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COLLAPSING VS. POSITIVE PINCHING

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Abstract

Let M be a closed simply connected manifold and $0 < \delta \leq 1$. Klingenberg and Sakai conjectured that there exists a constant $i_0 = i_0(M, \delta) > 0$ such that the injectivity radius of any Riemannian metric g on M with $\delta \leq K_g \leq 1$ can be estimated from below by i_0. We study this question by collapsing and Alexandrov space techniques. In particular we establish a bounded version of the Klingenberg-Sakai conjecture: Given any metric d_0 on M, there exists a constant $i_0 = i_0(M, d_0, \delta) > 0$, such that the injectivity radius of any δ-pinched d_0-bounded Riemannian metric g on M (i.e., $\mathrm{dist}_g \leq d_0$ and $\delta \leq K_g \leq 1$) can be estimated from below by i_0. We also establish a continuous version of the Klingenberg-Sakai conjecture, saying that a continuous family of metrics on M with positively uniformly pinched curvature cannot converge to a metric space of strictly lower dimension.

0 Introduction

The motivation for this work is a conjecture of Klingenberg and Sakai in positive curvature, concerning the existence of lower uniform bounds for the injectivity radius of δ-pinched manifolds in terms of the manifold and the pinching constant.

Conjecture 0.1 ([KS2]). *Let M be a closed simply connected manifold of dimension m and $0 < \delta \leq 1$. Then there exists $i_0 = i_0(M, \delta) > 0$ such that the injectivity radius i_g of any δ-pinched metric g on M, i.e., any Riemannian metric with sectional curvature $\delta \leq K_g \leq 1$, is bounded from below by $i_g \geq i_0$.*

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This conjecture is known to be true in the following cases:

- The dimension m is even and $\delta > 0$ arbitrary ([Kl1]).
  (The case dim $M = 2$ was solved by Pogorelov ([Po]) in 1946).
- The dimension m is odd and $\delta \geq 1/4 - \varepsilon$, $\varepsilon \approx 10^{-6}$ ([AM]).
  (For odd dimensions m, the case $\delta > 1/4$ was solved by Klingenberg ([Kl2]) in 1961, and the case $\delta \geq 1/4$ was solved independently by Cheeger-Gromoll ([CG]) and Klingenberg-Sakai ([KS1]) in 1980).
- M is three-dimensional and $\delta > 0$ arbitrary ([BT]) (see also [S]).
  (Here the conjecture is even true for Ricci pinching conditions, i.e., for any metric g on M^3 with $K_g \leq 1$ and $\mathrm{Ric}_g \geq \delta > 0$).

Except the Burago-Toponogov theorem, in all of the above results i_0 can in fact be chosen to be independent of the manifold M and the precise value of δ, namely, $i_0 = \pi$. However, the example of the Berger spheres (cf. [AM]) already shows that in general i_0 will depend on δ. Moreover, uniformly pinched collapsing sequences among the Aloff-Wallach, Eschenburg or Bazaikin spaces (see [AlW], [E], [B]) show that for small positive δ (in fact $\delta < 1/37$ will work, see [Pü]) there is, if one does not fix the topology of M, no chance for the conjecture to hold at all.

For odd $m > 3$ and $\delta > 0$ arbitrary the Klingenberg-Sakai conjecture is completely open. To describe our results on it, note first that in terms of Hausdorff convergence one can reformulate the Klingenberg-Sakai conjecture as follows:

Conjecture 0.1′. *Suppose that a compact simply connected manifold M admits a sequence of metrics $(g_n)_{n\in\mathbb{N}}$ with sectional curvature $\lambda \leq K_{g_n} \leq \Lambda$, such that, as $n \to \infty$, the sequence of metric spaces (M, g_n) Hausdorff converges to a compact metric space X of lower dimension (i.e., $\dim X < \dim M$). Then $\lambda \leq 0$ (i.e., these metrics cannot be uniformly positively pinched).*

Definition 0.2. *A sequence of metric spaces M_i is called stable if there is a topological space M and a sequence of metrics d_i on M such that (M, d_i) is isometric to M_i and such that the metrics d_i converge as functions on $M \times M$ to a continuous pseudometric.*

Theorem 0.3 (Stable Collapse). *Suppose that a compact (topological) manifold M admits a stable sequence of Riemannian metrics $(g_n)_{n\in\mathbb{N}}$ with sectional curvatures $\lambda \leq K_{g_n} \leq \Lambda$, such that, as $n \to \infty$, the metric spaces (M, g_n) Hausdorff converge to a compact metric space X of lower* *dimension. Then $\lambda \leq 0$ (i.e., these metrics cannot be uniformly positively pinched).*

Here is an equivalent, but "convergence-free", version of this result:

Theorem 0.3′ (Bounded version of the Klingenberg-Sakai conjecture). *Let M be a closed (topological) manifold and d_0 be a metric on M and $0 < \delta \leq 1$. Then there exists $i_0 = i_0(M, d_0, \delta) > 0$ such that the injectivity radius i_g of any δ-pinched d_0-bounded metric g on M, i.e., any Riemannian metric with sectional curvature $\delta \leq K_g \leq 1$ and $\mathrm{dist}_g(x, y) \leq d_0(x, y)$, is bounded from below by $i_g \geq i_0$.*

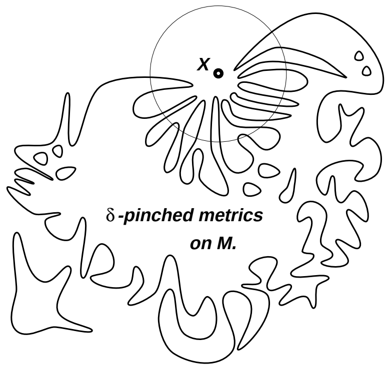
Recall that the Berger spheres constitute an example of a collapse of S^{2m+1} to $\mathbb{CP}^m$ by a continuous one parameter family of Riemannian metrics with positive curvature $0 < K \leq 1$. Our next result shows in particular that under the assumption of *positive pinching* $0 < \delta \leq K_g \leq 1$, such phenomena cannot occur, i.e., we have the following *continuous version of the Klingenberg-Sakai conjecture*:

Theorem 0.4 (Continuous Collapse). *Suppose that a compact manifold M admits a continuous one parameter family $(g_t)_{0<t\leq 1}$ of Riemannian metrics with sectional curvature $\lambda \leq K_{g_t} \leq \Lambda$, such that, as $t \to 0$, the family of metric spaces (M, g_t) Hausdorff converges to a compact metric space X of lower dimension. Then $\lambda \leq 0$ (i.e., these metrics cannot be uniformly positively pinched).*

Remark 0.5. Theorem 0.4 implies in particular the following: Suppose that the Klingenberg-Sakai conjecture is false, so that there exists a manifold M which, for some $\delta > 0$, admits a collapsing sequence $\{g_n\}$ of δ-pinched metrics. We may thus assume that the sequence of metric spaces (M, g_n) Hausdorff converges to a compact metric space X of lower dimension.

Then there exists an $\varepsilon = \varepsilon(M, \delta, X) > 0$, such that the intersection of the space of all δ-pinched metrics g on M with the Gromov-Hausdorff ε-neighborhood of X has an infinite number of connected components. Moreover, for each of these components its infimum distance to X is positive (see also the picture below).

Outline of the proofs. First note that since a compact positively curved manifold has finite fundamental group, it is enough to prove the Continuous Collapse theorem (as well as the Stable Collapse theorem) only in the simply connected case.



A. To prove the Stable Collapse theorem, we first establish

Theorem 0.6 (Gluing theorem). *Let M_n be a stable sequence of simply connected Riemannian manifolds with uniformly bounded sectional curvatures $\lambda \leq K(M_n) \leq \Lambda$ such that the sequence of metric spaces M_n Hausdorff converges to a compact metric space X of lower dimension. Then there exists a* noncompact *complete Alexandrov space $Y = Y(X, \{M_n\})$ with the same lower curvature bound λ.*

The Stable Collapse theorem then follows from the extension of Myers' theorem to Alexandrov spaces (see [BuGrPe] or [Pl]) which states that a complete Alexandrov space with lower positive curvature bound has finite diameter and hence is compact.

A′. In fact, the space Y which is constructed in the proof of the Gluing theorem will admit a free isometric action of $\mathbb{R}^{k'}$, $k' \geq 1$, such that $X = Y/\mathbb{R}^{k'}$. Note that for $\lambda \leq 0$ this construction does not yield much, since in this case $\mathbb{R} \times X$ always has curvature $\geq \lambda$.

The reader might also be interested in what this theorem gives us in the case of almost nonnegative pinching. The following result can be considered as a slight generalization of the Stable Collapse theorem, and follows from the proof of the Gluing theorem, see section 4.

Theorem 0.6a (Limit of Covering Geometry theorem). *Let M_n be a stable sequence of compact Riemannian m-manifolds with curvature bounds $-\epsilon_n^2 \leq K(M_n) \leq 1$ such that $\epsilon_n \to 0$ for $n \to \infty$ and such that the sequence of metric spaces M_n Hausdorff converges to a compact metric space X of lower dimension. Consider a sequence of points $p_n \in M_n$ and balls $B_n = B_{\pi/2} \in T_{p_n}$ which are equipped with the pullback metric under the exponential map $\exp_{p_n} : T_{p_n} \to M_n$. Then there is a converging subse-*

quence $B_n \to B$, where B has the same dimension as the manifolds M_n $(= m)$, and the following holds:

In a neighborhood of its center, the metric on B coincides with that of a metric product $\mathbb{R} \times N$, where N is a manifold with two-sided bounded curvature $0 \le K(N) \le 1$ in the sense of Alexandrov.

B. The Continuous Collapse theorem follows from the Stable Collapse theorem and the following result:

PROPOSITION 0.7. *Suppose that a simply connected manifold M admits a continuous one-parameter family of metrics $(g_t)_{0<t\le 1}$ with $\lambda \le K_{g_t} \le \Lambda$ such that, as $t \to 0$, the family of metric spaces $M_t = (M, g_t)$ Hausdorff converges to a compact metric space X of lower dimension. Then the family M_t contains a stable subsequence.*

C. The proof of the Gluing theorem is divided into two parts.

Recall that a metric g with bounded sectional curvature whose injectivity radii are small everywhere gives rise to a certain topological structure on M, a so-called N-structure of positive rank (cf. [CFGr] and section 1). In the case where M is simply connected and of bounded diameter, such a structure is actually given by an almost isometric smooth effective global torus action on M with empty fixed-point set, whose orbits, roughly speaking, contain the directions in which the injectivity radii of g are small.

The first part of the proof of the Gluing theorem consists of showing that if one applies to the metrics of a stable collapsing sequence the smoothing-averaging construction of Theorem 1.3 (see below), then the "collapsing" torus actions will be related in the following way:

PROPOSITION 0.8. *Let $\{M_n, g_n\}$ be a stable sequence of compact simply connected m-dimensional Riemannian manifolds with curvature $\lambda \le K(M_n) \le \Lambda$. Assume that, as $n \to \infty$, the sequence of metric spaces M_n Hausdorff converges to a compact metric space X of lower dimension.*

Then the sequence (M_n, g_n^ε), constructed in Theorem 1.3. has a subsequence (which we also denote by (M_n, g_n^ε)) which Hausdorff converges to a compact metric space X', and the Lipschitz distance between X and X' is $d_L(X, X') \le \varepsilon$.

Moreover there is a manifold M with an effective T^k action, a homeomorphism $h : M/T^k \to X'$, and homeomorphisms $h_n : M_n \to M$ so that the following holds: The mapping h_n conjugates the T^k action on M and the g_n^ε-isometric (collapsing) torus action on M_n, and the induced mappings

$h \circ \pi_{T^k} \circ h_n : (M_n, g_n^\varepsilon) \to X'$ (where $\pi_{T^k} : M \to M/T^k$ is the orbit space projection) are ε_n-almost isometries, where $\varepsilon_n \to 0$ as $n \to \infty$.

Roughly speaking, Proposition 0.8 says that for a stable sequence, after a suitable reparameterization one can assume that collapsing happens by shrinking invariant metrics along the orbits of a fixed T^k action on a fixed manifold M, whereas in directions not contained in these orbits the metrics only change very slightly. This kind of stability w.r.t. to the T^k action will be crucial for the construction of the Gluing theorem to work (see §2.8).

Note that the Riemannian metrics g_n^ε have almost the same pinching as g_n (see Theorem 1.3 below, or [R1]). Therefore by exchanging g_n with g_n^ε one does not lose much.

Also note that since h_n is in general not necessarily a diffeomorphism, the induced pullback metric is not necessarily Riemannian in the standard sense, i.e., it could happen that the pullback metrics are not compatible with the smooth structure on M. This will not pose problems, but we point out that by a different approach one could also obtain that h_n is smooth.

In Definition 0.2 one can actually use M with the metric $d_n(x,y) = \mathrm{dist}_{g_n}(h_n^{-1}(x), h_n^{-1}(y))$, so that the conclusion of Proposition 0.8 can be considered as a stronger version of Definition 0.2 for the case of bounded curvature. Using this definition and frame bundles one could also simplify the proof of the Stable Collapse theorem, but the general definition is needed for our proof of Proposition 0.7 and the Continuous Collapse theorem, see section 3.

C′. The second and most important part of the proof of the Gluing theorem is to construct the noncompact Alexandrov space Y, by using a special collapsing sequence as in Proposition 0.8.

The original idea of the second part of the proof the Gluing theorem is in fact very simple. Let us describe it here for the simplest example, a sequence of Berger spheres (S^3, g_n) collapsing to S^2:

For the Berger spheres, collapsing occurs along the S^1 orbits of the Hopf fibration $S^3 \to S^2 = S^3/S^1$.

Let us represent S^2 as two disks, D_1 and D_2, which are glued together along their boundaries. This corresponds to representing S^3 as the result of gluing together two solid tori, C_1 and C_2.

Consider their universal Riemannian coverings $(\widetilde{C}_1, g_n)$ and $(\widetilde{C}_2, g_n)$.

We cannot lift the gluing to glue $(\widetilde{C}_1, g_n)$ and $(\widetilde{C}_2, g_n)$ together, since the Euler class of the S^1 bundle $S^3 \to S^2$ is not zero. Here the Euler class can be geometrically interpreted as a mapping from $H_2(S^2)$ to the deck transformation group of $\widetilde{C}_i$.

Now consider the limits of the "tubes" $(\widetilde{C}_1, g_n)$ and $(\widetilde{C}_2, g_n)$ as $n \to \infty$. The marvelous fact is that the limit spaces can be glued *isometrically* together to obtain a noncompact space homeomorphic to $\mathbb{R} \times S^2$.

Here is the reason for this: Since the Euler class of this S^1 bundle is topologically fixed, and since the diameter of the S^1 orbits converge to zero, the geometric interpretation of the Euler class as an isometry of $(\widetilde{C}_i, g_n)$ will converge to the identity, and therefore the obstruction to gluing will in the limit disappear.

The remaining parts of the paper are organized as follows: In section 1, the relevant preliminaries are presented. Section 2 is concerned with the proof of the Gluing theorem modulo Proposition 0.8. In section 3, we prove Propositions 0.7. and 0.8. The proofs of the Stable Collapse theorem, the Continuous Collapse theorem and the Limit of Covering Geometry theorem are given in section 4.

We would like to thank Slava Matveyev for help with section 3.

1 Preliminaries

We gather here several notions and results about collapsed manifolds and Alexandrov spaces. As general references we mention [BuGrPe], [CFGr], [F], [GrLP], [R1] and [R2].

A Collapsed manifolds and N-structures. Let $M = (M^m, g)$ be a Riemannian manifold of dimension m and let $FM = F(M^m)$ denote its bundle of orthonormal frames. When fixing a bi-invariant metric on $O(m)$, the Levi-Civita connection of g gives rise to a canonical metric on FM, so that the projection $FM \to M$ becomes a Riemannian submersion and so that $O(m)$ acts on FM by isometries. Another fibration structure on FM is called $O(m)$ *invariant*, if the $O(m)$ action on FM preserves both its fibers and its structure group.

A *pure N-structure* on M^m is defined by an $O(m)$ invariant fibration, $\tilde{\eta} : FM \to B$, with fiber a nilmanifold isomorphic to $(N/\Gamma, \nabla^{\text{can}})$ and structural group contained in the group of affine automorphisms of the fiber, where N is a simply connected nilpotent group and ∇^{can} the canonical connection on N for which all left invariant vector fields are parallel. A pure N-structure on M induces, by $O(m)$-invariance, a partition of M into "orbits" of this structure (cf. [CFGr]), and is then said to have *positive rank* if all these orbits have positive dimension.

A pure N-structure $\tilde{\eta} : FM \to B$ over a Riemannian manifold (M, g) gives rise to a sheaf on FM whose local sections restrict to local right invariant vector fields on the fibers of $\tilde{\eta}$; see [CFGr]. If the local sections of this sheaf are local Killing fields for the metric g, then g is said to be *invariant* for the N-structure (and $\tilde{\eta}$ is then also sometimes referred to as *pure nilpotent Killing structure for* g).

In particular, if g is invariant for the N-structure, then a normal covering $\tilde{V}$ of a tubular neighborhood V of each orbit admits an isometric N-action.

Let Λ be the structural group of the covering $\pi : \tilde{V} \to V$. Then the action of Λ and N on $\tilde{V}$ generate an isometric action of some bigger Lie group H on $\tilde{V}$.

A manifold M with an N-structure and an invariant Riemannian metric g is called (ρ, k)-round (cf. [CFGr]), if for any point $p \in M$ there exist $V \supset B_\rho(p)$, $\pi : \tilde{V} \to V$ and a Lie group H as above (so that the identity component of H is our nilpotent group N), but which now also satisfy the following additional properties:

(i) the injectivity radius at $\pi^{-1}(p)$ of $\tilde{V}$ is $> \rho$;

(ii) $\#(H/N) = \#(\Lambda/\Lambda \cap N) \le k$.

Theorem 1.1 ([CFGr]). *Let, for $m \ge 2$ and $D > 0$, $\mathfrak{M}(m, D)$ denote the class of all m-dimensional compact connected Riemannian manifolds (M, g) with sectional curvature $|K_g| \le 1$ and diameter $\operatorname{diam}(g) \le D$.*

Then, given any $\varepsilon > 0$, there exists a positive number $v = v(m, D, \varepsilon) > 0$ such that if $(M, g) \in \mathfrak{M}(m, D)$ satisfies $\operatorname{vol}(g) < v$, then M^m admits a pure N-structure $\tilde{\eta} : FM \to B$ of positive rank so that

(a) *There is a smooth metric g^ε on M which is invariant for the N-structure $\tilde{\eta}$ and for which all fibers of $\tilde{\eta}$ have diameter less than ε, satisfying*

$$e^{-\varepsilon} g < g^\varepsilon < e^\varepsilon g \ , \quad |\nabla_g - \nabla_{g^\varepsilon}| < \varepsilon \, , \quad |\nabla^l_{g^\varepsilon} R_{g^\varepsilon}| < C(m, l, \varepsilon) \, ,$$
$$\text{where } C(m, 0, \varepsilon) = 1 \, ;$$

(b) *There exist constants $\rho = \rho(m, \varepsilon) > 0$ and $k = k(m, \varepsilon) \subset \mathbb{N}$ such that (M, g^ϵ) is (ρ, k)-round.*

Parts (a) and (c) of Theorem 1.1 follow from [CFGr, Theorems 1.3 and 1.7]. The fact that the N-structure $\tilde{\eta}$ in Theorem 1.1 is indeed a *pure* structure follows from the presence of a diameter bound (compare [F]).

Theorem 1.2 ([R1]). *Let the assumptions be as in Theorem 1.1. Then the nearby metric g^ε can in addition be chosen to satisfy*

$$\min K_g - \varepsilon \le K_{g^\varepsilon} \le \max K_g + \varepsilon \, .$$

The $O(m)$ invariance of a pure N-structure $\tilde{\eta} : FM \to B$ implies that the $O(m)$ action on FM descends to an $O(m)$ action on B and that the fibration on FM descends to a possibly singular fibration on M, $\eta : M^m \to B/O(m)$, such that the following diagram commutes.

$$\begin{array}{ccc} F(M^m) & \xrightarrow{\tilde{\eta}} & B \\ \Big\downarrow{\scriptstyle \pi_{O(m)}} & & \Big\downarrow{\scriptstyle \tilde{\pi}_{O(m)}} \\ M^m & \xrightarrow{\eta} & B/O(m) \end{array}$$

If $\pi_1(M^m)$ is finite, then the homotopy exact sequence shows that the fiber of a pure N-structure on M is a torus (see [R1]). If, in particular, M^m is simply connected, then since in this case the structure group of the torus fibration is trivial, a pure N-structure on a simply connected M is defined, up to an automorphism of a torus, by a global torus action.

Theorems 1.1, 1.2 and the above remark imply the following result, which we are going to use throughout this paper. (The first part of Theorem 1.3(c) below is a consequence of Perelman's Stability theorem (see [Pe1]).

Theorem 1.3. *Assume that (M_n, g_n) is a sequence of simply connected compact Riemannian m-manifolds with sectional curvature bounds $\lambda \leq K(g_n) \leq \Lambda$ and diameters $\operatorname{diam}(M_n) \leq D$ which collapses to an Alexandrov space X of dimension $m-k$. Then, given any $\varepsilon > 0$, for n sufficiently large ($n \geq n(\varepsilon)$) the following holds:*

(a) *There exists on the frame bundle FM_n of M_n an $O(m)$ invariant T^k fibration structure $T^k \to FM_n \to B_n$ for which the induced fibration on M_n is given by a smooth global effective T^k action with empty fixed-point set all of whose orbits have diameter less than ε;*

(b) *There exists on M_n a T^k invariant metric g_n^ε which satisfies*
$$e^{-\varepsilon} g_n < g_n^\varepsilon < e^\varepsilon g_n \,, \qquad \lambda - \varepsilon \leq K(g_n^\varepsilon) \leq \Lambda + \varepsilon \,;$$

(c) *The orbit space M_n/T^k is homeomorphic to X and, when equipped with the metric induced by g_n^ε, the Gromov-Hausdorff distance between X and M_n/T^k is less than ε;*

(d) *There exist constants $\rho = \rho(m,\varepsilon) > 0$ and $k = k(m,\varepsilon) \subset \mathbb{N}$ such that all (M, g_n^ϵ) are (ρ,k)-round, and the Lie group N in the definition of (ρ,k)-roundness (see above) is simply $\mathbb{R}^k = \widetilde{T}^k$.*

Remark 1.4. In fact, from [CFGr, §5] one can derive that k and ρ depend only on $\operatorname{Vol}(X)$ and $\operatorname{Diam}(X)$. This observation could even simplify our proofs, but we are not going to use it.

We will refer to the torus actions arising from Theorem 1.3(a) as the *collapsing torus actions associated to the (sufficiently collapsed) metrics* g_n.

B Hausdorff convergence and Alexandrov spaces. We will use the following definition of pointed Hausdorff convergence.

DEFINITION 1.5. *Let* Z_n *and* Z *be locally compact complete metric spaces with marked points (say,* $p_n \in Z_n$ *and* $p \in Z$*). We say that* Z_n *converges to* Z *in the Gromov-Hausdorff sense, if there is a metric* d *on the disjoint union of* $\{Z_n\}$ *and* Z *such that:*

(a) *Each* $(Z_n, d|_{Z_n})$ *is isometric to* Z_n*, and* $(Z, d|_Z)$ *is isometric to* Z*;*
(b) *For any* $R > 0$*, the balls* $B_n = B(p_n, R) \subset Z_n$ *converge to* $B = B(p, R) \subset Z$ *in the standard Hausdorff sense, i.e., for any* $\epsilon > 0$ *we have (with respect to the d-metric) that* $B \subset B_\epsilon(B_n)$ *and* $B_n \subset B_\epsilon(B)$*, when* n *is sufficiently large.*

Note that this definition can be easily adapted to the case where Z_n and Z are disjoint unions of a finite number of pointed metric spaces.

Recall that a length space is a metric space X where the distance between any two points is given by the infimum of the lengths of (continuous) curves that connect these points. A minimal geodesic $\overline{xy}$ between two points $x, y \in X$ is a constant speed curve from x to y whose length equals the distance $|xy|$ of these points. A triangle xyz in a length space X is given by three points $x, y, z \in X$ and three minimal geodesics $\overline{xy}, \overline{xz}, \overline{yz}$. If, for a real number κ, S_κ denotes the surface of constant curvature κ, a model triangle for a triangle xyz in X is a triangle $\tilde{x}\tilde{y}\tilde{z}$ in S_κ with the same side lengths. Comparison triangles exist and are unique if $\kappa \le 0$ or $|xy| + |xz| + |yz| < 2\pi/\sqrt{\kappa}$.

A length space X is called an *Alexandrov space with curvature* $\ge \kappa$ if each point $x \in X$ has a neighborhood U_x, such that for any points $a, b, c, d \in U_x$, the angles of the corresponding comparison triangles in S_κ satisfy the inequality $\tilde{\angle} bac + \tilde{\angle} cad + \tilde{\angle} dab \le 2\pi$.

Theorem 1.6 ([BuGrPe],[Pl]). *Let* X *be a complete Alexandrov space with curvature* $\ge \kappa > 0$*. Then* X *is compact with diameter* $\le \pi/\sqrt{\kappa}$*.*

To close this section, let us mention that all Alexandrov spaces that appear in this paper will be assumed to be locally compact, finite dimensional and complete.

2 Curvature Preserving Gluing and Stable Collapsing

In this section, we will prove the following result modulo Proposition 0.8.

Theorem 2.0 (Gluing theorem). *Let M_n be a stable sequence of simply connected Riemannian manifolds with uniformly bounded sectional curvatures $\lambda \leq K_{g_n} \leq \Lambda$ such that the sequence of metric spaces M_n Hausdorff converges to a compact metric space X of lower dimension. Then there exists a noncompact complete Alexandrov space $Y = Y(X, (g_n))$ with the same lower curvature bound λ.*

Since the proof of the Gluing theorem is rather long and technical, for convenience of the reader we break it into several steps:

After introducing some relevant terminology in (2.1), we will first give a proof of the Gluing theorem in the simplest case, when the associated T^k action is given by a circle action. This is done in 2.2. Then, after having explained the difficulties that arise in the case of a general T^k action (see 2.3), we will proceed to the proof of the general case.

The proof of the Gluing theorem, modulo some lemmas, is contained in sections 2.1-2.7. After an explaining remark given in section 2.8, in sections 2.9-2.12 we prove these lemmas.

2.1 Notation and conventions.

2.1.0. Let (M_n, g_n) satisfy the assumptions of the Gluing theorem.

First note that it is enough to construct $Y = Y(\varepsilon)$ only for a sequence (M_n, g_n^ε) as in Proposition 0.8. The space needed Y can then be obtained as a pointed Hausdorff limit of the spaces $Y(\varepsilon)$.

Thus from now on, let $g_n := g_n^\varepsilon$, $\lambda := \lambda - \varepsilon$, $\Lambda := \Lambda + \varepsilon$ and $X := X'$.

Let us reparameterize $M_n = (M_n, g_n)$ by $h_n^{-1} : M \to M_n$ (see Proposition 0.8). Then we can think about all mappings $h \circ \pi_{T^k} \circ h_n : (M_n, g_n) \to X$ as one fixed mapping $h \circ \pi_{T^k} : (M, g_n) \to X$.

(Here we keep the same notation for the Riemannian metric on M and for the pullback metric obtained from the homeomorphism h_n , but note again that the pullback metrics are not necessarily Riemannian in the standard sense.)

From now on, let $\pi_{T^k} := h \circ \pi_{T^k}$.

2.1.1. One can find a finite subgroup $F < T^k$ such that all isotropy groups of the $T^k := T^k/F$ action on $\bar{M} = M/F$ are connected. Let $\rho : \bar{M} \to X$ be the map induced by π_{T^k}. We will use the same notation for the metric on M and the induced metric on $\bar{M}$. One easily sees that $(\bar{M}, g_n)$ is an orbifold with the same curvature bounds as (M, g_n), i.e., on each orbifold chart the pulled back metric has curvature pinched between λ and Λ. As a factor of a space with lower curvature bound $(\bar{M}, g_n)$ has curvature $\geq \lambda$ in Alexandrov sense.

For a point $p \in \bar{M}$, let $I_p < T^k$ denote the isotropy group of the T^k action in p. Then I_p depends only on $\rho(p)$. Therefore, for I_p, we can (and will) also use the notation $I_{\rho(p)}$.

Let $\mathbb{R}^k$ be universal covering group of T^k. Let $\tilde{I}_p < \mathbb{R}^k = \widetilde{T}^k$ be the connected component of the identity of the preimage of $I_p < T^k$, and let $\Xi_p := \mathbb{Z}^k \cap \tilde{I}_p = \pi_1(I_p)$.

2.1.2. Choose a finite covering of X by contractible sufficiently small (exactly how small will be made clear in 2.11.2) closed sets with marked points (B_i, o_i), $i = 1, \dots, N$, such that the following conditions are satisfied:

(a) *The points o_i lie in interior of B_i, and on B_i, the isotropy group $I_i = I_{o_i}$ is maximal.*

(b) *All nonempty intersections $B_{ij} := B_i \cap B_j \neq \emptyset$ and $B_{ijk} := B_i \cap B_j \cap B_k \neq \emptyset$ are contractible, and inside each of these intersections there also exist points o_{ij}, o_{ijk} with maximal isotropy group $I_{ij} = I_{o_{ij}}$, $I_{ijk} = I_{o_{ijk}}$.*

(c) *There is a set B_r, $r \in \{1, \dots, N\}$, which contains only regular points (i.e., only points over which T^k acts freely).*

(The existence of such a covering can be derived from [Pe1] or [Pe2], see also 2.11.2.)

For any index $\alpha = i, ij, ijk$, with $i, j, k = 1, \dots, N$, set $C_\alpha := \rho^{-1}(B_\alpha)$ and let $\widetilde{C}_\alpha$ denote its universal covering space.

We now would first like to find metrics on $\widetilde{C}_\alpha$ which resemble the covering metrics induced by the metrics g_n, but which have nicer properties on the boundary. In order to do this, let us fix for each B_α an open set $B'_\alpha \supset B_\alpha$, so that B'_α is sufficiently close to B_α, so that B'_α is also contractible and so that B'_α also has the same maximal isotropy group I_α. Then $\widetilde{C}_\alpha$ can be viewed as a subset of $\widetilde{\rho^{-1}(B'_\alpha)}$.

Now consider $\widetilde{C}_\alpha$ with the metric induced from the covering metric of g_n on $\widetilde{\rho^{-1}(B'_\alpha)}$. Let us call the corresponding space $\widetilde{C}^n_\alpha$.

Thus in a neighborhood of any point, the metrics of the spaces $\widetilde{C}^n_\alpha$ coincide with the Riemannian covering metrics induced by the metrics g_n, which makes it possible to treat them like Riemannian metrics (see section 2.9). This construction helps to avoid possible difficulties on the boundary of $\widetilde{C}^n_\alpha$.

2.1.3. Then $\mathbb{R}^k$, the universal covering group of T^k, and $\mathbb{Z}^k = \pi_1(T^k) < \mathbb{R}^k$ act on each $\widetilde{C}_\alpha$, and, for all n, by isometry on each $\widetilde{C}^n_\alpha$. Moreover, for each α we have that $\widetilde{C}_\alpha/\mathbb{R}^k = B_\alpha$ (and this is exactly the reason why we needed to factorize along the finite group F!).

If we want to emphasize that a transformation $\xi \in \mathbb{Z}^k$ or $\mathbb{R}^k$ also defines an isometry of $\widetilde{C}^n_\alpha$, we will use the notation ξ^n, $\mathbb{Z}^k(n)$ and $\mathbb{R}^k(n)$.

Note that condition (b) above guarantees that for each index α, the group $\pi_1(C_\alpha)$ is isomorphic to $\pi_1(O_{o_\alpha}) = \pi_1(T^k)/\Xi_{o_\alpha}$.

The inclusions $C_{ij} \subset C_j$ and $C_{ijk} \subset C_{jk}$ of metric spaces define gluing maps

$$\phi_{ij} : C_{ij} \to C_j \quad \text{and} \quad \phi_{ijk} : C_{ijk} \to C_{jk}$$

which are local isometries for each g_n.

We lift those to the universal metric coverings to obtain mappings

$$\widetilde{\phi}_{ij} : \widetilde{C}_{ij} \to \widetilde{C}_j \quad \text{and} \quad \widetilde{\phi}_{ijk} : \widetilde{C}_{ijk} \to \widetilde{C}_{jk}\,.$$

(Note that these liftings are of course not unique.)

These mappings are (local!) isometries for all $\widetilde{C}^n_\alpha$, and they are easily seen to commute with the $\mathbb{R}^k$ action on each of the $\widetilde{C}^n_\alpha$. (Recall once more that if we want to emphasize that $\widetilde{\phi}_\alpha$ is also a local isometry of $\widetilde{C}^n_\alpha$, we will use the notation $\widetilde{\phi}^n_\alpha$.)

On $\widetilde{C}_{ijk}$, for a suitable element $\xi_{ijk} \in \mathbb{Z}^k \cong \pi_1(T^k)$, the local isometries $\widetilde{\phi}$ then satisfy a relation of the form

$$\widetilde{\phi}_{ji} \circ \widetilde{\phi}_{kji} = \widetilde{\phi}_{ki} \circ \widetilde{\phi}_{jki} \circ \xi_{ijk}\,.$$

For each ijk such that $C_i \cap C_j \cap C_k \neq \emptyset$, let us choose one such element ξ_{ijk}. We obtain a finite collection $\{\xi_{ijk}\} \subset \mathbb{Z}^k$.

The following lemma will be proved in section 2.11.2. and 2.12.2.

Lemma 2.1.4. *After passing to a subsequence if necessary, for any index α and for any sequence of reference points $p^n_\alpha \in \widetilde{C}_\alpha$ the sequence of pointed metric spaces $(\widetilde{C}^n_\alpha, p^n_\alpha)$ converges in the pointed Hausdorff distance to a pointed metric space $(\widetilde{C}^\infty_\alpha, p^\infty_\alpha)$. These limit spaces do not depend on the choice of reference points $p^n_\alpha \in \widetilde{C}^n_\alpha$ and have the same dimension as M.*

If in addition one of the following conditions holds:

(i) *$k = 1$, i.e., the T^k action on $\bar{M}$ is in fact a free S^1 action; or*
(ii) *T^k is generated by its isotropy subgroups on $\bar{M}$,*

then the reference points p^n_α can moreover be chosen in such a way that the local isometries

$$\widetilde{\phi}^n_{ij} : (\widetilde{C}^n_{ij}, p^n_{ij}) \to (\widetilde{C}^n_j, p^n_j) \quad \textit{and} \quad \widetilde{\phi}^n_{ijk} : (\widetilde{C}^n_{ijk}, p^n_{ijk}) \to (\widetilde{C}^n_{jk}, p^n_{jk})\,,$$

will converge to mappings

$$\widetilde{\phi}_{ij}^{\infty} : (\widetilde{C}_{ij}^{\infty}, p_{ij}^{\infty}) \to (\widetilde{C}_{j}^{\infty}, p_{j}^{\infty}) \quad \text{and} \quad \widetilde{\phi}_{ijk}^{\infty} : (\widetilde{C}_{ijk}^{\infty}, p_{ijk}^{\infty}) \to (\widetilde{C}_{jk}^{\infty}, p_{jk}^{\infty})\,;$$

where convergence is understood in the sense of the following definition:

Definition 2.1.5. *Let (Y_n, p_n) and (Z_n, q_n) be sequences of pointed metric spaces which for $n \to \infty$ Hausdorff converge to the pointed spaces (Y, p) and (Z, q), respectively. Let $f_n : Y_n \to Z_n$ be mappings from Y_n to Z_n. Then the sequence of mappings $(f_n)_{n\in\mathbb{N}}$ is said to converge to a limit map $f : Y \to Z$ from Y to Z, if for any point $y \in Y$ and any sequence of points (y_n) with $y_n \in Y_n$ that converges to y, the sequence $(f_n(y_n))$ converges to $f(y) \in Z$.*

Remark. The second part of Lemma 2.1.4 in case (ii) depends on Lemma 2.6.1 below, but there is no circle in the argument since we do not use this part of the lemma until section 2.6.

Remark. Note that when considering maps between pointed spaces we do NOT necessarily assume that these maps will preserve the base points.

Remark for easy-readers. For a first reading of the whole proof we suggest to make an extra assumption, namely, that there is an intrinsic background metric d_0 on M such that for all n it holds that $\mathrm{dist}_{g_n} \le d_0$. With this extra assumption the second part of the lemma follows from Arzela-Ascoli type of arguments. In fact (under this assumption), all points p_α^n can be chosen to be fixed points of $\tilde{C}_\alpha$.

2.2 We now first give the proof of the theorem in the simplest case, namely, in the case where the subsequence of associated collapsible T^k actions is given by a circle action, so that, in particular, X is an orbifold. As a concrete example, the reader may think of a sequence (S^3, g_n) of Berger spheres that collapses to S^2. In this case, the space Y is nothing but a metric product $Y = S^2 \times \mathbb{R}$ (see also section C′ in the introduction).

The proof of the Gluing theorem in the case of a circle action. Consider a compact simply connected manifold M. Let M satisfy the assumptions of the Gluing theorem. Assume that the collapsing torus action is given by some S^1 action on M for which all metrics g_n are invariant (see 2.1.0).

Choose a covering of the limit space X satisfying conditions (a), (b) and (c) (cf. section 2.1.2 above). As finite group $F < S^1$ we may now take the group which is generated by all isotropy groups. Then F is finite, since the S^1 action has empty fixed-point set, see 1.3(a). Let the isometric inclusions

$\phi_{ij}^n : C_{ij}^n \to C_j^n$ and $\phi_{ijk}^n : C_{ijk}^n \to C_{jk}^n$ and local isometries $\widetilde{\phi}_{ij}^n : \widetilde{C}_{ij}^n \to \widetilde{C}_j^n$ and $\widetilde{\phi_{ijk}^n} : \widetilde{C}_{ijk}^n \to \widetilde{C}_{jk}^n$ be defined as above.

Since the action of $S^1 := S^1/F$ on $\bar{M} = M/F$ is free, we may suppose that for each i, C_i and $\widetilde{C}_i$ are homeomorphic to $B_i \times S^1$ and $B_i \times \mathbb{R}$, respectively.

On $\widetilde{C}_{ijk} \cong B_{ijk} \times \mathbb{R}$, for a suitable element $\xi_{ijk} \in \mathbb{Z} \cong \pi_1(S^1)$, the local isometries $\widetilde{\phi}$ then satisfy a relation of the form

$$\widetilde{\phi}_{ji} \circ \widetilde{\phi}_{kji} = \widetilde{\phi}_{ki} \circ \widetilde{\phi}_{jki} \circ \xi_{ijk} \,. \tag{$*$}$$

When just viewed as deck transformations of $\widetilde{C}_{ijk}$, the mappings ξ_{ijk} of course do not depend on n at all. However, the essential point here is that the nontrivial elements in the finite collection $\{\xi_{ijk}^n\}$ of *isometries* of $\widetilde{C}_{ijk}^n$ also constitute the *obstruction to glue all* $\widetilde{C}_i^n$ *together by local isometries.*

REMARK 2.2.0. Obviously, in this special case the union of the ξ_{ijk} simply describes a combinatorial version of the Euler class of the circle bundle $\bar{M} \to \bar{M}/S^1$.

Thus, if all ξ_{ijk} were trivial, using the local isometries $\widetilde{\phi}$ we could glue from the tubes $\widetilde{C}_i^n \cong B_i \times \mathbb{R}$ a noncompact (and, obviously, complete) space Y which satisfied, in particular, the same lower curvature bound as g_n.

For fixed n this will not be the case. However, let us now look at what happens for $n \to \infty$, i.e., when M (together with $\bar{M}$) collapses to X.

First of all, by passing to a subsequence if necessary, we see (cf. Lemma 2.1.4 above) that for each index $\alpha = i, ij, ijk$, with $i, j, k = 1, \ldots, N$, the spaces $\widetilde{C}_\alpha^n$ converge in the (pointed) Gromov-Hausdorff topology to a limit space $\widetilde{C}_\alpha^\infty$ which, in our case, is homeomorphic to $B_\alpha \times \mathbb{R}$, and all local isometries $\widetilde{\phi}_{ij}^n, \widetilde{\phi}_{ijk}^n$ converge to local isometries $\widetilde{\phi}_{ij}^\infty : \widetilde{C}_{ij}^\infty \to \widetilde{C}_j^\infty$ and $\widetilde{\phi}_{ijk}^\infty : \widetilde{C}_{ijk}^\infty \to \widetilde{C}_{jk}^\infty$.

Now consider the sequence of isometries $\xi_{ijk}^n : \widetilde{C}_{ijk}^n \to \widetilde{C}_{ijk}^n$. As $n \to \infty$, all circle orbits *uniformly* collapse to points. Therefore for each ξ_{ijk} and any $x \in \widetilde{C}_{ijk}^n$ we have that the distances $|x\xi_{ijk}^n(x)|_{\widetilde{C}_{ijk}^n} \to 0$ as $n \to \infty$. Therefore each ξ_{ijk} converges to the identity mapping on $\widetilde{C}_{ijk}^\infty$. Thus on the limit spaces $\widetilde{C}_{ijk}^\infty$, the gluing obstruction relations $(*)$ reduce to a compatibility condition.

Therefore, we can use the collection of limit local isometries $\{\widetilde{\phi}_{ij}^\infty\}$, $\{\widetilde{\phi}_{ijk}^\infty\}$ to glue from the limit tubes $\widetilde{C}_i^\infty \cong B_i \times \mathbb{R}$ a space Y with curvature $K \geq \lambda$. Since Y is obviously noncompact, Theorem 2.0.1 is thus proved for the case of a circle action. $\square$

2.3 The difficulties that arise in the general situation and how to handle them. First of all, in the general situation we cannot assume that the given T^k action on M is free, even after factorizing the action by some finite group. As a consequence, we will have to consider additional obstructions that prevent us from gluing the tubes $\widetilde{C}_i^n$ by local isometries together (see the (S^3, T^2) example below).

Secondly, as opposed to the case of a collapse by an S^1 action, in the general situation some elements $\xi^n \in \mathbb{Z}^k(n)$ will have non-trivial limits.

To handle this problem, we will, before we glue, "mod out" the nontrivial limit group A (the small limit group of $\mathbb{R}^k(n)$, see Definition 2.6.3.1-2). The resulting factor spaces $Y_\alpha = \widetilde{C}_\alpha^\infty / A$ continue to be noncompact. Since on the Y_α gluing by local isometries is possible, we will obtain from them the desired space Y.

2.3.1. For the considerations to follow, the reader might find it convenient to have another concrete example in mind. The following (well-known one) gives a good picture of the general situation and will be referred to as the (S^3, T^2) example:

Example. Let $M = S^3 = \{(z_1, z_2) \in \mathbb{C}^2, |z_1|^2 + |z_2|^2 = 1\}$. Then the standard T^2 action on $\mathbb{C}^2$ by rotations induces on M a smooth T^2 action with empty fixed-point set which leaves the canonical metric on S^3 invariant. Now shrinking this metric tangentially to the orbits of a dense one parameter subgroup $R < T^2$ will produce on S^3 a sequence $(g_n)_{n\in\mathbb{N}}$ of T^2 invariant Riemannian metrics g_n with uniformly bounded curvatures $0 < K_{g_n} \le C$ such that for $n \to \infty$ the sequence of metric spaces (M, g_n) converges to the unit interval $X = M/T^2 = [0,1]$. (That all (M, g_n) have indeed positive curvature follows by the O'Neill formulas for example from the fact that each (M, g_n) can be expressed as a quotient of $(\mathbb{R}, g_{can}) \times (S^3, g_{can})$ by an isometric $\mathbb{R}$ action.)

Let $(B_1, o_1) = ([0, 2/3], 0)$ and $(B_2, o_2) = ([1/3, 1], 1)$, so that the induced covering of X satisfies conditions (a) and (b). (To formally satisfy also condition (c), one could set $(B_r, o_r) := (B_3, o_3) := ([1/3, 2/3], 1/2)$.) Then C_1 and C_2 are homeomorphic to solid tori, i.e., to a topological product $D^2 \times S^1$, while C_{12} is homeomorphic to the product $[1/3, 2/3] \times T^2$. In particular, the universal covering $\widetilde{C}_{12}$ of C_{12} is not contained in either $\widetilde{C}_1$ or $\widetilde{C}_2$.

The gluing maps $\widetilde{\phi}_{12} : \widetilde{C}_{12} \to \widetilde{C}_2$ and $\widetilde{\phi}_{21} : \widetilde{C}_{12} \to \widetilde{C}_1$ will then identify in $\widetilde{C}_{12}$ the orbits of two $\mathbb{Z}$-subgroups of $\mathbb{Z}^2 = \pi_1(T^2)$, this being the difference to the case of a free action. These subgroups then yield the *new obstacle* for gluing the tubes $\widetilde{C}_1$ and $\widetilde{C}_2$ together.

We now continue with the proof of the general case:

2.4 The construction of the obstruction set $\mathfrak{O}$. We will first construct the general obstruction set $\mathfrak{O}$. It will consist of a finite number of elements of the group $\mathbb{Z}^k = \pi_1(T^k)$ and of a finite number of subgroups of $\mathbb{Z}^k$:

Let us first fix pullback mappings $\widetilde{\phi}_{ij} : \widetilde{C}_{ij} \to \widetilde{C}_j$ and $\widetilde{\phi}_{ijk} : \widetilde{C}_{ijk} \to \widetilde{C}_{jk}$.

Recall that $\Xi_j := \tilde{I}_{o_j} \cap \mathbb{Z}^k = \pi_1(I_{o_j})$. Thus $\widetilde{\phi}_{ij}(p) = \widetilde{\phi}_{ij}(q)$ if and only if $p = \gamma q$ for some $\gamma \in \Xi_j$.

The collection of all groups Ξ_α will represent the first part of the obstruction set $\mathfrak{O}$.

Analogously to the S^1-case we can find for all i, j, k an element $\xi_{ijk} \in \pi_1(T^k) = \mathbb{Z}^k$ so that

$$\widetilde{\phi}_{ji} \circ \widetilde{\phi}_{kji} = \widetilde{\phi}_{ki} \circ \widetilde{\phi}_{jki} \circ \xi_{ijk} \,. \tag{$**$}$$

Equation $(**)$ can be viewed as a direct analogue of the obstruction relations $(*)$ (see 2.2) that we obtained above in the case of a free circle action.

The finite set $\mathfrak{O} := \{\Xi_\alpha\} \cup \{\xi_{ijk}\} \subset \mathbb{Z}^k$, which does not depend on n, now comprises all gluing obstructions.

Namely, note that if $\mathfrak{O}$ contains only trivial elements, then it is possible to glue all tubes $\widetilde{C}_i$ together. In particular, in this case all mappings $\widetilde{\phi}_{ij}$ and $\widetilde{\phi}_{ijk}$ are homeomorphisms onto their (respective) image.

Now we distinguish two cases:

2.5 Case 1. The isotropy groups of the T^k action on $\bar{M}$ do NOT generate T^k. Then there exists a $k-1$-dimensional subtorus $T^{k-1} \subset T^k$ that contains all isotropy groups. Consider the quotient space $M' := \bar{M}/T^{k-1}$. The sequence of T^k invariant Riemannian metrics g_n on $\bar{M}$ induces on M' a sequence of metrics d'_n that, for each n, turn M' into an Alexandrov space with uniform lower curvature bound $K \geq \lambda$. Moreover, the sequence (M', d'_n) then collapses by a free circle action to the Alexandrov space $X = M/T^k$. One can now apply the very same reasoning as in the proof of the Gluing theorem for circle actions to see that the limits of the corresponding deck transformation isometries are all trivial, and that we can glue a noncompact Alexandrov space Y with the same lower curvature bound λ. Since this observation already proves the whole theorem, our real concern is thus the next case.

(The only place which might deserve additional explanation here is the existence of the limit tubes $\widetilde{C'}^{\infty}_{\alpha}$, since in this case (M', d'_n) is formally just an Alexandrov space with a free S^1 action. But note that the new tubes can be obtained as factors of the old ones along an isometric $\mathbb{R}^{k-1} = \widetilde{T}^{k-1}$ action, i.e., $\widetilde{C'}^{n}_{\alpha} = \widetilde{C}^n_{\alpha}/\mathbb{R}^{k-1}$. Therefore their convergence follows from Lemma 2.1.4 just as well.)

2.6 Case 2. T^k is generated by the isotropy groups of the T^k action on $\bar{M}$. To continue the proof of Theorem 2.0.1 in this case, let us introduce some notation and formulate a lemma.

Let $p \in X$ be a regular point, i.e., a point for which the T^k action is free on the corresponding orbit $O_p \subset \bar{M}$. Let $\widetilde{O}^n_p$ be the universal metric covering of $O_p \subset \bar{M}$ with respect to the metric g_n.

As $\mathbb{R}^k$ acts by isometries, it follows that the intrinsic metric of $\widetilde{O}^n_p$ is flat. Now consider the homeomorphism $f : \mathbb{R}^k \to \widetilde{O}^n_p$ such that $f(\gamma) = \gamma q$, for some $q \in \widetilde{O}^n_p$. It gives us by pullback a Euclidean norm $z_n(p)$ on $\mathbb{R}^k$.

Now consider the regular tube $\widetilde{C}^n_r$ with central orbit $\widetilde{O}^n_{o_r} \subset \widetilde{C}^n_r$ that corresponds to (B_r, o_r) and the orbit O_{o_r} over o_r (see condition (c) in 2.1), and set $z_n := z_n(o_r)$.

The following lemma will be proved in section 2.9.

LEMMA 2.6.0. *There is a constant C such that for any sufficiently large n the following holds: For any α and any (not necessarily regular!) $q \in \widetilde{C}_\alpha$ and $\gamma \in \mathbb{R}^k$ one has that $|\gamma q\, q|_{\widetilde{C}^n_\alpha} \le C||\gamma||_{z_n}$.*

LEMMA 2.6.1. *Assume that T^k is generated by the isotropy groups of the T^k action on M. Then, given a background norm $||\cdot||$ on $\mathbb{R}^k > \mathbb{Z}^k$, there is $C > 0$ such that for all n and any $\gamma \in \mathbb{R}^k$ we have that $||\gamma||_{z_n} \le C||\gamma||$. Moreover, if in addition $\gamma \in \tilde{I}_\alpha$, we have that $||\gamma||_{z_n} \ge (1/C)||\gamma||$.*

The proof of this lemma will be given in section 2.10.

2.6.2 As a direct corollary of the last lemma, we obtain that there exists a subsequence of $(g_n)_{n\in\mathbb{N}}$ such that for any index α and any fixed element $\gamma \in \mathbb{Z}^k$, $\gamma^n \in \mathbb{Z}^k(n)$ converges to an isometry γ^∞ of each $\widetilde{C}^\infty_\alpha$.

This follows from the following observation: For each fixed element $\gamma \in \mathbb{Z}^k$, the sequence $(\gamma^n)_{n\in\mathbb{N}}$ of isometries $\gamma^n : \widetilde{C}^n_\alpha \to \widetilde{C}^n_\alpha$ is 1-Lipschitz, thus in particular equicontinuous. From Lemmas 2.6.1 and 2.6.0 we have $|x\,\gamma^n(x)|_{C^n_\alpha} \le C||\gamma||$. Thus by an Arzela-Ascoli argument a subsequence of (γ^n) converges to a mapping γ^∞ on each $\widetilde{C}^\infty_\alpha$. As a limit of isometries, γ^∞ is easily seen to be an isometry, too.

Note that from now on we may also use the second part of Lemma 2.1.4 in case (ii).

2.6.3. Now we give a definition of big (B-lim) and small (S-lim) limits for group actions. First, if a group Γ acts on a metric space Z and $\gamma \in \Gamma$, then by $\gamma : Z \to Z$ we will denote the mapping $z \to \gamma z$. In the following definition, convergence of mappings will be understood in the sense of Definition 2.1.5.

DEFINITION 2.6.3.1. *Let $\{Z_i\}$ be a sequence of metric spaces which converge in the Gromov-Hausdorff sense to a metric space Z. Assume that a group Γ_i acts on Z_i by isometries.*

Then we say that the groups Γ_i weakly converge if for any converging sequence of elements $\gamma_{i_k} \in \Gamma_{i_k}$ there is a sequence $\gamma_i \in \Gamma_i$ which converges to the same isometry on Z.

In this case the group of all such limits, acting on Z, will be called the big limit of Γ_i (B-lim(Γ_i)) of the groups Γ_i, i.e., B-lim$(\Gamma_i) \times Z \to Z$ and $\beta \in B$-lim(Γ) iff there exists a sequence $\gamma_i \in \Gamma_i$ which converges to β.

REMARK 2.6.3.1.1. Our definition of weak convergence might at first sight seem a little strange, but the following example should help to avoid confusion:

Let $Z_i = Z = \mathbb{R}^2$ and $\Gamma_i = \mathbb{R}$. Let Γ_i act on Z_i by horizontal translations if i is odd, and vertical translations if i is even. Then this sequence of actions does not converge weakly.

DEFINITION 2.6.3.2. *Let $\{Z_i\}$ be a sequence of metric spaces which converges in the Gromov-Hausdorff sense to a metric space Z. Assume that a group Γ acts on each Z_i by isometries.*

We say that the group Γ strongly converges *if for any fixed element $\gamma \in \Gamma$, the sequence of mappings $\gamma : Z_i \to Z_i$ converges.*

In this case the group of all such limits will be called the small limit of Γ (S-lim(Γ)), i.e., $\sigma \in S$-lim Γ if there is an element $\gamma \in \Gamma$ such that the mapping $\sigma : Y \to Y$ is a limit of a sequence of mappings $\gamma : Y_i \to Y_i$.

REMARK 2.6.3.3. To see the difference between the big limit and the small limit defined above consider the following example: Let $\mathbb{Z}$ act on $\mathbb{R}$ in the standard way by translations. Define a sequence of metrics g_n on $\mathbb{R}$ by rescaling the canonical metric g_1 on $\mathbb{R}$ with the factor $1/n$. Then S-lim$(\mathbb{Z})$ is trivial, whereas B-lim$(\mathbb{Z})$ is isomorphic to $\mathbb{R}$. Also note that our notion of big limit is related to the notion of limit group used in [FY].

Now note that we can (by passing to a subsequence if necessary) make the $\mathbb{R}^k(n)$ actions on $Z_n = \widetilde{C}_\alpha^n$ weakly converging for each α. Moreover, using Lemma 2.6.1, we may assume that they are strongly converging.

Recall that we assumed in (2.1.2, condition (c)) that one of the C_α (namely, C_r) contains only regular orbits.

Now take for Z_n in the definition above $Z_n := \widetilde{C}_r^n$ (only for one moment!). Then the big limit B-lim of the free $\mathbb{R}^k$ action on $\widetilde{C}_r^n$ is a free $\mathbb{R}^k$ action on $\widetilde{C}_r^\infty$.

Now take as Z_n the disjoint union of all $\widetilde{C}_\alpha^n$. Then the isometric effective $\mathbb{R}^k$ action on Z_n gives us as B-limit an isometric and effective $\mathbb{R}^k$ action on the disjoint union of all $\widetilde{C}_\alpha^\infty$. (Note however that Z_n is not connected, so that this action is effective does not imply that it is locally effective!)

Denote this group by $\mathbb{R}^k_{\mathrm{lim}}$.

Note that since the isometric effective $\mathbb{R}^k$ action on $\widetilde{C}_\alpha^n$ commutes with the locally isometric gluing maps $\widetilde{\phi}_{ij}^n$, the limit action of $\mathbb{R}^k_{\mathrm{lim}}$ on the disjoint union of all $\widetilde{C}_\alpha^\infty$ also commutes with the limit gluing mappings $\widetilde{\phi}_{ij}^\infty$.

Now consider $A := S\text{-lim}(\mathbb{R}^k)$. Then A is a connected subgroup of $\mathbb{R}^k_{\mathrm{lim}}$ (because it is a subspace generated by limits of a basis of $\mathbb{R}^k$), and therefore A is a linear subspace of $\mathbb{R}^k_{\mathrm{lim}}$. We have that A also acts by isometries on the disjoint union of all $\widetilde{C}_\alpha^\infty$. Moreover, since each element $\xi_{ijk} \subset \mathfrak{O}$ is contained in $\mathbb{Z}^k < \mathbb{R}^k$, we have that A contains all limits from (the second part of) the gluing obstruction set $\{\xi_{ijk}^\infty\} \subset \mathfrak{O}^\infty$.

The second part of Lemma 2.6.1 now guarantees that for each α the small limit of $\tilde{I}_\alpha$ is nondegenerate. In particular, it holds that $\tilde{I}_\alpha^\infty \overset{\text{def}}{=} B\text{-lim}(\tilde{I}_\alpha) = S\text{-lim}(\tilde{I}_\alpha) \subset A$. This implies that for each α we have that $\Xi_\alpha^\infty \overset{\text{def}}{=} B\text{-lim}(\Xi_\alpha) = S\text{-lim}(\Xi_\alpha) \subset \tilde{I}_\alpha^\infty \subset A$.

Note that since $S\text{-lim}(\Xi_\alpha)$ is not degenerate, we have that ϕ_α^∞ is still a local isometry.

Therefore $\mathfrak{O}^\infty = \{\Xi_\alpha^\infty\} \cup \{\xi_{ijk}^\infty\}$ is a collection of elements and subgroups of A.

Remark 2.6.3.4 (Example 2.3.1 continued). In the (S^3, T^2) case, one can check that the limit group A is isomorphic to $\mathbb{R}$, acting by rotation on $\widetilde{C}_1^\infty$ and $\widetilde{C}_2^\infty$ and by translation on $\widetilde{C}_{12}^\infty$.

2.7 The end of the proof of the gluing theorem. Let us now come to the last step in our construction, i.e., using the limit tubes $\widetilde{C}_\alpha^\infty$ to glue a noncompact Alexandrov space Y. Recall that for each index α, the limit tube $\widetilde{C}_\alpha^\infty$ has lower curvature bound $K \geq \lambda$ at each interior point. Since

A acts on each of the $\widetilde{C}_\alpha^\infty$ by isometries, the spaces $Y_\alpha := \widetilde{C}_\alpha^\infty/A$ satisfy in their interior the same lower curvature bounds (see [BuGrPe]).

Consider the collection of all $\widetilde{C}_\alpha^\infty$ with the limit gluing mappings $\widetilde{\phi}_{ij}^\infty, \widetilde{\phi}_{ijk}^\infty$. The obstruction set for this collection is exactly $\mathfrak{O}^\infty$.

Now, since all elements and all subgroups of $\mathfrak{O}^\infty$ are contained in A, the spaces Y_i have (with respect to the gluing mappings induced by $\widetilde{\phi}_{ij}^\infty$, $\widetilde{\phi}_{ijk}^\infty$) a trivial obstruction set. (Thus (see 2.4.) all corresponding gluing mappings are locally isometric embeddings and for them the incompatibility condition ((**) in 2.4) reduces to a compatibility relation.)

Therefore the spaces Y_i can be glued together by (local) isometries, and consequently we obtain a complete Alexandrov space Y with lower curvature bound $K \geq \lambda$.

To complete the proof of Theorem 2.0.1, it thus remains only to show that the space Y is indeed always noncompact. In the case of a free action, we have already seen that this is true, because in this situation the limit group A is trivial.

To prove our last assertion in full generality, first recall that since we have a collapse, as n goes to ∞, the diameter of each T^k orbit on (M, g_n) uniformly converges to 0.

By Lemma 2.6.1, we have that for a subsequence of $\{g_n\}$, there exists $z_{\lim}(x) = \lim_{n_i \to \infty} z_{n_i}(x)$. This gives us a norm on $R_{\lim}^k$.

Now let us choose a basis $\{e_i\}$ of $\mathbb{Z}^k < \mathbb{R}^k$. Then the limits of these basis elements, $\{e_i^\infty\}$, will be linearly dependent in $\mathbb{R}_{\lim}^k$, because the determinant of the basis $\{e_i\}$ in $(\mathbb{R}^k, z_n)$ can be interpreted as the volume of the orbit $O_{o_r}^n$ (see 2.6), and the volume of orbits goes to zero when their diameter does. Since A is the linear hull of $\{e_i^\infty\}$, we have that $A \subset R_{\lim}^k$ has positive codimension.

As we showed in 2.6.3 above, one has that $\tilde{I}_\alpha^\infty \subset A$ for any α, i.e., all isotropy groups of the $\mathbb{R}_{\lim}^k$ action are contained in A. Therefore the factor group $\mathbb{R}^{k'} = \mathbb{R}_{\lim}^k/A$ acts freely on Y. Since all orbits of this action are closed, our glued space Y is indeed noncompact and Theorem 2.0.1 is (modulo Proposition 0.8) proven. $\square$

REMARK 2.7.1 (Example 2.3.1 continued). In the (S^3, T^2) example, the glued space Y equals $Y = \mathbb{R} \times [0,1]$ equipped with the standard product metric, and Y is glued from $Y_1 = \mathbb{R} \times [0, 2/3]$ and $Y_2 = \mathbb{R} \times [1/3, 1]$.

2.8. Let us also make some remarks on what happens if one would try to apply the above arguments to more general cases.

Note that for our gluing construction to work, it is essential that the topology is fixed and that only the metrics g_n are allowed to vary.

For example, let $S^1 \to S^3 \to S^2$ be the Hopf fibration, and let $M_n = (S^3/\mathbb{Z}_n)$, where $\mathbb{Z}_n \subset S^1$, be a sequence of lens spaces collapsing to S^2. Then one still could choose a special sequence of gluing maps, such that all $\xi^n_{ijk} = 0$ except, say, ξ^n_{123}. In this case $\xi^n_{123} = ns_n$, where s_n is a shift of $\widetilde{C}^n_{123}$ corresponding to S^1 in $S^1 \to M_n \to S^2$. The elements ξ^n_{123} will converge in $\mathbb{R}^1_{\lim}$ to some nontrivial element $\xi^\infty_{123} \neq 0$. Therefore, to make our gluing construction work we have to factorize each C^∞_α by the subgroup $\mathbb{Z}\xi^\infty_{123} < \mathbb{R}_{\lim}$. As a result of the gluing construction one obtains just a compact space, namely, $Y = S^3 = \widetilde{M}_n$.

A little more interesting example is a sequence of simply connected Aloff-Wallach spaces $M_n := M_{n,n+1}$, which collapses with uniform positive pinching to $SU(3)/T^2$, and where the collapsible T^k actions are given by free circle actions. Then here one will find that, opposed to the case of a *fixed* circle action, the limit group is nontrivial. Moreover, in this case one could also find a special sequence of gluing maps such that all gluing maps will converge, and again the limits of the gluing obstacles will generate a discrete subgroup $\mathbb{Z}\xi^\infty < \mathbb{R}_{\lim}$. From the gluing construction one thus obtains just a compact space, namely, an Aloff-Wallach space, $Y = M_{1,1}$.

We now start to return our debts by giving proofs for Lemmas 2.1.4, 2.6.0, and 2.6.1.

2.9 The proof of Lemma 2.6.0. Let us show that there is a constant C such that for any sufficiently large n we have that for any α and any (not necessarily regular!) $q \in \widetilde{C}_\alpha$ and $\gamma \in \mathbb{R}^k$ it holds that $|\gamma q\, q|_{\widetilde{C}^n_\alpha} \leq C||\gamma||_{z_n}$.

Note that the point o_r is a definite distance away from the singularities in $(X, d_n) = (M, g_n)/T^k$ (see 2.1.2(c)).

Therefore for any point s in a closed neighborhood U of o_r we have that O_s has uniformly bounded second fundamental form w.r.t. each g_n. Therefore it is easy to see that for any such s, $z_n(s)$ is equivalent to $z_n(o_r)$.

Now connect $\rho(q)$ and o_r by a minimal geodesic c in (X, d_n), and consider on this geodesic a point $s \in U$ which is far enough from o_r. Take a horizontal geodesic $c^* \in (M, g_n)$ which projects to c and ends at q (by 'horizontal' we here understand 'having the same length as c'). Let s^* and o^*_r be points on c^* which correspond to s and o_r.

Note that any vector field on (M, g_n) which corresponds to an element of the Lie algebra of T^k is a Killing field, and any such field restricts to a Jacobi field on c^*.

Let v be a vector field on c^* which corresponds to an element $x \in \mathbb{R}^k$, where $\mathbb{R}^k$ is the Lie algebra of T^k. Then using curvature bounds it is easy to show that $|v_q| \le C(|v_{s^*}| + |v_{o_r^*}|) \le C'|v_{o_r^*}|$.

Integrating this inequality we get that $|\gamma q\, q|_{\widetilde{O}_q^n} \le C||\gamma||_{z_n}$. As $|\gamma q\, q|_{\widetilde{O}_q^n} \le |\gamma q\, q|_{\widetilde{C}_\alpha^n}$ the above assertion follows.

2.10 The proof of Lemma 2.6.1. Choose a basis (of $\mathbb{R}^k$) $e_1, \dots, e_k \in \mathbb{Z}^k < \mathbb{R}^k$ such that each e_i is tangent to an isotropy group of the T^k action. Then we can represent $\gamma \in \mathbb{R}^k$ as $\gamma = \Sigma_i a_i e_i$, where $a_i \in \mathbb{R}$, and (see §2.6) the z_n-norm of γ, $\|\gamma\|_{z_n}$, can be estimated by $\|\gamma\|_{z_n} \le \Sigma_i |a_i| \cdot \|e_i\|_{z_n}$.

Let us consider a fixed point of an isotropy group which is tangent to e_i. For the tangent space at this point, we have a linear representation of e_i in $\mathfrak{so}(m)$. For each g_n, the circle group $\exp(te_i)$, $t \in \mathbb{R}$, acts on the unit tangent vectors at this point by isometries.

Now we have that $\|e_i\|_{z_n} \le C(\mathrm{diam}(g_n), \max|K(g_n)|) \cdot \|e_i\|_{S^{m-1}}$, where S^{m-1} is the set of g_n-unit tangent vectors at this fix-point and $\|e_i\|_{S^{m-1}}$ is the maximal norm of the corresponding vector field on S^{m-1}.

All we need is a uniform bound for the numbers $\|e_i\|_{S^{m-1}}$, which does not depend on the metrics g_n.

We can represent $\exp(te_i)$ as an $m \times m$ matrix which contains on its diagonal 1's and rotation matrices of the following form:

$$\begin{pmatrix} \cos 2\pi n_{is} t & \sin 2\pi n_{is} t \\ -\sin 2\pi n_{is} t & \cos 2\pi n_{is} t \end{pmatrix}$$

It is easy to see that the collection of natural numbers $\{n_{is}\}$ only depends on the topology of this S^1-action, but NOT on the metric g_n.

Therefore the same is true for the norms $c_i := \|e_i\|_{S^{m-1}} = 2\pi \max n_{is}$. Consequently, one has that $\|\gamma\|_{z_n} \le C(\mathrm{diam}(g_n), \max|K(g_n)|) \cdot \Sigma_i |a_i| c_i$, which proves the first part of the lemma.

The same reasoning shows that if $\gamma \in \tilde{I}_\alpha$ then $\|\gamma\|_{z_n} \ge (1/C)\|\gamma\|$, which finishes the proof. □

2.11. We continue returning debts, this time for the first part of Lemma 2.1.4. First of all however we need to prove another lemma:

LEMMA 2.11.1. *Let $\mathcal{M}(R, \lambda_0, k)$ be a set of m-dimensional compact T^k spaces (W, T^k) with boundary and an isometric T^k action such that in the interior of each $W \in \mathcal{M}$, the sectional curvature K of W is bounded from below by $K \ge \lambda_0$ (in Alexandrov sense), such that the diameter of each T^k orbit is $\le R/2$ and such that for each W there exists a* central *T^k orbit with the following property: The distance of any interior point $x \in W$ to* *the central orbit is less than R, and the distance of any boundary point $x \in \partial W$ to the central T^k orbit is identical to R.*

Then $\mathcal{M}(R, \lambda_0, k)$ is precompact in the Hausdorff distance.

Proof of Lemma 2.11.1. For simplicity, we consider only the case where $\lambda_0 = 0$.

Let $\epsilon > 0$ be given, and consider a maximal collection $\{a_i\}$ in $W \in \mathcal{M}(R, 0, k)$ such that for all a_i, $\mathrm{dist}(a_i, \partial W) \geq \epsilon$ and $\mathrm{dist}(a_i a_j) \geq 2\epsilon$ for $i \neq j$.

Let $T^k \cdot c$ denote the central orbit of W, and let $V_0(\epsilon)$ denote the volume of a ball of radius ϵ centered on $T^k \cdot c$.

The Bishop-Gromov volume comparison theorem implies that for each a_i, the volume of an ϵ–ball $B_\epsilon(a_i)$ around a_i then satisfies the estimate

$$\begin{aligned}\mathrm{Vol}(B_\epsilon(a_i)) \geq \tfrac{\epsilon^m}{R^m} V_0(\epsilon) &\geq \tfrac{\epsilon^{2m}}{R^{2m}} \mathrm{Vol}(B_{R/2}(T^k \cdot c) \\ &\geq \tfrac{\epsilon^{2m}}{2^m R^{2m}} \mathrm{Vol}(W)\,.\end{aligned}$$

Therefore the cardinality of any such collection $\{a_i\}$ in W is bounded from above by a number which depends only on R, λ_0, k. Obviously $\{a_i\}$ is a 3ϵ-net, therefore the assertion that $\mathcal{M}(R, \lambda_0, k)$ is precompact now follows in exactly the same way as in the proof of Gromov's original compactness theorem (cf. [GrLP]). □

Remark 2.11.1.1. For Lemma 2.11.1 to hold, it is only essential that for each $W \in \mathcal{M}$ there exists some kind of "center", i.e., a subspace $C \subset W$ to which all boundary points $x \in \partial W$ have distance $\mathrm{dist}(x, C) = R$. However, this assumption is also a crucial one. Just think of an infinite sequence of "hairy" manifolds with uniformly bounded diameter, so that the Gromov-Hausdorff distance between each pair is at least 1.

2.11.2 Proof of the first part of Lemma 2.1.4. As our covering $\{B_i\}$ is sufficiently fine, we can assume that for sufficiently large n, the points inside C_α^n have distance $< R = R(\alpha)$ from some orbit O_x, where $x = x(\alpha) \in X$, and where R and x have the following properties:

(i) If $D_\alpha^n(R) = \bar{B}(O_x, R) \subset (\bar{M}, g_n)$ is the closed R-neighborhood of O_x, then the set of orbits in $D_\alpha^n(R)$ is contractible and x has maximal isotropy group in $D_\alpha^n(R)$;

(ii) $R \leq \rho$, where ρ is as in Theorem 1.3.

(To see this, it is enough to look at the factor space $(X, d_n) = (\bar{M}, g_n)/T^k$ and to use the fact that (X, d_n) is an Alexandrov space which is sufficiently close to (X, d_∞): First construct a covering of (X, d_∞) by contractible balls which agrees with the natural stratification, i.e., if a ball intersects a stratum then its center is on the closure of this stratum. Then for sufficiently large n we have that the corresponding covering with the same radii and centers is also a covering of (X, d_n). Therefore, if the covering $\{B_\alpha\}$ is fine enough, we have that each B_α is contained in a ball like this, and also B'_α (from 2.1.2) can be chosen to be in such a ball. The existence of such a covering follows from [Pe1] (and also [Pe2]).)

Now we will prove the existence of $\widetilde{D}^\infty_\alpha(R) = \lim_{n\to\infty} \widetilde{D}^n_\alpha(R)$.

Since the diameter of each T^k orbit goes to 0, it is easy to find a finite covering $E^n_\alpha \to D^n_\alpha(R)$ of $D^n_\alpha(R)$ for which all orbits have diameter $\leq R/2$, and such that for some $\epsilon > 0$ which does not depend on n, the injectivity radius of points lying on the central orbit will be $\geq \epsilon$. (The last assertion follows directly from Theorem 1.3(d)).

By Lemma 2.11.1, the sequence (E^n_α) has a limit E^∞_α. This compact limit space has the same dimension as C_α, since by the uniform lower bound for the injectivity radius for points on the central orbit and the upper curvature bound, also the volumes of the E^n_α are uniformly bounded from below (by some $v = v(R, |F|, \epsilon, \Lambda)$).

Let $\widetilde{E}^\infty_\alpha$ and $\widetilde{E}^n_\alpha$ be the universal metric coverings of E^∞_α and E^n_α (and $D^n_\alpha(R)$), respectively.

The convergence of E^n_α to E^∞_α induces, for n sufficiently large, isomorphisms $\pi_1(E^n_\alpha) \to \pi_1(E^\infty_\alpha)$. Therefore we have that the universal metric coverings $\widetilde{E}^n_\alpha$ converge in the pointed Gromov-Hausdorff topology to $\widetilde{E}^\infty_\alpha$. In particular, this implies that $\widetilde{D}^\infty_\alpha(R)$ exists and that it is isometric to $\widetilde{E}^\infty_\alpha$.

Now, to insure the existence of $\widetilde{C}^\infty_\alpha$, we use the fact that a subsequence of the preimages F^n_i and $F^{n\prime}_i$ of B_α and B'_α (see 2.1.2) inside E^n_α will converge to a subspaces F^∞_α and $F^{\infty\prime}_\alpha$ of E^∞_α. By lifting, as above, the respective Hausdorff approximations to the universal covering $\widetilde{C}_\alpha$, we see that a subsequence of the sequence $(\widetilde{C}^n_\alpha)$ converges to a limit space $\widetilde{F^\infty_\alpha}$ with the metric induced from $\widetilde{F^\infty_\alpha}'$. That this limit is independent of the choice of reference points and has the same dimension as $\widetilde{C}_\alpha$ then follows from the corresponding properties of $\widetilde{E}^\infty_\alpha$ established above.

Now step-by-step passing to subsequences of $\{g_n\}$ will give us a subsequence which we will again call $\{g_n\}$ such that the $\widetilde{C}_\alpha$ converge for each α. □

2.12. In this section we will prove the second part of Lemma 2.1.4.

Let us first explain why in general we cannot choose fixed reference points in $\widetilde{C}_\alpha$, so that the second part of Lemma 2.1.4 would be true.

EXAMPLE 2.12.1. We consider again the (S^3, T^2)-example, but now twist its parameterization.

Let the standard projection be $\rho : S^3 \to S^3/T^2 = [0,1]$. Consider a sequence of smooth functions $\tau_n : [0,1] \to \mathbb{R}^2 = \widetilde{T}^2$ (the universal covering group of T^2) which are zero near the endpoints, but for which $\tau_n(1/2)$ behaves very wildly when $n \to \infty$. Now consider the reparametrization $h_n : S^3 \to S^3$ by $x \to \tau_n(\rho(x))x$. These maps are diffeomorphisms because each T^2 fiber just is mapped to itself by translation. Now the diffeomorphisms h_n could be used in Definition 0.2 just as well as the identity map.

Assume that we want to choose p_1, p_2 and p_{12} such that $\rho(p_i)$ would be near the end points of $[0,1]$ and $\rho(p_{12}) = 1/2$.

Looking at $(\widetilde{C}_1^n, p_1), (\widetilde{C}_2^n, p_2)$ and $(\widetilde{C}_{12}^n, p_{12})$, we see that the function τ_n could be chosen that "bad" that all $|\widetilde{\phi}_{12}^n(p_{12})\, p_2|_{\widetilde{C}_2^n} \to \infty$ as well as $|\widetilde{\phi}_{21}^n(p_{12})\, p_1|_{\widetilde{C}_1^n} \to \infty$ when $n \to \infty$. Therefore there is no chance to have a limit for the gluing maps $\widetilde{\phi}_{ij}^n$.

(In this case one could easily avoid these problems by choosing all p_α to have $\rho(p_\alpha) = 1/2$. But an analogous example, where we would cover X without a common intersection of all tubes, would show that in general one cannot get rid of this trouble that easy.)

Now we start to construct these reference points.

2.12.2 Proof of the second part of Lemma 2.1.4. Note that X is simply connected (since M is, and X is homeomorphic to M/T^k). Remove from X an ϵ-neighborhood of its boundary (if any) and remove all singular sets of codimension ≥ 3. This remove can be done in such a way that the remaining part of X is still simply connected. Indeed, cutting neighborhoods of singular sets of codimension ≥ 3 corresponds to cutting from M a neighborhood of a collection of submanifolds of codimension ≥ 3. Therefore the remaining part is still simply connected and its factor along a T^k action is also simply connected. Finally, removing a neighborhood of the boundary obviously also preserves the property of being simply connected. (By [PePet, 3.1(2)], this remove can be done even in such a way that the remaining part of X will be a bi-Lipschitz manifold with boundary, where under a bi-Lipschitz manifold with boundary we understand a metric space such that each point has a neighborhood which is bi-Lipschitz homeomorphic to an open domain of $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n ; x_1 \geq 0\}$). Remove, if necessary, from this part neighborhoods of some other submanifolds so that the remaining part will have finite second homotopy group, and still be a bi-Lipschitz manifold with boundary. (That this can be done follows from standard techniques in topology.) Let us denote the remaining part of X by X_ϵ.

Let $\bar{M}_\epsilon \subset \bar{M}$ be the ρ-preimage of X_ϵ. The T^k action on $\bar{M}_\epsilon$ is free. (This is because the only singular sets which are left in X_ϵ have codimension 2, and since those sets are responsible for the finite isotropy groups in M, it follows that all isotropy groups in $\bar{M} = M/F$ are trivial, see 2.1.1).

Moreover, since $\pi_2(X_\epsilon)$ is finite, we have that the bundle $T^k \to \bar{M}_\epsilon \to X_\epsilon$ has zero Euler class. Therefore $\bar{M}_\epsilon$ is homeomorphic to $T^k \times X_\epsilon$. Now let us construct a metric on its universal covering $\widetilde{\bar{M}_\epsilon}$. To do this, we can assume that there is a finite collection $\{B_{r_k}\} \subset \{B_i\}$ such that $X_\varepsilon = \bigcup_k B_{r_k}$. Moreover, since over X_ε the T^k-fibration is topologically trivial, we may assume that all $\xi_{r_i r_j r_k} = 0$. Therefore we can glue all $\widetilde{C}^n_{r_k}$ together. Call this metric space $\widetilde{\bar{M}^n_\epsilon}$.

From this reasoning and the first part of Lemma 2.1.4 we thus obtain the existence of a limit space $\widetilde{\bar{M}^\infty_\epsilon} = \text{GH-lim}\, \widetilde{\bar{M}^n_\epsilon}$ (which is isometric to a space glued from the tubes $\widetilde{C}^\infty_{r_k}$ by ANY collection of gluing mappings for which the corresponding $\xi_{r_i r_j r_k}$ are trivial).

Let us choose a point $p \in \widetilde{\bar{M}^n_\epsilon}$. For some fixed $R \gg \text{diam}(X_\epsilon)$, consider the metric ball $B(p,R) \subset \widetilde{\bar{M}^n_\epsilon}$. Now R can be chosen that big that all orbits in $\widetilde{\bar{M}^n_\epsilon}$ will have nonempty intersection with $B(p,R)$ (for all n). Moreover, from the existence of $\widetilde{\bar{M}^\infty_\epsilon}$ it is easy to see that there exists $D' >> R$ such that for any two points in one orbit $x, y \in O_x \subset \widetilde{\bar{M}^n_\epsilon}$ we have that $|xy|_{O_x} \le D'$ if $|xy|_{\widetilde{\bar{M}^n_\epsilon}} \le 2R$ (for all n). I.e., the intersection of $B(p,R)$ with any orbit in $\widetilde{\bar{M}^n_\epsilon}$ has a uniform bound for the diameter (in the intrinsic metric of this orbit), which does not depend on n.

Now ϵ can be chosen that small that each B_α will contain some $x_\alpha \in X_\epsilon$ and that moreover $B_\alpha \cap X_\epsilon$ will be connected. For each α we fix a mapping m_α from $O_{x_\alpha} \subset \widetilde{\bar{M}_\epsilon}$ to the corresponding orbit $O'_{x_\alpha} \subset \widetilde{C}_\alpha$.

Take any point $q^n_\alpha \in O_{x_\alpha} \cap B(p,R) \subset \widetilde{\bar{M}^n_\epsilon}$ and set $p^n_\alpha := m_\alpha(q^n_\alpha) \in \widetilde{C}_\alpha$. (The set $B(p,R) \subset \widetilde{\bar{M}^n_\epsilon}$ depends on the metric and therefore on n, therefore q^n_α must depend on n.)

Then we can find (see below) a finite collection of elements $\{s_\alpha\} \subset \mathbb{Z}^k$ such that $|\widetilde{\phi}^n_{ij}(p^n_{ij})\, s_{ij} p^n_j|_{\widetilde{C}^n_j} \le C$ and $|\widetilde{\phi}^n_{ijk}(p^n_{ijk})\, s_{ijk} p^n_{jk}|_{\widetilde{C}^n_{jk}} \le C$ for some fixed $C > D'$ and all n, i, j, k.

This collection $\{s_\alpha\}$ will only depend on the choice of covering maps

$m_\alpha : O_{x_\alpha} \to O'_{x_\alpha}$ and on the gluing maps $\widetilde{\phi}_{ij}$. Since all these mappings do not depend on n, we have that the collection $\{s_\alpha\}$ also does not depend on n.

(Here let us explain how to find the collection of elements $\{s_\alpha\} \subset \mathbb{Z}^k$. First we construct the elements s_{ij}: Without loss of generality we can assume that there is k such that x_{ij} and x_j belong to the set B_{r_k}. The gluing map $\widetilde{\phi}_{r_k j} : \widetilde{C}_{r_k j} \to \widetilde{C}_j$ induces by restriction covering maps $m_\alpha : O_{x_j} \to O'_{x_j} \subset \widetilde{C}_j$ and $O_{x_{ij}} \to O''_{x_{ij}} \subset \widetilde{C}_j$ for the orbits over x_j as well as the orbits over x_{ij}.

Therefore, for appropriately chosen s'_{ij} and s''_{ij} in $\mathbb{Z}^k$ we have that $s'_{ij}\widetilde{\phi}_{r_n j}|_{O_{x_{ij}}} = \widetilde{\phi}_{ij} \circ m_{ij}$ and $\widetilde{\phi}_{r_n j}|_{O_{x_j}} = s''_{ij} m_j$.

Then it obviously holds that $|\widetilde{\phi}_{r_n j}(q^n_{ij})\, \widetilde{\phi}_{r_n j}(q^n_j)| \le C$ for some fixed C and all n,i,j. Therefore, by taking $s_{ij} = s'_{ij} s''_{ij}$ we have that $|\widetilde{\phi}^n_{ij}(p^n_{ij}) s_{ij} p^n_j|_{\widetilde{C}^n_j} \le C$. In the same way one can construct s_{ijk}.)

Now in the case of a circle action (case (i)) we have that since the diameters of the orbits uniformly converge to zero, for any fixed element $s \subset \mathbb{Z}^k$ we have that sp converges to p for any p in any $\widetilde{C}^n_\alpha$. In case (ii), by Lemma 2.6.1 we have that $|p\, sp|$ is uniformly bounded. Therefore in both cases it holds that $|\widetilde{\phi}^n_{ij}(p^n_{ij})\, p^n_j|_{\widetilde{C}^n_j} \le C'$ and $|\widetilde{\phi}^n_{ijk}(p^n_{ijk})\, p^n_{jk}|_{\widetilde{C}^n_{jk}} \le C'$ for some $C' < \infty$ and all n, i, j, k.

The mappings $\widetilde{\phi}^n_{ij}, \widetilde{\phi}^n_{ijk}$ are local isometries, in particular they are 1-Lipschitz. Thus by an Arzela-Ascoli argument we obtain the second part of Lemma 2.1.4. □

3 The Propositions on Continuous Collapse and Stability

In this section we prove the following claims:

Proposition 3.1.A (Proposition 0.7 in the Introduction). *Suppose that a simply connected manifold M admits a continuous one-parameter family of metrics $(g_t)_{0<t\le 1}$ with $\lambda \le K_{g_t} \le \Lambda$ such that, as $t \to 0$, the family of metric spaces $M_t = (M, g_t)$ Hausdorff converges to a compact metric space X of lower dimension. Then the family M_t contains a stable subsequence which converges to the same space X.*

Proposition 3.1.B (Proposition 0.8 in the Introduction). *Let $\{M_n, g_n\}$ be a stable sequence of compact simply connected m-dimensional Riemannian manifolds with curvature $\lambda \le K(M_n) \le \Lambda$. Assume that, as $n \to \infty$, the* *sequence of metric spaces M_n Hausdorff converges to a compact metric space X of lower dimension.*

Then the sequence (M_n, g_n^ε), constructed in Theorem 1.3. has a subsequence (which we also denote by (M_n, g_n^ε)) which Hausdorff converges to a compact metric space X', and the Lipschitz distance between X and X' is $d_L(X, X') \leq \varepsilon$.

Moreover there is a manifold M with an effective T^k action, a homeomorphism $h : M/T^k \to X'$, and homeomorphisms $h_n : M_n \to M$ so that the following holds: The mapping h_n conjugates the T^k action on M and the g_n^ε-isometric (collapsing) torus action on M_n, and the induced mappings $h \circ \pi_{T^k} \circ h_n : (M_n, g_n^\varepsilon) \to X'$ (where $\pi_{T^k} : M \to M/T^k$ is the orbit space projection) are ε_n-almost isometries, where $\varepsilon_n \to 0$ as $n \to \infty$.

Let us first gather some properties from Theorem 1.3 into the following definition:

Definition 3.2. *Let M be a compact simply connected manifold. Suppose that M admits a global smooth effective T^k-action without fixed points and a Riemannian metric g with $|K(g)| \leq 1$ and $\operatorname{diam}(g) \leq D$ such that*

(a) *the metric g is T^k-invariant;*
(b) *all T^k-orbits have diameter $\leq \epsilon$;*
(c) *the volume of the factor space M/T^k is $\operatorname{vol}(M/T^k) \geq V$.*

Then the T^k action will be called (ϵ, D, V)-collapse related *to (M, g).*

Now let M be as in the definition. Then $X = M/T^k$ is an Alexandrov space with curvature ≥ -1 (see section 1). X has a natural stratification, where each stratum is a connected component of subsets of X which have the same isotropy group $A \subset T^k$.

In all what follows, the notion of ϵ-almost isometry or ϵ-isometry will always be understood in the Gromov-Hausdorff sense. The proof of the next lemma uses a center of mass technique and may be compared with [CFGr, Thm 2.6].

Lemma 3.3. *There exists $c = c(V, D, m)$ and $\epsilon = \epsilon(V, D, m) << c$ with the following property: Let (M_i, T^k), $i \in \{1, 2\}$ be two simply connected Riemannian m-manifolds with torus actions which are (ϵ, D, V)-collapse related, and assume that there is a homeomorphism $I : M_1 \to M_2$ which is an ϵ-isometry. Then*

(a) *there is an automorphism $A : T^k \to T^k$ and a conjugation mapping $h : (M_1, T^k) \to (M_2, A(T^k))$ such that h is homotopy equivalent and 3ϵ-close to I.*

(b) *Let $X_i = M_i/T^k$. Since the map h (from (a)) is a conjugation mapping, one has an induced mapping $h' : X_1 \to X_2$.*

Then any homeomorphism $m_0 : X_1 \to X_2$ which preserves the natural stratifications of X_1, X_2 and which is c-close to h', can be lifted to a homeomorphism $m : M_1 \to M_2$, such that the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{m} & M_2 \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow{m_0} & X_2 \end{array}$$

Proof of part (a). We will obtain the mapping h by deforming the given homeomorphism $I : M_1 \to M_2$.

Let us first construct an automorphism $A : T_1^k \to T_2^k$.

Note that $d_{GH}(M_1, M_2) \le \epsilon$, and that $d_{GH}(M_i, X_i) \le \epsilon$, $i = 1, 2$. Thus $d_{GH}(X_1, X_2) \le 3\epsilon$. We will use a metric on disjoint union of X_1, X_2, M_1, M_2 such that $d(x, I(x)) = d(x, \pi_1(x)) = d(I(x), \pi_2(I(x)) = \epsilon$ for any $x \in M_1$.

Therefore we can choose ϵ so small that there is a regular point in $x \in M$ such that the 100ϵ-neighborhood of O_x contains only regular orbits and for any $r \le 100\epsilon$ the r-neighborhood of $\pi_1(x)$ in X_1 is homeomorphic to a standard ball. (This follows for instance from the existence of a regular point in an Alexandrov space and from the regularity of its small spherical neighborhood, see [Pe1] or [Pe2]).

Consider now the homeomorphisms

$$B_{96\epsilon}(T_1^k x) \xrightarrow{I} B_{98\epsilon}(T_2^k I(x)) \xrightarrow{I^{-1}} B_{100\epsilon}(T_1^k x)\,.$$

Since $B_r(T_1^k x)$ is a T^k-bundle over the ball $B_r(\pi_1(x))$, it is homeomorphic to $T^k \times B_r(\pi_1(x))$. Therefore the homeomorphisms above induce an isomorphism $A : \pi_1(T_1^k) \to \pi_1(T_2^k)$. The isomorphism A is uniquely extendible to an isomorphism $A : T_1^k \to T_2^k$, and this isomorphism does not depend on the choice of point.

Thus from now on we can think that the same torus T^k is acting on both manifolds M_1 and M_2. Now we modify a little the center of mass technique from [GroK]. Let (M, g) be a Riemannian manifold.

Definition 3.4. (i) Let (G, μ) be a connected space with probability measure and $\eta : G \to (M, g)$ be a continuous mapping. Let $p \in M$. Then a continuous mapping $N_p : G \to T_p(M)$ for which $\eta = \exp_p \circ N_p$ is called a *lifting of η at p*.

(ii) Assume that for any $x \in G$ there is a lifting $N^\circ_{\eta(x)}$ such that $N^\circ_{\eta(x)}(x) = 0 \in T_{\eta(x)}(M)$. Then the *lifted diameter* of the mapping η

(or $l\,\mathrm{diam}(\eta)$) is defined as the minimum of all numbers d, such that $|N^\circ_{\eta(x)}(y)| \le d$ for any $x, y \in G$.

(iii) A point p is called *center of mass* of the mapping $\eta : G \to (M, g)$ ($p = \mathcal{C}_G(\eta)$), if there is a lifting N_p which minimizes the integral $\int_G |N_p(x)|^2 d\mu(x)$ on the set of all liftings of η at all points in M where liftings are defined.

LEMMA 3.5. *Let M be a manifold with $|K(M)| \le 1$, let G be a connected space and let $\eta : G \to M$ be a mapping with $l\,\mathrm{diam}(\eta) < \pi/4$. Then the center of mass for this mapping is well defined.*

Proof. Consider the open manifold (B, g), where $B = B_\pi(0) \subset T_{\eta(x)}(M)$ and g is the pullback metric by the exponential mapping $\exp_{\eta(x)}$. Then B has curvature $|K_g| \le 1$, $B_{\pi/2}(0)$ is a convex subset of (B, g), and the injectivity radius of any point in $B_{\pi/2}(0)$ is at least $\pi/2$.

On (B, g) the exponential map $\exp_{\eta(x)} : B \to M$ is a local isometry. For $p^* \in B$ let $i_{p^*} : T_{p^*}(B) \to T_{\exp_{\eta(x)}(p^*)}(M)$ be the induced isometry of the tangent spaces.

Let p be a center of mass of the mapping $\eta : G \to (M, g)$, and let (as in 3.4) N_p be the corresponding (minimizing) lifting. We have that $|N_p(x)| < \pi/2$, because otherwise

$$\int_G |N_p(x)|^2 d\mu(x) \ge \int_G (\pi/4)^2 d\mu(x) > \int_G |N^\circ_{\eta(x)}(x)|^2 d\mu(x)\,.$$

Therefore one can lift p to a point $p^* \in B_{\pi/2}(0) \subset (B, g)$ (i.e., $\exp_{\eta(x)}(p^*) = p$) in such a way that $N^\circ_{\eta(x)} = \exp_{p^*} \circ i^{-1}_{p^*} \circ N_p$.

From here we have that p^* is the standard ([GroK]-) center of mass of the mapping $N^\circ_{\eta(x)} : G \to (B, g)$. Since p^* is well defined (see [GroK]), the same is true for OUR center of mass. $\square$

We now give an outline of arguments from [GroK] how to construct a conjugation map between two given T^k-actions, which is homotopic and 3ϵ-close to the mapping I:

First let us consider the mapping $\eta : M_1 \times T^k \to M_2$, $\eta_x(t) = t^{-1} I(tx)$.

From the construction of the isomorphism $T_1^k \to T_2^k$ one sees that $l\,\mathrm{diam}(\eta_x(T^k)) \le 3\epsilon$. Now consider the conjugation map $h : M_1 \to M_2$, $h(x) = \mathcal{C}_{T^k}(\eta_x(t))$, where $\mathcal{C}$ is the center of mass.

Part (a) of Lemma 3.3 is proved. $\square$

Note that the isomorphism $T_1^k \to T_2^k$ we constructed above is the only one which makes the lifted diameter $l\,\mathrm{diam}(\eta_x(T^k))$ small.

Also note that if we assume $d_{GH}(M_1, M_2) < \delta(M_1)$, one could improve the above result to obtain a conjugation *diffeomorphism*. To do this, it is necessary to construct a mapping I which would make the T^k-actions C^1-close. This is possible by applying harmonic charts and the above arguments plus [GroK].

Proof of part (b). Note that if we assume that the action is free, then in order to obtain (b) it is enough to show that two bundles π_2 and $m_0 \circ \pi_1$ have the same generalized Euler class. (I.e., since X_i is simply connected, this is just the mapping from the exact homotopy sequence $\pi_2(X_i) \overset{e}{\to} Z^k = \pi_1(T^k)$). But the Euler class is a homotopy invariant, therefore the constructed homotopy conjugation shows that the Euler classes are the same.

Now let us come back to the general case. First note that for the conjugation map h constructed in part (a) one has that the isotropy group of the image ($h(x) \in M_2$) is bigger than the isotropy group of the preimage ($x \in M_1$). Therefore we have the same collection of isotropy groups in M_1 and M_2, because one can repeat the same construction to obtain a conjugation map $M_2 \to M_1$. Hence we can find a finite subgroup $F < T^k$ such that the $T^k := T^k/F$ action on $\bar{M}_i = M_i/F$ has only connected isotropy groups.

Note also that the conjugation map $h : M_1 \to M_2$ induces a mapping $h' : X_1 \to X_2$ which is an almost isometry and homotopy equivalence, and from the above we have that a stratum $Str_1 \subset X_1$ corresponds to a stratum $Str_2 \subset X_2$ if

$$closure(Str_2) = h'(closure(Str_1)) \,.$$

Now let us choose a representation $T^k = S^1_1 \times S^1_2 \times \ldots \times S^1_k$. Consider then the sequence of factors $\bar{M}_i = \bar{M}_{i,k}, \bar{M}_{i,k-1}, \ldots, \bar{M}_{i,0} = X_i$, such that $\bar{M}_{i,l-1} = \bar{M}_{i,l}/S^1_l$.

Now assume that we already have a homeomorphism $m_{l-1} : \bar{M}_{1,l-1} \to \bar{M}_{2,l-1}$ which commutes with π_i and m_0. Let us construct $m_l : \bar{M}_{1,l} \to \bar{M}_{2,l}$.

Take the subset of $\bar{M}^*_{i,l}$ where the S^1_l-action is free, and let $\bar{M}^\circ_{i,l-1}$ be the projection of this subset to $\bar{M}_{i,l-1}$.

Now a free S^1 action with quotient space $M^\circ_{i,l-1}$ is completely determined by its Euler class.

From the existence of a homotopy conjugation almost isometry it is easy to see (see also Remark 3.6 below) that the Euler class is the same for each pair $\bar{M}^*_{1,l} \to \bar{M}^\circ_{1,l-1}$ and $\bar{M}^*_{2,l} \to \bar{M}^\circ_{2,l-1}$.

Since on the remaining part $\bar{M}_{i,l} \backslash \bar{M}^*_{i,l}$ our S^1_l-action is trivial, any homeomorphism $m_l : \bar{M}^*_{1,l} \to \bar{M}^*_{2,l}$ can be extended (uniquely) to a homeomorphism $m_l : \bar{M}_{1,l} \to \bar{M}_{2,l}$ (which commutes with m_{l-1}).

Therefore, given a homeomorphism $m_0 : X_1 \to X_2$, we can construct step by step a sequence of homeomorphisms $m_l : \bar{M}_{1,l} \to \bar{M}_{2,l}$ which will commute with each other. In particular, we have a homeomorphism $m_k : \bar{M}_1 \to \bar{M}_2$.

We only have to prove that we can lift m_k to a homeomorphism $m : M_1 \to M_2$. Now $M_i \to \bar{M}_i$ is a branched covering of $\bar{M}_i$ with structural group F. Therefore we only have to check that both of these mappings give the same mapping $\pi_1(\bar{M}_i^{\#}) \to F$, where $\bar{M}_i^{\#}$ is the image of the set of all regular points $M_i^{\#} \subset M_i$, i.e., $M_i^{\#}$ is the set of all points where the T^k-action is free. But this is again true because of the existence of a close homotopy equivalence conjugation (see Remark 3.6 below), so that the lemma is proved. □

REMARK 3.6. Formally the image of an element of the homotopy class of $\bar{M}^*_{1,l}$ or $M_1^{\#}$ could have points which are not contained in $\bar{M}^*_{2,l}$ or $M_2^{\#}$, respectively. However, as follows from [PePet, 3.1(2)], in any homotopy class we always can find an element which is sufficiently far from the corresponding singularities, and therefore its image will belong to the corresponding space.

Proof of Propositions 3.1.A and B. A. Let us choose a subsequence (M, g_n) of (M, g_t) such that $||xy|_{g_n} - |xy|_{g_{n+1}}| < \epsilon/2$, i.e., the identity map on M gives an $\epsilon/2$-Hausdorff approximation for (M, g_n) and (M, g_{n+1}). By applying the smoothing-averaging procedure for an appropriately chosen sequence $\epsilon_n \to 0$ (see Theorem 1.3), we obtain a sequence of metrics $\{g_n^{\varepsilon_n}\}$ with collapse related invariant T^kactions on M which collapses to the same space X, and which satisfies that $||xy|_{g_n^{\varepsilon_n}} - |xy|_{g_{n+1}^{\varepsilon_{n+1}}}| < \epsilon$.

Set $M_n = (M, g_n^{\varepsilon_n})$, and, to continue the proof of Proposition 3.1.A, skip to A & B below.

B. From the definition of stable collapse we have that there is a sequence of metrics d_n on M such that $M_n = (M, g_n)$ is isometric to (M, d_n) and such that d_n converge as functions on $M \times M$ to a continuous pseudometric d_∞ on M. By applying the smoothing-averaging procedure (see Theorem 1.3) for small and fixed ϵ, passing to a subsequence if necessary, we obtain a sequence of metrics $\{d_n^\varepsilon\}$ on M with collapse related invariant T^kactions on M so that $\{d_n^\varepsilon\}$ will converge to a pseudometric which is $e^{\pm\varepsilon}$-bi-Lipschitz equivalent to $\{d_\infty\}$. (In particular, the spaces (M, d_n^ε) will

Gromov-Hausdorff converge to a space X' such that $d_L(X, X') \leq \epsilon$.)

Set $M_n := (M, d_n^\varepsilon)$ and $X := X'$.

A & B. For each n now fix a homeomorphism $\chi_n : X_n = M_n/T^k \to X$ which is an ε_n-isometry, where $\varepsilon_n \to 0$ as $n \to \infty$ and which preserves the natural stratification. Such a homeomorphism exists from Perelman's stability theorem [Pe1]. From the fact that the closure of each stratum is an extremal subset (see [PePet]) it is easy to see that the homeomorphism constructed in [Pe1] preserves the stratification.

Now from part (b) of Lemma 3.3, we know that there is a conjugation homeomorphism $M_{n+1} \to M_n$ which commutes with $m_0 = \chi_{n+1}^{-1} \circ \chi_n$. By composition, we thus also have conjugation homeomorphisms $\hom_{n+i,n} : M_{n+i} \to M_n = M$ for some fixed n and all $i \in \mathbb{N}$, and these are the homeomorphisms needed in Proposition 3.1.B. It remains to finish the proof of Proposition 3.1.A:

A. As has already been mentioned above, the conclusion of Proposition 3.1.B is stronger than the definition of stable collapse, i.e., the metrics $d_n(x, y) = \mathrm{dist}_{g_n}(h_n^{-1}(x), h_n^{-1}(y))$ can be used in Definition 0.2. Therefore the sequence $(M, g_n^{\varepsilon_n})$ is stable, and since $e^{-\varepsilon_n} g_n < g_n^{\varepsilon_n} < e^{\varepsilon_n} g_n$, we have also proved Proposition 3.1.A. □

4 The Proofs of the Main Results

Theorem (Stable Collapse). *Suppose that a compact (topological) manifold M admits a stable sequence of Riemannian metrics $(g_n)_{n \in \mathbb{N}}$ with sectional curvatures $\lambda \leq K_{g_n} \leq \Lambda$, such that, as $n \to \infty$, the metric spaces (M, g_n) Hausdorff converge to a compact metric space X of lower dimension. Then $\lambda \leq 0$ (i.e., these metrics cannot be uniformly positively pinched).*

Proof. We argue by contradiction. Since a compact manifold with positive sectional curvature has finite fundamental group, by passing to its universal covering we may assume that M is simply connected. The Gluing Theorem 2.0 implies that in the case where the metrics in our theorem were uniformly positively pinched, there would exist a complete noncompact Alexandrov space with lower positive curvature bound. However, by Theorem 1.6 a complete Alexandrov space with strictly positive curvature has finite diameter and is compact. Thus the assumption of the existence of a sequence of uniformly positively pinched metrics leads to a contradiction. □

Theorem 0.6a (Limit of Covering Geometry theorem). *Let M_n be a stable sequence of compact Riemannian m-manifolds with curvature bounds $-\epsilon_n^2 \le K(M_n) \le 1$ such that $\epsilon_n \to 0$ for $n \to \infty$ and such that the sequence of metric spaces M_n Hausdorff converges to a compact metric space X of lower dimension. Consider a sequence of points $p_n \in M_n$ and balls $B_n = B_{\pi/2} \in T_{p_n}$ which are equipped with the pullback metric under the exponential map $\exp_{p_n} : T_{p_n} \to M_n$. Then there is a converging subsequence $B_n \to B$, where B has the same dimension as the manifolds M_n $(= m)$, and the following holds:*

In a neighborhood of its center, the metric on B coincides with that of a metric product $\mathbb{R} \times N$, where N is a manifold with two-sided bounded curvature $0 \le K(N) \le 1$ in the sense of Alexandrov.

Proof. Note that Theorem 0.6a is trivial if $|\pi_1(M)| = \infty$, since in this case $\widetilde{M}$ will converge to a noncompact nonnegatively curved Alexandrov space Y with group action Γ (the big limit of $\pi_1(M)$) such that Y/Γ is compact. Therefore there is a line in Y, so by Toponogov's splitting theorem for Alexandrov spaces (compare [BuGrPe]) Y splits and from this it follows that the same is true for the limit of covering geometry. (Note that we did not even use stability so far!)

If the fundamental group of M is finite, we can pass to its universal covering, where then we will also have a stable collapsing sequence. Therefore we can apply the above arguments to obtain the space Y.

Therefore, under the assumptions of Theorem 0.6a, we have (again by the splitting theorem for Alexandrov spaces) that Y splits isometrically as $\mathbb{R}^{k'} \times X$. Since A acts isometrically on $\widetilde{C}_\alpha^\infty$, this implies that all $\widetilde{C}_\alpha^\infty$ split isometrically, and therefore Theorem 0.6a follows. □

Theorem (Continuous Collapse). *Suppose that a compact manifold M admits a continuous one parameter family $(g_t)_{0<t\le 1}$ of Riemannian metrics with sectional curvature $\lambda \le K_{g_t} \le \Lambda$, such that, as $t \to 0$, the family of metric spaces (M, g_t) Hausdorff converges to a compact metric space X of lower dimension. Then $\lambda \le 0$ (i.e., these metrics cannot be uniformly positively pinched).*

Proof. As M has positive curvature, M has finite fundamental group. Therefore, if M admits a continuous collapse, then its universal covering $\widetilde{M}$ also admits a continuous collapse. Applying Proposition 3.1.A and after that the stable collapse theorem we obtain the continuous collapse theorem. □

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