Chapter 1

Isoperimetric inequality

AFTER THIS LINE READ AT YOUR OWN RISK!!!

For any plane figure F with perimeter ℓ, its area a satisfies the following inequality:

❶ $$4\pi \cdot a \leqslant \ell^2.$$

Moreover the equality holds if F is congruent to a round disc.

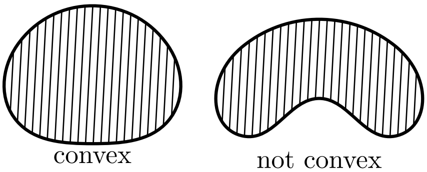
This is so-called *isoperimetric inequality* on the plane. Let us reformulate it without formulas, using the comparison language.

1.1. Isoperimetric inequality. *The area of plane figure bounded by a closed curve of length ℓ can not exceed the area of round disc with the same circumference ℓ. Moreover the equality holds only if the figure is congruent to the disc.*

The comparison reformulation has some advantages — it is more intuitive and it is also easier to generalize.

1.2. Exercise. *Come up with a formulation of isoperimetric inequality on the units sphere. Try to reformulate it as an algebraic inequality similar to* ❶*.*

Recall that a plane figure F is called *convex* if for every pair of points $x, y \in F$, the line segment $[x, y]$ that joins the pair of points lies also in F.



The following exercise reduces the isoperimetric inequality to the case of convex figures:

1.3. Exercise. *Assume F is a plane figure bounded by a closed curve of length ℓ. Show that there is a convex figure $F' \supset F$ bounded by a closed curve of length at most ℓ.*

The following problem is named after Dido, the legendary founder and first queen of Carthage.

1.4. Dido's problem. *A figure of the maximal area bounded by a straight line and a curve of given length with endpoints on that line is a half-disc.*

1.5. Exercise. *Show that Dido's problem follows from the isoperimetric inequality and the other way around.*

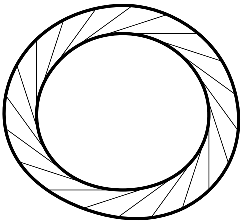
1.6. Exercise. *Use the isoperimetric inequality in the plane to show that inscribed convex polygons have maximal area among all the polygons with the given sides.*

1.7. Exercise. *Find the minimal length of curve that divides the unit square in the given ratio α.*

1.1 Lawlor's proof

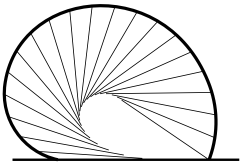
Here we present a sketch of the proof of Dido's problem based of the idea of Gary Lawlor in [1]. Before getting into the proof, try to solve the following exercise.

1.8. Exercise. *An old man walks along a trail around a convex meadows and pulls a brick on a rope of unit length (the rope is always strained). After walking around he noticed that the brick is at the same position as at the beginning. Show that the area between the trail and the path of the brick equals to the area of the unit disc.*

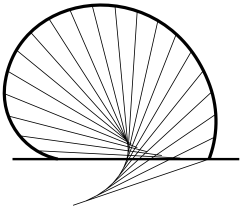


Sketch of the proof. Let F be a convex figure bounded by a line and a curve $\gamma(t)$ of length ℓ; we can assume that γ is a unit speed curve so the set of parameters is $[0, \ell]$.

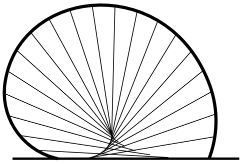
Imagine that we are walking along the curve with a stick of length r so that the other end of the stick drags on the flow. Assume that at initially $t = 0$ the stick points in the direction of $\gamma(\ell)$ — the other end of γ.



Note that is r is small then most of the time we drag the stick behind. Therefore at the end of walk the stick will make more than half turn and will point to the same side of the figure.



Let R be the radius of half-circle $\tilde{\gamma}(t)$ of length ℓ. Assume we walk along $\tilde{\gamma}$ with a stick as or length R the same way as described above. Note that the other end does not move (it always lies in the center) and the direction of stick changes with rate $\frac{1}{R}$. Note further for γ this rate would be at most $\frac{1}{R}$. Therefore after walking along γ, the stick of length R will rotate at most as much as if we would walk along $\tilde{\gamma}$.



It follows that there is a positive value $r \leqslant \leqslant R$ such that after walking along γ, a stick of length r will rotate exactly half turn, so it will point in the direction of $\gamma(0)$.

Let us show that the area of F can not exceed the area of half-disc D or radius r; since $r \leqslant R$, the latter implies Dido's problem.

Imagine that the stick is covered with paint and it paints the area below it. Notice that to color maxmimal area one has to move perpendicularly to the stick. Therefore the total area colored after walking along γ can not exeed the area of D.

It remains to show that all F is painted. Fix a point $p \in F$. Notice that at the beginning the point p lies on the left from the stick and at the end it lies on the right form it. Therefore it will be a moment of time t_0 when the sides change from left to right. At this time the point must be on the line containing the stick. Moreover if it lies on the extension then the sides change from right to left. Therefore p have to lie under the stick; that is, p is painted. $\square$

1.9. Exercise. *Find the places with cheating in the proof above and try to fix them.*

1.10. Exercise. *Read and understand the original proof of Gary Lawlor in [1].*

1.2 Length of curves

Recall that *real interval* is an arbitrary convex set of real line; that is, a set $\mathbb{I}$ such that $a, b \in \mathbb{I}$ and $a < x < b$ implies $x \in \mathbb{I}$. For example, given two real numbers $a < b$ we may consider the following intervals

$$[a,b] := \{\, x \in \mathbb{R} \mid a \leqslant x \leqslant b \,\}, \qquad [a,b) := \{\, x \in \mathbb{R} \mid a \leqslant x < b \,\},$$
$$(a,b] := \{\, x \in \mathbb{R} \mid a < x \leqslant b \,\}, \qquad (a,b) := \{\, x \in \mathbb{R} \mid a < x < b \,\}.$$

In addition to the bounded intervals described above, there are unbounded intervals

$$[a,\infty) := \{\, x \in \mathbb{R} \mid a \leqslant x \,\}, \qquad (a,\infty) := \{\, x \in \mathbb{R} \mid a < x \,\},$$
$$(-\infty,a] := \{\, x \in \mathbb{R} \mid a \geqslant x \,\}, \qquad (-\infty,a) := \{\, x \in \mathbb{R} \mid a > x \,\}.$$

Finally the whole real line is also an interval

$$(-\infty,\infty) = \mathbb{R}.$$

1.11. Definition. *A* plane curve *is a continuous mapping* $\alpha\colon \mathbb{I} \to \mathbb{R}^2$, *where* $\mathbb{I}$ *is a real interval and* $\mathbb{R}^2$ *is the Euclidean plane.*

If $\mathbb{I} = [a,b]$ *and*

$$\alpha(a) = p, \quad \alpha(b) = q,$$

we say that α *is a* curve from p to q.

1.12. Exercise. *Assume* $\alpha\colon [a,b] \to \mathbb{R}^2$ *is smooth curve, in particular the velocity vector* $\alpha'(t)$ *is defined. Show that*

$$\operatorname{length}\gamma = \int\limits_a^b |\gamma'(t)| \cdot dt.$$

A curve α called *simple* if it is described by an injective map; that is $\alpha(t) = \alpha(t')$ if and only if $t = t'$.

A curve $\alpha\colon [a,b] \to \mathbb{R}^2$ is called *closed* if $\alpha(a) = \alpha(b)$. A closed curve $\alpha\colon [a,b] \to \mathbb{R}^2$ is called simple if it is injective everywhere except the ends; that is, if $\alpha(t) = \alpha(t')$ for $t < t'$ then $t = a$ and $t' = b$.

1.13. Definition. *Let* $\alpha\colon \mathbb{I} \to \mathbb{R}^2$ *be a curve. Define* length *of* α *as*

$$\begin{aligned}\operatorname{length}\alpha = \sup\{&|\alpha(t_0) - \alpha(t_1)| + |\alpha(t_1) - \alpha(t_2)| + \dots \\ &\dots + |\alpha(t_{k-1}) - \alpha(t_k)|\}.\end{aligned}$$

where the supremum is taken over all k *and all sequences* $t_0 < t_1 < \dots < t_k$ *in* $\mathbb{I}$.

A curve is called rectifiable *if its length is finite.*

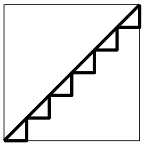
1.14. Exercise. *Construct a curve $\alpha\colon [0,1] \to \mathbb{R}^2$ which is not rectifiable.*

1.15. Semicontinuity of length. *Length is a lower semi-continuous functional on the space of curves $\alpha\colon \mathbb{I} \to \mathbb{R}^2$ with respect to point-wise convergence.*

In other words: assume that a sequence of curves $\alpha_n\colon \mathbb{I} \to \mathbb{R}^2$ converges point-wise to a curve $\alpha_\infty\colon \mathbb{I} \to \mathbb{R}^2$; i.e., for any fixed $t \in \mathbb{I}$, we have $\alpha_n(t) \to \alpha_\infty(t)$ as $n \to \infty$. Then

❷ $$\varliminf_{n\to\infty} \operatorname{length}\alpha_n \geqslant \operatorname{length}\alpha_\infty.$$

Note that the inequality ❷ might be strict. For example the diagonal of unit square α_∞ can be approximated by a sequence of stairs-like polygonal curves α_n with sides parallel to the sides of the square, α_6 is on the picture. In this case



$$\operatorname{length}\alpha_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length}\alpha_n = 2$$

for all n.

Proof. Fix $\varepsilon > 0$ and choose a sequence $t_0 < t_1 < \dots < t_k$ in $\mathbb{I}$ such that

$$\operatorname{length}\alpha_\infty - (|\alpha_\infty(t_0) - \alpha_\infty(t_1)| + |\alpha_\infty(t_1) - \alpha_\infty(t_2)| + \dots \\ \dots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|) < \varepsilon$$

Set

$$\begin{aligned} \Sigma_n &:= |\alpha_n(t_0) - \alpha_n(t_1)| + |\alpha_n(t_1) - \alpha_n(t_2)| + \dots \\ &\qquad \dots + |\alpha_n(t_{k-1}) - \alpha_n(t_k)|. \\ \Sigma_\infty &:= |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + |\alpha_\infty(t_1) - \alpha_\infty(t_2)| + \dots \\ &\qquad \dots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|. \end{aligned}$$

Note that $\Sigma_n \to \Sigma_\infty$ as $n \to \infty$ and $\Sigma_n \leqslant \operatorname{length}\alpha_n$ for each n. Hence

$$\varliminf_{n\to\infty} \operatorname{length}\alpha_n \geqslant \operatorname{length}\alpha_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get ❷. □

1.16. Exercise. *Assume a convex figure A bounded by a curve α lies in a figure B bounded by a curve β. Show that*

$$\operatorname{length}\alpha \leqslant \operatorname{length}\beta.$$

Appendix A

Semisolutions

Exercise 1.5. First let us show that Dido's problem follows from the isoperimetric inequality.

Assume F is a figure bounded by a straight line and a curve of length ℓ whose endpoints belong to that line. Let F' be the reflection of F in the line. Note that the union $G = F \cup F'$ is a figure bounded by a closed curve of length $2{\cdot}\ell$.

Applying the isoperimetric inequality, we get that the area of G can not exceed the area of round disc with the same circumference $2{\cdot}\ell$ and the equality holds only if the figure is congruent to the disc. Since F and F' are congruent, Dido's problem follows.

Now let us show that the isoperimetric inequality follows from the Dido's problem.

Assume G is a convex figure bounded by a closed curve of length $2{\cdot}\ell$. Cut G by a line that splits the perimeter in two equal parts — ℓ each. Denote by F and F' the two parts. Applying the Dido's problem for each part, we get that that are of each does not exceed the area of half-disc bounded by a half-circle. The two half-disc could be arranged into a round disc of circumference ℓ, hence the isoperimetric inequality follows.

Bibliography

[1] Gary Lawlor. “A new area-maximization proof for the circle”. *The Mathematical Intelligencer* 20.1 (1998), pp. 29–31.