Differential geometry of curves and surfaces: a working approach

Anton Petrunin and Sergio Zamora Barrera

Contents

Preface

These notes might be good for you if you plan to do differential geometry, or if you want to have a solid ground not to do it in the future.

Differential geometry does geometry on top of several branches of mathematics including real analysis, measure theory, calculus of variation, differential equations, elementary and convex geometry, topology, and more. This subject is wide even at the entrance. By that reason it is fun and pain to teach and to study.

We discuss smooth curves and surfaces — the main gate to differential geometry. This subject provides a collection of examples and ideas critical for further study. It is wise to become a master in this subject before making further steps — no need for a rush.

We give an idea about subject, keeping it elementary, visual, and virtually rigorous; we allow gaps that belong to other branches of mathematics (these subjects discussed briefly in the appendixes).

We try to minimize the distance between definitions and meaningful results. The highest points in these notes are the theorem of Vladimir Ionin and Herman Pestov about closed curves with bounded curvature, the theorem of Sergei Bernstein's on saddle graphs, and the theorem of Stephan Cohn-Vossen on existence of simple two-sided infinite geodesic on an open convex surface. These theorems give the first nontrivial examples of the so called *local to global theorems* — the hart of differential geometry.

These notes are based on lectures at MASS program (Mathematics Advanced Study Semesters at Pennsylvania State University) Fall semester 2018. Number of these topics were presented on the lectures of Yurii Burago who was teaching the first author at the Leningrad University.

An excellent minimalist introductory textbook with a more classical approach was written by Aleksei Chernavskii [14]; if you read Russian, use it as a complementary source for this course. For a deeper diving into the subject we recommend the textbook of Victor Toponogov [50].

Part I

Preliminaries

Chapter 1

Metric spaces

We assume that the reader is familiar with the notion of metric in Euclidean space. In this chapter briefly discuss its generalization and fix notations that will be used further.

Definitions

Metric is a function that returns a real value $\mathrm{dist}(x, y)$ for any pair x, y in a given nonempty set $\mathcal{X}$ and satisfies the following axioms for any triple x, y, z:

(a) Positiveness:
$$\mathrm{dist}(x, y) \geqslant 0.$$

(b) $x = y$ if and only if
$$\mathrm{dist}(x, y) = 0.$$

(c) Symmetry:
$$\mathrm{dist}(x, y) = \mathrm{dist}(y, x).$$

(d) Triangle inequality:
$$\mathrm{dist}(x, z) \leqslant \mathrm{dist}(x, y) + \mathrm{dist}(y, z).$$

A set with a metric is called *metric space* and the elements of the set are called *points*.

Shortcut for distance. Usually we consider only one metric on a set, therefore we can denote the metric space and its underlying set by the same letter, say $\mathcal{X}$. In this case we also use a shortcut notations

$|x-y|$ or $|x-y|_{\mathcal{X}}$ for the *distance* $\operatorname{dist}(x,y)$ *from* x *to* y *in* $\mathcal{X}$. For example, the triangle inequality can be written as

$$|x-z|_{\mathcal{X}} \leqslant |x-y|_{\mathcal{X}} + |y-z|_{\mathcal{X}}.$$

Examples. Euclidean space and plane as well as real line will be the most important examples of metric spaces for us. In these examples the introduced notation $|x-y|$ for the distance from x to y has perfect sense as a norm of the vector $x-y$. However, in general metric space the expression $x-y$ has no sense, but anyway we use expression $|x-y|$ for the distance.

If we say *plane* or *space* we mean *Eucledean* plane or space. However the plane (as well as the space) admits many other metrics, for example the so-called *Manhattan metric* from the following exercise.

1.1. Exercise. *Consider the function*

$$\operatorname{dist}(p,q) = |x_p - x_q| + |y_p - y_q|,$$

where $p = (x_p, y_p)$ *and* $q = (x_q, y_q)$ *are points in the coordinate plane* $\mathbb{R}^2$. *Show that* dist *is a metric on* $\mathbb{R}^2$.

Let us mention another example: the *discrete space* — arbitrary nonempty set $\mathcal{X}$ with the metric defined as $|x-y|_{\mathcal{X}} = 0$ if $x = y$ and $|x-y|_{\mathcal{X}} = 1$ otherwise.

Subspaces. Any subset of a metric space is also a metric space, by restricting the original metric to the subset; the obtained metric space is called a *subspace*. In particular, all subsets of Euclidean space are metric spaces.

Balls. Given a point p in a metric space $\mathcal{X}$ and a real number $R \geqslant 0$, the set of points x on the distance less then R (at most R) from p is called open (respectively closed) ball of radius R with center at p. The *open ball* is denoted as $B(p,R)$ or $B(p,R)_{\mathcal{X}}$; the second notation is used if we need to emphasize that the ball lies in the metric space $\mathcal{X}$. Formally speaking

$$B(p,R) = B(p,R)_{\mathcal{X}} = \{\, x \in \mathcal{X} \,:\, |x-p|_{\mathcal{X}} < R \,\}.$$

Analogously, the *closed ball* is denoted as $\bar{B}[p,R]$ or $\bar{B}[p,R]_{\mathcal{X}}$ and

$$\bar{B}[p,R] = \bar{B}[p,R]_{\mathcal{X}} = \{\, x \in \mathcal{X} \,:\, |x-p|_{\mathcal{X}} \leqslant R \,\}.$$

1.2. Exercise. *Let* $\mathcal{X}$ *be a metric space.*

(a) Show that if $\bar{B}[p,2] \subset \bar{B}[q,1]$ *for some points* $p, q \in \mathcal{X}$, *then* $\bar{B}[p,2] = \bar{B}[q,1]$.

(b) Construct a metric space $\mathcal{X}$ *with two points* p *and* q *such that the strict inclusion* $B(p,\tfrac{3}{2}) \subset B(q,1)$ *holds.*

Continuity

In this section we will extend standard notions from calculus to the metric spaces.

1.3. Definition. *Let $\mathcal{X}$ be a metric space. A sequence of points $x_1, x_2, \dots$ in $\mathcal{X}$ is called* convergent *if there is $x_\infty \in \mathcal{X}$ such that $|x_\infty - x_n| \to 0$ as $n \to \infty$. That is, for every $\varepsilon > 0$, there is a natural number N such that for all $n \geqslant N$, we have*

$$|x_\infty - x_n|_{\mathcal{X}} < \varepsilon.$$

In this case we say that the sequence (x_n) converges *to x_∞, or x_∞ is the* limit *of the sequence (x_n). Notationally, we write $x_n \to x_\infty$ as $n \to \infty$ or $x_\infty = \lim_{n\to\infty} x_n$.*

1.4. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called* continuous *if for any convergent sequence $x_n \to x_\infty$ in $\mathcal{X}$, we have $f(x_n) \to f(x_\infty)$ in $\mathcal{Y}$.*

Equivalently, $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$, there is $\delta > 0$ such that

$$|x - x'|_{\mathcal{X}} < \delta \quad \textit{implies} \quad |f(x) - f(x')|_{\mathcal{Y}} < \varepsilon.$$

1.5. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is* distance non-expanding map*; that is,*

$$|f(x) - f(x')|_{\mathcal{Y}} \leqslant |x - x'|_{\mathcal{X}}$$

for any $x, x' \in \mathcal{X}$. Show that f is continuous.

1.6. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A continous bijection $f\colon \mathcal{X} \to \mathcal{Y}$ is called a* homeomorphism *if its inverse $f^{-1}\colon \mathcal{Y} \to \mathcal{X}$ is also continuous.*

If there exists a homeomorphism $f\colon \mathcal{X} \to \mathcal{Y}$, we say that $\mathcal{X}$ is homeomorphic *to $\mathcal{Y}$, or $\mathcal{X}$ and $\mathcal{Y}$ are* homeomorphic*.*

If a metric space $\mathcal{X}$ is homeomorphic to a known space, for example plane, sphere, disc, circle and so on, we may also say that $\mathcal{X}$ is a *topological* plane, sphere, disc, circle and so on.

1.7. Definition. *A subset A of a metric space $\mathcal{X}$ is called* closed *if whenever a sequence (x_n) of points from A converges in $\mathcal{X}$, we have that $\lim_{n\to\infty} x_n \in A$.*

A set $\Omega \subset \mathcal{X}$ is called open *if for any $z \in \Omega$, there is $\varepsilon > 0$ such that $B(z, \varepsilon) \subset \Omega$.*

An open set Ω that contains a given point p is called *neighborhood of* p.

1.8. Exercise. *Let* Q *be a subset of a metric space* $\mathcal{X}$. *Show that* Q *is closed if and only if its complement* $\Omega = \mathcal{X} \backslash Q$ *is open.*

Chapter 2

Multivariable calculus

This material is discussed in any course of multivariable calculus, the classical book of Walter Rudin [46] is one of our favorites.

Regular value

A map $\boldsymbol{f}\colon \mathbb{R}^m \to \mathbb{R}^n$ can be thought as an array of functions

$$f_1, \dots, f_n\colon \mathbb{R}^m \to \mathbb{R}.$$

The map $\boldsymbol{f}$ is called *smooth* if each function f_i is smooth; that is, all partial derivatives of f_i are defined in the domain of definition of $\boldsymbol{f}$.

The Jacobian matrix of $\boldsymbol{f}$ at $\boldsymbol{x} \in \mathbb{R}^m$ is defined as

$$\mathrm{Jac}_{\boldsymbol{x}}\, \boldsymbol{f} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_m} \end{pmatrix};$$

we assume that the right hand side is evaluated at $\boldsymbol{x} = (x_1, \dots, x_m)$.

If the Jacobean matrix defines a surjective linear map $\mathbb{R}^m \to \mathbb{R}^n$ (that is, if $\mathrm{rank}(\mathrm{Jac}_{\boldsymbol{x}}\, \boldsymbol{f}) = n$) then we say that $\boldsymbol{x}$ is a *regular point* of $\boldsymbol{f}$.

If for some $\boldsymbol{y} \in \mathbb{R}^n$ each point $\boldsymbol{x}$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$ is regular, then we say that $\boldsymbol{y}$ is a *regular value* of $\boldsymbol{f}$. The following lemma states that *most* values of a smooth map are regular.

2.1. Sard's lemma. *Given a smooth map $\boldsymbol{f}\colon \Omega \to \mathbb{R}^n$ defined on an open set $\Omega \subset \mathbb{R}^m$, almost all values in $\mathbb{R}^n$ are regular.*

The words *almost all* means all values, with the possible exceptions belong to a set with vanishing *Lebesgue measure*. In particular if one chooses a random value equidistributed in an arbitrarily small ball $B \subset \mathbb{R}^n$, then it is a regular value of $\boldsymbol{f}$ with probability 1.

Note that if $m < n$, then $\boldsymbol{f}$ has no regular points. Therefore the only regular value of $\boldsymbol{f}$ are the points in the complement of the image $\operatorname{Im} \boldsymbol{f}$. The theorem states that in this case almost all points in $\mathbb{R}^n$, do *not* belong to $\operatorname{Im} \boldsymbol{f}$.

Inverse function theorem

The *inverse function theorem* gives a sufficient condition for a smooth map $\boldsymbol{f}$ to be invertible in a neighborhood of a given point $\boldsymbol{x}$. The condition is formulated in terms of the Jacobian matrix of $\boldsymbol{f}$ at $\boldsymbol{x}$.

The *implicit function theorem* is a close relative to the inverse function theorem; in fact it can be obtained as its corollary. It is used when we need to pass from parametric to implicit description of curves and surfaces.

Both theorems reduce the existence of a map satisfying certain equation to a question in linear algebra. We use these two theorems only for $n \leqslant 3$.

2.2. Inverse function theorem. *Let $\boldsymbol{f} = (f_1, \dots, f_n)\colon \Omega \to \mathbb{R}^n$ be a smooth map defined on an open set $\Omega \subset \mathbb{R}^n$. Assume that the Jacobian matrix* $\operatorname{Jac}_{\boldsymbol{x}} \boldsymbol{f}$ *is invertible at some point $\boldsymbol{x} \in \Omega$. Then there is a smooth map $\boldsymbol{h}\colon \Theta \to \mathbb{R}^n$ defined in an open neighborhood Θ of $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ that is a* local inverse of $\boldsymbol{f}$ at $\boldsymbol{x}$*; that is, there is a neighborhood $\Omega' \ni \boldsymbol{x}$ such that $\boldsymbol{f}$ defines a bijection $\Omega' \leftrightarrow \Theta$ and $\boldsymbol{h} \circ \boldsymbol{f}$ is an identity map on Ω'.*

More over if an Ω contains an ε-neighborhood of $\boldsymbol{x}$, and the first and second partial derivatives $\frac{\partial f_i}{\partial x_j}$, $\frac{\partial^2 f_i}{\partial x_j \partial x_k}$ are bounded by a constant C for all i, j, and k, then we can assume that Θ is a δ-neighborhood of $\boldsymbol{y}$, for some $\delta > 0$ that depends only on ε and C.

2.3. Implicit function theorem. *Let $\boldsymbol{f} = (f_1, \dots, f_n)\colon \Omega \to \mathbb{R}^n$ be a smooth map, defined on a open subset $\Omega \subset \mathbb{R}^{n+m}$, where $m, n \geqslant 1$. Let us consider $\mathbb{R}^{n+m}$ as a product space $\mathbb{R}^n \times \mathbb{R}^m$ with coodinates $x_1, \dots, x_n, y_1, \dots, y_m$. Consider the following matrix*

$$M = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{pmatrix}$$

formed by the first n columns of the Jacobian matrix. Assume M is invertible at some point $\boldsymbol{x} = (x_1, \dots, x_n, y_1, \dots y_m)$ in the domain of definition of $\boldsymbol{f}$ and $\boldsymbol{f}(\boldsymbol{x}) = 0$. Then there is a neighborhood $\Omega' \ni \boldsymbol{x}$ and a smooth function $\boldsymbol{h}\colon \mathbb{R}^m \to \mathbb{R}^n$ defined in a neighborhood $\Theta \ni 0$ such that for any $(x_1, \dots, x_n, y_1, \dots y_m) \in \Omega$, the equality

$$\boldsymbol{f}(x_1, \dots, x_n, y_1, \dots y_m) = 0$$

holds if and only if

$$(x_1, \dots x_n) = \boldsymbol{h}(y_1, \dots y_m).$$

Multiple integral

Set

$$\operatorname{jac}_{\boldsymbol{x}} \boldsymbol{f} := |\det[\operatorname{Jac}_{\boldsymbol{x}} \boldsymbol{f}]|;$$

that is, $\operatorname{jac}_{\boldsymbol{x}} \boldsymbol{f}$ is the absolute value of the determinant of the Jacobian matrix of $\boldsymbol{f}$ at $\boldsymbol{x}$.

The following theorem plays the role of a substitution rule for multiple variables.

2.4. Theorem. *Let $h\colon K \to \mathbb{R}$ be a bounded measurable function on a measurable subset $K \subset \mathbb{R}^n$. Assume $\boldsymbol{f}\colon K \to \mathbb{R}^n$ is an injective smooth map. Then*

$$\int\limits_{\boldsymbol{x} \in K} h(\boldsymbol{x}) \cdot \operatorname{jac}_{\boldsymbol{x}} \boldsymbol{f} = \int\limits_{\boldsymbol{y} \in \boldsymbol{f}(K)} h \circ \boldsymbol{f}^{-1}(\boldsymbol{y}).$$

Chapter 3

Differential equations

Initial value problem

The following theorem guarantees existence and uniqueness of solutions of an initial value problem for a system of ordinary differential equations

$$\begin{cases} x_1'(t) &= f_1(x_1, \dots, x_n, t), \\ & \vdots \\ x_n'(t) &= f_n(x_1, \dots, x_n, t), \end{cases}$$

where each $x_i = x_i(t)$ is a real valued function defined on a real interval $\mathbb{I}$ and each f_i is a smooth function defined on an open subset $\Omega \subset \mathbb{R}^n \times \mathbb{R}$.

The array of functions $(f_1, \dots, f_n)$ can be considered as one vector-valued function $\boldsymbol{f}\colon \Omega \to \mathbb{R}^n$ and the array $(x_1, \dots, x_n)$ can be considered as a vector $\boldsymbol{x} \in \mathbb{R}^n$. Therefore the system can be rewritten as one vector equation

$$\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}, t).$$

3.1. Theorem. *Suppose $\mathbb{I}$ is a real interval and $\boldsymbol{f}\colon \Omega \to \mathbb{R}^n$ is a smooth function defined on an open subset $\Omega \subset \mathbb{R}^n \times \mathbb{R}$. Then for any initial data $\boldsymbol{x}(t_0) = \boldsymbol{u}$ such that $(\boldsymbol{u}, t) \in \Omega$ the differential equation*

$$\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}, t)$$

has a unique solution $\boldsymbol{x}(t)$ defined at a maximal interval $\mathbb{J}$ that contains t_0. Moreover

(a) if $\mathbb{J} \neq \mathbb{R}$ (that is, if an end a of $\mathbb{J}$ is finite) then $\boldsymbol{x}(t)$ does not have a limit point in Ω as $t \to a$;

(b) the function $(\boldsymbol{u}, t_0, t) \mapsto \boldsymbol{x}(t)$ *has open domain of definition in* $\Omega \times \mathbb{R}$ *and it is smooth in this domain.*

Chapter 4

Real analysis

Lipschitz condition

Recall that a function f between metric spaces is called Lipschitz if there is a constant L such that

$$|f(x) - f(y)| \leqslant L{\cdot}|x - y|$$

for all values x and y in the domain of definition of f.

The following theorem makes possible to extend number of results about smooth function to Lipschitz functions. Recall that *almost all* means all values, with the possible exceptions belong to a set with vanishing *Lebesgue measure.*

4.1. Rademacher's theorem. *Let $f\colon [a,b] \to \mathbb{R}$ be a Lipschitz function. Then the derivative f' of f is a bounded measurable function defined almost everywhere in $[a,b]$ and it satisfies the fundamental theorem of calculus; that is, the following identity*

$$f(b) - f(a) = \int\limits_a^b f'(x){\cdot}dx,$$

holds if the integral is understood in the sense of Lebesgue.

The following theorem makes possible to extend many statements about continuous function to measurable functions.

4.2. Lusin's theorem. *Let $\varphi\colon [a,b] \to \mathbb{R}$ be a measurable function. Then for any $\varepsilon > 0$, there is a continuous function $\psi_\varepsilon\colon [a,b] \to \mathbb{R}$ that coincides with φ outside of a set of measure at most ε. Moreover, if φ is bounded above and/or below by some constants, then we may assume that so is ψ_ε.*

Convex functions

A function of two variables $(x, y) \mapsto f(x, y)$ is called convex if its epigraph $z \geqslant f(x, y)$ is a convex set. This is equivalent to the so-called *Jensen's inequality*

$$f\,(t{\cdot}x_1 + (1-t){\cdot}x_2) \le t{\cdot}f(x_1) + (1-t){\cdot}f(x_2)$$

for $t \in [0,1]$. If f is smooth, then the condition is equivalent to the following inequality for the second directional derivative:

$$D^2_w f \geqslant 0$$

for any vector $w \neq 0$ in the (x, y)-plane.

Uniform continuity and convergence

Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map between metric spaces. If for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x_1 - x_2|_{\mathcal{X}} < \delta \quad\Longrightarrow\quad |f(x_1) - f(x_2)|_{\mathcal{Y}} < \varepsilon,$$

then f is called *uniformly continuous.*

Evidently every uniformly continuous function is continuous; the converse does not hold. For example, the function $f(x) = x^2$ is continuous, but not uniformly continuous.

However if f is continuous and defined on a closed interval $[a, b]$, then f is uniformly continuous.

If the condition above holds for any function f_n in a sequence and δ depend solely on ε, then the sequence (f_n) is called uniformly equicontinuous. More precisely, a sequence of functions $f_n : \mathcal{X} \to \mathcal{Y}$ is called *uniformly equicontinuous* if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x_1 - x_2|_{\mathcal{X}} < \delta \quad\Longrightarrow\quad |f_n(x_1) - f_n(x_2)|_{\mathcal{Y}} < \varepsilon$$

for any n.

We say that a sequence of functions $f_i : \mathcal{X} \to \mathcal{Y}$ *converges uniformly* to a function $f_\infty : \mathcal{X} \to \mathcal{Y}$ if for any $\varepsilon > 0$, there is a natural number N such that for all $n \geq N$, we have $|f_\infty(x) - f_n(x)| < \varepsilon$ for all $x \in \mathcal{X}$.

4.3. Arzelá-Ascoli Theorem. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are compact metric spaces. Then any uniformly equicontinuous sequence of function $f_n\colon \mathcal{X} \to \mathcal{Y}$ has a subsequence that converges uniformly to a continuous function $f_\infty\colon \mathcal{X} \to \mathcal{Y}$.*

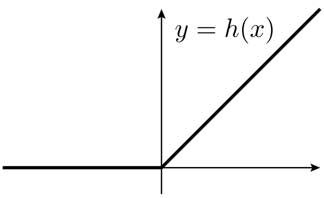
Cutoffs and mollifiers

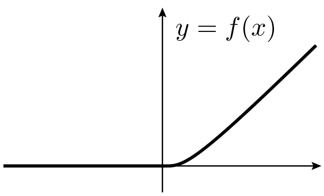
We use few examples of smooth functions that mimic behavior of some model functions.

For example, consider the function $h\colon t\mapsto \max\{\,0,t\,\}$ and the following function

$$f(t)=\begin{cases}0 & \text{if } t\leqslant 0,\\ \frac{t}{e^{1/t}} & \text{if } t>0.\end{cases}$$

Note that h and f behave alike — both vanish at $t\leqslant 0$ and grows to infinity for positive t. The function h is not smooth — its derivative at 0 is undefined. Unlike h, the function f is smooth. Indeed, the existence of all derivatives $f^{(n)}(x)$ at $x\ne 0$ is evident and direct calculations show that $f^{(n)}(0)=0$ for all n.





Other useful examples of that type are the so called *bell function* — a smooth function that is positive in an ε-neighborhood of zero and vanishing outside this neighborhood. An example of such function can be constructed based using the function f constructed above, say

$$b_\varepsilon(t)=c{\cdot}f(\varepsilon^2-t^2);$$

the constant c is chosen so that $\int b_\varepsilon=1$.

Another useful example is a sigmoid — nondecreasing function that vanish for $t\leqslant -\varepsilon$ and takes value 1 for any $t\geqslant\varepsilon$. For example the following function

$$\sigma_\varepsilon(t)=\int\limits_{-\infty}^{t} b_\varepsilon(x){\cdot}dx.$$

Chapter 5

Topology

The following material is covered in any reasonable introductory text to topology; one of our favorites is a textbook of Czes Kosniowski [29].

Continuous inverse

We sometimes use the following characterization of homeomorphisms between compact spaces.

5.1. Theorem. *A continuous bijection f between compact metric spaces has a continuous inverse; that is, f is a homeomorphism.*

Jordan's theorem

The first part of the following theorem was proved by Camille Jordan, the second part is due to Arthur Schoenflies.

5.2. Theorem. *The complement of any closed simple plane curve γ has exactly two connected components.*

Moreover, there is a homeomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps the unit circle to γ. In particular γ bounds a topological disc.

This theorem is known for its simple formulation and quite hard proof. By now many proofs of this theorem are known. For the first statement, a very short proof based on a somewhat developed technique is given by Patrick Doyle [17], among elementary proofs, one of my favorites is the proof given by Aleksei Filippov [19].

We use the following smooth analog of this theorem.

5.3. Theorem. *The complement of any closed simple smooth regular plane curve γ has exactly two connected components.*

Moreover the there is a diffeomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps the unit circle to γ.

The proof of this statement is much simpler. An amusing proof can be found in [13].

Connectedness

Recall that a continuous map α from the unit interval $[0,1]$ to a Euclidean space is called a *path*. If $p = \alpha(0)$ and $q = \alpha(1)$, then we say that α connects p to q.

A set X in the Euclidean space is called *path connected* if any two points $x, y \in X$ can be connected by a path lying in X.

A set X in the Euclidean space is called *connected* if one cannot cover X with two disjoint open sets V and W such that both intersections $X \cap V$ and $X \cap W$ are nonempty.

5.4. Proposition. *Any path connected set is connected. Moreover, any open connected set in the Euclidean space or plane is path connected.*

Given a point $x \in X$, the maximal connected subset of X containing x is called the *connected component* of x in X.

Chapter 6

Elementary geometry

Polygons

6.1. Theorem. *The sum of all the internal angles of a simple n-gon is $(n-2)\cdot\pi$.*

While this theorem is well known, it is hard to find a reference with a proof without cheating. So we present a proof here.

Proof. The proof is by induction on n. For $n = 3$ it says that sum of internal angles of a triangle is π, which is assumed to be known.

First let us show that for any $n \geqslant 4$, any n-gon has a diagonal that lies inside of it. Assume this holds true for all polygons with at most $n-1$ vertices.

Fix an n-gon P, $n \geqslant 4$. Applying a rotation if necessary, we can assume that all its vertexes have different x-coordinates. Let v be a vertex of P that minimizes the x-coordinate; denote by u and w its adjacent vertexes. Let us choose the diagonal uw if it lies in P. Otherwise the triangle $\triangle uvw$ contains another vertex of P. Choose a vertex s in the interior of $\triangle uvw$ that maximizes the distance to line uw. Note that the diagonal vs lies in P; if it is not the case, then vs crosses another side pq of P, one of the vertices p or q has larger distance to the line and it lies in the interior of $\triangle uvw$ — a contradiction.

Note that the diagonal divides P into two polygons, say Q and R, with smaller number of sides in each, say k and m respectively. Note that

❶ $$k + m = n + 2;$$

indeed each side of P appears once as a side of P or Q plus the diagonal appears twice — once as a side in Q and once as a side of R.

Observe that Q and R, which by the induction hypothesis are $(k-{}$ $-2)\cdot\pi$ and $(m-2)\cdot\pi$ respectively, and the sum of angles of P equals to the sum of angles in Q and Q. Finally by ❶ we have that

$$(k-2)\cdot\pi+(m-2)\cdot\pi=(n-2)\cdot\pi$$

which proves the theorem. □

Triangle inequality for angles

The following theorem says that the triangle inequality holds for angles between half-lines from a fixed point. In particular it implies that a sphere with the angle metric is a metric space.

6.2. Theorem. *The inequality*

$$\measuredangle aob+\measuredangle boc\geqslant\measuredangle aoc$$

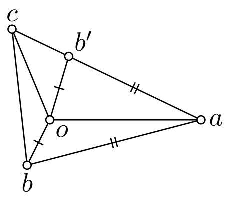
holds for any three half-lines oa, ob and oc in the Euclidean space.

We will use the following lemma in the proof; it says that the angle of a triangle monotonically depends on the opposite side, assuming the we keep the other two sides fixed. It follows directly from the cosine rule.

6.3. Lemma. *Let x, y, z, x', y' and z' be 6 points such that $|x-y|={}$ $=|x'-y'|>0$ and $|y-z|=|y'-z'|>0$. Then*

$$\measuredangle xyz\geqslant\measuredangle x'y'z'\quad\text{if and only if}\quad|x-z|\geqslant|x'-z'|.$$

Proof of 6.2. We can assume that $\measuredangle aob<\measuredangle aoc$; otherwise the statement is evident. In this case there is a half-line ob' in the angle aoc such that



$$\measuredangle aob=\measuredangle aob',$$

so in particular we have that

$$\measuredangle aob'+\measuredangle b'oc=\measuredangle aoc.$$

Without loss of generality we can assume that $|o-b|=|o-b'|$ and b' lies on a line segment ac, so

$$|a-b'|+|b'-c|=|a-c|.$$

Then by the triangle inequality

$$\text{❷} \qquad \begin{aligned} |a-b|+|b-c| &\geqslant |a-c| = \\ &= |a-b'|+|b'-c|. \end{aligned}$$

Note that in the triangles aob and aob' the side ao is shared, so $\measuredangle aob = \measuredangle aob'$ and $|o-b| = |o-b'|$. By side-angle-side congruence condition, we have that $\triangle aob \cong \triangle aob'$; in particular $|a-b'| = |a-b|$. Therefore from ❷ we get

$$|b-c| \geqslant |b'-c|.$$

Applying the angle monotonicity (6.3) we obtain

$$\measuredangle boc \geqslant \measuredangle b'oc.$$

Whence

$$\begin{aligned} \measuredangle aob + \measuredangle boc &\geqslant \measuredangle aob' + \measuredangle b'oc = \\ &= \measuredangle aoc. \end{aligned}$$

□

Convex sets

A set X in the Euclidean space is called *convex* if for any two points $x, y \in X$, any point z between x and y lies in X. It is called *strictly convex* if for any two points $x, y \in X$, any point z between x and y lies in the interior of X.

From the definition, it is easy to see that the intersection of an arbitrary family of convex sets is convex. The intersection of all convex sets containing X is called the *convex hull* of X; it is the minimal convex set containing the set X.

We will use the following corollary of the so called *hyperplane separation theorem*:

6.4. Lemma. *Let $K \subset \mathbb{R}^3$ be a closed convex set. Then for any point $p \notin K$ there is a plane Π that separates K from p; that is, K and p lie on opposite open half-spaces separated by Π.*

Chapter 7

Area

Area of spherical triangle

7.1. Lemma. *Let Δ be a spherical triangle; that is, Δ is the intersection of three closed half-spheres in the unit sphere $\mathbb{S}^2$. Then*

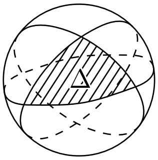
❶ $$\operatorname{area}\Delta = \alpha + \beta + \gamma - \pi,$$

where α, β and γ are the angles of Δ.

The value $\alpha + \beta + \gamma - \pi$ is called *excess* of the triangle Δ.

Proof. Recall that

❷ $$\operatorname{area}\mathbb{S}^2 = 4\cdot\pi.$$



Note that the area of a spherical slice S_α between two meridians meeting at angle α is proportional to α. Since for S_π is a half-sphere, from ❷, we get $\operatorname{area} S_\pi = 2\cdot\pi$. Therefore the coefficient is 2; that is,

❸ $$\operatorname{area} S_\alpha = 2\cdot\alpha.$$

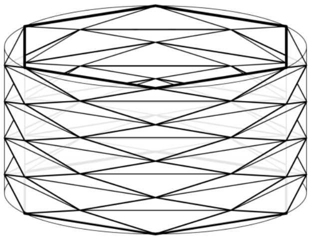
Extending the sides of Δ we get 6 slices: two S_α, two S_β and two S_γ which cover most of the sphere once, but the triangle Δ and its centrally symmetric copy Δ' are covered 3 times. It follows that

$$2\cdot \operatorname{area} S_\alpha + 2\cdot \operatorname{area} S_\beta + 2\cdot \operatorname{area} S_\gamma = \operatorname{area}\mathbb{S}^2 + 4\cdot \operatorname{area}\Delta.$$

Substituting ❷ and ❸ and simplifying, we get ❶. □

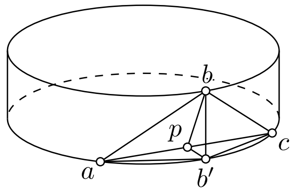
Schwarz's boot

Recall that we defined length of curve as the exact upper bound on the length of inscribed polygonal lines. It suggests to define area of a surface as a exact upper bound on the area of polyhedrons inscribed in the surface.



However as you will see from the following example, this idea fails badly even for cylindrical surface. Namely, we will show that if we define the area as the least upper bound of areas of inscribed polyhedral surfaces, then the area of the lateral surface of the cylinder must be infinite. The latter contradicts correct intuition that the area of this surface should be the product of the circumference of the base circle and the height of the cylinder.

Let us divide the cylinder into m equal cylinders by planes parallel to its base. This way we obtain $m+1$ circles on the lateral surface of a cylinder, including both bases. Further, let us divide each of these circles into n equal arcs in such a way that the dividing points of a circle will lie exactly above the midpoints of arcs on the circle under it. Consider all triangles formed by a chord of such arc and line segments connecting the ends of the chord with the points right above and right below the mid point of its arc. All the $2{\cdot}m{\cdot}n$ equal triangles form a polyhedral surface which is called Schwarz's boot. A Schwarz's boot for $m=8$ and $n=6$ is shown on the diagram.



Consider one of the triangle abc which form the Schwarz's boot. By construction, its base ac lies in a horizontal plane and the projection b' of b on this plane bisects the arc ac. Therefore the vertexes of the triangle $ab'c$ are the three consequent vertexes of a regular $2{\cdot}n$-gone inscribed in a unit circle. Denote the triangle $ab'c$ by s_n; clearly it depends only on n and the radius of the base circle.

Both triangles abc and $ab'c$ are isosceles with shares base ac. Note that the altitude bp is larger than the altitude $b'p$. Therefore

$$\begin{aligned}\operatorname{area}(\triangle abc) &= \tfrac{1}{2}{\cdot}|a-c|{\cdot}|b-p| > \\ &> \tfrac{1}{2}{\cdot}|a-c|{\cdot}|b'-p| = \\ &= \operatorname{area}(\triangle ab'c) = \\ &= s_n.\end{aligned}$$

In particular,

$$S_{m,n} > 2{\cdot}m{\cdot}n{\cdot}s_n,$$

where $S_{m,n}$ denotes the total area of the Schwarz's boot. Consider the pairs (m,n) such that m is much larger than n; namely $m > \frac{1}{s_n}$. Then

$$S_{m,n} > 2{\cdot}m{\cdot}n{\cdot}s_n > 2{\cdot}\tfrac{1}{s_n}{\cdot}n{\cdot}s_n = 2{\cdot}n.$$

Therefore $S_{m,n} \to \infty$ as $n \to \infty$.

Part II

Curves

Chapter 8

Definitions

Simple curves

In the following definition we use the notion of *metric space* which is discussed in Appendix 1. The Euclidean plane and space are the main examples of metric spaces that one should keep in mind.

Recall that a bijective continuous map $f\colon X \to Y$ between subsets of some metric spaces is called a *homeomorphism* if its inverse $f^{-1}\colon Y \to X$ is continuous.

8.1. Definition. *A connected subset γ in a metric space is called a* simple curve *if it is* locally *homeomorphic to a real interval.*

It turns out that any simple curve γ can be *parametrized* by a real interval or a circle. That is, there is a homeomorphism $G \to \gamma$ where G is a real interval (open, closed or semi-open) or the circle

$$\mathbb{S}^1 = \left\{\, (x, y) \in \mathbb{R}^2 \;:\; x^2 + y^2 = 1 \,\right\}.$$

We omit a proof of this statement; it is not hard, but would take us away from the subject. We hope however that this statement is intuitively obvious.

If G is an open interval or a circle we say that γ is a *curve without endpoints*, otherwise it is called a *curve with endpoints*. In the case when G is a circle we say that the curve is *closed*. When G is a closed interval, γ is called an *arc*.

A parametrization describes a curve completely. We will denote a curve and its parametrization by the same letter; for example, we may say a plane curve γ is given with a parametrization $\gamma\colon (a, b) \to \to \mathbb{R}^2$. Note, however, that any simple curve admits many different parametrization.

8.2. Exercise. *Find a continuous injective map $\gamma\colon(0,1)\to\mathbb{R}^2$ such that its image is not a simple curve.*

Parametrized curves

A *parameterized curve* is defined as a continuous map γ from a circle or a real interval (open, closed or semi-open) to a metric space. For a parameterized curve we do not assume that γ is injective; in other words a parameterized curve might have *self-intersections.*

If we say *curve* it means we do not want to specify whether it is a parameterized curve or a simple curve.

If the domain of a parameterized curve is the closed unit interval $[0,1]$, then it is also called a *path.* If in addition $p=\gamma(0)=\gamma(1)$, then γ is called a loop; in this case the point p is called the *base* of the loop.

8.3. Advanced exercise. *Let X be a subset of the plane. Suppose that two distinct points $p,q\in X$ can be connected by a path in X. Show that there is a simple arc in X connecting p to q.*

Smooth curves

Curves in Euclidean space or plane are called *space* or *plane curves*, respectively.

A parameterized space curve can be described by its coordinate functions

$$\gamma(t)=(x(t),y(t),z(t)).$$

Plane curves can be considered as a partial case of space curves with $z(t)\equiv 0$.

Recall that a real-to-real function is called *smooth* if its derivatives of all orders are defined everywhere in the domain of definition. If each of the coordinate functions $x(t),y(t)$ and $z(t)$ of a space curve γ are smooth, then the parametrized curve is called *smooth.*

If the *velocity vector*

$$\gamma'(t)=(x'(t),y'(t),z'(t))$$

does not vanish at any point, then the parameterized curve γ is called *regular.*

A simple space curve is called *smooth* (resp. *regular*) if it admits a smooth (resp. regular) parametrization. Regular smooth curves are among the main objects in differential geometry; colloquially, the term *smooth curve* is often used as a shortcut for *smooth regular curve.*

8.4. Exercise. *Let*

$$f(t)=\begin{cases}0 & \textit{if } t\leqslant 0,\\ \frac{t}{e^{1/t}} & \textit{if } t>0.\end{cases}$$

Show that $\alpha(t)=(f(t),f(-t))$ gives a smooth parametrization of a simple curve formed by the union of two half-axis in the plane.

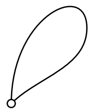
Show that any smooth parametrization of this curve has vanishing velocity vector at the origin. Conclude that this curve is not regular and smooth; that is, it does not admit a regular smooth parametrization.

8.5. Exercise. *Describe the set of real numbers ℓ such that the plane curve $\gamma_\ell(t)=(t+\ell\cdot\sin t,\ell\cdot\cos t)$, $t\in\mathbb{R}$ is*

(a) regular;
(b) simple.

Periodic parametrization

A natural way to describe a closed simple curve is as a *periodic* parameterized curve $\gamma\colon\mathbb{R}\to\mathcal{X}$; that is, a curve such that $\gamma(t+\ell)=\gamma(t)$ for a fixed period $\ell>0$ and all t. For example, the unit circle in the plane can be described by the $2\cdot\pi$-periodic parametrization $\gamma(t)=(\cos t,\sin t)$.



Any smooth regular closed curve can be described by a smooth regular loop. But in general the closed curve described by a smooth regular loop might fail to be regular at its base; an example is shown on the diagram.

Implicitly defined curves

Suppose $f\colon\mathbb{R}^2\to\mathbb{R}$ is a smooth function; that is, all its partial derivatives are defined in its domain of definition. Let $\gamma\subset\mathbb{R}^2$ be the set of solutions of the equation $f(x,y)=0$.

Assume γ is connected. According to the implicit function theorem (2.3), the set γ is a smooth regular simple curve if 0 is a *regular value* of f. This condition is equivalent to the gradient ∇f not vanishing at any point $p\in\gamma$. In other words, if $f(p)=0$, then $\frac{\partial f}{\partial x}(p)\neq 0$ or $\frac{\partial f}{\partial y}(p)\neq 0$.

Similarly, assume f,h is a pair of smooth functions defined in $\mathbb{R}^3$. The system of equations

$$\begin{cases}f(x,y,z)=0,\\ h(x,y,z)=0.\end{cases}$$

defines a regular smooth space curve if the set of solutions is connected and 0 is a regular value of the map $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined as

$$F\colon (x,y,z) \mapsto (f(x,y,z), h(x,y,z)).$$

In this case it means that the gradients ∇f and ∇h are linearly independent at any point $p \in \gamma$. In other words, the Jacobian matrix

$$\operatorname{Jac}_p F = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix}$$

for the map $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ has rank 2 at any point p such that $f(p) = h(p) = 0$.

The way described above to define a curve is called *implicit*; if a curve is defined by its parametrization, we say that it is *explicitly defined*. While implicit function theorem guarantees the existence of regular smooth parametrizations, when it comes to calculations, it is usualy easier to work directly with implicit representations.

8.6. Exercise. *Consider the set in the plane described by the equation*

$$y^2 = x^3.$$

Is it a simple curve? Is it a smooth regular curve?

8.7. Exercise. *Describe the set of real numbers ℓ such that the system of equations*

$$\begin{cases} x^2 + y^2 + z^2 & = 1 \\ x^2 + \ell\cdot x + y^2 & = 0 \end{cases}$$

describes a smooth regular curve.

Proper curves

A parametrized curve γ in a metric space $\mathcal{X}$ is called *proper* if for any compact set $K \subset X$, the inverse image $\gamma^{-1}(K)$ is compact.

For example, the curve $\gamma(t) = (e^t, 0, 0)$ defined on whole real line is not proper. Indeed, the half-line $(-\infty, 0]$ is not compact and it is the inverse image of unit closed ball around the origin.

Note that closed curves ans arcs are automatically proper since the parameter set is compact.

A simple curve is called proper if it admits a proper parametrization. It turns out that a simple curve is proper if and only if its image is a closed set. In particular, any implicitly defined plane or space curve is proper. We omit the proof of this statement, but it is not hard.

8.8. Exercise. *Use the Jordan's theorem (5.2) to show that any proper plane curve divides the plane in two connected components.*

Chapter 9

Length

Recall that a sequence

$$a = t_0 < t_1 < \dots < t_k = b.$$

is called a *partition* of the interval $[a, b]$.

9.1. Definition. *Let $\gamma\colon [a, b] \to \mathcal{X}$ be a curve in a metric space. The* length *of γ is defined as*

$$\begin{aligned}\operatorname{length}\gamma = \sup\{|\gamma(t_0) - \gamma(t_1)| + |\gamma(t_1) - \gamma(t_2)| + \dots \\ \dots + |\gamma(t_{k-1}) - \gamma(t_k)|\},\end{aligned}$$

where the supremum is taken over all partitions

$$a = t_0 < t_1 < \dots < t_k = b.$$

The length of γ is a nonnegative real number or infinity; the curve γ is called rectifiable *if its length is finite.*

The length of a closed curve is defined as the length of the corresponding loop. If a curve is defined on an open or semi-open interval, then its length is defined as the exact upper bound for lengths of all its arcs.

Suppose $\gamma\colon [a, b] \to \mathbb{R}^3$ is a parameterized space curve. For a partition $a = t_0 < t_1 < \dots < t_k = b$, set $p_i = \gamma(t_i)$. Then the polygonal line $p_0 \dots p_k$ is called *inscribed* in γ. If γ is closed, then $p_0 = p_k$, so the inscribed polygonal line is also closed.

9.2. Exercise. *Suppose $\gamma\colon [a, b] \to \mathbb{R}^3$ is a curve and the function $\varphi : [c, d] \to [a, b]$ is an monotonic continuous and $\varphi(c) = a$, $\varphi(d) = b$. Then the curve $\gamma \circ \varphi$ is calle a* reparametrization *of γ. Show that*

$$\operatorname{length}(\gamma \circ \varphi) = \operatorname{length}\gamma.$$

Note that the length of space curve γ can be defined as the exact upper bound of the lengths of polygonal lines $p_0 \dots p_k$ *inscribed* in γ.

9.3. Exercise. *Let $\alpha\colon [0,1] \to \mathbb{R}^3$ be a parametrization of a simple arc. Suppose a path $\beta\colon [0,1] \to \mathbb{R}^3$ has the same image as α; that is, $\beta([0,1]) = \alpha([0,1])$. Show that*

$$\operatorname{length}\beta \geqslant \operatorname{length}\alpha.$$

9.4. Exercise. *Assume $\gamma\colon [a,b] \to \mathbb{R}^3$ is a smooth curve. Show that*

(a) $\operatorname{length}\gamma \geqslant \int_a^b |\gamma'(t)|{\cdot}dt,$

(b) $\operatorname{length}\gamma \leqslant \int_a^b |\gamma'(t)|{\cdot}dt.$

Conclude that

$$\text{❶} \qquad \operatorname{length}\gamma = \int\limits_a^b |\gamma'(t)|{\cdot}dt.$$

9.5. Advanced exercises.

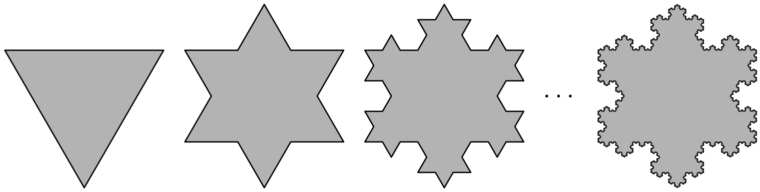
(a) Show that the formula ❶ *holds for any Lipschitz curve $\gamma\colon [a,b] \to \mathbb{R}^3$.*

(b) Construct a simple curve $\gamma\colon [a,b] \to \mathbb{R}^3$ such that $\gamma'(t) = 0$ almost everywhere. (In particular, the formula ❶ *does not hold for γ.).*

Nonrectifiable curves

A classical example of a nonrectifiable curve is the so-called *Koch snowflake*; it is a fractal curve that can be constructed as follows:

Start with an equilateral triangle, divide each side into three segments of equal length and add an equilateral triangle with base the middle segment. Repeat this construction recursively with the obtained polygons. Repeat this construction recursively to the obtained



polygons. The Koch snowflake is the boundary of the union of all the polygons. Three iterations and the resulting Koch snowflake are shown on the diagram.

9.6. Exercise.

(a) Show that the Koch snowflake is a closed simple curve; that is, it can be parameterized by a circle.

(b) Show that the Koch snowflake is not rectifiable.

Arc-length parametrization

We say that a curve γ has an *arc-length parametrization* (also called *natural parametrization*) if

$$t_2 - t_1 = \operatorname{length} \gamma|_{[t_1,t_2]}$$

for any two values of parameters $t_1 < t_2$; that is, the arc of γ from t_1 to t_2 has length $t_2 - t_1$.

Note that a smooth space curve $\gamma(t) = (x(t), y(t), z(t))$ is an arc-length parametrization if and only if it has unit velocity vector at all times; that is,

$$|\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = 1$$

for all t; by that reason smooth curves equipped with an arc-length parametrization are also called *unit-speed curves.* Note that smooth unit-speed curves are automatically regular.

Note that any rectifiable curve γ can be parameterized by its arc length. Indeed fix a value t_0 in the interval of parameters of γ and set

$$s(t) = \begin{cases} \operatorname{length} \gamma|_{[t_0,t]} & \text{if } t \geqslant t_0, \\ \operatorname{length} \gamma|_{[t,t_0]} & \text{if } t \leqslant t_0, \end{cases}$$

9.7. Proposition. *If $t \mapsto \gamma(t)$ is a smooth regular curve, then its arc-length parameterizaton is also smooth and regular. Moreover, the arc-length parameter s of γ can be written as an integral*

$$s(t) = \int_{t_0}^{t} |\gamma'(\tau)| \cdot d\tau. \qquad ❷$$

Proof. The function $t \mapsto s(t)$ defined by ❷ is a smooth increasing function. Further by fundamental theorem of calculus, $s'(t) = |\gamma'(t)|$. Therefore if γ is regular, then $s'(t) \neq 0$ for any parameter value t.

By inverse function theorem (2.2) the inverse function $s^{-1}(t)$ is also smooth and $|(\gamma\circ s^{-1})'|\equiv 1$. Therefore $\gamma\circ s^{-1}$ is a unit-speed reparametrization of γ. By construction $\gamma\circ s^{-1}$ is smooth and since $|(\gamma\circ s^{-1})'|\equiv 1$ it is regular. □

Most of the time we use s for an arc-length parameter of a curve.

9.8. Exercise. *Reparametrize the helix*

$$\gamma_{a,b}(t) = (a{\cdot}\cos t, a{\cdot}\sin t, b{\cdot}t)$$

by its arc-length.

We will be interested in the properties of curves that are invariant under a reparametrization. Therefore we can always assume that any given smooth regular curve comes with an arc-length parametrization. A nice property of arc-length parametrizations is that they are almost canonical — these parametrizations differ only by a sign and an additive constant. By that reason, it is easier to express parametrization-independent quatities using arc-length parametrizations. This observation will be used in the definition of curvature and torsion.

On the other hand, it is often impossible to find an arc-length parametrization in explicit form, which makes it hard to perform calculations; usually it is more convenient to use the original parametrization.

Convex curves

A simple plane curve is called *convex* if it bounds a convex region.

9.9. Proposition. *Assume a convex closed curve α lies inside the domain bounded by a simple closed plane curve β. Then*

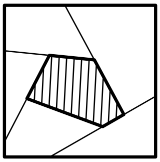
$$\operatorname{length}\alpha \leqslant \operatorname{length}\beta.$$

Let us denote by $\operatorname{perim} P$ the perimeter of a polygon P. Note that it is sufficient to show that for any polygon P inscribed in α there is a polygon Q inscribed in β such that $\operatorname{perim} P \leqslant \operatorname{perim} Q$.

Therefore it is sufficient to prove the following lemma.

9.10. Lemma. *Let P and Q be polygons. Assume P is convex and $Q\supset P$. Then*

$$\operatorname{perim} P \leqslant \operatorname{perim} Q.$$



Proof. Note that by the triangle inequality, the inequality

$$\operatorname{perim} P \leqslant \operatorname{perim} Q$$

holds if P can be obtained from Q by cutting it along a chord; that is, a line segment in Q that runs from boundary to boundary.

Note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \dots \subset P_n = Q$$

such that P_{i-1} obtained from P_i by cutting along a chord. Therefore

$$\begin{aligned} \operatorname{perim} P = \operatorname{perim} P_0 \leqslant \operatorname{perim} P_1 \leqslant \dots \\ \dots \leqslant \operatorname{perim} P_n = \operatorname{perim} Q \end{aligned}$$

and the lemma follows. □

9.11. Corollary. *Any convex closed plane curve is rectifiable.*

Proof. Any closed curve is bounded. Indeed, the curve can be described as an image of a loop $\alpha\colon [0,1] \to \mathbb{R}^2$, $\alpha(t) = (x(t), y(t))$. The coordinate functions $x(t)$ and $y(t)$ are continous functions defined on $[0,1]$. Therefore the absolute values of bothfunctions are bounded by some constant C. Therefore, α lies in the square defined by the inequalities $|x| \leqslant C$ and $|y| \leqslant C$.

By Proposition 9.9, the length of the curve cannot exceed the perimeter of the square — this is a finite number equal to $8\cdot C$. Whence the result. □

Recall that the convex hull of a set X is the smallest convex set that contains X; equivalently, the convex hull of X is the intersection of all convex sets containing X.

9.12. Exercise. *Let α be a simple closed plane curve. Denote by K the convex hull of α; let β be the boundary curve of K. Show that*

$$\operatorname{length} \alpha \geqslant \operatorname{length} \beta.$$

Try to show that the statement holds for arbitrary closed plane curves α, assuming only that K has nonempty interior.

Crofton formulas*

Consider a smooth plane curve $\gamma\colon [a,b] \to \mathbb{R}^2$. Given a unit vector u, denote by γ_u the curve that follows the orthogonal projection of γ to the line in the direction u; that is,

$$\gamma_u(t) = \langle u, \gamma(t)\rangle \cdot u.$$

Note that

$$|\gamma_u'(t)| = |\langle u, \gamma'(t)\rangle|$$

for any t. Note that for any plane vector w, the average magnitude of its projections is proportional to its magnitude; that is,

$$|w| = k\cdot\overline{|w_u|},$$

where $\overline{|w_u|}$ denotes the average value of $|w_u|$ for all unit vectors u. (The value k is the average value of $|\cos\varphi|$ for $\varphi\in[0,2\cdot\pi]$; it can be found by integration, but soon we will show another way to find it.)

If the curve γ is smooth, then according to Exercise 9.4

$$\begin{aligned}\operatorname{length}\gamma &= \int\limits_a^b |\gamma'(t)|\cdot dt = \\ &= \int\limits_a^b k\cdot\overline{|\gamma_u'(t)|}\cdot dt = \\ &= k\cdot\overline{\operatorname{length}\gamma_u}.\end{aligned}$$

This formula and its relatives are called *Crofton formulas.* Since k in universal for any curve, we can take γ to be the unit circle to compute k: the left hand side is $2\cdot\pi$. Note that for any unit vector u, the curve γ_u runs back and forth along an interval of length 2. Therefore $\operatorname{length}\gamma_u = 4$ and hence its average value is also 4. It follows that the coefficient k has to satisfy the equation $2\cdot\pi = k\cdot 4$; hence

$$\operatorname{length}\gamma = \tfrac{\pi}{2}\cdot\overline{\operatorname{length}\gamma_u}.$$

The Crofton's formula holds for arbitrary rectifiable curves, not necessary smooth; it can be proved using Exercise 9.5.

9.13. Exercise. *Show that any closed plane curve γ has length at least $\pi\cdot s$, where s is the average length of pojections of γ to lines. Moreover, equality holds if and only if γ is convex.*

Use this statement to give another solution to Exercise 9.12.

9.14. Advanced exercise. *Show that the length of a space curve is proportional to*

(a) the average length of its projections to all lines.

(b) the average length of its projections to all planes

Find the coefficients in each case.

Semicontinuity of length

Recall that the lower limit of a sequence of real numbers (x_n) is denoted by

$$\varliminf_{n\to\infty} x_n.$$

It is defined as the lowest partial limit; that is, the lowest possible limit of a subsequence of (x_n). The lower limit is defined for any sequence of real numbers and it lies in the exteded real line $[-\infty,\infty]$

9.15. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\gamma_n\colon[a,b]\to\mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty\colon[a,b]\to\mathcal{X}$; that is, for any fixed $t\in[a,b]$, $\gamma_n(t)\to\gamma_\infty(t)$ as $n\to\infty$. Then

❸
$$\varliminf_{n\to\infty}\operatorname{length}\gamma_n\geqslant\operatorname{length}\gamma_\infty.$$

Proof. Fix a partition $a=t_0<t_1<\dots<t_k=b$. Set

$$\begin{aligned}\Sigma_n&:=|\gamma_n(t_0)-\gamma_n(t_1)|+\dots+|\gamma_n(t_{k-1})-\gamma_n(t_k)|.\\ \Sigma_\infty&:=|\gamma_\infty(t_0)-\gamma_\infty(t_1)|+\dots+|\gamma_\infty(t_{k-1})-\gamma_\infty(t_k)|.\end{aligned}$$

Note that for each i we have

$$|\gamma_n(t_{i-1})-\gamma_n(t_i)|\to|\gamma_\infty(t_{i-1})-\gamma_\infty(t_i)|$$

and therefore

$$\Sigma_n\to\Sigma_\infty$$

as $n\to\infty$. Note that

$$\Sigma_n\leqslant\operatorname{length}\gamma_n$$

for each n. Hence

❹
$$\varliminf_{n\to\infty}\operatorname{length}\gamma_n\geqslant\Sigma_\infty.$$

If γ_∞ is rectifiable, we can assume that

$$\operatorname{length}\gamma_\infty<\Sigma_\infty+\varepsilon.$$

for any given $\varepsilon>0$. By ❹ it follows that

$$\varliminf_{n\to\infty}\operatorname{length}\gamma_n>\operatorname{length}\gamma_\infty-\varepsilon$$

for any $\varepsilon > 0$; whence ❸ follows.

It remains to consider the case when γ_∞ is not rectifiable; that is, $\operatorname{length}\gamma_\infty = \infty$. In this case we can choose a partition so that $\Sigma_\infty > L$ for any real number L. By ❹ it follows that

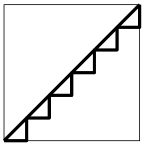
$$\varliminf_{n\to\infty} \operatorname{length}\gamma_n > L$$

for any given L; whence

$$\varliminf_{n\to\infty} \operatorname{length}\gamma_n = \infty$$

and ❸ follows. □

Note that the inequality ❸ might be strict. For example the diagonal γ_∞ of the unit square can be approximated by a stairs-like polygonal curves γ_n with sides parallel to the sides of the square (γ_6 is on the picture). In this case



$$\operatorname{length}\gamma_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length}\gamma_n = 2$$

for any n.

Length metric

Let $\mathcal{X}$ be a metric space. Given two points x, y in $\mathcal{X}$, denote by $d(x, y)$ the infimum of lengths of all paths connecting x to y; if there is no such path, then $d(x, y) = \infty$.

It is straightforward to see that the function d satisfies all the axioms of a metric except it might take infinite values. Therefore if any two points in $\mathcal{X}$ can be connected by a rectifiable curve, then d defines a new metric on $\mathcal{X}$; in this case d is called the *induced length metric*.

Evidently $d(x, y) \geqslant |x - y|$ for any pair of points $x, y \in \mathcal{X}$. If the equality holds for all pairs, then the metric $|{-}|$ is said to be a *length metric* and the space is called *length-metric space*.

Most of the time we consider length-metric spaces. In particular the Euclidean space is a length-metric space. A subspace A of a length-metric space $\mathcal{X}$ is not necessarily length-metric space; the induced length distance between points x and y in the subspace A will be denoted as $|x - y|_A$; that is, $|x - y|_A$ is the infimum of the lengths of paths in A from x to y.

9.16. Exercise. *Let $A\subset\mathbb{R}^3$ be a closed subset. Show that A is convex if and only if*

$$|x-y|_A=|x-y|_{\mathbb{R}^3}$$

for any $x,y\in A$

Spherical curves

Let us denote by $\mathbb{S}^2$ the unit sphere in the space; that is,

$$\mathbb{S}^2=\left\{\,(x,y,z)\in\mathbb{R}^3\;:\;x^2+y^2+z^2=1\,\right\}.$$

A space curve γ is called *spherical* if it runs in the unit sphere; that is, $|\gamma(t)|=1$, or equivalently, $\gamma(t)\in\mathbb{S}^2$ for any t.

Recall that $\measuredangle(u,v)$ denotes the angle between two vectors u and v.

9.17. Observation. *For any $u,v\in\mathbb{S}^2$, we have*

$$|u-v|_{\mathbb{S}^2}=\measuredangle(u,v)$$

Proof. The short arc γ of a great circle from u to v in $\mathbb{S}^2$ has length $\measuredangle(u,v)$. Therefore

$$|u-v|_{\mathbb{S}^2}\leqslant\measuredangle(u,v).$$

It remains to prove the opposite inequality. In other words, we need to show that given a polygonal line $\beta=p_0\dots p_n$ inscribed in γ there is a polygonal line $\beta_1=q_0\dots q_n$ inscribed in any given spherical path γ_1 connecting u to v such that

$$\text{length}\,\beta_1\geqslant\text{length}\,\beta. \tag{❺}$$

Define q_i as the first point on γ_1 such that $|u-p_i|=|u-q_i|$, but set $q_n=v$. Clearly β_1 is inscribed in γ_1 and according the triangle inequality for angles (6.2), we have that

$$\measuredangle(q_{i-1},q_i)\geqslant\measuredangle(p_{i-1},p_i).$$

Therefore

$$|q_{i-1}-q_i|\geqslant|p_{i-1}-p_i|$$

and ❺ follows. □

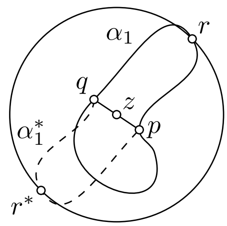
9.18. Hemisphere lemma. *Any closed spherical curve of length less than $2{\cdot}\pi$ lies in an open hemisphere.*

This lemma is a keystone in the proof of Fenchel's theorem (see 10.7). The lemma is not as simple as you might think — try to prove it yourself before reading the proof. The following proof is due to Stephanie Alexander.

Proof. Let γ be a closed curve in $\mathbb{S}^2$ of length $2\cdot\ell$, with $\ell < \pi$.

Let us divide γ into two arcs γ_1 and γ_2 of length ℓ, with endpoints p and q. Note that

$$\begin{aligned}\measuredangle(p,q) &\leqslant \operatorname{length}\gamma_1 = \\ &= \ell < \\ &< \pi.\end{aligned}$$



The north hemisphere corresponds to the disc and the south hemisphere to the complement of the disc.

Denote by z be the midpoint between p and q in $\mathbb{S}^2$; that is, z is the midpoint of the short arc of a great circle from p to q in $\mathbb{S}^2$. We claim that γ lies in the open north hemisphere with north pole at z. If not, γ intersects the equator in a point r. Without loss of generality we may assume that r lies on γ_1.

Rotate the arc γ_1 by the angle π around the line thru z and the center of the sphere. The obtained arc γ_1^* together with γ_1 forms a closed curve of length $2\cdot\ell$ passing thru r and its antipodal point r^*. Therefore

$$\tfrac{1}{2}\cdot\operatorname{length}\gamma = \ell \geqslant \measuredangle(r,r^*) = \pi,$$

a contradiction. $\square$

9.19. Exercise. *Describe a simple closed spherical curve that does not pass thru a pair of antipodal points and does not lie in any open hemisphere.*

9.20. Exercise. *Suppose that a closed simple spherical curve γ divides $\mathbb{S}^2$ into two regions of equal area. Show that*

$$\operatorname{length}\gamma \geqslant 2\cdot\pi.$$

9.21. Exercise. *Find a flaw in the solution of the following problem. Come up with a correct argument.*

Problem. Suppose that a closed plane curve γ has length at most 4. Show that γ lies in a unit disc.

Wrong solution. Note that it is sufficient to show that diameter of γ is at most 2; that is,

$$|p-q| \leqslant 2 \tag{❻}$$

for any two points p and q on γ.

The length of γ cannot be smaller than the closed inscribed polygonal line which goes from p to q and back to p. Therefore

$$2\cdot|p-q| \leqslant \operatorname{length}\gamma \leqslant 4;$$

whence ❻ follows. □

9.22. Advanced exercises. *Given points $v, w \in \mathbb{S}^2$, denote by w_v the closest point to w on the equator with pole at v; in other words, if $w^\perp$ is the projection of w to the plane perpendicular to v, then w_v is the unit vector in the direction of $w^\perp$. The vector w_v is defined if $w \neq \pm v$.*

(a) Show that for any spherical curve γ we have

$$\operatorname{length}\gamma = \overline{\operatorname{length}\gamma_v},$$

where $\overline{\operatorname{length}\gamma_v}$ denotes the average length of γ_v with v varying in $\mathbb{S}^2$. (This is a spherical analog of Crofton's formula.)

(b) Give another proof of the hemisphere lemma using part (9.22a).

Chapter 10

Curvature

Acceleration of a unit-speed curve

Recall that any regular smooth curve can be parameterized by its arc-length. The obtained parameterized curve, say γ, remains to be smooth and it has unit speed; that is, $|\gamma'(s)| = 1$ for all s. The following proposition states that in this case the acceleration vector stays perpendicular to the velocity vector.

10.1. Proposition. *Assume γ is a smooth unit-speed space curve. Then $\gamma'(s) \perp \gamma''(s)$ for any s.*

The scalar product (also known as dot product) of two vectors v and w will be denoted by $\langle \mathrm{v}, \mathrm{w} \rangle$. Recall that the derivative of a scalar product satisfies the product rule; that is, if $\mathrm{v} = \mathrm{v}(t)$ and $\mathrm{w} = \mathrm{w}(t)$ are smooth vector-valued functions of a real parameter t, then

$$\langle \mathrm{v}, \mathrm{w} \rangle' = \langle \mathrm{v}', \mathrm{w} \rangle + \langle \mathrm{v}, \mathrm{w}' \rangle.$$

Proof. The identity $|\gamma'| = 1$ can be rewritten as $\langle \gamma', \gamma' \rangle = 1$. Differentiating both sides,

$$2 \cdot \langle \gamma'', \gamma' \rangle = \langle \gamma', \gamma' \rangle' = 0,$$

whence $\gamma'' \perp \gamma'$. □

Curvature

For a unit-speed smooth space curve γ the magnitude of its acceleration $|\gamma''(s)|$ is called its *curvature* at the time s. If γ is simple, then we can say that $|\gamma''(s)|$ is the curvature at the point $p=\gamma(s)$ without ambiguity. The curvature is usually denoted by $\kappa(s)$ or $\kappa(s)_\gamma$ and in the case of simple curves it might be also denoted by $\kappa(p)$ or $\kappa(p)_\gamma$.

The curvature measures how fast the curve turns; if you drive along a plane curve, then curvature describes the position of your steering wheel at the given point (note that it does not depend on your speed).

In general, the term *curvature* is used for anything that measures how much a *geometric object* deviates from being *straight*; for curves, it measures how fast it deviates from a straight line.

10.2. Exercise. *Show that any regular smooth unit-speed spherical curve has curvature at least 1 at each time.*

Tangent indicatrix

Let γ be a regular smooth space curve. Let us consider another curve

❶ $$\mathrm{T}(t)=\tfrac{\gamma'(t)}{|\gamma'(t)|}$$

called *tangent indicatrix* of γ. Note that $|\mathrm{T}(t)|=1$ for any t; that is, T is a spherical curve.

If $s\mapsto\gamma(s)$ is a unit-speed parametrization, then $\mathrm{T}(s)=\gamma'(s)$. In this case we have the following expression for curvature:

$$\kappa(s)=|\mathrm{T}'(s)|=|\gamma''(s)|.$$

When γ is not necessarily parameterized by arc-length, then

❷ $$\kappa(t)=\frac{|\mathrm{T}'(t)|}{|\gamma'(t)|}.$$

Indeed, for an arc-length parametrization $s(t)$ we have $s'(t)=|\gamma'(t)|$. Therefore

$$\begin{aligned}\kappa(t)&=\left|\frac{d\mathrm{T}}{ds}\right|=\\&=|\tfrac{d\mathrm{T}}{dt}|/|\tfrac{ds}{dt}|=\\&=\frac{|\mathrm{T}'(t)|}{|\gamma'(t)|}.\end{aligned}$$

It follows that the indicatrix of a smooth regular curve γ is regular if the curvature of γ does not vanish.

10.3. Exercise. *Use the formulas* ❶ *and* ❷ *to show that for any smooth regular space curve γ we have the following expressions for its curvature:*

(a)

$$\kappa(t) = \frac{|\mathrm{W}(t)|}{|\gamma'(t)|^2},$$

where $\mathrm{W}(t)$ *denotes the projection of* $\gamma''(t)$ *to the plane normal to* $\gamma'(t)$*;*

(b)

$$\kappa(t) = \frac{|\gamma''(t) \times \gamma'(t)|}{|\gamma'(t)|^3},$$

where $\times$ *denotes the* vector product *(also known as* cross product*).*

10.4. Exercise. *Apply the formulas in the previous exercise to show that if* f *is a smooth real function, then its graph* $y = f(x)$ *has curvature*

$$\kappa(p) = \frac{|f''(x)|}{(1 + f'(x)^2)^{\frac{3}{2}}}$$

at the point $p = (x, f(x))$*.*

Tangent curves

Let γ be a smooth regular space curve and T its tangent indicatrix. The line thru $\gamma(t)$ in the direction of $\mathrm{T}(t)$ is called the *tangent line* at t.

The tangent line could be also defined as a unique line that has that has *first order of contact* with γ at s;

that is, $\rho(\ell) = o(\ell)$, where $\rho(\ell)$ denotes the distance from $\gamma(s + \ell)$ to the line.

We say that smooth regular curve γ_1 at s_1 is *tangent* to a smooth regular curve γ_2 at s_2 if $\gamma_1(s_1) = \gamma_2(s_2)$ and the tangent line of γ_1 at s_1 coincides with the tangent line of γ_2 at s_2; if both curves are simple we can also say that they are tangent at the point $p = \gamma_1(s_1) = \gamma_2(s_2)$ without ambiguity.

Total curvature

Let $\gamma\colon \mathbb{I} \to \mathbb{R}^3$ be a smooth unit-speed curve and T its tangent indicatrix. The integral

$$\Phi(\gamma) := \int\limits_{\mathbb{I}} \kappa(s){\cdot}ds$$

is called *total curvature of* γ.

When γ is not parameterized by arc-length, by a change of variables, the above integral takes the form

❸ $$\Phi(\gamma) := \int\limits_{\mathbb{I}} \kappa(\gamma(t))|\gamma'(t)|\cdot ds$$

10.5. Exercise. *Find the curvature of the helix*

$$\gamma_{a,b}(t) = (a\cdot \cos t, a\cdot \sin t, b\cdot t),$$

its tangent indicatrix and the total curvature of its arc for $t \in [0, 2\cdot\pi]$.

10.6. Observation. *The total curvature of a smooth regular curve is the length of its tangent indicatrix.*

Proof. Combine ❸ and ❷. □

10.7. Fenchel's theorem. *The total curvature of any closed regular space curve is at least $2\cdot\pi$.*

Proof. Fix a closed regular space curve γ; we can assume that it is described by a unit-speed loop $\gamma\colon [a, b] \to \mathbb{R}^3$; in this case $\gamma(a) = \gamma(b)$ and $\gamma'(a) = \gamma'(b)$.

Consider its tangent indicatrix $\mathrm{T} = \gamma'$. Recall that $|\mathrm{T}(s)| = 1$ for any s; that is, T is a closed spherical curve.

Let us show that T cannot lie in a hemisphere. Assume the contrary; without loss of generality we can assume that T lies in the north hemisphere defined by the inequality $z > 0$ in (x, y, z)-coordinates. It means that $z'(t) > 0$ for all t, where $\gamma(t) = (x(t), y(t), z(t))$. Therefore

$$z(b) - z(a) = \int\limits_a^b z'(s)\cdot ds > 0.$$

In particular, $\gamma(a) \neq \gamma(b)$, a contradiction.

Applying the observation (10.6) and the hemisphere lemma (9.18), we get

$$\Phi(\gamma) = \operatorname{length} \mathrm{T} \geqslant 2\cdot\pi.$$ □

10.8. Exercise. *Show that a closed space curve γ with curvature at most 1 cannot be shorter than the unit circle; that is,*

$$\operatorname{length} \gamma \geqslant 2\cdot\pi.$$

10.9. Advanced exercise. *Suppose that γ is a smooth regular space curve that does not pass thru the origin. Consider the spherical curve defined as $\sigma(t)=\frac{\gamma(t)}{|\gamma(t)|}$ for any t. Show that*

$$\operatorname{length}\sigma<\Phi(\gamma)+\pi.$$

Moreover, if γ is closed, then

$$\operatorname{length}\sigma\leqslant\Phi(\gamma).$$

Note that the last inequality gives an alternative proof of Fenchel's theorem. Indeed, without loss of generality we can assume that the origin lies on a chord of γ. In this case the closed spherical curve σ goes from a point to its antipode and comes back; it takes length π each way, whence

$$\operatorname{length}\sigma\geqslant 2\cdot\pi.$$

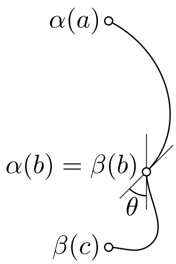
Piecewise smooth curves

Assume $\alpha\colon[a,b]\to\mathbb{R}^3$ and $\beta\colon[b,c]\to\mathbb{R}^3$ are two curves such that $\alpha(b)=\beta(b)$. Then one can combine these two curves into one $\gamma\colon[a,c]\to$ $\to\mathbb{R}^3$ by the rule

$$\gamma(t)=\begin{cases}\alpha(t)&\text{if}\quad t\leqslant b,\\ \beta(t)&\text{if}\quad t\geqslant b.\end{cases}$$

The obtained curve γ is called the *concatenation* of α and β. (The condition $\alpha(b)=\beta(b)$ ensures that the map $t\mapsto\gamma(t)$ is continuous.)

The same definition of concatenation can be applied if α and/or β are defied on semiopen intervals $(a,b]$ and/or $[b,c)$.



The concatenation can be also defined if the end point of the first curve coincides with the starting point of the second curve; if this is the case, then the time intervals of both curves can be shifted so that they fit together.

If in addition $\beta(c)=\alpha(a)$, then we can do cyclic concatination of these curves; this way we obtain a closed curve.

If $\alpha'(b)$ and $\beta'(b)$ are defined, then the angle $\theta=\measuredangle(\alpha'(b),\beta'(b))$ is called *external angle* of γ at time b. If $\theta=\pi$, then we say that γ has a *cusp* at time b.

Clearly, the assumption that the intervals $[a,b]$ and $[b,c]$ fit together is not essential, and we can concatenate any two curves α and β as long as the endpoint of α coincides with the starting point of β.

A space curve γ is called *piecewise smooth and regular* if it can be presented as an iterated concatination of a finite number of smooth regular curves; if γ is closed, then the concatination is assumed to be cyclic.

If γ is a concatination of smooth regular arcs $\gamma_1, \dots, \gamma_n$, then the total curvature of γ is defined as a sum of the total curvatures of γ_i and the external angles; that is,

$$\Phi(\gamma) = \Phi(\gamma_1) + \dots + \Phi(\gamma_n) + \theta_1 + \dots + \theta_{n-1}$$

where θ_i is the external angle at the joint between γ_i and γ_{i+1}; if γ is closed, then

$$\Phi(\gamma) = \Phi(\gamma_1) + \dots + \Phi(\gamma_n) + \theta_1 + \dots + \theta_n,$$

where θ_n is the external angle at the joint between γ_n and γ_1.

In particular, for a smooth regular loop $\gamma\colon [a,b] \to \mathbb{R}^3$, the total curvature of the corresponding closed curve $\hat\gamma$ is defined as

$$\Phi(\hat\gamma) := \Phi(\gamma) + \theta,$$

where $\theta = \measuredangle(\gamma'(a), \gamma'(b))$.

10.10. Generalized Fenchel's theorem. *Let γ be a closed piecewise smooth regular space curve. Then*

$$\Phi(\gamma) \geqslant 2\cdot\pi.$$

Proof. Suppose γ is a cyclic concatenation of n smooth regular arcs $\gamma_1, \dots, \gamma_n$. Denote by $\theta_1, \dots, \theta_n$ its external angles. We need to show that

$$\text{❹} \qquad \Phi(\gamma_1) + \dots + \Phi(\gamma_n) + \theta_1 + \dots + \theta_n \geqslant 2\cdot\pi.$$

Consider the tangent indicatrix $\mathrm{T}_1, \dots, \mathrm{T}_n$ for each arc $\gamma_1, \dots, \gamma_n$; these are smooth spherical arcs.

The same argument as in the proof of Fenchel's theorem, shows that the curves $\mathrm{T}_1, \dots, \mathrm{T}_n$ cannot lie in an open hemisphere.

Note that the spherical distance from the end point of T_i to the starting point of T_{i+1} is equal to the external angle θ_i (we enumerate the arcs modulo n, so $\gamma_{n+1} = \gamma_1$). Let us connect the end point of T_i to the starting point of T_{i+1} by a short arc of a great circle in the sphere. This way we get a closed spherical curve that is $\theta_1 + \dots + \theta_n$ longer then the total length of $\mathrm{T}_1, \dots, \mathrm{T}_n$.

Applying the hemisphee lemma (9.18) to the obtained closed curve, we get that

$$\operatorname{length} \mathrm{T}_1 + \dots + \operatorname{length} \mathrm{T}_n + \theta_1 + \dots + \theta_n \geqslant 2\cdot\pi.$$

□

Applying the observation (10.6), we get ❹.

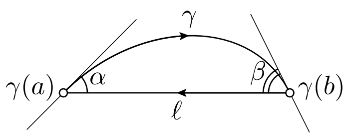
10.11. Chord lemma. *Let ℓ be the chord to a smooth regular arc $\gamma\colon [a,b] \to \mathbb{R}^3$. Assume γ meets ℓ at angles α and β at $\gamma(a)$ and $\gamma(b)$, respectively; that is*

$$\alpha = \measuredangle(\mathrm{w}, \gamma'(a)) \quad \textit{and} \quad \beta = \measuredangle(\mathrm{w}, \gamma'(b)),$$

where $\mathrm{w} = \gamma(b) - \gamma(a)$*. Then*

❺ $$\Phi(\gamma) \geqslant \alpha + \beta.$$

Proof. Let us parameterize the chord ℓ from $\gamma(b)$ to $\gamma(a)$ and consider the cyclic concatenation $\bar\gamma$ of γ and ℓ. The closed curve $\bar\gamma$ has two external angles $\pi - \alpha$ and $\pi - \beta$. Since the curvature of ℓ vanishes, we get



$$\Phi(\bar\gamma) = \Phi(\gamma) + (\pi - \alpha) + (\pi - \beta).$$

According to the generalized Fenechel's theorem (10.10), $\Phi(\bar\gamma) \geqslant 2\cdot\pi$; hence ❺ follows. □

10.12. Exercise. *Show that the estimate in the chord lemma is optimal. That is, given two points p, q and two unit vectors u, v in $\mathbb{R}^3$, show that there is a smooth regular curve γ that starts at p in the direction u and ends at q in the direction v such that $\Phi(\gamma)$ is arbitrarily close to $\measuredangle(\mathrm{w}, \mathrm{u}) + \measuredangle(\mathrm{w}, \mathrm{v})$, where $\mathrm{w} = q - p$.*

Polygonal lines

Polygonal lines are a particular case of piecewise smooth regular curves; each arc in its concatenation is a line segment. Since the curvature of a line segment vanishes, the total curvature of a polygonal line is the sum of its external angles.

10.13. Exercise. *Let a, b, c, d and x be distinct points in $\mathbb{R}^3$. Show that the total curvature of the polygonal line $abcd$ cannot exceed the total curvature of $abxcd$; that is,*

$$\Phi(abcd) \leq \Phi(abxcd).$$

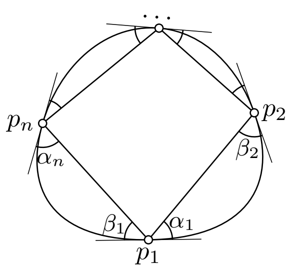
Use this statement to show that any closed polygonal line has curvature at least $2\cdot\pi$.

10.14. Proposition. *Assume a polygonal line $\beta = p_0 \dots p_n$ is inscribed in a smooth regular curve γ. Then*

$$\Phi(\gamma) \geqslant \Phi(\beta).$$

Moreover if γ is closed we allow the inscribed polygonal line β to be closed.

Proof. Since the curvature of line segments vanishes, the total curvature of polygonal line is the sum of external angles $\theta_i = \pi - \measuredangle p_{i-1}p_ip_{i+1}$.



Assume $p_i = \gamma(t_i)$. Set

$$\begin{aligned} \mathrm{w}_i &= p_{i+1} - p_i, & \mathrm{v}_i &= \gamma'(t_i), \\ \alpha_i &= \measuredangle(\mathrm{w}_i, \mathrm{v}_i), & \beta_i &= \measuredangle(\mathrm{w}_{i-1}, \mathrm{v}_i). \end{aligned}$$

In the case of a closed curve we use indexes modulo n, in particular $p_{n+1} = p_1$.

Note that $\theta_i = \measuredangle(\mathrm{w}_{i-1}, \mathrm{w}_i)$. By triangle inequality for angles 6.2, we get that

$$\theta_i \leqslant \alpha_i + \beta_i.$$

By the chord lemma, the total curvature of the arc of γ from p_i to p_{i+1} is at least $\alpha_i + \beta_{i+1}$.

Therefore if γ is a closed curve, we have

$$\begin{aligned} \Phi(\beta) &= \theta_1 + \dots + \theta_n \leqslant \\ &\leqslant \beta_1 + \alpha_1 + \dots + \beta_n + \alpha_n = \\ &= (\alpha_1 + \beta_2) + \dots + (\alpha_n + \beta_1) \leqslant \\ &\leqslant \Phi(\gamma). \end{aligned}$$

If γ is an arc, the argument is analogous:

$$\begin{aligned} \Phi(\beta) &= \theta_1 + \dots + \theta_{n-1} \leqslant \\ &\leqslant \beta_1 + \alpha_1 + \dots + \beta_{n-1} + \alpha_{n-1} \leqslant \\ &\leqslant (\alpha_0 + \beta_1) + \dots + (\alpha_{n-1} + \beta_n) \leqslant \\ &\leqslant \Phi(\gamma). \end{aligned}$$

□

10.15. Exercise.

(a) *Draw a smooth regular plane curve γ that has a self-intersection and such that $\Phi(\gamma) < 2\cdot\pi$.*

(b) *Show that if a smooth regular curve* $\gamma\colon [a,b]\to\mathbb{R}^3$ *has a self-intersection, then* $\Phi(\gamma)>\pi$.

10.16. Proposition. *The equality case in the Fenchel's theorem holds only for convex plane curves; that is, if the total curvature of a smooth regular space curve* γ *equals* $2{\cdot}\pi$, *then* γ *is a convex plane curve.*

The proof is an application of Proposition 10.14.

Proof. Consider an inscribed quadraliteral $abcd$ in γ. By the definition of total curvature, we have that

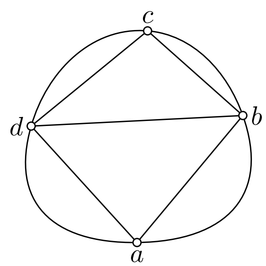
$$\begin{aligned}\Phi(abcd) &= (\pi-\measuredangle dab)+(\pi-\measuredangle abc)+(\pi-\measuredangle bcd)+(\pi-\measuredangle cda)=\\ &=4{\cdot}\pi-(\measuredangle dab+\measuredangle abc+\measuredangle bcd+\measuredangle cda))\end{aligned}$$

Note that

❻ $$\measuredangle abc\leqslant\measuredangle abd+\measuredangle dbc\quad\text{and}\quad\measuredangle cda\leqslant\measuredangle cdb+\measuredangle bda.$$

The sum of angles in any triangle is π, so combining these inequalities, we get that

$$\begin{aligned}\Phi(abcd)&\geqslant 4{\cdot}\pi-(\measuredangle dab+\measuredangle abd+\measuredangle bda)-(\measuredangle bcd+\measuredangle cdb+\measuredangle dbc)=\\ &=2{\cdot}\pi.\end{aligned}$$



By 10.14,

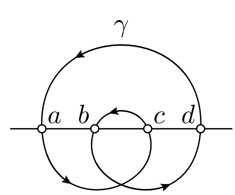
$$\Phi(abcd)\leqslant\Phi(\gamma)\leqslant 2{\cdot}\pi.$$

Therefore we have equalities in ❻. It means that the point d lies in the angle abc and the point b lies in the angle cda. That is, $abcd$ is a convex plane quadrilateral.

It follows that any quadrilateral inscribed in γ is a convex plane quadrilateral. Therefore all points of γ lie in a fixed plane and the domain bounded by γ in that plane is convex; that is, γ is a convex plane curve. □

10.17. Exercise. *Suppose that a closed curve* γ *crosses a line at four points* a, b, c *and* d. *Assume that these points appear on the line in the order* a, b, c, d *and they appear on the curve* γ *in the order* a, c, b, d. *Show that*

$$\Phi(\gamma)\geqslant 4{\cdot}\pi.$$



Lines crossing a curve at four points as in the above exercise are called *alternating quadrisecants.* It turns out that any *nontrivial knot* admits an alternating quadrisecant [16]; according to the exercise the latter implies the so-called *Fáry–Milnor theorem* — the total curvature any knot exceeds $4{\cdot}\pi$.

Bow lemma

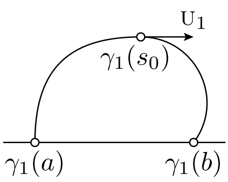
10.18. Lemma. *Let $\gamma_1\colon[a,b]\to\mathbb{R}^2$ and $\gamma_2\colon[a,b]\to\mathbb{R}^3$ be two smooth unit-speed curves. Suppose that $\kappa(s)_{\gamma_1}\geqslant\kappa(s)_{\gamma_2}$ for any s and the curve γ_1 is a simple arc of a convex curve; that is, it runs in the boundary of a covex plane figure. Then the distance between the ends of γ_1 cannot exceed the distance between the ends of γ_2; that is,*

$$|\gamma_1(b)-\gamma_1(a)|\leqslant|\gamma_2(b)-\gamma_2(a)|.$$

The following exercise states that the condition that γ_1 is a convex arc is necessary. It is instructive to do this exercise before reading the proof of the lemma.

10.19. Exercise. *Construct a simple smooth unit-speed plane curves $\gamma_1,\gamma_2\colon[a,b]\to\mathbb{R}^2$ such that that $\kappa(s)_{\gamma_1}>\kappa(s)_{\gamma_2}$ for any s and*

$$|\gamma_1(b)-\gamma_1(a)|>|\gamma_2(b)-\gamma_2(a)|.$$



Proof. Denote by T_1 and T_2 the tangent indicatrixes of γ_1 and γ_2, respectively.

Let $\gamma_1(s_0)$ be the point on γ_1 furthest to the line thru $\gamma(a)$ and $\gamma(b)$. Consider two unit vectors

$$\mathrm{u}_1=\mathrm{T}_1(s_0)=\gamma_1'(s_0)\quad\text{and}\quad\mathrm{u}_2=\mathrm{T}_2(s_0)=\gamma_2'(s_0).$$

By construction, the vector u_1 is parallel to $\gamma(b)-\gamma(a)$, in particular

$$|\gamma_1(b)-\gamma_1(a)|=\langle\mathrm{u}_1,\gamma_1(b)-\gamma_1(a)\rangle$$

Since γ_1 is an arc of a convex curve, its indicatrix T_1 runs in one direction along the unit circle. Suppose $s \leqslant s_0$, then

$$\begin{aligned}
\measuredangle(\gamma_1'(s), \mathrm{U}_1) &= \measuredangle(\mathrm{T}_1(s), \mathrm{T}_1(s_0)) = \\
&= \mathrm{length}(\mathrm{T}_1|_{[s,s_0]}) = \\
&= \int\limits_s^{s_0} |\mathrm{T}_1'(t)|\cdot dt = \\
&= \int\limits_s^{s_0} \kappa_1(t)\cdot dt \geqslant \\
&\geqslant \int\limits_s^{s_0} \kappa_2(t)\cdot dt = \\
&= \int\limits_s^{s_0} |\mathrm{T}_2'(t)|\cdot dt = \\
&= \mathrm{length}(\mathrm{T}_1|_{[s,s_0]}) \geqslant \\
&\geqslant \measuredangle(\mathrm{T}_2(s), \mathrm{T}_2(s_0)) = \\
&= \measuredangle(\gamma_2'(s), \mathrm{U}_2).
\end{aligned}$$

That is,

$$\measuredangle(\gamma_1'(s), \mathrm{U}_1) \geqslant \measuredangle(\gamma_2'(s), \mathrm{U}_2)$$

if $s \geqslant s_0$. The same argument shows that

❼
$$\measuredangle(\gamma_1'(s), \mathrm{U}_1) \geqslant \measuredangle(\gamma_2'(s), \mathrm{U}_2)$$

for $s \geqslant s_0$; therefore the inequality holds for any s.

Since U_1 is a unit vector parallel to $\gamma_1(b) - \gamma_1(a)$, we have that

$$|\gamma_1(b) - \gamma_1(a)| = \langle \mathrm{U}_1, \gamma_1(b) - \gamma_1(a)\rangle$$

and since U_2 is a unit vector, we have that

$$|\gamma_2(b) - \gamma_2(a)| \geqslant \langle \mathrm{U}_2, \gamma_2(b) - \gamma_2(a)\rangle$$

Integrating ❼, we get

$$\begin{aligned}|\gamma_1(b)-\gamma_1(a)| &= \langle \mathrm{u}_1, \gamma_1(b)-\gamma_1(a)\rangle = \\ &= \int\limits_a^b \langle \mathrm{u}_1, \gamma_1'(s)\rangle\cdot ds \leqslant \\ &\leqslant \int\limits_a^b \langle \mathrm{u}_2, \gamma_2'(s)\rangle\cdot ds = \\ &= \langle \mathrm{u}_2, \gamma_2(b)-\gamma_2(a)\rangle \leqslant \\ &\leqslant |\gamma_2(b)-\gamma_2(a)|.\end{aligned}$$

Hence the result. □

10.20. Exercise. *Let $\gamma\colon [a,b]\to\mathbb{R}^3$ be a smooth regular curve and $0<\theta\leqslant\frac{\pi}{2}$. Assume*

$$\Phi(\gamma)\leqslant 2\cdot\theta.$$

(a) Show that

$$|\gamma(b)-\gamma(a)| > \cos\theta\cdot\operatorname{length}\gamma.$$

(b) Use part (a) to give another solution of 10.15b.

(c) Show that the inequality in (a) is optimal; that is, given θ there is a smooth regular curve γ such that $\frac{|\gamma(b)-\gamma(a)|}{\operatorname{length}\gamma}$ is arbitrarily close to $\cos\theta$.

10.21. Exercise. *Let p and q be points in a unit circle dividing it in two arcs with lengths $\ell_1<\ell_2$. Suppose the space curve γ connects p to q and has curvature at most 1. Show that either*

$$\operatorname{length}\gamma\leqslant\ell_1 \quad \textit{or} \quad \operatorname{length}\gamma\geqslant\ell_2.$$

The following exercise generalizes 10.8.

10.22. Exercise. *Suppose $\gamma\colon [a,b]\to\mathbb{R}^3$ is a smooth regular loop with curvature at most 1. Show that*

$$\operatorname{length}\gamma\geqslant 2\cdot\pi.$$

DNA inequality*

Recall that the curvature of a spherical curve is at least 1 (Exercise 10.2). In particular, the length of a spherical curve cannot exceed its total curvature. The following theorem shows that the same inequality holds for *closed* curves in a unit ball.

10.23. Theorem. *Let γ be a smooth regular closed curve that lies in a unit ball. Then*

$$\Phi(\gamma) \geqslant \operatorname{length}\gamma.$$

This theorem was proved by Don Chakerian [11]; for plane curves it was proved earlier by István Fáry [18]. We present the proof given by Don Chakerian in [12]; few other proofs of this theorem are discussed by Serge Tabachnikov [48].

Proof. Without loss of generality we can assume the curve is described by a loop $\gamma\colon [0,\ell] \to \mathbb{R}^3$ parameterized by its arc-length, so $\ell = \operatorname{length}\gamma$. We can also assume that the origin is the center of the ball. It follows that

$$\langle\gamma'(s),\gamma'(s)\rangle = 1, \qquad |\gamma(s)| \leqslant 1$$

and in particular

❽
$$\begin{aligned}\langle\gamma''(s),\gamma(s)\rangle &\geqslant -|\gamma''(s)|\cdot|\gamma(s)| \geqslant \\ &\geqslant -\kappa(s)\end{aligned}$$

for all s. Since γ is a smooth closed curve, we have $\gamma'(0) = \gamma'(\ell)$ and $\gamma(0) = \gamma(\ell)$. Applying ❽, we get that

$$\begin{aligned}0 &= \langle\gamma(\ell),\gamma'(\ell)\rangle - \langle\gamma(0),\gamma'(0)\rangle = \\ &= \int_0^\ell \langle\gamma(s),\gamma'(s)\rangle'\cdot ds = \\ &= \int_0^\ell \langle\gamma'(s),\gamma'(s)\rangle\cdot ds + \int_0^\ell \langle\gamma(s),\gamma''(s)\rangle\cdot ds \geqslant \\ &\geqslant \ell - \Phi(\gamma),\end{aligned}$$

whence the result. □

Nonsmooth curves*

10.24. Theorem. *For any regular smooth space curve γ we have that*

$$\Phi(\gamma) = \sup\{\Phi(\beta)\},$$

where the supremum is taken over all polygonal lines β inscribed in γ (if γ is closed we assume that so is β).

This theorem is a refinement of Proposition 10.14. It shows that the following definition of total curvature of arbitrary curves, generalize the original definition that works only for (piecewise) smooth and regular curves.

We say that a parameterized curve is trivial if it is constant; that is, it stays at one point.

10.25. Definition. *The total curvature of a nontrivial parameterized space curve γ is the exact upper bound on the total curvatures of inscribed nondegenerate polygonal lines; if γ is closed, then we assume that the inscribed polygonal lines are closed as well.*

Proof of the theorem. Note that the inequality

$$\Phi(\gamma) \geqslant \Phi(\beta)$$

follows from 10.14; it remains to show

$$\Phi(\gamma) \leqslant \sup\{\Phi(\beta)\}. \quad ❾$$

Let $\gamma\colon [a,b] \to \mathbb{R}^3$ be a smooth curve. Fix a partition $a = t_0 < \dots < t_k = b$ and consider the corresponding inscribed polygonal line $\beta = p_0 \dots p_k$. (If γ is closed, then $p_0 = p_k$ and β is closed as well.)

Let $\tau = \xi_1 \dots \xi_k$ be a spherical polygonal line with the vertexes $\xi_i = \frac{p_i - p_{i-1}}{|p_i - p_{i-1}|}$. We can assume that τ has constant speed on each arc and $\tau(t_i) = \xi_i$ for each i. The spherical polygonal line τ will be called tangent indicatrix for β.

Consider a sequence of finer and finer partitions, denote by β_n and τ_n the corresponding inscribed polygonal lines and their tangent indicatrixes. Note that since γ is smooth, the idicatrixes τ_n converge pointwise to T — the tangent indicatrix of γ. By the semi-continuity of the length (9.15), we get that

$$\begin{aligned}\Phi(\gamma) &= \operatorname{length} \mathrm{T} \leqslant \\ &\leqslant \varliminf_{n\to\infty} \operatorname{length} \tau_n = \\ &= \varliminf_{n\to\infty} \Phi(\beta_n) \leqslant \\ &\leqslant \sup\{\Phi(\beta)\}.\end{aligned}$$

□

10.26. Exercise. *Show that the total curvature is lower semi-continuous with respect to pointwise convergence of curves. That is, if a sequence of curves* $\gamma_n\colon [a,b] \to \mathbb{R}^3$ *converges pointwise to a nontrivial curve* $\gamma_\infty\colon [a,b] \to \mathbb{R}^3$*, then*

$$\varliminf_{n\to\infty} \Phi(\gamma_n) \geqslant \Phi(\gamma_\infty).$$

10.27. Exercise. *Generalize Fenchel's theorem to all nontrivial closed space curves. That is, show that*

$$\Phi(\gamma) \geqslant 2{\cdot}\pi$$

for any closed space curve γ *(not necessary piecewise smooth and regular).*

10.28. Exercise.

(a) *Assume that a curve* $\gamma\colon [a,b] \to \mathbb{R}^3$ *has finite total curvature. Show that* γ *is rectifiable.*
(b) *Construct a rectifiable curve* $\gamma\colon [a,b] \to \mathbb{R}^3$ *that has infinite total curvature.*

For more on curves of finite total curvature read [3, 47].

DNA inequality revisited*

In this section we will give an alternative proof of the DNA inequality (10.23) that works for arbitrary, not necessarily smooth, curves. In the proof we use 10.25 to define the total curvature; according to 10.24, it is more general than the smooth definition given on page 45.

Alternative proof of 10.23. We will show that

$$\Phi(\gamma) > \operatorname{length}\gamma.$$

for any closed polygonal line $\gamma = p_1 \dots p_n$ in a unit ball. It implies the theorem since in any nontrivial closed curve we can inscribe a closed polygonal line with arbitrary close total curvature and length.

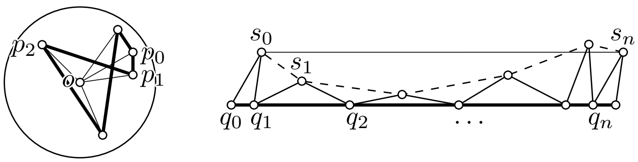
The indexes are taken modulo n, in particular $p_n = p_0$, $p_{n+1} = p_1$ and so on. Denote by θ_i the external angle of γ at p_i; that is,

$$\theta_i = \pi - \measuredangle p_{i-1}p_ip_{i+1}.$$

Denote by o the center of the ball. Consider a sequence of $n+1$ plane triangles

$$\begin{aligned}\triangle q_0 s_0 q_1 &\cong \triangle p_0 o p_1,\\ \triangle q_1 s_1 q_2 &\cong \triangle p_1 o p_2,\\ &\dots\\ \triangle q_n s_n q_{n+1} &\cong \triangle p_n o p_{n+1},\end{aligned}$$

such that the points $q_0, q_1, \dots, q_{n+1}$ lie on one line in that order and all the points $s_0, \dots, s_n$ lie on one side from this line.



Since $p_0 = p_n$ and $p_1 = p_{n+1}$, we have that

$$\triangle q_n s_n q_{n+1} \cong \triangle p_n o p_{n+1} = \triangle p_0 o p_1 \cong \triangle q_0 s_0 q_1,$$

so $s_0 q_0 q_n s_n$ is a parallelogram. Therefore

$$\begin{aligned}|s_0 - s_1| + \dots + |s_{n-1} - s_n| \geqslant |s_n - s_0| &=\\ &= |q_0 - q_n| =\\ &= |p_0 - p_1| + \dots + |p_{n-1} - p_n|\\ &= \operatorname{length}\gamma.\end{aligned}$$

Since $|q_i - s_{i-1}| = |q_i - s_i| = |p_i - o| \leqslant 1$, we have that

$$\measuredangle s_{i-1} q_i s_i > |s_{i-1} - s_i|$$

for each i. Therefore

$$\begin{aligned}\theta_i &= \pi - \measuredangle p_{i-1} p_i p_{i+1} \geqslant\\ &\geqslant \pi - \measuredangle p_{i-1} p_i o - \measuredangle o p_i p_{i+1} =\\ &= \pi - \measuredangle q_{i-1} q_i s_{i-1} - \measuredangle s_i q_i q_{i+1} =\\ &= \measuredangle s_{i-1} q_i s_i >\\ &> |s_{i-1} - s_i|.\end{aligned}$$

That is,

$$\theta_i > |s_{i-1} - s_i|$$

for each i.

It follows that

$$\begin{aligned}\Phi(\gamma) &= \theta_1 + \dots + \theta_n > \\ &> |s_0 - s_1| + \dots |s_{n-1} - s_n| \geqslant \\ &\geqslant \operatorname{length}\gamma.\end{aligned}$$

Hence the result. □

Let us mention the following closely related statement:

10.29. Theorem. *Suppose a closed regular smooth curve γ lies in a convex figure of perimeter $2\cdot\pi$. Then*

$$\Phi(\gamma) \geqslant \operatorname{length}\gamma.$$

This statement was conjectured by Serge Tabachnikov [48]. Despite the simplicity of the formulation, the proof is annoyingly difficult; it was proved by Jeffrey Lagarias and Thomas Richardson [30]; later a simpler proof was given by Alexander Nazarov and Fedor Petrov [39].

Chapter 11

Torsion

This chapter provides practice that might be useful, but most of the result in this chapter will not be used further in the sequel.

Frenet frame

Let γ be a smooth regular space curve. Without loss of generality, we may assume that γ has an arc-length parametrization, so the velocity vector $\mathrm{T}(s) = \gamma'(s)$ is unit.

Assume its curvature does not vanish at some time s; in other words, $\gamma''(s) \neq 0$. Then we can define the so-called *normal vector* at s as

$$\mathrm{N}(s) = \frac{\gamma''(s)}{|\gamma''(s)|}.$$

Note that

$$\mathrm{T}'(s) = \gamma''(s) = \kappa(s)\cdot\mathrm{N}(s).$$

According to 10.1, $\mathrm{N}(s) \perp \mathrm{T}(s)$. Therefore the vector product

$$\mathrm{B}(s) = \mathrm{T}(s) \times \mathrm{N}(s)$$

is a unit vector which makes the triple $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ an oriented orthonormal basis in $\mathbb{R}^3$; in particular, we have that

❶
$$\begin{aligned} \langle \mathrm{T}, \mathrm{T}\rangle &= 1, & \langle \mathrm{N}, \mathrm{N}\rangle &= 1, & \langle \mathrm{B}, \mathrm{B}\rangle &= 1, \\ \langle \mathrm{T}, \mathrm{N}\rangle &= 0, & \langle \mathrm{N}, \mathrm{B}\rangle &= 0, & \langle \mathrm{B}, \mathrm{T}\rangle &= 0. \end{aligned}$$

The orthonormal basis $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is called *Frenet frame* at s; the vectors in the frame are called *tangent*, *normal* and *binormal* respectively. Note that the frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is defined only if $k(s) \neq 0$.

The plane Π_s thru $\gamma(s)$ spanned by vectors $\mathrm{T}(s)$ and $\mathrm{N}(s)$ is called *osculating plane* at s; equivalently it can be defined as a plane thru $\gamma(s)$ that is perpendicular to the binormal vector $\mathrm{B}(s)$. This is the unique plane that has *second order of contact* with γ at s; that is, $\rho(\ell) = o(\ell^2)$, where $\rho(\ell)$ denotes the distance from $\gamma(s+\ell)$ to Π_s.

Torsion*

Let γ be a smooth unit-speed space curve and $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is its Frenet frame. The value

$$\tau(s) = \langle \mathrm{N}'(s), \mathrm{B}(s) \rangle$$

is called the *torsion* of γ at s.

Note that the torsion $\tau(s)$ is defined if $\kappa(s) \neq 0$. Indeed, if $\kappa(s) \neq 0$, then Frenet frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is defined at s. Moreover since the function $s \mapsto \kappa(s)$ is continuous, it must be positive in an open interval containing s; therefore the Frenet frame is also defined in this interval. Clearly $\mathrm{T}(s)$, $\mathrm{N}(s)$ and $\mathrm{B}(s)$ depend smoothly on s in their domains of definition. Therefore $\mathrm{N}'(s)$ is defined and so is the torsion $\tau(s) = \langle \mathrm{N}'(s), \mathrm{B}(s) \rangle$.

The torsion measures how fast the osculating plane rotates when one travels along γ.

11.1. Exercise. *Given real numbers a and b, calculate curvature and torsion of the helix*

$$\gamma_{a,b}(t) = (a\cdot \cos t, a\cdot \sin t, b\cdot t).$$

Conclude that for any $\kappa > 0$ and τ there is a helix with constant curvature κ and torsion τ.

Frenet formulas*

Assume the Frenet frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ of a curve γ is defined at s. Recall that

$$\mathrm{T}'(s) = \kappa(s)\cdot \mathrm{N}(s). \tag{❷}$$

It is convenient to write the remaining derivatives $\mathrm{N}'(s)$ and $\mathrm{B}'(s)$ in the frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$.

First let us show that

❸ $$\mathrm{N}'(s) = -\kappa(s)\cdot\mathrm{T}(s) + \tau(s)\cdot\mathrm{B}(s).$$

Since the frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is orthonormal, the above formula is equivalent to the following three identities:

❹ $$\langle \mathrm{N}', \mathrm{T}\rangle = -\kappa, \quad \langle \mathrm{N}', \mathrm{N}\rangle = 0, \quad \langle \mathrm{N}', \mathrm{B}\rangle = \tau,$$

The last identity follows from the definition of torsion. The second one comes from differentiating $\langle \mathrm{N}, \mathrm{N}\rangle = 1$ in ❶. Differentiating the identity $\langle \mathrm{T}, \mathrm{N}\rangle = 0$ in ❶; we get

$$\langle \mathrm{T}', \mathrm{N}\rangle + \langle \mathrm{T}, \mathrm{N}'\rangle = 0.$$

Applying ❷, we get the first one.

Differentiating the third identity in ❶, we get that $\mathrm{B}' \perp \mathrm{B}$. Further taking derivatives of the other identities with B in ❶, we get that

$$\begin{aligned} \langle \mathrm{B}', \mathrm{T}\rangle &= -\langle \mathrm{B}, \mathrm{T}'\rangle = -\kappa\cdot\langle \mathrm{B}, \mathrm{N}\rangle = 0 \\ \langle \mathrm{B}', \mathrm{N}\rangle &= -\langle \mathrm{B}, \mathrm{N}'\rangle = \tau \end{aligned}$$

Since the frame $\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)$ is orthonormal, it follows that

❺ $$\mathrm{B}'(s) = -\tau(s)\cdot\mathrm{N}(s).$$

The equations ❷, ❸ and ❺ are called Frenet formulas. All three can be written as one matrix identity:

$$\begin{pmatrix} \mathrm{T}' \\ \mathrm{N}' \\ \mathrm{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{T} \\ \mathrm{N} \\ \mathrm{B} \end{pmatrix}.$$

11.2. Exercise. *Deduce the formula ❺ from ❷ and ❸ by differentiating the identity* $\mathrm{B} = \mathrm{T} \times \mathrm{N}$.

11.3. Exercise. *Let γ be a regular space curve with nonvanishing curvature. Show that γ lies in a plane if and only if its torsion vanishes.*

11.4. Exercise. *Let γ be a smooth regular space curve, κ and τ its curvature and torsion, and* $\mathrm{T}, \mathrm{N}, \mathrm{B}$ *its Frenet frame. Show that*

$$\mathrm{B} = \frac{\gamma' \times \gamma''}{|\gamma' \times \gamma''|}.$$

Use this formula to show that

$$\tau = \frac{\langle \gamma' \times \gamma'', \gamma''' \rangle}{|\gamma' \times \gamma''|^2}.$$

Curves of constant slope*

We say that a smooth regular space curve γ has *constant slope* if its velocity vector makes a constant angle with a fixed direction. The following theorem was proved by Michel Ange Lancret [32] more than two centuries ago.

11.5. Theorem. *Let γ be a smooth regular curve; denote by κ and τ its curvature and torsion. Suppose $\kappa(s) > 0$ for all s. Then γ has constant slope if and only if the ratio $\frac{\tau}{\kappa}$ is constant.*

The theorem can be proved using the Frenet formulas. The following exercise will guide you thru the proof of the theorem.

11.6. Exercise. *Let γ be a smooth regular space curve with nonvanishing curvature, $\mathrm{T}, \mathrm{N}, \mathrm{B}$ its Frenet frame and κ, τ its curvature and torsion.*

(a) *Assume that $\langle \mathrm{W}, \mathrm{T} \rangle$ is constant for a fixed nonzero vector W. Show that*
$$\langle \mathrm{W}, \mathrm{N} \rangle = 0.$$
Use it to show that
$$\langle \mathrm{W}, -\kappa\cdot\mathrm{T} + \tau\cdot\mathrm{B} \rangle = 0.$$
Use these two identities to show that $\frac{\tau}{\kappa}$ is constant; it proves the "only if" part of the theorem.

(b) *Assume $\frac{\tau}{\kappa}$ is constant, show that the vector $\mathrm{W} = \frac{\tau}{\kappa}\cdot\mathrm{T} + \mathrm{B}$ is constant. Conclude that γ has constant slope; it proves the "if" part of the theorem.*

Let γ be a smooth unit-speed curve and s_0 a fixed real number. Then the curve
$$\alpha(s) = \gamma(s) + (s_0 - s)\cdot\gamma'(s)$$
is called the *evolvent* of γ. Note that if $\ell(s)$ denotes the tangent line to γ at s, then $\alpha(s) \in \ell(s)$ and $\alpha'(s) \perp \ell$ for all s.

11.7. Exercise. *Show that the evolvent of a constant slope curve is a plane curve.*

Spherical curves*

11.8. Theorem. *A smooth regular space curve γ lies in a unit sphere if and only if the following identity*
$$\left|\frac{\kappa'}{\tau}\right| = \kappa\cdot\sqrt{\kappa^2 - 1}.$$

holds for its curvature κ and torsion τ.

Note that the identity implicitely implies that the torsion τ of the curve is nonzero; otherwise the left hand side would be undefined while right hand side is defined. The proof is another application of the Frenet formulas; we present it in form of a guided exercise:

11.9. Exercise. *Suppose γ is a smooth unit-speed space curve. Denote by $\mathrm{T}, \mathrm{N}, \mathrm{B}$ its Frenet frame and by κ, τ its curvature and torsion.*

Assume that γ is spherical; that is, $|\gamma(s)| = 1$ for any s. Show that

(a) $\langle \mathrm{T}, \gamma\rangle = 0$; *conclude that* $\langle \mathrm{N}, \gamma\rangle^2 + \langle \mathrm{B}, \gamma\rangle^2 = 1$.

(b) $\langle \mathrm{N}, \gamma\rangle = -\frac{1}{\kappa}$;

(c) $\langle \mathrm{B}, \gamma\rangle' = \frac{\tau}{\kappa}$.

(d) *Use (c) to show that if γ is closed, then $\tau(s) = 0$ for some s.*

(e) *Use (a)–(c) to show that*

$$\left|\frac{\kappa'}{\tau}\right| = \kappa\cdot\sqrt{\kappa^2 - 1}.$$

It proves the "only if" part of the theorem.

Now assume that γ is a space curve that satisfies the idenity in (e).

(f) *Show that $p = \gamma + \frac{1}{\kappa}\cdot\mathrm{N} + \frac{\kappa'}{\kappa^2\cdot\tau}\cdot\mathrm{B}$ is constant; conclude that γ lies in the unit sphere centered at p.*

It proves the "if" part of the theorem.

For a unit-speed curve γ with nonzero curvature and torsion at s, the sphere Σ_s with center

$$p(s) = \gamma(s) + \tfrac{1}{\kappa(s)}\cdot\mathrm{N}(s) + \tfrac{\kappa'(s)}{\kappa^2(s)\cdot\tau(s)}\cdot\mathrm{B}(s)$$

and passing thru $\gamma(s)$ is called the *osculating sphere* of γ at s. This is the unique sphere that has *third order of contact* with γ at s; that is, $\rho(\ell) = o(\ell^3)$, where $\rho(\ell)$ denotes the distance from $\gamma(s+\ell)$ to Σ_s.

Fundamental theorem of space curves*

11.10. Theorem. *Let $\kappa(s)$ and $\tau(s)$ be two smooth real valued functions defined on a real interval $\mathbb{I}$. Suppose $\kappa(s) > 0$ for all s. Then there is a smooth unit-speed curve $\gamma\colon \mathbb{I} \to \mathbb{R}^3$ with curvature $\kappa(s)$ and torsion $\tau(s)$ for every s. Moreover γ is uniquely defined up to a rigid motion of the space.*

The proof is an application of the theorem on existence and uniqueness of a solution of ordinary differential equation (3.1).

Proof. Fix a parameter value s_0, a point $\gamma(s_0)$ and an oriented orthonormal frame $\mathrm{T}(s_0)$, $\mathrm{N}(s_0)$, $\mathrm{B}(s_0)$.

Consider the following system of differential equations

$$\begin{cases} \gamma' = \mathrm{T}, \\ \mathrm{T}' = \kappa\cdot\mathrm{N}, \\ \mathrm{N}' = -\kappa\cdot\mathrm{T} + \tau\cdot\mathrm{B}, \\ \mathrm{B}' = -\tau\cdot\mathrm{N}. \end{cases}$$

with the initial condition $\gamma(s_0)$ and an oriented orthonormal frame $\mathrm{T}(s_0)$, $\mathrm{N}(s_0)$, $\mathrm{B}(s_0)$. (The system of equations has four vector equations, so it can be rewritten as a system of 12 scalar equations.)

By 3.1, this system has a unique solution which is defined in a maximal subinterval $\mathbb{J} \subset \mathbb{I}$ containing s_0; we need to show that actually $\mathbb{J} = \mathbb{I}$.

Note that

$$\begin{aligned}
\langle \mathrm{T}, \mathrm{T}\rangle' &= 2\cdot\langle \mathrm{T}, \mathrm{T}'\rangle = 2\cdot\kappa\cdot\langle \mathrm{T}, \mathrm{N}\rangle = 0, \\
\langle \mathrm{N}, \mathrm{N}\rangle' &= 2\cdot\langle \mathrm{N}, \mathrm{N}'\rangle = -2\cdot\kappa\cdot\langle \mathrm{N}, \mathrm{T}\rangle + 2\cdot\tau\cdot\langle \mathrm{N}, \mathrm{B}\rangle = 0, \\
\langle \mathrm{B}, \mathrm{B}\rangle' &= 2\cdot\langle \mathrm{B}, \mathrm{B}'\rangle = -2\cdot\tau\langle \mathrm{B}, \mathrm{N}\rangle = 0, \\
\langle \mathrm{T}, \mathrm{N}\rangle' &= \langle \mathrm{T}', \mathrm{N}\rangle + \langle \mathrm{T}, \mathrm{N}'\rangle = \kappa\cdot\langle \mathrm{N}, \mathrm{N}\rangle - \kappa\cdot\langle \mathrm{T}, \mathrm{T}\rangle + \tau\cdot\langle \mathrm{T}, \mathrm{B}\rangle = 0, \\
\langle \mathrm{N}, \mathrm{B}\rangle' &= \langle \mathrm{N}', \mathrm{B}\rangle + \langle \mathrm{N}, \mathrm{B}'\rangle = 0, \\
\langle \mathrm{B}, \mathrm{T}\rangle' &= \langle \mathrm{B}', \mathrm{T}\rangle + \langle \mathrm{B}, \mathrm{T}'\rangle = -\tau\cdot\langle \mathrm{N}, \mathrm{T}\rangle + \kappa\cdot\langle \mathrm{B}, \mathrm{N}\rangle = 0.
\end{aligned}$$

That is, the values $\langle \mathrm{T}, \mathrm{T}\rangle$, $\langle \mathrm{N}, \mathrm{N}\rangle$, $\langle \mathrm{B}, \mathrm{B}\rangle$, $\langle \mathrm{T}, \mathrm{N}\rangle$, $\langle \mathrm{T}, \mathrm{N}\rangle$, $\langle \mathrm{B}, \mathrm{T}\rangle$ are constant functions of s. Since we choose $\mathrm{T}(s_0)$, $\mathrm{N}(s_0)$, $\mathrm{B}(s_0)$ to be an oriented orthonormal frame, we have that the triple $\mathrm{T}(s)$, $\mathrm{N}(s)$, $\mathrm{B}(s)$ is an oriented orthonormal for any s. In particular, $|\gamma'(s)| = 1$ for all s.

Assume $\mathbb{J} \subsetneq \mathbb{I}$. Then an end of $\mathbb{J}$, say a, lies in the interior of $\mathbb{I}$. By Theorem 3.1, at least one of the values $\gamma(s)$, $\mathrm{T}(s)$, $\mathrm{N}(s)$, $\mathrm{B}(s)$ escapes to infinity as $s \to a$. But this is impossible since the vectors $\mathrm{T}(s)$, $\mathrm{N}(s)$, $\mathrm{B}(s)$ remain unit and $|\gamma'(s)| = |\mathrm{T}(s)| = 1$ — a contradiction. Hence $\mathbb{J} = \mathbb{I}$.

It remains to prove the last statement.

Assume there are two curves γ_1 and γ_2 with the given curvature and torsion functions. Applying a motion of the space we can assume that the $\gamma_1(s_0) = \gamma_2(s_0)$ and the Frenet frames of the curves coincide at s_0. Then $\gamma_1 = \gamma_2$ by uniqueness of solutions of the system (3.1). □

11.11. Exercise. *Assume a curve $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ has constant curvature and torsion. Show that γ is a helix, possibly degenerate to a circle; that is, in a suitable coordinate system we have*

$$\gamma(t) = (a\cdot \cos t, a\cdot \sin t, b\cdot t)$$

for some constants a and b.

11.12. Advanced exercise. *Let γ be a smooth regular space curve such that the distance $|\gamma(t)-\gamma(t+\ell)|$ depends only on ℓ. Show that γ is a helix, possibly degenerate to a line or a circle.*

Chapter 12

Signed curvature

Suppose γ is a smooth unit-speed plane curve, so $\mathrm{T}(s) = \gamma'(s)$ is its unit tangent vector for any s.

Let us rotate $\mathrm{T}(s)$ by the angle $\frac{\pi}{2}$ counterclockwise; denote the obtained vector by $\mathrm{N}(s)$. The pair $\mathrm{T}(s), \mathrm{N}(s)$ is an oriented orthonormal frame in the plane which is analogous to the Frenet frame defined on page 61; we will keep the name *Frenet frame* for it.

Recall that $\gamma''(s) \perp \gamma'(s)$ (see 10.1). Therefore

❶ $$\mathrm{T}'(s) = k(s)\cdot\mathrm{N}(s).$$

for some real number $k(s)$; the value $k(s)$ is called *signed curvature* of γ at s. We may use notation $k(s)_\gamma$ if we need to specify the curve γ.

Note that

$$\kappa(s) = |k(s)|;$$

that is, up to sign, the signed curvature $k(s)$ equals the curvature $\kappa(s)$ of γ at s defined on page 43; the sign tells us in which direction it turns — if γ is turning left at time s, then $k(s) > 0$. If we want to emphasise that we are working with the *nonsigned* curvature of the curve, we call it *absolute curvature.*

Note that if we reverse the parametrization of γ or change the orientation of the plane, then the signed curvature changes its sign.

Since $\mathrm{T}(s), \mathrm{N}(s)$ is an orthonormal frame, we have

$$\langle \mathrm{T}, \mathrm{T}\rangle = 1, \qquad \langle \mathrm{N}, \mathrm{N}\rangle = 1, \qquad \langle \mathrm{T}, \mathrm{N}\rangle = 0,$$

Differentiating these idenitites we get

$$\langle \mathrm{T}', \mathrm{T}\rangle = 0, \qquad \langle \mathrm{N}', \mathrm{N}\rangle = 0, \qquad \langle \mathrm{T}', \mathrm{N}\rangle + \langle \mathrm{T}, \mathrm{N}'\rangle = 0,$$

By ❶, $\langle \mathrm{T}', \mathrm{N}\rangle = k$ and therefore $\langle \mathrm{T}, \mathrm{N}'\rangle = -k$. Whence we get

❷ $$\mathrm{N}'(s) = -k(s)\cdot \mathrm{T}(s).$$

The equations ❶ and ❷ are the Frenet formulas for plane curves. They can be written in matrix form as:

$$\begin{pmatrix} \mathrm{T}' \\ \mathrm{N}' \end{pmatrix} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{T} \\ \mathrm{N} \end{pmatrix}.$$

12.1. Exercise. *Let $\gamma_0\colon [a,b] \to \mathbb{R}^2$ be a smooth regular curve and* T *its tangent indicatrix. Consider another curve $\gamma_1\colon [a,b] \to \mathbb{R}^2$ defined by $\gamma_1(t) = \gamma_0(t) + \mathrm{T}(t)$. Show that*

$$\operatorname{length}\gamma_0 \leqslant \operatorname{length}\gamma_1.$$

The curves γ_0 and γ_1 in the exercise above describe the tracks of an idealized bicycle with distance 1 from the rear to the front wheel. Thus by the exercise, the front wheel must have longer track. For more on the geometry of bicycle tracks, see [21] and the references therein.

Fundamental theorem of plane curves

12.2. Theorem. *Let $k(s)$ be a smooth real valued function defined on a real interval $\mathbb{I}$. Then there is a smooth unit-speed curve $\gamma\colon \mathbb{I} \to \mathbb{R}^2$ with signed curvature $k(s)$. Moreover, γ is uniquely defined up to a rigid motion of the plane.*

This is the fundamental theorem of plane curves; it is a direct analog of 11.10 and it can be proved along the same lines. We present a slightly simpler proof.

Proof. Fix $s_0 \in \mathbb{I}$. Consider the function

$$\theta(s) = \int_{s_0}^{s} k(t)\cdot dt.$$

Note that by the fundamental theorem of calculus, we have $\theta'(s) = k(s)$ for all s.

Set

$$\mathrm{T}(s) = (\cos[\theta(s)], \sin[\theta(s)])$$

and let $\mathrm{N}(s)$ be its counterclockwise rotation by angle $\frac{\pi}{2}$; so

$$\mathrm{N}(s) = (-\sin[\theta(s)], \cos[\theta(s)]).$$

Consider the curve

$$\gamma(s) = \int_{s_0}^{s} \mathrm{T}(s)\cdot ds.$$

Since $|\gamma'| = |\mathrm{T}| = 1$, the curve γ is unit-speed and T, N is its Frenet frame.

Note that

$$\begin{aligned}\gamma''(s) = \mathrm{T}'(s) &= \\ &= (\cos[\theta(s)]', \sin[\theta(s)]') = \\ &= \theta'(s)\cdot(-\sin[\theta(s)], \cos[\theta(s)]) = \\ &= k(s)\cdot\mathrm{N}(s).\end{aligned}$$

So $k(s)$ is the signed curvature of γ at s. This proves the existence.

it remains to prove uniqueness. Assume γ_1 and γ_2 are two curves that satisfy the assumptions of the theorem. Applying a rigid motion, we can assume that $\gamma_1(s_0) = \gamma_2(s_0) = 0$ and the Frenet frame of both curves at s_0 is formed by the coordinate frame $(1,0), (0,1)$. Let us denote by $\mathrm{T}_1, \mathrm{N}_1$ and $\mathrm{T}_2, \mathrm{N}_2$ the Frenet frames of γ_1 and γ_2 respectively. The triples $\gamma_i, \mathrm{T}_i, \mathrm{N}_i$ satisfy the same system of ordinary differential equations

$$\begin{cases}\gamma_i' = \mathrm{T}_i, \\ \mathrm{T}_i' = k\cdot\mathrm{N}_i, \\ \mathrm{N}_i' = -k\cdot\mathrm{T}_i.\end{cases}$$

Moreover, they have the same initial values at s_0. By Theorem 3.1, $\gamma_1 = \gamma_2$. □

Note that from the proof of theorem we obtain the following corollary:

12.3. Corollary. *Let $\gamma\colon \mathbb{I} \to \mathbb{R}^2$ be a smooth unit-speed curve and $s_0 \in \mathbb{I}$. Denote by k the signed curvature of γ. Assume an oriented (x, y)-coordinate system is chosen in such a way that $\gamma(s_0)$ is the origin and $\gamma'(s_0)$ points in the direction of the x-axis. Then*

$$\gamma'(s) = (\cos[\theta(s)], \sin[\theta(s)]),$$

for all s, where

$$\theta(s) = \int_{s_0}^{s} k(t)\cdot dt.$$

Total signed curvature

Let $\gamma\colon\mathbb{I}\to\mathbb{R}^2$ be a smooth unit-speed plane curve. The *total signed curvature* of γ, denoted by $\Psi(\gamma)$, is defined as the integral of its signed curvature;

❸ $$\Psi(\gamma)=\int\limits_{\mathbb{I}}k(s)\cdot ds,$$

where k denotes the signed curvature of γ.

If $\mathbb{I}=[a,b]$, then

❹ $$\Psi(\gamma)=\theta(b)-\theta(a),$$

where θ is as in 12.3.

If γ is a piecewise smooth and regular plane curve, then we define its total signed curvature as the sum of the total signed curvatures of its arcs plus the sum of the signed external angles at its joints; it is positive if γ turns left, negative if γ turns right, 0 if it goes straight. It is undefined if it turns exactly backward; that is, the curve has a cusp. undefined if it turns exactly backward. That is, if γ is a concatenation of smooth and regular arcs $\gamma_1,\dots,\gamma_n$, then

$$\Psi(\gamma)=\Psi(\gamma_1)+\dots+\Psi(\gamma_n)+\theta_1+\dots+\theta_{n-1}$$

where θ_i is the signed external angle at the joint between γ_i and γ_{i+1}. If γ is closed, then the concatenation is cyclic and

$$\Psi(\gamma)=\Psi(\gamma_1)+\dots+\Psi(\gamma_n)+\theta_1+\dots+\theta_n,$$

where θ_n is the signed external angle at the joint between γ_n and γ_1.

Since $\left|\int k(s)\cdot ds\right|\leqslant\int|k(s)|\cdot ds$, we have

❺ $$|\Psi(\gamma)|\leqslant\Phi(\gamma)$$

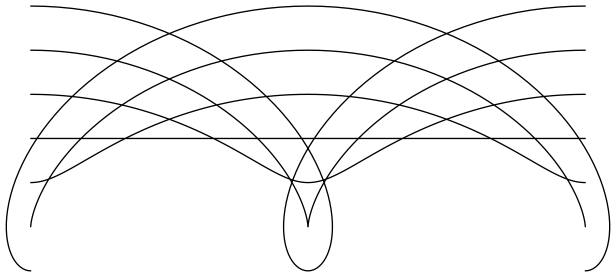
for any smooth regular plane curve γ; that is, total signed curvature can not exceed total curvature by absolute value. Note that the equality holds if and only if the signed curvature does not change the sign.

Trochoid is a curve traced out by a point fixed to a wheel as it rolls along a straight line.

12.4. Exercise. *Consider the family of* trochoids $\gamma_a\colon[0,2\cdot\pi]\to\mathbb{R}^2$ *defined by*

$$\gamma_a(t)=(t+a\cdot\sin t,a\cdot\cos t).$$

(a) Given $a\in\mathbb{R}$, find $\Psi(\gamma_a)$ if it is defined.



Trochoids γ_a for different values a.

(b) Given $a \in \mathbb{R}$, find $\Phi(\gamma_a)$.

12.5. Proposition. *The total signed curvature of any closed simple smooth regular plane curve γ is $\pm 2{\cdot}\pi$; it is $+2{\cdot}\pi$ if the region bounded by γ lies on the left from it and $-2{\cdot}\pi$ otherwise.*

Moreover the same statement holds for any closed piecewise simple smooth regular plane curve γ if its total signed curvature is defined.

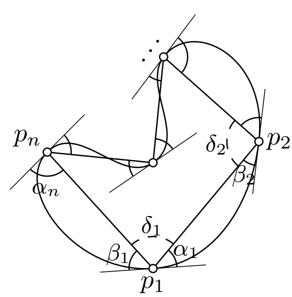
This proposition is called sometimes *Umlaufsatz*; it is a differential-geometric analog of the theorem about the sum of the internal angles of a polygon (6.1) which we use in the proof. A more conceptual proof was given by Heinz Hopf [26], [27, p. 42].

Proof. Without loss of generality we may assume that γ is oriented in such a way that the region bounded by γ lies on the left from it. We can also assume that it is parametrized by arc-length.

Consider a closed polygonal line $\beta = p_1 \dots p_n$ inscribed in γ. We can assume that the arcs between the vertexes are sufficiently small; in this case the polygonal line is simple and each arc γ_i from p_i to p_{i+1} has small total absolute curvature, say $\Phi(\gamma_i) < \pi$ for each i.

As usual we use indexes modulo n, in particular $p_{n+1} = p_1$. Assume $p_i = \gamma(t_i)$. Set

$$\mathrm{w}_i = p_{i+1} - p_i, \quad \mathrm{v}_i = \gamma'(t_i),$$
$$\alpha_i = \measuredangle(\mathrm{v}_i, \mathrm{w}_i), \quad \beta_i = \measuredangle(\mathrm{w}_{i-1}, \mathrm{v}_i),$$



where $\alpha_i, \beta_i \in (-\pi, \pi]$ are signed angles — α_i is positive if w_i points to the left from v_i.

By ❹, the value

❻ $$\Psi(\gamma_i) - \alpha_i - \beta_{i+1}$$

is a multiple of $2{\cdot}\pi$. Since $\Phi(\gamma_i) < \pi$, by the chord lemma (10.11), we also have that $|\alpha_i| + |\beta_i| < \pi$. By ❺, we have that $|\Psi(\gamma_i)| \leqslant \Phi(\gamma_i)$; therefore the value in ❻ vanishes. In other word, the equality

$$\Psi(\gamma_i) = \alpha_i + \beta_{i+1}$$

holds for each i.

Note that

❼ $$\delta_i = \pi - \alpha_i - \beta_i$$

is the internal angle of β at p_i; $\delta_i \in (0, 2{\cdot}\pi)$ for each i. Recall that the sum of the internal angles of an n-gon is $(n-2){\cdot}\pi$ (see 6.1); that is,

$$\delta_1 + \dots + \delta_n = (n-2){\cdot}\pi.$$

Therefore

❽ $$\begin{aligned}\Psi(\gamma) &= \Psi(\gamma_1) + \dots + \Psi(\gamma_n) = \\ &= (\alpha_1 + \beta_2) + \dots + (\alpha_n + \beta_1) = \\ &= (\beta_1 + \alpha_1) + \dots + (\beta_n + \alpha_n) = \\ &= (\pi - \delta_1) + \dots + (\pi - \delta_n) = \\ &= n{\cdot}\pi - (n-2){\cdot}\pi = \\ &= 2{\cdot}\pi.\end{aligned}$$

The case of piecewise smooth and regular curves is done the same way; we need to subdivide the arcs in the cyclic concatenation further to meet the requirement above and instead of equation ❼ we have

$$\delta_i = \pi - \alpha_i - \beta_i - \theta_i,$$

where θ_i is the signed external angle of γ at p_i; it vanishes if the curve γ is smooth at p_i. Therefore instead of equation ❽, we have

$$\begin{aligned}\Psi(\gamma) &= \Psi(\gamma_1) + \dots + \Psi(\gamma_n) + \theta_1 + \dots + \theta_n = \\ &= (\alpha_1 + \beta_2) + \dots + (\alpha_n + \beta_1) = \\ &= (\beta_1 + \alpha_1 + \theta_1) + \dots + (\beta_n + \alpha_n + \theta_n) = \\ &= (\pi - \delta_1) + \dots + (\pi - \delta_n) = \\ &= n{\cdot}\pi - (n-2){\cdot}\pi = \\ &= 2{\cdot}\pi.\end{aligned}$$

□

12.6. Exercise. *Draw a smooth regular closed plane curve γ such that*

(a) $\Psi(\gamma) = 0$;
(b) $\Psi(\gamma) = \Phi(\gamma) = 10\cdot\pi$;
(c) $\Psi(\gamma) = 2\cdot\pi$ and $\Phi(\gamma) = 4\cdot\pi$.

12.7. Exercise. *Let $\gamma\colon [a,b] \to \mathbb{R}$ be a smooth regular plane curve with Frenet frame T,N. Given a real parameter ℓ, consider the curve $\gamma_\ell(t) = \gamma(t) + \ell\cdot\mathrm{N}(t)$; it is called a* parallel curve of γ at signed distance ℓ.

(a) *Show that γ_ℓ is a regular curve if $\ell\cdot k(t) \neq 1$ for all t, where $k(t)$ denotes the signed curvature of γ.*

(b) *Set $L(\ell) = \operatorname{length}\gamma_\ell$. Show that*

$$L(\ell) = L(0) - \ell\cdot\Psi(\gamma) \tag{❾}$$

for all ℓ sufficiently close to 0.

(c) *Describe an example showing that formula ❾ does not hold for all ℓ.*

Osculating circline

12.8. Proposition. *Given a point p, a unit vector T and a real number k, there is a unique smooth unit-speed curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ that starts at p in the direction of T and has constant signed curvature k.*

Moreover, if $k = 0$, then $\sigma(s) = p + s\cdot\mathrm{T}$ which runs along a line; if $k \neq 0$, then σ runs around the circle of radius $\frac{1}{|k|}$ and center $p + \frac{1}{k}\cdot\mathrm{N}$, where T,N is an oriented orthonoral frame.

Further we will use the term *circline* for *a circle or a line*; these are the only plane curves with constant signed curvature.

Proof. The proof is done by a calculation based on 12.2 and 12.3.

Suppose $s_0 = 0$, choose a coordinate system such that p is its origin and T points in the direction of the x-axis. Therefore N points in the direction of the y-axis. Then

$$\begin{aligned}\theta(s) &= \int\limits_0^s k\cdot dt = \\ &= k\cdot s.\end{aligned}$$

Therefore

$$\sigma'(s) = (\cos[k\cdot s], \sin[k\cdot s]).$$

It remains to integrate the last identity. If $k = 0$, we get

$$\sigma(s) = (s, 0)$$

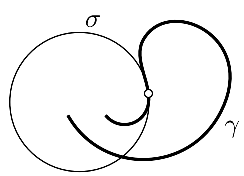
which describes the line $\sigma(s) = p + s\cdot\mathrm{T}$.

If $k \neq 0$, we get

$$\sigma(s) = (\tfrac{1}{k}\cdot\sin[k\cdot s], \tfrac{1}{k}\cdot(1 - \cos[k\cdot s])).$$

which is the circle of radius $r = \frac{1}{|k|}$ centered at $(0, \frac{1}{k}) = p + \frac{1}{k}\cdot\mathrm{N}$. $\square$

12.9. Definition. *Let γ be a smooth unit-speed plane curve; denote by $k(s)$ the signed curvature of γ at s.*



The unit-speed curve σ of constant signed curvature $k(s)$ that starts at $\gamma(s)$ in the direction $\gamma'(s)$ is called the osculating circline *of γ at s.*

The center and radius of the osculating circle at a given point are called center of curvature *and* radius of curvature *of the curve at that point.*

The osculating circle σ_s can be also defined as the unique circline that has *second order of contact* with γ at s; that is, $\rho(\ell) = o(\ell^2)$, where $\rho(\ell)$ denotes the distance from $\gamma(s+\ell)$ to σ_s.

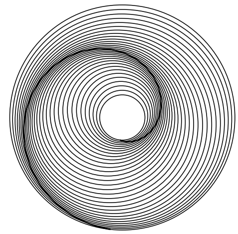
The following exercise would is recommended to the reader familiar with the notion of inversion.

12.10. Advanced exercise. *Suppose γ is a smooth regular plane curve that does not pass thru the origin. Let $\hat\gamma$ be the inversion of γ in the unit circle centered at the origin. Show that osculating circline of $\hat\gamma$ at s is the inversion of osculating circline of γ at s.*

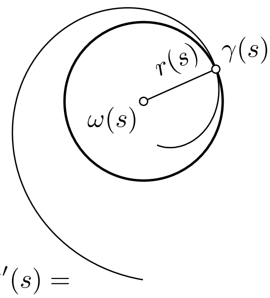
Spiral lemma

The following lemma was proved by Peter Tait [49] and later rediscovered by Adolf Kneser [28].

12.11. Lemma. *Assume that γ is a smooth regular plane curve with strictly decreasing positive signed curvature. Then the osculating circles of γ are nested; that is, if σ_s denoted the osculating circle of γ at s, then σ_{s_0} lies in the open disc bounded by σ_{s_1} for any $s_0 < s_1$.*



It turns out that the osculating circles of the curve γ give a peculiar foliation of an annulus by circles; it has the following property: if a smooth function is constant on each osculating circle it must be constant in the annulus [see 22, Lecture 10]. Also note that the curve γ is tangent to a circle of the foliation at each of its points. However, it does not run along any of those circles.



Proof. Let $\mathrm{T}(s), \mathrm{N}(s)$ be the Frenet frame, $\omega(s)$, $r(s)$ the center and radius of curvature of γ. By 12.8,

$$\omega(s) = \gamma(s) + r(s)\cdot\mathrm{N}(s).$$

Since $k > 0$, we have that $r(s)\cdot k(s) = 1$. Therefore applying Frenet formula ❷, we get that

$$\begin{aligned}\omega'(s) &= \gamma'(s) + r'(s)\cdot\mathrm{N}(s) + r(s)\cdot\mathrm{N}'(s) = \\ &= \mathrm{T}(s) + r'(s)\cdot\mathrm{N}(s) - r(s)\cdot k(s)\cdot\mathrm{T}(s) = \\ &= r'(s)\cdot\mathrm{N}(s).\end{aligned}$$

Since $k(s)$ is decreasing, $r(s)$ is increasing; therefore $r' \geqslant 0$. It follows that $|\omega'(s)| = r'(s)$ and $\omega'(s)$ points in the direction of $\mathrm{N}(s)$.

Since $\mathrm{N}'(s) = -k(s)\cdot\mathrm{T}(s)$, the direction of $\omega'(s)$ cannot have constant direction on a nontrivial interval; that is, the curve $s \mapsto \omega(s)$ contains no line segments. Therefore

$$\begin{aligned}|\omega(s_1) - \omega(s_0)| &< \mathrm{length}(\omega|_{[s_0,s_1]}) = \\ &= \int_{s_0}^{s_1} |\omega'(s)|\cdot ds = \\ &= \int_{s_0}^{s_1} r'(s)\cdot ds = \\ &= r(s_1) - r(s_0).\end{aligned}$$

In other words, the distance between the centers of σ_{s_1} and σ_{s_0} is strictly less than the difference between their radii. Therefore the osculating circle at s_0 lies inside the osculating circle at s_1 without touching it. □

The curve $s \mapsto \omega(s)$ is called the *evolute* of γ; it traces the centers of curvature of the curve. The evolute of γ can be written as

$$\omega(t) = \gamma(t) + \tfrac{1}{k(t)}\cdot\mathrm{N}(t)$$

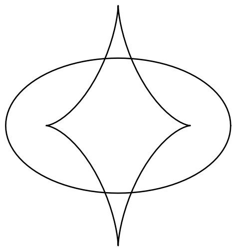
and in the proof we showed that $(\frac{1}{k})'\cdot\mathrm{N}$ is its velocity vector.

12.12. Exercise. *Show that the stretched astroid*

$$\omega(t) = (\tfrac{a^2-b^2}{a}\cdot\cos^3 t, \tfrac{b^2-a^2}{b}\cdot\sin^3 t)$$

is an evolute of the ellipse defined by

$$\gamma(t) = (a\cdot\cos t, b\cdot\sin t).$$



The following theorem states formally that *if you drive on the plane and turn the steering wheel to the left all the time, then you will not be able to come back to the same place.*

12.13. Theorem. *Assume γ is a smooth regular plane curve with positive and strictly monotonic signed curvature. Then γ is simple.*

The same statement holds without assuming positivity of curvature; the proof requires only minor modifications.

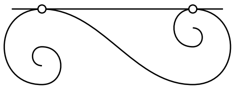
Proof of 12.13. Note that $\gamma(s)$ lies on the osculating circle σ_s of γ at s. If $s_1\neq s_0$, then by lemma 12.11, σ_{s_0} does not intersect σ_{s_1}. Therefore $\gamma(s_1)\neq\gamma(s_0)$, hence the result. □

12.14. Exercise. *Show that a 3-dimensional analog of the theorem does not hold. That is, there are self-intersecting smooth regular space curves with strictly monotonic curvature.*

12.15. Exercise. *Assume that γ is a smooth regular plane curve with positive strictly monotonic signed curvature.*

(a) Show that no line can be tangent to γ at two distinct points.

(b) Show that no circle can be tangent to γ at three distinct points.



Note that part (12.15a) does not hold if we allow the curvature to be negative; an example is shown on the diagram.

Chapter 13

Supporting curves

Cooriented tangent curves

Suppose γ_1 and γ_2 are smooth regular plane curves. Recall that the curves γ_1 and γ_2 are tangent at the time parameters t_1 and t_2 if $\gamma_1(t_1) = \gamma_2(t_2)$ and they share the tangent line at these time parameters.

In this case the point $p = \gamma_1(t_1) = \gamma_2(t_2)$ is called a *point of tangency* of the curves. If both curves are simple, then we may say that γ_1 and γ_2 are tangent at the point p without ambiguity.

Note that if γ_1 and γ_2 are tangent at the time parameters t_1 and t_2, then the velocity vectors $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ are parallel. If $\gamma_1'(t_1)$

and $\gamma_2'(t_2)$ point in the same direction we say that the curves are *cooriented*, if these directions are opposite, we say that the curves are *controriented*.

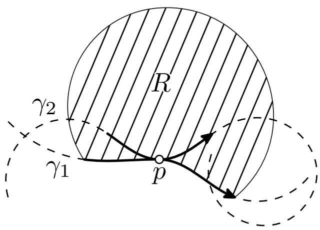
Note that reverting the parametrization of one of the curves, cooriented curves become counteroriented and vice versa; so we can always assume that the curves are cooriented at a given point of tangency.

Supporting curves

Let γ_1 and γ_2 be two smooth regular plane curves that share a point

$$p = \gamma_1(t_1) = \gamma_2(t_2)$$

which is not an endpoint of any of the curves. Suppose that there is $\varepsilon > 0$ such that the arc $\gamma_2|_{[t_2-\varepsilon,t_2+\varepsilon]}$ lies in a closed plane region R with the arc $\gamma_1|_{[t_1-\varepsilon,t_1+\varepsilon]}$ in its boundary, then we say that γ_1 *locally supports* γ_2 at the time parameters t_1 and t_2. If both curves are simple, then we also could say that γ_1 *locally supports* γ_2 at the point p without ambiguity.



If γ_1 is simple proper curve, so it divides the plane into two closed region that lie on left and right from γ_1, then we say that γ_1 *globally supports* γ_2 at t_2 if γ_2 runs in one of these closed regions and $\gamma_2(t_2)$ lies on γ.

If γ_2 is a closed curve so it divides the plane into two regions a bounded inside and unbounded outside. In this case we say if γ_1 supports γ_2 *from inside* (*from outside*) if γ_1 supports γ_2 and in the region inside it (respectively outside it).

Note that if γ_1 and γ_2 share a point $p = \gamma_1(t_1) = \gamma_2(t_2)$ and not tangent at t_1 and t_2, then γ_2 crosses γ_1 at t_2 moving from one of its sides to the other. It follows that γ_1 can not locally support γ_2 at the time parameters t_1 and t_2. Whence we get the following:

13.1. Definition-Observation. *Let γ_1 and γ_2 be two smooth regular plane curves. Suppose γ_1 locally supports γ_2 at time parameters t_1 and t_2. Then γ_1 is tangent to γ_2 at t_1 and t_2.*

In particular, we could say if γ_1 and γ_2 are coorineted or contrоriented at the time parameters t_1 and t_2. If the curves are cooriented and the region R in the definition of supporting curves lie on the right (left) from the arc of γ_1, then we say that γ_1 supports γ_2 from the left (respectively right).

If the curves on the diagram oriented according the arrows, then γ_1 supports γ_2 from the right at p (as well as γ_2 supports γ_1 from the left at p).

We say that a smooth regular plane curve γ has a *vertex* at s if the signed curvature function is critical at s; that is, if $k'(s)_\gamma = 0$. If γ is simple we could say that the point $p = \gamma(s)$ is a vertex of γ without ambiguity.

13.2. Exercise. *Assume that osculating circle σ_s of a smooth regular simple plane curve γ locally supports γ at $p = \gamma(s)$. Show that p is a vertex of γ.*

Supporting test

The following proposition resembles the second derivative test.

13.3. Proposition. *Let γ_1 and γ_2 be two smooth regular plane curves.*

Suppose γ_1 locally supports γ_2 from the left (right) at the time parameters t_1 and t_2. Then

$$k_1(t_1) \leqslant k_2(t_2) \quad (\textit{respectively} \quad k_1(t_1) \geqslant k_2(t_2)).$$

where k_1 and k_2 denote the signed curvature of γ_1 and γ_2 respectively.

A partial converse also holds. Namely, if γ_1 and γ_2 tangent and cooriented at the time parameters t_1 and t_2 then γ_1 locally supports γ_2 from the left (right) at the time parameters t_1 and t_2 if

$$k_1(t_1) < k_2(t_2) \quad (\textit{respectively} \quad k_1(t_1) > k_2(t_2)).$$

Proof. Without loss of generality, we can assume that $t_1 = t_2 = 0$, the shared point $\gamma_1(0) = \gamma_2(0)$ is the origin and the velocity vectors $\gamma_1'(0)$, $\gamma_2'(0)$ point in the direction of x-axis.

Note that small arcs of $\gamma_1|_{[-\varepsilon,+\varepsilon]}$ and $\gamma_2|_{[-\varepsilon,+\varepsilon]}$ can be described as a graph $y = f_1(x)$ and $y = f_2(x)$ for smooth functions f_1 and f_2 such that $f_i(0) = 0$ and $f_i'(0) = 0$. Note that $f_1''(0) = k_1(0)$ and $f_2''(0) = k_2(0)$ (see 10.4)

Clearly, γ_1 supports γ_2 from the left (right) if

$$f_1(x) \leqslant f_2(x) \quad (\text{respectively} \quad f_1(x) \geqslant f_2(x))$$

for all sufficiently small values x. Applying the second derivative test, we get the result. □

13.4. Advanced exercise. *Let γ_0 and γ_1 be two smooth unit-speed simple plane curves that are tangent and cooriented at the point $p = \gamma_0(0) = \gamma_1(0)$. Assume $k_0(s) \leqslant k_1(s)$ for any s. Show that γ_0 locally supports γ_1 from the right at p.*

Give an example of two proper curves γ_0 and γ_1 satisfying the above condition such that γ_0 does not globally support γ_1 at p.

Note that according to the DNA inequality (10.23) for any closed smooth regular curve that runs in a unit disc, the average of its absolute curvature at least 1; in particular it has a point with absolute curvature at lest 1. The following exercise says that the last statement holds for loops.

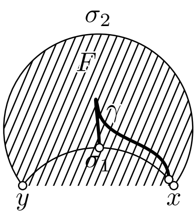
13.5. Exercise. *Assume a closed smooth regular plane loop γ runs in a unit disc. Show that there is a point on γ with absolute curvature at least 1.*

13.6. Exercise. *Assume a closed smooth regular plane curve γ runs between parallel lines on distance 2 from each other. Show that there is a point on γ with absolute curvature at least 1.*

Try to prove the same for a smooth regular plane loop.

13.7. Exercise. *Assume a closed smooth regular plane curve γ runs inside of a triangle $\triangle$ with inradius 1; that is, the inscribed circle of $\triangle$ has radius 1. Show that there is a point on γ with absolute curvature at least* 1.

The exercise above is a baby case of a 13.15.



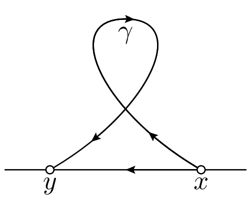
13.8. Exercise. *Let F be a plane figure bounded by two circle arcs σ_1 and σ_2 of signed curvature 1 that run from x to y. Suppose σ_1 is a shorter than σ_2. Assume a simple arc γ runs in F and has the end points on σ_1. Show that the absolute curvature of γ is at least 1 at some parameter value.*

Convex curves

Recall that a plane curve is convex if it bounds a convex region.

13.9. Proposition. *Suppose that a closed simple plane curve γ bounds a figure F. Then F is convex if and only if the signed curvature of γ does not change sign.*

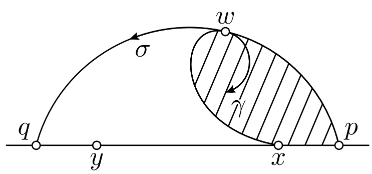
13.10. Lens lemma. *Let γ be a smooth regular simple plane curve that runs from x to y. Assume that γ runs on the right side (left side) of the oriented line xy and only its endpoints x and y lie on the line. Then γ has a point with positive (respectively negative) signed curvature.*



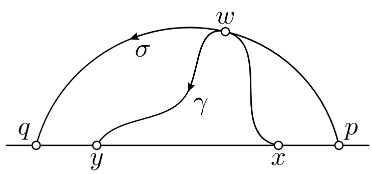
Note that the lemma fails for curves with self-intersections; the curve γ on the diagram always turns right, so it has negative curvature everywhere, but it lies on the right side of the line xy.

Proof. Choose points p and q on the line xy so that the points p, x, y, q appear in that order. We can assume that p and q lie sufficiently far from x and y, so that the half-disc with diameter pq contains γ.

Consider the smallest disc segment with chord $[pq]$ that contains γ. Note that its arc σ supports γ at some point $w = \gamma(t_0)$.



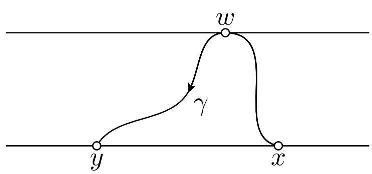
Let us parameterise σ from p to q. Note that the γ and σ are tangent and cooriented at w. If not, then the arc of γ from w to y would be trapped in the curvelinear triangle xwp bounded by arcs of σ, γ and the line segment $[px]$. But this is impossible since y does not belong to this triangle.



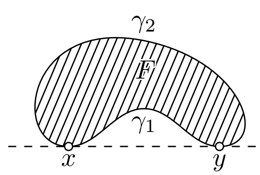
It follows that σ supports γ at t_0 from the right. By 13.3,

$$k(t_0)_\gamma \geqslant k_\sigma > 0.$$

. □



Remark. Instead of taking the minimal disc segment, one can take a point w on γ that maximizes the distance to the line xy. The same argument shows that the curvature at w is nonnegative, which is slightly weaker than the required positive curvature.



Proof of 13.9. If F is convex, then every tangent line of γ supports γ. If a point moves along γ, the figure F has to stay on one side from its tangent line; that is, we can assume that each tangent line supports γ on one side, say on the right. Since line has vanishing curvature, the supporting test (13.3) implies that $k \geqslant 0$ at each point.

Denote by K the convex hull of F. If F is not convex, then F is a proper subset of K. Therefore ∂K contains a line segment that is not a part of ∂F. In other words, there is a line that supports γ at two points, say x and y that divide γ in two arcs γ_1 and γ_2, both distinct from the line segment $[x, y]$.

Note the one of the arcs γ_1 or γ_2 is parametrized from x to y and the other from y to x. Passing to a smaller arc if necessary we can ensure that only its endpoints lie on the line. Applying the lens lemma, we get that the arcs γ_1 and γ_2 contain points with curvatures of opposite signs.

That is, if F is not convex, then curvature of γ changes sign. Equivalently: if curvature of γ does not change sign, then F is convex. □

13.11. Exercise. *Suppose γ is a smooth regular simple closed convex plane curve of diameter larger than 2. Show that γ has a point with absolute curvature less than 1.*

13.12. Exercise. *Suppose γ is a simple smooth regular curve in the plane with positive curvature. Assume γ crosses a line ℓ at the points $p_1, p_2, \dots p_n$ and these points appear on γ in the same order.*

(a) Show that p_2 cannot lie between p_1 and p_3 on ℓ.

(b) Show that if p_3 lies between p_1 and p_2 on ℓ, then the points appear on ℓ in the following order:

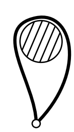
$$p_1, p_3, \dots, p_4, p_2.$$

(c) Describe all possible orders of p_i on ℓ.

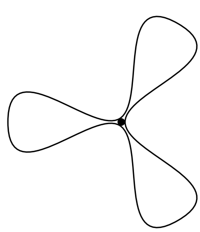
Moon in a puddle

The following theorem is a slight generalization of the theorem proved by Vladimir Ionin and German Pestov in [42]. For convex curves, this result was known earlier [8, §24].

13.13. Theorem. *Assume γ is a simple closed smooth regular plane loop with absolute curvature bounded by 1. Then it surrounds a unit disc.*



This theorem gives a simple but nontrivial example of the so-called *local to global theorems* — based on some local data (in this case the curvature of a curve) we conclude a global property (in this case existence of a large disc surrounded by the curve).



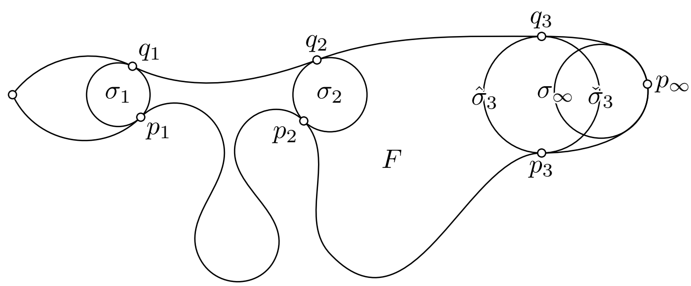
A straightforward approach would be to start with some disc in the region bounded by the curve and blow it up to maximize its radius. However, as one may see from the diagram it does not always lead to a solution — a closed plane curve of curvature at most 1 may surround a disc of radius smaller than 1 that cannot be enlarged continuously.

13.14. Key lemma. *Assume γ is a simple closed smooth regular plane loop. Then at one point of γ (distinct from its base) its osculating circle σ globally support γ from the inside.*

First let us show that the theorem follows from the lemma.

Proof of 13.13 modulo 13.14. Since γ has absolute curvature at most 1, each osculating circle has radius at least 1. According to the key lemma one of the osculating circles σ globally support γ from inside. In particular σ lies inside of γ, whence the result. □

Proof of 13.14. Denote by F the closed region surrounded by γ. We need to show that one osculating circle lies completely in F. Assume contrary; that is, the osculating circle at each point $p \in \gamma$ does not lie in F.

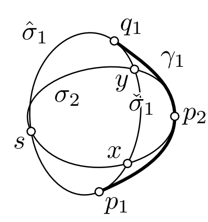


Given a point $p \in \gamma$ let us consider the maximal circle σ that lies completely in F and tangent to γ at p. Note that σ has to touch γ at another point; otherwise if we increase its radius r slightly the resulting circle will still stay in F. The circle σ will be called the *incircle* of F at p.

Fix a point p_1 and let σ_1 be the incircle at p_1. Denote by γ_1 an arc of γ from p_1 to a first point q_1 on σ_1. Denote by $\hat\sigma_1$ and $\check\sigma_1$ two arcs of σ_1 from p_1 to q_1 such that the cyclic concatenation of $\hat\sigma_1$ and γ_1 surrounds $\check\sigma_1$.

Let p_2 be the midpoint of γ_1 and σ_2 be the incircle at p_2.

Note that σ_2 cannot intersect $\hat\sigma_1$. Otherwise, if σ_2 intersects $\hat\sigma_1$ at some point s, then σ_2 has to have two more common points with $\check\sigma_1$, say x and y — one for each arc of σ_2 from p_2 to s. Therefore $\sigma_1 = \sigma_2$ as two circles with three common points: s, x, and y. On the other hand, by construction, we have that $p_2 \in \sigma_2$ and $p_2 \notin \sigma_1$ — a contradiction.



Two ovals on the diagram pretend to be circles.

Recall that σ_2 has to touch γ at another point. From above it follows that it can only touch γ_1 and therefore we can choose an arc $\gamma_2 \subset \gamma_1$ that runs from p_2 to a first point q_2 on σ_2. Note that by construction we have that

❶ $$\operatorname{length}\gamma_2 < \tfrac{1}{2}\cdot \operatorname{length}\gamma_1.$$

Repeating this construction recursively, we get an infinite sequence of arcs $\gamma_1 \supset \gamma_2 \supset \ldots$; by ❶, we also get that

$$\operatorname{length}\gamma_n \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore the intersection

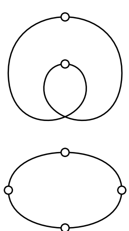
$$\bigcap_n \gamma_n$$

contains a single point; denote it by p_∞.

Let σ_∞ be the incircle at p_∞; it has to touch γ at another point, say q_∞. The same argument as above shows that $q_\infty \in \gamma_n$ for any n. It follows that $q_\infty = p_\infty$ — a contradiction. $\square$

13.15. Exercise. *Assume that a closed smooth regular curve γ lies in a figure F bounded by a closed simple plane curve. Suppose that R is the maximal radius of discs that lies in F. Show that absolute curvature of γ is at least $\frac{1}{R}$ at some parameter value.*

Four-vertex theorem



Recall that a vertex of a smooth regular curve is defined as a critical point of its signed curvature; in particular, any local minimum (or maximum) of the signed curvature is a vertex. For example, every point of a circle is its vertex.

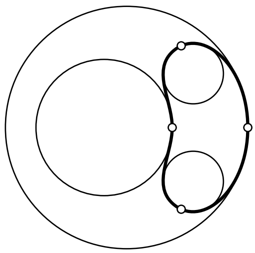
13.16. Four-vertex theorem. *Any smooth regular simple plane curve has at least four vertices.*

Evidently any closed smooth regular curve has at least two vertexes — where the minimum and the maximum of the curvature are attained. On the diagram the vertexes are marked; the first curve has one self-intersection and exactly two vertexes; the second curve has exactly four vertexes and no self-intersections.

The four-vertex theorem was first proved by Syamadas Mukhopadhyaya [38] for convex curves. It has a number of different proofs and generalizations. One of my favorite proofs was given by Robert Osserman [40]. We give another proof based on the key lemma in the previous section. It proves the following stronger statement.

13.17. Theorem. *Any smooth regular simple plane curve has is globally supported by its osculating circle at least at 4 distinct points; two from inside and two from outside.*

Proof of 13.16 modulo 13.17. First note that if an osculating circline σ at a point p supports γ locally, then p is a vertex. Indeed, if p is not a vertex, then a small arc around p has monotonic curvature. Applying the spiral lemma (12.11) we get that the osculating circles at this arc are nested. In particular the curve γ crosses σ at p and therefore σ is does not locally support γ at p. □



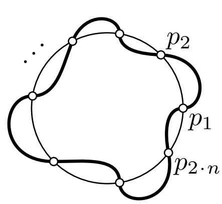
Proof of 13.17. According to key lemma (13.14), there is a point $p \in \gamma$ such that its osculating circle supports γ from inside. The curve γ can be considered as a loop with the base at p. Therefore the key lemma implies existence of another point $q \in \gamma$ with the same property.

It shows the existence of two osculating circles that support γ from inside; it remains to show existence of two osculating circles that support γ from outside.

In order to get the osculating circles supporting γ from outside, one can repeat the proof of key lemma taking instead of incircle the circline of maximal signed curvature that supports the curve from outside, assuming that γ is oriented so that the region on the left from it is bounded.

(Alternatively, if one applies to γ an inverse with the center inside γ, then the obtained curve γ_1 also has two osculating circles that support γ_1 from inside. According to 12.10, these osculating circlines are inverses of the circlines osculating to γ. Note that the rigion lying inside of γ is mapped to the region outside of γ_1 and the other way around. Therefore these two circlines correspond to the osculating circlines supporting γ from outside.) □

13.18. Advanced exercise. *Suppose γ is a closed simple smooth regular plane curve and σ is a circle. Assume γ crosses σ at the points $p_1, \dots, p_{2\cdot n}$ and these points appear in the same cycle order on γ and on σ. Show that γ has at least $2\cdot n$ vertexes.*



Construct an example of a closed simple smooth regular plane curve γ with only 4 vertexes that crosses a given circle at arbitrarily many points.

Part III

Surfaces

Chapter 14

Definitions

Topological surfaces

We will be mostly interested in smooth regular surfaces defined in the following section. However few times we will use the following general definition.

A connected subset Σ in the Euclidean space $\mathbb{R}^3$ is called a *topological surface* (more precisely an *embedded surface without boundary*) if any point of $p \in \Sigma$ admits a neighborhood W in Σ that can be parameterized by an open subset in the Euclidean plane; that is, there is an injective continuous map $U \to W$ from an open set $U \subset \mathbb{R}^2$ such that its inverse $W \to U$ is also continuous.

Smooth surfaces

Recall that a function f of two variables x and y is called *smooth* if all its partial derivatives $\frac{\partial^{m+n}}{\partial x^m \partial y^n} f$ are defined and are continuous in the domain of definition of f.

A connected set $\Sigma \subset \mathbb{R}^3$ is called a *smooth surface* (or more precisely *smooth regular embedded surface*) if it can be described locally as a graph of a smooth function in an appropriate coordinate system.

More precisely, for any point $p \in \Sigma$ one can choose a coordinate system (x, y, z) and a neighborhood $U \ni p$ such that the intersection $W = U \cap \Sigma$ is formed by a graph $z = f(x, y)$ of a smooth function f defined in an open domain of the (x, y)-plane.

Examples. The simplest example of a smooth surface is the (x, y)-plane

$$\Pi = \left\{ (x, y, z) \in \mathbb{R}^3 \, : \, z = 0 \right\}.$$

The plane Π is a surface since it can be described as the graph of the function $f(x, y) = 0$.

All other planes are smooth surfaces as well since one can choose a coordinate system so that it becomes the (x, y)-plane. We may also present a plane as a graph of a linear function $f(x, y) = a{\cdot}x + b{\cdot}y + c$ for some constants a, b and c (assuming the plane is not perpendicular to the (x, y)-plane).

A more interesting example is the unit sphere

$$\mathbb{S}^2 = \left\{\, (x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 = 1 \,\right\}.$$

This set is not the graph of any function, but $\mathbb{S}^2$ is locally a graph; it can be covered by the following 6 graphs:

$$\begin{aligned} z = f_\pm(x, y) &= \pm\sqrt{1 - x^2 - y^2}, \\ y = g_\pm(x, z) &= \pm\sqrt{1 - x^2 - z^2}, \\ x = h_\pm(y, z) &= \pm\sqrt{1 - y^2 - z^2}, \end{aligned}$$

where each function $f_\pm, g_\pm, h_\pm$ is defined in an open unit disc. Any point $p \in \mathbb{S}^2$ lies in one of these graphs therefore $\mathbb{S}^2$ is a surface. Since each function is smooth, so is $\mathbb{S}^2$.

Surfaces with boundary

A connected subset in a surface that is bounded by one or more curves is called *surface with boundary*; the curves form the *boundary line* of the surface.

When we say *surface* we usually mean a *smooth regular surface without boundary*; we may use the terms *surface without boundary* if we need to emphasise it; otherwise we may use the term *surface with possibly nonempty boundary.*

Proper, closed and open surfaces

If the surface Σ is formed by a closed set, then it is called *proper*. For example, for any smooth function f, defined on whole plane, its graph $z = f(x, y)$ is a proper surface. The sphere $\mathbb{S}^2$ gives another example of proper surface.

On the other hand, the open disc

$$\left\{\, (x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 < 1,\; z = 0 \,\right\}$$

is not proper; this set is neither open nor closed.

A compact surface without boundary is called *closed* (this term is closely related to *closed curve* but has nothing to do with *closed set*).

A proper noncompact surface without boundary is called *open* (again the term *open set* is not relevant).

For example, the paraboloid $z = x^2 + y^2$ is an open surface; sphere $\mathbb{S}^2$ is a closed surface.

Note that any proper surface without boundary is either closed or open.

The following claim is a two-dimesional analog of 8.8; hopefully it is intuitively obvious. Its proof is not at all trivial; a standard proof uses the so-called *Alexander's duality* which is a classical technique in algebraic topology. We omit its proof since it would take us far away from the main subject.

14.1. Claim. *The complement of any proper topological surface without boundary (or, equivalently any open or closed topological surface) has exactly two connected components.*

Implicitly defined surfaces

14.2. Proposition. *Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose that 0 is a regular value of f; that is, $\nabla_p f \neq 0$ if $f(p) = 0$. Then any connected component Σ of the set of solutions of the equation $f(x, y, z) = 0$ is a surface.*

Proof. Fix $p \in \Sigma$. Since $\nabla_p f \neq 0$ we have

$$\tfrac{\partial f}{\partial x}(p) \neq 0, \quad \tfrac{\partial f}{\partial y}(p) \neq 0, \quad \text{or} \quad \tfrac{\partial f}{\partial z}(p) \neq 0.$$

We may assume $\frac{\partial f}{\partial z}(p) \neq 0$; otherwise permute the coordinates x, y, z.

The implicit function theorem (2.3) implies that a neighborhood of p in Σ is the graph $z = h(x, y)$ of a smooth function h defined on an open domain in $\mathbb{R}^2$. It remains to apply the definition of smooth surface (page 88). □

14.3. Exercise. *Describe the set of real numbers ℓ such that the equation*

$$x^2 + y^2 - z^2 = \ell$$

describes a smooth regular surface.

Local parametrizations

Let U be an open domain in $\mathbb{R}^2$ and $s\colon U \to \mathbb{R}^3$ be a smooth map. We say that s is regular if its Jacobian has maximal rank; in this case it means that the vectors $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$ are linearly independent at any $(u,v) \in U$; equivalently $\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v} \neq 0$, where $\times$ denotes the vector product.

14.4. Proposition. *If $s\colon U \to \mathbb{R}^3$ is a smooth regular embedding of an open connected set $U \subset \mathbb{R}^2$, then its image $\Sigma = s(U)$ is a smooth surface.*

Proof. Set

$$s(u,v) = (x_s(u,v), y_s(u,v), z_s(u,v)).$$

Since s is regular, its Jacobian matrix

$$\operatorname{Jac} s = \begin{pmatrix} \frac{\partial x_s}{\partial u} & \frac{\partial x_s}{\partial v} \\ \frac{\partial y_s}{\partial u} & \frac{\partial y_s}{\partial v} \\ \frac{\partial z_s}{\partial u} & \frac{\partial z_s}{\partial v} \end{pmatrix}$$

has rank two at any pint $(u,v) \in U$.

Fix a point $p \in \Sigma$; by shifting the coordinate system we may assume that p is the origin. Permuting the coordinates x, y, z if necessary, we may assume that the matrix

$$\begin{pmatrix} \frac{\partial x_s}{\partial u} & \frac{\partial x_s}{\partial v} \\ \frac{\partial y_s}{\partial u} & \frac{\partial y_s}{\partial v} \end{pmatrix},$$

is invertible. Note that this is the Jacobian matix of the map

$$(u,v) \mapsto (x_s(u,v), y_s(u,v)).$$

The inverse function theorem implies that there is a smooth regular function h defined on an open set $W \ni 0$ in the (x,y)-plane such that

$$(x_s \circ h)(x,y) = x \quad \text{and} \quad (y_s \circ h)(x,y) = y$$

for any $(x,y) \in W$. It follows that the graph $z = z_s \circ h(x,y)$ for $(x,y) \in W$ is a subset in Σ. Clearly this graph is open in Σ. Since p is arbitrary, we get that Σ is a surface. $\square$

If we have s and Σ as in the proposition, then we say that s is a *smooth parametrization* of the surface Σ.

Not all the smooth surfaces can be described by such a parametrization; for example the sphere $\mathbb{S}^2$ cannot. But any smooth surface Σ admits a local parametrization; that is, any point $p \in \Sigma$ admits an open neighborhood $W \subset \Sigma$ with a smooth regular parametrization s. In this case any point in W can be described by two parameters, usually denoted by u and v, which are called *local coordinates* at p. The map s is called a *chart* of Σ.

If W is a graph $z = h(x, y)$ of a smooth function h, then the map

$$s\colon (u, v) \mapsto (u, v, h(u, v))$$

is a chart. Indeed, s has an inverse $(u, v, h(u, v)) \mapsto (u, v)$ which is continuous; that is, s is an embedding. Further, $\frac{\partial s}{\partial u} = (1, 0, \frac{\partial h}{\partial u})$ and $\frac{\partial s}{\partial v} = (0, 1, \frac{\partial h}{\partial v})$. Whence $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$ are linearly independent; that is, s is a regular map.

Note that from 14.4, we obtain the following corollary.

14.5. Corollary. *A connected set $\Sigma \subset \mathbb{R}^3$ is a smooth regular surface if and only if Σ has a local parametrization by a smooth regular map at any point $p \in \Sigma$.*

14.6. Exercise. *Consider the following map*

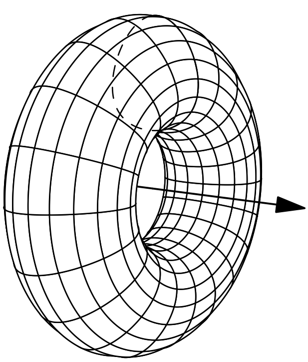
$$s(u, v) = \left(\tfrac{2\cdot u}{1+u^2+v^2}, \tfrac{2\cdot v}{1+u^2+v^2}, \tfrac{2}{1+u^2+v^2}\right).$$

Show that s is a chart of the unit sphere centered at $(0, 0, 1)$; describe the image of s.

The map s in the exercise can be visualized using the following map

$$(u, v, 1) \mapsto \left(\tfrac{2\cdot u}{1+u^2+v^2}, \tfrac{2\cdot v}{1+u^2+v^2}, \tfrac{2}{1+u^2+v^2}\right)$$

which is called *stereographic projection* from the plane $z = 1$ to the unit sphere with center at $(0, 0, 1)$ Note that the point $(u, v, 1)$ and its image $\left(\tfrac{2\cdot u}{1+u^2+v^2}, \tfrac{2\cdot v}{1+u^2+v^2}, \tfrac{2}{1+u^2+v^2}\right)$ lie on one half-line starting at the origin.



Let $\gamma(t) = (x(t), y(t))$ be a plane curve. Recall that the *surface of revolution* of the curve γ around the x-axis can be described as the image of the map

$$(t, \theta) \mapsto (x(t), y(t)\cdot \cos\theta, y(t)\cdot \sin\theta).$$

For fixed t or θ the obtained curves are called *meridian* or respectively *parallel* of the surface; note that parallels are formed by circles in the plane perpendicular to the axis of rotation.

14.7. Exercise. *Assume γ is a closed simple smooth regular plane curve that does not intersect the x-axis. Show that surface of revolution of γ around the x-axis is a smooth regular surface.*

Global parametrizations

A surface can be described by an embedding from a known surface to the space.

For example, consider the ellipsoid

$$\Sigma_{a,b,c} = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ \tfrac{x^2}{a^2} + \tfrac{y^2}{b^2} + \tfrac{z^2}{c^2} = 1 \right\}$$

for some positive numbers a, b, and c. Note that by 14.2, $\Sigma_{a,b,c}$ is a smooth regular surface. Indeed, set $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$, then

$$\nabla f(x, y, z) = (\tfrac{2}{a^2}\cdot x, \tfrac{2}{b^2}\cdot y, \tfrac{2}{c^2}\cdot z).$$

Therefore $\nabla f \neq 0$ if $f = 1$; that is, 1 is a regular value of f.

Note that $\Sigma_{a,b,c}$ can be defined as the image of the map $s\colon \mathbb{S}^2 \to \mathbb{R}^3$, defined as the restriction of the following map to the unit sphere $\mathbb{S}^2$:

$$(x, y, z) \mapsto (a\cdot x, b\cdot y, c\cdot z).$$

For a surface Σ, a map $s\colon \Sigma \to \mathbb{R}^3$ is called a *smooth parametrized surface* if for any chart $f\colon U \to \Sigma$ the composition $s \circ f$ is smooth and regular; that is, all partial derivatives $\frac{\partial^{m+n}}{\partial u^m \partial v^n}(s \circ f)$ exist and are continuous in the domain of definition and the following two vectors $\frac{\partial}{\partial u}(s \circ f)$ and $\frac{\partial}{\partial v}(s \circ f)$ are linearly independent.

If in addition the map $s\colon \Sigma \to \mathbb{R}^3$ is an embedding, then the image $\Sigma' = s(\Sigma)$ is a smooth surface. The later follows since for any chart $f\colon U \to \Sigma$ the composition $s \circ f\colon U \to \Sigma'$ is a chart of Σ'. In this case the map s is called *diffeomorphism* from Σ to Σ'; the surfaces Σ to Σ' are called *diffeomorphic* is there is a diffeomorphism $s\colon \Sigma \to \Sigma'$.

14.8. Advanced exercise. *Show that the surfaces Σ and Θ are diffeomorphic if*

(a) Σ and Θ obtained from the plane by removing n points.
(b) Σ and Θ are open convex subset of a plane bounded by a smooth curves.
(c) Σ and Θ are open convex subset of a plane.
(d) Σ and Θ are open star-shaped subset of a plane.

Evidently the parametric definition includes the embedded surfaces defined previously — as the domain of parameters we can take the surface itself and the identity map as s. But parametrized surfaces are more general, in particular they might have self-intersections.

If Σ is a known surface for example a sphere or a plane, the parameterized surface $s\colon \Sigma \to \mathbb{R}^3$ might be called by the same name. For example, any embedding $s\colon \mathbb{S}^2 \to \mathbb{R}^3$ might be called a *topological sphere* and if s is smooth and regular, then it might be called *smooth sphere*. (A smooth regular map $s\colon \mathbb{S}^2 \to \mathbb{R}^3$ which is not necessary an embedding is called a *smooth immersion*, so we can say that s describes a *smooth immersed sphere*.) Similarly an embedding $s\colon \mathbb{R}^2 \to \mathbb{R}^3$ might be called *topological plane*, and if s is smooth, it might be called *smooth plane*.

Chapter 15

First order structure

Tangent plane

15.1. Definition. *Let Σ be a smooth surface. A vector* w *is a* tangent vector *of Σ at p if and only if there is a curve γ that runs in Σ and has* w *as a velocity vector at p; that is, $p = \gamma(t)$ and $\mathrm{w} = \gamma'(t)$ for some t.*

15.2. Proposition-Definition. *Let Σ be a smooth surface and $p \in \Sigma$. Then the set of tangent vectors of Σ at p forms a plane; this plane is called* tangent plane *of Σ at p.*

Moreover if $s\colon U \to \Sigma$ is a local chart and $p = s(u_p, v_p)$, then the tangent plane of Σ at p is spanned by vectors $\frac{\partial s}{\partial u}(u_p, v_p)$ and $\frac{\partial s}{\partial v}(u_p, v_p)$.

The tangent plane to Σ at p is usually denoted by T_p or $\mathrm{T}_p\Sigma$. Tangent plane T_p might be considered as a linear subspace of $\mathbb{R}^3$ or as a parallel plane passing thru p. In the latter case it can be interpreted as the best approximation at p of the surface Σ by a plane; it has *first order of contact* with Σ at p; that is, $\rho(q) = o(|p - q|)$, where $q \in \Sigma$ and $\rho(q)$ denotes the distance from q to T_p.

Proof. Fix a chart s at p. Assume γ is a smooth curve that starts at p. Without loss of generality, we can assume that γ is covered by the chart; in particular, there are smooth functions $u(t)$ and $v(t)$ such that

$$\gamma(t) = s(u(t), v(t)).$$

Applying chain rule, we get

$$\gamma' = \tfrac{\partial s}{\partial u}\cdot u' + \tfrac{\partial s}{\partial v}\cdot v';$$

that is, γ' is a linear combination of $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$.

Since the smooth functions $u(t)$ and $v(t)$ can be chosen arbitrary, any linear combination of $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$ is a tangent vector at the corresponding point. □

15.3. Exercise. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function with a regular value 0 and Σ be a surface described as a connected component of the set of solutions $f(x, y, z) = 0$. Show that the tangent plane $\mathrm{T}_p\Sigma$ is perpendicular to the gradient $\nabla_p f$ at any point $p \in \Sigma$.*

15.4. Exercise. *Let Σ be a smooth surface and $p \in \Sigma$. Fix an (x, y, z)-coordinates. Show that a neighborhood of p in Σ is a graph $z = f(x, y)$ of a smooth function f defined on an open subset in the (x, y)-plane if and only if the tangent plane T_p is not* vertical*; that is, if T_p is not perpendicular to the (x, y)-plane.*

Directional derivative

In this section we extend the definition of directional derivative to smooth functions defined on smooth surfaces.

First let recall the standard definition of directional derivative.

Suppose f is a function defined at a point p in the space, and w is a vector. Then directional derivative $D_{\mathrm{w}}f(p)$ is defined as the derivative at zero $h'(0)$ of the function

$$h(t) = f(p + t{\cdot}\mathrm{w}).$$

15.5. Proposition-Definition. *Let Σ be a smooth regular surface and f is a smooth function defined on Σ. Suppose γ is a smooth curve in Σ that starts at p with the velocity vector $\mathrm{w} \in \mathrm{T}_p$; that is, $\gamma(0) = p$ and $\gamma'(0) = \mathrm{w}$. Then the derivative $(f \circ \gamma)'(0)$ depends only on f, p and w; it is called* directional derivative of f along w at p *and denoted by*

$$D_{\mathrm{w}}f, \quad (D_{\mathrm{w}})f(p), \quad \text{or} \quad (D_{\mathrm{w}})f(p)_\Sigma$$

— we may omit p and Σ if it is clear from the context.

Moreover, if $(u, v) \mapsto s(u, v)$ is a local chart at $p = s(u_p, v_p)$, and $\mathrm{w} = a{\cdot}\frac{\partial s}{\partial u}(u_p, v_p) + b{\cdot}\frac{\partial s}{\partial v}(u_p, v_p)$, then

$$D_{\mathrm{w}}f(p) = a{\cdot}\tfrac{\partial f \circ s}{\partial u}(u_p, v_p) + b{\cdot}\tfrac{\partial f \circ s}{\partial v}(u_p, v_p).$$

Note that our definition agrees with standard definition of directional derivative if Σ is a plane. Indeed, in this case $\gamma(t) = p + \mathrm{w}{\cdot}t$ is a curve in Σ that starts at p with the velocity vector w. For a general surface the point $p+\mathrm{w}\cdot t$ might not lie on the surface; therefore the function f might be undefined at this point; therefore the standard definition does not work.

Proof. Without loss of generality, we may assume that γ is covered by the chart s; if not we can chop γ. In this case

$$\gamma(t) = s(u(t), v(t))$$

for some smooth functions u, v defined in a neighborhood of 0 such that $u(0) = u_p$ and $v(0) = v_p$.

Applying the chain rule, we get that

$$\gamma'(0) = u'(0)\cdot\tfrac{\partial s}{\partial u}(u_p, v_p) + v'(0)\cdot\tfrac{\partial s}{\partial v}(u_p, v_p).$$

Since $\mathrm{w} = \gamma'(0)$ and the vectors $\frac{\partial s}{\partial u}$, $\frac{\partial s}{\partial v}$ are linearly independent, we get that $a = u'(0)$ and $b = v'(0)$.

Applying the chain rule again, we get that

❶ $$(f\circ\gamma)'(0) = a\cdot\tfrac{\partial f\circ s}{\partial u}(u_p, v_p) + b\cdot\tfrac{\partial f\circ s}{\partial v}(u_p, v_p).$$

Notice that the left hand side in ❶ does not depend on the choice of the chart s and the right hand side depends only on p, w, f, and s. It follows that $(f\circ\gamma)'(0)$ depends only on p, w and f.

The last statement follows from ❶. □

Linearization

Any smooth map θ from a surface Σ to $\mathbb{R}^3$ can be described by its coordinate functions $\theta(p) = (\theta_x(p), \theta_y(p), \theta_z(p))$. To take a directional derivative of the map we should take the directional derivative of each of its coordinate function.

$$D_{\mathrm{w}}\theta := (D_{\mathrm{w}}\theta_x, D_{\mathrm{w}}\theta_y, D_{\mathrm{w}}\theta_z).$$

Assume θ is a smooth map from one smooth surface Σ_0 to another Σ_1 and $p\in\Sigma_0$. Note that $D_{\mathrm{w}}\theta(p)\in \mathrm{T}_{\theta(p)}\Sigma_1$ for any $\mathrm{w}\in\mathrm{T}_p$. Indeed choose a curve γ_0 in Σ_0 such that $\gamma_0(0) = p$ and $\gamma_0'(0) = \mathrm{w}$. Observe that $\gamma_1 = \theta\circ\gamma_0$ is a smooth curve in Σ_1 and by the definition directional derivative, we have $D_{\mathrm{w}}\theta(p) = \gamma_1'(0)$. It remains to note that $\gamma_1(0) = \theta(p)$ and therefore its velocity $\gamma_1'(0)$ is in $\mathrm{T}_{\theta(p)}\Sigma_1$.

Recall that 15.5 implies that $L_p\theta\colon \mathrm{w}\mapsto D_{\mathrm{w}}\theta$ defines a linear map $L_p\theta\colon \mathrm{T}_p\Sigma_0\to\mathrm{T}_{\theta(p)}\Sigma_1$; that is,

$$D_{c\cdot\mathrm{w}}\theta = c\cdot D_{\mathrm{w}}\theta(p) \quad\text{and}\quad D_{\mathrm{v}+\mathrm{w}}\theta = D_{\mathrm{v}}\theta(p) + D_{\mathrm{w}}\theta(p)$$

for any $c \in \mathbb{R}$ and $\mathrm{v}, \mathrm{w} \in \mathrm{T}_p$. Th map $L_p\theta$ is called a *linearization* (or *differential*) of θ at p.

The linear map $L_p\theta$ can be written as a $2{\times}2$-matrix M in orthonormal basises of T_p and $\mathrm{T}_{\theta(p)}\Sigma_1$. Set $\operatorname{jac}_p \theta = |\det M|$; this value does not depend on the choice orthonormal basises in T_p and $\mathrm{T}_{\theta(p)}\Sigma_1$.

Let $\theta_1 \colon \Sigma_1 \to \Sigma_2$ be another smooth map between smooth surfaces Σ_1 and Σ_2. Suppose that $p_1 = \theta(p) \in \Sigma_1$; observe that

$$L_p(\theta_1 \circ \theta) = L_{p_1}\theta_1 \circ L_p\theta.$$

It follows that

❷ $$\operatorname{jac}_p(\theta_1 \circ \theta) = \operatorname{jac}_{p_1} \theta_1 \cdot \operatorname{jac}_p \theta.$$

If Σ_0 is a domain in the (u, v)-plane, then the value $\operatorname{jac}_p \theta$ can be found using the following formulas

$$\begin{aligned}\operatorname{jac} s &= \left|\tfrac{\partial s}{\partial v} \times \tfrac{\partial s}{\partial u}\right| = \\ &= \sqrt{\langle\tfrac{\partial s}{\partial u}, \tfrac{\partial s}{\partial u}\rangle\cdot\langle\tfrac{\partial s}{\partial v}, \tfrac{\partial s}{\partial v}\rangle - \langle\tfrac{\partial s}{\partial u}, \tfrac{\partial s}{\partial v}\rangle^2} = \\ &= \sqrt{\det[\operatorname{Jac}^\top s\cdot \operatorname{Jac} s]}.\end{aligned}$$

where $\operatorname{Jac} s$ denotes the Jacobean matrix of s; it is a $2{\times}3$ matrix with column vectors $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$.

The value $\operatorname{jac}_p \theta$ has the following geometric meaning: if P_0 is a region in T_p and $P_1 = (L_p\theta)(P_0)$, then

$$\operatorname{area} P_1 = \operatorname{jac}_p \theta\cdot \operatorname{area} P_0.$$

This identity will become important in the definition of surface area.

Surface integral and area

Let Σ be a smooth surface and $h \colon \Sigma \to \mathbb{R}$ be a smooth function. Let us define the integral $\int_R h$ of the function h along a region $R \subset \Sigma$. (This definition can be applied to any Borel set of $R \subset \Sigma$.)

Assume that there is a chart $(u, v) \mapsto s(u, v)$ of Σ defined on an open set $U \subset \mathbb{R}^2$ such that $R \subset s(U)$. In this case set

❸ $$\int\limits_R h := \iint\limits_{s^{-1}(R)} h \circ s(u, v)\cdot \operatorname{jac}_{(u,v)} s\cdot du\cdot dv.$$

($\operatorname{jac}_{(u,v)} s$ is defined in previous section.)

By the substitution rule (2.4), the right hand side in ❸ does not depend on the choice of s. That is, if $s_1\colon U_1\to\Sigma$ is another chart such that $s_1(U_1)\supset R$, then

$$\iint\limits_{s^{-1}(R)} h\circ s(u,v)\cdot \operatorname{jac}_{(u,v)} s\cdot du\cdot dv = \iint\limits_{s_1^{-1}(R)} h\circ s_1(u,v)\cdot \operatorname{jac}_{(u,v)} s_1\cdot du\cdot dv.$$

In other words, the defining identity ❸ makes sense.

A general region R can be subdivided into regions $R_1, R_2\dots$ such that each R_i lies in the image of some chart. After that one could define the integral along R as the sum

$$\int\limits_R h := \int\limits_{R_1} h + \int\limits_{R_2} h + \dots$$

It is straightforward to check that the value $\int_R h$ does not depend on the choice of subdivision.

The area of a region R in a smooth surface Σ is defined as the surface integral

$$\operatorname{area} R = \int\limits_R 1.$$

The following proposition provides a substitution rule for surface integral.

15.6. Proposition. *Suppose $\theta\colon \Sigma_0\to\Sigma_1$ is a smooth parameterization of a smooth surface Σ_1 by a smooth surface Σ_0. Then for any region $R\subset\Sigma_0$ and any smooth function $f\colon\Sigma_1\to\mathbb{R}$ we have*

$$\int\limits_R (f\circ\theta)\cdot \operatorname{jac}\theta = \int\limits_{\theta(R)} f.$$

In particular, if $f\equiv 1$, we have

$$\int\limits_R \operatorname{jac}\theta = \operatorname{area}[\theta(R)].$$

Proof. Follows from ❷ and the definiton of surface integral. □

Remark. The notion of area of surface is closely related to length of curve. However, to define length we uses a different idea — it was defined as the least upper bound on the lengths of inscribed polygonal lines. It turns out that analogous definition does not work even for very simple surfaces. Section 7 describes the so called *Schwarz's boot* — a classical example of that type.

Tangent vectors as functionals*

In this section we introduce a more conceptual way to define tangent vectors. We will not use this approach in the sequel, but it is better to know about it.

A tangent vector $\mathrm{w} \in \mathrm{T}_p$ to a smooth surface Σ defines a linear functional[1] D_{w} that takes a smooth function φ on Σ and spits the directional derivative $D_{\mathrm{w}}\varphi$. It is straightforward to check that the functional D obeys the product rule:

❹ $$D_{\mathrm{w}}(\varphi{\cdot}\psi) = (D_{\mathrm{w}}\varphi){\cdot}\psi(p) + \varphi(p){\cdot}(D_{\mathrm{w}}\psi).$$

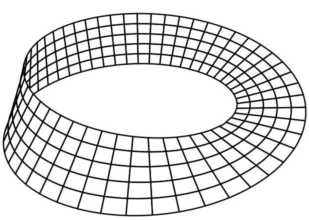
It turns out that the tangent vector w is completely determined by the functional D_{w}. Moreover tangent vectors at p can be defined as linear functionals on the space of smooth functions that satisfy the product rule ❹.

This new definition is less intuitive, but it is more convenient to use since it grabs the key algebraic property of tangent vectors. (It is an art to make a right definition.) Many statements admit simpler proofs with this approach, for example linearity of the map $\mathrm{w} \mapsto D_{\mathrm{w}}f$ becomes a tautology.

Normal vector and orientation

A unit vector that is normal to T_p is usually denoted by ν_p; it is uniquely defined up to sign.

A surface Σ is called *oriented* if it is equipped with a unit normal vector field ν; that is, a continuous map $p \mapsto \nu_p$ such that $\nu_p \perp \mathrm{T}_p$ and $|\nu_p| = 1$ for any p. The choice of the field ν is called the *orientation* on Σ. A surface Σ is called *orientable* if it can be oriented. Note that each orientable surface admits two orientations ν and $-\nu$.



Möbius strip shown on the diagram gives an example of a nonorientable surface — there is no choice of normal vector field that is continuous along the middle of the strip (it changes the sign if you try to go around).

Note that each surface is locally orientable. In fact each chart $f(u, v)$ admits

[1] Term *functional* is used for functions that take a function as an argument and return a number.

an orientation

$$\nu = \frac{\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}}{\left|\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}\right|}.$$

Indeed the vectors $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are tangent vectors at p; since they are linearly independent, their vector product does not vanish and it is perpendicular to the tangent plane. Evidently $(u, v) \mapsto \nu(u, v)$ is a continuous map. Therefore ν is a unit normal field.

15.7. Exercise. *Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function with a regular value 0 and Σ is a surface described as a connected component of the set of solutions $h(x, y, z) = 0$. Show that Σ is orientable.*

Recall that any proper surface without boundary in the Euclidean space divides it into two connected components (14.1). Therefore we can choose the unit normal field on any smooth proper surfaces that points into one of the components of the complement. Therefore we obtain the following observation.

15.8. Observation. *Any smooth open or closed surface in Euclidean space is oriented.*

In particular it follows that the Möbius strip cannot be extended to an open or closed smooth surface without boundary.

Spherical map

Let Σ be a smooth oriented surface with unit normal field ν. The map $\nu \colon \Sigma \to \mathbb{S}^2$ defined by $p \mapsto \nu_p$ is called *spherical map* or *Gauss map.*

For surfaces, the spherical map plays essentially the same role as the tangent indicatrix for curves.

Sections

15.9. Advanced exercise. *Show that any closed set in a plane can appear as an intersection of this plane with an open smooth regular surface.*

The exercise above says that plane sections of a smooth regular surface might look complicated. The following lemma makes it possible to perturb the plane so that the section becomes nice.

15.10. Lemma. *Let Σ be a smooth regular surface. Suppose $f \colon \mathbb{R}^3 \to$ $\to \mathbb{R}$ is a smooth function. Then for any constant r_0 there is an arbitrarily close value r such that each connected component of the* *intersection of the level set* $L_r = f^{-1}\{r\}$ *with* Σ *is a smooth regular curve.*

Proof. The surface Σ can be covered by a countable set of charts $s_i\colon U_i \to \Sigma$. Note that the composition $f \circ s_i$ is a smooth function for any i. By Sard's lemma (2.1), almost all real numbers r are regular values of each $f \circ s_i$.

Fix such a value r sufficiently close to r_0 and consider the level set L_r described by the equation $f(x, y, z) = r$. Any point in the intersection $\Sigma \cap L_r$ lies in the image of one of the charts. From above it admits a neighborhood which is a regular smooth curve; hence the result. □

15.11. Corollary. *Let* Σ *be a smooth surface. Then for any plane* Π *there is a parallel plane* Π' *that lies arbitrary close to* Π *and such that the intersection* $\Sigma \cap \Pi$ *is a union of disjoint smooth curves.*

Chapter 16

Curvatures

Tangent-normal coordinates

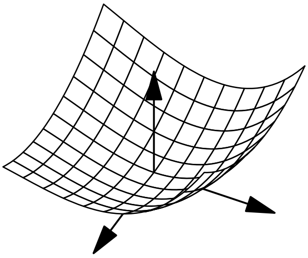
Fix a point p in a smooth oriented surface Σ. Consider a coordinate system (x, y, z) with origin at p such that the (x, y)-plane coincides with T_p and the z-axis in the direction of the normal vector ν_p. By 15.4, we can present Σ locally around p as a graph of a smooth function f. Note that f satisfies the following additional properties:

$$f(0,0) = 0, \qquad (\tfrac{\partial}{\partial x} f)(0,0) = 0, \qquad (\tfrac{\partial}{\partial y} f)(0,0) = 0.$$

The first equality holds since $p = (0, 0, 0)$ lies on the graph and the last two equalities mean that the tangent plane at p is horizontal.

Set

$$\begin{aligned} \ell &= (\tfrac{\partial^2}{\partial x^2} f)(0,0), \\ m &= (\tfrac{\partial^2}{\partial x \partial y} f)(0,0) = (\tfrac{\partial^2}{\partial y \partial x} f)(0,0), \\ n &= (\tfrac{\partial^2}{\partial y^2} f)(0,0). \end{aligned}$$



The Taylor series for f at $(0, 0)$ up to the second oreder term can be then written as

$$f(x, y) = \tfrac{1}{2}(\ell{\cdot}x^2 + 2{\cdot}m{\cdot}x{\cdot}y + n{\cdot}y^2) + o(x^2 + y^2).$$

Note that values ℓ, m, and n are completely determined by this equation. The so-called *osculating paraboloid*

$$z = \tfrac{1}{2}{\cdot}(\ell{\cdot}x^2 + 2{\cdot}m{\cdot}x{\cdot}y + n{\cdot}y^2)$$

gives the best approximation of the surface at p; it has *second order of contact* with Σ at p.

Note that

$$\ell\cdot x^2 + 2\cdot m\cdot x\cdot y + n\cdot y^2 = \langle M_p\cdot\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right), \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\rangle,$$

where

❶ $$M_p = \begin{pmatrix}\ell & m\\ m & n\end{pmatrix};$$

it is called the Hessian matrix of f at $(0,0)$.

Principle curvatures

Note that tangent-normal coordinates give an almost canonical coordinate system in a neighborhood of p; it is unique up to a rotation of the (x,y)-plane. Rotating the (x,y)-plane results in the rewriting the matrix M_p in the new basis.

Since the Hessian matrix M_p is symmetric, it is diagonalizable by orthogonal matrices. That is, by rotating the (x,y)-plane we can assume that $m=0$ in ❶. In this case

$$M_p = \begin{pmatrix}k_1 & 0\\ 0 & k_2\end{pmatrix},$$

the diagonal components k_1 and k_2 of M_p are called *principle curvatures* of Σ at p; they might also be denoted as $k_1(p)$ and $k_2(p)$, or $k_1(p)_\Sigma$ and $k_2(p)_\Sigma$; if we need to emphasise that we calculate it at the point p for the surface Σ. We will always assume that $k_1\leqslant k_2$.

Note that if $x=f(x,y)$ is a local graph representation of Σ in these coordinates, then

$$f(x,y) = \tfrac12\cdot(k_1\cdot x^2 + k_2\cdot y^2) + o(x^2+y^2).$$

The principle curvatures can be also defined as the eigenvalues of the Hessian matrix M_p. The eigendirections of M_p are called *principle directions* of Σ at p. Note that if $k_1(p)\neq k_2(p)$, then p has exactly two principle directions, which are perpendicular to each other; if $k_1(p) = k_2(p)$ then all tangent directions at p are principle.

Note that if we revert the orientation of Σ, then the principle curvatures at each point switch their signs and indexes.

A smooth regular curve on a surface Σ that always runs in the principle directions is called a *line of curvature* of Σ.

16.1. Exercise. *Assume that a smooth surface Σ is mirror symmetric with respect to a plane Π. Suppose that Σ and Π intersect along a curve γ. Show that γ is a line of curvature of Σ.*

More curvatures

Fix an oriented smooth surface Σ and a point $p\in\Sigma$.

The product
$$K=k_1(p)\cdot k_2(p)$$
is called Gauss curvature at p. We may denote it by $K(p)$ or $K(p)_\Sigma$ if we need to specify the point p and the surface Σ.

Since the product of principle values equals to determinant, the Gauss curvature equals to the determinant of the Hessian matrix $M_p = \left(\begin{smallmatrix}\ell & m\\ n & n\end{smallmatrix}\right)$; that is,
$$K=\ell\cdot n-m^2.$$

16.2. Exercise. *Show that any surface with positive Gauss curvature is orientable.*

The sum
$$H(p)=k_1(p)+k_2(p)$$
is called *mean curvature*[1] at p. We may also denote it by $H(p)_\Sigma$. The mean curvature can be also interpreted as the trace of the Hessian matrix M_p.

Note that reverting orientaion of Σ does not change Gauss curvatre and changes sign of mean curvature. In particular, the Gauss curvature is well defined for nonoriented surface Σ and p.

A surface with vanishing mean curvature is called *minimal*; the reason for such a name will be given in Proposition 17.11.

Shape operator

Let p be a point on a smooth oriented surface Σ. Suppose $z=f(x,y)$ is a local description of Σ in a tangent-normal coordinates at p and
$$M_p=\begin{pmatrix}\ell & m\\ m & n\end{pmatrix};$$
is the Hessian matrix of f at $(0,0)$; the components ℓ, m, and n are defined on page 103.

The multiplication by the Hessian matrix defies the so called *shape operator*
$$S\colon \left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)\mapsto M_p\cdot\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right);$$

[1]Some authors define mean curvature as $\frac12\cdot(k_1(p)+k_2(p))$ — the mean value of the principle curvatures. It is suits the name better, but not as convenient when it comes to calculations.

it is a linear operator $S\colon \mathrm{T}_p \to \mathrm{T}_p$. For a point $p \in \Sigma$ the shape operator of a tangent vector $\mathrm{w} \in \mathrm{T}_p$ will be denoted by $S(\mathrm{w})$ if it is clear from the context which base point p and which surface we are working with; otherwise we may use notations

$$S_p(\mathrm{w}) \quad \text{or even} \quad S_p(\mathrm{w})_\Sigma.$$

Since M_p is symmetric, S is *self-adjoint*; that is

$$\langle S(\mathrm{v}), \mathrm{w}\rangle = \langle \mathrm{v}, S(\mathrm{w})\rangle$$

for any $\mathrm{v}, \mathrm{w} \in \mathrm{T}_p$. Note also that principle curvatures of Σ at p are the eigenvalues of S_p and the principle directions are the directions of principle vectors of S_p.

The shape operator S_p is defined using the Hessian matrix M_p, that depends on the choice of basis in T_p; the following proposition implies in particular that S_p does not depend on the choice of basis.

16.3. Proposition. *Let p be a point on a smooth oriented surface Σ. Suppose Σ is described locally as a graph $z = f(x, y)$ in a tangent-normal coordinates at p. Then*

$$\langle S(\mathrm{v}), \mathrm{w}\rangle = D_\mathrm{w} D_\mathrm{v} f(0, 0)$$

for any $\mathrm{v}, \mathrm{w} \in \mathrm{T}_p$. Moreover S is unique linear operator $\mathrm{T}_p \to \mathrm{T}_p$ that satisfies the above condition.

Here $D_\mathrm{v} f$ denoted directional derivative of f along vector v; that is, if $\varphi(t) = f(q + \mathrm{v}\cdot t)$, then $D_\mathrm{v} f(q) = \varphi'(0)$.

Proof. Suppose $\mathrm{v} = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ and $\mathrm{v} = \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)$, then

$$D_\mathrm{v} = a\cdot \tfrac{\partial}{\partial x} + b\cdot \tfrac{\partial}{\partial y}, \qquad\qquad D_\mathrm{w} = c\cdot \tfrac{\partial}{\partial x} + d\cdot \tfrac{\partial}{\partial y}.$$

Therefore

$$D_\mathrm{w} D_\mathrm{v} f = a\cdot c\cdot \tfrac{\partial^2 f}{\partial^2 x} + b\cdot c\cdot \tfrac{\partial^2 f}{\partial x \partial y} + a\cdot d\cdot \tfrac{\partial^2 f}{\partial y \partial x} + b\cdot d\cdot \tfrac{\partial^2 f}{\partial^2 y}$$

evaluating this expression at $(0, 0)$ we get

$$\begin{aligned} D_\mathrm{w} D_\mathrm{v} f(0,0) &= a\cdot c\cdot \ell + b\cdot c\cdot m + a\cdot d\cdot m + b\cdot d\cdot n = \\ &= \langle M_p\cdot \mathrm{v}, \mathrm{w}\rangle = \langle \mathrm{v}, M_p\cdot \mathrm{w}\rangle = \\ &= \langle S(\mathrm{v}), \mathrm{w}\rangle = \langle \mathrm{v}, S(\mathrm{w})\rangle. \end{aligned}$$

□

16.4. Corollary. *Let p be a point on a smooth oriented surface Σ. Suppose Σ is described locally as a graph $z = f(x, y)$ in a tangent-normal coordinates at p. Denote by I, J and K the standard basis in the (x, y, z)-coordinates. Then*

$$\begin{aligned} &\langle S(\mathrm{I}), \mathrm{I}\rangle = \ell, && \langle S(\mathrm{I}), \mathrm{J}\rangle = m, && \langle S(\mathrm{I}), \mathrm{K}\rangle = 0, \\ &\langle S(\mathrm{J}), \mathrm{I}\rangle = m, && \langle S(\mathrm{J}), \mathrm{J}\rangle = n, && \langle S(\mathrm{J}), \mathrm{K}\rangle = 0, \end{aligned}$$

where ℓ, m, and n are the components of the Hessian matrix of f at $(0, 0)$ defined on page 103.

Proof. Note that $D_{\mathrm{I}} = \frac{\partial}{\partial x}$ and $D_{\mathrm{J}} = \frac{\partial}{\partial y}$. It remains to use 16.3 and the expressions for ℓ, m, and n (see page 103). □

In the following proposition we use the notion of directional derivative defined in 15.5.

16.5. Proposition. *Let Σ be a smooth surface with unit normal field ν. Suppose $p \in \Sigma$ and $S\colon \mathrm{T}_p \to \mathrm{T}_p$ is the shape operator at p. Then*

❷ $$S(\mathrm{W}) = -D_{\mathrm{W}}\nu$$

for any $\mathrm{W} \in \mathrm{T}_p$. Equivalently

❸ $$S = -L_p\nu,$$

where $L_p\nu$ denotes linearization of the spherical map $\nu\colon \Sigma \to \mathbb{S}^2$ at p.

The reason for minus sign in ❷ and ❸is the same as in the formula for curvature of plane curve in its Frenet frame: $\mathrm{N}' = -k\cdot\mathrm{T}$. The proof is done by straightforward calculations.

Proof of 16.5. Let Σ be a smooth surface with unit normal field ν. Suppose $(u, v) \mapsto s(u, v)$ is a local chart of Σ at p. Since ν is unit we have the identity

$$1 = \langle \nu\circ s, \nu\circ s\rangle.$$

Note that the vectors $\frac{\partial s}{\partial u}(u, v)$ and $\frac{\partial s}{\partial v}(u, v)$ are tangent to Σ at $s(u, v)$. Since $\nu_p \perp \mathrm{T}_p$ for any $p \in \Sigma$, we have two more identities:

$$0 = \langle \tfrac{\partial}{\partial u} s, \nu\circ s\rangle, \qquad\qquad 0 = \langle \tfrac{\partial}{\partial u} s, \nu\circ s\rangle\rangle.$$

Taking partial derivatives of these there identities and applying the product rule, we get the following six identities:

$$
\begin{aligned}
0 &= \tfrac{\partial}{\partial u}\langle \nu\circ s, \nu\circ s\rangle = 2\cdot\langle \tfrac{\partial}{\partial u}\nu\circ s, \nu\circ s\rangle,\\
0 &= \tfrac{\partial}{\partial v}\langle \nu\circ s, \nu\circ s\rangle = 2\cdot\langle \tfrac{\partial}{\partial v}\nu\circ s, \nu\circ s\rangle,\\
0 &= \tfrac{\partial}{\partial u}\langle \tfrac{\partial}{\partial u}s, \nu\circ s\rangle = \langle \tfrac{\partial^2}{\partial u^2}s, \nu\circ s\rangle + \langle \tfrac{\partial}{\partial u}s, \tfrac{\partial}{\partial u}\nu\circ s\rangle,\\
0 &= \tfrac{\partial}{\partial v}\langle \tfrac{\partial}{\partial u}s, \nu\circ s\rangle = \langle \tfrac{\partial^2}{\partial v\partial u}s, \nu\circ s\rangle + \langle \tfrac{\partial}{\partial u}s, \tfrac{\partial}{\partial v}\nu\circ s\rangle\\
0 &= \tfrac{\partial}{\partial u}\langle \tfrac{\partial}{\partial v}s, \nu\circ s\rangle = \langle \tfrac{\partial^2}{\partial u\partial v}s, \nu\circ s\rangle + \langle \tfrac{\partial}{\partial v}s, \tfrac{\partial}{\partial u}\nu\circ s\rangle,\\
0 &= \tfrac{\partial}{\partial v}\langle \tfrac{\partial}{\partial v}s, \nu\circ s\rangle = \langle \tfrac{\partial^2}{\partial v^2}s, \nu\circ s\rangle + \langle \tfrac{\partial}{\partial v}s, \tfrac{\partial}{\partial v}\nu\circ s\rangle.
\end{aligned}
$$

Now suppose $z = f(x,y)$ be a local description of Σ in the tangent-normal coordinates at p. Note that

$$s(u,v) = (u,v,f(u,v))$$

describes a chart of Σ at p.

Denote by I, J and K the standard basis in the (x,y,z)-coordinates. Note that $s(0,0) = p$ and

$$\tfrac{\partial}{\partial u}s(0,0) = \mathrm{I}, \qquad \tfrac{\partial}{\partial v}s(0,0) = \mathrm{J}, \qquad \nu\circ s(0,0) = \mathrm{K},$$

In particular $D_{\mathrm{I}}\nu = \tfrac{\partial}{\partial u}\nu\circ s(0,0)$ and $D_{\mathrm{J}}\nu = \tfrac{\partial}{\partial v}\nu\circ s(0,0)$. Further,

$$\tfrac{\partial^2}{\partial u^2}s(0,0) = \ell\cdot\mathrm{K}, \qquad \tfrac{\partial^2}{\partial v\partial u}s(0,0) = m\cdot\mathrm{K}, \qquad \tfrac{\partial^2}{\partial v^2}s(0,0) = n\cdot\mathrm{K},$$

where ℓ, m, and n are the components of the Hessian matrix of f at $(0,0)$ defined on page 103.

Evaluating the above 6 identities at $(u,v) = (0,0)$, we get that

$$
\begin{aligned}
&\langle -D_{\mathrm{I}}\nu, \mathrm{I}\rangle = \ell, && \langle -D_{\mathrm{I}}, \mathrm{J}\rangle = m, && \langle -D_{\mathrm{I}}\nu, \mathrm{K}\rangle = 0,\\
&\langle -D_{\mathrm{J}}\nu, \mathrm{I}\rangle = m, && \langle -D_{\mathrm{J}}\nu, \mathrm{J}\rangle = n, && \langle -D_{\mathrm{J}}\nu, \mathrm{K}\rangle = 0,
\end{aligned}
$$

That is, $-D_{\mathrm{I}}\nu$ and $-D_{\mathrm{J}}\nu$ satisfy the same equalities as $S(\mathrm{I})$ and $S(\mathrm{J})$in 16.4. Since these equalities define S completely, ❷ follows. □

16.6. Corollary. *Let Σ be a smooth surface with orientation defined by unit normal field ν. Suppose the spherical map $\nu\colon \Sigma \to \mathbb{S}^2$ is injective. Then*

$$\int\limits_{\Sigma} |K| = \operatorname{area}[\nu(\Sigma)].$$

Proof. Observe that the tangent planes $\mathrm{T}_p\Sigma = \mathrm{T}_{\nu(p)}\mathbb{S}^2$ are parallel for any $p \in \Sigma$. Indeed both of these planes are perpendicular to $\nu(p)$.

Choose an orthonormal basis of T_p with in principle directions, so the shape operator can be expressed by matrix $\left(\begin{smallmatrix} k_1 & 0 \\ 0 & k_2 \end{smallmatrix}\right)$.

Since by 16.5, $S_p = -L_p\nu$, we have

$$\operatorname{jac}_p \nu = |\det\left(\begin{smallmatrix} k_1 & 0 \\ 0 & k_2 \end{smallmatrix}\right)| = |K(p)|.$$

By 15.6, the statement follows. □

16.7. Exercise. *Suppose that $(u,v) \mapsto s(u,v)$ is a chart of a smooth surface Σ with unit normal field ν. Show that*

$$\begin{aligned} \langle S(\tfrac{\partial s}{\partial u}), \tfrac{\partial s}{\partial u}\rangle &= \langle \tfrac{\partial^2 s}{\partial u^2}, \nu\rangle, & \langle S(\tfrac{\partial s}{\partial u}), \tfrac{\partial s}{\partial v}\rangle &= \langle \tfrac{\partial^2 s}{\partial u \partial v}, \nu\rangle, \\ \langle S(\tfrac{\partial s}{\partial v}), \tfrac{\partial s}{\partial u}\rangle &= \langle \tfrac{\partial^2 s}{\partial u \partial v}, \nu\rangle, & \langle S(\tfrac{\partial s}{\partial v}), \tfrac{\partial s}{\partial v}\rangle &= \langle \tfrac{\partial^2 s}{\partial v^2}, \nu\rangle \end{aligned}$$

for any (u,v).

16.8. Exercise. *Let Σ be a smooth oriented surface with the unit normal field ν. Suppose that Σ has unit principle curvatures at any point.*

(a) Show that $S_p(w) = w$ for any $p \in \Sigma$ and $w \in \mathrm{T}_p\Sigma$.

(b) Show that $p+\nu_p$ is constant; that is, the point $c = p+\nu_p$ does not depend on $p \in \Sigma$. Conclude that Σ is a part of the unit sphere centered at c.

16.9. Exercise. *Assume that smooth surfaces Σ_1 and Σ_2 intersect at constant angle along a smooth regular curve γ. Show that if γ is a curvature line in Σ_1, then it is also a curvature line in Σ_2.*

Conclude that if a smooth surface Σ intersects a plane or sphere along a smooth curve γ, then γ is a curvature line of Σ.

Curve in a surface

Suppose γ is a regular smooth curve in a smooth oriented surface Σ. As usual we denote by ν the unit normal field on Σ.

Without loss of generality we may assume that γ is unit-speed; in this case $\mathrm{T}(s) = \gamma'(s)$ is its tangent indicatrix. Let us use a shortcut notation $\nu(s) = \nu(\gamma(s))$. Note that the unit vectors $\mathrm{T}(s)$ and $\nu(s)$ are orthogonal; therefore there is a unique unit vector $\mu(s)$ such that $\mathrm{T}(s), \mu(s), \nu(s)$ is an oriented orthonormal basis. Since $\mathrm{T}(s) \perp \nu(s)$, the vector $\mu(s)$ is tangent to Σ at $\gamma(s)$. In fact $\mu(s)$ is a counterclockwise rotation of $\mathrm{T}(s)$ by right angle in the tangent plane $\mathrm{T}_{\gamma(s)}$. This vector can be also defined as a vector product $\mu(s) = \nu(s) \times \mathrm{T}(s)$.

Since γ is unit-speed, we have that $\gamma'' \perp \gamma'$; (see 10.1). Therefore the acceleration of γ can be written as a linear combination of μ and ν; that is,

$$\gamma''(s) = k_g(s)\cdot\mu(s) + k_n(s)\cdot\nu(s).$$

The values $k_g(s)$ and $k_n(s)$ are called *geodesic* and *normal curvature* of γ at s. Since the frame $\mathrm{T}(s), \mu(s), \nu(s)$ is orthonormal, these curvatures can be also written as the following scalar products:

$$\begin{aligned} k_g(s) &= \langle\gamma''(s), \mu(s)\rangle = \\ &= \langle\mathrm{T}'(s), \mu(s)\rangle. \\ k_n(s) &= \langle\gamma''(s), \nu(s)\rangle = \\ &= \langle\mathrm{T}'(s), \nu(s)\rangle. \end{aligned}$$

Since $0 = \langle\mathrm{T}(s), \nu(s)\rangle$ we have that

$$\begin{aligned} 0 &= \langle\mathrm{T}(s), \nu(s)\rangle = \\ &= \langle\mathrm{T}'(s), \nu(s)\rangle + \langle\mathrm{T}(s), \nu'(s)\rangle = \\ &= k_n(s) + \langle\mathrm{T}(s), D_{\mathrm{T}(s)}\nu\rangle. \end{aligned}$$

Applying 16.5, we get the following:

16.10. Proposition. *Assume γ is a smooth unit-speed curve in a smooth surface Σ. Suppose $p = \gamma(s_0)$ and $\mathrm{v} = \gamma'(s_0)$. Then*

$$k_n(s_0) = \langle S_p(\mathrm{v}), \mathrm{v}\rangle,$$

where k_n denotes the normal curvature of γ at s_0 and S_p is the shape operator at p.

Note that according to the proposition, the normal curvature of regular smooth curve in Σ is completely determined by the velocity vector v at the point p. By that reason the normal curvature is also denoted by k_{v}; according to the proposition,

$$k_{\mathrm{v}} = \langle S_p(\mathrm{v}), \mathrm{v}\rangle$$

for any unit vector v in T_p.

Let p be a point on a smooth surface Σ. Assume we choose the tangent-normal coordinates at p so that the Hessian matrix is diagonalized, we can assume that

$$M_p = \begin{pmatrix} k_1(p) & 0 \\ 0 & k_2(p) \end{pmatrix}.$$

Consider a vector $\mathrm{w} = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ in the (x, y)-plane. Then

$$\begin{aligned}\langle S(\mathrm{w}), \mathrm{w}\rangle &= \langle M_p\cdot\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\rangle = \\ &= a^2\cdot k_1(p) + b^2\cdot k_2(p).\end{aligned}$$

If w is unit, then $a^2 + b^2 = 1$ which implies the following:

16.11. Observation. *For any point p on an oriented smooth surface Σ, the principle curvatures $k_1(p)$ and $k_2(p)$ are respectively minimum and maximum of the normal curvatures at p. Moreover, if θ is the angle between a unit vector $\mathrm{w} \in \mathrm{T}_p$ and the first principle direction at p, then*

$$k_{\mathrm{w}}(p) = k_1(p)\cdot(\cos\theta)^2 + k_2(p)\cdot(\sin\theta)^2.$$

The last identity is called *Euler's formula.*

16.12. Meusnier's theorem. *Let γ be a regular smooth curve that runs in a smooth oriented surface Σ. Suppose $p = \gamma(t_0)$ and $\mathrm{v} = \gamma'(t_0)$ and $\alpha = \measuredangle(\nu(p), \mathrm{N}(t_0))$; that is, α is the angle between the unit normal to Σ at p and the unit normal vector in the Frenet frame of γ at t_0. Then the following identity holds for the curvature $\kappa(t_0)$ and the normal curvature $k_n(t_0)$ of γ at t_0:*

$$\kappa(t_0)\cdot\cos\alpha = k_n(t_0).$$

Proof. Since $\gamma'' = \mathrm{T}' = \kappa\cdot\mathrm{N}$, we get that

$$\begin{aligned}k_n(t_0) &= \langle\gamma'', \nu\rangle = \\ &= \kappa(t_0)\cdot\langle\mathrm{N}, \nu\rangle = \\ &= \kappa(t_0)\cdot\cos\alpha.\end{aligned}$$

□

The theorem above, as well as the statement in the following exercise are proved by Jean Baptiste Meusnier [37].

16.13. Exercise. *Let Σ be a smooth surface, $p \in \Sigma$ and $\mathrm{v} \in \mathrm{T}_p\Sigma$ is a unit vector. Assume that $k_{\mathrm{v}}(p) \neq 0$; that is, the normal curvature of Σ at p in the direction of v does not vanish.*

Show that the osculating circles at p of smooth regular curves in Σ that run in the direction v sweep out a sphere S with center $p + \frac{1}{k_{\mathrm{v}}}\cdot\nu$ and radius $r = \frac{1}{|k_{\mathrm{v}}|}$.

16.14. Exercise. *Let $\gamma(s) = (x(s), y(s))$ be a smooth unit-speed simple plane curve in the upper half-plane. Suppose that Σ is the surface of revolution of γ with respect to the x-axis.*

(a) Show that parallels and meridians form lines of curvature on Σ.
(b) Show that

$$\frac{|x'(s)|}{y(s)} \quad \text{and} \quad \frac{-y''(s)}{|x'(s)|}$$

are principle curvatures of Σ at $(x(s), y(s), 0)$ in the direction of parallel and meridian respectively.

16.15. Exercise. *Show that* catenoid *defined implicitly by equation*

$$(\operatorname{ch} z)^2 = x^2 + y^2$$

is a minimal surface.

16.16. Exercise. *Show that* helicoid *defined by the following parametric equation*

$$s(u, v) = (u \cdot \sin v, u \cdot \cos v, v)$$

is a minimal surface.

Lagunov's example

16.17. Exercise. *Assume V is a body in $\mathbb{R}^3$ bounded by a smooth surface of revolution with principle curvatures at most 1 in absolute value. Show that V contains a unit ball.*

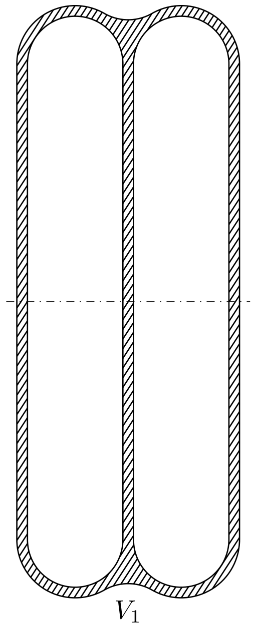
The following question is a 3-dimensional analog of the moon in a puddle problem (13.14).

16.18. Question. *Assume a set $V \subset \mathbb{R}^3$ is bounded by a closed connected surface Σ with principle curvatures bounded in absolute value by 1. Is it true that V contains a ball of radius 1?*

According to 16.17, the answer is "yes" for surfaces of revolution. We also have "yes" for convex surfaces; see 18.7. It turns out that in general the answer is "no". The following example was constructed by Vladimir Lagunov [31].

Construction. Let us start with a body of revolution V_1 with cross section shown on the diagram. The boundary curve of the cross section consists of 6 long vertical line segments smoothly jointed into 3 closed simple curves. The boundary of V_1 has 3 components, each of which is a smooth sphere.

We assume that the curves have curvature at most 1. Moreover with the exception for almost vertical parts, the curve have absolute curvature about 1 all the time. The only thick part in V is the place



where all three boundary components come together; the remaining part of V is assumed to be very thin. It could be arranged that the radius r of the maximal ball in V is just a little bit above $r_2 = \frac{2}{\sqrt{3}} - 1$. ($r_2$ is the radius of the smallest circle tangent to three unit circles that are tangent to each other.) In particular, we may assume that $r < \frac{1}{6}$.

Exercise 16.14 gives formulas for the principle curvatures of the boundary of V; which imply that both principle curvatures are at most 1 by absolute value.

It remains to modify V_1 to make its boundary connected without alloing larger balls inside.

Note that each sphere in the boundary contains two flat discs; they come into pairs closely lying to each other. Let us drill thru two of such pairs and reconnect the holes by another body of revolution whose axis is shifted but stays parallel to the axis of V_1. Denote the obtained body by V_2; its cross section of the obtained body is shown on the diagram.

Then repeat the operation for the other two pairs. Denote the obtained body by V_3; the cross section of the obtained body is shown on the diagram.

Note that the boundary of V_3 is connected. Assuming that the holes are large, its boundary can be made so that its principle curvatures are still at most 1; the latter can be proved the same way as for V_1. □

Note that the surface of V_3 in the Lagunov's example has genus 2; that is, it can be parameterized by a sphere with two handles.

Indeed, the boundary of V_1 consists of three smooth spheres.

When we drill a hole, we make one hole in two spheres and two holes in one shpere. We reconnect two spheres by a tube and obtain one sphere. Connecting the two holes of the other sphere by a tube we get a torus; it is on the right side on the picture of V_2. That is, the boundary of V_2 is formed by one sphere and one torus.

To construct V_3 from V_2, we make a torus from the remaining sphere and connect it to the other torus by a tube. This way we get a sphere with two handles; that is, it has genus 2.

16.19. Exercise. *Modify Lagunov's construction to make the boundary surface a sphere with 4 handles.*

Variations*

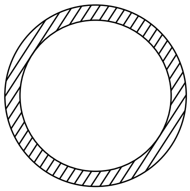
In this section we will discuss few results related to 16.18.

Recall that r_2 is the radius of the smallest circle tangent to three unit circles that are tangent to each other. Let r_3 be the radius of the smallest sphere tangent to four unit spheres that are tangent to each other. Direct calculations show that

$$r_2 = \tfrac{2}{\sqrt{3}} - 1 < \tfrac{1}{6} \quad \text{and} \quad r_3 = \sqrt{\tfrac{3}{2}} - 1 > \tfrac{1}{5}.$$

The following example shows that the bound obtained in the construction of the Lagunov's example is optimal.

16.20. Advanced exercise. *Suppose a connected body $V \subset \mathbb{R}^3$ is bounded by a finite number of closed smooth surfaces with principle curvatures bounded in absolute value by 1. Assume that V does not contain a ball of radius r_2. Show that its boundary has two components of the same topological type; that is, both can be written in parametric form with the same parameter domain.*



For example consider the region between two large concentric spheres with almost equal radiuses. This region can be made arbitrary thin and the curvature of the boundary can be made arbitrary close to zero.

Note that Lagunov's example shows that the estimate is sharp.

16.21. Very advanced exercise. *Suppose a body $V \subset \mathbb{R}^3$ is bounded by a smooth sphere with principle curvatures bounded in absolute value by 1. Show that V contains a ball of radius r_3.*

Show that this bound is sharp; that is there are examples of V as above that contain a ball of maximal radius arbitrary close to r_3

Chapter 17

Area

Flux and area

Let U be a vector field defined on a smooth oriented surface Σ with unit normal filed ν. Recall that *flux* of U thru Σ is defined by the integral

$$\mathrm{flux}_{\mathrm{U}}\,\Sigma = \int\limits_{\Sigma} \langle \mathrm{U}, \nu\rangle.$$

17.1. Observation. *Let* U *be a vector field defined on a smooth oriented surface* Σ*. Assume* $|\mathrm{U}| \leqslant 1$ *at every point. Then*

$$\mathrm{flux}_{\mathrm{U}}\,\Sigma \leqslant \mathrm{area}\,\Sigma.$$

Proof. Since $|\mathrm{U}| \leqslant 1$ and $|\nu| = 1$ at every point, we have that

$$\langle \mathrm{U}, \nu\rangle \leqslant 1$$

Therefore

$$\mathrm{flux}_{\mathrm{U}}\,\Sigma = \int\limits_{\Sigma} \langle \mathrm{U}, \nu\rangle \leqslant \int\limits_{\Sigma} 1 = \mathrm{area}\,\Sigma.$$

□

Divergence and curl

Consider a smooth vector field U defined on a domain Ω in $\mathbb{R}^3$. Recall that divergence of U is defined as $\operatorname{div}\mathrm{U} = \langle \nabla, \mathrm{U}\rangle$. In other words, if I, J, and K denote the elements of the standard basis of $\mathbb{R}^3$ and $\mathrm{U} = P{\cdot}\mathrm{I} + Q{\cdot}\mathrm{J} + R{\cdot}\mathrm{K}$ for some smooth functions P, Q, R, then

$$\operatorname{div}\mathrm{U} = \tfrac{\partial P}{\partial x} + \tfrac{\partial Q}{\partial y} + \tfrac{\partial R}{\partial z}.$$

Divergence is a function on Ω; its value at a point p is denoted by $\operatorname{div}_p \mathrm{U}$.

The following two theorems are assumed to be known:

17.2. Divergence theorem. *If a piecewise smooth surface Σ bounds a body V in $\mathbb{R}^3$, then*

$$\operatorname{flux}_{\mathrm{U}}\Sigma = \int\limits_{p\in V} \operatorname{div}_p \mathrm{U},$$

assuming that the orientation on Σ is defined by a unit normal field that points out of V.

Given a vector field U, set $\operatorname{curl}\mathrm{U} := \nabla \times \mathrm{U}$; that is, if we write the field U in the standard basis

$$\mathrm{U} = P{\cdot}\mathrm{I} + Q{\cdot}\mathrm{J} + R{\cdot}\mathrm{J},$$

then

$$\operatorname{curl}\mathrm{U} := \left(\tfrac{\partial R}{\partial y} - \tfrac{\partial Q}{\partial z}\right){\cdot}\mathrm{I} + \left(\tfrac{\partial P}{\partial z} - \tfrac{\partial R}{\partial x}\right){\cdot}\mathrm{J} + \left(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\right){\cdot}\mathrm{K}$$

17.3. Curl theorem. *Let Σ be a compact oriented surface bounded by a curve γ. Assume that $\gamma\colon [0,\ell] \to \mathbb{R}^3$ is parameterized by its arc-length and oriented in such a way that Σ lies on left from it. Then, for any smooth vector field U defined in a neighborhood of Σ, we have*

$$\operatorname{flux}_{\operatorname{curl}\mathrm{U}}\Sigma = \int\limits_0^\ell \langle \mathrm{U}, \gamma'(t)\rangle{\cdot}dt.$$

17.4. Exercise. *Let Σ be a compact smooth surface with boundary $\partial\Sigma$ in the (x,y)-plane. Denote by Δ the compact region in the (x,y)-plane bounded by the $\partial\Sigma$. Suppose that $\mathrm{I}, \mathrm{J}, \mathrm{K}$ is the standard basis in the space.*

(a) Assume that all the points of Σ lie in the upper half-space of the (x,y)-plane. Use the divergence theorem (17.2) and the observation for the constant vector field K to show that

$$\operatorname{area}\Sigma \geqslant \operatorname{area}\Delta.$$

(b) *Use the curl theorem (17.3) to prove the inequality in 17.4a for arbitrary smooth surface Σ with boundary $\partial\Sigma$ in the (x,y)-plane, without assuming that the remaining points lie in the upper half-space.*

Further we denote by $\mathrm{I}, \mathrm{J}, \mathrm{K}$ the standard basis in a (x,y,z)-coordinate system of $\mathbb{R}^3$.

17.5. Exercise. *Let Σ be a smooth closed surface that lies between parallel planes on distance 1 from the (x,y)-plane. Suppose that Σ bounds a region R. Use the divergence theorem and the observation for the vector field $\mathrm{U} = z{\cdot}\mathrm{K}$ to show that*

$$\operatorname{area}\Sigma > \operatorname{vol} R.$$

17.6. Exercise. *Let Σ be a smooth closed surface that lies inside the infinite cylinder $x^2 + y^2 \leqslant 1$. Suppose that Σ bounds a region R. Use the divergence theorem and the observation for the vector field $\mathrm{I}, \mathrm{J}, \mathrm{K}$ to show that*

$$\operatorname{area}\Sigma > 2{\cdot} \operatorname{vol} R.$$

17.7. Corollary. *Let Σ and Σ' be a compact surfaces with identical boundary lines; that is, $\partial\Sigma = \partial\Sigma'$. Suppose that Σ is oriented and its unit normal field ν can be extended to a vector field U such that $|\mathrm{U}| = 1$ at every point and $\operatorname{div}\mathrm{U} = 0$ at every point between Σ and Σ'. Then*

$$\operatorname{area}\Sigma' \geqslant \operatorname{area}\Sigma.$$

The vector field U as in the corollary is called *calibration* of Σ

Mean curvature and divergence

The following lemma will be used to construct unit vector fields with vanishing or positive divergence.

17.8. Lemma. *Let Σ be a smooth orented surface, ν be the unit normal field on Σ. Suppose that a unit vector field U is a smooth extension of ν in a neighborhood of Σ. Then at any point $p \in \Sigma$,*

$$\operatorname{div}_p \mathrm{U} + H(p) = 0,$$

where $H(p)$ is the mean curvature of Σ at p.

Proof. Consider the tangent-normal coordinates (x,y,z) of Σ at p such that the x- and y-coordinates point in the principle directions of Σ. If $\mathrm{I},\mathrm{J},\mathrm{K}$ denotes the standard basis, then

$$S_p(\mathrm{I}) = k_1\cdot\mathrm{I}, \qquad\qquad S_p(\mathrm{J}) = k_2\cdot\mathrm{J},$$

where S_p stands for the shape operator of Σ at p.

Proposition 16.5 implies that

$$\frac{\partial\mathrm{U}}{\partial x} = \frac{\partial\nu}{\partial x} = -S_p(\mathrm{I}), \quad \text{and} \quad \frac{\partial\mathrm{U}}{\partial y} = \frac{\partial\nu}{\partial y} = -S_p(\mathrm{J}).$$

It follows that

❶ $$\langle\tfrac{\partial\mathrm{U}}{\partial x},\mathrm{I}\rangle = -k_1, \quad \text{and} \quad \langle\tfrac{\partial\mathrm{U}}{\partial y},\mathrm{J}\rangle = -k_2.$$

Further, since U is unit, we have $\langle\mathrm{U},\mathrm{U}\rangle\equiv 1$. Taking derivative of this identity, we get

$$\begin{aligned} 0 &= \tfrac{\partial}{\partial z}\langle\mathrm{U},\mathrm{U}\rangle = \\ &= 2\cdot\langle\tfrac{\partial}{\partial z}\mathrm{U},\mathrm{U}\rangle. \end{aligned}$$

In particular,

❷ $$\langle\tfrac{\partial}{\partial z}\mathrm{U},\mathrm{K}\rangle_p = 0.$$

Let us write the field U in the standard basis $\mathrm{U} = u_1\cdot\mathrm{I} + u_2\cdot\mathrm{J} + u_3\cdot\mathrm{K}$. Since the basis $\mathrm{I},\mathrm{J},\mathrm{K}$ is orthonormal, the functions u_1,u_2,u_3 can be defined by

$$u_1 = \langle\mathrm{U},\mathrm{I}\rangle, \qquad u_2 = \langle\mathrm{U},\mathrm{J}\rangle, \qquad u_3 = \langle\mathrm{U},\mathrm{K}\rangle.$$

Applying the definition of divergence and using ❶ and ❷, we obtain

$$\begin{aligned} \operatorname{div}\mathrm{U} &= \tfrac{\partial u_1}{\partial x} + \tfrac{\partial u_2}{\partial y} + \tfrac{\partial u_3}{\partial z} = \\ &= \tfrac{\partial}{\partial x}\langle\mathrm{U},\mathrm{I}\rangle + \tfrac{\partial}{\partial y}\langle\mathrm{U},\mathrm{J}\rangle + \tfrac{\partial}{\partial z}\langle\mathrm{U},\mathrm{K}\rangle = \\ &= \langle\tfrac{\partial\mathrm{U}}{\partial x},\mathrm{I}\rangle + \langle\tfrac{\partial\mathrm{U}}{\partial y},\mathrm{J}\rangle + \langle\tfrac{\partial\mathrm{U}}{\partial z},\mathrm{K}\rangle \\ &= -k_1(p) - k_2(p) + 0 \\ &= -H(p) \end{aligned}$$

□

Let V be a body in $\mathbb{R}^3$ bounded by a closed smooth surface Σ; assume Σ is equipped with orientation defined by unit normal field ν that points outside V. We say that V is *mean-convex* if the mean curvature of Σ is nonpositive.

17.9. Exercise. *Let V be a mean-convex body in $\mathbb{R}^3$ bounded by a closed smooth surface Σ. Denote by W the outer region of Σ, it is a complement of the interior of V.*

(a) *Suppose V is star-shaped; that is, there is a point $p \in V$ such that for any other point $x \in V$ the line segment $[p,x]$ lies in V. Construct a unit vector field* u *on W such that* $\operatorname{div}\mathrm{u} \geqslant 0$ *at every point in W and the restriction of* u *to Σ is a normal field that points in W.*

(b) *Suppose that another body V' is bounded by a closed smooth surface Σ' and $V' \supset V$. Use part (a) and the divergence theorem to show that*

$$\operatorname{area}\Sigma' \geqslant \operatorname{area}\Sigma$$

if V is star-shaped.

(c) *Construct a non-star-shaped mean-convex body V bounded by a smooth surface such that the inequality in17.9a does not hold for some body $V' \supset V$ with smooth boundary Σ'.*

Area-minimizing surfaces

A smooth surface Σ is called *area-minimizing* if for any compact surfaces Δ in Σ with boundary line $\partial\Delta$ the following inequality

$$\operatorname{area}\Delta \leqslant \operatorname{area}\Delta'$$

holds for any other smooth compact surface Δ' with the same boundary line; that is, if $\partial\Delta' = \partial\Delta$.

17.10. Exercise. *Suppose Σ is a compact area-minimizing surface with boundary line $\partial\Sigma$. Let p be a point in Σ. Show that if the ball $B(p,r)$ does not intersect $\partial\Sigma$, then*

$$\operatorname{area}[\Sigma \cap B(p,r)] \leqslant 2{\cdot}\pi{\cdot}r^2;$$

that is, this area can not exceed half of the area of sphere of radius r.

Recall that a surface with vanishing mean curvature is called *minimal.*

17.11. Proposition. *Any area-minimizing surface is minimal.*

Proof. Assume Σ is a surface with nonzero mean curvature at some point p. Without loss of generality we may assume that it is positive, otherwise switch the orientation.

Let $z = f(x,y)$ be a local description of Σ in the tangent-normal coordinates at p. Denote by $H(x,y)$ and $\nu(x,y)$ the mean curvature and the unit normal vector of the graph at the point $(x,y,f(x,y))$. Passing to a smaller domain of f, we can assume that

$$H(x,y) > 0$$

for any $(x, y) \in \operatorname{Dom} f$.

Consider the vector field U on the domain $\Omega = \mathbb{R} \times \operatorname{Dom} f$ defined by $\mathrm{U}(x, y, z) = \nu(x, y)$. Note that

❸ $$(\operatorname{div} \mathrm{U})(x, y, z) + H(x, y) = 0.$$

Indeed, by construction, U is invariant with respect to shifts of Ω up or down. In particular the divergence $\operatorname{div} \mathrm{U}$ does only on x and y. Therefore it is sufficient to show ❸ for points on the graph $z = f(x, y)$. The latter follows from Lemma 17.8.

Fix a closed ε-neighborhood D_ε of the origin in the (x, y)-plane; we can assume that D_ε lies in the domain of f. Choose a smooth function $(x, y) \mapsto h(x, y)$ defined on D_ε in its interior and vanishes on its boundary; for example, $h = \varepsilon^2 - x^2 - y^2$ will do. Set

$$f_t(x, y) = f(x, y) + t{\cdot}h(x, y).$$

Denote by Δ_t be the graph $z = f_t(x, y)$. Set $a(t) = \operatorname{area} \Delta_t$ and $b(t) = \operatorname{flux}_{\mathrm{U}} \Delta_t$. Observe that both functions $t \mapsto a(t)$ and $t \mapsto b(t)$ are smooth.

By construction of U, we have that $a(0) = b(0)$. By Observation 17.1, we have that $a(t) \geqslant b(t)$ for any t. Therefore

❹ $$a'(0) = b'(0).$$

Fix $t > 0$. Let Θ_t be the domain squeezed between Δ and Δ_t; that is,

$$\Theta_t = \{\, (x, y, z) \;:\; f(x, y) < z < f_t(x, y) \,\}.$$

By divergence theorem, we have

$$\begin{aligned} b(t) - b(0) &= \iiint\limits_{\Theta_t} \operatorname{div} \mathrm{U}{\cdot}dx{\cdot}dy{\cdot}dz = \\ &= -\iiint\limits_{\Theta_t} H(x, y){\cdot}dx{\cdot}dy{\cdot}dz = \\ &= t{\cdot}\left[-\iint\limits_{\Delta} H(x, y){\cdot}h(x, y){\cdot}dx{\cdot}dy\right]. \end{aligned}$$

Since $H > 0$ and $h > 0$ in the interior of Δ, the last integral is positive. It follows that $b(t)$ is a linear function with negative slope. By ❹, $a'(0) = b'(0) < 0$. In particular,

$$\operatorname{area} \Delta_t < \operatorname{area} \Delta$$

for small $t > 0$; that is, Σ is not area-minimizing. $\square$

The following two exercises show that minimal surface might be not area-minimizing. Recall that catenoid and helicoid are minimal; see exercises 16.15 and 16.16. The following exercise state that sufficienlty large piece of these surfaces are not area-minimizing.

17.12. Exercise. *Show that the* catenoid

$$(\operatorname{ch} z)^2 = x^2 + y^2.$$

is not area-minimizing.

17.13. Exercise. *Show that the* helicoid

$$s(u, v) = (u \cdot \sin v, u \cdot \cos v, v).$$

is not a area-minimizing.

The following theorem provides a condition on minimal surface that guarantees that it is area-minimizing.

17.14. Theorem. *Let Σ be a graph $z = f(x, y)$ of a smooth function f defined on an open convex set in the (x, y)-plane. Suppose Σ is minimal, then it is area-minimizing.*

In particular, any minimal surface Σ is locally area-minimizing*; that is, some neighborhood of any point p in Σ is area-minimizing.*

We omit its proof altho it is not hard; it can be build on the ideas from the solutions of 17.9 and 17.11.

Chapter 18

Supporting surfaces

Definitions

Assume two orientable surfaces Σ_1 and Σ_2 have a common point p. If there is a neighborhood U of p such that $\Sigma_1 \cap U$ lies on one side from Σ_2 in U, then we say that Σ_2 *locally supports* Σ_1 at p.

Let us describe Σ_2 locally at p as a graph $z = f_2(x, y)$ in a tangent-normal coordinates at p. If Σ_2 *locally supports* Σ_1 at p, then we may assume that all points of Σ_1 near p lie above the graph $z = f_2(x, y)$. In particular the tangent plane of Σ_1 at p is horizontal; that is, the tangent planes of Σ_1 and Σ_2 at p coincide.

It follows that, we can assume that Σ_1 and Σ_2 are *cooriented* at p; that is, they have common unit normal vector at p. If not, we can revert the orientation of one of the surfaces.

If Σ_2 locally supports Σ_1 and cooriented at p, then we can say that Σ_1 supports Σ_2 from *inside* or from *outside*, assuming that the normal vector points *inside* the domain bounded by surface Σ_2 in U.

More precisely, we can use for Σ_1 and Σ_2 a common tangent-normal coordinate system at p. This way we write Σ_1 and Σ_2 locally as graphs: $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. Then Σ_1 locally supports Σ_2 from inside (from outside) if $f_1(x, y) \geqslant f_2(x, y)$ (respectively $f_1(x, y) \leqslant f_2(x, y)$) for (x, y) in a sufficiently small neighborhood of the origin.

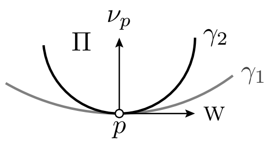
Note that Σ_1 locally supports Σ_2 from inside at the point p is equivalent to Σ_2 locally supports Σ_1 from outside. Further if we revert the orientation of both surfaces, then supporting from inside becomes supporting from outside and the other way around.

18.1. Proposition. *Let Σ_1 and Σ_2 be oriented surfaces. Assume Σ_1 locally supports Σ_2 from inside at the point p (equivalently Σ_2 locally* *supports Σ_1 from outside). Then*

$$k_1(p)_{\Sigma_1} \geqslant k_1(p)_{\Sigma_2} \quad \textit{and} \quad k_2(p)_{\Sigma_1} \geqslant k_2(p)_{\Sigma_2}.$$

18.2. Exercise. *Give an example of two surfaces Σ_1 and Σ_2 that have common point p with common unit normal vector ν_p such that $k_1(p)_{\Sigma_1} > k_1(p)_{\Sigma_2}$ and $k_2(p)_{\Sigma_1} > k_2(p)_{\Sigma_2}$, but Σ_1 does not support Σ_2 locally at p.*

Proof. We can assume that Σ_1 and Σ_2 are graphs $z = f_1(x, y)$ and $z = f_2(x, y)$ in a common tangent-normal coordinates at p, so we have $f_1 \geqslant f_2$.



Fix a unit vector $\mathrm{w} \in \mathrm{T}_p\Sigma_1 = \mathrm{T}_p\Sigma_2$. Consider the plane Π passing thru p and spanned by the normal vector ν_p and w. Let γ_1 and γ_2 be the curves of intersection of Σ_1 and Σ_2 with Π.

Let us orient Π so that the common normal vector ν_p for both surfaces at p points to the left from w. Further, let us parametrize both curves so that they are running in the direction of w at p and therefore cooriented. Note that in this case the curve γ_1 supports the curve γ_2 from the right.

By 13.3, we have the following inequality for the normal curvatures of Σ_1 and Σ_2 at p in the direction of w:

❶ $$k_{\mathrm{w}}(p)_{\Sigma_1} \geqslant k_{\mathrm{w}}(p)_{\Sigma_2}.$$

According to 16.11,

$$k_1(p)_{\Sigma_i} = \min\{\, k_{\mathrm{w}}(p)_{\Sigma_i} \;:\; \mathrm{w} \in \mathrm{T}_p, |\mathrm{w}| = 1 \,\}$$

for $i = 1, 2$. Choose w so that $k_1(p)_{\Sigma_1} = k_{\mathrm{w}}(p)_{\Sigma_1}$. Then by ❶, we have that

$$\begin{aligned} k_1(p)_{\Sigma_1} &= k_{\mathrm{w}}(p)_{\Sigma_1} \geqslant \\ &\geqslant k_{\mathrm{w}}(p)_{\Sigma_2} \geqslant \\ &\geqslant \min_{\mathrm{v}}\{\, k_{\mathrm{v}}(p)_{\Sigma_2} \,\} = \\ &= k_1(p)_{\Sigma_2}; \end{aligned}$$

here we assumed that $\mathrm{v} \in \mathrm{T}_p$ and $\mathrm{v}| = 1$. That is, $k_1(p)_{\Sigma_1} \geqslant k_1(p)_{\Sigma_2}$.

Similarly, by 16.11, we have that

$$k_2(p)_{\Sigma_i} = \max_{\mathrm{w}}\{\, k_{\mathrm{w}}(p)_{\Sigma_i} \,\}.$$

Let us fix W so that $k_2(p)_{\Sigma_2} = k_{\mathrm{W}}(p)_{\Sigma_2}$. Then

$$\begin{aligned} k_2(p)_{\Sigma_2} &= k_{\mathrm{W}}(p)_{\Sigma_2} \leqslant \\ &\leqslant k_{\mathrm{W}}(p)_{\Sigma_1} \leqslant \\ &\leqslant \max_{\mathrm{V}} \{\, k_{\mathrm{V}}(p)_{\Sigma_1} \,\} = \\ &= k_2(p)_{\Sigma_1}; \end{aligned}$$

that is, $k_2(p)_{\Sigma_1} \geqslant k_2(p)_{\Sigma_2}$. □

18.3. Corollary. *Let Σ_1 and Σ_2 be oriented surfaces. Assume Σ_1 locally supports Σ_2 from inside at the point p. Then*
(a) $H(p)_{\Sigma_1} \geqslant H(p)_{\Sigma_2}$;
(b) If $k_1(p)_{\Sigma_2} \geqslant 0$, then $K(p)_{\Sigma_1} \geqslant K(p)_{\Sigma_2}$.

Proof. Part (a) follows from 18.1 and the definition of mean curvature

$$H(p)_{\Sigma_i} = k_1(p)_{\Sigma_i} + k_2(p)_{\Sigma_i}.$$

(b). Since $k_2(p)_{\Sigma_i} \geqslant k_1(p)_{\Sigma_i}$ and $k_1(p)_{\Sigma_2} \geqslant 0$, we get that all the principle curvatures $k_1(p)_{\Sigma_1}$, $k_1(p)_{\Sigma_2}$, $k_2(p)_{\Sigma_1}$, and $k_2(p)_{\Sigma_2}$ are non-negative. By 18.1, it implies that

$$\begin{aligned} K(p)_{\Sigma_1} &= k_1(p)_{\Sigma_1} \cdot k_2(p)_{\Sigma_1} \geqslant \\ &\geqslant k_1(p)_{\Sigma_2} \cdot k_2(p)_{\Sigma_2} = \\ &= K(p)_{\Sigma_2}. \end{aligned}$$

□

18.4. Exercise. *Show that any closed surface in a unit ball has a point with Gauss curvature at least 1.*

18.5. Exercise. *Show that any closed surface that lies on the distance at most 1 from a straight line has a point with Gauss curvature at least 1.*

Convex surfaces

A proper surface without boundary that bounds a convex region is called *convex.*

18.6. Exercise. *Show that Gauss curvature of any convex smooth surface is nonnegative at each point.*

18.7. Exercise. *Assume R is a convex body in $\mathbb{R}^3$ bounded by a surface with principle curvatures at most 1. Show that R contains a unit ball.*

Recall that a region R in the Euclidean space is called *strictly convex* if for any two points $x, y \in R$, any point z between x and y lies in the interior of R.

Clearly any open convex set is strictly convex; the cube (as well as any convex polyhedron) gives an example of a convex set which is not strictly convex. It is easy to see that a closed convex region is strictly convex if and only if its boundary does not contain a line segment.

18.8. Lemma. *Let $z = f(x, y)$ be the local description of a smooth surface Σ in a tangent-normal coordinates at some point $p \in \Sigma$. Assume both principle curvatures of Σ are positive at p. Then the function f is strictly convex in a neighborhood of the origin and has a local minimum at the origin.*

In particular the tangent plane T_p locally supports Σ from outside at p.

Proof. Since both principle curvatures are positive, we have that

$$D^2_{\mathrm{w}} f(0,0) = \langle S_p(\mathrm{w}), \mathrm{w}\rangle > 0$$

for any unit tangent vector $\mathrm{w} \in \mathrm{T}_p\Sigma$ (which is the (x, y)-plane).

Since the set of unit vectors is compact, we have that

$$D^2_{\mathrm{w}} f(0,0) > \varepsilon$$

for some fixed $\varepsilon > 0$ and any unit tangent vector $\mathrm{w} \in \mathrm{T}_p\Sigma$.

By continuity of the function $(x, y, \mathrm{w}) \mapsto D^2_{\mathrm{w}} f(x, y)$, we have that $D^2_{\mathrm{w}} f(x, y) > 0$ if $\mathrm{w} \neq 0$ and (x, y) lies in a neighborhood of the origin. That is, f is a strictly convex function in a neighborhood of the origin in the (x, y)-plane.

Finally since $\nabla f(0,0) = 0$ and f is strictly convex in a neighborhood of the origin it has a strict local minimum at the origin. □

18.9. Exercise. *Let Σ be a smooth surface (without boundary) with positive Gauss curvature. Show that any connected component of intersection of Σ with a plane Π is either a single point or a smooth regular plane that can be parameterized so that it has positive signed curvature.*

The following theorem gives a global description of surfaces with positive Gauss curvature.

18.10. Theorem. *Suppose Σ is a proper smooth surface with positive Gauss curvature. Then Σ bounds a strictly convex region.*

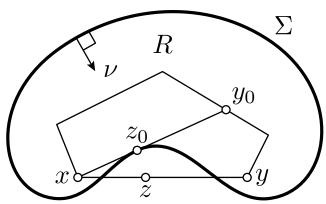
Note that in the proof we have to use that surface is a connected set; otherwise a pair of disjoint spheres which bound two disjoint balls would give a counterexample.

Proof. Since the Gauss curvature is positive, we can choose unit normal field ν on Σ so that the principle curvatures are positive at any point. Denote by R the region bounded by Σ that lies on the side of ν; that is, ν points inside of R at any point of Σ. (The region R exists by 14.1.)

Fix $p \in \Sigma$; let $z = f(x, y)$ be a local description of Σ in the tangent-normal coordinates at p. By 18.8, f is strictly convex in a neighborhood of the origin. In particular the intersection of a small ball centered at p with the epigraph $z \geqslant f(x, y)$ is strictly convex. In other words, R is *locally strictly convex*; that is, for any point $p \in R$, the intersection of R with a small ball centered at p is strictly convex.

Since Σ is connected, so is R; moreover any two points in the interior of R can be connected by a polygonal line in the interior of R.

Assume the interior of R is not convex; that is, there are points $x, y \in R$ and a point z between x and y that does not lie in the interior of R. Consider a polygonal line β from x to y in the interior of R. Let y_0 be the first point on β such that the chord $[x, y_0]$ touches Σ at some point, say z_0.



Since R is locally strictly convex, $R \cap B(z_0, \varepsilon)$ is strictly convex for all sufficiently small $\varepsilon > 0$. On the other hand z_0 lies between two points in the intersection $[x, y_0] \cap B(z_0, \varepsilon)$. Since $[x, y_0] \subset R$, we arrived to a contradiction.

Therefore the interior of R is a convex set. Note that the region R is the closure of its interior, therefore R is convex as well.

Since R is locally strictly convex, its boundary Σ contains no line segments. Therefore R is strictly convex. $\square$

Note that the proof above implies that *any connected locally convex region is convex.*

18.11. Exercise. *Assume that a closed surface Σ surrounds a unit circle. Show that Gauss curvature of Σ is at most 1 at some point.*

18.12. Exercise. *Let Σ be a closed smooth surface of diameter at least π; that is, there is a pair of points $p, q \in \Sigma$ such that $|p - q| \geqslant \pi$. Show that Σ has a point with Gauss curvature at most 1.*

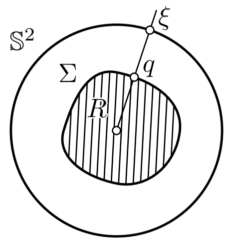
18.13. Theorem. *Suppose Σ is a smooth convex surface.*

(a) If Σ is compact then it is a smooth sphere; that is, Σ admits a smooth regular parametrization by $\mathbb{S}^2$.

(b) If Σ is open then there is a coordinate system such that Σ is a graph $z = f(x,y)$ of a convex function f defined on a convex open region of the (x,y)-plane.

The following exercises will guide you thru the proof of both parts of the theorem.

18.14. Exercise. *Assume a convex compact region R contains the origin in its interior and bounded by a smooth surface Σ.*

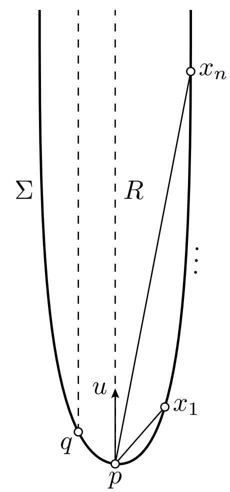


(a) Show that any half-line that starts at the origin intersects Σ at a single point; that is, there is a positive function $\rho\colon \mathbb{S}^2 \to \mathbb{R}$ such that Σ is formed by points $q = \rho(\xi)\cdot\xi$ for $\xi \in \mathbb{S}^2$.

(b) Show that $\rho\colon \mathbb{S}^2 \to \mathbb{R}$ is a smooth function.

(c) Conclude that $\xi \mapsto \rho(\xi)\cdot\xi$ is a smooth regular parametrization $\mathbb{S}^2 \to \Sigma$.

18.15. Exercise. *Assume a strictly convex closed noncompact region R contains the origin in its interior and bounded by a smooth surface Σ.*



(a) Show that R contains a half-line ℓ.

(b) Show that any line parallel to ℓ intersects Σ at most at one point.

(c) Consider (x,y,z)-coordinate system such that the z-axis points in the direction of ℓ. Show that projection of Σ to the (x,y) plane is an open convex set; denote it by Ω.

(d) Conclude that Σ is a graph $z = f(x,y)$ of a convex function f defined on Ω.

18.16. Exercise. *Show that any open surface Σ with positive Gauss curvature is a topological plane; that is, there is an embedding $\mathbb{R}^2 \to \mathbb{R}^3$ with image Σ.*

Try to show that Σ is a smooth plane; that is, the embedding f can be made smooth and regular.

18.17. Exercise. *Show that any open smooth surface Σ with positive Gauss curvature lies inside of an infinite circular cone.*

18.18. Exercise. *Suppose Σ is a smooth convex surface. Show that*

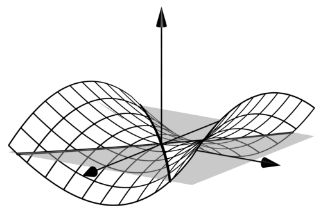
(a) *If Σ is closed, then the spherical map $\nu\colon \Sigma \to \mathbb{S}^2$ is a bijection. Conclude that*

$$\int\limits_{\Sigma} K = 4\cdot\pi.$$

(b) *If Σ is open, then the spherical map $\nu\colon \Sigma \to \mathbb{S}^2$ maps Σ ineffectively into a subset of a hemisphere. Conclude that*

$$\int\limits_{\Sigma} K \leqslant 2\cdot\pi.$$

Saddle surfaces



A surface is called *saddle* if its Gauss curvature at each point is nonpositive; in other words principle curvatures at each point have opposite signs or one of them is zero.

If the Gauss curvature is negative at each point, then the surface is called *strictly saddle*; equivalently it means that the principle curvatures have opposite signs at each point. Note that in this case the tangent plane does not support the surface even locally — moving along the surface in the principle directions at a given point, one goes above and below the tangent plane at this point.

18.19. Exercise. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth positive function. Show that the surface of revolution of the graph $y = f(x)$ around the x-axis is saddle if and only if f is convex; that is, if $f''(x) \geqslant 0$ for any x.*

A surface Σ is called *ruled* if for every point $p \in \Sigma$ there is a line segment $\ell_p \subset \Sigma_p$ thru p that is infinite or has its endpoint(s) on the boundary line of Σ.

18.20. Exercise. *Show that any ruled surface Σ is saddle.*

18.21. Exercise. *Suppose Σ is an open saddle surface. Show that for any point $p \in \Sigma$ there is a curve $\gamma\colon [0,\infty) \to \Sigma$ that starts at p and monotonically escapes to infinity; that is, the function $t \mapsto |\gamma(t)|$ is increasing and $|\gamma(t)| \to \infty$ as $t \to \infty$.*

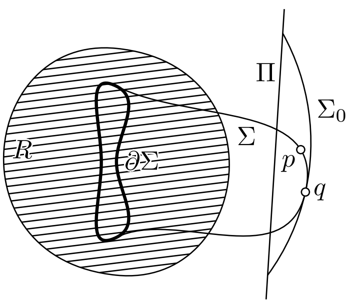
A tangent direction on a smooth surface with vanishing normal curvature is called *asymptotic*. A smooth regular curve that always runs in an asymptotic direction is called an *asymptotic line*.

18.22. Advanced exercise. *Let γ be a closed smooth asymptotic line in a graph $z = f(x, y)$ of a smooth function f. Assume Σ is strictly saddle in a neighborhood of γ. Show that the projection of γ to the (x, y)-plane cannot be star-shaped.*

Hats

Note that a closed surface cannot be saddle. Indeed consider a smallest sphere that contains a closed surface Σ inside; it supports Σ at some point p and at this point the principle curvature must have the same sign. The following more general statement is proved using the same idea.

18.23. Lemma. *Assume Σ is a compact saddle surface and its boundary line lies in a convex closed region R. Then whole surface Σ lies in R.*



Proof. Arguing by contradiction, assume there is point $p \in \Sigma$ that does not lie in R. Let Π be a plane that separates p from R; it exists by 6.4. Denote by Σ' the part of Σ that lies with p on the same side from Π.

Since Σ is compact, it is surrounded by a sphere; let σ be the circle of intersection of this sphere and Π. Consider the smallest spherical dome Σ_0 with boundary σ that surrounds Σ'.

Note that Σ_0 supports Σ at some point q. Without loss of generality we may assume that Σ_0 and Σ are cooriented at q and Σ_0 has positive principle curvatures. In this case Σ_0 supports Δ from outside. By we have 18.3, $K(q)_\Sigma \geqslant K(q)_{\Sigma_0} > 0$ — a contradiction. □

Note that if we assume that Σ is strictly saddle, then we could arrive to a contradiction by taking a point $q \in \Sigma$ on the maximal distance from R.

18.24. Exercise. *Let Δ be a compact smooth regular saddle surface with boundary and $p \in \Delta$. Suppose that the boundary line of Δ*

lies in the unit sphere centered at p. Show that if Δ is a disc, then $\operatorname{length}(\partial\Delta) \geqslant 2{\cdot}\pi$.

Show that the statement does not hold without assuming that Δ is a disc.

If Δ is as in the exercise, then in fact area $\Delta \geqslant \pi$. The proof of this statement can be obtained by applying the so called *coarea fromula* together with the inequality in the exercise.

18.25. Exercise. *Show that an open saddle surface cannot lie inside of an infinite circular cone.*

A disc Δ in a surface Σ is called a *hat* of Σ if its boundary line $\partial\Delta$ lies in a plane Π and the remaining points of Δ lie on one side of Π.

18.26. Proposition. *A smooth surface Σ is saddle if and only if it has no hats.*

Note that a saddle surface can contain a closed plane curve. For example the hyperboloid $x^2 + y^2 - z^2 = 1$ contains the unit circle in the (x, y)-plane. However, according to the proposition (as well as the lemma), a plane curve cannot bound a disc (as well any compact set) in a saddle surface.

Proof. Since plane is convex, the “only if” part follows from 18.23; it remains to prove the “if” part.

Assume Σ is not saddle; that is, it has a point p with strictly positive Gauss curvature; or equivalently, the principle curvatures $k_1(p)$ and $k_2(p)$ have the same sign.

Let $z = f(x, y)$ be a graph representation of Σ in the tangent-normal coordinates at p. By 18.8, f is convex in a small neighborhood of $(0, 0)$. In particular the set F_ε defined by the inequality $f(x, y) \leqslant \varepsilon$ is convex for sufficiently small $\varepsilon > 0$; in particular, it is a topological disc. Note that $(x, y) \mapsto (x, y, f(x, y))$ is a homeomorphism from F_ε to

$$\Delta_\varepsilon = \left\{\, (x, y, f(x, y)) \in \mathbb{R}^3 \;:\; f(x, y) \leqslant \varepsilon \,\right\};$$

so Δ_ε is a topological disc for any sufficiently small $\varepsilon > 0$. Note that the boundary line of Δ_ε lies on the plane $z = \varepsilon$ and whole disc lies below it; that is, Δ_ε is a hat of Σ. □

The following exercise shows that Δ_ε is in fact a smooth disc. It can be used to prove slightly stronger version of 18.26; namely one can the disc in the definition of hat is a smooth disc.

18.27. Exercise. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth strictly convex function with minimum at the origin. Show that the set F_ε in the graph $z = f(x, y)$ defined by the inequality $f(x, y) \leqslant \varepsilon$ is a smooth disc* *for any $\varepsilon > 0$; that is, there is a diffeomorphism $\mathbb{D} \to F_\varepsilon$, where $\mathbb{D} = \{\, (x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \leqslant 1 \,\}$ is the unit disc.*

18.28. Exercise. *Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an affine transformation; that is, L is an invertable map $\mathbb{R}^3 \to \mathbb{R}^3$ that sends any plane to a plane. Show that for any saddle surface Σ the image $L(\Sigma)$ is also a saddle surface.*

Saddle graphs

The following theorem was proved by Sergei Bernstein [6].

18.29. Theorem. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Assume its graph $z = f(x,y)$ is a strictly saddle surface in $\mathbb{R}^3$. Then f is not bounded; that is, there is no constant C such that $|f(x,y)| \leqslant C$ for any $(x,y) \in \mathbb{R}^2$.*

The theorem states that a saddle graph cannot lie between parallel horizontal planes; applying 18.28 we get that saddle graphs cannot lie between parallel planes, not necessarily horizontal. The following exercise shows that the theorem does not hold for saddle surfaces which are not graphs.

18.30. Exercise. *Construct an open strictly saddle surface that lies between parallel planes.*

Since $\exp(x - y^2) > 0$, the following exercise shows that there are strictly saddle graphs with functions bounded on one side; that is, both (upper and lower) bounds are needed in the proof of Bernshtein's theorem.

18.31. Exercise. *Show that the graph $z = \exp(x - y^2)$ is strictly saddle.*

Note that according to 18.23, there are no proper saddle surfaces in a parallelepiped that boundary line lies on one of its faces. The following lemma gives an analogous statement for a parallelepiped with an infinite side.

18.32. Lemma. *There is no proper strictly saddle smooth surface that lies on bounded distance from a line and has its boundary line in a plane.*

Proof. Note that in a suitable coordinate system, the statement can be reformulated the following way: *There is no proper strictly saddle* *smooth surface with the boundary line in the (x, y)-plane that lies in a region of the following form:*

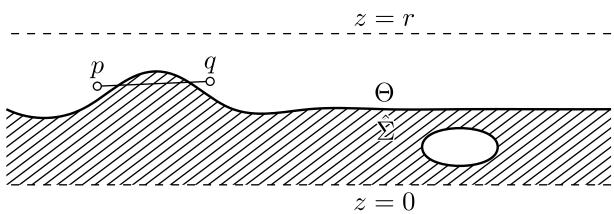
$$R = \{\, (x, y, z) \in \mathbb{R}^3 \,:\, 0 \leqslant z \leqslant r, 0 \leqslant y \leqslant r \,\}.$$

Let us prove this statement.

Assume contrary, let Σ be such a surface. Consider the projection $\hat{\Sigma}$ of Σ to the (x, z)-plane. It lies in the upper half-plane and below the line $z = r$.

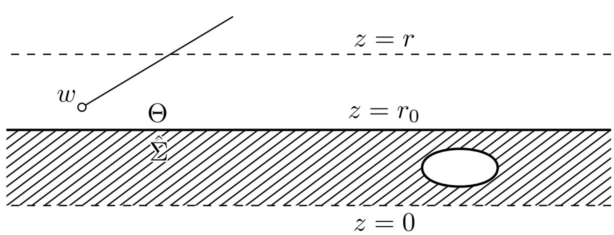
Consider the open upper half-plane $H = \{\, (x, z) \in \mathbb{R}^2 \,:\, z > 0 \,\}$. Let Θ be the connected component of the complement $H \backslash \hat{\Sigma}$ that contains all the points above the line $z = r$.

Note that Θ is convex. If not, then there is a line segment $[pq]$ for some $p, q \in \Theta$ that cuts from $\hat{\Sigma}$ a compact piece. Consider the plane



Π thru $[pq]$ that is perpendicular to the (x, z)-plane. Note that Π cuts from Σ a compact region Δ. By general position argument (see 15.10) we can assume that Δ is a compact surface with boundary line in Π and the remaining part of Δ lies on one side from Π. Since the plane Π is convex, this statement contradicts 18.23.

Summarizing, Θ is an open convex set of H that contains all points above $z = r$. By convexity, together with any point w, the set Θ contains all points on the half-lines that point up from it. Whence it contain all points with z-coordinate larger than the z-coordinate of w. Since Θ is open it can be described by inequality $z > r_0$. It follows that

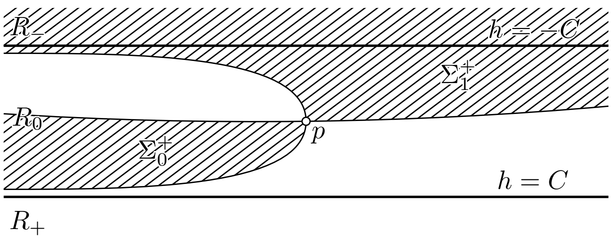


the plane $z = r_0$ supports Σ at some point (in fact at many points). By 18.1, the latter is impossible — a contradiction. $\square$

Proof of 18.29. Denote by Σ the graph $z = f(x, y)$. Assume contrary; that is, Σ lies between two planes $z = \pm C$.

Note that the function f cannot be constant. It follows that the tangent plane T_p at some point $p \in \Sigma$ is not horizontal.

Denote by Σ^+ the part of Σ that lies above T_p. Note that it has at least two connected components which are approaching p from both sides in the principle direction with positive principle curvature. Indeed if there would be a curve that runs in Σ^+ and approaches p from both sides, then it would cut a disc from Σ with boundary line above T_p and some points below it; the latter contradicts 18.23.



The surface Σ seeing from above.

Summarizing, Σ^+ has at least two connected components, denote them by Σ_0^+ and Σ_1^+. Let $z = h(x, y) = a{\cdot}x + b{\cdot}y + c$ be the equation of T_p. Note that Σ^+ contains all points in the region

$$R_- = \{\, (x, y, f(x, y)) \in \Sigma \; : \; h(x, y) < -C \,\}$$

which is a connected set and no points in

$$R_+ = \{\, (x, y, f(x, y)) \in \Sigma \; : \; h(x, y) > C \,\}$$

Whence one of the connected components, say Σ_0^+, lies in

$$R_0 = \{\, (x, y, f(x, y)) \in \Sigma \; : \; |h(x, y)| \leqslant C \,\}.$$

This set lies on a bounded distance from the line of intersection of T_p with the (x, y)-plane.

Moving the plane T_p slightly upward, we can cut from Σ_0^+ a proper surface with boundary line lying in this plane (see 15.10). The obtained surface is still on a bounded distance to a line which is impossible by 18.32. $\square$

The following exercise gives a condition that guarantees that a saddle surface is a graph; it can be used in combination with Bernshtein's theorem.

18.33. Advanced exercise. *Let Σ be a smooth strictly saddle disk in $\mathbb{R}^3$. Assume that the orthogonal projection to the (x,y)-plane maps the boundary line of Σ injectively to a convex closed curve. Show that the orthogonal projection to the (x,y)-plane is injective on Σ.*

In particular, Σ is the graph $z = f(x,y)$ of a function f defined on a convex figure in the (x,y)-plane.

Remarks

Note that Bernstein's theorem and the lemma in its proof do not hold for nonstrictly saddle surfaces; counterexamples can be found among infinite cylinders over smooth regular curves. In fact it can be shown that these are the only counterexamples; a proof is based on the same idea, but more technical.

By 18.26, saddle surfaces can be defined as smooth surfaces without hats. This definition can be used for arbitrary surfaces, not necessarily smooth. Some results, for example Bernshtein's characterization of saddle graphs can be extended to generalized saddle surfaces, but this class of surfaces is far from being understood. Some nontrivial properties were proved by Samuil Shefel [**shefel**]; see also [2, Capter 4].

Part IV

Geodesics

Chapter 19

Shortest paths

Shortest paths

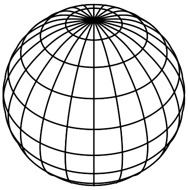
Let p and q be two points on a surface Σ. Recall that $|p-q|_\Sigma$ denotes the induced length distance from p to q; that is, the exact lower bound on lengths of paths in Σ from p to q.

Note that if Σ is smooth, then any two points in Σ can be joined by a piecewise smooth path. Since any such path is rectifiable, the value $|p-q|_\Sigma$ is finite for any pair $p, q \in \Sigma$.

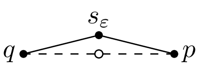
A path γ from p to q in Σ that minimizes the length is called a *shortest path* from p to q.

The image of a shortest path between p and q in Σ is usually denoted by $[pq]$ or by $[pq]_\Sigma$. In general there might be no shortest path between two given points on the surface and it might be many of them; this is shown in the following two examples.

Usually, if we write $[pq]_\Sigma$, then we assume that a shortest path exists and we made a choice of one of them.



Nonuniqueness. There are plenty of shortest paths between the poles on the sphere — each meridian is a shortest path.



Nonexistence. Let Σ be the (x,y)-plane with removed origin. Consider two points $p = (1,0,0)$ and $q = (-1,0,0)$ in Σ.

Note that $|p-q|_\Sigma = 2$. Indeed, given $\varepsilon > 0$, consider the point $s_\varepsilon = (0,\varepsilon,0)$. Observe that the polygonal path $ps_\varepsilon q$ lies in Σ and its length $2\cdot\sqrt{1+\varepsilon^2}$ approaches 2 as $\varepsilon \to 0$. It follows that

$|p-q|_\Sigma \leqslant 2$. On the other hand $|p-q|_\Sigma \geqslant |p-q|_{\mathbb{R}^3} = 2$; therefore $|p-q|_\Sigma = 2$.

It follows that a shortest path from p to q, if it exists, must have length 2. By triangle inequality any curve of length 2 from p to q must run along the line segment $[pq]$; in particular it must pass thru the origin. Since the origin does not lie in Σ, there is no shortest from p to q in Σ

19.1. Proposition. *Any two points in a proper smooth surface can be joined by a shortest path.*

Proof. Fix a proper smooth surface Σ with two points p and q. Set $\ell = |p-q|_\Sigma$.

By the definition of induced length metric, there is a sequence of paths γ_n from p to q in Σ such that

$$\operatorname{length}\gamma_n \to \ell \quad \text{as} \quad n \to \infty.$$

Without loss of generality, we may assume that $\operatorname{length}\gamma_n < \ell + 1$ for any n and each γ_n is parameterized proportional to its arc-length. In particular each path $\gamma_n\colon [0,1] \to \Sigma$ is $(\ell+1)$-Lipschitz; that is,

$$|\gamma(t_0) - \gamma(t_1)| \leqslant (\ell+1)\cdot|t_0 - t_1|$$

for any $t_0, t_1 \in [0,1]$.

Note that the image of γ_n lies in the closed ball $\bar{B}[p, \ell+1]$ for any n. It follows that the coordinate functions of γ_n are uniformly equicontinuous and uniformly bounded. By 4.3, we can pass to a converging subsequence of γ_n; denote by $\gamma_\infty\colon [0,1] \to \mathbb{R}^3$ its limit.

As a limit of uniformly continuous sequence, γ_∞ is continuous; that is, γ_∞ is a path. Evidently γ_∞ runs from p to q; in particular

$$\operatorname{length}\gamma_\infty \geqslant \ell.$$

Since Σ is a closed set, γ_∞ lies in Σ. Finally, by 9.15,

$$\operatorname{length}\gamma_\infty \leqslant \ell.$$

That is, $\gamma_\infty = \ell$ or, equivalently, γ_∞ is a shortest path from p to q. $\square$

Closest point projection

19.2. Lemma. *Let R be a closed convex set in $\mathbb{R}^3$. Then for every point $p \in \mathbb{R}^3$ there is a unique point $\bar{p} \in R$ that minimizes the distance to R; that is, $|p - \bar{p}| \leqslant |p - x|$ for any point $x \in R$.*

Moreover the map $p\mapsto\bar p$ is short; that is,

$$❶ \qquad |p-q|\geqslant|\bar p-\bar q|$$

for any pair of points $p,q\in\mathbb{R}^3$.

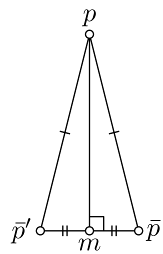
The map $p\mapsto\bar p$ is called the *closest point projection*; it maps the Euclidean space to R. Note that if $p\in R$, then $\bar p=p$.

Proof. Fix a point p and set

$$\ell=\inf\{\,|p-x|\;:\;x\in R\,\}.$$

Choose a sequence $x_n\in R$ such that $|p-x_n|\to\ell$ as $n\to\infty$.

Without loss of generality, we can assume that all the points x_n lie in a ball of radius $\ell+1$ centered at p. Therefore we can pass to a partial limit $\bar p$ of x_n; that is, $\bar p$ is a limit of a subsequence of x_n. Since R is closed, $\bar p\in R$. By construction



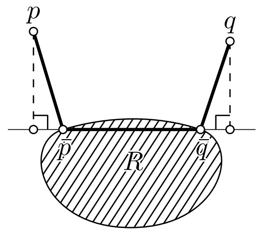
$$\begin{aligned}|p-\bar p|&=\lim_{n\to\infty}|p-x_n|=\\&=\ell.\end{aligned}$$

Hence the existence follows.

Assume there are two distinct points $\bar p,\bar p'\in R$ that minimize the distance to p. Since R is convex, their midpoint $m=\tfrac12\cdot(\bar p+\bar p')$ lies in R. Note that $|p-\bar p|=|p-\bar p'|=\ell$; that is, $\triangle p\bar p\bar p'$ is isosceles and therefore $\triangle p\bar p m$ is right with the right angle at m. Since a leg of a right triangle is shorter than its hypotenuse, we have $|p-m|<\ell$ — a contradiction.

It remains to prove inequality ❶.

We can assume that $\bar p\neq\bar q$, otherwise there is nothing to prove. Note that if $p\neq\bar p$ (that is, if $p\notin R$), then $\angle p\bar p\bar q$ is right or obtuse. Otherwise there would be a point x on the line segment $[\bar q\bar p]$ that is closer to p than $\bar p$. Since R is convex, the line segment $[\bar q\bar p]$ and therefore x lie in R. Hence $\bar p$ is not closest to p — a contradiction.



The same way we can show that if $q\neq\bar q$, then $\angle q\bar q\bar p$ is right or obtuse.

We have to consider the following 4 cases: (1) $p\neq\bar p$ and $q\neq\bar q$, (2) $p=\bar p$ and $q\neq\bar q$, (3) $p\neq\bar p$ and $q=\bar q$, (4) $p=\bar p$ and $q=\bar q$. In all these cases the obtained angle estimates imply that the orthogonal projection of the line segment $[pq]$ to the line $\bar{p}\bar{q}$ contains the line segment $[\bar{p}\bar{q}]$. In particular

$$|p-q| \geqslant |\bar{p}-\bar{q}|.$$

□

19.3. Corollary. *Assume a surface Σ bounds a closed convex region R and $p, q \in \Sigma$. Denote by W the outer closed region of Σ; in other words W is the union of Σ and the complement of R. Then*

$$\operatorname{length}\gamma \geqslant |p-q|_\Sigma$$

for any path γ in W from p to q. Moreover if γ does not lie in Σ, then the inequality is strict.

Proof. The first part of the corollary follows from the lemma and the definition of length. Indeed consider the closest point projection $\bar{\gamma}$ of γ. Note that $\bar{\gamma}$ lies in Σ and connects p to q therefore

$$\operatorname{length}\bar{\gamma} \geqslant |p-q|_\Sigma.$$

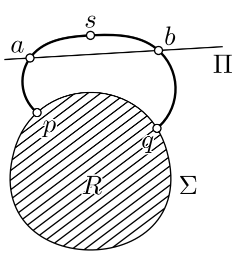
To prove the first statemnt, it is sufficient to show that

❷ $$\operatorname{length}\gamma \geqslant \operatorname{length}\bar{\gamma}.$$

Consider an inscribed polygonal line $p_0 \dots p_n$ in γ. Denote by $\bar{p}_i$ the closest point projection of p_i to R. Note that the polygonal line $\bar{p}_0 \dots \bar{p}_n$ is inscribed in $\bar{\gamma}$; moreover any inscribed polygonal line in $\bar{\gamma}$ can appear this way. By 19.2 $|p_i - p_{i-1}| \geqslant |\bar{p}_i - \bar{p}_{i-1}|$ for any i. Therefore

$$\operatorname{length} p_0 \dots p_n \geqslant \operatorname{length} \bar{p}_0 \dots \bar{p}_n.$$

Taking least upper bound of each side of the inequality for all inscribed polygonal lines $p_0 \dots p_n$ in γ, we get ❷.



It remains to prove the second statement. Suppose that there is a point $s = \gamma(t_1) \notin \Sigma$; note that $s \notin R$. By the separation lemma (6.4) there is a plane Π that cuts s from Σ. The curve γ must intersect at least at two points: one point before t_1 and one after; let $a = \gamma(t_0)$ and $b = \gamma(t_2)$ be these points. Note that the arc of γ from a to b is strictly longer that $|a-b|$; indeed its length is at least $|a-s| + |s-b|$ and $|a-s| + |s-b| > |a-b|$ since $s \notin [ab]$.

Remove from γ the arc from a to b and glue in the line segment $[ab]$; denote the obtained curve by γ_1. From above, we have that

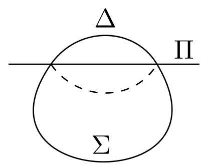
$$\operatorname{length}\gamma > \operatorname{length}\gamma_1$$

Note that γ_1 runs in W. Therefore by the first part of corollary, we have

$$\operatorname{length}\gamma_1 \geqslant |p-q|_\Sigma.$$

Whence the second statement follows. $\square$

19.4. Exercise. *Suppose Σ is a closed smooth surface that bounds a convex region R in $\mathbb{R}^3$ and Π is a plane that cuts a hat Δ from Σ. Assume that the reflection of the interior of Δ across Π lies in the interior of R. Show that Δ is* convex *with respect to the intrinsic metric of Σ; that is, if both ends of a shortest path in Σ lie in Δ, then the entire path lies in Δ.*



Let us define the *intrinsic diameter* of a closed surface Σ as the exact upper bound on the lengths of shortest paths in the surface.

19.5. Exercise. *Assume that a closed smooth surface Σ with positive Gauss curvature lies in a unit ball.*

(a) Show that the intrinsic diameter of Σ cannot exceed π.

(b) Show that the area of Σ cannot exceed $4{\cdot}\pi$.

Chapter 20

Geodesics

Definition

A smooth curve γ on a smooth surface Σ is called *geodesic* if for any t, the acceleration $\gamma''(t)$ is perpendicular to the tangent plane $\mathrm{T}_{\gamma(t)}$.

20.1. Exercise. *Show that the helix*

$$\gamma(t) = (\cos t, \sin t, a \cdot t)$$

is a geodesic on the cylindrical surface described by the equation $x^2 + + y^2 = 1$.

20.2. Exercise. *Assume that a smooth surface Σ is mirror symmetric with respect to a plane Π. Suppose that Σ and Π intersect along a smooth regular curve γ. Show that γ parameterized by its arc-length is a geodesic on Σ.*

Recall that asymptotic line is defined on page 130.

20.3. Exercise. *Suppose that a curve γ is a geodesic and, at the same time, is an asymptotic line on a smooth surface Σ. Show that γ is a line segment.*

Physically, geodesics can be understood as the trajectories of a particle that slides on Σ without friction. Indeed, since there is no friction, the force that keeps the particle on Σ must be perpendicular to Σ. Therefore, by the second Newton's laws of motion, we get that the acceleration γ'' is perpendicular to $\mathrm{T}_{\gamma(t)}$.

Existence

The following lemma and proposition can be also interpreted physically; lemma follow from the conservation of energy and the proposition gives smooth dependence of trajectory of a particle depending on its initial position and velocity.

20.4. Lemma. *Any geodesic has constant speed.*

More precisely, if γ is a geodesic on a smooth surface, then $|\gamma'|$ is constant.

Proof. Since $\gamma'(t)$ is a tangent vector at $\gamma(t)$, we have that $\gamma''(t) \perp \perp \gamma'(t)$, or equivalently $\langle\gamma'',\gamma'\rangle = 0$ for any t. Whence

$$\begin{aligned}\langle\gamma',\gamma'\rangle' &= 2{\cdot}\langle\gamma'',\gamma'\rangle = \\ &= 0.\end{aligned}$$

That is, $|\gamma'(t)|^2 = \langle\gamma'(t),\gamma'(t)\rangle$ is constant. □

20.5. Proposition. *Let Σ be a smooth surface without boundary. Given a tangent vector v to Σ at a point p there is a unique geodesic $\gamma\colon \mathbb{I} \to \Sigma$ defined on a maximal open interval $\mathbb{I} \ni 0$ that starts at p with velocity vector v; that is, $\gamma(0) = p$ and $\gamma'(0) = \mathrm{v}$.*

Moreover

(a) the map $(p,\mathrm{v},t) \mapsto \gamma(t)$ is smooth in its domain of definition.

(b) if Σ is proper, then $\mathbb{I} = \mathbb{R}$; that is, the maximal interval is whole real line.

The proof of this proposition relies on the existence and uniqueness of the initial value problem (3.1).

20.6. Lemma. *Let f be a smooth function defined on an open domain in $\mathbb{R}^2$. A smooth curve $t \mapsto \gamma(t) = (x(t), y(t), z(t))$ is the geodesic in a graph $z = f(x,y)$ if and only if $z(t) = f(x(t),y(t))$ for any t and the functions $t \mapsto x(t)$ and $t \mapsto y(t)$ satisfy a differential equation*

$$\begin{cases} x'' = g(x,y,x',y'), \\ y'' = h(x,y,x',y'), \end{cases}$$

where the functions g and h are smooth functions of four variables that determined by f.

The proof of the lemma is done by means of direct calculations.

Proof. In the following calculations, we often omit the arguments — we may write x instead of $x(t)$ and f instead of $f(x,y)$ or $f(x(t),y(t))$ and so on.

First let us calulate $z''(t)$ in terms of f, $x(t)$, and $y(t)$.

$$\begin{aligned} z''(t) &= f(x(t),y(t))'' = \\ &= \left(\tfrac{\partial f}{\partial x}\cdot x' + \tfrac{\partial f}{\partial y}\cdot y'\right)' = \\ &= \tfrac{\partial^2 f}{\partial x^2}\cdot(x')^2 + \tfrac{\partial f}{\partial x}\cdot x'' + \tfrac{\partial^2 f}{\partial y^2}\cdot(y')^2 + \tfrac{\partial f}{\partial y}\cdot y''. \end{aligned}$$

Now observe that the equation

❶ $$\gamma''(t) \perp \mathrm{T}_{\gamma(t)}$$

means that γ'' is perpendicular to two basis vectors in $\mathrm{T}_{\gamma(t)}$. Therefore the vector equation ❶ can be rewritten as the following system of two real equations

$$\begin{cases} \langle \gamma''(t), \tfrac{\partial s}{\partial x}\rangle = 0, \\ \langle \gamma''(t), \tfrac{\partial s}{\partial y}\rangle = 0, \end{cases}$$

where $s(x,y) := (x,y,f(x,y))$, $x = x(t)$, and $y = y(t)$.

Observe that $\frac{\partial s}{\partial x} = (1,0,\frac{\partial f}{\partial x})$ and $\frac{\partial s}{\partial y} = (0,1,\frac{\partial f}{\partial y})$. Since $\gamma'' = (x'',y'',z'')$, we can rewrite the system the following way.

$$\begin{cases} x'' + \tfrac{\partial f}{\partial x}\cdot z'' = 0, \\ y'' + \tfrac{\partial f}{\partial y}\cdot z'' = 0, \end{cases}$$

It remains use expression ❷ for z'', combine like terms and simplify. $\square$

Proof of 20.5. Let $z = f(x,y)$ be a description of Σ in a tangent-normal coordinates at p. By Lemma 20.6 the condition $\gamma''(t) \perp \mathrm{T}_{\gamma(t)}$ can be written as a second order differential equation. Applying the existence and uniqueness of the initial value problem (3.1) we get existence and uniqueness of geodesic γ in a in a small interval $(-\varepsilon,\varepsilon)$ for some $\varepsilon > 0$.

Let us extend γ to a maximal open interval $\mathbb{I}$. Suppose there is another geodesic γ_1 with the same initial data that is defined on a maximal open interval $\mathbb{I}_1$. Suppose γ_1 splits from γ at some time $t_0 > 0$; that is, γ_1 coincides with γ on the interval $[0,t_0)$, but they are different on the interval $[0,t_0+\varepsilon)$ for any $\varepsilon > 0$. By continuity $\gamma_1(t_0) = \gamma(t_0)$ and $\gamma'(t_0) = \gamma'(t_0)$. Applying uniqueness of the initial value problem (3.1) again, we get that γ_1 coincides with γ in a small neighborhood of t_0 — a contradiction.

The case $t_0 < 0$ can be proved along the same lines. It follows that γ_1 coincides with γ.

Part *(a)* follows since the solution of the initial value problem depends smoothly on the initial data (3.1).

Suppose *(b)* does not hold; that is, the maximal interval $\mathbb{I}$ is a proper subset of the real line $\mathbb{R}$. Without loss of generality we may assume that $b = \sup \mathbb{I} < \infty$.

By 20.4 $|\gamma'|$ is constant, in particular $t \mapsto \gamma(t)$ is a uniformly continuous function. Therefore the limit

$$q = \lim_{t \to b} \gamma(t)$$

is defined. Since Σ is a proper surface, $q \in \Sigma$.

Applying the argument above in a tangent-normal coordinates at q shows that γ can be extended as a geodesic behind q. Therefore $\mathbb{I}$ is not a maximal interval — a contradiction. □

Exponential map

Let Σ be smooth regular surface and $p \in \Sigma$. Given a tangent vector $\mathrm{v} \in \mathrm{T}_p$ consider a geodesic γ_{v} in Σ that runs from p with the initial velocity v; that is, $\gamma(0) = p$ and $\gamma'(0) = \mathrm{v}$.

The point $q = \gamma_{\mathrm{v}}(1)$ is called *exponential map* of v, or briefly $q = \exp_p \mathrm{v}$. (There is a reason to call this map *exponential*, but it will take us too far from the subject.) By 20.5, the map $\exp_p \colon \mathrm{T}_p \to \Sigma$ is smooth and defined in a neighborhood of zero in T_p; moreover, if Σ is proper, then $\exp_p$ is defined on the whole space T_p.

Note that the exponential map $\exp_p$ is defined on the tangent plane T_p, which is a smooth surfaces, and its target is another smooth surface Σ. Observe that one can identify the plane T_p with its tangent plane $\mathrm{T}_0\mathrm{T}_p$ so the linearization $L_0 \exp_p$ maps T_p to itself. Further note that by the definition of exponential map we have that $L_0 \exp_p(\mathrm{v}) = \mathrm{v}$ for any $\mathrm{v} \in \mathrm{T}_p$.

The above discussion is summarized in the following statement.

20.7. Observation. *Let Σ be a smooth surface and $p \in \Sigma$. Then the exponential map $\exp_p \colon \mathrm{T}_p \to \Sigma$ is smooth and its linearization $L_0(\exp_p) \colon \mathrm{T}_p \to \mathrm{T}_p$ is the identity map.*

20.8. Proposition. *Let Σ be smooth surface and $p \in \Sigma$. Then the exponential map $\exp_p \colon \mathrm{T}_p \to \Sigma$ is a smooth regular parametrization of a neighborhood of p in Σ by a neighborhood of 0 in the tangent plane T_p.*

Moreover for any $p \in \Sigma$ there is $\varepsilon > 0$ such that for any $q \in \Sigma$ such that $|q - p|_\Sigma < \varepsilon$ the map $\exp_q \colon \mathrm{T}_q \to \Sigma$ is a smooth regular parametrization of the ε-neighborhood of q in Σ by the ε-neighborhood of zero in the tangent plane T_q.

The proposition follows from the observation and the inverse function theorem (2.2).

Proof. Let $z = f(x, y)$ be a local graph representation of Σ in the tangent-normal coordinates at p. Note that (x, y)-plane coincides with the tangent plane T_p.

Denote by s be the composition of the exponential map $\exp_p$ with the orthogonal projection $(x, y, z) \mapsto (x, y)$. By 20.7, the linearization $L_0 s$ is the identity. Applying the inverse function theorem (2.2) we get the first part of the proposition.

The second part can be proved along the same lines, using the second part the inverse function theorem (2.2); it guarantees that the size of the neighborhood in T_q for all points q sufficiently close to p. $\square$

Shortest paths are geodesics

20.9. Proposition. *Let Σ be a smooth regular surface. Then any shortest path γ in Σ parameterized proportional to its arc-length is a geodesic in Σ. In particular γ is a smooth curve.*

A partial converse to the first statement also holds: a sufficiently short arc of any geodesic is a shortest path. More precisely, any point p in Σ has a neighborhood U such that any geodesic that lies completely in U is a shortest path.

A geodesic might not form a shortest path, but if this is the case, then it is called a *minimizing geodesic.* Note that according to the proposition, any shortest path is a reparametrization of a minimizing geodesic.

A formal proof will be given much latter; see page 179. The following informal physical explanation might be sufficiently convincing. In fact, if one assumes that γ is smooth, then it is easy to convert this explanation into a rigorous proof.

Informal explanation. Let us think about a shortest path γ as of stable position of a stretched elastic thread that is forced to lie on a frictionless surface. Since it is frictionless, the force density $N(t)$ that keeps the geodesic γ in the surface must be proportional to the normal vector to the surface at $\gamma(t)$.

The tension in the thread has to be the same at all points (otherwise the thread would move back or forth and it would not be stable). Denote the tension by T.

We can assume that γ has unit speed; in this case the net force from tension to the arc $\gamma_{[t_0,t_1]}$ is $T\cdot(\gamma'(t_1) - \gamma'(t_0))$. Hence the density of net force from tension at t_0 is

$$F(t_0) = \lim_{t_1 \to t_0} T \cdot \frac{\gamma'(t_1) - \gamma'(t_0)}{t_1 - t_0} = \\ = T \cdot \gamma''(t_0).$$

According to the second Newton's law of motion, we have

$$F(t) + N(t) = 0;$$

which implies that $\gamma''(t) \perp \mathrm{T}_{\gamma(t)}\Sigma$. $\square$

Note that according to the proposition above, any shortest path parameterized by its arc-length is a smooth curve. This observation should help to solve in the following two exercises.

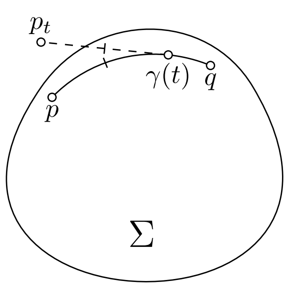
20.10. Exercise. *Show that two shortest paths can cross each other at most once. More precisely, if two shortest paths have two distinct common points p and q, then either these points are the ends of both shortest paths or both shortest paths contain an arc from p to q.*

Show by example that nonoverlapping geodesics can cross each other an arbitrary number of times.

20.11. Exercise. *Assume that a smooth regular surface Σ is mirror symmetric with respect to a plane Π. Show that no shortest path α in Σ can* cross Π *more than once.*

In other words, if you travel along α, then you change the sides of Π at most once.

20.12. Advanced exercise. *Let Σ be a smooth closed strictly convex surface in $\mathbb{R}^3$ and $\gamma\colon [0,\ell] \to \Sigma$ be a unit-speed minimizing geodesic. Set $p = \gamma(0)$, $q = \gamma(\ell)$ and*



$$p_t = \gamma(t) - t \cdot \gamma'(t),$$

where $\gamma'(t)$ denotes the velocity vector of γ at t.

Show that for any $t \in (0,\ell)$, one cannot see *q from p_t; that is, the line segment $[p_t q]$ intersects Σ at a point distinct from q.*

Show that the statement does not hold without assuming that γ is minimizing.

Liberman's lemma

The following lemma is a smooth analog of lemma proved by Joseph Liberman [35].

20.13. Liberman's lemma. *Assume γ is a unit-speed geodesic on the graph $z = f(x, y)$ of a smooth convex function f defined on an open subset of the plane. Suppose that $\gamma(t) = (x(t), y(t), z(t))$ Then $t \mapsto z(t)$ is a convex function; that is, $z''(t) \geqslant 0$ for any t.*

Proof. Choose the orientation on the graph so that the unit normal vector ν always points up; that is, it has positive z-coordinate.

Since γ is a geodesic, we have $\gamma''(t) \perp \mathrm{T}_{\gamma(t)}$, or equivalently $\gamma''(t)$ is proportional to $\nu_{\gamma(t)}$ for any t. Further

$$\gamma''(t) = k(t)\cdot\nu_{\gamma(t)},$$

where $k(t)$ is the normal curvature at $\gamma(t)$ in the direction of $\gamma'(t)$.

Therefore

$$❷ \qquad z''(t) = \cos(\theta_\gamma(t))\cdot k(t)\cdot\nu_{\gamma(t)},$$

where $\theta_\gamma(t)$ denotes the angle between $\nu_{\gamma(t)}$ and the z-axis.

Since ν points up, we have $\theta_\gamma(t) < \frac{\pi}{2}$, or equivalently

$$\cos(\theta_\gamma(t)) > 0$$

for any t.

Since f is convex, we have that tangent plane supports the graph from below at any point; in particular $k(t) \geqslant 0$ for any t. It follows that the right hand side in ❷ is nonnegative; whence the statement follows. $\square$

20.14. Exercise. *Assume γ is a unit-speed geodesic on a smooth convex surface Σ and p in the interior of a convex set bounded by Σ. Set $\rho(t) = |p - \gamma(t)|^2$. Show that $\rho''(t) \leqslant 2$ for any t.*

Bound on total curvature

20.15. Theorem. *Assume Σ is a graph $z = f(x, y)$ of a convex ℓ-Lipschitz function f defined on an open set in the (x, y)-plane. Then the total curvature of any geodesic in Σ is at most $2\cdot\ell$.*

The above theorem was proved by Vladimir Usov [51], an amusing generalization was found by David Berg [5].

Proof. Let $\gamma(t) = (x(t), y(t), z(t))$ be a unit-speed geodesic on Σ. According to Liberman's lemma $z(t)$ is convex.

Since the slope of f is at most ℓ, we have

$$|z'(t)| \leqslant \tfrac{\ell}{\sqrt{1+\ell^2}}.$$

If γ is defined on the interval $[a, b]$, then

$$\begin{aligned}\int\limits_a^b z''(t) = z'(b) - z'(a) \leqslant \\ \leqslant 2\cdot\tfrac{\ell}{\sqrt{1+\ell^2}}.\end{aligned}$$

Further, note that z'' is the projection of γ'' to the z-axis. Since f is ℓ-Lipschitz, the tangent plane $\mathrm{T}_{\gamma(t)}\Sigma$ cannot have slope greater than ℓ for any t. Because γ'' is perpendicular to that plane, we have that

$$|\gamma''(t)| \leqslant z''(t)\cdot\sqrt{1+\ell^2}.$$

Recall that $\Phi(\gamma)$ denotes the total curvature of curve γ. It follows that

$$\begin{aligned}\Phi(\gamma) &= \int\limits_a^b |\gamma''(t)|\cdot dt \leqslant \\ &\leqslant \sqrt{1+\ell^2}\cdot \int\limits_a^b z''(t)\cdot dt \leqslant \\ &\leqslant 2\cdot\ell.\end{aligned}$$

$\square$

20.16. Exercise. *Note that the graph $z = \ell\cdot\sqrt{x^2+y^2}$ with removed origin is a smooth surface; denote it by Σ. Show that any both side infinite geodesic γ has total curvature exactly $2\cdot\ell$.*

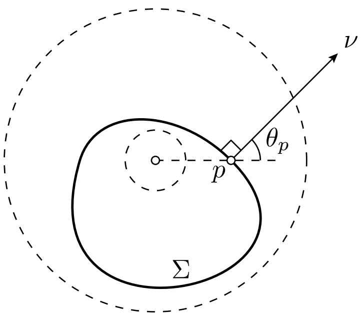
Note that the function $f(x, y) = \ell\cdot\sqrt{x^2+y^2}$ is ℓ-Lipschitz. The graph $z = f(x, y)$ in the exercise can be smoothed in a neighborhood of the origin while keeping it convex. It follows that the estimate in the Usov's theorem is optimal.

20.17. Exercise. *Assume f is a convex $\frac{3}{2}$-Lipschitz function defined on the (x, y)-plane. Show that any geodesic γ on the graph $z = f(x, y)$ is simple; that is, it has no self-intersections.*

Construct a convex 2-Lipschitz function defined on the (x, y)-plane with a nonsimple geodesic γ on its graph $z = f(x, y)$.

20.18. Theorem. *Suppose a smooth surface Σ bounds a convex set K in the Euclidean space. Assume $B(0,\varepsilon)\subset K\subset B(0,1)$. Then the total curvatures of any shortest path in Σ can be bounded in terms of ε.*

The following exercise will guide you thru the proof of the theorem.



20.19. Exercise. *Let Σ be as in the theorem and γ be a unit-speed shortest path in Σ. Denote by ν_p the unit normal vector that points outside of Σ; denote by θ_p the angle between ν_p and the direction from the origin to a point $p\in\Sigma$. Set $\rho(t)=$ $=|\gamma(t)|^2$; let $k(t)$ be the curvature of γ at t.*

(a) Show that $\cos\theta_p\geqslant\varepsilon$ for any $p\in\Sigma$.

(b) Show that $|\rho'(t)|\leqslant 2$ for any t.

(c) Show that

$$\rho''(t)=2-2\cdot k(t)\cdot\cos\theta_{\gamma(t)}\cdot|\gamma(t)|$$

for any t.

(d) Use the closest-point projection from the unit sphere to Σ to show that

$$\operatorname{length}\gamma\leqslant\pi.$$

(e) Use the statements above to conclude that

$$\Phi(\gamma)\leqslant\frac{100}{\varepsilon^2}.$$

Note that the obtained bound on total curvature goes to infinity as $\varepsilon\to 0$. In fact there is a bound independent of ε [33].

Chapter 21

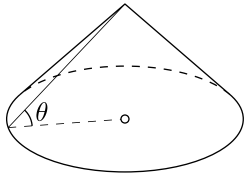
Parallel transport

We start to study the *intrinsic geometry* of surfaces. A property is called *intrinsic* if it is defined in terms of measuring things inside the surface, for example length of curves or angles between the curves that lie in the surface. Otherwise, if a definition of property use ambient space, then it is called *extrinsic*.

For instance, a shortest path is an object of intrinsic geometry of a surface, while definition of geodesic is not intrinsic — it requires acceleration which needs the ambient space. However, from Proposition 20.9 it follows that there is an intrinsic definition of geodesics as *locally shortest paths with constant-speed parameterization*.

The following exercise should help you to be in the right mood for this; it might look like a tedious problem in calculus, but actually it is an easy problem in geometry.

21.1. Exercise. *There is a mountain of frictionless ice formed by a cone with a circular base. Suppose that a cowboy stands at the bottom and he wants to climb the mountain; he throws up his lasso which slips neatly over the top of the cone, pulls it tight and starts to climb. If the angle of inclination θ is large, there is no problem; the lasso grips tight and up he goes. On the other hand if θ is small, the lasso slips off as soon as the cowboy pulls on it.*



What is the critical angle θ_0 at which the cowboy can no longer climb the ice-mountain?

Parallel fields

Let Σ be a smooth surface in the Euclidean space and $\gamma\colon[a,b]\to\Sigma$ be a smooth curve. A smooth vector-valued function $t\mapsto \mathrm{v}(t)$ is called a *tangent field* on γ if the vector $\mathrm{v}(t)$ lies in the tangent plane $\mathrm{T}_{\gamma(t)}\Sigma$ for each t.

A tangent field $\mathrm{v}(t)$ on γ is called *parallel* if $\mathrm{v}'(t)\perp \mathrm{T}_{\gamma(t)}$ for any t.

In general the family of tangent planes $\mathrm{T}_{\gamma(t)}\Sigma$ is not parallel. Therefore one cannot expect to have a truly parallel family $\mathrm{v}(t)$ with $\mathrm{v}'\equiv 0$. The condition $\mathrm{v}'(t)\perp \mathrm{T}_{\gamma(t)}$ means that the family is as parallel as possible — it rotates together with the tangent plane, but does not rotate inside the plane.

Note that by the definition of geodesic, the velocity field $\mathrm{v}(t) = \gamma'(t)$ of any geodesic γ is parallel on γ.

21.2. Exercise. *Let Σ be a smooth regular surface in the Euclidean space, $\gamma\colon[a,b]\to\Sigma$ a smooth curve and $\mathrm{v}(t)$, $\mathrm{w}(t)$ parallel vector fields along γ.*

(a) Show that $|\mathrm{v}(t)|$ is constant.

(b) Show that the angle $\theta(t)$ between $\mathrm{v}(t)$ and $\mathrm{w}(t)$ is constant.

Parallel transport

Assume $p=\gamma(a)$ and $q=\gamma(b)$. Given a tangent vector $\mathrm{v}\in\mathrm{T}_p$ there is unique parallel field $\mathrm{v}(t)$ along γ such that $\mathrm{v}(a)=\mathrm{v}$. The latter follows from 3.1; the uniqueness also follows from Exercise 21.2.

The vector $\mathrm{w}=\mathrm{v}(b)\in\mathrm{T}_q$ is called the *parallel transport* of v along γ in Σ.

The parallel transport is denoted by ι_γ; so we can write $\mathrm{w}=\iota_\gamma(\mathrm{v})$ or we can write $\mathrm{w}=\iota_\gamma(\mathrm{v})_\Sigma$ if we need to emphasize that we consider surface Σ. From the Exercise 21.2, it follows that parallel transport $\iota_\gamma\colon\mathrm{T}_p\to\mathrm{T}_q$ is an isometry. In general, the parallel transport $\iota_\gamma\colon\mathrm{T}_p\to\mathrm{T}_q$ depends on the choice of γ; that is, for another curve γ_1 connecting p to q in Σ, the parallel transport $\iota_{\gamma_1}\colon\mathrm{T}_p\to\mathrm{T}_q$ might be different.

Suppose that γ_1 and γ_2 are two smooth curves in smooth surfaces Σ_1 and Σ_2. Denote by $\nu_i\colon\Sigma_i\to\mathbb{S}^2$ the Gauss maps of Σ_1 and Σ_2. If $\nu_1\circ\gamma_1(t)=\nu_2\circ\gamma_2(t)$ for any t, then we say that curves γ_1 and γ_2 have *identical spherical images* in Σ_1 and Σ_2 respectively.

In this case tangent plane $\mathrm{T}_{\gamma_1(t)}\Sigma_1$ is parallel to $\mathrm{T}_{\gamma_2(t)}\Sigma_2$ for any t and so we can identify $\mathrm{T}_{\gamma_1(t)}\Sigma_1$ and $\mathrm{T}_{\gamma_2(t)}\Sigma_2$. In particular if $\mathrm{v}(t)$ is a tangent vector field along γ_1, then it is also a tangent vector field along γ_2. Moreover $\mathrm{v}'(t)\perp\mathrm{T}_{\gamma_1(t)}\Sigma_1$ is equivalent to $\mathrm{v}'(t)\perp\mathrm{T}_{\gamma_2(t)}\Sigma_2$; that is, if $\mathrm{v}(t)$ is a parallel vector field along γ_1, then it is also a parallel vector field along γ_2.

The dicussion above leads to the following observation that will play key role in the sequel.

21.3. Observation. *Let γ_1 and γ_2 be two smooth curves in smooth surfaces Σ_1 and Σ_2. Suppose that γ_1 and γ_2 have identical spherical images in Σ_1 and Σ_2 respectively. Then the parallel transport ι_{γ_1} and ι_{γ_2} are identical.*

21.4. Exercise. *Let Σ_1 and Σ_2 be two surfaces with common curve γ. Suppose that Σ_1 bounds a region that contains Σ_2. Show that the parallel translations along γ in Σ_1 coincides the parallel translations along γ in Σ_2.*

The following physical interpretation of parallel translation was suggested by Mark Levi [34]; it might help to build right intuition.

Think of walking along γ and carrying a perfectly balanced bike wheel keeping its axis normal to Σ and touching only the axis. It should be physically evident that if the wheel is non-spinning at the starting point p, then it will not be spinning after stopping at q. (Indeed, by pushing the axis one cannot produce torque to spin the wheel.) The map that sends the initial position of the wheel to the final position is the parallel transport ι_γ.

The observation above essentially states that *moving axis of the wheel without changing its direction does not change the direction of the wheel's spikes.*

On a more formal level, one can choose a partition $a = t_0 < \dots <$ $< t_n = b$ of $[a, b]$ and consider the sequence of orthogonal projections $\varphi_i\colon \mathrm{T}_{\gamma(t_{i-1})} \to \mathrm{T}_{\gamma(t_i)}$. For a fine partition, the composition

$$\varphi_n \circ \dots \circ \varphi_1 \colon T_p \to \mathrm{T}_q$$

gives an approximation of ι_γ. Each φ_i does not increase the magnitude of a vector and neither the composition. It is straightforward to see that if the partition is sufficiently fine, then it is almost isometry; in particular it almost preserves the magnitudes of tangent vectors.

21.5. Exercise. *Construct a loop γ in $\mathbb{S}^2$ with base at p such that the parallel transport $\iota_\gamma\colon \mathrm{T}_p \to \mathrm{T}_p$ is not the identity map.*

Geodesic curvature

Plane is the simplest example of smooth surface. Earlier we introduced signed curvature of a plane curve. Let us introduce the so called *geodesic curvature* — an analogous notion for a smooth curve γ in general oriented smooth surface Σ.

Let $\nu\colon \Sigma \to \mathbb{S}^2$ be the spherical map that defines the orientation on Σ. Without loss of generality we can assume that γ has unit speed. Then for any t the vectors $\nu(t) = \nu(\gamma(t))_\Sigma$ and the velocity vector $\mathrm{T}(t) = \gamma'(t)$ are unit vectors that are normal to each other. Denote by $\mu(t)$ the unit vector that is normal to both $\nu(t)$ and $\mathrm{T}(t)$ that points to the left from γ; that is, $\mu = \nu \times \mathrm{T}$. Note that the triple $\mathrm{T}(t), \mu(t), \nu(t)$ is an oriented orthonormal basis for any t.

Since γ is unit-speed, the acceleration $\gamma''(t)$ is perpendicular to $\mathrm{T}(t)$; therefore at any parameter value t, we have

$$\gamma''(t) = k_g(t)\cdot\mu(t) - k_n(t)\cdot\nu(t),$$

for some real numbers $k_n(t)$ and $k_g(t)$. The numbers $k_n(t)$ and $k_g(t)$ are called *normal* and *geodesic curvature* of γ at t respectively; we may write $k_n(t)_\Sigma$ and $k_g(t)_\Sigma$ if we need to emphasize that we work in the surface Σ.

Note that the geodesic curvature vanishes if γ is a geodesic. It measures how much a given curve diverges from being a geodesic; it is positive if γ turns left and negative if γ turns right.

21.6. Exercise. *Let γ be a smooth regular curve in a smooth surface Σ. Show that γ is a geodesic if and only if it has constant speed and vanishing geodesic curvature.*

Total geodesic curvature

The total geodesic curvature is defined as integral

$$\Psi(\gamma) := \int\limits_{\mathbb{I}} k_g(t)\cdot dt,$$

assuming that γ is a smooth unit-speed curve defined on the real interval $\mathbb{I}$.

Note that if Σ is a plane and γ lies in Σ, then geodesic curvature of γ equals to signed curvature and therefore total geodesic curvature equals to the total signed curvature. By that reason we use the same notation $\Psi(\gamma)$ as for total signed curvature; if we need to emphasize that we consider γ as a curve in Σ, we write $\Psi(\gamma)_\Sigma$.

If γ is a piecewise smooth regular curve in Σ, then its total geodesic curvature is defined as a sum of all total geodesic curvature of its arcs and the sum signed exterior angles of γ at the joints. More precisely, if γ is a concatenation of smooth regular curves $\gamma_1, \dots, \gamma_n$, then

$$\Psi(\gamma) = \Psi(\gamma_1) + \dots + \Psi(\gamma_n) + \theta_1 + \dots + \theta_{n-1},$$

where θ_i is the signed external angle at the joint γ_i and γ_{i+1}; it is positive if we turn left and negative if we turn right, it is undefined if we turn to the opposite direction. If γ is closed, then

$$\Psi(\gamma) = \Psi(\gamma_1) + \dots + \Psi(\gamma_n) + \theta_1 + \dots + \theta_n,$$

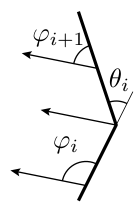
where θ_n is the signed external angle at the joint γ_n and γ_1.

If each arc γ_i in the concatenation is a geodesic, then γ is called *broken geodesic*. In this case $\Psi(\gamma_i) = 0$ for each i and therefore the total geodesic curvature of γ is the sum of its signed external angles.

21.7. Proposition. *Assume γ is a closed broken geodesic in a smooth oriented surface Σ that starts and ends at the point p. Then the parallel transport $\iota_\gamma \colon T_p \to \mathrm{T}_p$ is a rotation of the plane T_p clockwise by angle $\Psi(\gamma)$.*

Moreover, the same statement holds for smooth closed curves and piecewise smooth curves.

Proof. Assume γ is a cyclic concatenation of geodesics $\gamma_1, \dots, \gamma_n$. Fix a tangent vector v at p and extend it to a parallel vector field along γ. Since $\mathrm{w}_i(t) = \gamma_i'(t)$ is parallel along γ_i, the angle φ_i between v and w_i stays constant on each γ_i.



If θ_i denotes the external angle at this vertex of switch from γ_i to γ_{i+1}, we have that

$$\varphi_{i+1} = \varphi_i - \theta_i \pmod{2\cdot\pi}.$$

Therefore after going around we get that

$$\varphi_{n+1} - \varphi_1 = -\theta_1 - \dots - \theta_n = -\Psi(\gamma).$$

Hence the first statement follows.

For the smooth unit-speed curve $\gamma \colon [a, b] \to \Sigma$, the proof is analogous. If $\varphi(t)$ denotes the angle between $\mathrm{v}(t)$ and $\mathrm{w}(t) = \gamma'(t)$, then

$$\varphi'(t) + k_g(t) \equiv 0$$

Whence the angle of rotation

$$\begin{aligned}\varphi(b)-\varphi(a) &= \int\limits_a^b \varphi'(t)\cdot dt = \\ &= -\int\limits_a^b k_g\cdot dt = \\ &= -\Psi(\gamma)\end{aligned}$$

The case of piecewise regular smooth curve is a straightforward combination of the above two cases. $\square$

Chapter 22

Gauss–Bonnet formula

The following theorem was proved by Carl Friedrich Gauss [23] for geodesic triangles; Pierre Bonnet and Jacques Binet independently generalized the statement for arbitrary curves.

22.1. Theorem. *Let Δ be a topological disc in a smooth oriented surface Σ bounded by a simple piecewise smooth and regular curve $\partial\Delta$ that is oriented in such a way that Δ lies on its left. Then*

➊ $$\Psi(\partial\Delta) + \int\limits_{\Delta} K = 2{\cdot}\pi,$$

where K denotes the Gauss curvature of Σ.

We will give an informal proof of 22.1 in a leading partial case based on the bike wheel interpretation described above. A formal computational proof will be given much latter; see page 184.

Before going into the proofs, we suggest to solve the following exercises using the Gauss–Bonnet formula.

22.2. Exercise. *Assume γ is a closed simple curve with constant geodesic curvature 1 in a smooth closed surface Σ with positive Gauss curvature. Show that*

$$\operatorname{length}\gamma \leqslant 2{\cdot}\pi;$$

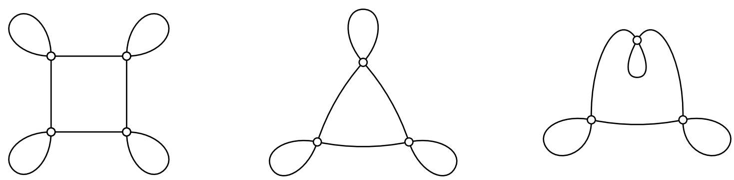
that is, the length of γ cannot exceed the length of the unit circle in the plane.

22.3. Exercise. *Let γ be a closed simple geodesic on a smooth closed surface Σ with positive Gauss curvature. Assume $\nu\colon \Sigma \to \mathbb{S}^2$ is a Gauss map. Show that the curve $\alpha = \nu \circ \gamma$ divides the sphere into regions of equal area.*

Conclude that

$$\operatorname{length}\alpha \geqslant 2{\cdot}\pi.$$

22.4. Exercise. *Let Σ be a smooth regular sphere with positive Gauss curvature and $p \in \Sigma$. Suppose γ is a closed geodesic that is covered by one chart. Can it happen that in this chart, the curve γ looks like one the curves on the following diagrams?*



(easy) *(tricky)* *(advanced)*

The following exercise gives the optimal bound on Lipschitz constant of a convex function that guarantees that its geodesics have no self-intersections; compare to 20.17.

22.5. Exercise. *Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a $\sqrt{3}$-Lipschitz smooth convex function. Show that any geodesic in the surface defined by the graph $z = f(x, y)$ has no self-intersections.*

22.6. Exercise. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Suppose its graph Σ is saddle surface. Show that any two points in Σ are connected by a unique geodesic.*

Spherical case

Note that if Σ is a plane, then the Gauss curvatue vanished; therefore the Gauss–Bonnet formula ❶ can be written as

$$\Psi(\partial\Delta) = 2{\cdot}\pi,$$

and it follows from 12.5.

If Σ is the unit sphere, then $K \equiv 1$; in this case Theorem 22.1 can be formulated the following way:

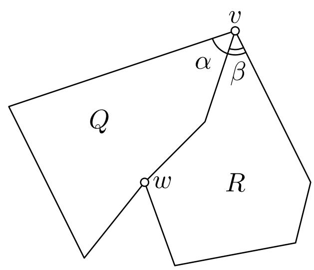
22.7. Proposition. *Let P be a spherical polygon bounded by a simple closed broken geodesic ∂P. Assume ∂P is oriented such that P lies on the left from ∂P. Then*

$$\Psi(\partial P) + \operatorname{area} P = 2{\cdot}\pi.$$

Moreover the same formula holds for any spherical region P bounded by piecewise smooth simple closed curve ∂P.

This proposition will be used in the informal proof given below.

Sketch of proof. If the contour $\partial\Delta$ of a spherical triangle with angles α, β and γ is oriented such that the triangle lies on the left, then its external angles are $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$. Therefore the total geodesic curvature of $\partial\Delta$ is $\Psi(\partial\Delta)=3{\cdot}\pi-\alpha-\beta-\gamma$ which implies the statement in this case.



Now suppose that a spherical polygon is P divided in two polygons Q and R by polygonal line between vertexes v and w then

$$\Psi(\partial P)+2{\cdot}\pi=\Psi(\partial Q)+\Psi(\partial R).$$

Indeed, for the internal angles Q and R at v are α and β, then their external angles are $\pi-\alpha$ and $\pi-\beta$ respectfully. The internal angle of P in this case is $\alpha+\beta$ and its external angle is $\pi-\alpha-\beta$ Clearly we have that

$$(\pi-\alpha)+(\pi-\beta)=(\pi-\alpha-\beta)+\pi;$$

that is, the sum of external angles of Q and R at v is π plus the external angle of P at v. The same holds for the external angles at w and the rest of the external angles of P appear once on Q or R. Therefore if the proposition holds for Q and R, then it holds for P.

Iterating this sonstruction we can reduce the problem to the case of spherical triangle which is already proved.

The second statement can be proved by approximation. One has to show that the total geodesic curvature of an inscribed broken geodesic approximates the total geodesic curvature of the original curve. We omit the proof of the latter statement, but it can be done along the same lines as 10.24. □

22.8. Exercise. *Assume γ is a simple piecewise smooth loop on $\mathbb{S}^2$ that divides its area into two equal parts. Denote by p the base point of γ. Show that the parallel transport $\iota_\gamma\colon \mathrm{T}_p\mathbb{S}^2\to\mathrm{T}_p\mathbb{S}^2$ is the identity map.*

Intuitive proof

We will suppose that it is intuitively clear that moving the axis of the wheel without changing its direction does not change the direction of the wheel's spikes.

More precisely, assume we keep the axis of a non-spinning bike wheel and perform the following two experiments:

(i) We move it around and bring the axis back to the original position. As a result the wheel might turn by some angle; let us measure this angle.

(ii) We move the direction of the axis the same way as before without moving the center of the wheel. After that we measure the angle of rotation.

Then the resulting angles in these two experiments is the same.

Consider a surface Σ with a Gauss map $\nu\colon \Sigma \to \mathbb{S}^2$. Note that for any point p on Σ, the tangent plane $\mathrm{T}_p\Sigma$ is parallel to the tangent plane $\mathrm{T}_{\nu(p)}\mathbb{S}^2$; so we can identify these tangent spaces. From the experiments above, we get the following:

22.9. Lemma. *Suppose α is a piecewise smooth regular curve in a smooth regular surface Σ which has a Gauss map $\nu\colon \Sigma \to \mathbb{S}^2$. Then the parallel transport along α in Σ coincides with the parallel transport along the curve $\beta = \nu\circ\alpha$ in $\mathbb{S}^2$.*

Proof of partial case of 22.1. We will prove the formula for proper surface Σ with positive Gauss curvature. In this case, by 16.6 the formula can be rewritten as

$$\Psi(\partial\Delta) + \mathrm{area}[\nu(\Delta)] = 2\cdot\pi. \qquad ❷$$

(This case is leading — the general case can be proved similarly, but one has to use the signed area counted with multiplicity).

Fix $p \in \partial\Delta$; assume the loop α runs along $\partial\Delta$ so that Δ lies on the left from it. Consider the parallel translation $\iota\colon \mathrm{T}_p \to \mathrm{T}_p$ along α. According to 21.7, ι is a clockwise rotation by angle $\Psi(\alpha)_\Sigma$.

Set $\beta = \nu\circ\alpha$. According to 22.9, ι is also parallel translation along β in $\mathbb{S}^2$. In particular ι is a clockwise rotation by angle $\Psi(\beta)_{\mathbb{S}^2}$. By 22.7

$$\Psi(\beta)_{\mathbb{S}^2} + \mathrm{area}[\nu(\Delta)] = 2\cdot\pi.$$

Therefore ι is a counterclockwise rotation by $\mathrm{area}[\nu(\Delta)]$

Summarizing, the clockwise rotation by $\Psi(\alpha)_\Sigma$ is identical to a counterclockwise rotation by $\mathrm{area}[\nu(\Delta)]$. The rotations are identical if the angles are equal modulo $2\cdot\pi$. Therefore

$$\Psi(\partial\Delta)_\Sigma + \mathrm{area}[\nu(\Delta)] = 2\cdot\pi\cdot n \qquad ❸$$

for an integer n.

It remains to show that $n = 1$. By 12.5, this is so for a topological disc in a plane. One can think of a general disc Δ as about a result of a continuous deformation of a plane disc. The integer n cannot change in the process of deformation since the left hand side in ❸ is continuous along the deformation; whence $n = 1$ for the result of the deformation.

Let us redo the last argument formally. Assume that Δ lies in a local graph realization $z = f(x, y)$ of Σ. Consider one parameter family Σ_t of graphs $z = t\cdot f(x, y)$ and denote by Δ_t the corresponding disc in Σ_t, so $\Delta_1 = \Delta$ amd Δ_0 is its projection to the (x, y)-plane. Since Σ_0 is a plane domain, we have $\mathrm{area}[\nu_0(\Delta_0)] = 0$. Therefore by 12.5 we gave

$$\Psi(\partial\Delta_0)_{\Sigma_0} + \mathrm{area}[\nu_0(\Delta_0)] = 2\cdot\pi.$$

Note that

$$\Psi(\partial\Delta_t)_{\Sigma_t} + \mathrm{area}[\nu_t(\Delta_t)]$$

depends continuously on t. According to ❸, its value is a multiple of $2\cdot\pi$; therefore it has to be constant. Whence the Gauss–Bonnet formula follows.

If Δ does not lie in one graph, then one could divide it into smaller discs, apply the formula for each and sum up the result. The proof is done along the same lines as 22.7. □

Simple geodesic

The following theorem provides an interesting application of Gauss–Bonnet formula; it is proved by Stephan Cohn-Vossen [Satz 9 in 15].

22.10. Theorem. *Any open smooth regular surface with positive Gauss curvature has a simple two-sided infinite geodesic.*

22.11. Lemma. *Suppose Σ is an open surface in with positive Gauss curvature in the Euclidean space. Then there is a convex function f defined on a convex open region of (x, y)-plane such that Σ can be presented as a graph $z = f(x, y)$ in some (x, y, z)-coordinate system of the Euclidean space.*

Moreover

❹ $$\iint\limits_{\Sigma} G \leqslant 2\cdot\pi.$$

Proof. The surface Σ is a boundary of an unbounded closed convex set K.

Fix $p \in \Sigma$ and consider a sequence of points x_n such that $|x_n - p| \to$ $\to \infty$ as $n \to \infty$. Set $u_n = \frac{x_n - p}{|x_n - p|}$; the unit vector in the direction from p to x_n. Since the unit sphere is compact, we can pass to a subsequence of (x_n) such that u_n converges to a unit vector u.

Note that for any $q \in \Sigma$, the directions $v_n = \frac{x_n - q}{|x_n - q|}$ converge to u as well. The half-line from q in the direction of u lies in K. Indeed any point on the half-line is a limit of points on the line segments $[qx_n]$; since K is closed, all of these poins lie in K.

Let us choose the z-axis in the direction of u. Note that line segments can not lie in Σ, otherwise its Gauss curvature would vanish. It follows that any vertical line can intersect Σ at most at one point. That is, Σ is a graph of a function $z = f(x, y)$. Since K is convex, the function f is convex and it is defined in a region Ω which is convex. The domain Ω is the projection of Σ to the (x, y)-plane. This projection is injective and by the inverse function theorem, it maps open sets in Σ to open sets in the plane; hence Ω is open.

It follows that the outer normal vectors to Σ at any point, points to the south hemisphere $\mathbb{S}^2_- = \left\{ (x, y, z) \in \mathbb{S}^2 \,:\, z < 0 \right\}$. Therefore the area of the spherical image of Σ is at most $\operatorname{area} \mathbb{S}^2_- = 2\cdot\pi$. The area of this image is the integral of the Gauss curvature along Σ. That is,

$$\begin{aligned} \iint\limits_{\Sigma} G &= \operatorname{area}[\nu(\Sigma)] \leqslant \\ &\leqslant \operatorname{area} \mathbb{S}^2_- = \\ &= 2\cdot\pi, \end{aligned}$$

where $\nu(p)$ denotes the outer unit normal vector at p. Hence ❹ follows. $\square$

Proof of 22.10. Let Σ be an open surface in with positive Gauss curvature and γ a two-sided infinite geodesic in Σ. The following is the key statement in the proof.

22.12. Claim. *The geodesic γ contains at most one simple loop.*

Assume γ has a simple loop ℓ. By Lemma 22.11, Σ is parameterized by a open convex region Ω in the plane; therefore ℓ bounds a disc in Σ; denote it by Δ. If φ is the angle at the base of the loop, then by Gauss–Bonnet,

$$\iint\limits_{\Delta} G = \pi + \varphi.$$

By Lemma 22.11, $\varphi < \pi$; that is, γ has no concave simple loops

Assume γ has two simple loops, say ℓ_1 and ℓ_2 that bound discs Δ_1 and Δ_2. Then the disks Δ_1 and Δ_2 have to overlap, otherwise the curvature of Σ would exceed $2\cdot\pi$.

We may assume that $\Delta_1 \not\subset \Delta_2$; the loop ℓ_2 appears after ℓ_1 on γ and there are no other simple loops between them. In this case, after going around ℓ_1 and before closing ℓ_2, the curve γ must enter Δ_1 creating a concave loop. The latter contradicts the above observation.

If a geodesic γ has a self-intersection, then it contains a simple loop. From above, there is only one such loop; it cuts a disk from Σ and goes around it either clockwise or counterclockwise. This way we divide all the self-intersecting geodesics into two sets which we will call *clockwise* and *counterclockwise*.

Note that the geodesic $t \mapsto \gamma(t)$ is clockwise if and only if the same geodesic traveled backwards $t \mapsto \gamma(-t)$ is counterclockwise. By shooting unit-speed geodesics in all directions at a given point $p = \gamma(0)$, we get a one parameter family of geodesics γ_s for $s \in [0, \pi]$ connecting the geodesic $t \mapsto \gamma(t)$ with the $t \mapsto \gamma(-t)$; that is, $\gamma_0(t) = \gamma(t)$ and $\gamma_\pi(t) = \gamma(-t)$. It follows that there are geodesics which aren't clockwise nor counterclockwise. Those geodesics have no self-intersections. $\square$

Chapter 23

Comparison

Model triangles and angles

Recall that a shortest path between points x and y in a surface Σ will be denoted as $[xy]$ or $[xy]_\Sigma$, and $|x-y|_\Sigma$ denotes the *intrinsic distance* from x to y in Σ.

A *geodesic triangle* in a surface Σ is a triple of points $x, y, z \in \Sigma$ with choice of minimizing geodesics $[xy]$, $[yz]$ and $[zx]$. The points x, y, z are called *vertexes* of the geodesic triangle, the minimizing geodesics $[xy]$, $[yz]$ and $[zx]$ are called its sides; the triangle itself is denoted by $[xyz]$.

A triangle $[\tilde{x}\tilde{y}\tilde{z}]$ in the plane $\mathbb{R}^2$ is called *model triangle* of the triangle $[xyz]$, briefly $[\tilde{x}\tilde{y}\tilde{z}] = \tilde{\triangle}xyz$, if its corresponding sides are equal; that is,

$$|\tilde{x}-\tilde{y}|_{\mathbb{R}^2} = |x-y|_\Sigma, \quad |\tilde{y}-\tilde{z}|_{\mathbb{R}^2} = |y-z|_\Sigma, \quad |\tilde{z}-\tilde{x}|_{\mathbb{R}^2} = |z-x|_\Sigma.$$

A pair of minimizing geodesics $[xy]$ and $[xz]$ starting from one point x is called *hinge* and denoted as $[x\,{}^y_z]$. The angle between these geodesics at x is denoted by $\measuredangle[x\,{}^y_z]$. The corresponding angle $\measuredangle[\tilde{x}\,{}^{\tilde{y}}_{\tilde{z}}]$ in the model triangle $[\tilde{x}\tilde{y}\tilde{z}] = \tilde{\triangle}xyz$ is denoted by $\tilde{\measuredangle}(x\,{}^y_z)$.

Formulation

Part *(a)* of the following theorem is called *Toponogov comparison theorem* and sometimes *Alexandrov comparison theorem*; it is was proved by Paolo Pizzetti [44] and latter independently by Alexandr Alexandrov [4]; generalizations were obtained by Victor Toponogov [50], Mikhael Gromov, Yuri Burago and Grigory Perelman [7].

Part *(b)* is called *Cartan–Hadamard theorem*; it was proved by Hans von Mangoldt [36] and generalized by Elie Cartan [10], Jacques Hadamard [25], Herbert Busemann [9], Willi Rinow [45], Mikhael Gromov [24, p. 119], Stephanie Alexander and Richard Bishop [1].

A surface Σ is called *simply connected* if any closed simple curve in Σ bounds a disc. Equivalently any closed curve in Σ can be continuously deformed into a *trivial curve*; that is, a curve that stands at one point all the time.

Observe that a plane or a sphere are examples of simply connected surfaces, while torus or cylinder are not simply connected.

23.1. Comparison theorem. *Let Σ be a proper smooth regular surface.*

(a) If Σ has nonnegative Gauss curvature, then

$$\measuredangle[x\,{}^{y}_{z}] \geqslant \tilde\measuredangle(x\,{}^{y}_{z})$$

for any geodesic triangle $[xyz]$.

(b) If Σ is simply connected and has nonpositive Gauss curvature, then

$$\measuredangle[x\,{}^{y}_{z}] \leqslant \tilde\measuredangle(x\,{}^{y}_{z})$$

for any geodesic triangle $[xyz]$.

Let us make two remarks about the statement.

First, the angle $\measuredangle[x\,{}^{y}_{z}]$ is a number in the interval $[0,\pi]$. If the triangle $[xyz]$ bounds a disc Δ and θ is the external angle at x which used in Gauss–Bonnet formula, then $\measuredangle[x\,{}^{y}_{z}] = |\pi - \theta|$. The corresponding internal angle might be $\measuredangle[x\,{}^{y}_{z}]$ or $2\cdot\pi - \measuredangle[x\,{}^{y}_{z}]$ depending on which side lies the disc Δ.

- ◇ Since the angles of any plane triangle sum up to π, the part (23.1 *a*) of the theorem implies that angles of any triangle in a surface with nonnegative Gauss curvature have sum at least π.
- ◇ The triangle may not bound a disc[1], but if it does, then by Gauss–Bonnet formula the sum of its *internal* angles is at least π.

These two statements are closely related, but they are not equivalent. Indeed, if α is the angle in the comparison theorem, then the internal angle might be α or $2\cdot\pi-\alpha$; while Gauss–Bonnet formula gives a lower bound on the sum of internal angles it does not forbid that each of these angles is close to $2\cdot\pi$. However the latter is impossible by the comparison theorem.

[1]For example equator on the cyclinder is formed by a geodesic triangle that does not bound a disc.

Second, note that without condition that Σ is simply connected, the statement *(23.1b)* does not hold. For example the equator $z=0$ of the hyperboloid (which is not simply connected)

$$\{\,(x,y,z)\in\mathbb{R}^3\;:\;x^2+y^2-z^2=1\,\}$$

forms a triangle with all angles π, which contradict the comparison in *(23.1b)*.

23.2. Exercise. *Let p and q be points on a closed convex surface Σ that lie on maximal intrinsic distance from each other; that is, $|p-q|_\Sigma\geqslant|x-y|_\Sigma$ for any $x,y\in\Sigma$. Show that*

$$\tilde\measuredangle(x\,{}^p_q)\geqslant\tfrac{\pi}{3}$$

for any point $x\in\Sigma\backslash\{p,q\}$.

Local comparisons

First we prove the following local version of comparison theorem and then use it to prove the global version.

23.3. Theorem. *The comparison theorem (23.1) holds in a small neighborhood of any point.*

That is, if Σ be a smooth regular surface without boundary, then any point $p\in\Sigma$ admits a neighborhood $U\ni p$ such that

(a) If Σ has nonnegative Gauss curvature, then

$$\measuredangle[x\,{}^y_z]\geqslant\tilde\measuredangle(x\,{}^y_z)$$

for any geodesic triangle $[xyz]$ in U.

(b) If Σ has nonpositive Gauss curvature, then

$$\measuredangle[x\,{}^y_z]\leqslant\tilde\measuredangle(x\,{}^y_z)$$

for any geodesic triangle $[xyz]$ in U.

Note that we can assume that U is simply connected therefore this condition is not necessary to include in part *(**??**)*.

Proof. Assume $y=\exp_x v$ and $z=\exp_x w$ for two small vectors $v,w\in\mathrm{T}_x$. Note that

$$\begin{aligned}\measuredangle[x\,{}^v_w]_{\mathrm{T}_x}&=\measuredangle[x\,{}^y_z]_\Sigma,\\|x-v|_{\mathrm{T}_x}&=|x-y|_\Sigma,\\|x-w|_{\mathrm{T}_x}&=|x-z|_\Sigma.\end{aligned}$$

(a). Consider the line segment $\tilde\gamma$ joining v to w in the tangent plane T_x and set $\gamma=\exp_x\circ\tilde\gamma$. By ???, we have

$$\operatorname{length}\gamma\leqslant\operatorname{length}\tilde\gamma.$$

Since $|v-w|_{\mathrm{T}_x}=\operatorname{length}\tilde\gamma$ and $|y-z|_\Sigma\leqslant\operatorname{length}\gamma$, we get

$$|v-w|_{\mathrm{T}_x}\geqslant|y-z|_\Sigma.$$

Therefore

$$\tilde\measuredangle(x\,{}^{z}_{y})\geqslant\measuredangle[x\,{}^{z}_{y}].$$

(b). Consider a minimizing geodesic γ joining y to z in Σ and let $\tilde\gamma$ be the corresponding curve joining v to w in T_x; that is, $\gamma=\exp_x\circ\tilde\gamma$. By ???, we have

$$\operatorname{length}\gamma\geqslant\operatorname{length}\tilde\gamma.$$

Since $|v-w|_{\mathrm{T}_x}\leqslant\operatorname{length}\tilde\gamma$ and $|y-z|_\Sigma=\operatorname{length}\gamma$, we get

$$|v-w|_{\mathrm{T}_x}\geqslant|y-z|_\Sigma.$$

Therefore

$$\tilde\measuredangle(x\,{}^{z}_{y})\geqslant\measuredangle[x\,{}^{z}_{y}].$$ ☐

Alexandrov's lemma

It this section we prove the following lemma in the plane geometry.

23.4. Lemma. *Assume $[pxyz]$ and $[p'x'y'z']$ be two quadraliterals in the plane with equal corresponding sides. Assume that the sides $[x'y']$ and $[y'z']$ extend each other; that is, y' lies on the line segment $[x'z']$. Then the following expressions have the same signs:*

(i) $|p-y|-|p'-y'|$;

(ii) $\measuredangle[x\,{}^{p}_{y}]-\measuredangle[x'\,{}^{p'}_{y'}]$;

(iii) $\pi-\measuredangle[y\,{}^{p}_{x}]-\measuredangle[y\,{}^{p}_{z}]$;

Proof. In the proof we use the following *monotonicity property*: if two sides adjacent to an angle in a plane triangle are fixed, then the angle is increases if the opposite side increase.

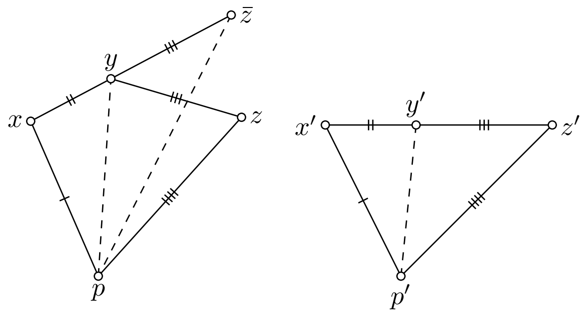
Take a point $\bar z$ on the extension of $[xy]$ beyond y so that $|y-\bar z|=|y-z|$ (and therefore $|x-\bar z|=|x'-z'|$).

From monotonicity, the following expressions have the same sign:

(i) $|p-y|-|p'-y'|$;

(ii) $\measuredangle[x\,{}^{y}_{p}]-\measuredangle[x'\,{}^{y'}_{p'}]=\measuredangle[x\,{}^{\bar z}_{p}]-\measuredangle[x'\,{}^{z'}_{p'}]$;

(iii) $|p-\bar z|-|p'-z'|$;



(iv) $\measuredangle[y\,{}^{\bar z}_{p}] - \measuredangle[y'\,{}^{z'}_{p'}]$;
The statement follows since

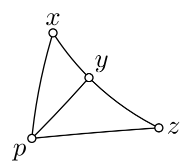
$$\measuredangle[y'\,{}^{z'}_{p'}] + \measuredangle[y'\,{}^{x'}_{p'}] = \pi$$

and

$$\measuredangle[y\,{}^{\bar z}_{p}] + \measuredangle[y\,{}^{x}_{p}] = \pi. \qquad \square$$

Further we will use the following reformulation of this lemma that is using language of comparison triangles and angles.

23.5. Reformulation. *Assume $[pxz]$ be a triangle in a surface Σ and the point y lies on the side $[xz]$. Consider its model triangle $[\tilde p\tilde x\tilde z] = \tilde\triangle pxz$ and let $\tilde y$ be the corresponding point on the side $[\tilde x\tilde z]$. Then the following expressions have the same signs:*



(i) $|p-y|_\Sigma - |\tilde p - \tilde y|_{\mathbb{R}^2}$;
(ii) $\tilde\measuredangle(x\,{}^{p}_{y}) - \tilde\measuredangle(x\,{}^{p}_{z})$;
(iii) $\pi - \tilde\measuredangle(y\,{}^{p}_{x}) - \tilde\measuredangle(y\,{}^{p}_{z})$;

Reformulations of comparison

In this section we formulate conditions equivalent to the conclusion of the comparison theorem (23.1).

A triangle $[xyz]$ in a surface is called *fat* (or, respectively, *thin*) if for any two points p and q on the sides of the triangle and the corresponding points $\tilde p$ and $\tilde q$ on the sides of its model triangle $[\tilde x\tilde y\tilde z] = \tilde\triangle xyz$ we have $|p-q| \geqslant |\tilde p - \tilde q|$ (or respectively $|p-q| \leqslant |\tilde p - \tilde q|$).

23.6. Proposition. *Let Σ be a proper smooth regular surface. Then the following three conditions are equivalent:*
(i^+) For any geodesic triangle $[xyz]$ in Σ we have

$$\measuredangle[x\,{}^{y}_{z}] \geqslant \tilde\measuredangle(x\,{}^{y}_{z}).$$

(ii⁺) *For any geodesic triangle* $[pxz]$ *in* Σ *and* y *on the side* $[xz]$ *we have*

$$\tilde\measuredangle(x\,{}^p_y) \geqslant \tilde\measuredangle(x\,{}^p_z).$$

(iii⁺) *Any geodesic triangle in* Σ *is fat.*

Similarly, following three conditions are equivalent:

(i⁻) *For any geodesic triangle* $[xyz]$ *in* Σ *we have*

$$\measuredangle[x\,{}^y_z] \leqslant \tilde\measuredangle(x\,{}^y_z).$$

(ii⁻) *For any geodesic triangle* $[pxz]$ *in* Σ *and* y *on the side* $[xz]$ *we have*

$$\tilde\measuredangle(x\,{}^p_y) \leqslant \tilde\measuredangle(x\,{}^p_z).$$

(iii⁻) *Any geodesic triangle in* Σ *is thin.*

Proof. We will prove the implications *(i⁺)⇒(ii⁺)⇒(iii⁺)⇒(i⁺)*. The implications *(i⁻)⇒(ii⁻)⇒(iii⁻)⇒(i⁻)* can be done the same way.

(i⁺)⇒(ii⁺). Note that $\measuredangle[y\,{}^p_x] + \measuredangle[y\,{}^p_z] = \pi$. By *(i⁺)*,

$$\tilde\measuredangle(y\,{}^p_x) + \tilde\measuredangle(y\,{}^p_z) \leqslant \pi.$$

It reamains to apply Alexandrov's lemma (23.5).

(ii⁺)⇒(iii⁺). Applying *(i⁺)* twice, first for $y \in [xz]$ and then for $w \in [px]$, we get that

$$\tilde\measuredangle(x\,{}^w_y) \geqslant \tilde\measuredangle(x\,{}^p_y) \geqslant \tilde\measuredangle(x\,{}^p_z)$$

and therefore

$$|w - y|_\Sigma \geqslant |\tilde w - \tilde y|_{\mathbb{R}^2},$$

where $\tilde w$ and $\tilde y$ are the points corresponding to w and y points on the sides of the model triangle. Hence the implication follows.

(iii⁺)⇒(i⁺). Since the triangle is fat, we have

$$\tilde\measuredangle(x\,{}^w_y) \geqslant \tilde\measuredangle(x\,{}^p_z)$$

for any $w \in\,]xp]$ and $y \in\,]xz]$. Note that $\tilde\measuredangle(x\,{}^w_y) \to \measuredangle[x\,{}^p_z]$ as $w, y \to x$, whence the implication follows. $\square$

In the following exercises you can apply the globalization theorem.

23.7. Exercise. *Let* Σ *be a closed (or open) regular surface and with nonnegative Gauss curvature. Show that for any four distinct points the following inequality holds:*

$$\tilde\measuredangle(p\,{}^x_y) + \tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^z_x) \leqslant 2\cdot\pi.$$

23.8. Exercise. *Let Σ be a open smooth regular surface and γ be a unit-speed geodesic in Σ and $p \in \Sigma$.*

Consider the function

$$h(t) = |p - \gamma(t)|^2_\Sigma - t^2.$$

(a) Show that if the Gauss curvature of Σ is nonnegative, then h is a concave function.
(b) Show that if Σ is simply connected and the Gauss curvature of Σ is nonpositive, then h is a convex function.

23.9. Exercise. *Let $\tilde{x}_1 \dots \tilde{x}_n$ be a convex plane polygon and $x_1 \dots x_n$ be a broken geodesic in an open simply connected surface Σ with nonpositive curvature. Assume that $|x_i - x_{i-1}|_\Sigma = |\tilde{x}_i - \tilde{x}_{i-1}|_{\mathbb{R}^2}$ and $\measuredangle[x_{i-1}\,{}^{x_i}_{x_{i+1}}] \geqslant \measuredangle[x_{i-1}\,{}^{x_i}_{x_{i+1}}]$ for each i. Show that*

$$|x_1 - x_n|_\Sigma \geqslant |\tilde{x}_1 - \tilde{x}_n|_{\mathbb{R}^2}.$$

For $\Sigma = \mathbb{R}^2$, the exercise above is the so called *arm lemma*; you can use it without proof.

23.10. Exercise. *Let x' and y' be the midpoints of minimizing geodesics $[px]$ and $[py]$ in an open smooth regular surface Σ.*
(a) Show that if the Gauss curvature of Σ is nonnegative, then

$$2{\cdot}|x' - y'|_\Sigma \geqslant |x - y|_\Sigma.$$

(b) Show that if Σ is simply connected and has nonpositive Gauss curvature, then

$$2{\cdot}|x' - y'|_\Sigma \leqslant |x - y|_\Sigma.$$

Nonnegative curvature

In this section we will prove part *(23.1a)* of the comparison theorem (23.1) assuming that Σ is compact; the general case requre only minor modifications.

Since Σ is compact, from the local theorem (23.3), we get that there is $\varepsilon > 0$ such that the inequality

$$\measuredangle[x\,{}^y_z] \geqslant \tilde{\measuredangle}(x\,{}^y_z).$$

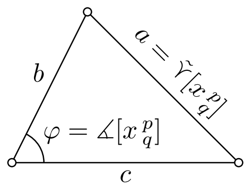
holds for any hinge $[x\,{}^y_z]$ such that $|x - y| + |x - z| < \varepsilon$. The following lemma states that in this case the same holds for any hinge $[x\,{}^y_z]$ such that $|x - y| + |x - z| < \frac{3}{2}{\cdot}\varepsilon$. Applying the lemma few times we will get that the comparison holds for arbitrary hinge, which will prove part *(23.1a)*.

23.11. Key lemma. *Let Σ be an open smooth regular surface. Assume that the comparison*

$$❶ \qquad \measuredangle[x\,{}^y_z] \geqslant \tilde\measuredangle(x\,{}^y_z)$$

holds for any hinge $[x\,{}^y_z]$ with $|x-y|+|x-z|<\tfrac23\cdot\ell$. Then the comparison ❶ *holds for any hinge $[x\,{}^y_z]$ with $|x-y|+|x-z|<\ell$.*

Proof. Given a hinge $[x\,{}^p_q]$ consider a triangle in the plane with angle $\measuredangle[x\,{}^p_q]$ and two adjacent sides $|x-p|$ and $|x-q|$. Let us denote by $\tilde\Upsilon[x\,{}^p_q]$ the third side of this triangle; let us call it *model side* of the hinge.



Note that the inequalities

$$\measuredangle[x\,{}^p_q] \geqslant \tilde\measuredangle(x\,{}^p_q) \quad\text{and}\quad \tilde\Upsilon[x\,{}^p_q] \geqslant |p-q|$$

are equivalent. So it is sufficient to prove that

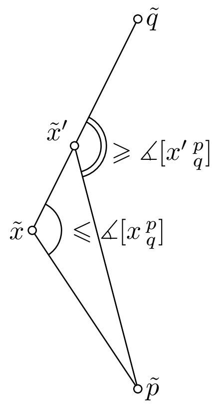
$$❷ \qquad \tilde\Upsilon[x\,{}^p_q] \geqslant |p-q|.$$

for any hinge $[x\,{}^p_q]$ with $|x-p|+|x-q|<\ell$.

Given a hinge $[x\,{}^p_q]$ such that

$$\tfrac23\cdot\ell \leqslant |p-x|+|x-q| < \ell,$$

let us construct a new smaller hinge $[x'\,{}^p_q]$; that is,



$$❸ \qquad |p-x|+|x-q| \geqslant |p-x'|+|x'-q|$$

and such that

$$❹ \qquad \tilde\Upsilon[x\,{}^p_q] \geqslant \tilde\Upsilon[x'\,{}^p_q].$$

Assume $|x-q| \geqslant |x-p|$, otherwise switch the roles of p and q in the following construction. Take $x' \in [xq]$ such that

$$❺ \qquad |p-x|+3\cdot|x-x'| = \tfrac23\cdot\ell$$

Choose a geodesic $[x'p]$ and consider the hinge $[x'\,{}^p_q]$ fromed by $[x'p]$ and $[x'q] \subset [xq]$. Then ❸ follows since the length of $[x'p]$ can not exceed the total length of $[x'x]$ and $[x'p]$.

Further, note that $|p-x|+|x-x'|, |p-x'|+|x'-x| < \tfrac{2}{3}\cdot\ell$. In particular,

❻ $$\measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde\measuredangle(x\,{}^{p}_{x'}) \quad\text{and}\quad \measuredangle[x'\,{}^{p}_{x}] \geqslant \tilde\measuredangle(x'\,{}^{p}_{x}).$$

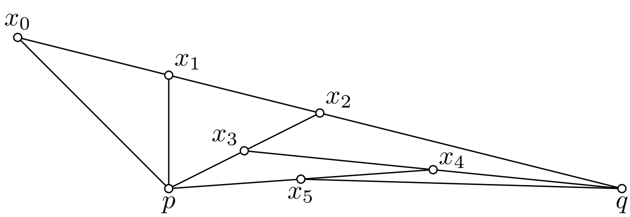
Consider the model triangle $[\tilde x\tilde x'\tilde p] = \tilde\triangle xx'p$. Take $\tilde q$ on the extension of $[\tilde x\tilde x']$ beyond x' such that $|\tilde x-\tilde q| = |x-q|$ (and therefore $|\tilde x'-\tilde q| = |x'-q|$). From ❻,

$$\measuredangle[x\,{}^{p}_{q}] = \measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde\measuredangle(x\,{}^{p}_{x'}) \quad\Rightarrow\quad \tilde\Upsilon[x\,{}^{q}_{p}] \geqslant |\tilde p-\tilde q|.$$

Since $\measuredangle[x'\,{}^{p}_{x}] + \measuredangle[x'\,{}^{p}_{q}] = \pi$, ❻ implies

$$\pi - \tilde\measuredangle(x'\,{}^{p}_{x}) \geqslant \pi - \measuredangle[x'\,{}^{p}_{x}] \geqslant \measuredangle[x'\,{}^{p}_{q}].$$

Therefore $|\tilde p-\tilde q| \geqslant \tilde\Upsilon[x'\,{}^{q}_{p}]$ and ❹ follows.



Set $x_0 = x$. Let us apply inductively the above construction to get a sequence of hinges $[x_n\,{}^{p}_{q}]$ with $x_{n+1} = x_n'$. By ❹ and triangle inequality, both sequences

$$s_n = \tilde\Upsilon[x_n\,{}^{p}_{q}] \quad\text{and}\quad r_n = |p-x_n| + |x_n-q|$$

are nonincreasing.

The sequence might terminate at some n only if $r_n < \tfrac{2}{3}\cdot\ell$. In this case, by the assumptions of the lemma,

$$s_n = \tilde\Upsilon[x_n\,{}^{p}_{q}] \geqslant |p-q|.$$

Since sequence s_n is nonincreasing;

$$s_0 = \tilde\Upsilon[x\,{}^{p}_{q}] \geqslant |p-q|,$$

whence inequality ❷ follows.

If the sequence does not terminate, then $r_n \geqslant \tfrac{2}{3}\cdot\ell$ for all n. Since (r_n) is nonincreasing, $r_n \to r \geqslant |p-q|_\Sigma$ as $n\to\infty$.

Let us show that $\measuredangle[x_n\,{}^{p}_{q}] \to \pi$ as $n\to\infty$.

Indeed assume $\measuredangle[x_n\,{}^{p}_{q}] \leqslant \pi - \varepsilon$ for some $\varepsilon > 0$. Without loss of generality we can assume that $x_{n+1} \in [x_n q]$; otherwise switch p and q further. Note that $|x_n - x_{n+1}|, |p - x_n| > \frac{\ell}{100}$. Therefore by comparison

$$|p - x_{n+1}| < \tilde{\Upsilon}[x_n\,{}^{p}_{x_{n+1}}] < |p - x_n| + |x_n - x_{n+1}| - \delta$$

for some fixed $\delta = \delta(\varepsilon) > 0$. Therefore $r_n - r_{n+1} > \delta$. The latter can not hold for large n, otherwise the sequence r_n would not converge.

It follows that for any $\varepsilon > 0$ we have that $\measuredangle[x_n\,{}^{p}_{q}] > \pi - \varepsilon$ for all large n; that is, $\measuredangle[x_n\,{}^{p}_{q}] \to \pi$ as $n \to \infty$.

Since $\measuredangle[x_n\,{}^{p}_{q}] \to \pi$, we have $s_n - r_n \to 0$ as $n \to \infty$; that is, $s_n \to r$.

Since the sequence (s_n) is nonincreasing and $r \geqslant |p - q|$, we get

$$s_n \geqslant |p - q|$$

for any n. In particular

$$\tilde{\Upsilon}[s\,{}^{p}_{q}] = s_0 \geqslant |p - q|,$$

so we obtain ❷. □

23.12. Exercise. *Assume a disc Δ lies in open smooth regular surface Σ with nonnegative Gauss curvature and bounded by a closed broken geodesic $x_1 \dots x_n$ with positive exterior angles; that is, when you travel along the boundary, you always turn to the side where Δ is.*

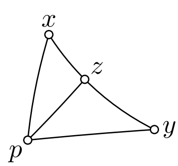
Show that there is a convex plane polygon $\tilde{x}_1 \dots \tilde{x}_n$ which sides are equal to the corresponding sides of $x_1 \dots x_n$ and with internal angles at not bigger than in the corresponding angles of $x_1 \dots x_n$.

Inheritance lemma

The following lemma will play key role in the proof of part *(23.1b)* of the comparison theorem (23.1).

23.13. Inheritance Lemma. *Assume that a triangle $[pxy]$ in a surface Σ* decomposes *into two triangles $[pxz]$ and $[pyz]$; that is, $[pxz]$ and $[pyz]$ have common side $[pz]$, and the sides $[xz]$ and $[zy]$ together form the side $[xy]$ of $[pxy]$.*

If both triangles $[pxz]$ and $[pyz]$ are thin, then so is $[pxy]$.



We shall need the following lemma in plane geometry.

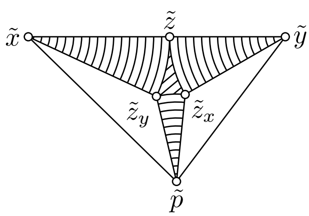
23.14. Lemma. *Let* $\blacktriangle\tilde p\tilde x\tilde y$ *be a solid plane triangle; that is,* $\blacktriangle\tilde p\tilde x\tilde y = \mathrm{Conv}\{\tilde p,\tilde x,\tilde y\}$. *Given* $\tilde z\in[\tilde x\tilde y]$, *consider points* $\dot p,\dot x,\dot z,\dot y$ *in the plane such that*

$$\begin{aligned}&|\dot p-\dot x|=|\tilde p-\tilde x|, &&|\dot p-\dot y|=|\tilde p-\tilde y|, &&|\dot p-\dot z|\leqslant|\tilde p-\tilde z|,\\ &|\dot x-\dot z|=|\tilde x-\tilde z|, &&|\dot y-\dot z|=|\tilde y-\tilde z|,\end{aligned}$$

where points $\dot x$ *and* $\dot y$ *lie on either side of* $[\dot p\dot z]$. *Then there is a short map*

$$F\colon \blacktriangle\tilde p\tilde x\tilde y\to\blacktriangle\dot p\dot x\dot z\cup\blacktriangle\dot p\dot y\dot z$$

that maps $\tilde p$, $\tilde x$, $\tilde y$ *and* $\tilde z$ *to* $\dot p$, $\dot x$, $\dot y$ *and* $\dot z$ *respectively.*



Proof. Note that

$$\begin{aligned}|\dot x-\dot y|&\leqslant|\dot x-\dot z|+|\dot z-\dot y|=\\&=|\tilde x-\tilde z|+|\tilde z-\tilde y|\\&=|\tilde x-\tilde y|.\end{aligned}$$

Applying monotonicity propery, we get that

$$\measuredangle[\dot p\,{}^{\dot x}_{\dot y}]\leqslant\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}].$$

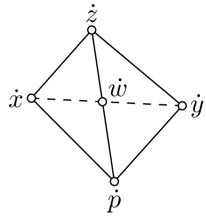
It follows that there are nonoverlapping triangles $[\tilde p\tilde x\tilde z_y]\cong[\dot p\dot x\dot z]$ and $[\tilde p\tilde y\tilde z_x]\cong[\dot p\dot y\dot z]$ inside triangle $[\tilde p\tilde x\tilde y]$.

Connect points in each pair $(\tilde z,\tilde z_x)$, $(\tilde z_x,\tilde z_y)$ and $(\tilde z_y,\tilde z)$ with arcs of circles centered at $\tilde y$, $\tilde p$, and $\tilde x$ respectively. Define F as follows.

- ◇ Map $\blacktriangle\tilde p\tilde x\tilde z_y$ isometrically onto $\blacktriangle\dot p\dot x\dot y$; similarly map $\blacktriangle\tilde p\tilde y\tilde z_x$ onto $\blacktriangle\dot p\dot y\dot z$.
- ◇ If a point w lies in one of the three circular sectors, say at distance r from center of the circle, let $F(w)$ be the point on the corresponding segment $[\dot p\dot z]$, $[\dot x\dot z]$ or $[\dot y\dot z]$ whose distance from the lefthand endpoint of the segment is r.
- ◇ Finally, if w lies in the remaining curvilinear triangle $\tilde z\tilde z_x\tilde z_y$, set $F(w)=\dot z$.

By construction, F satisfies the remaining conditions of the lemma. □

Proof of the inheritance lemma (23.13). Construct model triangles $[\dot p\dot x\dot z]=\tilde\triangle(pxz)$ and $[\dot p\dot y\dot z]=\tilde\triangle(pyz)$ so that $\dot x$ and $\dot y$ lie on opposite sides of $[\dot p\dot z]$.



Suppose

$$\tilde{\measuredangle}(z\,{}^{p}_{x}) + \tilde{\measuredangle}(z\,{}^{p}_{y}) < \pi.$$

Then for some point $\dot w \in [\dot p \dot z]$, we have

$$|\dot x - \dot w| + |\dot w - \dot y| < |\dot x - \dot z| + |\dot z - \dot y| = |x - y|.$$

Let $w \in [pz]$ correspond to $\dot w$; that is, $|z - w| = |\dot z - \dot w|$. Since $[pxz]$ and $[pyz]$ are thin, we have

$$|x - w| + |w - y| < |x - y|,$$

contradicting the triangle inequality.

Thus

$$\tilde{\measuredangle}(z\,{}^{p}_{x}) + \tilde{\measuredangle}(z\,{}^{p}_{y}) \geqslant \pi.$$

By Alexandrov's lemma (23.5), this is equivalent to

❼ $$\tilde{\measuredangle}(x\,{}^{p}_{z}) \leqslant \tilde{\measuredangle}(x\,{}^{p}_{y}).$$

Let $[\tilde p \tilde x \tilde y] = \tilde\triangle(pxy)$ and $\tilde z \in [\tilde x \tilde y]$ correspond to z; that is, $|x - z| = |\tilde x - \tilde z|$. Inequality ❼ is equivalent to $|p - z| \leqslant |\tilde p - \tilde z|$. Hence Lemma 23.14 applies; let $F\colon \blacktriangle\tilde p \tilde x \tilde y \to \blacktriangle\dot p \dot x \dot z \cup \blacktriangle\dot p \dot y \dot z$ be the provided short map.

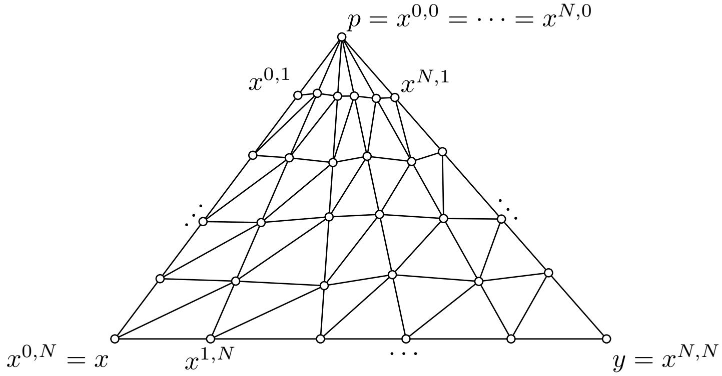
Fix v, w on the sides of $[pxy]$; let $\tilde v, \tilde w$ be the corresponding points on the sides of the model triangle $[\tilde p \tilde x \tilde y] = \tilde\triangle pxy$ and $\dot v, \dot w$ be the corresponding points on the sides of the model triangles $[\dot p \dot x \dot z] = \tilde\triangle pxz$ and $[\dot p \dot y \dot z] = \tilde\triangle pyz$. Denote by ℓ the length of shortest curve from $\dot v$ to $\dot w$ in $\blacktriangle\dot p \dot x \dot z \cup \blacktriangle\dot p \dot y \dot z$. Since F is short, $|\tilde v - \tilde w|_{\mathbb{R}^2} \geqslant \ell$. Since both triangles $[pxz]$ and $[pyz]$ are thin, $\ell \geqslant |v - w|_\Sigma$.

It follows that $|\tilde v - \tilde w|_{\mathbb{R}^2} \geqslant |v - w|_\Sigma$ for any v and w; that is, the triangle $[pxy]$ is thin. $\square$

Nonpositive curvature

Assume Σ is an open smooth regular surface with nonpositive curvature. As it follow from Exercise 22.6 any two points x and y in Σ are joined by unique geodesic $[xy]$.

Note that the geodesic $[xy]$ depends continuously on its endpoints x and y. That is, if $\gamma_{[xy]}\colon [0,1] \to \Sigma$ is the constant speed parametrization of $[xy]$ from x to y, then the map $(x, y, t) \mapsto \gamma_{[xy]}(t)$ is continuous in three arguments. Indeed, assume contrary, that is, $x_n \to x$, $y_n \to y$ and $t_n \to t$ as $n \to \infty$ and $\gamma_{[x_n y_n]}(t_n)$ does not converge to $\gamma_{[xy]}(t)$. Then we can pass to a subsequence such that $\gamma_{[x_n y_n]}(t_n)$ converges to



a point distinct from $w \neq \gamma_{[xy]}(t)$. Note that $w \notin [xy]$. Therefore there will be two distinct geodesics from x to y; one is $[xy]$ and the other is the limit of $[x_n y_n]$ which passes thru w.

Proof of part (23.1b) of the comparison theorem (23.1). Fix a triangle $[pxy]$; by Proposition 23.6, it is sufficient to show that the triangle $[pxy]$ is thin.

Fix large integer N and divide $[xy]$ by points $x = x^{0,N}, \dots, x^{N,N} = y$ into N equal parts. Further divide each geodesic $[p\, x^{i,N}]$ into N equal parts by points $p = x^{i,0}, \dots, x^{i,N}$. Since the geodesic depends continuously on its end points, we can assume that each triangle $[x^{i,j}\, x^{i,j+1}\, x^{i+1,j+1}]$ and $[x^{i,j}\, x^{i+1,j}\, x^{i+1,j+1}]$ is small; in particular, by local comparison (23.3), each of these triangles is thin.

Now we show that the thin property propagates to $[pxy]$ by repeated application of the inheritance lemma (23.13):

- ◇ First, for fixed i, sequentially applying the lemma shows that the triangles $[x\, x^{i,1}\, x^{i+1,2}]$, $[x\, x^{i,2}\, x^{i+1,2}]$, $[x\, x^{i,2}\, x^{i+1,3}]$, and so on are thin.

In particular, for each i, the long triangle $[x\, x^{i,N}\, x^{i+1,N}]$ is thin.

- ◇ Applying the lemma again shows that the triangles $[x\, x^{0,N}\, x^{2,N}]$, $[x\, x^{0,N}\, x^{3,N}]$, and so on are thin.

In particular, $[pxy] = [p\, x^{0,N}\, x^{N,N}]$ is thin. □

23.15. Exercise. *Assume γ_1 and γ_2 be two geodesics in an open smooth regular simply connected surface Σ with nonpositive Gauss curvature. Show that the function*

$$h(t) = |\gamma_1(t) - \gamma_2(t)|_\Sigma$$

is convex.

Chapter 24

Semigeodesic chart

This chapter contains formal computational proofs of several statements discussed above; the calculations performed in so called semigeodesic charts — a special type of charts on smooth surface.

Polar coordinates

The property of exponential map in 20.8 can be used to define *polar coordinates* in a smooth surface Σ with respect to a point $p \in \Sigma$.

Namely, fix polar coordinates (θ, r) on tangent plane T_p. If $\mathrm{v} \in \mathrm{T}_p$ has coordinates (θ, r), then we say that $q = \exp_p \mathrm{v}$ is the point in Σ with the polar coordinates (θ, r), or briefly $q = w_p(r, \theta)$. Note that according to Proposition 20.8 polar coordinates behave usual way in a neighborhood of p; that is, there is $r_0 >$ such that if $r_1, r_2 < r_0$ then $w_p(r_1, \theta_1) = w_p(r_2, \theta_2)$ if and only if $r_1 = r_2 = 0$ or $r_1 = r_2$ and $\theta_1 = \theta_2 + 2{\cdot}n{\cdot}\pi$ for an integer n.

The following statement is known as *Gauss lemma*, it will play key role in the proof of 20.9.

24.1. Lemma. *Let $w_p(\theta, r)$ describes polar coordinates in a neingborhood of point p of smooth surface Σ. Then*

$$\tfrac{\partial w_p}{\partial \theta} \perp \tfrac{\partial w_p}{\partial r}$$

for any r and θ.

Proof. Note that by the definition of exponential map, for a fixed θ, the curve $\gamma_\theta(t) = w(\theta, t)$ is a unit-speed geodesic that starts at p; in particular we have the following two identities:

(i) Since the geodesic has unit speed we have $|\frac{\partial}{\partial r}w_p| = |\gamma_\theta'(r)| = 1$. In particular,

$$\tfrac{\partial}{\partial\theta}\langle\tfrac{\partial w_p}{\partial r}, \tfrac{\partial w_p}{\partial r}\rangle = 0$$

(ii) Since $\frac{\partial^2 w_p}{\partial r^2} = \gamma_\theta''(r) \perp \mathrm{T}_{\gamma_\theta(r)}$, we have

$$\langle\tfrac{\partial^2 w_p}{\partial r^2}, \tfrac{\partial w_p}{\partial\theta}\rangle = 0$$

Therefore

$$\begin{aligned}\tfrac{\partial}{\partial r}\langle\tfrac{\partial w_p}{\partial\theta}, \tfrac{\partial w_p}{\partial r}\rangle &= \langle\tfrac{\partial^2 w_p}{\partial\theta\partial r}, \tfrac{\partial w_p}{\partial r}\rangle + \langle\tfrac{\partial w_p}{\partial\theta}, \tfrac{\partial^2 w_p}{\partial r^2}\rangle = \\ &= \tfrac12\cdot\tfrac{\partial}{\partial\theta}\langle\tfrac{\partial w_p}{\partial r}, \tfrac{\partial w_p}{\partial r}\rangle = \\ &= 0\end{aligned}$$

That is $\langle\frac{\partial w_p}{\partial\theta}, \frac{\partial w_p}{\partial r}\rangle$ does not depend on r.

Note that $w_p(\theta, 0) = p$ for any θ. Therefore $\frac{\partial w_p}{\partial\theta} = 0$ and in particular

$$\langle\tfrac{\partial w_p}{\partial\theta}, \tfrac{\partial w_p}{\partial r}\rangle = 0$$

if $r = 0$. Since $\langle\frac{\partial w_p}{\partial\theta}, \frac{\partial w_p}{\partial r}\rangle$ does not depend on r, we get

$$\langle\tfrac{\partial w_p}{\partial\theta}, \tfrac{\partial w_p}{\partial r}\rangle = 0$$

everywhere. Whence the statement follows. □

Shortest paths are geodesics: a formal proof

The following proof relies on the construction of polar coordinates in a smooth surface see Appendix 24 and the Gauss lemma 24.1.

Proof of 20.9. Let $\gamma\colon [0,\ell] \to \Sigma$ be a shortest path parameterized by arc-length. Suppose $\ell = \operatorname{length}\gamma$ is sufficiently small, so γ can be described in the polar coordinates at p; say $\gamma(t) = w_p(\theta(t), r(t))$ for some functions $t \mapsto \theta(t)$ and $t \mapsto r(t)$ such that $r(0) = 0$.

Note that by chain rule, we have

❶ $$\gamma'(t) = \tfrac{\partial w_p}{\partial\theta}\cdot\theta'(t) + \tfrac{\partial w_p}{\partial r}\cdot r'(t)$$

if the left side is defined. By Gauss lemma 24.1, $\frac{\partial w_p}{\partial\theta} \perp \frac{\partial w_p}{\partial r}$ and by definition of polar coordinates $|\frac{\partial w_p}{\partial r}| = 1$. Therefore ❶ implies

❷ $$|\gamma'(t)| \geqslant r'(t).$$

for any t where $\gamma'(t)$ is defined.

Since γ parameterized by arc-length, we have

$$|\gamma(t_2) - \gamma(t_1)| \leqslant |t_2 - t_1|.$$

In particular, γ is Lipschitz. Therefore by Rademacher's theorem (4.1) the derivative γ' is defined almost everywhere. By 9.5a, we have that

$$\begin{aligned}\operatorname{length}\gamma &= \int\limits_0^\ell |\gamma'(t)|\cdot dt \geqslant \\ &\geqslant \int\limits_0^\ell r'(t)\cdot dt = \\ &= r(\ell).\end{aligned}$$

Note that by the definition of polar coordinates, there is a geodesic of length $r(\ell)$ from $p = \gamma(0)$ to $q = \gamma(\ell)$. Since γ is a shortest path, we have therefore $r(\ell) = \ell$ and moreover $r(t) = t$ for any t. This equality holds if and only if we have equality in ❷ for almost all t. The latter implies that γ is a geodesic.

It remains to prove the partial converse.

Fix a point $p \in \Sigma$. Let $\varepsilon > 0$ be as in 20.8. Assume a geodesic γ of length less than ε from p to q does not minimize the length between its endpoints. Then there is a shortest path from p to q, which becomes a geodesic if parameterized by its arc-length. That is, there are two geodesics from p to q of length smaller than ε. In other words there are two vectors $\mathrm{v}, \mathrm{w} \in \mathrm{T}_p$ such that $|\mathrm{v}| < \varepsilon$, $|\mathrm{w}| < \varepsilon$ and $q = \exp_p \mathrm{v} = \exp_p \mathrm{w}$. But according to 20.8, the exponential map is injective in ε-neighborhood of zero — a contradiction. □

Gauss curvature

A chart $(u,v) \mapsto s(u,v)$ of smooth surface Σ is called *semigeodesic* if $|\frac{\partial s}{\partial u}| = 1$ and $\frac{\partial s}{\partial u} \perp \frac{\partial s}{\partial v}$ for any (u,v).

24.2. Proposition. *Any point p on a smooth surface Σ can be covered by a semigeodesic chart.*

Proof. By Gauss lemma (24.1) polar coordinates is an example of a semigeodesic chart. It remains to find a point $q \neq p$ such that polar coordinates on Σ with center at q cover p; any sufficiently close point will do the trick. □

A chart $(u,v) \mapsto s(u,v)$ is called *orthogonal* if $\frac{\partial s}{\partial u} \perp \frac{\partial s}{\partial v}$ for any (u,v). Note that any semigeodesic chart is orthogonal.

24.3. Lemma. *Let $(u,v)\mapsto s(u,v)$ be a orthogonal chart of smooth surface Σ. Then*

$$K=\tfrac{1}{a\cdot b}\cdot\left(\tfrac{\partial}{\partial u}\left(\tfrac{1}{a}\cdot\tfrac{\partial}{\partial u}b\right)+\tfrac{\partial}{\partial v}\left(\tfrac{1}{b}\cdot\tfrac{\partial}{\partial v}a\right)\right) \tag{❸}$$

where $a=a(u,v):=|\tfrac{\partial s}{\partial u}|$, $b=b(u,v):=|\tfrac{\partial s}{\partial v}|$, and $K=K(u,v)$ denotes the Gauss curvature of Σ at $s(u,v)$.

In particular, if $(u,v)\mapsto s(u,v)$ is a semigeodesic chart, then $a\equiv 1$ and therefore

$$K=\tfrac{1}{b}\cdot\tfrac{\partial^2 b}{\partial u^2}.$$

Proof. Set $\mathrm{U}(u,v):=\tfrac{1}{a}\cdot\tfrac{\partial s}{\partial u}$ and $\mathrm{V}(u,v):=\tfrac{1}{b}\cdot\tfrac{\partial s}{\partial v}$. Recall that $\nu(u,v)$ denotes the unit normal vector to Σ at $s(u,v)$.

Observe that $\mathrm{U},\mathrm{V},\nu$ form an orthonormal frame for any (u,v). Without loss of generality we may assume that the frame $\mathrm{U},\mathrm{V},\nu$ is oriented; that is, $\nu=\mathrm{U}\times\mathrm{V}$ for any (u,v).

Suppose that $\ell=\ell(u,v)$, $m=m(u,v)$, and $n=n(u,v)$ be the components of the matrix describing the shape operator in the frame U,V. Since U,V is an orthonormal frame, by 16.7 we have

$$\begin{aligned}\tfrac{1}{a^2}\cdot\langle\tfrac{\partial^2 s}{\partial u^2},\nu\rangle&=\langle S(\mathrm{U}),\mathrm{U}\rangle=\ell, &\tfrac{1}{a\cdot b}\cdot\langle\tfrac{\partial^2 s}{\partial u\partial v},\nu\rangle&=\langle S(\mathrm{U}),\mathrm{V}\rangle=m,\\ \tfrac{1}{a\cdot b}\cdot\langle\tfrac{\partial^2 s}{\partial v\partial u},\nu\rangle&=\langle S(\mathrm{V}),\mathrm{U}\rangle=m, &\tfrac{1}{b^2}\cdot\langle\tfrac{\partial^2 s}{\partial v^2},\nu\rangle&=\langle S(\mathrm{V}),\mathrm{V}\rangle=n.\end{aligned} \tag{❹}$$

First let us show that the statement follows from the following identities:

$$\begin{aligned}\tfrac{\partial}{\partial u}\mathrm{U}&=-\tfrac{1}{b}\cdot\tfrac{\partial a}{\partial v}\cdot\mathrm{V}+a\cdot\ell\cdot\nu, &\tfrac{\partial}{\partial u}\mathrm{V}&=\tfrac{1}{b}\cdot\tfrac{\partial a}{\partial v}\cdot\mathrm{U}+a\cdot m\cdot\nu\\ \tfrac{\partial}{\partial v}\mathrm{U}&=-\tfrac{1}{a}\cdot\tfrac{\partial b}{\partial u}\cdot\mathrm{V}+b\cdot m\cdot\nu, &\tfrac{\partial}{\partial v}\mathrm{V}&=\tfrac{1}{a}\cdot\tfrac{\partial b}{\partial u}\cdot\mathrm{U}+b\cdot n\cdot\nu\end{aligned} \tag{❺}$$

Indeed, recall that the Gauss curvature equals to the determinant of the matrix $\left(\begin{smallmatrix}\ell & m\\ n & n\end{smallmatrix}\right)$; that is, $K=\ell\cdot n-m^2$. Therefore

$$\begin{aligned}\langle\tfrac{\partial}{\partial u}\mathrm{U},\tfrac{\partial}{\partial v}\mathrm{V}\rangle-\langle\tfrac{\partial}{\partial v}\mathrm{U},\tfrac{\partial}{\partial u}\mathrm{V}\rangle&=a\cdot b\cdot(\ell\cdot n-m^2)=\\ &=a\cdot b\cdot K.\end{aligned}$$

On the other hand

$$\begin{aligned}\langle\tfrac{\partial}{\partial u}\mathrm{U},\tfrac{\partial}{\partial v}\mathrm{V}\rangle-\langle\tfrac{\partial}{\partial v}\mathrm{U},\tfrac{\partial}{\partial u}\mathrm{V}\rangle&=\left(\tfrac{\partial}{\partial v}\langle\tfrac{\partial}{\partial u}\mathrm{U},\mathrm{V}\rangle-\langle\tfrac{\partial^2}{\partial u\partial v}\mathrm{U},\mathrm{V}\rangle\right)-\\ &-\left(\tfrac{\partial}{\partial u}\langle\tfrac{\partial}{\partial v}\mathrm{U},\mathrm{V}\rangle-\langle\tfrac{\partial^2}{\partial u\partial v}\mathrm{U},\mathrm{V}\rangle\right)=\\ &=\tfrac{\partial}{\partial v}\left(-\tfrac{1}{b}\cdot\tfrac{\partial a}{\partial v}\right)-\tfrac{\partial}{\partial u}\left(\tfrac{1}{a}\cdot\tfrac{\partial b}{\partial u}\right)\end{aligned}$$

The identity ❸ follows since the left hand sides in the last two equations are identical.

It remains to prove ❺. Since the frame $\textsc{u}, \textsc{v}, \nu$ is orthonormal, first two vector identities are equivalent to the following six real identities:

❻
$$\begin{aligned}
&\langle \tfrac{\partial}{\partial u}\textsc{u}, \textsc{u}\rangle = 0, && \langle \tfrac{\partial}{\partial u}\textsc{v}, \textsc{u}\rangle = \tfrac{1}{b}\cdot\tfrac{\partial a}{\partial v},\\
&\langle \tfrac{\partial}{\partial u}\textsc{u}, \textsc{v}\rangle = -\tfrac{1}{b}\cdot\tfrac{\partial a}{\partial v}, && \langle \tfrac{\partial}{\partial u}\textsc{v}, \textsc{v}\rangle = 0,\\
&\langle \tfrac{\partial}{\partial u}\textsc{u}, \nu\rangle = a\cdot\ell, && \langle \tfrac{\partial}{\partial u}\textsc{v}, \nu\rangle = a\cdot m.
\end{aligned}$$

By taking partial derivatives of the identities $\langle\textsc{u},\textsc{u}\rangle = 1$, $\langle\textsc{u},\textsc{v}\rangle = 0$, and $\langle\textsc{v},\textsc{v}\rangle = 1$, we get

$$\langle \tfrac{\partial}{\partial u}\textsc{u}, \textsc{u}\rangle = 0, \qquad \langle \tfrac{\partial}{\partial u}\textsc{v}, \textsc{v}\rangle = 0, \qquad \langle \tfrac{\partial}{\partial u}\textsc{v}, \textsc{u}\rangle = -\langle \textsc{v}, \tfrac{\partial}{\partial u}\textsc{u}\rangle.$$

Further, observe that

❼
$$\begin{aligned}
\tfrac{\partial}{\partial u}\textsc{v} &= \tfrac{\partial}{\partial v}\big(\tfrac{1}{b}\cdot\tfrac{\partial s}{\partial v}\big) =\\
&= \tfrac{1}{b}\cdot\tfrac{\partial^2 s}{\partial u\partial v} + \tfrac{\partial}{\partial u}\big(\tfrac{1}{b}\big)\cdot\tfrac{\partial s}{\partial v}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle \tfrac{\partial}{\partial u}\textsc{v}, \textsc{u}\rangle &= \tfrac{1}{a\cdot b}\cdot\langle \tfrac{\partial^2 s}{\partial v\partial u}, \tfrac{\partial s}{\partial u}\rangle =\\
&= \tfrac{1}{2\cdot a\cdot b}\cdot\tfrac{\partial}{\partial v}\langle \tfrac{\partial s}{\partial u}, \tfrac{\partial s}{\partial u}\rangle =\\
&= \tfrac{1}{2\cdot a\cdot b}\cdot\tfrac{\partial a^2}{\partial v} =\\
&= \tfrac{1}{b}\cdot\tfrac{\partial a}{\partial v}
\end{aligned}$$

and

$$\langle \textsc{v}, \tfrac{\partial}{\partial u}\textsc{u}\rangle = -\tfrac{1}{b}\cdot\tfrac{\partial a}{\partial v}.$$

Since $0 = \tfrac{\partial}{\partial u}\langle\textsc{v},\textsc{u}\rangle = \langle \tfrac{\partial}{\partial u}\textsc{v}, \textsc{u}\rangle + \langle \textsc{v}, \tfrac{\partial}{\partial u}\textsc{u}\rangle$, we get

$$\langle \tfrac{\partial}{\partial u}\textsc{v}, \textsc{u}\rangle = -\langle \textsc{v}, \tfrac{\partial}{\partial u}\textsc{u}\rangle = \tfrac{1}{b}\cdot\tfrac{\partial a}{\partial v}.$$

Applying ❹ and ❼, we get

$$\begin{aligned}
\langle \tfrac{\partial}{\partial u}\textsc{u}, \nu\rangle &= \tfrac{1}{a}\cdot\langle \tfrac{\partial^2 s}{\partial u^2}, \nu\rangle = a\cdot\ell,\\
\langle \tfrac{\partial}{\partial u}\textsc{v}, \nu\rangle &= \tfrac{1}{a}\cdot\langle \tfrac{\partial^2 s}{\partial u\partial v}, \nu\rangle = a\cdot m,
\end{aligned}$$

that implies the last two equalities in ❻. Therefore the first two identities in ❺ are proved. The other two identities in ❺ can be proved along the same lines. □

The remarkable theorem

Let Σ_1 and Σ_2 be two smooth regular surfaces in the Euclidean space. A map $f\colon \Sigma_1 \to \Sigma_2$ is called length-preserving if for any curve γ_1 in Σ_1 the curve $\gamma_2 = f\circ\gamma_1$ in Σ_2 has the same length. If in addition f is smooth and bijective, then it is called *intrinsic isometry.*

A simple example of intrinsic isometry can obtained by warping a plane into a cylinder. The following exercise produce slightly more interesting example.

24.4. Exercise. *Suppose $\gamma(t) = (x(t), y(t))$ is a smooth unit-speed curve in the plane such that $y(t) = a\cdot\cos t$. Let Σ_γ be the surface of revolution of γ around the x-axis. Show that a small open domain in Σ_γ admits a smooth length-pereserving map to the unit sphere.*

Conclude that any round disc Δ in $\mathbb{S}^2$ of intrinsic radius smaller than $\frac{\pi}{2}$ admits a smooth length preserving deformation; that is, there is one parameter family of surfaces with boundary Δ_t, such that $\Delta_0 = \Delta$ and Δ_t is not congruent to Δ_0 for any $t \ne 0$.[1]

24.5. Theorem. *Suppose $f\colon \Sigma_1 \to \Sigma_2$ is an intrinsic isometry between two smooth regular surfaces in the Euclidean space; $p_1 \in \Sigma_1$ and $p_2 = f(p_1) \in \Sigma_1$. Then*

$$K(p_1)_{\Sigma_1} = K(p_2)_{\Sigma_2};$$

that is, the Gauss curvature of Σ_1 at p_1 is the same as the Gauss curvature of Σ_2 at p_2.

This theorem was proved by Carl Friedrich Gauss [23] who called it *Remarkable theorem* (Theorema Egregium). The theorem is indeed remarkable because the Gauss curvature is defined as a product of principle curvatures which might be different at these points; however, according to the theorem, their product can not change. In other words, the Gaussian curvature is an *intrinsic invariant.*

In fact Gauss curvature of the surface at the given point can be found *intrinsically*, by measuring the lengths of curves in the surface. For example, Gauss curvature $K(p)$ in the following formula for the circumference $c(r)$ of a geodesic circle centered at p in a surface:

$$c(r) = 2\cdot\pi\cdot r - \tfrac{\pi}{3}\cdot K(p)\cdot r^3 + o(r^3).$$

Note that the theorem implies there is no smooth length-preserving map that sends an open region in the unit sphere to the plane.[2] It

[1] In fact any disc in $\mathbb{S}^2$ of intrinsic radius smaller than π admits a smooth length preserving deformation.

[2] There are plenty of non-smooth length-preserving maps from the sphere to the plane; see [43] and the references there in.

follows since the Gauss curvature of the plane is zero and the unit sphere has Gauss curvature 1. In other words, there is no map of a region on Earth without distortion.

Proof.

Gauss–Bonnet formula: a formal proof

The following proof relies on two statements.

First, the following identity in polar coordinates $(r,\theta)\mapsto w_p(r,\theta)$ on a smooth surface:

$$K=\tfrac{1}{b}\cdot\tfrac{\partial^2 b}{\partial\theta^2},$$

where $b=|\tfrac{\partial w_p}{\partial\theta}|$; see 24.3.

Second, is the Green formula which can be formulated the following way. Let γ be a piecewise smooth simple closed curve in a plane that bounds a compact region D. Suppose that γ is oriented in such a way that D lies on the left from γ. Then for any two smooth functions P and Q defined on D we have

$$\iint\limits_D(\tfrac{\partial Q}{\partial x}-\tfrac{\partial P}{\partial y})\cdot dx\cdot dy=\int\limits_\gamma(P\cdot dx+Q\cdot dy).$$

Note that Green and similar to Gauss–Bonnet formulas are similar — they relate the integral along a disc and its boundary curve. So it should be not surprising Green helps to prove Gauss–Bonnet.

24.6. Lemma. *Let* $\mathrm{u},\mathrm{v},\mathrm{w}$ *be two tangent vector fields along a piecewise smooth curve* γ *on a smooth surface* Σ. *Suppose that* γ *is parametrized by length and at each point the frame* $\mathrm{u}(s),\mathrm{v}(s)$ *is oriented and orthonormal for any* s*; that is,* $|\mathrm{u}|=|\mathrm{v}|=1$ *and* v *obtained from* u *by counterclockwise rotation of the tangent plane at each point. Further suppose that* w *is a parallel vector field. Then there is a smooth function* $\varphi(s)$ *such that* $\mathrm{u}(s)$ *is a counterclockwise rotation of* $\mathrm{w}(s)$ *and*

$$\varphi'(s)=\langle\mathrm{u}'(s),\mathrm{v}(s)\rangle$$

for any s.

Moreover if $\gamma\colon[a,b]\to\Sigma$ *is a loop based at a point* p *then the parallel transport* $\iota\colon\mathrm{T}_p\to\mathrm{T}_p$ *along* γ *is a counterclockwise roatation by angle*

$$\omega=\varphi(a)-\varphi(b)=-\int\limits_a^b\varphi'(s)\cdot ds.$$

24.7. Corollary. *Suppose that γ is a piecewise smooth loop on a smooth surface Σ.*

Part V

Semisolutions

1.1. Check all the conditions in the definition of metric, page 6.

1.2; *(a).* Observe that $|p-q|_{\mathcal{X}} \leqslant 1$. Apply the triangle inequality to show that $|p-x|_{\mathcal{X}} \leqslant 2$ for any $x \in B[q,1]$. Make a conclusion.

(b). Take $\mathcal{X}$ to be a half-line $[0,\infty)$ with the standard metric; $p=0$ and $q=\frac{4}{5}$.

1.5. Show that the conditions in 1.4 hold for $\delta=\varepsilon$.

1.8. Suppose the complement $\Omega=\mathcal{X}\backslash Q$ is open. Then for each point $p \in \Omega$ there is $\varepsilon>0$ such that $|p-q|_{\mathcal{X}}>\varepsilon$ for any $q \in Q$. It follows that p is not a limit point of any sequence $q_n \in Q$. That is, any limit of points in Q lies in Q which by definition means that Q is closed.

Now suppose $\Omega=\mathcal{X}\backslash Q$ is not open. Then there is a point $p \in \Omega$ such that for any natural n there is a point $q_n \in Q$ such that $|p-q_n|_{\mathcal{X}} < \frac{1}{n}$; in particular $q_n \to p$ and $n \to \infty$. Since $p \notin Q$, we get that Q is not closed.

8.2. The image of γ might have a shape of digit 8 or 9.

8.3. Let α be a path connecting p to q.

Passing to a subinterval if necessary, we can assume that $\alpha(t) \neq p, q$ for $t \neq 0,1$.

An open set Ω in $(0,1)$ will be called *suitable* if for any connected component (a,b) of Ω we have $\alpha(a)=\alpha(b)$. Show that the union of nested suitable sets is suitable. Therefore we can find a maximal suitable set $\hat{\Omega}$.

Define $\beta(t)=\alpha(a)$ for any t in a connected component $(a,b) \subset \Omega$. Note that for any $x \in [0,1]$ the set $\beta^{-1}\{\beta(x)\}$ is connected.

It remains to re-parametrize β to make it injective. In other words we need to construct a non-decreasing surjective function $\tau\colon [0,1] \to [0,1]$ such that $\tau(t_1)=\tau(t_2)$ if and only if there is a connected component (a,b) such that $t_1, t_2 \in [a,b]$. The construction is similar to the construction of devil's staircase.

8.4. Denote the union of two half-axis by L. Observe that $f(t) \to \infty$ as $t \to \infty$. Since $f(0)=0$, the intermediate value theorem implies that $f(t)$ takes all nonnegative values for $t \geqslant 0$. Use it to show that L is the range of α.

Note that the function f is smooth. Indeed, the existence of all derivatives $f^{(n)}(x)$ at $x \neq 0$ is evident and direct calculations show that $f^{(n)}(0)=0$ for all n. Therefore $t \mapsto \alpha(t)=(f(t), f(-t))$ is smooth as well.

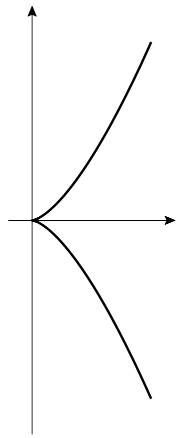
Further, show that the function f is strictly increasing for $t>0$, and, moreover, if $0<t_0<t_1$, then $0<f(t_0)<f(t_1)$. Use it to show that the maps $t \mapsto \alpha(t)$ is injective.

Summarizing we get that α is a smooth parametrization of L.

Now suppose $\beta\colon t \mapsto (x(t), y(t))$ is a smooth parameterization of L. Without loss of generality we may assume that $x(0) = y(0) = 0$. Note that $x(t) \geqslant 0$ for any t therefore $x'(0) = 0$. The same way we get that $y'(0) = 0$. That is, $\beta'(0) = 0$; so L does not admit a smooth regular parameterization.

8.5. Apply the definitions. For *(a)* you need to check that $\gamma'_\ell \neq 0$. For *(b)* you need to check that $\gamma_\ell(t_0) = \gamma(t_1)$ only if $t_0 = t_1$.

8.6. This is so called *semicubical parabola*; it is shown on the diagram. Try to argue similarly to 8.4.



8.7. For $\ell = 0$ the system describes a pair of points $(0, 0, \pm 1)$, so we can assume that $\ell \neq 0$. Note that first equation desribes the unit sphere centered at the origin and the second equation describes a cylinder over the circle in the (x, y)-plane with diameter with opposite points $(0, 0)$ and $(0, \ell)$.

For $\ell \neq 0$, find the gradients ∇f and ∇h for the functions

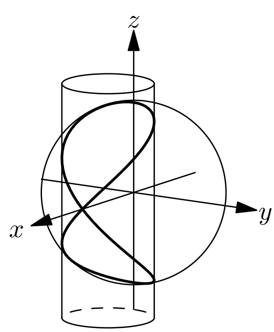
$$f(x, y, z) = x^2 + y^2 + z^2 - 1$$
$$h(x, y, z) = x^2 + \ell{\cdot}x + y^2$$

and show that they are linearly dependent only on the x-axis. Conclude that for $\ell \neq \pm 1$ each connected component of the set of solutions is a smooth regular curve.

Show that

- ◇ if $|\ell| < 1$, then the set has two connected components with $z > 0$ and $z < 0$.
- ◇ if $|\ell| \geqslant 1$, then the set is connected.

Note that the condition on gradients provides only sufficient condition. Therefore he case $\ell = \pm 1$ has to be checked by hands. In this case a neighborhood of $(\pm 1, 0, 0)$ does not admit a smooth regular parametrization — try to prove it. The case $\ell = 0$ shown on the diagram.



Remark. In the case $\ell = \pm 1$ it is called *Viviani's curve.* It admits the following smooth regular parameterization with a self-intersection:

$$t \mapsto (\pm(\cos t)^2, \cos t{\cdot}\sin t, \sin t).$$

8.8. Without loss of generality we may assume that the origin does not lie on the curve.

Show that inversion of the plane $(x,y)\mapsto(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2})$ maps our curve maps to a closed curve with removed origin. Apply Jordan's theorem for the obtained curve and use the inversion again.

9.2. Observe that if $c=\tau_0<\dots<\tau_n=d$ is a partition of $[c,d]$ if and only if $t_i=\varphi(\tau_i)$ is a partition of $[a,b]$ and apply the definition of length (9.1).

9.3. Fix a partition $0=t_0<\dots<t_n=1$ of $[0,1]$. Set $\tau_0=0$ and $\tau_i=\max\{\,\tau\in[0,1]\;:\;\beta(\tau_i)=\alpha(t_i)\,\}$. Show that (τ_i) is a partition of $[0,1]$; that is, $0=\tau_0<\tau_1<\dots<\tau_n=1$.

By construction

$$\begin{aligned}|\alpha(t_0)-\alpha(t_1)|+|\alpha(t_1)-\alpha(t_2)|+\dots+|\alpha(t_{n-1})-\alpha(t_n)|=\\ =|\beta(\tau_0)-\beta(\tau_1)|+|\beta(\tau_1)-\beta(\tau_2)|+\dots+|\beta(\tau_{n-1})-\beta(\tau_n)|.\end{aligned}$$

Since the partition (t_i) is arbitrary, we get

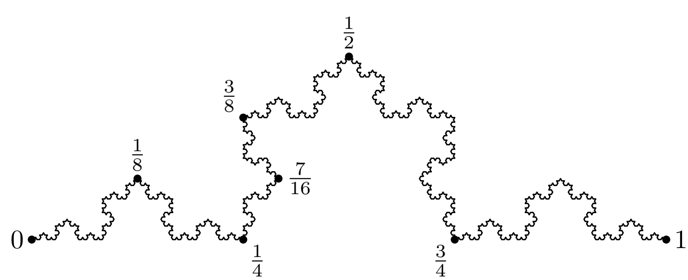
$$\operatorname{length}\beta\geqslant\operatorname{length}\alpha.$$

Remark. Note that the partition (τ_i) is not arbitrary, therefore the inequality might be strict; it might happen if β runs back and forth along α.

9.4. For (9.4a), apply the fundamental theorem of calculus for each segment in a given partition. For (9.4b) consider a partition such that the velocity vector $\alpha'(t)$ is nearly constant on each of its segments.

9.5. Use theorems of Rademacher and Lusin (4.1 and 4.2).

9.6; *(a).* Look at the diagram and guess the parameterization of an



arc of the snow flake by $[0,1]$. Extend it to whole snow flake. Show that it is indeed describes an embedding of the circle in the plane.

(b). Suppose that $\gamma\colon [0,1]\to\mathbb{R}^2$ is a rectifiable curve and γ_k be a scaled copy of γ with factor $k>0$; that is $\gamma_k(t)=k{\cdot}\gamma(t)$ for any t. Show that

$$\operatorname{length}\gamma_k = k{\cdot}\operatorname{length}\gamma.$$

Now suppose the arc γ of the Koch snowflake shown on the diagram is rectifiable; denote its length by ℓ. Evidently $\ell>0$. Observe that it γ can be divided into 4 arcs each of which is a scaled copy of γ with factor $\frac13$. It follows that $\ell=\frac43{\cdot}\ell$ — a contradiction.

9.8. We have to assume that $a\ne 0$ or $b\ne 0$; otherwise we get a constant curve.

Show that the curve has constant velocity $|\gamma'(t)|\equiv\sqrt{a^2+b^2}$. Therefore

$$s=\tfrac{t}{\sqrt{a^2+b^2}}$$

is an arc-length parameter.

9.12. Choose a closed polygonal line $p_1\dots p_n$ inscribed in β. By 9.11, we can assume that it length is arbitrary close to the length of β; that is, given $\varepsilon>0$

$$\operatorname{length}(p_1\dots p_n)>\operatorname{length}\beta-\varepsilon.$$

Show that we may assume in addition that each point p_i lies on α.

Observe that since α is simple, the points $p_1,\dots,p_n$ appear on α in the same cyclic order; that is, the polygonal line $p_1\dots p_n$ is also inscribed in α. In particular

$$\operatorname{length}\alpha\geqslant\operatorname{length}(p_1\dots p_n).$$

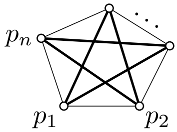
It follows that

$$\operatorname{length}\alpha>\operatorname{length}\beta-\varepsilon.$$

for any $\varepsilon>0$. Whence

$$\operatorname{length}\alpha\geqslant\operatorname{length}\beta.$$

If α has self-intersections, then the points $p_1,\dots,p_n$ might appear on α in a different order, say $p_{i_1},\dots,p_{i_n}$. Apply triangle inequality to show that



$$\operatorname{length}(p_{i_1}\dots p_{i_n})\geqslant\operatorname{length}(p_1\dots p_n)$$

and use it to modify the proof above.

9.13. Denote by ℓ_u the line segment obtained orthogonal projection of γ to the line in the direction u. Note that γ_u runs back and forth along ℓ_u, we get

$$\operatorname{length}\gamma_u\geqslant 2{\cdot}\operatorname{length}\ell_u.$$

Applying the Crofton formula, we get that

$$\operatorname{length}\gamma \geqslant \pi\cdot\overline{\operatorname{length}\ell_u}.$$

In the case of equality, the curve γ_u runs exactly back and forth along ℓ_u without additional zigzags for almost all (and therefore for all) u.

Let K be a closed set bounded by γ. Observe that the last statement implies that every line may intersect K only along a closed segment. In other words K is convex.

9.14. The proof is identical to the proof of the standard Crofton formula. To find the coefficient one has to find average length of projection of unit vector to a line. Which can be done by integration.

$$\frac{1}{k_a} = \frac{1}{\operatorname{area}\mathbb{S}^2}\cdot\int\limits_{\mathbb{S}^2}|x|; \qquad \frac{1}{k_b} = \frac{1}{\operatorname{area}\mathbb{S}^2}\cdot\int\limits_{\mathbb{S}^2}\sqrt{1-x^2}.$$

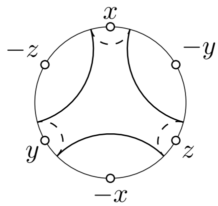
The answers are $k_a = 2$ and $k_b = \frac{4}{\pi}$.

9.16. The "only-if" part is trivial. To show the "if" part, assume A is not convex; that is, there are points $x, y \in A$ and a point $z \notin A$ that lies between x and y.

Since A is closed, its complement is open. That is, the ball $B(z,\varepsilon)$ does not intersect A for some $\varepsilon > 0$.

Show that there is $\delta > 0$ such that any path of length at most $|z-y|_{\mathbb{R}^3}+\delta$ pass thru $B(z,\varepsilon)$. It follows that $|z-y|_A \geqslant |z-y|_{\mathbb{R}^3}+\delta$, in particular$|z-y|_A \neq |z-y|_{\mathbb{R}^3}$.

9.19. The spherical curve shown on the diagram does not have antipodal pairs of points. However it has three points x, y, z on one of its sides and their antipodal points $-x, -y, -z$ on the other. Show that this property is sufficient to conclude that the curve does not lie in any hemisphere.



9.20. Assume contrary, then by the hemisphere lemma (9.18) γ lies in an open hemisphere. In particular it cannot divide $\mathbb{S}^2$ into two regions of equal area — a contradiction.

9.21. The very first sentence is wrong — it is *not* sufficient to show that diameter iis at most 2. For example an equilateral triangle with circumradius slightly above 1 may have diameter (which is defined as the maximal distance between its points) slightly bigger than $\sqrt{3}$, so it can be made smaller that 2.

On the other hand, it is easy to modify the proof of the hemisphere lemma (9.18) to get a correct solution. That is, (1) choose two points p and q on γ that divide it into two arcs of the same length; (2) set z to be a midpoint of p and q, and (3) show that γ lies in the unit disc centered at z.

9.22. For *(a)*, modify the proof of the original Crofton formula [see page 36].

(b). Assume $\operatorname{length}\gamma < 2\cdot\pi$. By *(a)*,

$$\overline{\operatorname{length}\gamma_u} < 2\cdot\pi.$$

Therefore we can choose u so that

$$\operatorname{length}\gamma_u < 2\cdot\pi.$$

Observe that γ_u runs in a semicircle h and therefore γ lies in a hemisphere with h as a diameter.

10.2. Differentiate the identity $\langle\gamma(s),\gamma(s)\rangle = 1$ a couple of times.

10.3. Prove and use the following identities:

$$\gamma''(t) - \gamma''(t)^{\perp} = \tfrac{\gamma'(t)}{|\gamma'(t)|}\cdot\langle\gamma''(t), \tfrac{\gamma'(t)}{|\gamma'(t)|}\rangle,$$
$$|\gamma'(t)| = \sqrt{\langle\gamma'(t),\gamma'(t)\rangle}.$$

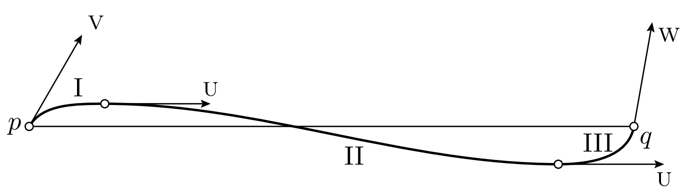
10.4. Apply 10.3a for the parameterization $t\mapsto(t,f(t))$.

10.5. Show that $\gamma''_{a,b}\perp\gamma'_{a,b}$ and apply 10.3a.

10.8. Apply Fenchel's theorem.

10.9. Assume that γ is unit-speed; show that $|\sigma'|\leqslant\kappa+\theta'$, where $\theta(s)=\measuredangle(\gamma(s),\gamma'(s))$.

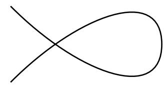
10.12. Set $\alpha=\measuredangle(\mathrm{W},\mathrm{U})$ and $\beta=\measuredangle(\mathrm{W},\mathrm{V})$. Try to guess the example from the diagram.



The shown curve is divided into three arcs: I, II, and III. Arc I turns from V to U; it has total curvature α. Analogously the arc III turns from U to W and has total curvature β. Arc II goes very close and almost parallel to the chord pq and its total curvature can be made arbitrary small.

10.13. Use that exterior angle of a triangle equals to the sum of the two remote interior angles; for the second part apply the induction on number of vertexes.

10.15. An example for *(a)* is shown on the diagram.



(b). Assume x is a point of self-intersection. Show that we may choose two points y and z on γ so that the triangle xyz iz nondegenerate. In particular, $\measuredangle xyz + \measuredangle yzx < \pi$, or, equivalently,

$$\Phi(xyzx) > \pi;$$

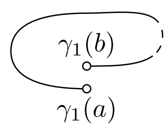
here we assume that $xyzx$ is polygonal line which is not closed. It remains to apply 10.14.

10.17. Observe that

$$\Phi(acbd) = 4{\cdot}\pi,$$

here we assume that $acbd$ denotes the closed polygonal line. It remains to apply 10.14.

10.19. Start with the curve γ_1 shown on the diagram. To obtain γ_2, slightly unbend (that is, decrease the curvature of) the dashed arc of γ_1.



10.20. Choose a value $s_0 \in [a, b]$ that splits the total curvature into two equal parts, θ in each. Observe that $\measuredangle(\gamma'(s_0), \gamma'(s)) \leqslant \theta$ for any s. Use this inequality the same way as in the proof of the bow lemma.

10.21. Let $\ell = \operatorname{length}\gamma$. Suppose $\ell_1 < \ell < \ell_2$. Let γ_1 be an arc of unit circle with length ℓ.

Show that the distance between the ends of γ_1 is smallet than $|p-q|$ and apply the bow lemma (10.18).

10.22. If $\operatorname{length}\gamma < 2{\cdot}\pi$, apply the bow lemma (10.18) to γ and an arc of unit circle of the same length.

10.26. Modify the proof of semi-continuity of length (9.15).

10.27. Choose two distinct points p and q on γ. Consider a *diangle pq* (that is, a closed polygonal line with two vertexes p and q). Observe that both external angles of the diangle have measure π. Therefore the total curvature of diangle is $2{\cdot}\pi$.

It remains to apply the definition of total curvature for arbitrary curves (10.25).

10.28; *(a).* Observe that if $\Phi(\gamma) \leqslant 1$, then

$$\text{❽} \qquad \operatorname{length}\gamma \leqslant 2\cdot|\gamma(b) - \gamma(a)|.$$

Indeed, if $\Phi(\gamma) \leqslant 1$, then from the definition of total curvature [10.25] it follows that for any polygonal line $p0 \dots p_n$ inscribed in γ the angle between any pair of edges does noe exceed 1. Let q_i be the orthogonal projection of p_i to the line of its first edge p_0p_1. By the angle estimate, we have that the points $q_0, \dots, q_n$ appear on the line in the same order and

$$|q_i - q_{i-1}| > \tfrac{1}{2}\cdot|p_i - p_{i-1}|.$$

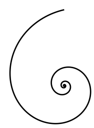
Therefore

$$\begin{aligned} |\gamma(b) - \gamma(a)| &= |p_n - p_0| \geqslant \\ &\geqslant |q_n - q_0| = \\ &= |q_n - q_{n-1}| + \dots + |q_1 - q_0| > \\ &> \tfrac{1}{2}\cdot(|p_n - p_{n-1}| + \dots + |p_1 - p_0|). \end{aligned}$$

Recall that length of γ is defined as the exact upper bound for the sum $|p_n - p_{n-1}| + \dots + |p_1 - p_0|$. Therefore ❽ follows.

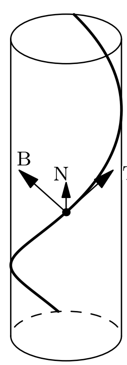
In general, if $\Phi(\gamma)$ is bounded, we can subdivide γ into arcs with total curvature at most 1, apply the above argument to each of arc and sum up the results.

(b). A logarithmic spiral is an example; it can be defined in polar coordinates by $\gamma(t) = (t, a\cdot \ln t)$ if $t > 0$ and $\gamma(0)$ is the origin; we may assume that $t \in [0, 1]$.



It remains to show that indeed, γ has infinite total curvature, but finite length.

11.1. The arc-length parameter s is already found in 9.8. It remains to find Frenet frame and calculate curvature and torsion. The latter can be done by straightforward calculations.



If the calculations done right, then you should see that curvature κ and torsion τ do not depend on time and given. Moreover for any $\kappa > 0$ and τ one can find a and b so that the helix $\gamma_{a,b}$ has curvature κ and torsion τ.

One may also see it geometrically using that the helix is maped to itself by one parameter family of glide rotations around z-axis. Therefore, for the t-parametrization, Frenet frame rotates around z-axis with the angular velocity 1. It remains rewrite it for the arc-length parametrization and note that

$$\begin{aligned}\mathrm{T}(0) &= (0, \cos\theta, \sin\theta),\\ \mathrm{N}(0) &= (-1, 0, 0),\\ \mathrm{B}(0) &= (0, \sin\theta, -\cos\theta),\end{aligned}$$

where $\operatorname{tg}\theta = b/a$ if $a > 0$.

11.2. By product rule, we get

$$\begin{aligned}\mathrm{B}' &= (\mathrm{T}\times\mathrm{N})' =\\ &= \mathrm{T}'\times\mathrm{N} + \mathrm{T}\times\mathrm{N}'.\end{aligned}$$

It remains to substitute the values from ❷ and ❸ and simplify.

11.3. Show and use that the binormal vector is constant.

11.4. Observe that $\frac{\gamma'\times\gamma''}{|\gamma'\times\gamma''|}$ is a unit vector perpendicular to the plane spanned by γ' and γ'', so, up to sign, it has to be equal to B. It remains to check that the sign is right.

11.6, *(a)*. Observe that $\langle \mathrm{W}, \mathrm{T}\rangle' = 0$. Show that it implies that

$$\langle \mathrm{W}, \mathrm{N}\rangle = 0.$$

Further observe that $\langle \mathrm{W}, \mathrm{N}\rangle' = 0$. Show that it implies that

$$\langle \mathrm{W}, -\kappa\cdot\mathrm{T} + \tau\cdot\mathrm{B}\rangle = 0.$$

(a). Show that $\mathrm{W}' = 0$; it implies that $\langle \mathrm{W}, \mathrm{T}\rangle = \frac{\tau}{\kappa}$. In particular, the velocity vector of γ makes a constant angle with W; that is, γ has constant slope.

11.7. Show that $\langle w, \alpha\rangle$ is constant if γ makes constant angle with a fixed vector w and α is the evolvent of γ.

11.9. Suppose $\langle \mathrm{W}, \mathrm{T}\rangle$ is a constant. Show that $\langle \mathrm{W}, \alpha\rangle' = 0$. It follows that $\langle \mathrm{W}, \alpha\rangle$ is a constant, so α lies in a plane perpendicular to W.

11.11. Use the second statement in 11.1.

11.12. Note that the function

$$\rho(\ell) = |\gamma(t+\ell) - \gamma(t)|^2 = \langle \gamma(t+\ell) - \gamma(t), \gamma(t+\ell) - \gamma(t)\rangle$$

is smooth and does not depend on t. Express speed, curvature and torsion of γ in terms of derivatives $\rho^{(n)}(0)$ and apply 11.11.

12.1. Without loss of generality, we may assume that γ_0 is parameterized by its arc-length. Then

$$|\gamma_1'| = |\gamma_0' + \mathrm{T}'| = |\mathrm{T} + \kappa\cdot\mathrm{N}| = \sqrt{1+\kappa^2} \geqslant 1 = |\gamma_0'|;$$

that is, $|\gamma_1'(t)| \geqslant |\gamma_0'(t)|$ for any $t \in [a,b]$ The statement follows since

$$\operatorname{length}\gamma_i = \int\limits_a^b |\gamma_i'(t)|\cdot dt.$$

12.4. Observe that

$$\gamma_a'(t) = (1 + a\cdot\cos t, -a\cdot\sin t);$$

that is γ_a' runs clockwise along a circle with center at $(1,0)$ and radius 1. If $|a| > 1$ then $\mathrm{T}_a(t) = \gamma_a'/|\gamma_a'|$ runs clockwise and makes full turn in time $2\cdot\pi$. It follows that if $|a| > 1$, then

$$\Psi(\gamma_a) = -2\cdot\pi, \quad \Phi(\gamma)_a = |\Psi(\gamma_a)| = 2\cdot\pi.$$

If $|a| < 1$, set $\theta_a = \arcsin a$. Note that $\mathrm{T}_a(t) = \gamma_a'/|\gamma_a'|$ starts with the horizontal direction $\mathrm{T}_a(0) = (1,0)$, turns monotonically to angle θ_a, then monotonically to $-\theta_a$ and then monotonically to back to $\mathrm{T}_a(2\cdot\pi) = (1,0)$. It follows that if $|a| > 1$, then

$$\Psi(\gamma_a) = 0, \quad \Phi(\gamma_a) = 4\cdot\theta_a.$$

In the cases $a = -1$ the velocity $\gamma_{-1}'(t)$ vanish at $t = 0$ and $2\cdot\pi$. Nevertheless, the curve admits a smooth regular parametrization — show it. In this case $\Psi(\gamma_{-1}) = -\pi$ and $\Phi(\gamma_{-1}) = \pi$.

In the cases $a = 1$ the velocity $\gamma_1'(t)$ vanish at $t = \pi$. At $t = \pi$ the curve has a cusp; that is

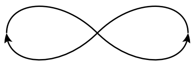
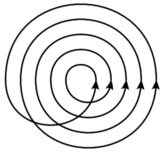
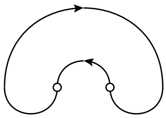
$$\lim_{t\to\pi-} \mathrm{T}_1(t) = -\lim_{t\to\pi+} \mathrm{T}_1(t).$$

So $\gamma_1(t)$ has undefined total signed curvature. The curve is a joint of two smooth arcs with external angle π, the total curvature of each arc is $\frac{\pi}{2}$, so $\Phi(\gamma_1) = \frac{\pi}{2} + \pi + \frac{\pi}{2} = 2\cdot\pi$.

12.6. The two marked points in the last example (for part *(c)*) have parallel tangent lines.

12.7; *(a)*. Show that

$$\gamma_\ell'(t) = (1 - \ell\cdot k(t))\cdot\gamma'(t).$$

Since γ is regular, $\gamma' \neq 0$. Therefore if $\gamma_\ell'(t) = 0$, then $\ell{\cdot}k(t) = 1$.

(b). Observe that we can assume that γ is parameterized by its arc-length, so $\gamma'(t) = \mathrm{T}(t)$. Suppose $|\ell| < \frac{1}{\kappa(t)}$ for any t. Then

$$|\gamma_\ell'(t)| = (1 - \ell{\cdot}k(t)).$$

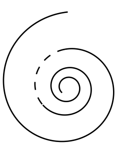
$$\begin{aligned} L(\ell) &= \int\limits_a^b (1 - \ell{\cdot}k(t)){\cdot}dt = \\ &= \int\limits_a^b 1{\cdot}dt - \ell{\cdot}\int\limits_a^b k(t)){\cdot}dt = \\ &= L(0) + \ell{\cdot}\Psi(\gamma). \end{aligned}$$

(c). Consider the unit circle $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2{\cdot}\pi]$ and γ_ℓ for $\ell = 2$.

12.10. Use the definition of osculating circle via order of contact and that inversion maps circles to circlines.

12.12. Find $\mathrm{T}(t)$ and $\mathrm{N}(t)$. Use the formula in 10.3*b* to calculate curvature $\kappa(t)$. Apply the formula given right before the exercise.

12.14. Start with a plane spiral curve as shown on the diagram. Increase the torsion of the dashed arc without changing curvature until it a self-intersection appears.



12.15. Observe that if a line or circle is tangent to γ, then it is tangent to osculating circle at the same point and apply the spiral lemma (12.11).

13.2. Apply the spiral lemma (12.11).

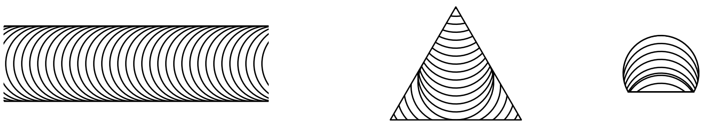
13.4. Consider the coordinate system with p as the origin and x-axis as the common tangent line to γ_1 and γ_2. We may assume that γ_i are defined in $(-\varepsilon, \varepsilon)$ for some small $\varepsilon > 0$, so that they run almost horizontally to the right.

Given $t \in [0,1]$ consider the curve γ_t that is tangent and cooriented to the x axis at $\gamma_t(0)$ and has signed curvature defined by $k_t(s) = (1-t)\cdot k_0(s) + t\cdot k_1(s)$. It exists by 12.2.

Fix $s \approx 0$. Consider the curve $\alpha_s\colon t \mapsto \gamma_t(s)$. Show that α_s moves almost vertically up, while γ_t moves almost horizontally to the right. Conclude that in a small neighborhood of p, the curve γ_1 lies above γ_0. Whence the statement follows.

13.5. Let reduce the radius of the circle until it touches γ. Observe that the circle supports γ and apply 13.3.

13.6, 13.7 and 13.8. Observe that one of the arcs of curvature 1 in the families shown on the diagram suppots γ and apply 13.3. To do



the second part in 13.6, use shown family and another family of arcs curved in the opposite direction.

Remark. Exercise 13.15 is a more general.

13.11. Note that we can assume that γ bounds a convex figure F, otherwise by 13.9 its curvature changes the sign and therefore it has zero curvature at some point. Choose two points x and y surrounded by γ such that $|x-y|>2$, look at the maximal lens bounded by two arcs with common chord xy that lies in F and apply supporting test (13.3).

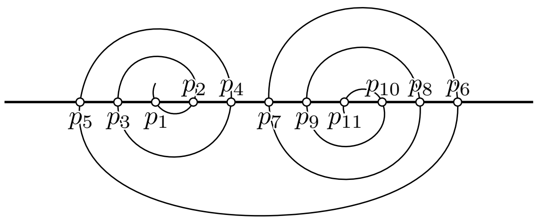
13.12; *(a).* Apply the lens lemma to show that if p_2 lies between p_1 and p_3, then the curvature of γ switches its sign.

(b). Show that in this p_4 lies between p_2 and p_3; further p_5 lies between p_3 and p_4; and so on.

(c). According to *(a)*, the point p_3 might lie between p_1 and p_2 and then further order is determined uniquely or p_1 lies between p_2 and p_3. In the latter case we have two choices, either p_4 lies between p_2 and p_3 and then further order is determined uniquely or p_2 lies between p_3 and p_4. In the latter case we get a choice again.

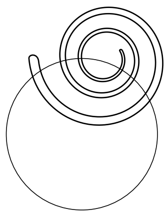
Assume we make a first choice on the step number k. Without loss of generality we may assume that p_k lies to the right from p_{k-2}. Then we have the following order:

$$p_{k-2}, p_{p-4}, \dots, p_{p-5}, p_{k-3}, p_k, p_{k+2}, \dots, p_{k+1}, p_{k-1}.$$



The case $k = 7$ shown on the diagram.

13.15. Note that γ contains a simple loop; apply to it 13.13.



13.18. Repeat the proof of theorem for each cyclic concatenation of an arc of γ from p_i to p_{i+1} with large arc of the circle.

An example for the second part can be guessed from the diagram.

14.3. Denote the set of solutions by Φ_ℓ.

Show that $\nabla_p f = 0$ if and only if $p = (0, 0, 0)$. Use 14.2 to conclude that if $\ell \neq 0$, then Φ_ℓ is a union of disjoint smooth regular surfaces.

Show that Φ_ℓ is connected if and only if $\ell \leqslant 0$. It follows that if $\ell < 0$, then Φ_ℓ is a smooth regular surface and if $\ell > 0$ then it is not.

The case $\ell = 0$ has to be done by hands — it does not satisfy the sufficient condition in 14.2, but it does not solely imply that Φ_0 is not a smooth surface.

Show that any neighborhood of origin in Φ_0 can not be described by a graph in any coordinate system; so by the definition (page 88) Φ_0 is not a smooth surface.

14.6. First check that the image of s lies in the unit sphere centered at $(0, 0, 1)$; that is show that

$$\left(\tfrac{2\cdot u}{1+u^2+v^2}\right)^2 + \left(\tfrac{2\cdot v}{1+u^2+v^2}\right)^2 + \left(\tfrac{2}{1+u^2+v^2} - 1\right)^2 = 1.$$

for any u and v.

Further, show that the map

$$(x, y, z) \mapsto \left(\tfrac{2\cdot x}{x^2+y^2+z^2}, \tfrac{2\cdot y}{x^2+y^2+z^2}\right)$$

describe the inverse map which is continuous away from the origin. In particular, s is an embedding that covers whole sphere except the origin.

It remains to show that s is regular; that is, $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$ are linearly independent at all points of the (u, v)-plane.

14.7. Set

$$s\colon (t,\theta)\mapsto (x(t), y(t)\cdot\cos\theta, y(t)\cdot\sin\theta).$$

Show that s is regular; that is, $\frac{\partial s}{\partial t}$ and $\frac{\partial s}{\partial \theta}$ are linearly independent. (It might help to observe that $\frac{\partial s}{\partial t}\perp\frac{\partial s}{\partial \theta}$).

Show that s is local embedding; that is, any (t_0,θ_0) admits a neighborhood U in the (t,θ)-plane such that the restriction $s|_U$ has a continuous inverse. It remains to apply 14.5.

14.8. The solutions of these exercises are build on the following general construction known as the *Moser trick*.

Suppose that U_t is a smooth vector field on a plane. Consider the ordinary differential equation $x'(t)=\mathrm{U}_t(x(t))$. Consider the map $\iota\colon x(0)\mapsto x(1)$ where $x(t)$ is a solution of the equation. The map ι is called *flow* of vector field U_t for the time interval $[0,1]$. Observe that according to 3.1 the map ι is smooth in its domain of definition. Moreover the same holds for its inverse; indeed ι^{-1} is the flow for of the vector field $-\mathrm{U}_{1-t}$. That is, ι is a diffeomorphism from the domain of definition to its image.

Therefore, in order to construct a diffeomorphism from one open subset of the plane to another it sufficient to construct a smooth vector field such that flow maps one set to the other; such map is automatically a diffeomorphism.

(a). Suppose $\Sigma=\mathbb{R}^2\backslash p_1,\dots,p_n$ and $\Theta=\mathbb{R}^2\backslash q_1,\dots,q_n$. Choose smooth paths $\gamma_i\colon[0,1]\to\mathbb{R}^2$ such that $\gamma_i(0)=p_i$, $\gamma_i(1)=q_i$, and $\gamma_i(t)\neq\gamma_j(t)$ if $i\neq j$.

Choose a smooth vector field V_t such that $\mathrm{V}_t(\gamma_i(t))=\gamma_i'(t)$ for any i and t. We can assume in addition that V_t vanish outside of a sufficiently large disc; it can be arranged by a multiplying the vector field to a function $\sigma_1(R-|x|)$; see page 17.

It remains to apply the Moser trick to the constructed vector field.

(b). Without loss of generality we can assume that the origin belongs to both Σ and Θ. Observe that the boundary curves of Σ and Θ can be written in polar coordinates as $(\theta, f(\theta)$ and $(\theta, g(\theta)$ for smooth functions $f,g\colon\mathbb{S}^1\to\mathbb{R}$.

Denote by U_θ the unit vector written as $(\theta,1)$ in the polar coordinates. Consider the vector field $[g(\theta)-f(\theta)]\cdot\mathrm{U}_\theta$ it is defined Show that there is a vector field V defined on $\mathbb{R}^2\backslash\{0\}$ that flows $\partial\Sigma$ to $\partial\Theta$; in fact such vector field can be found among radial fields

Observe that the boundary curves can be written in polar coordinates as $(\theta,\rho_i(\theta)$ for smooth functions $\rho_i\colon\mathbb{S}^1\to\mathbb{R}$.

???

15.3. Let γ be a smooth curve in Σ. Observe that $f\circ\gamma(t)\equiv 0$. Differentiate this identity and apply the definition of tangent vector (15.1).

15.4. Assume a neighborhood of p in Σ is a graph $z=f(x,y)$. In this case $s\colon(u,v)\mapsto(u,v,f(u,v))$ is a smooth chart at p. Show that the plane spanned by $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$ is not vertical; together with 15.2 it proves the if part.

To prove the only-if part, fix a chart

$$s\colon(u,v)\mapsto(x(u,v),y(u,v),z(u,v))$$

at p and apply the inverse function theorem for the map $(u,v)\mapsto$ $\mapsto(x(u,v),y(u,v))$.

15.7. Show that $\nu=\frac{\nabla h}{|\nabla h|}$ defines a unit normal field on Σ.

15.9. Fix a closed set A in the (x,y)-plane. Show that there is a smooth nonnegative function $(x,y)\mapsto f(x,y)$ such that $(x,y)\in A$ if and only if $f(x,y)=0$. Observe that the graph $z=f(x,y)$ describes a required surface.

16.1. Fix a point p on γ. Since Σ is mirror symmetric with respect to Π, so is the tangent plane T_p.

Choose (x,y)-coordinates on T_p so that the x-axis is the intersection $\Pi\cap\mathrm{T}_p$. Suppose that the osculating paraboloid is described by the graph

$$z=\tfrac{1}{2}\cdot(\ell\cdot x^2+2\cdot m\cdot x\cdot y+n\cdot y^2)$$

Since Σ is mirror symmetric, so is the paraboloid; that is, changing y to $(-y)$ does not change the value $\ell\cdot x^2+2\cdot m\cdot x\cdot y+n\cdot y^2$. In other words $m=0$, or equivalently, the x-axis points in the direction of curvature.

16.2. Note that the principle curvatures have the same sign at each point. Therefore we can choose a unit normal ν at each point so that both principle curvatures are positive. Show that it defines a field on the surface.

16.7.

16.8; *(a).* Observe that Σ has unit Hessian matrix at each point and apply the definition of shape operator.

(b). Fix a chart s in Σ and show that

$$\tfrac{\partial}{\partial u}(s(u,v)+\nu(u,v))=\tfrac{\partial}{\partial v}(s(u,v)+\nu(u,v))=0.$$

Make a conclusion.

16.9. We can assume that γ is parameterized by its arc-length. Denote by $\nu_1(s)$ and $\nu_2(s)$ the unit normal vectors to Σ_1 and Σ_2 at $\gamma(s)$. Since γ is a curvature line in Σ_1, we have ν_1' is proportional to γ'; in particular

$$\langle \nu_1', \nu_2 \rangle = 0.$$

Note that $\langle \nu_1(t), \nu_2(t) \rangle$ is constant. By taking its derivative and applying the above identity show that

$$\langle \nu_1, \nu_2' \rangle = 0.$$

Conclude that ν_2' is proportional to γ' and therefore γ is a curvature line in Σ_2.

16.13. Use Meusnier's theorem (16.12), to find center and radius of curvature of γ in terms of normal curvature of γ at p; make a conclusion.

16.14. Use 16.1 and Meusnier's theorem (16.12).

16.15. Use 16.14.

16.16. Apply Meusnier's theorem (16.12) to show that the coordinate curves $\alpha_v\colon u \mapsto s(u,v)$ and $\beta_u\colon v \mapsto s(u,v)$ are asymptotic; that is they have vanishing normal curvature.

Observe that these two families are orthogonal to each other. Therefore the Hessian matrix in the frame $\frac{\partial s}{\partial u}/|\frac{\partial s}{\partial u}|$ and $\frac{\partial s}{\partial v}/|\frac{\partial s}{\partial v}|$ will have zeros on the diagonal. Apply that mean curvature is the trace of the Hessian matrix.

16.17. Use 16.1 and 13.13.

16.19. Drill an extra hole or combine two examples together.

16.20. Let us define *cut locus* of V as a closure of the set of points $x \in V$ such that there are at least two points on ∂V that minimize the distance to x.

Denote by L the cut locus L of V.

Choose a connected component Σ of the boundary ∂V. Show that L is a smooth surface and the closest point projection $L \to \Sigma$ is a smooth regular parameterization. In particular there is unique point on Σ that minimize the distance to a given point $x \in L$. It follows that there is another component Σ' with the same property.

Finally show that ∂V can not be more than two components.

16.21. Read about Bing's two-room house. Try to thicken it to construct the needed example.

Assume V does not contain a ball of radius r_3. Show that its cut locus L of V is formed by a few smooth surfaces that meets by three at their boundary points. Show that L is not simply connected that is there is a loop in L that can not be deformed continuously to a trivial loop. Conclude that V is not simply connected.

Finally show that if V is bounded by sphere, then V is simply connected — a contradiction.

17.4; *(a).* Since $\operatorname{div}\mathrm{K} = 0$, applying the divergence theorem to the domain bounded by Δ and Σ, we get that

$$\operatorname{flux}_{\mathrm{K}}\Sigma = \operatorname{flux}_{\mathrm{K}}\Delta.$$

Since $|\mathrm{K}| = 1$, we have

$$\operatorname{flux}_{\mathrm{K}}\Sigma \leqslant \operatorname{area}\Sigma.$$

It remains to observe that

$$\operatorname{flux}_{\mathrm{K}}\Delta = \operatorname{area}\Delta.$$

(b). Consider the field $\mathrm{U} = x\cdot\mathrm{J}$. Observe that $\mathrm{K} = \operatorname{curl}\mathrm{U}$. Therefore by the curl theorem (17.3) we have

$$\operatorname{flux}_{\mathrm{K}}\Sigma = \int_0^{\ell} \langle \mathrm{U}, \gamma'(t)\rangle\cdot dt = \operatorname{flux}_{\mathrm{K}}\Delta,$$

where $\gamma\colon [0,\ell] \to \mathbb{R}^3$ is the common boundary of Σ and Δ parameterized by its arc-length and with the right choice of orientation.

The same argument as in *(a)* finishes the proof.

17.5. Observe that $\operatorname{div}\mathrm{U} = 1$. Applying the divergence theorem and the observation (17.1) we get

$$\operatorname{vol} R = \iiint_R \operatorname{div}\mathrm{U}\cdot dx\cdot dy\cdot dz = \operatorname{flux}_{\mathrm{U}}\Sigma \leqslant \operatorname{area}\Sigma.$$

17.9; *(a).* We may assume that p is the origin of $\mathbb{R}^3$. Let us extend the outer normal field ν to Σ to a field U defined in $\mathbb{R}^3\backslash\{0\}$ by

$$\mathrm{U}(\lambda\cdot q) = \nu(q)$$

for any $\lambda > 0$ and $q \in \Sigma$.

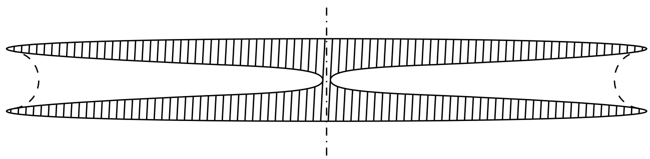
Observe that $\mathrm{U}(\lambda\cdot q)$ is normal to the surface

$$\lambda\cdot\Sigma = \{\,\lambda\cdot q \;:\; q \in \Sigma\,\}.$$

By 17.8, we have
$$\operatorname{div} \mathrm{U} = -H(\lambda{\cdot}q)_{\lambda\cdot\Sigma} \geqslant 0.$$

(b). Apply the divergence theorem for the region squeezed between Σ and Σ'.

(c). An example they can be found among bodies of revolution as shown on the picture. The gutter in the middle can be chosen to be taken to be a catenoid which is a minimal surface; see 16.15. Show



that if the gutter is sufficiently deep, then the surface of revolution of the dashed line can be made smaller.

17.10.

17.12. Consider the region of catenoid in the cylinder $x^2 + y^2 \leqslant R^2$ for $R > 1$. It is bounded by two circles defined by the equations

$$\begin{cases} x^2 + y^2 \leqslant R^2, \\ z = \pm r \end{cases}$$

where $\operatorname{ch} r = R$.

Show that if R is large, the area of this region is at least $\pi{\cdot}R^2$ — the area of disc of radius R.

Observe that the lateral surface of cylinder bounded by the two circles is $4{\cdot}\pi{\cdot}R{\cdot}r$. Since $r/R \to 0$ as $R \to infty$, we get that this surface has smaller area then region of catenoid.

17.13. Show that for large R, the area of the helicoid in the cylinder defined by $|z| \leqslant R$ and $x^2 + y^2 \leqslant R^2$ exceeds the area of the surface of cylinder. Make a conclusion.

18.2. Choose curvatures such that

$$k_2(p)_{\Sigma_1} > k_2(p)_{\Sigma_2} > k_1(p)_{\Sigma_1} > k_1(p)_{\Sigma_2}$$

and suppose that the first principle direction of Σ_1 coincides with the second principle direction of Σ_2 and the other way around.

18.4 and 18.5. Apply the same reasoning as in the problems 13.6–13.8, but use families of spheres instead.

18.6. Show and use that any tangent plane T_p supports Σ at p.

18.7. Assume a maximal ball in V touches its boundary at the points p and q. Consider the projection of V to a plane thru p, q and the center of the ball.

18.9. Suppose that a point p lies in the intersection $\Pi \cap \Sigma$.

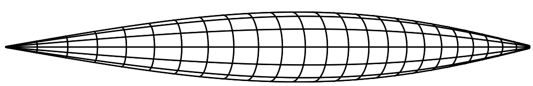
Show that if the tangent plane $\mathrm{T}_p\Sigma$ is parallel to Π, then p is an isolated point of the intersection $\Pi \cap \Sigma$.

It follows that if γ is a connected component of the intersection $\Pi \cap \Sigma$ that is not an isolated point, then at each point p on γ the tangent plane $\mathrm{T}_p\Sigma$ is transversal to Π. Apply the implicit function theorem to show that γ is a smooth regular curve.

Finally, observe that curvature of γ can not be smaller than normal curvature of Σ in the same direction. Whence γ has no points with vanishing curvature. Therefore for a right choice of orientation of γ, we its signed curvature is positive.

18.11. Look for a supporting spherical dome with the unit circle as the boundary.

18.12. Note that we can assume that the surface has positive Gauss curvature, otherwise the statement is evident. Therefore the surface bounds a convex region that contains a line segment of length π.



Observe that the Gauss curvature of the surface of revolution of the graph $y = a \cdot \sin x$ for $x \in (0, \pi)$ cannot exceed 1 (Use 10.4 and **??**). Try to support the surface Σ from inside by a surface of revolution of the described type.

Remark. In fact if Gauss curvature of Σ is at least 1, then the intrinsic diameter of Σ can not exceed π. The latter means that any two points in Σ can be connected by a path that lies in Σ and has length at most π.

18.14.

18.15.

18.16. By 18.15d, Σ is parameterized by an open convex plane domain Ω. It remains to show that Ω can parameterize the whole plane.

We may assume that the origin of the plane leis in Ω. Show that in this case the boundary of Ω can be written in polar coordinates as $(\theta, f(\theta))$ where $f\colon \mathbb{S}^1 \to \mathbb{R}$ is a positive continuous function. Then homeomorphism to the plane can be described in polar coordinate by changing only the radial coordinate; for example as $(\theta,r)\mapsto(\theta,\frac{1}{f(\theta)-r}-\frac{1}{f(\theta)})$.

To do the second part ane may apply 14.8c.

18.17. Choose a coordinate system so that (x,y)-plane supports Σ at the origin, so Σ lies in the upper half-space.

Show that there is $\varepsilon>0$ such that any line starting from the origin with slope at most ε may intersect Σ only in the unit ball centered at the origin; we may assume that ε is small, say $\varepsilon<1$. Consider the cone formed by half-lines from the origin with slope ε shifted down by 10 and observe that entire surface lies in this cone.

18.18. Choose distinct points $p,q\in\Sigma$. Apply 18.10 to show that the angle $\measuredangle(\nu(p),p-q)$ is acute and $\measuredangle(\nu(q),p-q)$ is obtuse. Conclude that $\nu(p)\ne\nu(q)$; that is, $\nu\colon\Sigma\to\mathbb{S}^2$ is injective.

(a). Given a unit vector u, consider a point $p\in\Sigma$ that maximize the scalar product $\langle p,\mathrm{u}\rangle$. Show that $\nu(p)=\mathrm{u}$. Conclude that the spherical map $\nu\colon\Sigma\to\mathbb{S}^2$ is onto. It follows that $\nu\colon\Sigma\to\mathbb{S}^2$ is bijection.

Applying 16.6, we get that

$$\begin{aligned}\int_\Sigma K&=\operatorname{area}\mathbb{S}^2=\\&=4\cdot\pi.\end{aligned}$$

(b). Choose an (x,y,z)-coordinate system provided by 18.15d. Observe that for any p the normal vector $\nu(p)$ forms an obtuse angle with the direction of z-axis. It follows that the image $\nu(\Sigma)$ lies in the south hemisphere.

Applying 16.6, we get that

$$\begin{aligned}\int_\Sigma K&\leqslant\tfrac12\cdot\operatorname{area}\mathbb{S}^2=\\&=2\cdot\pi.\end{aligned}$$

18.19. Use 16.14.

18.20. Prove and use that each point $p\in\Sigma$ has a direction with vanishing normal curvature.

18.21.

18.22. Denote by Π_t the tangent plane to Σ at $\gamma(t)$ and by ℓ_t the tangent line of γ at time t.

Since γ is asymptotic, the plane Π_t rotates around ℓ_t as t changes. Since Σ is saddle, the speed of rotation cannot vanish.[3]

Note that Π_t is a graph of a linear function, say h_t, defined on the (x,y)-plane. Denote by $\bar\ell_t$ the projection of ℓ_t to the (x,y)-plane. The described rotation of Π_t can be expressed algebraicaly: the derivative $\frac{d}{dt}h_t(w)$ vanishes at the point w if and only if $w\in\bar\ell_t$ and the derivative changes sign if w changes the side of $\bar\ell_t$.

Denote by $\bar\gamma$ the projection of γ to the (x,y)-plane. If $\bar\gamma$ is star shaped with respect to a point w, then w cannot cross $\bar\ell_t$. Therefore the function $t\mapsto h_t(w)$ is monotone on $\mathbb{S}^1$. Observing that this function cannot be constant, we arrive to a contradiction.

Soruse: The problem is discuss by Dmitri Panov [41].

18.24. Use the 18.23 and the hemisphere lemma (9.18).

18.25. Assume Σ is an open saddle surface that lies in a cone K. Show that there is a plane Π that cuts Σ and cuts from K a compact region. Observe that Π cuts from Σ a compact region.

By 15.10 one can move plane Π slightly so that it cuts from Σ a compact surface with boundary. Apply 18.23.

18.27. Observe that it is sufficient to construct a smooth parametrization of F_ε by a closed hemisphere.

Consider the radial projection of F_ε to the sphere Σ with the center at $p=(0,0,\varepsilon)$; that is a point $q\in F_\varepsilon$ is mapped to a point $s(q)$ on the sphere that lies on the ray pq.

Show that s is a diffeomorphism from F_ε to a south hemisphere of Σ.

18.28. Apply 18.26.

18.30. Look for an example among the surfaces of revolution and use 16.14.

18.31. Look at the sections of the graph by planes parallel to the (x,y)-plane and to the (x,z)-plane, then apply Meusnier's theorem (16.12).

18.33. Suppose that orthogonal projection of Σ to the (x,y)-plane is not injective. Show that there is a point $p\in\Sigma$ with vertical tangent plane; that is $\mathrm{T}_p\Sigma$ is perpendicular to the (x,y)-plane.

Let Γ_p be the connected component of p in the intersection of Σ and T_p. Show that (1) Γ_p is a tree, (2) each vertex of Γ_p has degree 4

[3] In fact by the Beltrami–Enneper theorem, if γ has unit speed, then the speed of rotation is $\pm\sqrt{-K}$, where K is the Gauss curvature which cannot vanish on a saddle surface.

or 1, (3) p is a vertex of degree 4, and (4) each endvertex lies on the bounary of Σ.

Observe that T_p meets the boundary of Σ at two points, therefore Γ has 2 endpoints. Use it to get a contradiction.

19.4. Assume the contrary, then there is a minimizing geodesic $\gamma \not\subset \Delta$ with ends p and q in Δ.

Without loss of generality, we may assume that only one arc of γ lies outside of Δ. Reflection of this arc with respect to Π together with the remaining part of γ forms another curve $\hat{\gamma}$ from p to q; it runs partly along Σ and partly outside Σ, but does not get inside Σ. Note that

$$\operatorname{length}\hat{\gamma} = \operatorname{length}\gamma.$$

Denote by $\bar{\gamma}$ the closest point projection of $\hat{\gamma}$ on Σ. Note that the curve $\bar{\gamma}$ lies in Σ, it has the same ends as γ, and by 19.3

$$\operatorname{length}\bar{\gamma} < \operatorname{length}\gamma.$$

This means that γ is not length minimizing, a contradiction.

19.5. Use 19.2.

20.2. Denote by μ a unit vector perpendicular to Π. Since γ lies in Π, we have that γ'' is parallel to Π, or equivalently $\gamma'' \perp \mu$. Since γ is unit speed, 10.1 implies that $\gamma'' \perp \gamma'$.

Since Σ is mirror symmetric with respect to a plane Π, the tangent palne $\mathrm{T}_{\gamma(t)}\Sigma$ is also mirror symmetric with respect to a plane Π. It follows that $\mathrm{T}_{\gamma(t)}\Sigma$ is spanned by μ and $\gamma'(t)$. Therefore $\gamma'' \perp \mu$ and $\gamma'' \perp \gamma'$ imply $\gamma'' \perp \mathrm{T}_{\gamma(t)}\Sigma$; that is, γ is a geodesic.

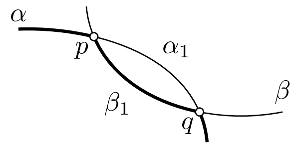
20.1. Show that $\nu_{\gamma(t)} = (\cos t, \sin t, 0)$. Calculate $\gamma''(t)$ and show that it is proportional to $\nu_{\gamma(t)}$. Note that the latter is equivalent to $\gamma''(t) \perp \mathrm{T}_{\gamma(t)}$.

20.3. Without loss of generality, we can assume that γ has unit speed. By the definition of geodesic, we have that $\gamma''(s) \perp \mathrm{T}_{\gamma(s)}$. Therefore

$$\gamma''(s) = k_n(s)\cdot\nu_{\gamma(s)},$$

where $k_n(s)$ is the normal curvature of γ at time s. Since γ is asymptotic, $k_n(s) \equiv 0$; that is, $\gamma''(s) \equiv 0$, therefore γ' is constant and γ runs along a line segment.

20.10. Assume that two shortest paths α and β have two common point p and q. Denote by α_1 and β_1 the arcs of α and β from p to q. Suppose that α_1 is distinct from β_1.

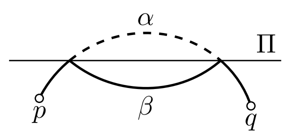


Note that α_1 and β_1 are shortest paths with the same endpoints; in particular they have the same length. Exchanging α_1 in α to β_1 produces a shortest path, say $\hat{\alpha}$, that is distinct from α. By 20.9, $\hat{\alpha}$ is a geodesic.

Suppose α_1 is a proper subarc of α; that is, if $\alpha_1 \neq \alpha$, or, equivalently, either p or q is not an endpoint of α. Then α and $\hat{\alpha}$ share one point and velocity vector at this point. By 20.5 α coincides with $\hat{\alpha}$ — a contradiction.

It follows that p and q are the end points of α. The same way we can show that p and q are the end points of β.

20.11. Assume a shortest path α changes the sides of Π at least twice. In this case there is an arc α_1 of α that lies on one side on Π and has ends on Π and these ends are distinct from the ends of α.



Let us remove the arc α_1 from α and exchange it to the reflection of α_1 across Π. Note that the obtained curve, say β, lies in the surface; it has the same length as α, and it connects the same pair of points, say p and q. Therefore β is another shortest path from p to q that is distinct from α.

By 20.9, α and β are geodesics. Since α and β have a common subarc, they share one point and velocity vector at this point; by 20.5 α coinsides with β — a contradiction.

20.12. Show that the concatenation of the line segment $[p_t, \gamma(t)]$ and the arc $\gamma|_{[t,\ell]}$ is a shortest path in the closed region W outside of Σ.

20.14. Equip Σ with unit normal field ν that points inside. Denote by $k(t)$ the normal curvature of Σ at $\gamma(t)$ in the direction of $\gamma'(t)$. Since Σ is convex $k(t) \geqslant 0$ for any t.

Since γ is geodesic, we have $\gamma''(t) = k(t)\cdot\nu_{\gamma(t)}$.

Since γ has unit speed, $\langle\gamma'(t), \gamma'(t)\rangle = 1$ for any t.

Without loss of generality, we can assume that p is the origin of $\mathbb{R}^3$. Since p is inside Σ, we have that $\langle\gamma(t), \nu_{\gamma(t)}\rangle \leqslant 0$ for any t. It follows that

$$\langle\gamma''(t), \gamma(t)\rangle = k(t)\cdot\langle\gamma(t), \nu_{\gamma(t)}\rangle \leqslant 0$$

for any t.

Applying the above estimates, we get that

$$\begin{aligned}\rho''(t) &= \langle\gamma(t), \gamma(t)\rangle'' = \\ &= 2\cdot\langle\gamma''(t), \gamma(t)\rangle + 2\cdot\langle\gamma'(t), \gamma'(t)\rangle \leqslant \\ &\leqslant 2.\end{aligned}$$

20.16. Suppose $\gamma(t) = (x(t), y(t), z(t))$. Show that

❾ $$|\gamma''(t)| = z''(t)\cdot\sqrt{1+\ell^2}$$

for any t.

Observe that $z'(t) \to \pm\frac{\ell}{\sqrt{1+\ell^2}}$ as $t \to \pm\infty$. Conclude that

❿ $$\int\limits_{-\infty}^{+\infty} z''(t)\cdot dt = \tfrac{2\cdot\ell}{\sqrt{1+\ell^2}}.$$

By ❾ and ❿, we have

$$\begin{aligned}\Phi(\gamma) &= \int\limits_{-\infty}^{+\infty} |\gamma''(t)|\cdot dt = \\ &= \sqrt{1+\ell^2}\cdot\int\limits_{-\infty}^{+\infty} z''(t)\cdot dt = \\ &= 2\cdot\ell.\end{aligned}$$

20.17. Use 20.15 and 10.15.

The suggested argument does not give the optimal bound for the Lipschitz constant that guarantees that γ is simple, but later (see 22.5) we will show that the exact bound is $\sqrt{3} = \operatorname{tg}\frac{\pi}{3}$ — it is the same as in the exercise about mountain of with the shape of a perfect cone; see 21.1.

21.1. Cut the lateral surface of the mountain by a line from the cowboy to the top, unfold it on the plane and try to figure out what is the image of the strained lasso.

Since the distance between points, can not be bigger than length of a path connecting them, this statement implies the problem.

Source: I learned this problem from Joel Fine, who attributed it so Frederic Bourgeois [20].

21.2; *(a).* Show and use that $\langle \mathrm{v}(t), \mathrm{v}'(t)\rangle = 0$.

(b) Show that $|\mathrm{v}(t)|$, $|\mathrm{w}(t)|$, and $\langle \mathrm{v}(t), \mathrm{w}(t)\rangle = 0$, are constants; it can be done the same way as *(a).* Then use the formula

$$\langle \mathrm{v}(t), \mathrm{w}(t)\rangle = |\mathrm{v}(t)|\cdot|\mathrm{w}(t)|\cdot\cos\theta.$$

21.4. Observe Σ_1 supports Σ_2 at any point of γ. Conclude that γ identical spherical images in Σ_1 and Σ_2 and apply Observation 21.3.

21.5. Consider triangle that coordinate octant cuts form the sphere and try to argue that parallel transport around it rotates the tangent plane by angle $\frac{\pi}{2}$.

21.6. Suppose $\mathrm{T}(t), \mu(t), \nu(t)$ is the frame as in the definition of geodesic curvature.

If γ is a geodesic, then by 20.4, it has constant speed. Applying scaling, we may assume that the speed is 1. In this case

$$\gamma''(t) = k_g(t)\cdot\mu(t) - k_n(t)\cdot\nu(t).$$

Since $\gamma''(t) \perp \mathrm{T}_{\gamma(t)}$ we get that $k_g = 0$. That proves the "only if" part.

Now assume that γ has constant speed. Againg, applying scaling, we may assume that the speed is 1. In this case

$$\gamma''(t) = -k_n(t)\cdot\nu(t).$$

Therefore $\gamma''(t) \perp \mathrm{T}_{\gamma(t)}$.

22.8. Apply 21.7 and 22.7.

22.2. By 18.14, Σ is a smooth sphere. By Jordan theorem (5.2) the curve γ divides Σ into two discs. Let us denote by Δ the disc that lies on the left from γ.

Observe that $\Psi(\gamma) = \operatorname{length}\gamma$ and apply the Gauss–Bonnet formula (22.1) for Δ.

22.3. Apply 16.6, 18.18*a*, and the Gauss–Bonnet formula (22.1). To prove the last statement, apply 9.20.

22.5. Note that it is sufficient to show that the surface has no geodesic loops. Estimate the integral of Gauss curvature of whole surface and a disc in it surrounded by a geodesic loop.

22.4; *(easy).* Consider the 4 regions bounded by loops. Apply Gauss–Bonnet formula (22.1) to show that the integral of Gauss curvature on each of these region exceeds π. Therefore

$$\int\limits_{\Sigma} K > 4\cdot\pi.$$

The latter contradicts 18.18*a*.

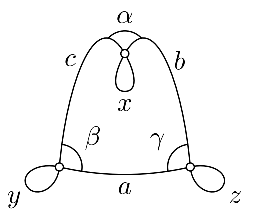
(tricky). Denote by α, β, and γ the angles of the triangle. Apply the Gauss–Bonnet formula (22.1) to show that the loops surround regions with integral of Gauss curvature $\pi + \alpha$, $\pi + \beta$, and $\pi + \gamma$ respectively.

Apply the Gauss–Bonnet formula for the triangular region to show that $\alpha+\beta+\gamma>\pi$. It follows that

$$\int\limits_{\Sigma} K > (\pi+\alpha)+(\pi+\beta)+(\pi+\gamma)>4\cdot\pi$$

which contradicts 18.18a.

(advanced). Assume a convex surface Σ with such geodesic exists; suppose that arcs and angles are labeled by their lengths.



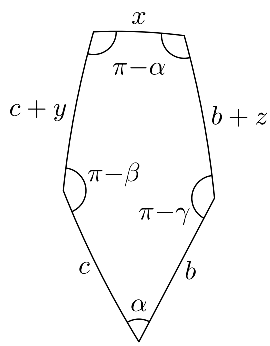
Apply Gauss–Bonnet formula to show that

$$2\cdot\alpha<\beta+\gamma$$

and

$$2\cdot\beta+2\cdot\gamma<\pi+\alpha.$$

Conclude that $\alpha<\frac{\pi}{3}$.



Consider the part of geodesic without the arc a. Pass to its convex hull; denote its surface by Σ'.

Note that Σ' is divided by the curve into 4 parts, one pentagon and three monogons. Each of these pieces can be developed in the plane, moreover the resulting figure is convex.

Consider the plane figure that corresponds to the pentagon. Its sides are formed by convex curves with marked lengths, the angles of the pentagon are marked as well. It remains to show that there is no pentagon with these properties.

23.2.

24.4.

Bibliography

[1] S. Alexander and R. Bishop. "The Hadamard–Cartan theorem in locally convex metric spaces". *Enseign. Math. (2)* 36.3-4 (1990), pp. 309–320.

[2] S. Alexander, V. Kapovitch, and A. Petrunin. *An invitation to Alexandrov geometry: CAT (0) spaces.*

[3] A. D. Aleksandrov and Yu. G. Reshetnyak. "Rotation of a curve in an n-dimensional Euclidean space". *Sibirsk. Mat. Zh.* 29.1 (1988), pp. 3–22.

[4] A. D. Alexandrov. *Vnutrennyaya Geometriya Vypuklyh Poverhnosteı*. OGIZ, Moscow-Leningrad,] 1948, p. 387.

[5] I. D. Berg. "An estimate on the total curvature of a geodesic in Euclidean 3-space-with-boundary." *Geom. Dedicata* 13 (1982), pp. 1–6.

[6] S. N. Bernstein. "Sur un théoréme de géométrie et son application aux équations aux dérivées partielles du type elliptique". *Comm. de la Soc. Math de Kharkov (2eme ser.), 15 (1915–17), 38–45.* 26 (1927). See also: Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differential gleichungen vom elliptischen Typus, Math. Zeit, 26, pp. 551–558.

[7] Yu. Burago, M. Gromov, and G. Perelman. "A. D. Aleksandrov spaces with curvatures bounded below". *Uspekhi Mat. Nauk* 47.2(284) (1992), pp. 3–51, 222.

[8] W. Blaschke. *Kreis und Kugel.* Verlag von Veit & Comp., Leipzig, 1916.

[9] H. Busemann. "Spaces with non-positive curvature". *Acta Math.* 80 (1948), pp. 259–310.

[10] É. Cartan. *Leçons sur la Géométrie des Espaces de Riemann.* 1928.

[11] G. D. Chakerian. "An inequality for closed space curves". *Pacific J. Math.* 12 (1962), pp. 53–57.

[12] G. D. Chakerian. "An inequality for closed space curves". *Pacific J. Math.* 12 (1962), pp. 53–57.

[13] Gregory R. Chambers and Yevgeny Liokumovich. "Converting homotopies to isotopies and dividing homotopies in half in an effective way". *Geom. Funct. Anal.* 24.4 (2014), pp. 1080–1100.

[14] . . *2*. 2012.

[15] S. Cohn-Vossen. „Totalkrümmung und geodätische Linien auf einfachzusammenhängenden offenen vollständigen Flächenstücken". . . 1(43).2 (1936), S. 139–164.

[16] E. Denne. *Alternating quadrisecants of knots.* Thesis (Ph.D.)–University of Illinois at Urbana-Champaign. ProQuest LLC, Ann Arbor, MI, 2004, p. 119.

[17] P. H. Doyle. "Plane separation". *Proc. Cambridge Philos. Soc.* 64 (1968), p. 291.

[18] István Fáry. "Sur certaines inégalites géométriques". *Acta Sci. Math. Szeged* 12.Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A (1950), pp. 117–124.

[19] A. F. Filippov. "An elementary proof of Jordan's theorem". *Uspehi Matem. Nauk (N.S.)* 5.5(39) (1950), pp. 173–176.

[20] J. Fine. *One-step problems in geometry.* MathOverflow. URL:https://mathoverflow.net/q/8378 (version: 2009-12-09). eprint: `https://mathoverflow.net/q/8378`.

[21] R. Foote, M. Levi, and S. Tabachnikov. "Tractrices, bicycle tire tracks, hatchet planimeters, and a 100-year-old conjecture". *Amer. Math. Monthly* 120.3 (2013), pp. 199–216.

[22] D. Fuchs and S. Tabachnikov. *Mathematical omnibus.* Thirty lectures on classic mathematics. American Mathematical Society, Providence, RI, 2007.

[23] K. F. Gauss. "Disquisitiones generales circa superficies curvas". *Commentationes Societatis Regiae Scientiarum Gottingensis recentiores* 6 (classis mathematicae) (1828). [Translated to English in *General investigations of curved surfaces* 1902.], pp. 99–146.

[24] M. Gromov. "Hyperbolic groups". *Essays in group theory.* Vol. 8. Math. Sci. Res. Inst. Publ. Springer, New York, 1987, pp. 75–263.

[25] J. Hadamard. "Sur certaines propriétés des trajectoires en dynamique". *J. math. pures appl.* 3 (1897), pp. 331–387.

[26] Heinz Hopf. "Über die Drehung der Tangenten und Sehnen ebener Kurven". *Compositio Math.* 2 (1935), pp. 50–62.

[27] Heinz Hopf. *Differential geometry in the large.* Second. Vol. 1000. Lecture Notes in Mathematics. Notes taken by Peter Lax and John W. Gray, With a preface by S. S. Chern, With a preface by K. Voss. Springer-Verlag, Berlin, 1989, pp. viii+184.

[28] A. Kneser. „Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie." *Heinrich Weber Festschrift.* 1912.

[29] Czes Kosniowski. *A first course in algebraic topology.* Cambridge University Press, Cambridge-New York, 1980, pp. viii+269.

[30] J. Lagarias and T. Richardson. "Convexity and the average curvature of plane curves". *Geom. Dedicata* 67.1 (1997), pp. 1–30.

[31] V. N. Lagunov. "On the largest sphere contained in a closed surface". *Sibirsk. Mat. Ž.* 1 (1960), pp. 205–232.

[32] Michel Ange Lancret. "Mémoire sur les courbes à double courbure". *Mémoires présentés à l'Institut des Sciences, Lettres et Arts, par divers savants, et lus dans ses assemblées. Sciences mathématiques et physiques.* 1 (1802), pp. 416–454.

[33] N. Lebedeva and A. Petrunin. "On the total curvature of minimizing geodesics on convex surfaces". *Algebra i Analiz* 29.1 (2017). Reprinted in St. Petersburg Math. J. **2**9 (2018), no. 1, 139–153, pp. 189–208.

[34] M. Levi. "A "bicycle wheel" proof of the Gauss–Bonnet theorem". *Exposition. Math.* 12.2 (1994), pp. 145–164.

[35] J. Liberman. "Geodesic lines on convex surfaces". *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 32 (1941), pp. 310–313.

[36] H. von Mangoldt. "Ueber diejenigen Punkte auf positiv gekrümmten Flächen, welche die Eigenschaft haben, dass die von ihnen ausgehenden geodätischen Linien nie aufhören, kürzeste Linien zu sein". *J. Reine Angew. Math.* 91 (1881), pp. 23–53.

[37] J. B. Meusnier. "Mémoire sur la courbure des surfaces". *Mem des savan etrangers* 10.1776 (1785), pp. 477–510.

[38] S. Mukhopadhyaya. "New methods in the geometry of a plane arc". *Bull Calcutta Math Soc* 1 (1909), pp. 31–37.

[39] A. I. Nazarov and F. V. Petrov. "On a conjecture of S. L. Tabachnikov". *Algebra i Analiz* 19.1 (2007), pp. 177–193.

[40] R. Osserman. "The four-or-more vertex theorem". *Amer. Math. Monthly* 92.5 (1985), pp. 332–337.

[41] D. Panov. "Parabolic curves and gradient mappings". *Proc. Steklov Inst. Math.* 2(221) (1998), pp. 261–278.

[42] G. Pestov and V. Ionin. "On the largest possible circle imbedded in a given closed curve". *Dokl. Akad. Nauk SSSR* 127 (1959), pp. 1170–1172.

[43] A. Petrunin and A. Yashinski. "Piecewise isometric mappings". *Algebra i Analiz* 27.1 (2015), pp. 218–247.

[44] P. Pizzetti. "Paragone fra due triangoli a lati uguali". *Atti della Reale Accademia dei Lincei, Rendiconti (5) Classe di Scienze Fisiche, Matematiche e Naturali* 16 (1) (1907), pp. 6–11.

[45] W. Rinow. *Die innere Geometrie der metrischen Räume*. Die Grundlehren der mathematischen Wissenschaften, Bd. 105. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961, pp. xv+520.

[46] Walter Rudin. *Principles of mathematical analysis*. 1976.

[47] John M. Sullivan. "Curves of finite total curvature". *Discrete differential geometry*. Vol. 38. Oberwolfach Semin. 2008, pp. 137–161.

[48] Serge Tabachnikov. "The tale of a geometric inequality". *MASS selecta*. Amer. Math. Soc., Providence, RI, 2003, pp. 257–262.

[49] P. G. Tait. "Note on the circles of curvature of a plane curve." *Proc. Edinb. Math. Soc.* 14 (1896), p. 26.

[50] V. A. Toponogov. "On convexity of Riemannian spaces of positive curvature". *Dokl. Akad. Nauk SSSR (N.S.)* 115 (1957), pp. 674–676.

[51] V. V. Usov. "The length of the spherical image of a geodesic on a convex surface". *Sibirsk. Mat. Ž.* 17.1 (1976), pp. 233–236.