Invitation to comparison geometry

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Chapter 1

Metric spaces

Metric is a function that returns a real value $\mathrm{dist}(x, y)$ for any pair x, y in a given nonempty set $\mathcal{X}$ and satisfies the following axioms for any triple x, y, z:

(a) Positiveness:
$$\mathrm{dist}(x, y) \geqslant 0.$$

(b) $x = y$ if and only if
$$\mathrm{dist}(x, y) = 0.$$

(c) Symmetry:
$$\mathrm{dist}(x, y) = \mathrm{dist}(y, x).$$

(d) Triangle inequality:
$$\mathrm{dist}(x, z) \leqslant \mathrm{dist}(x, y) + \mathrm{dist}(y, z).$$

A set with a metric is called *metric space* and the elements of the set are called *points.*

Shortcut for distance. Usually we consider only one metric on a set, therefore we can denote the metric space and its underlying set by the same letter, say $\mathcal{X}$. In this case we also use a shortcut notations $|x - y|$ or $|x - y|_{\mathcal{X}}$ for the *distance* $\mathrm{dist}(x, y)$ *from* x *to* y *in* $\mathcal{X}$. For example, the triangle inequality can be written as
$$|x - z|_{\mathcal{X}} \leqslant |x - y|_{\mathcal{X}} + |y - z|_{\mathcal{X}}.$$

Examples. Euclidean space and plane as well as real line will be the most important examples of metric spaces for us. In these examples the introduced notation $|x-y|$ for the distance from x to y has perfect sense as a norm of the vector $x - y$. However, in general metric space the expression $x-y$ has no sense, but anyway we use expression $|x-y|$ for the distance.

If we say *plane* or *space* we mean *Eucledean* plane or space. However the plane (as well as the space) admits many other metrics, for example the so called Manhattan metric from the following exercise.

1.1. Exercise. *Consider the function*

$$\operatorname{dist}(p,q) = |x_p - x_q| + |y_p - y_q|,$$

where $p = (x_p, y_p)$ and $q = (x_q, y_q)$ are points in the coordinate plane $\mathbb{R}^2$. Show that dist *is a metric on $\mathbb{R}^2$.*

Let us mention another example: the *discrete space* — arbitrary nonempty set $\mathcal{X}$ with the metric defined as $|x-y|_{\mathcal{X}} = 0$ if $x = y$ and $|x-y|_{\mathcal{X}} = 1$ otherwise.

Subspaces. Any subset of a metric space is also a metric space, by restricting the original metric to the subset; the obtained metric space is called a *subspace.* In particular, all subsets of Euclidean space are metric spaces.

Balls. Given a point p in a metric space $\mathcal{X}$ and a real number $R \geqslant 0$, the set of points x on the distance less then R (or at most R) from p is called open (or correspondingly closed) ball of radius R with center at p. The *open ball* is denoted as $B(p,R)$ or $B(p,R)_{\mathcal{X}}$; the second notation is used if we need to emphasize that the ball lies in the metric space $\mathcal{X}$. Formally speaking

$$B(p,R) = B(p,R)_{\mathcal{X}} = \{\, x \in \mathcal{X} \;:\; |x-p|_{\mathcal{X}} < R \,\}.$$

Analogously, the *closed ball* is denoted as $\bar{B}[p,R]$ or $\bar{B}[p,R]_{\mathcal{X}}$ and

$$\bar{B}[p,R] = \bar{B}[p,R]_{\mathcal{X}} = \{\, x \in \mathcal{X} \;:\; |x-p|_{\mathcal{X}} \leqslant R \,\}.$$

1.2. Exercise. *Let $\mathcal{X}$ be a metric space.*

(a) Show that if $\bar{B}[p,2] \subset \bar{B}[q,1]$ for some points $p, q \in \mathcal{X}$, then $\bar{B}[p,2] = \bar{B}[q,1]$.

(b) Construct a metric space $\mathcal{X}$ with two points p and q such that $B(p,\frac{3}{2}) \subset B(q,1)$ and the inclusions is strict.

Calculus

In this section we will extend standard notions from calculus to the metric spaces.

1.3. Definition. *Let $\mathcal{X}$ be a metric space. A sequence of points $x_1, x_2, \dots$ in $\mathcal{X}$ is called* convergent *if there is $x_\infty \in \mathcal{X}$ such that $|x_\infty - x_n| \to 0$ as $n \to \infty$. That is, for every $\varepsilon > 0$, there is a natural number N such that for all $n \geqslant N$, we have*

$$|x_\infty - x_n| < \varepsilon.$$

In this case we say that the sequence (x_n) converges *to x_∞, or x_∞ is the* limit *of the sequence (x_n). Notationally, we write $x_n \to x_\infty$ as $n \to \infty$ or $x_\infty = \lim_{n\to\infty} x_n$.*

1.4. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called* continuous *if for any convergent sequence $x_n \to x_\infty$ in $\mathcal{X}$, we have $f(x_n) \to f(x_\infty)$ in $\mathcal{Y}$.*

Equivalently, $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$, there is $\delta > 0$ such that

$$|x - x'|_{\mathcal{X}} < \delta \quad \textit{implies} \quad |f(x) - f(x')|_{\mathcal{Y}} < \varepsilon.$$

1.5. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is* distance non-expanding map*; that is,*

$$|f(x) - f(x')|_{\mathcal{Y}} \leqslant |x - x'|_{\mathcal{X}}$$

for any $x, x' \in \mathcal{X}$. Show that f is continuous.

1.6. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A continous bijection $f\colon \mathcal{X} \to \mathcal{Y}$ is called a* homeomorphism *if its inverse $f^{-1}\colon \mathcal{Y} \to \mathcal{X}$ is also continuous.*

If there exists a homeomorphism $f\colon \mathcal{X} \to \mathcal{Y}$, we say that $\mathcal{X}$ is homeomorphic *to $\mathcal{Y}$, or $\mathcal{X}$ and $\mathcal{Y}$ are* homeomorphic.

If a metric space $\mathcal{X}$ is homeomorphic to a known space, for example plane, sphere, disc, circle and so on, we may also say that $\mathcal{X}$ is a *topological* plane, sphere, disc, circle and so on.

1.7. Definition. *A subset A of a metric space $\mathcal{X}$ is called* closed *if whenever a sequence (x_n) of points from A converges in $\mathcal{X}$, we have that $\lim_{n\to\infty} x_n \in A$.*

A set $\Omega \subset \mathcal{X}$ is called open *if for any $z \in \Omega$, there is $\varepsilon > 0$ such that $B(z, \varepsilon) \subset \Omega$.*

An open set Ω that contains a given point p is called *neighborhood of p*.

1.8. Exercise. *Let Q be a subset of a metric space $\mathcal{X}$. Show that A is closed if and only if its complement $\Omega = \mathcal{X} \backslash Q$ is open.*

Chapter 2

Curves

Paths. Let $\mathcal{X}$ be a metric space. A continuous map $f\colon [0,1]\to\mathcal{X}$ is called a *path*. If $p=f(0)$ and $q=f(1)$, then we say that f *connects* p *to* q.

If any two points in $\mathcal{X}$ can be connected by a path then $\mathcal{X}$ is called *path connected*. Similarly, a subset $A\subset\mathcal{X}$ is called *path connected* if any two points $p,q\in A$ can be connected by a path that runs in A; equivalently, the subspace A is path connected.

Simple curves.

2.1. Definition. *A path connected subset γ in a metric space is called a* simple curve *if it is* locally *homeomorphic to a real interval; that is, any point $p\in\gamma$ has a neighborhood $U\ni p$ such that the intersection $U\cap\gamma$ is homeomorphic to a real interval.*

It turns out that any curve γ admits a homeomorphism from a real interval or a circle; that is, there is a continuous bijection $G\to\gamma$ with continuous inverse; here (and further) G denotes a circle or real interval. We omit a proof of this statement, but it is not hard.

The homeomorphism $G\to\gamma$ as above is called *parametrization* of γ. The parametrization completely defines the curve. Often will use the same letter for curve and its parametrization, so we can say curve γ has parametrization $\gamma\colon G\to\mathcal{X}$. Note however that any curve admits many different parametrization.

2.2. Exercise. *Find a continuous injective map $\gamma\colon [0,1)\to\mathbb{R}^2$ such that its image is not a simple curve.*

Hint: The image of γ should have a shape of digit 9.

If G is a circle, then the curve $\gamma\colon G\to\mathcal{X}$ is called *closed*. If G is a real interval, then we may say that γ is an *arc*.

Parameterized curves. A *parameterized curve* is defined as a continuous map $\gamma\colon G\to\mathcal{X}$. For a parameterized curve we do not assume that γ is injective; in other words the parameterized curve might have self-intersections.

2.3. Advanced exercise. *Let $\alpha\colon[0,1]\to\mathcal{X}$ be a path from p to q. Assume $p\ne q$. Show that there is a simple path connecting from p to q in $\mathcal{X}$.*

Smooth curves

A curve in the Euclidean space or plane, called *space* or *plane curve* correspondingly.

A space curve can be described by its coordinate functions

$$\gamma(t)=(x(t),y(t),z(t)).$$

Plane curves can be considered as a partial case of space curves with $z(t)\equiv 0$.

If each of the coordinate functions $x(t),y(t),z(t)$ of the space curve γ is a smooth (that is, it has derivatives of all orders everywhere in its domain) then the parameterized curve is called *smooth*.

If the *velocity vector*

$$\gamma'(t)=(x'(t),y'(t),z'(t))$$

does not vanish at all points, then the parameterized curve γ is called *regular*.

A simple space curve is called *smooth and regular* if it admits a smooth and regular parametrization correspondingly. Regular smooth curves are among the main objects in differential geometry; the term *smooth curve* often used for *smooth regular curve*.

2.4. Exercise. *The function*

$$f(t)=\begin{cases}0 & \text{if } t\leqslant 0,\\ \frac{t}{e^{1/t}} & \text{if } t>0.\end{cases}$$

is smooth.[1]

Show that $\gamma(t)=(f(t),f(-t))$ gives a smooth parametrization of the curve S formed by the union of two half-axis in the plane.

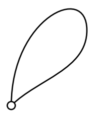
[1] The existence of all derivatives $f^{(n)}(x)$ at $x\ne 0$ is evident and direct calculations show that $f^{(n)}(0)=0$ for any n.

Show that any smooth parametrization of S has vanishing velocity vector at the origin. Conclude that the curve S is not regular and smooth.

2.5. Exercise. *Describe the set of real numbers a such that the plane curve $\gamma_a(t) = (t + a\cdot \sin t, a\cdot \cos t)$, $t \in \mathbb{R}$ is*
(a) regular;
(b) simple.

Loops and periodic parametrization. A closed simple curve can be described as an image of a parameterized curve $\gamma\colon [0,1] \to \mathcal{X}$ such that $p = \gamma(0) = \gamma(1)$; such curves are called *loops*; the point p in this case is called *base* of the loop.

However, it is more natural to present it as a *periodic* parameterized cure $\gamma\colon \mathbb{R} \to \mathcal{X}$; that is, there is a constant ℓ such that $\gamma(t + \ell) = \gamma(t)$ for any t. For example the unit circle in the plane can be described by $2\cdot\pi$-periodic parametrization $\gamma(t) = (\cos t, \sin t)$.



Any smooth regular closed curve can be described by a smooth regular loop. But in general the closed curve that described by a smooth regular loop might fail to be smooth and regular — it might fail to be smooth at its base; an example shown on the diagram.

Implicitly defined curves

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function; that is, all its partial derivatives defined in its domain of definition. Consider the set S of solution of equation $f(x, y) = 0$ in the plane.

Assume S is path connected. According to implicit function theorem (A.2), the set S is a smooth regular simple curve if 0 is a *regular value* of f. In this case it means that the gradient ∇f does not vanish at any point $p \in S$. In other words, if $f(p) = 0$, then $\frac{\partial f}{\partial x}(p) \neq 0$ or $\frac{\partial f}{\partial y}(p) \neq 0$.

Similarly, assume f, h is a pair of smooth functions defined in $\mathbb{R}^3$. The system of equations $f(x, y, z) = h(x, y, z) = 0$ defines a regular smooth space curve if the set of solutions is path connected and 0 is a regular value of the map $F\colon (x, y, z) \mapsto (f(x, y, z), h(x, y, z))$. In this case it means that the gradients ∇f and ∇h are linearly independent at any point $p \in S$. In other words, if $f(p) = 0$, then at the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}$$

for the map $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ has rank 2 at p.

The described way to define a curve is called *implicit*; if a curve is defined by its parametrization, we say that it is *explicitly defined*. While implicit function theorem guarantees the existence of regular smooth parametrizations, do not expect it to be in a closed form. When it comes to calculations, usually it is easier to work directly with implicit presentation.

2.6. Exercise. *Consider the set in the plane described by the equation*

$$y^2 = x^3.$$

Is it a simple curve? and if “yes”, is it a smooth regular curve?

2.7. Exercise. *Describe the set of real numbers a such that the system of equations*

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ x^2 + a{\cdot}x + y^2 &= 0 \end{aligned}$$

describes a smooth regular curve.

Chapter 3

Length

Recall that a sequence

$$a = t_0 < t_1 < \dots < t_k = b.$$

is called a *partition* of the interval $[a,b]$.

3.1. Definition. *Let $\alpha\colon [a,b] \to \mathcal{X}$ be a curve in a metric space. The* length *of a α is defined as*

$$\begin{aligned}\operatorname{length}\alpha = \sup\{|\alpha(t_0)-\alpha(t_1)| + |\alpha(t_1)-\alpha(t_2)| + \dots\\ \dots + |\alpha(t_{k-1})-\alpha(t_k)|\},\end{aligned}$$

where the exact upper bound is taken over all partitions

$$a = t_0 < t_1 < \dots < t_k = b.$$

The length of α is a nonnegative real number or infinity; the curve α is called rectifiable *if its length is finite.*

The length of a closed curve is defined as the length of a corresponding loop. If a curve is defined on a open or closed-open interval then its length is defined as the exact upper bound for lengths of all its closed arcs.

If α is a space curve, then the above definition says that it length is the exact upper bound of the lengths of polygonal lines $p_0 \dots p_k$ *inscribed* in the curve, where $p_i = \alpha(t_i)$ for a partition $a = t_0 < t_1 < \dots < t_k = b$. If α is closed then $p_0 = p_k$ and therefore the inscribed polygonal line is also closed.

3.2. Exercise. *Let $\alpha\colon [0,1] \to \mathbb{R}^3$ be a simple curve. Suppose a parametrized curve $\beta\colon [0,1] \to \mathbb{R}^3$ has that same image as α; that is*

$\beta([0,1])=\alpha([0,1])$. *Show that*

$$\operatorname{length}\beta \geqslant \operatorname{length}\alpha.$$

3.3. Exercise. *Assume* $\alpha\colon [a,b]\to\mathbb{R}^3$ *is a smooth curve. Show that*

(a) $\operatorname{length}\alpha \geqslant \int_a^b |\alpha'(t)|\cdot dt,$

(b) $\operatorname{length}\alpha \leqslant \int_a^b |\alpha'(t)|\cdot dt.$

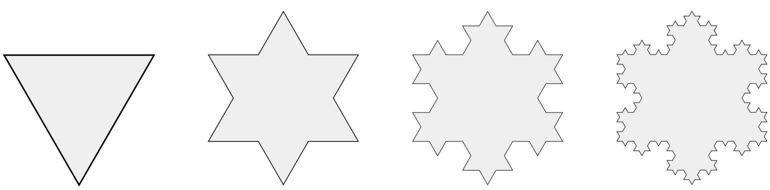
Conclude that

❶ $$\operatorname{length}\alpha = \int\limits_a^b |\alpha'(t)|\cdot dt.$$

Hints: For (a), apply the fundamental theorem of calculus for each segment in a given partition. For (b) consider a partition such that the velocity vector $\alpha'(t)$ is nearly constant on each of its segments.

Nonrectifiable curves. A classical example of a nonrectifiable curve is the so called *Koch snowflake*; it is a fractal curve that can be constructed the following way:

Start with an equilateral triangle, divide each of its side into three segments of equal length and add an equilateral triangle with base at the middle segment. Repeat this construction recursively to the obtained polygons. Few first iterations of the construction are shown



on the diagram. The Koch snowflake is the boundary of the union of all the polygons.

3.4. Exercise.

(a) Show that Koch snowflake is a closed simple curve; that is, it admits a homeomorphism to a circle.

(b) Show that Koch snowflake is not rectifiable.

Arc length parametrization

We say that a parametrized curve γ has an *arc length parametrization*[1] if for any two values of parameters $t_1 < t_2$, the value $t_2 - t_1$ is the length of $\gamma|_{[t_1,t_2]}$; that is, the closed arc of γ from t_1 to t_2.

Note that a smooth space curve $\gamma(t) = (x(t), y(t), z(t))$ has arc length parametrization if and only if it has unit velocity vector at all times; that is

$$|\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = 1;$$

by that reason arc length parametrization of smooth curves with also called *unit-speed curves.* Note that smooth unit-speed curves are automatically regular.

Any rectifiable curve can be parameterized by arc length. For a parametrized smooth curve γ, the arc length parameter s can be written as an integral

$$s(t) = \int\limits_{t_0}^{t} |\gamma'(\tau)|\cdot d\tau.$$

Note that $s(t)$ is a smooth increasing function. Further by fundamental theorem of calculus, $s'(t) = |\gamma'(t)|$. Therefore if γ is regular, then $s'(t) \neq 0$ for any parameter value t. By inverse function theorem (A.1) the inverse function $s^{-1}(t)$ is also smooth. Therefore $\gamma \circ s^{-1}$ — the reparametrization of γ by arclength s — remains smooth and regular.

Most of the time we use s for an arc length parameter of a curve.

3.5. Exercise. *Reparametrize the helix $\gamma_a(t) = (a\cdot \cos t, a\cdot \sin t, t)$ by arc length.*

We will be interested in the properties of curves that are invariant under a reparametrization. Therefore we can always assume that the given smooth regular curve comes with a arc length parametrization. A good property of arc length parametrizations is that it is almost canonical — these parametrizations differ only by a sign and additive constant. On the other hand, often it is impossible to find an arc length parametrization in a closed form which makes it hard to use it calculations; usually it is more convenient to use the original parametrization.

[1]which is also called *natural parametrization*

Convex curves

A simple plane curve is called *convex* if it bounds a convex region.

3.6. Proposition. *Assume a convex closed curve α lies inside the domain bounded by a closed simple plane curve β. Then*

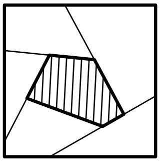
$$\operatorname{length}\alpha \leqslant \operatorname{length}\beta.$$

Note that it is sufficient to show that for any polygon P inscribed in α there is a polygon Q inscribed in β with $\operatorname{perim} P \leqslant \operatorname{perim} Q$, where $\operatorname{perim} P$ denotes the perimeter of P.

Therefore it is sufficient to prove the following lemma.

3.7. Lemma. *Let P and Q be polygons. Assume P is convex and $Q \supset P$. Then*

$$\operatorname{perim} P \leqslant \operatorname{perim} Q.$$



Proof. Note that by the triangle inequality, the inequality

$$\operatorname{perim} P \leqslant \operatorname{perim} Q$$

holds if P can be obtained from Q by cutting it along a chord; that is, a line segment with ends on the boundary of Q that lies in Q.

Note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \dots \subset P_n = Q$$

such that P_{i-1} obtained from P_i by cutting along a chord. Therefore

$$\begin{aligned}\operatorname{perim} P = \operatorname{perim} P_0 \leqslant \operatorname{perim} P_1 \leqslant \dots\\ \dots \leqslant \operatorname{perim} P_n = \operatorname{perim} Q\end{aligned}$$

and the lemma follows. □

3.8. Corollary. *Any convex closed plane curve is rectifiable.*

Proof. Any closed curve is bounded; that is, it lies in a sufficiently large square. Indeed the curve can be described as an image of a loop $\alpha\colon [0,1] \to \mathbb{R}^2$, $\alpha(t) = (x(t), y(t))$. The coordinate functions $x(t)$ and $y(t)$ are continous functions defined on $[0,1]$. Therefore the absolute values of both of these functions are bounded by some constant C. That is α lies in the square defined by the inequalities $|x| \leqslant C$ and $|y| \leqslant C$.

By Proposition 3.6, the length of the curve can not exceed the perimeter of the square $8{\cdot}C$, whence the result. □

Recall that convex hull of a set X is the smallest convex set that contains X; in other words convex hull is the intersection of all convex sets containing X.

3.9. Exercise. *Let α be a closed simple plane curve. Denote by K the convex hull of α; let β be the boundary curve of K. Show that*

$$\operatorname{length}\alpha \geqslant \operatorname{length}\beta.$$

Try to show that the statement holds for arbitrary closed plane curve α, assuming that X has nonempty interior.

Crofton formulas*

Consider a plane curve $\alpha\colon [a,b] \to \mathbb{R}^2$. Given a unit vector u, denote by α_u the curve that follows orthogonal projections of α to the line in the direction u; that is

$$\alpha_u(t) = \langle u, \alpha(t)\rangle{\cdot}u.$$

Note that

$$|\alpha'(t)| = |\langle u, \alpha'(t)\rangle|$$

for any t. Note that for any plane vector the magnitude of its average projection is proportional to its magnitude with coefficient; that is,

$$|w| = k{\cdot}\overline{|w_u|},$$

where $\overline{|w_u|}$ denotes the average value of $|w_u|$ for all unit vectors u. (The value k is the average value of $|\cos\varphi|$ for $\varphi \in [0, 2{\cdot}\pi]$; it can be found by integration, but soon we will show another way to find it.)

If the curve α is smooth, then according to Exercise 3.3

$$\begin{aligned}\operatorname{length}\alpha &= \int\limits_a^b |\alpha'(t)|{\cdot}dt = \\ &= \int\limits_a^b k{\cdot}\overline{|\alpha_u'(t)|}{\cdot}dt = \\ &= k{\cdot}\overline{\operatorname{length}\alpha_u}.\end{aligned}$$

This formula and its relatives are called Crofton formulas. To find the coefficient k one can apply it for the unit circle: the left hand side is $2{\cdot}\pi$ — this is the length of unit circle. Note that for any unit vector u, the curve α_u runs back and forth along an interval of length 2. Therefore $\operatorname{length}\alpha_u = 4$ and hence its average value is also 4. It follows that the coefficient k has to satisfy the equation $2{\cdot}\pi = k{\cdot}4$; whence

$$\operatorname{length}\alpha = \tfrac{\pi}{2}{\cdot}\overline{\operatorname{length}\alpha_u}.$$

The Crofton's formula holds for arbitrary rectifiable curves, not necessary smooth; it can be proved using Exercises 3.12.

3.10. Exercise. *Assume that closed plane curve α has length at least $\pi{\cdot}s$, where s is the average of pojections of α to lines. Moreover the equality holds if and only if α is convex.*

Use this statement to solve of Exercise 3.9.

3.11. Advanced exercise. *Show that the length of space curve is proportional to the average length of its projections to all lines and to planes. Find the coefficients in each case.*

3.12. Advanced exercises.

(a) *Show that the formula ❶ holds for any Lipschitz curve $\alpha\colon [a,b]\to\to\mathbb{R}^3$.*

(b) *Construct a simple curve $\alpha\colon [a,b]\to\mathbb{R}^3$ such that the velocity vector $\alpha'(t)$ is defined and bounded for almost all $t\in[a,b]$, but the formula ❶ does not hold.*

Hint: Use theorems of Rademacher and Lusin (A.3 and A.4).

Semicontinuity of length

Recall that the lower limit of a sequence of real numbers (x_n) is denoted by

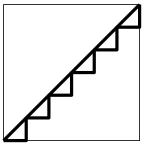
$$\varliminf_{n\to\infty} x_n.$$

It is defined as the lowest partial limit; that is, the lowest possible limit of a subsequence of (x_n). The lower limit is defined for any sequence of real numbers and it lies in the exteded real line $[-\infty,\infty]$

3.13. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\alpha_n\colon [a,b]\to\mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\alpha_\infty\colon [a,b]\to\mathcal{X}$; that is, $\alpha_n(t)\to\alpha_\infty(t)$ for any fixed $t\in[a,b]$ as $n\to\infty$. Then

❷ $$\varliminf_{n\to\infty}\operatorname{length}\alpha_n\geqslant\operatorname{length}\alpha_\infty.$$



Note that the inequality ❷ might be strict. For example the diagonal α_∞ of the unit square can be approximated by a sequence of stairs-like polygonal curves α_n with sides parallel to the sides of the square (α_6 is on the picture). In this case

$$\operatorname{length}\alpha_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length}\alpha_n = 2$$

for any n.

Proof. Fix a partition $a = t_0 < t_1 < \dots < t_k = b$. Set

$$\begin{aligned}\Sigma_n &:= |\alpha_n(t_0) - \alpha_n(t_1)| + \dots + |\alpha_n(t_{k-1}) - \alpha_n(t_k)|.\\ \Sigma_\infty &:= |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + \dots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|.\end{aligned}$$

Note that $\Sigma_n \to \Sigma_\infty$ as $n \to \infty$ and $\Sigma_n \leqslant \operatorname{length}\alpha_n$ for each n. Hence

❸ $$\varliminf_{n\to\infty} \operatorname{length}\alpha_n \geqslant \Sigma_\infty.$$

If α_∞ is rectifiable, we can assume that

$$\operatorname{length}\alpha_\infty < \Sigma_\infty + \varepsilon.$$

for any given $\varepsilon > 0$. By ❸ it follows that

$$\varliminf_{n\to\infty} \operatorname{length}\alpha_n > \operatorname{length}\alpha_\infty - \varepsilon$$

for any $\varepsilon > 0$; whence ❷ follows.

It remains to consider the case when α_∞ is not rectifiable; that is $\operatorname{length}\alpha_\infty = \infty$. In this case we can choose a partition so that $\Sigma_\infty > L$ for any real number L. By ❸ it follows that

$$\varliminf_{n\to\infty} \operatorname{length}\alpha_n > L$$

for any L; whence $\varliminf_{n\to\infty} = \infty$ and ❷ follows. $\square$

Length metric

Let $\mathcal{X}$ be a metric space. Given two points x, y in $\mathcal{X}$, denote by $d(x, y)$ the exact lower bound for lengths of all paths connecting x to y; if there is no such path we assume that $d(x, y) = \infty$.

Note that function d satisfies all the axioms of metric except it might take infinite value. Therefore if any two points in $\mathcal{X}$ can be connected by a rectifiable curve, then d defines a new metric on $\mathcal{X}$; in this case d is called *induced length metric.*

Evidently $d(x,y) \geqslant |x-y|$ for any pair of points $x,y \in \mathcal{X}$. If the equality holds for any pair, then then the metric is called *length metric* and the space is called *length-metric space.*

Most of the time we consider length-metric spaces. In particular the Euclidean space is a length-metric space. A subspaces A of length-metric space $\mathcal{X}$ might be not a lenght-metric space; the induced length distance between points x and y in the subspace A will be denoted as $|x-y|_A$; that is $|x-y|_A$ is the exact lower bound for the length of paths in A.

3.14. Exercise. *Let $A \subset \mathbb{R}^3$ be a closed subset. Show that A is convex if and only if*

$$|x-y|_A = |x-y|_{\mathbb{R}^3}.$$

3.15. Exercise. *Let us denote by $\mathbb{S}^1$ the unit circle in the plane; that is,*

$$\mathbb{S}^1 = \left\{\, (x,y) \in \mathbb{R}^3 \;:\; x^2+y^2=1 \,\right\}.$$

Show that

$$|u-v|_{\mathbb{S}^1} = \measuredangle(u,v) := \arccos\langle u,v\rangle$$

for any $u,v \in \mathbb{S}^1$.

Spherical curves

A space curve γ is called *spherical* if it runs in the unit sphere; that is, $|\gamma(t)| = 1$ for any t.

3.16. Exercise. *Let us denote by $\mathbb{S}^2$ the unit sphere in the space; that is,*

$$\mathbb{S}^2 = \left\{\, (x,y,z) \in \mathbb{R}^3 \;:\; x^2+y^2+z^2=1 \,\right\}.$$

Show that

$$|u-v|_{\mathbb{S}^2} = \measuredangle(u,v) := \arccos\langle u,v\rangle$$

for any $u,v \in \mathbb{S}^2$.

Hint: Use Exercise 3.15 and the following map $f\colon (r,\theta,\varphi) \mapsto (r,\theta,0)$ in spherical coordinates. Note that f is distance nonexpanding and it maps $\mathbb{R}^3$ to a half-plane and $\mathbb{S}^2$ to one of its meridians.

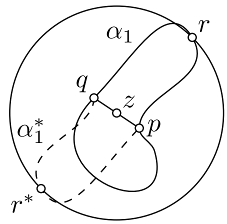
3.17. Hemisphere lemma. *Any closed curve of length $< 2{\cdot}\pi$ in $\mathbb{S}^2$ lies in an open hemisphere.*

This lemma is a keystone in the proof of Fenchel's theorem given below. The lemma is not as simple as you might think — try to prove it yourself. I learned the following proof from Stephanie Alexander.

Proof. Let α be a closed curve in $\mathbb{S}^2$ of length $2{\cdot}\ell$.

Assume $\ell < \pi$.

Let us divide α into two arcs α_1 and α_2 of length ℓ, with endpoints p and q. According to Exercise 3.16, $\measuredangle(p,q) \leqslant \ell < \pi$. Denote by z be the midpoint between p and q in $\mathbb{S}^2$; that is z is the midpoint of an equator arc from p to q. We claim that α lies in the open north hemisphere with north pole at z. If not, α intersects the equator in a point, say r. Without loss of generality we may assume that r lies on α_1.



The north hemisphere corresponds to the disc and the south hemisphere to the complement of the disc.

Rotate the arc α_1 by angle π around the line thru z and the center of the sphere. The obtained arc α_1^* together with α_1 forms a closed curve of length $2{\cdot}\ell$ that passes thru r and its antipodal point r^*. Therefore

$$\tfrac{1}{2}\cdot \operatorname{length}\alpha = \ell \geqslant \measuredangle(r,r^*) = \pi,$$

a contradiction. ☐

3.18. Exercise. *Describe a simple closed spherical curve that does not pass thru a pair of antipodal points and does not lie in any hemisphere.*

3.19. Exercise. *Suppose that a closed simple spherical curve α divides $\mathbb{S}^2$ into two regions of equal area. Show that*

$$\operatorname{length}\alpha \geqslant 2{\cdot}\pi.$$

3.20. Exercise. *Consider the following problem, find a flaw in the given solution. Come up with a correct argument.*

Problem. Suppose that a closed plane curve α has length at most 4. Show that α lies in a unit disc.

Wrong solution. Note that it is sufficient to show that diameter of α is at most 2; that is, the distance between any two pairs of points p and q of α an not exceed 2.

The length of α can not be smaller then the closed inscribed polygonal line which goes from p to q and back to p. Therefore

$$2\cdot|p-q| \leqslant \operatorname{length}\alpha \leqslant 4. \qquad \square$$

3.21. Advanced exercises. *Given points $v, w \in \mathbb{S}^2$, denote by w_v the closest point to w_u on the equator with pole at v; in other words, it $w^\perp$ is the projection of w to the plane perpendicular to v, then w_v is the unit vector in the direction of $w^\perp$. The vector w_v is defined if $w \neq \pm v$.*

1. *Show that for any spherical curve α we have that*

$$\operatorname{length}\alpha = \overline{\operatorname{length}\alpha_v},$$

   where $\overline{\operatorname{length}\alpha_v}$ denotes the average length for all $v \in \mathbb{S}^2$. (This is a spherical analog of Crofton's formula.)

2. *Give another proof of hemisphere lemme using part (1).*

Chapter 4

Curvature of curves

In general the term *curvature* is used for something that measures how much a geometric object deviates from being *straight*; in particular the curvature of a curve suppose to measure how fast it deviates from a straight line.

Acceleration of unit-speed curve

Recall that any regular smooth curve can be parametrized by its length. The obtained curve γ remains to be smooth and it has unit speed; that is, $|\gamma'(s)| = 1$ for all s.

The following proposition essentailly states that the acceleration vector is perpendicular to the velocity vector if the speed remains constant.

4.1. Proposition. *Assume γ be a smooth unit-speed space curve. Then $\gamma'(s) \perp \gamma''(s)$ for any s.*

The scalar product (also known as dot product) of two vectors v and w will be denoted by $\langle v, w\rangle$. Recall that the derivative of a scalar product satisfies the product rule; that is if $v = v(t)$ and $w = w(t)$ are smooth vector-valued functions of a real parameter t, then

$$\langle v, w\rangle' = \langle v', w\rangle + \langle v, w'\rangle.$$

Proof. Since $|\gamma'(s)| = 1$, we have

$$\langle \gamma'(s), \gamma'(s)\rangle = 1.$$

Differentiating both sides we get

$$2{\cdot}\langle \gamma''(s), \gamma'(s)\rangle = 0,$$

hence the result. □

Curvature

For a unit speed space curve γ the magnitude of its acceleration $|\gamma''(s)|$ is called its *curvature* at s. If γ is simple, then we can say that $|\gamma''(s)|$ is the curvature at the point $p=\gamma(s)$ without ambiguity. The curvature is usually denoted by $k(s)$ and in the latter case it might be also denoted by $k(p)$.

Informally the curvature measures how fast the curve turns; if you drive along a plane curve, curvature tells how much to turn the steering wheel at the given point (note that it does not depend on your speed).

4.2. Exercise. *Show that any regular smooth spherical curve has curvature at least 1 at each point.*

Hint: Differentiate the identity $\langle\gamma(s),\gamma(s)\rangle=1$ a couple of times.

Tangent indicatrix

It was convenient to use arc length parametrization to define curvature, but for finding curvature it more convenient to use the original description of curvature via tangent indicatrix described below.

Let γ be a regular smooth space curve. Let us consider another curve

❶ $$\tau(t)=\tfrac{\gamma'(t)}{|\gamma'(t)|}$$

that is called *tangent indicatrix* of γ. Note that $|\tau(t)|=1$ for any t; that is, τ is a spherical curve that points in the direction tangent to γ at each time.

If γ has a unit speed parametrization, then $\tau(t)=\gamma'(t)$. In this case we have the following expression for curvature: $k(t)=|\tau'(t)|=|\gamma''(t)|$.

In general case we have

❷ $$k(t)=\tfrac{|\tau'(t)|}{|\gamma'(t)|}.$$

Indeed, for an arc length parametrization $s(t)$ we have $s'(t)=|\gamma'(t)|$. Therefore

$$\begin{aligned}k(t)&=|\tfrac{d\tau}{ds}|=\\&=|\tfrac{d\tau}{dt}|/|\tfrac{ds}{dt}|=\\&=\tfrac{|\tau'(t)|}{|\gamma'(t)|}.\end{aligned}$$

4.3. Exercise. *Use the formulas ❶ and ❷ to show that for any smooth regular space curve γ we have the following expressions for its curvature:*

(a)

$$k(t) = \frac{|\gamma''(t)^\perp|}{|\gamma'(t)|^2},$$

where $\gamma''(t)^\perp$ denotes the projection of $\gamma''(t)$ to the perpendicular plane to $\gamma'(t)$;

(b)

$$k(t) = \frac{|\gamma''(t) \times \gamma'(t)|}{|\gamma'(t)|^3}.$$

Hint: Prove and use the following identities:

$$\gamma''(t) - \gamma''(t)^\perp = \tfrac{\gamma'(t)}{|\gamma'(t)|} \cdot \langle \gamma''(t), \tfrac{\gamma'(t)}{|\gamma'(t)|} \rangle,$$
$$|\gamma'(t)| = \sqrt{\langle \gamma'(t), \gamma'(t) \rangle}.$$

4.4. Exercise. *Apply the formulas in the previous exercise to show that if f is a smooth real function, then its graph $\Gamma_f = \{\,(x, f(x)) \in \in \mathbb{R}^2\,\}$ has curvature*

$$k(p) = \frac{|f''(x)|}{(1 + f'(x)^2)^{\frac{3}{2}}}$$

at the point $p = (x, f(x))$.

Total curvature

Let $\gamma\colon [a, b] \to \mathbb{R}^3$ be a regular smooth curve and τ its *tangent indicatrix*. Recall that without loss of generality we can assume that γ has a unit speed parametrization; in this case $\tau(t) = \gamma'(t)$ and hence

$$k(t) := |\gamma''(t)| = |\tau'(t)|,$$

that is, the curvature of γ at t is the speed of the *tangent indicatrix* τ at t.

The integral

$$\Phi(\gamma) := \int\limits_a^b k(t) \cdot dt$$

is called *total curvature of* γ.

4.5. Exercise. *Find the curvature of helix $\gamma_a = (a\cdot\cos t, a\cdot\sin t, t)$, its tangent indicatrix and the total curvature of its arc $t \in [0, 2\cdot\pi]$.*

4.6. Observation. *The total curvature of a smooth regular curve is the length of its tangent indicatrix.*

Proof. It is sufficient to prove the observation for a unit-speed space curve $\gamma\colon [a,b] \to \mathbb{R}^3$. Denote by τ its tangent indicatrix. Then

$$\begin{aligned}\Phi(\gamma) &:= \int\limits_a^b k(t)\cdot dt = \\ &= \int\limits_a^b |\tau'(t)|\cdot dt = \\ &= \operatorname{length}\tau.\end{aligned}$$

□

4.7. Fenchel's theorem. *The total curvature of of any closed regular space curve is at least $2\cdot\pi$.*

Proof. Fix a closed regular space curve γ; we can assume that it has a unit-speed parametrization $\gamma\colon [a,b] \to \mathbb{R}^3$, so $\gamma(a) = \gamma(b)$.

Consider its tangent indicatrix $\tau = \gamma'$. By fundamental theorem of calculus we have that

❸ $$\int\limits_a^b \tau(t)\cdot dt = \gamma(b) - \gamma(a) = 0.$$

Recall that $|\tau(t)| = 1$ for any t; that is, τ is a spherical curve. Let us show that τ can not lie in a hemisphere. Assume contrary, denote by u the center of the hemisphere; so $\langle u, \tau(t)\rangle > 0$ for any t. From ❸, we get that

$$\begin{aligned}0 &= \left\langle u, \int\limits_a^b \tau(t)\cdot dt \right\rangle = \\ &= \int\limits_a^b \langle u, \tau(t)\rangle\cdot dt > \\ &> 0.\end{aligned}$$

— a contradiction.

Applying the observation (4.6) and the hemisphere lemma (3.17), we get that

$$\Phi(\gamma) = \operatorname{length}\tau \geqslant 2\cdot\pi.$$

□

Appendix A

Preliminaries

In this chapter we state and discuss results from different branches of mathematics which were used further in the book. The reader is not expected to know proofs of these statements, but it is better to check that his intuition agrees with each.

Multivariable calculus

A map $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^k$ can be thought as array of functions

$$f_1, \dots, f_k \colon \mathbb{R}^n \to \mathbb{R}.$$

The map $\boldsymbol{f}$ is called *smooth* if each function f_i is smooth; that is, all partial derivatives of f_i are defined in the domain of definition of $\boldsymbol{f}$.

Inverse function theorem gives a sufficient condition for a smooth function to be invertible in a neighborhood of a given point p in its domain. The condition is formulated in terms of partial derivative of f_i at p.

Implicit function theorem is a close relative to inverse function theorem; in fact it can be obtained as its corollary. It is used for instance when we need to pass from parametric to implicit description of curves and surface.

Both theorems reduce the existence of a map satisfying certain equation to a question in linear algebra. We use these two theorems only for $n \leqslant 3$.

These two theorems are discussed in any course of multivariable calculus, the classical book of Walter Rudin [3] is one of my favorites.

1.1. Inverse function theorem. *Let* $\boldsymbol{f} = (f_1, \dots, f_n)\colon \mathbb{R}^n \to \mathbb{R}^n$

be a smooth map. Assume that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is invertible at some point p in the domain of definition of $\boldsymbol{f}$. Then there is a smooth function $\boldsymbol{h}\colon \mathbb{R}^m \to \mathbb{R}^n$ defined is a neighborhood Ω_q of $q = \boldsymbol{f}(p)$ that is local inverse of $\boldsymbol{f}$ at p*; that is, there are neighborhoods $\Omega_p \ni p$ such that $\boldsymbol{f}$ defines a bijection $\Omega_p \to \Omega_q$ and $\boldsymbol{f}(x) = y$ if and only if $x = \boldsymbol{h}(y)$ for any $x \in \Omega_p$ and any $y \in \Omega_q$.*

1.2. Implicit function theorem. *Let $\boldsymbol{f} = (f_1, \dots, f_n)\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a smooth map, $m, n \geqslant 1$. Let us consider $\mathbb{R}^{n+m}$ as a product space $\mathbb{R}^n \times \mathbb{R}^m$ with coodinates $x_1, \dots, x_n, y_1, \dots, y_m$. Consider the following matrix*

$$M = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

formed by first n columns of the Jacobian matrix. Assume M is invertible at some point p in the domain of definition of $\boldsymbol{f}$ and $\boldsymbol{f}(p) = 0$. Then there is a neighborhood $\Omega_p \ni p$ and smooth function $\boldsymbol{h}\colon \mathbb{R}^m \to \mathbb{R}^n$ defined is a neighborhood $\Omega_0 \ni 0$ that for any $(x_1, \dots, x_n, y_1, \dots y_m) \in \Omega_p$ the equality

$$\boldsymbol{f}(x_1, \dots, x_n, y_1, \dots y_m) = 0$$

holds if and only if

$$(x_1, \dots x_n) = \boldsymbol{h}(y_1, \dots y_m).$$

If the assumption in the theorem holds for any point p such that $\boldsymbol{f}(p) = 0$, then we say that 0 is a regular value of $\boldsymbol{f}$. *Sard's theorem* states that most of the values of smooth map are regular; in particular generic smooth function satisfies the assumption of the theorem.

Real analysis

Recall that a function f is called Lipschitz if there is a constant L such that

$$|f(x) - f(y)| \leqslant L{\cdot}|x - y|$$

for values x and y in the domain of definition of f. This definition works for maps between metric spaces, but we will use it for real-to-real functions only.

1.3. Rademacher's theorem. *Let $f\colon [a,b] \to \mathbb{R}$ be a Lipschitz function. then derivative $f'(x)$ is defined for alomst all $x \in [a,b]$. Moreover the derivative f' is a bounded measurable function defined almost everywhere in $[a,b]$ and it satisfies the fundamental theorem of calculus; that is, the following identity*

$$f(b) - f(a) = \int\limits_a^b f'(x)\cdot dx,$$

holds if the integral understood in the sense of Lebesgue.

It is often helps to work with measurable functions; it makes possible to extend many statements about continuous function to measurable functions.

1.4. Lusin's theorem. *Let $\varphi\colon [a,b] \to \mathbb{R}$ be a measurable function. Then for any $\varepsilon > 0$, there is a continuous function $\psi_\varepsilon\colon [a,b] \to \mathbb{R}$ that coincides with φ outside of a set of measure at most ε. Moreover, φ is bounded above and/or below by some constants then we can assume that so is ψ_ε.*

Fundamental theorem of ODE

Picard theorem or the fundamental theorem of ordinary differential equations; it guarantees existance and uniqueness of a solution of an initial value problem for a system of ordinary differential equations

$$\begin{cases} x_1' = f_1(x_1, \dots, x_n), \\ \quad \dots \\ x_n' = f_n(x_1, \dots, x_n), \end{cases}$$

The array functions $(f_1, \dots, f_n)$ can be considered as one vector-valued function $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ and the array $(x_1, \dots, x_n)$ can be considered as a vector $\boldsymbol{x} \in \mathbb{R}^n$. Therefore the system can be rewritten as one vector equation

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}).$$

We use only the following partial case of this theorem.

1.5. Theorem. *Suppose $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. Then for any initial data $\boldsymbol{x}(0) = \boldsymbol{u}$ the differential equation*

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$$

has a unique solution $\boldsymbol{x}(t)$ defined at some open interval containing zero. Moreover the function $(\boldsymbol{u}, t) \mapsto \boldsymbol{x}(t)$ is smooth.

Topology

The following theorem is known for simple formulation and quite hard proof. The first part of the theorem is proved by Camille Jordan, the second part is due to Arthur Schoenflies.

1.6. Theorem. *The complement of any closed simple γ plane curve has exactly two connected components. One of these components is bounded B and the other U is unbounded.*

Moreover the union $B \cup \gamma$ is a topological disc.

By now many proofs of this theorem are known. A very short proof based on somewhat developed technique is given by Patrick Doyle [4], among elementary proofs, one of my favorites is the proof given by Aleksei Filippov [5].

Appendix B

Homework assignments

HWA-01. Exercises: 1.2, 2.4, 2.5, 2.7, 3.14.

HWA-02. Exercises: 3.4(b), 3.5, 3.9, 3.15, 3.20.

HWA-03. Exercises: 3.16, 3.18, 3.19, 4.3a, 4.5.

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