

Invitation to comparison geometry

Anton Petrunin and Sergio Zamora Barrera

Contents

1	Isoperimetric inequality	4
1.1	Lawlor's proof	5
2	Length	7
2.1	Length of curve	7
2.2	Semicontinuity of length	9
2.3	Axioms of length	10
2.4	Crofton formula	13
2.5	Applications	15
3	Total curvature	18
3.1	Smooth regular curves	18
3.2	General definition	19
3.3	Crofton again	22
3.4	DNA inequality	22
3.5	Curves of finite total curvature	24
3.6	Total signed curvature	25
4	Fáry–Milnor theorem	27
4.1	Tame knots	27
4.2	Fáry–Milnor theorem	28
4.3	Milnor's proof	29
4.4	Fáry's proof	30
4.5	Proof of Alexander and Bishop	32
5	Osculating circlines	35
5.1	Acceleration of unit-speed curve	35
5.2	Signed curvature	36
5.3	Osculating circline	36
5.4	Spiral theorem	37

6	Supporting circlines	39
6.1	Definitions	39
6.2	Supporting test	40
6.3	Lens lemma	41
6.4	Convexity and inflection points	42
6.5	Four-vertex theorem	44
6.6	Moon in a puddle	46
7	Surfaces	49
7.1	Embedded surfaces	49
7.2	Immersed surfaces	50
7.3	Tangent plane	51
7.4	Curvatures	51
7.5	Boundaries with bounded curvature	52
A	Semisolutions	55
	Bibliography	58

Chapter 1

Isoperimetric inequality

For any plane figure F with perimeter ℓ , its area a satisfies the following inequality:

$$\textcircled{1} \qquad 4\pi \cdot a \leq \ell^2.$$

Moreover the equality holds iff F is congruent to a round disk.

This is the so-called *isoperimetric inequality* on the plane. Let us restate it without formulas, using the comparison language.

1.1. Isoperimetric inequality. *The area of a plane figure bounded by a closed curve of length ℓ can not exceed the area of a round disk with the same circumference ℓ . Moreover the equality holds only if the figure is congruent to the disk.*

The comparison reformulation has some advantages — it is more intuitive and it is also easier to generalize.

1.2. Exercise. *Come up with a formulation of the isoperimetric inequality on the unit sphere. Try to reformulate it as an algebraic inequality similar to $\textcircled{1}$.*

Recall that a plane figure F is called *convex* if for every pair of points $x, y \in F$, the line segment $[x, y]$ that joins the pair of points also lies in F .



The following exercise reduces the isoperimetric inequality to the case of convex figures:

1.3. Exercise. Assume F is a plane figure bounded by a closed curve of length ℓ . Show that there is a convex figure $F' \supset F$ bounded by a closed curve of length at most ℓ .

The following problem is named after Dido, the legendary founder and first queen of Carthage.

1.4. Dido's problem. The figure of maximal area bounded by a straight line and a curve of given length with endpoints on that line is a half-disk.

1.5. Exercise. Show that Dido's problem follows from the isoperimetric inequality and the other way around.

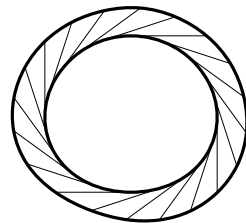
1.6. Exercise. Use the isoperimetric inequality in the plane to show that among all the polygons with given sides, the convex polygons inscribed in circles have maximal area.

1.7. Exercise. Find the minimal length of a curve that divides the unit square in a given ratio α .

1.1 Lawlor's proof

Here we present a sketch of the proof of Dido's problem based on the idea of Gary Lawlor in [1]. Before getting into the proof, try to solve the following exercise.

1.8. Exercise. An old man walks along a trail around a convex meadows and pulls a brick tied to a rope of unit length (the rope is always strained). After walking around he noticed that the brick is at the same position as at the beginning. Show that the area between the trail and the path of the brick equals the area of the unit disk.



Sketch of the proof. Let F be a convex figure bounded by a line and a curve $\gamma(t)$ of length ℓ ; we can assume that γ is a unit speed curve so the set of parameters is $[0, \ell]$.

Imagine that we are walking along the curve with a stick of length r so that the other end of the stick drags as we walk. Assume that initially at $t = 0$ the stick points in the direction of $\gamma(\ell)$ — the other end of γ .

Note that if r is small then most of the time we drag the stick behind. Therefore at the end of the walk the stick will have made more than half turn and will point to the same side of the figure.

Let R be the radius of the half-circle $\tilde{\gamma}(t)$ of length ℓ . Assume we walk along $\tilde{\gamma}$ with a stick of length R the same way as described above. Note that the other end does not move (it always lies in the center) and the direction of the stick changes with rate $\frac{1}{R}$. Note further that for γ this rate is at most $\frac{1}{R}$. Therefore after walking along γ , the stick of length R will rotate at most as much as if we walk along $\tilde{\gamma}$.

It follows that there is a positive value $r \leq R$ such that after walking along γ with a stick of length r , it will rotate exactly half turn, so at the end it will point towards $\gamma(0)$.

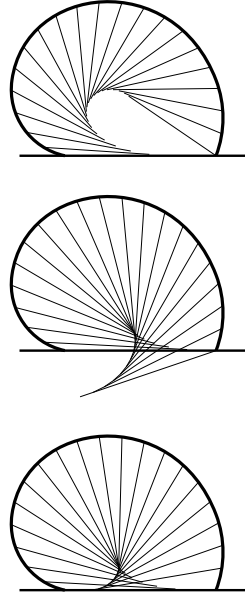
Imagine that the stick of length r is covered with red paint and it paints the area below it. If we move with velocity v then the angular velocity (radians per second) of the stick is at most $\frac{v}{r}$ and the equality holds if we move perpendicularly to the stick. Therefore we paint at a rate of at most $\frac{1}{2}v \cdot r$.

If we do the same for the half-disk of radius R and a stick of length R with blue paint, then we paint the area of the disk without overlap with the rate $\frac{1}{2}v \cdot R$. Since $r \leq R$, the total red-painted area can not exceed the blue-painted area, that is, D .

It remains to show that all F is red-painted. Fix a point $p \in F$. Notice that at the beginning the point p lies on the left from the stick and at the end it lies on the right from it. Therefore there will be a moment t_0 when the side changes from left to right. At this time the point must be on the line containing the stick. Moreover, if it lies on the extension then the side changes from right to left. Therefore p has to lie under the stick; that is, p is painted. □

1.9. Exercise. Find the steps with cheating in the above proof and try to fix them.

1.10. Exercise. Read and understand the original proof of Gary Lawlor in [1].



Chapter 2

Length

The material of this and the following chapters overlaps largely with [2, Chapter 5].

2.1 Length of curve

2.1. Definition. Consider a plane curve $\alpha: [a, b] \rightarrow \mathbb{R}^2$; a continuous mapping from the real interval $[a, b]$ to the Euclidean plane \mathbb{R}^2 .

If $\alpha(a) = p$ and $\alpha(b) = q$, we say that α is a curve from p to q .

A curve $\alpha: [a, b] \rightarrow \mathbb{R}^2$ is called closed if $\alpha(a) = \alpha(b)$.

A curve α is called simple if it is described by an injective map; that is $\alpha(t) = \alpha(t')$ if and only if $t = t'$. However, a closed curve $\alpha: [a, b] \rightarrow \mathbb{R}^2$ is called simple if it is injective everywhere except at the ends; that is, if $\alpha(t) = \alpha(t')$ for $t < t'$ then $t = a$ and $t' = b$.

A closed curve is called convex if it bounds a convex region.

2.2. Advanced exercise. Let $\alpha: [0, 1] \rightarrow \mathbb{R}^2$ from p to q . Assume $p \neq q$. Show that there is a simple curve $\beta: [0, 1] \rightarrow \mathbb{R}^2$ from p to q that runs in the image of α ; that is for any $t \in [0, 1]$ there is $t' \in [0, 1]$ such that $\beta(t) = \alpha(t')$.

Recall that a sequence

$$a = t_0 < t_1 < \cdots < t_k = b.$$

is called a *partition* of the interval $[a, b]$.

2.3. Definition. Let $\alpha: [a, b] \rightarrow \mathbb{R}^2$ be a curve. The length of α is defined as

$$\text{length } \alpha = \sup\{|\alpha(t_0) - \alpha(t_1)| + |\alpha(t_1) - \alpha(t_2)| + \dots \\ \dots + |\alpha(t_{k-1}) - \alpha(t_k)|\}.$$

where the exact upper bound is taken over all partitions

$$a = t_0 < t_1 < \dots < t_k = b.$$

Note that $\text{length } \alpha \in [0, \infty]$; the curve α is called *rectifiable* if its length is finite.

Informally, one could say that the length of a curve is the exact upper bound of the lengths of polygonal lines *inscribed* in the curve.

2.4. Exercise. Assume $\alpha: [a, b] \rightarrow \mathbb{R}^2$ is a smooth curve, in particular the velocity vector $\alpha'(t)$ is defined and depends continuously on t . Show that

$$\text{length } \alpha = \int_a^b |\alpha'(t)| \cdot dt.$$

2.5. Exercise. Construct a nonrectifiable curve $\alpha: [0, 1] \rightarrow \mathbb{R}^2$.

A closed simple plane curve is called *convex* if it bounds a convex figure.

2.6. Proposition. Assume a convex figure A bounded by a curve α lies inside a figure B bounded by a curve β . Then

$$\text{length } \alpha \leq \text{length } \beta.$$

Note that it is sufficient to show that for any polygon P inscribed in α there is a polygon Q inscribed in β with $\text{perim } P \leq \text{perim } Q$, where $\text{perim } P$ denotes the perimeter of P .

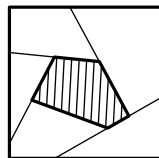
Therefore it is sufficient to prove the following lemma.

2.7. Lemma. Let P and Q be polygons. Assume P is convex and $Q \supset P$. Then $\text{perim } P \leq \text{perim } Q$.

Proof. Note that by the triangle inequality, the inequality

$$\text{perim } P \leq \text{perim } Q$$

holds if P can be obtained from Q by cutting it along a chord; that is, a line segment with ends on the boundary of Q that lies in Q .



Note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \cdots \subset P_n = Q$$

such that P_{i-1} obtained from P_i by cutting along a chord. Therefore

$$\begin{aligned} \text{perim } P &= \text{perim } P_0 \leq \text{perim } P_1 \leq \cdots \\ &\leq \text{perim } P_n = \text{perim } Q \end{aligned}$$

and the lemma follows. \square

2.8. Corollary. *Any convex closed curve is rectifiable.*

Proof. Any closed curve is bounded; that is, it lies in a sufficiently large square.

By Proposition 2.6, the length of the curve can not exceed the perimeter of the square, hence the result. \square

2.2 Semicontinuity of length

Recall that the lower limit of a sequence of real numbers (x_n) is denoted by

$$\varliminf_{n \rightarrow \infty} x_n.$$

It is defined as the lowest partial limit; that is, the lowest possible limit of a subsequence of (x_n) . The lower limit is defined for any sequence of real numbers and it lies in the extended real line $[-\infty, \infty]$

2.9. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\alpha_n: [a, b] \rightarrow \mathbb{R}^2$ converges pointwise to a curve $\alpha_\infty: [a, b] \rightarrow \mathbb{R}^2$; that is, $\alpha_n(t) \rightarrow \alpha_\infty(t)$ for any fixed $t \in [a, b]$ as $n \rightarrow \infty$. Then

$$\textcircled{1} \quad \varliminf_{n \rightarrow \infty} \text{length } \alpha_n \geq \text{length } \alpha_\infty.$$

Note that the inequality $\textcircled{1}$ might be strict. For example the diagonal α_∞ of the unit square

can be approximated by a sequence of stairs-like polygonal curves α_n with sides parallel to the sides of the square (α_6 is on the picture). In this case

$$\text{length } \alpha_\infty = \sqrt{2} \quad \text{and} \quad \text{length } \alpha_n = 2$$



for any n .

Proof. Fix $\varepsilon > 0$ and choose a partition $a = t_0 < t_1 < \cdots < t_k = b$ such that

$$\text{length } \alpha_\infty < |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + \cdots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)| + \varepsilon.$$

Set

$$\begin{aligned}\Sigma_n &:= |\alpha_n(t_0) - \alpha_n(t_1)| + \cdots + |\alpha_n(t_{k-1}) - \alpha_n(t_k)|. \\ \Sigma_\infty &:= |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + \cdots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|.\end{aligned}$$

Note that $\Sigma_n \rightarrow \Sigma_\infty$ as $n \rightarrow \infty$ and $\Sigma_n \leq \text{length } \alpha_n$ for each n . Hence

$$\lim_{n \rightarrow \infty} \text{length } \alpha_n \geq \text{length } \alpha_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get **1**. □

2.3 Axioms of length

Concatenation. Assume $\alpha: [a, b] \rightarrow \mathbb{R}^2$ and $\beta: [b, c] \rightarrow \mathbb{R}^2$ are two curves such that $\alpha(b) = \beta(b)$. Then one can combine these two curves into one $\gamma: [a, c] \rightarrow \mathbb{R}^2$ by the rule $\gamma(t) = \alpha(t)$ for $t \leq b$ and $\gamma(t) = \beta(t)$ for $t \geq b$. The obtained curve γ is called the *concatenation* of α and β and is denoted as $\gamma = \alpha * \beta$.

Note that

$$\text{length}(\alpha * \beta) = \text{length } \alpha + \text{length } \beta$$

for any two curves α and β such that the concatenation $\alpha * \beta$ is defined.

Reparametrization. Assume $\alpha: [a, b] \rightarrow \mathbb{R}^2$ is a curve and $\tau: [c, d] \rightarrow [a, b]$ is a continuous strictly monotonic onto map. Consider the curve $\alpha': [c, d] \rightarrow \mathbb{R}^2$ defined by $\alpha' = \alpha \circ \tau$. The curve α' is called a *reparametrization* of α .

Note that

$$\text{length } \alpha' = \text{length } \alpha$$

whenever α' is a reparametrization of α .

2.10. Proposition. Let ℓ be a functional that returns a value in $[0, \infty]$ for any curve $\alpha: [a, b] \rightarrow \mathbb{R}$.

Assume it satisfies the following properties:



(i) (Normalization) If $\alpha: [a, b] \rightarrow \mathbb{R}^2$ is a linear curve,¹ then

$$\ell(\alpha) = |\alpha(a) - \alpha(b)|.$$

(ii) (Additivity) If the concatenation $\alpha * \beta$ is defined, then

$$\ell(\alpha * \beta) = \ell(\alpha) + \ell(\beta).$$

(iii) (Motion invariance) The functional ℓ is invariant with respect to the motions of the plane; that is, if m is an isometry of the plane, then

$$\ell(m \circ \alpha) = \ell(\alpha)$$

for any curve α .

(iv) (Reparametrization invariance) If α' is a reparametrization of a curve α then

$$\ell(\alpha') = \ell(\alpha).$$

(In fact linear reparametrizations will be sufficient.)

(v) (Semi-continuity) If a sequence of curves $\alpha_n: [a, b] \rightarrow \mathbb{R}^2$ converges pointwise to a curve $\alpha_\infty: [a, b] \rightarrow \mathbb{R}^2$, then

$$\liminf_{n \rightarrow \infty} \ell(\alpha_n) \geq \ell(\alpha_\infty).$$

Then

$$\textcircled{2} \quad \ell(\alpha) = \text{length } \alpha$$

for any plane curve α .

Proof. Note that from normalization and additivity, the identity

$$\textcircled{3} \quad \ell(\beta) = \text{length } \beta$$

holds for any polygonal line β that is linear on each edge.

Note that the following two inequalities

$$\textcircled{4} \quad \ell(\alpha) \leq \text{length } \alpha$$

$$\textcircled{5} \quad \ell(\alpha) \geq \text{length } \alpha$$

imply $\textcircled{2}$; we will prove them separately.

Fix a curve $\alpha: [a, b] \rightarrow \mathbb{R}^2$ and a partition $a = t_0 < t_1 < \dots < t_k = b$. Consider the curve $\beta: [a, b] \rightarrow \mathbb{R}^2$ defined as the linear

¹That is $\alpha = w + v \cdot t$ for some vectors w and v .

segment from $\alpha(t_i)$ to $\alpha(t_{i+1})$ on each interval $t \in [t_i, t_j]$. By the definition of length,

$$\text{length } \beta \leq \text{length } \alpha.$$

Since the map $\alpha: [a, b] \rightarrow \mathbb{R}^2$ is continuous, one can find a sequence of partitions of $[a, b]$ such that the corresponding curves β_n converge to α pointwise. Applying the semi-continuity of ℓ , ❸ and the definition of length, we get that

$$\begin{aligned} \ell(\alpha) &\leq \varliminf_{n \rightarrow \infty} \ell(\beta_n) = \\ &= \varliminf_{n \rightarrow \infty} \text{length } \beta_n \leq \\ &\leq \text{length } \alpha. \end{aligned}$$

Hence ❹ follows.

Note that a curve $\alpha: [a, b] \rightarrow \mathbb{R}^2$ with a partition $a = t_0 < t_1 < \dots < t_k = b$ can be considered as a concatenation

$$\alpha = \alpha_1 * \alpha_2 * \dots * \alpha_k$$



where α_i is the restriction of α to $[t_{i-1}, t_i]$.

Observe that there is a sequence of motions m_i of the plane so that

$$m_i \circ \alpha(t_i) = m_{i+1} \circ \alpha(t_i)$$

for any i and the points

$$m_1 \circ \alpha(t_0), m_1 \circ \alpha(t_1), \dots, m_k \circ \alpha(t_k)$$

lie in that order on a single line. For the concatenation

$$\gamma = (m_1 \circ \alpha_1) * (m_2 \circ \alpha_2) * \dots * (m_k \circ \alpha_k)$$

we have

$$\ell(\gamma) = \ell(\alpha).$$

Assume α is rectifiable. In this case we can find a sequence of partitions of $[a, b]$ such that reparametrizations of γ_n converge to a linear segment γ'_∞ ; denote these reparametrizations by γ'_n . Also, $\text{length } \gamma'_\infty = \text{length } \alpha$; indeed, since γ'_∞ is linear,

$$\begin{aligned} \text{length } \gamma'_\infty &= |\gamma'_\infty(a) - \gamma'_\infty(b)| = \\ &= \lim_{n \rightarrow \infty} \Sigma_n = \\ &= \text{length } \alpha. \end{aligned}$$

where Σ_n is the sum in the definition of length for the n -th partition. Hence it is sufficient to choose a sequence of partitions such that $\Sigma_n \rightarrow \text{length } \alpha$.

Applying additivity, invariance of ℓ with respect to motions and reparametrizations, we get that

$$\begin{aligned}\ell(\alpha) &= \lim_{n \rightarrow \infty} \ell(\gamma_n) = \\ &= \lim_{n \rightarrow \infty} \ell(\gamma'_n) \geq \\ &\geq \ell(\gamma'_\infty) = \\ &= \text{length } \alpha.\end{aligned}$$

Hence ⑤ follows.

If α is not rectifiable, a similar construction produces an approximation of an arbitrary long line segment. (We need to run zig-zag to reduce the distance $|\gamma'_\infty(a) - \gamma'_\infty(b)|$.) It follows that

$$\ell(\alpha) \geq |\gamma'_\infty(a) - \gamma'_\infty(b)|.$$

Since $|\gamma'_\infty(a) - \gamma'_\infty(b)|$ can take arbitrary large values, we get $\ell(\alpha) = \infty$. \square

2.11. Exercise. *Construct a functional ℓ that satisfies all the conditions in Proposition 2.10 except the semi-continuity.*

2.4 Crofton formula

Let α be a plane curve and u a unit vector. Denote by α_u the orthogonal projection of α to a line ℓ in the direction of u ; that is, $\alpha_u(t) \in \ell$ and $\alpha(t) - \alpha_u(t) \perp \ell$ for any t .

2.12. Crofton formula. *The length of any plane curve α is proportional to the average of the lengths of its projections α_u for all unit vectors u . Moreover for any plane curve α we have*

$$\text{length } \alpha = \frac{\pi}{2} \cdot \overline{\text{length } \alpha_u},$$

where $\overline{\text{length } \alpha_u}$ denotes the average value of $\text{length } \alpha_u$.

Proof. First let us show that the formula

$$\text{⑥} \quad \text{length } \alpha = k \cdot \overline{\text{length } \alpha_u},$$

holds for some fixed coefficient k . It will follow once we show that both sides of the formula satisfy the length axioms in 2.10.

The normalization can be achieved by adjusting k .

The semi-continuity of the right hand side follows since length α_u is semi-continuous and therefore the average has to be semi-continuous.

It is straightforward to check the remaining properties.

It remains to find k . Let us apply the formula 6 to the unit circle. The circle has length $2\cdot\pi$ and its projection to any line has length 4 — it is a segment of length 2 traveled back and forth. Evidently the average value is also 4, so

$$2\cdot\pi = k\cdot 4,$$

hence $k = \frac{\pi}{2}$. □

Reformulation via number of intersections. Given a unit vector u and a real number ρ , consider the line of vectors w on the plane satisfying the equation

$$\langle u, w \rangle = \rho,$$

where $\langle u, w \rangle$ denotes the scalar product. Any line on the plane admits exactly two such presentations with pairs (u, ρ) and $(-u, -\rho)$. A pair (u, ρ) describes uniquely an *oriented* line — that is a line with a chosen unit normal vector.

Fix a unit vector u_0 and denote by $u(\varphi)$ the result of rotating u_0 counterclockwise by the angle φ . Denote by $\ell(\varphi, \rho)$ the oriented line associated to the pair $(u(\varphi), \rho)$. To describe any line, we need a pair $(\varphi, \rho) \in (-\pi, \pi] \times \mathbb{R}$.

For a curve α , set $n_\alpha(\varphi, \rho)$ to be the number of parameter values t such that $\alpha(t)$ lies on the line $\ell(\varphi, \rho)$. The value $n_\alpha(\varphi, \rho)$ is a non-negative integer or ∞ . Note that if α is a simple curve, then $n_\alpha(\ell)$ is the number of intersections of α with ℓ .

2.13. Another Crofton formula. *For any curve α ,*

$$\text{length } \alpha = \frac{1}{4} \cdot \iint_{(-\pi, \pi] \times \mathbb{R}} n_\alpha(\rho, \varphi) \cdot d\rho \cdot d\varphi.$$

the integral is to be understood in the sense of Lebesgue.

By definition of average value,

$$\overline{\text{length } \alpha_u} = \frac{1}{2\cdot\pi} \cdot \int_{-\pi}^{\pi} \text{length } \alpha_{u(\varphi)} \cdot d\varphi.$$

Therefore the proof of this reformulation of the Crofton follows from the following observation.

2.14. Observation. If $u = u(\varphi)$, then

$$\text{length } \alpha_u = \int_{\mathbb{R}} n_\alpha(\rho, \varphi) \cdot d\rho;$$

The proof is straightforward for those who understand Lebesgue integral.

Variations. The same argument can be used to derive other formulas of the same type. For example.

Recall that a big circle in a sphere is the intersection of the sphere with a plane passing thru its center. For example, the equator as well as the meridians are big circles.

2.15. Spherical Crofton formula. *The length of any curve α in the unit sphere is π times the average number of its crossings with big circles.*

More presciently, given a unit vector u , denote by $n_\alpha(u)$ the number of crossings of α and the equator with pole at u . Then

$$\text{length } \alpha = \pi \cdot \overline{n_\alpha(u)}.$$

Equivalently,

$$\text{length } \alpha = \overline{\text{length } \alpha_u},$$

where α_u denotes the curve obtained by closest point projection of α to the equator with pole at u .

2.16. Exercise. *Come up with Crofton formulas for curves in the Euclidean space via projections to lines and to planes. Find the coefficients in those formulas.*

2.5 Applications

Alternative proof of Proposition 2.6. Note that

$$\text{length } \beta_u \geq \text{length } \alpha_u$$

for any unit vector u . Indeed α_u runs back and forth along a line segment and β_u has to run at least that much.

It follows that

$$\overline{\text{length } \beta_u} \geq \overline{\text{length } \alpha_u}.$$

It remains to apply the Crofton formula. □

Recall that the diameter of a plane figure F is defined as the least upper bound on the distances between pairs of its points; that is,

$$\text{diam } F = \sup \{ |x - y| : x, y \in F \}.$$

The equilateral triangle with side 1 gives an example of a convex figure of diameter 1 that cannot be covered by a round disc of diameter 1.

2.17. Exercise. Assume F is a convex figure of diameter 1 and D is the round disc of diameter 1. Show that

$$\text{perim } F \leq \text{perim } D.$$

A convex figure F has constant width a if the orthogonal projection of F to any line has length a . There are many non-circular shapes of constant width. A nontrivial example is the Reuleaux triangle shown on the picture; it is the intersection of three round disks of the same radius, each having its center on the boundary of the other two. The following exercise is the so called Barbier's theorem.



2.18. Exercise. Show that figures with constant width a have the same perimeter (which equals $\pi \cdot a$ — the perimeter of the round disc of diameter a).

2.19. Exercise. Let γ be a closed curve in the unit sphere of length shorter than $2 \cdot \pi$. Show that γ lies in a hemisphere.

2.20. Exercise. Let α be a closed curve of length π . Show that it lies between a pair of parallel lines at distance 1 from each other.

2.21. Exercise. A spaceship flies around a nonrotating planet of unit radius and comes back to the original position; it was able to take a picture of every point on the surface of the planet.

Try to use the Crofton formulas to get a lower bound on the length of its trajectory (does not need to be exact, but should be larger than $2 \cdot \pi$).

What do you think could be the shortest trajectory?

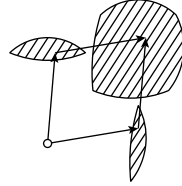
The Hausdorff distance $d_H(F, G)$ between two closed bounded sets F and G in the plane is defined as the exact lower bound on $\varepsilon > 0$ such that the ε -neighborhood of F contains G and the ε -neighborhood of G contains F .

2.22. Exercise. Assume F and G are two closed convex figures on the plane such that $d_H(F, G) < \varepsilon$. Show that

$$|\text{perim } F - \text{perim } G| < 2 \cdot \pi \cdot \varepsilon.$$

Given two sets A and B on the plane, the set C is called their *Minkowski sum* (briefly $C = A + B$) if C is formed by adding each vector in A to each vector in B ; that is,

$$C = \{a + b : a \in A, b \in B\}.$$



Note that if A and B are convex then so is $C = A + B$.

Indeed, A is convex if and only if for any pair of points $a_0, a_1 \in A$ and any $t \in [0, 1]$, the point $a_t = (1 - t) \cdot a_0 + t \cdot a_1$ belongs to A . Similarly, B is convex if and only if for any pair of points $b_0, b_1 \in B$ and any $t \in [0, 1]$, the point $b_t = (1 - t) \cdot b_0 + t \cdot b_1$ belongs to B .

Fix a pair of points $c_0, c_1 \in C$; by the definition of Minkowski sum, there are two pairs of points $a_0, a_1 \in A$ and $b_0, b_1 \in B$ such that $c_0 = a_0 + b_0$ and $c_1 = a_1 + b_1$. Then

$$\begin{aligned} c_t &= (1 - t) \cdot c_0 + t \cdot c_1 = \\ &= (1 - t) \cdot (a_0 + b_0) + t \cdot (a_1 + b_1) = \\ &= [(1 - t) \cdot a_0 + t \cdot a_1] + [(1 - t) \cdot b_0 + t \cdot b_1] = \\ &= a_t + b_t. \end{aligned}$$

That is, $c_t \in C$ for any $t \in [0, 1]$, hence the result.

2.23. Exercise. Show that

$$\text{perim}(A + B) = \text{perim } A + \text{perim } B$$

for any pair of convex figures in the plane.

2.24. Exercise. Use Exercise 2.23 and Lemma 2.7 to give another solution of Exercise 2.22.

2.25. Exercise. Let γ be a curve that lies in a convex figure F in the plane.

Let γ be a curve that lies inside a convex figure F on the plane. Assume that

$$2 \cdot \text{length } \gamma \geq n \cdot \text{perim } F$$

for some integer n . Show that there is a line ℓ that intersects γ in at least n distinct points.

Chapter 3

Total curvature

3.1 Smooth regular curves

Here we introduce the so called *total curvature of a curve*. In general the term *curvature* is used for something that measures how much a geometric object deviates from being straight; total curvature is not an exception — as you will see, if the total curvature of a curve is zero, then the curve runs along a straight line.

Let $\alpha: [a, b] \rightarrow \mathbb{R}^3$ be a *smooth regular curve* — smooth means that the velocity vector $\alpha'(t)$ is defined and is continuous with respect to t , and regular means that $\alpha'(t) \neq 0$ for all t . If the curve α is closed then we assume in addition that $\alpha'(a) = \alpha'(b)$.

Denote by $\tau(t)$ the unit vector in the direction of $\alpha'(t)$; that is, $\tau(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$. Then $\tau: [a, b] \rightarrow \mathbb{S}^2$ is another curve which is called *tangent indicatrix* of α . The length of τ is called the *total curvature of α* ; that is,

$$\text{TotCurv } \alpha := \text{length } \tau.$$

3.1. Exercise. *Show that*

$$\text{TotCurv } \alpha \geq 2 \cdot \pi$$

for any smooth closed regular curve α .

Moreover, the equality holds if and only if α is a closed convex curve that lies in a plane.

The above exercise is the so called Fenchel's theorem.

3.2 General definition

The total curvature of a polygonal line is defined as the sum of its external angles.

More precisely, for a polygonal line $\beta = p_0 \dots p_n$, the external angle at the vertex p_i is defined as $\alpha_i = \pi - \angle p_{i-1}p_i p_{i+1}$. The total curvature of the polygonal line $\beta = p_0 \dots p_n$ is defined as the sum

$$\text{TotCurv } \beta = \alpha_1 + \dots + \alpha_{n-1};$$

it is defined if the polygonal line is *nondegenerate*; that is, $p_{i-1} \neq p_i$ for any i .

If the polygonal line $p_0 \dots p_n$ is closed; that is $p_0 = p_{n+1}$ you add one more angle

$$\alpha_0 + \alpha_1 + \dots + \alpha_{n-1},$$

where $\alpha_0 = \pi - \angle p_n p_0 p_1$.

One can define the tangent indicatrix of a polygonal line β as a spherical polygonal line (each edge is an arc of a big circle in the sphere) whose vertexes are the unit vectors ξ_1, \dots, ξ_n in the directions of $p_1 - p_0, p_2 - p_1, \dots, p_n - p_{n-1}$ correspondingly; if the polygonal line is closed then we add one more vertex ξ_0 in the direction of $p_0 - p_n$ and two more edges $\xi_0 \xi_1$ and $\xi_n \xi_0$ so the indicatrix of a closed polygonal line is a closed spherical polygonal line.

Note that the total curvature of a polygonal line is the length of its tangent indicatrix.

3.2. Exercise. Let a, b, c, d and x be distinct points in \mathbb{R}^3 . Show that

$$\text{TotCurv } abcd \geq \text{TotCurv } abxcd.$$

3.3. Exercise. Use Exercise 3.2 to prove an analog of Fenchel's theorem (Exercise 3.1) for closed polygonal lines.

We gave two definitions of total curvature: the first one is given in Section 3.1 via the tangent indicatrix — it works for smooth regular curves; the second, via external angles — it works for polygonal lines. The latter can be used to define total curvature of arbitrary curves.

Let $\alpha: [a, b] \rightarrow \mathbb{R}^3$ be a curve and $a = t_0 < \dots < t_n = b$ a partition. Set $p_i = \alpha(t_i)$. Then the polygonal line $p_0 \dots p_n$ is said to be inscribed in α .

3.4. Definition. The total curvature of a nonconstant curve α is the exact upper bound on the total curvatures of inscribed nondegenerate polygonal lines; if α is closed then we assume that the inscribed polygonal lines are closed as well.

We need to assume that the curve is nonconstant, otherwise it does not admit inscribed polygonal lines that are not trivial.

3.5. Exercise. Show that the total curvature is lower semi-continuous with respect to pointwise convergence of curves. That is, if a sequence of curves $\alpha_n: [a, b] \rightarrow \mathbb{R}^3$ converges pointwise to a curve $\alpha_\infty: [a, b] \rightarrow \mathbb{R}^3$, then

$$\liminf_{n \rightarrow \infty} \text{TotCurv } \alpha_n \geq \text{TotCurv } \alpha_\infty.$$

Hint: Modify the proof of semi-continuity of length (Theorem 2.9).

The following definition tells that the two definitions agree.

3.6. Theorem. For smooth regular curves the two definitions of total curvature agree; that is, for any regular curve, the length of its tangent indicatrix is equal to the exact upper bound on the total curvatures of inscribed nondegenerate polygonal lines.

Note that from the theorem and Exercise 3.3, we get a generalization of Fenchel's theorem (Exercise 3.1) — it works for arbitrary closed curves, not necessary smooth and regular.

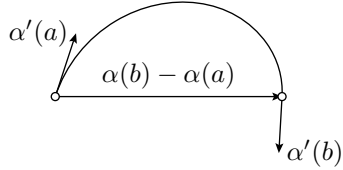
3.7. Lemma. Let $\alpha: [a, b] \rightarrow \mathbb{R}^3$ be a smooth regular curve. Consider three unit vectors λ , μ and ν in the directions of $\alpha'(a)$, $\alpha(b) - \alpha(a)$ and $\alpha'(b)$ correspondingly. Then

$$\text{TotCurv } \alpha \geq \angle(\lambda, \mu) + \angle(\mu, \nu).$$

Proof. The tangent indicatrix τ runs from λ to ν in the unit sphere \mathbb{S}^2 .

Note that τ can not be separated from μ by an equator. Indeed the vector

$$\alpha(b) - \alpha(a) = \int_a^b \alpha'(t) \cdot dt$$



points in the same direction as μ . Therefore if the indicatrix $\tau = \frac{\alpha'}{|\alpha'|}$ lies in a hemisphere then μ lies in the same hemisphere.

Fix an equator ℓ in general position. If ℓ intersects the spherical polygonal line $\lambda\mu\nu$ at one point, then ℓ separates λ from ν and therefore it must intersect τ . If ℓ intersects the spherical polygonal line $\lambda\mu\nu$ at two points, then ℓ separates μ from λ and ν and therefore it must intersect τ at least twice — τ must cross ℓ and then come back. It follows that for almost all equators the number of intersections with

the spherical polygonal line $\lambda\mu\nu$ can not exceed the number of intersections with τ . By the spherical Crofton formula (2.15), τ is longer than the spherical polygonal line $\lambda\mu\nu$. But the polygonal line $\lambda\mu\nu$ has length $\angle(\lambda, \mu) + \angle(\mu, \nu)$, hence the result. \square

Let us sketch an alternative proof of the lemma which is built on Fenchel's theorem.

Alternative proof of the lemma. Note that the curve α can be extended to a smooth regular closed curve $\hat{\alpha}$ by an arc β that starts from $\alpha(b)$ in the same direction as α . Then turns and joins the segment $[\alpha(b), \alpha(a)]$, runs along the segment until it is close to $\alpha(a)$ turns and smoothly joins α at $\alpha(a)$.

Note that the total curvature of β can be made arbitrarily close to $2\pi - \angle(\lambda, \mu) - \angle(\mu, \nu)$. Indeed, β needs a bit more than $\pi - \angle(\mu, \nu)$ to turn and join the segment $[\alpha(b), \alpha(a)]$ and bit more than $\pi - \angle(\lambda, \mu)$ to turn and join the segment α .

By Fenchel's theorem,

$$\text{TotCurv } \hat{\alpha} \geq 2\pi.$$

Evidently

$$\text{TotCurv } \hat{\alpha} = \text{TotCurv } \alpha + \text{TotCurv } \beta,$$

hence the lemma follows. \square

Proof of 3.6. Let $\alpha: [a, b] \rightarrow \mathbb{R}^3$ be a smooth curve. Fix a partition $a = t_0 < \dots < t_n = b$ and consider the corresponding inscribed polygonal line $\beta = w_0 \dots w_n$. Let $\chi = \xi_1 \dots \xi_n$ be its tangent indicatrix — this is a spherical polygonal line; we assume that $\chi(t_i) = \xi_i$ and it has constant speed on each arc.

Consider a sequence of finer and finer partitions, denote by β_n and χ_n the corresponding inscribed polygonal line and its tangent indicatrix; since α is smooth, the χ_n converge pointwise to τ — the tangent indicatrix of α . By semi-continuity of the length functional, we get

$$\begin{aligned} \text{TotCurv } \alpha &= \text{length } \tau \leq \\ &\leq \varliminf_{n \rightarrow \infty} \text{length } \chi_n = \\ &= \varliminf_{n \rightarrow \infty} \text{TotCurv } \beta_n \leq \\ &\leq \sup\{\text{TotCurv } \beta\}, \end{aligned}$$

where the last supremum is taken over all partitions and their corresponding inscribed polygonal lines β .

It remains to prove that

$$\textcircled{1} \quad \text{TotCurv } \alpha \geq \text{TotCurv } \beta,$$

for any polygonal line β inscribed in α . Let ζ_i be the unit vector in the direction of $\alpha'(t_i)$. Consider the spherical polygonal line $\gamma = \zeta_0 \xi_1 \zeta_1 \xi_2 \dots \xi_n \zeta_n$; recall that $\chi = \xi_0 \dots \xi_n$. By the triangle inequality,

$$\text{length } \gamma \geq \text{length } \chi = \text{TotCurv } \beta.$$

By Lemma 3.7,

$$\text{TotCurv } \alpha \geq \text{length } \gamma,$$

hence $\textcircled{1}$ follows. \square

3.3 Crofton again

Given a curve α in \mathbb{R}^3 and a unit vector u , denote by α_{u^\perp} and α_u the projection of α to the plane perpendicular to u and the line parallel to u correspondingly.

To prove the following proposition, apply the spherical Crofton formula to the tangent indicatrix of α .

3.8. Proposition. *Let α be a polygonal line in \mathbb{R}^3 . Show that*

$$\begin{aligned} \text{TotCurv } \alpha &= \overline{\text{TotCurv } \alpha_{u^\perp}} = \\ &= \overline{\text{TotCurv } \alpha_u}. \end{aligned}$$

Note that since the curve α_u runs back and forth along one line, every time it changes direction contributes π to the total curvature of α_u . Therefore the total curvature of α_u is $n \cdot \pi$, where n is the number of changes of direction. Since n has to be even, $\text{TotCurv } \alpha_u$ may take values $2 \cdot \pi$, $4 \cdot \pi$, $6 \cdot \pi$ and so on.

3.9. Exercise. *Use the proposition and the observation above to give yet another proof of Fenchel's theorem (Exercise 3.1).*

3.4 DNA inequality

3.10. Theorem. *Let α be a closed curve that lies in a unit disc. Then*

$$\text{TotCurv } \alpha \geq \text{length } \alpha.$$

Note that if $\text{length } \alpha \leq 2 \cdot \pi$, then Fenchel's theorem gives a better estimate, for longer curves this gives something new.

Proof. Assume α is a polygonal line.

Fix a unit vector u . Note that the curve α_u can run at most length 2 in one direction; therefore the number of turns has to be at least $\frac{1}{2} \cdot \text{length } \alpha$. Since each turn of α_u contributes π to its total curvature, we get

$$\text{TotCurv } \alpha_u \geq \frac{\pi}{2} \cdot \text{length } \alpha_u.$$

The same inequality holds for the average values of left and right hand sides; that is,

$$\overline{\text{TotCurv } \alpha_u} \geq \frac{\pi}{2} \cdot \overline{\text{length } \alpha_u}.$$

Applying the Crofton's formula and Proposition 3.8 we get the result.

It remains to reduce the general case to polygonal lines. Given $\varepsilon > 0$, we choose an inscribed polygonal line β such that

$$\text{length } \alpha < \text{length } \beta + \varepsilon.$$

By the definition of total curvature (3.4) and from the first part of the proof

$$\begin{aligned} \text{TotCurv } \alpha &\geq \text{TotCurv } \beta \geq \\ &\geq \text{length } \beta > \\ &> \text{length } \alpha - \varepsilon. \end{aligned}$$

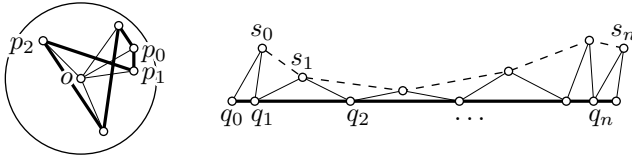
The statement follows since ε was arbitrary. \square

Alternative proof. The same argument as above shows that it is sufficient to consider a closed polygonal line $\beta = p_0 p_1 \dots p_{n-1}$ in the unit disc. We assume that $p_n = p_0$, $p_{n+1} = p_1$ and so on. Denote by α_i the external angle at p_i .

Denote by o the center of the disc. Consider a sequence of triangles

$$\triangle q_0 q_1 s_0 \cong \triangle p_0 p_1 o, \triangle q_1 q_2 s_1 \cong \triangle p_1 p_2 o, \dots$$

such that the points $q_0, q_1 \dots$ lie on one line in that order and all the s_i 's lie on one side from this line.



Note that

$$|s_n - s_0| = \text{length } \beta.$$

Therefore

$$|s_0 - s_1| + \cdots + |s_{n-1} - s_n| \geq \text{length } \beta.$$

Note that

$$|q_i - s_{i-1}| = |q_i - s_i| = |p_i - o| \leq 1$$

and

$$\angle s_{i-1} q_i s_i \leq \alpha_i$$

for each i . Therefore

$$|s_{i-1} - s_i| < \alpha_i$$

for each i .

It follows that

$$\begin{aligned} \text{TotCurv } \beta &= \alpha_1 + \cdots + \alpha_n \geq \\ &\geq |s_0 - s_1| + \cdots + |s_{n-1} - s_n| \geq \\ &\geq \text{length } \beta. \end{aligned}$$

Hence the result. \square

With minor modifications both proofs given above work in the 3-dimensional case (and in higher dimensions). The following more general result was proved by Jeffrey Lagarias and Thomas Richardson in [3], an other proof is given by Alexander Nazarov and Fedor Petrov in [4].

3.11. Theorem. *Let α be a closed curve that lies in a convex plane figure bounded by curve γ . Then average curvature of α is not less than average curvature of γ . Since $\text{TotCurv } \gamma = 2\pi$, it can be written as*

$$\frac{\text{TotCurv } \alpha}{\text{length } \alpha} \geq \frac{2\pi}{\text{length } \gamma}.$$

3.5 Curves of finite total curvature

3.12. Exercise. *Assume that a curve $\alpha: [a, b] \rightarrow \mathbb{R}^3$ has finite total curvature. Show that α is rectifiable.*

We say that a curve $\alpha: [a, b] \rightarrow \mathbb{R}^3$ *does not stop* if α is not constant on any subinterval of $[a, b]$.

3.13. Exercise. *Assume that the curve α does not stop and its total curvature is less than π . Show that α is simple; that is, it has no self-intersections.*

3.14. Exercise-definition. Assume that a curve $\alpha: [a, b] \rightarrow \mathbb{R}^3$ does not stop and has finite total curvature. Show that the direction of exit and entrance is defined for any point.

That is for any $t_0 \in [a, b)$ the unit vector

$$v(\varepsilon) = \frac{\alpha(t_0 + \varepsilon) - \alpha(t_0)}{|\alpha(t_0 + \varepsilon) - \alpha(t_0)|}$$

converges as $\varepsilon \rightarrow 0^+$; its limit is called the direction of exit and it will be denoted by $\alpha^+(t_0)$

Analogously, for any $t_0 \in (a, b]$ the unit vector

$$w(\varepsilon) = \frac{\alpha(t_0 - \varepsilon) - \alpha(t_0)}{|\alpha(t_0 - \varepsilon) - \alpha(t_0)|}$$

converges as $\varepsilon \rightarrow 0^+$; its limit is called the direction of entrance and it will be denoted by $\alpha^-(t_0)$.

3.15. Exercise. Assume that a curve $\alpha: [a, b] \rightarrow \mathbb{R}^3$ does not stop and has finite total curvature. Show that

$$\alpha^+(t) = -\alpha^-(t)$$

at all $t \in [a, b]$ except possibly on a countable subset.

3.16. Exercise. Assume a sequence of curves $\alpha_n: [a, b] \rightarrow \mathbb{R}^3$ converges to a curve $\alpha_\infty: [a, b] \rightarrow \mathbb{R}^3$ and

$$\lim_{n \rightarrow \infty} \text{length } \alpha_n > \text{length } \alpha_\infty.$$

Show that

$$\text{TotCurv } \alpha_n \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

3.6 Total signed curvature

Let us define the *total signed curvature* of a polygonal line in the plane as the sum of the signed external angles; the external angle has positive sign if the line turns left and negative sign if the line turns right; the signed external angle is undefined if a pair of adjacent edges overlap; that is if at one vertex the polygonal line turns in the exact opposite direction. In particular the total signed curvature is defined for any simple polygonal line in the plane.

3.17. Exercise. *Assume that the total signed curvature of a closed polygonal line in the plane is defined. Show that it is a multiple of $2\cdot\pi$.*

3.18. Exercise. *Show that the total signed curvature of any closed simple polygonal line in the plane is $\pm 2\cdot\pi$.*

Chapter 4

Fáry–Milnor theorem

4.1 Tame knots

It is tricky to make a formal definition that captures the intuitive meaning of *knot*. An attempt to define knots as simple closed curves leads to pathological examples as the one show on the diagram — these are the so called *wild knots*. If one adds that the curve has to



be smooth and regular, then these examples disappear, but it is still tricky to give right definition of *deformation* — the following diagram shows that it can not be defined as a continuous family of closed simple



smooth regular curves. Observe that all curves on the diagram are smooth and regular for all times including the last moment.

We define a *knot* (more precicely *tame knot*) as a simple closed polygonal line in the Euclidean space \mathbb{R}^3 .

The notation $\triangle abc$ is used for the triangle abc ; that is, a polygonal line with three edges and vertexes a , b and c . Let us denote by $\blacktriangle abc$ the convex hull of the points a , b and c ; $\blacktriangle abc$ is the solid triangle with the vertexes a , b and c . The points a , b and c are assumed to be distinct, but they might lie on one line; that is, for us a degenerate triangle is a legitimate triangle.

We define a *triangular isotopy of a knot* to be the generation of a new knot from the original one by means of the following two operations:

Assume $[pq]$ is an edge of the knot and x is a point such that the solid triangle $\blacktriangle pqx$ has no common points with the knot except for the edge $[pq]$. Then we can replace the edge $[pq]$ in the knot by the two adjacent edges $[px]$ and $[xq]$.

We can also perform the inverse operation. That is, if for two adjacent edges $[px]$ and $[xq]$ of a knot the triangle $\blacktriangle pqx$ has no common points with the knot except for the points on the edges $[px]$ and $[xq]$, then we can replace the two adjacent edges $[px]$ and $[xq]$ by the edge $[pq]$.

Polygons that arise from one another by a finite sequence of triangular isotopies are called *isotopic*.

A knot that is not isotopic to a triangle is called nontrivial.

The trefoil knot shown on the diagram gives a simple example of nontrivial knot. A proof that the trefoil knot is nontrivial can be found in any textbook on knot theory, we do not give it here. The most elementary and visual proof is based on the so called *tricolorability* of knot diagrams.



4.1. Exercise. Let x and y be two points on the adjacent edges $[p_1p_2]$ and $[p_2p_3]$ of a knot $\beta = p_1p_2p_3 \dots p_n$. Assume that the solid triangle $\blacktriangle xp_2y$ intersects β only along $[xp_2] \cup [p_2y]$. Show that the knot $\beta' = p_1xyp_3 \dots p_n$ is isotopic to β .

4.2 Fáry–Milnor theorem

We will give some proofs of the following theorem.

4.2. Theorem. The total curvature of any nontrivial knot is at least $4 \cdot \pi$.

The famous Fáry–Milnor theorem states that the inequality is strict; that is, the total curvature of any nontrivial knot *exceeds* $4 \cdot \pi$. It

is easy to construct a trefoil knot with total curvature arbitrary close to 4π ; therefore this result is optimal.

The question was raised by Karol Borsuk [5] and answered independently by István Fáry and John Milnor [6, 7]; later other proofs were found.

4.3 Milnor's proof

In the proof we will use the following fact.

4.3. Proposition. *Assume that a height function $(x, y, z) \rightarrow z$ has only one local minimum and one local maximum on a closed simple polygonal line and all the vertexes of the polygonal line are at different height. Then the line is a trivial knot.*

The proof is a simple application of the definition of isotopy, given in the previous section.

Proof. Let $\beta = p_1 \dots p_n$ be the closed simple polygonal line such that the height function $(x, y, z) \rightarrow z$ has one local minimum one local maximum. Note that on each of the two arcs of β from the min-vertex to the max-vertex the height function increases monotonically.

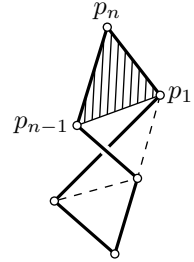
Consider the three vertexes with the largest height; they have to include the max-vertex and two more. Note that these three vertexes are consequent in the polygonal line; without loss of generality we can assume that they are p_{n-1}, p_n, p_1 .

Note that the solid triangle $\blacktriangle p_{n-1}p_np_1$ does not intersect any edge β except the two adjacent edges $[p_{n-1}p_n] \cup [p_np_1]$. Indeed, if $\blacktriangle p_{n-1}p_np_1$ intersects $[p_1p_2]$, then, since p_2 lies below $\blacktriangle p_{n-1}p_np_1$, $[p_1p_2]$ must intersect $[p_{n-1}p_n]$ which is impossible since β is simple. The same way one can show that $\blacktriangle p_{n-1}p_np_1$ can not intersect $[p_{n-2}p_{n-1}]$. The remaining edges lie below $\blacktriangle p_{n-1}p_np_1$, hence they can not intersect this triangle.

Applying a triangular isotopy, to $\blacktriangle p_{n-1}p_np_1$ we get a closed simple polygonal line $\beta' = p_1 \dots p_{n-1}$ which is isotopic to β .

Since all the vertexes p_i have different height, the assumption of the proposition holds for β' .

Repeating this procedure $n - 3$ times we get a triangle. Hence β is a trivial knot. \square



Milnor's proof of 4.2. Let α be a simple closed polygonal line. Assume

its total curvature is less than 4π . Then by Proposition 3.8,

$$\text{TotCurv } \alpha_u < 4\pi$$

for some unit vector u . Moreover, we can assume that u points in a generic direction; that is, u is not perpendicular to any edge or diagonal of α .

The total curvature of α_u is π times the number of turns of α_u which has to be an even number. It follows that the number of turns of α_u is at most 2; it cannot be less than 2 for a generic direction, therefore it is exactly 2.

That is, if we rotate the space so that u points upward, then the height function has exactly one minimum and one maximum; by Proposition 4.3, α is a trivial knot — hence the result. \square

4.4 Fáry's proof

Let us give a sketch of another proof, based on the original idea of István Fáry.

Fáry's proof of 4.2. Consider a projection of the knot to a plane in general position. That is, we assume that the self-intersections of the projection are at most double and the projection of each edge is not degenerate. The obtained closed polygonal line $\beta = p_1 p_2 \dots p_n$ divides the plane into domains, one of which is unbounded, denote it by U , and the others are bounded.



First note that all domains can be colored in a chessboard order; that is, they can be colored in black and white in such a way that domains with common borderline get different colors. If the unbounded domain is colored in white and every other domain is colored in black then one can untie the knot by flipping these domains one by one.

4.4. Exercise. Give a formal proof of the last statement; that is, show that if the only unbounded domain is white then β is isotopic to a triangle.

Therefore among the bounded domains there is a white domain, denote it by D . The domain D cannot adjoin U , since they have the same color. Fix a point o in this domain.



For each i , set

$$\begin{aligned}\varphi_i &= \pi - \angle p_{i-1}p_i p_{i+1}, \\ \psi_i &= \angle p_{i-1}op_i, \\ \theta_i &= \angle op_i p_{i+1}.\end{aligned}$$

Here indexes are taken modulo n ; in particular, $p_n = p_0$.

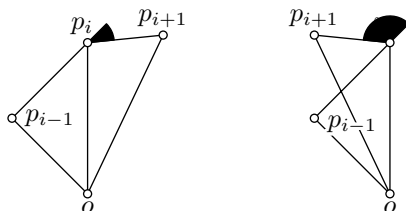
Note that φ_i is the external angle at p_i ; therefore

$$\text{TotCurv } \beta = \varphi_1 + \cdots + \varphi_n$$

Direct calculations show that

$$\varphi_i \geq \psi_i + \theta_{i-1} - \theta_i.$$

In the two pictures below, φ_i is the solid angle and the angles ψ_i , θ_{i-1} and θ_i are just as drawn. We have equality on the first picture and strict inequality on the second picture.



It follows that

$$\varphi_1 + \cdots + \varphi_n \geq \psi_1 + \cdots + \psi_n.$$

The last sum is the total angle at which β is seen from o counted with multiplicity. The boundary of D contributes at least $2\cdot\pi$ to this sum and the boundary of U contributes with other $2\cdot\pi$; since their boundaries do not overlap we get

$$\psi_1 + \cdots + \psi_n \geq 4\cdot\pi,$$

hence the result.

This is true for the projection of the knot to any plane in general position. The remaining planes contribute nothing to the average value. Therefore by Proposition 3.8, the total curvature of the original knot is at least $4\cdot\pi$. \square

4.5. Exercise. *Construct a closed smooth simple curve with total curvature arbitrarily close to $2\cdot\pi$ such that its projection to any plane has at least 10 self-intersections.*

4.5 Proof of Alexander and Bishop

Here we sketch a proof of the Fáry–Milnor theorem given by of Stephanie Alexander and Richard Bishop in [8].

The proof is elementary, but not simple (elementary does not mean simple, it means only that it does not use much theory). It is based on the following two facts that we are already familiar with:

- ◊ If a closed polygonal line β' is inscribed in a closed polygonal line β then

$$\text{TotCurv } \beta' \leq \text{TotCurv } \beta.$$

- ◊ The total curvature of a doubly covered bigon is $4 \cdot \pi$; that is,

$$\text{TotCurv } \beta = 4 \cdot \pi$$

if $\beta = pqpq$ for two distinct points p and q . Similarly if a quadrilateral is sufficiently close to a doubly covered bigon, then its total curvature is close to $4 \cdot \pi$.

Proof. Let $\beta = p_1 \dots p_n$ be a closed polygonal line that is not a trivial knot; that is, one can not get a triangle from β by applying a sequence of triangular isotopies defined in the previous section.

We proceed by induction on the number $n \geq 3$. In the base case $n = 3$ the polygonal line β is a triangle. Therefore, by definition, β is a trivial knot — nothing to show.

Consider the smallest n for which the statement fails; that is, there is a closed simple polygonal line $\beta = p_1 \dots p_n$ that is not a trivial knot and such that

$$\text{①} \quad \text{TotCurv } \beta < 4 \cdot \pi.$$

We use the indexes modulo n ; that is, $p_0 = p_n$, $p_1 = p_{n+1}$ and so on. Without loss of generality, we may assume that β is in general position; that is, no four vertexes of β lie on one plane.

Set $\beta_0 = \beta$. If the solid triangle $\blacktriangle p_0 p_1 p_2$ intersects β_0 only in the two adjacent edges, then applying the corresponding triangular isotopy, we get a knot β'_0 with $n - 1$ edges that is inscribed in β_0 . Therefore

$$\text{TotCurv } \beta_0 \geq \text{TotCurv } \beta'_0.$$

On the other hand, by the induction hypothesis

$$\text{TotCurv } \beta'_0 \geq 4 \cdot \pi,$$

which contradicts ①.

Choose the first point w'_1 on the edge $[p_1 p_2]$ so that the line segment $[p_0 w'_1]$ intersects β_0 . Denote a point of intersection by y_1 .

Choose a point w_1 on $[p_1p_2]$ a bit before w'_1 (below we explain how close). Denote by x_1 the point on $[p_0w_1]$ that minimizes the distance to y_1 . This way we get a closed polygonal line $\beta_1 = w_1p_2 \dots p_n$ with two marked points x_1 and y_1 . Denote by m_1 the number of edges in the arc $x_1w_1 \dots y_1$ of β_1 .



By Exercise 4.1, β_1 is isotopic to β_0 ; in particular β_1 is a nontrivial knot.

Now let us repeat the procedure for the adjacent edges $[w_1p_2]$ and $[p_2p_3]$ of β_1 . If the solid triangle $\blacktriangle w_1p_2p_3$ intersects β_1 only at these two adjacent edges, then we get a contradiction with the induction hypothesis the same way as before. Otherwise we get a new knot $\beta_2 = w_1w_2p_3 \dots p_n$ with two marked points x_2 and y_2 . Denote by m_2 the number of edges in the broken line $x_2w_2 \dots y_2$.

Note that the points x_1, x_2, y_1, y_2 can not appear on β_2 in the same cyclic order; otherwise the broken line $x_1x_2y_1y_2$ can be made to be arbitrary close to a doubly covered bigon which again contradicts ❶.¹

Therefore we can assume that the arc $x_2w_2 \dots y_2$ lies inside the arc $x_1w_1 \dots y_1$ in β_2 and therefore $m_1 > m_2$.

Continuing this procedure we get a sequence of polygonal lines $\beta_i = w_1 \dots w_i p_{i+1} p_n$ with marked points x_i and y_i such that the number of edges m_i from x_i to y_i decreases as i increases. Clearly $m_i > 1$ for any i and $m_1 < n$. Therefore it requires less than n steps to get a contradiction with the induction hypothesis. \square

4.6. Exercise. Suppose that a closed curve α crosses a line at four points a, b, c and d . Assume that the points a, b, c and d appear on the line in that order and they appear on the curve α in the order a, c, b, d . Show that



$$\text{TotCurv } \alpha \geq 4 \cdot \pi.$$

A line crossing a knot at four points as in the exercise is called *alternating quadrisecants*. It turns out that any nontrivial knot admits

¹More precisely, the choice of w_1 has to be made so that the distance $|x_1 - y_1|$ would be much less than all the distances between y_1 and any point $z \in \beta \cap \blacktriangle p_1p_2p_3$, so we have

$$\angle y_1 z x_1 < \frac{\varepsilon}{10},$$

where $\varepsilon = 4 \cdot \pi - \text{TotCurv } \beta$. In this case, since $y_2 \in \beta \cap \blacktriangle p_1p_2p_3$ and x_2 can be taken arbitrary close to y_2 , we have

$$\text{TotCurv } x_1x_2y_1y_2 > 4 \cdot \pi - \varepsilon = \text{TotCurv } \beta$$

which can not happen since $x_1x_2y_1y_2$ is inscribed in β .

an alternating quadrisecants [9]; it provides yet another proof of the Fáry–Milnor theorem.

4.7. Advanced exercise. *Show that given any real number Φ there is a knot β such that any knot isotopic to β has total curvature at least Φ .*

Hint: Use that there are knots with arbitrary large *bridge number*, see for example [10] and the references therein.

AFTER THIS LINE READ AT YOUR OWN RISK!!!

Chapter 5

Osculating circlines

5.1 Acceleration of unit-speed curve

Any regular smooth curve can be parametrized by its length. The obtained curve α has unit speed; that is, $|\alpha'(t)| = 1$ for any t . A curve with such parametrization is called *unit-speed* curve or a curve with a *natural parametrization*.

It is straightforward to show any smooth regular curve remains smooth (and surely regular) if equipped with a natural parametrization; here smooth means that all derivatives $\alpha^{(n)}(t)$ are defined for any n and all values of t in the domain of definition of α .

The following proposition essentially states that vector of acceleration is perpendicular to the velocity vector if the speed remains constant.

5.1. Proposition. *Assume $\alpha: [a, b] \rightarrow \mathbb{R}^2$ be a smooth unit-speed curve. Then*

$$\alpha'(t) \perp \alpha''(t)$$

for any t .

The scalar product (also known as dot product) of two vectors v and w will be denoted by $\langle v, w \rangle$. Recall that for derivative of scalar product the product rule holds; that is if $v = v(t)$ and $w = w(t)$ are smooth vector-valued functions of real argument t , then

$$\langle v, w \rangle' = \langle v', w \rangle + \langle v, w' \rangle.$$

Proof. Since $|\alpha'(t)| = 1$, we have

$$\langle \alpha'(t), \alpha'(t) \rangle = 1.$$

Taking derivative of both sides we get

$$2 \cdot \langle \alpha''(t), \alpha'(t) \rangle = 0,$$

hence the result. \square

5.2 Signed curvature

Given a vector $v \in \mathbb{R}^2$ denote by $i \cdot v$ the vector obtained from v by the counterclockwise rotation by $\frac{\pi}{2}$. (The “multiplication” by i agrees with the multiplication by imaginary unit if one use complex coordinates on the plane $z = x + i \cdot y$.)

Suppose $\alpha: [a, b] \rightarrow \mathbb{R}^2$ is a smooth unit-speed curve. Recall that curvature of α at t can be defined as $|\alpha''(t)|$.

The *signed curvature* $\kappa_\alpha(t)$ is uniquely defined by the identity

$$\alpha''(t) = \kappa_\alpha(t) \cdot i \cdot \alpha'(t).$$

Note that by Proposition 5.1 this equation has a solution. Since $|\alpha'(t)| = 1$ we have that $|\kappa_\alpha(t)| = |\alpha''(t)|$ for any t .

The signed curvature measures how fast the direction $\tau(t) = \alpha'(t)$ rotates; if is positive if it turns left and negative if it turns right; if the curve goes straight then its curvature vanish.

5.3 Osculating circline

It is straightforward to prove the following statement.

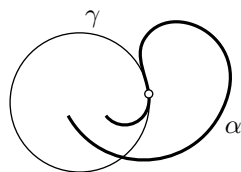
5.2. Proposition. *Given a point p , a unit vector u and a real number κ there is unique smooth unit-speed curve $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$ that starts at p in the direction of u and has constant signed curvature κ .*

Moreover if $\kappa = 0$, then γ runs along a line $\gamma = p + t \cdot u$ and if $\kappa \neq 0$, then γ runs around a circle of radius $\frac{1}{|\kappa|}$ with center at $p + \frac{i}{\kappa} \cdot u$.

Further we will use term *circline* for *circle or line*.

5.3. Definition. *Let α be a smooth unit-speed plane curve; denote by $\kappa_\alpha(t)$ the signed curvature of α at t .*

A unit-speed plane curve γ of constant signed curvature $\kappa_\alpha(t_0)$ that starts at $\alpha(t_0)$ and runs in the direction $\alpha'(t_0)$ is called osculating circline of α at t_0 .



The center and radius of the osculating circle at a given point are called *center of curvature* and *radius of curvature* of the curve at that point.

5.4 Spiral theorem

The following theorem states that if you drive on the plane and turn the steering wheel to the right all the time, then you will not be able to come back to the same place. This theorem was proved by Peter Tait [see 11] and later rediscovered by Adolf Kneser [see 12].

5.4. Theorem. *Assume α is a smooth regular plane curve with strictly monotonic curvature. Then α is simple.*

The same statement holds for signed curvature; the proof requires only minor modifications.

5.5. Exercise. *Show that a 3-dimensional analog of the theorem does not hold. That is, there are self-intersecting smooth regular space curves with strictly monotonic curvature.*

The proof of theorem is based on the following lemma.

5.6. Lemma. *Assume that α is a smooth regular plane curve with strictly decreasing positive signed curvature. Then the osculating circles of α are nested; that is, if γ_t denoted the osculating circle of α at t , then γ_{t_0} lies in the open disc bounded by γ_{t_1} for any $t_0 < t_1$.*

The osculating circles of the curve α give a peculiar foliation of an annulus by circles; it has the following property: if a smooth function is constant on each osculating circle it must be constant in the annulus [see 2, Lecture 10]. Also note that the curve α is tangent to the circle of the foliation at each of its point, however it does not run along a circle.

Proof. Let $z(t)$ be curvature center and

$$r(t) = \frac{1}{\kappa_\alpha(t)}$$

is radius of curvature of α at t . Note that

$$z(t) = \alpha(t) + r(t) \cdot i \cdot \alpha'(t).$$

Therefore

$$\begin{aligned} z'(t) &= \alpha'(t) + r'(t) \cdot i \cdot \alpha'(t) + r(t) \cdot i \cdot \alpha''(t) = \\ &= \alpha'(t) + r'(t) \cdot i \cdot \alpha'(t) + r(t) \cdot i \cdot \kappa_\alpha(t) \cdot i \cdot \alpha'(t) = \\ &= \alpha'(t) + r'(t) \cdot i \cdot \alpha'(t) - \alpha'(t) = \\ &= r'(t) \cdot i \cdot \alpha'(t). \end{aligned}$$

Since $\kappa_\alpha(t)$ is decreasing, $r(t)$ is increasing; therefore $r' \geq 0$. It follows that $|z'(t)| = r'(t)$ and $z'(t) \perp \alpha'(t)$.



Note that the curve $z(t)$ does not have straight arcs; therefore

$$\begin{aligned}
 |z(t_1) - z(t_0)| &< \int_{t_0}^{t_1} |z'(t)| \cdot dt = \\
 (*) \qquad \qquad \qquad &= \int_{t_0}^{t_1} r'(t) \cdot dt = \\
 &= r(t_1) - r(t_0).
 \end{aligned}$$

By (*), the osculating circle at t_0 lies inside of the osculating circle at t_1 without touching it. \square

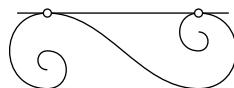
Proof of 5.4. Note that $\alpha(t) \in \gamma_t$ for any t . Applying the lemma we get $\alpha(t_1) \neq \alpha(t_0)$ if $t_1 \neq t_0$. Hence the result. \square

The lemma can be used to solve the following exercise.

5.7. Exercise. Assume that α is a smooth regular plane curve with strictly monotonic curvature.

- (a) Show that no line can be tangent to α at two distinct points.
- (b) Show that no circle can be tangent to α at three distinct points.

Note that part (a) does not hold for smooth regular plane curve with strictly monotonic *signed* curvature; an example is shown on the diagram.



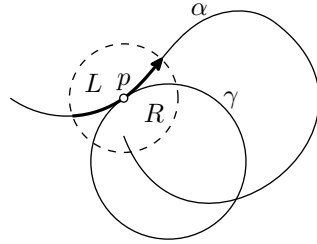
Chapter 6

Supporting circlines

6.1 Definitions

Suppose $\alpha: [a, b] \rightarrow \mathbb{R}^2$ be a smooth unit-speed plane curve and $t_0 \in (a, b)$.

A circline γ supports α at t_0 if $\alpha(t_0) \in \gamma$ and γ lies locally on one side of α . If $p = \gamma(t_0)$ for a single value t_0 , then we can also say γ *supports* α at p without ambiguity.



More precisely assume that there is a spherical neighborhood $U \ni p$ such that for some interval $[a', b'] \ni t_0$ the arc $\bar{\alpha} = \alpha|_{[a', b']}$ has no self-intersection and runs from boundary to boundary of U . In this case α divides U into two sets L and R ; L lies on the left and R lies on the right from $\bar{\alpha}$. If $\gamma \cap U$ contains only points of $\bar{\alpha}$ and R , we say that γ *supports* α on the right; if $\gamma \cap U$ contains only points of $\bar{\alpha}$ and L , we say that γ *supports* α on the left.

Note that a circle supports itself on the right and left at the same time.

Suppose a unit-speed circline γ supports a smooth unit-speed plane curve α at t_0 . Without loss of generality we can assume that $\gamma(0) = \alpha(t_0)$. Then $\gamma'(0) = \pm \alpha'(t_0)$. If not, then the curve α would cross γ transversely and therefore could not stay at the same side for values close to t_0 . Therefore reverting the parametrization of γ if necessary we may (and further will) assume that

$$\gamma'(0) = \alpha'(t_0)$$

holds for any supporting circline γ to α at t_0 .

6.2 Supporting test

The following proposition resembles the second derivative test.

6.1. Proposition. *Assume γ is a circle that supports α at t_0 from right (correspondingly left). Then*

$$\kappa(t_0) \geq \kappa \quad (\text{correspondingly } \kappa(t_0) \leq \kappa).$$

where κ is the signed curvature of γ and $\kappa(t_0)$ is the signed curvature of α at t_0 .

A partial converse also holds. Namely suppose a unit-speed circline γ with signed curvature κ starts at $\alpha(t_0)$ in the direction $\alpha'(t_0)$. Then γ supports α at t_0 from the right (correspondingly left) if

$$\kappa(t_0) > \kappa \quad (\text{correspondingly } \kappa(t_0) < \kappa).$$

Proof. We prove only case $\kappa > 0$. The 2 remaining cases $\kappa = 0$ and $\kappa < 0$ can be done essentially same way.

Since $\kappa \neq 0$, the curve γ is a circle (it can not be a line). According to Proposition 5.2, γ has radius $\frac{1}{\kappa}$ and it is centered at

$$z = \alpha(t_0) + \frac{i}{\kappa} \cdot \alpha'(t_0).$$

Consider the function

$$f(t) = |z - \alpha(t)|^2 - \frac{1}{\kappa^2}.$$

Note that $f(t) \leq 0$ (correspondingly $f(t) \geq 0$) if and only if $\alpha(t)$ lies on the closed left (correspondingly right) side from γ . It follows that

◇ if γ supports α at t_0 from right, then

$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) \leq 0;$$

◇ if γ supports α at t_0 from left, then

$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) \geq 0;$$

◇ if

$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) < 0,$$

then γ supports α at t_0 from right;

◇ if

$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) > 0,$$

then γ supports α at t_0 from left;

Direct calculations show that

$$\begin{aligned} f(t_0) &= 0; \\ f'(t_0) &= \langle z - \alpha(t), z - \alpha(t) \rangle' |_{t=t_0} = \\ &= -2 \cdot \langle \alpha'(t_0), z - \alpha(t_0) \rangle = \\ &= -2 \cdot \langle \alpha'(t_0), \frac{i}{\kappa} \cdot \alpha'(t_0) \rangle = \\ &= 0; \\ f''(t_0) &= \langle z - \alpha(t), z - \alpha(t) \rangle'' |_{t=t_0} = \\ &= 2 \cdot (\langle \alpha'(t_0), \alpha'(t_0) \rangle - \langle \alpha''(t_0), z - \alpha(t_0) \rangle) = \\ &= 2 \cdot \left(1 - \kappa \cdot \frac{1}{\kappa(t_0)} \right) \end{aligned}$$

Hence the result. \square

6.2. Exercise. Assume α is a closed smooth unit-speed plane curve that runs in a unit disk. Show that there is a point on α with curvature at least 1.

Give two proofs, one based on DNA inequality 3.10 and the other based on Proposition 6.1.

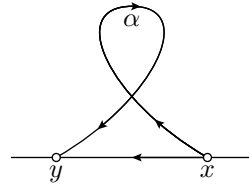
6.3 Lens lemma

6.3. Lemma. Let α be a smooth regular simple curve that runs from x to y . Assume that α runs on right side (correspondingly left side) of the oriented line xy and only its end points x and y lie on the line. Then α has a point with positive (correspondingly negative) curvature.

Note that the lemma fails for curves with self-intersections; the curve α on the diagram turns always right, so it has negative curvature everywhere and it lies on the right side of the line xy .

Proof. Choose points p and q on ℓ so that the points p, x, y, q appear in the same order on ℓ .

Consider the smallest disc segment with chord $[pq]$ that contains α . Note that its arc γ supports at a point $s = \alpha(t_0)$.

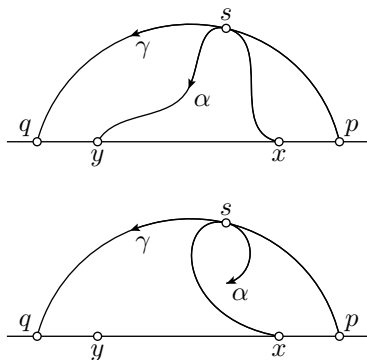


Note that the $\alpha'(t_0)$ is tangent to γ at s . Moreover $\alpha'(t_0)$ points in the direction of q ; that is, if we go along γ in the direction of $\alpha'(t_0)$ then we have to start at p and end at q . If the direction is opposite, then the arc of α from s to y would be trapped in the curvilinear triangle xsp bounded by arcs of γ , α and the line segment $[px]$. But this is impossible since y does not belong to this triangle.

It follows that γ supports α at t_0 from right. By Proposition 6.1,

$$\kappa(t_0) \geq \kappa,$$

where $\kappa(t_0)$ is signed curvature of α at t_0 and κ is the curvature of γ . Evidently $\kappa > 0$, hence the result. \square



6.4 Convexity and inflection points

6.4. Exercise. Assume α is a closed regular simple plane curve with positive signed curvature. Show that α bounds a convex set.¹

6.5. Exercise. Assume a smooth regular curve α has curvature at most 1 at any point (that is, $|\kappa_\alpha(t)| \leq 1$ for any t). Show that each of two unit circles tangent to γ at t_0 is supporting.

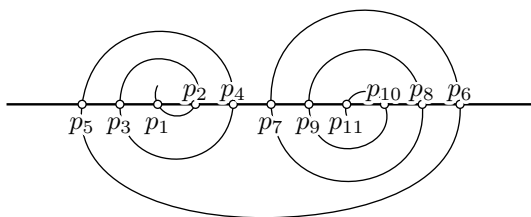
Moreover there is $\varepsilon > 0$ ($\varepsilon = \frac{1}{2}$ will do) such that any arc of length $< \varepsilon$ of the curve α that starts at $p = \alpha(t_0)$ can not enter the unit circle tangent to α at p .

6.6. Exercise. Suppose α is a simple smooth regular curve in the plane with positive curvature. Assume α crosses a line ℓ at the points p_1, p_2, \dots, p_n and these points appear on α in the same order.

- Show that p_2 can not lie between p_1 and p_3 on ℓ .
- Show that if p_3 lies between p_1 and p_2 on ℓ then these points appear on ℓ in the following order:

$$p_1, p_3, \dots, p_4, p_2.$$

- Try to describe all possible orders in the case if p_1 lies between p_2 and p_3 (see the diagram).

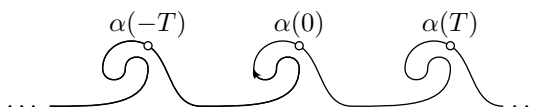


Recall that $\text{Conv } X$ denotes the convex hull of set X ; that is, $\text{Conv } X$ is the intersection of all convex sets containing X .

6.7. Exercise. Suppose α is a simple smooth regular curve in the plane with positive curvature. Then the boundary of $\text{Conv } \alpha$ is formed by an arc of α and a line segment connecting the ends of its arc.

6.8. Exercise. Suppose α is a simple smooth regular curve in the plane. Show that α lies on one side from one of its tangent lines.

If curvature of a curve α vanish at t_0 , then we say that t_0 is inflection value of parameter, and $p = \alpha(t_0)$ is its inflection point; the later convention might be ambiguous only if α has self-intersection at p . In other words, t_0 is inflection value if osculating circle at t_0 coincides with the tangent line.



6.9. Exercise. Let $\alpha: \mathbb{R} \rightarrow \mathbb{R}^2$ be a smooth simple regular plane curve. Assume α is periodic in the following sense: there is $T > 0$ and a vector $v \in \mathbb{R}^2$ such that

$$\alpha(t + T) = \alpha(t) + v.$$

Show that α has at least 2 inflection points in the interval $[0, T]$.

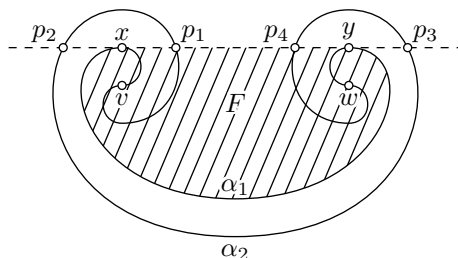
6.10. Theorem. Let α be a closed simple smooth regular plane curve. Assume α has exactly two inflection points which divide α in two arcs α_1 and α_2 . Then

$$\text{Conv } \alpha_1 \supset \alpha_2 \quad \text{or} \quad \text{Conv } \alpha_2 \supset \alpha_1.$$

¹Hint: show that any tangent line to α is supporting.

Proof. Let us denote the inflection points by v and w and orient the arcs α_1 and α_2 from v to w ; we can assume that both arcs α_1 and α_2 have positive curvature.

Set $F = \text{Conv } \alpha_1$. By Exercise 6.7, F is bounded by an arc $\bar{\alpha}_1$ of α_1 from x to y and the line segment xy . We may assume that the line xy is horizontal, x lies on the left from y and so α_1 lies below the line.



If $F \not\supset \alpha_2$, then α_2 runs outside of F , so it has to cross the line segment xy . Denote by p_1, p_2, \dots the points of intersection of α_2 with the line xy as they appear on α_2 .

Since α_2 has positive curvature, p_1 lies on left from p_0 . If p_2 lies between x and p_1 then the curve α_2 is trapped in the region bounded by the arc xvp_2 of α and the line segment p_2x ; therefore it can not reach w — a contradiction. Therefore p_2 lies on the extension of yx behind x .

Further p_3 lies on right from p_2 . If p_3 lies between p_1 and x , then α_2 is trapped in the region bounded by the arc xvp_3 of α and the line segment p_3x — a contradiction again. Therefore p_3 lies on the extension of xy behind y and the arc p_2p_3 of α surrounds F . Whence

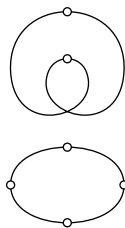
$$\text{Conv } \alpha_2 \supset \alpha_1.$$

□

6.5 Four-vertex theorem

A vertex of a smooth regular curve is defined as a critical point of its curvature; in particular, any point of local minimum (or maximum) is a vertex.

6.11. Exercise. Assume an osculating circle of curve α at t_0 supports α at t_0 . Show that t_0 is a vertex of α .



6.12. Four-vertex theorem. Any smooth regular simple plane curve has at least four vertices.

Evidently any closed curve have to have at least two vertexes — one minimum and one maximum. On the diagram the vertexes are marked; the first curve has one self-intersection and exactly two vertexes; he second curve has exactly four vertexes and no self-intersections.

The four-vertex theorem was first proved by Syamadas Mukhopadhyaya [13] for convex curves. By now it has number of different proofs and generalizations. We will present a proof given by Robert Osserman [14].

Proof. Fix a simple smooth regular closed plane curve α .

Suppose that $2 \cdot n$ points $p_1, s_1, \dots, p_n, s_n$ appear on a closed curve α in the same cyclic order. Fix a real number κ . Assume that the curvature of α at p_i is at least κ and its curvature at s_i is at most κ . Then each of n arcs $p_n p_1, p_1 p_2, \dots, p_{n-1} p_n$ of α has a minimum point of curvature in its interior. Similarly each of n arcs $s_n s_1, s_1 s_2, \dots, s_{n-1} s_n$ of α has a maximum point of curvature.

In the case if some of these local minimum coincide with a local maximum, an arc around this point has constant curvature; in this case all these points are vertexes and we have infinite number of them. If they all different, then we have least $2 \cdot n$ vertexes.

Therefore it is sufficient to show that

1 *there are at least 4 points p_1, s_1, p_2, s_2 with the described properties for some κ .*

Note that

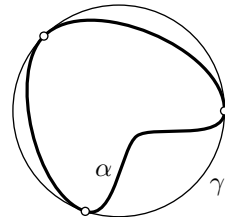
2 *α admits unique circumscribed circle γ ; that is, a circle of minimal radius that encloses α .*

Denote by r the least lower bound of circles that enclose α . We can choose a sequence of circles γ_n enclosing α such that their radiuses $r_n \rightarrow r$. Note that all the centers of γ_i lie on bounded distance from α . Therefore passing to a subsequence we can assume that centers of γ_n converge to a point o . Note that the circle γ with center o and radius r encloses α ; hence the existence of circumscribed circle follows.

If there are two distinct circumscribed circle, then α lies in the intersection of the discs bounded by these circles. But this intersection is enclosed in a circle of smaller radius — a contradiction. Hence Claim **2** follows.

3 *Assume γ is the circumscribed circle of α . Then γ touches α at least 2 points which divide the γ in arcs no longer than semicircle.*

If it would not be the case then one could move γ slightly keeping its radius the same so that γ will not touch α at all. But



in this case α could be enclosed by a circle of smaller radius — a contradiction.

Let us orient α and γ counterclockwise. Then at the common points the directions of α and γ coincide. Note that these points appear on α and γ in the same order; otherwise α could not be simple.

Denote by κ the signed curvature of γ , since it is oriented counterclockwise, $\kappa = \frac{1}{r} > 0$.

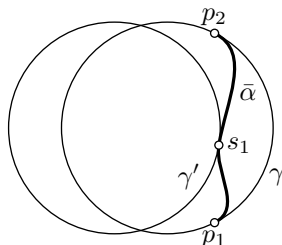
Fix two common points p_1 and p_2 of α and γ . By Proposition 6.1, the curvature of α at p_1 and p_2 is at least κ . Let $\bar{\alpha}$ be the arc of α from p_1 to p_2 .

We can assume that the circle is centered at the origin and the points p_1 and p_2 lie on the same vertical line in the right halfplane of the coordinate plane.

Let o be the center of γ . Consider an other circle γ' with the same radius r and the center o' moved to the left along x -axis to the last moment it intersects $\bar{\alpha}$.

Denote by s_1 a common point of $\bar{\alpha}$ and γ' ; we can assume that s_1 is not an end point of $\bar{\alpha}$. At the point s_1 the directions of $\bar{\alpha}$ and γ' coincide, otherwise α could not be simple — the same argument is used in the proof of Lemma 6.3. Therefore γ' supports α from left at s_1 . By Proposition 6.1, the curvature of α at s_1 is at most κ .

By repeating the same argument for an other pair of points p_2, p_3 ,² we prove Claim ❶. Whence the theorem follows. \square



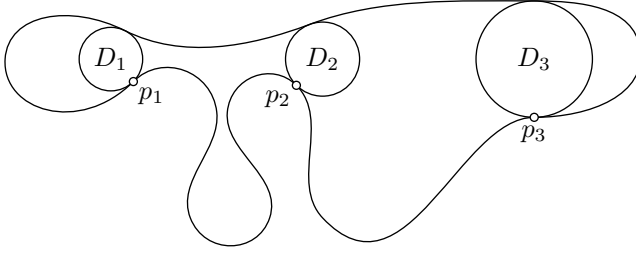
6.13. Exercise. Show that any smooth regular curve of constant width has at least 6 vertexes.

6.6 Moon in a puddle

The following theorem was proved by Vladimir Ionin and Pestov German Pestov [15]. It gives a first nontrivial example of the so called “local to global theorems” — based on some local data (in this case curvature of curve) we conclude its global property (in this case existence of large disc surrounded by the curve). For convex curves, the result was known much earlier [16, §24].

6.14. Theorem. Assume α is a simple closed smooth regular plane curve of curvature bounded by 1. Then it surrounds a unit disc.

²If p_1 and p_2 divide γ into two semicircles, then we can take $p_3 = p_1$.



Proof. Denote by F the closed region surrounded by α .

Fix $p_1 \in \alpha$. Consider the disc D_1 of maximal radius that is tangent to α at p_1 and lies completely in F .

If the radius of D_1 is at least 1, then the problem is solved. Otherwise note that p_1 is an isolated point of the intersection $D_p \cap \alpha$. Moreover according to Exercise 6.5, there is a fixed value $\varepsilon > 0$ such that any arc of α that starts at p_1 can not end in D_1 .

Consider an arc α_1 of α that runs from p_1 to the first point in D_1 . Denote by F_1 the set bounded by α_1 and a part of boundary of D_1 that containing D_1 . From above $\text{length } \alpha_1 > \varepsilon$.

Let p_2 be the midpoint of α_1 . Let D_2 be the disc of maximal radius that is tangent to α_1 at p_2 and lies completely in F_1 . The disc D_2 is touching the boundary ∂F_1 at other points, dividing it in at least two arcs.

Note that D_2 can not touch the boundary of D_1 , otherwise it would lie in D_1 which is impossible. Therefore at least one of these arcs, say α_2 , do not contain the common boundary F_1 and D_1 . Note that

$$\text{length } \alpha_2 < \frac{1}{2} \cdot \text{length } \alpha_1.$$

Again, if the radius of D_2 is at least 1, then the theorem is proved. By Exercise 6.5, it happens if $\text{length } \alpha_2 < \varepsilon$. If the radius is smaller than 1, denote by F_2 be the set that is bounded by α_2 and a part of boundary of D_2 and contains D_2 and repeat the construction several times.

Since the length of arcs gets at least twice as small on each step, after several steps the obtained disc D_n will lie completely in F_{n-1} and therefore in F . \square

A straightforward modification of the proof above gives the following.

6.15. Theorem. *Suppose α is a closed simple smooth regular plane curve. Denote by F and G the two closed domains bounded by α , say F is bounded and G is unbounded. Then α has at least 2 osculating circlines that lie in F and 2 osculating circlines that lie in G .*

Note that Theorem 6.14 as well as the Four-vertex theorem follow from Theorem 6.15; the first implication is evident and the second follow from Exercise 6.11.

Chapter 7

Surfaces

7.1 Embedded surfaces

Recall that a function f of two variables x and y is called smooth if all its partial derivatives $\frac{\partial^{m+n}}{\partial x^m \partial y^n} f$ are defined and continuous in the domain of definition of f .

A subset Σ is called *smooth surface* (or more precisely *smooth regular embedded surface*) if it can be described locally as a graph of smooth function in appropriate coordinate systems. More precisely, any point $p \in \Sigma$ admits a neighborhood U such that in some coordinate system (x, y, z) , the intersection $W = U \cap \Sigma$ can be written as a graph $z = f(x, y)$ of a smooth function f defined in an open domain of (x, y) -plane.

If Σ is compact, then it is called *closed surface* (the term *closed set* is not directly relevant).

If Σ is closed and noncompact, then it is called *open surface* (again the term *open set* is not relevant).

A closed subset in a surface that is bounded by one or more smooth curves is called *surface with boundary*; in this case the collection of curves is called *boundary line* of the surface.

Examples. For simplest example of surface is the (x, y) -plane

$$\Pi = \{ (x, y, z) \in \mathbb{R}^3 : z = 0 \}.$$

It can be described as the graph of the function $f(x, y) = 0$; hence Π is a surface.

All other planes are surfaces as well since one can choose a coordinate system so that it becomes (x, y) -plane or we can rewrite as a graph of linear function $f(x, y) = a \cdot x + b \cdot y + c$ for some constants a , b and c .

A more interesting example is the unit sphere

$$\mathbb{S}^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}.$$

This set is not a graph of any function, but \mathbb{S}^2 can be covered by 6 graphs

$$\begin{aligned} z &= f_{\pm}(x, y) = \pm \sqrt{1 - x^2 - y^2}, \\ y &= g_{\pm}(x, z) = \pm \sqrt{1 - x^2 - z^2}, \\ x &= h_{\pm}(y, z) = \pm \sqrt{1 - y^2 - z^2}; \end{aligned}$$

each function $f_{\pm}, g_{\pm}, h_{\pm}$ is defined in open unit disc. Therefore the unit sphere is a smooth surface.

7.2 Immersed surfaces

Parametrizations. A surface can be described by a map from a known surface to the space. For example the ellipsoid

$$\Sigma_{a,b,c} = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$$

for some positive numbers a, b, c can be defined as the image of a map $s: \mathbb{S}^2 \rightarrow \mathbb{R}^3$ that is the restriction of the map $(x, y, z) \mapsto (a \cdot x, b \cdot y, c \cdot z)$ to the unit sphere \mathbb{S}^2 . The map s has to be smooth and regular as defined below.

Assume \mathbb{S}^2 is written locally as a graph $z = f(x, y)$ in some coordinate system.

The map $s: \mathbb{S}^2 \rightarrow \mathbb{R}^3$ is smooth if the composition $s \circ f$ has all partial derivatives $\frac{\partial^{m+n}}{\partial x^m \partial y^n}(s \circ f)$ are defined and continuous in the domain of definition of f .

The map $s: \mathbb{S}^2 \rightarrow \mathbb{R}^3$ is regular if the vectors $\frac{\partial}{\partial x}(s \circ f)$ and $\frac{\partial}{\partial y}(s \circ f)$ are linearly independent at each point of the domain of f .

Evidently the parametric definition includes the embedded surfaces defined above — as a set of parameters we can take the surface itself and the identity map as s .

Immersed surfaces. The parametric definition allows the surfaces to have self-intersections and therefore more general. The surfaces with possible self-intersections are can called *immersed*.

In the described example \mathbb{S}^2 is the *domain of parameters* of the surface. We can say that the surface $\Sigma_{a,b,c}$ is a *sphere* since it has sphere as the domain of parameters.

We may use other domains of parameters, torus or sphere with two handles or for surfaces with boundary, disc, annulus, or Möbius band and so on.

The set of parameters can be more complicated, for example projective plane — sphere where opposite points are identified; such set of parameters can not be realized as an embedded surface in \mathbb{R}^3 , but it can be embedded in a higher dimensional Euclidean space.

7.3 Tangent plane

Let $z = f(x, y)$ be a local graph realization of a surface. Assume $p = (x_p, y_p, z_p)$ lies on this graph, so $z_p = f(x_p, y_p)$. The plane passing thru p and spanned by two vectors $(\frac{\partial}{\partial x}f)(x_p, y_p)$ and $(\frac{\partial}{\partial y}f)(x_p, y_p)$ is called *tangent plane* of Σ at p . It can be interpreted as the best approximation of the surface Σ by a plane at p .

It is straightforward to check that tangent plane does not depend of the local presentation of Σ by a graph.

The unit normal vector to the tangent plane at p usually denoted by ν_p ; it is uniquely defined up to sign. Likely you know that

$$\nu_p = \frac{[(\frac{\partial}{\partial x}f)(x_p, y_p), (\frac{\partial}{\partial y}f)(x_p, y_p)]}{|[(\frac{\partial}{\partial x}f)(x_p, y_p), (\frac{\partial}{\partial y}f)(x_p, y_p)]|},$$

where $[v, w]$ denotes the vector product of vectors v and w , but we will not need this formula.

Special coordinate system. If one choose the coordinate system so that the origin is at p and the (x, y) -plane coincide with the tangent plane then the function f satisfies the following additional properties:

$$f(0, 0) = 0, \quad (\frac{\partial}{\partial x}f)(0, 0) = 0, \quad (\frac{\partial}{\partial y}f)(0, 0) = 0.$$

The first equality holds since $p = (0, 0, 0)$ lies on the surface and the last two equalities mean that the tangent plane at p is horizontal.

This gives almost canonical coordinate system in a neighborhood of p ; it is unique up to rotation of the (x, y) -plane and switching the sign of z -coordinate.

7.4 Curvatures

Hessian. Fix a point p on a smooth surface Σ and the associated special coordinate system.

Consider the Hessian matrix

$$M_p = \begin{pmatrix} (\frac{\partial^2}{\partial x^2} f)(0,0) & (\frac{\partial^2}{\partial x \partial y} f)(0,0) \\ (\frac{\partial^2}{\partial y \partial x} f)(0,0) & (\frac{\partial^2}{\partial y^2} f)(0,0) \end{pmatrix}.$$

This is symmetric matrix, therefore by rotation of (x, y) -plane, we can make it diagonal; that is we can assume that $(\frac{\partial^2}{\partial x \partial y} f)(0,0) = 0$. Then the diagonal elements are called *principle curvatures* of Σ at p ; they defied up to sign; They are denoted as $k_1(p)$ and $k_2(p)$. The principle curvatures can be also defined as the eigenvalues of M_p .

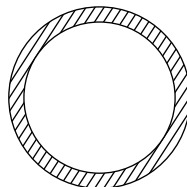
The determinant of M_p is $k_1(p) \cdot k_2(p)$; it is called *Gauss curvature* of Σ at p . The trace of M_p is $k_1(p) + k_2(p)$; it is called *mean curvature* of Σ at p .

Form the discussion above, we get that Gauss curvature and up to sign principle curvatures and mean curvature do not depend only on Σ and p , but not on the choice of the coordinate system.

7.5 Boundaries with bounded curvature

Note that there sets in \mathbb{R}^3 bounded by a closed surface Σ with principle curvatures at most 1 by absolute value that do not contain a ball of radius 1.

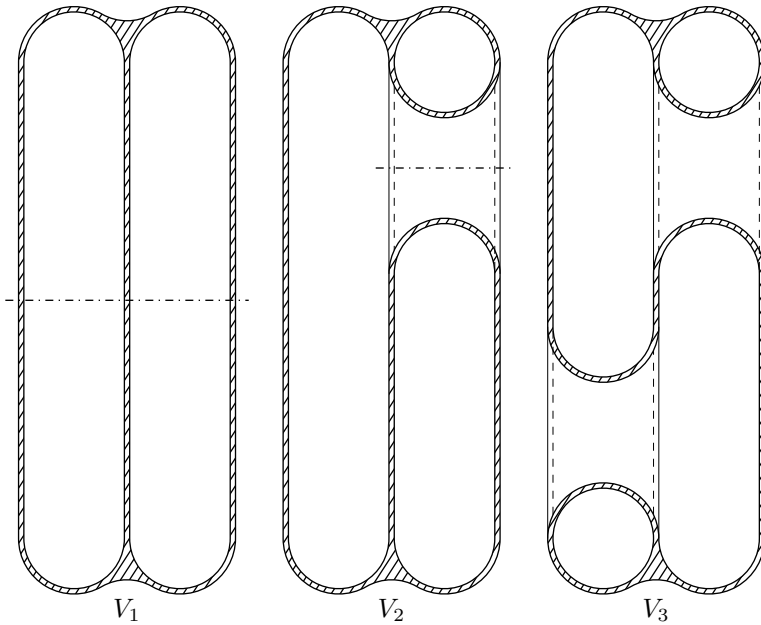
For example the region between two spheres with large close to each other radiiuses. This region can be made arbitrary thin and the curvature of the boundary can be made arbitrary close to zero.



7.1. Advanced exercise. Suppose a set $V \subset \mathbb{R}^3$ is bounded by a closed surface Σ with principle curvatures at most 1 by absolute value. Assume that V does not contain a ball of radius $\frac{1}{100}$. Show that Σ has two components of the same topological type; that is, both can be written in a parametric form with the same parameter domain.

The same example would work for the curves if we allow boundary of the plane figure to be not connected. The following question might look like a right 3-dimensional analog of the moon in a puddle problem (6.14).

7.2. Question. Assume a set $V \subset \mathbb{R}^3$ is bounded by a closed connected surface Σ of bounded curvature. Is it true that V contains a ball of radius 1?



It turns out that the answer is still “no”, the following example was constructed by Vladimir Lagunov [17].

Construction. Let us start with a body of revolution V_1 with the cross section shown on the diagram. The boundary curve of the cross section is made by 6 vertical line segments that smoothly jointed into 3 closed simple curves. The boundary of V_1 has 3 components, each of which is a sphere.

A simple computation shows that if the curvature of all curves is at most 1 then the boundary surface of V_1 has principle curvatures at most 1 by absolute value.

At most of the places V_1 can be made arbitrary thin, the only thick place is where all three spheres come together; it could be arranged that the radius of the maximal ball just a bit above

$$r_2 = \frac{2}{\sqrt{3}} - 1 < \frac{1}{6}.$$

This the radius of the smaller circle tangent to three unit circles that tangent to each other.

It remains to modify V_1 to make its boundary connected without getting larger balls inside.

Note that each sphere in the boundary contains two flat discs; they come into pairs close lying to each other. Let us drill thru two of such

pairs and reconnect the holes by an other body of revolution which axis is shifted but stays parallel to the axis of V_1 . Denote the obtained body by V_2 ; its cross section of the obtained body is shown on the diagram.

Then repeat the operation for the other two pairs. Denote the obtained body by V_3 ; its cross section of the obtained body is shown on the diagram.

It is easy to see that the boundary of V_3 is connected and assuming that the holes are large its boundary can be made so that its principle curvatures is still at most 1. \square

In fact the boundary of V_3 is a sphere with two handles. Indeed, the boundary of V_2 is three spheres, when we drill a hole, we make one hole in two spheres and two holes in one sphere. We reconnect two spheres by a tube and obtain one sphere and connect two holes of one sphere by a tube we get a torus. at the second operation we make a torus from the sphere and connect it to the other torus by tube. This way we get a sphere with two handles.

7.3. Exercise. Assume V is a body of revolution in \mathbb{R}^3 and its boundary is a connected surface with principle curvatures at most 1 by absolute value. Show that V contains a unit ball.

7.4. Exercise. Modify Lagunov's construction so that the boundary surface would be a sphere with 4 handles.

Hint: Combine two examples together using the same construction (or use the spherical annulus together with Lagunov's example).

7.5. Advanced exercise. Assume a set $V \subset \mathbb{R}^3$ is bounded by a sphere Σ with principle curvatures at most 1. Show that it contains a ball of radius $r_3 = \sqrt{\frac{3}{2}} - 1 > \frac{1}{5}$, which is the radius of smaller sphere that tangent to three unit sphere that are tangent to each other.

Appendix A

Semisolutions

Exercise 1.5. First let us show that Dido's problem follows from the isoperimetric inequality.

Assume F is a figure bounded by a straight line and a curve of length ℓ whose endpoints belong to that line. Let F' be the reflection of F in the line. Note that the union $G = F \cup F'$ is a figure bounded by a closed curve of length $2 \cdot \ell$.

Applying the isoperimetric inequality, we get that the area of G can not exceed the area of round disc with the same circumference $2 \cdot \ell$ and the equality holds only if the figure is congruent to the disc. Since F and F' are congruent, Dido's problem follows.

Now let us show that the isoperimetric inequality follows from the Dido's problem.

Assume G is a convex figure bounded by a closed curve of length $2 \cdot \ell$. Cut G by a line that splits the perimeter in two equal parts — ℓ each. Denote by F and F' the two parts. Applying the Dido's problem for each part, we get that that are of each does not exceed the area of half-disc bounded by a half-circle. The two half-disc could be arranged into a round disc of circumference ℓ , hence the isoperimetric inequality follows.

Exercise 2.16. Let $\alpha: [a, b] \rightarrow \mathbb{R}^3$ be a curve. Given a unit vector u , denote by α_u the projection of α on a line in the direction of u ; denote by α_{u^\perp} the of α on a plane perpendicular to u .

Two formulas

$$\text{length } \alpha = k \cdot \overline{\text{length } \alpha_u}$$

and

$$\text{length } \alpha = k' \cdot \overline{\text{length } \alpha_{u^\perp}}$$

can be proved the same way as the Crofton's formula in the plane.

It remains to find the coefficients k and k' . It is sufficient to calculate the average projection of unit segment to a line and to a plane. We need to find two integrals

$$k = \oint_{\mathbb{S}^2} |x| \cdot d \text{ area}$$

and

$$k' = \oint_{\mathbb{S}^2} \sqrt{1 - x^2} \cdot d \text{ area},$$

where $\mathbb{S}^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$ is the unit sphere in the Euclidean space and \oint denotes the average value — since the area of unit sphere is $4 \cdot \pi$, we have

$$\oint_{\mathbb{S}^2} f(x, y, z) \cdot d \text{ area} = \frac{1}{4 \cdot \pi} \cdot \int_{\mathbb{S}^2} f(x, y, z) \cdot d \text{ area}$$

Note that in the cylindrical coordinates

$$(x, \varphi = \arctan \frac{y}{z}, \rho = \sqrt{y^2 + z^2}),$$

we have $d \text{ area} = dx \cdot d\varphi$. Therefore

$$k = \oint_{[-1, 1]} |x| \cdot dx = \frac{1}{2}$$

and

$$k' = \oint_{[-1, 1]} \sqrt{1 - x^2} \cdot dx = \frac{\pi}{4}.$$

Comment. Note that $\frac{k'}{k} = \frac{\pi}{2}$ is the coefficient in the 2-dimensional Crofton formula. This is not a coincidence — think about it.

Exercise 3.1. Assume contrary, that is there is a closed smooth regular curve α such that $\text{TotCurv } \alpha < 2 \cdot \pi$.

The tangent indicatrix τ of α is a curve in a sphere; by the definition of total curvature, the length of τ is the total curvature of α ; in particular

$$\text{length } \tau < 2 \cdot \pi.$$

By Exercise 2.19, τ lies in an open hemisphere. If u is the center of the hemisphere, then

$$\langle u, \tau(t) \rangle > 0 \quad \text{and therefore} \quad \langle u, \alpha'(t) \rangle > 0$$

for any t . Therefore the function $t \mapsto \langle u, \alpha(t) \rangle$ is strictly increasing. In particular, if α is defined on the time interval $[a, b]$, then

$$\langle u, \alpha(a) \rangle < \langle u, \alpha(b) \rangle.$$

But α is closed; that is $\alpha(a) = \alpha(b)$ — a contradiction.

Now let us prove the equality case. First note that it is sufficient to show that τ runs around an equator.

Assume τ is not an equator, from above we know that τ can not lie in an open hemisphere. Note that we can shorten τ by a small chord. The obtained curve τ' is shorter than $2 \cdot \pi$ and therefore lies in an open hemisphere. Applying this construction for shorter and shorter chord and passing to the limit we get that τ lies in closed hemisphere. Denote its center by u as before, then

$$\langle u, \tau(t) \rangle \geq 0 \quad \text{and therefore} \quad \langle u, \alpha'(t) \rangle \geq 0$$

for any t . Since α is closed we have that $\langle u, \alpha(t) \rangle$ is constant; that is, runs in a plane perpendicular to u and τ lies in an equator perpendicular to u .

So τ is a curve that runs along equator, has length $2 \cdot \pi$ and does not lie in a open hemisphere. Since τ is not an equator, it have to run along half-equator back and forth. In this case τ lies in an other closed hemisphere and has some points in its interior. The latter contradicts closeness of α the same way as above.

Bibliography

- [1] G. Lawlor. “A new area-maximization proof for the circle”. *The Mathematical Intelligencer* 20.1 (1998), pp. 29–31.
- [2] D. Fuchs and S. Tabachnikov. *Mathematical omnibus*. Thirty lectures on classic mathematics. American Mathematical Society, Providence, RI, 2007.
- [3] J. Lagarias and T. Richardson. “Convexity and the average curvature of plane curves”. *Geom. Dedicata* 67.1 (1997), pp. 1–30.
- [4] A. I. Nazarov and F. V. Petrov. “On a conjecture of S. L. Tabachnikov”. *Algebra i Analiz* 19.1 (2007), pp. 177–193.
- [5] K. Borsuk. “Sur la courbure totale des courbes fermées”. *Ann. Soc. Polon. Math.* 20 (1947), 251–265 (1948).
- [6] I. Fáry. “Sur la courbure totale d’une courbe gauche faisant un nœud”. *Bull. Soc. Math. France* 77 (1949), pp. 128–138.
- [7] J. Milnor. “On the total curvature of knots”. *Ann. of Math. (2)* 52 (1950), pp. 248–257.
- [8] S. Alexander and R. Bishop. “The Fary-Milnor theorem in Hadamard manifolds”. *Proc. Amer. Math. Soc.* 126.11 (1998), pp. 3427–3436.
- [9] E. Denne. *Alternating quadrisecants of knots*. Thesis (Ph.D.)—University of Illinois at Urbana-Champaign. ProQuest LLC, Ann Arbor, MI, 2004, p. 119.
- [10] J. Schultens. “Additivity of bridge numbers of knots”. *Math. Proc. Cambridge Philos. Soc.* 135.3 (2003), pp. 539–544.
- [11] P. G. Tait. “Note on the circles of curvature of a plane curve.” *Proc. Edinb. Math. Soc.* 14 (1896), p. 26.
- [12] A. Kneser. „Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie.“ *Heinrich Weber Festschrift*. 1912.
- [13] S. Mukhopadhyaya. “New methods in the geometry of a plane arc”. *Bull. Calcutta Math Soc* 1 (1909), pp. 31–37.
- [14] R. Osserman. “The four-or-more vertex theorem”. *Amer. Math. Monthly* 92.5 (1985), pp. 332–337.
- [15] G. Pestov and V. Ionin. “On the largest possible circle imbedded in a given closed curve”. *Dokl. Akad. Nauk SSSR* 127 (1959), pp. 1170–1172.
- [16] W. Blaschke. *Kreis und Kugel*. Verlag von Veit & Comp., Leipzig, 1916.
- [17] V. N. Lagunov. “On the largest sphere contained in a closed surface”. *Sibirsk. Mat. Ž.* 1 (1960), pp. 205–232.