| = |y - x'| = |x' - y'| = |y' - x|$. Show that there are 4 distinct isometries (including identity) which send A to it-self.

Use these isometires to construct 3 distinct flexible octahedra with vertices x, y, x', y' and a given vertex z in general position.

19.C. Let $f, g \colon \mathbb{S}^2 \to \mathbb{R}^3$ be two maps which are linear on a each triangle of a a triangulation $\mathcal{T}$ of $\mathbb{S}^2$. Assume that the induced metrics on $\mathbb{S}^2$ for f and g coincide. Let us consider $\mathbb{R}^3$ as a subspace in $\mathbb{R}^6$. Show that there is a continuous one-parameter family of maps $h_t \colon \mathbb{S}^2 \to \mathbb{R}^6$, $t \in [0,1]$ such that for

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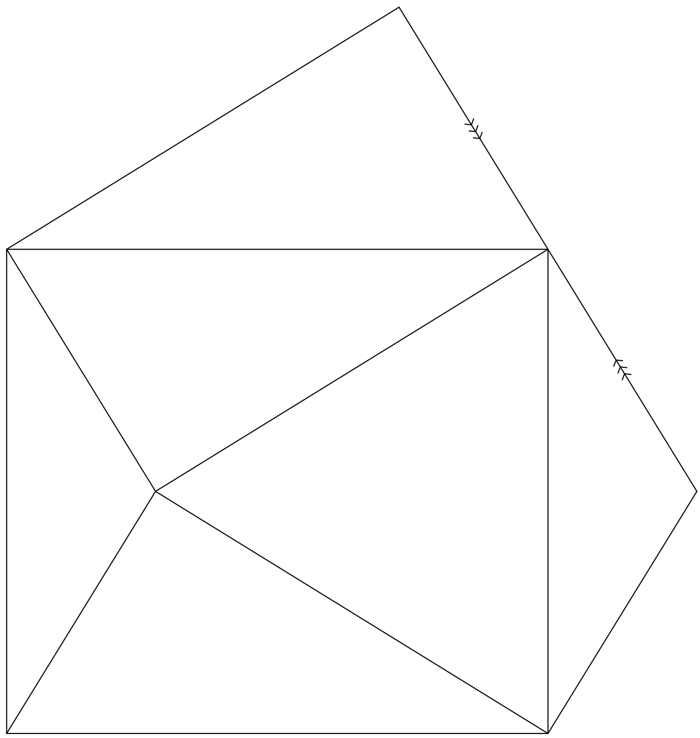
[2] Two triangulations $\mathcal{T}$ and $\mathcal{T}'$ are the same combinatorially if there is a bijection $f : V \to V'$ between their vertex sets such that

i) there is an edge from v to w in $\mathcal{T}$ if and only if there is an edge from $f(v)$ to $f(w)$ in $\mathcal{T}'$;

ii) there is a triangle connecting u, v, w in $\mathcal{T}$ if and only if there is a triangle connecting $f(u), f(v), f(w)$ in $\mathcal{T}'$.

any t, the map h_t is linear on each triangle of $\mathcal{T}$, the metric induced on $\mathbb{S}^2$ by h_t does not depend on t and $f = h_0$, $g = f_1$.



Chapter 20

Details

20.1 Lebesgue's number

The results in this section will not be used directly further in the lectures. We present it only to give a connection to standard definition of comapct space. In addition, a similar argument will be used once in the proof of Hopf–Rinow theorem (4.3.5).

Let X be a metric space, a collection of open subsets $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is called *open cover* of X if

$$X = \bigcup_{\alpha\in A} U_\alpha.$$

20.1.1. Lebesgue's number lemma. *Let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of a compact metric space X. Then there is an $\varepsilon > 0$ (it is called a* Lebesgue number *of the cover) such that for any $x \in X$ the ball $B_\varepsilon(x) \subset U_\alpha$ for some $\alpha \in \mathcal{A}$.*

Proof. Given $x \in X$, denote by $\varrho(x)$ the maximal value $R > 0$ such that $B_R(x) \subset U_\alpha$ for some $\alpha \in \mathcal{A}$. Clearly $\varrho(x) > 0$ for any $x \in X$.

Without loss of generality, we may assume $\varrho(x) < \infty$ for one (and therefore any) $x \in X$. Otherwise the conclusion of the lemma holds for arbitrary $\varepsilon > 0$.

Note

$$|\varrho(x) - \varrho(y)| \leqslant |x - y|$$

for any $x, y \in X$; in particular ϱ is continuous. Then the conclusion of the lemma holds for

$$\varepsilon = \tfrac{1}{2} \cdot \min_{x \in X}\{\varrho(x)\}.$$ □

As a corollary of Lebesgue's number lemma, we obtain an alternative definition of compact metric space using open coverings. This definition is the standard definition of compact spaces. Since it use only the notion of open sets, it can be generalized to so called topological spaces.

20.1.2. Theorem. *A metric space X is compact if and only if any open cover of X contains a finite subcover.*

I.e., for any open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of X there is a finite set $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathcal{A}$ such that

$$X = \bigcup_{i=1}^{n} U_\alpha.$$

Proof; "if"-part. First let us show that X is complete. Assume contrary; i.e., there is a Cauchy sequence (x_n) which is not converging. Set $r_n = \sup_{m \geqslant n}\{|x_n - x_m|_X\}$ and $U_n = X \backslash \bar{B}_{r_n}(x_n)$. Since x_n does not converge, we have

$$\bigcap_{n=1} \bar{B}_{r_n}(x_n) = \varnothing,$$

or equivalently $\{U_n\}_{n=1}^{\infty}$ is a cover of X. On the other hand it is easy to see that any finite sub-collection of $\{U_n\}_{n=1}^{\infty}$ does not contain x_n for all large n, a contradiction.

Fix $\varepsilon > 0$ and consider cover of X by open balls $\{B_\varepsilon(x)\}_{x \in X}$. Note that if $\{B_\varepsilon(x_i)\}_{i=1}^{n}$ is a finite subcover then $\{x_1, x_2, \dots, x_n\}$ forms an ε-net in X. Apply Theorem 1.4.8.

"only if"-part. Let $\varepsilon > 0$ be a Lebesgue's number of the covering. Choose a finite $\frac{\varepsilon}{2}$-net $\{x_1, x_2, \dots, x_n\}$ of X. Clearly

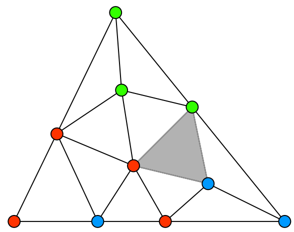
$$\text{❶} \qquad \bigcup_{i=1}^{n} B_\varepsilon(x_i) = X.$$

For each x_i choose U_{α_i} such that $U_{\alpha_i} \supset B_\varepsilon(x_n)$. From ❶,

$$\bigcup_{i=1}^{n} U_{\alpha_i} = X.$$ □

20.2 Domain invariance

In the proof of Domain invariance theorem, we will use the following lemma in combinatoric topology.



20.2.1. Sperner's lemma. *Let $\mathcal{T}$ be a triangulation of m-dimensional simplex Δ. Assume that we color all the vertices of $\mathcal{T}$ in $m+1$ colors in such a way that each vertex of Δ colored in different colors and the vertices of $\mathcal{T}$ which lie on a face F of Δ is colored in the color of the vertices of F.*

Then among the m-dimensional simplexes of $\mathcal{T}$ there is at least one which vertices are colored in all the $m+1$ different colors.

If you google Sperner's lemma you will find a dousen of nice proofs, and it is also fun to make one your-self.

Chapter 21

Hints and solutions

Chapter 1

Exercise 1.1.2. Note that it is sufficient to show that for any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$|\boldsymbol{x} + \boldsymbol{y}| \leqslant |\boldsymbol{x}| + |\boldsymbol{y}|.$$

Take the square of the left and right parts, simplify, and google Cauchy–Bunyakovsky inequality.

Exercise 1.4.6. Arguing by contadiction we may find a point which is of distance at least ε from any element of the ε-packing. This contradicts the maximality of the packing.

Exercise 1.4.7. Assume B is a $(2{\cdot}\varepsilon)$-packing with m elements. Since A is an ε-net, for each $b \in B$ we can choose an element $f(b) \in A$ such that $|f(b) - b| \leqslant \varepsilon$. Note that

$$|f(b) - f(b'| \geqslant |b - b'| - 2{\cdot}\varepsilon > 0$$

if $b \neq b'$. I.e., $f\colon B \to A$ is injective and hence $m \leqslant n$.

Problem 1.B.

First solution.???

Second solution.[1] Given any pair of points $x, y \in X$, set $x_n = f^n(x)$ and $y_n = f^n(y)$. Since X is compact, one can choose an increasing sequence of integers

[1]This proof was suggested by Travis Morrison, one of the students in the MASS program 2011.

n_k such that both sequences $(x_{n_i})_{i=1}^\infty$ and $(y_{n_i})_{i=1}^\infty$ converge. In particular, both of these sequences are Cauchy; i.e.,

$$|x_{n_i} - x_{n_j}|, |y_{n_i} - y_{n_j}| \to 0$$

as $\min\{i, j\} \to \infty$. Since f is noncontracting, we get

$$|x - x_{|n_i - n_j|}| \leqslant |x_{n_i} - x_{n_j}|.$$

It follows that there is a sequence $m_i \to \infty$ such that

❶ $$x_{m_i} \to x \quad \text{and} \quad y_{m_i} \to y \quad \text{as} \quad k \to \infty.$$

Let $\ell_n = |x_n - y_n|$. Since f is noncontracting, (ℓ_n) is a nondecreasing sequence. On the other hand, from ❶, it follows that $\ell_{m_i} \to |x - y| = \ell_0$ as $i \to \infty$; i.e., (ℓ_n) is a constant sequece. In particular, $\ell_0 = \ell_1$ for any x and $y \in X$; i.e., f is distance preserving map.

Therefore $f(X)$ is isometric to X. From ❶, we get that $f(X)$ is everywhere dense. Since X is compact, we get that $f(X)$ is closed, and hence $f(X) = X$.

Problem 1.A.

Problem ??. The answer is not unique. The set $B = \{\frac{1}{2}, 4, 8\}$ will do; $|A - B|_{\mathcal{H}(\mathbb{R})} = \frac{1}{2}$.

Problem 1.C. Use the triangle inequality to show that K_x is distance non-expanding. To show that K_x is distance non-contracting, note that

$$|z - y| = \operatorname{dist}_z(y) - \operatorname{dist}_y(y).$$

Problem 1.D. Let X be a compact metric space. First show that there is a countable everywhere dense set $\{x_1, x_2, \dots\}$ in X. Then consider the map $f\colon X \to \mathcal{F}(\mathbb{N})$ defined by $f\colon x \mapsto (|x - x_1|, |x - x_2|, \dots)$. The same argument as in Problem 1.C shows that the map is distance non-expanding. To show that K_x is distance non-contracting, note that

$$|z - y| = \lim x_{n_i} \to y \operatorname{dist}_z(x_{n_i}) - \operatorname{dist}_y(x_{n_i}).$$

Problem 1.E. Use Exercise 1.3.3.

Problem 1.F. Assuming contrary, one can construct a sequence of points x_n such that

$$\varrho(x_{n+1}) \leqslant \tfrac{99}{100} \cdot \varrho(x_n) \quad \text{and} \quad |x_{n+1} - x_n| < \varrho(x_n).$$

Show that x_n is a Cauchy sequence and look at its limit.

Problem 1.G. Let Q be a compact ε-net in X. Applying Theorem 1.4.8, choose a finite ε-net A in Q and note that A is a $2\cdot\varepsilon$-net in X. Then apply Theorem 1.4.8 again.

Chapter 2

Exercise 2.2.2. The "only if" part is trivial. To prove the "if" part, note that from the triange inequality, we have

$$\operatorname{dist}_A \leqslant \operatorname{dist}_B \pm R.$$

Exercise 2.2.3. The triangle inequality and symmetry are evident. It only remains to show that $|A-B|_{\mathcal{H}(X)} = 0$ implies $A = B$ (see Definition 1.1.1).

If $|A-B|_{\mathcal{H}(X)} = 0$, then the closure of A contains B and vise versa. The sets A and B are compact and therefore closed (see page 9). Hence $A = B$.

Exercise 2.2.4. Let A and B be two compact subsets of X. Let $x, y \in A$ be a pair of points of maximal distance apart.

If $|A-B|_{\mathcal{H}(X)} \leqslant \varepsilon$ then there are points $x', y' \in B$ such that $|x-x'|, |y-y'| \leqslant \varepsilon$. Hence

$$|x-y| \leqslant |x'-y'| + 2\cdot\varepsilon.$$

Therefore $\operatorname{diam} A \leqslant \operatorname{diam} B + 2\cdot\varepsilon$. Switching A and B, we get

$$|\operatorname{diam} B - \operatorname{diam} A| \leqslant 2\cdot\varepsilon.$$

In particular, diam is 2-Lipschitz functuon on $\mathcal{H}(X)$.

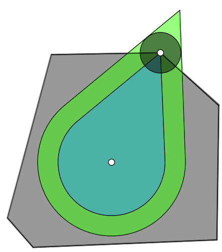
Exercise 2.2.7. The sets K_n are compact and therefore closed. Therefore their intresection K_∞ is also closed. Since $K_\infty \subset K_1$, we have that K_∞ is compact.

To show that $K_\infty \neq \varnothing$, choose a point $x_n \in K_n$ and let x_∞ be a partial limit of the sequence x_n. For any fixed m, we have $x_n \in K_m$ for all large n. Therefore $x_\infty \in K_m$ for any m; i.e., $x_\infty \in K_\infty$.

For the last part of the exercise, one could take the subsets $K_n = [n, \infty)$ of real line.

Exercise 2.3.3. Let K_n be a sequence of convex compact figures which converges to K_∞ in the Hausdorff sense. It is sufficient to show that for any two points $x_\infty, y_\infty \in K_\infty$, its midpoint $z_\infty = \frac{x_\infty + y_\infty}{2}$ lies in K_∞.

Clearly, we can choose points $x_n, y_n \in K_n$ such that $x_n \to x_\infty$ and $y_n \to y_\infty$. Since K_n is convex, we have $z_n = \frac{x_n+y_n}{2} \in K_n$. Clearly, $z_n \to z_\infty$; hence $z_\infty \in K_\infty$.



Exercise 2.3.7. Given $x \in P$, consider the "teardrop" D_x formed by the convex hull of x and $\bar{B}_R(0)$. Note that $D_x \subset P$ and $(1+\varepsilon)\cdot D_x \supset \bar{B}_{R\cdot\varepsilon}(x)$.

Exercise 2.3.8. Prove that

$$\operatorname{perim} F \geqslant 2 \cdot \operatorname{diam} F$$

for any convex figure F in the plane. Use this statement and Exercise 2.2.4 in addition to the arguments for nondegenerate case.

Exercise 2.4.1.???

Problem 9.0.2. Since K is compact, the function dist_x admits a minimum K.

Assume there are two distinct points of minimal distance, say $\bar{x}$ and $\bar{x}'$. From convexity, their midpoint $z = \frac{\bar{x}+\bar{x}'}{2}$ lies in K. Clearly $|x-z| < |x-\bar{x}| = |x-\bar{x}'|$, a contradiction.

Given $x, y \in \mathbb{R}^m$, we need to show that

$$|\bar{x} - \bar{y}| \leqslant |x - y|. \tag{❶}$$

In the case where $x, y \in K$, we have $\bar{x} = x$ and $\bar{y} = y$, hence ❶ follows.

If $x \in K$, $y \notin K$ then $\bar{x} = x$. We may assume that $\bar{y} \neq \bar{x}$, otherwise ❶ is trivial. Note that

$$\angle \bar{y}\bar{x}x \geqslant \tfrac{\pi}{2}. \tag{❷}$$

If not, moving $\bar{y}$ along $[\bar{y}\bar{x}]$ would decrease the distance to y. From convexity of K, we have $[\bar{y}\bar{x}] \subset K$, hence a contradiction. Clearly, ❷ imples ❶

It remains to consider the case $x, y \notin K$. As above we may assume that $\bar{y} \neq \bar{x}$. Further the same argument as above shows that

$$\angle \bar{y}\bar{x}x, \angle \bar{x}\bar{y}y \geqslant \tfrac{\pi}{2}. \tag{❸}$$

Straightforward calculations show that ❸ imply ❶. The latter is also a partial case of the Arm Lemma (14.5.1).

Problem 2.B. Both statements do not hold. For the first one take A to be the unit square in the plane and B a finite ε-net in A. In this case, $|A-B|_{\mathcal{H}(\mathbb{R}^2)} \leqslant \varepsilon$. On the other hand, $\partial B = B$ and $|B - A|_{\mathcal{H}(\mathbb{R}^2)} \geqslant \frac{1}{2} - \varepsilon$.

For the second part of problem, one can take the unit square A and $B = \partial A$, then $\partial B = B$.

(In fact one can construct an arbitrary number of compact sets with the same boundary. Google “Lakes of Wada” to see such examples.)

Problem 2.E. Given a set of points X, consider a maximal set F, which includes X as a subset. Let us show that F is a figure of constant width.

Note that F is an intersection of a family of unit discs; in particular F is convex. If F lies between two lines of distance less than 1, then F has to lie in an intersection of two discs tangent to these lines. In this case, the center of one of these discs can be added to F so that the diameter does not exceed 1, a contradiction.

Any figure of constant width contain a pair of points of distance 1. Then it contains the intersection of all unit discs which contain these two points. The set is a digon with sides formed by arcs of unit circle; the area of the digon is much bigger than 1/100.

Assume $\operatorname{diam} F \geqslant 10000$. We can assume that the points in F which realize the diameter have the form $(0,0)$ and $(x,0)$. If F can not be broken into two parts, then one can find 50000 points of the form $(2,y_1)$ $(4,y_2),\dots,(10000,y_{50000})$ in F. Each lies in a figure of width 1 and from above their total area is more that 50000/100.

Problem 2.F. Note that given $\varepsilon>0$, the set

$$\bigcup_{i=1}^{n} K_i$$

forms an ε-net in Q for all large enough n. Then apply Problem 1.G.

To generalize Lemma 2.2.8, note that we can pass to the subspace Q in X.

Chapter 3

Exercise 3.1.5. ???

Exercise 3.1.6. First do the same for two two-point sets in the plane equipped with the Manhattan metric; it is defined on page 5. Note that an isometry must map midpoints to midpoints.

Exercise 3.2.2. From the definition of ε-isometry, we have

$$\begin{aligned}|g(f(x))-g(f(x'))|_Z &\lesseqgtr |f(x)-f(x')|_Y\pm\varepsilon \lesseqgtr\\ &\lesseqgtr |x-x'|_X\pm 2{\cdot}\varepsilon.\end{aligned}$$

It remains to show that $(g\circ f)(X)$ is a $3{\cdot}\varepsilon$-net in Z. Given $z\in Z$, choose $y\in Y$ and subsequently $x\in X$ such that

$$|g(y)-z|_Z\leqslant\varepsilon,\quad |f(x)-y|_Y\leqslant\varepsilon.$$

Then

$$\begin{aligned}|g(f(x)) - z|_Z &\leqslant |g(f(x)) - g(y)|_Z + |g(y) - z|_Z \\ &\leqslant |f(x) - y|_Y + \varepsilon + \varepsilon \\ &\leqslant 3{\cdot}\varepsilon.\end{aligned}$$

Exercise 3.2.3. If $f\colon X \to Y$ is an ε-isometry, then for any $y \in Y$, the set $\{\, x \in X \mid |f(x) - y| \leqslant \varepsilon \,\}$ is nonempty. By the Axiom of Choice, we can choose such an x for each y, producing a function $g\colon Y \to X$ with the property that

$$|f(g(y)) - y|_Y \leqslant \varepsilon, \qquad \text{for all } y \in Y.$$

Then

$$\begin{aligned}|g(y) - g(y')|_X &< |f(g(y)) - f(g(y'))|_Y + \varepsilon \\ &\leqslant |y - y'|_Y + |f(g(y)) - y|_Y + |f(g(y')) - y'|_Y + \varepsilon \\ &\leqslant |y - y'|_Y + 3{\cdot}\varepsilon,\end{aligned}$$

and in a similar fashion,

$$\begin{aligned}|y - y'|_Y &\leqslant |f(g(y)) - f(g(y'))|_Y + |f(g(y)) - y|_Y + |f(g(y')) - y'|_Y \\ &< |g(y) - g(y')|_X + 3{\cdot}\varepsilon.\end{aligned}$$

Finally,

$$\begin{aligned}|g(f(x)) - x|_X &< |f(g(f(x))) - f(x)|_Y + \varepsilon \\ &\leqslant 2{\cdot}\varepsilon\end{aligned}$$

shows that $g(Y)$ is a $2{\cdot}\varepsilon$-net in X (hence it is a $3{\cdot}\varepsilon$-net in X.)

Exercise 3.2.4. Embed X and Y in some space Z such that $|X - Y|_{\mathcal{H}(Z)} < \varepsilon$. Given $x \in X$, the set $\{\, y \in Y \mid |x - y|_Z < \varepsilon \,\}$ is therefore nonempty. We can then (using the Axiom of Choice) choose an element y in this set and define $f(x) = y$. The function $f\colon X \to Y$ has the property that

$$|f(x) - x|_Z < \varepsilon, \qquad \text{for all } x \in X.$$

By the triangle inequality,

$$\begin{aligned}|f(x) - f(x')|_Y &\leqslant |x - x'|_Z + |f(x) - x|_Z + |f(x') - x'|_Z \\ &< |x - x'|_X + 2{\cdot}\varepsilon\end{aligned}$$

and similarly

$$\begin{aligned}|x-x'|_X &\leqslant |f(x)-f(x')|_Z+|f(x)-x|_Z+|f(x')-x'|_Z\\ &< |f(x)-f(x')|_Y+2\cdot\varepsilon.\end{aligned}$$

Lastly, we show that $f(X)$ is a $2\cdot\varepsilon$-net in Y. Given $y\in Y$, there exists $x\in X$ with $|x-y|_Z<\varepsilon$ because $|X-Y|_{\mathcal{H}(Z)}<\varepsilon$. Then

$$\begin{aligned}|f(x)-y|_Y &\leqslant |f(x)-x|_Z+|x-y|_Z\\ &< 2\cdot\varepsilon.\end{aligned}$$

Exercise 3.2.6.

Exercise 3.3.1. We need to prove two inequalities: $|X-P|_{\mathcal{M}}\leqslant\frac{\operatorname{diam}X}{2}$ and $|X-P|_{\mathcal{M}}\geqslant\frac{\operatorname{diam}X}{2}$.

To prove the first one, we shall simply construct a metric on the set

$$W=P\sqcup X=\{p\}\sqcup X$$

by setting $|p-x|_W=\frac{\operatorname{diam}X}{2}$ for any $x\in X$ and $|x-y|_W=|x-y|_X$ for any $x,y\in X$. It is straightforward to check that this indeed defines a metric and $|P-X|_{\mathcal{H}(W)}=\frac{\operatorname{diam}X}{2}$.

To prove the second inequality, choose points $x,y\in X$ so that $|x-y|_X=\operatorname{diam}X$. Assume X is a subset of a space Z such that $|\{p\}-X|_{\mathcal{H}(Z)}<r$. Then $|p-x|,|p-y|<r$ and therefore $r>\frac{\operatorname{diam}X}{2}$.

Exercise 3.3.2. Let P be a one-point space. According to Exercise 3.3.1, $|X-P|_{\mathcal{M}}=\frac{\operatorname{diam}X}{2}$. From the triangle inequality,we have

$$||X-P|_{\mathcal{M}}-|X-P|_{\mathcal{M}}|\leqslant|X-Y|_{\mathcal{M}},$$

hence, the result.

Exercise 3.3.3. Let $x_1,\dots,x_N\in X_\infty$ be a maximal ε-packing. If $|X_n-X_\infty|_{\mathcal{M}}<r$, then we can embed X_n and X_∞ in some space Z such that there exist points $y_1,\dots,y_N\in X_n$ with $|x_i-y_i|_Z<r$ for each i. It follows from the triangle inequality that

$$|y_i-y_j|\geqslant|x_i-x_j|-|x_i-y_i|-|x_j-y_j|>|x_i-x_j|-2r$$

for all i,j. Since $|x_i-x_j|>\varepsilon$ for all $i\neq j$, and since there are only finitely many x_i, there is a $\delta>0$ such that $|x_i-x_j|>\varepsilon+\delta$ for all $i\neq j$. It follows from the above calculation that if $r < \delta/2$, then $|y_i - y_j| > \varepsilon$ for all $i \neq j$. Hence, $y_1, \ldots, y_N$ is an ε-packing in X_n (not necessarily maximal.) Thus

$$\operatorname{pack}_\varepsilon X_n \geqslant \operatorname{pack}_\varepsilon X_\infty.$$

Exercise 3.5.2. Consider the set $\mathcal{Q}$ of all metric spaces with at most two points.

Problem 3.A.

(i) The minimal Hausdorff distance is $1/\sqrt{3}$ and is achieved by taking the point to be the center of the equilateral triangle.

(ii) See Exercise 3.3.1.

Problem 3.B. If $A \subseteq X$ and $B \subseteq Y$ are ε-nets such that A is isometric to B, then $|A - B|_{\mathcal{M}} = 0$. Check that $|X - A|_{\mathcal{H}(X)} \leqslant \varepsilon$, which shows $|X - A|_{\mathcal{M}} \leqslant \varepsilon$ (and similarly $|Y - B|_{\mathcal{M}} \leqslant \varepsilon$.) By the triangle inequality,

$$|X - Y|_{\mathcal{M}} \leqslant |X - A|_{\mathcal{M}} + |A - B|_{\mathcal{M}} + |Y - B|_{\mathcal{M}} \leqslant 2{\cdot}\varepsilon.$$

??? For the second question...

Problem 3.C.

(i) Since $B_{\operatorname{diam} X}(x) \supset X$ for any $x \in X$, we have $\operatorname{rad} X \leqslant \operatorname{diam} X$. If there exist $R > 0$ and $x \in X$ such that $B_R(x) \supset X$, then for any $y, z \in X$,

$$|y - z| \leqslant |x - y| + |x - z| \leqslant 2{\cdot}R.$$

Taking the supremum over all $y, z \in X$ gives $\frac{1}{2}{\cdot}\operatorname{diam} X \leqslant R$, and taking the infimum over all such R gives $\frac{1}{2}{\cdot}\operatorname{diam} X \leqslant \operatorname{rad} X$.

(ii) It suffices to show $\operatorname{rad} X \leqslant \operatorname{rad} Y + 2{\cdot}|X - Y|_{\mathcal{M}}$. Embed X and Y in a space Z and let $D = |X - Y|_{\mathcal{H}(Z)}$. Let $y \in Y$ be such that $B_R(y) \supset Y$ for some $R > 0$. Then there exists $x \in X$ such that $|x - y|_Z \leqslant D$. For any other $x' \in X$, we can similarly find $y' \in Y$ with $|x' - y'|_Z \leqslant D$, and then

$$|x - x'|_X \leqslant |x - y|_Z + |y - y'|_Y + |x' - y'|_Z \leqslant R + 2{\cdot}D.$$

In other words, $B_{R+2\cdot D}(x) \supset X$ and consequently $\operatorname{rad} X \leqslant R + 2{\cdot}D$. Taking the infimum over all such R gives $\operatorname{rad} X \leqslant \operatorname{rad} Y + 2{\cdot}D$. Taking the infimum over all such embeddings of X and Y into Z yields

$$\operatorname{rad} X \leqslant \operatorname{rad} Y + 2{\cdot}|X - Y|_{\mathcal{M}}.$$

Problem 3.D. ???

Problem 3.E. Part *a* follows directly from Gromov's compactness theorem (3.5.1) and Theorem 1.4.8*b*.

b). Applying Gromov's compactness theorem (3.5.1), we get that there is a sequence $\varepsilon_n \to 0+$ such that if $[Z] \in K$ then $\mathrm{pack}_{\varepsilon_n} Z \leqslant n$ for each n .

It remains to construct a compact space X such that if $\mathrm{pack}_{\varepsilon_n} Z \leqslant n$ for each n then there is a distance non-contracting map $Z \to X$. As a set of points in X, take all infinite sequences of positive integers $(k_1, k_2, \dots)$ such that $k_n \leqslant n$ for each n. Given two such sequences $\boldsymbol{k} = (k_1, k_2, \dots)$ and $\boldsymbol{k}' = (k_1', k_2', \dots)$ set

➊ $$|\boldsymbol{k} - \boldsymbol{k}'|_X = 2{\cdot}\varepsilon_{n-1},$$

where n is the smallest number such that $k_n \neq k_n'$.

To check that X is compact, we apply diagonal procedure ...???

It remains to construct a distance non-contracting map $Z \to X$. According to Exercise 1.4.6, we can choose an ε_n-net $a_{n,1}, a_{n,2}, \dots, a_{m,n}$ in Z with $m \leqslant n$. Given a point $z \in Z$, consider sequence k_n of integers such that $|a_{k_n,n} - z| < 2{\cdot}\varepsilon_n$. Note that a sequence $\boldsymbol{k} = (k_1, k_2, \dots)$ describes a point in X. From ➊, it follows that the map $Z \to X$ defined as $z \mapsto \boldsymbol{k}$ is distance non-contracting.

Problem 3.F. ???

Chapter 4

Exercise 4.1.3.

Exercise 4.2.2.

Exercise 4.2.3. Consider the "comb space" $X \subset \mathbb{R}^2$ which is the union of the line segment from $(0,0)$ to $(1,0)$, the line segment from $(0,0)$ to $(0,1)$, and the line segment from $(1/n, 0)$ to $(1/n, 1)$ for each $n \in \mathbb{N}$. Any two points in X can be connected by a curve of length at most 3, so $\hat{d}$ is a finite metric. However, any d-open ball about a point $(0, y)$ will be disconnected, whereas $\hat{d}$-open balls are always path-connected.

Exercise 4.3.2.

Exercise 4.3.4. The open interval $(0,1)$ is locally compact, but not complete.

Exercise 4.6.1.

Problem 4.A. By (???) it suffices to show that for any two $X, Y \in \mathcal{H}(\mathbb{R}^2)$ and any $\varepsilon > 0$, there is $Z \in \mathcal{H}(\mathbb{R}^2)$ such that

$$|X - Z|_{\mathcal{H}(\mathbb{R}^2)} < \frac{1}{2}\cdot|X - Y|_{\mathcal{H}(\mathbb{R}^2)} + \varepsilon, \quad |Y - Z|_{\mathcal{H}(\mathbb{R}^2)} < \frac{1}{2}\cdot|X - Y|_{\mathcal{H}(\mathbb{R}^2)} + \varepsilon.$$

By Theorem 1.4.8, every compact set has a finite ε-net. It is readily checked that the Hausdorff distance between a set and its ε-net is less than ε. So by the triangle inequality for the Hausdorff metric, it suffices to prove the above result in the case where X and Y are finite sets.

Suppose $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$. For every $x_i \in X$, choose $y \in Y$ of minimal distance to x_i and let $a_i = \frac{x_i+y}{2}$ be their midpoint. Similarly, for every $y_j \in Y$, choose $x \in X$ of minimal distanct to y_j and let $b_j = \frac{y_j+x}{2}$. Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ and set $Z = A \cup B$. Note that because Z is a finite set, it is compact.

We will show that $|X-Z|_{\mathcal{H}(\mathbb{R}^2)} \leqslant \frac{1}{2}\cdot|X-Y|_{\mathcal{H}(\mathbb{R}^2)}$ and by a similar argument, $|Y - Z|_{\mathcal{H}(\mathbb{R}^2)} \leqslant \frac{1}{2}\cdot|X - Y|_{\mathcal{H}(\mathbb{R}^2)}$. Let $R > 0$ be such that

$$\bigcup_{x\in X} B_R(x) \supset Y, \quad \bigcup_{y\in Y} B_R(y) \supset X.$$

It follows that

$$\bigcup_{z\in Z} B_{\frac{1}{2}\cdot R}(z) \supset X$$

because for every $x \in X$, there is $y \in Y$ and $z \in Z$ with $|x - z| = \frac{1}{2}\cdot|x - y|$, and moreover y can be taken to be of minimal distance to x, so that $|x - y| \leqslant R$. On the other hand, for every $z \in Z$, there is $x \in X$ and $y \in Y$ such that $|x - z| = \frac{1}{2}\cdot|x - y| \leqslant \frac{1}{2}\cdot\cdot R$, again by minimality. Thus,

$$\bigcup_{x\in X} B_{\frac{1}{2}\cdot R}(x) \supset Z.$$

So $|X - Z|_{\mathcal{H}(\mathbb{R}^2)} \leqslant R$, and by taking an infimum over all such R,

$$|X - Z|_{\mathcal{H}(\mathbb{R}^2)} \leqslant \tfrac{1}{2}\cdot|X - Y|_{\mathcal{H}(\mathbb{R}^2)}.$$

Problem 4.B. Repeat the argument of Problem 4.A to show that $\mathcal{H}(\mathcal{F}(\mathbb{N}))$ is a length space. The result then follows from Proposition 3.1.4.

Problem 4.C. Use the distance non-expanding map as in Problem 9.0.2.

Problem 4.D.

Problem 4.E.

Problem 4.F.

Chapter 5

Problem 5.A. Look at the natural triangulation of the boundary of convex polyhedra with all the vertices on the curve $t \mapsto (t, t^2, t^3, t^4)$ in $\mathbb{R}^4$.

???

Problem 5.B. Let k be the number of vertices of P. Let us apply induction on k; the base case $k = 3$ is trivial.

Choose vertices b of P which minimize the x coordinate. Note that the polygon P is convex at b.

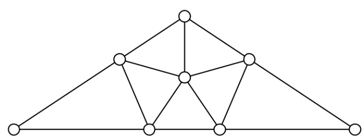
Let a and c be the vertices of P which are the right and left neighbors of b. If the interior of segment $[ac]$ lies completely in P, then after cutting the triangle $[abc]$ from P we get a polygon P' with smaller number of vertices; applying the induction hypothesis, we get a triangulation of P' which together with $[abc]$ gives a triangulation of P.

If the interior of $[ac]$ does not lie inside of P, then triangle $[abc]$ contains yet an other vertex of P. Let d be a vertex of P in the triangle $[abc]$ which lies on the maximal distance from the line (ac). Note that the interior of the segment $[bd]$ lies in the interior of P; so $[bd]$ cuts P into two polygons say Q and R with smaller number of vertices in each. It remains to apply the induction hypothesis of Q and R.

Comment. Note that if P is a convex polygon with vertices $a_1, a_2, \dots, a_k$ then cutting P along the segments $[a_1a_3]$, $[a_1a_4], \dots, [a_1a_{k-1}$ gives a triangulation of P.

Problem 5.C.

Problem 6.0.5. The solution should be clear from the picture. In fact any 2-dimensional polyhedral space admits a triangulation all of which triangles are acute, see [21].



Problem 5.F. Google Delaunay triangulation.

Problem 5.G. Since P is compact, we can choose a finite cover of P by open balls $B(x_i, r_i)$, such that for each i the ball $B(x_i, 7{\cdot}r_i)$ forms a cone neighborhood of x_i. Assume further that $i \in \{1, 2, \dots, n\}$.

Given $z \in P$ consider the set

$$F_z = \{\, i \in \{1, 2, \dots, n\} \mid z \in B(x_i, r_i) \,\}.$$

Set

$$\square_z = \operatorname{Conv} \{\, x_i \mid i \in F_z \,\}.$$

Note that it is sufficient to show that

$$P = \bigcup_{z \in P} \square_z.$$

First let us use induction to show that

❶ $\square_z \subset P$ *for any* $z \in P$.

Without loss of generality we may assume that $F_z = \{1, 2, \dots, k\}$ for some $k \leqslant n$ and $r_1 \leqslant r_2 \leqslant \dots r_k$. Set

$$A_i = \operatorname{Conv}\{\, x_j \mid j \leqslant i \,\};$$

so $\square_z = A_k$. Clearly, $A_1 = \{x_1\} \subset P$ and

❷
$$\begin{aligned} A_i &= \operatorname{Conv}(A_{i-1} \cup \{x_i\}) = \\ &= \{\, t{\cdot}x_i + (1-t){\cdot}y \in \mathbb{R}^m \mid y \in A_{i-1},\ t \in [0,1] \,\} \end{aligned}$$

for any $i \leqslant k$.

Note that $B_{3 \cdot r_i}(x_i) \supset A_{i-1}$ for any $i \leqslant k$. From ❷ we have

$$\begin{aligned} A_{i-1} \subset P \ &\Rightarrow\ A_{i-1} \subset B_{3 \cdot r_i}(x_i) \cap K_{x_i} \ \Rightarrow \\ &\Rightarrow\ A_i \subset B_{3 \cdot r_i}(x_i) \cap K_{x_i} \ \Rightarrow\ A_i \subset P. \end{aligned}$$

Hence ❶ follows.

It remains to show that

❸
$$P \subset \bigcup_{z \in P} \square_z.$$

Assume contrary. Let $Q = \bigcup_{z \in P} \square_z$ and $z \in P$ be a point which maiximize the distance to

It remains to show that for any $y \in P$ there is $z \in P$ such that

For each i consider funnction $f_i(z) = |x_i z|^2 - r_i^2$. We will call $f_i(z)$ *power of* z *with respect to sphere of radius* r_i *centered* at x_i. Clearly $f_i(z) < 0$ if and only if $z \in B(x_i, r_i)$. Consider corresponding Voronoi domain

$$V_i = \{\, z \in \mathbb{R}^m \mid f_i(z) \leqslant f_j(z) \text{ for any } i \,\}.$$

Note that each V_i is an intersection of $n - 1$ half-spaces

$$H_{i,j} = \{\, z \in \mathbb{R}^m \mid f_i(z) \leqslant f_j(z) \,\}.$$

Set $f = \min_i f_i$; it is a continuous function on X. Note that $f < 0$, in particular $V_i \subset B(x_i, r_i)$ for each i.

Note that if z is a point of local maximum of f then $\square_z \ni z$. Indeed, assume contrary; i.e., $z \notin \square_z$. Let $z^* \in \square_z$ be the closest point to z (it exists by ???). The

We may assume that radii are chousen generically; i.e. if $z \in \cap_{i \in Q} V_i$ for some index set Q then the functions $\{\, f_i \mid i \in Q \,\}$ are linearly independent in arbitrary neighborhood of z.

Consider nerve of covering $\{V_i\}$ of X; it is an abstract simplicial complex $\mathcal{S}$ with set of vertexes in the index set of x_i and an index subset Q forms a simplex $\bigcap_{i \in Q} V_i \neq \varnothing$.

Notice that vertexes of any simplex Δ^k in $\mathcal{S}$ can be reordered as $i_0, i_1, \dots, i_k$ on such a way that $r_{i_0} \leqslant r_{i_1} \leqslant \ldots \leqslant r_{i_k}$. Clearly $x_{i_n} \in B(x_{i_m}, 3{\cdot}r_{i_m})$ for all $n \leqslant m$. Let us construct a map $\Delta^k \to X$.

1. map $i_0 \mapsto x_{i_0}$;
2. map $i_1 \mapsto x_{i_1}$ and use cone structure in $B(x_{i_1}, 3{\cdot}r_{i_1})$ to extend it linearly to 1-simplex $i_0 i_1$;
3. map $i_2 \mapsto x_{i_2}$ and use cone structure in $B(x_{i_2}, 3{\cdot}r_{i_2})$ to extend it linearly to 2-simplex $i_0 i_1 i_2$;
4. and so on.

It is straightforward to check that simplex with metric induced by this map is isometric to a simplex in Euclidean space. Further, this map agree on intersections of different simplexes of $\mathcal{S}$, hence we get a map $\iota\colon \mathcal{S} \to X$.

It only remains to show that $\iota(\mathcal{S}) = X$. Assume contrary; i.e., $\Omega = X \backslash \iota(\mathcal{S}) \neq \neq \varnothing$. For each $x \in X$ choose a closest point $x^* \in \iota(\mathcal{S})$. Note that

$$f(x) > f(x^*)$$

for all $x \in \Omega$ sufficiently close to $\iota(\mathcal{S})$. It follows that there is a point $x_0 \in \Omega$, of local maximum of f.

Let Q be the a subset of the index set, such that $i \in Q$ if and only if $V_i \ni x_0$; denote by Δ the simplex corresponding to Q. Sinse x_0 is a maximum point of f, we get $x_0 \in \Delta$, a contradiction. $\square$

Chapter ??

Exercise 6.0.6. First note that if $x \in V_i$ then any geodesic $[z_i x]$ lies in V_i. Indeed, if $y \in [z_i x]$ then for any j we have

$$\begin{aligned}|z_i - y| &= |z_i - x| - |y - x| \leqslant \\ &\leqslant |z_j - x| - |y - x| \leqslant \\ &\leqslant |z_j - y|;\end{aligned}$$

i.e., $y \in V_i$.

Therefore if $V_i \not\subset S_i$ then there is a triangle Δ of the triangulation such that $z_i \in \Delta$ and V_i contains a point x on a side e of Δ which does not contain z_i.

The smallest distance from x to on of z_j on e is smaller than ε. Hence $|z_i - x| < \varepsilon$???

Exercise 6.0.1.

Exercise ??.

Exercise ??.

Exercise 7.0.2.

Exercise ??.

Exercise 8.0.3. Choose a sufficiently fine triangulation of P, say the diameter of each triangle is less than ε. If $\{a_1, a_2, \dots, a_n\}$ is the set of vertices for this triangulation, take $b_i = f(a_i)$ and apply Brehm's extension theorem. We obtain a map $f_\varepsilon \colon P \to \mathbb{R}^m$ which coincides with f on the set $\{a_1, a_2, \dots, a_n\}$.

Then the statement follows since the triangulation is fine and both f and f_ε are distance non-expanding.

Exercise 8.0.5.

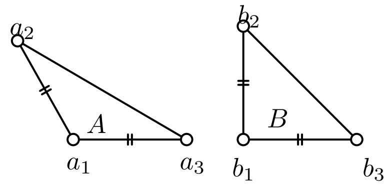
Exercise ??.

Exercise ??.

Exercise 9.0.3.

Problem ??.

Problem 7.0.5. Let A be as in Brehm's theorem. According to Problem 9.0.2, there is a distance non-expanding map $h \colon \mathbb{R}^2 \to A$ such that $h(a) = a$ for any $a \in A$. Taking the composition $f \circ h$ we get the needed distance non-expanding map $\mathbb{R}^2 \to \mathbb{R}^2$.



Problem 7.0.3. The second inequality does not hold in general. This can be seen in the picture.

The first inequality follows easily from two famous theorems in discreet geometry, one is Alexander's theorem [2] and the other is the Kneser–Poulsen conjecture which was solved in 2-dimensional case by Bezdek and Connelly in [4]. (The reduction to each of these theorems takes one line, but might be not completely evident.) I encourage you to read these papers, they are totally beautiful. You will be surprised to learn that to solve this 2-dimensional problem it is convenient to work in 4-dimensional space, the reason can be seen in Problem 7.0.4. Here I present an other solution from [18], it is not nearly as nice, but more elementary.

Given a finite collection of points $a_1, a_2, \dots, a_n$, Set

$$\ell(a_1, \dots, a_n) = \operatorname{perim} \operatorname{Conv}(a_1, \dots, a_n).$$

Then $\operatorname{perim} A \geqslant \operatorname{perim} B$ can be written as

$$\ell(a_1, \dots, a_n) \geqslant \ell(b_1, \dots, b_n).$$

Applying Brehm's extension theorem (7.0.1) we get a piecewise distance preserving map $f\colon A \to \mathbb{R}^2$ such that $f(a_i) = b_i$ for each i.

Assume contrary; i.e.,

❶ $$\ell(a_1, \dots, a_n) < \ell(b_1, \dots, b_n).$$

We can assume that $\{a_1, a_2, \dots, a_n\}$ and f are chosen in such a way that

❷ *The number n takes the minimal value for which* ❶ *can hold.*

❸ *If $x_1, \dots, x_n \in A$ and $y_i = f(x_i)$ then*

$$\ell(y_1, \dots, y_n) - \ell(x_1, \dots, x_n) \leqslant \ell(b_1, \dots, b_n) - \ell(a_1, \dots, a_n).$$

To meet Condition ❸, one has to take instead of $a_1, \dots, a_n$ an n-point subset $x_1, \dots, x_n \in A$ for which the difference taking the maximal value. It is possible since A is compact.

Note that all b_i are distinct vertices of B. Indeed, assume b_n lies inside or on side of B or $b_n = b_i$ for some $i \neq n$. Then

$$\ell(b_1, \dots, b_{n-1}) = \ell(b_1, \dots, b_{n-1}, b_n),$$
$$\ell(a_1, \dots, a_{n-1}) \leqslant \ell(a_1, \dots, a_{n-1}, a_n).$$

This contradicts Condition ❷.

By $\measuredangle a_i$, we will denote the angle of A at a_i, and by $\measuredangle b_i$ the angle of B at b_i. Let us show that

❹ $$\measuredangle b_i \geqslant \measuredangle a_i$$

Move a_i with unit speed inside A along the bisector of the angle, then the value $\ell(a_1, a_2, \dots, a_n)$ decrease with rate $2{\cdot}\cos\frac{\alpha}{2}$. The point $b_i = f(a_i)$ will also move with unit speed; it is not hard to see that at the value $\ell(b_1, \dots, b_n)$ can not decrease faster than with $2{\cdot}\cos\frac{\beta}{2}$. By Condition ❸, the difference

$$\ell(b_1, \dots, b_n) - \ell(a_1, \dots, a_n)$$

can not increase. Therefore $2{\cdot}\cos\frac{\measuredangle b_i}{2} \geqslant 2{\cdot}\cos\frac{\measuredangle a_i}{2}$; hence ❹.

Applying the theorem about sum of angles of n-gon to ❹, we get that all a_i are vertices of A. (We also get $\measuredangle b_i = \measuredangle a_i$, but we will not need it.)

We can assume that a_i are labeled in the cyclic order as they appear on the boundary of A. In this case

$$\ell(a_1, \dots, a_n) = |a_1 - a_2| + \dots + |a_{n-1} - a_n| + |a_n - a_1|$$

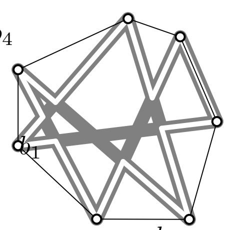
Since $|b_i - b_j| \leqslant |a_i - a_j|$ for all i and j, we get

$$|b_1 - b_2| + \dots + |b_n - b_1| \leqslant |a_1 - a_2| + \dots + |a_n - a_1|.$$

Finally, note that

$$\ell(b_1, \dots, b_n) \leqslant |b_1 - b_2| + \dots + |b_n - b_1|.$$

In other words the ℓ of the vertices of a closed broken line can not exceed the total length of the broken line. The later statement should be evident from the picture.



red $\geqslant$ green $\geqslant$ black.

Therefore

$$\ell(A'_1, A'_2, \dots, A'_n) \leqslant \ell(a_1, a_2, \dots, a_n),$$

a contradiction.

Problem 7.0.4. Here is an example of such curves.

$$\alpha_i(t) = \left(\frac{a_i + b_i}{2} + \cos(\pi{\cdot}t){\cdot}\frac{a_i - b_i}{2},\ \sin(\pi{\cdot}t){\cdot}\frac{a_i - b_i}{2}\right).$$

It is straightforward to check that $\ell_{i,j}$ are monotonic.

Chapter 19

Exercise 19.1.1.

Problem 19.A.

Problem 19.B.

Problem 19.C. Apply the same construction as in the proof of Alexander's theorem (18.1.5).

Chapter 14

Exercise 14.3.2.

Exercise 5.6.1.

Exercise 5.6.4. Let k, l and m be the number of vertices, edges, and triangles respectively in the triangulation $\mathcal{T}$. Use that sum of angles of in any triangle is π together with Euler's formula $k - l + m = 2$ and the identity $3{\cdot}m = 2{\cdot}l$.

Exercise 14.3.1. Let K_p be the intersection of K with a small sphere centered at p. Note that K_p is a convex spherical polygon.

Applying the idea in the proof of Lemma 2.3.5, we get that perimeter of K_p is not bigger than perimeter of half-sphere. Hence the statement follow.

Problem 14.A.

Problem 14.B.

Problem 14.C.

Problem 14.D.

Problem 14.G.

Problem 14.E.

Chapter 15

Exercise 15.3.2. Let us first show ❷. Assume $P' \in \,]\mathbf{P}_k[$ and $|P' - P|_{\mathcal{M}} < \frac{\delta}{2}$, let $f\colon P' \to P$ be a $2{\cdot}\delta$-isometry; it exists according to Exercise 3.2.4. For ε and δ as in Exercise 15.3.2, we can label the vertices of P' by v_1', $v_2', \dots, v_k'$ so that $|f(v_i') - v_i| < \varepsilon$. In particular

$$|v_i' - v_j'|_{P'} \lessgtr |v_i - v_j|_P \pm 3{\cdot}\varepsilon$$

for any i and j.

Let us join a pair of vertices (v_i', v_j') by a geodesic in P' if the corresponding pair (v_i, v_j) is connected by an edge in $\mathcal{T}$.

Notice that if δ is small enough the constructed geodesics intersect only at the common end points. Indeed, assume contrary holds for arbitrary small $\varepsilon > 0$; i.e., we have a sequence of polyhedral spaces $P_n \in]\mathbf{P}_k[$ such that $P_n \xrightarrow[\text{GH}]{} P$ as $n \to \infty$. Let α_n and β_n be geodesics which at p_n in P_n. Let p be a partial limit of $f_n(p_n) \in P$. Note that p lies on a geodesic between two pairs of vertices, a contradiction.

In follows that these geodesics appear as edges of a triangulation of P'. It means that the constructed geodesics subdivide P' in flat triangles; so we obtain a triangulation $\mathcal{T}'$ of P' with the same combinatorics as $\mathcal{T}$.

Note if $[P'] \in \mathbf{P}_k$ is sufficiently close to $[P]$ then P' can be triangulated by the same simplicial complex as P in such a way that the lengths of all corresponding edges in P and P' are sufficiently close. ???

Note that the length of the edges and triangulation completely describe the polyhedral space. Thus we get a parametrization of a neighborhood of $[P]$ by an open set in $\mathbb{R}^l$, where l is the number of edges in the triangulation. Recall that according to Euler's formula (5.5.1), $l = 3{\cdot}k - 6$. Hence the lemma follows.

Now assume P is a realizable space. Let K be a convex polyhedron with the surface isometric to P. By Lemmas ??? and ???, we can choose neighborhoods $W \ni \bar{P}$ in $\mathbf{P}_k$ and $V \ni \bar{K}$ in $\mathbf{K}_k$ which homeomorphic to open sets in $\mathbb{R}^{2\cdot k-6}$. Since Φ_k is continuous, we may assume that $\Phi_k(V) \subset W$. Applying Domain invariance theorem, we get $\Phi_k(V)$ is open in W. Hence $\Phi_k(V)$ contains a small ball around $\bar{P}$. Hence the result.

Problem ??.

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Chapter ??

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Chapter 16

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Chapter 17

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Chapter 12

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Chapter 18

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