Differential geometry of curves and surfaces: a working approach

Anton Petrunin and Sergio Zamora Barrera

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Preface

These notes are based on lectures at MASS program (Mathematics Advanced Study Semesters at Pennsylvania State University) Fall semester 2018.

The course is designed for those who plan to do differential geometry in the future, or at least who want to have a solid ground to decide not to do it.

The differential geometry of curves and surfaces is a classical subject that is introductory to differential geometry. This subject provides a collection of examples critical for further study, so it does not make sense to do differential geometry until one is a master in curves and surfaces.

Differential geometry does geometry on top of several branches of mathematics including real analysis, differential equations, topology and few other branches of geometry, including elementary and convex geometry. The subject of differential geometry is huge, it is easy to imagine two professional differential geometers who can not find a single subject in the field which they are both slightly interested in. These are two reasons why it is hard to study and hard teach.

Part I

Curves

Chapter 1

Definitions

Simple curves

Recall that that a bijective continuous map $f\colon X\to Y$ between subsets of some metric spaces is called *homeomorphism* if its inverse $f^{-1}\colon Y\to$ $\to X$ is continuous.

1.1. Definition. *A connected subset γ in a metric space is called a* simple curve *if it is* locally *homeomorphic to a real interval; that is, any point $p\in\gamma$ has a neighborhood $U\ni p$ such that the intersection $U\cap\gamma$ is homeomorphic to an open real interval.*

It turns out that any curve can be *parameterized* by an open real interval or a circle. That is, for any curve γ there is a homeomorphism $(a,b)\to\gamma$ or $\mathbb{S}^1\to\gamma$ where $\mathbb{S}^1$ denotes the unit circle; that is

$$\mathbb{S}^1=\left\{\,(x,y)\in\mathbb{R}^2\;:\;x^2+y^2=1\,\right\}.$$

We omit a proof of this statement; it is not hard, but would take us away from the subject. We hope however that this is intuitively obvious.

Curves that admit a parametrization by a circle are called *closed.* The subsets of curves bounded from one or two sides by points are called *curves with endpoints.* If it has two endpoints, then it is called *arc*; note that any arc can be parameterized by a closed interval. A curve as well as a curve with endpoint(s) can be regarded as a curve; if we need to emphasize that we work with a genuine curve we may say a *curve without endpoints.*

A parametrization describes a curve completely. Often we will denote a curve and its parametrization by the same letter; for example, we may say a plane curve γ is given with a parametrization

$\gamma\colon (a,b) \to \mathbb{R}^2$. Note however that any curve admits many different parametrization.

1.2. Exercise. *Find a continuous injective map $\gamma\colon (0,1) \to \mathbb{R}^2$ such that its image is not a simple curve.*

Parameterized curves

A *parameterized curve* is defined as a continuous map γ from a circle or a real interval (open, closed or semi-open) to a metric space. For a parameterized curve we do not assume that γ is injective; in other words the parameterized curve might have *self-intersections.*

If we say curve it means we do not want to specify whether it is a parameterized curve or a simple curve.

If the domain of a parameterized curve is the closed unit interval $[0,1]$, then it is also called a *path.* If in addition $p = \gamma(0) = \gamma(1)$, then γ is called a loop; the point p in this case is called *base* of the loop.

1.3. Advanced exercise. *Let X be a subset of the plane. Suppose that two distinct points $p, q \in X$ can be connected by a path in X. Show that there is a simple arc in X connecting p to q.*

Smooth curves

A curve in the Euclidean space or plane, is called *space* or *plane curve* correspondingly.

A parameterized space curve can be described by its coordinate functions

$$\gamma(t) = (x(t), y(t), z(t)).$$

Plane curves can be considered as a partial case of space curves with $z(t) \equiv 0$.

Recall that a real-to-real function is called *smooth* if its derivatives of all orders are defined everywhere in the domain of definition. If each of the coordinate functions $t \mapsto x(t), t \mapsto y(t)$ and $t \mapsto z(t)$ of the space curve γ is a smooth, then the parameterized curve is called *smooth.*

If the *velocity vector*

$$\gamma'(t) = (x'(t), y'(t), z'(t))$$

does not vanish at all points, then the parameterized curve γ is called *regular.*

A simple space curve is called *smooth and regular* if it admits a smooth and regular parametrization. Regular smooth curves are among the main objects in differential geometry; colloquially, the term *smooth curve* often used as a shortcut for *smooth regular curve.*

1.4. Exercise. *Note that the function*

$$f(t)=\begin{cases}0 & \text{if } t\leqslant 0,\\ \frac{t}{e^{1/t}} & \text{if } t>0.\end{cases}$$

is smooth. Indeed, the existence of all derivatives $f^{(n)}(x)$ at $x\neq 0$ is evident and direct calculations show that $f^{(n)}(0)=0$ for any n.

Show that $\gamma(t)=(f(t),f(-t))$ gives a smooth parametrization of a simple curve formed by the union of two half-axis in the plane.

Show that any smooth parametrization of this curve has vanishing velocity vector at the origin. Conclude that this curve is not regular and smooth; that is it does not admit a regular smooth parametrization.

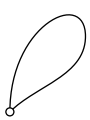
1.5. Exercise. *Describe the set of real numbers ℓ such that the plane curve $\gamma_\ell(t)=(t+\ell\cdot\sin t,\ell\cdot\cos t)$, $t\in\mathbb{R}$ is*

(a) regular;

(b) simple.

Periodic parametrization

Note that any closed simple curve can be described as an image of a loop. However, it is more natural to present it as a *periodic* parameterized curve $\gamma\colon\mathbb{R}\to\mathcal{X}$; that is, such that $\gamma(t+\ell)=\gamma(t)$ for a fixed period ℓ and any t. For example the unit circle in the plane can be described by $2{\cdot}\pi$-periodic parametrization $\gamma(t)=(\cos t,\sin t)$.



Any smooth regular closed curve can be described by a smooth regular loop. But in general the closed curve that described by a smooth regular loop might fail to be smooth at its base; an example is shown on the diagram.

Implicitly defined curves

Let $f\colon\mathbb{R}^2\to\mathbb{R}$ be a smooth function; that is, all its partial derivatives defined in its domain of definition. Consider the set γ of solution of equation $f(x,y)=0$ in the plane.

Assume γ is connected. According to implicit function theorem (A.10), the set γ is a smooth regular simple curve if 0 is a *regular* *value* of f. In this case it means that the gradient ∇f does not vanish at any point $p\in\gamma$. In other words, if $f(p)=0$, then $\frac{\partial f}{\partial x}(p)\ne 0$ or $\frac{\partial f}{\partial y}(p)\ne 0$.

Similarly, assume f,h is a pair of smooth functions defined in $\mathbb{R}^3$. The system of equations

$$\begin{cases} f(x,y,z)=0,\\ h(x,y,z)=0.\end{cases}$$

defines a regular smooth space curve if the set of solutions is connected and 0 is a regular value of the map $F\colon\mathbb{R}^3\to\mathbb{R}^2$ defined as

$$F\colon(x,y,z)\mapsto(f(x,y,z),h(x,y,z)).$$

In this case it means that the gradients ∇f and ∇h are linearly independent at any point $p\in\gamma$. In other words, the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial f}{\partial z}\end{pmatrix}$$

for the map $F\colon\mathbb{R}^3\to\mathbb{R}^2$ has rank 2 at any p such that $f(p)=h(p)=0$.

The described way to define a curve is called *implicit*; if a curve is defined by its parametrization, we say that it is *explicitly defined.* While implicit function theorem guarantees the existence of regular smooth parametrizations, do not expect it to be in a closed form. When it comes to calculations, usually it is easier to work directly with implicit representation.

1.6. Exercise. *Consider the set in the plane described by the equation*

$$y^2=x^3.$$

Is it a simple curve? and if "yes", is it a smooth regular curve?

1.7. Exercise. *Describe the set of real numbers ℓ such that the system of equations*

$$\begin{cases} x^2+y^2+z^2 &=1\\ x^2+\ell\cdot x+y^2 &=0\end{cases}$$

describes a smooth regular curve.

Proper curves

A parametrized curve γ in a metric space $\mathcal{X}$ is called *proper* if for any compact set K the inverse image $\gamma^{-1}(K)$ is compact.

For example curve $\gamma(t) = (e^t, 0, 0)$ defined on whole real line is not proper. Indeed the half-line $(-\infty, 0]$ is not compact and it is the inverse image of unit closed ball around the origin.

Note that any closed curve as well arc are proper curves since its parameter set is compact.

A simple curve is called proper it it admits a proper parametrization. It turns out that simple curve is proper if and only if its image is a closed set. In particular any implicitly defined plane or space curve is proper. We omit the proof of this statement, but it is not hard.

1.8. Exercise. *Use the Jordan's theorem (A.19) to show that any proper plane curve divides the plane in two connected components.*

Chapter 2

Length

Recall that a sequence

$$a = t_0 < t_1 < \dots < t_k = b.$$

is called a *partition* of the interval $[a, b]$.

2.1. Definition. *Let $\gamma\colon [a, b] \to \mathcal{X}$ be a curve in a metric space. The* length *of γ is defined as*

$$\begin{aligned}\operatorname{length}\gamma = \sup\{|\gamma(t_0) - \gamma(t_1)| + |\gamma(t_1) - \gamma(t_2)| + \dots \\ \dots + |\gamma(t_{k-1}) - \gamma(t_k)|\},\end{aligned}$$

where the exact upper bound is taken over all partitions

$$a = t_0 < t_1 < \dots < t_k = b.$$

The length of γ is a nonnegative real number or infinity; the curve γ is called rectifiable *if its length is finite.*

The length of a closed curve is defined as the length of a corresponding loop. If a curve is defined on a open or semi-open interval, then its length is defined as the exact upper bound for lengths of all its closed arcs.

If γ is a space curve, then the above definition says that it length is the exact upper bound of the lengths of polygonal lines $p_0 \dots p_k$ *inscribed* in the curve, where $p_i = \gamma(t_i)$ for a partition $a = t_0 < t_1 < \dots < t_k = b$. If γ is closed then $p_0 = p_k$ and therefore the inscribed polygonal line is also closed.

2.2. Exercise. *Let $\alpha\colon [0, 1] \to \mathbb{R}^3$ be a parametrization of a simple closed arc. Suppose a path $\beta\colon [0, 1] \to \mathbb{R}^3$ has that same image as α;*

that is, $\beta([0,1]) = \alpha([0,1])$*. Show that*

$$\operatorname{length}\beta \geqslant \operatorname{length}\alpha.$$

2.3. Exercise. *Assume* $\gamma\colon [a,b] \to \mathbb{R}^3$ *is a smooth curve. Show that*

(a) $\operatorname{length}\gamma \geqslant \int_a^b |\gamma'(t)|\cdot dt$*,*

(b) $\operatorname{length}\gamma \leqslant \int_a^b |\gamma'(t)|\cdot dt$*.*

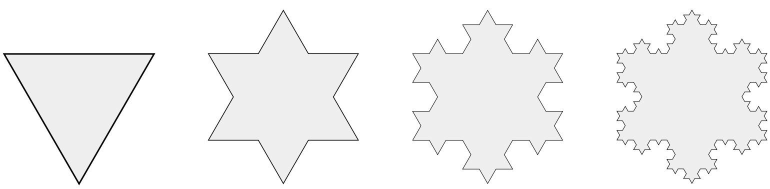
Conclude that

❶ $$\operatorname{length}\gamma = \int\limits_a^b |\gamma'(t)|\cdot dt.$$

Nonrectifiable curves

A classical example of a nonrectifiable curve is the so-called *Koch snowflake*; it is a fractal curve that can be constructed the following way:

Start with an equilateral triangle, divide each of its side into three segments of equal length and add an equilateral triangle with base at the middle segment. Repeat this construction recursively to the obtained polygons. Few first iterations of the construction are shown



on the diagram. The Koch snowflake is the boundary of the union of all the polygons.

2.4. Exercise.

(a) Show that Koch snowflake is a closed simple curve; that is, it can be parameterized by a circle.

(b) Show that Koch snowflake is not rectifiable.

Arc length parametrization

We say that a curve γ has an *arc-length parametrization* (also called *natural parametrization*) if for any two values of parameters $t_1 < t_2$, the value $t_2 - t_1$ is the length of $\gamma|_{[t_1,t_2]}$; that is, the closed arc of γ from t_1 to t_2.

Note that a smooth space curve $\gamma(t) = (x(t), y(t), z(t))$ has arc-length parametrization if and only if it has unit velocity vector at all times; that is,

$$|\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = 1;$$

by that reason smooth curves equipped with arc-length parametrization also called *unit-speed curves.* Note that smooth unit-speed curves are automatically regular.

Any rectifiable curve can be parameterized by arc length. For a parametrized smooth curve γ, the arc-length parameter s can be written as an integral

$$s(t) = \int\limits_{t_0}^{t} |\gamma'(\tau)|\cdot d\tau.$$

Note that $s(t)$ is a smooth increasing function. Further by fundamental theorem of calculus, $s'(t) = |\gamma'(t)|$. Therefore if γ is regular, then $s'(t) \neq 0$ for any parameter value t. By inverse function theorem (A.9) the inverse function $s^{-1}(t)$ is also smooth. Therefore $\gamma \circ s^{-1}$ — the reparametrization of γ by arc length s — remains smooth and regular.

Most of the time we use s for an arc-length parameter of a curve.

2.5. Exercise. *Reparametrize the helix*

$$\gamma_{a,b}(t) = (a\cdot \cos t, a\cdot \sin t, b\cdot t)$$

by its arc length.

We will be interested in the properties of curves that are invariant under a reparametrization. Therefore we can always assume that the given smooth regular curve comes with an arc-length parametrization. A good property of arc-length parametrizations is that it is almost canonical — these parametrizations differ only by a sign and additive constant. By that reason, it is easier to express parametrization-independent quatities using arc-length parametrizations; this will be usefull in the definition of curvature and torsion.

On the other hand, often it is impossible to find an arc-length parametrization in a closed form which makes it hard to use it calculations; usually it is more convenient to use the original parametrization.

Convex curves

A simple plane curve is called *convex* if it bounds a convex region.

2.6. Proposition. *Assume a convex closed curve α lies inside the domain bounded by a closed simple plane curve β. Then*

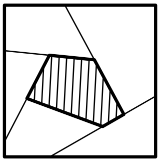
$$\operatorname{length}\alpha \leqslant \operatorname{length}\beta.$$

Note that it is sufficient to show that for any polygon P inscribed in α there is a polygon Q inscribed in β with $\operatorname{perim} P \leqslant \operatorname{perim} Q$, where $\operatorname{perim} P$ denotes the perimeter of P.

Therefore it is sufficient to prove the following lemma.

2.7. Lemma. *Let P and Q be polygons. Assume P is convex and $Q \supset P$. Then*

$$\operatorname{perim} P \leqslant \operatorname{perim} Q.$$



Proof. Note that by the triangle inequality, the inequality

$$\operatorname{perim} P \leqslant \operatorname{perim} Q$$

holds if P can be obtained from Q by cutting it along a chord; that is, a line segment with ends on the boundary of Q that lies in Q.

Note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \cdots \subset P_n = Q$$

such that P_{i-1} obtained from P_i by cutting along a chord. Therefore

$$\begin{aligned}\operatorname{perim} P = \operatorname{perim} P_0 \leqslant \operatorname{perim} P_1 \leqslant \dots \\ \dots \leqslant \operatorname{perim} P_n = \operatorname{perim} Q\end{aligned}$$

and the lemma follows. □

2.8. Corollary. *Any convex closed plane curve is rectifiable.*

Proof. Any closed curve is bounded; that is, it lies in a sufficiently large square. Indeed the curve can be described as an image of a loop $\alpha\colon [0,1] \to \mathbb{R}^2$, $\alpha(t) = (x(t), y(t))$. The coordinate functions $x(t)$ and $y(t)$ are continous functions defined on $[0,1]$. Therefore the absolute values of both of these functions are bounded by some constant C. That is, α lies in the square defined by the inequalities $|x| \leqslant C$ and $|y| \leqslant C$.

By Proposition 2.6, the length of the curve cannot exceed the perimeter of the square $8{\cdot}C$, whence the result. □

Recall that convex hull of a set X is the smallest convex set that contains X; in other words convex hull is the intersection of all convex sets containing X.

2.9. Exercise. *Let α be a closed simple plane curve. Denote by K the convex hull of α; let β be the boundary curve of K. Show that*

$$\operatorname{length}\alpha \geqslant \operatorname{length}\beta.$$

Try to show that the statement holds for arbitrary closed plane curve α, assuming only that X has nonempty interior.

Crofton formulas*

Consider a smooth plane curve $\gamma\colon [a,b] \to \mathbb{R}^2$. Given a unit vector u, denote by γ_u the curve that follows orthogonal projections of γ to the line in the direction u; that is,

$$\gamma_u(t) = \langle u, \gamma(t)\rangle{\cdot}u.$$

Note that

$$|\gamma_u'(t)| = |\langle u, \gamma'(t)\rangle|$$

for any t. Note that for any plane vector the magnitude of its average projection is proportional to its magnitude with coefficient; that is,

$$|w| = k{\cdot}\overline{|w_u|},$$

where $\overline{|w_u|}$ denotes the average value of $|w_u|$ for all unit vectors u. (The value k is the average value of $|\cos\varphi|$ for $\varphi \in [0, 2{\cdot}\pi]$; it can be found by integration, but soon we will show another way to find it.)

If the curve γ is smooth, then according to Exercise 2.3

$$\begin{aligned}\operatorname{length}\gamma &= \int_a^b |\gamma'(t)|{\cdot}dt = \\ &= \int_a^b k{\cdot}\overline{|\gamma_u'(t)|}{\cdot}dt = \\ &= k{\cdot}\overline{\operatorname{length}\gamma_u}.\end{aligned}$$

This formula and its relatives are called *Crofton formulas.* To find the coefficient k one can apply it for the unit circle: the left hand side is $2{\cdot}\pi$ — this is the length of unit circle. Note that for any unit vector u, the curve γ_u runs back and forth along an interval of length 2. Therefore $\operatorname{length}\gamma_u = 4$ and hence its average value is also 4. It follows that the coefficient k has to satisfy the equation $2{\cdot}\pi = k{\cdot}4$; whence

$$\operatorname{length}\gamma = \tfrac{\pi}{2}{\cdot}\overline{\operatorname{length}\gamma_u}.$$

The Crofton's formula holds for arbitrary rectifiable curves, not necessary smooth; it can be proved using Exercises 2.12.

2.10. Exercise. *Show that any closed plane curve γ has length at least $\pi{\cdot}s$, where s is the average of pojections of γ to lines. Moreover the equality holds if and only if γ is convex.*

Use this statement to give another solution of Exercise 2.9.

2.11. Exercise. *Show that the length of space curve is proportional to the average length of its projections to all lines and to planes. Find the coefficients in each case.*

2.12. Advanced exercises.

(a) *Show that the formula ❶ holds for any Lipschitz curve $\gamma\colon [a,b]\to\to\mathbb{R}^3$.*

(b) *Construct a simple curve $\gamma\colon [a,b]\to\mathbb{R}^3$ such that the velocity vector $\gamma'(t)$ is defined and bounded for almost all $t\in[a,b]$, but the formula ❶ does not hold.*

Semicontinuity of length

Recall that the lower limit of a sequence of real numbers (x_n) is denoted by

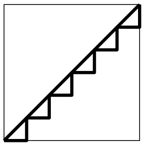
$$\varliminf_{n\to\infty} x_n.$$

It is defined as the lowest partial limit; that is, the lowest possible limit of a subsequence of (x_n). The lower limit is defined for any sequence of real numbers and it lies in the exteded real line $[-\infty,\infty]$

2.13. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\gamma_n\colon [a,b]\to\mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty\colon [a,b]\to\mathcal{X}$; that is, $\gamma_n(t)\to\gamma_\infty(t)$ for any fixed $t\in[a,b]$ as $n\to\infty$. Then

❷ $$\varliminf_{n\to\infty}\operatorname{length}\gamma_n \geqslant \operatorname{length}\gamma_\infty.$$



Note that the inequality ❷ might be strict. For example the diagonal γ_∞ of the unit square can be approximated by a sequence of stairs-like polygonal curves γ_n with sides parallel to the sides of the square (γ_6 is on the picture). In this case

$$\operatorname{length}\gamma_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length}\gamma_n = 2$$

for any n.

Proof. Fix a partition $a = t_0 < t_1 < \dots < t_k = b$. Set

$$\begin{aligned}\Sigma_n &:= |\gamma_n(t_0) - \gamma_n(t_1)| + \dots + |\gamma_n(t_{k-1}) - \gamma_n(t_k)|.\\ \Sigma_\infty &:= |\gamma_\infty(t_0) - \gamma_\infty(t_1)| + \dots + |\gamma_\infty(t_{k-1}) - \gamma_\infty(t_k)|.\end{aligned}$$

Note that for each i we have

$$|\gamma_n(t_{i-1}) - \gamma_n(t_i)| \to |\gamma_\infty(t_{i-1}) - \gamma_\infty(t_i)|$$

and therefore

$$\Sigma_n \to \Sigma_\infty$$

as $n \to \infty$. Note that

$$\Sigma_n \leqslant \operatorname{length}\gamma_n$$

for each n. Hence

❸
$$\varliminf_{n\to\infty} \operatorname{length}\gamma_n \geqslant \Sigma_\infty.$$

If γ_∞ is rectifiable, we can assume that

$$\operatorname{length}\gamma_\infty < \Sigma_\infty + \varepsilon.$$

for any given $\varepsilon > 0$. By ❸ it follows that

$$\varliminf_{n\to\infty} \operatorname{length}\gamma_n > \operatorname{length}\gamma_\infty - \varepsilon$$

for any $\varepsilon > 0$; whence ❷ follows.

It remains to consider the case when γ_∞ is not rectifiable; that is, $\operatorname{length}\gamma_\infty = \infty$. In this case we can choose a partition so that $\Sigma_\infty > L$ for any real number L. By ❸ it follows that

$$\varliminf_{n\to\infty} \operatorname{length}\gamma_n > L$$

for any given L; whence

$$\varliminf_{n\to\infty} \operatorname{length}\gamma_n = \infty$$

and ❷ follows. □

Length metric

Let $\mathcal{X}$ be a metric space. Given two points x, y in $\mathcal{X}$, denote by $d(x, y)$ the exact lower bound for lengths of all paths connecting x to y; if there is no such path we assume that $d(x, y) = \infty$.

Note that function d satisfies all the axioms of metric except it might take infinite value. Therefore if any two points in $\mathcal{X}$ can be connected by a rectifiable curve, then d defines a new metric on $\mathcal{X}$; in this case d is called *induced length metric.*

Evidently $d(x, y) \geqslant |x - y|$ for any pair of points $x, y \in \mathcal{X}$. If the equality holds for any pair, then then the metric is called *length metric* and the space is called *length-metric space.*

Most of the time we consider length-metric spaces. In particular the Euclidean space is a length-metric space. A subspaces A of length-metric space $\mathcal{X}$ might be not a length-metric space; the induced length distance between points x and y in the subspace A will be denoted as $|x - y|_A$; that is, $|x - y|_A$ is the exact lower bound for the length of paths in A.

2.14. Exercise. *Let $A \subset \mathbb{R}^3$ be a closed subset.Show that A is convex if and only if*

$$|x - y|_A = |x - y|_{\mathbb{R}^3}$$

for any $x, y \in A$

Spherical curves

Let us denote by $\mathbb{S}^2$ the unit sphere in the space; that is,

$$\mathbb{S}^2 = \left\{\, (x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 = 1 \,\right\}.$$

A space curve γ is called *spherical* if it runs in the unit sphere; that is, $|\gamma(t)| = 1$ or equivalently $\gamma(t) \in \mathbb{S}^2$ for any t.

Recall that $\measuredangle(u, v)$ denotes the angle between two vectors u and v.

2.15. Observation. *For any $u, v \in \mathbb{S}^2$, we have that*

$$|u - v|_{\mathbb{S}^2} = \measuredangle(u, v)$$

Proof. The short arc γ of a grt circle from u to v in $\mathbb{S}^2$ has length $\measuredangle(u, v)$; that is, $|u - v|_{\mathbb{S}^2} \leqslant \measuredangle(u, v)$.

It remains to prove the opposite inequality. In other words, we need to sho that given an polygonal line $\beta = p_0 \dots p_n$ inscribed in γ

there is a polygonal line $\beta_1 = q_0 \dots q_n$ inscribed in any given spherical path γ_1 connecting u to v such that

$$\text{❹} \qquad \operatorname{length}\beta_1 \geqslant \operatorname{length}\beta.$$

Define q_i as the first point on γ_1 such that $|u - p_i| = |u - q_i|$, but set $q_n = v$. Clearly β_1 is inscribed in γ_1 and according the triangle inequality for angles (A.24), we have that

$$\measuredangle(q_{i-1}, q_i) \geqslant \measuredangle(p_{i-1}, p_i).$$

Therefore

$$|q_{i-1} - q_i| \geqslant |p_{i-1} - p_i|$$

and ❹ follows. □

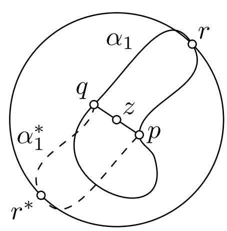
2.16. Hemisphere lemma. *Any closed spherical curve of length less than* $2{\cdot}\pi$ *lies in an open hemisphere.*

This lemma is a keystone in the proof of Fenchel's theorem given below (see 3.7). The lemma is not as simple as you might think — try to prove it yourself. The following proof is due to Stephanie Alexander.

Proof. Let γ be a closed curve in $\mathbb{S}^2$ of length $2{\cdot}\ell$. Assume $\ell < \pi$.

Let us divide γ into two arcs γ_1 and γ_2 of length ℓ, with endpoints p and q. Note that

$$\begin{aligned}\measuredangle(p,q) &\leqslant \operatorname{length}\gamma_1 = \\ &= \ell < \\ &< \pi.\end{aligned}$$



The north hemisphere corresponds to the disc and the south hemisphere to the complement of the disc.

Denote by z be the midpoint between p and q in $\mathbb{S}^2$; that is, z is the midpoint of the short arc of a great circle from p to q in $\mathbb{S}^2$. We claim that γ lies in the open north hemisphere with north pole at z. If not, γ intersects the equator in a point, say r. Without loss of generality we may assume that r lies on γ_1.

Rotate the arc γ_1 by angle π around the line thru z and the center of the sphere. The obtained arc γ_1^* together with γ_1 forms a closed curve of length $2{\cdot}\ell$ that passes thru r and its antipodal point r^*. Therefore

$$\tfrac{1}{2}{\cdot}\operatorname{length}\gamma = \ell \geqslant \measuredangle(r, r^*) = \pi,$$

a contradiction. □

2.17. Exercise. *Describe a simple closed spherical curve that does not pass thru a pair of antipodal points and does not lie in any hemisphere.*

2.18. Exercise. *Suppose that a closed simple spherical curve γ divides $\mathbb{S}^2$ into two regions of equal area. Show that*

$$\operatorname{length}\gamma \geqslant 2\cdot\pi.$$

2.19. Exercise. *Consider the following problem, find a flaw in the given solution. Come up with a correct argument.*

Problem. Suppose that a closed plane curve γ has length at most 4. Show that γ lies in a unit disc.

Wrong solution. Note that it is sufficient to show that diameter of γ is at most 2; that is,

❺ $$|p-q| \leqslant 2$$

for any two points p and q on γ.

The length of γ cannot be smaller then the closed inscribed polygonal line which goes from p to q and back to p. Therefore

$$2\cdot|p-q| \leqslant \operatorname{length}\gamma \leqslant 4;$$

whence ❺ follows. □

2.20. Advanced exercises. *Given points $v, w \in \mathbb{S}^2$, denote by w_v the closest point to w_u on the equator with pole at v; in other words, if $w^\perp$ is the projection of w to the plane perpendicular to v, then w_v is the unit vector in the direction of $w^\perp$. The vector w_v is defined if $w \neq \pm v$.*

1. *Show that for any spherical curve γ we have that*

$$\operatorname{length}\gamma = \overline{\operatorname{length}\gamma_v},$$

where $\overline{\operatorname{length}\gamma_v}$ denotes the average length for all $v \in \mathbb{S}^2$. (This is a spherical analog of Crofton's formula.)

2. *Give another proof of hemisphere lemme using part (1).*

Chapter 3

Curvature

Acceleration of a unit-speed curve

Recall that any regular smooth curve can be parametrized by its arc length. The obtained parametrized curve, say γ, remains to be smooth and it has unit speed; that is, $|\gamma'(s)| = 1$ for all s. The following proposition states that in this case the acceleration vector stays perpendicular to the velocity vector.

3.1. Proposition. *Assume γ is a smooth unit-speed space curve. Then $\gamma'(s) \perp \gamma''(s)$ for any s.*

The scalar product (also known as dot product) of two vectors v and w will be denoted by $\langle \mathrm{v}, \mathrm{w}\rangle$. Recall that the derivative of a scalar product satisfies the product rule; that is, if $\mathrm{v} = \mathrm{v}(t)$ and $\mathrm{w} = \mathrm{w}(t)$ are smooth vector-valued functions of a real parameter t, then

$$\langle \mathrm{v}, \mathrm{w}\rangle' = \langle \mathrm{v}', \mathrm{w}\rangle + \langle \mathrm{v}, \mathrm{w}'\rangle.$$

Proof. The identity $|\gamma'| = 1$ can be rewritten as $\langle \gamma', \gamma'\rangle = 1$. Therefore

$$2\cdot\langle \gamma'', \gamma'\rangle = \langle \gamma', \gamma'\rangle' = 0,$$

whence $\gamma'' \perp \gamma'$. □

Curvature

For a unit-speed smooth space curve γ the magnitude of its acceleration $|\gamma''(s)|$ is called its *curvature* at the time s. If γ is simple, then we can say that $|\gamma''(s)|$ is the curvature at the point $p = \gamma(s)$ without ambiguity. The curvature is usually denoted by $\kappa(s)$ or $\kappa(s)_\gamma$ and in the latter case it might be also denoted by $\kappa(p)$ or $\kappa(p)_\gamma$.

The curvature measures how fast the curve turns; if you drive along a plane curve, curvature tells how much to turn the steering wheel at the given point (note that it does not depend on your speed).

In general, the term *curvature* is used for different types of geometric objects, and it always measures how much it deviates from being *straight*; for curves, it measures how fast it deviates from a straight line.

3.2. Exercise. *Show that any regular smooth spherical curve has curvature at least 1 at each time.*

Tangent indicatrix

Let γ be a regular smooth space curve. Let us consider another curve

$$❶ \qquad \mathrm{T}(t) = \tfrac{\gamma'(t)}{|\gamma'(t)|}$$

that is called *tangent indicatrix* of γ. Note that $|\mathrm{T}(t)| = 1$ for any t; that is, T is a spherical curve.

If γ has a unit-speed parametrization, then $\mathrm{T}(t) = \gamma'(t)$. In this case we have the following expression for curvature: $\kappa(t) = |\mathrm{T}'(t)| = |\gamma''(t)|$.

In general case we have

$$❷ \qquad \kappa(t) = \frac{|\mathrm{T}'(t)|}{|\gamma'(t)|}.$$

Indeed, for an arc-length parametrization $s(t)$ we have $s'(t) = |\gamma'(t)|$. Therefore

$$\begin{aligned}\kappa(t) &= \left|\frac{d\mathrm{T}}{ds}\right| = \\ &= |\tfrac{d\mathrm{T}}{dt}|/|\tfrac{ds}{dt}| = \\ &= \frac{|\mathrm{T}'(t)|}{|\gamma'(t)|}.\end{aligned}$$

It follows that indicatrix of a smooth regular curve γ is regular if the curvature of γ does not vanish.

3.3. Exercise. *Use the formulas ❶ and ❷ to show that for any smooth regular space curve γ we have the following expressions for its curvature:*

(a)
$$\kappa(t) = \frac{|\mathrm{W}(t)|}{|\gamma'(t)|^2},$$
where $\mathrm{W}(t)$ *denotes the projection of* $\gamma''(t)$ *to the normal plane of* $\gamma'(t)$*;*

(b)
$$\kappa(t) = \frac{|\gamma''(t) \times \gamma'(t)|}{|\gamma'(t)|^3},$$
where $\times$ *denotes the* vector product *(also known as* cross product*).*

3.4. Exercise. *Apply the formulas in the previous exercise to show that if f is a smooth real function, then its graph $y = f(x)$ has curvature*
$$\kappa(p) = \frac{|f''(x)|}{(1 + f'(x)^2)^{\frac{3}{2}}}$$
at the point $p = (x, f(x))$.

Tangent curves

Let γ be a smooth regular space curve and T its tangent indicatrix. The line thru $\gamma(t)$ in the direction of $\mathrm{T}(t)$ is called *tangent line* at t. It could be also defined as a unique line that has that has *first order of contact* with γ at s; that is, $\rho(\ell) = o(\ell)$, where $\rho(\ell)$ denotes the distance from $\gamma(s + \ell)$ to the line.

We say that smooth regular curve γ_1 at s_1 is *tangent* to a smooth regular curve γ_2 at s_2 if $\gamma_1(s_1) = \gamma_2(s_2)$ and the tangent line of γ_1 at s_1 coinside with the tangent line of γ_2 at s_2; if both of the curves are simple we can also say that they are tangent at the point $p = \gamma_1(s_1) = \gamma_2(s_2)$ without ambiguity.

Total curvature

Let $\gamma\colon \mathbb{I} \to \mathbb{R}^3$ be a regular smooth curve and T its tangent indicatrix. Recall that without loss of generality we can assume that γ has a unit-speed parametrization; in this case $\mathrm{T}(s) = \gamma'(s)$ and hence
$$\begin{aligned}\kappa(s) &:= |\gamma''(s)| = \\ &= |\mathrm{T}'(s)|;\end{aligned}$$
that is, the curvature of γ at time s is the speed of the tangent indicatrix T at the same time moment.

The integral

$$\Phi(\gamma) := \int\limits_{\mathbb{I}} \kappa(s)\cdot ds$$

is called *total curvature of* γ.

3.5. Exercise. *Find the curvature of the helix*

$$\gamma_{a,b}(t) = (a\cdot \cos t, a\cdot \sin t, b\cdot t),$$

its tangent indicatrix and the total curvature of its arc $t \in [0, 2\cdot\pi]$.

3.6. Observation. *The total curvature of a smooth regular curve is the length of its tangent indicatrix.*

Proof. It is sufficient to prove the observation for a unit-speed space curve $\gamma\colon \mathbb{I} \to \mathbb{R}^3$. Denote by T its tangent indicatrix. Then

$$\begin{aligned}\Phi(\gamma) := \int\limits_{\mathbb{I}} \kappa(s)\cdot ds &= \\ = \int\limits_{\mathbb{I}} |\gamma''(s)|\cdot ds &= \\ = \int\limits_{\mathbb{I}} |\mathrm{T}'(s)|\cdot ds &= \\ = \operatorname{length} \mathrm{T}.\end{aligned}$$

□

3.7. Fenchel's theorem. *The total curvature of any closed regular space curve is at least* $2\cdot\pi$.

Proof. Fix a closed regular space curve γ; we can assume that it is described by a loop $\gamma\colon [a, b] \to \mathbb{R}^3$; in this case $\gamma(a) = \gamma(b)$ and $\gamma'(a) = \gamma'(b)$.

Consider its tangent indicatrix $\mathrm{T} = \gamma'$. Recall that $|\mathrm{T}(s)| = 1$ for any s; that is, T is a closed spherical curve.

Let us show that T cannot lie in a hemisphere. Assume the contrary; without loss of generality we can assume that T lies in the north hemisphere defined by the inequality $z > 0$ in (x, y, z)-coordinates. It means that $z'(t) > 0$ at any time, where $\gamma(t) = (x(t), y(t), z(t))$. Therefore

$$z(b) - z(a) = \int\limits_a^b z'(s)\cdot ds > 0.$$

In particular, $\gamma(a) \neq \gamma(b)$, a contradiction.

Applying the observation (3.6) and the hemisphere lemma (2.16), we get that

$$\Phi(\gamma) = \operatorname{length} \mathrm{T} \geqslant 2\cdot\pi. \qquad \square$$

3.8. Exercise. *Show that a closed space curve γ with curvature at most 1 cannot be shorter than the unit circle; that is,* $\operatorname{length}\gamma \geqslant 2\cdot\pi$.

3.9. Advanced exercise. *Suppose that γ is a smooth regular space curve that does not pass thru the origin. Consider the spherical curve defined as $\sigma(t) = \frac{\gamma(t)}{|\gamma(t)|}$ for any t. Show that*

$$\operatorname{length}\sigma < \Phi(\gamma) + \pi.$$

Moreover, if γ is closed, then

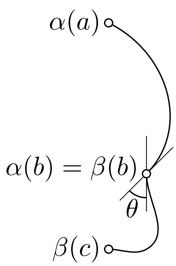
$$\operatorname{length}\sigma \leqslant \Phi(\gamma).$$

Note that the last inequality gives an alternative proof of Fenchel's theorem. Indeed, without loss of generality we can assume that the origin lies on a chord of γ. In this case the closed spherical curve σ goes from a point to its antipode and comes back; it takes length π each way, whence

$$\operatorname{length}\sigma \geqslant 2\cdot\pi.$$

Piecewise smooth curves

Assume $\alpha\colon [a,b] \to \mathbb{R}^3$ and $\beta\colon [b,c] \to \mathbb{R}^3$ are two curves such that $\alpha(b) = \beta(b)$. Then one can combine these two curves into one $\gamma\colon [a,c] \to \mathbb{R}^3$ by the rule



$$\gamma(t) = \begin{cases} \alpha(t) & \text{if} \quad t \leqslant b, \\ \beta(t) & \text{if} \quad t \geqslant b. \end{cases}$$

The obtained curve γ is called the *concatenation* of α and β. (The condition $\alpha(b) = \beta(b)$ ensures that the map $t \mapsto \gamma(t)$ is continuous.)

The same definition of cancatination can be applied if α and/or β are defied on semiopen intervals $(a,b]$ and/or $[b,c)$.

The concatenation can be also defined if the end point of the first curve coincides with the starting point of the second curve; if this is the case, then the time intervals of both curves can be shifted so that they fit together.

If in addition $\beta(c)=\alpha(a)$ then we can do cyclic concatination of these curves; this way we obtain a closed curve.

If $\alpha'(b)$ and $\beta'(b)$ are defined then the angle $\theta=\measuredangle(\alpha'(b),\beta'(b))$ is called *external angle* of γ at time b.

A space curve γ is called *piecewise smooth and regular* if it can be presented as a concatination of finite number of smooth regular curves; if γ is closed, then the concatination is assumed to be cyclic.

If γ is a concatination of smooth regular arcs $\gamma_1,\dots,\gamma_n$, then the total curvature of γ is defined as a sum of the total curvatures of γ_i and the external angles; that is,

$$\Phi(\gamma)=\Phi(\gamma_1)+\dots+\Phi(\gamma_n)+\theta_1+\dots+\theta_{n-1}$$

where θ_i is the external angle at the joint between γ_i and γ_{i+1}; if γ is closed, then

$$\Phi(\gamma)=\Phi(\gamma_1)+\dots+\Phi(\gamma_n)+\theta_1+\dots+\theta_n,$$

where θ_n is the external angle at the joint between γ_n and γ_1.

3.10. Generalized Fenchel's theorem. *Let γ be a closed piecewise smooth regular space curve. Then*

$$\Phi(\gamma)\geqslant 2\cdot\pi.$$

Proof. Suppose γ is a cyclic concatenation of n smooth regular arcs $\gamma_1,\dots,\gamma_n$. Denote by $\theta_1,\dots,\theta_n$ its external angles. We need to show that

$$❸ \qquad \Phi(\gamma_1)+\dots+\Phi(\gamma_n)+\theta_1+\dots+\theta_n\geqslant 2\cdot\pi.$$

Consider the tangent indicatrix $\mathrm{T}_1,\dots,\mathrm{T}_n$ for each arc $\gamma_1,\dots,\gamma_n$; these are smooth spherical arcs.

The same argument as in the proof of Fenchel's theorem, shows that the curves $\mathrm{T}_1,\dots,\mathrm{T}_n$ cannot lie in an open hemisphere.

Note that the spherical distance from the end point of T_i to the starting point of T_{i+1} is equal to the external angle θ_i (we enumerate the arcs modulo n, so $\gamma_{n+1}=\gamma_1$). Therefore if we connect the end point of T_i to the starting point of T_{i+1} by a short arc of a great circle in the sphere, then we add $\theta_1+\dots+\theta_n$ to the total length of $\mathrm{T}_1,\dots,\mathrm{T}_n$.

Applying the hemisphee lemma (2.16) to the obtained closed curve, we get that

$$\operatorname{length}\mathrm{T}_1+\dots+\operatorname{length}\mathrm{T}_n+\theta_1+\dots+\theta_n\geqslant 2\cdot\pi.$$

Applying the observation (3.6), we get ❸. □

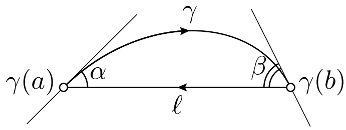
3.11. Chord lemma. *Let ℓ be the chord to a smooth regular arc $\gamma\colon [a,b]\to\mathbb{R}^3$. Assume γ meets ℓ at angles α and β at its ends; that is*

$$\alpha = \measuredangle(\mathrm{w},\gamma'(a)) \quad \textit{and} \quad \beta = \measuredangle(\mathrm{w},\gamma'(b)),$$

where $\mathrm{w} = \gamma(b)-\gamma(a)$. *Then*

$$\Phi(\gamma) \geqslant \alpha+\beta.$$

Proof. Let us parameterize the chord ℓ from $\gamma(b)$ to $\gamma(a)$ and consider the cyclic concatenation $\bar\gamma$ of γ and ℓ. The closed curve $\bar\gamma$ has two external angles $\pi-\alpha$ and $\pi-\beta$. Since curvature of ℓ vanish, we get that



$$\Phi(\bar\gamma) = \Phi(\gamma) + (\pi-\alpha) + (\pi-\beta).$$

According to the generalized Fenechel's theorem (3.10),

$$\Phi(\bar\gamma) \geqslant 2\cdot\pi;$$

hence the result. □

3.12. Exercise. *Show that the estimate in the chord lemma is optimal. That is, given two points p,q and two nonzero vectors u,v in $\mathbb{R}^3$, show that there is a smooth regular curve γ that starts at p in the direction of* u *and ends at q in the direction of* v *such that $\Phi(\gamma)$ is arbitrarily close to* $\measuredangle(\mathrm{w},\mathrm{u})+\measuredangle(\mathrm{w},\mathrm{v})$, *where* $\mathrm{w}=q-p$.

Polygonal lines

Polygonal lines are partial case of piecewise smooth regular curves; each arc in its concatenation is a line segment. Since the curvature of a line segment vanish, the total curvature of polygonal line is the sum of its external angles.

3.13. Exercise. *Let a,b,c,d and x be distinct points in $\mathbb{R}^3$. Show that the total curvature of polygonal line $abcd$ cannot exceed the total curvature of $abxcd$; that is,*

$$\Phi(abcd) \le \Phi(abxcd).$$

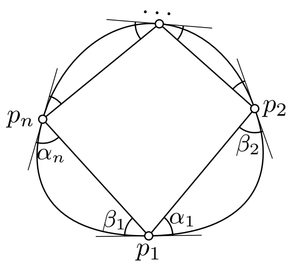
Use this statement to show that any closed polygonal line has curvature at least $2\cdot\pi$.

3.14. Proposition. *Assume a polygonal line $\beta = p_1 \dots p_n$ is inscribed in a smooth regular curve γ. Then*

$$\Phi(\gamma) \geqslant \Phi(\beta).$$

Moreover if γ is closed we can assume that the inscribed polygonal line β is also closed.

Proof. Since the curvature of line segments vanishes, the total curvature of polygonal line is the sum of external angles $\theta_i = \pi - \measuredangle p_{i-1}p_ip_{i+1}$.



Assume $p_i = \gamma(t_i)$. Set

$$\begin{aligned} \mathrm{w}_i &= p_{i+1} - p_i, & \mathrm{v}_i &= \gamma'(t_i), \\ \alpha_i &= \measuredangle(\mathrm{w}_i, \mathrm{v}_i), & \beta_i &= \measuredangle(\mathrm{w}_{i-1}, \mathrm{v}_i). \end{aligned}$$

In case of closed curve we use indexes modulo n, in particular $p_{n+1} = p_1$.

Note that $\theta_i = \measuredangle(\mathrm{w}_{i-1}, \mathrm{w}_i)$. By triangle inequality for angles A.24, we get that

$$\theta_i \leqslant \alpha_i + \beta_i.$$

By the chord lemma, the total curvature of the arc of γ from p_i to p_{i+1} is at least $\alpha_i + \beta_{i+1}$.

Therefore if γ is a closed curve, we have

$$\begin{aligned} \Phi(\beta) &= \theta_1 + \dots + \theta_n \leqslant \\ &\leqslant \beta_1 + \alpha_1 + \dots + \beta_n + \alpha_n = \\ &= (\alpha_1 + \beta_2) + \dots + (\alpha_n + \beta_1) \leqslant \\ &\leqslant \Phi(\gamma). \end{aligned}$$

If γ is an arc, the argument is analogous:

$$\begin{aligned} \Phi(\beta) &= \theta_2 + \dots + \theta_{n-1} \leqslant \\ &\leqslant \beta_2 + \alpha_2 + \dots + \beta_{n-1} + \alpha_{n-1} \leqslant \\ &\leqslant (\alpha_1 + \beta_2) + \dots + (\alpha_{n-1} + \beta_n) \leqslant \\ &\leqslant \Phi(\gamma). \end{aligned}$$

□

3.15. Exercise.

(a) *Draw a smooth regular plane curve γ which has a self-intersection, such that $\Phi(\gamma) < 2\cdot\pi$.*

(b) *Show that if a smooth regular curve $\gamma\colon [a,b] \to \mathbb{R}^3$ has a self-intersection, then $\Phi(\gamma) > \pi$.*

3.16. Proposition. *The equality case in the Fenchel's theorem holds only for convex plane curves; that is, if the total curvature of a smooth regular space curve γ equals $2\cdot\pi$, then γ is a convex plane curve.*

The proof is an application of Proposition 3.14.

Proof. Consider an inscribed quadraliteral $abcd$ in γ. By the definition of total curvature, we have that

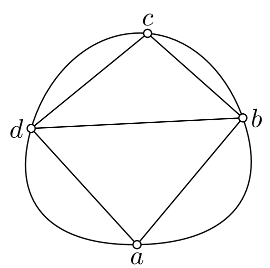
$$\begin{aligned}\Phi(abcd) &= (\pi - \measuredangle dab) + (\pi - \measuredangle abc) + (\pi - \measuredangle bcd) + (\pi - \measuredangle cda) = \\ &= 4\cdot\pi - (\measuredangle dab + \measuredangle abc + \measuredangle bcd + \measuredangle cda))\end{aligned}$$

Note that

$$\measuredangle abc \leqslant \measuredangle abd + \measuredangle dbc \quad \text{and} \quad \measuredangle cda \leqslant \measuredangle cdb + \measuredangle bda. \tag{❹}$$

The sum of angles in any triangle is π. Therefore combining these inequalities, we get that

$$\begin{aligned}\Phi(abcd) &\geqslant 4\cdot\pi - (\measuredangle dab + \measuredangle abd + \measuredangle bda) - (\measuredangle bcd + \measuredangle cdb + \measuredangle dbc) = \\ &= 2\cdot\pi.\end{aligned}$$



By 3.14,

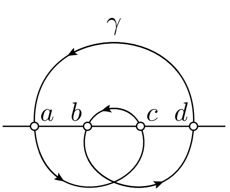
$$\Phi(abcd) \leqslant \Phi(\gamma) \leqslant 2\cdot\pi.$$

Therefore we have equalities in ❹. It means that the point d lies in the angle abc and the point b lies in the angle cda. That is, $abcd$ is a convex plane quadraliteral.

It follows that any quadraliteral inscribed in γ is convex plane quadraliteral. Therefore all points of γ lie in one plane and the domain bounded by γ is convex; that is, γ is a convex plane curve. □

3.17. Exercise. *Suppose that a closed curve γ crosses a line at four points a, b, c and d. Assume that these points appear on the line in the order a, b, c, d and they appear on the curve γ in the order a, c, b, d. Show that*

$$\Phi(\gamma) \geqslant 4\cdot\pi.$$

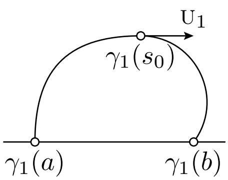


Lines crossing a curve at four points as in the exercise are called *alternating quadrisecants*. It turns out that any *nontrivial knot* admits an alternating quadrisecant [1]; according to the exercise the latter implies the so-called *Fáry–Milnor theorem* — the total curvature any knot exceeds $4\cdot\pi$.

Bow lemma

3.18. Lemma. *Let $\gamma_1\colon [a,b]\to\mathbb{R}^2$ and $\gamma_2\colon [a,b]\to\mathbb{R}^3$ be two smooth unit-speed curves; denote by $\kappa_1(s)$ and $\kappa_2(s)$ their curvatures at s. Suppose that $\kappa_1(s)\geqslant\kappa_2(s)$ for any s and the curve γ_1 is a simple arc of a convex curve; that is, it runs in the boundary of a covex plane figure. Then the distance between the ends of γ_1 cannot exceed the distance between the ends of γ_2; that is,*

$$|\gamma_1(b)-\gamma_1(a)|\leqslant|\gamma_2(b)-\gamma_2(a)|.$$



Proof. Denote by T_1 and T_2 the tangent indicatrixes of γ_1 and γ_2 correspondingly.

Let $\gamma_1(s_0)$ be the point on γ_1 that maximize the distance to the line thru $\gamma(a)$ and $\gamma(b)$. Consider two unit vectors

$$\mathrm{u}_1=\mathrm{T}_1(s_0)=\gamma_1'(s_0)\quad\text{and}\quad\mathrm{u}_2=\mathrm{T}_2(s_0)=\gamma_2'(s_0).$$

By construction the vector u_1 is parallel to $\gamma(b)-\gamma(a)$ in particular

$$|\gamma_1(b)-\gamma_1(a)|=\langle\mathrm{u}_1,\gamma_1(b)-\gamma_1(a)\rangle$$

Since γ_1 is an arc of convex curve, its indicatrix T_1 runs in one direction along the unit circle. Suppose $s\leqslant s_0$, then

$$\begin{aligned}
\measuredangle(\gamma_1'(s),\mathrm{u}_1)&=\measuredangle(\mathrm{T}_1(s),\mathrm{T}_1(s_0))=\\
&=\operatorname{length}(\mathrm{T}_1|_{[s,s_0]})=\\
&=\int_s^{s_0}|\mathrm{T}_1'(t)|\cdot dt=\\
&=\int_s^{s_0}\kappa_1(t)\cdot dt\geqslant\\
&\geqslant\int_s^{s_0}\kappa_2(t)\cdot dt=\\
&=\int_s^{s_0}|\mathrm{T}_2'(t)|\cdot dt=\\
&=\operatorname{length}(\mathrm{T}_1|_{[s,s_0]})\geqslant\\
&\geqslant\measuredangle(\mathrm{T}_2(s),\mathrm{T}_2(s_0))=\\
&=\measuredangle(\gamma_2'(s),\mathrm{u}_2).
\end{aligned}$$

That is,

$$\measuredangle(\gamma_1'(s), \mathrm{u}_1) \geqslant \measuredangle(\gamma_2'(s), \mathrm{u}_2)$$

if $s \geqslant s_0$. The same argument shows that

$$❺ \qquad \measuredangle(\gamma_1'(s), \mathrm{u}_1) \geqslant \measuredangle(\gamma_2'(s), \mathrm{u}_2)$$

for $s \geqslant s_0$; therefore the inequality holds for any s.

Since u_1 is a unit vector parallel to $\gamma_1(b) - \gamma_1(a)$, we have that

$$|\gamma_1(b) - \gamma_1(a)| = \langle \mathrm{u}_1, \gamma_1(b) - \gamma_1(a) \rangle$$

and since u_2 is a unit vector, we have that

$$|\gamma_2(b) - \gamma_2(a)| \geqslant \langle \mathrm{u}_2, \gamma_2(b) - \gamma_2(a) \rangle$$

Integrating ❺, we get that

$$\begin{aligned} |\gamma_1(b) - \gamma_1(a)| &= \langle \mathrm{u}_1, \gamma_1(b) - \gamma_1(a) \rangle = \\ &= \int_a^b \langle \mathrm{u}_1, \gamma_1'(s) \rangle \cdot ds \leqslant \\ &\leqslant \int_a^b \langle \mathrm{u}_2, \gamma_2'(s) \rangle \cdot ds = \\ &= \langle \mathrm{u}_2, \gamma_2(b) - \gamma_2(a) \rangle \leqslant \\ &\leqslant |\gamma_2(b) - \gamma_2(a)|. \end{aligned}$$

Hence the result. □

3.19. Exercise. *Let $\gamma\colon [a, b] \to \mathbb{R}^3$ be a smooth regular curve and $0 < \theta \leqslant \frac{\pi}{2}$. Suppose*

$$\Phi(\gamma) \leqslant 2 \cdot \theta.$$

(a) Show that

$$|\gamma(b) - \gamma(a)| > \cos\theta \cdot \operatorname{length}\gamma.$$

(b) Use part (a) to give another solution of 3.15b.

(c) Show that the inequality in (a) is optimal; that is, given θ there is a smooth regular curve γ such that $\frac{|\gamma(b) - \gamma(a)|}{\operatorname{length}\gamma}$ is arbitrarily close to $\cos\theta$.

3.20. Exercise. *Suppose that two points p and q lie on a unit circle dividing it in two arcs with lengths $\ell_1 < \ell_2$. Show that if a curve γ runs from p to q and has curvature at most 1, then either*

$$\operatorname{length}\gamma \leqslant \ell_1 \quad \textit{or} \quad \operatorname{length}\gamma \geqslant \ell_2.$$

The following exercise generalizes 3.8.

3.21. Exercise. *Suppose $\gamma\colon [a,b] \to \mathbb{R}^3$ is a smooth regular loop with curvature at most 1. Show that*

$$\operatorname{length}\gamma \geqslant 2\cdot\pi.$$

DNA inequality*

Recall that curvature of a spherical curve is at least 1 (Exercise 3.2). In particular the length of spherical curve cannot exceed its total curvature. The following theorem shows that the same inequality holds for *closed* curves in a unit ball.

3.22. Theorem. *Let γ be a smooth regular closed curve that lies in a unit ball. Then*

$$\Phi(\gamma) \geqslant \operatorname{length}\gamma.$$

This theorem was proved by Don Chakerian [2]; for plane curves it was proved earlier by István Fáry [3]. We present the proof given by Don Chakerian in [4]; few other proofs of this theorem are discussed by Serge Tabachnikov [5].

Proof. Without loss of generality we can assume the curve is described by a loop $\gamma\colon [0,\ell] \to \mathbb{R}^3$ parameterized by its arc length, so $\ell = \operatorname{length}\gamma$. We can also assume that the origin is the center of the ball. It follows that

$$\langle\gamma'(s),\gamma'(s)\rangle = 1, \qquad |\gamma(s)| \leqslant 1$$

and in particular

$$\begin{aligned}\langle\gamma''(s),\gamma(s)\rangle &\geqslant -|\gamma''(s)|\cdot|\gamma(s)| \geqslant \\ &\geqslant -\kappa(s)\end{aligned} \tag{❻}$$

for any s, where $\kappa(s) = |\gamma''(s)|$ is the curvature of γ at s.

Since γ is a smooth closed curve, we have that $\gamma'(0) = \gamma'(\ell)$ and

$\gamma(0)=\gamma(\ell)$. Applying ❻, we get that

$$\begin{aligned}0&=\langle\gamma(\ell),\gamma'(\ell)\rangle-\langle\gamma(0),\gamma'(0)\rangle=\\&=\int\limits_0^\ell\langle\gamma(s),\gamma'(s)\rangle'\cdot ds=\\&=\int\limits_0^\ell\langle\gamma'(s),\gamma'(s)\rangle\cdot ds+\int\limits_0^\ell\langle\gamma(s),\gamma''(s)\rangle\cdot ds\geqslant\\&\geqslant\ell-\Phi(\gamma),\end{aligned}$$

whence the result. □

Nonsmooth curves*

3.23. Theorem. *For any regular smooth space curve γ we have that*

$$\Phi(\gamma)=\sup\{\Phi(\beta)\},$$

where the least upper bound is taken for all polygonal lines β inscribed in γ (if γ is closed we assume that so is β).

Proof. Note that the inequality

$$\Phi(\gamma)\geqslant\Phi(\beta)$$

follows from 3.14; it remains to show

❼ $$\Phi(\gamma)\leqslant\sup\{\Phi(\beta)\}.$$

Let $\gamma\colon[a,b]\to\mathbb{R}^3$ be a smooth curve. Fix a partition $a=t_0<$ $<\dots<t_k=b$ and consider the corresponding inscribed polygonal line $\beta=p_0\dots p_k$. (If γ is closed, then $p_0=p_k$ and β is closed as well.)

Let $\tau=\xi_1\dots\xi_k$ be a spherical polygonal line with the vertexes $\xi_i=\frac{p_i-p_{i-1}}{|p_i-p_{i-1}|}$. We can assume that τ has constant speed on each arc and $\tau(t_i)=\xi_i$ for each i. The spherical polygonal line τ will be called tangent indicatrix for β.

Consider a sequence of finer and finer partitions, denote by β_n and τ_n the corresponding inscribed polygonal lines and their tangent indicatrixes. Note that since γ is smooth, the idicatrixes τ_n converge pointwise to T — the tangent indicatrix of γ. By semi-continuity of the length (2.13), we get that

$$\begin{aligned}\Phi(\gamma) &= \operatorname{length} \mathrm{T} \leqslant \\ &\leqslant \varliminf_{n\to\infty} \operatorname{length} \tau_n = \\ &= \varliminf_{n\to\infty} \Phi(\beta_n) \leqslant \\ &\leqslant \sup\{\Phi(\beta)\},\end{aligned}$$

where the least upper bound is taken over all partitions and their corresponding inscribed polygonal lines β; whence ❼ follows. □

The theorem above can be used to define total curvature for arbitrary curves, not necessary (piecewise) smooth and regular. We say that a parameterized curve is trivial if it is constant; that is, it stays at one point.

3.24. Definition. *The total curvature of a nontrivial parameterized space curve γ is the exact upper bound on the total curvatures of inscribed nondegenerate polygonal lines; if γ is closed then we assume that the inscribed polygonal lines are closed as well.*

3.25. Exercise. *Show that the total curvature is lower semi-continuous with respect to pointwise convergence of curves. That is, if a sequence of curves $\gamma_n\colon [a,b] \to \mathbb{R}^3$ converges pointwise to a nontrivial curve $\gamma_\infty\colon [a,b] \to \mathbb{R}^3$, then*

$$\varliminf_{n\to\infty} \Phi(\gamma_n) \geqslant \Phi(\gamma_\infty).$$

3.26. Exercise. *Generalize Fenchel's theorem all nontrivial closed space curves. That is, show that*

$$\Phi(\gamma) \geqslant 2\cdot\pi$$

for any closed space curve γ(not necessary piecewise smooth and regular).

3.27. Exercise. *Assume that a curve $\gamma\colon [a,b] \to \mathbb{R}^3$ has finite total curvature. Show that γ is rectifiable.*

Construct a rectifiable curve $\gamma\colon [a,b] \to \mathbb{R}^3$ that has infinite total curvature.

A good survey on curves of finite total curvature is written by John Sullivan [6].

DNA inequality revisited*

In this section we will give an alternative proof of 3.22 that works for arbitrary, not necessarily smooth, curves. In the proof we use 3.24 to define for the total curvature; according to 3.23, it is more general than the smooth definition given on page 24.

Alternative proof of 3.22. We will show that

$$\Phi(\gamma) > \operatorname{length}\gamma.$$

for any closed polygonal line $\gamma = p_1 \dots p_n$ in a unit ball. It implies the theorem since in any nontrivial closed curve we can inscribe a closed polygonal line with arbitrary close total curvature and length.

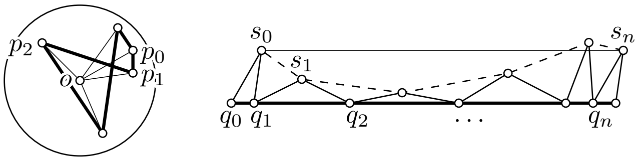
The indexes are taken modulo n, in particular $p_n = p_0$, $p_{n+1} = p_1$ and so on. Denote by θ_i the external angle of γ at p_i; that is,

$$\theta_i = \pi - \measuredangle p_{i-1}p_ip_{i+1}.$$

Denote by o the center of the ball. Consider a sequence of $n+1$ plane triangles

$$\begin{aligned}
\triangle q_0 s_0 q_1 &\cong \triangle p_0 o p_1,\\
\triangle q_1 s_1 q_2 &\cong \triangle p_1 o p_2,\\
&\dots\\
\triangle q_n s_n q_{n+1} &\cong \triangle p_n o p_{n+1},
\end{aligned}$$

such that the points $q_0, q_1 \dots$ lie on one line in that order and all the points $s_0, \dots, s_n$ lie on one side from this line.



Since $p_0 = p_n$ and $p_1 = p_{n+1}$, we have that

$$\triangle q_n s_n q_{n+1} \cong \triangle p_n o p_{n+1} = \triangle p_0 o p_1 \cong \triangle q_0 s_0 q_1,$$

so $s_0 q_0 q_n s_n$ is a parallelogram. Therefore

$$\begin{aligned}
|s_0 - s_1| + \dots + |s_{n-1} - s_n| \geqslant |s_n - s_0| &=\\
= |q_0 - q_n| &=\\
= |p_0 - p_1| + \dots + |p_{n-1} - p_n|&\\
= \operatorname{length}\gamma.&
\end{aligned}$$

Since $|q_i - s_{i-1}| = |q_i - s_i| = |p_i - o| \leqslant 1$, we have that

$$\measuredangle s_{i-1} q_i s_i > |s_{i-1} - s_i|$$

for each i. Therefore

$$\begin{aligned}\theta_i = \pi - \measuredangle p_{i-1} p_i p_{i+1} \geqslant \\ \geqslant \pi - \measuredangle p_{i-1} p_i o - \measuredangle o p_i p_{i+1} = \\ = \pi - \measuredangle q_{i-1} q_i s_{i-1} - \measuredangle s_i q_i q_{i+1} = \\ = \measuredangle s_{i-1} q_i s_i > \\ > |s_{i-1} - s_i|.\end{aligned}$$

That is,

$$\theta_i > |s_{i-1} - s_i|$$

for each i.

It follows that

$$\begin{aligned}\Phi(\gamma) = \theta_1 + \dots + \theta_n > \\ > |s_0 - s_1| + \dots |s_{n-1} - s_n| \geqslant \\ \geqslant \operatorname{length} \gamma.\end{aligned}$$

Hence the result. □

Let us mention the following closely related statement:

3.28. Theorem. *Suppose a closed regular smooth curve γ lies in a convex figure with the perimeter $2{\cdot}\pi$. Then*

$$\Phi(\gamma) \geqslant \operatorname{length} \gamma.$$

This statement was conjectured by Serge Tabachnikov [5]. Despite the simplicity of the formulation, the proof is annoyingly difficult; it was proved by Jeffrey Lagarias and Thomas Richardson [7]; latter a simpler proof was given by Alexander Nazarov and Fedor Petrov [8].

Chapter 4

Torsion

This chapter provides practice that might be useful, but most of the result in this chapter will not be used further in the sequel.

Frenet frame

Let γ be a smooth regular space curve. Without loss of generality, we may assume that γ has an arc-length parametrization, so the velocity vector $\mathrm{T}(s) = \gamma'(s)$ is unit.

Assume its curvature does not vanish at some time moment s; in other words, $\gamma''(s) \neq 0$. Then we can define the so-called *normal vector* at s as

$$\mathrm{N}(s) = \frac{\gamma''(s)}{|\gamma''(s)|}.$$

Note that

$$\mathrm{T}'(s) = \gamma''(s) = \kappa(s)\cdot\mathrm{N}(s).$$

According to 3.1, $\mathrm{N}(s) \perp \mathrm{T}(s)$. Therefore the vector product

$$\mathrm{B}(s) = \mathrm{T}(s) \times \mathrm{N}(s)$$

is a unit vector which makes the triple $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ an oriented orthonormal basis in $\mathbb{R}^3$; in particular, we have that

❶
$$\begin{aligned} &\langle \mathrm{T}, \mathrm{T}\rangle = 1, \quad \langle \mathrm{N}, \mathrm{N}\rangle = 1, \quad \langle \mathrm{B}, \mathrm{B}\rangle = 1,\\ &\langle \mathrm{T}, \mathrm{N}\rangle = 0, \quad \langle \mathrm{N}, \mathrm{B}\rangle = 0, \quad \langle \mathrm{B}, \mathrm{T}\rangle = 0. \end{aligned}$$

The orthonormal basis $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is called *Frenet frame* at s; the vectors in the frame are called *tangent*, *normal* and *binormal* correspondingly. Note that the frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is defined only if $k(s) \neq 0$.

The plane Π_s thru $\gamma(s)$ spanned by vectors $\mathrm{T}(s)$ and $\mathrm{N}(s)$ is called *osculating plane* at s; equivalently it can be defined as a plane thru $\gamma(s)$ that is perpendicular to the binormal vector $\mathrm{B}(s)$. This a unique plane that has *second order of contact* with γ at s; that is, $\rho(\ell) = o(\ell^2)$, where $\rho(\ell)$ denotes the distance from $\gamma(s+\ell)$ to Π_s.

Torsion*

Let γ be a smooth unit-speed space curve and $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is its Frenet frame. The value

$$\tau(s) = \langle \mathrm{N}'(s), \mathrm{B}(s) \rangle$$

is called *torsion* of γ at s.

Note that the torsion $\tau(s)$ is defined if $\kappa(s) \neq 0$. Indeed, if $\kappa(s) \neq 0$ then Frenet frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is defined at s. Moreover since the function $s \mapsto \kappa(s)$ is continuous, it must be positive in an open interval containing s; therefore Frenet frame is also defined in this interval. Clearly $\mathrm{T}(s)$, $\mathrm{N}(s)$ and $\mathrm{B}(s)$ depend smoothly on s in their domains of definition. Therefore $\mathrm{N}'(s)$ is defined and so is the torsion $\tau(s) = \langle \mathrm{N}'(s), \mathrm{B}(s) \rangle$.

The torsion measures how fast the osculating plane rotated when one travels along γ.

4.1. Exercise. *Given real numbers a and b, calculate curvature and torsion of the helix*

$$\gamma_{a,b}(t) = (a \cdot \cos t, a \cdot \sin t, b \cdot t).$$

Conclude that for any $\kappa > 0$ and τ there is a helix with constant curvature κ and torsion τ.

Frenet formulas*

Assume the Frenet frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ of curve γ is defined at s. Recall that

$$\mathrm{T}'(s) = \kappa(s) \cdot \mathrm{N}(s). \qquad ❷$$

Let us write the remaining derivatives $\mathrm{N}'(s)$ and $\mathrm{B}'(s)$ in the frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$.

First let us show that

$$\text{❸} \qquad \mathrm{N}'(s) = -\kappa(s)\cdot\mathrm{T}(s) + \tau(s)\cdot\mathrm{B}(s).$$

Since the frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is orthonormal it is equivalent to the following three identities:

$$\text{❹} \qquad \langle \mathrm{N}', \mathrm{T}\rangle = -\kappa, \quad \langle \mathrm{N}', \mathrm{N}\rangle = 0, \quad \langle \mathrm{N}', \mathrm{B}\rangle = \tau,$$

The last identity follows from the definition of torsion. Differentiating $\langle \mathrm{N}, \mathrm{N}\rangle = 1$ in ❶, we get that

$$2\cdot\langle \mathrm{N}', \mathrm{N}\rangle = 0;$$

whence the second identity in ❹ follows. Differentiating the identity $\langle \mathrm{T}, \mathrm{N}\rangle = 0$ in ❶; we get that

$$\langle \mathrm{T}', \mathrm{N}\rangle + \langle \mathrm{T}, \mathrm{N}'\rangle = 0.$$

Applying ❷, we get that

$$\begin{aligned}\langle \mathrm{N}', \mathrm{T}\rangle &= -\langle \mathrm{T}', \mathrm{N}\rangle = \\ &= -\kappa\cdot\langle \mathrm{N}, \mathrm{N}\rangle = \\ &= -\kappa.\end{aligned}$$

It proves the first equality in ❹; whence ❸ follows.

Taking derivatives of the third identity in ❶, we get that $\mathrm{B}' \perp \mathrm{B}$. Further taking derivatives of the other identities with B in ❶, we get that

$$\begin{aligned}\langle \mathrm{B}', \mathrm{T}\rangle &= -\langle \mathrm{B}, \mathrm{T}'\rangle = -\kappa\cdot\langle \mathrm{B}, \mathrm{N}\rangle = 0\\ \langle \mathrm{B}', \mathrm{N}\rangle &= -\langle \mathrm{B}, \mathrm{N}'\rangle = \tau\end{aligned}$$

Since the frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is orthonormal, it follows that

$$\text{❺} \qquad \mathrm{B}'(s) = -\tau(s)\cdot\mathrm{N}(s).$$

The equations ❷, ❸ and ❺ are called Frenet formulas. All three can be written as one matrix identity:

$$\begin{pmatrix}\mathrm{T}'\\ \mathrm{N}'\\ \mathrm{B}'\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\cdot\begin{pmatrix}\mathrm{T}\\ \mathrm{N}\\ \mathrm{B}\end{pmatrix}.$$

4.2. Exercise. *Deduce the formula ❺ from ❷ and ❸ by differentiating the identity* $\mathrm{B} = \mathrm{T}\times\mathrm{N}$.

4.3. Exercise. *Let γ be a regular space curve with nonvanishing curvature. Show that γ lies in a plane if and only if its torsion vanishes.*

4.4. Exercise. *Let γ be a smooth regular space curve, κ and τ its curvature and torsion, and $\mathrm{T}, \mathrm{N}, \mathrm{B}$ its Frenet frame. Show that*

$$\mathrm{B} = \frac{\gamma' \times \gamma''}{|\gamma' \times \gamma''|}.$$

Use this formula to show that

$$\tau = \frac{\langle \gamma' \times \gamma'', \gamma''' \rangle}{|\gamma' \times \gamma''|^2}.$$

Curves of constant slope*

We say that a smooth regular space curve γ has *constant slope* if its velocity vector makes a constant angle with a fixed direction. The following theorem was proved by Michel Ange Lancret [9] more than two centuries ago.

4.5. Theorem. *Let γ be a smooth regular curve; denote by κ and τ its curvature and torsion. Suppose $\kappa(s) > 0$ at any s. Then γ has constant slope if and only if the ratio $\frac{\tau}{\kappa}$ is constant.*

The theorem can be proved using the Frenet formulas. The following exercise will guide you thru the proof of the theorem.

4.6. Exercise. *Suppose that γ is a smooth regular space curve with nonvanishing curvature, $\mathrm{T}, \mathrm{N}, \mathrm{B}$ is its Frenet frame and κ, τ are its curvature and torsion.*

(a) Assume that $\langle \mathrm{W}, \mathrm{T} \rangle$ is constant for a fixed nonzero vector W. Show that

$$\langle \mathrm{W}, \mathrm{N} \rangle = 0.$$

Use it to show that

$$\langle \mathrm{W}, -\kappa \cdot \mathrm{T} + \tau \cdot \mathrm{B} \rangle = 0.$$

Use these two identities to show that $\frac{\tau}{\kappa}$ is constant; it proves the "only if" part of the theorem.

(b) Assume that $\frac{\tau}{\kappa}$ is constant, show that the vector $\mathrm{W} = \frac{\tau}{\kappa} \cdot \mathrm{T} + \mathrm{B}$ is constant. Conclude that γ has constant slope; it proves the "if" part of the theorem.

Assume γ is a smooth unit-speed curve and s_0 is a fixed real number. Then the curve

$$\alpha(s) = \gamma(s) + (s_0 - s)\cdot\gamma'(s)$$

is called *evolvent* of γ. Note that if $\ell(s)$ denotes the tangent line of γ at s, then $\alpha(s) \in \ell(s)$ and $\alpha'(s) \perp \ell$ for any s.

4.7. Exercise. *Show that evolvent of a constant slope curve is a plane curve.*

Spherical curves*

4.8. Theorem. *A smooth regular space curve γ lies in a unit sphere if and only if the following identity*

$$\left|\frac{\kappa'}{\tau}\right| = \kappa\cdot\sqrt{\kappa^2 - 1}.$$

holds for its curvature κ and torsion τ.

Note that the identity implicitely implies that the torsion τ of the curve is nonzero; otherwise the left hand side would be undefined while right hand side is defined. The proof is another application of Frenet formulas; we present it in a form of guided exercise:

4.9. Exercise. *Suppose γ is a smooth unit-speed space curve. Denote by $\mathrm{T}, \mathrm{N}, \mathrm{B}$ its Frenet frame and by κ, τ its curvature and torsion.*

Assume that γ is spherical; that is, $|\gamma(s)| = 1$ for any s. Show that

(a) $\langle \mathrm{T}, \gamma\rangle = 0$; *conclude that* $\langle \mathrm{N}, \gamma\rangle^2 + \langle \mathrm{B}, \gamma\rangle^2 = 1$.
(b) $\langle \mathrm{N}, \gamma\rangle = -\frac{1}{\kappa}$;
(c) $\langle \mathrm{B}, \gamma\rangle' = \frac{\tau}{\kappa}$.
(d) Use (c) to show that if γ is closed, then $\tau(s) = 0$ for some s.
(e) Use (a)–(c) to show that

$$\left|\frac{\kappa'}{\tau}\right| = \kappa\cdot\sqrt{\kappa^2 - 1}.$$

It proves the "only if" part of the theorem.

Now assume that γ is a space curve that satisfies the idenity in (e).

(f) Show that $p = \gamma + \frac{1}{\kappa}\cdot\mathrm{N} + \frac{\kappa'}{\kappa^2\cdot\tau}\cdot\mathrm{B}$ is constant; conclude that γ lies in a unit sphere centered at p.

It proves the "if" part of the theorem.

For a unit-speed curve γ with nonzero curvature and torsion at s, the sphere Σ_s with the center

$$p(s) = \gamma(s) + \tfrac{1}{\kappa(s)}\cdot\mathrm{N}(s) + \tfrac{\kappa'(s)}{\kappa^2(s)\cdot\tau(s)}\cdot\mathrm{B}(s)$$

that pass thru $\gamma(s)$ is called *osculating sphere* of γ at s. This a unique sphere that has *third order of contact* with γ at s; that is, $\rho(\ell) = o(\ell^3)$, where $\rho(\ell)$ denotes the distance from $\gamma(s+\ell)$ to Σ_s.

Fundamental theorem of space curves*

4.10. Theorem. *Let $\kappa(s)$ and $\tau(s)$ be two smooth real valued functions defined on a real interval $\mathbb{I}$. Suppose $\kappa(s) > 0$ for any s. Then there is a smooth unit-speed curve $\gamma\colon \mathbb{I} \to \mathbb{R}^3$ with curvature $\kappa(s)$ and torsion $\tau(s)$ for every s. Moreover γ is uniquely defined up to a rigid motion of the space.*

The proof is an application of the theorem on existence and uniqueness of a solution of ordinary differential equation (A.14).

Proof. Fix a parameter value s_0, a point $\gamma(s_0)$ and an oriented orthonormal frame $\mathrm{T}(s_0)$, $\mathrm{N}(s_0)$, $\mathrm{B}(s_0)$.

Consider the following system of differential equations

$$\begin{cases} \gamma' = \mathrm{T}, \\ \mathrm{T}' = \kappa\cdot\mathrm{N}, \\ \mathrm{N}' = -\kappa\cdot\mathrm{T} + \tau\cdot\mathrm{B}, \\ \mathrm{B}' = -\tau\cdot\mathrm{N}. \end{cases}$$

with the initial condition $\gamma(s_0)$ and an oriented orthonormal frame $\mathrm{T}(s_0)$, $\mathrm{N}(s_0)$, $\mathrm{B}(s_0)$. (The system of equations has four vector equations, so it can be rewritten as a system of 12 scalar equations.)

By A.14, this system has a unique solution which is defined in a maximal subinterval $\mathbb{J} \subset \mathbb{I}$ containing s_0; we need to show that actually $\mathbb{J} = \mathbb{I}$.

Note that

$$\begin{aligned}
\langle \mathrm{T}, \mathrm{T}\rangle' &= 2\cdot\langle \mathrm{T}, \mathrm{T}'\rangle = 2\cdot\kappa\cdot\langle \mathrm{T}, \mathrm{N}\rangle = 0, \\
\langle \mathrm{N}, \mathrm{N}\rangle' &= 2\cdot\langle \mathrm{N}, \mathrm{N}'\rangle = -2\cdot\kappa\cdot\langle \mathrm{N}, \mathrm{T}\rangle + 2\cdot\tau\cdot\langle \mathrm{N}, \mathrm{B}\rangle = 0, \\
\langle \mathrm{B}, \mathrm{B}\rangle' &= 2\cdot\langle \mathrm{B}, \mathrm{B}'\rangle = -2\cdot\tau\langle \mathrm{B}, \mathrm{N}\rangle = 0, \\
\langle \mathrm{T}, \mathrm{N}\rangle' &= \langle \mathrm{T}', \mathrm{N}\rangle + \langle \mathrm{T}, \mathrm{N}'\rangle = \kappa\cdot\langle \mathrm{N}, \mathrm{N}\rangle - \kappa\cdot\langle \mathrm{T}, \mathrm{T}\rangle + \tau\cdot\langle \mathrm{T}, \mathrm{B}\rangle = 0, \\
\langle \mathrm{N}, \mathrm{B}\rangle' &= \langle \mathrm{N}', \mathrm{B}\rangle + \langle \mathrm{N}, \mathrm{B}'\rangle = 0, \\
\langle \mathrm{B}, \mathrm{T}\rangle' &= \langle \mathrm{B}', \mathrm{T}\rangle + \langle \mathrm{B}, \mathrm{T}'\rangle = -\tau\cdot\langle \mathrm{N}, \mathrm{T}\rangle + \kappa\cdot\langle \mathrm{B}, \mathrm{N}\rangle = 0.
\end{aligned}$$

That is, the values $\langle \mathrm{T}, \mathrm{T}\rangle$, $\langle \mathrm{N}, \mathrm{N}\rangle$, $\langle \mathrm{B}, \mathrm{B}\rangle$, $\langle \mathrm{T}, \mathrm{N}\rangle$, $\langle \mathrm{T}, \mathrm{N}\rangle$, $\langle \mathrm{B}, \mathrm{T}\rangle$ are constant functions of s. Since we choose $\mathrm{T}(s_0)$, $\mathrm{N}(s_0)$, $\mathrm{B}(s_0)$ to be an oriented orthonormal frame, we have that the $\mathrm{T}(s)$, $\mathrm{N}(s)$, $\mathrm{B}(s)$ is oriented orthonormal for any s. In particular $|\gamma'(s)| = 1$ for any s.

Assume $\mathbb{J} \neq \mathbb{I}$. Then an end of $\mathbb{J}$, say a, lies in the interior of $\mathbb{I}$. By Theorem A.14, at least one of the values $\gamma(s)$, $\mathrm{T}(s)$, $\mathrm{N}(s)$, $\mathrm{B}(s)$ escapes to infinity as $s \to a$. But this is impossible since the vectors $\mathrm{T}(s)$, $\mathrm{N}(s)$, $\mathrm{B}(s)$ remain unit and $|\gamma'(s)| = |\mathrm{T}(s)| = 1$ — a contradiction. Whence $\mathbb{J} = \mathbb{I}$.

It remains to prove the last statement.

Assume there are two curves γ_1 and γ_2 with the given curvature and torsion functions. Applying a motion of the space we can assume that the $\gamma_1(s_0) = \gamma_2(s_0)$ and the Frenet frames of the curves coincide at s_0. Then $\gamma_1 = \gamma_2$ by uniqueness of solution of the system (A.14). That is, the curve is unique up to a rigid motion of the space. $\square$

4.11. Exercise. *Assume a curve $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ has constant curvature and torsion. Show that γ is a helix, possibly degenerate to a circle; that is, in a suitable coordinate system we have*

$$\gamma(t) = (a\cdot \cos t, a\cdot \sin t, b\cdot t)$$

for some constants a and b.

4.12. Advanced exercise. *Let γ be a smooth regular space curve such that the distance $|\gamma(t) - \gamma(t+\ell)|$ depends only on ℓ. Show that γ is a helix, possibly degenerate to a line or a circle.*

Chapter 5

Plane curves

Signed curvature

Suppose γ is a smooth unit-speed plane curve, so $\mathrm{T}(s) = \gamma'(s)$ is its unit tangent vector for any s.

Let us rotate $\mathrm{T}(s)$ by angle $\frac{\pi}{2}$ counterclockwise; denote the obtained vector by $\mathrm{N}(s)$. The pair $\mathrm{T}(s), \mathrm{N}(s)$ is an oriented orthonormal frame in the plane which is analogous to the Frenet frame defined on page 38; we will keep the name *Frenet frame* for it.

Recall that $\gamma''(s) \perp \gamma'(s)$ (see 3.1). Therefore

❶ $$\mathrm{T}'(s) = k(s)\cdot\mathrm{N}(s).$$

for some real number $k(s)$; the value $k(s)$ is called *signed curvature* of γ at s. We may use notation $k(s)_\gamma$ if we need to specify the curve γ.

Note that
$$\kappa(s) = |k(s)|;$$
that is, up to sign, the signed curvature $k(s)$ equals to the curvature $\kappa(s)$ of γ at s defined on page 21; the sign tells which direction it turns — if γ turns left, then $k > 0$. If we want to emphasise that we work with *nonsigned* curvature of the curve, we call it *absolute curvature*.

Note that if we reverse the parametrization of γ or change the orientation of the plane, then the signed curvature changes its sign.

Since $\mathrm{T}(s), \mathrm{N}(s)$ is an orthonormal frame, we have that

$$\langle \mathrm{T}, \mathrm{T}\rangle = 1, \qquad \langle \mathrm{N}, \mathrm{N}\rangle = 1, \qquad \langle \mathrm{T}, \mathrm{N}\rangle = 0,$$

Differentiating these idenitites we get that

$$\langle \mathrm{T}', \mathrm{T}\rangle = 0, \qquad \langle \mathrm{N}', \mathrm{N}\rangle = 0, \qquad \langle \mathrm{T}', \mathrm{N}\rangle + \langle \mathrm{T}, \mathrm{N}'\rangle = 0,$$

By ❶, $\langle \mathrm{T}', \mathrm{N}\rangle = k$ and therefore $\langle \mathrm{T}, \mathrm{N}'\rangle = -k$. Whence we get

❷ $$\mathrm{N}'(s) = -k(s)\cdot \mathrm{T}(s).$$

The equations ❶ and ❷ are Frenet formulas for plane curves. They could be also written in a matrix form:

$$\begin{pmatrix} \mathrm{T}' \\ \mathrm{N}' \end{pmatrix} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{T} \\ \mathrm{N} \end{pmatrix}.$$

5.1. Exercise. *Let $\gamma_0\colon [a,b] \to \mathbb{R}^2$ be a smooth regular curve and* T *its tangent indicatrix. Consider another curve $\gamma_1\colon [a,b] \to \mathbb{R}^2$ defined by $\gamma_1(t) = \gamma_0(t) + \mathrm{T}(t)$. Show that*

$$\operatorname{length}\gamma_0 \leqslant \operatorname{length}\gamma_1.$$

The curves γ_0 and γ_1 in the exercise above describe tracks of idealized bicycle with the distance 1 from rear to front wheel. Thus by the exercise, the front wheel have to have the longer track. For more on geometry of bicycle tracks see [10] and the references there in.

Fundamental theorem of plane curves

5.2. Theorem. *Let $k(s)$ be a smooth real valued function defined on a real interval $\mathbb{I}$. Then there is a smooth unit-speed curve $\gamma\colon \mathbb{I} \to \mathbb{R}^2$ with signed curvature $k(s)$ at every s. Moreover γ is uniquely defined up to a rigid motion of the plane.*

This is the fundamental theorem of plane curves; it is direct analog of 4.10 and it can be proved along the same lines. We give a slightly simpler proof.

Proof. Fix $s_0 \in \mathbb{I}$. Consider the function

$$\theta(s) = \int\limits_{s_0}^{s} k(t)\cdot dt.$$

Note that by the fundamental theorem of calculus, we have $\theta'(s) = k(s)$ for any s.

Set

$$\mathrm{T}(s) = (\cos[\theta(s)], \sin[\theta(s)])$$

and let $\mathrm{N}(s)$ be its counterclockwise rotation by angle $\frac{\pi}{2}$; so

$$\mathrm{N}(s) = (-\sin[\theta(s)], \cos[\theta(s)]).$$

Consider the curve

$$\gamma(s) = \int_{s_0}^{s} \mathrm{T}(s)\cdot ds.$$

Since $|\gamma'| = |\mathrm{T}| = 1$, the curve γ is unit-speed and T, N is its Frenet frame.

Note that

$$\begin{aligned}\gamma''(s) = \mathrm{T}'(s) &= \\ &= (\cos[\theta(s)]', \sin[\theta(s)]') = \\ &= \theta'(s)\cdot(-\sin[\theta(s)], \cos[\theta(s)]) = \\ &= k(s)\cdot\mathrm{N}(s).\end{aligned}$$

That is, $k(s)$ is the signed curvature of γ at s.

The existence is proved; it remains to prove uniqueness.

Assume γ_1 and γ_2 are two curves that satisfy the assumptions of the theorem. Applying a rigid motion, we can assume that $\gamma_1(s_0) = \gamma_2(s_0) = 0$ and the Frenet frame of both curves at s_0 is formed by the coordinate frame $(1,0), (0,1)$. Let us denote by $\mathrm{T}_1, \mathrm{N}_1$ and $\mathrm{T}_2, \mathrm{N}_2$ the Frenet frames of γ_1 and γ_2 correspondingly. The triples $\gamma_i, \mathrm{T}_i, \mathrm{N}_i$ satisfy the same system system of ordinary differential equations

$$\begin{cases}\gamma_i' = \mathrm{T}_i,\\ \mathrm{T}_i' = k\cdot\mathrm{N}_i,\\ \mathrm{N}_i' = -k\cdot\mathrm{T}_i.\end{cases}$$

Motreover, they have the same the initial values at s_0. Therefore $\gamma_1 = \gamma_2$. □

Note that from the proof of theorem we obtain the following corollary:

5.3. Corollary. *Suppose $\gamma\colon \mathbb{I} \to \mathbb{R}^2$ is a smooth unit-speed curve and $s_0 \in \mathbb{I}$. Denote by k the signed curvature of γ. Assume an oriented (x,y)-coordinate system on is chosen in such a way that $\gamma(s_0)$ is the origin and $\gamma'(s_0)$ points in the direction of x-axis. Then*

$$\gamma'(s) = (\cos[\theta(s)], \sin[\theta(s)])$$

where

$$\theta(s) = \int_{s_0}^{s} k(t)\cdot dt.$$

Total signed curvature

Let $\gamma\colon \mathbb{I}\to\mathbb{R}^2$ be a smooth unit-speed plane curve. The integral of its signed curvature is called *total signed curvature* and it denoted by $\Psi(\gamma)$; so

❸ $$\Psi(\gamma)=\int\limits_{\mathbb{I}} k(s)\cdot ds,$$

where k denotes signed curvature of γ.

If $\mathbb{I}=[a,b]$, then
$$\Psi(\gamma)=\theta(b)-\theta(a),$$

where θ is as in 5.3.

If γ is a piecewise smooth and regular plane curve, then we define its total signed curvature as the sum of total signed curvatures of its arcs plus the sum of signed external angles at the joints; it is positive if γ turns left, negative if γ turns right, 0 if it goes straight and undefined if it turns backward. That is, if γ is a concatenation of smooth and regular arcs $\gamma_1,\dots,\gamma_n$ then

$$\Psi(\gamma)=\Psi(\gamma_1)+\dots+\Psi(\gamma_n)+\theta_1+\dots+\theta_{n-1}$$

where θ_i is the signed external angle at the joint between γ_i and γ_{i+1}. If γ is closed, then the concatenation is cyclic and

$$\Psi(\gamma=\Psi(\gamma_1)+\dots+\Psi(\gamma_n)+\theta_1+\dots+\theta_n,$$

where θ_n is the signed external angle at the joint between γ_n and γ_1.

Since $\left|\int k(s)\cdot ds\right|\leqslant\int|k(s)|\cdot ds$, we have that

❹ $$|\Psi(\gamma)|\leqslant\Phi(\gamma)$$

for any smooth regular plane curve γ; that is, total signed curvature can not exceed total curvature by absolute value.

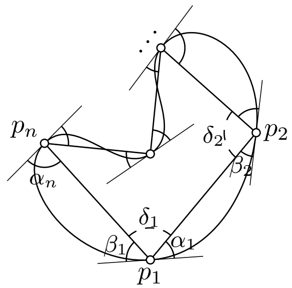
5.4. Proposition. *The total signed curvature of any closed simple smooth regular plane curve γ is $\pm2\cdot\pi$; it is $+2\cdot\pi$ if the region bounded by γ lies on the left from it and $-2\cdot\pi$ otherwise.*

Moreover the same statement holds for any closed piecewise simple smooth regular plane curve γ if its total signed curvature is defined.

This proposition is called sometimes *Umlaufsatz*; it is a differential-geometric analog of the theorem about sum of the internal angles of a polygon (A.23) which we use in the proof. A more conceptual proof was given by Heinz Hopf [11], [12, p. 42].

Proof. Without loss of generality we may assume that γ is oriented in such a way that the region bounded by γ lies on the left from it. We can also assume that it parametrized by arc length.

Consider a closed polygonal line $p_1 \dots p_n$ inscribed in γ. We can assume that the arcs between the vertexes are sufficiently small; in this case the polygonal line is simple and each arc γ_i from p_i to p_{i+1} have small total absolute curvature, say $\Phi(\gamma_i) < \pi$ for each i.



As usual we use indexes modulo n, in particular $p_{n+1} = p_1$. Assume $p_i = \gamma(t_i)$. Set

$$\begin{aligned} \mathrm{w}_i &= p_{i+1} - p_i, & \mathrm{v}_i &= \gamma'(t_i), \\ \alpha_i &= \measuredangle(\mathrm{v}_i, \mathrm{w}_i), & \beta_i &= \measuredangle(\mathrm{w}_{i-1}, \mathrm{v}_i), \end{aligned}$$

where $\alpha_i, \beta_i \in (-\pi, \pi]$ are signed angles — α_i is positive if w_i points to the left from v_i.

By ❸, the value

❺ $$\Psi(\gamma_i) - \alpha_i - \beta_{i+1}$$

is a multiple of $2{\cdot}\pi$. Since $\Phi(\gamma_i) < \pi$, by chord lemma (3.11), we also have that $|\alpha_i| + |\beta_i| < \pi$. By ❹, we have that $|\Psi(\gamma_i)| \leqslant \Phi(\gamma_i)$; therefore the value in ❺ vanishes, or equivalently

$$\Psi(\gamma_i) = \alpha_i + \beta_{i+1}$$

for each i.

Note that

❻ $$\delta_i = \pi - \alpha_i - \beta_i$$

is the internal angle of $p_1 \dots p_n$ at p_i; $\delta_i \in (0, 2{\cdot}\pi)$ for each i. Recall that the sum of internal angles of an n-gon is $(n-2){\cdot}\pi$ (see A.23); that is,

$$\delta_1 + \dots + \delta_n = (n-2){\cdot}\pi.$$

Therefore

❼ $$\begin{aligned} \Psi(\gamma) &= \Psi(\gamma_1) + \dots + \Psi(\gamma_n) = \\ &= (\alpha_1 + \beta_2) + \dots + (\alpha_n + \beta_1) = \\ &= (\beta_1 + \alpha_1) + \dots + (\beta_n + \alpha_n) = \\ &= (\pi - \delta_1) + \dots + (\pi - \delta_n) = \\ &= n{\cdot}\pi - (n-2){\cdot}\pi = \\ &= 2{\cdot}\pi. \end{aligned}$$

The piecewise smooth and regular curve is done the same way; we need to subdivide the arcs in the cyclic concatenation further to meet the requirement above and instead of equation ❻ we have

$$\delta_i = \pi - \alpha_i - \beta_i - \theta_i,$$

where θ_i is the signed external angle at p_i; it vanishes if the curve γ is smooth at p_i. Therefore instead of equation ❼, we have

$$\begin{aligned}\Psi(\gamma) &= \Psi(\gamma_1) + \dots + \Psi(\gamma_n) + \theta_1 + \dots + \theta_n = \\ &= (\alpha_1 + \beta_2) + \dots + (\alpha_n + \beta_1) = \\ &= (\beta_1 + \alpha_1 + \theta_1) + \dots + (\beta_n + \alpha_n + \theta_n) = \\ &= (\pi - \delta_1) + \dots + (\pi - \delta_n) = \\ &= n{\cdot}\pi - (n-2){\cdot}\pi = \\ &= 2{\cdot}\pi.\end{aligned}$$

□

5.5. Exercise. *Draw a smooth regular closed plane curve with zero total signed curvature.*

5.6. Exercise. *Let $\gamma\colon [a,b] \to \mathbb{R}$ be a smooth regular plane curve with Frenet frame T, N. Given a real parameter ℓ, consider the curve $\gamma_\ell(t) = \gamma(t) + \ell{\cdot}\mathrm{N}(t)$; it is called a* parallel curve of γ on the signed distance ℓ.

(a) Show that γ_ℓ is a regular curve if $\ell{\cdot}k(t) \neq 1$ for any t, where $k(t)$ denotes the signed curvature of γ.

(b) Set $L(\ell) = \operatorname{length}\gamma_\ell$. Show that

$$L(\ell) = L(0) - \ell{\cdot}\Psi(\gamma)$$

for all ℓ sufficiently close to 0. Describe an example showing that this formula does not hold for all ℓ.

Osculating circline

5.7. Proposition. *Given a point p, a unit vector T and a real number k, there is a unique smooth unit-speed curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ that starts at p in the direction of T and has constant signed curvature k.*

Moreover, if $k = 0$, then $\sigma(s) = p + s{\cdot}\mathrm{T}$ which runs along the line; if $k \neq 0$, then σ runs around the circle of radius $\frac{1}{|k|}$ and center $p + \frac{1}{k}{\cdot}\mathrm{N}$, where T, N is an oriented orthonoral frame.

Further we will use the term *circline* for *a circle or a line*; these are the only plane curves with constant signed curvature.

Proof. The proof is done by calculation based on 5.2 and 5.3.

Suppose $s_0 = 0$, choose coordinate system such that p is its origin, T points in the direction of x-axis and therefore N points in the direction of y-axis. Then

$$\begin{aligned}\theta(s) &= \int_0^s k\cdot dt = \\ &= k\cdot s.\end{aligned}$$

Therefore

$$\sigma'(s) = (\cos[k\cdot s], \sin[k\cdot s]).$$

It remains to integrate the last identity. If $k = 0$, we get

$$\sigma(s) = (s, 0)$$

which describes the line $\sigma(s) = p + s\cdot\mathrm{T}$.

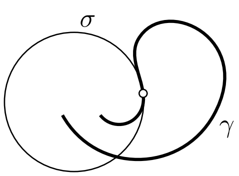
If $k \neq 0$, we get

$$\sigma(s) = (\tfrac{1}{k}\cdot\sin[k\cdot s], \tfrac{1}{k}\cdot(1 - \cos[k\cdot s])).$$

which is the circle of radius $r = \frac{1}{|k|}$ centered at $(0, \frac{1}{k}) = p + \frac{1}{k}\cdot\mathrm{N}$. □

5.8. Definition. *Let γ be a smooth unit-speed plane curve; denote by $k(s)$ the signed curvature of γ at s.*

The unit-speed curve σ of constant signed curvature $k(s)$ that starts at $\gamma(s)$ in the direction $\gamma'(s)$ is called the osculating circline *of γ at s.*



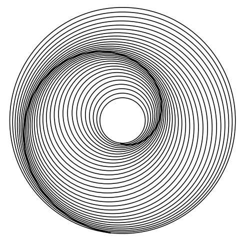
The center and radius of the osculating circle at a given point are called *center of curvature* and *radius of curvature* of the curve at that point.

The osculating circle σ_s can be also defined as the (necessarily unique) circline that has *second order of contact* with γ at s; that is, $\rho(\ell) = o(\ell^2)$, where $\rho(\ell)$ denotes the distance from $\gamma(s+\ell)$ to σ_s.

Spiral lemma

The following lemma was proved by Peter Tait [13] and later rediscovered by Adolf Kneser [14].

5.9. Lemma. *Assume that γ is a smooth regular plane curve with strictly decreasing positive signed curvature. Then the osculating circles of γ are nested; that is, if σ_s denoted the osculating circle of γ at s, then σ_{s_0} lies in the open disc bounded by σ_{s_1} for any $s_0 < s_1$.*



It turns out that osculating circles of the curve γ give a peculiar foliation of an annulus by circles; it has the following property: if a smooth function is constant on each osculating circle it must be constant in the annulus [see 15, Lecture 10]. Also note that the curve γ is tangent to a circle of the foliation at each of its points. However, it does not run along a circle.

Proof. Let $\mathrm{T}(s), \mathrm{N}(s)$ be the Frenet frame, $\omega(s)$ the curvature center and $r(s)$ the radius of curvature of γ at s. By 5.7,

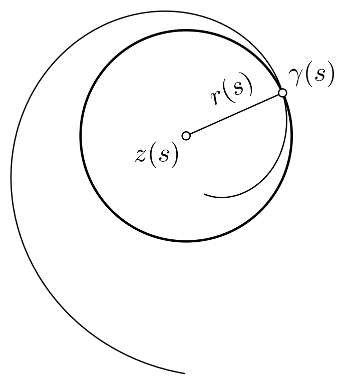
$$\omega(s) = \gamma(s) + r(s)\cdot \mathrm{N}(s).$$

Since $k > 0$, we have that $r(s)\cdot k(s) = 1$. Therefore applying Frenet formula ❷, we get that

$$\begin{aligned}\omega'(s) &= \gamma'(s) + r'(s)\cdot\mathrm{N}(s) + r(s)\cdot\mathrm{N}'(s) = \\ &= \mathrm{T}(s) + r'(s)\cdot\mathrm{N}(s) - r(s)\cdot k(s)\cdot\mathrm{T}(s) = \\ &= r'(s)\cdot\mathrm{N}(s).\end{aligned}$$

Since $k(s)$ is decreasing, $r(s)$ is increasing; therefore $r' \geqslant 0$. It follows that $|\omega'(s)| = r'(s)$ and $\omega'(s)$ points in the direction of $\mathrm{N}(s)$.

Since $\mathrm{N}'(s) = -k(s)\cdot\mathrm{T}(s)$, the direction of $\omega'(s)$ cannot have constant direction on a nontrivial interval; that is, the curve $s \mapsto \omega(s)$ contains no line segments. It follows that



$$\begin{aligned}|\omega(s_1) - \omega(s_0)| &< \mathrm{length}(\omega|_{[s_0,s_1]}) = \\ &= \int_{s_0}^{s_1} |\omega'(s)|\cdot ds = \\ &= \int_{s_0}^{s_1} r'(s)\cdot ds = \\ &= r(s_1) - r(s_0).\end{aligned}$$

In other words, the distance between the centers of σ_{s_1} and σ_{s_0} is strictly less than the difference between their radiuses. Therefore the osculating circle at s_0 lies inside the osculating circle at s_1 without touching it. $\square$

The curve $s \mapsto \omega(s)$ is called *evolute* of γ; it traces the centers of curvature of the curve. The evolute of γ can be written as

$$\omega(t) = \gamma(t) + \tfrac{1}{k(t)}\cdot\mathrm{N}(t)$$

and in the proof we showed that $(\frac{1}{k})' \cdot \mathrm{N}$ is its velocity vector.

5.10. Exercise. *Show that the stretched astroid*

$$\alpha(t) = (\tfrac{a^2-b^2}{a} \cdot \cos^3 t, \tfrac{b^2-a^2}{b} \cdot \sin^3 t)$$

is an evolute of the ellipse $\gamma(t) = (a \cdot \cos t, b \cdot \sin t)$.

The following theorem states formally that *if you drive on the plane and turn the steering wheel to the right all the time, then you will not be able to come back to the same place.*

5.11. Theorem. *Assume γ is a smooth regular plane curve with positive and strictly monotonic signed curvature. Then γ is simple.*

The same statement holds without assuming positivity of curvature; the proof requires only minor modifications.

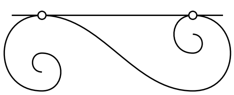
Proof of 5.11. Note that $\gamma(s)$ lies on the osculating circle σ_s of γ at s. If $s_1 \neq s_0$, then by lemma σ_{s_0} does not intersect σ_{s_1}. Therefore $\gamma(s_1) \neq \gamma(s_0)$, hence the result. $\square$

5.12. Exercise. *Show that a 3-dimensional analog of the theorem does not hold. That is, there are self-intersecting smooth regular space curves with strictly monotonic curvature.*

5.13. Exercise. *Assume that γ is a smooth regular plane curve with positive strictly monotonic signed curvature.*

(a) Show that no line can be tangent to γ at two distinct points.

(b) Show that no circle can be tangent to γ at three distinct points.



Note that part (a) does not hold if we allow the curvature to be negative; an example is shown on the diagram.

Supporting curves

Suppose γ_1 and γ_2 are smooth regular plane curves. Recall that the curves are tangent if $\gamma_1(t_1) = \gamma_2(t_2)$ for some time parameters t_1 and t_2 and they share the tangent line at these time parameters; that is, the tangent lines of γ_1 at t_1 coincides with the tangent line γ_2 at t_2.

Suppose γ is a smooth regular plane curve. Recall that a circline σ is tangent to γ at t_0 if $\gamma(t_0) = \sigma(t_1)$ for some t_1 and they share the tangent at these time parameters; that is, the tangent lines of γ at t_0 coincides with the tangent line σ at t_1.

We can (and often will) assume that tangent circline is *cooriented* with the curve; that is, the tangent vectors $\gamma'(t_0)$ and $\sigma'(t_1)$ point in the same direction. If not we can reverse the parametrization of σ. If both curves are given with arc-length parametrization, then coorientation means that $\gamma'(t_0) = \sigma'(t_1)$.

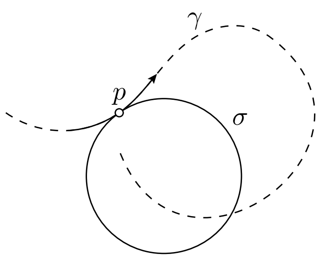
If γ is simple we can say that σ is tangent to γ at the point $p = \gamma(t_0)$ without ambiguity.

A circline σ supports γ at t_0 if $\gamma(t_0) \in \sigma$ and γ lies on one side of σ. (We assume that t_0 is not an end point of the interval of parameters.) If $p = \sigma(t_0)$ for a single value t_0, then we can also say *σ supports γ at p* without ambiguity.

Note that if σ supports γ at t_0, then σ is tangent to γ at t_0. Indeed, if it is not the case, then $\gamma'(t_0)$ would point inside or outside of σ; therefore γ would cross σ from one side to another. Therefore we can assume that σ is cooriented with γ at t_0. In this case we say that σ supports γ from the left (right) if γ lies on the right (correspondingly left) side from σ.

Note that a circle supports itself on the right and left at the same time at any point.

We say that a circle σ *locally supports* a curve γ at s if it supports its arc $\gamma|_{(s-\varepsilon,s+\varepsilon)}$ for some $\varepsilon > 0$. The same definitions for local support on the left and right are applied.



The circle σ on the diagram locally supports curve γ on the right at p, but does not support it globally — since γ crosses σ at a latter time.

We say that a smooth regular plane curve γ has a *vertex* at s if the signed curvature function is critical at s; that is, if $k'(s)_\gamma = 0$. If γ is simple we could say that the point $p = \gamma(s)$ is a vertex of γ without ambiguity.

5.14. Exercise. *Assume that osculating circle σ_s of a smooth regular plane curve γ supports γ at s. Show that γ has a vertex at s.*

Supporting test

The following proposition resembles the second derivative test.

5.15. Proposition. *Assume σ is a circline that locally supports γ at t_0 from the right (correspondingly left). Suppose σ is cooriented to γ at t_0. Then*

$$k(t_0)_\gamma \geqslant k_\sigma \quad (\textit{correspondingly} \quad k(t_0)_\gamma \leqslant k_\sigma).$$

where k_σ is the signed curvature of σ and $k(t_0)_\gamma$ is the signed curvature of γ at t_0.

A partial converse also holds. Namely, suppose a unit-speed circline σ with signed curvature k_σ starts at $\gamma(t_0)$ in the direction $\gamma'(t_0)$. Then σ locally supports γ at t_0 from the right (correspondingly left) if

$$k(t_0)_\gamma > k_\sigma \quad (\textit{correspondingly} \quad k(t_0)_\gamma < k_\sigma).$$

Proof. We prove only the case $k_\sigma > 0$. The 2 remaining cases $k_\sigma = 0$ and $k_\sigma < 0$ can be done essentially the same way.

Since $k_\sigma \neq 0$, the curve σ is a circle. According to Proposition 5.7, σ has radius $r_\sigma = \frac{1}{k_\sigma}$ and it is centered at

$$z = \gamma(t_0) + r \cdot \mathrm{N}(t_0).$$

Consider the function

$$f(t) = |z - \gamma(t)|^2 - \tfrac{1}{k_\sigma^2}.$$

Note that $f(t) \leqslant 0$ (correspondingly $f(t) \geqslant 0$) if and only if $\gamma(t)$ lies on the closed left (correspondingly right) side from σ. It follows that

- if σ locally supports γ at t_0 from the right, then

$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) \leqslant 0;$$

- if σ locally supports γ at t_0 from the left, then

$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) \geqslant 0;$$

- if

$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) < 0,$$

  then σ locally supports γ at t_0 from the right;
- if

$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) > 0,$$

  then σ locally supports γ at t_0 from the left;

Direct calculations show that

$$
\begin{aligned}
f(t_0) &= 0;\\
f'(t_0) &= \langle z-\gamma(t), z-\gamma(t)\rangle'|_{t=t_0} =\\
&= -2\cdot\langle\gamma'(t_0), z-\gamma(t_0)\rangle =\\
&= -2\cdot r\cdot\langle\gamma'(t_0), \mathrm{N}(t_0)\rangle =\\
&= 0;\\
f''(t_0) &= \langle z-\gamma(t), z-\gamma(t)\rangle''|_{t=t_0} =\\
&= 2\cdot\left(\langle\gamma'(t_0),\gamma'(t_0)\rangle - \langle\gamma''(t_0), z-\gamma(t_0)\rangle\right) =\\
&= 2\cdot\left(\langle\mathrm{T}(t_0),\mathrm{T}(t_0)\rangle - r\cdot k(t_0)_\gamma\cdot\langle\mathrm{N}(t_0),\mathrm{N}(t_0)\rangle\right) =\\
&= 2\cdot\left(1-\frac{k(t_0)_\gamma}{k_\sigma}\right).
\end{aligned}
$$

Hence the result. □

5.16. Exercise. *Assume a closed smooth regular plane curve γ runs between parallel lines on distance 2 from each other. Show that there is a point on γ with absolute curvature at least 1.*

5.17. Exercise. *Assume a closed smooth regular plane curve γ runs inside of a triangle $\triangle$ with inradius 1; that is, the inscribed circle of $\triangle$ has radius 1. Show that there is a point on γ with absolute curvature at least* 1*.*

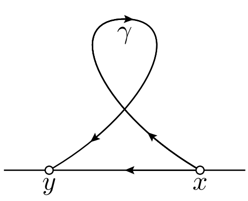
Convex curves

Recall that a plane curve is convex if it bounds a convex region.

5.18. Proposition. *Suppose that a closed simple curve γ bounds a figure F. Then F is convex if and only if the signed curvature of γ does not change sign.*

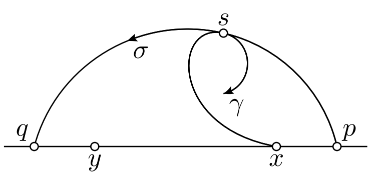
5.19. Lens lemma. *Let γ be a smooth regular simple curve that runs from x to y and distinct from the line segment from x to y. Assume that γ runs on the closed right side (correspondingly left side) of the oriented line xy and only its end points x and y lie on the line. Then γ has a point with positive (correspondingly negative) signed curvature.*

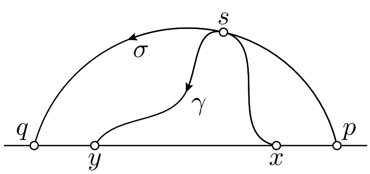
Note that the lemma fails for curves with self-intersections; the curve γ on the diagram always turns right, so it has negative curvature everywhere, but it lies on the right side of the line xy.



Proof. Choose points p and q on ℓ so that the points p, x, y, q appear in the same order on ℓ. We can assume that p and q lie sufficiently far from x and y, so the half-disc with diameter pq contains γ.

Consider the smallest disc segment with chord $[pq]$ that contains γ. Note that its arc σ supports γ at a point $s = \gamma(t_0)$.





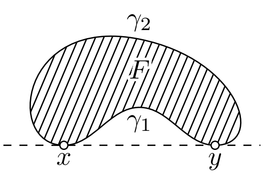
Note that the $\gamma'(t_0)$ is tangent to σ at s. Let us parameterise σ from p to q. Then γ and σ are cooriented as s. If not, then the arc of γ from s to y would be trapped in the curvelinear triangle xsp bounded by arcs of σ, γ and the line segment $[px]$. But this is impossible since y does not belong to this triangle.

It follows that σ supports γ at t_0 from the right. By 5.15,

$$k(t_0)_\gamma \geqslant k_\sigma,$$

Evidently $k_\sigma > 0$, hence the result. $\square$

Remark. Instead of taking minimal disc segment, one can take a point s on γ that maximize the distance to the line xy. The same argument shows that curvature at s is nonnegative, which is slightly weaker than the required positive curvature.



Proof of 5.18. If F is convex, then every tangent line of γ supports γ. If a point moves along γ, the figure F has to stay on one side from its tangent line; that is, we can assume that each tangent line supports γ on one side, say on the right. Applying the supporting test (5.15), we get that $k \geqslant 0$ at each point.

Now assume F is not convex. Then there is a line that supports γ at two points, say x and y that divide γ in two arcs γ_1 and γ_2, both distinct from the line segment xy. Note the one of the arcs is parametrized from x to y and the other from y to x. Applying the lens lemma, we get that the arcs γ_1 and γ_2 contain points with curvatures of opposite signs.

That is, if F is not convex, then curvature of γ changes sign. Equivalently: if curvature of γ does not change sign then F is convex. $\square$

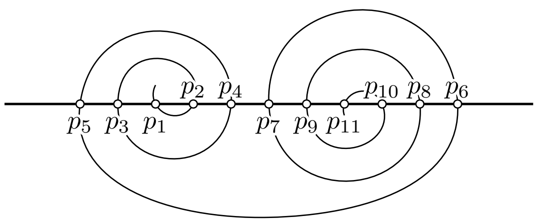
5.20. Exercise. *Suppose γ is a smooth regular simple closed convex plane curve of diameter bigger than 2. Show that γ has a point with absolute curvature less than 1.*

5.21. Exercise. *Suppose γ is a simple smooth regular curve in the plane with positive curvature. Assume γ crosses a line ℓ at the points $p_1, p_2, \dots p_n$ and these points appear on γ in that same order.*

(a) Show that p_2 cannot lie between p_1 and p_3 on ℓ.

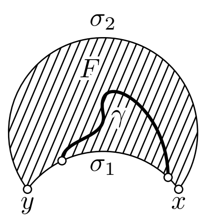
(b) Show that if p_3 lies between p_1 and p_2 on ℓ then the points appear on ℓ in the following order:

$$p_1, p_3, \dots, p_4, p_2.$$



(c) Try to describe all possible orders when p_1 lies between p_2 and p_3 (see the diagram).

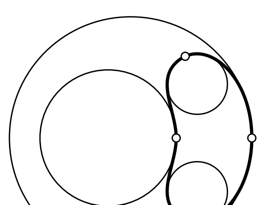
5.22. Exercise. *Let F be a plane figure bounded by two circle arcs σ_1 and σ_2 of signed curvature 1 that run from x to y. Suppose σ_1 is a shorter than σ_2. Assume a simple arc γ runs in F and has the end points on σ_1. Show that the absolute curvature of γ is at least 1 at some parameter value.*



Moon in a puddle

5.23. Theorem. *Assume γ is a simple closed smooth regular plane curve. Then at least two of its osculating circles support γ from the left and at least two from the right.*

The diagram shows for supporting osculating circles, two from inside and two outside the curve for the given curve.

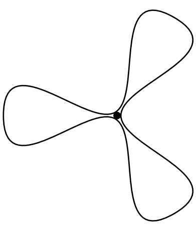


The above theorem is a slight generalization of the following theorem proved by Vladimir Ionin and German Pestov in [16]:

5.24. Theorem. *Assume γ is a simple closed smooth regular plane curve of absolute curvature bounded by 1. Then it surrounds a unit disc.*

This theorem is a direct corollary of 5.23; indeed, since absolute curvature is bounded by 1, every osculating circle has radius at least 1 and by 5.23 two pf these circles are surrounded by γ.

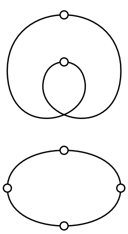
This theorem gives a simple but nontrivial example of the so-called *local to global theorems* — based on some local data (in this case the curvature of a curve) we conclude a global property (in this case existence of a large disc surrounded by the curve). For convex curves, this result was known earlier [17, §24].



A straightforward approach to the latter theorem would be to start with some disc in the region bounded by the curve and blow it up to maximize its radius. However, as one may see from the diagram it does not always lead to a solution a closed plane curve of absolute curvature bounded by 1 may surround a disc of radius smaller than 1 that cannot be enlarged continuously.

Recall that a vertex of a smooth regular curve is defined as a critical point of its signed curvature; in particular, any local minimum (or maximum) of the signed curvature is a vertex.

According to 5.14, if an osculating circle supports the curve at the same point p, then p is a vertex. Therefore 5.23 implies existence of 4 vertexes of γ. That is, we proved the following theorem:



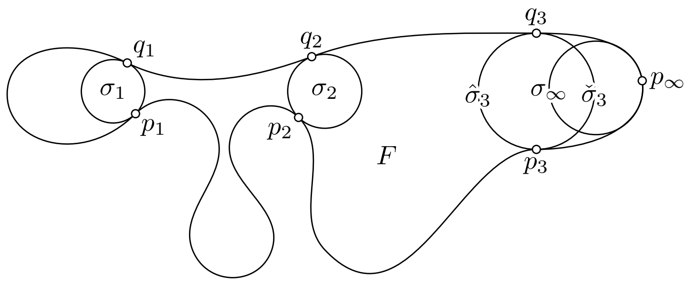
5.25. Four-vertex theorem. *Any smooth regular simple plane curve has at least four vertices.*

Evidently any closed curve has at least two vertexes — where the minimum and the maximum of the curvature are attained. On the diagram the vertexes are marked; the first curve has one self-intersection and exactly two vertexes; the second curve has exactly four vertexes and no self-intersections.

The four-vertex theorem was first proved by Syamadas Mukhopadhyaya [18] for convex curves. By now it has a large number of different proofs and generalizations. One of my favorite proofs was given by Robert Osserman [19]; the proof of Vladimir Ionin and German Pestov given below is even better.

Proof of 5.23. Denote by F the closed region surrounded by γ; as usual we parametrize γ so that F lies on the left from it.

First let us show that one osculating circle is supporting γ from the left; that is, it lies completely in F — this is the main part of the proof.



Assume contrary; that is, the osculating circle at each point $p \in \gamma$ does not lie in F. For each point $p \in \gamma$ let us consider the maximal circle σ that lies completely in F and tangent to γ at p; in other words, σ has minimal signed curvature among these circles. Note that σ has to touch γ at another point; otherwise we could increase its radius slightly while keeping the circle in F.

Fix a point p_1 and let σ_1 be the corresponding circle. Denote by γ_1 an arc of γ from p_1 to a first point q_1 on σ_1. Denote by $\hat\sigma_1$ and $\check\sigma_1$ two arcs of σ_1 from p_1 to q_1 such that the cyclic concatenation of $\hat\sigma_1$ and γ_1 surrounds $\check\sigma_1$.

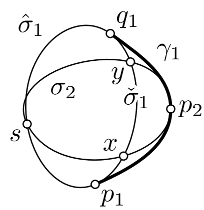
Let p_2 be the midpoint of γ_1 and σ_2 be the corresponding circle.

Note that σ_2 cannot intersect $\hat\sigma_1$. Otherwise, if σ_2 intersects $\hat\sigma_1$ at some point s, then σ_2 has two more common points with $\check\sigma_1$ — x and y, one for each arc of σ_2 from p_2 to s. That is, σ_1 and σ_2 have common point s, x and y. Therefore $\sigma_1 = \sigma_2$ as two circles with three common points. On the other hand, by construction $p_2 \in \sigma_2$ and $p_2 \notin \sigma_1$ — a contradiction.

Recall that σ_2 has to touch γ at another point. From above it follows that it can only touch γ_1 and therefore we can choose an arc $\gamma_2 \subset \gamma_1$ that runs from p_2 to a first point q_2 on σ_2. Note that by construction we have that

❽ $$\operatorname{length}\gamma_2 < \tfrac{1}{2}\cdot\operatorname{length}\gamma_1.$$

Let us repeat this construction recessively. We get an infinite sequence of arcs $\gamma_1 \supset \gamma_2 \supset$



Two ovals on the diagram pretend to be circles.

.... By ❽, we also get that

$$\operatorname{length}\gamma_n \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore the intersection

$$\bigcap_n \gamma_n$$

contains a single point; denote it by p_∞.

Let σ_∞ be the corresponding circle at p_∞; it has to touch γ at another point q_∞. The same argument as above shows that $q_\infty \in \gamma_n$ for any n. It follows that $q_\infty = p_\infty$ — a contradiction.

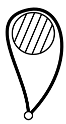
Now suppose that there is only one point $q \in \gamma$ at which osculating circle supports γ from the left. In this case for any point $p \ne q$ on γ the corresponding circle touches γ at another point.

Chose a point $p_1 \ne q$ on γ, take its corresponding circle σ_1 and note that there are two choices for arc γ_1 one of which does not contain q. Repeating the same construction starting from γ_1 we also arrive to a contradiction.

It remains to show that existence of a pair of osculating circlines that support γ form the right. This is done the same way, one only has to change the definition of corresponding circline — given $p \in \gamma$, it has to be the circline of maximal signed curvature that supports γ from the right at p. We leave it as an exercise:

5.26. Exercise. *List the necessary changes in the proof above for the existence of circlines that support γ form the right.* □

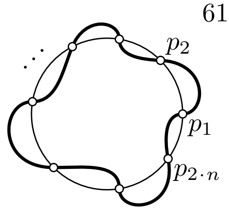
Theorem 5.24 admits the following generalization:



5.27. Theorem. *Let γ be a smooth regular simple plane loop. Suppose that absolute curvature of γ does not exceed 1. Then γ surrounds a unit circle.*

5.28. Exercise. *Describe the modifications in the proof of 5.23 which are necessary to prove 5.27.*

5.29. Exercise. *Assume that a closed smooth regular curve γ lies in a figure F bounded by a closed simple plane curve. Suppose that R is the maximal radius of discs that lies in F. Show that absolute curvature of γ is at least $\frac{1}{R}$ at some parameter value.*



5.30. Advanced exercise. *Suppose γ is a closed simple smooth regular plane curve and σ is a circle. Assume γ crosses σ at the points $p_1, \dots, p_{2\cdot n}$ and these points appear in the same cycle order on γ and on σ. Show that osculating circles at n distinct points of γ lie inside γ and that osculating circles at other n distinct points of γ lie outside of γ. In particular the curve γ has at least $2\cdot n$ vertexes.*

Construct an example of a closed simple smooth regular plane curve γ with only 4 vertexes that crosses a given circle at arbitrarily many points.

Recall that the *inverse* of a point x with respect to the unit circle centered at the origin is the point $\hat{x} = \frac{x}{|x|^2}$.

5.31. Advanced exercise. *Suppose γ is a smooth regular plane curve that does not pass thru the origin. Let $\hat{\gamma}$ be the inversion of γ in the unit circle centered at the origin. Show that osculating circline of $\hat{\gamma}$ at s is the inversion of osculating circline of γ at s. Conclude that every vertex of $\hat{\gamma}$ is the inversion of a vertex of γ.*

Note that the exercise provides an alternative way to finish the proof of 5.23 — once we proved the existence of two osculating circles that support γ from the left, we can apply to γ inversion with the center surrounded by γ. In this case the curve γ is mapped to a curve $\hat{\gamma}$, the domain inside γ is mapped to the domain outside $\hat{\gamma}$ and the other way around. It follows that if an osculating circle supports the obtained curve $\hat{\gamma}$ on the right then its inversion supports γ from the left and the other way around. That is, from the existance of two supporting circles on the left we also get the existance of two supporting circles on the right.

Part II

Surfaces

Chapter 6

Definitions

Topological surfaces

Few times we will need the following general definition.

A path connected subset Σ in a metric space is called a *surface* (more precisely an *embedded surface without boundary*) if any point of $p \in \Sigma$ admits a neighborhood W in Σ which is *homeomorphic* to an open subset in the Euclidean plane; that is, if there is an injective continuous map $U \to W$ from an open set $U \subset \mathbb{R}^2$ such that its inverse $W \to U$ is also continuous.

However, as well as in the case of curves, we will be mostly interested in smooth surfaces in the Euclidean space describe in the following section.

Smooth surfaces

Recall that a function f of two variables x and y is called *smooth* if all its partial derivatives $\frac{\partial^{m+n}}{\partial x^m \partial y^n} f$ are defined and are continuous in the domain of definition of f.

A path connected set $\Sigma \subset \mathbb{R}^3$ is called a *smooth surface* (or more precisely *smooth regular embedded surface*) if it can be described locally as a graph of a smooth function in an appropriate coordinate system.

More precisely, for any point $p \in \Sigma$ one can choose a coordinate system (x, y, z) and a neighborhood $U \ni p$ such that the intersection $W = U \cap \Sigma$ is formed by a graph $z = f(x, y)$ of a smooth function f defined in an open domain of the (x, y)-plane.

Examples. The simplest example of a surface is the (x, y)-plane

$$\Pi = \left\{ (x, y, z) \in \mathbb{R}^3 \;:\; z = 0 \right\}.$$

The plane Π is a surface since it can be described as the graph of the function $f(x,y)=0$.

All other planes are surfaces as well since one can choose a coordinate system so that it becomes the (x,y)-plane. We can also present a plane as a graph of a linear function $f(x,y)=a\cdot x+b\cdot y+c$ for some constants a, b and c (assuming the plane is not perpendicular to the (x,y)-plane).

A more interesting example is the unit sphere

$$\mathbb{S}^2=\left\{\,(x,y,z)\in\mathbb{R}^3\;:\;x^2+y^2+z^2=1\,\right\}.$$

This set is not the graph of any function, but $\mathbb{S}^2$ is locally a graph; in fact it can be covered by 6 graphs:

$$\begin{aligned}
z&=f_\pm(x,y)=\pm\sqrt{1-x^2-y^2},\\
y&=g_\pm(x,z)=\pm\sqrt{1-x^2-z^2},\\
x&=h_\pm(y,z)=\pm\sqrt{1-y^2-z^2};
\end{aligned}$$

each function $f_\pm,g_\pm,h_\pm$ is defined in an open unit disc. That is, $\mathbb{S}^2$ is a smooth surface.

Surfaces with boundary. A connected subset in a surface that is bounded by one or more curves is called *surface with boundary*; in this case the collection of curves is called the *boundary line* of the surface.

When we say *surface* we usually mean a *surface without boundary*; we may use the term *surface with possibly nonempty boundary* if we need to talk about surfaces with and without boundary.

More conventions. If the surface Σ is formed by a closed set, then it is called *proper.* For example, paraboloids

$$z=x^2+y^2,\qquad z=x^2-y^2$$

or sphere

$$x^2+y^2+z^2=1$$

are proper surfaces, while the open disc in a plane

$$\left\{\,(x,y,z)\in\mathbb{R}^3\;:\;x^2+y^2<1,z=0\,\right\}$$

is a surface which is not proper.

If compact surface without boundary is called *closed* (this term is closely related to *closed curve* but has nothing to do with *closed set*).

A proper noncompact surface without boundary is called *open* (again the term *open set* is not relevant).

For example, any paraboloid is an open surface; sphere is a closed surface.

Local parametrizations

Let U be an open domain in $\mathbb{R}^2$ and $s\colon U \to \mathbb{R}^3$ be a smooth map. We say that s is regular if its Jacobian has maximal rank; in this case it means that the vectors $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$ are linearly independent at any $(u, v) \in U$ or equivalently $\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v} \neq 0$, where $\times$ denotes the vector product.

6.1. Proposition. *If $s\colon U \to \mathbb{R}^3$ is a smooth regular embedding of an open connected set $U \subset \mathbb{R}^2$, then its image $\Sigma = s(U)$ is a smooth surface.*

Proof. Let $s(u, v) = (s_1(u, v), s_2(u, v), s_3(u, v))$. Since s is regular the Jacobian matrix

$$\begin{pmatrix} \frac{\partial s_1}{\partial u} & \frac{\partial s_1}{\partial v} \\ \frac{\partial s_2}{\partial u} & \frac{\partial s_2}{\partial v} \\ \frac{\partial s_3}{\partial u} & \frac{\partial s_3}{\partial v} \end{pmatrix}$$

has rank two.

Fix a point $p \in \Sigma$; by shifting the coordinate system we may assume that p is the origin. Permuting the coordinates x, y, z if necessary, we may assume that the matrix

$$\begin{pmatrix} \frac{\partial s_1}{\partial u} & \frac{\partial s_1}{\partial v} \\ \frac{\partial s_2}{\partial u} & \frac{\partial s_2}{\partial v} \end{pmatrix}$$

is invertible. Let $\bar{s}\colon U \to \mathbb{R}^2$ be the projection of s to the (x, y)-coordinate plane; that is, $\bar{s}(u, v) = (s_1(u, v), s_2(u, v))$. Note that the 2×2-matrix above is the Jacobian matrix of $\bar{s}$.

The inverse function theorem implies that there is a smooth regular function h defined on an open set $W \ni 0$ in the (x, y)-plane such that $h(0, 0) = (0, 0)$ and $\bar{s} \circ h$ is the identity map.

Note that the graph $z = s_3 \circ h(x, y)$ for $(x, y) \in W$ is a subset in Σ. Indeed, if $(u, v) = h(x, y)$, then $x = s_1(u, v)$ and $y = s_2(u, v)$. Therefore the identity $z = s_3 \circ h(x, y)$ can be rewritten as $(x, y, z) = s(u, v)$.

Clearly the graph $z = s_3 \circ h(x, y)$ for $(x, y) \in W$ is open in Σ; that is, Γ a neighborhood of p in Σ that can be described as a graph of a smooth function $f_3 \circ h\colon W \to \mathbb{R}$. Since p is arbitrary, we get that Σ is a surface. $\square$

If we have s and Σ as in the proposition, then we say that s is a *parametrization* of the surface Σ.

Not all the smooth surfaces can be described by such a parametrization; for example the sphere $\mathbb{S}^2$ cannot. But any smooth surface Σ admits a local parametrization; that is, any point $p \in \Sigma$ admits an open neighborhood $W \subset \Sigma$ with a smooth regular parametrization s. In this case any point in W can be described by two parameters, usually denoted by u and v, which are called *local coordinates* at p. The map s is called a *chart* of Σ.

If W is a graph $z = h(x, y)$ then the map $s\colon (u, v) \mapsto (u, v, h(u, v))$ is a chart. Indeed, s has an inverse $(u, v, h(u, v)) \mapsto (u, v)$ which is continuous; that is, s is an embedding. Further, $\frac{\partial s}{\partial u} = (1, 0, \frac{\partial h}{\partial u})$ and $\frac{\partial s}{\partial v} = (0, 1, \frac{\partial h}{\partial v})$, whence $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$ are linearly independent.

Note that from 6.1, we obtain the following corollary.

6.2. Corollary. *A path connected set $\Sigma \subset \mathbb{R}^3$ is a smooth regular surface if at any point $p \in \Sigma$ it has a local parametrization by a smooth regular map.*

6.3. Exercise. *Consider the following map*

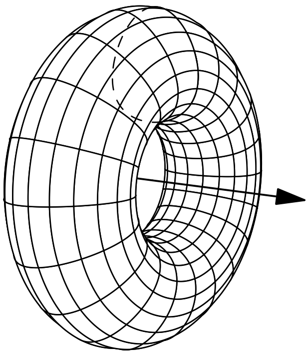
$$s(u, v) = (\tfrac{2\cdot u}{1+u^2+v^2}, \tfrac{2\cdot v}{1+u^2+v^2}, \tfrac{2}{1+u^2+v^2}).$$

Show that s is a chart of the unit sphere centered at $(0, 0, 1)$; describe the image of s.

The map

$$(u, v, 1) \mapsto (\tfrac{2\cdot u}{1+u^2+v^2}, \tfrac{2\cdot v}{1+u^2+v^2}, \tfrac{2}{1+u^2+v^2})$$

is called *stereographic projection.* Note that the point $(u, v, 1)$ and its image $(\tfrac{2\cdot u}{1+u^2+v^2}, \tfrac{2\cdot v}{1+u^2+v^2}, \tfrac{2}{1+u^2+v^2})$ lie on one half-line starting at the origin.



Let $\gamma(t) = (x(t), y(t))$ be a plane curve. Recall that the image of the map

$$(t, \theta) \mapsto (x(t), y(t)\cdot\cos\theta, y(t)\cdot\sin\theta)$$

is called the *surface of revolution* of the curve γ around the x-axis. For fixed t or θ the obtained curves are called meridian or correspondingly parallel of the surface of revolution; note that parallels are formed by circles in the plane perpendicular to the axis of rotation.

6.4. Exercise. *Assume γ is a closed simple smooth regular plane curve that does not intersect the x-axis. Show that surface of revolution of γ around the x-axis is a smooth regular surface.*

Golbal parametrizations

A surface can be described by an embedding from a known surface to the space. For example the ellipsoid

$$\Sigma_{a,b,c} = \left\{ (x,y,z) \in \mathbb{R}^3 \ : \ \tfrac{x^2}{a^2} + \tfrac{y^2}{b^2} + \tfrac{z^2}{c^2} = 1 \right\}$$

for some positive numbers a, b, c can be defined as the image of the map $s\colon \mathbb{S}^2 \to \mathbb{R}^3$, defined as the restriction of the map $(x,y,z) \mapsto$ $\mapsto (a\cdot x, b\cdot y, c\cdot z)$ to the unit sphere $\mathbb{S}^2$.

For a surface Σ, a map $s : \Sigma \to \mathbb{R}^3$ is called a *smooth parametrized surface* if for any chart $f\colon U \to \Sigma$ the composition $s \circ f$ is smooth and regular; that is, all partial derivatives $\frac{\partial^{m+n}}{\partial u^m \partial v^n}(s \circ f)$ exist and are continuous in the domain of definition and the following two vectors $\frac{\partial}{\partial u}(s \circ f)$ and $\frac{\partial}{\partial v}(s \circ f)$ are linearly independent.

Evidently the parametric definition includes the embedded surfaces defined previously — as the domain of parameters we can take the surface itself and the identity map as s. But parametrized surfaces are more general, in particular they might have self-intersections.

If Σ is a known surface for example a sphere or a plane, the parametrized surface $s\colon \Sigma \to \mathbb{R}^3$ might be called by the same name. For example, any embedding $s\colon \mathbb{S}^2 \to \mathbb{R}^3$ might be called a topological sphere and if s is smooth and regular, then it might be called smooth sphere. (A smooth regular map $s\colon \mathbb{S}^2 \to \mathbb{R}^3$ which is not necessary an embedding is called a *smooth regular immersion*, so we can say that s describes a smooth immersed sphere.) Similarly an embedding $s\colon \mathbb{R}^2 \to \mathbb{R}^3$ might be called topological plane, and if s is smooth, it might be called smooth plane.

Implicitly defined surfaces

6.5. Proposition. *Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose that 0 is a regular value of f; that is, $\nabla_p f \neq 0$ if $f(p) = 0$. Then any path connected component Σ of the set of solutions of the equation $f(x,y,z) = 0$ is a surface.*

Proof. Fix $p \in \Sigma$. Since $\nabla_p f \neq 0$ we have

$$\tfrac{\partial f}{\partial x}(p) \neq 0, \quad \tfrac{\partial f}{\partial y}(p) \neq 0, \quad \text{or} \quad \tfrac{\partial f}{\partial z}(p) \neq 0.$$

We may assume $\frac{\partial f}{\partial z}(p) \neq 0$; otherwise permute the coordinates x, y, z.

The implicit function theorem (A.10) implies that a neighborhood of p in Σ is the graph $z = h(x,y)$ of a smooth function h defined on an open domain in $\mathbb{R}^2$. It remains to apply the definition of smooth surface (page 63). $\square$

6.6. Exercise. *Describe the set of real numbers a such that the equation*

$$x^2 + y^2 - z^2 = a$$

describes a smooth regular surface.

Tangent plane

Let $z = f(x, y)$ be a local graph realization of a surface. Assume that a point $p = (x_p, y_p, z_p)$ lies on this graph, so $z_p = f(x_p, y_p)$. The plane spanned by the vectors $(1, 0, (\frac{\partial}{\partial x} f)(x_p, y_p))$ and $(0, 1, (\frac{\partial}{\partial y} f)(x_p, y_p))$ is called the *tangent plane* of Σ at p. The tangent plane to Σ at p is usually denoted by T_p or $\mathrm{T}_p\Sigma$. Vectors in T_p are called *tangent vectors* of Σ at p.

Tangent plane T_p might be considered as a linear subspace of $\mathbb{R}^3$ or as a parallel plane passing thru p. In the latter case it can be interpreted as the best approximation of the surface Σ by a plane at p; it has *first order of contact* with Σ at p; that is, $\rho(q) = o(|p-q|)$, where $q \in \Sigma$ and $\rho(q)$ denotes the distance from q to T_p.

6.7. Proposition. *Let Σ be a smooth surface. A vector w is a tangent vector of Σ at p if and only if there is a curve γ that runs in Σ and has w as a velocity vector at p.*

Note that according to the proposition the tangent plane $\mathrm{T}_p\Sigma$ can be defined as the set of all velocity vectors at p of smooth parameterized curves that run in Σ. In particular the tangent plane to a surface at a given point does not depend on the choice of its local graph representation.

Proof. We can assume that Σ is a graph $z = f(x, y)$; otherwise pass to a local presentation of Σ around p.

"Only if" part. Suppose that $(x(t), y(t))$ denotes the projection of $\gamma(t)$ to the (x, y)-plane. Since γ runs in Σ, we have that

$$\gamma(t) = \big(x(t), y(t), f(x(t), y(t))\big).$$

Therefore

$$\begin{aligned}\gamma' &= (x', y', \tfrac{\partial f}{\partial x}(x, y)\cdot x' + \tfrac{\partial f}{\partial y}(x, y)\cdot y') = \\ &= x'\cdot(1, 0, (\tfrac{\partial}{\partial x} f)(x, y)) + y'\cdot(0, 1, (\tfrac{\partial}{\partial y} f)(x, y)).\end{aligned}$$

That is, $\gamma'(t) \in \mathrm{T}_{\gamma(t)}$ for any t.

"If" part. Without loss of generality we can assume that p is the origin. Fix a tangent vector

$$w = a{\cdot}(1, 0, (\tfrac{\partial}{\partial x} f)(0, 0)) + b{\cdot}(0, 1, (\tfrac{\partial}{\partial y} f)(0, 0))$$

and consider the curve $\gamma(t) = (a{\cdot}t, b{\cdot}t, f(a{\cdot}t, b{\cdot}t))$. By construction γ runs in Σ and the direct calculations show that $\gamma'(0) = w$. $\square$

6.8. Exercise. *Assume $f\colon U \to \mathbb{R}^3$ is a smooth regular chart of a surface Σ and $p = f(u_0, v_0)$. Show that the tangent plane $\mathrm{T}_p\Sigma$ is spanned by vectors $\frac{\partial f}{\partial u}(u_0, v_0)$ and $\frac{\partial f}{\partial v}(u_0, v_0)$.*

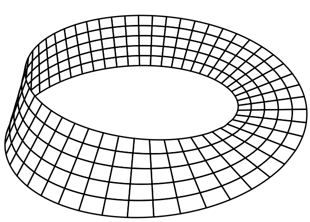
6.9. Exercise. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function with a regular value* 0 *and Σ is a surface described as a connected component of the set of solutions $f(x, y, z) = 0$. Show that the tangent plane $\mathrm{T}_p\Sigma$ is perpendicular to the gradient $\nabla_p f$ at any point $p \in \Sigma$.*

6.10. Exercise. *Let Σ be a smooth surface and $p \in \Sigma$. Fix an (x, y, z)-coordinates. Show that a neighborhood of p in Σ is a graph $z = f(x, y)$ of a smooth function f defined on an open subset in the (x, y)-plane if and only if the tangent plane T_p is not a* vertical plane*; that is, if the projection of T_p to the (x, y)-plane does not degenerate to a line.*

Normal vector and orientation

A unit vector that is normal to T_p is usually denoted by n_p; it is uniquely defined up to sign.

A surface Σ is called *oriented* if it is equipped with a unit normal vector field n; that is, a continuous map $p \mapsto n_p$ such that $n_p \perp \mathrm{T}_p$ and $|n_p| = 1$ for any p. The choice of the field n is called the *orientation* on Σ. A surface Σ is called *orientable* if it can be oriented. Note that each orientable surface admits two orientations n and $-n$.



Möbius strip shown on the diagram gives an example of a nonorientable surface — there is no choice of normal vector field that is continuous along the middle of the strip, when you go around it changes the sign.

Note that each surface is locally orientable. In fact each chart $f(u, v)$ admits an orientation

$$n = \frac{\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}}{\left|\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}\right|}.$$

Indeed the vectors $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are tangent vectors at p; since they are linearly independent, their vector product does not vanish and it is perpendicular to the tangent plane. Therefore $n(u, v)$ is a unit normal vector at $f(u, v)$; evidently $(u, v) \mapsto n(u, v)$ is a continuous map.

6.11. Exercise. *Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function with a regular value 0 and Σ is a surface described as a connected component of the set of solutions $h(x, y, z) = 0$. Show that Σ is orientable.*

The following claim should be intuitively obvious. Its proof is not at all trivial; a standard proof uses the so-called *Alexander's duality* which is a classical technique in algebraic topology.

6.12. Claim. *The complement of any open or closed surface in the Euclidean space has exactly two connected components.*

Note that if a proper surface is smooth, we can choose the unit normal field on it that points into one of the components of the complement. Therefore we obtain the following observation.

6.13. Observation. *Any smooth open or closed surface in Euclidean space is oriented.*

In particular it follows that the Möbius strip cannot be extended to an open or closed smooth surface without boundary.

Plane sections

6.14. Advanced exercise. *Show that any closed set in the plane can appear as an intersection of this plane with an open smooth regular surface.*

As a consequence of the exercise above, the plane sections of a smooth regular surface might look complicated. The following lemma makes it possible to perturb the plane so that the section becomes nice.

6.15. Lemma. *Let Σ be a smooth regular surface. Then for any plane Π there is an arbitrarily close parallel plane Π' such that each connected component of the intersection $\Sigma \cap \Pi'$ is a smooth regular curve.*

Proof. Assume Π is described by equation $f(x,y,z)=r_0$, where

$$f(x,y,z)=a{\cdot}x+b{\cdot}y+c{\cdot}z.$$

The surface Σ can be covered by a countable set of charts $s_i\colon U_i\to\Sigma$. Note that the composition $f\circ s_i$ is a smooth function. By Sard's lemma (A.11), almost all real numbers r are regular values of each $f\circ s_i$.

Fix such value r sufficiently close to r_0 and consider the plane Π' described by the equation $f(x,y,z)=r$. Note that $\Pi'\parallel\Pi$ and is arbitrarily close to it. Any point in the intersection $\Sigma\cap\Pi'$ lies in the image of one of the charts. From above it admits a neighborhood which is a regular smooth curve; hence the result. $\square$

Chapter 7

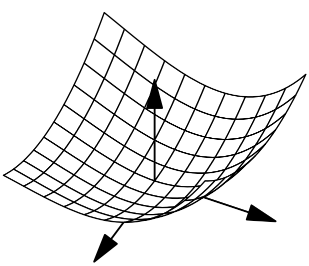
Curvatures

Tangent-normal coordinates

Fix a point p in a smooth surface Σ. Consider a coordinate system (x,y,z) with origin at p such that the (x,y)-plane coincides with T_p. By 6.10, we can present Σ locally around p as a graph of a function f. Note that f satisfies the following additional properties:

$$f(0,0)=0, \qquad (\tfrac{\partial}{\partial x}f)(0,0)=0, \qquad (\tfrac{\partial}{\partial y}f)(0,0)=0.$$

The first equality holds since $p=(0,0,0)$ lies on the graph and the last two equalities mean that the tangent plane at p is horizontal.



Consider the Hessian matrix

❶ $$M_p=\begin{pmatrix}\ell & m\\ m & n\end{pmatrix},$$

where

$$\begin{aligned}\ell&=(\tfrac{\partial^2}{\partial x^2}f)(0,0),\\ m&=(\tfrac{\partial^2}{\partial x\partial y}f)(0,0)=(\tfrac{\partial^2}{\partial y\partial x}f)(0,0),\\ n&=(\tfrac{\partial^2}{\partial y^2}f)(0,0).\end{aligned}$$

The components of the matrix describe the surface at up to the second order at p. In fact the so-called *osculating paraboloid*

$$z=\tfrac12(\ell{\cdot}x^2+2{\cdot}m{\cdot}x{\cdot}y+n{\cdot}y^2)$$

gives the best approximation of the surface at p; it has *second order of contact* with Σ at p.

Given two vectors v, w in the (x,y)-plane, consider the value

$$\mathrm{II}_p(v,w) := (D_w D_v f)(0,0),$$

where D denotes the directional derivative. The function $(v,w) \mapsto$ $\mapsto \mathrm{II}_p(v,w)$ is called the *second fundamental form* at p; it takes two tangent vectors v and w at p and spits out the real number $\mathrm{II}_p(v,w)$.

The second fundamental form can be written in terms of the Hessian matrix. Indeed if $w = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)$, then

$$D_w = a\cdot\frac{\partial}{\partial x} + b\cdot\frac{\partial}{\partial y} \quad \text{and} \quad D_v = c\cdot\frac{\partial}{\partial x} + d\cdot\frac{\partial}{\partial y}.$$

Therefore

❷
$$\begin{aligned}\mathrm{II}_p(w,v) &:= (D_w D_v f)(0,0) = \\ &= a\cdot c\cdot\ell + (a\cdot d + b\cdot c)\cdot m + b\cdot d\cdot n = \\ &= \langle M_p\cdot w, v\rangle = \\ &= \langle M_p\cdot v, w\rangle.\end{aligned}$$

Note that from ❷ it follows that II_p is symmetric; that is,

❸
$$\mathrm{II}_p(v,w) = \mathrm{II}_p(w,v)$$

for any two tangent vectors $v, w \in \mathrm{T}_p$.

Principle curvatures

Note that tangent-normal coordinates give an almost canonical coordinate system in a neighborhood of p; it is unique up to a rotation of the (x,y)-plane and a switching of the sign of the z-coordinate. Rotating the (x,y)-plane is equivalent to changing its orthonormal basis, which results in the rewriting the Hessian matrix in this new basis.

Since Hessian matrix is symmetric, it is diagonalizable by orthogonal matrices. That is, by rotating the (x,y)-plane we can assume that $m = 0$ in ❶. In this case the diagonal components of M_p are called *principle curvatures* of Σ at p; they are uniquely defined up to sign; they are denoted as $k_1(p)$ and $k_2(p)$, or $k_1(p)_\Sigma$ and $k_2(p)_\Sigma$ if we need to emphasize that these are the curvatures of the surface Σ. We will always assume that $k_1 \leqslant k_2$.

The principle curvatures can be also defined as the eigenvalues of M_p; the eigendirections of M_p are called *principle directions* of Σ at p. Note that if $k_1(p) \neq k_2(p)$ then p has exactly two principle directions, which are perpendicular to each other.

Note that if we revert the orientation of Σ, then the principle curvatures at each point switch their signs and indexes.

Normal curvature

Assume we choose the coordinates in the (x,y)-plane so that the Hessian matrix is diagonalized, we can assume that

$$M_p = \begin{pmatrix} k_1(p) & 0 \\ 0 & k_2(p) \end{pmatrix}.$$

According to ❷, the second directional derivative for a vector $w = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ in the (x,y)-plane can be written as

$$(D_w^2 f)(0,0) = a^2 \cdot k_1(p) + b^2 \cdot k_2(p).$$

If w is unit, then the second directional derivative $D_w^2 f(0,0)$ can be intepreted as the signed curvature of the curve formed by the intersection of Σ with the plane thru p spanned by n_p and w. In this case $\mathrm{II}_p(w,w) = D_w^2 f(0,0)$ is called *normal curvature* in the direction w; it is denoted by $k_w(p)$ or $k_w(p)_\Sigma$.

Since $|w| = 1$, we have $a^2 + b^2 = 1$ which implies the following:

7.1. Observation. *For any point p on an oriented smooth surface Σ, the principle curvatures $k_1(p)$ and $k_2(p)$ are correspondingly minimum and maximum of the normal curvatures at p. Moreover, if θ is the angle between a unit vector $w \in \mathrm{T}_p$ and the first principle direction at p, then*

$$k_w(p) = k_1(p)\cdot\cos^2\theta + k_2(p)\cdot\sin^2\theta.$$

The last identity is the so-called *Euler's formula.*

A smooth regular curve on a surface Σ that always runs in the principle directions is called a *line of curvature* of Σ.

7.2. Exercise. *Assume that a smooth surface Σ is mirror symmetric with respect to a plane Π. Suppose that Σ and Π intersect along a curve γ. Show that γ is a line of curvature of Σ.*

7.3. Exercise. *Assume V is a body of revolution in $\mathbb{R}^3$ and its boundary is a smooth surface with principle curvatures at most 1 in absolute value. Show that V contains a unit ball.*

More curvatures

Fix an oriented smooth surface Σ and a point $p \in \Sigma$.

The product

$$K(p) = k_1(p)\cdot k_2(p)$$

is called Gauss curvature at p. We may denote it by $K(p)_\Sigma$ if we need to emphasize that this is curvature of Σ. The Gauss curvature can be also interpreted as determinant of the Hessian matrix M_p.

The sum

$$H(p) = \tfrac{1}{2}\cdot(k_1(p) + k_2(p))$$

is called mean curvature at p. We may denote it by $H(p)_\Sigma$ if we need to emphasize that this is curvature of Σ. The mean curvature can be also interpreted as half of the trace of the Hessian matrix M_p.

Note that the Gauss curvature depends only on Σ and p, and not on the choice of the coordinate system. The same is true up to sign for the mean curvature — it changes the sign if we revert the orientation of the surface.

7.4. Exercise. *Show that any surface with positive Gauss curvature is orientable.*

Supporting surfaces

Assume two oriented surfaces Σ_1 and Σ_2 have a common point p. If there is a neighborhood U of p such that $\Sigma_1 \cap U$ lies on one side from Σ_2 in U, then we say that Σ_2 *locally supports* Σ_1 at p.

7.5. Exercise. *Let Σ_1 and Σ_2 be two smooth surfaces. Assume Σ_2* locally supports *Σ_1 at a point p. Show that $\mathrm{T}_p\Sigma_1 = \mathrm{T}_p\Sigma_2$; that is, the tangent planes of Σ_1 and Σ_2 at p coincide.*

By the exercise, we can assume that Σ_1 and Σ_2 are cooriented at p; that is, they have common unit normal vector at p. If not we can revert the orientation of one of the surfaces.

If Σ_1 and Σ_2 are cooriented at p, then we can say that Σ_1 locally supports Σ_2 from *inside* or from *outside*, assuming that the normal vector points *inside* the domain bounded by surface Σ_2 in U.

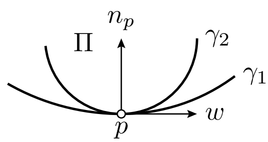
More precisely, we can use for Σ_1 and Σ_2 one tangent-normal coordinate system at p, assuming that the z-axis points in the direction of the unit normal vector n_p to both surfaces. This way we write Σ_1 and Σ_2 locally as graphs: $z = f_1(x,y)$ and $z = f_2(x,y)$ correspondingly. Then Σ_1 locally supports Σ_2 from inside (from outside) if $f_1(x,y) \geqslant f_2(x,y)$ (correspondingly $f_1(x,y) \leqslant f_2(x,y)$) for (x,y) in a sufficiently small neighborhood of the origin.

Note that Σ_1 locally supports Σ_2 from inside at the point p is equivalent to Σ_2 locally supports Σ_1 from outside. Further if we revert the orientation of both surfaces then supporting from inside becomes supporting from outside and the other way around.

7.6. Proposition. *Let Σ_1 and Σ_2 be oriented surfaces. Assume Σ_1 locally supports Σ_2 from inside at the point p (equivalently Σ_2 locally supports Σ_1 from outside). Then $k_1(p)_{\Sigma_1} \geqslant k_1(p)_{\Sigma_2}$ and $k_2(p)_{\Sigma_1} \geqslant$ $\geqslant k_2(p)_{\Sigma_2}$.*

7.7. Exercise. *Give an example of two surfaces Σ_1 and Σ_2 that have common point p with common unit normal vector n_p such that $k_1(p)_{\Sigma_1} > k_1(p)_{\Sigma_2}$ and $k_2(p)_{\Sigma_1} > k_2(p)_{\Sigma_2}$, but Σ_1 does not support Σ_2 locally at p.*

Proof. We can assume that Σ_1 and Σ_2 are graphs $z = f_1(x, y)$ and $z = f_2(x, y)$ in a common tangent-normal coordinates at p, so we have $f_1 \geqslant f_2$.



Fix a unit vector $w \in \mathrm{T}_p\Sigma_1 = \mathrm{T}_p\Sigma_2$. Consider the plane Π passing thru p and spanned by the normal vector n_p and w. Let γ_1 and γ_2 be the curves of intersection of Σ_1 and Σ_2 with Π.

Let us orient Π so that n_p points to the left from w. Further, let us parametrize both curves so that they are running in the direction of w at p and therefore cooriented. In this case the curve γ_1 supports the curve γ_2 from the right.

Therefore we have the following inequality for the normal curvatures of Σ_1 and Σ_2 at p in the direction of w:

❹ $$k_w(p)_{\Sigma_1} \geqslant k_w(p)_{\Sigma_2}.$$

According to 7.1,

$$k_1(p)_{\Sigma_i} = \min\{\, k_w(p)_{\Sigma_i} \;:\; w \in \mathrm{T}_p, |w| = 1 \,\}$$

for $i = 1, 2$. Choose w so that $k_1(p)_{\Sigma_1} = k_w(p)_{\Sigma_1}$. Then by ❹, we have that

$$\begin{aligned} k_1(p)_{\Sigma_1} &= k_w(p)_{\Sigma_1} \geqslant \\ &\geqslant k_w(p)_{\Sigma_2} \geqslant \\ &\geqslant \min\{\, k_w(p)_{\Sigma_2} \,\} = \\ &= k_1(p)_{\Sigma_2}; \end{aligned}$$

that is, $k_1(p)_{\Sigma_1} \geqslant k_1(p)_{\Sigma_2}$.

Similarly, by 7.1, we have that

$$k_2(p)_{\Sigma_i} = \max\{\, k_w(p)_{\Sigma_i} \,\}.$$

Let us fix w so that $k_2(p)_{\Sigma_2} = k_w(p)_{\Sigma_2}$. Then

$$\begin{aligned} k_2(p)_{\Sigma_2} &= k_w(p)_{\Sigma_2} \leqslant \\ &\leqslant k_w(p)_{\Sigma_1} \leqslant \\ &\leqslant \max\{\, k_w(p)_{\Sigma_1} \,\} = \\ &= k_2(p)_{\Sigma_1}; \end{aligned}$$

that is, $k_2(p)_{\Sigma_1} \geqslant k_2(p)_{\Sigma_2}$. □

7.8. Corollary. *Let Σ_1 and Σ_2 be oriented surfaces. Assume Σ_1 locally supports Σ_2 from inside at the point p. Then*

(a) $H(p)_{\Sigma_1} \geqslant H(p)_{\Sigma_2}$;
(b) If $k_1(p)_{\Sigma_2} \geqslant 0$, then $K(p)_{\Sigma_1} \geqslant K(p)_{\Sigma_2}$.

Proof. By (7.6), we get that $k_1(p)_{\Sigma_1} \geqslant k_1(p)_{\Sigma_2}$ and $k_2(p)_{\Sigma_2} \geqslant k_2(p)_{\Sigma_2}$. Therefore part (a) follows since

$$H(p)_{\Sigma_i} = \tfrac{1}{2}\cdot(k_1(p)_{\Sigma_i} + k_2(p)_{\Sigma_i})).$$

(b). Since $k_2(p)_{\Sigma_i} \geqslant k_1(p)_{\Sigma_i}$ and $k_1(p)_{\Sigma_2} \geqslant 0$, we get that all the principle curvatures $k_1(p)_{\Sigma_1}$, $k_1(p)_{\Sigma_1}$ and $k_2(p)_{\Sigma_1}$ and $k_2(p)_{\Sigma_2}$ are nonnegative. Whence

$$\begin{aligned} K(p)_{\Sigma_1} &= k_1(p)_{\Sigma_1}\cdot k_2(p)_{\Sigma_1} \geqslant \\ &\geqslant k_1(p)_{\Sigma_2}\cdot k_2(p)_{\Sigma_2} = \\ &= K(p)_{\Sigma_2}. \end{aligned}$$

□

7.9. Exercise. *Show that any closed surface has a point with positive Gauss curvature.*

7.10. Exercise. *Assume that a closed surface Σ surrounds a unit disc. Show that Gauss curvature of Σ is at most 1 at some point.*

Try to prove the same assuming that Σ surrounds a unit circle only.

Curve in a surface

Recall that the second fundamental form II_p is defined on page 73.

7.11. Proposition. *Suppose γ is a smooth curve in a smooth oriented surface Σ with a unit normal field n. Then, the following identity holds for any time parameter t:*

$$\langle \gamma''(t), \mathrm{N}_{\gamma(t)}\rangle = \mathrm{II}_{\gamma(t)}(\gamma'(t), \gamma'(t)).$$

Proof. Fix a parameteer value t_0; set $p = \gamma(t_0)$, $v = \gamma'(t_0)$ and $a = \gamma''(t_0)$; so we need to show that

❺ $$\langle a, n_p\rangle = \mathrm{II}_p(v, v).$$

Let $z = f(x, y)$ be the local representaion of Σ in the tangent-normal coordinates at p; we assume that n points in the direction of n_p.

Without loss of generality may assume that γ runs in the graph $z = f(x, y)$; so

$$\gamma(t) = (x(t), y(t), f(x(t), y(t)))\,.$$

Then

$$\begin{aligned}
\gamma' &= (x', y', \tfrac{\partial f}{\partial x}\cdot x' + \tfrac{\partial f}{\partial y}\cdot y');\\
\gamma'' &= (x'', y'', \tfrac{\partial^2 f}{\partial x^2}\cdot (x')^2 + 2\cdot \tfrac{\partial^2 f}{\partial x\partial y}\cdot x'\cdot y' + \tfrac{\partial^2 f}{\partial y^2}\cdot (y')^2 + \tfrac{\partial f}{\partial x}\cdot x'' + \tfrac{\partial f}{\partial y}\cdot y'').
\end{aligned}$$

Recall that $p = \gamma(t_0) = (0, 0, 0)$ and

$$f(0, 0) = 0, \qquad \tfrac{\partial f}{\partial x}(0, 0) = 0, \qquad \tfrac{\partial f}{\partial y}(0, 0) = 0.$$

Therefore

$$\begin{aligned}
v &= (x', y', 0)\,(t_0);\\
a &= \left(x'', y'', \tfrac{\partial^2 f}{\partial x^2}\cdot (x')^2 + 2\cdot \tfrac{\partial^2 f}{\partial x\partial y}\cdot x'\cdot y' + \tfrac{\partial^2 f}{\partial y^2}\cdot (y')^2\right)(t_0).
\end{aligned}$$

Note that

$$\mathrm{II}_p(v, v) = \left(\tfrac{\partial^2 f}{\partial x^2}\cdot (x')^2 + 2\cdot \tfrac{\partial^2 f}{\partial x\partial y}\cdot x'\cdot y' + \tfrac{\partial^2 f}{\partial y^2}\cdot (y')^2\right)(t_0);$$

that is, the z-coordinate of the acceleration a equals $\mathrm{II}_p(v, v)$ which is equivalent to ❺. □

7.12. Corollary. *Let γ be a regular smooth curve that runs in a smooth surface Σ. Suppose $p = \gamma(t_0)$ and $w = \gamma'(t_0)$. Denote by α the angle between the unit normal to Σ at p and the unit normal vector in the Frenet frame of γ. Then the following identity holds for the curvature $\kappa(t_0)_\gamma$ of γ at p and the normal curvature $k_w(p)$ of Σ at p in the direction of w:*

$$\kappa(t_0)_\gamma\cdot \cos\alpha = k_w(p).$$

Proof. Denote by n the unit normal vector to Σ at p and by N the unit normal vector in the Frenet frame of γ. Note that $\cos\alpha = \langle n, \mathrm{N}\rangle$.

Applying 7.11, we get that

$$\begin{aligned} k_w(p) = \mathrm{II}_p(w,w) &= \\ &= \langle \gamma'', n\rangle = \\ &= \kappa(t_0)_\gamma \cdot \langle \mathrm{N}, n\rangle = \\ &= \kappa(t_0)_\gamma \cdot \cos\alpha. \end{aligned}$$

□

The corollary above, as well as the statement in the following exercise are proved by Jean Baptiste Meusnier [20].

7.13. Exercise. *Let Σ be a smooth surface, $p \in \Sigma$ and $w \in \mathrm{T}_p\Sigma$ is a unit vector. Assume that $k_w(p) \neq 0$; that is, the normal curvature of Σ at p in the direction of w does not vanish.*

Show that the osculating circles at p of smooth regular curves in Σ that run in the direction w sweep out a sphere.

7.14. Exercise. *Let $\gamma(t) = (x(t), y(t))$ be a smooth unit-speed simple plane curve in the upper half-plane. Suppose that Σ is the surface of revolution of γ with respect to the x-axis.*

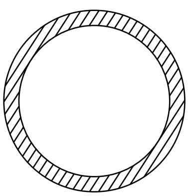
Express the principle curvatures of Σ at $(x(t), y(t), 0)$ in terms of $y(t)$, $y'(t)$ and $y''(t)$. Conclude that $-\frac{y''(t)}{y(t)}$ is the Gauss curvature of Σ at $(x(t), y(t), 0)$.

7.15. Exercise. *Assume that a regular smooth curve γ lies in a surface of positive Gauss curvature. Show that curvature of γ does not vanish at any value.*

Chapter 8

Bounded principle curvatures

Note that there sets in $\mathbb{R}^3$ bounded by a closed surface Σ with principle curvatures at most 1 by absolute value that do not contain a ball of radius 1.



For example the region between two large concentric spheres with almost equal raduses. This region can be made arbitrary thin and the curvature of the boundary can be made arbitrary close to zero.

The same example works in the plane — a pair of circles with arbitrary small curvature can bound an arbitrary thin region.

8.1. Advanced exercise. *Suppose a set $V \subset \mathbb{R}^3$ is bounded by a closed surface Σ with principle curvatures bounded in absolute value by 1. Assume that V does not contain a ball of radius $\frac{1}{100}$. Show that Σ has two components of the same topological type; that is, both can be written in parametric form with the same parameter domain.*

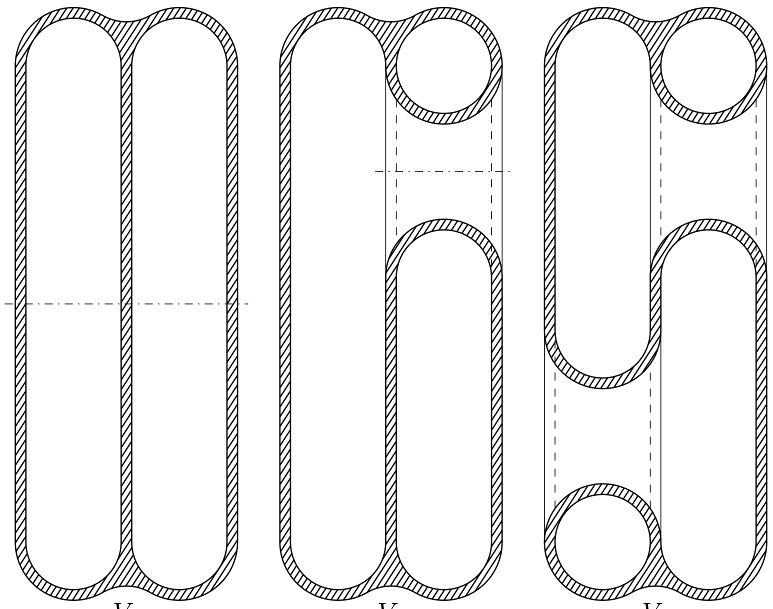
The same example would work for curves if we allow the boundary of the plane figure to be not connected. The following question might look like a right 3-dimensional analog of the moon in a puddle problem (5.23).

Lagunov's example

8.2. Question. *Assume a set $V \subset \mathbb{R}^3$ is bounded by a closed connected surface Σ with principle curvatures bounded in absolute value by 1. Is it true that V contains a ball of radius 1?*

It turns out that the answer is "no". The following example was constructed by Vladimir Lagunov [21].

Construction. Let us start with a body of revolution V_1 with cross section shown on the diagram. The boundary curve of the cross section consists of 6 vertical line segments smoothly jointed into 3 closed simple curves. The boundary of V_1 has 3 components, each of which is a sphere.



A simple computation shows that if the curvature of all curves is at most 1 then the boundary surface of V_1 has priniple curvatures at most 1 in absolute value.

At most of the places V_1 can be made arbitrary thin, the only thick place is where all tree spheres come together; it could be arranged that the radius of the maximal ball is just a little bit above

$$r_2 = \tfrac{2}{\sqrt{3}} - 1 < \tfrac{1}{6}.$$

This is the radius of the smallest circle tangent to three unit circles that are tangent to each other.

It remains to modify V_1 to make its boundary connected without alloing larger balls inside.

Note that each sphere in the boundary contains two flat discs; they come into pairs close lying to each other. Let us drill thru two of such pairs and reconnect the holes by another body of revolution whose axis is shifted but stays parallel to the axis of V_1. Denote the obtained body by V_2; its cross section of the obtained body is shown on the diagram.

Then repeat the operation for the other two pairs. Denote the obtained body by V_3; the cross section of the obtained body is shown on the diagram.

It is easy to see that the boundary of V_3 is connected and assuming that the holes are large its boundary can be made so that its principle curvatures are still at most 1. □

8.3. Claim. *The surface of V_3 has genus 2; that is it can be parameterized by a sphere with two handles.*

Proof. Note that the boundary of V_1 consists of three spheres.

When we drill a hole, we make one hole in two spheres and two holes in one shpere. We reconnect two spheres by a tube and obtain one sphere. Connecting the two holes of the other sphere by a tube we get a torus. That is, the boundary of V_2 is formed by one sphere and one torus.

To construct V_3 from V_2, we make a torus from the remaining sphere and connect it to the torus by a tube. This way we get a sphere with two handles; that is, it has genus 2. □

8.4. Exercise. *Assume V is a body of revolution in $\mathbb{R}^3$ and its boundary is a connected surface with principle curvatures at most 1 in absolute value. Show that V contains a unit ball.*

8.5. Exercise. *Assume V is a convex body in $\mathbb{R}^3$ bounded by a surface with principle curvatures at most 1. Show that V contains a unit ball.*

8.6. Exercise. *Modify Lagunov's construction to make the boundary surface a sphere with 4 handles.*

8.7. Advanced exercise. *Show that the bound in the Lagunov's example is optimal. That is, if a body $V \subset \mathbb{R}^3$ is bounded by a connected surface Σ with principle curvatures at most 1, then V contains a ball of radius r_2.*

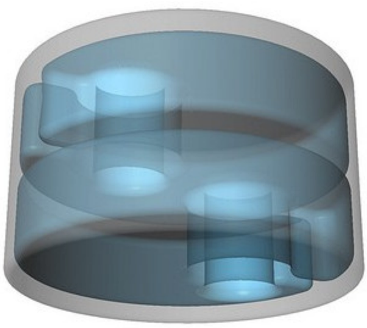
On embedded sphere

8.8. Advanced exercise. *Note that the body V in the example of Lagunov is constructed by thikening a surface having a singular curve where the surface self-intersects at an angle of* $120°$. *Show that this way one can not obtain a body bounded by a sphere.*

In fact one can show that if a body $V \subset \mathbb{R}^3$ is bounded by a sphere Σ with principle curvatures at most 1, then V contains a ball of radius $r_3 = \sqrt{\frac{3}{2}} - 1 > \frac{1}{5}$, which is the radius of the smallest sphere tangent to three unit spheres that are tangent to each other. Moreover, this bound is optimal.

An example of such a body can be obtained by thickening the so called Bing's house. It is certainly a surface whose singularities are formed by three curves meeting at two points; four ends at each point. The remaining surface of Bing's house is smooth and has bounded principle curvatures; we can assume that they are bounded by an arbitrary small number.

At the singular curves the three pieces of surface have to cross at angles $\frac{2}{3}\cdot\pi$ and at the sigular points 6 pieces of surface should come together forming 6 tringles with vertex in the center of a regular tetrahedron and the bases at its 6 edges. Thickening of a sufficiently large Bing's house of that type produces the optimal bound r_3 on the maximal ball that it contains.



The thickening of Bing's house shown on the picture can not give the optimal bound, but still it can produce an example of an embedded sphere that does not surround a ball of radius 1.

This picture is very similar to the Lagunov's example described above — it can be obtained by filling the rings in the section of V_3 by thickened discs.

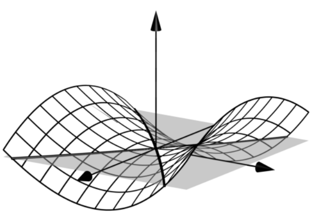
This picture was taken from a post of Ken Baker [22]; this post has many other beautiful pictures that help to visualize Bing's house.

Chapter 9

Saddle surfaces

Definitions

A surface is called *saddle* if its Gauss curvature at each point is nonpositive; in other words principle curvatures at each point have opposite signs or one of them is zero.



If the Gauss curvature is negative at each point, then the surface is called *strictly saddle*; equivalently it means that the principle curvatures have opposite signs at each point. Note that in this case the tangent plane does not support the surface even locally — moving along the surface in the principle directions at a given point, one goes above and below the tangent plane at this point.

9.1. Exercise. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth positive function. Show that the surface of revolution of the graph $y = f(x)$ around the x-axis is saddle if and only if f is convex; that is, if $f''(x) \geqslant 0$ for any x.*

A surface Σ is called *ruled* if for every point $p \in \Sigma$ there is a line segment $\ell_p \subset \Sigma_p$ thru p that is infinite or has its endpoint(s) on the boundary line of Σ.

9.2. Exercise. *Show that any ruled surface Σ is saddle.*

9.3. Exercise. *Suppose Σ is an open saddle surface. Show that for any point $p \in \Sigma$ there is a curve $\gamma\colon [0, \infty) \to \Sigma$ that starts at p and monotonically escapes to infinity; that is, $t \mapsto |\gamma(t)|$ is an increasing function and $|\gamma(t)| \to \infty$ as $t \to \infty$.*

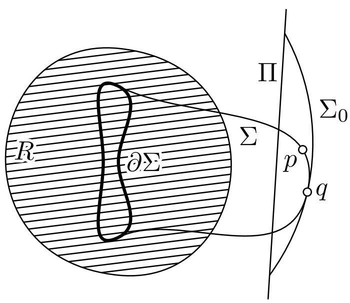
A tangent direction on a smooth surface with vanishing normal curvature is called *asymptotic*. A smooth regular curve that always runs in an asymptotic direction is called an asymptotic line *asymptotic line*.

9.4. Advanced exercise. *Let $\Sigma \subset \mathbb{R}^3$ be the graph $z = f(x, y)$ of a smooth function f and γ be a closed smooth asymptotic line in Σ. Assume Σ is strictly saddle in a neighborhood of γ. Show that the projection of γ to the (x, y)-plane cannot be star-shaped.*

Hats

Note that a closed surface cannot be saddle. Indeed consider a smallest sphere that contains a closed surface Σ inside; it supports Σ at some point p and at this point the principle curvature must have the same sign. The following more general statement is proved using the same idea.

9.5. Lemma. *Assume Σ is a compact saddle surface and its boundary line lies in a convex closed region R. Then $\Sigma \subset R$.*



Proof. Assume contrary; that is, there is point $p \in \Sigma$ that does not lie in R. Let Π be a plane that separates p from R; it exists by A.22. Denote by Σ' the part of Σ that lies with p on the same side from Π.

Since Σ is compact, it is surrounded by a sphere S; let σ be the circle of intersection of S and Π. Consider the smallest spherical dome Σ_0 with boundary σ that includes Σ'.

Note that Σ_0 supports Σ at some point q. Without loss of generality we may assume that Σ_0 and Σ are cooriented at q and Σ_0 has positive principle curvatures. In this case Σ_0 supports Δ from outside. By 7.8, $K(q)_\Sigma \geqslant K(q)_{\Sigma_0} > 0$ — a contradiction. $\square$

Note that if we assume that Σ is strictly saddle, then we could arrive to a contradiction by taking a point q on the same side with p and on the maximal distance from Π.

9.6. Exercise. *Let Δ be a smooth regular saddle disc and $p \in \Delta$. Assume that the boundary line $\partial\Delta$ lies in the unit sphere centered at p. Show that* $\operatorname{length}(\partial\Delta) \geqslant 2\cdot\pi$.

9.7. Exercise. *Show that an open saddle surface cannot lie inside of an infinite circular cone.*

A disc Δ in a surface Σ is called a *hat* of Σ if its boundary line $\partial\Delta$ lies in a plane Π and the remaining points of Δ lie on one side of Π.

9.8. Proposition. *A smooth surface Σ is saddle if and only if it has no hats.*

Note that a saddle surface can contain a closed plane curve. For example the hyperboloid $x^2 + y^2 - z^2 = 1$ contains the unit circle in the (x, y)-plane centered at the origin. However, a plane curve cannot bound a disc (as well any compact set) in a saddle surface.

Proof. Since plane is a convex set, the "only if" part follows from 9.5; it remains to prove the "if" part.

Assume Σ is not saddle; that is, it has a point p with strictly positive Gauss curvature; or equivalently, the principle curvatures $k_1(p)$ and $k_2(p)$ have the same sign.

Let $z = f(x, y)$ be a graph representation of Σ in the tangent-normal coordinates at p. Without loss of generality we may assume that both principle curvatures are positive, or equivalently the

$$D_w^2 f(0,0) = \mathrm{II}_p(w,w) > 0$$

for any unit tangent vector $w \in \mathrm{T}_p\Sigma$ (which is the (x, y)-plane).

Since the set of unit vectors is compact, we have that

$$D_w^2 f(0,0) > \varepsilon$$

for some fixed $\varepsilon > 0$ and any unit tangent vector $w \in \mathrm{T}_p\Sigma$. By continuity of the function $(x, y, w) \mapsto D_w^2 f(x, y)$, we have that $D_w^2 f(x, y) > 0$ for (x, y) in a neighborhood of the origin. That is, f is a strictly convex function in a neighborhood of the origin in the (x, y)-plane. In particular the set

$$\Delta_\varepsilon = \left\{\, (x, y, f(x, y)) \in \mathbb{R}^3 \ :\ f(x, y)) \leqslant \varepsilon \,\right\}$$

is a disc for sufficiently small $\varepsilon > 0$ (see 9.10). Note that its boundary line lies on the plane $z = \varepsilon$ and whole disc lies below it; that is, Δ_ε is a hat. $\square$

Note that we proved the following lemma; it will be useful later.

9.9. Lemma. *Let $z = f(x, y)$ be the local description of a smooth surface Σ in a tangent-normal coordinates at some point $p \in \Sigma$. Assume both principle curvatures of Σ are positive at p. Then the function f is strictly convex in a neighborhood of the origin.*

In the proof above we assumed that the statement in following exercise is evident.

9.10. Exercise. *Let Δ_ε be as in the proof above. Show that Δ_ε is a smooth disc; that is, Δ_ε is the image of regular embedding $\mathbb{D} \to \mathbb{R}^3$, where $\mathbb{D} = \left\{ (x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \leqslant 1 \right\}$.*

9.11. Exercise. *Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation; that is, $T(x, y, z) = (x, y, z)\cdot A$ for an invertible $3{\times}3$-matrix A. Show that for any saddle surface Σ the image $T(\Sigma)$ is also a saddle surface.*

Saddle graphs

The following theorem was proved by Sergei Bernstein [23].

9.12. Theorem. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Assume its graph $z = f(x, y)$ is a strictly saddle surface in $\mathbb{R}^3$. Then f is not bounded; that is, there is no constant C such that $|f(x, y)| \leqslant C$ for any $(x, y) \in \mathbb{R}^2$.*

Before going into the proof let us discuss some examples.

Note that the theorem states that a saddle graph cannot lie between parallel horizontal planes; applying 9.11 we get that saddle graphs cannot lie between parallel planes, not necessarily horizontal. The following exercise shows that the theorem does not hold for saddle surfaces which are not graphs.

9.13. Exercise. *Construct an open strictly saddle surface that lies between parallel planes.*

The following exercise shows that there are saddle graphs with functions bounded on one side; that is, both (upper and lower) bounds are needed in the proof of Bernshtein's theorem.

9.14. Exercise. *Show that there are positive functions with strictly saddle graphs. In fact the graph $z = \exp(x - y^2)$ is strictly saddle.*

Note that according to 9.5, there are no proper saddle surfaces in a parallelepiped that boundary line lies on one of its faces. The following lemma gives an analogous statement for a parallelepiped with an infinite side.

9.15. Lemma. *There is no proper strictly saddle smooth surface with the boundary line in a plane that lies on bounded distance from a line.*

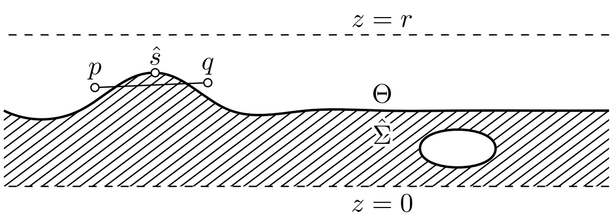
Proof. Note that in a suitable coordinate system, the statement can be reformulated the following way: *There is no proper strictly saddle* *smooth surface with the boundary line in the (x, y)-plane that lies in a region of the following form:*

$$R = \{ (x, y, z) \in \mathbb{R}^3 \,:\, 0 \leqslant z \leqslant r, 0 \leqslant y \leqslant r \}.$$

Further we will prove this statement.

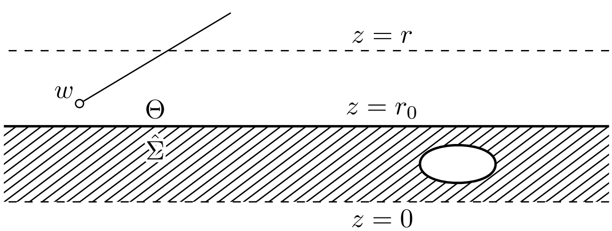
Assume contrary, let Σ be such a surface. Consider the projection $\hat{\Sigma}$ of Σ to the (x, z)-plane. It lies in the upper half-plane and below the line $z = r$.

Consider the open upper half-plane $H = \{ (x, z) \in \mathbb{R}^2 \,:\, z > 0 \}$. Let Θ be the connected component of the complement $H \backslash \hat{\Sigma}$ that contains all the points above the line $z = r$.



Note that Θ is convex. If not then a line segment $[pq]$ for some $p, q \in \Theta$ cuts from $\hat{\Sigma}$ a compact piece. Consider the plane Π thru $[pq]$ that is perpendicular to the (x, z)-plane. Note that π cuts from Σ a compact region Δ. By general position argument 6.15, we can assume that Δ is a compact surface with boundary line in Π and the remaining part of Δ lies on one side from Π. Since the plane Π is convex, this statement contradicts 9.5.

Summarizing, Θ is an open convex set of H that contains all points above $z = r$. By convexity, together with any point w, the set Θ contains all points on the half-lines that point up from it. Whence it contain all points with z-coordinate larger than the z-coordinate of w. Since Θ is open it can be described by inequality $z > r_0$. It follows that

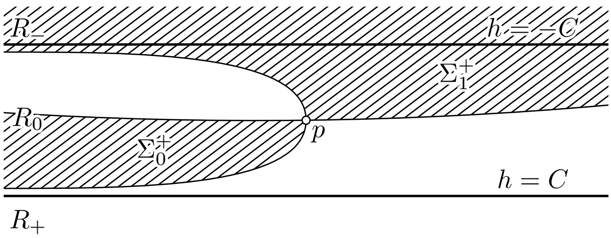


the plane $z = r_0$ supports Σ at some point (in fact at many points). By 7.6, the latter is impossible — a contradiction. $\square$

Proof of 9.12. Denote by Σ the graph $z = f(x, y)$. Assume contrary; that is, Σ lies between two planes $z = \pm C$.

Note f cannot be constant. It follows that the tangent plane T_p at some point $p \in \Sigma$ is not horizontal.

Denote by Σ^+ the part of Σ that lies above T_p. Note that it has at least two connected components which are approaching p from both sides in the principle direction with positive principle curvature. Indeed if there would be a curve that runs in Σ^+ and approaches p from both sides then it would cut a disc from Σ with boundary line above T_p and some points below it; the latter contradicts 9.5.



The surface Σ seeing from above.

Summarizing, Σ^+ has at least two connected components, denote them by Σ_0^+ and Σ_1^+. Let $z = h(x, y) = a{\cdot}x + b{\cdot}y + c$ be the equation of T_p. Note that Σ^+ contains all points in the region

$$R_- = \{\, (x, y, f(x, y)) \in \Sigma \;:\; h(x, y) < C \,\}$$

which is a connected set and no points in

$$R_+ = \{\, (x, y, f(x, y)) \in \Sigma \;:\; h(x, y) > C \,\}$$

Whence one of the connected components, say Σ_0^+, lies in

$$R_0 = \{\, (x, y, f(x, y)) \in \Sigma \;:\; |h(x, y)| \leqslant C \,\}.$$

This set lies on a bounded distance from the line of intersection of T_p with the (x, y)-plane.

Moving the plane T_p slightly upward, we can cut from Σ_0^+ is a proper surface with boundary line lying in this plane (see 6.15). The obtained surface is still on a bounded distance to a line which is impossible by 9.15. $\square$

The following exercise gives a condition that guarantees that a saddle surface is a graph; it can be used in combination with Bernshtein's theorem.

9.16. Advanced exercise. *Let Σ be a smooth saddle disk in $\mathbb{R}^3$. Assume that the orthogonal projection to the (x,y)-plane maps the boundary line of Σ injectively to a convex closed curve. Show that the orthogonal projection to the (x,y)-plane is injective on Σ.*

In particular, Σ is the graph $z = f(x,y)$ of a function f defined on a convex figure in the (x,y)-plane.

Remarks

Note that Bernstein's theorem and the lemma in its proof do not hold for saddle surfaces; counterexamples can be found among infinite cylinders over smooth regular curves. In fact it can be shown that these are the only counterexamples; a proof is based on the same idea, but more technical.

By 9.8, saddle surfaces can be defined as smooth surfaces without hats. This definition can be used for arbitrary surfaces, not necessarily smooth. Some results, for example Bernshtein's characterization of saddle graphs can be extended to generalized saddle surfaces, but this class of surfaces is far from being understood. Some nontrivial properties were proved by Samuil Shefel [24]; see also [25, Capter 4].

Chapter 10

Positive Gauss curvature

Convexity

10.1. Exercise. *Suppose that an oriented surface Σ bounds a convex region R.*

(a) Show that Gauss curvature of Σ is nonnegative at each point.

(b) Show that for any point $p \in \Sigma$ and q in the interior of R we have that

$$\langle n_p, q - p \rangle > 0,$$

where n_p is the unit normal vector at p that points in R.

Recall that a region R in the Euclidean space is called *strictly convex* if for any two points $x, y \in R$, any point z between x and y lies in the interior of R.

Clearly any open convex set is strictly convex; the cube (as well as any convex polyhedron) gives an example of a convex set which is not strictly convex. It is easy to see that a convex region is strictly convex if and only if its boundary does not contain a line segment.

The following theorem gives a global description of surfaces with positive Gauss curvature.

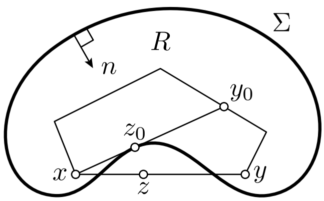
10.2. Theorem. *Assume Σ is an open or closed smooth surface with positive Gauss curvature. Then Σ bounds a strictly convex region.*

Note that in the proof we have to use that surface is a connected set; otherwise a pair of disjoint spheres which bound two disjoint balls would give a counterexample.

Proof. Since the Gauss curvature is positive, we can choose unit normal field n on Σ so that the principle curvatures are positive at any point. Let R be the region bounded by Σ that lies on the side of n; that is, n points inside of R at any point of Σ.

Fix $p \in \Sigma$; let $z = f(x, y)$ be a local description of Σ in the tangent-normal coordinates at p. By 9.9, f is strictly convex in a neighborhood of the origin. In particular the intersection of a small ball centered at p with the epigraph $z \geqslant f(x, y)$ is strictly convex. In other words, R is *locally strictly convex*; that is, for any point $p \in R$, the intersection of R with a small ball centered at p is strictly convex.

Since Σ is connected, so is R; moreover any two points in the interior of R can be connected by a polygonal line in the interior of R.



Assume the interior of R is not convex; that is, there are points $x, y \in R$ and a point z between x and y that does not lie in the interior of R. Consider a polygonal line β from x to y in the interior of R. Let y_0 be the first point on β such that the chord $[x, y_0]$ touches Σ at some point, say z_0.

Since R is locally strictly convex, $R \cap B(z_0, \varepsilon)$ is strictly convex for all sufficiently small $\varepsilon > 0$. On the other hand z_0 lies between two points in the intersection $[x, y_0] \cap B(z_0, \varepsilon)$. Since $[x, y_0] \subset R$, we arrived to a contradiction.

Therefore the interior of R is a convex set. Note that the region R is the closure of its interior, therefore R is convex as well.

Since R is locally strictly convex, its boundary Σ contains no line segments. Therefore R is strictly convex. $\square$

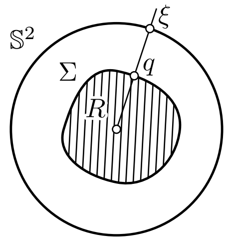
In fact only minor modifications of the proof above imply that any connected locally convex region is convex.

Closed surfaces

10.3. Lemma. *Assume Σ is a closed smooth surface with positive Gauss curvature. Then Σ is a smooth sphere; that is, Σ admits a smooth regular parametrization by $\mathbb{S}^2$.*

Proof. Without loss of generality we can assume that the origin lies in the interior of the convex region R bounded by Σ.

By convexity of R, any half-line starting at the origin intersects Σ at a single point; that is, there is a positive function $\rho\colon \mathbb{S}^2 \to \mathbb{R}$ such that Σ is formed by points $q = \rho(\xi)\cdot\xi$ for $\xi \in \mathbb{S}^2$.



Let us show that ρ is a smooth function. Fix a point $p = (x_p, y_p, z_p)$ on Σ. Consider a local implicit description of Σ at p as a solution of equation $h(x, y, z) = 0$ with nonvanishing gradient; so $h(p) = 0$. Note that for any point q in a neighborhood of p, we have that

$$h(q) = 0 \iff q \in \Sigma \iff q = \rho(\xi)\cdot\xi$$

for some $\xi \in \mathbb{S}^2$. In other words h defines implicitly ρ as in the implicit function theorem.

Recall that $\nabla_p h \perp \mathrm{T}_p$. Since the origin lies in the interior of R, it cannot lie on T_p; that is, $\langle \nabla_p f, p\rangle \neq 0$; or equivalently $D_\eta h(p) \neq 0$, where $\eta = \frac{p}{|p|}$ is the unit vector in the direction of p and D denotes the directional derivative.

Fix a (u, v) chart s on $\mathbb{S}^2$ in a neighborhood of η; note that the map

$$S\colon (u, v, w) \mapsto w\cdot s(u, v)$$

is smooth and regular for $w > 0$; that is, the vectors

$$\tfrac{\partial S}{\partial u}, \quad \tfrac{\partial S}{\partial v}, \quad \tfrac{\partial S}{\partial w} = w$$

are linearly independent. Note that the function $h \circ S$ is smooth and $\frac{\partial h\circ S}{\partial \rho}(p) = D_\eta h(p) \neq 0$. Applying implicit function theorem, we get that ρ is smooth in a neighborhood of η; since η is arbitrary, ρ is smooth on whole $\mathbb{S}^2$. □

If one only needs to show that Σ is a topological sphere, then one only needs to show that ρ is continuous. The latter is a consequence of another classical result in topology — the so-called *closed graph theorem.*

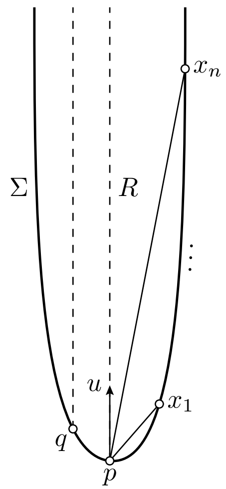
10.4. Exercise. *Let Σ be a closed smooth surface of diameter at least π; that is there is a pair of points $p, q \in \Sigma$ such that $|p - q| \geqslant \pi$. Show that Σ has a point with arbitrarily small Gauss curvature at most 1.*

Open surfaces

10.5. Lemma. *Suppose Σ is an open surface with positive Gauss curvature. Then there is a coordinate system such that Σ is a graph $z = f(x, y)$ of a convex function f defined on a convex open region of the (x, y)-plane.*

Proof. The surface Σ is a boundary of an unbounded closed convex region R.

We can assume that the origin lies on Σ. Consider a sequence of points $x_n \in \Sigma$ such that $|x_n| \to \infty$ as $n \to \infty$. Set $u_n = \frac{x_n}{|x_n|}$; this is the unit vector in the direction from x_n. Since the unit sphere is compact, we can pass to a subsequence of (x_n) such that u_n converges to a unit vector u.



Note that for any $q \in \Sigma$, the directions $v_n = \frac{x_n - q}{|x_n - q|}$ converge to u as well. Indeed, the sequences of vectors $\frac{x_n-q}{|x_n-q|}$, $\frac{x_n-q}{|x_n|}$ and $\frac{x_n}{|x_n|}$ have the same limit since $\frac{|q|}{|x_n|} \to 0$ and $\frac{|x_n|}{|x_n-q|} \to 1$ as $n \to \infty$; the latter follows since

$$\begin{aligned} \left|\tfrac{|x_n|}{|x_n-q|} - 1\right| &= \left|\tfrac{|x_n|-|x_n-q|}{|x_n-q|} - 1\right| \leqslant \\ &\leqslant \tfrac{|p-q|}{|x_n-q|}. \end{aligned}$$

Morover, the half-line from q in the direction of u lies in R. Indeed any point on the half-line is a limit of points on the line segments $[q, x_n]$; since R is closed, all of these points lie in R.

Let us choose the z-axis in the direction of u. Note that there is no point $p \in \Sigma$ with vertical tangent plane. Otherwise Σ would contain a vertical half-line starting from p and therefore Gauss curvature of Σ would vanish. It follows that any vertical line can intersect Σ at most at one point. That is, Σ is a graph $z = f(x, y)$ of a smooth convex function f.

Let Ω be the domain of definition of f. Note that Ω is the projection of Σ to the (x, y)-plane which is the same as the projection of R. Since R is convex, so is Ω.

Since no tangent plane is vertical, the projection from Σ to the (x, y)-plane is regular. By inverse function theorem, the set Ω is open. $\square$

10.6. Exercise. *Show that any open surface Σ with positive Gauss curvature is a topological plane; that is, there is an embedding $\mathbb{R}^2 \to \mathbb{R}^3$ with image Σ.*

Try to show that Σ is a smooth plane; that is, the embedding f can be made smooth and regular.

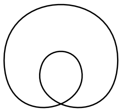
10.7. Exercise. *Show that any open smooth surface Σ with positive Gauss curvature lies inside of an infinite circular cone.*

Immersed surfaces

It seems that first formulation and proof of the following theorem was given by James Stoker [26] who attributed it to Jacques Hadamard, who proved a closely relevant statement in [27, item 23].

10.8. Theorem. *Any closed connected immersed surface with positive Gauss curvature is embedded.*

In other words such surface can not have self-intersections. Note that an analogous statement does not hold in the plane; on the diagram you can see a closed curve with a self-intersection and positive curvature at all points. Exercise **??** gives a condition that guarantees simplicity of a locally convex plane curve; it will be used in the following proof.



Before going into the proof, note that theorems 10.8 and 10.2 imply the following:

10.9. Corollary. *Any closed connected immersed surface with positive Gauss curvature is an embedded sphere that bounds a convex region.*

In the following sections we will give one complete proof and sketch an alternative proof.

The first proof uses a *Morse-type argument* for the height function; that is, we study how the part of the surface that lies below a plane changes when we move the plane upward. Little more careful analysis of this changes would imply the corollary above directly, without using Theorem 10.2.

The sketch use equidistants surfaces and the Gauss map. We will not proove a topological statement relying on intuition.

In the proof we abuse notation slightly; we say a *point of the immersed surface* instead of a *point in the parameter domain of the immersed surface*. So each point of self-intersection is considered as two or more "distinct" points of the surface.

10.1 Morse-type proof

Let Σ be a closed surface with positive Gauss curvature, possibly with self-intersections.

Fix a horizontal plane Π_h defined by the equation $z = h$ in an (x, y, z)-coordinate system. Note that the intersection $W_h = \Sigma \cap \Pi_h$ is formed by a finite collection of closed curves and isolated points.

(These curves and isolated points might intersect in the Euclidean space, but they are disjoint in the domain of parameters of Σ.)

Indeed, if $\mathrm{T}_p = \Pi_h$, then, since the principle curvatures are positive, p is a local minimum or local maximum of the height function. In both cases, p is an isolated point of W_h in Σ. If the tangent plane T_p is not Π_h, then it is not perpendicular to (x,z)-plane or (y,z)-plane. Therefore by Proposition **??**, the surface can be written locally as a graph $x = f(y,z)$ or $y = f(x,z)$; in both cases p lies on the curve $x = f(y,h)$ or correspondingly $y = f(x,h)$.

Summarizing, the closed set $W_h \subset \Sigma$ locally looks like a curve or an isolated point. Since Σ is compact, so is W. Therefore W is a finite disjoint collection of isolated points and closed simple curves in Σ.

Assume α_{h_0} is a closed curve in W_{h_0}. Note that its neighborhood is swept by curves α_h in W_h for $h \approx h_0$. Indeed a neighbohood of α_{h_0} in Σ can be covered by a finite number of graphs of the type $x = f(y,z)$ (or $y = f(x,z)$) and the curves α_h can be described locally as curves of the form $t \mapsto (f(t,h), t, h)$ (or correspondingly $t \mapsto (t, f(t,h), h)$) for $h \approx h_0$.

As α_h is the intersection of a locally convex surface with a plane, the curvature of α_h has fixed sign; so if we choose an orientation for the curves properly, we can assume that they all have positive curvature.

The family α_h depends smoothly on h and the same holds for its tangent indicatrix. Therefore the total signed curvature K_h of α_h depends continuously on h. If $K_h = 2{\cdot}\pi$ for some h, then $K_h = 2{\cdot}\pi$ for every h. It follows since, the function $h \mapsto K_h$ is continuous and its value is a multiple of $2{\cdot}\pi$. In this case, by Exercise **??**, all curves α_h are simple and each one bounds a convex region in the plane Π_h.

Summarizing, if one of the curves in the constructed family α_h is simple, then each curve in the family is simple and each α_h bounds a convex region in the plane Π_h.

Choose a point $p \in \Sigma$ that minimizes the height function z. Without loss of generality we may assume that p is the origin and therefore the surface lies in the upper half-space.

Fix $h > 0$. The intersection of the set $z \leqslant h$ with the surface may contain several connected components; one of them contains p, denote this component by Σ_h.[1]

From above, Σ_h is a surface with possibly nonempty boundary. Indeed it might be bounded only by few closed curves in W_h; any isolated point of W_h either lies in Σ_h together with its neighborhood or does not lie in Σ_h.

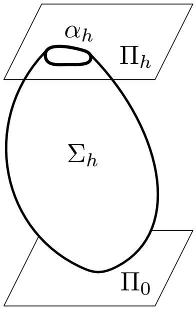
[1] These components might intersect in the space, but they are disjoint in the domain of parameters. Note also that from the corollary, it follows that there is only one component Σ_h, but we can not use it before the theorem is proved.

Note that for small values of h, the surface Σ_h is an embedded disc. Indeed, if $z = f(x, y)$ is a graph represetation of Σ around p, then Σ_h is a graph of f over

$$\Delta = \left\{ (x, y) \in \mathbb{R}^2 \ : \ f(x, y) \leqslant h \right\}.$$

Since the Gauss curvature is positive, the function f is convex and therefore Δ is convex and bounded by a smooth curve; any such set can be parameterized by a disc.

Let $H > 0$ be the maximal value such that Σ_h has no self-intersections for any $h < H$. For a sequence $h_n \to H^-$, choose a point q_n on the boundary of Σ_h and pass to a partial limit q of q_n in Σ; that is, q is a limit of a subsequence of (q_n).



If the tangent plane at q is *not* horizontal, then there is a closed curve α_H in Σ that passes thru q and lies on the plane $z = H$. From the above discussion, the curve α_H, as well as all α_h with $h \approx H$ are closed embedded convex curves. Hence Σ_h has no self-intersections for some $h > H$ — a contradiction.

If the tangent plane at q is horizontal, then the surface Σ_H has no boundary. Since Σ is connected, $\Sigma_H = \Sigma$. Since Σ_h has no self-intersections for $h < H$, we get that Σ is an embedded surface. $\square$

10.10. Exercise. *Modify the proof of the theorem to show that any open immersed surface with positive Gauss curvature is embedded.*

10.2 Proof via equidistant surfaces

Recall that a surface Σ is called *orientable* if one can choose at each point p of the surface a unit normal vector n_p in such a way that the function $p \mapsto n_p$ is continuous in every chart of Σ. For immersed surfaces we should say that n is a continuous function defined on the parameter domain of the surface. The map n is called a *Gauss map* of the surface.

10.11. Claim. *Assume Σ is a closed immersed surface with positive Gauss curvature, then it is orientable.*

Proof. Indeed we can choose the unit normal vector n_p in such a way that both principle curvatures are positive. In this case the surface lies locally on the side of tangent plane T_p which is opposite from n_p.

Evidently this choice is continuous. □

The unit normal described in the proof of the claim will be called *outer normal*.

10.12. Lemma. *Assume Σ is a closed connected immersed surface with positive Gauss curvature. Then the Gauss map $n\colon \Sigma \to \mathbb{S}^2$ has a smooth regular inverse; in particular, Σ is a sphere.*

This lemma follows from two facts: (1) if Gauss curvature does not vanish then the Gauss map is regular, in particular this map has a local inverse at each point and (2) the sphere $\mathbb{S}^2$ is *simply connected*; that is, $\mathbb{S}^2$ is connected any closed curve in $\mathbb{S}^2$ can be deformed continuously into a trivial curve that stays at one point. The proof is standard in topology, we hope that the statement is intuitively obvious. The reader might be able to reinvent the theory by trying to prove that if the map $\varphi\colon \mathbb{S}^2 \to \mathbb{S}^2$ is smooth and regular then it has an inverse.

Equidistant surfaces. Assume $n\colon \Sigma \to \mathbb{S}^2$ is a Gauss map of a smooth surface Σ. Fix a real number R and consider the map $h_R\colon \Sigma \to \to \mathbb{R}^3$ defined by $h_R\colon p \mapsto p + R\cdot n_p$. The map h_R describe the so called *equidistant surface*; it is smooth by definition, but in general it does not have to be regular.

10.13. Lemma. *Suppose $n\colon \Sigma \to \mathbb{S}^2$ is a Gauss map of a surface Σ. Assume the corresponding principle curvatures are nonnegative at all points. Then the equidistant surface Σ_R is regular and its principle curvatures are positive and strictly smaller than $\frac{1}{R}$.*

Proof. To prove regularity, let us use the special representation of Σ as a graph $z = f(x, y)$ with the x and y axis in the principle directions of Σ at p.[2]

Due to the choice of directions of x and y axis, for the Gauss map $g(x, y)$, we have

$$\begin{aligned} \tfrac{\partial}{\partial x} g(0,0) &= (k_1, 0, 0), \\ \tfrac{\partial}{\partial y} g(0,0) &= (0, k_2, 0). \end{aligned}$$

Then $h_R = h_0 + R\cdot g$; therefore

$$\begin{aligned} \tfrac{\partial}{\partial x} h_R &= (1 + R\cdot k_1, 0, 0), \\ \tfrac{\partial}{\partial y} h_R &= (0, 1 + R\cdot k_2, 0) \end{aligned}$$

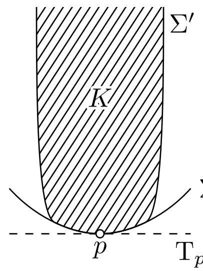
[2]If we assume that n_p points in the direction of the z-axis, then Σ is given in parametric form $h_0\colon (x, y) \mapsto (x, y, f(x, y))$, where $f = -\frac{k_1}{2}\cdot x^2 - \frac{k_2}{2}\cdot y^2 + o(x^2 + y^2)$.

which are linearly independent if $R \geqslant 0$ and $k_1, k_2 \geqslant 0$.

If Σ bounds a convex closed set K. Then Σ_R bounds K_R — the closed R-neighborhood of K; that is, K_R is the set of all points at distance at most R from K.

Since Σ is smooth it is supported at each point p from inside by a small ball $B_\varepsilon(o)$. Then the ball $B_{R+\varepsilon}(o)$ lies in K_R and touches its boundary at the point corresponding to p. Hence the principle curvatures at p are at least $\frac{1}{R+\varepsilon}$.

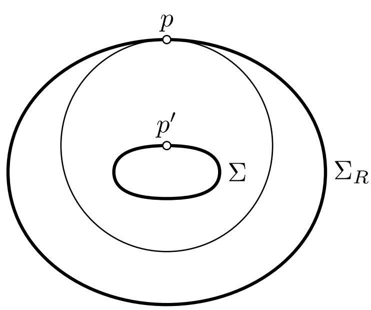
In the general case, a local chart of Σ can be modified so that it has a piece of the original surface around p and bounds a convex set. Here is one way to do this:



Choose a smooth function $\varphi(x)$ that is convex increasing and such that for sufficiently small $\varepsilon > 0$ we have $\varphi(x) = x$ if $x < \varepsilon$ and $\varphi(x) \to \infty$ as $x \to 2{\cdot}\varepsilon$. (Such functions do exist; moreover an explicit formula can be written, but we leave it without a proof.)

Assume $z = f(x,y)$ is a special representation of Σ around p by some convex function f. Direct computations show that $h = \varphi \circ f(x,y)$ is still convex. The surface Σ' described as the graph $z = h(x,y)$ bounds a convex closed set K and the part of Σ' described by the parameters $\{\,(x,y) \,:\, f(x,y) < \varepsilon\,\}$ coincide with a neighborhood of p in Σ. Hence the general case follows. $\square$

Proof assembling. Let $s\colon \mathbb{S}^2 \to \mathbb{R}^3$ be the parametization of Σ provided by Lemma 10.12. Then the equidistant surface Σ_R can be parametrized by $s_R(u) = s(u) + R{\cdot}u$ for $u \in \mathbb{S}^2$. Rescaling s_R by a factor of $\frac{1}{r}$ we get the map $u \mapsto \frac{1}{R}{\cdot}s(u) + u$ which converges smoothly to the indentity map on the sphere $\mathbb{S}^2$. Therefore Σ_R is embedded for sufficiently large R.



Applying Theorem 10.2, we get that Σ_R bounds a convex set.

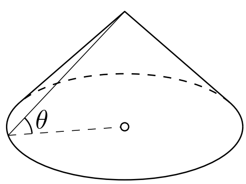
By Lemma 10.13, the principle curvatures of Σ_R are smaller than $\frac{1}{R}$. Therefore the same idea as in Exercise 8.5 shows that any point p of Σ_R can be supported by a ball of radius R from inside. Note that the center p' of such ball has to lie on Σ; indeed it lies at distance R in the normal direction. In other words, the map

$s_0(u) = s_R(u) - R{\cdot}u$ is injective, or
equivalently Σ has no self-intersection. $\square$

Chapter 11

Geodesics

We start to study the intrinsic geometry of surfaces. The following exercise should help you to be in the right mood for this; it might look like a tedious problem in calculus, but actually it is an easy problem in geometry.



11.1. Exercise. *There is a mountain of frictionless ice with the shape of a perfect cone with a circular base. A cowboy is at the bottom and he wants to climb the mountain. So, he throws up his lasso which slips neatly over the top of the cone, he pulls it tight and starts to climb. If the angle of inclination θ is large, there is no problem; the lasso grips tight and up he goes. On the other hand if θ is small, the lasso slips off as soon as the cowboy pulls on it.*

What is the critical angle θ_0 at which the cowboy can no longer climb the ice-mountain?

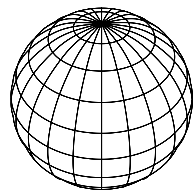
Shortest paths

Let p and q be two points on a surface Σ. Recall that $|p-q|_\Sigma$ denotes the induced length distance from p to q; that is, the exact lower bound on lengths of paths in Σ from p to q.

Note that if Σ is smooth, then any two points in Σ can be joined by a piecewise smooth path. Since any such path is rectifiable, the value $|p-q|_\Sigma$ is finite for any pair of points $p, q \in \Sigma$.

A path γ from p to q in Σ that minimizes the length is called a *shortest path* from p to q.

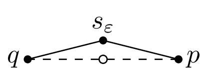
The image of a shortest path between p and q in Σ is usually denoted by $[p,q]_\Sigma$. In general there might be no shortest path between two given points on the surface and it might be many of them; this is shown in the following two examples. However if we write $[p,q]_\Sigma$, then we assume that a shortest path exists and we made a choice of one of them.



Nonuniqueness. There are plenty of shortest paths between the poles on the sphere — each meridian is a shortest path.

Nonexistence. Let Σ be the (x,y)-plane with removed origin. Consider two points $p=(1,0,0)$ and $q=(-1,0,0)$ in Σ.

Note that $|p-q|_\Sigma=2$. Indeed, given $\varepsilon>0$, consider the point $s_\varepsilon=(0,\varepsilon,0)$. Note that the polygonal path $ps_\varepsilon q$ lies in Σ and its length $2\cdot\sqrt{1+\varepsilon^2}$ approaches 2 as $\varepsilon\to 0$. It follows that $|p-q|_\Sigma\leqslant 2$. On the other hand $|p-q|_\Sigma\geqslant|p-q|_{\mathbb{R}^3}=2$; that is, $|p-q|_\Sigma=2$.



Therefore a shortest path from p to q (if it exists) must have length 2. By triangle inequality any curve of length 2 from p to q must run along the line segment $[p,q]$; in particular it must pass thru the origin. Since the origin does not lie in Σ, there is no shortest from p to q in Σ

11.2. Proposition. *Any two points in a proper smooth surface can be joined by a shortest path.*

Proof. Fix a proper smooth surface Σ with two points p and q. Set $\ell=|p-q|_\Sigma$.

By the definition of induced length metric, there is a sequence of paths γ_n from p to q in Σ such that

$$\operatorname{length}\gamma_n\to\ell\quad\text{as}\quad n\to\infty.$$

Without loss of generality, we may assume that $\operatorname{length}\gamma_n<\ell+1$ for any n and each γ_n is parameterized proportional to its arc length. In particular each path $\gamma_n\colon[0,1]\to\Sigma$ is $(\ell+1)$-Lipschitz; that is,

$$|\gamma(t_0)-\gamma(t_1)|\leqslant(\ell+1)\cdot|t_0-t_1|$$

for any $t_0,t_1\in[0,1]$. Further the image of γ_n lies in the closed ball $\bar B[p,\ell+1]$ for any n. It follows that the coordinate functions of γ_n are uniformly equicontinuous and uniformly bounded. By A.17, we can pass to a converging subsequence of γ_n; denote by $\gamma_\infty\colon[0,1]\to\mathbb{R}^3$ its limit. As a limit of uniformly continuous sequence, γ_∞ is continuous; that is, γ_∞ is a path. Evidently γ_∞ runs from p to q. Since Σ is a closed set, γ_∞ lies in Σ. Finally, by 2.13,

$$\gamma_\infty \leqslant \ell;$$

that is, γ_∞ is a shortest path from p to q. □

Closest point projection

11.3. Lemma. *Let R be a closed convex set in $\mathbb{R}^3$. Then for every point $p \in \mathbb{R}^3$ there is a unique point $\bar{p} \in R$ that minimizes the distance $|p - x|$ among all points $x \in R$.*

Moreover the map $p \mapsto \bar{p}$ is short; that is,

➊
$$|p - q| \geqslant |\bar{p} - \bar{q}|$$

for any pair of points $p, q \in \mathbb{R}^3$.

The map $p \mapsto \bar{p}$ is called the *closest point projection*; it maps the Euclidean space to R. Note that if $p \in R$, then $\bar{p} = p$.

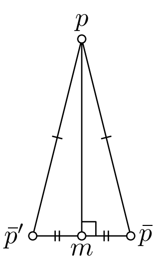
Proof. Fix a point p and set

$$\ell = \inf \{\, |p - x| \;:\; x \in R \,\}.$$

Choose a sequence $x_n \in R$ such that $|p - x_n| \to \ell$ as $n \to \infty$.

Without loss of generality, we can assume that all the points x_n lie in a ball of radius $\ell + 1$ centered at p. Therefore we can pass to a partial limit $\bar{p}$ of x_n; that is, $\bar{p}$ is a limit of a subsequence of x_n. Since R is closed $\bar{p} \in R$. By construction

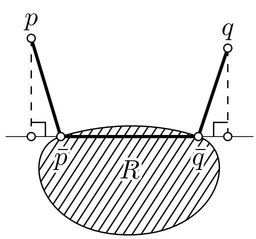
$$\begin{aligned} |p - \bar{p}| &= \lim_{n\to\infty} |p - x_n| = \\ &= \ell. \end{aligned}$$



Hence the existence follows.

Assume there are two distinct points $\bar{p}, \bar{p}' \in R$ that minimize the distance to p. Since R is convex, their midpoint $m = \frac{1}{2}\cdot(\bar{p} + \bar{p}')$ lies in R. Note that $|p - \bar{p}| = |p - \bar{p}'| = \ell$; that is, $\triangle p\bar{p}\bar{p}'$ is isosceles and therefore $\triangle p\bar{p}m$ is right with the right angle at m. Since a leg of a right triangle is shorter than its hypotenuse, we have $|p - m| < \ell$ — a contradiction.

It remains to prove inequality ➊.



We can assume that $\bar{p} \neq \bar{q}$, otherwise there is nothing to prove. Note that if $p \neq \bar{p}$ (that is, if $p \notin R$), then $\angle p\bar{p}\bar{q}$ is right or obtuse. Otherwise there would be a point x on the line segment $[\bar{q}, \bar{p}]$ that is closer to p than $\bar{p}$. Since R is convex, the line segment $[\bar{q}, \bar{p}]$ and therefore x lie in R. Hence $\bar{p}$ is not closest to p — a contradiction.

The same way we can show that if $q \neq \bar{q}$, then $\angle q\bar{q}\bar{p}$ is right or obtuse.

We have to consider the following 4 cases: (1) $p \neq \bar{p}$ and $q \neq \bar{q}$, (2) $p = \bar{p}$ and $q \neq \bar{q}$, (3) $p \neq \bar{p}$ and $q = \bar{q}$, (4) $p = \bar{p}$ and $q = \bar{q}$. In all these cases the obtained angle estimates imply that the orthogonal projection of the line segment $[p, q]$ to the line $\bar{p}\bar{q}$ contains the line segment $[\bar{p}, \bar{q}]$. In particular

$$|p - q| \geqslant |\bar{p} - \bar{q}|. \qquad \square$$

11.4. Corollary. *Assume a surface Σ bounds a closed convex region R and $p, q \in \Sigma$. Denote by W the outer closed region of Σ; in other words W is the union of Σ and the complement of R. Then for any curve γ in W that runs from p to q we have*

$$\operatorname{length} \gamma \geqslant |p - q|_\Sigma.$$

Moreover if γ does not lie in Σ, then the inequality is strict.

Proof. The first part of the corollary follows from the lemma and the definition of length. Indeed consider the closest point projection $\bar{\gamma}$ of γ. Note that $\bar{\gamma}$ lies in Σ and connects p to q therefore

$$\operatorname{length} \bar{\gamma} \geqslant |p - q|_\Sigma.$$

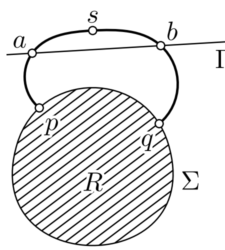
Consider an inscribed polygonal line $p_0 \dots p_n$ in γ. Denote by $\bar{p}_i$ the closest point projection of p_i to R. Note that the polygonal line $\bar{p}_0 \dots \bar{p}_n$ is inscribed in $\bar{\gamma}$; moreover any inscribed polygonal line in $\bar{\gamma}$ can appear this way. By 11.3 $|p_i - p_{i-1}| \geqslant |\bar{p}_i - \bar{p}_{i-1}|$ for any i. Therefore

$$\operatorname{length} p_0 \dots p_n \geqslant \operatorname{length} \bar{p}_0 \dots \bar{p}_n.$$

Taking least upper bound of each side of the inequality for all inscribed polygonal lines $p_0 \dots p_n$ in γ, we get

$$\operatorname{length} \gamma \geqslant \operatorname{length} \bar{\gamma}.$$

Whence the first statement follows.



To prove the second statement, note that if $s = \gamma(t_1) \notin \Sigma$, then $s \notin R$. Hence there is a plane Π that cuts s from Σ. The curve γ must intersect at least at two points: one point before t_1 and one after; let $a = \gamma(t_0)$ and $b = \gamma(t_2)$ be these points. Note that the arc of γ from a to b is strictly longer that $|a-b|$; indeed on the way γ visits s that is not on the plane Π and therefore not on the lie segment $[a, b]$.

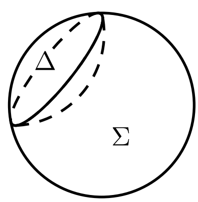
Remove from γ the arc from a to b and glue in the line segment $[a, b]$; denote the obtained curve by γ_1. From above,

$$\operatorname{length} \gamma > \operatorname{length} \gamma_1$$

Note that γ_1 runs in W. Therefore by the first part of corollary, we have

$$\operatorname{length} \gamma_1 \geqslant |p - q|_\Sigma.$$

Whence the second statement follows. □



11.5. Exercise. *Suppose Σ is a closed smooth surface that bounds a convex region R in $\mathbb{R}^3$ and Π is a plane that cuts a hat Δ from Σ. Assume that the reflection of the interior of Δ with respect to Π lies in the interior of R. Show that Δ is* convex *with respect to the intrinsic metric of Σ; that is, if both ends of a shortest path in Σ lie in Δ, then the entire geodesic lies in Δ.*

Let us define the *intrinsic diameter* of a closed surface Σ as the exact upper bound on the lengths of shortest paths in the surface.

11.6. Exercise. *Assume that a closed smooth surface Σ with positive Gauss curvature lies in a unit ball. Show that the intrinsic diameter of Σ cannot exceed π.*

Geodesics

A smooth curve γ on a smooth surface Σ is called *geodesic* if its acceleration $\gamma''(t)$ is perpendicular to the tangent plane $\mathrm{T}_{\gamma(t)}$ for each t.

Geodesics can be understood as the trajectories of a particle that slides on Σ without friction. In this case the force that keeps the particle on Σ must be perpendicular to Σ. By the second Newton's laws of motion, we get that the acceleration γ'' is perpendicular to $\mathrm{T}_{\gamma(t)}$.

11.7. Exercise. *Assume that a smooth surface Σ is mirror symmetric with respect to a plane Π. Suppose that Σ and Π intersect along a curve γ. Show that γ is a geodesic of Σ.*

11.8. Exercise. *Show that the helix*

$$\gamma(t) = (\cos t, \sin t, a\cdot t)$$

is a geodesic on the cylindrical surface described by the equation $x^2 + y^2 = 1$.

Recall that asymptotic line is defined on page 85.

11.9. Exercise. *Show that if a curve that is a geodesic and an asymptotic line on a smooth surface, then the curve is a line segment.*

11.10. Lemma. *Any geodesic γ has constant speed; that is, $|\gamma'(t)|$ is constant.*

Proof. Since $\gamma'(t)$ is a tangent vector at $\gamma(t)$, we have that $\gamma''(t) \perp \perp \gamma'(t)$, or equivalently $\langle\gamma'', \gamma'\rangle = 0$ for any t. Whence

$$\begin{aligned}\langle\gamma', \gamma'\rangle' &= 2\cdot\langle\gamma'', \gamma'\rangle = \\ &= 0.\end{aligned}$$

That is, $|\gamma'(t)|^2 = \langle\gamma'(t), \gamma'(t)\rangle$ is constant. □

11.11. Proposition. *Let Σ be a smooth surface without boundary. Given a tangent vector v to Σ at a point p there is a unique geodesic $\gamma\colon \mathbb{I} \to \Sigma$ defined on a maximal open interval $\mathbb{I} \ni 0$ that starts at p with velocity vector v; that is, $\gamma(0) = p$ and $\gamma'(0) = v$.*

Moreover

(a) the map $(p, v, t) \mapsto \gamma(t)$ is smooth in its domain of definition.

(b) if Σ is proper, then $\mathbb{I} = \mathbb{R}$; that is, the maximal interval is whole real line.

Sketch of proof. The first part of the proposition and part (a) follows from existence and uniqueness of a solution of initial value problem (A.14). One only needs to rewrite the condition $\gamma''(t) \perp \mathrm{T}_{\gamma(t)}$ as a differential equation $\gamma''(t) = \mathrm{II}_{\gamma(t)}(\gamma'(t), \gamma'(t))$.

The part (b) follows from 11.10. Indeed by A.14, if the maximal interval is not whole real line, then the curve γ must escape to infinity. But the latter is impossible since γ runs with constant speed. □

Exponential map

Let Σ be smooth regular surface and $p \in \Sigma$. Given a tangent vector $v \in \mathrm{T}_p$ consider a geodesic γ_v in Σ that runs from p with the initial velocity v; that is, $\gamma(0) = p$ and $\gamma'(0) = v$.

The point $q = \gamma_v(1)$ is called *exponential map* of v, or briefly $q = \exp_p v$. (There is a reason to call this map *exponential*, but it will take us too far from the subject.) By 11.11, the map $\exp_p\colon \mathrm{T}_p \to \Sigma$ is smooth and defined in a neighborhood of zero in T_p; moreover, if Σ is proper, then $\exp_p$ is defined on the whole space T_p.

Note that the Jacobian of $\exp_p$ at zero is the identity matrix. Indeed, let $z = f(x, y)$ be a local graph representation of Σ in the tangent-normal coordinates. The tangent plane at p is the (x, y)-plane. Let γ_x and γ_y be the geodesics starting from p in the directions $(1, 0, 0)$ and $(0, 1, 0)$ correspondingly. A general tangent vector can be written as $v = (x, y, 0)$. Note that $\frac{\partial \exp_p}{\partial x}(0, 0) = \gamma_x'(0) = (1, 0, 0)$ and $\frac{\partial \exp_p}{\partial x}(0, 0) = \gamma_y'(0) = (0, 1, 0)$. That is, the Jacobian matrix of $\exp_p$ at $(0, 0)$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It follows that the Jacobian matrix of the projection of $\exp_p$ to the (x, y)-plane is the identity matrix. Therefore by the inverse function theorem (A.9), we get the following statement:

11.12. Proposition. *Let Σ be smooth surface and $p \in \Sigma$. Then the exponential map $\exp_p\colon \mathrm{T}_p \to \Sigma$ is a smooth regular parametrization of a neighborhood of p in Σ by a neighborhood of 0 in the tangent plane T_p.*

Moreover for any $p \in \Sigma$ there is $\varepsilon > 0$ such that for any $x \in \Sigma$ such that $|x - p|_\Sigma < \varepsilon$ the map $\exp_x\colon \mathrm{T}_x \to \Sigma$ is a smooth regular parametrization of the ε-neighborhood of x in Σ by the ε-neighborhood of zero in the tangent plane T_x.

Shortest paths are geodesics

11.13. Claim. *Let Σ be a smooth regular surface. Then any shortest path γ in Σ parameterized proportional to its length is a geodesic in Σ. In particular γ is a smooth curve.*

A partial converse to the first statement also holds: a sufficiently short arc of any geodesic is a shortest path. More precisely, given a smooth surface Σ there is a positive function ρ on Σ such that if a *geodesic γ starts at $p \in \Sigma$ and has length at most $\rho(p)$ then it is a shortest path.*

A geodesic might not form a shortest path, but if this is the case, then it is called *minimizing geodesic*. Note that according to the claim, any shortest path is a reparametrization of a minimizing geodesic.

This claim provides connection between intrinsic geometry of the surface and its extrinsic geometry. This connection will be important later; in particular it will play the key role in the proof of the so-called *remarkable theorem* (13.19).

Intrinsic means that it can be expressed in terms of measuring things inside the surface, for example length of curves or angles between the curves that lie in the surface. Extrinsic means that we have to use ambient space in order to measure it.

For instance, a shortest path is an object of intrinsic geometry of a surface, while definition of geodesic is not intrinsic — it requires acceleration which needs the ambient space. Note that there is a smooth bijection between the cylinder $z = x^2$ and the plane $z = 0$ that preserves the lengths of all curves; in other words the cylinder can be *unfolded* on the plane. Such a bijection sends geodesics in the cylinder to geodesics on the plane and the other way around; however a geodesic on the cylinder might have nonvanishing second derivative while geodesics on the plane are straight lines with vanishing second derivative.

Informal sketch. The smoothness should be intuitively obvious; at least the curve should be twice differentiable otherwise it can be shortened.

Let us give an informal physical explanation why $\gamma''(t) \perp \mathrm{T}_{\gamma(t)}\Sigma$. One may think about the geodesic γ as of stable position of a stretched elastic thread that is forced to lie on a frictionless surface. Since it is frictionless, the force density $N(t)$ that keeps the geodesic γ in the surface must be proportional to the normal vector to the surface at $\gamma(t)$.

The tension in the thread has to be the same at all points (otherwise the thread would move back or forth and it would not be stable). Denote by T the tension. We can assume that γ has unit speed, In this case the net force from tension to the arc $\gamma_{[t_0,t_1]}$ is $T\cdot(\gamma'(t_1) - \gamma'(t_0))$. Hence the density of net force from tension at t is $F(t) = T\cdot\gamma''(t)$. According to the second Newton's law of motion, we have

$$F(t) + N(t) = 0;$$

which implies that $\gamma''(t)$ is perpendicular to $\mathrm{T}_{\gamma(t)}\Sigma$.

Fix a point $p \in \Sigma$. Let $\varepsilon > 0$ be as in 14.3. Assume a geodesic γ of length less than ε from p to q does not minimize the length between its endpoints. Then there is a shortest path from p to q, which becomes a geodesic if parameterized by its arc length. That is, there are two geodesics from p to q of length smaller than ε. In other words there are two vectors $v, w \in \mathrm{T}_p$ such that $|v| < \varepsilon$, $|w| < \varepsilon$ and $q = \exp_p v = \exp_p w$. But according to 14.3, the exponential map is injective in ε-neighborhood of zero — a contradiction. $\square$

11.14. Exercise. *Show that two shortest paths can cross each other at most once. More precisely, if two shortest paths have two distinct common points p and q then either these points are the ends of both shortest paths or both shortest paths contain an arc from p to q.*

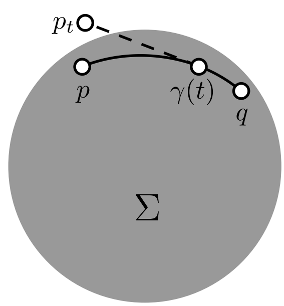
Show by example that nonoverlapping geodesics can cross each other an arbitrary number of times.

11.15. Exercise. *Assume that a smooth regular surface Σ is mirror symmetric with respect to a plane Π. Show that no shortest path in Σ can cross Π more than once.*

11.16. Advanced exercise. *Let Σ be a smooth closed strictly convex surface in $\mathbb{R}^3$ and $\gamma\colon [0,\ell] \to \Sigma$ be a unit-speed minimizing geodesic. Set $p = \gamma(0)$, $q = \gamma(\ell)$ and*

$$p_t = \gamma(t) - t{\cdot}\gamma'(t),$$

where $\gamma'(t)$ denotes the velocity vector of γ at t.



Show that for any $t \in (0,\ell)$, one cannot see *q from p_t; that is, the line segment $[p_t, q]$ intersects Σ at a point distinct from q.*

Show that the statement does not hold without assuming that γ is minimizing.

Liberman's lemma

The following lemma is a smooth analog of lemma proved by Joseph Liberman [28].

11.17. Liberman's lemma. *Assume γ is a geodesic on the graph $z = f(x, y)$ of a smooth convex function f defined on an open subset* *of the plane. Suppose that $\gamma(t) = (x(t), y(t), z(t))$ Then $t \mapsto z(t)$ is a convex function; that is, $z''(t) \geqslant 0$ for any t.*

Proof. Choose the orientation on the graph so that the unit normal vector n always points up; that is, it has positive z-coordinate.

Since γ is a geodesic, we have $\gamma''(t) \perp \mathrm{T}_{\gamma(t)}$, or equivalently $\gamma''(t)$ is proportional to $n_{\gamma(t)}$ for any t. By 7.11, we have

$$\langle \gamma''(t), n_{\gamma(t)} \rangle = \mathrm{II}_{\gamma(t)}(\gamma'(t), \gamma'(t));$$

hence

$$\gamma''(t) = n_{\gamma(t)} \cdot \mathrm{II}_{\gamma(t)}(\gamma'(t), \gamma'(t))$$

for any t.

Therefore

❷ $$z''(t) = \cos(\theta_\gamma(t)) \cdot n_{\gamma(t)} \cdot \mathrm{II}_{\gamma(t)}(\gamma'(t), \gamma'(t)),$$

where $\theta_\gamma(t)$ denotes the angle between $n_{\gamma(t)}$ and the z-axis.

Since n points up, we have $\theta_\gamma(t) < \frac{\pi}{2}$, or equivalently

$$\cos(\theta_\gamma(t)) > 0$$

for any t.

Since f is convex, we have that tangent plane supports the graph from below at any point; in particular $\mathrm{II}_{\gamma(t)}(\gamma'(t), \gamma'(t)) \geqslant 0$. It follows that the right hand side in ❷ is nonnegative; whence the statement follows. □

11.18. Exercise. *Assume γ is a unit-speed geodesic on a smooth convex surface Σ and p in the interior of a convex set bounded by Σ. Set $\rho(t) = |p - \gamma(t)|^2$. Show that $\rho''(t) \leqslant 2$ for any t.*

Bound on total curvature

11.19. Theorem. *Assume Σ is a graph $z = f(x, y)$ of a convex ℓ-Lipschitz function f defined on an open set in the (x, y)-plane. Then the total curvature of any geodesic in Σ is at most $2 \cdot \ell$.*

The above theorem was proved by Vladimir Usov [29], later David Berg [30] pointed out that the same proof works for geodesics in closed epigraphs of ℓ-Lipschitz functions which are not necessary convex; that is, sets of the type

$$W = \{\, (x, y, z) \in \mathbb{R}^3 \,:\, z \geqslant f(x, y) \,\}$$

Proof. Let $\gamma(t) = (x(t), y(t), z(t))$ be a unit-speed geodesic on Σ. According to Liberman's lemma $z(t)$ is convex.

Since the slope of f is at most ℓ, we have

$$|z'(t)| \leqslant \tfrac{\ell}{\sqrt{1+\ell^2}}.$$

If γ is defined on the interval $[a, b]$, then

$$\begin{aligned}\int\limits_a^b z''(t) = z'(b) - z'(a) \leqslant \\ \leqslant 2\cdot\tfrac{\ell}{\sqrt{1+\ell^2}}.\end{aligned}$$

Further, note that z'' is the projection of γ'' to the z-axis. Since f is ℓ-Lipschitz, the tangent plane $\mathrm{T}_{\gamma(t)}\Sigma$ cannot have slope greater than ℓ for any t. Because γ'' is perpendicular to that plane,

$$|\gamma''(t)| \leqslant z''(t)\cdot\sqrt{1+\ell^2}.$$

Recall that $\Phi(\gamma)$ denotes the total curvature of curve γ. It follows that

$$\begin{aligned}\Phi(\gamma) &= \int\limits_a^b |\gamma''(t)|\cdot dt \leqslant \\ &\leqslant \sqrt{1+\ell^2}\cdot\int\limits_a^b z''(t)\cdot dt \leqslant \\ &\leqslant 2\cdot\ell.\end{aligned}$$

□

11.20. Exercise. *Note that the graph $z = \ell\cdot\sqrt{x^2+y^2}$ with removed origin is a smooth surface; denote it by Σ. Show that it has an both side infinite geodesic γ with total curvature exactly $2\cdot\ell$.*

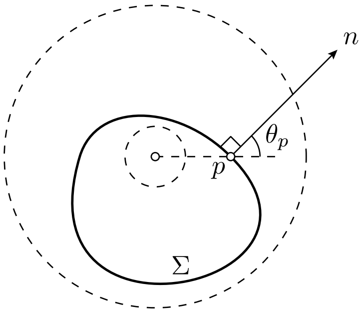
Note that the function $f(x, y) = \ell\cdot\sqrt{x^2+y^2}$ is ℓ-Lipschitz. The graph $z = f(x, y)$ in the exercise can be smoothed in a neighborhood of the origin while keeping it convex. It follows that the estimate in the Usov's theorem is optimal.

11.21. Exercise. *Assume f is a convex $\tfrac{3}{2}$-Lipschitz function defined on the (x, y)-plane. Show that any geodesic γ on the graph $z = f(x, y)$ is simple; that is, it has no self-intersections.*

Construct a convex 2-Lipschitz function defined on the (x, y)-plane with a nonsimple geodesic γ on its graph $z = f(x, y)$.

11.22. Theorem. *Suppose a smooth surface Σ bounds a convex set K in the Euclidean space. Assume $B(0,\varepsilon) \subset K \subset B(0,1)$. Then the total curvatures of any shortest path in Σ can be bounded in terms of ε.*

The following exercise will guide you thru the proof of the theorem.



11.23. Exercise. *Let Σ be as in the theorem and γ be a unit-speed shortest path in Σ. Denote by n_p the unit normal vector that points outside of Σ; denote by θ_p the angle between n_p and the direction from the origin to a point $p \in \Sigma$. Set $\rho(t) = |\gamma(t)|^2$; let $k(t)$ be the curvature of γ at t.*

(a) *Show that $\cos\theta_p \geqslant \varepsilon$ for any $p \in \Sigma$.*

(b) *Show that $|\rho'(t)| \leqslant 2$ for any t.*

(c) *Show that*
$$\rho''(t) = 2 - 2\cdot k(t)\cdot \cos\theta_{\gamma(t)}\cdot|\gamma(t)|$$
for any t.

(d) *Use the closest-point projection from the unit sphere to Σ to show that*
$$\operatorname{length}\gamma \leqslant \pi.$$

(e) *Use the the statements above to conclude that*
$$\Phi(\gamma) \leqslant \frac{100}{\varepsilon^2}.$$

Note that our bound on total curvature given above goes to infinity as $\varepsilon \to 0$, but in fact there is a bound independent of ε; it is good of any closed convex surface [31].

Chapter 12

Spherical map

Differential

Let $f\colon \Sigma \to \mathbb{R}^3$ be a smooth map defined on a smooth surface Σ; that is, for any chart s of Σ the composition $f \circ s$ is smooth.

Given a smooth curve γ in Σ consider the smooth curve $\hat{\gamma} = f \circ \gamma$. Assume γ starts at a point $p \in \Sigma$ with velocity vector $v \in \mathrm{T}_p\Sigma$; that is, $p = \gamma(0)$ and $v = \gamma'(0)$. The differential of v at p is defined as

❶ $$d_p f(v) = \hat{\gamma}'(0).$$

The domain of definition of $d_p f$ is the tangent plane $\mathrm{T}_p\Sigma$. The differential is an operator that produces a map $d_p f\colon \mathrm{T}_p\Sigma \to \mathbb{R}^3$ for given smooth map $f\colon \Sigma \to \mathbb{R}^3$ and $p \in \Sigma$.

Note that the value $d_p f(v)$ does not depend on the choice of γ; that is, if γ_1 is another curve in Σ such that $\gamma_1(0) = p$ and $\gamma_1'(0) = v$, then

❷ $$(f \circ \gamma)'(0) = (f \circ \gamma_1)'(0).$$

Indeed, $v = \gamma_1'(0) = \gamma'(0)$; therefore we have that

$$|\gamma_1(\varepsilon) - \gamma(\varepsilon)| = o(\varepsilon).$$

Since f is smooth,

$$|f \circ \gamma_1(\varepsilon) - f \circ \gamma(\varepsilon)| = o(\varepsilon);$$

whence ❷ follows.

Note that if Σ is a plane, then

$$d_p f(v) = D_v f,$$

where D_v denotes the direction derivative. For general surface $D_v f$ is not well defined since the point $p + t{\cdot}v$ may not lie in Σ for small values t.

12.1. Exercise. *Assume f is a smooth map from one surface Σ_0 to another Σ_1 and $p \in \Sigma_0$. Show that the range of $d_p f$ lies in the tangent plane $\mathrm{T}_{f(p)}\Sigma_1$.*

12.2. Proposition. *The differential is a linear map. That is, for any smooth map $f\colon \Sigma \to \mathbb{R}^3$ defined on a smooth surface Σ and $p \in \Sigma$, the map $d_p f\colon \mathrm{T}_p \to \mathbb{R}^3$ is linear.*

Proof. Fix a chart $(u, v) \mapsto s(u, v)$ on Σ that covers a neighborhood of p. Without loss of generality we may assume that $p = s(0, 0)$.

Any smooth curve γ that starts at p can be written locally in the chart as $\gamma(t) = s(u(t), v(t))$; since γ stats at p, we have $u(0) = v(0) = 0$. Applying the chain rule, we get

$$\begin{aligned} \gamma'(0) &= \tfrac{\partial s}{\partial u}(0,0){\cdot}u'(0) + \tfrac{\partial s}{\partial v}(0,0){\cdot}v'(0),\\ (f\circ\gamma)'(0) &= \tfrac{\partial f\circ s}{\partial u}(0,0){\cdot}u'(0) + \tfrac{\partial f\circ s}{\partial v}(0,0){\cdot}v'(0). \end{aligned}$$

The statement follows since $d_p(\gamma'(0)) = (f\circ\gamma)'(0)$. □

A more conceptual way*

In this section we introduce a more conceptual way to define tangent vectors. We will not use this approach in the sequel, but it is better to know about it.

A tangent vector $v \in \mathrm{T}_p$ to a smooth surface Σ is a linear functional[1] that takes a smooth function φ on Σ, spits out a real number denoted by $v\varphi$ and satisfies the product rule:

❸ $$v(\varphi{\cdot}\psi) = (v\varphi){\cdot}\psi(p) + \varphi(p){\cdot}(v\psi).$$

If v is a velocity vector of a smooth curve γ that starts at p, then you can set

❹ $$v\varphi = (\varphi\circ\gamma)'(0).$$

It follows that any velocity vector $v = \gamma'(0)$ is also a tangent vector in the new definition. It is not hard to check the converse as well; that is,

[1]Term *functional* is used for functions that take a function as an argument and return a number.

to any linear functional v satisfying the product rule there is a curve γ such that ❹ holds.

The new definition is less intuitive, but it is more convenient to use since it grabs the key algebraic property of tangent vectors. For example the differential could be defined using the following identity:

$$(d_p f(v))\varphi := v(\varphi \circ f);$$

that is, the result of the functional $d_p f(v)$ on the function $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$ is defined to be the same as the result of the functional v on the function $\varphi \circ f\colon \Sigma \to \mathbb{R}$.

Many statements admit simpler proofs with this approach, for example 12.2 becomes a tautology.

Spherical map

Let Σ be a smooth oriented surface with unit normal field n. The map $n\colon \Sigma \to \mathbb{S}^2$ defined by $p \mapsto n_p$ is called *spherical map* or *Gauss map.*

For surfaces, the spherical map plays essentially the same role as the tangent indicatrix for curves.

Recall that if $(u, v) \mapsto s(u, v)$ is a chart on Σ then

$$n(u, v) = \pm\frac{\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}}{\left|\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}\right|}.$$

Since s is a chart $\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v} \neq 0$, therefore the spherical map is smooth.

In particular the differential of the spherical map is defined at any point of Σ.

Shape operator

Suppose Σ is an oriented surface with the unit normal field n; in other words $n\colon \Sigma \to \mathbb{S}^2$ is its spherical map.

Fix a point $p \in \Sigma$. The *shape operator* at p is defined as

❺ $$S_p := -d_p n.$$

The shape operator S_p is defined on the tangent plane $\mathrm{T}_p\Sigma$ and it returns a vector in the same plane (otherwise we could not call it an *operator*). The latter is shown in the following proposition, which also gives a reason for the change of sign in ❺.

12.3. Proposition. *Suppose Σ is an oriented surface. Then for any $p \in \Sigma$ and any $v \in \mathrm{T}_p$ we have that $S_p(v) \in \mathrm{T}_p$. Moreover*

$$\langle S_p(v), w\rangle = \langle S_p(w), v\rangle = \mathrm{II}_p(v, w)$$

for any $v, w \in \mathrm{T}_p$.

Proof. Assume an oriented surface Σ is written locally as a graph $z = f(x,y)$ in the tangent-normal coordinates at $p \in \Sigma$. As usual we assume that the normal vector n_p points in the direction of the z-axis, in this case the normal vector at any point of the graph points up; that is, its z-coordinate is positive.

Consider the corresponding chart of Σ:

$$s(x,y) = (x, y, f(x,y)).$$

Denote by $n(x,y)$ the unit normal vector at $s(x,y)$; it is a shortcut notation for $n_{s(x,y)}$.

Note that $\frac{\partial s}{\partial x}(0,0) = (1,0,0)$ and $\frac{\partial s}{\partial y}(0,0) = (0,1,0)$. For a tangent vector

$$v = (a\cdot\tfrac{\partial s}{\partial x} + b\cdot\tfrac{\partial s}{\partial y})(0,0) = (a,b,0) \in \mathrm{T}_p$$

we have that

❻
$$\begin{aligned} S_p(v) &= -D_v n(0,0) = \\ &= -(a\cdot\tfrac{\partial n}{\partial x} + b\cdot\tfrac{\partial n}{\partial y})(0,0), \end{aligned}$$

where D_v denotes the directional derivative along a vector v in the (x,y)-plane which is T_p.

Indeed, the first equality follows from ❺ and the definition of differential ❶ applied for the curve $\gamma(t) = (a\cdot t, b\cdot t, f(a\cdot t, b\cdot t))$ at $t = 0$ and the second follow since $D_v = a\cdot\frac{\partial}{\partial x} + b\cdot\frac{\partial}{\partial y}$.

Taking partial derivatives of $\langle n, n\rangle = 1$, we get that

$$\begin{aligned} 0 &= \tfrac{\partial}{\partial x}\langle n, n\rangle = \\ &= 2\cdot\langle\tfrac{\partial n}{\partial x}, n\rangle. \end{aligned}$$

That is, $\langle\frac{\partial n}{\partial x}, n\rangle = 0$ and the same way we get $\langle\frac{\partial n}{\partial y}, n\rangle = 0$. By ❻ it follows that $S_p(v) \perp n_p$, or equivalently $S_p(v) \in \mathrm{T}_p$ for any $v \in \mathrm{T}_p$.

Further, since $\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y} \in \mathrm{T}_{s(x,y)}\Sigma$ and $n(x,y) \perp \mathrm{T}_{s(x,y)}\Sigma$, we have that

$$\langle n, \tfrac{\partial s}{\partial x}\rangle \equiv 0 \quad \text{and} \quad \langle n, \tfrac{\partial s}{\partial y}\rangle \equiv 0.$$

Taking a derivative of these identities, we get that

❼
$$\begin{aligned} \langle\tfrac{\partial n}{\partial x}, \tfrac{\partial s}{\partial x}\rangle + \langle n, \tfrac{\partial^2 s}{\partial x^2}\rangle &\equiv 0, \\ \langle\tfrac{\partial n}{\partial y}, \tfrac{\partial s}{\partial x}\rangle + \langle n, \tfrac{\partial^2 s}{\partial y\partial x}\rangle &\equiv 0, \\ \langle\tfrac{\partial n}{\partial x}, \tfrac{\partial s}{\partial y}\rangle + \langle n, \tfrac{\partial^2 s}{\partial x\partial y}\rangle &\equiv 0, \\ \langle\tfrac{\partial n}{\partial y}, \tfrac{\partial s}{\partial y}\rangle + \langle n, \tfrac{\partial^2 s}{\partial y^2}\rangle &\equiv 0, \end{aligned}$$

Fix two vectors $v = (a, b, 0)$ and $w = (c, d, 0)$ in T_p (which is the (x, y)-plane). Since $n(0, 0) = (0, 0, 1)$ we get

$$f(x, y) \equiv \langle n(0, 0), s(x, y)\rangle.$$

Therefore by ❻, ❼ and ❷ we get that

$$\begin{aligned}
\langle S_p(v), w\rangle &= -\langle a\cdot\tfrac{\partial n}{\partial x} + b\cdot\tfrac{\partial n}{\partial y}, c\cdot\tfrac{\partial s}{\partial x} + d\cdot\tfrac{\partial s}{\partial y}\rangle(0,0) = \\
&= -\big(a\cdot c\cdot\langle\tfrac{\partial n}{\partial x}, \tfrac{\partial s}{\partial x}\rangle + a\cdot d\cdot\langle\tfrac{\partial n}{\partial x}, \tfrac{\partial s}{\partial y}\rangle + \\
&\quad + b\cdot c\cdot\langle\tfrac{\partial n}{\partial y}, \tfrac{\partial s}{\partial x}\rangle + b\cdot d\cdot\langle\tfrac{\partial n}{\partial y}, \tfrac{\partial s}{\partial y}\rangle\big)(0,0) = \\
&= \big(a\cdot c\cdot\langle n, \tfrac{\partial^2 s}{\partial x^2}\rangle + a\cdot d\cdot\langle n, \tfrac{\partial^2 s}{\partial x\partial y}\rangle + \\
&\quad + b\cdot c\cdot\langle n, \tfrac{\partial^2 s}{\partial y\partial x}\rangle + b\cdot d\cdot\langle n, \tfrac{\partial^2 s}{\partial y^2}\rangle\big)(0,0) = \\
&= \big(a\cdot c\cdot\tfrac{\partial^2 f}{\partial x^2} + a\cdot d\cdot\tfrac{\partial^2 f}{\partial x\partial y} + \\
&\quad + b\cdot c\cdot\tfrac{\partial^2 f}{\partial y\partial x} + b\cdot d\cdot\tfrac{\partial^2 f}{\partial y^2}\big)(0,0) = \\
&= \mathrm{II}_p(v, w).
\end{aligned}$$

It remains to apply ❸ on page 73. □

Recall that

$$\mathrm{II}_p(w, v) = \langle M_p\cdot v, w\rangle,$$

where M_p is the Hessian matrix at p. It follows that

$$S_p(v) = M_p\cdot v$$

if v is written in the standard basis of the (x, y)-plane. Whence we get the following theorem.

12.4. Theorem. *Let Σ be a smooth oriented surface and $p \in \Sigma$. A nonzero tangent vector $v \in \mathrm{T}_p$ points in a principle direction at p if and only if $S_p(v) \parallel v$ and if so, then the unique coefficient k such that $S_p(v) = k\cdot v$ is the principle curvature in this direction.*

In particular $K(p) = \det S_p$ and $H(p) = \operatorname{trace} S_p$.

12.5. Exercise. *Let Σ be a smooth oriented surface with the unit normal field n. Suppose that Σ has unit normal curvature at any point in any direction.*

(a) Show that $S_p(w) = w$ for any $p \in \Sigma$ and $w \in \mathrm{T}_p\Sigma$.

(b) Show that $p+n_p$ is constant; that is, the point $c = p+n_p$ does not depend on $p \in \Sigma$. Conclude that Σ is a part of the unit sphere centered at c.

12.6. Exercise. *Assume that smooth surfaces Σ_1 and Σ_2 intersect at constant angle along a smooth regular curve γ. Show that if γ is a curvature line in Σ_1, then it is also a curvature line in Σ_2.*

Conclude that if a smooth surface Σ intersects a plane or sphere along a smooth curve γ, then γ is a curvature line of Σ.

12.7. Exercise. *Suppose that a geodesic γ on the surface Σ is also a curvature line. Show that γ lies in a plane.*

Area

Let Σ be a smooth surface and $h\colon \Sigma \to \mathbb{R}$ be a smooth function. Let us define the integral $\int_R h$ of the function h along a region $R \subset \Sigma$.

First assume that there is a chart $(u,v) \mapsto s(u,v)$ of Σ defined on an open set $U \subset \mathbb{R}^2$ such that $R \subset s(U)$. In this case set

$$❽ \qquad \int\limits_R h := \iint\limits_{s^{-1}(R)} h \circ s(u,v) \cdot \left|\tfrac{\partial s}{\partial v}(u,v) \times \tfrac{\partial s}{\partial u}(u,v)\right| \cdot du \cdot dv.$$

By substitution rule for multiple variables (A.12), the right hand side in ❽ does not depend on choice of s; that is, if $s_1\colon U_1 \to \Sigma$ is another chart such that $s_1(U_1) \supset R$, then

$$\iint\limits_{s^{-1}(R)} h \circ s \cdot \left|\tfrac{\partial s}{\partial v} \times \tfrac{\partial s}{\partial u}\right| \cdot du \cdot dv = \iint\limits_{s_1^{-1}(R)} h \circ s_1 \cdot \left|\tfrac{\partial s_1}{\partial v} \times \tfrac{\partial s_1}{\partial u}\right| \cdot du \cdot dv.$$

(In fact the factor $\left|\frac{\partial s}{\partial v} \times \frac{\partial s}{\partial u}\right|$ is chosen so to meet this property.)

For a general region R one could subdivide it into regions $R_1, R_2 \dots$ such that each R_i lies in the image of some chart. After that one could define the integral along R as the sum

$$\int\limits_R h = \int\limits_{R_1} h + \int\limits_{R_2} h + \dots$$

The area of R is defined as the integral

$$\operatorname{area} R = \int\limits_R 1.$$

Spherical image

12.8. Theorem. *Let Σ be an oriented proper surface without boundary and with positive Gauss curvature. Then the spherical map $n\colon \Sigma \to \mathbb{S}^2$ is injective and*

$$\int\limits_R K = \operatorname{area}[n(R)]$$

for any region R in Σ.

Proof. Lets show that the spherical map $n\colon \Sigma \to \mathbb{S}^2$ is injective. Fix two distinct points $p, q \in \Sigma$. Recall that Σ bounds a strictly convex region. Therefore the n_p makes an obtuse angle with the line segment $[p, q]$. The same way we can show that n_q makes an obtuse angle with the line segment $[q, p]$. In other words the projections of n_p and n_q on the line pq point in the opposite directions. In particular $n_p \neq n_q$; that is, the spherical map is injective.

Note that it is sufficient to prove the identity assuming that the region R is covered by one chart $(u, v) \mapsto s(u, v)$ of Σ; if not cut R into smaller regions and sum up the results. Applying the definition of integral, we have the following expression for the left hand side

$$\int\limits_R K := \iint\limits_{s^{-1}(R)} K[s(u,v)]\cdot|\tfrac{\partial s}{\partial v}(u,v) \times \tfrac{\partial s}{\partial u}(u,v)|\cdot du\cdot dv.$$

Applying the definition of area, we have the following expression for the right hand side

$$\operatorname{area}[n(R)] := \iint\limits_{s^{-1}(R)} |\tfrac{\partial n\circ s}{\partial v}(u,v) \times \tfrac{\partial n\circ s}{\partial u}(u,v)|\cdot du\cdot dv.$$

Therefore it is sufficient to show that

❾ $$\tfrac{\partial n\circ s}{\partial v}(u,v) \times \tfrac{\partial n\circ s}{\partial u}(u,v) = K[s(u,v)]\cdot \tfrac{\partial s}{\partial v}(u,v) \times \tfrac{\partial s}{\partial u}(u,v)$$

for any (u, v) in the domain of definition.

Fix a point $p = s(u, v)$. Recall that

$$\tfrac{\partial n\circ s}{\partial u} = S_p(\tfrac{\partial s}{\partial u}) \quad \text{and} \quad \tfrac{\partial n\circ s}{\partial v} = S_p(\tfrac{\partial s}{\partial v}).$$

Therefore

$$\tfrac{\partial n\circ s}{\partial v} \times \tfrac{\partial n\circ s}{\partial u} = \det S_p\cdot \tfrac{\partial s}{\partial v} \times \tfrac{\partial s}{\partial u}.$$

Since $K(p) = \det S_p$, ❾ follows. □

12.9. Exercise. *Let Σ be a closed surface with positive Gauss curvature. Show that*

$$\int\limits_{\Sigma} K = 4{\cdot}\pi.$$

12.10. Exercise. *Let Σ be an open surface with positive Gauss curvature. Show that*

$$\int\limits_{\Sigma} K \leqslant 2{\cdot}\pi.$$

Chapter 13

Parallel transport

Parallel fields

Let Σ be a smooth surface in the Euclidean space and $\gamma\colon [a,b]\to\Sigma$ be a smooth curve. A smooth vector-valued function $t\mapsto v(t)$ is called a *tangent field* on γ if the vector $v(t)$ lies in the tangent plane $\mathrm{T}_{\gamma(t)}\Sigma$ for each t.

A tangent field $v(t)$ on γ is called *parallel* if $v'(t)\perp\mathrm{T}_{\gamma(t)}$ for any t.

In general the family of tangent planes $\mathrm{T}_{\gamma(t)}\Sigma$ is not parallel. Therefore one cannot expect to have a truly parallel family $v(t)$ with $v'\equiv 0$. The condition $v'(t)\perp\mathrm{T}_{\gamma(t)}$ means that this family is as parallel as possible — it rotates together with the tangent plane, but does not rotate inside the plane.

Note that by the definition of geodesic, the velocity field $v(t) = \gamma'(t)$ of any geodesic γ is parallel along γ.

13.1. Exercise. *Let Σ be a smooth regular surface in the Euclidean space, $\gamma\colon [a,b]\to\Sigma$ a smooth curve and $v(t)$, $w(t)$ parallel vector fields along γ.*

(a) Show that $|v(t)|$ is constant.

(b) Show that the angle $\theta(t)$ between $v(t)$ and $w(t)$ is constant.

Parallel transport

Assume $p=\gamma(a)$ and $q=\gamma(b)$. Given a tangent vector $v\in\mathrm{T}_p$ there is unique parallel field $v(t)$ along γ such that $v(a)=v$. The latter follows from A.14; the uniqueness also follows from Exercise 13.1.

The vector $v(b)\in\mathrm{T}_q$ is called the *parallel transport* of v along γ and denoted as $\iota_\gamma(v)$.

From the Exercise 13.1, it follows that parallel transport $\iota_\gamma\colon \mathrm{T}_p \to \to \mathrm{T}_q$ is an an isometry; it depends on the choice of γ — for another curve γ_1 connecting p to q in Σ, the parallel transport $\iota_{\gamma_1}\colon \mathrm{T}_p \to \mathrm{T}_q$ might be different.

To interpret the parallel transport physically, think of walking along γ and carrying a perfectly balanced bike wheel in such a way that you touch only its axis and keep it normal to Σ. It should be physically evident that if the wheel is non-spinning at the starting point p, then it will not be spinning after stopping at q. (Indeed, by pushing the axis one cannot produce torque to spin the wheel.) The map that sends the initial position of the wheel to the final position is the parallel transport ι_γ.

This physical interpretation was suggested by Mark Levi [32]; it will be used further.

On a more formal level, one can choose a partition $a = t_0 < \ldots < < t_n = b$ of $[a, b]$ and consider the sequence of orthogonal projections $\varphi_i\colon \mathrm{T}_{\gamma(t_{i-1})} \to \mathrm{T}_{\gamma(t_i)}$. For a fine partition, the composition

$$\varphi_n \circ \cdots \circ \varphi_1\colon T_p \to \mathrm{T}_q$$

gives an approximation of ι_γ. Each φ_i does not increase the magnitude of a vector and neither the composition. It is straightforward to see that if if the partition is sufficiently fine, then it is almost isometry; in particular it almost preserves the magnitudes of tangent vectors.

13.2. Exercise. *Let γ be a smooth closed loop with base point p on a smooth oriented surface Σ with the unit normal field n. Suppose that the spherical image of γ lies in a great circle. Show that the parallel translation $\iota_\gamma\colon \mathrm{T}_p \to \mathrm{T}_p$ along γ is the identity map.*

Geodesic curvature

Plane is the simplest example of smooth surface. Earlier we introduced signed curvature of a plane curve. For a smooth curve γ in general oriented smooth surface Σ the analogous notion is called *geodesic curvature* which we are about to introduce.

Let $n\colon \Sigma \to \mathbb{S}^2$ be the spherical map that defines the orientation on Σ. Without loss of generality we can assume that γ has unit speed. Then for any t the vectors $n(t) = n(\gamma(t))_\Sigma$ and the velocity vector $\mathrm{T}(t) = \gamma'(t)$ are unit vectors that are normal to each other. Denote by $\mu(t)$ the unit vector that is normal to both $n(t)$ and $\mathrm{T}(t)$ that points to the left from γ; that is, $\mu = n \times \mathrm{T}$. Note that the triple $\mathrm{T}(t), \mu(t), n(t)$ is an oriented orthonormal basis for any t.

Since γ is unit-speed, the acceleration $\gamma''(t)$ is perpendicular to $\mathrm{T}(t)$; therefore at any parameter value t, we have

$$\gamma''(t) = k_g(t)\cdot\mu(t) - k_n(t)\cdot n(t),$$

for some real numbers $k_n(t)$ and $k_g(t)$. The numbers $k_n(t)$ and $k_g(t)$ are called *normal* and *geodesic curvature* of γ at t correspondingly.

Note that the geodesic curvature vanishes if γ is a geodesic. It measures how much a given curve diverges from being a geodesic; it is positive if γ turns left and negative if γ turns right.

Total geodesic curvature

The total geodesic curvature is defined as integral

$$\Psi(\gamma) := \int\limits_{\mathbb{I}} k_g(t)\cdot dt,$$

assuming that γ is defined on the real interval $\mathbb{I}$. Note that if Σ is a plane and γ lies in Σ then geodesic curvature of γ equals to signed curvature and therefore total geodesic curvature equals to the total signed curvature. By that reason we use the same notation $\Psi(\gamma)$ as for total signed curvature; if we need to emphasize that we consider γ as a curve in Σ, we write $\Psi(\gamma)_\Sigma$.

If γ is a piecewise smooth regular curve in Σ, then its total geodesic curvature is defined as a sum of all total geodesic curvature of its arcs and the sum signed exterior angles of γ at the joints. More precisely, if γ is a concatenation of smooth regular curves $\gamma_1, \dots, \gamma_n$ then

$$\Psi(\gamma) = \Psi(\gamma_1) + \dots + \Psi(\gamma_n) + \theta_1 + \dots + \theta_{n-1},$$

where θ_i is the signed external angle at the joint γ_i and γ_{i+1}; it is positive if we turn left and negative if we turn right, it is undefined if we turn to the opposite direction. If γ is closed, then

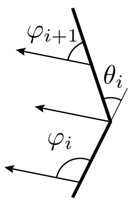
$$\Psi(\gamma) = \Psi(\gamma_1) + \dots + \Psi(\gamma_n) + \theta_1 + \dots + \theta_n,$$

where θ_n is the signed external angle at the joint γ_n and γ_1.

Note that if each arc γ_i in the concatenation is a geodesic, then γ is called *broken geodesic*. Note that in this case $\Psi(\gamma_i) = 0$ for each i and therefore the total geodesic curvature of γ is the sum of its signed external angles.

13.3. Proposition. *Assume γ is a closed broken geodesic in a smooth oriented surface Σ that starts and ends at the point p. Then the parallel* *transport* $\iota_\gamma\colon T_p \to \mathrm{T}_p$ *is a rotation of the the plane* T_p *clockwise by angle* $\Psi(\gamma)$.

Moreover, the same statement holds for smooth closed curves and piecewise smooth curves.



Proof. Assume γ is a cyclic concatenation of geodesics $\gamma_1,\dots,\gamma_n$. Fix a tangent vector v at p and extend it to a parallel vector field along γ. Since $w_i(t)=\gamma_i'(t)$ is parallel along γ_i, the angle φ_i between v and w_i stays constant on each γ_i.

If θ_i denotes the external angle at this vertex of switch from γ_i to γ_{i+1}, we have that

$$\varphi_{i+1}=\varphi_i-\theta_i \pmod{2\cdot\pi}.$$

Therefore after going around we get that

$$\varphi_{n+1}-\varphi_1=-\theta_1-\dots-\theta_n=-\Psi(\gamma).$$

Hence the the first statement follows.

For the smooth unit-speed curve $\gamma\colon[a,b]\to\Sigma$, the proof is analogous. If $\varphi(t)$ denotes the angle between $v(t)$ and $w(t)=\gamma'(t)$, then

$$\varphi'(t)+k_g(t)\equiv 0$$

Whence the angle of rotation

$$\begin{aligned}\varphi(b)-\varphi(a)&=\int_a^b\varphi'(t)\cdot dt=\\&=-\int_a^b k_g\cdot dt=\\&=-\Psi(\gamma)\end{aligned}$$

The case of piecewise regular smooth curve is a straightforward combination of the above two cases. □

Spherical area

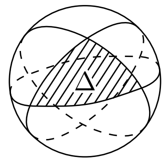
13.4. Lemma. *Let* Δ *be a spherical triangle; that is,* Δ *is the intersection of three closed half-spheres in the unit sphere* $\mathbb{S}^2$. *Then*

❶ $$\operatorname{area}\Delta=\alpha+\beta+\gamma-\pi,$$

where α, β *and* γ *are the angles of* Δ.

The value $\alpha+\beta+\gamma-\pi$ is called *excess* of the triangle Δ.

Proof. Recall that



❷ $$\operatorname{area}\mathbb{S}^2 = 4{\cdot}\pi.$$

Note that the area of a spherical slice S_α between two meridians meeting at angle α is proportional to α. Since for S_π is a half-sphere, from ❷, we get $\operatorname{area} S_\pi = 2{\cdot}\pi$. Therefore the coefficient is 2; that is,

❸ $$\operatorname{area} S_\alpha = 2{\cdot}\alpha.$$

Extending the sides of Δ we get 6 slices: two S_α, two S_β and two S_γ which cover most of the sphere once, but the triangle Δ and its centrally symmetric copy Δ' are covered 3 times. It follows that

$$2{\cdot}\operatorname{area} S_\alpha + 2{\cdot}\operatorname{area} S_\beta + 2{\cdot}\operatorname{area} S_\gamma = \operatorname{area}\mathbb{S}^2 + 4{\cdot}\operatorname{area}\Delta.$$

Substituting ❷ and ❸ and simplifying, we get ❶. □

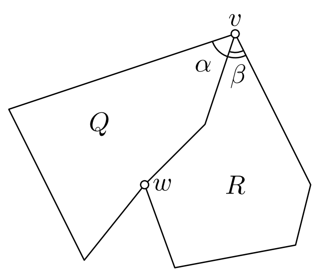
If the contour $\partial\Delta$ of a spherical triangle with angles α, β and γ is oriented such that the triangle lies on the left, then its external angles are $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$. Therefore the total geodesic curvature of $\partial\Delta$ is $\Psi(\partial\Delta) = 3{\cdot}\pi-\alpha-\beta-\gamma$. The identity ❶ can be rewritten as

❹ $$\Psi(\partial\Delta) + \operatorname{area}\Delta = 2{\cdot}\pi.$$

The formula ❹ holds for an arbitrary spherical polygon bounded by a simple broken geodesic. The latter can be proved by triangulating the poygon, applying the formula for each triangle in the triangulation and summing up the results.

If a spherical polygon is P divided in two polygons Q and R by polygonal line between vertexes v and w then

$$\Psi(\partial P) + 2{\cdot}\pi = \Psi(\partial Q) + \Psi(\partial R).$$



Indeed, for the internal angles Q and R at v are α and β, then their external angles are $\pi-\alpha$ and $\pi-\beta$ respectfully. The internal angle of P in this case is $\alpha+\beta$ and its external angle is $\pi-\alpha-\beta$ Clearly we have that

$$(\pi-\alpha) + (\pi-\beta) = (\pi-\alpha-\beta) + \pi;$$

that is, the sum of external angles of Q and R at v is π plus the external angle of P at v. The same holds for the external angles at

w and the rest of the external angles of P appear once on Q or R. Therefore if the formula ❹ holds for Q and R, then it holds for P.

The following proposition gives a spherical analog of 5.4.

13.5. Proposition. *Let P be a spherical polygon bounded by a simple closed broken geodesic ∂P. Assume ∂P is oriented such that P lies on the left from ∂P. Then*

$$\Psi(\partial P) + \operatorname{area} P = 2\cdot\pi.$$

Moreover the same formula holds for any spherical region P bounded by piecewise smooth simple closed curve ∂P.

Sketch of proof. The proof of the first statement is given above.

The second statement can be proved by approximation. One has to show that the total geodesic curvature of an inscribed broken geodesic approximates the total geodesic curvature of the original curve. We omit the proof of the latter statement, but it can be done along the same lines as 3.23. □

13.6. Exercise. *Assume γ is a simple piecewise smooth loop on $\mathbb{S}^2$ that divides its area into two equal parts. Denote by p the base point of γ. Show that $\iota_\gamma\colon \mathrm{T}_p\mathbb{S}^2 \to \mathrm{T}_p\mathbb{S}^2$ is the identity map.*

Gauss–Bonnet formula

13.7. Theorem. *Let Δ be a topological disc in a smooth oriented surface Σ bounded by a simple piecewise smooth and regular curve $\partial\Delta$ that is oriented in such a way that Δ lies on its left. Then*

❺ $$\Psi(\partial\Delta) + \int\limits_\Delta K = 2\cdot\pi,$$

where K denotes the Gauss curvature of Σ.

For geodesic triangles this theorem was proved by Carl Friedrich Gauss [33]; Pierre Bonnet and Jacques Binet independently generalized the statement for arbitrary curves.

Note that if Σ is a plane, then the Gauss curvatue vanished; therefore the statement of theorem follows from 5.4.

If Σ is the unit sphere then $K \equiv 1$ and therefore formula ❶ can be rewritten as

$$\Psi(\partial\Delta) + \operatorname{area}\Delta = 2\cdot\pi,$$

which follows from 13.5.

We will give an informal proof of 13.15 in a partial case based on the bike wheel interpretation described above. We suppose that it is intuitively clear that moving the axis of the wheel without changing its direction does not change the direction of the wheel's spikes.

More precisely, assume we keep the axis of a non-spinning bike wheel and perform the following two experiments:

(i) We move it around and bring the axis back to the original position. As a result the wheel might turn by some angle; let us measure this angle.

(ii) We move the direction of the axis the same way as before without moving the center of the wheel. After that we measure the angle of rotation.

Then the resulting angles in these two experiments is the same.

Consider a surface Σ with a Gauss map $n\colon \Sigma \to \mathbb{S}^2$. Note that for any point p on Σ, the tangent plane $\mathrm{T}_p\Sigma$ is parallel to the tangent plane $\mathrm{T}_{n(p)}\mathbb{S}^2$; so we can identify these tangent spaces. From the experiments above, we get the following:

13.8. Lemma. *Suppose α is a piecewise smooth regular curve in a smooth regular surface Σ which has a Gauss map $n\colon \Sigma \to \mathbb{S}^2$. Then the parallel transport along α in Σ coincides with the parallel transport along the curve $\beta = n\circ\alpha$ in $\mathbb{S}^2$.*

Proof of partial case of 13.15. We will prove the formula for proper surface Σ with positive Gauss curvature. In this case, by 12.8 the formula can be rewritten as

$$\Psi(\partial\Delta) + \operatorname{area}[n(\Delta)] = 2\cdot\pi. \tag{6}$$

The general case can be proved similarly, but one has to use the area formula (A.13) and oriented area surrounded by a spherical curve.

Fix $p \in \partial\Delta$; assume the loop α runs along $\partial\Delta$ so that Δ lies on the left from it. Consider the parallel translation $\iota\colon \mathrm{T}_p \to \mathrm{T}_p$ along α. According to 13.3, ι is a clockwise rotation by angle $\Psi(\alpha)_\Sigma$.

Set $\beta = n\circ\alpha$. According to 13.16, ι is also parallel translation along β in $\mathbb{S}^2$. In particular ι is a clockwise rotation by angle $\Psi(\beta)_{\mathbb{S}^2}$. By 13.5

$$\Psi(\beta)_{\mathbb{S}^2} + \operatorname{area}[n(\Delta)] = 2\cdot\pi.$$

Therefore ι is a counterclockwise rotation by $\operatorname{area}[n(\Delta)]$

Summarizing, the clockwise rotation by $\Psi(\alpha)_\Sigma$ is identical to a counterclockwise rotation by $\operatorname{area}[n(\Delta)]$. The rotations are identical if the angles are equal modulo $2\cdot\pi$. Therefore

$$\Psi(\partial\Delta)_\Sigma + \operatorname{area}[n(\Delta)] = 2\cdot\pi\cdot n \tag{7}$$

for an integer n.

It remains to show that $n = 1$. By ❼, this is so for a topological disc in a plane. One can think of a general disc Δ as about a result of a continuous deformation of a plane disc. The integer n cannot change in the process of deformation since the left hand side in ❼ is continuous along the deformation.

Let us redo the last argument more formally.

First assume that Δ lies in a local graph realization $z = f(x, y)$ of Σ. Consider one parameter family Σ_t of graphs $z = t{\cdot}f(x, y)$ and denote by Δ_t the corresponding disc in Σ_t, so $\Delta_1 = \Delta$ amd Δ_0 is its projection to the (x, y)-plane. Since Σ_0 is a plane domain, we have $\operatorname{area}[n_0(\Delta_0)] = 0$. Therefore by 5.4 we gave

$$\Psi(\partial\Delta_0)_{\Sigma_0} + \operatorname{area}[n_0(\Delta_0)] = 2{\cdot}\pi.$$

Note that

$$\Psi(\partial\Delta_t)_{\Sigma_t} + \operatorname{area}[n_t(\Delta_t)]$$

depends continuously on t. According to ❼, its value is a multiple of $2{\cdot}\pi$; therefore it has to be constant. Whence the Gauss–Bonnet formula follows.

If Δ does not lie in one graph, then one could divide it into smaller discs, apply the formula for each and sum up the result. The proof is done along the same lines as 13.5. □

13.9. Exercise. *Assume γ is a closed simple curve with constant geodesic curvature 1 in a smooth closed surface Σ with positive Gauss curvature. Show that*

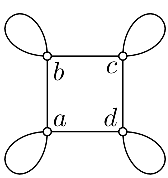
$$\operatorname{length}\gamma \leqslant 2{\cdot}\pi;$$

that is, the length of γ cannot exceed the length of the unit circle in the plane.

13.10. Exercise. *Let γ be a closed simple geodesic on a smooth closed surface Σ with positive Gauss curvature. Assume $n\colon \Sigma \to \mathbb{S}^2$ is a Gauss map. Show that the curve $\alpha = n \circ \gamma$ divides the sphere into regions of equal area.*

Conclude that

$$\operatorname{length}\alpha \geqslant 2{\cdot}\pi.$$

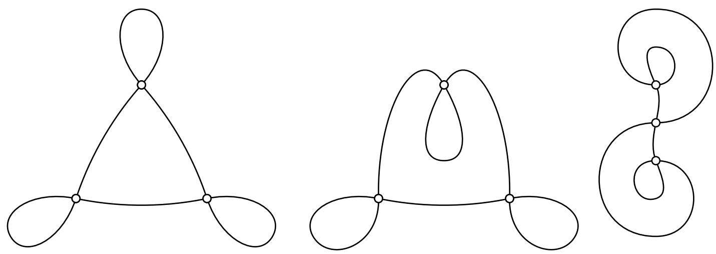


13.11. Exercise. *Let Σ be a smooth closed surface with a closed geodesic γ. Assume γ has exactly 4 self-intersection at the points a, b, c and d that appear on γ in the order a, a, b, b, c, c, d, d. Show that Σ cannot have positive Gauss curvature.*

The following exercise gives the optimal bound on Lipschitz constant of a convex function that guarantees that its geodesics have no self-intersections; compare to 11.21.

13.12. Exercise. *Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a $\sqrt{3}$-Lipschitz smooth convex function. Show that any geodesic in the surface defined by the graph $z = f(x, y)$ has no self-intersections.*

13.13. Advanced exercise. *Let Σ be a smooth regular sphere with positive Gauss curvature and $p \in \Sigma$. Suppose γ is a closed geodesic that does not pass thru p. Assume $\Sigma\backslash\{p\}$ parametrized by the plane. Can it happen that in this parametrization, γ looks like one of the curves on the diagram? Say as much as possible about possible/impossible*



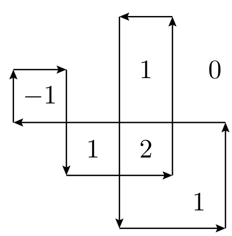
diagrams of that type.

Signed area

The formula ❹ holds modulo $2{\cdot}\pi$ for any closed broken geodesic, if one use *signed area* surrounded by curve instead of usual area; that is, we count area of the regions taking into account how many times the curve goes around the region.

Namely, we have to choose a *south pole* and state that its region has zero multiplicity. When you cross the curve the mulitplicity changes by ± 1; we add 1 if the curve crosses your path from left to right and we subtract 1 otherwise. The signed area surrounded by a closed curve is the sum of area of all the regions counted with multiplicities.

Here is an example of a broken line with multiplicities assuming that the big region has the south pole inside.



This signed-area formula can be proved in a similar way: Apply the formula for each triangle with vertex at the north pole and base at each edge of the broken geodesic. Sum the resulting identities taking each with a sign: plus

if the triangle lies on the left from the edge and
minus if the triangle lies on the right from edge.

Choosing a different pole will change all the coefficients by the same number. So the resulting formula holds only modulo the area of $\mathbb{S}^2$, which is $4\cdot\pi$ — this will not destroy identity modulo $2\cdot\pi$.

Furthermore, by approximation, the signed-area formula holds for any reasonable curve, say piecewise smooth regular curves on the sphere. Summarizing, we hope the discussion above convinced the reader that the following statement holds.

A domain Δ in a surface is called a *disc* (or more precisely *topological disc*) if it is bounded by a closed simple curve and can be parameterized by a unit plane disc

$$\mathbb{D} = \left\{ (x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leqslant 1 \right\}.$$

That is, there is a continuous bijection $\mathbb{D} \to \Delta$.

13.14. Proposition. *For any closed piecewise smooth regular curve α on the sphere, we have that*

$$\Psi(\alpha) + \operatorname{area}\alpha = 0 \pmod{2\cdot\pi},$$

where $\operatorname{area}\alpha$ *denotes the signed area surrounded by α and $\Psi(\alpha)$ the total geodesic curvature of α.*

Moreover, if α is a simple curve that bounds a disc Δ on the left from it, then we have

$$\Psi(\alpha) + \operatorname{area}\Delta = 2\cdot\pi.$$

13.1 Gauss–Bonnet formula

13.15. Theorem. *Let Δ be a disc in a smooth oriented surface Σ bounded by a simple piecewise smooth and regular curve $\partial\Delta$ that is oriented in such a way that Δ lies on its left. Then*

❶ $$\Psi(\partial\Delta) + \iint\limits_{\Delta} G = 2\cdot\pi,$$

where G denotes the Gauss curvature of Σ.

For geodesic triangles this theorem was proved by Carl Friedrich Gauss [33]; Pierre Bonnet and Jacques Binet independently generalized the statement for arbitrary curves. The modern formulation described below was given by Wilhelm Blaschke.

Remarks; (1). For a general compact domain Δ (not necessary a disc) we have that

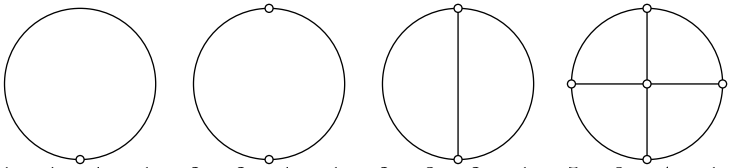
❷ $$\Psi(\partial\Delta) + \iint\limits_{\Delta} G = 2{\cdot}\pi{\cdot}\chi(\Delta),$$

where $\chi(\Delta)$ is the so called *Euler's characteristic* of Δ. The Euler's characteristic is *topological invariant*, in particular preserved in a continuous deformation.

If a surface Σ (possibly with boundary) can be divided into f discs by drawing e edges connecting v vertexes, then

$$\chi(\Sigma) = v - e + f.$$

For example the disc $\mathbb{D}$ has Euler's characteristic 1; it can be divided



$1 - 1 + 1 = 1$; $2 - 2 + 1 = 1$; $2 - 3 + 2 = 1$; $5 - 8 + 4 = 1$.

into discs many ways, but each time we have $v - e = f = 1$. The latter agrees with ❶ and ❷. It is useful to know that $\chi(\mathbb{S}^2) = 2$; $\chi(\mathbb{T}^2) = 0$ where $\mathbb{T}^2$ denotes torus; $\chi(S_g) = 2 - 2{\cdot}g$, where S_g is a surface of genus g; that is, sphere with g handles.

(2). Note that if Σ is a plane then a geodesic in Σ are formed by line segments. In this case the statement of theorem follows from Exercise **??**.

(3). If Σ is the unit sphere then $G \equiv 1$ and therefore formula ❶ can be rewritten as

$$\Psi(\partial\Delta) + \operatorname{area}\Delta = 2{\cdot}\pi,$$

which follows from Proposition 13.14.

We will give an informal proof of 13.15 based on the bike wheel interpretation described above. We suppose that it is intuitively clear that moving the axis of the wheel without changing its direction does not change the direction of the wheel's spikes.

More precisely, assume we keep the axis of a non-spinning bike wheel and perform the following two experiments:

(i) We move it around and bring the axis back to the original position. As a result the wheel might rotate by some angle; let us measure this angle.

(ii) We move the direction of the axis the same way as before without moving the center of the wheel. After that we measure the angle of rotation.

Then the resulting angle in these two experiments is the same.

Consider a oriented smooth surface Σ with the spherical; map $n\colon \Sigma \to \mathbb{S}^2$. Note that for any point p on Σ, the tangent plane $\mathrm{T}_p\Sigma$ is parallel to the tangent plane $\mathrm{T}_{n(p)}\mathbb{S}^2$; so we can identify these tangent spaces. From the experiments above, we get the following:

13.16. Lemma. *Suppose α is a piecewise smooth regular curve in a smooth regular surface Σ which has a Gauss map $n\colon \Sigma \to \mathbb{S}^2$. Then the parallel transport along α in Σ coincides with the parallel transport along the curve $\beta = n \circ \alpha$ in $\mathbb{S}^2$.*

13.17. Exercise. *Let Σ be a smooth closed surface with positive Gauss curvature. Given a line ℓ denote by ω_ℓ the closed curve formed by points with tangent planes parallel to ℓ.*[1] *Show that parallel transport around ω_ℓ is the identity map.*

Now we are ready to prove the theorem.

Proof of 13.15. Let α be the boundary $\partial\Delta$ parameterized in such a way that Δ lies on the left from it. Assume p is the point where α starts and ends.

Set $\beta = n \circ \gamma$ and $q = n(p)$, so the spherical curve β starts and ends at q.

By Lemma 13.16 the parallel transport along α in Σ coincides with the parallel transport along the curve β in $\mathbb{S}^2$. By Proposition 13.3, it follows that

$$\Psi(\alpha, \Sigma) = \Psi(\beta, \mathbb{S}^2) \pmod{2\cdot\pi}.$$

By Proposition 13.14,

$$\Psi(\beta, \mathbb{S}^2) + \operatorname{area}\beta = 0 \pmod{2\cdot\pi}.$$

Therefore

$$\Psi(\alpha, \Sigma) + \operatorname{area}\beta = 0 \pmod{2\cdot\pi}.$$

Recall that the shape operator $s_p\colon T_p\Sigma \to \mathrm{T}_{n(p)}\mathbb{S}^2 = T_p\Sigma$ is the Jacobian of the Gauss map $n\colon \Sigma \to \mathbb{S}^2$ at the point p. In appropriately chosen coordinates in T_p, the shape operator can be presented by a diagonal matrix $\left(\begin{smallmatrix} k_1 & 0 \\ 0 & k_2 \end{smallmatrix}\right)$, where k_1 and k_2 are the principle curvatures at p. Therefore, the determinant of s_p is the Gauss curvature at p.

[1] Equivalently the normal vector at any point of ω_ℓ is perpendicular to ℓ. If the light falls on Σ from one side parallel to ℓ, then ω_ℓ divides the bright and dark sides of Σ.

If Σ is a closed surface with positive Gauss curvature, then the Gauss map $n\colon \Sigma \to \mathbb{S}^2$ is a smooth bijection. Therefore

$$\iint\limits_{\Delta} G = \operatorname{area}[n(\Delta)].$$

In the general case we have to count the area $n(\Delta)$ taking orientation and multiplicity of the Gauss map into account. In this case

$$\iint\limits_{\Delta} G = \operatorname{area}\beta,$$

where $\operatorname{area}\beta$ is the signed area surrounded by β; it is defined above. Therefore

❸ $$\Psi(\alpha,\Sigma) + \iint\limits_{\Delta} G = 0 \pmod{2\cdot\pi}.$$

If Δ is a disc in the plane then Gauss curvature vanishes and by Exercise **??**, we have

$$\Psi(\partial\Delta) + \iint\limits_{\Delta} G = 2\cdot\pi.$$

Assunme that Σ_t is a smooth one parameter family of surfaces with a one parameter family of discs $\Delta_t \subset \Sigma_t$ and α_t is the boundary $\partial\Delta_t$ parameterized in such a way that Δ_t lies on the left from it. The value

$$f(t) = \Psi(\alpha_t) + \iint\limits_{\Delta} G$$

is continuous in t and by ❸ it has to be constant.

If Σ_0 is a plane, then

$$\Psi(\partial\Delta_0) + \iint\limits_{\Delta_0} G = 2\cdot\pi.$$

Intuitively it is clear that any disc can be obtained as a resultof continuous deformation of plane disc. Therefore

$$\Psi(\partial\Delta_1) + \iint\limits_{\Delta_1} G = 2\cdot\pi$$

for arbitrary disc Δ_1; whence ❶ follows. $\square$

13.2 The remarkable theorem

Let Σ_1 and Σ_2 be two smooth regular surfaces in the Euclidean space. A map $f\colon \Sigma_1 \to \Sigma_2$ is called length-preserving if for any curve γ_1 in Σ_1 the curve $\gamma_2 = f\circ\gamma_1$ in Σ_2 has the same length. If in addition f is smooth and bijective then it is called *intrinsic isometry.*

A simple example of intrinsic isometry can obtained by warping a plane into a cylinder. The following exercise produce slightly more interesting example.

13.18. Exercise. *Suppose $\gamma(t) = (x(t), y(t))$ is a smooth unit-curve in the plane such that $y(t) = a\cdot\cos t$. Let Σ_γ be the surface of revolution of γ around the x-axis. Show that a small open domain in Σ_γ admits a smooth length-pereserving map to the unit sphere.*

Conclude that any round disc Δ in $\mathbb{S}^2$ of intrinsic radius smaller than $\frac{\pi}{2}$ admits a smooth length preserving deformation; that is, there is one parameter family of surfaces with boundary Δ_t, such that $\Delta_0 = \Delta$ and Δ_t is not congruent to Δ_0 for any $t \neq 0$.[2]

13.19. Theorem. *Suppose $f\colon \Sigma_1 \to \Sigma_2$ is an intrinsic isometry between two smooth regular surfaces in the Euclidean space; $p_1 \in \Sigma_1$ and $p_2 = f(p_1) \in \Sigma_1$. Then*

$$G(p_1)_{\Sigma_1} = G(p_2)_{\Sigma_2};$$

that is, the Gauss curvature of Σ_1 at p_1 is the same as the Gauss curvature of Σ_2 at p_2.

This theorem was proved by Carl Friedrich Gauss [33] who called it *Remarkable theorem* (Theorema Egregium). The theorem is indeed remarkable because the Gauss curvature is defined as a product of principle curvatures which might be different at these points; however, according to the theorem, their product can not change.

In fact Gauss curvature of the surface at the given point can be found *intrinsically*, by measuring the lengths of curves in the surface. For example, Gauss curvature $G(p)$ in the following formula for the circumference $c(r)$ of a geodesic circle centered at p in a surface:

$$c(r) = 2\cdot\pi\cdot r - \tfrac{\pi}{3}\cdot G(p)\cdot r^3 + o(r^3).$$

Note that the theorem implies there is no smooth length-preserving map that sends an open region in the unit sphere to the plane.[3] It

[2] In fact any disc in $\mathbb{S}^2$ of intrinsic radius smaller than π admits a smooth length preserving deformation.

[3] There are plenty of non-smooth length-preserving maps from the sphere to the plane; see [34] and the references there in.

follows since the Gauss curvature of the plane is zero and the unit sphere has Gauss curvature 1. In other words, there is no map of a region on Earth without distortion.

Proof. Set $g_1 = G(p_1)_{\Sigma_1}$ and $g_2 = G(p_2)_{\Sigma_2}$; we need to show that

❶ $$g_1 = g_2.$$

Suppose Δ_1 is a small geodesic triangle in Σ_1 that contains p_1. Set $\Delta_2 = f(\Delta_1)$. We may assume that the Gauss curvature is almost constant in Δ_1 and Δ_2; that is, given $\varepsilon > 0$, we can assume that

❷ $$\begin{aligned} |G(x_1)_{\Sigma_1} - g_1| &< \varepsilon, \\ |G(x_2)_{\Sigma_2} - g_2| &< \varepsilon \end{aligned}$$

for any $x_1 \in \Delta_1$ and $x_2 \in \Delta_2$.

Since f is length-preserving the triangles Δ_2 is geodesic and

❸ $$\operatorname{area}\Delta_1 = \operatorname{area}\Delta_2.$$

Moreover, triangles Δ_1 and Δ_2 have the same corresponding angles; denote them by α, β and γ.

By Gauss–Bonnet formula, we get that

❹ $$\iint\limits_{\Delta_1} G_{\Sigma_1} = \alpha + \beta + \gamma - \pi = \iint\limits_{\Delta_2} G_{\Sigma_2}.$$

By ❷,

$$\left| g_1 - \frac{1}{\operatorname{area}\Delta_1} \cdot \iint\limits_{\Delta_1} G_{\Sigma_1} \right| < \varepsilon,$$
$$\left| g_2 - \frac{1}{\operatorname{area}\Delta_2} \cdot \iint\limits_{\Delta_2} G_{\Sigma_2} \right| < \varepsilon.$$

By ❸ and ❹,

$$\frac{1}{\operatorname{area}\Delta_1} \cdot \iint\limits_{\Delta_1} G_{\Sigma_1} = \frac{1}{\operatorname{area}\Delta_2} \cdot \iint\limits_{\Delta_2} G_{\Sigma_2},$$

therefore

$$|g_1 - g_2| < 2\cdot\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, ❶ follows. □

13.3 Simple geodesic

The following theorem provides an interesting application of Gauss–Bonnet formula; it is proved by Stephan Cohn-Vossen [Satz 9 in 35].

13.20. Theroem. *Any open smooth regular surface with positive Gauss curvature has a simple two-sided infinite geodesic.*

13.21. Lemma. *Suppose Σ is an open surface in with positive Gauss curvature in the Euclidean space. Then there is a convex function f defined on a convex open region of (x,y)-plane such that Σ can be presented as a graph $z=f(x,y)$ in some (x,y,z)-coordinate system of the Euclidean space.*

Moreover

❶ $$\iint\limits_{\Sigma} G\leqslant 2\cdot\pi.$$

Proof. The surface Σ is a boundary of an unbounded closed convex set K.

Fix $p\in\Sigma$ and consider a sequence of points x_n such that $|x_n-p|\to\to\infty$ as $n\to\infty$. Set $u_n=\frac{x_n-p}{|x_n-p|}$; the unit vector in the direction from p to x_n. Since the unit sphere is compact, we can pass to a subsequence of (x_n) such that u_n converges to a unit vector u.

Note that for any $q\in\Sigma$, the directions $v_n=\frac{x_n-q}{|x_n-q|}$ converge to u as well. The half-line from q in the direction of u lies in K. Indeed any point on the half-line is a limit of points on the line segments $[q,x_n]$; since K is closed, all of these poins lie in K.

Let us choose the z-axis in the direction of u. Note that line segments can not lie in Σ, otherwise its Gauss curvature would vanish. It follows that any vertical line can intersect Σ at most at one point. That is, Σ is a graph of a function $z=f(x,y)$. Since K is convex, the function f is convex and it is defined in a region Ω which is convex. The domain Ω is the projection of Σ to the (x,y)-plane. This projection is injective and by the inverse function theorem, it maps open sets in Σ to open sets in the plane; hence Ω is open.

It follows that the outer normal vectors to Σ at any point, points to the south hemisphere $\mathbb{S}^2_-=\{\,(x,y,z)\in\mathbb{S}^2\;:\;z<0\,\}$. Therefore the area of the spherical image of Σ is at most $\operatorname{area}\mathbb{S}^2_-=2\cdot\pi$. The area of this image is the integral of the Gauss curvature along Σ. That is,

$$\begin{aligned}\iint\limits_{\Sigma} G &= \operatorname{area}[n(\Sigma)] \leqslant \\ &\leqslant \operatorname{area}\mathbb{S}^2_- = \\ &= 2{\cdot}\pi,\end{aligned}$$

where $n(p)$ denotes the outer unit normal vector at p. Hence ❶ follows. □

Proof of 13.20. Let Σ be an open surface in with positive Gauss curvature and γ a two-sided infinite geodesic in Σ. The following is the key statement in the proof.

13.22. Claim. *The geodesic γ contains at most one simple loop.*

Assume γ has a simple loop ℓ. By Lemma 13.21, Σ is parameterized by a open convex region Ω in the plane; therefore ℓ bounds a disc in Σ; denote it by Δ. If φ is the angle at the base of the loop, then by Gauss–Bonnet,

$$\iint\limits_{\Delta} G = \pi + \varphi.$$

By Lemma 13.21, $\varphi < \pi$; that is, γ has no concave simple loops

Assume γ has two simple loops, say ℓ_1 and ℓ_2 that bound discs Δ_1 and Δ_2. Then the disks Δ_1 and Δ_2 have to overlap, otherwise the curvature of Σ would exceed $2{\cdot}\pi$.

We may assume that $\Delta_1 \not\subset \Delta_2$; the loop ℓ_2 appears after ℓ_1 on γ and there are no other simple loops between them. In this case, after going around ℓ_1 and before closing ℓ_2, the curve γ must enter Δ_1 creating a concave loop. The latter contradicts the above observation.

If a geodesic γ has a self-intersection, then it contains a simple loop. From above, there is only one such loop; it cuts a disk from Σ and goes around it either clockwise or counterclockwise. This way we divide all the self-intersecting geodesics into two sets which we will call *clockwise* and *counterclockwise*.

Note that the geodesic $t \mapsto \gamma(t)$ is clockwise if and only if the same geodesic traveled backwards $t \mapsto \gamma(-t)$ is counterclockwise. By shooting unit-speed geodesics in all directions at a given point $p = \gamma(0)$, we get a one parameter family of geodesics γ_s for $s \in [0, \pi]$ connecting the geodesic $t \mapsto \gamma(t)$ with the $t \mapsto \gamma(-t)$; that is, $\gamma_0(t) = \gamma(t)$ and $\gamma_\pi(t) = \gamma(-t)$. It follows that there are geodesics which aren't clockwise nor counterclockwise. Those geodesics have no self-intersections. □

Chapter 14

Local comparison

14.1 First variation formula

14.1. Proposition. *Assume $(s,t)\mapsto w(s,t)$ be a local parametrization of an oriented smooth regular surface Σ such that $\frac{\partial}{\partial s}w\perp\frac{\partial}{\partial t}w$, $|\frac{\partial}{\partial s}w|=1$ and the vector $\frac{\partial}{\partial s}w$ points to the right from $\frac{\partial}{\partial t}w$ at any parameter value (s,t).*

Fix a closed real interval $[a,b]$ and consider a one parameter family of curves $\sigma_s\colon[a,b]\to\Sigma$ defined as the coordinate lines $\sigma_s(t)=w(s,t)$. Set $\ell(s)=\operatorname{length}\sigma_s$. Then

$$\ell'(s)=\Theta_{\sigma_s}$$

for any s.

The proof is done by direct calculations.

Proof. Since $\frac{\partial}{\partial s}w\perp\frac{\partial}{\partial t}w$, we have that

$$\langle\tfrac{\partial}{\partial s}w,\tfrac{\partial}{\partial t}w\rangle=0$$

and therefore

$$\langle\tfrac{\partial^2}{\partial s\partial t}w,\tfrac{\partial}{\partial t}w\rangle+\langle\tfrac{\partial}{\partial s}w,\tfrac{\partial^2}{\partial t^2}w\rangle=\tfrac{\partial}{\partial t}\langle\tfrac{\partial}{\partial s}w,\tfrac{\partial}{\partial t}w\rangle=0.$$

Note that $|\gamma_s'(t)| = |\frac{\partial}{\partial t} w(s,t)|$ and therefore

$$\begin{aligned}
\tfrac{\partial}{\partial s}|\gamma_s'(t)| &= \tfrac{\partial}{\partial s}\sqrt{\langle \tfrac{\partial}{\partial t} w(s,t), \tfrac{\partial}{\partial t} w(s,t)\rangle} = \\
&= \frac{\langle \frac{\partial^2}{\partial s \partial t} w(s,t), \frac{\partial}{\partial t} w(s,t)\rangle}{\sqrt{\langle \frac{\partial}{\partial t} w(s,t), \frac{\partial}{\partial t} w(s,t)\rangle}} = \\
&= -\frac{\langle \frac{\partial}{\partial s} w, \frac{\partial^2}{\partial t^2} w\rangle}{|\gamma_s'(t)|} = \\
&= -\frac{\langle \frac{\partial}{\partial s} w, \gamma_s''(t)\rangle}{|\gamma_s'(t)|}.
\end{aligned}$$

The values $\ell(s)$ do not change if we reparametrize γ_s, so we can assume that for a fixed value s the curve σ_s is unit-speed. Since $|\frac{\partial}{\partial s} w| = 1$ and $\frac{\partial}{\partial s} w$ points to the right from $\frac{\partial}{\partial t} w = \gamma_s'(t)$, the last expression equals to $k_g(s,t)$, where $k_g(s,t)$ denotes the geodesic curvature of σ_s at t. Therefore, for this particular s we have

$$\begin{aligned}
\ell'(s) &= \int_a^b \tfrac{\partial}{\partial s}|\gamma_s'(t)|\cdot dt = \\
&= \int_a^b k_g(s,t)\cdot dt = \\
&= \Theta_{\sigma_s}.
\end{aligned}$$

Since the left hand side and the right hands side of this formula do not depend on the parametrization of σ_s, this formula holds for all s.[1] $\square$

The parametrization of a surface satisfying the conditions in the proposition are called *semigeodesic coordinates.* The following exercise explains the reason for this name.

14.2. Exercise. *Assume $(s,t) \mapsto w(s,t)$ be a local parametrization of an oriented smooth regular surface Σ as in the proposition above. Show that for any fixed t the curve $\gamma_t(s) = w(s,t)$ is a geodesic.*

[1] One may avoid passing the a unit-speed parametrization by using the following formula for geodesic curvature which holds for any regular parametrization:

$$k_g(t,s) = \langle n(\sigma_s(t)), [\sigma_s'(t), \sigma_s''(t)]\rangle / |\sigma_s'(t)|^3;$$

it saves thinking but makes the calculations longer.

14.2 Exponential map

Let Σ be smooth regular surface and $p \in \Sigma$. Given a tangent vector $v \in \mathrm{T}_p$ consider a geodesic γ_v in Σ that runs from p with the initial velocity v; that is, $\gamma(0) = p$ and $\gamma'(0) = v$.

The point $q = \gamma_v(1)$ is called *exponential map* of v, or briefly $q = \exp_p v$. The map $\exp_p \colon \mathrm{T}_p \to \Sigma$ is defined in a neighborhood of zero. We assume that it is intuitively obvious that the map $\exp_p$ is smooth; formally it follows since the solution of the initial value problem for the equation $\gamma_v''(t) \perp \mathrm{T}_{\gamma_v(t)}$ which describes the geodesic γ_v smoothly depend on the initial data v. Note that the Jacobian of $\exp_p$ at zero is the identity matrix. Therefore frm the inverse function theorem we get the following statement:

14.3. Proposition. *Let Σ be smooth regular surface and $p \in \Sigma$. Then the exponential map $\exp_p \colon \mathrm{T}_p \to \Sigma$ is a smooth regular parametrization of a neighborhood of p in Σ by a neighborhood of 0 in the tangent plane T_p.*

Moreover for any $p \in \Sigma$ there is $\varepsilon > 0$ such that for any $x \in \Sigma$ such that $|x - p|_\Sigma < \varepsilon$ the map $\exp_x \colon \mathrm{T}_x \to \Sigma$ is a smooth regular parametrization of the ε-neighborhood of x in Σ by the ε-neighborhood of zero in the tangent plane T_x.

Note that if there are two minimizing geodesics between two points x and y in a surface, then there are two distinct vectors $v, v' \in \mathrm{T}_x$ such that $y = \exp_x v = \exp_x v'$. Therefore by the above proposition we get the following:

14.4. Corollary. *Let Σ be a smooth regular surface. Then for any point $p \in \Sigma$ there is $\varepsilon > 0$ such that any two points x and y in the ε-neightbohood of p in Σ can be connected by a unique minimizing geodesic $[xy]_\Sigma$.*

14.3 Polar coordinates

Proposition 14.3 implies existence of polar coordinates in a neighborhood of any point in p in Σ. That is, any point x in Σ sufficiently close to p can be uniquely described by the distance $|x - p|_\Sigma$ and the direction from p to x.

Assume (θ, r) are the described polar coordinates at p. Namely, assume $\tilde w(\theta, r)$ denotes the tangent vector at p with polar coordinates (θ, r) and $w(\theta, r) = \exp_p[\tilde w(\theta, r)]$. By the definition of exponential map, for a fixed θ, the curve $\gamma_\theta(t) = w(\theta, t)$ is a unit-speed geodesic that starts at p; in particular $|\frac{\partial}{\partial r} w| = |\gamma_\theta'(r)| = 1$ and $\gamma_\theta''(r) \perp \mathrm{T}_{\gamma_\theta(r)}$.

The curve $\sigma_r(t)=w(t,r)$ is a parametrization of the circle of radius r and center at p in Σ; that is, if $q=\sigma_r(t)$, then $|q-p|_\Sigma=r$. If the latter is not the case, then a minimizing geodesic $[pq]_\Sigma$ would be shorter than r and therefore q would not be described uniquely in the polar coordinates.

Note that $\tfrac{\partial}{\partial r}w\perp\tfrac{\partial}{\partial\theta}w$ if $r>0$; otherwise for small $\varepsilon>0$ the intrinsic distance from p to $w(\theta\pm\varepsilon,r)$ would be shorter than r, which contradicts the previous statement.

14.5. Proposition. *Let $w(\theta,r)$ and $\tilde w(\theta,r)$ be the polar coordinates of a surface Σ at p and its tangent plane T_p at zero, so $w(\theta,r)=$ $=\exp_p[\tilde w(\theta,r)]$. Given a real interval $[a,b]$ consider the one parameter families of circular arcs $\sigma_r\colon[a,b]\to\Sigma$ and $\tilde\sigma_r\colon[a,b]\to\mathrm{T}_p$ $\sigma_r(t)=$ $=w(t,r)$ and $\tilde\sigma_r(t)=\tilde w(t,r)$. Set $\ell(r)=\operatorname{length}\sigma_r$ and $\tilde\ell(r)=$ $=\operatorname{length}\tilde\sigma_r$.*[2]

(i) If the Gauss curvature of Σ is nonnegative, then

$$\ell(r)\leqslant\tilde\ell(r)$$

for all small $r>0$.

(ii) If the Gauss curvature of Σ is nonpositive, then

$$\ell(r)\geqslant\tilde\ell(r)$$

for all small $r>0$.

Taking a limit as $b\to a$, we obtain the following corollary.

14.6. Corollary. *Let $w(\theta,r)$ and $\tilde w(\theta,r)$ be the polar coordinates of a surface Σ at p and its tangent plane T_p at zero, so $w(\theta,r)=$ $=\exp_p[\tilde w(\theta,r)]$.*

(i) If the Gauss curvature of Σ is nonnegative, then

$$|\tfrac{\partial}{\partial\theta}w|\leqslant|\tfrac{\partial}{\partial\theta}\tilde w|$$

for all small $r>0$.

(ii) If the Gauss curvature of Σ is nonpositive, then

$$|\tfrac{\partial}{\partial\theta}w|\geqslant|\tfrac{\partial}{\partial\theta}\tilde w|$$

for all small $r>0$.

Proof. From the above discussion, the polar coordinates $w(\theta,r)$ are semigeodesic; that is, $w(\theta,r)$ satisfies the conditions in the first variation formula (14.1). In particular if $\ell(r)=\operatorname{length}\sigma_r$, then

$$\ell'(r)=\Theta_{\sigma_r}$$

[2] Note that angular measure of $\tilde\sigma_r$ is $b-a$; therefore $\tilde\ell(r)=r\cdot(b-a)$.

for any $r > 0$.

By Gauss–Bonnet formula, the last identity can be rewritten as

❶ $$\ell'(r) = 2{\cdot}(b-a) - \iint\limits_{\Delta_r} G,$$

where Δ_r is the sector in Σ in the polar coordinates at p

$$\{\, w(t,s) \; : \; a \leqslant t \leqslant b, \; 0 \leqslant s \leqslant r \,\};$$

which is bounded by two geodesics from p with angle $b-a$ and a circular arc that meets these geodesics at right angle.

Since the plane has vanishing Gauss curvature, we have

❷ $$\tilde\ell'(r) = 2{\cdot}(b-a),$$

which agrees with the formula for the length of the arc $\tilde\ell(r) = 2{\cdot}\pi{\cdot}r$.

If the Gauss curvature of Σ is nonnegative, the equations ❶ and ❷ imply that

$$\ell'(r) \leqslant \tilde\ell'(r)$$

for any small r.

If the Gauss curvature of Σ is nonnegative, the same equations imply that

$$\ell'(r) \geqslant \tilde\ell'(r)$$

for any small r.

Since $\ell(0) = \tilde\ell(0)$, integrating the inequalities proves both statements. □

The following exercise provides a stronger statement. It almost follow from the proof above, but one has to make an extra observation.

14.7. Exercise. *Assume Σ is a smooth regular surface and $p \in \Sigma$, denote by $\ell(r)$ the circumference of the circle with the center at p and radius r in Σ and let $\tilde\ell(r) = 2{\cdot}\pi{\cdot}r$ the circumference of the plane circle of radius r.*

(i) Show that if Gauss curvature of Σ is nonnegative, then the function $r \mapsto \ell(r)$ is concave for small $r > 0$. Conclude that the function $r \mapsto \frac{\ell(r)}{\tilde\ell(r)}$ is nonincreasing for small $r > 0$.

(ii) Show that if Gauss curvature of Σ is nonpositive, then the function $r \mapsto \ell(r)$ is convex for small $r > 0$. Conclude that the function $r \mapsto \frac{\ell(r)}{\tilde\ell(r)}$ is nondecresing for small $r > 0$.

14.4 Local comparison

The following proposition is a special case of the so a comparison theorem, proved by Harry Rauch [36].

14.8. Theorem. *Let Σ be a smooth regular surface and $p \in \Sigma$. Assume $\tilde\gamma\colon [a,b]$ is a curve the tangent plane $\mathrm{T}_p\Sigma$ that runs in a sufficiently small neighborhood of the origin; consider the curve*

$$\gamma = \exp_p \circ\gamma$$

in Σ.

(i) If Gauss curvature of Σ is nonnegative, then

$$\operatorname{length}\gamma \leqslant \operatorname{length}\tilde\gamma$$

(ii) If Gauss curvature of Σ is nonpositive, then

$$\operatorname{length}\gamma \geqslant \operatorname{length}\tilde\gamma.$$

The proof is a direct application of Corollary 14.6.

Proof. Let us denote $\tilde w(\theta,r)$ and $w(\theta,r)$ the polar coordinates of T_p and Σ at p Recall that

$$\begin{aligned}
&\tfrac{\partial\tilde w}{\partial\theta} \perp \tfrac{\partial\tilde w}{\partial r}; &\qquad& |\tfrac{\partial\tilde w}{\partial r}| = 1;\\
&\tfrac{\partial w}{\partial\theta} \perp \tfrac{\partial w}{\partial r}; && |\tfrac{\partial w}{\partial r}| = 1;
\end{aligned}$$

By Corollary 14.6, we also have

$$|\tfrac{\partial\tilde w}{\partial\theta}| \geqslant |\tfrac{\partial\tilde w}{\partial\theta}|; \qquad\qquad |\tfrac{\partial\tilde w}{\partial\theta}| \leqslant |\tfrac{\partial\tilde w}{\partial\theta}|;$$

if Gauss curvature is nonnegative or nonpositive correspondingly.

It is sufficient to show that

$$\text{❶}\qquad |\gamma'(t)| \leqslant |\tilde\gamma'(t)| \quad \text{or, correspondingly} \quad |\gamma'(t)| \geqslant |\tilde\gamma'(t)|$$

for any t.

Note that both curves $\gamma(t)$ and $\tilde\gamma(t)$ described the same way in the polar coordinates; denote these coordinates by $(\theta(t), r(t))$. Then

$$\begin{aligned}
\gamma'(t)|^2 &= |\tfrac{\partial w}{\partial\theta}\cdot\theta'(t) + \tfrac{\partial w}{\partial r}\cdot r'(t)|^2 =\\
&= |\tfrac{\partial w}{\partial\theta}|^2\cdot|\theta'(t)|^2 + |r'(t)|^2
\end{aligned}$$

The same way

$$\tilde\gamma'(t)|^2 = |\tfrac{\partial\tilde w}{\partial\theta}|^2\cdot|\theta'(t)|^2 + |r'(t)|^2;$$

hence ❶ follows. □

Chapter 15

Global comparison

15.1 Formulation

A minimizing geodesic between points x and y in a surface Σ will be denoted as $[xy]$ or $[xy]_\Sigma$; the latter notation is used if we need to emphasise that the geodesic lies in Σ. If we write $[xy]$, then we assume that a minimizing geodesic exists and we made a choice of one of them.

In general minimizing geodesic might be not unique for example any meridian in the sphere is a minimizing geodesic between its poles. If Σ is proper, then a minimizing geodesic always exists.

A *geodesic triangle* in a surface Σ is a triple of points $x, y, z \in \Sigma$ with choice of minimizing geodesics $[xy]$, $[yz]$ and $[zx]$. The points x, y, z are called *vertexes* of the geodesic triangle, the minimizing geodesics $[xy]$, $[yz]$ and $[zx]$ are called its sides; the triangle itself is denoted by $[xyz]$.

The length of one (and therefore any) minimizing geodesic $[xy]_\Sigma$ will be denoted by $|x-y|_\Sigma$; it is called *intrinsic distance* from x to y in Σ. If defined, then $|x-y|_\Sigma$ is the exact lower bound on the lengths of curves from x to y in Σ.

A triangle $[\tilde x\tilde y\tilde z]$ in the plane $\mathbb{R}^2$ is called *model triangle* of the triangle $[xyz]$, briefly $[\tilde x\tilde y\tilde z] = \tilde\triangle xyz$, if its corresponding sides are equal; that is,

$$|\tilde x-\tilde y|_{\mathbb{R}^2} = |x-y|_\Sigma, \quad |\tilde y-\tilde z|_{\mathbb{R}^2} = |y-z|_\Sigma, \quad |\tilde z-\tilde x|_{\mathbb{R}^2} = |z-x|_\Sigma.$$

A pair of minimizing geodesics $[xy]$ and $[xz]$ starting from one point x is called *hinge* and denoted as $[x\,{}^y_z]$. The angle between these geodesics at x is denoted by $\measuredangle[x\,{}^y_z]$. The corresponding angle $\measuredangle[\tilde x\,{}^{\tilde y}_{\tilde z}]$ in the model triangle $[\tilde x\tilde y\tilde z] = \tilde\triangle xyz$ is denoted by $\tilde\measuredangle(x\,{}^y_z)$.

A surface Σ is called *simply connected* if any closed simple curve in Σ bounds a disc. Equivalently any closed curve in Σ can be continuously deformed into a trivial curve (trivial means that it stays at one point). A plane or sphere are examples of simply connected surfaces, while torus or cylinder are not simply connected.

15.1. Comparison theorem. *Let Σ be a proper smooth regular surface with a geodesic triangle $[xyz]$.*

(i) If Σ has nonnegative Gauss curvature, then

$$\measuredangle[x\,{}^y_z] \geqslant \tilde\measuredangle(x\,{}^y_z).$$

(ii) If Σ is simply connected and has nonpositive Gauss curvature, then

$$\measuredangle[x\,{}^y_z] \leqslant \tilde\measuredangle(x\,{}^y_z).$$

Let us make few remarks on the formulation.

The angle $\measuredangle[x\,{}^y_z]$ is a number in the interval $[0,\pi]$. If the triangle $[xyz]$ bounds a disc Δ and θ is the external angle at x which used in Gauss–Bonnet formula, then $\measuredangle[x\,{}^y_z] = |\pi - \theta|$. The corresponding internal angle might be $\measuredangle[x\,{}^y_z]$ or $2\cdot\pi - \measuredangle[x\,{}^y_z]$ depending on which side lies the disc Δ.

- ⋄ Since the angles of any plane triangle sum up to π, the part (i) of the theorem implies that angles of any triangle in a surface with nonnegative Gauss curvature have sum at least π.
- ⋄ The triangle may not bound a disc[1], but if it does, then by Gauss–Bonnet formula the sum of its *internal* angles is at least π.

These two statements are closely related, but they are not the same. Note that if α is the angle in the comparison theorem, then the internal angle might be α or $2\cdot\pi-\alpha$; while Gauss–Bonnet formula gives a lower bound on the sum of internal angles it does not forbid that each of these angles is close to $2\cdot\pi$. However the latter is impossible by the comparison theorem.

First note that without condition that Σ is simply connected, the statement *(ii)* does not hold. For example the equator $z = 0$ of the hyperboloid (which is not simply connected)

$$\{\,(x,y,z)\in\mathbb{R}^3 \,:\, x^2+y^2-z^2=1\,\}$$

forms a triangle with all angles π, which contradict the comparison in *(ii)*.

15.2. Exercise. *Let Σ be an open smooth regular simply connected surface with nonpositive Gauss curvature. Show that any two points in Σ are connected by a unique geodesic.*

[1] For example equator on the cyclinder is formed by a geodesic triangle that does not bound a disc.

15.2 Names and history

Part *(i)* of this theorem is called *Toponogov comparison theorem*; it is was proved by Paolo Pizzetti [37] and latter independently by Alexandr Alexandrov [38]; generalizations were obtained by Victor Toponogov [39], Mikhael Gromov, Yuri Burago and Grigory Perelman [40].

Part *(ii)* is called *Cartan–Hadamard theorem*; it was proved by Hans von Mangoldt [41] and generalized by Elie Cartan [42], Jacques Hadamard [27], Herbert Busemann [43], Willi Rinow in [44], Mikhael Gromov [45, p. 119], Stephanie Alexander and Richard Bishop in [46].

15.3 Local part

First we prove the following local version of comparison theorem and then use it to prove the global version.

15.3. Theorem. *The comparison theorem (15.1) holds in a small neighborhood of any point.*

That is, if Σ be a smooth regular surface without boundary, then any point $p \in \Sigma$ admits a neighborhood $U \ni p$ such that

(i) If Σ has nonnegative Gauss curvature, then for any geodesic triangle $[xyz]$ in U we have

$$\measuredangle[x\,{}^{y}_{z}] \geqslant \tilde\measuredangle(x\,{}^{y}_{z}).$$

(ii) If Σ has nonpositive Gauss curvature, then for any geodesic triangle $[xyz]$ in U we have

$$\measuredangle[x\,{}^{y}_{z}] \leqslant \tilde\measuredangle(x\,{}^{y}_{z}).$$

Note that we can assume that U is simply connected therefore this condition is not necessary to include in part *(ii)*.

Proof. Assume $y = \exp_x v$ and $z = \exp_x w$ for two small vectors $v, w \in \mathrm{T}_x$. Note that

$$\begin{aligned}\measuredangle[x\,{}^{v}_{w}]_{\mathrm{T}_x} &= \measuredangle[x\,{}^{y}_{z}]_{\Sigma},\\ |x-v|_{\mathrm{T}_x} &= |x-y|_{\Sigma},\\ |x-w|_{\mathrm{T}_x} &= |x-z|_{\Sigma}.\end{aligned}$$

If the Gauss curvature is nonnegative, consider the line segment $\tilde\gamma$ joining v to w in the tangent plane T_x and set $\gamma = \exp_x \circ\tilde\gamma$. By Rauch comparison theorem (14.8), we have

$$\operatorname{length}\gamma \leqslant \operatorname{length}\tilde\gamma.$$

Since $|v-w|_{\mathrm{T}_x} = \operatorname{length}\tilde\gamma$ and $|y-z|_\Sigma \leqslant \operatorname{length}\gamma$, we get

$$|v-w|_{\mathrm{T}_x} \geqslant |y-z|_\Sigma.$$

Therefore

$$\tilde\measuredangle(x\,{}^z_y) \geqslant \measuredangle[x\,{}^z_y].$$

If the Gauss curvature is nonpositive, consider a minimizing geodesic γ joining y to z in Σ and let $\tilde\gamma$ be the corresponding curve joining v to w in T_x; that is, $\gamma = \exp_x\circ\tilde\gamma$. By Rauch comparison theorem (14.8), we have

$$\operatorname{length}\gamma \geqslant \operatorname{length}\tilde\gamma.$$

Since $|v-w|_{\mathrm{T}_x} \leqslant \operatorname{length}\tilde\gamma$ and $|y-z|_\Sigma = \operatorname{length}\gamma$, we get

$$|v-w|_{\mathrm{T}_x} \geqslant |y-z|_\Sigma.$$

Therefore

$$\tilde\measuredangle(x\,{}^z_y) \geqslant \measuredangle[x\,{}^z_y].$$ □

15.4 Alexandrov's lemma

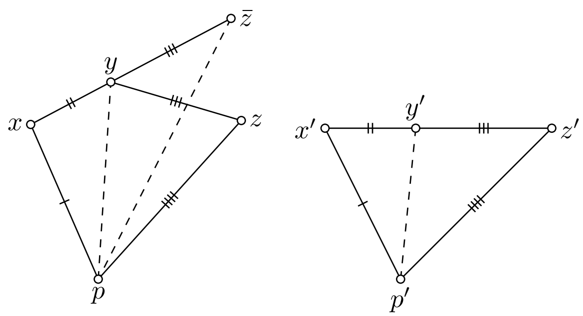
It this section we prove the following lemma in the plane geometry.

15.4. Lemma. *Assume $[pxyz]$ and $[p'x'y'z']$ be two quadraliterals in the plane with equal corresponding sides. Assume that the sides $[x'y']$ and $[y'z']$ extend each other; that is, y' lies on the line segment $[x'z']$. Then the following expressions have the same signs:*

(i) $|p-y|-|p'-y'|$;

(ii) $\measuredangle[x\,{}^p_y]-\measuredangle[x'\,{}^{p'}_{y'}]$;

(iii) $\pi-\measuredangle[y\,{}^p_x]-\measuredangle[y\,{}^p_z]$;



Proof. In the proof we use the following *monotonicity property*: if two sides adjacent to an angle in a plane triangle are fixed, then the angle is increases if the opposite side increase.

Take a point $\bar z$ on the extension of $[xy]$ beyond y so that $|y-\bar z| = |y-z|$ (and therefore $|x-\bar z| = |x'-z'|$).

From monotonicity, the following expressions have the same sign:

(i) $|p-y| - |p'-y'|$;

(ii) $\measuredangle[x\,{}^{y}_{p}] - \measuredangle[x'\,{}^{y'}_{p'}] = \measuredangle[x\,{}^{\bar z}_{p}] - \measuredangle[x'\,{}^{z'}_{p'}]$;

(iii) $|p-\bar z| - |p'-z'|$;

(iv) $\measuredangle[y\,{}^{\bar z}_{p}] - \measuredangle[y'\,{}^{z'}_{p'}]$;

The statement follows since

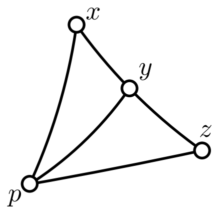
$$\measuredangle[y'\,{}^{z'}_{p'}] + \measuredangle[y'\,{}^{x'}_{p'}] = \pi$$

and

$$\measuredangle[y\,{}^{\bar z}_{p}] + \measuredangle[y\,{}^{x}_{p}] = \pi. \qquad \square$$

Further we will use the following reformulation of this lemma that is using language of comparison triangles and angles.

15.5. Reformulation. *Assume $[pxz]$ be a triangle in a surface Σ and the point y lies on the side $[xz]$. Consider its model triangle $[\tilde p\tilde x\tilde z] = \tilde\triangle pxz$ and let $\tilde y$ be the corresponding point on the side $[\tilde x\tilde z]$. Then the following expressions have the same signs:*



(i) $|p-y|_\Sigma - |\tilde p-\tilde y|_{\mathbb{R}^2}$;

(ii) $\tilde\measuredangle(x\,{}^{p}_{y}) - \tilde\measuredangle(x\,{}^{p}_{z})$;

(iii) $\pi - \tilde\measuredangle(y\,{}^{p}_{x}) - \tilde\measuredangle(y\,{}^{p}_{z})$;

15.5 Reformulations of comparison

In this section we formulate conditions equivalent to the conclusion of the comparison theorem (15.1).

A triangle $[xyz]$ in a surface is called *fat* (or correspondingly *thin*) if for any two points p and q on the sides of the triangle and the corresponding points $\tilde p$ and $\tilde q$ on the sides of its model triangle $[\tilde x\tilde y\tilde z] = \tilde\triangle xyz$ we have $|p-q| \geqslant |\tilde p-\tilde q|$ (or correspondingly $|p-q| \leqslant |\tilde p-\tilde q|$).

15.6. Proposition. *Let Σ be a proper smooth regular surface. Then the following three conditions are equivalent:*

(i^+) *For any geodesic triangle $[xyz]$ in Σ we have*

$$\measuredangle[x\,{}^{y}_{z}] \geqslant \tilde\measuredangle(x\,{}^{y}_{z}).$$

(ii^+) *For any geodesic triangle $[pxz]$ in Σ and y on the side $[xz]$ we have*

$$\tilde\measuredangle(x\,{}^{p}_{y}) \geqslant \tilde\measuredangle(x\,{}^{p}_{z}).$$

(iii$^+$) Any geodesic triangle in Σ is fat.

Similarly, following three conditions are equivalent:

(i$^-$) For any geodesic triangle $[xyz]$ in Σ we have

$$\measuredangle[x\,{}^y_z] \leqslant \tilde\measuredangle(x\,{}^y_z).$$

(ii$^-$) For any geodesic triangle $[pxz]$ in Σ and y on the side $[xz]$ we have

$$\tilde\measuredangle(x\,{}^p_y) \leqslant \tilde\measuredangle(x\,{}^p_z).$$

(iii$^-$) Any geodesic triangle in Σ is thin.

Proof. We will prove the implications *(i$^+$)$\Rightarrow$(ii$^+$)$\Rightarrow$(iii$^+$)$\Rightarrow$(i$^+$).* The implications *(i$^-$)$\Rightarrow$(ii$^-$)$\Rightarrow$(iii$^-$)$\Rightarrow$(i$^-$)* can be done the same way.

(i$^+$)$\Rightarrow$(ii$^+$). Note that $\measuredangle[y\,{}^p_x] + \measuredangle[y\,{}^p_z] = \pi$. By *(i$^+$)*,

$$\tilde\measuredangle(y\,{}^p_x) + \tilde\measuredangle(y\,{}^p_z) \leqslant \pi.$$

It reamains to apply Alexandrov's lemma (15.5).

(ii$^+$)$\Rightarrow$(iii$^+$). Applying *(i$^+$)* twice, first for $y \in [xz]$ and then for $w \in [px]$, we get that

$$\tilde\measuredangle(x\,{}^w_y) \geqslant \tilde\measuredangle(x\,{}^p_y) \geqslant \tilde\measuredangle(x\,{}^p_z)$$

and therefore

$$|w - y|_\Sigma \geqslant |\tilde w - \tilde y|_{\mathbb{R}^2},$$

where $\tilde w$ and $\tilde y$ are the points corresponding to w and y points on the sides of the model triangle. Hence the implication follows.

(iii$^+$)$\Rightarrow$(i$^+$). Since the triangle is fat, we have

$$\tilde\measuredangle(x\,{}^w_y) \geqslant \tilde\measuredangle(x\,{}^p_z)$$

for any $w \in\,]xp]$ and $y \in\,]xz]$. Note that $\tilde\measuredangle(x\,{}^w_y) \to \measuredangle[x\,{}^p_z]$ as $w, y \to x$, whence the implication follows. $\square$

In the following exercises you can apply the globalization theorem.

15.7. Exercise. *Let Σ be a closed (or open) regular surface and with nonnegative Gauss curvature. Show that for any four distinct points the following inequality holds:*

$$\tilde\measuredangle(p\,{}^x_y) + \tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^z_x) \leqslant 2{\cdot}\pi.$$

15.8. Exercise. *Let Σ be a open smooth regular surface and γ be a unit-speed geodesic in Σ and $p \in \Sigma$.*

Consider the function

$$h(t) = |p - \gamma(t)|^2_\Sigma - t^2.$$

(a) *Show that if the Gauss curvature of Σ is nonnegative then h is a concave function.*
(b) *Show that if Σ is simply connected and the Gauss curvature of Σ is nonpositive then h is a convex function.*

15.9. Exercise. *Let $\tilde{x}_1 \dots \tilde{x}_n$ be a convex plane polygon and $x_1 \dots x_n$ be a broken geodesic in an open simply connected surface Σ with nonpositive curvature. Assume that $|x_i - x_{i-1}|_\Sigma = |\tilde{x}_i - \tilde{x}_{i-1}|_{\mathbb{R}^2}$ and $\measuredangle[x_{i-1}\,{}^{x_i}_{x_{i+1}}] \geqslant \measuredangle[x_{i-1}\,{}^{x_i}_{x_{i+1}}]$ for each i. Show that*

$$|x_1 - x_n|_\Sigma \geqslant |\tilde{x}_1 - \tilde{x}_n|_{\mathbb{R}^2}.$$

For $\Sigma = \mathbb{R}^2$, the exercise above is the so called *arm lemma*; you can use it without proof.

15.10. Exercise. *Let x' and y' be the midpoints of minimizing geodesics $[px]$ and $[py]$ in an open smooth regular surface Σ.*
(a) *Show that if the Gauss curvature of Σ is nonnegative, then*

$$2{\cdot}|x' - y'|_\Sigma \geqslant |x - y|_\Sigma.$$

(b) *Show that if Σ is simply connected and has nonpositive Gauss curvature, then*

$$2{\cdot}|x' - y'|_\Sigma \leqslant |x - y|_\Sigma.$$

15.6 Nonnegative curvature

In this section we will prove part *(i)* of the comparison theorem (15.1) assuming that Σ is compact; the general case requre only minor modifications.

Since Σ is compact, from the local theorem (15.3), we get that there is $\varepsilon > 0$ such that the inequality

$$\measuredangle[x\,{}^y_z] \geqslant \tilde{\measuredangle}(x\,{}^y_z).$$

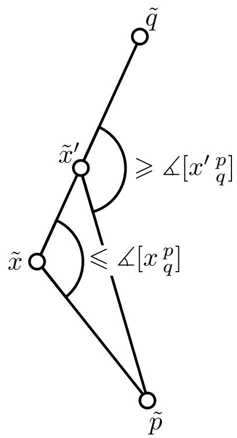
holds for any hinge $[x\,{}^y_z]$ such that $|x - y| + |x - z| < \varepsilon$. The following lemma states that in this case the same holds for any hinge $[x\,{}^y_z]$ such that $|x - y| + |x - z| < \frac{3}{2}{\cdot}\varepsilon$. Applying the lemma few times we will get that the comparison holds for arbitrary hinge, which will prove part *(i)*.

15.11. Key lemma. *Let Σ be an open smooth regular surface. Assume that the comparison*

❶ $$\measuredangle[x\,{}^y_z] \geqslant \tilde{\measuredangle}(x\,{}^y_z)$$

holds for any hinge $[x\,{}^{y}_{z}]$ with $|x-y|+|x-z|<\tfrac{2}{3}\cdot\ell$. Then the comparison ❶ *holds for any hinge $[x\,{}^{y}_{z}]$ with $|x-y|+|x-z|<\ell$.*

Proof. Given a hinge $[x\,{}^{p}_{q}]$ consider a triangle in the plane with angle $\measuredangle[x\,{}^{p}_{q}]$ and two adjacent sides $|x-p|$ and $|x-q|$. Let us denote by $\tilde{\Upsilon}[x\,{}^{p}_{q}]$ the third side of this triangle; let us call it *model side* of the hinge.



Note that the inequalities

$$\measuredangle[x\,{}^{p}_{q}] \geqslant \tilde{\measuredangle}(x\,{}^{p}_{q}) \quad\text{and}\quad \tilde{\Upsilon}[x\,{}^{p}_{q}] \geqslant |p-q|$$

are equivalent. So it is sufficient to prove that

❷ $$\tilde{\Upsilon}[x\,{}^{p}_{q}] \geqslant |p-q|.$$

for any hinge $[x\,{}^{p}_{q}]$ with $|x-p|+|x-q|<\ell$.

Given a hinge $[x\,{}^{p}_{q}]$ such that

$$\tfrac{2}{3}\cdot\ell \leqslant |p-x|+|x-q| < \ell,$$

let us construct a new smaller hinge $[x'\,{}^{p}_{q}]$; that is,

❸ $$|p-x|+|x-q| \geqslant |p-x'|+|x'-q|$$

and such that

❹ $$\tilde{\Upsilon}[x\,{}^{p}_{q}] \geqslant \tilde{\Upsilon}[x'\,{}^{p}_{q}].$$

Assume $|x-q| \geqslant |x-p|$, otherwise switch the roles of p and q in the following construction. Take $x' \in [xq]$ such that

❺ $$|p-x|+3\cdot|x-x'| = \tfrac{2}{3}\cdot\ell$$

Choose a geodesic $[x'p]$ and consider the hinge $[x'\,{}^{p}_{q}]$ fromed by $[x'p]$ and $[x'q] \subset [xq]$. Then ❸ follows since the length of $[x'p]$ can not exceed the total length of $[x'x]$ and $[x'p]$.

Further, note that $|p-x|+|x-x'|, |p-x'|+|x'-x| < \tfrac{2}{3}\cdot\ell$. In particular,

❻ $$\measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde{\measuredangle}(x\,{}^{p}_{x'}) \ \text{ and }\ \measuredangle[x'\,{}^{p}_{x}] \geqslant \tilde{\measuredangle}(x'\,{}^{p}_{x}).$$

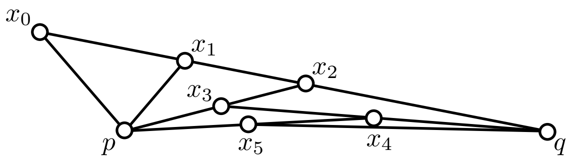
Consider the model triangle $[\tilde{x}\tilde{x}'\tilde{p}] = \tilde{\triangle}xx'p$. Take $\tilde{q}$ on the extension of $[\tilde{x}\tilde{x}']$ beyond x' such that $|\tilde{x}-\tilde{q}| = |x-q|$ (and therefore $|\tilde{x}'-\tilde{q}| = |x'-q|$). From ❻,

$$\measuredangle[x\,{}^{p}_{q}] = \measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde{\measuredangle}(x\,{}^{p}_{x'}) \quad\Rightarrow\quad \tilde{\Upsilon}[x\,{}^{q}_{p}] \geqslant |\tilde{p}-\tilde{q}|.$$

Since $\measuredangle[x'\,{}^{p}_{x}] + \measuredangle[x'\,{}^{p}_{q}] = \pi$, ❻ implies

$$\pi - \tilde\measuredangle(x'\,{}^{p}_{x}) \geqslant \pi - \measuredangle[x'\,{}^{p}_{x}] \geqslant \measuredangle[x'\,{}^{p}_{q}].$$

Therefore $|\tilde p - \tilde q| \geqslant \tilde\Upsilon[x'\,{}^{q}_{p}]$ and ❹ follows.



Set $x_0 = x$. Let us apply inductively the above construction to get a sequence of hinges $[x_n\,{}^{p}_{q}]$ with $x_{n+1} = x'_n$. By ❹ and triangle inequality, both sequences

$$s_n = \tilde\Upsilon[x_n\,{}^{p}_{q}] \quad \text{and} \quad r_n = |p - x_n| + |x_n - q|$$

are nonincreasing.

The sequence might terminate at some n only if $r_n < \frac{2}{3}\cdot\ell$. In this case, by the assumptions of the lemma,

$$s_n = \tilde\Upsilon[x_n\,{}^{p}_{q}] \geqslant |p - q|.$$

Since sequence s_n is nonincreasing;

$$s_0 = \tilde\Upsilon[x\,{}^{p}_{q}] \geqslant |p - q|,$$

whence inequality ❷ follows.

If the sequence does not terminate, then $r_n \geqslant \frac{2}{3}\cdot\ell$ for all n. Since (r_n) is nonincreasing, $r_n \to r \geqslant |p - q|_\Sigma$ as $n \to \infty$.

Let us show that $\measuredangle[x_n\,{}^{p}_{q}] \to \pi$ as $n \to \infty$.

Indeed assume $\measuredangle[x_n\,{}^{p}_{q}] \leqslant \pi - \varepsilon$ for some $\varepsilon > 0$. Without loss of generality we can assume that $x_{n+1} \in [x_n q]$; otherwise switch p and q further. Note that $|x_n - x_{n+1}|, |p - x_n| > \frac{\ell}{100}$. Therefore by comparison

$$|p - x_{n+1}| < \tilde\Upsilon[x_n\,{}^{p}_{x_{n+1}}] < |p - x_n| + |x_n - x_{n+1}| - \delta$$

for some fixed $\delta = \delta(\varepsilon) > 0$. Therefore $r_n - r_{n+1} > \delta$. The latter can not hold for large n, otherwise the sequence r_n would not converge.

It follows that for any $\varepsilon > 0$ we have that $\measuredangle[x_n\,{}^{p}_{q}] > \pi - \varepsilon$ for all large n; that is, $\measuredangle[x_n\,{}^{p}_{q}] \to \pi$ as $n \to \infty$.

Since $\measuredangle[x_n\,{}^{p}_{q}] \to \pi$, we have $s_n - r_n \to 0$ as $n \to \infty$; that is, $s_n \to r$.

Since the sequence (s_n) is nonincreasing and $r \geqslant |p - q|$, we get

$$s_n \geqslant |p - q|$$

for any n. In particular

$$\tilde{\Upsilon}[s\,{}^p_q] = s_0 \geqslant |p - q|,$$

so we obtain ❷. □

15.12. Exercise. *Assume a disc Δ lies in open smooth regular surface Σ with nonnegative Gauss curvature and bounded by a closed broken geodesic $x_1 \dots x_n$ with positive exterior angles; that is, when you travel along the boundary, you always turn to the side where Δ is.*

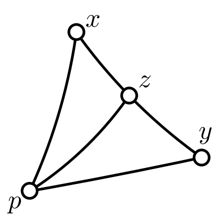
Show that there is a convex plane polygon $\tilde{x}_1 \dots \tilde{x}_n$ which sides are equal to the corresponding sides of $x_1 \dots x_n$ and with internal angles at not bigger than in the corresponding angles of $x_1 \dots x_n$.

15.7 Inheritance lemma

The following lemma will play key role in the proof of part *(ii)* of the comparison theorem (15.1).

15.13. Inheritance Lemma. *Assume that a triangle $[pxy]$ in a surface Σ* decomposes *into two triangles $[pxz]$ and $[pyz]$; that is, $[pxz]$ and $[pyz]$ have common side $[pz]$, and the sides $[xz]$ and $[zy]$ together form the side $[xy]$ of $[pxy]$.*

If both triangles $[pxz]$ and $[pyz]$ are thin, then so is $[pxy]$.



We shall need the following lemma in plane geometry.

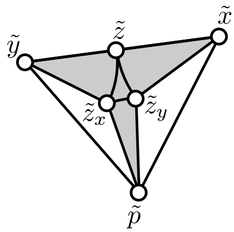
15.14. Lemma. *Let $\blacktriangle\tilde{p}\tilde{x}\tilde{y}$ be a solid plane triangle; that is, $\blacktriangle\tilde{p}\tilde{x}\tilde{y} = \mathrm{Conv}\{\tilde{p}, \tilde{x}, \tilde{y}\}$. Given $\tilde{z} \in [\tilde{x}\tilde{y}]$, consider points $\dot{p}, \dot{x}, \dot{z}, \dot{y}$ in the plane such that*

$$\begin{aligned} &|\dot{p} - \dot{x}| = |\tilde{p} - \tilde{x}|, && |\dot{p} - \dot{y}| = |\tilde{p} - \tilde{y}|, && |\dot{p} - \dot{z}| \leqslant |\tilde{p} - \tilde{z}|, \\ &|\dot{x} - \dot{z}| = |\tilde{x} - \tilde{z}|, && |\dot{y} - \dot{z}| = |\tilde{y} - \tilde{z}|, \end{aligned}$$

where points $\dot{x}$ and $\dot{y}$ lie on either side of $[\dot{p}\dot{z}]$. Then there is a short map

$$F\colon \blacktriangle\tilde{p}\tilde{x}\tilde{y} \to \blacktriangle\dot{p}\dot{x}\dot{z} \cup \blacktriangle\dot{p}\dot{y}\dot{z}$$

that maps $\tilde{p}$, $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ to $\dot{p}$, $\dot{x}$, $\dot{y}$ and $\dot{z}$ respectively.



Proof. Note that

$$\begin{aligned}|\dot{x}-\dot{y}| \leqslant |\dot{x}-\dot{z}|+|\dot{z}-\dot{y}| = \\ = |\tilde{x}-\tilde{z}|+|\tilde{z}-\tilde{y}| \\ = |\tilde{x}-\tilde{y}|.\end{aligned}$$

Applying monotonicity propery, we get that

$$\measuredangle[\dot{p}\,{}^{\dot{x}}_{\dot{y}}] \leqslant \measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{y}}].$$

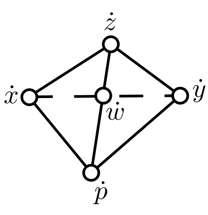
It follows that there are nonoverlapping triangles $[\tilde{p}\tilde{x}\tilde{z}_y] \cong [\dot{p}\dot{x}\dot{z}]$ and $[\tilde{p}\tilde{y}\tilde{z}_x] \cong [\dot{p}\dot{y}\dot{z}]$ inside triangle $[\tilde{p}\tilde{x}\tilde{y}]$.

Connect points in each pair $(\tilde{z},\tilde{z}_x)$, $(\tilde{z}_x,\tilde{z}_y)$ and $(\tilde{z}_y,\tilde{z})$ with arcs of circles centered at $\tilde{y}$, $\tilde{p}$, and $\tilde{x}$ respectively. Define F as follows.

- ◇ Map $\blacktriangle\tilde{p}\tilde{x}\tilde{z}_y$ isometrically onto $\blacktriangle\dot{p}\dot{x}\dot{y}$; similarly map $\blacktriangle\tilde{p}\tilde{y}\tilde{z}_x$ onto $\blacktriangle\dot{p}\dot{y}\dot{z}$.
- ◇ If a point w lies in one of the three circular sectors, say at distance r from center of the circle, let $F(w)$ be the point on the corresponding segment $[\dot{p}\dot{z}]$, $[\dot{x}\dot{z}]$ or $[\dot{y}\dot{z}]$ whose distance from the lefthand endpoint of the segment is r.
- ◇ Finally, if w lies in the remaining curvilinear triangle $\tilde{z}\tilde{z}_x\tilde{z}_y$, set $F(w)=\dot{z}$.

By construction, F satisfies the remaining conditions of the lemma. □

Proof of Inheritance lemma 15.13. Construct model triangles $[\dot{p}\dot{x}\dot{z}] = = \tilde{\triangle}(pxz)$ and $[\dot{p}\dot{y}\dot{z}] = \tilde{\triangle}(pyz)$ so that $\dot{x}$ and $\dot{y}$ lie on opposite sides of $[\dot{p}\dot{z}]$.



Suppose

$$\tilde{\measuredangle}(z\,{}^{p}_{x}) + \tilde{\measuredangle}(z\,{}^{p}_{y}) < \pi.$$

Then for some point $\dot{w} \in [\dot{p}\dot{z}]$, we have

$$|\dot{x}-\dot{w}|+|\dot{w}-\dot{y}| < |\dot{x}-\dot{z}|+|\dot{z}-\dot{y}| = |x-y|.$$

Let $w \in [pz]$ correspond to $\dot{w}$; that is, $|z-w| = |\dot{z}-\dot{w}|$. Since $[pxz]$ and $[pyz]$ are thin, we have

$$|x-w|+|w-y| < |x-y|,$$

contradicting the triangle inequality.

Thus

$$\tilde{\measuredangle}(z\,{}^{p}_{x}) + \tilde{\measuredangle}(z\,{}^{p}_{y}) \geqslant \pi.$$

By Alexandrov's lemma (15.5), this is equivalent to

❶ $$\tilde\measuredangle(x\,{}^{p}_{z}) \leqslant \tilde\measuredangle(x\,{}^{p}_{y}).$$

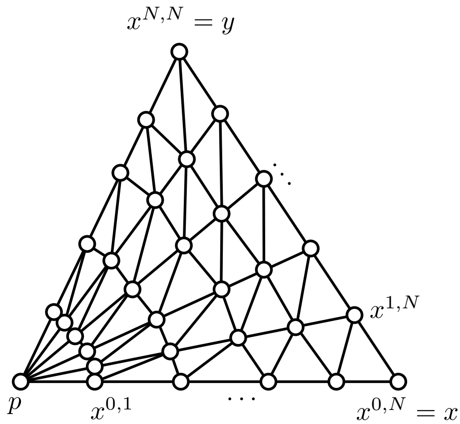
Let $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)$ and $\tilde z \in [\tilde x\tilde y]$ correspond to z; that is, $|x - z| = |\tilde x - \tilde z|$. Inequality ❶ is equivalent to $|p - z| \leqslant |\tilde p - \tilde z|$. Hence Lemma 15.14 applies; let $F\colon \blacktriangle\tilde p\tilde x\tilde y \to \blacktriangle\dot p\dot x\dot z \cup \blacktriangle\dot p\dot y\dot z$ be the provided short map.

Fix v, w on the sides of $[pxy]$; let $\tilde v, \tilde w$ be the corresponding points on the sides of the model triangle $[\tilde p\tilde x\tilde y] = \tilde\triangle pxy$ and $\dot v, \dot w$ be the corresponding points on the sides of the model triangles $[\dot p\dot x\dot z] = \tilde\triangle pxz$ and $[\dot p\dot y\dot z] = \tilde\triangle pyz$. Denote by ℓ the length of shortest curve from $\dot v$ to $\dot w$ in $\blacktriangle\dot p\dot x\dot z \cup \blacktriangle\dot p\dot y\dot z$. Since F is short, $|\tilde v - \tilde w|_{\mathbb{R}^2} \geqslant \ell$. Since both triangles $[pxz]$ and $[pyz]$ are thin, $\ell \geqslant |v - w|_\Sigma$.

It follows that $|\tilde v - \tilde w|_{\mathbb{R}^2} \geqslant |v - w|_\Sigma$ for any v and w; that is, the triangle $[pxy]$ is thin. □

15.8 Nonpositive curvature

Assume Σ is an open smooth regular surface with nonpositive curvature. As it follow from Exercise 15.2 any two points x and y in Σ are joined by unique geodesic $[xy]$.



Note that the geodesic $[xy]$ depends continuously on its endpoints x and y. That is, if $\gamma_{[xy]}\colon [0,1] \to \Sigma$ is the constant speed parametrization of $[xy]$ from x to y, then the map $(x, y, t) \mapsto \gamma_{[xy]}(t)$ is continuous in three arguments. Indeed, assume contrary, that is, $x_n \to x$, $y_n \to y$ and $t_n \to t$ as $n \to \infty$ and $\gamma_{[x_ny_n]}(t_n)$ does not converge to $\gamma_{[xy]}(t)$. Then we can pass to a subsequence such that $\gamma_{[x_ny_n]}(t_n)$ converges to a point distinct from $w \neq \gamma_{[xy]}(t)$. Note that $w \notin [xy]$. Therefore there will be two distinct geodesics from x to y; one is $[xy]$ and the other is the limit of $[x_ny_n]$ which passes thru w.

Proof of part (ii) of the comparison theorem (15.1). Fix a triangle $[pxy]$; by Proposition 15.6, it is sufficient to show that the triangle $[pxy]$ is thin.

Fix large integer N and divide $[xy]$ by points $x = x^{0,N}, \dots, x^{N,N} = y$ into N equal parts. Further divide each geodesic $[p\,x^{i,N}]$ into N equal parts by points $p = x^{i,0}, \dots, x^{i,N}$. Since the geodesic depends continuously on its end points, we can assume that each triangle $[x^{i,j}\,x^{i,j+1}\,x^{i+1,j+1}]$ and $[x^{i,j}\,x^{i+1,j}\,x^{i+1,j+1}]$ is small; in particular, by local comparison (15.3), each of these triangles is thin.

Now we show that the thin property propagates to $[pxy]$ by repeated application of the inheritance lemma (15.13):

- ⋄ First, for fixed i, sequentially applying the lemma shows that the triangles $[x\,x^{i,1}\,x^{i+1,2}]$, $[x\,x^{i,2}\,x^{i+1,2}]$, $[x\,x^{i,2}\,x^{i+1,3}]$, and so on are thin.

In particular, for each i, the long triangle $[x\,x^{i,N}\,x^{i+1,N}]$ is thin.

- ⋄ Applying the lemma again shows that the triangles $[x\,x^{0,N}\,x^{2,N}]$, $[x\,x^{0,N}\,x^{3,N}]$, and so on are thin.

In particular, $[pxy] = [p\,x^{0,N}\,x^{N,N}]$ is thin. □

15.15. Exercise. *Assume γ_1 and γ_2 be two geodesics in an open smooth regular simply connected surface Σ with nonpositive Gauss curvature. Show that the function*

$$h(t) = |\gamma_1(t) - \gamma_2(t)|_\Sigma$$

is convex.

Appendix A

Review

Here we state and discuss results from different branches of mathematics which were used further in the book. The reader is not expected to know proofs of these statements, but it is better to check that his intuition agrees with each.

A.1 Metric spaces

Metric is a function that returns a real value $\mathrm{dist}(x,y)$ for any pair x,y in a given nonempty set $\mathcal{X}$ and satisfies the following axioms for any triple x,y,z:

(a) Positiveness:

$$\mathrm{dist}(x,y) \geqslant 0.$$

(b) $x = y$ if and only if

$$\mathrm{dist}(x,y) = 0.$$

(c) Symmetry:

$$\mathrm{dist}(x,y) = \mathrm{dist}(y,x).$$

(d) Triangle inequality:

$$\mathrm{dist}(x,z) \leqslant \mathrm{dist}(x,y) + \mathrm{dist}(y,z).$$

A set with a metric is called *metric space* and the elements of the set are called *points*.

Shortcut for distance. Usually we consider only one metric on a set, therefore we can denote the metric space and its underlying set by the same letter, say $\mathcal{X}$. In this case we also use a shortcut notations

$|x-y|$ or $|x-y|_{\mathcal{X}}$ for the *distance* $\operatorname{dist}(x,y)$ *from* x *to* y *in* $\mathcal{X}$. For example, the triangle inequality can be written as

$$|x-z|_{\mathcal{X}} \leqslant |x-y|_{\mathcal{X}} + |y-z|_{\mathcal{X}}.$$

Examples. Euclidean space and plane as well as real line will be the most important examples of metric spaces for us. In these examples the introduced notation $|x-y|$ for the distance from x to y has perfect sense as a norm of the vector $x-y$. However, in general metric space the expression $x-y$ has no sense, but anyway we use expression $|x-y|$ for the distance.

If we say *plane* or *space* we mean *Eucledean* plane or space. However the plane (as well as the space) admits many other metrics, for example the so-called *Manhattan metric* from the following exercise.

A.1. Exercise. *Consider the function*

$$\operatorname{dist}(p,q) = |x_p - x_q| + |y_p - y_q|,$$

where $p=(x_p,y_p)$ *and* $q=(x_q,y_q)$ *are points in the coordinate plane* $\mathbb{R}^2$*. Show that* dist *is a metric on* $\mathbb{R}^2$*.*

Let us mention another example: the *discrete space* — arbitrary nonempty set $\mathcal{X}$ with the metric defined as $|x-y|_{\mathcal{X}}=0$ if $x=y$ and $|x-y|_{\mathcal{X}}=1$ otherwise.

Subspaces. Any subset of a metric space is also a metric space, by restricting the original metric to the subset; the obtained metric space is called a *subspace*. In particular, all subsets of Euclidean space are metric spaces.

Balls. Given a point p in a metric space $\mathcal{X}$ and a real number $R\geqslant 0$, the set of points x on the distance less then R (or at most R) from p is called open (or correspondingly closed) ball of radius R with center at p. The *open ball* is denoted as $B(p,R)$ or $B(p,R)_{\mathcal{X}}$; the second notation is used if we need to emphasize that the ball lies in the metric space $\mathcal{X}$. Formally speaking

$$B(p,R) = B(p,R)_{\mathcal{X}} = \{\, x\in\mathcal{X} \;:\; |x-p|_{\mathcal{X}} < R \,\}.$$

Analogously, the *closed ball* is denoted as $\bar{B}[p,R]$ or $\bar{B}[p,R]_{\mathcal{X}}$ and

$$\bar{B}[p,R] = \bar{B}[p,R]_{\mathcal{X}} = \{\, x\in\mathcal{X} \;:\; |x-p|_{\mathcal{X}} \leqslant R \,\}.$$

A.2. Exercise. *Let* $\mathcal{X}$ *be a metric space.*

(a) Show that if $\bar{B}[p,2]\subset\bar{B}[q,1]$ *for some points* $p,q\in\mathcal{X}$*, then* $\bar{B}[p,2]=\bar{B}[q,1]$*.*

(b) Construct a metric space $\mathcal{X}$ *with two points* p *and* q *such that* $B(p,\tfrac{3}{2})\subset B(q,1)$ *and the inclusions is strict.*

A.2 Continuity

In this section we will extend standard notions from calculus to the metric spaces.

A.3. Definition. *Let $\mathcal{X}$ be a metric space. A sequence of points $x_1, x_2, \dots$ in $\mathcal{X}$ is called* convergent *if there is $x_\infty \in \mathcal{X}$ such that $|x_\infty - x_n| \to 0$ as $n \to \infty$. That is, for every $\varepsilon > 0$, there is a natural number N such that for all $n \geqslant N$, we have*

$$|x_\infty - x_n| < \varepsilon.$$

In this case we say that the sequence (x_n) converges *to x_∞, or x_∞ is the* limit *of the sequence (x_n). Notationally, we write $x_n \to x_\infty$ as $n \to \infty$ or $x_\infty = \lim_{n\to\infty} x_n$.*

A.4. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called* continuous *if for any convergent sequence $x_n \to x_\infty$ in $\mathcal{X}$, we have $f(x_n) \to f(x_\infty)$ in $\mathcal{Y}$.*

Equivalently, $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$, there is $\delta > 0$ such that

$$|x - x'|_{\mathcal{X}} < \delta \quad \textit{implies} \quad |f(x) - f(x')|_{\mathcal{Y}} < \varepsilon.$$

A.5. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is* distance non-expanding map*; that is,*

$$|f(x) - f(x')|_{\mathcal{Y}} \leqslant |x - x'|_{\mathcal{X}}$$

for any $x, x' \in \mathcal{X}$. Show that f is continuous.

A.6. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A continous bijection $f\colon \mathcal{X} \to \mathcal{Y}$ is called a* homeomorphism *if its inverse $f^{-1}\colon \mathcal{Y} \to \mathcal{X}$ is also continuous.*

If there exists a homeomorphism $f\colon \mathcal{X} \to \mathcal{Y}$, we say that $\mathcal{X}$ is homeomorphic *to $\mathcal{Y}$, or $\mathcal{X}$ and $\mathcal{Y}$ are* homeomorphic*.*

If a metric space $\mathcal{X}$ is homeomorphic to a known space, for example plane, sphere, disc, circle and so on, we may also say that $\mathcal{X}$ is a *topological* plane, sphere, disc, circle and so on.

A.7. Definition. *A subset A of a metric space $\mathcal{X}$ is called* closed *if whenever a sequence (x_n) of points from A converges in $\mathcal{X}$, we have that $\lim_{n\to\infty} x_n \in A$.*

A set $\Omega \subset \mathcal{X}$ is called open *if for any $z \in \Omega$, there is $\varepsilon > 0$ such that $B(z, \varepsilon) \subset \Omega$.*

An open set Ω that contains a given point p is called *neighborhood of* p.

A.8. Exercise. *Let Q be a subset of a metric space $\mathcal{X}$. Show that A is closed if and only if its complement $\Omega = \mathcal{X}\backslash Q$ is open.*

A.3 Regular values

A map $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^k$ can be thought as array of functions

$$f_1, \dots, f_k\colon \mathbb{R}^n \to \mathbb{R}.$$

The map $\boldsymbol{f}$ is called *smooth* if each function f_i is smooth; that is, all partial derivatives of f_i are defined in the domain of definition of $\boldsymbol{f}$.

Inverse function theorem gives a sufficient condition for a smooth function to be invertible in a neighborhood of a given point p in its domain. The condition is formulated in terms of partial derivative of f_i at p.

Implicit function theorem is a close relative to inverse function theorem; in fact it can be obtained as its corollary. It is used for instance when we need to pass from parametric to implicit description of curves and surface.

Both theorems reduce the existence of a map satisfying certain equation to a question in linear algebra. We use these two theorems only for $n \leqslant 3$.

These two theorems are discussed in any course of multivariable calculus, the classical book of Walter Rudin [47] is one of my favorites.

A.9. Inverse function theorem. *Let $\boldsymbol{f} = (f_1, \dots, f_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Assume that the Jacobian matrix*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is invertible at some point p in the domain of definition of $\boldsymbol{f}$. Then there is a smooth function $\boldsymbol{h}\colon \mathbb{R}^m \to \mathbb{R}^n$ defined is a neighborhood Ω_q of $q = \boldsymbol{f}(p)$ that is local inverse of $\boldsymbol{f}$ at p*; that is, there are neighborhoods $\Omega_p \ni p$ such that $\boldsymbol{f}$ defines a bijection $\Omega_p \to \Omega_q$ and $\boldsymbol{f}(x) = y$ if and only if $x = \boldsymbol{h}(y)$ for any $x \in \Omega_p$ and any $y \in \Omega_q$.*

A.10. Implicit function theorem. *Let $\boldsymbol{f} = (f_1, \dots, f_n)\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a smooth map, $m, n \geqslant 1$. Let us consider $\mathbb{R}^{n+m}$ as a product* *space* $\mathbb{R}^n \times \mathbb{R}^m$ *with coodinates* $x_1, \dots, x_n, y_1, \dots, y_m$. *Consider the following matrix*

$$M = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

formed by first n *columns of the Jacobian matrix. Assume* M *is invertible at some point* p *in the domain of definition of* $\boldsymbol{f}$ *and* $\boldsymbol{f}(p) = 0$. *Then there is a neighborhood* $\Omega_p \ni p$ *and smooth function* $\boldsymbol{h}\colon \mathbb{R}^m \to \mathbb{R}^n$ *defined is a neighborhood* $\Omega_0 \ni 0$ *that for any* $(x_1, \dots, x_n, y_1, \dots y_m) \in \Omega_p$ *the equality*

$$\boldsymbol{f}(x_1, \dots, x_n, y_1, \dots y_m) = 0$$

holds if and only if

$$(x_1, \dots x_n) = \boldsymbol{h}(y_1, \dots y_m).$$

If the assumption in the theorem holds for any point p such that $\boldsymbol{f}(p) = 0$, then we say that 0 is a regular value of $\boldsymbol{f}$. The following lemma states that most of the values of smooth map are regular; in particular generic smooth function satisfies the assumption of the theorem.

A.11. Sard's lemma. *Almost all values of a smooth map* $\boldsymbol{f}\colon U \to \to \mathbb{R}^m$ *defined on an open set* $U \subset \mathbb{R}^n$ *are regular.*

The words *almost all* means that with exception o set of zero Lebesgue measure. In particular if one chooses a random value equidistributed in arbitrarily small ball $B \subset \mathbb{R}^m$ then it is a regular value of f with probability 1.

A.4 Multiple integral

The following theorem is a substitution rule for multiple variables.

A.12. Theorem. *Let* $K \subset \mathbb{R}^n$ *be a compact set and* $h\colon K \to \mathbb{R}$ *be a bounded measurable function. Assume* $\boldsymbol{f}\colon K \to \mathbb{R}^n$ *is an injective smooth map. Then*

$$\int\limits_K h(\boldsymbol{x}) \cdot |J_{\boldsymbol{f}}(\boldsymbol{x})| \cdot d\boldsymbol{x} = \int\limits_{\boldsymbol{f}(K)} h \circ \boldsymbol{f}^{-1}(\boldsymbol{y}) \cdot d\boldsymbol{y},$$

where $J_{\boldsymbol{f}}(\boldsymbol{x})$ *denotes the Jacobian of* $\boldsymbol{f}$ *at* $\boldsymbol{x}$*; that is, the determinant of the Jacobian matrix of* $\boldsymbol{f}$ *at* $\boldsymbol{x}$.

A.13. Area formula. *Let $\boldsymbol{f}\colon K\to\mathbb{R}^n$ be a smooth map defined on a compact set $K\subset\mathbb{R}^n$; denote by $J_{\boldsymbol{f}}$ the Jacobian of $\boldsymbol{f}$. Then for any function $h\colon K\to\mathbb{R}$*

$$\int\limits_K h(\boldsymbol{x})\cdot|J_{\boldsymbol{f}}(\boldsymbol{x})|\cdot d\boldsymbol{x}=\int\limits_{\boldsymbol{f}(K)} H_K(\boldsymbol{y})\cdot d\boldsymbol{y},$$

where

$$H_K(\boldsymbol{y})=\sum_{\substack{\boldsymbol{x}\in K\\ \boldsymbol{f}(\boldsymbol{x})=\boldsymbol{y}}} h(\boldsymbol{x}).$$

(The integrals are understood in the sense of Lebesgue.)

Let us sketch the proof of area formula using Sard's lemma (A.11) and the substitution rule (A.12).

Sketch of proof. Denote by $S\subset K$ the set of critical points of $\boldsymbol{f}$; that is, $\boldsymbol{x}\in S$ if $J_{\boldsymbol{f}}(\boldsymbol{x})=0$. By Sard's lemma, $\boldsymbol{f}(S)$ *has vanishing measure.* Note that

$$\int\limits_S h(\boldsymbol{x})\cdot|J_{\boldsymbol{f}}(\boldsymbol{x})|\cdot d\boldsymbol{x}=0$$

since $J_{\boldsymbol{f}}(\boldsymbol{x})=0$ and

$$\int\limits_{\boldsymbol{f}(S)} H_S(\boldsymbol{y})\cdot d\boldsymbol{y}$$

since $\boldsymbol{f}(S)$ has vanishing measure. In particular,

$$\int\limits_S h(\boldsymbol{x})\cdot|J_{\boldsymbol{f}}(\boldsymbol{x})|\cdot d\boldsymbol{x}=\int\limits_{\boldsymbol{f}(S)} H_S(\boldsymbol{y})\cdot d\boldsymbol{y};$$

that is the area formula holds for S.

It remains to prove that

$$\int\limits_{K\backslash S} h(\boldsymbol{x})\cdot|J_{\boldsymbol{f}}(\boldsymbol{x})|\cdot d\boldsymbol{x}=\int\limits_{\boldsymbol{f}(K\backslash S)} H_{K\backslash S}(\boldsymbol{y})\cdot d\boldsymbol{y}.$$

Since $J_{\boldsymbol{f}}(\boldsymbol{x})\neq 0$ for any $\boldsymbol{x}\in K\backslash S$, by inverse function theorem, the restriction of $\boldsymbol{f}$ to a neighborhood $U\ni\boldsymbol{x}$ has a smooth inverse. Therefore for any compact set $K'\subset U$ we have that

$$\int\limits_{K'} h(\boldsymbol{x})\cdot|J_{\boldsymbol{f}}(\boldsymbol{x})|\cdot d\boldsymbol{x}=\int\limits_{\boldsymbol{f}(K_1)} h(\boldsymbol{f}^{-}1(\boldsymbol{y}))\cdot d\boldsymbol{y}.$$

It remains to subdivide K_1 into a countable collection of subsets of that type and sum up the corresponding formulas. $\square$

A.5 Initial value problem

The following theorem guarantees existence and uniqueness of a solution of an initial value problem for a system of ordinary differential equations

$$\begin{cases} x_1'(t) &= f_1(x_1, \dots, x_n, t), \\ &\dots \\ x_n'(t) &= f_n(x_1, \dots, x_n, t), \end{cases}$$

where each $x_i = x_i(t)$ is a real valued function defined on a real interval $\mathbb{I}$ and each f_i is a smooth function defined on $\mathbb{R}^n \times \mathbb{I}$.

The array functions $(f_1, \dots, f_n)$ can be considered as one vector-valued function $\boldsymbol{f} \colon \mathbb{R}^n \times \mathbb{I} \to \mathbb{R}^n$ and the array $(x_1, \dots, x_n)$ can be considered as a vector $\boldsymbol{x} \in \mathbb{R}^n$. Therefore the system can be rewritten as one vector equation

$$\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}, t).$$

A.14. Theorem. *Suppose $\mathbb{I}$ is a real interval and $\boldsymbol{f} \colon \mathbb{R}^n \times \mathbb{I} \to \mathbb{R}^n$ is a smooth function. Then for any initial data $\boldsymbol{x}(t_0) = \boldsymbol{u}$ the differential equation*

$$\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}, t)$$

has a unique solution $\boldsymbol{x}(t)$ defined at a maximal subinterval $\mathbb{J}$ of $\mathbb{I}$ that contains t_0. Moreover

(a) if $\mathbb{J} \neq \mathbb{I}$ (that is, if an end a of $\mathbb{J}$ lies in the interior of $\mathbb{I}$) then $\boldsymbol{x}(t)$ diverges for $t \to a$;

(b) the function $(\boldsymbol{u}, t_0, t) \mapsto \boldsymbol{x}(t)$ is smooth.

A.6 Lipschitz condition

Recall that a function f is called Lipschitz if there is a constant L such that

$$|f(x) - f(y)| \leqslant L \cdot |x - y|$$

for values x and y in the domain of definition of f. This definition works for maps between metric spaces, but we will use it for real-to-real functions only.

A.15. Rademacher's theorem. *Let $f \colon [a, b] \to \mathbb{R}$ be a Lipschitz function. then derivative $f'(x)$ is defined for alomst all $x \in [a, b]$. Moreover the derivative f' is a bounded measurable function defined almost everywhere in $[a, b]$ and it satisfies the fundamental theorem of* *calculus; that is, the following identity*

$$f(b) - f(a) = \int\limits_a^b f'(x){\cdot}dx,$$

holds if the integral understood in the sense of Lebesgue.

It is often helps to work with measurable functions; it makes possible to extend many statements about continuous function to measurable functions.

A.16. Lusin's theorem. *Let $\varphi\colon [a,b] \to \mathbb{R}$ be a measurable function. Then for any $\varepsilon > 0$, there is a continuous function $\psi_\varepsilon\colon [a,b] \to \mathbb{R}$ that coincides with φ outside of a set of measure at most ε. Moreover, φ is bounded above and/or below by some constants then we can assume that so is ψ_ε.*

A.7 Uniform continuity

Let f be a real function defined on a real interval. If for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x_1 - x_2|_X < \delta \quad\Longrightarrow\quad |f(x_1) - f(x_2)|_Y < \varepsilon,$$

then f is called *uniformly continuous.*

Evidently every uniformly continuous function is continuous; the converse does not hold. For example, the function $f(x) = x^2$ is continuous, but not uniformly continuous. Indeed, in this case

$$\text{❶}\qquad |f(x_1) - f(x_2)| = |x_1 + x_2|{\cdot}|x_1 - x_2|$$

If $|x_1 - x_2|$ is arbitrarily small positive value, then one is free to chose $|x_1 + x_2|$ sufficiently large, so that the product in ❶ is bigger that any given number. However if f is continuous and defined on a closed interval $[a,b]$, then f is uniformly continuous

If the condition above holds for any function f_n in a sequence and δ depend solely on ε, then the sequence (f_n) is called uniformly equicontinuous. More precisely, a sequence of functions $f_n : X \to Y$ is called *uniformly equicontinuous* if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x_1 - x_2|_X < \delta \quad\Longrightarrow\quad |f_n(x_1) - f_n(x_2)|_Y < \varepsilon$$

for any n. The following lemma is a partial case of [47, Theorem 7.25].

A.17. Lemma. *Any uniformly equicontinuous sequence of function $f_n\colon [a,b] \to [c,d]$ has a subsequence that converges pointwise to a continuous function.*

A.8 Jordan's theorem

We sometimes characterization of homeomorphism.

A.18. Theorem. *A continuous bijection f between compact metric spaces has continuous inverse; that is, f is a homeomorphism.*

The first part of the following theorem is proved by Camille Jordan, the second part is due to Arthur Schoenflies.

A.19. Theorem. *The complement of any closed simple plane curve γ has exactly two connected components.*

Moreover the there is a homeomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps the unit circle to γ. In particular γ bounds a topological disc.

This theorem is known for simple formulation and quite hard proof. By now many proofs of this theorem are known. For the first statement, a very short proof based on somewhat developed technique is given by Patrick Doyle [48], among elementary proofs, one of my favorites is the proof given by Aleksei Filippov [49].

We use the following smooth analog of this theorem.

A.20. Theorem. *The complement of any closed simple smooth regular plane curve γ has exactly two connected components.*

Moreover the there is a diffeomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps the unit circle to γ.

The proof of this statement is much simpler. An amusing proof of can be found in [50].

A.9 Connectedness

Recall that a continuous map from the unit interval $[0,1]$ to a space is called a *path.*

A set X in the Euclidean space is called *path connected* if any two points $x, y \in X$ can be connected by a path lying in X; that is, there is a path $\alpha\colon [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

A set X in the Euclidean space is called *connected* if one cannot cover X by two disjoint open sets V and W such that both intersections $X \cap V$ and $X \cap W$ are nonempty.

A.21. Proposition. *Any path connected set is connected. Moreover any open connected set in the Euclidean space or plane is path connected.*

Given a point $x \in X$ the maximal connected subset containing x is called *connected component* of x in X.

A.10 Convexity

A set X in the Euclidean space is called *convex* if for any two points $x, y \in X$, any point z between x and y lies in X. It is called *strictly convex* if for any two points $x, y \in X$, any point z between x and y lies in the interior of X.

From the definition, it is easy to see that intersection of arbitrary family of convex sets is convex. The intersection of all convex sets containing X is called *convex hull* of X; it is the minimal convex set containing the given set X.

A.22. Lemma. *Let $K \subset \mathbb{R}^3$ be a closed convex subset. Then for any point $p \notin K$ there is a plane Π that separates K from p; that is, K and p lie on the opposite open half-spaces of Π.*

A function of two variables $(x, y) \mapsto f(x, y)$ is called convex if its epigraph $z \geqslant f(x, y)$ is a convex set. This is equivalent to the so-called *Jensen's inequality*

$$f\left(t{\cdot}x_1 + (1-t){\cdot}x_2\right) \le t{\cdot}f(x_1) + (1-t){\cdot}f(x_2)$$

for $t \in [0, 1]$. If f is smooth, then the condition is equivalent to the following inequality for second directional derivative:

$$D_w^2 f \geqslant 0$$

for any vector $w \neq 0$ in the (x, y)-plane.

A.11 Elementary geometry

A.23. Theorem. *The sum of sum of all the internal angles of a simple n-gon is $(n-2){\cdot}\pi$.*

Proof. The proof is by induction on n. For $n = 3$ it says that sum of internal angles of a triangle is π, which is assumed to be known.

First let us show that for any $n \geqslant 4$, any n-gon has a diagonal that lies inside of it. Assume this is holds true for all polygons with at most $n - 1$ vertex.

Fix an n-gon P, $n \geqslant 4$. Applying rotation if necessary, we can assume that all its vertexes have different x-coordinates. Let v be a vertex of P that minimizes the x-coordinate; denote by u and w its adjacent vertexes. Let us choose the diagonal uw if it lies in P. Otherwise the triangle $\triangle uvw$ contains another vertex of P. Choose a vertex s in the interior of $\triangle uvw$ that maximizes the distance to line uw. Note that the diagonal vs lyes in P; if it is not the case then vs crosses another side pq of P, one of the vertexes p or q has larger distance to the line and it lies in the interior of $\triangle uvw$ — a contradiction.

Note that the diagonal divides P into two polygons, say Q and R, with smaller number of sides in each, say k and m correspondingly. Note that

❶ $$k + m = n + 2;$$

indeed each side of P appears once as a side of P or Q plus the diagonal appears twice — once as a side in Q and once as a side of R. Note that the sum of angles of P is the sum of angles of Q and R, which by the induction hypothesis are $(k-2)\cdot\pi$ and $(m-2)\cdot\pi$ correspondingly. It remains to note that ❶ implies

$$(k-2)\cdot\pi + (m-2)\cdot\pi = (n-2)\cdot\pi.$$ □

A.12 Triangle inequality for angles

The following theorem says that triangle inequality holds for angles between half-lines from a fixed point. In particular it implies that a unit sphere with angle metric is a metric space.

A.24. Theorem. *The inequality*

$$\measuredangle aob + \measuredangle boc \geqslant \measuredangle aoc$$

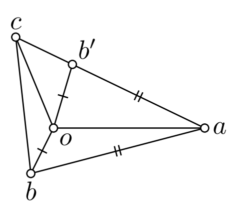
holds for any three half-lines oa, ob and oc in the Euclidean space.

The following lemma says that angle of a triangle monotonically depends on the opposit side, assuming the we keep the remaining two sides fixed. It is a simple statement in elementary geometry; in particular it follows directly from the cosine rule.

A.25. Lemma. *Let x, y, z, x', y' and z' be 6 points such that $|x-y| = |x'-y'| > 0$ and $|y-z| = |y'-z'| > 0$. Then*

$$\measuredangle xyz \geqslant \measuredangle x'y'z' \quad \textit{if and only if} \quad |x-z| \geqslant |x'-y'|.$$

Proof of A.24. We can assume that $\measuredangle aob < \measuredangle aoc$; otherwise the statement is evident. In this case there is a half-line ob' in the angle aoc such that



$$\measuredangle aob = \measuredangle aob',$$

so in particular we have that

$$\measuredangle aob' + \measuredangle b'oc = \measuredangle aoc.$$

Without loss of generality we can assume that $|o-b| = |o-b'|$ and b' lies on a line segment ac, so

$$|a-b'| + |b'-c| = |a-c|.$$

Then by triangle inequality

❶
$$\begin{aligned}|a-b| + |b-c| \geqslant |a-c| = \\ = |a-b'| + |b'-c|.\end{aligned}$$

Note that in the triangles aob and aob' the side ao is shared, $\measuredangle aob = \measuredangle aob'$ and $|o-b| = |o-b'|$. By side-angle-side congruence condition, we have that $\triangle aob \cong \triangle aob'$; in particular $|a-b'| = |a-b|$. Therefore from ❶ we have that

$$|b-c| \geqslant |b'-c|.$$

Applying the angle monotonicity (A.25) we get that

$$\measuredangle boc \geqslant \measuredangle b'oc.$$

Whence

$$\begin{aligned}\measuredangle aob + \measuredangle boc \geqslant \measuredangle aob' + \measuredangle b'oc = \\ = \measuredangle aoc.\end{aligned}$$

□

Appendix B

Semisolutions

Exercise 1.2. The image of γ might have a shape of digit 8 or 9.

Exercise 2.3. For (a), apply the fundamental theorem of calculus for each segment in a given partition. For (b) consider a partition such that the velocity vector $\alpha'(t)$ is nearly constant on each of its segments.

Advanced exercise 2.12. Use theorems of Rademacher and Lusin (A.15 and A.16).

Exercise 3.2. Differentiate the identity $\langle\gamma(s),\gamma(s)\rangle = 1$ a couple of times.

Exercise 3.3. Prove and use the following identities:

$$\begin{gathered}\gamma''(t) - \gamma''(t)^{\perp} = \tfrac{\gamma'(t)}{|\gamma'(t)|}\cdot\langle\gamma''(t),\tfrac{\gamma'(t)}{|\gamma'(t)|}\rangle,\\ |\gamma'(t)| = \sqrt{\langle\gamma'(t),\gamma'(t)\rangle}.\end{gathered}$$

Advanced exercise 3.9. Assume that γ is unit-speed; show that $|\sigma'| \leqslant \kappa + \theta'$, where $\theta(s) = \measuredangle(\gamma(s),\gamma'(s))$.

Exercise 3.13. Use that exterior angle of a triangle equals to the sum of the two remote interior angles; for the second part apply the induction on number of vertexes.

Exercise 3.19. Choose a value $s_0 \in [a,b]$ that splits the total curvature into two equal parts, θ in each. Observe that $\measuredangle(\gamma'(s_0),\gamma'(s)) \leqslant \theta$ for any s. Use this inequality the same way as in the proof of the bow lemma.

Exercise 3.25. Modify the proof of semi-continuity of length (2.13).

Exercise 4.3. Show and use that the binormal vector is constant.

Exercise 4.7. Show that $\langle w, \alpha\rangle$ is constant if γ makes constant angle with a fixed vector w and α is the evolvent of γ.

Exercise 4.11. Use the second statement in 4.1.

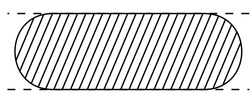
Advanced exercise 4.12. Note that the function

$$\rho(\ell) = |\gamma(t+\ell) - \gamma(t)|^2 = \langle \gamma(t+\ell) - \gamma(t), \gamma(t+\ell) - \gamma(t)\rangle$$

is smooth and does not depend on t. Express speed, curvature and torsion of γ in terms of derivatives $\rho^{(n)}(0)$ and apply 4.11.

Exercise 5.14. Apply the spiral lemma (5.9).

Exercise 5.16. Note that the curve lies in a figure F as on the diagram. More precisely, F is formed by a rectangle with pair of bases on the lines and two half discs attached to the sides of length 2. Look at the right most position of F that still contains the curve.



Exercise 5.20. Note that we can assume that γ bounds a convex figure F, otherwise by 5.18 its curvature changes the sign and therefore it has zero curvature at some point. Choose two points x and y surrounded by γ such that $|x-y| > 2$, look at the maximal lens bounded by two arcs with common chord xy that lies in F and apply supporting test (5.15).

Exercise 5.29. Note that γ contains a simple loop; apply to it 5.27.

Exercise 5.29. Note that γ contains a simple loop; apply to it 5.27.

Exercise 5.30. Repeat the proof of theorem for each cyclic concatenation of an arc of γ from p_i to p_{i+1} with large arc of the circle.

Exercise 5.31. Use the definition of osculating circle via order of contact and that inversion maps circles to circlines.

Exercise 6.11. Show that $n = \frac{\nabla h}{|\nabla h|}$ defines a unit normal field on Σ.

Exercise 7.3. Use 7.2 and 5.24.

Exercise 7.9. Consider the minimal sphere that encloses the surface.

Exercise 7.10. Look for a supporting spherical dome with the unit circle as the boundary.

Exercise 7.14. Use 7.2 and 7.12.

Exercise 8.5. Assume a maximal ball in V touches its boundary at the points p and q. Consider the projection of V to a plane thru p, q and the center of the ball.

Exercise 8.6. Drill an extra hole or combine two examples together.

Exercise 9.1. Use 7.14.

Exercise 9.2. Prove and use that each point $p \in \Sigma$ has a direction with vanishing normal curvature.

Exercise 9.6. Use the 9.5 and the hemisphere lemma (2.16).

Exercise 9.10. Observe that it is sufficient to construct a smooth parametrization of Δ_ε by a closed hemisphere. To do this repeat the argument in 10.3 with the center at a point surrounded by the boundary line of Δ_ε in its plane.

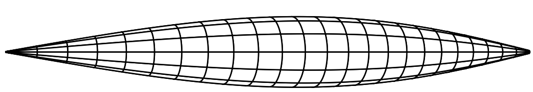
Exercise 9.13. Look for an example among the surfaces of revolution and use 7.14.

Exercise 9.14. Look at the sections of the graph by planes parallel to the (x, y)-plane and to the (x, z)-plane, then apply Meusnier's theorem apply Meusnier's theorem (7.12).

Exercise 10.1. Show and use that any tangent plane T_p supports Σ at p.

Exercise 10.4. Note that we can assume that the surface has positive Gauss curvature, otherwise the statement is evident. Therefore the surface bounds a convex region that contains a line segment of length π.

Observe that the Gauss curvature of the surface of revolution of the graph $y = a\cdot \sin x$ for $x \in (0, \pi)$ cannot exceed 1 (Use 3.4 and 7.12). Try to support the surface Σ from inside by a surface of revolution of the described type with large R.



Exercise 11.1. Cut the lateral surface of the mountain by a line from the cowboy to the top, unfold it on the plane and try to figure out what is the image of the strained lasso.

Advanced exercise 11.16. Show that the concatenation of the line segment $[p_t, \gamma(t)]$ and the arc $\gamma|_{[t,\ell]}$ is a shortest path in the closed region W outside of Σ.

Exercise 11.21. Use 11.19 and 3.15.

The suggested argument does not give the optimal bound for the Lipschitz constant that guarantees that γ is simple, but later (see 13.12) we will show that the exact bound is $\sqrt{3} = \operatorname{tg} \frac{\pi}{3}$ — it is the same as in the exercise about mountain of with the shape of a perfect cone; see 11.1.

Exercise 12.6. Denote by $n_1(t)$ and $n_2(t)$ the unit normal vectors to Σ_1 and Σ_2 at $\gamma(t)$. Note that $\langle n_1(t), n_2(t)\rangle$ is constant; take it derivative and apply 12.4.

Exercise 12.10. Use 13.21.

Exercise 13.2. Denote by w the pole of the equator; show and use that w is a parallel vector field along γ.

Exercise 13.11. Estimate integral of Gauss curvature bounded by a simple geodesic loop and apply 12.9.

Exercise 13.12. Note that it is sufficient to show that the surface has no geodesic loops. Estimate the integral of Gauss curvature of whole surface and a disc in it surrounded by a geodesic loop.

Exercise 11.6. Use 11.3.

Exercise 14.2. Note that in order to show that $\gamma_t''(s) \perp \mathrm{T}_{\gamma_t(s)}$, it is sufficient to show that $\langle \frac{\partial^2}{\partial s^2} w, \frac{\partial}{\partial t} w\rangle = 0$.

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