Invitation to comparison geometry

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Chapter 1

Metric spaces

Metric is a function that returns a real value $\operatorname{dist}(x,y)$ for any pair x,y in a given nonempty set $\mathcal{X}$ and satisfies the following axioms for any triple x,y,z:

(a) Positiveness:
$$\operatorname{dist}(x,y) \geqslant 0.$$

(b) $x=y$ if and only if
$$\operatorname{dist}(x,y) = 0.$$

(c) Symmetry:
$$\operatorname{dist}(x,y) = \operatorname{dist}(y,x).$$

(d) Triangle inequality:
$$\operatorname{dist}(x,z) \leqslant \operatorname{dist}(x,y) + \operatorname{dist}(y,z).$$

A set with a metric is called *metric space* and the elements of the set are called *points*.

Shortcut for distance. Usually we consider only one metric on a set, therefore we can denote the metric space and its underlying set by the same letter, say $\mathcal{X}$. In this case we also use a shortcut notations $|x-y|$ or $|x-y|_{\mathcal{X}}$ for the *distance* $\operatorname{dist}(x,y)$ *from* x *to* y *in* $\mathcal{X}$. For example, the triangle inequality can be written as
$$|x-z|_{\mathcal{X}} \leqslant |x-y|_{\mathcal{X}} + |y-z|_{\mathcal{X}}.$$

Examples. Euclidean space and plane as well as real line will be the most important examples of metric spaces for us. In these examples the introduced notation $|x-y|$ for the distance from x to y has perfect sense as a norm of the vector $x-y$. However, in general metric space the expression $x-y$ has no sense, but anyway we use expression $|x-y|$ for the distance.

If we say *plane* or *space* we mean *Eucledean* plane or space. However the plane (as well as the space) admits many other metrics, for example the so called Manhattan metric from the following exercise.

1.1. Exercise. *Consider the function*

$$\operatorname{dist}(p,q)=|x_p-x_q|+|y_p-y_q|,$$

where $p=(x_p,y_p)$ and $q=(x_q,y_q)$ are points in the coordinate plane $\mathbb{R}^2$. Show that dist *is a metric on $\mathbb{R}^2$.*

Let us mention another example: the *discrete space* — arbitrary nonempty set $\mathcal{X}$ with the metric defined as $|x-y|_{\mathcal{X}}=0$ if $x=y$ and $|x-y|_{\mathcal{X}}=1$ otherwise.

Subspaces. Any subset of a metric space is also a metric space, by restricting the original metric to the subset; the obtained metric space is called a *subspace*. In particular, all subsets of Euclidean space are metric spaces.

Balls. Given a point p in a metric space $\mathcal{X}$ and a real number $R\geqslant 0$, the set of points x on the distance less then R (or at most R) from p is called open (or correspondingly closed) ball of radius R with center at p. The *open ball* is denoted as $B(p,R)$ or $B(p,R)_{\mathcal{X}}$; the second notation is used if we need to emphasize that the ball lies in the metric space $\mathcal{X}$. Formally speaking

$$B(p,R)=B(p,R)_{\mathcal{X}}=\{\,x\in\mathcal{X}\;:\;|x-p|_{\mathcal{X}}<R\,\}.$$

Analogously, the *closed ball* is denoted as $\bar B[p,R]$ or $\bar B[p,R]_{\mathcal{X}}$ and

$$\bar B[p,R]=\bar B[p,R]_{\mathcal{X}}=\{\,x\in\mathcal{X}\;:\;|x-p|_{\mathcal{X}}\leqslant R\,\}.$$

1.2. Exercise. *Let $\mathcal{X}$ be a metric space.*

(a) Show that if $\bar B[p,2]\subset\bar B[q,1]$ for some points $p,q\in\mathcal{X}$, then $\bar B[p,2]=\bar B[q,1]$.

(b) Construct a metric space $\mathcal{X}$ with two points p and q such that $B(p,\tfrac32)\subset B(q,1)$ and the inclusions is strict.

Calculus

In this section we will extend standard notions from calculus to the metric spaces.

1.3. Definition. *Let $\mathcal{X}$ be a metric space. A sequence of points $x_1, x_2, \dots$ in $\mathcal{X}$ is called* convergent *if there is $x_\infty \in \mathcal{X}$ such that $|x_\infty - x_n| \to 0$ as $n \to \infty$. That is, for every $\varepsilon > 0$, there is a natural number N such that for all $n \geqslant N$, we have*

$$|x_\infty - x_n| < \varepsilon.$$

In this case we say that the sequence (x_n) converges *to x_∞, or x_∞ is the* limit *of the sequence (x_n). Notationally, we write $x_n \to x_\infty$ as $n \to \infty$ or $x_\infty = \lim_{n\to\infty} x_n$.*

1.4. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called* continuous *if for any convergent sequence $x_n \to x_\infty$ in $\mathcal{X}$, we have $f(x_n) \to f(x_\infty)$ in $\mathcal{Y}$.*

Equivalently, $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$, there is $\delta > 0$ such that

$$|x - x'|_{\mathcal{X}} < \delta \quad \textit{implies} \quad |f(x) - f(x')|_{\mathcal{Y}} < \varepsilon.$$

1.5. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is* distance non-expanding map*; that is,*

$$|f(x) - f(x')|_{\mathcal{Y}} \leqslant |x - x'|_{\mathcal{X}}$$

for any $x, x' \in \mathcal{X}$. Show that f is continuous.

1.6. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A continous bijection $f\colon \mathcal{X} \to \mathcal{Y}$ is called a* homeomorphism *if its inverse $f^{-1}\colon \mathcal{Y} \to \mathcal{X}$ is also continuous.*

If there exists a homeomorphism $f\colon \mathcal{X} \to \mathcal{Y}$, we say that $\mathcal{X}$ is homeomorphic *to $\mathcal{Y}$, or $\mathcal{X}$ and $\mathcal{Y}$ are* homeomorphic.

If a metric space $\mathcal{X}$ is homeomorphic to a known space, for example plane, sphere, disc, circle and so on, we may also say that $\mathcal{X}$ is a *topological* plane, sphere, disc, circle and so on.

1.7. Definition. *A subset A of a metric space $\mathcal{X}$ is called* closed *if whenever a sequence (x_n) of points from A converges in $\mathcal{X}$, we have that $\lim_{n\to\infty} x_n \in A$.*

A set $\Omega \subset \mathcal{X}$ is called open *if for any $z \in \Omega$, there is $\varepsilon > 0$ such that $B(z, \varepsilon) \subset \Omega$.*

An open set Ω that contains a given point p is called *neighborhood of p*.

1.8. Exercise. *Let Q be a subset of a metric space $\mathcal{X}$. Show that A is closed if and only if its complement $\Omega = \mathcal{X} \backslash Q$ is open.*

Chapter 2

Curves

Paths. Let $\mathcal{X}$ be a metric space. A continuous map $f\colon[0,1]\to\mathcal{X}$ is called a *path*. If $p=f(0)$ and $q=f(1)$, then we say that f *connects* p *to* q.

If any two points in $\mathcal{X}$ can be connected by a path then $\mathcal{X}$ is called *path connected*. Similarly, a subset $A\subset\mathcal{X}$ is called *path connected* if any two points $p,q\in A$ can be connected by a path that runs in A; equivalently, the subspace A is path connected.

Simple curves.

2.1. Definition. *A path connected subset γ in a metric space is called a* simple curve *if it is* locally *homeomorphic to a real interval; that is, any point $p\in\gamma$ has a neighborhood $U\ni p$ such that the intersection $U\cap\gamma$ is homeomorphic to a real interval.*

It turns out that any curve γ admits a homeomorphism from a real interval or a circle; that is, there is a continuous bijection $G\to\gamma$ with continuous inverse; here (and further) G denotes a circle or real interval. We omit a proof of this statement, but it is not hard.

The homeomorphism $G\to\gamma$ as above is called *parametrization* of γ. The parametrization completely defines the curve. Often will use the same letter for curve and its parametrization, so we can say curve γ has parametrization $\gamma\colon G\to\mathcal{X}$. Note however that any curve admits many different parametrization.

2.2. Exercise. *Find a continuous injective map $\gamma\colon[0,1)\to\mathbb{R}^2$ such that its image is not a simple curve.*

Hint: The image of γ should have a shape of digit 9.

If G is a circle, then the curve $\gamma\colon G\to\mathcal{X}$ is called *closed*. If G is a real interval, then we may say that γ is an *arc*.

Parameterized curves. A *parameterized curve* is defined as a continuous map $\gamma\colon G\to\mathcal{X}$. For a parameterized curve we do not assume that γ is injective; in other words the parameterized curve might have self-intersections.

2.3. Advanced exercise. *Let $\alpha\colon[0,1]\to\mathcal{X}$ be a path from p to q. Assume $p\ne q$. Show that there is a simple path connecting from p to q in $\mathcal{X}$.*

Smooth curves

A curve in the Euclidean space or plane, called *space* or *plane curve* correspondingly.

A space curve can be described by its coordinate functions

$$\gamma(t)=(x(t),y(t),z(t)).$$

Plane curves can be considered as a partial case of space curves with $z(t)\equiv 0$.

If each of the coordinate functions $x(t),y(t),z(t)$ of the space curve γ is a smooth (that is, it has derivatives of all orders everywhere in its domain) then the parameterized curve is called *smooth*.

If the *velocity vector*

$$\gamma'(t)=(x'(t),y'(t),z'(t))$$

does not vanish at all points, then the parameterized curve γ is called *regular*.

A simple space curve is called *smooth and regular* if it admits a smooth and regular parametrization correspondingly. Regular smooth curves are among the main objects in differential geometry; the term *smooth curve* often used for *smooth regular curve*.

2.4. Exercise. *The function*

$$f(t)=\begin{cases}0 & \text{if } t\leqslant 0,\\ \frac{t}{e^{1/t}} & \text{if } t>0.\end{cases}$$

is smooth.[1]

Show that $\gamma(t)=(f(t),f(-t))$ gives a smooth parametrization of the curve S formed by the union of two half-axis in the plane.

[1] The existence of all derivatives $f^{(n)}(x)$ at $x\ne 0$ is evident and direct calculations show that $f^{(n)}(0)=0$ for any n.

Show that any smooth parametrization of S has vanishing velocity vector at the origin. Conclude that the curve S is not regular and smooth.

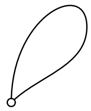
2.5. Exercise. *Describe the set of real numbers a such that the plane curve $\gamma_a(t) = (t + a\cdot \sin t, a\cdot \cos t)$, $t \in \mathbb{R}$ is*

(a) regular;

(b) simple.

Loops and periodic parametrization. A closed simple curve can be described as an image of a parameterized curve $\gamma\colon [0,1] \to \mathcal{X}$ such that $p = \gamma(0) = \gamma(1)$; such curves are called *loops*; the point p in this case is called *base* of the loop.

However, it is more natural to present it as a *periodic* parameterized cure $\gamma\colon \mathbb{R} \to \mathcal{X}$; that is, there is a constant ℓ such that $\gamma(t+\ell) = \gamma(t)$ for any t. For example the unit circle in the plane can be described by $2\cdot\pi$-periodic parametrization $\gamma(t) = (\cos t, \sin t)$.



Any smooth regular closed curve can be described by a smooth regular loop. But in general the closed curve that described by a smooth regular loop might fail to be smooth and regular — it might fail to be smooth at its base; an example shown on the diagram.

Implicitly defined curves

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function; that is, all its partial derivatives defined in its domain of definition. Consider the set S of solution of equation $f(x,y) = 0$ in the plane.

Assume S is path connected. According to implicit function theorem (A.2), the set S is a smooth regular simple curve if 0 is a *regular value* of f. In this case it means that the gradient ∇f does not vanish at any point $p \in S$. In other words, if $f(p) = 0$, then $\frac{\partial f}{\partial x}(p) \neq 0$ or $\frac{\partial f}{\partial y}(p) \neq 0$.

Similarly, assume f, h is a pair of smooth functions defined in $\mathbb{R}^3$. The system of equations $f(x,y,z) = h(x,y,z) = 0$ defines a regular smooth space curve if the set of solutions is path connected and 0 is a regular value of the map $F\colon (x,y,z) \mapsto (f(x,y,z), h(x,y,z))$. In this case it means that the gradients ∇f and ∇h are linearly independent at any point $p \in S$. In other words, if $f(p) = 0$, then at the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}$$

for the map $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ has rank 2 at p.

The described way to define a curve is called *implicit*; if a curve is defined by its parametrization, we say that it is *explicitly defined.* While implicit function theorem guarantees the existence of regular smooth parametrizations, do not expect it to be in a closed form. When it comes to calculations, usually it is easier to work directly with implicit presentation.

2.6. Exercise. *Consider the set in the plane described by the equation*

$$y^2 = x^3.$$

Is it a simple curve? and if "yes", is it a smooth regular curve?

2.7. Exercise. *Describe the set of real numbers a such that the system of equations*

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ x^2 + a{\cdot}x + y^2 &= 0 \end{aligned}$$

describes a smooth regular curve.

Chapter 3

Length

Recall that a sequence

$$a = t_0 < t_1 < \dots < t_k = b.$$

is called a *partition* of the interval $[a, b]$.

3.1. Definition. *Let $\alpha\colon [a, b] \to \mathcal{X}$ be a curve in a metric space. The* length *of a α is defined as*

$$\begin{aligned}\operatorname{length}\alpha = \sup\{|\alpha(t_0) - \alpha(t_1)| + |\alpha(t_1) - \alpha(t_2)| + \dots \\ \dots + |\alpha(t_{k-1}) - \alpha(t_k)|\},\end{aligned}$$

where the exact upper bound is taken over all partitions

$$a = t_0 < t_1 < \dots < t_k = b.$$

The length of α is a nonnegative real number or infinity; the curve α is called rectifiable *if its length is finite.*

The length of a closed curve is defined as the length of a corresponding loop. If a curve is defined on a open or closed-open interval then its length is defined as the exact upper bound for lengths of all its closed arcs.

If α is a space curve, then the above definition says that it length is the exact upper bound of the lengths of polygonal lines $p_0 \dots p_k$ *inscribed* in the curve, where $p_i = \alpha(t_i)$ for a partition $a = t_0 < t_1 < \dots < t_k = b$. If α is closed then $p_0 = p_k$ and therefore the inscribed polygonal line is also closed.

3.2. Exercise. *Let $\alpha\colon [0, 1] \to \mathbb{R}^3$ be a simple curve. Suppose a parametrized curve $\beta\colon [0, 1] \to \mathbb{R}^3$ has that same image as α; that is*

$\beta([0,1]) = \alpha([0,1])$. *Show that*

$$\operatorname{length}\beta \geqslant \operatorname{length}\alpha.$$

3.3. Exercise. *Assume* $\alpha\colon [a,b] \to \mathbb{R}^3$ *is a smooth curve. Show that*

(a) $\operatorname{length}\alpha \geqslant \int_a^b |\alpha'(t)|\cdot dt$,

(b) $\operatorname{length}\alpha \leqslant \int_a^b |\alpha'(t)|\cdot dt$.

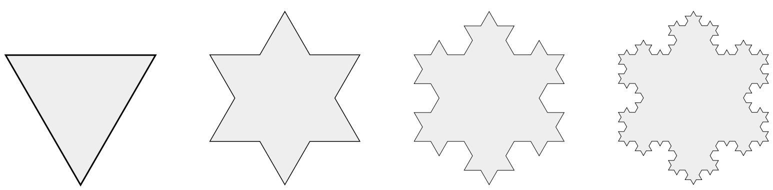
Conclude that

❶
$$\operatorname{length}\alpha = \int\limits_a^b |\alpha'(t)|\cdot dt.$$

Hints: For (a), apply the fundamental theorem of calculus for each segment in a given partition. For (b) consider a partition such that the velocity vector $\alpha'(t)$ is nearly constant on each of its segments.

Nonrectifiable curves. A classical example of a nonrectifiable curve is the so called *Koch snowflake*; it is a fractal curve that can be constructed the following way:

Start with an equilateral triangle, divide each of its side into three segments of equal length and add an equilateral triangle with base at the middle segment. Repeat this construction recursively to the obtained polygons. Few first iterations of the construction are shown



on the diagram. The Koch snowflake is the boundary of the union of all the polygons.

3.4. Exercise.

(a) Show that Koch snowflake is a closed simple curve; that is, it admits a homeomorphism to a circle.

(b) Show that Koch snowflake is not rectifiable.

Arc length parametrization

We say that a parametrized curve γ has an *arc length parametrization*[1] if for any two values of parameters $t_1 < t_2$, the value $t_2 - t_1$ is the length of $\gamma|_{[t_1,t_2]}$; that is, the closed arc of γ from t_1 to t_2.

Note that a smooth space curve $\gamma(t) = (x(t), y(t), z(t))$ has arc length parametrization if and only if it has unit velocity vector at all times; that is

$$|\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = 1;$$

by that reason arc length parametrization of smooth curves with also called *unit-speed curves.* Note that smooth unit-speed curves are automatically regular.

Any rectifiable curve can be parameterized by arc length. For a parametrized smooth curve γ, the arc length parameter s can be written as an integral

$$s(t) = \int\limits_{t_0}^{t} |\gamma'(\tau)|{\cdot}d\tau.$$

Note that $s(t)$ is a smooth increasing function. Further by fundamental theorem of calculus, $s'(t) = |\gamma'(t)|$. Therefore if γ is regular, then $s'(t) \neq 0$ for any parameter value t. By inverse function theorem (A.1) the inverse function $s^{-1}(t)$ is also smooth. Therefore $\gamma \circ s^{-1}$ — the reparametrization of γ by arclength s — remains smooth and regular.

Most of the time we use s for an arc length parameter of a curve.

3.5. Exercise. *Reparametrize the helix*

$$\gamma_{a,b}(t) = (a{\cdot}\cos t, a{\cdot}\sin t, b{\cdot}t)$$

by arc length.

We will be interested in the properties of curves that are invariant under a reparametrization. Therefore we can always assume that the given smooth regular curve comes with a arc length parametrization. A good property of arc length parametrizations is that it is almost canonical — these parametrizations differ only by a sign and additive constant. On the other hand, often it is impossible to find an arc length parametrization in a closed form which makes it hard to use it calculations; usually it is more convenient to use the original parametrization.

[1] which is also called *natural parametrization*

Convex curves

A simple plane curve is called *convex* if it bounds a convex region.

3.6. Proposition. *Assume a convex closed curve α lies inside the domain bounded by a closed simple plane curve β. Then*

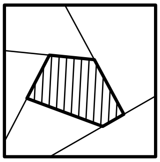
$$\operatorname{length}\alpha \leqslant \operatorname{length}\beta.$$

Note that it is sufficient to show that for any polygon P inscribed in α there is a polygon Q inscribed in β with $\operatorname{perim} P \leqslant \operatorname{perim} Q$, where $\operatorname{perim} P$ denotes the perimeter of P.

Therefore it is sufficient to prove the following lemma.

3.7. Lemma. *Let P and Q be polygons. Assume P is convex and $Q \supset P$. Then*

$$\operatorname{perim} P \leqslant \operatorname{perim} Q.$$



Proof. Note that by the triangle inequality, the inequality

$$\operatorname{perim} P \leqslant \operatorname{perim} Q$$

holds if P can be obtained from Q by cutting it along a chord; that is, a line segment with ends on the boundary of Q that lies in Q.

Note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \dots \subset P_n = Q$$

such that P_{i-1} obtained from P_i by cutting along a chord. Therefore

$$\begin{aligned}\operatorname{perim} P = \operatorname{perim} P_0 \leqslant \operatorname{perim} P_1 \leqslant \dots\\ \dots \leqslant \operatorname{perim} P_n = \operatorname{perim} Q\end{aligned}$$

and the lemma follows. □

3.8. Corollary. *Any convex closed plane curve is rectifiable.*

Proof. Any closed curve is bounded; that is, it lies in a sufficiently large square. Indeed the curve can be described as an image of a loop $\alpha\colon [0,1] \to \mathbb{R}^2$, $\alpha(t) = (x(t), y(t))$. The coordinate functions $x(t)$ and $y(t)$ are continous functions defined on $[0,1]$. Therefore the absolute values of both of these functions are bounded by some constant C. That is α lies in the square defined by the inequalities $|x| \leqslant C$ and $|y| \leqslant C$.

By Proposition 3.6, the length of the curve can not exceed the perimeter of the square $8{\cdot}C$, whence the result. □

Recall that convex hull of a set X is the smallest convex set that contains X; in other words convex hull is the intersection of all convex sets containing X.

3.9. Exercise. *Let α be a closed simple plane curve. Denote by K the convex hull of α; let β be the boundary curve of K. Show that*

$$\operatorname{length}\alpha \geqslant \operatorname{length}\beta.$$

Try to show that the statement holds for arbitrary closed plane curve α, assuming that X has nonempty interior.

Crofton formulas*

Consider a plane curve $\alpha\colon [a,b] \to \mathbb{R}^2$. Given a unit vector u, denote by α_u the curve that follows orthogonal projections of α to the line in the direction u; that is

$$\alpha_u(t) = \langle u, \alpha(t)\rangle{\cdot}u.$$

Note that

$$|\alpha'(t)| = |\langle u, \alpha'(t)\rangle|$$

for any t. Note that for any plane vector the magnitude of its average projection is proportional to its magnitude with coefficient; that is,

$$|w| = k{\cdot}\overline{|w_u|},$$

where $\overline{|w_u|}$ denotes the average value of $|w_u|$ for all unit vectors u. (The value k is the average value of $|\cos\varphi|$ for $\varphi \in [0, 2{\cdot}\pi]$; it can be found by integration, but soon we will show another way to find it.)

If the curve α is smooth, then according to Exercise 3.3

$$\begin{aligned}\operatorname{length}\alpha &= \int\limits_a^b |\alpha'(t)|{\cdot}dt = \\ &= \int\limits_a^b k{\cdot}\overline{|\alpha_u'(t)|}{\cdot}dt = \\ &= k{\cdot}\overline{\operatorname{length}\alpha_u}.\end{aligned}$$

This formula and its relatives are called Crofton formulas. To find the coefficient k one can apply it for the unit circle: the left hand side is $2\cdot\pi$ — this is the length of unit circle. Note that for any unit vector u, the curve α_u runs back and forth along an interval of length 2. Therefore $\operatorname{length}\alpha_u = 4$ and hence its average value is also 4. It follows that the coefficient k has to satisfy the equation $2\cdot\pi = k\cdot 4$; whence

$$\operatorname{length}\alpha = \tfrac{\pi}{2}\cdot\overline{\operatorname{length}\alpha_u}.$$

The Crofton's formula holds for arbitrary rectifiable curves, not necessary smooth; it can be proved using Exercises 3.12.

3.10. Exercise. *Assume that closed plane curve α has length at least $\pi\cdot s$, where s is the average of pojections of α to lines. Moreover the equality holds if and only if α is convex.*

Use this statement to solve of Exercise 3.9.

3.11. Advanced exercise. *Show that the length of space curve is proportional to the average length of its projections to all lines and to planes. Find the coefficients in each case.*

3.12. Advanced exercises.

(a) *Show that the formula ➊ holds for any Lipschitz curve $\alpha\colon [a,b]\to \to\mathbb{R}^3$.*

(b) *Construct a simple curve $\alpha\colon [a,b]\to\mathbb{R}^3$ such that the velocity vector $\alpha'(t)$ is defined and bounded for almost all $t\in[a,b]$, but the formula ➊ does not hold.*

Hint: Use theorems of Rademacher and Lusin (A.4 and A.5).

Semicontinuity of length

Recall that the lower limit of a sequence of real numbers (x_n) is denoted by

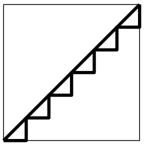
$$\varliminf_{n\to\infty} x_n.$$

It is defined as the lowest partial limit; that is, the lowest possible limit of a subsequence of (x_n). The lower limit is defined for any sequence of real numbers and it lies in the exteded real line $[-\infty,\infty]$

3.13. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\alpha_n\colon [a,b]\to\mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\alpha_\infty\colon[a,b]\to\mathcal{X}$; that is, $\alpha_n(t)\to\alpha_\infty(t)$ for any fixed $t\in[a,b]$ as $n\to\infty$. Then

➋ $$\varliminf_{n\to\infty}\operatorname{length}\alpha_n \geqslant \operatorname{length}\alpha_\infty.$$



Note that the inequality ❷ might be strict. For example the diagonal α_∞ of the unit square can be approximated by a sequence of stairs-like polygonal curves α_n with sides parallel to the sides of the square (α_6 is on the picture). In this case

$$\operatorname{length}\alpha_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length}\alpha_n = 2$$

for any n.

Proof. Fix a partition $a = t_0 < t_1 < \dots < t_k = b$. Set

$$\Sigma_n := |\alpha_n(t_0) - \alpha_n(t_1)| + \dots + |\alpha_n(t_{k-1}) - \alpha_n(t_k)|.$$
$$\Sigma_\infty := |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + \dots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|.$$

Note that $\Sigma_n \to \Sigma_\infty$ as $n \to \infty$ and $\Sigma_n \leqslant \operatorname{length}\alpha_n$ for each n. Hence

❸ $$\varliminf_{n\to\infty} \operatorname{length}\alpha_n \geqslant \Sigma_\infty.$$

If α_∞ is rectifiable, we can assume that

$$\operatorname{length}\alpha_\infty < \Sigma_\infty + \varepsilon.$$

for any given $\varepsilon > 0$. By ❸ it follows that

$$\varliminf_{n\to\infty} \operatorname{length}\alpha_n > \operatorname{length}\alpha_\infty - \varepsilon$$

for any $\varepsilon > 0$; whence ❷ follows.

It remains to consider the case when α_∞ is not rectifiable; that is $\operatorname{length}\alpha_\infty = \infty$. In this case we can choose a partition so that $\Sigma_\infty > L$ for any real number L. By ❸ it follows that

$$\varliminf_{n\to\infty} \operatorname{length}\alpha_n > L$$

for any L; whence

$$\varliminf_{n\to\infty} \operatorname{length}\alpha_n = \infty$$

and ❷ follows. □

Length metric

Let $\mathcal{X}$ be a metric space. Given two points x, y in $\mathcal{X}$, denote by $d(x, y)$ the exact lower bound for lengths of all paths connecting x to y; if there is no such path we assume that $d(x, y) = \infty$.

Note that function d satisfies all the axioms of metric except it might take infinite value. Therefore if any two points in $\mathcal{X}$ can be connected by a rectifiable curve, then d defines a new metric on $\mathcal{X}$; in this case d is called *induced length metric.*

Evidently $d(x,y) \geqslant |x-y|$ for any pair of points $x, y \in \mathcal{X}$. If the equality holds for any pair, then then the metric is called *length metric* and the space is called *length-metric space.*

Most of the time we consider length-metric spaces. In particular the Euclidean space is a length-metric space. A subspaces A of length-metric space $\mathcal{X}$ might be not a lenght-metric space; the induced length distance between points x and y in the subspace A will be denoted as $|x-y|_A$; that is $|x-y|_A$ is the exact lower bound for the length of paths in A.

3.14. Exercise. *Let $A \subset \mathbb{R}^3$ be a closed subset. Show that A is convex if and only if*

$$|x-y|_A = |x-y|_{\mathbb{R}^3}.$$

3.15. Exercise. *Let us denote by $\mathbb{S}^1$ the unit circle in the plane; that is,*

$$\mathbb{S}^1 = \left\{\, (x,y) \in \mathbb{R}^3 \;:\; x^2+y^2=1 \,\right\}.$$

Show that

$$|u-v|_{\mathbb{S}^1} = \measuredangle(u,v) := \arccos\langle u,v\rangle$$

for any $u, v \in \mathbb{S}^1$.

Spherical curves

A space curve γ is called *spherical* if it runs in the unit sphere; that is, $|\gamma(t)| = 1$ for any t.

3.16. Exercise. *Let us denote by $\mathbb{S}^2$ the unit sphere in the space; that is,*

$$\mathbb{S}^2 = \left\{\, (x,y,z) \in \mathbb{R}^3 \;:\; x^2+y^2+z^2=1 \,\right\}.$$

Show that

$$|u-v|_{\mathbb{S}^2} = \measuredangle(u,v) := \arccos\langle u,v\rangle$$

for any $u, v \in \mathbb{S}^2$.

Hint: Use Exercise 3.15 and the following map $f\colon (r,\theta,\varphi) \mapsto (r,\theta,0)$ in spherical coordinates. Note that f is distance nonexpanding and it maps $\mathbb{R}^3$ to a half-plane and $\mathbb{S}^2$ to one of its meridians.

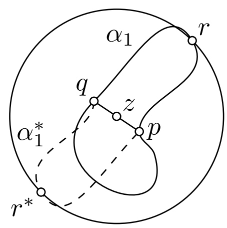
3.17. Hemisphere lemma. *Any closed curve of length $<2{\cdot}\pi$ in $\mathbb{S}^2$ lies in an open hemisphere.*

This lemma is a keystone in the proof of Fenchel's theorem given below. The lemma is not as simple as you might think — try to prove it yourself. I learned the following proof from Stephanie Alexander.

Proof. Let α be a closed curve in $\mathbb{S}^2$ of length $2{\cdot}\ell$.

Assume $\ell<\pi$.

Let us divide α into two arcs α_1 and α_2 of length ℓ, with endpoints p and q. According to Exercise 3.16, $\measuredangle(p,q)\leqslant\ell<\pi$. Denote by z be the midpoint between p and q in $\mathbb{S}^2$; that is z is the midpoint of an equator arc from p to q. We claim that α lies in the open north hemisphere with north pole at z. If not, α intersects the equator in a point, say r. Without loss of generality we may assume that r lies on α_1.



The north hemisphere corresponds to the disc and the south hemisphere to the complement of the disc.

Rotate the arc α_1 by angle π around the line thru z and the center of the sphere. The obtained arc α_1^* together with α_1 forms a closed curve of length $2{\cdot}\ell$ that passes thru r and its antipodal point r^*. Therefore

$$\tfrac{1}{2}\cdot\operatorname{length}\alpha=\ell\geqslant\measuredangle(r,r^*)=\pi,$$

a contradiction. $\square$

3.18. Exercise. *Describe a simple closed spherical curve that does not pass thru a pair of antipodal points and does not lie in any hemisphere.*

3.19. Exercise. *Suppose that a closed simple spherical curve α divides $\mathbb{S}^2$ into two regions of equal area. Show that*

$$\operatorname{length}\alpha\geqslant 2{\cdot}\pi.$$

3.20. Exercise. *Consider the following problem, find a flaw in the given solution. Come up with a correct argument.*

Problem. Suppose that a closed plane curve α has length at most 4. Show that α lies in a unit disc.

Wrong solution. Note that it is sufficient to show that diameter of α is at most 2; that is, the distance between any two pairs of points p and q of α an not exceed 2.

The length of α can not be smaller then the closed inscribed polygonal line which goes from p to q and back to p. Therefore

$$2\cdot|p-q| \leqslant \operatorname{length}\alpha \leqslant 4.$$

□

3.21. Advanced exercises. *Given points $v, w \in \mathbb{S}^2$, denote by w_v the closest point to w_u on the equator with pole at v; in other words, it $w^{\perp}$ is the projection of w to the plane perpendicular to v, then w_v is the unit vector in the direction of $w^{\perp}$. The vector w_v is defined if $w \neq \pm v$.*

1. *Show that for any spherical curve α we have that*

$$\operatorname{length}\alpha = \overline{\operatorname{length}\alpha_v},$$

   where $\overline{\operatorname{length}\alpha_v}$ denotes the average length for all $v \in \mathbb{S}^2$. (This is a spherical analog of Crofton's formula.)
2. *Give another proof of hemisphere lemme using part (1).*

Chapter 4

Space curves

Acceleration of unit-speed curve

Recall that any regular smooth curve can be parametrized by its arc length. The obtained curve, say γ, remains to be smooth and it has unit speed; that is, $|\gamma'(s)| = 1$ for all s.

The following proposition states that the acceleration vector is perpendicular to the velocity vector if the speed remains constant.

4.1. Proposition. *Assume γ is a smooth unit-speed space curve. Then $\gamma'(s) \perp \gamma''(s)$ for any s.*

The scalar product (also known as dot product) of two vectors v and w will be denoted by $\langle v, w\rangle$. Recall that the derivative of a scalar product satisfies the product rule; that is if $v = v(t)$ and $w = w(t)$ are smooth vector-valued functions of a real parameter t, then

$$\langle v, w\rangle' = \langle v', w\rangle + \langle v, w'\rangle.$$

Proof. The identity $|\gamma'| = 1$ can be rewritten as $\langle \gamma', \gamma'\rangle = 1$. Therefore

$$2\cdot\langle \gamma'', \gamma'\rangle = \langle \gamma', \gamma'\rangle' = 0,$$

whence $\gamma'' \perp \gamma'$. $\square$

Curvature

For a unit speed smooth space curve γ the magnitude of its acceleration $|\gamma''(s)|$ is called its *curvature* at the time s. If γ is simple, then we can say that $|\gamma''(s)|$ is the curvature at the point $p = \gamma(s)$ without ambiguity. The curvature is usually denoted by $k(s)$ or $k_\gamma(s)$ and in the latter case it might be also denoted by $k(p)$ or $k_\gamma(p)$.

The curvature measures how fast the curve turns; if you drive along a plane curve, curvature tells how much to turn the steering wheel at the given point (note that it does not depend on your speed). In general, the term *curvature* is used for different types of geometric objects, and it always measures how much it deviates from being *straight*; for curves, it it measures how fast it deviates from a straight line.

4.2. Exercise. *Show that any regular smooth spherical curve has curvature at least 1 at each time.*

Hint: Differentiate the identity $\langle\gamma(s),\gamma(s)\rangle = 1$ a couple of times.

Tangent indicatrix

Let γ be a regular smooth space curve. Let us consider another curve

➊ $$\tau(t) = \tfrac{\gamma'(t)}{|\gamma'(t)|}$$

that is called *tangent indicatrix* of γ. Note that $|\tau(t)| = 1$ for any t; that is, τ is a spherical curve.

The line thru $\gamma(s)$ in the direction of $\tau(s)$ is called *thangent line* at s.

If γ has a unit speed parametrization, then $\tau(t) = \gamma'(t)$. In this case we have the following expression for curvature: $k(t) = |\tau'(t)| = |\gamma''(t)|$.

In general case we have

➋ $$k(t) = \tfrac{|\tau'(t)|}{|\gamma'(t)|}.$$

Indeed, for an arc length parametrization $s(t)$ we have $s'(t) = |\gamma'(t)|$. Therefore

$$\begin{aligned} k(t) &= |\tfrac{d\tau}{ds}| = \\ &= |\tfrac{d\tau}{dt}|/|\tfrac{ds}{dt}| = \\ &= \tfrac{|\tau'(t)|}{|\gamma'(t)|}. \end{aligned}$$

Note that indicatrix of a smooth regular curve γ is regular if the curvature of γ does not vanish.

4.3. Exercise. *Use the formulas* ➊ *and* ➋ *to show that for any smooth regular space curve γ we have the following expressions for its curvature:*

(a)
$$k(t)=\frac{|\gamma''(t)^\perp|}{|\gamma'(t)|^2},$$
where $\gamma''(t)^\perp$ denotes the projection of $\gamma''(t)$ to the normal plane of $\gamma'(t)$;

(b)
$$k(t)=\frac{|\gamma''(t)\times\gamma'(t)|}{|\gamma'(t)|^3},$$
where $\times$ denotes the vector product *(also known as* cross product*).*

Hint: Prove and use the following identities:
$$\begin{aligned}\gamma''(t)-\gamma''(t)^\perp&=\tfrac{\gamma'(t)}{|\gamma'(t)|}\cdot\langle\gamma''(t),\tfrac{\gamma'(t)}{|\gamma'(t)|}\rangle,\\ |\gamma'(t)|&=\sqrt{\langle\gamma'(t),\gamma'(t)\rangle}.\end{aligned}$$

4.4. Exercise. *Apply the formulas in the previous exercise to show that if f is a smooth real function, then its graph $y=f(x)$ has curvature*
$$k(p)=\frac{|f''(x)|}{(1+f'(x)^2)^{\frac{3}{2}}}$$
at the point $p=(x,f(x))$.

Total curvature

Let $\gamma\colon\mathbb{I}\to\mathbb{R}^3$ be a regular smooth curve and τ its tangent indicatrix. Recall that without loss of generality we can assume that γ has a unit speed parametrization; in this case $\tau(s)=\gamma'(s)$ and hence
$$k(s):=|\gamma''(s)|=|\tau'(s)|;$$
that is, the curvature of γ at time s is the speed of the tangent indicatrix τ at the same time moment.

The integral
$$\Phi(\gamma):=\int\limits_{\mathbb{I}}k(s)\cdot ds$$
is called *total curvature of γ*.

4.5. Exercise. *Find the curvature of the helix*
$$\gamma_{a,b}(t)=(a\cdot\cos t,a\cdot\sin t,b\cdot t),$$

its tangent indicatrix and the total curvature of its arc $t \in [0, 2{\cdot}\pi]$.

4.6. Observation. *The total curvature of a smooth regular curve is the length of its tangent indicatrix.*

Proof. It is sufficient to prove the observation for a unit-speed space curve $\gamma\colon \mathbb{I} \to \mathbb{R}^3$. Denote by τ its tangent indicatrix. Then

$$\begin{aligned}\Phi(\gamma) &:= \int\limits_{\mathbb{I}} k(s){\cdot}ds = \\ &= \int\limits_{\mathbb{I}} |\tau'(s)|{\cdot}ds = \\ &= \operatorname{length}\tau.\end{aligned}$$

□

4.7. Fenchel's theorem. *The total curvature of any closed regular space curve is at least $2{\cdot}\pi$.*

Proof. Fix a closed regular space curve γ; we can assume that it is described by a loop $\gamma\colon [a,b] \to \mathbb{R}^3$; in this case $\gamma(a) = \gamma(b)$ and $\gamma'(a) = \gamma'(b)$.

Consider its tangent indicatrix $\tau = \gamma'$. Recall that $|\tau(s)| = 1$ for any s; that is, τ is a closed spherical curve.

Let us show that τ can not lie in a hemisphere. Assume the contrary; without loss of generality we can assume that τ lies in the north hemisphere defined by the inequality $z > 0$ in (x,y,z)-coordinates. It means that $z'(t) > 0$ at any time, where $\gamma(t) = (x(t), y(t), z(t))$. Therefore

$$z(b) - z(a) = \int\limits_a^b z'(s){\cdot}ds > 0.$$

In particular, $\gamma(a) \neq \gamma(b)$, a contradiction.

Applying the observation (4.6) and the hemisphere lemma (3.17), we get that

$$\Phi(\gamma) = \operatorname{length}\tau \geqslant 2{\cdot}\pi.$$

□

4.8. Exercise. *Show that a closed space curve γ with curvature at most 1 can not be shorter than the unit circle; that is,* $\operatorname{length}\gamma \geqslant 2{\cdot}\pi$.

4.9. Advanced exercise. *Suppose that γ is a smooth regular space curve that does not pass thru the origin. Consider the spherical curve defined as $\sigma(t) = \frac{\gamma(t)}{|\gamma(t)|}$ for any t. Show that*

$$\operatorname{length}\sigma < \Phi(\gamma) + \pi.$$

Moreover, if γ is closed, then

$$\operatorname{length}\sigma \leqslant \Phi(\gamma).$$

Note that the last inequality gives an alternative proof of Fenchel's theorem. Indeed, without loss of generality we can assume that the origin lies on a chord of γ; in this case the spherical curve σ passes thru a apir of antipodal points in $\mathbb{S}^2$ whence $\operatorname{length}\sigma \geqslant 2\cdot\pi$.

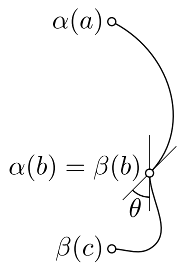
Piecewise smooth curves

Assume $\alpha\colon [a,b] \to \mathbb{R}^3$ and $\beta\colon [b,c] \to \mathbb{R}^3$ are two curves such that $\alpha(b) = \beta(b)$. Then one can combine these two curves into one $\gamma\colon [a,c] \to \to \mathbb{R}^3$ by the rule

$$\gamma(t) = \begin{cases} \alpha(t) & \text{if} \quad t \leqslant b, \\ \beta(t) & \text{if} \quad t \geqslant b. \end{cases}$$

The obtained curve γ is called the *concatenation* of α and β.

The same definition of cancatination can be applied if α and/or β are defied on semiopen intervals $(a,b]$ and/or $[b,c)$.



The concatenation can be also defined if the end point of the first curve coincides with the starting point of the second curve; if this is the case, the time intervals of both curves can be shifted so that they fit together.

If in addition $\beta(c) = \alpha(a)$ then we can do cyclic concatination of these curves; this way we obtain a closed curve.

If $\alpha'(b)$ and $\beta'(b)$ are defined then the angle $\theta = \measuredangle(\alpha'(b), \beta'(b))$ is called *external angle* of γ at time b.

A space curve γ is called *piecewise smooth and regular* if it can be presented as a concatination of finite number of smooth regular curves; if γ is closed, then the concatination is assumed to be cyclic.

If γ is a concatination of smooth regular arcs $\gamma_1, \dots, \gamma_n$, then the total curvature of γ is defined as a sum of total curvatures of γ_i and the external angles; that is,

$$\Phi(\gamma) = \Phi(\gamma_1) + \dots + \Phi(\gamma_n) + \theta_1 + \dots + \theta_{n-1}$$

where θ_i is the external angle at the joint γ_i and γ_{i+1}; if γ is closed, then

$$\Phi(\gamma) = \Phi(\gamma_1) + \dots + \Phi(\gamma_n) + \theta_1 + \dots + \theta_n,$$

where θ_n is the external angle at the joint γ_n and γ_1.

4.10. Generalized Fenchel's theorem. *Let γ be a closed piecewise smooth regular space curve. Then*

$$\Phi(\gamma) \geqslant 2{\cdot}\pi.$$

Proof. Suppose γ is a cyclic concatenation of n smooth regular arcs $\gamma_1, \dots, \gamma_n$. Denote by $\theta_1, \dots, \theta_n$ its external angles. We need to show that

❸ $$\Phi(\gamma_1) + \dots + \Phi(\gamma_n) + \theta_1 + \dots + \theta_n \geqslant 2{\cdot}\pi.$$

Consider the tangent indicatrix $\tau_1, \dots, \tau_n$ for each arc $\gamma_1, \dots, \gamma_n$; these are smooth spherical arcs.

The same argument as in the proof of Fenchel's theorem, shows that the curves $\tau_1, \dots, \tau_n$ can not lie in an open hemisphere.

Note that the spherical distance from the end point of τ_i to the starting point of τ_{i+1} is equal to the external angle θ_i (we enumerate modulo n, so $\gamma_{n+1} = \gamma_1$). Therefore if we connect the end point of τ_i to the starting point of τ_{i+1} by a short arc of a great circle in the sphere, then we add $\theta_1 + \dots + \theta_n$ to the total length of $\tau_1, \dots, \tau_n$.

Applying the hemisphee lemma (3.17) to the obtained closed curve, we get that

$$\operatorname{length}\tau_1 + \dots + \operatorname{length}\tau_n + \theta_1 + \dots + \theta_n \geqslant 2{\cdot}\pi.$$

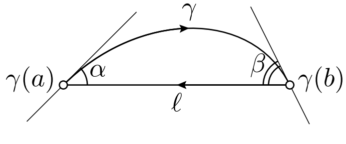
Applying the observation (4.6), we get ❸. □

4.11. Chord lemma. *Let ℓ be the chord to a smooth regular arc $\gamma\colon [a,b] \to \mathbb{R}^3$. Assume γ meets ℓ at angles α and β at its ends; that is*

$$\alpha = \measuredangle(w, \gamma'(a)) \quad \textit{and} \quad \beta = \measuredangle(w, \gamma'(b)),$$

where $w = \gamma(b) - \gamma(a)$. Then

$$\Phi(\gamma) \geqslant \alpha + \beta.$$



Proof. Let us parameterize the chord ℓ from $\gamma(b)$ to $\gamma(a)$ and consider the cyclic concatenation $\bar\gamma$ of γ and ℓ. The closed curve $\bar\gamma$ has two external angles $\pi - \alpha$ and $\pi - \beta$. Since curvature of ℓ vanish, we get that

$$\Phi(\bar\gamma) = \Phi(\gamma) + (\pi - \alpha) + (\pi - \beta).$$

According to the generalized Fenechel's theorem (4.10),

$$\Phi(\bar\gamma) \geqslant 2\cdot\pi;$$

hence the result. □

4.12. Exercise. *Show that the estimate in the chord lemma is optimal.*

That is, given two points p, q and two nonzero vectors u, v in $\mathbb{R}^3$. Show that there is a smooth regular curve γ that starts at p in the direction of u and ends at q in the direction of v such that $\Phi(\gamma)$ is arbitrary close to $\measuredangle(w, u) + \measuredangle(w, v)$, where $w = q - p$.

Polygonal lines

Polygonal lines are partial case of piecewise smooth regular curves; each arc in its concatenation is a line segment. Since the curvature of a line segment vanish, the total curvature of polygonal line is the sum of its external angles.

4.13. Exercise. *Let a, b, c, d and x be distinct points in $\mathbb{R}^3$. Show that the total curvature of polygonal line $abcd$ can not exceed the the total curvature of $abxcd$; that is,*

$$\Phi(abcd) \le \Phi(abxcd).$$

Use this statement to show that any closed polygonal line has curvature at least $2\cdot\pi$.

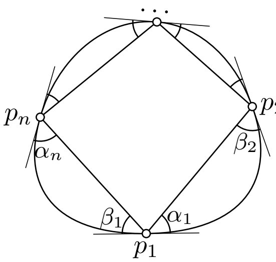
Hint: Use that exterior angle of a triangle equals to the sum of the two remote interior angles; for the second part apply the induction on number of vertexes.

4.14. Proposition. *Assume a polygonal line $\hat\gamma = p_1 \dots p_n$ is inscribed in a smooth regular curve γ. Then*

$$\Phi(\gamma) \geqslant \Phi(\hat\gamma).$$

Moreover if γ is closed we can assume that the inscribed polygonal line $\hat\gamma$ is also closed.

Proof. Since the curvature of line segments vanishes, the total curvature of polygonal line is the sum of external angles $\theta_i = \pi - \measuredangle p_{i-1}p_ip_{i+1}$.



Assume $p_i = \gamma(t_i)$. Set

$$w_i = p_{i+1} - p_i, \qquad v_i = \gamma'(t_i),$$
$$\alpha_i = \measuredangle(w_i, v_i), \qquad \beta_i = \measuredangle(w_{i-1}, v_i).$$

In case of closed curve we use indexes modulo n, in particular $p_{n+1} = p_1$.

Note that $\theta_i = \measuredangle(w_{i-1}, w_i)$. Therefore

$$\theta_i \leqslant \alpha_i + \beta_i.$$

By the chord lemma, the total curvature of the arc of γ from p_i to p_{i+1} is at least $\alpha_i + \beta_{i+1}$.

Therefore if γ is a closed curve, we have

$$\begin{aligned}\Phi(\hat\gamma) &= \theta_1 + \dots + \theta_n \leqslant \\ &\leqslant \beta_1 + \alpha_1 + \dots + \beta_n + \alpha_n = \\ &= (\alpha_1 + \beta_2) + \dots + (\alpha_n + \beta_1) \leqslant \\ &\leqslant \Phi(\gamma).\end{aligned}$$

If γ is an arc, the argument is analogous:

$$\begin{aligned}\Phi(\hat\gamma) &= \theta_2 + \dots + \theta_{n-1} \leqslant \\ &\leqslant \beta_2 + \alpha_2 + \dots + \beta_{n-1} + \alpha_{n-1} \leqslant \\ &\leqslant (\alpha_1 + \beta_2) + \dots + (\alpha_{n-1} + \beta_n) \leqslant \\ &\leqslant \Phi(\gamma).\end{aligned}$$

□

4.15. Exercise.

(a) Draw a smooth regular plane curve γ which has a self-intersection, such that $\Phi(\gamma) < 2\cdot\pi$.

(b) Show that if a smooth regular curve $\gamma\colon [a,b] \to \mathbb{R}^3$ has a self-intersection, then $\Phi(\gamma) > \pi$.

4.16. Proposition. *The equality case in the Fenchel's theorem holds only for convex plane curves; that is, if the total curvature of a smooth regular space curve γ is equal to $2\cdot\pi$, then it is a convex plane curve.*

The proof is an application of Proposition 4.14.

Proof. Consider an inscribed quadraliteral $abcd$ in γ. By the definition of total curvature, we have that

$$\begin{aligned}\Phi(abcd) &= (\pi - \measuredangle dab) + (\pi - \measuredangle abc) + (\pi - \measuredangle bcd) + (\pi - \measuredangle cda) = \\ &= 4\cdot\pi - (\measuredangle dab + \measuredangle abc + \measuredangle bcd + \measuredangle cda))\end{aligned}$$

Note that

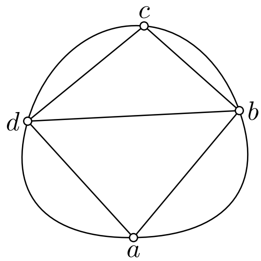
$$\text{❹} \qquad \measuredangle abc \leqslant \measuredangle abd + \measuredangle dbc \quad \text{and} \quad \measuredangle cda \leqslant \measuredangle cdb + \measuredangle bda.$$

The sum of angles in any triangle is π. Therefore combining these inequalities, we get that

$$\Phi(abcd) \geqslant 4\cdot\pi - (\measuredangle dab + \measuredangle abd + \measuredangle bda) - (\measuredangle bcd + \measuredangle cdb + \measuredangle dbc) = \\ = 2\cdot\pi.$$

By 4.14,

$$\Phi(abcd) \leqslant \Phi(\gamma) \leqslant 2\cdot\pi.$$

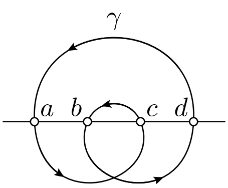


Therefore we have equalities in ❹. It means that the point d lies in the angle abc and the point b lies in the angle cda. That is, $abcd$ is a convex plane quadralit-eral.

It follows that any quadraliteral inscribed in γ is convex plane quadraliteral. Therefore all points of γ lie in one plane and the domain bounded by γ is convex; that is, γ is a convex plane curve. $\square$

4.17. Exercise. *Suppose that a closed curve γ crosses a line at four points a, b, c and d. Assume that these points appear on the line in the order a, b, c, d and they appear on the curve γ in the order a, c, b, d. Show that*

$$\Phi(\gamma) \geqslant 4\cdot\pi.$$

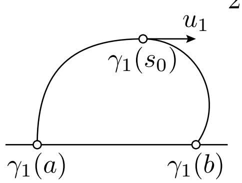


A line crossing a curve at four points as in the exercise is called *alternating quadrisecants.* It turns out that any *nontrivial knot* admits an alternating quadrisecants [1]; it implies the so called Fáry–Milnor theorem — *the total curvature any knot exceeds* $4\cdot\pi$.

Bow lemma

4.18. Lemma. *Let $\gamma_1\colon [a,b] \to \mathbb{R}^2$ and $\gamma_2\colon [a,b] \to \mathbb{R}^3$ be two smooth unit-speed curves; denote by $k_1(s)$ and $k_2(s)$ their curvatures at s. Suppose that $k_1(s) \geqslant k_2(s)$ for any s and the curve γ_1 is a simple arc of a convex curve; that is, it runs in the boundary of a covex plane figure. Then the distance between the ends of γ_1 can not axceed the distance between the ends of γ_2; that is,*

$$|\gamma_1(b) - \gamma_1(a)| \leqslant |\gamma_2(b) - \gamma_2(a)|.$$



Proof. Denote by τ_1 and τ_2 the tangent indicatrixes of γ_1 and γ_2 correspondingly.

Let $\gamma_1(s_0)$ be the point on γ_1 that maximize the distance to the line thru $\gamma(a)$ and $\gamma(b)$. Consider two unit vectors

$$u_1 = \tau_1(s_0) = \gamma_1'(s_0) \quad \text{and} \quad u_2 = \tau_2(s_0) = \gamma_2'(s_0).$$

By construction the vector u_1 is parallel to $\gamma(b) - \gamma(a)$ in particular

$$|\gamma_1(b) - \gamma_1(a)| = \langle u_1, \gamma_1(b) - \gamma_1(a)\rangle$$

Since γ_1 is an arc of convex curve, it indicatrix $\tau(s)$ runs in one direction along the unit circle. Suppose $s \leqslant s_0$, then

$$\begin{aligned}
\measuredangle(\gamma_1'(s), u_1) &= \measuredangle(\tau_1(s), \tau_1(s_0)) = \\
&= \operatorname{length}(\tau_1|_{[s,s_0]}) = \\
&= \int\limits_s^{s_0} |\tau_1'(t)|\cdot dt = \\
&= \int\limits_s^{s_0} k_1(t)\cdot dt \geqslant \\
&\geqslant \int\limits_s^{s_0} k_2(t)\cdot dt = \\
&= \int\limits_s^{s_0} |\tau_2'(t)|\cdot dt = \\
&= \operatorname{length}(\tau_1|_{[s,s_0]}) \geqslant \\
&\geqslant \measuredangle(\tau_2(s), \tau_2(s_0)) = \\
&= \measuredangle(\gamma_2'(s), u_2),
\end{aligned}$$

The same argument shows that

$$\measuredangle(\gamma_1'(s), u_1) \geqslant \measuredangle(\gamma_2'(s), u_2)$$

for $s \geqslant s_0$; therefore the inequality holds for any s. Since the vectors $\gamma_1'(s), u_1, \gamma_2'(s), u_2$ are unit, it follwos that

$$\langle\gamma_1'(s), u_1\rangle \leqslant \langle\gamma_2'(s), u_2\rangle.$$

Integrating the last inequality, we get that

$$\begin{aligned}|\gamma_1(b)-\gamma_1(a)| &= \langle u_1,\gamma_1(b)-\gamma_1(a)\rangle = \\ &= \int\limits_a^b \langle u_1,\gamma_1'(s)\rangle\cdot ds \leqslant \\ &\leqslant \int\limits_a^b \langle u_2,\gamma_2'(s)\rangle\cdot ds = \\ &= \langle u_2,\gamma_2(b)-\gamma_2(a)\rangle \leqslant \\ &\leqslant |\gamma_2(b)-\gamma_2(a)|.\end{aligned}$$

Hence the result. □

4.19. Exercise. *Let $\gamma\colon[a,b]\to\mathbb{R}^3$ be a smooth regular curve and $0<\theta\leqslant\frac{\pi}{2}$. Suppose*

$$\Phi(\gamma)\leqslant 2\cdot\theta.$$

(a) Show that

$$|\gamma(b)-\gamma(a)|>\cos\theta\cdot\operatorname{length}\gamma.$$

(b) Use part (a) to give another solution of 4.15b.

(c) Show that the inequality in (a) is optimal; that is, given θ there is a smooth regular curve γ such that $\frac{|\gamma(b)-\gamma(a)|}{\operatorname{length}\gamma}$ is arbitrary close to $\cos\theta$.

Hint: Choose a value $s_0\in[a,b]$ that splits the total curvature into two equal parts, θ in each. Observe that $\measuredangle(\gamma'(s_0),\gamma'(s))\leqslant\theta$ for any s. Use this inequality the same way as in the proof of the bow lemma.

4.20. Exercise. *Suppose that two points p and q lie on a unit circle and dividing it in two arcs with lengths $\ell_1<\ell_2$. Show that if a curve γ runs from p to q and has curvature at most 1, then either*

$$\operatorname{length}\gamma\leqslant\ell_1 \quad \textit{or} \quad \operatorname{length}\gamma\geqslant\ell_2.$$

DNA inequality*

Recall that curvature of a spherical curve is at least 1 (Exercise 4.2). In particular the length of spherical curve can not exceed its total curvature. The following theorem shows that the same inequality holds for *closed* curves in a unit ball.

4.21. Theorem. *Let γ be a smooth regular closed curve that lies in a unit ball. Then*

$$\Phi(\gamma) \geqslant \operatorname{length}\gamma.$$

Proof. Without loss of generality we can assume the curve is described by a loop $\gamma\colon[0,\ell]\to\mathbb{R}^3$ parameterized by its arc length, so $\ell=\operatorname{length}\gamma$. We can also assume that the origin is the center of the ball. It follows that

$$\langle\gamma'(s),\gamma'(s)\rangle = 1, \qquad |\gamma(s)|\leqslant 1$$

and in particular

$$\begin{aligned}\langle\gamma''(s),\gamma(s)\rangle &\geqslant -|\gamma''(s)|\cdot|\gamma(s)|\geqslant\\ &\geqslant -k(s)\end{aligned}$$

for any s, where $k(s)=|\gamma''(s)|$ is the curvature of γ at s.

Since γ is closed, we have that $\gamma'(0)=\gamma'(\ell)$ and $\gamma(0)=\gamma(\ell)$. Therefore

$$\begin{aligned}0&=\langle\gamma(\ell),\gamma'(\ell)\rangle-\langle\gamma(0),\gamma'(0)\rangle=\\ &=\int_0^\ell\langle\gamma(s),\gamma'(s)\rangle'\cdot ds=\\ &=\int_0^\ell\langle\gamma'(s),\gamma'(s)\rangle\cdot ds+\int_0^\ell\langle\gamma(s),\gamma''(s)\rangle\cdot ds\geqslant\\ &\geqslant\ell-\Phi(\gamma),\end{aligned}$$

whence the result. □

This theorem was proved by Don Chakerian [2, 3]; for plane curves it was proved earlier by István Fáry [4]. Few proofs of this theorem are discussed by Serge Tabachnikov [5]. He also conjectured the following closely related statement:

4.22. Theorem. *Suppose a closed regular smooth curve γ lies in a convex figure with the perimeter $2\cdot\pi$. Then*

$$\Phi(\gamma) \geqslant \operatorname{length}\gamma.$$

It was proved by Jeffrey Lagarias and Thomas Richardson [6]; latter a simpler proof was given by Alexander Nazarov and Fedor Petrov [7]. Despite the simplicity of the statement, its proof is annoyingly difficult; we do not present it here.

Nonsmooth curves*

4.23. Theorem. *For any regular smooth space curve γ we have that*

$$\Phi(\gamma) = \sup\{\Phi(\beta)\},$$

where the least upper bound is taken for all polygonal lines β inscribed in γ; if γ is closed we assume that so is β.

Proof. Note that the inequality

$$\Phi(\gamma) \geqslant \Phi(\beta)$$

follows from 4.14; it remains to show

❺ $$\Phi(\gamma) \leqslant \sup\{\Phi(\beta)\}.$$

Let $\gamma\colon [a,b] \to \mathbb{R}^3$ be a smooth curve. Fix a partition $a = t_0 < \,< \dots < t_n = b$ and consider the corresponding inscribed polygonal line $\beta = p_0 \dots p_n$. (If γ is closed, then $p_0 = p_n$ and β is closed as well.)

Let $\chi = \xi_1 \dots \xi_n$ be a spherical polygonal line with the vertexes $\xi_i = \frac{p_i - p_{i-1}}{|p_i - p_{i-1}|}$. We can assume that χ has constant speed on each arc and $\chi(t_i) = \xi_i$ for each i. The spherical polygonal line χ will be called tangent indicatrix for β.

Consider a sequence of finer and finer partitions, denote by β_n and χ_n the corresponding inscribed polygonal lines and their tangent indicatrixes. Note that since γ is smooth, the idicatrixes χ_n converge pointwise to τ — the thangent indicatrix of γ. By semi-continuity of the length (3.13), we get that

$$\begin{aligned}\Phi(\gamma) &= \operatorname{length}\tau \leqslant \\ &\leqslant \varliminf_{n\to\infty} \operatorname{length}\chi_n = \\ &= \varliminf_{n\to\infty} \Phi(\beta_n) \leqslant \\ &\leqslant \sup\{\Phi(\beta)\},\end{aligned}$$

where the last supremum is taken over all partitions and their corresponding inscribed polygonal lines β; whence ❺ follows. □

The theorem above can be used to define total curvature for arbitrary curves, not necessary (piecewise) smooth and regular. We say that a parameterized curve is trivial if it is constant; that is, it stays at one point.

4.24. Definition. *The total curvature of a nontrivial parameterized space curve γ is the exact upper bound on the total curvatures of inscribed nondegenerate polygonal lines; if γ is closed then we assume that the inscribed polygonal lines are closed as well.*

4.25. Exercise. *Show that the total curvature is lower semi-continuous with respect to pointwise convergence of curves. That is, if a sequence of curves $\gamma_n\colon [a,b] \to \mathbb{R}^3$ converges pointwise to a nontrivial curve $\gamma_\infty\colon [a,b] \to \mathbb{R}^3$, then*

$$\varliminf_{n\to\infty} \Phi(\gamma_n) \geqslant \Phi(\gamma_\infty).$$

Hint: Modify the proof of semi-continuity of length (Theorem 3.13).

4.26. Exercise. *Show that Fenchel's theorem holds for any nontrivial closed curve γ; that is,*

$$\Phi(\gamma) \geqslant 2{\cdot}\pi.$$

4.27. Exercise. *Assume that a curve $\gamma\colon [a,b] \to \mathbb{R}^3$ has finite total curvature. Show that γ is rectifiable.*

Construct a rectifiable curve $\gamma\colon [a,b] \to \mathbb{R}^3$ that has infinite total curvature.

A good survey on curves of finite total curvature is written by John Sullivan [8].

Chapter 5

Torsion

Frenet frame

Let γ be a smooth regular space curve. Without loss of generality, we may assume that γ has arc length parametrization, so the velocity vector $\tau(s) = \gamma'(s)$ is unit.

Assume its curvature does not vanish at some time moment s; in other words, $\gamma''(s) \neq 0$. Then we can define the so called *normal vector* at s as

$$\nu(s) = \frac{\gamma''(s)}{|\gamma''(s)|}.$$

Note that

$$\tau'(s) = \gamma''(s) = k(s)\cdot\nu(s).$$

According to 4.1, $\nu(s) \perp \tau(s)$. Therefore the vector product

$$\beta(s) = \tau(s) \times \nu(s)$$

is a unit vector which makes the triple $\tau(s), \nu(s), \beta(s)$ an oriented orthonormal basis in $\mathbb{R}^3$; in particular, we have that

❶
$$\begin{aligned} \langle\tau,\tau\rangle &= 1, & \langle\nu,\nu\rangle &= 1, & \langle\beta,\beta\rangle &= 1, \\ \langle\tau,\nu\rangle &= 0, & \langle\nu,\beta\rangle &= 0, & \langle\beta,\tau\rangle &= 0. \end{aligned}$$

The orthonormal basis $\tau(s), \nu(s), \beta(s)$ is called *Frenet frame* at s; the vectors in the frame are called *tangent*, *normal* and *binormal* correspondingly. Note that the frame $\tau(s), \nu(s), \beta(s)$ is defined only if $k(s) \neq 0$.

The plane thru $\gamma(s)$ spanned by vectors $\tau(s)$ and $\nu(s)$ is called *osculating plane* at s; equivalently it can be defined as a plane thru

$\gamma(s)$ that is perpendicular to the binormal vector $\beta(s)$. This a unique plane that has *second order of contact* with γ at s; that is, $\rho(\ell) = o(\ell^2)$, where $\rho(\ell)$ denotes the distance from $\gamma(s+\ell)$ to the osculating plane at s.

Torsion

Let γ be a smooth unit-speed space curve and $\tau(s), \nu(s), \beta(s)$ is its Frenet frame. The value

$$\varkappa(s) = \langle \nu'(s), \beta(s) \rangle$$

is called *torsion* of γ at s.

Note that the torsion $\varkappa(s)$ is defined at each s with nonzero curvature. Indeed, if $k(s) \neq 0$ then Frenet frame $\tau(s), \nu(s), \beta(s)$ is defined at s. Moreover since the function $s \mapsto k(s)$ is continuous, it must be positive in an open interval containing s; therefore Frenet frame is also defined in this interval. Clearly $\tau(s)$, $\nu(s)$ and $\beta(s)$ depend smoothly on s in their domains of definition. Therefore $\nu'(s)$ is defined and so is the torsion $\varkappa(s)$.

The torsion measures how fast the osculating plane rotated when one travels along γ.

5.1. Exercise. *Given real numbers a and b, calculate curvature and torsion of the helix*

$$\gamma_{a,b}(t) = (a \cdot \cos t, a \cdot \sin t, b \cdot t).$$

Conclude that for any $k > 0$ and $\varkappa$ there is a helix with constant curvature k and torsion $\varkappa$.

Frenet formulas

Assume the Frenet frame $\tau(s), \nu(s), \beta(s)$ of curve γ is defined at s. Recall that

❷ $$\tau'(s) = k(s) \cdot \nu(s).$$

Let us write the remaining derivatives $\nu'(s)$ and $\beta'(s)$ in the frame $\tau(s), \nu(s), \beta(s)$.

First let us show that

❸ $$\nu'(s) = -k(s) \cdot \tau(s) + \varkappa(s) \cdot \beta(s).$$

Since the frame $\tau(s), \nu(s), \beta(s)$ is orthonormal it is equivalent to the following three identities:

$$\langle \nu', \tau\rangle = -k, \qquad \langle \nu', \nu\rangle = 0, \qquad \langle \nu', \beta\rangle = \kappa,$$

The last identity follows from the definition of torsion. Differentiating $\langle \nu, \nu\rangle = 1$ in ❶, we get that

$$2\cdot\langle \nu', \nu\rangle = 0;$$

whence the second identity follows. Differentiating the identity $\langle \tau, \nu\rangle = = 0$ in ❶; we get that

$$\langle \tau', \nu\rangle + \langle \tau, \nu'\rangle = 0.$$

Applying ❷, we get that

$$\begin{aligned}\langle \nu', \tau\rangle &= -\langle \tau', \nu\rangle = \\ &= -k\cdot\langle \nu, \nu\rangle = \\ &= -k.\end{aligned}$$

It proves the first equality $\langle \nu', \tau\rangle = -k$; whence ❸ follows.

Taking derivatives of the third identity in ❶, we get that $\beta' \perp \beta$. Further taking derivatives of the other identities with β in ❶, we get that

$$\begin{aligned}\langle \beta', \tau\rangle &= -\langle \beta, \tau'\rangle = -\langle \beta, k\cdot\nu\rangle = 0\\ \langle \beta', \nu\rangle &= -\langle \beta, \nu'\rangle = \kappa\end{aligned}$$

That is,

❹ $$\beta'(s) = -\kappa(s)\cdot\nu(s).$$

The equations ❷, ❸ and ❹ are called Frenet formulas. All three can be written as one matrix identity:

$$\begin{pmatrix}\tau'\\ \nu'\\ \beta'\end{pmatrix} = \begin{pmatrix}0 & k & 0\\ -k & 0 & \kappa\\ 0 & -\kappa & 0\end{pmatrix}\cdot\begin{pmatrix}\tau\\ \nu\\ \beta\end{pmatrix}.$$

5.2. Exercise. *Deduce the formula ❹ from ❷ and ❸ by differentiating the identity $\beta = \tau \times \nu$.*

5.3. Exercise. *Let γ be a regular space curve with nonvanishing curvature. Show that γ lies in a plane if and only if its torsion vanishes.*

Hint: Show and use that the binormal vector is constant.

Curves of constant slope

We say that a smooth regular space curve γ has *constant slope* if its velocity vector makes a constant angle with a fixed direction. The following theorem was proved by Michel Ange Lancret [9] more than two century ago.

5.4. Theorem. *Let γ be a smooth regular curve with nonvanishing curvature. Then γ has constant slope if and only if the ratio $\frac{\kappa}{k}$ is constant; here k and κ denote the curvature and torsion of γ.*

The following exercise will guide you thru the proof of the theorem.

5.5. Exercise. *Suppose that γ is a smooth regular space curve with nonvanishing curvature, τ, ν, β is its Frenet frame and k, κ are its curvature and torsion.*

(a) *Assume that $\langle w, \tau\rangle$ is constant for a fixed nonzero vector w. Show that*
$$\langle w, \nu\rangle = 0.$$
Use it to show that
$$\langle w, -k{\cdot}\tau + \kappa{\cdot}\beta\rangle = 0.$$
Use these two identities to show that $\frac{\kappa}{k}$ is constant; it proves the "only if" part of the theorem.

(b) *Assume that $\frac{\kappa}{k}$ is constant, show that the vector $w = \frac{\kappa}{k}{\cdot}\tau + \beta$ is constant. Conclude that γ is has constant slope; it proves the "if" part of the theorem.*

Assume γ is a smooth unit speed curve and s_0 is a fixed real number. Then the curve
$$\alpha(s) = \gamma(s) + (s_0 - s){\cdot}\gamma'(s)$$
is called *evolvent* of γ. Note that if $\ell(s)$ denotes the tangent line of γ at s, then $\alpha(s) \in \ell(s)$ and $\alpha'(s) \perp \ell$ for any s.

5.6. Exercise. *Show that evolvent of a constant slope curve lies in a plane.*

Hint: Show that $\langle w, \alpha\rangle$ is constant if γ makes constant angle with a fixed vector w and α is the evolvent of γ.

Fundamental theorem of curves

5.7. Theorem. *Let $k(s)$ and $\kappa(s)$ be two smooth real valued functions defined on a real interval $\mathbb{I}$. Suppose $k(s) > 0$ for any s. Then there is a smooth unit-speed curve $\gamma\colon \mathbb{I} \to \mathbb{R}^3$ with curvature $k(s)$ and torsion $\kappa(s)$ for every s. Moreover γ is uniquely defined up to a rigid motion of the space.*

Proof. Fix a parameter value s_0, a point $\gamma(s_0)$ and an oriented orthonormal frame $\tau(s_0)$, $\nu(s_0)$, $\beta(s_0)$. Consider the following initial value problem:

$$\begin{cases} \gamma' = \tau, \\ \tau' = k\cdot\nu, \\ \nu' = -k\cdot\tau + \kappa\cdot\beta, \\ \beta' = -\kappa\cdot\nu. \end{cases}$$

It has four vector equations, so it can be rewritten as a system of 12 scalar equations. By A.3, it has a solution which is defined in a maximal subinterval $\mathbb{J}$ containing s_0.

Note that

$$\begin{aligned} \langle\tau,\tau\rangle' &= 2\cdot\langle\tau,\tau'\rangle = 2\cdot k\cdot\langle\tau,\nu\rangle = 0, \\ \langle\nu,\nu\rangle' &= 2\cdot\langle\nu,\nu'\rangle = -2\cdot k\cdot\langle\nu,\tau\rangle + 2\cdot\kappa\cdot\langle\nu,\beta\rangle = 0, \\ \langle\beta,\beta\rangle' &= 2\cdot\langle\beta,\beta'\rangle = -2\cdot\kappa\langle\beta,\nu\rangle = 0, \\ \langle\tau,\nu\rangle' &= \langle\tau',\nu\rangle + \langle\tau,\nu'\rangle = k\cdot\langle\nu,\nu\rangle - k\cdot\langle\tau,\tau\rangle + \kappa\cdot\langle\tau,\beta\rangle = 0, \\ \langle\nu,\beta\rangle' &= \langle\nu',\beta\rangle + \langle\nu,\beta'\rangle = 0, \\ \langle\beta,\tau\rangle' &= \langle\beta',\tau\rangle + \langle\beta,\tau'\rangle = -\kappa\cdot\langle\nu,\tau\rangle + k\cdot\langle\beta,\nu\rangle = 0. \end{aligned}$$

That is, the values $\langle\tau,\tau\rangle$, $\langle\nu,\nu\rangle$, $\langle\beta,\beta\rangle$, $\langle\tau,\nu\rangle$, $\langle\tau,\nu\rangle$, $\langle\beta,\tau\rangle$ are constant functions of s. Since we choose $\tau(s_0)$, $\nu(s_0)$, $\beta(s_0)$ to be an oriented orthonormal frame, we have that the $\tau(s)$, $\nu(s)$, $\beta(s)$ is oriented orthonormal for any s.

In particular $|\gamma'(s)| = 1$ for any s.

Assume $\mathbb{J} \ne \mathbb{I}$. Then an end of $\mathbb{J}$, say a, lies in the interior of $\mathbb{I}$. By Theorem A.3, at least one of the values $\gamma(s)$, $\tau(s)$, $\nu(s)$, $\beta(s)$ escapes to infinity as $s \to a$. But this is impossible since the vectors $\tau(s)$, $\nu(s)$, $\beta(s)$ remain unit and $|\gamma'(s)| = |\tau(s)| = 1$ — a contradiction. Whence $\mathbb{J} = \mathbb{I}$.

Assume there are two curves γ_1 and γ_2 with the given curvature and torsion functions. Applying a motion of the space we can assume that the $\gamma_1(s_0) = \gamma_2(s_0)$ and the Frenet frames of the curves coincide at s_0. Then $\gamma_1 = \gamma_2$ by uniqueness of solution of the system (A.3). That is, the curve is unique up to a rigid motion of the space. □

5.8. Exercise. *Assume a curve $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ has constant curvature and torsion. Show that γ is a helix, possibly degenerate to a circle; that is, in a suitable coordinate system we have*

$$\gamma(t) = (a\cdot \cos t, a\cdot \sin t, b\cdot t)$$

for some constants a and b.

Hint: Use the second statement in Exercise 5.1. Use

5.9. Advanced exercise. *Let γ be a smooth regular space curve such that the distance $|\gamma(t) - \gamma(t+\ell)|$ depends only on ℓ. Show that γ is a helix, possibly degenerate to a line or a circle.*

Hint: Note that

$$\rho(\ell) = |\gamma(t+\ell) - \gamma(t)|^2 = \langle \gamma(t+\ell) - \gamma(t), \gamma(t+\ell) - \gamma(t)\rangle$$

is a smooth function that does not depend on t. Express speed, curvature and torsion of γ in terms of derivatives $\rho^{(n)}(0)$.

Chapter 6

Plane curves

Signed curvature

Suppose γ is a smooth unit-speed plane curve, so $\tau(s) = \gamma'(s)$ is its unit tangent vector.

Let us rotate $\tau(s)$ by angle $\frac{\pi}{2}$ counterclockwise; denote the obtained vector by $\nu(s)$. The pair $\tau(s), \nu(s)$ is an oriented orthonormal frame in the plane which is analogous to the Frenet frame for space curves; we will keep the name *Frenet frame* for it.

Recall that $\gamma''(s) \perp \gamma'(s)$ (4.1). Therefore

❶ $$\tau'(s) = k(s){\cdot}\nu(s).$$

for some real number $k(s)$; the value $k(s)$ is called *signed curvature* of γ at s. Note that up to sign it equals to the curvature of γ at s as it defined o page 20; the sign tells which direction γ turns — if it turns left, then it is positive.

Note that if we reverse the parametrization of γ or change the orientation of the plane, then the signed curvature changes its sign.

Since $\tau(s), \nu(s)$ is an orthonormal frame, we get that

$$\langle \tau, \tau \rangle = 1, \qquad \langle \nu, \nu \rangle = 1, \qquad \langle \tau, \nu \rangle = 0,$$

Differentiating these idenitites we get that

$$\langle \tau', \tau \rangle = 0, \qquad \langle \nu', \nu \rangle = 0, \qquad \langle \tau', \nu \rangle + \langle \tau, \nu' \rangle = 0,$$

By ❶, $\langle \tau', \nu \rangle = k$ and therefore $\langle \tau, \nu' \rangle = -k$. Whence we get

❷ $$\nu'(s) = -k(s){\cdot}\tau(s).$$

The equations ❶ and ❷ are Frenet formulas for plane curves. They could be also written in a matrix form:

$$\begin{pmatrix} \tau' \\ \nu' \end{pmatrix} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \cdot \begin{pmatrix} \tau \\ \nu \end{pmatrix}.$$

The following theorem is the fundamental theorem of plane curves; it is direct analog of 5.7 and it can be proved along the same lines.

6.1. Theorem. *Let $k(s)$ be two smooth real valued function defined on a real interval $\mathbb{I}$. Then there is a smooth unit-speed curve $\gamma\colon \mathbb{I} \to \mathbb{R}^2$ with signed curvature $k(s)$ at every s. Moreover γ is uniquely defined up to a rigid motion of the plane.*

Total signed curvature

6.2. Lemma. *Suppose $\gamma\colon [a,b] \to \mathbb{R}^2$ is a smooth unit-speed curve. Then there is a smooth function $\theta\colon [a,b] \to \mathbb{R}$ such that*

$$\gamma'(s) = (\cos[\theta(s)], \sin[\theta(s)]) \quad \textit{and} \quad \theta'(s) = k(s)$$

for any s, where $k(s)$ denotes the signed curvature of γ.

Proof. Since γ is unit-speed, $\gamma'(a) = (\cos\theta_0, \sin\theta_0)$ for some θ_0. Set

$$\theta(s) = \theta_0 + \int\limits_a^s k(t)\cdot dt;$$

by the fundamental theorem of calculus, we have $\theta'(s) = k(s)$ for any s.

Set $\tau(s) = (\cos[\theta(s)], \sin[\theta(s)])$ and let $\nu(s)$ be its counterclockwise rotation by angle $\frac{\pi}{2}$; so $\tau(s) = (-\sin[\theta(s)], \cos[\theta(s)])$. Note that

$$\begin{aligned}
\tau'(s) &= (\cos[\theta(s)]', \sin[\theta(s)]') = \\
&= \theta'(s)\cdot(-\sin[\theta(s)], \cos[\theta(s)]) = \\
&= k(s)\cdot\nu(s) \\
\nu'(s) &= (-\sin[\theta(s)]', \cos[\theta(s)]') = \\
&= \theta'(s)\cdot(-\cos[\theta(s)], \sin[\theta(s)]) = \\
&= -k(s)\cdot\nu(s)
\end{aligned}$$

That is, τ and ν satisfy the Frenet formulas ❶ and ❷ for γ. By construction $\tau(a), \nu(a)$ is the Frenet frame at a; therefore $\tau(s), \nu(s)$ is the Frenet frame at any s. In particular,

$$\gamma'(s) = \tau(s) = (\cos[\theta(s)], \sin[\theta(s)])$$

for any s. □

Let γ be a smooth unit-speed plane curve. The integral of its signed curvature is called *total signed curvature* and it denoted by $\Psi(\gamma)$; so if θ and γ is as in 6.2, then

❸
$$\Psi(\gamma) = \int\limits_a^b k(s){\cdot}ds = \theta(b) - \theta(a).$$

Since $\left|\int k(s){\cdot}ds\right| \leqslant \int |k(s)|{\cdot}ds$, we have that

❹
$$|\Psi(\gamma)| \leqslant \Phi(\gamma)$$

for any smooth regular plane curve γ.

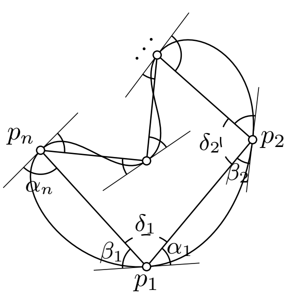
6.3. Proposition. *The total signed curvature of any closed simple smooth plane curve γ is $\pm 2{\cdot}\pi$; it is $+2{\cdot}\pi$ if the region bounded by γ lies on the left from it and $-2{\cdot}\pi$ otherwise.*

This proposition is a differential-geometric analog of the theorem about sum of the internal angles of a polygon (A.7) which we use in the proof. A more conceptual proof was given by Heinz Hopf [10, p. 42], [11, p. 42].

Proof. Without loss of generality we may assume that γ is oriented in such a way that the region bounded by γ lies on the left from it. We can also assume that it parametrized by arc length.

Consider a closed polygonal line $p_1 \dots p_n$ inscribed in γ. We can assume that the arcs between the vertexes are sufficiently small; in this case the polygonal line is simple and each arc γ_i from p_i to p_{i+1} have small total curvature, say $\Phi(\gamma_i) < \pi$ for each i. (As usual we use indexes modulo n, in particular $p_{n+1} = p_1$.)

Assume $p_i = \gamma(t_i)$. Set



$$\begin{aligned} w_i &= p_{i+1} - p_i, & v_i &= \gamma'(t_i), \\ \alpha_i &= \measuredangle(v_i, w_i), & \beta_i &= \measuredangle(w_{i-1}, v_i), \end{aligned}$$

where $\alpha_i, \beta_i \in (-\pi, \pi]$ are oriented angles.

By ❸, the value

$$\Psi(\gamma_i) - \alpha_i - \beta_{i+1}$$

is a multiple of $2{\cdot}\pi$. Since $\Phi(\gamma_i) < \pi$, by chord lemma (4.11), we have that $|\alpha_i| + |\beta_i| < \pi$. By ❹, we have that $|\Psi(\gamma_i)| \leqslant \Phi(\gamma_i)$; therefore

$$\Psi(\gamma_i) = \alpha_i + \beta_{i+1}.$$

for each i.

Note that $\delta_i = \pi - \alpha_i - \beta_i$ is the internal angle of $p_1 \dots p_n$ at p_i; $\delta_i \in (0, 2{\cdot}\pi$ for each i. Recall that the sum of internal angles of an n-gon is $(n-2){\cdot}\pi$ (see A.7); that is,

$$\delta_1 + \dots + \delta_n = (n-2){\cdot}\pi.$$

Therefore

$$\begin{aligned}\Psi(\gamma) &= \Psi(\gamma_1) + \dots + \Psi(\gamma_n) = \\ &= (\alpha_1 + \beta_2) + \dots + (\alpha_n + \beta_1) = \\ &= (\beta_1 + \alpha_1) + \dots + (\beta_n + \alpha_n) = \\ &= (\pi - \delta_1) + \dots + (\pi - \delta_n) = \\ &= n{\cdot}\pi - (n-2){\cdot}\pi = \\ &= 2{\cdot}\pi.\end{aligned}$$

□

6.4. Exercise. *Draw a smooth regular closed plane curve with vanishing total signed curvature.*

Osculating circline

As a direct corollary of Theorem 6.1, we get the following:

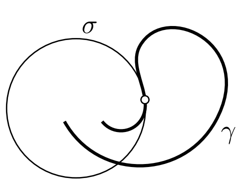
6.5. Proposition. *Given a point p, a unit vector τ and a real number k, there is a unique smooth unit-speed curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ that starts at p in the direction of τ and has constant signed curvature k.*

Moreover, if $k = 0$, then $\sigma = p + s{\cdot}\tau$ which runs along the line; if $k \ne 0$, then σ runs around the circle of radius $\frac{1}{|k|}$ and center $p + \frac{1}{k}{\cdot}\nu$, where τ, ν is an oriented orthonoral frame.

Further we will use the term *circline* for *a circle or a line.*

6.6. Definition. *Let γ be a smooth unit-speed plane curve; denote by $k(s)$ the signed curvature of γ at s.*

For $s_0 \in [a,b]$, the unit-speed curve σ of constant signed curvature $k(s_0)$ that starts at $\gamma(s_0)$ in the direction $\gamma'(s_0)$ is called the osculating circline *of γ at s_0.*



The center and radius of the osculating circle at a given point are called *center of curvature* and *radius of curvature* of the curve at that point.

This is a unique circline that has *second order of contact* with γ at s; that is, $\rho(\ell) = o(\ell^2)$, where $\rho(\ell)$ denotes the distance from $\gamma(s+\ell)$ to the osculating circline at s.

Spiral lemma

The following lemma was proved by Peter Tait [12] and later rediscovered by Adolf Kneser [13].

6.7. Lemma. *Assume that γ is a smooth regular plane curve with strictly decreasing positive signed curvature. Then the osculating circles of γ are nested; that is, if σ_s denoted the osculating circle of γ at s, then σ_{s_0} lies in the open disc bounded by σ_{s_1} for any $s_0 < s_1$.*

It turnes out that osculating circles of the curve γ give a peculiar foliation of an annulus by circles; it has the following property: if a smooth function is constant on each osculating circle it must be constant in the annulus [see 14, Lecture 10]. Also note that the curve γ is tangent to a circle of the foliation at each of its points. However, it does not run along a circle.

Proof. Let $\tau(s), \nu(s)$ be the Frenet frame, $z(s)$ the curvature center and $r(s)$ the radius of curvature of γ at s. Recall that $r(s){\cdot}k(s) = 1$. By 6.5,

$$z(s) = \gamma(s) + r(s){\cdot}\nu(s).$$

Applying Frenet formula ❷, we get that

$$\begin{aligned} z'(s) &= \gamma'(s) + r'(s){\cdot}\nu(s) + r(s){\cdot}\nu'(s) = \\ &= \tau(s) + r'(s){\cdot}\nu(s) - r(s){\cdot}k(s){\cdot}\tau(s) = \\ &= r'(s){\cdot}\nu(s). \end{aligned}$$

Since $k(s)$ is decreasing, $r(s)$ is increasing; therefore $r' \geqslant 0$. It follows that $|z'(s)| = r'(s)$ and $z'(s)$ points in the direction of $\nu(s)$.

Since $\nu'(s) = -k(s){\cdot}\tau(s)$, the directiion of $z'(s)$ always rotates; that is, the curve $s \mapsto z(s)$ contains no line segments. It follows that

$$\begin{aligned} |z(s_1) - z(s_0)| &< \operatorname{length}(z|_{[s_0,s_1]}) = \\ &= \int_{s_0}^{s_1} |z'(s)|{\cdot}ds = \\ &= \int_{s_0}^{s_1} r'(s){\cdot}ds = \\ &= r(s_1) - r(s_0). \end{aligned} \tag{$*$}$$

In other words, the distance between the centers of σ_{s_1} and σ_{s_0} is strictly less than the difference between their radiuses. Therefore the osculating circle at s_0 lies inside the osculating circle at s_1 without touching it. □

The following theorem has the following intuitive formulation: *if you drive on the plane and turn the steering wheel to the right all the time, then you will not be able to come back to the same place.*

6.8. Theorem. *Assume γ is a smooth regular plane curve with strictly monotonic curvature. Then γ is simple.*

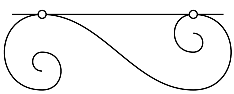
Proof of 6.8. Note that $\gamma(s) \in \sigma_s$ for any s. Applying the lemma we get $\gamma(s_1) \neq \gamma(s_0)$ if $s_1 \neq s_0$. Hence the result. □

The same statement also holds for signed curvature; the proof requires only minor modifications.

6.9. Exercise. *Show that a 3-dimensional analog of the theorem does not hold. That is, there are self-intersecting smooth regular space curves with strictly monotonic curvature.*

6.10. Exercise. *Assume that γ is a smooth regular plane curve with positive strictly monotonic signed curvature.*

(a) Show that no line can be tangent to γ at two distinct points.
(b) Show that no circle can be tangent to γ at three distinct points.



Note that part (a) does not hold if we alow the curvature to be negative; an example is shown on the diagram.

We say that a smooth regular plane curve γ has a *vertex* at s if the signed curvature function has extremal at s; that is, if $k'_\gamma(s) = 0$. If γ is simple we could say that the point $p = \gamma(s)$ is a vertex of γ without ambiguity.

6.11. Exercise. *Assume that a smooth regular plane curve γ runs on one side of its osculating circle at s. Show that γ has a vertex at s.*

Appendix A

Review

Here we state and discuss results from different branches of mathematics which were used further in the book. The reader is not expected to know proofs of these statements, but it is better to check that his intuition agrees with each.

Multivariable calculus

A map $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^k$ can be thought as array of functions

$$f_1, \dots, f_k\colon \mathbb{R}^n \to \mathbb{R}.$$

The map $\boldsymbol{f}$ is called *smooth* if each function f_i is smooth; that is, all partial derivatives of f_i are defined in the domain of definition of $\boldsymbol{f}$.

Inverse function theorem gives a sufficient condition for a smooth function to be invertible in a neighborhood of a given point p in its domain. The condition is formulated in terms of partial derivative of f_i at p.

Implicit function theorem is a close relative to inverse function theorem; in fact it can be obtained as its corollary. It is used for instance when we need to pass from parametric to implicit description of curves and surface.

Both theorems reduce the existence of a map satisfying certain equation to a question in linear algebra. We use these two theorems only for $n \leqslant 3$.

These two theorems are discussed in any course of multivariable calculus, the classical book of Walter Rudin [15] is one of my favorites.

A.1. Inverse function theorem. *Let* $\boldsymbol{f} = (f_1, \dots, f_n)\colon \mathbb{R}^n \to \mathbb{R}^n$

be a smooth map. Assume that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is invertible at some point p *in the domain of definition of* $\boldsymbol{f}$*. Then there is a smooth function* $\boldsymbol{h}\colon \mathbb{R}^m \to \mathbb{R}^n$ *defined is a neighborhood* Ω_q *of* $q = \boldsymbol{f}(p)$ *that is* local inverse of $\boldsymbol{f}$ at p*; that is, there are neighborhoods* $\Omega_p \ni p$ *such that* $\boldsymbol{f}$ *defines a bijection* $\Omega_p \to \Omega_q$ *and* $\boldsymbol{f}(x) = y$ *if and only if* $x = \boldsymbol{h}(y)$ *for any* $x \in \Omega_p$ *and any* $y \in \Omega_q$*.*

A.2. Implicit function theorem. *Let* $\boldsymbol{f} = (f_1, \dots, f_n)\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ *be a smooth map,* $m, n \geqslant 1$*. Let us consider* $\mathbb{R}^{n+m}$ *as a product space* $\mathbb{R}^n \times \mathbb{R}^m$ *with coodinates* $x_1, \dots, x_n, y_1, \dots, y_m$*. Consider the following matrix*

$$M = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

formed by first n *columns of the Jacobian matrix. Assume* M *is invertible at some point* p *in the domain of definition of* $\boldsymbol{f}$ *and* $\boldsymbol{f}(p) = 0$*. Then there is a neighborhood* $\Omega_p \ni p$ *and smooth function* $\boldsymbol{h}\colon \mathbb{R}^m \to \mathbb{R}^n$ *defined is a neighborhood* $\Omega_0 \ni 0$ *that for any* $(x_1, \dots, x_n, y_1, \dots y_m) \in \Omega_p$ *the equality*

$$\boldsymbol{f}(x_1, \dots, x_n, y_1, \dots y_m) = 0$$

holds if and only if

$$(x_1, \dots x_n) = \boldsymbol{h}(y_1, \dots y_m).$$

If the assumption in the theorem holds for any point p such that $\boldsymbol{f}(p) = 0$, then we say that 0 is a regular value of $\boldsymbol{f}$. *Sard's theorem* states that most of the values of smooth map are regular; in particular generic smooth function satisfies the assumption of the theorem.

Initial value problem

The following theorem guarantees existence and uniqueness of a solution of an initial value problem for a system of ordinary differential equations

$$\begin{cases} x_1'(t) &= f_1(x_1, \dots, x_n, t), \\ & \dots \\ x_n'(t) &= f_n(x_1, \dots, x_n, t), \end{cases}$$

where each $x_i = x_i(t)$ is a real valued function defined on a real interval $\mathbb{I}$ and each f_i is a smooth function defined on $\mathbb{R}^n \times \mathbb{I}$.

The array functions $(f_1, \dots, f_n)$ can be considered as one vector-valued function $\boldsymbol{f}\colon \mathbb{R}^n \times \mathbb{I} \to \mathbb{R}^n$ and the array $(x_1, \dots, x_n)$ can be considered as a vector $\boldsymbol{x} \in \mathbb{R}^n$. Therefore the system can be rewritten as one vector equation

$$\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}, t).$$

A.3. Theorem. *Suppose $\mathbb{I}$ is a real interval and $\boldsymbol{f}\colon \mathbb{R}^n \times \mathbb{I} \to \mathbb{R}^n$ is a smooth function. Then for any initial data $\boldsymbol{x}(t_0) = \boldsymbol{u}$ the differential equation*

$$\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}, t)$$

has a unique solution $\boldsymbol{x}(t)$ defined at a maximal subinterval $\mathbb{J}$ of $\mathbb{I}$ that contains t_0. Moreover

(a) if $\mathbb{J} \neq \mathbb{I}$, that is, if an end a of $\mathbb{J}$ lies in the interior of $\mathbb{I}$, then $\boldsymbol{x}(t)$ diverges for $t \to a$;

(b) the function $(\boldsymbol{u}, t_0, t) \mapsto \boldsymbol{x}(t)$ is smooth.

Real analysis

Recall that a function f is called Lipschitz if there is a constant L such that

$$|f(x) - f(y)| \leqslant L \cdot |x - y|$$

for values x and y in the domain of definition of f. This definition works for maps between metric spaces, but we will use it for real-to-real functions only.

A.4. Rademacher's theorem. *Let $f\colon [a, b] \to \mathbb{R}$ be a Lipschitz function. then derivative $f'(x)$ is defined for alomst all $x \in [a, b]$. Moreover the derivative f' is a bounded measurable function defined almost everywhere in $[a, b]$ and it satisfies the fundamental theorem of calculus; that is, the following identity*

$$f(b) - f(a) = \int\limits_a^b f'(x) \cdot dx,$$

holds if the integral understood in the sense of Lebesgue.

It is often helps to work with measurable functions; it makes possible to extend many statements about continuous function to measurable functions.

A.5. Lusin's theorem. *Let $\varphi\colon [a,b] \to \mathbb{R}$ be a measurable function. Then for any $\varepsilon > 0$, there is a continuous function $\psi_\varepsilon\colon [a,b] \to \mathbb{R}$ that coincides with φ outside of a set of measure at most ε. Moreover, φ is bounded above and/or below by some constants then we can assume that so is ψ_ε.*

Topology

The first part of the following theorem is proved by Camille Jordan, the second part is due to Arthur Schoenflies.

A.6. Theorem. *The complement of any closed simple γ plane curve has exactly two connected components.*

Moreover the there is a homeomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps the unit circle to γ. In particular one of the components is a topological disc.

This theorem is known for simple formulation and quite hard proof. By now many proofs of this theorem are known. For the first statement, a very short proof based on somewhat developed technique is given by Patrick Doyle [16], among elementary proofs, one of my favorites is the proof given by Aleksei Filippov [17].

An amusing proof of general statement for smooth regular curves can be found in [18]; we use this theorem mostly in this (much simpler) case.

Elementary geometry

A.7. Theorem. *The sum of sum of all the internal angles of a simple n-gon is $(n-2)\cdot\pi$.*

Proof. The proof is by induction on n. For $n = 3$ it says that sum of internal angles of a triangle is π, which is assumed to be known.

First let us show that for any $n \geqslant 4$, any n-gon has a diagonal that lies inside of it. Assume this is holds true for all polygons with at most $n-1$ vertex.

Fix an n-gon P, $n \geqslant 4$. Applying rotation if necessary, we can assume that all its vertexes have different x-coordinates. Let v be a vertex of P that minimize the x-coordinate; denote by u and w

its adjacent vertexes. Let us choose the diagonal uw if it lies in P. Otherwise the triangle $\triangle uvw$ contains another vertex of P. Choose a vertex s in the interior of $\triangle uvw$ that maximize the distance to line uw. Note that the diagonal vs lyes in P; if it is not the case then vs crosses another side pq of P, one of the vertexes p or q has larger distance to the line and it lies in the interior of $\triangle uvw$ — a contradiction.

Note that the diagonal divides P into two polygons, say Q and R, with smaller number of sides in each, say k and m correspondingly. Note that

$$k + m = n + 2; \tag{❶}$$

indeed each side of P appears once as a side of P or Q plus the diagonal appears twice — once as a side in Q and once as a side of R. Note that the sum of angles of P is the sum of angles of Q and R, which by the induction hypothesis are $(k-2)\cdot\pi$ and $(m-2)\cdot\pi$ correspondingly. It remains to note that ❶ implies

$$(k-2)\cdot\pi + (m-2)\cdot\pi = (n-2)\cdot\pi. \qquad \square$$

Appendix B

Homework assignments

HWA-01. Exercises: 1.2, 2.4, 2.5, 2.7, 3.14.

HWA-02. Exercises: 3.4(b), 3.5, 3.9, 3.15, 3.20.

HWA-03. Exercises: 3.16, 3.18, 3.19, 4.3a, 4.5.

HWA-04. Exercises: 4.2, 4.4, 4.15, 4.17 + 1.12.17 in the Toponogov's book.

HWA-05. Exercises: 4.8, 4.12, 4.13, 5.5, 6.4.

Bibliography

[1] E. Denne. *Alternating quadrisecants of knots*. Thesis (Ph.D.)–University of Illinois at Urbana-Champaign. ProQuest LLC, Ann Arbor, MI, 2004, p. 119.

[2] G. D. Chakerian. "An inequality for closed space curves". *Pacific J. Math.* 12 (1962), pp. 53–57.

[3] G. D. Chakerian. "An inequality for closed space curves". *Pacific J. Math.* 12 (1962), pp. 53–57.

[4] István Fáry. "Sur certaines inégalites géométriques". *Acta Sci. Math. Szeged* 12.Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A (1950), pp. 117–124.

[5] Serge Tabachnikov. "The tale of a geometric inequality". *MASS selecta*. Amer. Math. Soc., Providence, RI, 2003, pp. 257–262.

[6] J. Lagarias and T. Richardson. "Convexity and the average curvature of plane curves". *Geom. Dedicata* 67.1 (1997), pp. 1–30.

[7] A. I. Nazarov and F. V. Petrov. "On a conjecture of S. L. Tabachnikov". *Algebra i Analiz* 19.1 (2007), pp. 177–193.

[8] John M. Sullivan. "Curves of finite total curvature". *Discrete differential geometry*. Vol. 38. Oberwolfach Semin. 2008, pp. 137–161.

[9] Michel Ange Lancret. "Mémoire sur les courbes à double courbure". *Mémoires présentés à l'Institut des Sciences, Lettres et Arts, par divers savants, et lus dans ses assemblées. Sciences mathématiques et physiques.* 1 (1802), pp. 416–454.

[10] Heinz Hopf. "Über die Drehung der Tangenten und Sehnen ebener Kurven". *Compositio Math.* 2 (1935), pp. 50–62.

[11] Heinz Hopf. *Differential geometry in the large*. Second. Vol. 1000. Lecture Notes in Mathematics. Notes taken by Peter Lax and John W. Gray, With a preface by S. S. Chern, With a preface by K. Voss. Springer-Verlag, Berlin, 1989, pp. viii+184.

[12] P. G. Tait. "Note on the circles of curvature of a plane curve." *Proc. Edinb. Math. Soc.* 14 (1896), p. 26.

[13] A. Kneser. „Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie." *Heinrich Weber Festschrift*. 1912.

[14] D. Fuchs and S. Tabachnikov. *Mathematical omnibus*. Thirty lectures on classic mathematics. American Mathematical Society, Providence, RI, 2007.

[15] Walter Rudin. *Principles of mathematical analysis*. 1976.

[16] P. H. Doyle. "Plane separation". *Proc. Cambridge Philos. Soc.* 64 (1968), p. 291.

[17] A. F. Filippov. "An elementary proof of Jordan's theorem". *Uspehi Matem. Nauk (N.S.)* 5.5(39) (1950), pp. 173–176.

[18] Gregory R. Chambers and Yevgeny Liokumovich. "Converting homotopies to isotopies and dividing homotopies in half in an effective way". *Geom. Funct. Anal.* 24.4 (2014), pp. 1080–1100.