Subharmonic functions on Alexandrov space

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Introduction. Here we establish basic properties of harmonic functions on Alexandrov space and as an aplication we obtain such classical results as estimates of eigen values for an Alexandrov space and isoperimetric inequality of Gromov-Levy type.

At the same time author hopes that harmonic function will give a way to establish a good analizis on Alexandrov spaces with curvature bounded below, as it already happened for spaces with biliteraly bounded curvature (see [N]).

§0 Definition and basic properties of harmonic and subharmonic functions

0.1. Definitions. Let $\Omega \subset M$ be an open domain, $\dim M = n$. The closer of set of Lipschitz functions in norm

$$\|f\| = \int_\Omega (f^2 + |\mathrm{grad} f|^2) dh_n,$$

is called *Sobolev space* $W_{1,2}(\Omega)$.

The closer of Lipschitz functions with zero on $\partial\Omega$ in norm

$$\|f\| = \int_\Omega (f^2 + |\mathrm{grad} f|^2) dh_n,$$

is called *Sobolev space with zero on the boundary* $\mathring{W}_{1,2}(\Omega)$.

Two functions $f, g \in W_{1,2}(\Omega)$ have the *same boundary values* if $f - g \in \mathring{W}_{1,2}(\Omega)$.

0.2. Definition. Let $f \in W_{1,2}(\Omega)$ we say that f is a λ-*subharmonic* function if for any positive function $\alpha \in \mathring{W}_{1,2}(\Omega)$

$$\int_\Omega \langle \mathrm{grad} f, \mathrm{grad}\alpha \rangle + \lambda \alpha dh_n \geq 0.$$

If the last inequality is exact for any such α then f is λ-*harmonic.*

In other words $f \in W_{1,2}(\Omega)$ is λ-*subharmonic* if it minimize energy function

$$E_\lambda = \int_\Omega (\lambda g + |\mathrm{grad} g|^2) dh_n$$

in the set of functions $g \le f$, $g \in W_{1,2}(\Omega)$ and has the same boundary values as f.

0.3. Lemma. Let $f : \Omega \to R$ be a harmonic function and $C \subset \Omega$ be a compact subset. Then there is a constant c such that $|f|\big|_C < c$. and c depends only on distance from C to the boudury of Ω and $\|f\|_{1,2}$.

Proof. In order to prove Lemma we need the folowing result

0.4. Lemma *(Rough isoperimetric inequality)* Let M be a compact Alexandrov space then for any $v <$Vol(M) there is a costant $c = c(M, v) < \infty$ such that if a subset $C \subset M$ has volume $< v$ then

$$\mathrm{Vol}_n^{1/n}(C) \le c\mathrm{Vol}_{n-1}^{1/n-1}(\partial C).$$

Proof of 0.4 We will use the followin Theorem

Theorem. (Perelman) For any $p \in M$ there is $R > 0$ such that for any $r < R$ the ball $B_p(r)$ is C-bi-Lipschitz homeomorphic to $B(r) \subset C_p$, where C depends only on Σ_p.

(unfortunately this result is not published, but one could get the idea from [P...] where a homeomorphism (not bi-Lipschitz one) is constucted)

Assume we have proved the rough isoperimetric inequality for dim $< n$, let us prove for dimension $= n$.

We can cover M by finite balls each of them is bi-Lipschitz homeomorphic to a ball in $C_{p_i} = C(\Sigma_{p_i})$. From the assuption we have a rough isoperimetric inequality for each Σ_{p_i}. simple culculations shows that it gives a rough isoperimetric inequality for each $B_{p_i}(r_i)$ and it easy to see that there is a constant k such that for some i we have $\mathrm{Vol}(B_{p_i}) \cap C > k\mathrm{Vol}(C)$ and $\mathrm{Vol}(B_{p_i})\backslash C > k\mathrm{Vol}(M\backslash C)$. Therefore applying the rough isoperimetric inequality for this B_i weare getting the needed inequality for M. ♠

Now it is easy to see (or well known) that as a corollary of lemma one has the following inequality for function $u \in \overset{\circ}{W}_{1,2}(\Omega)$

$$\|u\|_{L_2(\Omega)} \leq c mes^{\frac{2}{n}}(\Omega)\|\nabla u\|.$$

The rest of the proof is a coppy of the prof of Theorem 13.1 section 3 of [LU]. All arguments in this proof could be aplied for Alexandrov space exept the inequality above which is poved there with some use that u defined on R^n ♠

§1 Laplacian of a semiconvex function and its applications.

Let $f : \Omega \to R$ be a λ-convex function. Let G_t be the associated gradient flow (see [...]). Take a Lipschitz nonnegative function $\alpha : \Omega \to R$ such that $\alpha|_{\partial\Omega} = 0$. Then

$$\int_\Omega \langle \mathrm{grad} f, \mathrm{grad}\alpha \rangle dh_n = \frac{d}{dt}\int \alpha d G_t^{-1} h_n \geq -n\lambda \int \alpha dh_n$$

because as it proved in [...] $|G_t(x)G_t(y)| \leq e^{\lambda t}|xy|$.

That means that there is a sign-measure which we call $\mathrm{Lap} f$ such that for any Lipschitz function $\alpha : \Omega \to R$ such that $\alpha|_{\partial\Omega} = 0$ we have

$$\int_\Omega \langle \mathrm{grad} f, \mathrm{grad}\alpha \rangle dh_n = \int_\Omega \alpha d \mathrm{Lap} f$$

Now let us represent this Laplacian as a sum of its regular and singular part $\mathrm{Lap} f = \triangle f h_n + \mathrm{Lap}_s f$ where $\triangle f$ is some function on Ω. From above $\mathrm{Lap}_s f \geq 0$. Function f is semiconvex, therefore according to [P...] it has a representation

$$f(y) = f(x) + \langle \mathrm{grad} f, \log_x y \rangle + H(\log_x y) + o(|xy|^2)$$

at almost any point x. Therefore it can be defined "formal" Laplacian as $\mathrm{Tr}(H)$.

Proposition. $\triangle f = \mathrm{Tr}(H)$ a.e..

Proof. Let $\omega \subset \Omega$ be a domain which admits a DC coordinate system $\beta : \omega \to U \subset R^n$. By [P...] $f \circ \beta$ has second derivatives a.e. and g_{ij} has first derivatives a.e.. Now

$$\int_\omega \langle \mathrm{grad} f, \mathrm{grad}\alpha \rangle dh_n = \int_U g^{ij} \frac{\partial f \circ \beta}{\partial x_i} \frac{\partial \alpha \circ \beta}{\partial x_j} \det{}^{-1}(g^{ij}) dh_n$$

Therefore $\triangle f = -\det(g^{ij} \frac{\partial}{\partial x_j} (\det^{-1}(g^{ij}) g^{ij} \frac{\partial f \circ \beta}{\partial x_i})$ a.e.. The last expression survives under smooth changing of coordinates and if we change system such that at point x we have $g^{ij}(x) = \delta^{ij}$ and $\frac{\partial g^{ij}}{\partial x_j} = 0$ then it easy to see that it is $\sum_i \frac{\partial^2 f \circ \beta'}{\partial x_i^2} = \mathrm{Tr}(H)$.

Corollary. For any point $p \in M$ there is a neighborhood $U \ni p$ such that there is a 1-subharmonic function $f : U \to R$ such that $f(p) = 0$, and for $x \in U \backslash p$ we have

$$0 < c < \frac{f(x)}{|px|^2} < C < \infty.$$

Proof. Let us consider a collection of points q_i such that for some $r > 0$ $|pq_i| = r$ and for any direction $\sigma \in \Sigma_p$ there is i such that $\angle(\sigma, q_i') < \pi/4$. Let us find constants a and b such that for function $\phi(x) = a + b/x^{n-2} + x^2$ we have $\phi(r) = o$ and $\phi'(r) = 0$. Set $\phi^*(x) = \phi(x)$ for $x \leq r$ and $\phi^*(x) = 0$ for $x \geq r$.

Take $f = \min_i \{\phi^* \circ \mathrm{dist}_{q_i}\}$. As a minimum of semiconvex functions f is semiconvex and direct applying of Lemma ... shows that f meets all our conditions in a small neighborhood of p.

Let $\phi_{k,n} : R_+ \to R$ be the function such that

$$f_p = \phi_{k,n} \circ \mathrm{dist}_p$$

is a harmonic function on S_k and $x^{n-2}\phi_{k,n} \to 0$ when $x \to 0$. (for $k = 0$ we have $\phi_n = -x^{2-n}$ for $n > 2$, $\phi_n = \log x$ for $n = 1$ and $\phi_n = x$ for $n = 1$.

Proposition. Let M be an Alexandrov space with curvature $\geq k$ then for any point $p \in M$ the function $f_p = \phi_{k,n} \circ \mathrm{dist}_p$ is a subharmonic function on $M \backslash p$.

Proof. Direct calculations.

Let f be a harmonic function then from 0.3 it is easy to see that for any $p \in \Omega$ and any $r > 0$ such that $B_r(p) \subset \Omega$ we have a representation $f(p) = \int_{S_r}(p) f(x) d\mu_{p,r}$ where $\mu \in L^2$, a positive probability measure on S_r.

Lemma.

$$\mathrm{Vol}(\Sigma_p) r^{n-1} \mu_{p,r} > h_{n-1}.$$

Proof. Let us consider function $g = 1/|px|^{n-2}$ it is a subharmonic function on $M \backslash p$. Direct applying of the definition for regularization of g instead of α we obtain the result.

By technical reason we will need an other kernel

$$\nu_{p,r} = \frac{1}{nr^n} \int_0^r r^{n-1} \mu_{p,r} dr.$$

Corollary.

$$\mathrm{Vol}(B_1(o) \subset C_p) r^n \nu_{p,r} > h_n|_{B_r(p)}.$$

Corollary. Let $f : \Omega \to R$ be a λ-subharmonic function and $p \in \Omega$. Then there is a constant $c = c(p, \lambda)$ such that

$$\int_{B_p(r)} f d\nu_{p,r} \geq f(p) - c(p, \lambda) r^2$$

Proof. For $\lambda \geq 0$ we can easily assume $c = 0$. If $\lambda < 0$ consider $|\lambda|$-subharmonic function $|\lambda| f_p$, where f_p is a function from Lemma ... . The sum $f + |\lambda| f_p$ is subharmonic therefore

$$\int_{B_p(r)} f + |\lambda| f_p d\nu_{p,r} \geq f(p).$$

Hence

$$\int_{B_p(r)} f d\nu_{p,r} \geq f(p) - |\lambda| \int_{B_p(r)} f_p d\nu_{p,r} = \geq$$

$$\geq f(p) - c(p,\lambda) r^2 \spadesuit.$$

Definition. A point $x \in M$ is called smooth if there is a map $G : U_x \to R^n$ such that $|g_{ij}(G(y)) - \delta_{ij}| = o(|xy|)$

According to [Per D 4.4] almost any point in Alexandrov space is smooth. In particular

$$||yz| - |G(x)G(y)|| = o(|xy|^2 + |xz|^2).$$

Therefore

$$\frac{\mathrm{Vol}(B_x(r)) - r^n c_n}{r^n} = \frac{\mathrm{Vol}(B_x(r)) - \mathrm{Vol}(B_{G(x)})}{r^n} = o(r)$$

Lemma. Let $f : \Omega \to R$ be a subharmonic function and $x \in \Omega$ be a smooth point and we have

$$\limsup_{y \to x} \frac{f(y) - f(x)}{|yx|} < c < \infty.$$

Then

$$\int_{B_x(r)} f(x) - f(y) dh_n(y) < o(r^{n+2})$$

Proof.

$$0 \geq \int_{B_x(r)} f(x) - f(y) d\nu_{x,r}(y) =$$

$$= \frac{1}{c_n r^n} \int_{B_x(r)} f(x) - f(y) dh_n(y) + \int_{B_x(r)} f(x) - f(y) d\big(\nu_{x,r} - \frac{1}{c_n r^n} h_n\big)(y) \geq$$

$$\geq \frac{1}{c_n r^n} \int_{B_x(r)} f(x) - f(y) dh_n(y) - O(r)o(r) \spadesuit.$$

§2 Harmonic functions are Lipschitz.

Theorem. Let Ω be an open domain in Alexandrov space M. Let $f : \Omega \to R$ be a harmonic function. Then for any compact subset $C \subset \Omega$ there is a constant $L < \infty$ such that $f|_C$ is a Lipschitz function with constant L.

The theorem is a corollary of the following

Let $\phi : [0,1) \to R$ be an increasing convex C^∞ function such that $\lim_{x\to 1} \phi(x) = \infty$. Set

$$f_t : \Omega_t \to R, \quad f_t(x) = \max_{y \in B_(x,t)} f(y) - t\phi(\frac{|xy|}{t}).$$

where Ω_t is the maximal domain of x such that the last maximum does not admitted on the boundary of Ω. Note that $f_{t+\tau} = (f_t)_\tau$ and $\Omega_{t+\tau} = (\Omega_t)_\tau$.

Key Lemma. Let $f : \Omega \to R$ be subharmonic. Then $f_t : \Omega_t \to R$ is a subharmonic function.

Let us show first how to reduce the theorem to the Lemma.

Proof of the Theorem. Take ϕ such that $\phi(x) = 0$ for $x \leq 1 - \alpha$. Take a harmonic function f. We have that f_t is a subharmonic for any $t > 0$. Therefore $(f_t - f)/t$ is a subharmonic function.

Take an $\epsilon > 0$ such that all f_t for $0 < t < \epsilon$ are defined and finite in closed ϵ-neighborhood of C. Set $C' =$cl$(B_{\epsilon/2}(C))$ and $C'' =$cl$(B_\epsilon(C))$, $(C \subset C' \subset C'' \subset \Omega)$.

It is easy to see that for any point x we have

$$\limsup_{y\to x} \frac{f_t(y) - f_t(x)}{|xy|} \geq \limsup_{t\to t_0} (f_t - f_{t_0})/(t - t_0) \geq$$

$$\geq (1-\alpha) \limsup_{y\to x} \frac{f_t(y) - f_t(x)}{|xy|}.$$

As a maximum of locally Lipschitz functions f_t are Lipschitz on C'' for $t > 0$. Hence from above $\|(f_t)'_t\|_{L^2, C'} \leq E_o(f_t|_{C'})$.

On the other hand let us consider a Lipschitz function h such that $h|_{C'} = -c$ and $h|_{\partial C''} = c$. Let $A_t = x \in C''; f_t(x) > h(x)$. Then if for all $t \in [0, \epsilon)$ we have $c > \sup_{x \in C''} |f_t|$ then $C' \subset A_t$. Therefore by the definition of subharmonic function

$$E_o(f_t|_{C'}) \leq E_\lambda(f_t|_{A_t}) + c|\lambda| h_n(A_t) \leq E_\lambda(h|_{A_t}) + c|\lambda| h_n(C'') \leq c_1$$

Hence $\|f_t - f\|_{L^2, C'}/t \leq c_1$. By 0.3 we have that for some constant $|f_t - f|/t \leq l$ on C. In particular $|\text{grad} f(x)| < l/(1-\alpha) = L$ for $x \in C$ and therefore f is L-Lipschitz on C ♠.

From now on we start some preparation for the proof of Key Lemma.

Assume $g_0, g_1, g_2, ..., g_n$ is a collection of functions on a domain Ω and $x = (x^0, x^1, x^2, ..., x^n) \in R^{n+1}$. Set $\langle x, g\rangle = x^0 g_0 + x^1 g_1 + ... + x^n g_n$, $\langle x, g\rangle : \Omega \to R$. Let us define mapping $q : R^n = \{x \in R^{n+1}; x^0 = 1\} \to \Omega$ such that $q(x)$ is a maximum point of function $\langle x, g\rangle$ with domain of definition such that $\langle x, g\rangle$ does not admit a maximum on the boundary. Note that q is not uniquely defined.

Lemma. Let g_0 be a λ-subharmonic function which has a maximum strictly inside of an open domain Ω and $g_1, g_2, ...g_n$ be a coordinate system on this domain of convex functions. Then there is $\epsilon > 0$ such that q is well defined on $(0, \epsilon)^n$ and $\delta(x)$-expanding map, i.e. $|q(x)q(y)| > \delta(x)|xy|$, $\delta : (0, \epsilon) \to R$ is a positive measurable function.

In particular $\text{Im}(q)$ has nonzero Hausdorff measure.

Proof. Let it be wrong then there is a point $q = q(x)$ such that for any $\epsilon > 0$ there is a sequence $q_k = q(x_k)$ such that $q_k \to q$ and $|q_k q|/|x_k - x| < \epsilon$.

That means

$$\langle x, g\rangle(q) - \langle x, g\rangle \geq 0$$

$$\langle x_k, g\rangle(q_k) - \langle x_k, g\rangle \geq 0$$

Therefore

$$\langle x, g\rangle(q) - \langle x, g\rangle \geq \langle x_k - x, g\rangle - \langle x_k, g\rangle(q_k) + \langle x, g\rangle(q) \geq$$

$$\geq \langle x_k - x, g\rangle - \langle x_k - x, g\rangle(q_k)$$

Further as $x_i \geq 0$ we have that $-(\langle x, g\rangle(q) - \langle x, g\rangle)$ is λ-subharmonic. Therefore by Lemma ... there is a constant $c = c(q, \lambda)$ such that for sufficiently small $r > 0$

$$\int_{B_r(q)} \langle x, g\rangle(q) - \langle x, g\rangle d\nu_{q,r} \leq cr^2.$$

Hence for positive part of function we have

$$cr^2 \geq \int_{B_r(q)} (\langle\langle x_k - x, g\rangle - \langle x_k - x, g\rangle(q_k))^+ d\nu_{q,r} \geq$$

$$\geq \int_{B_r(q)} (\langle x_k - x, g\rangle - \langle x_k - x, g\rangle(q))^+ d\nu_{q,r} + \langle x_k - x, g\rangle(q_k) - \langle x_k - x, g\rangle(q) \geq$$

$$\geq c_1|x_k - x|r - c_2|x_k - x||q_k q|$$

where c, c_1 and c_2 are positive constants. Set $r_k = \lambda|q_k q|$, we get

$$\lambda^2 c|q_k q|^2 \geq (\lambda c_1 - c_2)|x_k - x||q_k q|.$$

Hence

$$|q_k q| \geq \frac{\lambda c_1 - c_2}{\lambda^2 c}|x_k - x|$$

the constant in the last inequality is positive for sufficiently big λ, a contradiction.

The last part of the Lemma is a direct corollary of the first part ♠.

Let us note that if f is a semiconcave function then all f_t are semiconcave and by ..... we can estimate Laplacian of f_t. Indeed the singular part is nonnegative as f_t semiconcave and regular part is nonnegative from the Second variation formula.

Proof of the Key Lemma. Assume f_t is not a subharmonic function for some $t > 0$. As a maximum of semiconvex functions all f_t for $t > 0$ are semiconvex, and as for any $t, \tau > 0$ we have $f_{t+\tau} = (f_t)_\tau$ we have that for arbitrary small t, function f_t is not subharmonic.

Take a small ball $B_\epsilon(p)$ such that $\mathrm{Lap}(B_\epsilon(p)) < 0$, $B_\epsilon(p) \subset \Omega_t$.

Let us construct a harmonic function $f' : B_p(\epsilon) \to R$ with $f'|_{\partial B_p(\epsilon)} = -f_t|_{\partial B_p(\epsilon)}$....

It is easy to see that for some point $y \in B_p(\epsilon)$ we have $f'(y) > -f_t(y)$. Therefore the maximum point of $F(x,y) = f(x) + f'(y) - t\phi(\frac{|xy|}{t})$ is reached strictly inside $B_p(\epsilon) \times (B_p(\epsilon + t) \bigcap \Omega)$.

One can take t and ϵ so small that $B_p(\epsilon + t)$ will be inside of some domain where finite strong convex function h is defined see [PP ...]. Let $f_1 = f + \epsilon' h$ and $f_1' = f + \epsilon' h$ where $\epsilon' > 0$ is so small that $F_1 = f_1(x) + f_1'(y) - t\phi(\frac{|xy|}{t})$ is having maximum strictly inside of $B_p(\epsilon) \times B_p(\epsilon + t)$.

Let (x^*, y^*) be a maximum point of F_1.

Proposition. Both points x^* and y^* are regular.

Proof of Proposition. Indeed, F_2 has the maximum at (x^*, y^*), therefore function

$$f_1(x) - t\phi(\frac{|xy^*|}{t}) = f(x) + \epsilon' h(x) - t\phi(\frac{|xy^*|}{t})$$

has a maximum at x^*.

Assume x^* is a singular point (i.e. $C_{x^*} \neq R^n$). From the maximum property we have

$$\int_{B_r(x^*)} f(x) + \epsilon' h(x) - t\phi(\frac{|xy^*|}{t}) d\nu_{x^*,r} \leq f(x^*) + \epsilon' h(x^*) - t\phi(\frac{|x^*y^*|}{t})$$

function $f(x) - t\phi(\frac{|xy^*|}{t})$ is semisubharmonic in a small neighborhood of x^*. Therefore by Lemma...

$$\int_{B_r(x^*)} f(x^*) - t\phi(\frac{|x^*y^*|}{t}) - (f(x) - t\phi(\frac{|xy^*|}{t}))d\nu_r \leq O(r^2).$$

On the other hand as h is a Lipschitz function

$$\int_{B_{x^*}(r)} h(x) d\nu_r = h(x^*) + \int_{B_{x^*}(r)} h(x) - h(x^*) d\nu_r =$$

$$= h(x^*) + \int_{B_{x^*}(r)} h(x) - h(x^*) d\nu_r' + \int_{B_{x^*}(r)} h(x) - h(x^*) d(\nu_r - \nu_r') =$$

$$= h(x^*) - cr + o(r),$$

$c > 0$. Therefore we get

$$\int_{B_{x^*}(r)} f(x)+\epsilon' h(x)-t\phi(\frac{|xy^*|}{t})d\nu_r \geq f(x^*)+\epsilon' h(x^*)-t\phi(\frac{|x^*y^*|}{t})+cr+o(r),$$

a contradiction ♠.

Set $f_2(x) = f_1(x) - \epsilon''|xx^*|^2$ and $f_2'(y) = f_1'(y) - \epsilon''|yy^*|^2$, for some positive $\epsilon'' \ll \epsilon'$. Then $F_2 = f_2(x) + f_2'(y) - t\phi(\frac{|xy|}{t})$ has unique strong maximum at (x^*, y^*) and functions f_2 and f_2' are δ-subharmonic for some $\delta > 0$.

Now as x^* and y^* are regular points one can consider a coordinate system in neighborhoods of them defined by convex functions $g_1, g_2, .., g_n$ for x^* and $g_{n+1}, g_{n+2}, .., g_{2n}$ for y^*.

Then there is $\epsilon > 0$ such that if $0 < x_i < \epsilon$

then

$$F_2(x,y) + x_1 g_1(x) + x_2 g_2(x) + ... + x_n g_n(x) + x_{n+1} g_{n+1}(y) + ... + x_{2n} g_{2n}(y)$$

admits a minimum inside of the domain.

Set $g_0 = F_2$ and apply Lemma ... we obtain that $h_{2n}(\mathrm{Im}(q)) > 0$.

Therefore there exists a collection $x_1, x_2, ..., x_{2n}$ such that

$$F_2(x,y) + x_1 g_1(x) + x_2 g_2(x) + ... + x_n g_n(x) + x_{n+1} g_{n+1}(y) + ... + x_{2n} g_{2n}(y)$$

has a maximum at (x°, y°) and this maximum points x° and y° are smooth and the shortest path $x^\circ y^\circ$ can be prolonged in both sides. Indeed as it proved in [Pet Th...] $h_n(\mathrm{Cutloc}(p)) = 0$. Therefore h_{2n} of pair of points $(x, y) \in M$ such that a shortest path xy in M can not be prolonged on one end is zero.

Set

$$f_3 = f_2 + x_1 g_1 + x_2 g_2 + ... + x_n g_n$$

$$f_3' = f_2' + x_{n+1} g_{n+1} + x_{n+2} g_{n+2} + ... + x_{2n} g_{2n}$$

and

$$F_3(x,y) = f_3(x) + f_3'(y) - t\phi(\frac{|xy|}{t})$$

Note that f_3 and f_3' are δ-subharmonic functions, ($\delta > 0$) and F_3 has a maximum at (x°, y°).

Let us choose a mapping $G : B_x(\epsilon) \to R^n$ as in [Per D 4.4]. Let us construct "exponential mapping" at x, $e : B_o(\epsilon) \subset R^n \to M$ such that $|xe(v)| = |v|$ and direction from $G(x)$ to $G(e(x))$ coincider with direction v. That is easy to see that $|\exp(v)e(v)| = o(|v^2|)$ (the proof is coincider with one in [Per D 4.4] but at one place one should change word geodesic to quasigeodesic). This give us right to use e instead of exp in the second variation formula, see [Pet ...]. Indeed

$$|e_{x^\circ}(\epsilon v)e_{y^\circ}(\epsilon T(v))| = |\exp_{x^\circ}(\epsilon v)\exp_{y^\circ}(\epsilon T(v))| + o(\epsilon^2).$$

It is easy to see that $(1 - o(v))h_n \circ e < h_n < (1 + o(v))h_n \circ e$

Proposition.

$$\int_{B(e_{x^\circ}^{-1}(x^\circ),r)} f_3(e(v)) - f_3(x^\circ) dh_n(v) =$$

$$= \int_{B(x^\circ,r)} f_3(x) - f_3(x^\circ) dh_n(x) + o(r^{n+2}).$$

Proof. As $F_3 = f_3(x) + f_3'(y) - t\phi(\frac{|xy|}{t})$ has a maximum at (x°, y°) we have that function $f_3(x) - t\phi(\frac{|xy^\circ|}{t}))$ has a maximum at x°.

Therefore $\alpha(x) = f_3(x) - t\phi(\frac{|xy^\circ|}{t}) - f_3(x^\circ) + t\phi(\frac{|x^\circ y^\circ|}{t} \geq 0$ is a semisubharmonic function. It is easy that $\beta(x) = t\phi(\frac{|x^\circ y^\circ|}{t}) - t\phi(\frac{|xy^\circ|}{t})$ is a locally Lipschitz function. Further

$$\left| \int_{B(e_{x^\circ}^{-1}(x^\circ),r)} f_3(e(v)) - f_3(x^\circ) dh_n(v) - \int_{B(x^\circ,r)} f_3(x) - f_3(x^\circ) dh_n(x) \right| =$$

$$= \left| \int_{B(x^\circ,r)} f_3(x) - f_3(x^\circ) d(h_n \circ e^{-1} - h_n)(x) \right| \leq$$

$$\leq \left| \int_{B(x^\circ,r)} \alpha(x) d(h_n \circ e^{-1} - h_n)(x) \right| + \left| \int_{B(x^\circ,r)} \beta(x) d(h_n \circ e^{-1} - h_n) \right| \leq$$

$$\leq o(r^{n+1}) \left| \int_{B(x^\circ,r)} \alpha(x) d\nu_{x^\circ,r} \right| + O(r) o(r) O(r^n) =$$

$$= o(r^{n+1}) O(r^2) + o(r^{n+2}) = o(r^{n+2}) \ \spadesuit.$$

Now from the maximum property

$$F_3(x^\circ, y^\circ) \geq F_3(e_{x^\circ}(\epsilon v), e_{y^\circ}(\epsilon T(v)).$$

Therefore

$$0 \leq \frac{1}{c_n r^n} \int_{B(r)} F_3(x^\circ, y^\circ) - F_3(e_{x^\circ}(\epsilon v), e_{y^\circ}(\epsilon T(v) =$$

$$= \frac{1}{c_n r^n} \Bigg(\int_{B_{x^\circ}(r)} f_3(x) dh_n(x) + \int_{B_{y^\circ}(r)} f_3'(y) dh_n(y) -$$

$$- \int_{B(1)} t\phi(\frac{|e_{x^\circ}(rv) e_{y^\circ}(rT(v))|}{t}) + o(r^{n+2}) \Bigg) <$$

$$< -\delta r^2 + o(r^2)$$

The last expression is negative for suficently small r, a contrudiction $\spadesuit$.

REFERENCES

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