Chapter 1

Isoperimetric inequality

For any plane figure F with perimeter ℓ, its area a satisfies the following inequality:

❶ $$4\pi\cdot a \leqslant \ell^2.$$

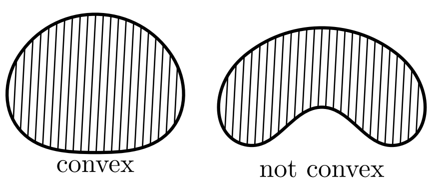
Moreover the equality holds iff F is congruent to a round disk.

This is the so-called *isoperimetric inequality* on the plane. Let us restate it without formulas, using the comparison language.

1.1. Isoperimetric inequality. *The area of a plane figure bounded by a closed curve of length ℓ can not exceed the area of a round disk with the same circumference ℓ. Moreover the equality holds only if the figure is congruent to the disk.*

The comparison reformulation has some advantages — it is more intuitive and it is also easier to generalize.

1.2. Exercise. *Come up with a formulation of the isoperimetric inequality on the unit sphere. Try to reformulate it as an algebraic inequality similar to* ❶.



Recall that a plane figure F is called *convex* if for every pair of points $x, y \in F$, the line segment $[x, y]$ that joins the pair of points also lies in F.

The following exercise reduces the isoperimetric inequality to the case of convex figures:

1.3. Exercise. *Assume F is a plane figure bounded by a closed curve of length ℓ. Show that there is a convex figure $F' \supset F$ bounded by a closed curve of length at most ℓ.*

The following problem is named after Dido, the legendary founder and first queen of Carthage.

1.4. Dido's problem. *The figure of maximal area bounded by a straight line and a curve of given length with endpoints on that line is a half-disk.*

1.5. Exercise. *Show that Dido's problem follows from the isoperimetric inequality and the other way around.*

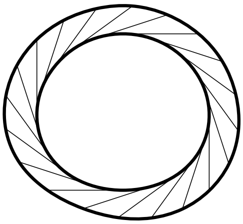
1.6. Exercise. *Use the isoperimetric inequality in the plane to show that among all the polygons with given sides, the convex polygons inscribed in circles have maximal area.*

1.7. Exercise. *Find the minimal length of a curve that divides the unit square in a given ratio α.*

1.1 Lawlor's proof

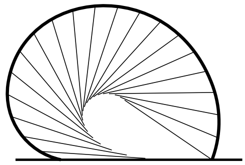
Here we present a sketch of the proof of Dido's problem based of the idea of Gary Lawlor in [1]. Before getting into the proof, try to solve the following exercise.

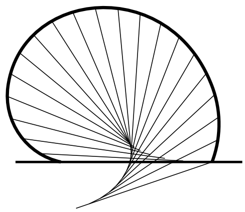
1.8. Exercise. *An old man walks along a trail around a convex meadows and pulls a brick tied to a rope of unit length (the rope is always strained). After walking around he noticed that the brick is at the same position as at the beginning. Show that the area between the trail and the path of the brick equals the area of the unit disk.*

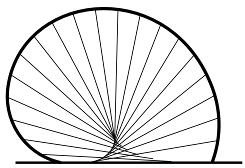


Sketch of the proof. Let F be a convex figure bounded by a line and a curve $\gamma(t)$ of length ℓ; we can assume that γ is a unit speed curve so the set of parameters is $[0, \ell]$.

Imagine that we are walking along the curve with a stick of length r so that the other end of the stick drags as we walk. Assume that initially at $t = 0$ the stick points in the direction of $\gamma(\ell)$ — the other end of γ.







Note that if r is small then most of the time we drag the stick behind. Therefore at the end of the walk the stick will have made more than half turn and will point to the same side of the figure.

Let R be the radius of the half-circle $\tilde{\gamma}(t)$ of length ℓ. Assume we walk along $\tilde{\gamma}$ with a stick of length R the same way as described above. Note that the other end does not move (it always lies in the center) and the direction of the stick changes with rate $\frac{1}{R}$. Note further that for γ this rate is at most $\frac{1}{R}$. Therefore after walking along γ, the stick of length R will rotate at most as much as if we walk along $\tilde{\gamma}$.

It follows that there is a positive value $r \leqslant \leqslant R$ such that after walking along γ with a stick of length r, it will rotate exactly half turn, so at the end it will point towards $\gamma(0)$.

Imagine that the stick of length r is covered with red paint and it paints the area below it. If we move with velocity v then the angular velocity (radians per second) of the stick is at most $\frac{v}{r}$ and the equality holds if we move perpendicularly to the stick. Therefore we paint at a rate of at most $\frac{1}{2}v{\cdot}r$.

If we do the same for the half-disk of radius R and a stick of length R with blue paint, then we paint the area of the disk without overlap with the rate $\frac{1}{2}v{\cdot}R$. Since $r \leqslant R$, the total red-painted area can not exceed the blue-painted area, that is, D.

It remains to show that all F is red-painted. Fix a point $p \in F$. Notice that at the beginning the point p lies on the left from the stick and at the end it lies on the right form it. Therefore there will be a moment t_0 when the side changes from left to right. At this time the point must be on the line containing the stick. Moreover, if it lies on the extension then the side changes from right to left. Therefore p has to lie under the stick; that is, p is painted.

$\square$

1.9. Exercise. *Find the steps with cheating in the above proof and try to fix them.*

1.10. Exercise. *Read and understand the original proof of Gary Lawlor in [1].*

Chapter 2

Length

The material of this and the following chapters overlaps largely with [2, Chapter 5].

2.1 Length of curve

2.1. Definition. *Consider a* plane curve $\alpha\colon [a,b] \to \mathbb{R}^2$*; a continuous mapping from the real interval* $[a,b]$ *to the Euclidean plane* $\mathbb{R}^2$*.*

If $\alpha(a) = p$ *and* $\alpha(b) = q$*, we say that* α *is a* curve from p to q*.*

A curve $\alpha\colon [a,b] \to \mathbb{R}^2$ *is called* closed *if* $\alpha(a) = \alpha(b)$*.*

A curve α *is called* simple *if it is described by an injective map; that is* $\alpha(t) = \alpha(t')$ *if and only if* $t = t'$*. However, a closed curve* $\alpha\colon [a,b] \to \mathbb{R}^2$ *is called simple if it is injective everywhere except at the ends; that is, if* $\alpha(t) = \alpha(t')$ *for* $t < t'$ *then* $t = a$ *and* $t' = b$*.*

A closed curve is called convex *if it bounds a convex region.*

2.2. Advanced exercise. *Let* $\alpha\colon [0,1] \to \mathbb{R}^2$ *from* p *to* q*. Assume* $p \neq q$ *Show that there is a simple curve* $\beta\colon [0,1] \to \mathbb{R}^2$ *from* p *to* q *that runs in the image of* α*; that is for any* $t \in [0,1]$ *there is* $t' \in [0,1]$ *such that* $\beta(t) = \alpha(t')$*.*

Recall that a sequence

$$a = t_0 < t_1 < \cdots < t_k = b.$$

is called a *partition* of the interval $[a,b]$.

2.3. Definition. *Let* $\alpha\colon [a,b] \to \mathbb{R}^2$ *be a curve. The* length *of* α *is defined as*

$$\operatorname{length}\alpha = \sup\{|\alpha(t_0)-\alpha(t_1)| + |\alpha(t_1)-\alpha(t_2)| + \dots$$
$$\dots + |\alpha(t_{k-1})-\alpha(t_k)|\}.$$

where the exact upper bound is taken over all partitions

$$a = t_0 < t_1 < \dots < t_k = b.$$

Note that $\operatorname{length}\alpha \in [0,\infty]$*; the curve* α *is called* rectifiable *if its length is finite.*

Informally, one could say that the length of a curve is the exact upper bound of the lengths of polygonal lines *inscribed* in the curve.

2.4. Exercise. *Assume* $\alpha\colon [a,b]\to\mathbb{R}^2$ *is a smooth curve, in particular the velocity vector* $\alpha'(t)$ *is defined and depends continuously on* t*. Show that*

$$\operatorname{length}\alpha = \int\limits_a^b |\alpha'(t)|\cdot dt.$$

2.5. Exercise. *Construct a nonrectifiable curve* $\alpha\colon [0,1]\to\mathbb{R}^2$*.*

A closed simple plane curve is called *convex* if it bounds a convex figure.

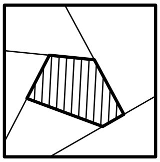
2.6. Proposition. *Assume a convex figure* A *bounded by a curve* α *lies inside a figure* B *bounded by a curve* β*. Then*

$$\operatorname{length}\alpha \leqslant \operatorname{length}\beta.$$

Note that it is sufficient to show that for any polygon P inscribed in α there is a polygon Q inscribed in β with $\operatorname{perim}P \leqslant \operatorname{perim}Q$, where $\operatorname{perim}P$ denotes the perimeter of P.

Therefore it is sufficient to prove the following lemma.

2.7. Lemma. *Let* P *and* Q *be polygons. Assume* P *is convex and* $Q\supset P$*. Then* $\operatorname{perim}P \leqslant \operatorname{perim}Q$*.*



Proof. Note that by the triangle inequality, the inequality

$$\operatorname{perim}P \leqslant \operatorname{perim}Q$$

holds if P can be obtained from Q by cutting it along a chord; that is, a line segment with ends on the boundary of Q that lies in Q.

Note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \dots \subset P_n = Q$$

such that P_{i-1} obtained from P_i by cutting along a chord. Therefore

$$\begin{aligned}\operatorname{perim} P = \operatorname{perim} P_0 \leqslant \operatorname{perim} P_1 \leqslant \dots\\ \dots \leqslant \operatorname{perim} P_n = \operatorname{perim} Q\end{aligned}$$

and the lemma follows. □

2.8. Corollary. *Any convex closed curve is rectifiable.*

Proof. Any closed curve is bounded; that is, it lies in a sufficiently large square.

By Proposition 2.6, the length of the curve can not exceed the perimeter of the square, hence the result. □

2.2 Semicontinuity of length

Recall that the lower limit of a sequence of real numbers (x_n) is denoted by

$$\varliminf_{n\to\infty} x_n.$$

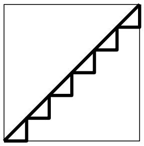
It is defined as the lowest partial limit; that is, the lowest possible limit of a subsequence of (x_n). The lower limit is defined for any sequence of real numbers and it lies in the exteded real line $[-\infty, \infty]$

2.9. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\alpha_n\colon [a,b] \to \mathbb{R}^2$ converges pointwise to a curve $\alpha_\infty\colon [a,b] \to \mathbb{R}^2$; that is, $\alpha_n(t) \to \alpha_\infty(t)$ for any fixed $t \in [a,b]$ as $n \to \infty$. Then

❶ $$\varliminf_{n\to\infty} \operatorname{length} \alpha_n \geqslant \operatorname{length} \alpha_\infty.$$

Note that the inequality ❶ might be strict. For example the diagonal α_∞ of the unit square



can be approximated by a sequence of stairs-like polygonal curves α_n with sides parallel to the sides of the square (α_6 is on the picture). In this case

$$\operatorname{length} \alpha_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length} \alpha_n = 2$$

for any n.

Proof. Fix $\varepsilon > 0$ and choose a partition $a = t_0 < t_1 < \dots < t_k = b$ such that

$$\operatorname{length}\alpha_\infty < |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + \dots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)| + \varepsilon.$$

Set

$$\begin{aligned}\Sigma_n &:= |\alpha_n(t_0) - \alpha_n(t_1)| + \dots + |\alpha_n(t_{k-1}) - \alpha_n(t_k)|.\\ \Sigma_\infty &:= |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + \dots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|.\end{aligned}$$

Note that $\Sigma_n \to \Sigma_\infty$ as $n \to \infty$ and $\Sigma_n \leqslant \operatorname{length}\alpha_n$ for each n. Hence

$$\varliminf_{n\to\infty} \operatorname{length}\alpha_n \geqslant \operatorname{length}\alpha_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get ❶. □

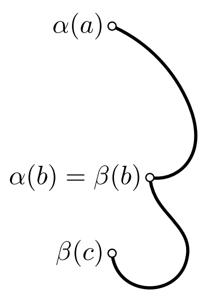
2.3 Axioms of length

Concatenation. Assume $\alpha\colon [a,b] \to \mathbb{R}^2$ and $\beta\colon [b,c] \to \mathbb{R}^2$ are two curves such that $\alpha(b) = \beta(b)$. Then one can combine these two curves into one $\gamma\colon [a,c] \to \mathbb{R}^2$ by the rule $\gamma(t) = \alpha(t)$ for $t \leqslant b$ and $\gamma(t) = \beta(t)$ for $t \geqslant b$. The obtained curve γ is called the *concatenation* of α and β and is denoted as $\gamma = \alpha * \beta$.

Note that

$$\operatorname{length}(\alpha * \beta) = \operatorname{length}\alpha + \operatorname{length}\beta$$

for any two curves α and β such that the concatenation $\alpha * \beta$ is defined.



Reparametrization. Assume $\alpha\colon [a,b] \to \mathbb{R}^2$ is a curve and $\tau\colon [c,d] \to [a,b]$ is a continuous strictly monotonic onto map. Consider the curve $\alpha'\colon [c,d] \to \mathbb{R}^2$ defined by $\alpha' = \alpha \circ \tau$. The curve α' is called a *reparametrization* of α.

Note that

$$\operatorname{length}\alpha' = \operatorname{length}\alpha$$

whenever α' is a reparametrization of α.

2.10. Proposition. *Let ℓ be a functional that returns a value in $[0,\infty]$ for any curve $\alpha\colon [a,b] \to \mathbb{R}$.*

Assume it satisfies the following properties:

(i) *(Normalization) If* $\alpha\colon [a,b]\to\mathbb{R}^2$ *is a linear curve,*[1] *then*
$$\ell(\alpha)=|\alpha(a)-\alpha(b)|.$$

(ii) *(Additivity) If the concatenation* $\alpha*\beta$ *is defined, then*
$$\ell(\alpha*\beta)=\ell(\alpha)+\ell(\beta).$$

(iii) *(Motion invariance) The functional* ℓ *is invariant with respect to the motions of the plane; that is, if* m *is an isometry of the plane, then*
$$\ell(m\circ\alpha)=\ell(\alpha)$$
for any curve α.

(iv) *(Reparametrization invariance) If* α' *is a reparametrization of a curve* α *then*
$$\ell(\alpha')=\ell(\alpha).$$
(In fact linear reparametrizations will be sufficient.)

(v) *(Semi-continuity) If a sequence of curves* $\alpha_n\colon [a,b]\to\mathbb{R}^2$ *converges pointwise to a curve to a curve* $\alpha_\infty\colon [a,b]\to\mathbb{R}^2$*, then*
$$\varliminf_{n\to\infty}\ell(\alpha_n)\geqslant\ell(\alpha_\infty).$$

Then

❷ $$\ell(\alpha)=\operatorname{length}\alpha$$

for any plane curve α.

Proof. Note that from normalization and additivity, the identity

❸ $$\ell(\beta)=\operatorname{length}\beta$$

holds for any polygonal line β that is linear on each edge.

Note that the following two inequalities

❹ $$\ell(\alpha)\leqslant\operatorname{length}\alpha$$

❺ $$\ell(\alpha)\geqslant\operatorname{length}\alpha$$

imply ❷; we will prove them separately.

Fix a curve $\alpha\colon [a,b]\to\mathbb{R}^2$ and a partition $a=t_0<t_1<\ldots<$ $<t_k=b$. Consider the curve $\beta\colon [a,b]\to\mathbb{R}^2$ defined as the linear

[1] That is $\alpha=w+v{\cdot}t$ for some vectors w and v.

segment from $\alpha(t_i)$ to $\alpha(t_{i+1})$ on each interval $t \in [t_i, t_j]$. By the definition of length,

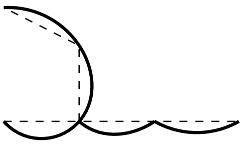
$$\operatorname{length}\beta \leqslant \operatorname{length}\alpha.$$

Since the map $\alpha\colon [a,b] \to \mathbb{R}^2$ is continuous, one can find a sequence of partitions of $[a,b]$ such that the corresponding curves β_n converge to α pointwise. Applying the semi-continuity of ℓ, ❸ and the definition of length, we get that

$$\begin{aligned}\ell(\alpha) \leqslant \varliminf_{n\to\infty} \ell(\beta_n) &= \\ &= \varliminf_{n\to\infty} \operatorname{length}\beta_n \leqslant \\ &\leqslant \operatorname{length}\alpha.\end{aligned}$$

Hence ❹ follows.

Note that a curve $\alpha\colon [a,b] \to \mathbb{R}^2$ with a partition $a = t_0 < t_1 < \ldots < t_k = b$ can be considered as a concatenation



$$\alpha = \alpha_1 * \alpha_2 * \cdots * \alpha_k$$

where α_i is the restriction of α to $[t_{i-1}, t_i]$.

Observe that there is a sequence of motions m_i of the plane so that

$$m_i \circ \alpha(t_i) = m_{i+1} \circ \alpha(t_i)$$

for any i and the points

$$m_1 \circ \alpha(t_0), m_1 \circ \alpha(t_1), \ldots m_k \circ \alpha(t_k)$$

lie in that order on a single line. For the concatenation

$$\gamma = (m_1 \circ \alpha_1) * (m_2 \circ \alpha_2) * \cdots * (m_k \circ \alpha_k)$$

we have

$$\ell(\gamma) = \ell(\alpha).$$

Note that one can find a sequence of partitions of $[a,b]$ such that reparametrizations of γ_n converge to a linear segment γ'_∞; denote these reparametrizations by γ'_n. Also, $\operatorname{length}\gamma'_\infty = \operatorname{length}\alpha$; indeed, since γ'_∞ is linear,

$$\begin{aligned}\operatorname{length}\gamma'_\infty &= |\gamma'_\infty(a) - \gamma'_\infty(b)| = \\ &= \lim_{n\to\infty} \Sigma_n = \\ &= \operatorname{length}\alpha.\end{aligned}$$

where Σ_n is the sum in the definition of length for the n-th partition. Hence it is sufficient to choose a sequence of partitions such that $\Sigma_n \to$ $\to \operatorname{length}\alpha$.

Applying additivity, invariance of ℓ with respact to motions and reparametizations, we get that

$$\begin{aligned}\ell(\alpha) &= \lim_{n\to\infty} \ell(\gamma_n) = \\ &= \lim_{n\to\infty} \ell(\gamma_n') \geqslant \\ &\geqslant \ell(\gamma_\infty') = \\ &= \operatorname{length}\alpha.\end{aligned}$$

Hence ❺ follows. □

2.11. Exercise. *Construct a functional ℓ that satisfies all the conditions in Proposition 2.10 except the semi-continuity.*

2.4 Crofton formula

Let α be a plane curve and u a unit vector. Denote by α_u the orthogonal projection of α to a line ℓ in the direction of u; that is, $\alpha_u(t) \in \ell$ and $\alpha(t) - \alpha_u(t) \perp \ell$ for any t.

2.12. Crofton formula. *The length of any plane curve α is proportional to the average of the lengths of its projections α_u for all unit vectors u. Moreover for any plane curve α we have*

$$\operatorname{length}\alpha = \tfrac{\pi}{2}\cdot\overline{\operatorname{length}\alpha_u},$$

where $\overline{\operatorname{length}\alpha_u}$ *denotes the average value of* $\operatorname{length}\alpha_u$.

Proof. First let us show that the formula

❻ $$\operatorname{length}\alpha = k\cdot\overline{\operatorname{length}\alpha_u},$$

holds for some fixed coefficient k. It will follow once we show that both sides of formula satisfy the length axioms in 2.10.

The normalization can be achieved by adjusting k.

The semi-continuity of the right hand side follows since $\operatorname{length}\alpha_u$ is semi-continuous and therefore the average has to be semi-continuous.

It is straightforward to check the remaining properties.

It remains to find k. Let us apply the formula ❻ to the unit circle. The circle has length $2\cdot\pi$ and its projection to any line has length 4

— it is a segment of length 2 traveled back and forth. Evidently the average value is also 4, so

$$2{\cdot}\pi = k{\cdot}4,$$

hence $k = \frac{\pi}{2}$. □

Reformulation via number of intersections. Given a unit vector u and a real number ρ, consider the line of vectors w on the plane satisfying the equation

$$\langle u, w\rangle = \rho,$$

where $\langle u, w\rangle$ denotes the scalar product. Any line on the plane admits exactly two such presentations with pairs (u, ρ) and $(-u, -\rho)$. A pair (u, ρ) describes uniquely an *oriented* line — that is a line with a chosen unit normal vector.

Fix a unit vector u_0 and denote by $u(\varphi)$ the result of rotating u_0 counterclockwise by the angle φ. Denote by $\ell(\varphi, \rho)$ the oriented line asociated to the pair $(u(\varphi), \rho)$. To describe any line, we need a pair $(\varphi, \rho) \in (-\pi, \pi] \times \mathbb{R}$.

For a curve α, set $n_\alpha(\varphi, \rho)$ to be the number of parameter values t such that $\alpha(t)$ lies on the line $\ell(\varphi, \rho)$. The value $n_\alpha(\varphi, \rho)$ is a non-negative integer or ∞. Note that if α is a simple curve, then $n_\alpha(\ell)$ is the number of intersections of α with ℓ.

2.13. Another Crofton formula. *For any curve α,*

$$\operatorname{length}\alpha = \tfrac{1}{4}\cdot \iint\limits_{(-\pi,\pi]\times\mathbb{R}} n_\alpha(\rho, \varphi){\cdot}d\rho{\cdot}d\varphi.$$

the integral is to be understood in the sense of Lebesgue.

By definition of average value,

$$\overline{\operatorname{length}\alpha_u} = \frac{1}{2{\cdot}\pi}\cdot\int\limits_{-\pi}^{\pi} \operatorname{length}\alpha_{u(\varphi)}{\cdot}d\varphi.$$

Therefore the proof of this reformulation of the Crofton follows from the following observation.

2.14. Observation. *If $u = u(\varphi)$, then*

$$\operatorname{length}\alpha_u = \int\limits_{\mathbb{R}} n_\alpha(\rho, \varphi){\cdot}d\rho;$$

The proof is straightforward for those who understand Lebesgue integral.

Variations. The same argument can be used to derive other formulas of the same type. For example.

Recall that a big circle in a sphere is the intersection of the sphere with a plane passing thru its center. For example, the equator as well as the meridians are big circles.

2.15. Spherical Crofton formula. *The length of any curve α in the unit sphere is π times the average number of its crossings with big circles.*

More presciently, given a unit vector u, denote by $n_\alpha(u)$ the number of crossings of α and the equator with pole at u. Then

$$\operatorname{length}\alpha = \pi\cdot\overline{n_\alpha(u)}.$$

Equivalently,

$$\operatorname{length}\alpha = \overline{\operatorname{length}\alpha_u},$$

where α_u denotes the curve obtianed by closest point projection of α to the equator with pole at u.

2.16. Exercise. *Come up with Crofton formulas for curves in the Euclidean space via projections to lines and to planes. Find the coefficients in those formulas.*

2.5 Applications

Alternative proof of Proposition 2.6. Note that

$$\operatorname{length}\beta_u \geqslant \operatorname{length}\alpha_u$$

for any unit vector u. Indeed α_u runs back and forth along a line segment and β_u has to run at least that much.

It follows that

$$\overline{\operatorname{length}\beta_u} \geqslant \overline{\operatorname{length}\alpha_u}.$$

It remains to apply the Crofton formula. □

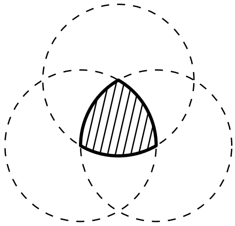
Recall that the diameter of a plane figure F is defined as the least upper bound on the distances between pairs of its points; that is,

$$\operatorname{diam}F = \sup\{\,|x-y| \,:\, x,y\in F\,\}.$$

The equilateral triangle with side 1 gives an example of a convex figure of diameter 1 that cannot be covered by a round disc of diameter 1.

2.17. Exercise. *Assume F is a convex figure of diameter 1 and D is the round disc of diameter 1. Show that*

$$\operatorname{perim} F \leqslant \operatorname{perim} D.$$



A convex figure F has constant width a if the orthogonal projection of F to any line has length a. There are many non-circular shapes of constant width. A nontrivial example is the Reuleaux triangle shown on the picture; it is the intersection of three round disks of the same radius, each having its center on the boundary of the other two. The following exercise is the so called Barbier's theorem.

2.18. Exercise. *Show that figures with constant width a have the same perimeter (which equals $\pi\cdot a$ — the perimeter of the round disc of diameter a).*

2.19. Exercise. *Let γ be a closed curve in the unit sphere of length shorter than $2\cdot\pi$. Show that γ lies in a hemisphere.*

2.20. Exercise. *Let α be a closed curve of length π. Show that it lies between a pair of parallel lines at distance 1 from each other.*

2.21. Exercise. *A spaceship flies around a nonrotating planet of unit radius and comes back to the original position; it was able to take a picture of every point on the surface of the planet.*

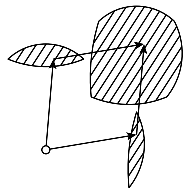
Try to use the Crofton formulas to get a lower bound on the length of its trajectory (does not need to be exact, but should be larger than $2\cdot\pi$).

What do you think could be the shortest trajectory?

The Hausdorff distance $d_H(F,G)$ between two closed bounded sets F and G in the plane is defined as the exact lower bound on $\varepsilon > 0$ such that the ε-neightborhood of F contains G and the ε-neightborhood of G contains F.

2.22. Exercise. *Assume F and G are two closed convex figures on the plane such that $d_H(F,G) < \varepsilon$. Show that*

$$|\operatorname{perim} F - \operatorname{perim} G| < 2\cdot\pi\cdot\varepsilon.$$



Given two sets A and B on the plane, the set C is called their *Minkowski sum* (briefly $C = A + B$) if C is formed by adding each vector in A to each vector in B; that is,

$$C = \{\, a + b \,:\, a \in A,\ b \in B \,\}.$$

Note that if A and B are convex then so is $C = A + B$.

Indeed, A is convex if and only if for any pair of points $a_0, a_1 \in A$ and any $t \in [0,1]$, the point $a_t = (1-t)\cdot a_0 + t\cdot a_1$ belongs to A. Similarly, B is convex if and only if for any pair of points $b_0, b_1 \in B$ and any $t \in [0,1]$, the point $b_t = (1-t)\cdot b_0 + t\cdot b_1$ belongs to A.

Fix a pair of points $c_0, c_1 \in C$; by the definition of Minkowski sum, there are two pairs of points $a_0, a_1 \in A$ and $b_0, b_1 \in B$ such that $c_0 = a_0 + b_0$ and $c_1 = a_1 + b_1$. Then

$$\begin{aligned} c_t &= (1-t)\cdot c_0 + t\cdot c_1 = \\ &= (1-t)\cdot(a_0 + b_0) + t\cdot(a_1 + b_1) = \\ &= [(1-t)\cdot a_0 + t\cdot a_1] + [(1-t)\cdot b_0 + t\cdot b_1] = \\ &= a_t + b_t. \end{aligned}$$

That is, $c_t \in C$ for any $t \in [0,1]$, hence the result.

2.23. Exercise. *Show that*

$$\operatorname{perim}(A + B) = \operatorname{perim} A + \operatorname{perim} B$$

for any pair of convex figures in the plane.

2.24. Exercise. *Use Exercise 2.23 and Lemma 2.7 to give another solution of Exercise 2.22.*

2.25. Exercise. *Let γ be a curve that lies in a convex figure F in the plane.*

Let γ be a curve that lies inside a convex figure F on the plane. Assume that

$$2\cdot \operatorname{length}\gamma \geqslant n\cdot \operatorname{perim} F$$

for some integer n. Show that there is a line ℓ that intersects γ in at least n distinct points.

Chapter 3

Total curvature

3.1 Smooth regular curves

Here we introduce the so called *total curvature of a curve.* In general the term *curvature* is used for something that measures how much a geometric object deviates from being straight; total curvature is not an exception — as you will see, if the total curvature of a curve is zero, then the curve runs along a straight line.

Let $\alpha\colon [a,b] \to \mathbb{R}^3$ be a *smooth regular* curve — smooth means that the velocity vector $\alpha'(t)$ is defined and is continuous with respect to t, and regular means that $\alpha'(t) \neq 0$ for all t. If the curve α is closed then we assume in addition that $\alpha'(a) = \alpha'(b)$.

Denote by $\tau(t)$ the unit vector in the direction of $\alpha'(t)$; that is, $\tau(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$. Then $\tau\colon [a,b] \to \mathbb{S}^2$ is an other curve which is called *tangent indicatrix* of α. The length of τ is called the *total curvature of* α; that is,

$$\operatorname{TotCurv}\alpha := \operatorname{length}\tau.$$

3.1. Exercise. *Show that*

$$\operatorname{TotCurv}\alpha \geqslant 2\cdot\pi$$

for any smooth closed regular curve α.

Moreover, the equality holds if and only if α is a closed convex curve that lying in a plane.

The above exercise is the so called Fenchel's theorem.

3.2 General definition

The total curvature of a polygonal line is defined as the sum of its external angles.

More precisely, for a polygonal line $\beta = p_0 \dots p_n$, the external angle at the vertex p_i is defined as $\alpha_i = \pi - \measuredangle p_{i-1} p_i p_{i+1}$. The total curvature of the polygonal line $\beta = p_0 \dots p_n$ is defined as the sum

$$\operatorname{TotCurv} \beta = \alpha_1 + \dots + \alpha_{n-1};$$

it is defined if the polygonal line is *nondegenerate*; that is, $p_{i-1} \neq p_i$ for any i.

If the polygonal line $p_0 \dots p_n$ is closed; that is $p_0 = p_{n+1}$ you add one more angle

$$\alpha_0 + \alpha_1 + \dots + \alpha_{n-1},$$

where $\alpha_0 = \pi - \measuredangle p_n p_0 p_1$.

One can define the tangent indicatrix of a polygonal line β as a spherical polygonal line (each edge is an arc of a big circle in the sphere) whose vertexes are the unit vectors $\xi_1, \dots, \xi_n$ in the directions of $p_1 - p_0$, $p_2 - p_1, \dots, p_n - p_{n-1}$ correspondingly; if the polygonal line is closed then we add one more vertex ξ_0 in the direction of $p_0 - p_n$ and two more edges $\xi_0 \xi_1$ and $\xi_n \xi_0$ so the indicatrix of a closed polygonal line is a closed spherical polygonal line.

Note that the total curvature of a polygonal line is the length of its tangent indicatrix.

3.2. Exercise. *Let a, b, c, d and x be distinct points in $\mathbb{R}^3$. Show that*

$$\operatorname{TotCurv} abcd \geqslant \operatorname{TotCurv} abxcd.$$

3.3. Exercise. *Use Exercise 3.2 to prove an analog of Fenchel's theorem (Exercise 3.1) for closed polygonal lines.*

We gave two definitions of total curvature: the first one is given in Section 3.1 via the tangent indicatrix — it works for smooth regular curves; the second, via external angles — it works for polygonal lines. The latter can be used to define total curvature of arbitrary curves.

Let $\alpha\colon [a, b] \to \mathbb{R}^3$ be a curve and $a = t_0 < \dots < t_n = b$ a partition. Set $p_i = \alpha(t_i)$. Then the polygonal line $p_0 \dots p_n$ is said to be inscribed in α.

3.4. Definition. *The total curvature of a nonconstant curve α is the exact upper bound on the total curvatures of inscribed nondegenerate polygonal lines; if α is closed then we assume that the inscribed polygonal lines are closed as well.*

We need to assume that the curve is nonconstant, otherwise it does not admit inscribed polygonal lines that are not trivial.

3.5. Exercise. *Show that the total curvature is lower semi-continuous with respect to pointwise convergence of curves. That is, if a sequence of curves* $\alpha_n\colon [a,b]\to\mathbb{R}^3$ *converges pointwise to a curve* $\alpha_\infty\colon [a,b]\to$ $\to\mathbb{R}^3$, *then*

$$\varliminf_{n\to\infty}\operatorname{TotCurv}\alpha_n\geqslant\operatorname{TotCurv}\alpha_\infty.$$

Hint: Modify the proof of semi-continuity of length (Theorem 2.9).

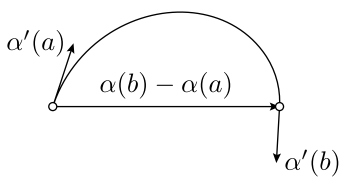
The following definition tells that the two definitions agree.

3.6. Theorem. *For smooth regular curves the two definitions of total curvature agree; that is, for any regular curve, the length of its tangent indicatrix is equal to the exact upper bound on the total curvatures of inscribed nondegenerate polygonal lines.*

Note that from the theorem and Exercise 3.3, we get a generalization of Fenchel's theorem (Exercise 3.1) — it works for arbtrary closed curves, not necessary smooth and regular.

3.7. Lemma. *Let* $\alpha\colon [a,b]\to\mathbb{R}^3$ *be a smooth regular curve. Consider three unit vectors* λ, μ *and* ν *in the directions of* $\alpha'(a)$, $\alpha(b)-\alpha(a)$ *and* $\alpha'(b)$ *correspondingly. Then*

$$\operatorname{TotCurv}\alpha\geqslant\measuredangle(\lambda,\mu)+\measuredangle(\mu,\nu).$$



Proof. The tangent indicatrix τ runs from λ to ν in the unit sphere $\mathbb{S}^2$.

Note that τ can not be separated from μ by an equator. Indeed the vector

$$\alpha(b)-\alpha(a)=\int_a^b\alpha'(t)\cdot dt$$

points in the same direction as μ. Therefore if the indicatrix $\tau=\frac{\alpha'}{|\alpha'|}$ lies in a hemisphere then μ lies in the same hemisphere.

Fix an equator ℓ in general position. If ℓ intersects the spherical polygonal line $\lambda\mu\nu$ at one point, then ℓ separates λ from ν and therefore it must intersect τ. If ℓ intersects the spherical polygonal line $\lambda\mu\nu$ at two points, then ℓ separates μ from λ and ν and therefore it must intersect τ at least twice — τ must cross ℓ and then come back. It follows that for almost all equators the number of intesections with the spherical polygonal line $\lambda\mu\nu$ can not exceed the number of intersections with τ. By the spherical Crofton formula (2.15), τ is longer than the spherical polygonal line $\lambda\mu\nu$. But the polygonal line $\lambda\mu\nu$ has length $\measuredangle(\lambda,\mu)+\measuredangle(\mu,\nu)$, hence the result. $\square$

Let us sketch an alternative proof of the lemma which is built on Fenchel's theorem.

Alternative proof of the lemma. Note that the curve α can be extended to a smooth regular closed curve $\hat\alpha$ by an arc β that starts from $\alpha(b)$ in the same direction as α. Then turns and joins the segment $[\alpha(b),\alpha(a)]$, runs along the segment until it is close to $\alpha(a)$ turns and smoothly joints α at $\alpha(a)$.

Note that the total curvature of β can be made arbitrarily close to $2\cdot\pi-\measuredangle(\lambda,\mu)-\measuredangle(\mu,\nu)$. Indeed, β needs a bit more than $\pi-\measuredangle(\mu,\nu)$ to turn an join the segment $[\alpha(b),\alpha(a)]$ and bit more than $\pi-\measuredangle(\lambda,\mu)$ to turn an join the segment α.

By Fenchel's theorem,

$$\operatorname{TotCurv}\hat\alpha\geqslant 2\cdot\pi.$$

Evidently

$$\operatorname{TotCurv}\hat\alpha=\operatorname{TotCurv}\alpha+\operatorname{TotCurv}\beta,$$

hence the lemma follows. $\square$

Proof of 3.6. Let $\alpha\colon[a,b]\to\mathbb{R}^3$ be a smooth curve. Fix a partition $a=t_0<\dots<t_n=b$ and consider the corresponding inscribed polygonal line $\beta=w_0\dots w_n$. Let $\chi=\xi_1\dots\xi_n$ be its tangent indicatrix — this is a spherical polygonal line; we assume that $\chi(t_i)=\xi_i$ and it has constant speed on each arc.

Consider a sequence of finer and finer partitions, denote by β_n and χ_n the corresponding inscribed polygonal line and its tangent indicatrix; since α is smooth, the χ_n converge pointwise to τ — the thangent indicatrix of α. By semi-continuity of the length functional, we get

$$\begin{aligned}\operatorname{TotCurv}\alpha&=\operatorname{length}\tau\leqslant\\&\leqslant\varliminf_{n\to\infty}\operatorname{length}\chi_n=\\&=\varliminf_{n\to\infty}\operatorname{TotCurv}\beta_n\leqslant\\&\leqslant\sup\{\operatorname{TotCurv}\beta\},\end{aligned}$$

where the last supremum is taken over all partitions and their corresponding inscribed polygonal lines β.

It remains to prove that

$$\text{❶} \qquad \operatorname{TotCurv}\alpha \geqslant \operatorname{TotCurv}\beta,$$

for any polygonal line β inscribed in α. Let ζ_i be the unit vector in the direction of $\alpha'(t_i)$. Consider the spherical polygonal line $\gamma = \zeta_0\xi_1\zeta_1\xi_2\dots\xi_n\zeta_n$; recall that $\chi = \xi_0\dots\xi_n$. By the triangle inequality,

$$\operatorname{length}\gamma \geqslant \operatorname{length}\chi = \operatorname{TotCurv}\beta.$$

By Lemma 3.7,

$$\operatorname{TotCurv}\alpha \geqslant \operatorname{length}\gamma,$$

hence ❶ follows. □

3.3 Crofton again

Given a curve α in $\mathbb{R}^3$ and a unit vector u, denote by $\alpha_{u^\perp}$ and α_u the projection of α to the plane perpendicular to u and the line parallel to u correspondingly.

To prove the following proposition, apply the spherical Crofton formula to the tangent indicatrix of α.

3.8. Proposition. *Let α be a polygonal line in $\mathbb{R}^3$. Show that*

$$\begin{aligned}\operatorname{TotCurv}\alpha &= \overline{\operatorname{TotCurv}\alpha_{u^\perp}} = \\ &= \overline{\operatorname{TotCurv}\alpha_u}.\end{aligned}$$

Note that since the curve α_u runs back and forth along one line, every time it changes direction contributes π to the total curvature of α_u. Therefore the total curvature of α_u is $n\cdot\pi$, where n is the number of changes of direction. Since n has to be even, $\operatorname{TotCurv}\alpha_u$ may take values $2\cdot\pi$, $4\cdot\pi$, $6\cdot\pi$ and so on.

3.9. Exercise. *Use the proposition and the observation above to give yet another proof of Fenchel's theorem (Exercise 3.1).*

3.4 DNA inequality

3.10. Theorem. *Let α be a closed curve that lies in a unit disc. Then*

$$\operatorname{TotCurv}\alpha \geqslant \operatorname{length}\alpha.$$

Note that if $\operatorname{length}\alpha \leqslant 2\cdot\pi$, then Fenchel's theorem gives a better estimate, for longer curves this gives something new.

Proof. Assume α is a polygonal line.

Fix a unit vector u. Note that the curve α_u can run at most length 2 in one direction; therefore the number of turns has to be at least $\frac{1}{2}\cdot\operatorname{length}\alpha$. Since each turn of α_u contributes π to its total curvature, we get

$$\operatorname{TotCurv}\alpha_u \geqslant \tfrac{\pi}{2}\cdot\operatorname{length}\alpha_u.$$

The same inequality holds for the average values of left and right hand sides; that is,

$$\overline{\operatorname{TotCurv}\alpha_u} \geqslant \tfrac{\pi}{2}\cdot\overline{\operatorname{length}\alpha_u}.$$

Applying the Crofton's formula and Proposition 3.8 we get the result.

It remains to reduce the general case to polygonal lines. Given $\varepsilon > 0$, we choose an inscribed polygonal line β such that

$$\operatorname{length}\alpha < \operatorname{length}\beta + \varepsilon.$$

By the definition of total curvature (3.4) and from the first part of the proof

$$\begin{aligned}\operatorname{TotCurv}\alpha \geqslant \operatorname{TotCurv}\beta &\geqslant \\ &\geqslant \operatorname{length}\beta > \\ &> \operatorname{length}\alpha - \varepsilon.\end{aligned}$$

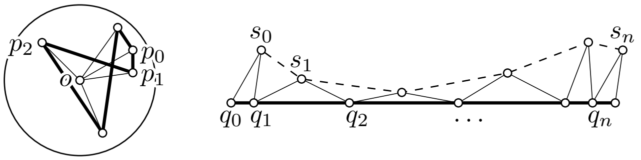
The statement follows since ε was arbitrary. □

Alternative proof. The same argument as above shows that it is sufficient to consider a closed polygonal line $\beta = p_0p_1\dots p_{n-1}$ in the unit disc. We assume that $p_n = p_0$, $p_{n+1} = p_1$ and so on. Denote by α_i the external angle at p_i.

Denote by o the center of the disc. Consider a sequence of triangles

$$\triangle q_0q_1s_0 \cong \triangle p_0p_1o,\ \ \triangle q_1q_2s_1 \cong \triangle p_1p_2o, \dots$$

such that the points $q_0, q_1 \dots$ lie on one line in that order and all the s_i's lie on one side from this line.



Note that

$$|s_n - s_0| = \operatorname{length}\beta.$$

Therefore
$$|s_0 - s_1| + \dots + |s_{n-1} - s_n| \geqslant \operatorname{length}\beta.$$

Note that

$$|q_i - s_{i-1}| = |q_i - s_i| = |p_i - o| \leqslant 1$$

and
$$\measuredangle s_{i-1} q_i s_i \leqslant \alpha_i$$

for each i. Therefore
$$|s_{i-1} - s_i| < \alpha_i$$

for each i.

It follows that

$$\begin{aligned}\operatorname{TotCurv}\beta &= \alpha_1 + \dots + \alpha_n \geqslant \\ &\geqslant |s_0 - s_1| + \dots |s_{n-1} - s_n| \geqslant \\ &\geqslant \operatorname{length}\beta.\end{aligned}$$

Hence the result. □

3.5 Curves of finite total curvature

3.11. Exercise. *Assume that a curve $\alpha\colon [a,b] \to \mathbb{R}^3$ has finite total curvature. Show that α is rectifiable.*

We say that a curve $\alpha\colon [a,b] \to \mathbb{R}^3$ *does not stop* if α is not constant on any subinterval of $[a,b]$.

3.12. Exercise. *Assume that the curve α does not stop and its total curvature is less than π. Show that α is simple; that is, it has no self-intersections.*

3.13. Exercise-definition. *Assume that a curve $\alpha\colon [a,b] \to \mathbb{R}^3$ does not stop and has finite total curvature. Show that the direction of exit and entrance is defined for any point.*

That is for any $t_0 \in [a,b)$ the unit vector

$$v(\varepsilon) = \frac{\alpha(t_0 + \varepsilon) - \alpha(t_0)}{|\alpha(t_0 + \varepsilon) - \alpha(t_0)|}$$

converges as $\varepsilon \to 0^+$; its limit is called the direction of exit and it will be denoted by $\alpha^+(t_0)$

Analogously, for any $t_0 \in (a,b]$ the unit vector

$$w(\varepsilon) = \frac{\alpha(t_0-\varepsilon)-\alpha(t_0)}{|\alpha(t_0-\varepsilon)-\alpha(t_0)|}$$

converges as $\varepsilon \to 0^+$; its limit is called the direction of entrance and it will be denoted by $\alpha^-(t_0)$.

3.14. Exercise. *Assume that a curve $\alpha\colon [a,b] \to \mathbb{R}^3$ does not stop and has finite total curvature. Show that*

$$\alpha^+(t) = -\alpha^-(t)$$

at all $t \in [a,b]$ except possibly on a countable subset.

3.15. Exercise. *Assume a sequence of curves $\alpha_n\colon [a,b] \to \mathbb{R}^3$ converges to a curve $\alpha_\infty\colon [a,b] \to \mathbb{R}^3$ and*

$$\varliminf_{n\to\infty} \operatorname{length}\alpha_n > \operatorname{length}\alpha_\infty.$$

Show that

$$\operatorname{TotCurv}\alpha_n \to \infty \quad \textit{as} \quad n \to \infty.$$

3.6 Total signed curvature

Let us define the *total signed curvature* of a polygonal line in the plane as the sum of the signed external angles; the external angle has positive sign if the line turns left and negative sign if the line turns right; the signed external angle is undefined if a pair of adjacent edges overlap; that is if at one vertex the polygonal line turns in the exact opposite direction. In particular the total signed curvature is defined for any simple polygonal line in the plane.

3.16. Exercise. *Assume that the total signed curvature of a closed polygonal line in the plane is defined. Show that it is a multiple of $2\cdot\pi$.*

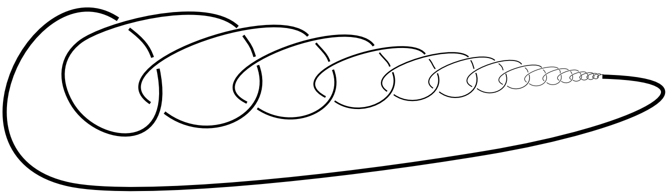
3.17. Exercise. *Show that the total signed curvature of any closed simple polygonal line in the plane is $\pm 2\cdot\pi$.*

Chapter 4

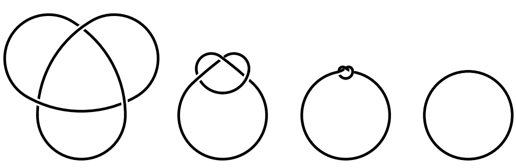
Fáry–Milnor theorem

4.1 Tame knots

It is tricky to make a formal definition that captures the intuitive meaning of *knot*. An attempt to define knots as simple closed curves leads to pathological examples as the one show on the diagram — these are the so called *wild knots*. If one adds that the curve has to



be smooth and regular, then these examples disappear, but it is still tricky to give right definition of *deformation* — the following diagram shows that it can not be defined as a continuous family of closed simple



smooth regular curves. Observe that all curves on the diagram are smooth and regular for all times including the last moment.

We define a *knot* (more precicely *tame knot*) as a simple closed polygonal line in the Euclidean space $\mathbb{R}^3$.

The notation $\triangle abc$ is used for the triangle abc; that is, a polygonal line with three edges and vertexes a, b and c. Let us denote by $\blacktriangle abc$ the convex hull of the points a, b and c; $\blacktriangle abc$ is the solid triangle with the vertexes a, b and c. The points a, b and c are assumed to be distinct, but they might lie on one line; that is, for us a degenerate triangle is a legitimate triangle.

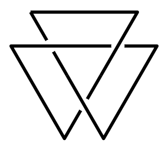
We define a *triangular isotopy of a knot* to be the generation of a new knot from the original one by means of the following two operations:

Assume $[pq]$ is an edge of the knot and x is a point such that the solid triangle $\blacktriangle pqx$ has no common points with the knot except for the edge $[pq]$. Then we can replace the edge $[pq]$ in the knot by the two adjacent edges $[px]$ and $[xq]$.

We can also perform the inverse operation. That is, if for two adjacent edges $[px]$ and $[xq]$ of a knot the triangle $\blacktriangle pqx$ has no common points with the knot except for the points on the edges $[px]$ and $[xq]$, then we can replace the two adjacent edges $[px]$ and $[xq]$ by the edge $[pq]$.

Polygons that arise from one another by a finite sequence of triangular isotopies are called *isotopic*.

A knot that is not isotopic to a triangle is called nontrivial.



The trefoil knot shown on the diagram gives a simple example of nontrivial knot. A proof that the trefoil knot is nontrivial can be found in any textbook on knot theory, we do not give it here. The most elementary and visual proof is based on the so called *tricolorability* of knot diagrams.

4.1. Exercise. *Let x and y be two points on the adjacent edges $[p_1p_2]$ and $[p_2p_3]$ of a knot $\beta = p_1p_2p_3\dots p_n$. Assume that the solid triangle $\blacktriangle xp_2y$ intersects β only along $[xp_2]\cup[p_2y]$. Show that the knot $\beta' = p_1xyp_3\dots p_n$ is isotopic to β.*

4.2 Fáry–Milnor theorem

We will give some proofs of the following theorem.

4.2. Theorem. *The total curvature of any nontrival knot is at least $4\cdot\pi$.*

The famous Fáry–Milnor theorem states that the inequality is strict; that is, the total curvature of any nontrival knot *exceeds* $4\cdot\pi$. It is easy to construct a trefoil knot with total curvature arbitrary close to $4{\cdot}\pi$; therefore this result is optimal.

The question was raised by Karol Borsuk [3] and answered independently by István Fáry and John Milnor [4, 5]; later other proofs were found.

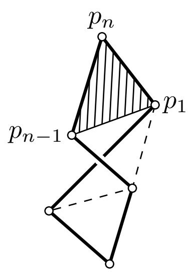
4.3 Milnor's proof

In the proof we will use the following fact.

4.3. Proposition. *Assume that a height function $(x, y, z) \to z$ has only one local minimum and one local maximum on a closed simple polygonal line and all the vertexes of the polygonal line are at different height. Then the line is a trivial knot.*

The proof is a simple application of the definition of isotopy, given in the previous section.

Proof. Let $\beta = p_1 \dots p_n$ be the closed simple polygonal line such that the height function $(x, y, z) \to z$ has one local minimum one local maximum. Note that on each of the two arcs of β from the min-vertex to the max-vertex the height function increases monotonically.



Consider the three vertexes with the largest height; they have to include the max-vertex and two more. Note that these three vertexes are consequent in the polygonal line; without loss of generality we can assume that they are p_{n-1}, p_n, p_1.

Note that the solid triangle $\blacktriangle p_{n-1}p_np_1$ does not intersect any edge β except the two adjacent edges $[p_{n-1}p_n] \cup [p_np_1]$. Indeed, if $\blacktriangle p_{n-1}p_np_1$ intersects $[p_1p_2]$, then, since p_2 lies below $\blacktriangle p_{n-1}p_np_1$, $[p_1p_2]$ must intersect $[p_{n-1}p_n]$ which is impossible since β is simple. The same way one can show that $\blacktriangle p_{n-1}p_np_1$ can not intersect $[p_{n-2}p_{n-1}]$. The remaining edges lie below $\blacktriangle p_{n-1}p_np_1$, hence they can not intersect this triangle.

Applying a triangular isotopy, to $\blacktriangle p_{n-1}p_np_1$ we get a closed simple polygonal line $\beta' = p_1 \dots p_{n-1}$ which is isotopic to β.

Since all the vertexes p_i have different height, the assumption of the proposition holds for β'.

Repeating this procedure $n - 3$ times we get a triangle. Hence β is a trivial knot. □

Milnor's proof of 4.2. Let α be a simple closed polygonal line. Assume its total curvature is less that $4{\cdot}\pi$. Then by Proposition 3.8,

$$\operatorname{TotCurv}\alpha_u < 4{\cdot}\pi$$

for some unit vector u. Moreover, we can assume that u points in a generic direction; that is, u is not perpendicular to any edge or diagonal of α.

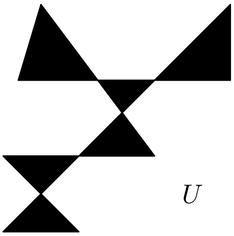
The total curvature of α_u is π times the number of turns of α_u which has to be an even number. It follows that the number of turns of α_u is at most 2; it cannot be less than 2 for a generic direction, therefore it is exactly 2.

That is, if we rotate the space so that u points upward, then the height function has exactly one minimum and one maximum; by Proposition 4.3, α is a trivial knot — hence the result. □

4.4 Fáry's proof

Let us give a sketch of another proof, based on the original idea of István Fáry.

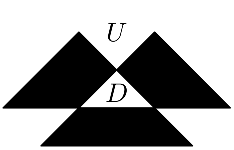
Fáry's proof of 4.2. Consider a projection of the knot to a plane in general position. That is, we assume that the self-intersections of the projection are at most double and the projection of each edge is not degenerate. The obtained closed polygonal line $\beta = p_1p_2\dots p_n$ divides the plane into domains, one of which is unbounded, denote it by U, and the others are bounded.



First note that all domains can be colored in a chessboard order; that is, they can be colored in black and white in such a way that domains with common borderline get different colors. If the unbounded domain is colored in white and every other domain is colored in black then one can untie the knot by flipping these domains one by one.

4.4. Exercise. *Give a formal proof of the last statement; that is, show that if the only undbounded domain is white then β is isotopic to a triangle.*

Therefore among the bounded domains there is a white domain, denote it by D. The domain D cannot adjoin U, since they have the same color. Fix a point o in this domain.



For each i, set

$$\begin{aligned}
\varphi_i &= \pi - \measuredangle p_{i-1}p_ip_{i+1},\\
\psi_i &= \measuredangle p_{i-1}op_i,\\
\theta_i &= \measuredangle op_ip_{i+1}.
\end{aligned}$$

Here indexes are taken modulo n; in particular, $p_n = p_0$.

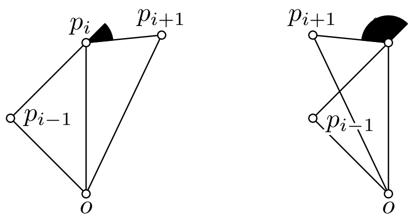
Note that φ_i is the external angle at p_i; therefore

$$\operatorname{TotCurv}\beta = \varphi_1 + \dots + \varphi_n$$

Direct calculations show that

$$\varphi_i \geqslant \psi_i + \theta_{i-1} - \theta_i.$$

In the two pictures below, φ_i is the solid angle and the angles ψ_i, θ_{i-1} and θ_i are just as drawn. We have equality on the first picture and strict inequality on the second picture.



It follows that

$$\varphi_1 + \dots + \varphi_n \geqslant \psi_1 + \dots + \psi_n.$$

The last sum is the total angle at which β is seen from o counted with multiplicity. The boundary of D contributes at least $2{\cdot}\pi$ to this sum and the boundary of U contributes with other $2{\cdot}\pi$; since their boundaries do not overlap we get

$$\psi_1 + \dots + \psi_n \geqslant 4{\cdot}\pi,$$

hence the result.

This is true for the projection of the knot to any plane in general position. The remaining planes contribute nothing to the average value. Therefore by Proposition 3.8, the total curvature of the original knot is at least $4{\cdot}\pi$. □

4.5. Exercise. *Construct a closed smooth simple curve with total curvature arbitrarily close to $2{\cdot}\pi$ such that its projection to any plane has at least 10 self-intersections.*

4.5 Proof of Alexander and Bishop

Here we sketch a proof of the Fáry–Milnor theorem given by of Stephanie Alexander and Richard Bishop in [6].

The proof is elementary, but not simple (elementary does not mean simple, it means only that it does not use much theory). It is based on the following two facts that we are already familiar with:

◇ If a closed polygonal line β' is inscribed in a closed polygonal line β then
$$\operatorname{TotCurv}\beta' \leqslant \operatorname{TotCurv}\beta.$$

◇ The total curvature of a doubly covered bigon is $4{\cdot}\pi$; that is,
$$\operatorname{TotCurv}\beta = 4{\cdot}\pi$$
if $\beta = pqpq$ for two distinct points p and q. Similarly if a quadrilateral is sufficiently close to a doubly covered bigon, then its total curvature is close to $4{\cdot}\pi$.

Proof. Let $\beta = p_1\dots p_n$ be a closed polygonal line that is not a trivial knot; that is, one can not get a triangle from β by applying a sequence of triangular isotopies defined in the previous section.

We proceed by induction on the number $n \geqslant 3$. In the base case $n = 3$ the polygonal line β is a triangle. Therefore, by definition, β is a trivial knot — nothing to show.

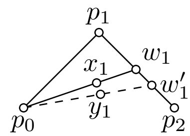
Consider the smallest n for which the statement fails; that is, there is a closed simple polygonal line $\beta = p_1\dots p_n$ that is not a trivial knot and such that

❶ $$\operatorname{TotCurv}\beta < 4{\cdot}\pi.$$

We use the indexes modulo n; that is, $p_0 = p_n$, $p_1 = p_{n+1}$ and so on. Without loss of generality, we may assume that β is in general position; that is, no four vertexes of β lie on one plane.

Set $\beta_0 = \beta$. If the solid triangle $\blacktriangle p_0p_1p_2$ intersects β_0 only in the two adjacent edges, then applying the corresponding triangular isotopy, we get a knot β_0' with $n-1$ edges that is inscribed in β_0. Therefore
$$\operatorname{TotCurv}\beta_0 \geqslant \operatorname{TotCurv}\beta_0'.$$
On the other hand, by the induction hypothesis
$$\operatorname{TotCurv}\beta_0' \geqslant 4{\cdot}\pi,$$
which contradicts ❶.

Choose the first point w_1' on the edge $[p_1p_2]$ so that the line segment $[p_0w_1']$ intersects β_0. Denote a point of intersection by y_1.



Choose a point w_1 on $[p_1p_2]$ a bit before w_1' (below we explain how close). Denote by x_1 the point on $[p_0w_1]$ that minimizes the distance to y_1. This way we get a closed polygonal line $\beta_1 = w_1p_2\dots p_n$ with two marked points x_1 and y_1. Denote by m_1 the number of edges in the arc $x_1w_1\dots y_1$ of β_1.

By Exercise 4.1, β_1 is isotopic to β_0; in particular β_1 is a nontrivial knot.

Now let us repeat the procedure for the adjacent edges $[w_1p_2]$ and $[p_2p_3]$ of β_1. If the solid triangle $\blacktriangle w_1p_2p_3$ intersects β_1 only at these two adjacent edges, then we get a contradiction with the induction hypothesis the same way as before. Otherwise we get a new knot $\beta_2 = w_1w_2p_3\dots p_n$ with two marked points x_2 and y_2. Denote by m_2 the number of edges in the broken line $x_2w_2\dots y_2$.

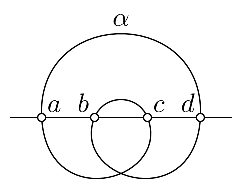
Note that the points x_1, x_2, y_1, y_2 can not appear on β_2 in the same cyclic order; otherwise the broken line $x_1x_2y_1y_2$ can be made to be arbitrary close to a doubly covered bigon which again contradicts ❶.[1]

Therefore we can assume that the arc $x_2w_2\dots y_2$ lies inside the arc $x_1w_1\dots y_1$ in β_2 and therefore $m_1 > m_2$.

Continuing this procedure we get a sequence of polygonal lines $\beta_i = {}$ $= w_1\dots w_ip_{i+1}p_n$ with marked points x_i and y_i such that the number of edges m_i from x_i to y_i decreases as i increases. Clearly $m_i > 1$ for any i and $m_1 < n$. Therefore it requires less than n steps to get a contradiction with the induction hypothesis. □

4.6. Exercise. *Suppose that a closed curve α crosses a line at four points a, b, c and d. Assume that the points a, b, c and d appear on the line in that order and they appear on the curve α in the order a, c, b, d. Show that*

$$\operatorname{TotCurv}\alpha \geqslant 4\cdot\pi.$$



A line crossing a knot at four points as in the exercise is called *alternating quadrisecants.* It turns out that any nontrivial knot admits

[1] More precisely, the choice of w_1 has to be made so that the distance $|x_1 - y_1|$ would be much less that all the distances between y_1 and any point $z \in \beta\cap\blacktriangle p_1p_2p_3$, so we have

$$\measuredangle y_1zx_1 < \tfrac{\varepsilon}{10},$$

where $\varepsilon = 4\cdot\pi - \operatorname{TotCurv}\beta$. In this case, since $y_2 \in \beta \cap \blacktriangle p_1p_2p_3$ and x_2 can be taken arbitrary close to y_2, we have

$$\operatorname{TotCurv} x_1x_2y_1y_2 > 4\cdot\pi - \varepsilon = \operatorname{TotCurv}\beta$$

which can not happen since $x_1x_2y_1y_2$ is inscribed in β.

an alternating quadrisecants [7]; it provides yet another proof of the Fáry–Milnor theorem.

4.7. Advanced exercise. *Show that given any real number Φ there is a knot β such that any knot isotopic to β has total curvature at least Φ.*

Hint: Use that there are knots with arbitrary large *bridge number*, see for example [8] and the references therein.

AFTER THIS LINE READ AT YOUR OWN RISK!!!

Chapter 5

Plane curves

5.1 Unit-speed curves

Any regular smooth curve can be parametrized by its length. The obtained curve α (that is the constructed reparametrization of the given curve) has unit speed; that is, $|\alpha'(t)| = 1$ for any t. A curve with such parametrization is called *unit-speed* curve or a curve with a *natural parametrization.*

It is straightforward to show any smooth regular curve remains smooth (and regular) if equipped with a natural parametrization; here smooth means that all derivatives $\alpha^{(n)}(t)$ are defined for all values of t in the domain of definition and any n.

5.1. Proposition. *Assume $\alpha\colon [a,b] \to \mathbb{R}^2$ be a smooth unit-speed curve. Then*

$$\alpha'(t) \perp \alpha''(t)$$

for any t.

The scalar product (also known as dot product) of two vectors v and w will be denoted by $\langle v, w\rangle$. Recall that for derivative of scalar product the product rule holds; that is if $v = v(t)$ and $w = w(t)$ are smooth vector-valued functions of real argument t, then

$$\langle v, w\rangle' = \langle v', w\rangle + \langle v, w'\rangle.$$

Proof. Since $|\alpha'(t)| = 1$, we have

$$\langle \alpha'(t), \alpha'(t)\rangle = 1.$$

Taking derivative of both sides we get

$$2{\cdot}\langle \alpha''(t), \alpha'(t)\rangle = 0,$$

hence the result. □

5.2 Signed curvature

Given a vector $v \in \mathbb{R}^2$ denote by $i{\cdot}v$ the vector obtained from v by the counterclockwise rotation by $\frac{\pi}{2}$. (The "multiplication" by i agrees with the miultiplication by imaginary unit if one use complex coordinates on the plane $z = x + i{\cdot}y$.)

Assume $\alpha\colon [a,b] \to \mathbb{R}^2$ be a smooth unit-speed curve. Recall that curvature of α at t can be defined as $|\alpha''(t)|$.

The *signed curvature* $\kappa(t)$ is uniquely defined by the identity

$$\alpha''(t) = \kappa(t){\cdot}i{\cdot}\alpha'(t).$$

Note that by Proposition 5.1 this equation has a solution. Since $|\alpha'(t)| = 1$ we have that $|\kappa(t)| = |\alpha''(t)|$ for any t.

The signed curvature measures how fast the direction $\tau(t) = \alpha'(t)$ rotates; if is positive if it turns left and negative if it turns right; if the curve goes straight then its curvature vanish.

5.3 Osculating circline

It is straightforward to prove the following statement.

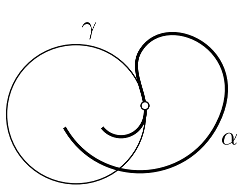
5.2. Proposition. *Given a point p, a unit vector u and a real number κ there is unique smooth unit-speed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ that starts at p in the direction of u and has constant signed curvature κ.*

Moreover if $\kappa = 0$, then γ is a line $\gamma = p + t{\cdot}u$ and if $\kappa \neq 0$, then γ runs around a circle of radius $\frac{1}{|\kappa|}$ with center at $p + \frac{i}{\kappa}{\cdot}u$.

Further we will use term *circline* for *circle or line.*

5.3. Definition. *Let α be a smooth unit-speed plane curve; denote by $\kappa(t)$ the signed curvature of α at t.*

A unit-speed plane curve γ of constant signed curvature $\kappa(t_0)$ that starts at $\alpha(t_0)$ and runs in the direction $\alpha'(t_0)$ is called osculating circline *of α at t_0.*



The center and radius of the osculating circle at a given point are called *center of curvature* and *radius of curvature* of the curve at that point.

5.4 Spiral

The following problem states that if you drive on the plane and turn the steering wheel to the right all the time, then you will not be able to come back to the same place. This theorem was proved by Peter Tait [see 9] and later rediscovered by Adolf Kneser [see 10].

5.4. Theorem. *Assume γ is a smooth regular plane curve with strictly monotonic curvature. Then γ has no self-intersections.*

The proof is based on observation that the osculating circles of γ are nested. The same statement holds for signed curvature; the proof requires only minor modifications.

Proof. Without loss of generality we may assume that the curve γ is parametrized by its length and its curvature $\kappa(t)$ decreases and stays positive.

Let $z(t)$ be curvature center and $r(t) = \frac{1}{\kappa(t)}$ is radius of curvature of γ at t. Note that

$$z(t) = \gamma(t) + \tfrac{\gamma''(t)}{|\gamma''(t)|^2}, \qquad r(t) = \tfrac{1}{|\gamma''(t)|}.$$

Straightforward calculations show that

$$|z'(t)| = r'(t).$$

and $z'(t) \perp \gamma'(t)$. It follows that the curve $z(t)$ does not have straight arcs; therefore

$$\begin{aligned} |z(t_1) - z(t_0)| &< \int\limits_{t_0}^{t_1} |z'(t)|\cdot dt = \\ &= \int\limits_{t_0}^{t_1} r'(t)\cdot dt = \\ &= r(t_1) - r(t_0). \end{aligned} \tag{$*$}$$

By $(*)$, the osculating circle at t_0 lies inside of the osculating circle at t_1 without touching it. In particular, $\gamma(t_1) \neq \gamma(t_0)$ if $t_1 > t_0$. $\square$

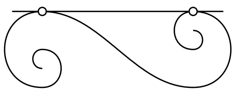
The osculating circles of the curve give a peculiar decomposition of an annulus into circles; it has the following property: if a smooth function is constant on each osculating circle it must be constant in the annulus [see 2, Lecture 10].

5.5. Exercise. *Show that a 3-dimensional analog of the theorem does not hold. That is, there are self-intersecting smooth regular space curves with strictly monotonic curvature.*

5.6. Exercise. *Assume that γ is a smooth regular plane curve with strictly monotonic curvature.*

(a) *Show that no line can be tangent to γ at two distinct points.*

(b) *Show that no circle can be tangent to γ at three distinct points.*



Note that part (a) does not hold for smooth regular plane curve with strictly monotonic *signed* curvature; an example is shown on the diagram.

Note that if the curve $\gamma(t)$ is defined for $t \in [0, \infty)$ and its curvature converges to ∞ as $t \to \infty$, then the problem implies the convergence of $\gamma(t)$ as $t \to \infty$. The latter could be considered as a continuous analog of the Leibniz's test for alternating series.

5.5 Supporting circlines

Suppose $\alpha\colon [a, b] \to \mathbb{R}^2$ be a smooth unit-speed plane curve and $t_0 \in$ $\in (a, b)$.

A unit-speed circline γ supports α at t_0 if $\alpha(t_0)$ lies on γ and the points $\alpha(t)$ lie on one closed side of γ for all values t sufficiently close to t_0.

The following claim resembles the first derivative test.

5.7. Claim. *Suppose a unit-speed circline γ supports a smooth unit-speed plane curve α at t_0 from right (correspondingly left). Without loss of generality we can assume that $\gamma(0) = \alpha(t_0)$.*

Denote by κ the signed curvature of γ and by $\kappa(t_0)$ the signed curvature if α at t_0. Then $\gamma'(0) = \pm\alpha'(t_0)$.

Otherwise the curve α would cross γ transversely and therefore could not stay at the same side for values close to t_0.

Reverting the parametrization of γ if necessary we may (and further will) assume that

$$\gamma'(0) = \alpha'(t_0)$$

holds for any supporting circline γ to α at t_0. In this case we say that γ supports α from rigth (correspondingly left) if α lies on left (correspondingly right) side of γ.

The following proposition resembles the second derivative test.

5.8. Proposition. *Assume γ is a circle that that supports α at t_0 from rigth (correspondingly left). Then*

$$\kappa(t_0) \geqslant \kappa \quad (\textit{correspondingly} \quad \kappa(t_0) \leqslant \kappa).$$

where κ is the signed curvature of γ and $\kappa(t_0)$ is the signed curvature of α at t_0.

A partial converse also holds. Namely suppose a unit-speed circline γ with signed curvature κ starts at $\alpha(t_0)$ in the direction $\alpha'(t_0)$. Then γ supports α at t_0 from the right (correspondingly left) if

$$\kappa(t_0) > \kappa \quad (\textit{correspondingly} \quad \kappa(t_0) < \kappa).$$

Proof. We prove only case $\kappa > 0$. The 2 remaining cases $\kappa = 0$ and $\kappa < 0$ can be done essentially same way.

Since $\kappa \neq 0$, the curve γ is a circle (it can not be a line). According to Proposition 5.2, γ has radius $\frac{1}{\kappa}$ and it is centered at

$$z = \alpha(t_0) + \tfrac{i}{\kappa}\cdot\alpha'(t_0).$$

Consider the function

$$f(t) = |z - \alpha(t)|^2 - \tfrac{1}{\kappa^2}.$$

Note that $f(t) \leqslant 0$ (correspondingly $f(t) \geqslant 0$) if an only if $\alpha(t)$ lies on the closed left (correspondingly right) side from γ. It follow that

- ◇ if γ supports α at t_0 from right, then
$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) \leqslant 0;$$
- ◇ if γ supports α at t_0 from left, then
$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) \geqslant 0;$$
- ◇ if
$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) < 0,$$
then γ supports α at t_0 from right;
- ◇ if
$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) > 0,$$
then γ supports α at t_0 from left;

Direct calculations show that

$$\begin{aligned}
f(t_0) &= 0;\\
f'(t_0) &= \langle z - \alpha(t), z - \alpha(t)\rangle'|_{t=t_0} =\\
&= -2\cdot\langle\alpha'(t_0), z - \alpha(t_0)\rangle =\\
&= -2\cdot\langle\alpha'(t_0), \tfrac{i}{\kappa}\cdot\alpha'(t_0)\rangle =\\
&= 0;\\
f''(t_0) &= \langle z - \alpha(t), z - \alpha(t)\rangle''|_{t=t_0} =\\
&= 2\cdot\left(\langle\alpha'(t_0), \alpha'(t)\rangle - \langle\alpha''(t_0), z - \alpha(t)\rangle\right) =\\
&= 2\cdot\left(1 - \kappa\cdot\frac{1}{\kappa(t_0)}\right)
\end{aligned}$$

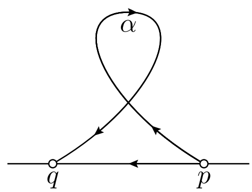
Hence the result. □

5.9. Exercise. *Assume α is a closed smooth unit-speed plane curve that runs in a unit disk. Show that there is a point on α with curvature at least* 1*.*

Give two proofs, one based on DNA inequality 3.10 and the other based on Proposition 5.8.

5.10. Lemma. *Let α be a smooth regular simple curve that runs from p to q. Assume that α runs on right side (correspondingly left side) of the oriented line pq and only its end points p and q lie on the line. Then α has a point with positive (correspondingly negative) curvature.*

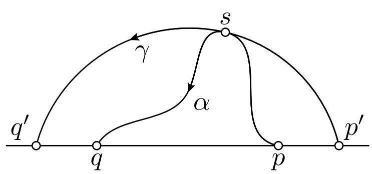
Note that the lemma fails for curves with self-intersections; the curve α on the diagram has negative curvature everywhere and it lies on the right side of the line pq.

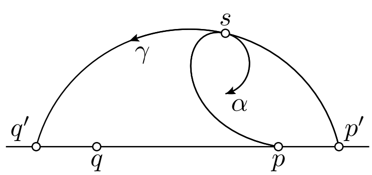


Proof. Choose points p' and q' on ℓ so that the points p', p, q, q' appear in the same order on ℓ.

Consider the smallest disc segment with chord $[p'q']$ that contains α. Note that its arc γ supports at a point $s = \alpha(t_0)$.

Note that the $\alpha'(t_0)$ is tangent to γ at s. Moreover $\alpha'(t_0)$ points in the direction of q'; that is, if we go along γ in the direction of $\alpha'(t_0)$ then we have to start at p' and end at q'. If the direction is opposite, then the arc of α from s to q would be trapped in the curvelinear triangle psp' bounded by arcs of γ, α and the line segment $[p'p]$. But this is impossible since q does not belong to this triangle.





It follows that γ supports α at t_0 from right. By Proposition 5.8,

$$\kappa(t_0) \geqslant \kappa,$$

where $\kappa(t_0)$ is signed curvature of α at t_0 and κ is the curvature of γ. Evidently $\kappa > 0$, hence the result. □

5.11. Exercise. *Assume that a bounded domain D in the plane is bounded by a closed regular smooth simple plane curve α. Show that*

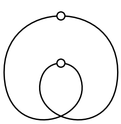
D is convex if and only if the signed curvature of α does not change the sign.

Chapter 6

Four-vertex theorem

A vertex of a smooth regular curve is defined as a critical point of its curvature; in particular, any point of local minimum (or maximum) is a vertex.

6.1. Four-vertex theorem. *Any smooth regular simple plane curve has at least four vertices.*



Evidently any closed curve have to have at least two vertexes — one minimum and one maximum. The curve on the diagram has one self-intersection and exactly two vertexes, they are marked on the diagram. There are smooth regular simple plane curves with exactly 4 vertexes, ellipse for example.

The four-vertex theorem was first proved by Syamadas Mukhopadhyaya [11] for convex curves. By now it has number of different proofs and generalizations.

We will present a proof given by Robert Osserman [12].

Proof. Fix a smooth regular closed plane curve α.

Suppose that $2{\cdot}n$ points $p_1, s_1, \dots, p_n, s_n$ appear on a closed curve α in the same cyclic order. Fix a real number κ. Assume that the curvature of α at p_i is at least κ and its curvature at s_i is smaller than κ. Then each of n arcs $p_n p_1$, $p_1 p_2, \dots p_{n-1} p_n$ of α has a local minimum. Similarly each of n arcs $s_n s_1$, $s_1 s_2, \dots s_{n-1} s_n$ of α has a local maximum. In particular it has at least $2{\cdot}n$ vertexes. It is sufficient to show that

❶ *there are at least 4 points p_1, s_1, p_2, s_2 with the described condition.*

Note that

❷ *α admits unique* circumscribed circle *γ; that is, γ is a circle of minimal radius that encloses α.*

Indeed, denote by r the least lower bound of circles that enclose α. We can choose a sequence of circles γ_n enclosing α such that their radiuses $r_n \to r$. Note that all the centers of γ_i lie on bounded distance from α. Therefore passing to a subsequence we can assume that centers of γ_n converge to a point o. Note that the circle γ with center o and radius r encloses α; hence the existence of circumscribed circle follows.

If there are two distinct circumscribed circle, then α lies in the intersection of the discs bounded by these circles. But this intersection is enclosed in a circle of smaller radius — a contradiction.

❸ *Assume γ is the circumscribed circle of α. Then γ touches α at least* 2 *points which divide the γ in arcs no longer than semicircle.*

Indeed if it would not be the case then one could move γ slightly keeping its radius the same so that γ will not touch α at all. But in this case α could be enclosed by a circle of smaller radius — a contradiction.

Let us orient α and γ counterclockwise. Then at the common points the directions of α and γ coincide. Note that these points appear on α and γ in the same order; otherwise α could not be simple.

Denote by κ the signed curvature of γ, since it is oriented counterclockwise, $\kappa = \frac{1}{r} > 0$.

Fix two common points p_1 and q of α and γ. By Proposition 5.8, the curvature of α at p_1 and q is at least κ.

Denote by $\bar\gamma$ the arc of γ from p_1 to q. Assume $\bar\gamma$ does not exceed semicircle. Let $\bar\alpha$ be the corresponding arc of α from p_1 to q.

Consider the minimal lens $L_{p_1,q}$ with vertexes p_1 and q that contain $\bar\alpha$; that is a closed region bounded by two arcs from p_1 to q. Note that one of the arcs has to be $\bar\gamma$; denote the other arc by $\bar\gamma'$ and its curvature by κ'.

If the lens $L_{p_1,q}$ degenerates to $\bar\gamma$, then $\bar\alpha = \bar\gamma$ and all the points on α have curvature κ. In particular each point of $\bar\alpha$ is a vertex and the theorem follows.

If $L_{p_1,q}$ is nondegenerate, then

❹ $$\kappa' < \kappa.$$

This inequality is evident if $\kappa' \leqslant 0$. If $\kappa' > 0$ then $\kappa' = \frac{1}{r'}$ where r' is the radius of γ'. Since γ does not exceed semicircle, we have that $r' > r$ and therefore ❹ follows.

Note that there is a point s on $\bar\alpha$ common with $\bar\gamma'$; otherwise $L_{p_1,q}$ would not be minimal. Moreover at the point s the directions of $\bar\alpha$ and

$\bar{\gamma}'$ coinside, otherwise α could not be simple — the same argument is used in the proof of Lemma 5.10.

By Proposition 5.8, the curvature of α at s is at most κ'.

After repeat the same argument for an other pair of points p_2, p_3 (possibly $p_3 = p_1$), the claim ❶ and therefore the theorem follows. □

6.2. Exercise. *Show that any smooth regular curve of constant width has at least 6 vertexes.*

Appendix A

Semisolutions

Exercise 1.5. First let us show that Dido's problem follows from the isoperimetric inequality.

Assume F is a figure bounded by a straight line and a curve of length ℓ whose endpoints belong to that line. Let F' be the reflection of F in the line. Note that the union $G = F \cup F'$ is a figure bounded by a closed curve of length $2\cdot\ell$.

Applying the isoperimetric inequality, we get that the area of G can not exceed the area of round disc with the same circumference $2\cdot\ell$ and the equality holds only if the figure is congruent to the disc. Since F and F' are congruent, Dido's problem follows.

Now let us show that the isoperimetric inequality follows from the Dido's problem.

Assume G is a convex figure bounded by a closed curve of length $2\cdot\ell$. Cut G by a line that splits the perimeter in two equal parts — ℓ each. Denote by F and F' the two parts. Applying the Dido's problem for each part, we get that that are of each does not exceed the area of half-disc bounded by a half-circle. The two half-disc could be arranged into a round disc of circumference ℓ, hence the isoperimetric inequality follows.

Exercise 2.16. Let $\alpha\colon [a,b] \to \mathbb{R}^3$ be a curve. Given a unit vector u, denote by α_u the projection of α on a line in the direction of u; denote by $\alpha_{u^\perp}$ the of α on a plane perpendicular to u.

Two formulas

$$\operatorname{length}\alpha = k\cdot\overline{\operatorname{length}\alpha_u}$$

and

$$\operatorname{length}\alpha = k'\cdot\overline{\operatorname{length}\alpha_{u^\perp}}$$

can be proved the same way as the Crofton's formula in the plane.

It remains to find the coefficients k and k'. It is sufficient to calculate the average projection of unit segment to a line and to a plane. We need to find two integrals

$$k = \oint\limits_{\mathbb{S}^2} |x|\cdot d\,\mathrm{area}$$

and

$$k' = \oint\limits_{\mathbb{S}^2} \sqrt{1-x^2}\cdot d\,\mathrm{area},$$

where $\mathbb{S}^2 = \left\{\,(x,y,z)\in\mathbb{R}^3 \;:\; x^2+y^2+z^2=1\,\right\}$ is the unit sphere in the Euclidean space and $\oint$ denotes the average value — since the area of unit sphere is $4\cdot\pi$, we have

$$\oint\limits_{\mathbb{S}^2} f(x,y,z)\cdot d\,\mathrm{area} = \tfrac{1}{4\cdot\pi}\cdot \int\limits_{\mathbb{S}^2} f(x,y,z)\cdot d\,\mathrm{area}$$

Note that in the cylindrical coordinates

$$(x, \varphi = \arctan\tfrac{y}{z}, \rho = \sqrt{y^2+z^2}),$$

we have $d\,\mathrm{area} = dx\cdot d\varphi$. Therefore

$$k = \oint\limits_{[-1,1]} |x|\cdot dx = \tfrac{1}{2}$$

and

$$k' = \oint\limits_{[-1,1]} \sqrt{1-x^2}\cdot dx = \tfrac{\pi}{4}.$$

Comment. Note that $\frac{k'}{k} = \frac{\pi}{2}$ is the coefficient in the 2-dimensional Crofton formula. This is not a coincidence — think about it.

Exercise 3.1. Assume contrary, that is there is a closed smooth regular curve α such that $\operatorname{TotCurv}\alpha < 2\cdot\pi$.

The tangent indicatrix τ of α is a curve in a sphere; by the definition of total curvature, the length of τ is the total curvature of α; in particular

$$\operatorname{length}\tau < 2\cdot\pi.$$

By Exercise 2.19, τ lies in an open hemisphere. If u is the center of the hemisphere, then

$$\langle u, \tau(t)\rangle > 0 \quad \text{and therefore} \quad \langle u, \alpha'(t)\rangle > 0$$

for any t. Therefore the function $t \mapsto \langle u, \alpha(t)\rangle$ is strictly increasing. In particular, if α is defined on the time interval $[a, b]$, then

$$\langle u, \alpha(a)\rangle < \langle u, \alpha(a)\rangle.$$

But α is closed; that is $\alpha(a) = \alpha(b)$ — a contradiction.

Now let us prove the equality case. First note that it is sufficient to show that τ runs around an equator.

Assume τ is not an equator, from above we know that τ can not lie in an open hemisphere. Note that we can shorten τ by a small chord. The obtained curve τ' is shorter than $2{\cdot}\pi$ and therefore lies in an open hemisphere. Applying this construction for shorter and shorter chord and passing to the limit we get that τ lies in closed hemisphere. Denote its center by u as before, then

$$\langle u, \tau(t)\rangle \geqslant 0 \quad \text{and therefore} \quad \langle u, \alpha'(t)\rangle \geqslant 0$$

for any t. Since α is closed we have that $\langle u, \alpha(t)\rangle$ is constant; that is, runs in a plane perpendicular to u and τ lies in an equator perpendicular to u.

So τ is a curve that runs along equator, has length $2{\cdot}\pi$ and does not lie in a open hemisphere. Since τ is not an equator, it have to run along half-equator back and forth. In this case τ lies in an other closed hemisphere and has some points in its interior. The latter contradicts closeness of α the same way as above.

Appendix B

Tickets

1. One theoretical question:
   - ◇ Definition of length and its semicontinuity;
   - ◇ Axioms of length and Crofton's formula;
   - ◇ Formulations and variations of Crofton's formula;
   - ◇ Total curvature, Fenchel's theorem and equivalence of two definitions;
   - ◇ Knots and Fáry–Milnor theorem (any proof);
   - ◇ Osculating circles and the spiral theorem;
   - ◇ Supporting circles and Four-vertex theorem.
2. One exercise.
3. An additional problem for perfect score.

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