Chapter 1

Isoperimetric inequality

For any plane figure F with perimeter ℓ, its area a satisfies the following inequality:

❶ $$4\pi\cdot a \leqslant \ell^2.$$

Moreover the equality holds iff F is congruent to a round disk.

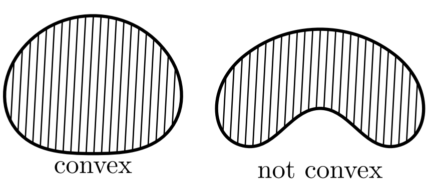
This is the so-called *isoperimetric inequality* on the plane. Let us restate it without formulas, using the comparison language.

1.1. Isoperimetric inequality. *The area of a plane figure bounded by a closed curve of length ℓ can not exceed the area of a round disk with the same circumference ℓ. Moreover the equality holds only if the figure is congruent to the disk.*

The comparison reformulation has some advantages — it is more intuitive and it is also easier to generalize.

1.2. Exercise. *Come up with a formulation of the isoperimetric inequality on the unit sphere. Try to reformulate it as an algebraic inequality similar to* ❶.

Recall that a plane figure F is called *convex* if for every pair of points $x, y \in F$, the line segment $[x, y]$ that joins the pair of points lies also in F.



The following exercise reduces the isoperimetric inequality to the case of convex figures:

1.3. Exercise. *Assume F is a plane figure bounded by a closed curve of length ℓ. Show that there is a convex figure $F' \supset F$ bounded by a closed curve of length at most ℓ.*

The following problem is named after Dido, the legendary founder and first queen of Carthage.

1.4. Dido's problem. *The figure of maximal area bounded by a straight line and a curve of given length with endpoints on that line is a half-disk.*

1.5. Exercise. *Show that Dido's problem follows from the isoperimetric inequality and the other way around.*

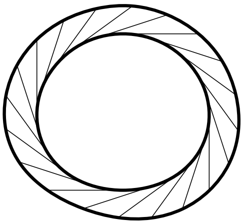
1.6. Exercise. *Use the isoperimetric inequality in the plane to show that among all the polygons with given sides, the convex polygons inscribed in a circle have maximal area.*

1.7. Exercise. *Find the minimal length of a curve that divides the unit square in a given ratio α.*

1.1 Lawlor's proof

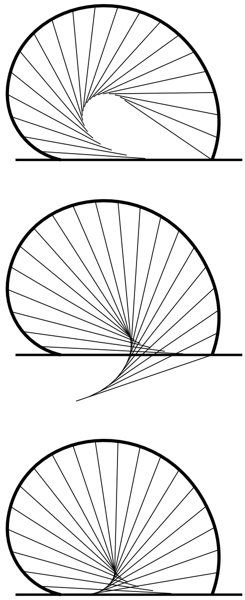
Here we present a sketch of the proof of Dido's problem based of the idea of Gary Lawlor in [1]. Before getting into the proof, try to solve the following exercise.

1.8. Exercise. *An old man walks along a trail around a convex meadows and pulls a brick tied to a rope of unit length (the rope is always strained). After walking around he noticed that the brick is at the same position as at the beginning. Show that the area between the trail and the path of the brick equals the area of the unit disk.*



Sketch of the proof. Let F be a convex figure bounded by a line and a curve $\gamma(t)$ of length ℓ; we can assume that γ is a unit speed curve so the set of parameters is $[0, \ell]$.

Imagine that we are walking along the curve with a stick of length r so that the other end of the stick drags as we walk. Assume that initially at $t = 0$ the stick points in the direction of $\gamma(\ell)$ — the other end of γ.



Note that if r is small then most of the time we drag the stick behind. Therefore at the end of the walk the stick will have made more than half turn and will point to the same side of the figure.

Let R be the radius of the half-circle $\tilde{\gamma}(t)$ of length ℓ. Assume we walk along $\tilde{\gamma}$ with a stick of length R the same way as described above. Note that the other end does not move (it always lies in the center) and the direction of the stick changes with rate $\frac{1}{R}$. Note further that for γ this rate is at most $\frac{1}{R}$. Therefore after walking along γ, the stick of length R will rotate at most as much as if we walk along $\tilde{\gamma}$.

It follows that there is a positive value $r \leqslant \leqslant R$ such that after walking along γ with a stick of length r, it will rotate exactly half turn, so at the end it will point towards $\gamma(0)$.

Imagine that the stick of length r is covered with red paint and it paints the area below it. If we move with velocity v then the angular velocity (radians per second) of the stick is at most $\frac{v}{r}$ and the equality holds if we move perpendicularly to the stick. Therefore we paint at a rate of at most $\frac{1}{2}v{\cdot}r$.

If we do the same for the half-disk of radius R and a stick of length R with blue paint, then we paint the area of the disk without overlap with the rate $\frac{1}{2}v{\cdot}R$. Since $r \leqslant R$, the total red-painted area can not exceed the blue-painted area, that is, D.

It remains to show that all F is red-painted. Fix a point $p \in F$. Notice that at the beginning the point p lies on the left from the stick and at the end it lies on the right form it. Therefore there will be a moment t_0 when the side changes from left to right. At this time the point must be on the line containing the stick. Moreover, if it lies on the extension then the side changes from right to left. Therefore p has to lie under the stick; that is, p is painted.

$\square$

1.9. Exercise. *Find the steps with cheating in the above proof and try to fix them.*

1.10. Exercise. *Read and understand the original proof of Gary Lawlor in [1].*

Chapter 2

Length

The material of this and the following chapters overlaps largely with [2, Chapter 5].

2.1 Length of curve

2.1. Definition. *Consider a* plane curve $\alpha\colon [a,b]\to\mathbb{R}^2$*; a continuous mapping from the real interval* $[a,b]$ *to the Euclidean plane* $\mathbb{R}^2$.

If $\alpha(a)=p$ *and* $\alpha(b)=q$*, we say that* α *is a* curve from p to q.

A curve $\alpha\colon [a,b]\to\mathbb{R}^2$ *is called* closed *if* $\alpha(a)=\alpha(b)$.

A curve α *is called* simple *if it is described by an injective map; that is* $\alpha(t)=\alpha(t')$ *if and only if* $t=t'$*. However, a closed curve* $\alpha\colon [a,b]\to\mathbb{R}^2$ *is called simple if it is injective everywhere except at the ends; that is, if* $\alpha(t)=\alpha(t')$ *for* $t<t'$ *then* $t=a$ *and* $t'=b$.

2.2. Exercise. *Let* $\alpha\colon [0,1]\to\mathbb{R}^2$ *from* p *to* q*. Assume* $p\ne q$ *Show that there is a simple curve* $\beta\colon [0,1]\to\mathbb{R}^2$ *from* p *to* q *that runs in the image of* α*; that is for any* $t\in[0,1]$ *there is* $t'\in[0,1]$ *such that* $\beta(t)=\alpha(t')$.

Recall that a sequence

$$a=t_0<t_1<\dots<t_k=b.$$

is called a *partition* of the interval $[a,b]$.

2.3. Definition. *Let* $\alpha\colon [a,b]\to\mathbb{R}^2$ *be a curve. The* length *of* α *is defined as*

$$\begin{aligned}\operatorname{length}\alpha = \sup\{|\alpha(t_0)-\alpha(t_1)| + |\alpha(t_1)-\alpha(t_2)| + \dots \\ \dots + |\alpha(t_{k-1})-\alpha(t_k)|\}.\end{aligned}$$

where the exact upper bound is taken over all partitions

$$a = t_0 < t_1 < \dots < t_k = b.$$

Note that $\operatorname{length}\alpha \in [0,\infty]$*; the curve* α *is called* rectifiable *if its length is finite.*

Informally, one could say that the length of a curve is the exact upper bound of the lengths of polygonal lines *inscribed* in the curve.

2.4. Exercise. *Assume* $\alpha\colon [a,b] \to \mathbb{R}^2$ *is a smooth curve, in particular the velocity vector* $\alpha'(t)$ *is defined and depends continuously on* t*. Show that*

$$\operatorname{length}\alpha = \int\limits_a^b |\alpha'(t)|\cdot dt.$$

2.5. Exercise. *Construct a nonrectifiable curve* $\alpha\colon [0,1] \to \mathbb{R}^2$*.*

A closed simple plane curve is called *convex* if it bounds a convex figure.

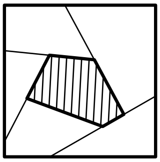
2.6. Proposition. *Assume a convex figure* A *bounded by a curve* α *lies inside a figure* B *bounded by a curve* β*. Then*

$$\operatorname{length}\alpha \leqslant \operatorname{length}\beta.$$

Note that it is sufficient to show that for any polygon P inscribed in α there is a polygon Q inscribed in β with $\operatorname{perim} P \leqslant \operatorname{perim} Q$, where $\operatorname{perim} P$ denotes the perimeter of P.

Therefore it is sufficient to prove the following lemma.

2.7. Lemma. *Let* P *and* Q *be polygons. Assume* P *is convex and* $Q \supset P$*. Then* $\operatorname{perim} P \leqslant \operatorname{perim} Q$*.*



Proof. Note that by the triangle inequality, the inequality

$$\operatorname{perim} P \leqslant \operatorname{perim} Q$$

holds if P can be obtained from Q by cutting it along a chord; that is, a line segment with ends on the boundary of Q that lies in Q.

Note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \dots \subset P_n = Q$$

such that P_{i-1} obtained from P_i by cutting along a chord. Therefore

$$\begin{aligned}\operatorname{perim} P = \operatorname{perim} P_0 \leqslant \operatorname{perim} P_1 \leqslant \dots \\ \dots \leqslant \operatorname{perim} P_n = \operatorname{perim} Q\end{aligned}$$

and the lemma follows. □

2.8. Corollary. *Any convex closed curve is rectifiable.*

Proof. Fix a curve $\alpha\colon [a,b] \to \mathbb{R}^2$. Note that α is bounded; indeed

Any closed curve is bounded; that is, it lies in a sufficiently large square.

By Proposition 2.6, the length of the curve can not exceed the perimeter of the square, hence the result. □

2.2 Semicontinuity of length

Recall that the lower limit of a sequence of real numbers (x_n) is denoted by

$$\varliminf_{n\to\infty} x_n.$$

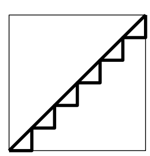
It is defined as the lowest partial limit; that is, the lowest possible limit of a subsequence of (x_n). The lower limit is defined for any sequence of real numbers and it lies in the exteded real line $[-\infty,\infty]$

2.9. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\alpha_n\colon [a,b] \to \mathbb{R}^2$ converges pointwise to a curve $\alpha_\infty\colon [a,b] \to \mathbb{R}^2$; that is, $\alpha_n(t) \to \alpha_\infty(t)$ for any fixed $t \in [a,b]$ as $n \to \infty$. Then

❶ $$\varliminf_{n\to\infty} \operatorname{length} \alpha_n \geqslant \operatorname{length} \alpha_\infty.$$

Note that the inequality ❶ might be strict. For example the diagonal α_∞ of the unit square



can be approximated by a sequence of stairs-like polygonal curves α_n with sides parallel to the sides of the square (α_6 is on the picture). In this case

$$\operatorname{length} \alpha_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length} \alpha_n = 2$$

for any n.

Proof. Fix $\varepsilon > 0$ and choose a partition $a = t_0 < t_1 < \dots < t_k = b$ such that

$$\operatorname{length} \alpha_\infty < |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + \dots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)| + \varepsilon.$$

Set

$$\begin{aligned} \Sigma_n &:= |\alpha_n(t_0) - \alpha_n(t_1)| + \dots + |\alpha_n(t_{k-1}) - \alpha_n(t_k)|. \\ \Sigma_\infty &:= |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + \dots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|. \end{aligned}$$

Note that $\Sigma_n \to \Sigma_\infty$ as $n \to \infty$ and $\Sigma_n \leqslant \operatorname{length} \alpha_n$ for each n. Hence

$$\varliminf_{n\to\infty} \operatorname{length} \alpha_n \geqslant \operatorname{length} \alpha_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get ❶. □

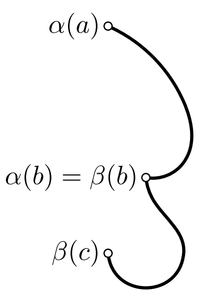
2.3 Axioms of length

Concatenation. Assume $\alpha\colon [a,b] \to \mathbb{R}^2$ and $\beta\colon [b,c] \to \mathbb{R}^2$ are two curves such that $\alpha(b) = \beta(b)$. Then one can combine these two curves into one $\gamma\colon [a,c] \to \mathbb{R}^2$ by the rule $\gamma(t) = \alpha(t)$ for $t \leqslant b$ and $\gamma(t) = \beta(t)$ for $t \geqslant b$. The obtained curve γ is called the *concatenation* of α and β and is denoted as $\gamma = \alpha * \beta$.

Note that

$$\operatorname{length}(\alpha * \beta) = \operatorname{length} \alpha + \operatorname{length} \beta$$

for any two curves α and β such that the concatenation $\alpha * \beta$ is defined.



Reparametrization. Assume $\alpha\colon [a,b] \to \mathbb{R}^2$ is a curve and $\tau\colon [c,d] \to [a,b]$ is a continuous strictly monotonic onto map. Consider the curve $\alpha'\colon [c,d] \to \mathbb{R}^2$ defined by $\alpha' = \alpha \circ \tau$. The curve α' is called a *reparametrization* of α.

Note that

$$\operatorname{length} \alpha' = \operatorname{length} \alpha$$

whenever α' is a reparametrization of α.

2.10. Proposition. *Let ℓ be a functional that returns a value in $[0,\infty]$ for any curve $\alpha\colon [a,b] \to \mathbb{R}$.*

Assume it satisfies the following properties:

(i) *(Normalization) If* $\alpha\colon [a,b]\to\mathbb{R}^2$ *is a linear curve,*[1] *then*

$$\ell(\alpha)=|\alpha(a)-\alpha(b)|.$$

(ii) *(Additivity) If the concatenation* $\alpha*\beta$ *is defined, then*

$$\ell(\alpha*\beta)=\ell(\alpha)+\ell(\beta).$$

(iii) *(Motion invariance) The functional* ℓ *is invariant with respect to the motions of the plane; that is, if* m *is an isometry of the plane, then*

$$\ell(m\circ\alpha)=\ell(\alpha)$$

for any curve α*.*

(iv) *(Reparametrization invariance) If* α' *is a reparametrization of a curve* α *then*

$$\ell(\alpha')=\ell(\alpha).$$

(In fact linear reparametrizations will be sufficient.)

(v) *(Semi-continuity) If a sequence of curves* $\alpha_n\colon [a,b]\to\mathbb{R}^2$ *converges pointwise to a curve to a curve* $\alpha_\infty\colon [a,b]\to\mathbb{R}^2$*, then*

$$\varliminf_{n\to\infty}\ell(\alpha_n)\geqslant\ell(\alpha_\infty).$$

Then

❷ $$\ell(\alpha)=\operatorname{length}\alpha$$

for any plane curve α*.*

Proof. Note that from normalization and additivity, the identity

❸ $$\ell(\beta)=\operatorname{length}\beta$$

holds for any polygonal line β that is linear on each edge.

Note that the following two inequalities

❹ $$\ell(\alpha)\leqslant\operatorname{length}\alpha$$

❺ $$\ell(\alpha)\geqslant\operatorname{length}\alpha$$

imply ❷; we will prove them separately.

Fix a curve $\alpha\colon [a,b]\to\mathbb{R}^2$ and a partition $a=t_0<t_1<\ldots<$ $<t_k=b$. Consider the curve $\beta\colon [a,b]\to\mathbb{R}^2$ defined as the linear

[1] That is $\alpha=w+v\cdot t$ for some vectors w and v.

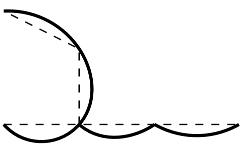
segment from $\alpha(t_i)$ to $\alpha(t_{i+1})$ on each interval $t \in [t_i, t_j]$. By the definition of length,

$$\operatorname{length}\beta \leqslant \operatorname{length}\alpha.$$

Since the map $\alpha\colon [a,b] \to \mathbb{R}^2$ is continuous, one can find a sequence of partitions of $[a,b]$ such that the corresponding curves β_n converge to α pointwise. Applying the semi-continuity of ℓ, ❸ and the definition of length, we get that

$$\begin{aligned}\ell(\alpha) &\leqslant \varliminf_{n\to\infty} \ell(\beta_n) = \\ &= \varliminf_{n\to\infty} \operatorname{length}\beta_n \leqslant \\ &\leqslant \operatorname{length}\alpha.\end{aligned}$$

Hence ❹ follows.



Note that a curve $\alpha\colon [a,b] \to \mathbb{R}^2$ with a partition $a = t_0 < t_1 < \ldots < t_k = b$ can be considered as a concatenation

$$\alpha = \alpha_1 * \alpha_2 * \cdots * \alpha_k$$

where α_i is the restriction of α to $[t_{i-1}, t_i]$.

Observe that there is a sequence of motions m_i of the plane so that

$$m_i \circ \alpha(t_i) = m_{i+1} \circ \alpha(t_i)$$

for any i and the points

$$m_1 \circ \alpha(t_0), m_1 \circ \alpha(t_1), \ldots m_k \circ \alpha(t_k)$$

lie in that order on a single line. For the concatenation

$$\gamma = (m_1 \circ \alpha_1) * (m_2 \circ \alpha_2) * \cdots * (m_k \circ \alpha_k)$$

we have

$$\ell(\gamma) = \ell(\alpha).$$

Note that one can find a sequence of partitions of $[a,b]$ such that reparametrizations of γ_n converge to a linear segment γ'_∞; denote these reparametrizations by γ'_n. Also, $\operatorname{length}\gamma'_\infty = \operatorname{length}\alpha$; indeed, since γ'_∞ is linear,

$$\begin{aligned}\operatorname{length}\gamma'_\infty &= |\gamma'_\infty(a) - \gamma'_\infty(b)| = \\ &= \lim_{n\to\infty} \Sigma_n = \\ &= \operatorname{length}\alpha.\end{aligned}$$

where Σ_n is the sum in the definition of length for the n-th partition. Hence it is sufficient to choose a sequence of partitions such that $\Sigma_n \to$ $\to \operatorname{length} \alpha$.

Applying additivity, invariance of ℓ with respact to motions and reparametizations, we get that

$$\begin{aligned} \ell(\alpha) &= \lim_{n\to\infty} \ell(\gamma_n) = \\ &= \lim_{n\to\infty} \ell(\gamma'_n) \geqslant \\ &\geqslant \ell(\gamma'_\infty) = \\ &= \operatorname{length} \alpha. \end{aligned}$$

Hence ❺ follows. □

2.11. Exercise. *Construct a functional ℓ that satisfies all the conditions in Proposition 2.10 except the semi-continuity.*

2.4 Crofton formula

Let α be a plane curve and u a unit vector. Denote by α_u the orthogonal projection of α to a line ℓ in the direction of u; that is, $\alpha_u(t) \in \ell$ and $\alpha(t) - \alpha_u(t) \perp \ell$ for any t.

2.12. Crofton formula. *The length of any plane curve α is proportional to the average of the lengths of its projections α_u for all unit vectors u. Moreover for any plane curve α we have*

$$\operatorname{length} \alpha = \tfrac{\pi}{2}\cdot\overline{\operatorname{length} \alpha_u},$$

where $\overline{\operatorname{length} \alpha_u}$ *denotes the average value of* $\operatorname{length} \alpha_u$.

Proof. First let us show that the formula

❻ $$\operatorname{length} \alpha = k\cdot\overline{\operatorname{length} \alpha_u},$$

holds for some fixed coefficient k. It will follow once we show that both sides of formula satisfy the length axioms in 2.10.

The normalization can be achieved by adjusting k.

The semi-continuity of the right hand side follows since $\operatorname{length} \alpha_u$ is semi-continuous and therefore the average has to be semi-continuous.

It is straightforward to check the remaining properties.

It remains to find k. Let us apply the formula ❻ to the unit circle. The circle has length $2\cdot\pi$ and its projection to any line has length 4

— it is a segment of length 2 traveled back and forth. Evidently the average value is also 4, so

$$2{\cdot}\pi = k{\cdot}4,$$

hence $k = \frac{\pi}{2}$. □

Reformulation via number of intersections. Given a unit vector u and a real number ρ, consider the line of vectors w on the plane satisfying the equation

$$\langle u, w\rangle = \rho,$$

where $\langle u, w\rangle$ denotes the scalar product. Any line on the plane admits exactly two such presentations with pairs (u, ρ) and $(-u, -\rho)$. A pair (u, ρ) describes uniquely an *oriented* lines — that is a line with a chosen unit normal vector.

Fix a unit vector u_0 and denote by $u(\varphi)$ the result of rotating u_0 counterclockwise by the angle φ.Denote by $\ell(\varphi, \rho)$ the oriented line asociated to the pair $(u(\varphi), \rho)$. To describe any line, we need a pair $(\varphi, \rho) \in (-\pi, \pi] \times \mathbb{R}$.

For a curve α, set $n_\alpha(\varphi, \rho)$ to be the number of parameter values t such that $\alpha(t)$ lies on the line $\ell(\varphi, \rho)$. The value $n_\alpha(\varphi, \rho)$ is a non-negative integer or ∞. Note that if α is a simple curve, then $n_\alpha(\ell)$ is the number of intersections of α with ℓ.

2.13. An other Crofton formula. *For any curve α,*

$$\operatorname{length}\alpha = \tfrac{1}{4}\cdot \iint\limits_{(-\pi,\pi]\times\mathbb{R}} n_\alpha(\rho, \varphi)\cdot d\rho\cdot d\varphi.$$

the integral is to be understood in the sense of Lebesgue.

By definition of average value,

$$\overline{\operatorname{length}\alpha_u} = \frac{1}{2\cdot\pi}\cdot \int\limits_{-\pi}^{\pi} \operatorname{length}\alpha_{u(\varphi)}\cdot d\varphi.$$

Therefore the proof of this reformulation of the Crofton follows from the following observation.

2.14. Observation. *If $u = u(\varphi)$, then*

$$\operatorname{length}\alpha_u = \int\limits_{\mathbb{R}} n_\alpha(\rho, \varphi)\cdot d\rho;$$

The proof is straightforward for those who understand Lebesgue integral.

Variations. The same argument can be used to derive other formulas of the same type. For example.

Recall that a big circle in a sphere is the intersection of the sphere with a plane passing thru its center. For example, the equator as well as the meridian are big circles.

2.15. Spherical Crofton formula. *The length of any curve α in the unit sphere is π times the average number of its crossings with big circles.*

More presciently, given a unit vector u, denote by $n_\alpha(u)$ the number of crossings of α and the equator with pole at u. Then

$$\operatorname{length}\alpha = \overline{n_\alpha(u)}.$$

Equivalently,

$$\operatorname{length}\alpha = \overline{\operatorname{length}\alpha_u},$$

where α_u denotes the curve obtianed by closest point projection of α to the equator with pole at u.

2.16. Exercise. *Come up with a Crofton formulas for curves in the Euclidean space via projections to lines and to planes. Find the coefficients in these formulas.*

2.5 Applications

Alternative proof of Proposition 2.6. Note that

$$\operatorname{length}\beta_u \geqslant \operatorname{length}\alpha_u$$

for any unit vector u. Indeed α_u runs back and forth along a line segment and β_u has to run at least that much.

It follows that

$$\overline{\operatorname{length}\beta_u} \geqslant \overline{\operatorname{length}\alpha_u}.$$

It remains to apply the Crofton formula. □

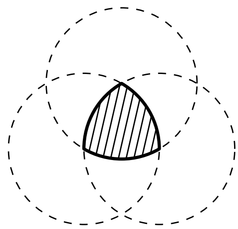
Recall that the diameter of a plane figure F is defined as the least upper bound on the distances between pairs of its points; that is,

$$\operatorname{diam} F = \sup\{\, |x-y| \;:\; x, y \in F\,\}.$$

The equilateral triangle with side 1 gives an example of a convex figure of diameter 1 that cannot be covered by a round disc of diameter 1.

2.17. Exercise. *Assume F is a convex figure of diameter 1 and D is the round disc of diameter 1. Show that*

$$\operatorname{perim} F \leqslant \operatorname{perim} D.$$



A convex figure F has constant width a if the orthogonal projection of F to any line has length a. There are many non-circular shapes of constant width. A nontrivial example is the Reuleaux triangle shown on the picture; it is the intersection of three round disks of the same radius, each having its center on the boundary of the other two. The following exercise is the so called Barbier's theorem.

2.18. Exercise. *Show that figures with constant width a have the same perimeter (which equals $\pi\cdot a$ — the perimeter of the round disc of diameter a).*

2.19. Exercise. *Let γ be a closed curve in the unit sphere of length shorter than $2\cdot\pi$. Show that γ lies in a hemisphere.*

2.20. Exercise. *Let α be a closed curve of length π. Show that it lies between a pair of parallel lines at distance* 1 *from each other.*

2.21. Exercise. *A spaceship flies around a nonrotating planet of unit radius and comes back to the original position; it was able to make a picture of every point on the surface of the planet.*

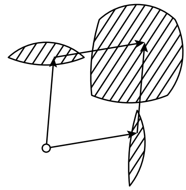
Try to use the Crofton formulas to get a lower bound on the length of its trajectory (does not need to be exact, but should be larger than $2\cdot\pi$).

What do you think could be the shortest trajectory?

The Hausdorff distance $d_H(F,G)$ between two closed bounded sets F and G in the plane is defined as the exact lower bound on $\varepsilon > 0$ such that the ε-neightborhood of F contains G and the ε-neightborhood of G contains F.

2.22. Exercise. *Assume F and G are two closed convex figures on the plane such that $d_H(F,G) < \varepsilon$. Show that*

$$|\operatorname{perim} F - \operatorname{perim} G| < 2\cdot\pi\cdot\varepsilon.$$



Given two sets A and B on the plane, the set C is called their *Minkowski sum* (briefly $C = A + B$) if C is formed by adding each vector in A to each vector in B; that is,

$$C = \{\, a + b \,:\, a \in A,\ b \in B \,\}.$$

Note that if A and B are convex then so is $C = A + B$.

Indeed, A is convex if and only if for any pair of points $a_0, a_1 \in A$ and any $t \in [0,1]$, the point $a_t = (1-t)\cdot a_0 + t\cdot a_1$ belongs to A. Similarly, B is convex if and only if for any pair of points $b_0, b_1 \in B$ and any $t \in [0,1]$, the point $b_t = (1-t)\cdot b_0 + t\cdot b_1$ belongs to A.

Fix a pair of points $c_0, c_1 \in C$; by the definition of Minkowski sum, there are two pairs of points $a_0, a_1 \in A$ and $b_0, b_1 \in B$ such that $c_0 = a_0 + b_0$ and $c_1 = a_1 + b_1$. Then

$$\begin{aligned} c_t &= (1-t)\cdot c_0 + t\cdot c_1 = \\ &= (1-t)\cdot(a_0 + b_0) + t\cdot(a_1 + b_1) = \\ &= [(1-t)\cdot a_0 + t\cdot a_1] + [(1-t)\cdot b_0 + t\cdot b_1] = \\ &= a_t + b_t. \end{aligned}$$

That is, $c_t \in C$ for any $t \in [0,1]$, hence the result.

2.23. Exercise. *Show that*

$$\operatorname{perim}(A + B) = \operatorname{perim} A + \operatorname{perim} B$$

for any pair of convex figures in the plane.

2.24. Exercise. *Use Exercise 2.23 and Lemma 2.7 to give another solution of Exercise 2.22.*

2.25. Exercise. *Let γ be a curve that lies in a convex figure F in the plane.*

Let γ be a curve that lies inside a convex figure F on the plane. Assume that

$$2\cdot \operatorname{length}\gamma \geqslant n\cdot \operatorname{perim} F$$

for some integer n. Show that there is a line ℓ that intersects γ in at least n distinct points.

AFTER THIS LINE READ AT YOUR OWN RISK!!!

Chapter 3

Total curvature

3.1 Smooth regular curves

Here we introduce the so called *total curvature of curve*. In general term *curvature* is used for something that measures how a geometric object deviates from being a straight; total curvature is not an exception — as you will see if the total curvature of a curve is vanishing then the curve runs along a straight line.

Let $\alpha\colon [a,b] \to \mathbb{R}^3$ be a *smooth regular* curve — smooth means that the velocity vector $\alpha'(t)$ is defined and continuous with respect to t and regular means that $\alpha'(t) \ne 0$ for any t. If the curve α is closed then we assume in addition that $\alpha'(a) = \alpha'(b)$.

Denote by $\tau(t)$ the unit vector in the direction of $\alpha'(t)$; that is, $\tau(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$. The $\tau\colon [a,b] \to \mathbb{S}^2$ is an other curve which is called *tangent indicatrix* of α. The length of τ is called *total curvature of* α; that is,

$$\operatorname{TotCurv}\alpha := \operatorname{length}\tau.$$

3.1. Exercise. *Show that*

$$\operatorname{TotCurv}\alpha \geqslant 2{\cdot}\pi$$

for any smooth closed regular curve α.

Moreover, the equality holds if and only if α is a closed and convex curve that lies in a plane.

The above exercise is the so called Fenchel's theorem.

3.2 General definition

The total curvature of a polygonal line is defined as the sum of its external angles.

More precisely, for a polygonal line $\beta = p_0 \dots p_n$, the external angle at the vertex p_i is defined as $\alpha_i = \pi - \measuredangle p_{i-1} p_i p_{i+1}$. The total curvature of the polygonal line $\beta = p_0 \dots p_n$ is defined as the sum

$$\operatorname{TotCurv} \beta = \alpha_1 + \dots + \alpha_{n-1};$$

it is defined if the polygonal line is *nondegenerate*; that is, $p_{i-1} \neq p_i$ for any i.

If the polygonal line $p_0 \dots p_n$ is closed; that is $p_0 = p_n$ you add one more angle

$$\alpha_0 + \alpha_1 + \dots + \alpha_{n-1},$$

where $\alpha_0 = \pi - \measuredangle p_n p_0 p_1$.

One can define the tangent indicatrix for a polygonal line β as a spherical polygonal line (each edge is an arc of big circle of the sphere) that vertexes are unit vectors $\xi_1, \dots, \xi_n$ in the directions of $p_1 - p_0$, $p_2 - p_1, \dots, p_n - p_{n-1}$ correspondingly; if the polygonal line is closed then we add one more vertex ξ_0 in the directions of $p_0 - p_n$ and two more edges $\xi_0 \xi_1$ and $\xi_n \xi_0$ so the indicatrix of closed polygonal line is a closed spherical polygonal line.

Note that the total curvature of polygonal line is the length of its tangent indicatrix.

3.2. Exercise. *Let a, b, c, d and x be distinct points in $\mathbb{R}^3$. Show that*

$$\operatorname{TotCurv} abcd \geqslant \operatorname{TotCurv} abxcd.$$

3.3. Exercise. *Use Exercise 3.2 to prove an analog of Fenchel's theorem (Exercise 3.1) for closed polygonal lines.*

Let $\alpha\colon [a,b] \to \mathbb{R}^3$ be a curve and $a = t_0 < \dots < t_n = b$ a partition. Set $p_i = \alpha(t_i)$. Then the polygonal line $p_0 \dots p_n$ is called inscribed in α.

We gave two definitions of total curvature: the first one is given in Section 3.1 via tangent indicatrix — it works for smooth regular curves; the second, via external angles — it works for polygonal lines. The latter can be used to define total curvature of arbitrary curves.

3.4. Definition. *The total curvature of a nonconstant curve α is the exact upper bound on the total curvatures of inscribed nondegenerate polygonal lines; if the curve is closed then we assume that the inscribed polygonal lines are closed as well.*

We need to assume that the curve is nonconstant, otherwise it does not admit inscribed polygonal lines that are not trivial.

3.5. Exercise. *Show that the total curvature is lower semi-continuous with respect to pointwise convergence of curves. That is, if a sequence of curves* $\alpha_n\colon [a,b] \to \mathbb{R}^3$ *converges pointwise to a curve* $\alpha_\infty\colon [a,b] \to \to \mathbb{R}^3$*, then*

$$\varliminf_{n\to\infty} \operatorname{TotCurv} \alpha_n \geqslant \operatorname{TotCurv} \alpha_\infty.$$

Hint: Modify the proof of semi-continuity of length (Theorem 2.9).

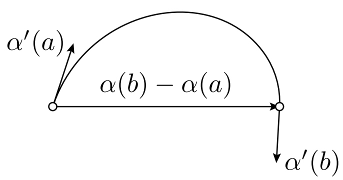
The following definition tells that these two definitions agree.

3.6. Theorem. *For smooth regular curves the two definitions of total curvature agree; that is, for any regular curve, the length of its tangent indicatrix is equal to the exact upper bound on the total curvatures of inscribed nondegenerate polygonal lines.*

Note that from the theorem and Exercise 3.3, we get a generalization of Fenchel's theorem (Exercise 3.1) — it works for arbtrary closed curves, not necessary smooth and regular.

3.7. Lemma. *Let* $\alpha\colon [a,b] \to \mathbb{R}^3$ *be a smooth regular curve. Consider three unit vectors* λ, μ *and* ν *in the directions of* $\alpha'(a)$, $\alpha(b) - \alpha(a)$ *and* $\alpha'(b)$ *correspondingly. Then*

$$\operatorname{TotCurv} \alpha \geqslant \measuredangle(\lambda,\mu) + \measuredangle(\mu,\nu).$$



Proof. The tangent indicatrix τ runs from λ to ν in the unit sphere $\mathbb{S}^2$.

Note that τ can not be separated from μ by an equator. Indeed the vector

$$\alpha(b) - \alpha(a) = \int\limits_a^b \alpha'(t)\cdot dt$$

points in the same direction as μ. Therefore if the indicatrix $\tau = \frac{\alpha'}{|\alpha'|}$ lies in a hemisphere then μ lies in the same hemisphere.

Fix an equator ℓ in general position. If ℓ intersects the spherical polygonal line $\lambda\mu\nu$ at one point, then ℓ separates λ from ν and therefore it must intersect τ. If ℓ intersects the spherical polygonal line $\lambda\mu\nu$ at two points, then ℓ separates μ from λ and ν and therefore it must intersect τ at two points — τ must cross ℓ and then come back. It follows that for almost all equators the number of intesections with the spherical polygonal line $\lambda\mu\nu$ can not exceed the number of intersections with τ. By the spherical Crofton formula (2.15), τ is longer than the spherical polygonal line $\lambda\mu\nu$. But the polygonal line $\lambda\mu\nu$ has length $\measuredangle(\lambda,\mu)+\measuredangle(\mu,\nu)$, hence the result. $\square$

Let us sketch an alternative proof of the lemma which is built on Fenchel's theorem.

An alternative proof of the lemma. Note that the curve α can be extended to a smooth regular closed curve $\hat\alpha$ by an arc β that starts from $\alpha(b)$ in the same direction as α turns and joins the segment $[\alpha(b),\alpha(a)]$ runs along the segment close to $\alpha(a)$ turns and joints α smoothly at $\alpha(a)$.

Note that the total curvature of β can be made arbitrary close to $2\cdot\pi-\measuredangle(\lambda,\mu)-\measuredangle(\mu,\nu)$. Indeed, β needs a bit more than $\pi-\measuredangle(\mu,\nu)$ to turn an join the segment $[\alpha(b),\alpha(a)]$ and bit more than $\pi-\measuredangle(\lambda,\mu)$ to turn an join the segment α.

By Fenchel's theorem,

$$\operatorname{TotCurv}\hat\alpha\geqslant 2\cdot\pi.$$

Evidently

$$\operatorname{TotCurv}\hat\alpha=\operatorname{TotCurv}\alpha+\operatorname{TotCurv}\beta,$$

hence the lemma follows. $\square$

Proof of 3.6. Let $\alpha\colon[a,b]\to\mathbb{R}^3$ be a smooth curve. Fix a partition $a=t_0<\dots<t_n=b$ and consider the corresponding inscribed polygonal line $\beta=w_0\dots w_n$. Let $\chi=\xi_1\dots\xi_n$ be its tangent indicatrix — this is a spherical polygonal line; we assume that $\chi(t_i)=\xi_i$ and it has constant speed on each arc.

Consider a sequence of finer and finer partitions, denote by β_n and χ_n the correspoponging inscribed polygonal line and its tangent indicatrix; since α is smooth, the χ_n converges pointwise to the τ — the thangent indicatrix of α. By semi-continuity of length functional, we get that

$$\begin{aligned}\operatorname{TotCurv}\alpha&=\operatorname{length}\tau\leqslant\\&\leqslant\varliminf_{n\to\infty}\operatorname{length}\chi_n=\\&=\varliminf_{n\to\infty}\operatorname{TotCurv}\beta_n\leqslant\\&\leqslant\sup\{\operatorname{TotCurv}\beta\},\end{aligned}$$

where the list upper bound is taken for all partitions and corresponding inscribed polygonal lines β.

It remains to prove that

$$❶ \qquad \operatorname{TotCurv}\alpha \geqslant \operatorname{TotCurv}\beta,$$

for any polygonal line β inscribed in α. Let ζ_i be the unit vector in the direction of $\alpha'(t_i)$. Consider the spherical polygonal line $\gamma = \zeta_0\xi_1\zeta_1\xi_2\dots\xi_n\zeta_n$; recall that $\chi = \xi_0\dots\xi_n$. By triangle inequality,

$$\operatorname{length}\gamma \geqslant \operatorname{length}\chi = \operatorname{TotCurv}\beta.$$

By Lemma 3.7,

$$\operatorname{TotCurv}\alpha \geqslant \operatorname{length}\gamma,$$

hence ❶ follows. □

3.3 Crofton again

Given a curve α in $\mathbb{R}^3$ and a unit vector u, denote by $\alpha_{u^\perp}$ and α_u the projection of α to the plane perpendicular to u and the line parallel to u correspondingly.

To prove the following proposition, apply the spherical Crofton formula to the tangent indicatrix of α.

3.8. Proposition. *Let α be a polygonal line in $\mathbb{R}^3$. Show that*

$$\begin{aligned}\operatorname{TotCurv}\alpha &= \overline{\operatorname{TotCurv}\alpha_{u^\perp}} = \\ &= \overline{\operatorname{TotCurv}\alpha_u}.\end{aligned}$$

Note that since the curve α_u runs back and forth along one line. Each change of its direction contributes π to the total curvature of α_u. Therefore the total curvature of α_u is $n\cdot\pi$, where n is the number of switches of the direction. Since n has to be even, $\operatorname{TotCurv}\alpha_u$ may take values $2\cdot\pi$, $4\cdot\pi$, $6\cdot\pi$ and so on.

3.9. Exercise. *Use the proposition and the observation above to give yet an other proof of Fenchel's theorem (Exercise 3.1).*

3.4 Applications

3.10. Theorem. *The total curvature of any nontrival knot is at least $4\cdot\pi$.*

The famous Fary–Milnor theorem states that the inequality is strict; that is, the total curvature of any nontrival knot *exceeds* $4\cdot\pi$. It is easy to construct a trefoil knot with total curvature arbitrary close to $4\cdot\pi$; therefore this result is optimal.

In the proof we will use the following fact: *if a height function has only one local minimum and one local maximum on a closed simple polygonal line then the line is a trivial knot.* It is easy to prove assuming that we gave the definitions of nontrivial knot. Roughly, if a height function has only one local minimum and one local maximum, then at each intermediate height, there are exactly two points of the curve. Connecting each such pair with a straight segment, we obtain a disk bounded by the knot. Therefore the know is trivial.

A standard introduction to knot theory defines knots as simple closed polygonal lines, so in the proof we use this agreement; alternatively one can define knot as a closed smooth regular curve, but this approach requires more work.

Proof. Let α be a simple closed polygonal line. Assume its total curvature is less that $4\cdot\pi$. Then by Proposition 3.8,

$$\operatorname{TotCurv}\alpha_u < 4\cdot\pi$$

for some unit vector u. Moreover, we can assume that u points in a generic direction; that is, u is not perpendicular to any edge of α.

The total curvature of α_u is π times the number of turns of α_u which has to be an even number. It follows that number of turns of α_u is at most 2; it can not be less than 2 for generic direction and therefore it is 2. That is, if we rotate the space so that u pints to the top, than the height function has exactly one minimum and one maximum; by the fact stated above α is a trivial knot — hence the result. □

3.11. Exercise. *Construct a closed smooth simple curve with total curvature arbitrary close to* $2\cdot\pi$ *such that its projection to any plane has at least* 10 *self-intersections.*

3.12. DNA inequality. *Let* α *be a closed curve that lies in a unit disc. Then*

$$\operatorname{TotCurv}\alpha \geqslant \operatorname{length}\alpha.$$

Note that if $\operatorname{length}\alpha \leqslant 2\cdot\pi$, then Fenchel's theorem gives a better estimate, for longer curves it gives something new.

Proof. Assume α is a polygonal line.

Fix a unit vector u. Note that the curve α_u can run at most length 2 in one direction; therefore the number of turns has to be at least

$\frac{1}{2} \cdot \operatorname{length} \alpha$. Since each turn of α_u contributes π to its total curvature, we get

$$\operatorname{TotCurv} \alpha_u \geqslant \tfrac{\pi}{2} \cdot \operatorname{length} \alpha_u.$$

The same inequality holds for the average values of left and right hand sides; that is,

$$\overline{\operatorname{TotCurv} \alpha_u} \geqslant \tfrac{\pi}{2} \cdot \overline{\operatorname{length} \alpha_u}.$$

It remains to apply the Crofton's formula and Proposition 3.8.

It remains to reduce the general case to polygonal lines. Let us inscribe a polygonal line β in α with length sufficiently close to length of α; that is, given $\varepsilon > 0$, we choose an inscribed polygonal line β such that

$$\operatorname{length} \alpha < \operatorname{length} \beta + \varepsilon.$$

By the definition of total curvature (3.4) and from above

$$\begin{aligned} \operatorname{TotCurv} \alpha &\geqslant \operatorname{TotCurv} \beta \geqslant \\ &\geqslant \operatorname{length} \beta > \\ &> \operatorname{length} \alpha - \varepsilon. \end{aligned}$$

The statement follows since ε is arbitrary positive number. □

3.13. Exercise. *Assume that a curve $\alpha\colon [a,b] \to \mathbb{R}^3$ has finite total curvature. Show that α is rectifiable.*

We say that a curve $\alpha\colon [a,b] \to \mathbb{R}^3$ *does not stop* if α is not constant on any subinterval of $[a,b]$.

3.14. Exercise. *Assume that the curve α does not stop and its the total curvature is less than π. Show that α is simple; that is it has no self-intersections.*

3.15. Exercise-definition. *Assume that a curve $\alpha\colon [a,b] \to \mathbb{R}^3$ does not stop and has finite total curvature. Show that the direction of exit and entrance is defined for any point.*

That is for any $t_0 \in [a,b)$ the unit vector

$$v(\varepsilon) = \frac{\alpha(t_0+\varepsilon) - \alpha(t_0)}{|\alpha(t_0+\varepsilon) - \alpha(t_0)|}$$

converges as $\varepsilon \to 0^+$; its limit is called the direction of exit and it will be denoted by $\alpha^+(t_0)$

Analogously, for any $t_0 \in (a,b]$ the unit vector

$$w(\varepsilon) = \frac{\alpha(t_0-\varepsilon) - \alpha(t_0)}{|\alpha(t_0-\varepsilon) - \alpha(t_0)|}$$

converges as $\varepsilon \to 0^+$*; its limit is called the direction of entrance and it will be denoted by* $\alpha^-(t_0)$.

3.16. Exercise. *Assume that a curve* $\alpha\colon [a,b] \to \mathbb{R}^3$ *does not stop and has finite total curvature. Show that*

$$\alpha^+(t) = -\alpha^-(t)$$

at all $t \in [a,b]$ *except a countable subset.*

Appendix A

Semisolutions

Exercise 1.5. First let us show that Dido's problem follows from the isoperimetric inequality.

Assume F is a figure bounded by a straight line and a curve of length ℓ whose endpoints belong to that line. Let F' be the reflection of F in the line. Note that the union $G = F \cup F'$ is a figure bounded by a closed curve of length $2{\cdot}\ell$.

Applying the isoperimetric inequality, we get that the area of G can not exceed the area of round disc with the same circumference $2{\cdot}\ell$ and the equality holds only if the figure is congruent to the disc. Since F and F' are congruent, Dido's problem follows.

Now let us show that the isoperimetric inequality follows from the Dido's problem.

Assume G is a convex figure bounded by a closed curve of length $2{\cdot}\ell$. Cut G by a line that splits the perimeter in two equal parts — ℓ each. Denote by F and F' the two parts. Applying the Dido's problem for each part, we get that that are of each does not exceed the area of half-disc bounded by a half-circle. The two half-disc could be arranged into a round disc of circumference ℓ, hence the isoperimetric inequality follows.

Exercise 2.16. Let $\alpha\colon [a,b] \to \mathbb{R}^3$ be a curve. Given a unit vector u, denote by α_u the projection of α on a line in the direction of u; denote by $\alpha_{u^\perp}$ the of α on a plane perpendicular to u.

Two formulas

$$\operatorname{length}\alpha = k{\cdot}\overline{\operatorname{length}\alpha_u}$$

and

$$\operatorname{length}\alpha = k'{\cdot}\overline{\operatorname{length}\alpha_{u^\perp}}$$

can be proved the same way as the Crofton's formula in the plane.

It remains to find the coefficients k and k'. It is sufficient to calculate the average projection of unit segment to a line and to a plane. We need to find two integrals

$$k = \fint_{\mathbb{S}^2} |x|\cdot d\,\mathrm{area}$$

and

$$k' = \fint_{\mathbb{S}^2} \sqrt{1-x^2}\cdot d\,\mathrm{area},$$

where $\mathbb{S}^2 = \{\,(x,y,z)\in\mathbb{R}^3 \,:\, x^2+y^2+z^2=1\,\}$ is the unit sphere in the Euclidean space and $\fint$ denotes the average value — since the area of unit sphere is $4\cdot\pi$, we have

$$\fint_{\mathbb{S}^2} f(x,y,z)\cdot d\,\mathrm{area} = \tfrac{1}{4\cdot\pi}\cdot \int_{\mathbb{S}^2} f(x,y,z)\cdot d\,\mathrm{area}$$

Note that in the cylindrical coordinates

$$(x, \varphi = \arctan\tfrac{y}{z}, \rho = \sqrt{y^2+z^2}),$$

we have $d\,\mathrm{area} = dx\cdot d\varphi$. Therefore

$$k = \fint_{[-1,1]} |x|\cdot dx = \tfrac12$$

and

$$k' = \fint_{[-1,1]} \sqrt{1-x^2}\cdot dx = \tfrac{\pi}{4}.$$

Comment. Note that $\frac{k'}{k} = \frac{\pi}{2}$ is the coefficient in the 2-dimensional Crofton formula. This is not a coincidence — think about it.

Exercise 3.1. Assume contrary, that is there is a closed smooth regular curve α such that $\mathrm{TotCurv}\,\alpha < 2\cdot\pi$.

The tangent indicatrix τ of α is a curve in a sphere; by the definition of total curvature, the length of τ is the total curvature of α; in particular

$$\mathrm{length}\,\tau < 2\cdot\pi.$$

By Exercise 2.19, τ lies in an open hemisphere. If u is the center of the hemisphere, then

$$\langle u, \tau(t)\rangle > 0 \quad \text{and therefore} \quad \langle u, \alpha'(t)\rangle > 0$$

for any t. Therefore the function $t \mapsto \langle u, \alpha(t)\rangle$ is strictly increasing. In particular, if α is defined on the time interval $[a, b]$, then

$$\langle u, \alpha(a)\rangle < \langle u, \alpha(a)\rangle.$$

But α is closed; that is $\alpha(a) = \alpha(b)$ — a contradiction.

Now let us prove the equality case. First note that it is sufficient to show that τ runs around an equator.

Assume τ is not an equator, from above we know that τ can not lie in an open hemisphere. Note that we can shorten τ by a small chord. The obtained curve τ' is shorter than $2{\cdot}\pi$ and therefore lies in an open hemisphere. Applying this construction for shorter and shorter chord and passing to the limit we get that τ lies in closed hemisphere. Denote its center by u as before, then

$$\langle u, \tau(t)\rangle \geqslant 0 \quad \text{and therefore} \quad \langle u, \alpha'(t)\rangle \geqslant 0$$

for any t. Since α is closed we have that $\langle u, \alpha(t)\rangle$ is constant; that is, runs in a plane perpendicular to u and τ lies in an equator perpendicular to u.

So τ is a curve that runs along equator, has length $2{\cdot}\pi$ and does not lie in a open hemisphere. Since τ is not an equator, it have to run along half-equator back and forth. In this case τ lies in an other closed hemisphere and has some points in its interior. The latter contradicts closeness of α the same way as above.

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