

# Invitation to comparison geometry

Anton Petrunin and Sergio Zamora Barrera

# Contents

<b>1</b>	<b>Metric spaces</b>	<b>3</b>
<b>2</b>	<b>Curves</b>	<b>6</b>
<b>3</b>	<b>Length</b>	<b>10</b>
<b>A</b>	<b>Preliminaries</b>	<b>18</b>
<b>B</b>	<b>Homework assignments</b>	<b>21</b>
	<b>Bibliography</b>	<b>22</b>

# Chapter 1

## Metric spaces

*Metric* is a function that returns a real value  $\text{dist}(x, y)$  for any pair  $x, y$  in a given nonempty set  $\mathcal{X}$  and satisfies the following axioms for any triple  $x, y, z$ :

(a) Positiveness:

$$\text{dist}(x, y) \geq 0.$$

(b)  $x = y$  if and only if

$$\text{dist}(x, y) = 0.$$

(c) Symmetry:

$$\text{dist}(x, y) = \text{dist}(y, x).$$

(d) Triangle inequality:

$$\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z).$$

A set with a metric is called *metric space* and the elements of the set are called *points*.

**Shortcut for distance.** Usually we consider only one metric on a set, therefore we can denote the metric space and its underlying set by the same letter, say  $\mathcal{X}$ . In this case we also use a shortcut notations  $|x - y|$  or  $|x - y|_{\mathcal{X}}$  for the *distance*  $\text{dist}(x, y)$  from  $x$  to  $y$  in  $\mathcal{X}$ . For example, the triangle inequality can be written as

$$|x - z|_{\mathcal{X}} \leq |x - y|_{\mathcal{X}} + |y - z|_{\mathcal{X}}.$$

**Examples.** Euclidean space and plane as well as real line will be the most important examples of metric spaces for us. In these examples the introduced notation  $|x - y|$  for the distance from  $x$  to  $y$  has perfect sense as a norm of the vector  $x - y$ . However, in general metric space

the expression  $x - y$  has no sense, but anyway we use expression  $|x - y|$  for the distance.

If we say *plane* or *space* we mean *Euclidean* plane or space. However the plane (as well as the space) admits many other metrics, for example the so called Manhattan metric from the following exercise.

**1.1. Exercise.** Consider the function

$$\text{dist}(p, q) = |x_p - x_q| + |y_p - y_q|,$$

where  $p = (x_p, y_p)$  and  $q = (x_q, y_q)$  are points in the coordinate plane  $\mathbb{R}^2$ . Show that  $\text{dist}$  is a metric on  $\mathbb{R}^2$ .

Let us mention another example: the *discrete space* — arbitrary nonempty set  $\mathcal{X}$  with the metric defined as  $|x - y|_{\mathcal{X}} = 0$  if  $x = y$  and  $|x - y|_{\mathcal{X}} = 1$  otherwise.

**Subspaces.** Any subset of a metric space is also a metric space, by restricting the original metric to the subset; the obtained metric space is called a *subspace*. In particular, all subsets of Euclidean space are metric spaces.

**Balls.** Given a point  $p$  in a metric space  $\mathcal{X}$  and a real number  $R \geq 0$ , the set of points  $x$  on the distance less then  $R$  (or at most  $R$ ) from  $p$  is called open (or correspondingly closed) ball of radius  $R$  with center at  $p$ . The *open ball* is denoted as  $B(p, R)$  or  $B(p, R)_{\mathcal{X}}$ ; the second notation is used if we need to emphasize that the ball lies in the metric space  $\mathcal{X}$ . Formally speaking

$$B(p, R) = B(p, R)_{\mathcal{X}} = \{x \in \mathcal{X} : |x - p|_{\mathcal{X}} < R\}.$$

Analogously, the *closed ball* is denoted as  $\bar{B}[p, R]$  or  $\bar{B}[p, R]_{\mathcal{X}}$  and

$$\bar{B}[p, R] = \bar{B}[p, R]_{\mathcal{X}} = \{x \in \mathcal{X} : |x - p|_{\mathcal{X}} \leq R\}.$$

**1.2. Exercise.** Let  $\mathcal{X}$  be a metric space.

(a) Show that if  $\bar{B}[p, 2] \subset \bar{B}[q, 1]$  for some points  $p, q \in \mathcal{X}$ , then  $\bar{B}[p, 2] = \bar{B}[q, 1]$ .

(b) Construct a metric space  $\mathcal{X}$  with two points  $p$  and  $q$  such that  $B(p, \frac{3}{2}) \subset B(q, 1)$  and the inclusions is strict.

## Calculus

In this section we will extend standard notions from calculus to the metric spaces.

**1.3. Definition.** Let  $\mathcal{X}$  be a metric space. A sequence of points  $x_1, x_2, \dots$  in  $\mathcal{X}$  is called *convergent* if there is  $x_\infty \in \mathcal{X}$  such that  $|x_\infty - x_n| \rightarrow 0$  as  $n \rightarrow \infty$ . That is, for every  $\varepsilon > 0$ , there is a natural number  $N$  such that for all  $n \geq N$ , we have

$$|x_\infty - x_n| < \varepsilon.$$

In this case we say that the sequence  $(x_n)$  converges to  $x_\infty$ , or  $x_\infty$  is the limit of the sequence  $(x_n)$ . Notationally, we write  $x_n \rightarrow x_\infty$  as  $n \rightarrow \infty$  or  $x_\infty = \lim_{n \rightarrow \infty} x_n$ .

**1.4. Definition.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be metric spaces. A map  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is called *continuous* if for any convergent sequence  $x_n \rightarrow x_\infty$  in  $\mathcal{X}$ , we have  $f(x_n) \rightarrow f(x_\infty)$  in  $\mathcal{Y}$ .

Equivalently,  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is continuous if for any  $x \in \mathcal{X}$  and any  $\varepsilon > 0$ , there is  $\delta > 0$  such that

$$|x - x'|_{\mathcal{X}} < \delta \text{ implies } |f(x) - f(x')|_{\mathcal{Y}} < \varepsilon.$$

**1.5. Exercise.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be metric spaces  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is distance non-expanding map; that is,

$$|f(x) - f(x')|_{\mathcal{Y}} \leq |x - x'|_{\mathcal{X}}$$

for any  $x, x' \in \mathcal{X}$ . Show that  $f$  is continuous.

**1.6. Definition.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be metric spaces. A continuous bijection  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is called a *homeomorphism* if its inverse  $f^{-1}: \mathcal{Y} \rightarrow \mathcal{X}$  is also continuous.

If there exists a homeomorphism  $f: \mathcal{X} \rightarrow \mathcal{Y}$ , we say that  $\mathcal{X}$  is homeomorphic to  $\mathcal{Y}$ , or  $\mathcal{X}$  and  $\mathcal{Y}$  are homeomorphic.

If a metric space  $\mathcal{X}$  is homeomorphic to a known space, for example plane, sphere, disc, circle and so on, we may also say that  $\mathcal{X}$  is a *topological* plane, sphere, disc, circle and so on.

**1.7. Definition.** A subset  $A$  of a metric space  $\mathcal{X}$  is called *closed* if whenever a sequence  $(x_n)$  of points from  $A$  converges in  $\mathcal{X}$ , we have that  $\lim_{n \rightarrow \infty} x_n \in A$ .

A set  $\Omega \subset \mathcal{X}$  is called *open* if for any  $z \in \Omega$ , there is  $\varepsilon > 0$  such that  $B(z, \varepsilon) \subset \Omega$ .

An open set  $\Omega$  that contains a given point  $p$  is called *neighborhood* of  $p$ .

**1.8. Exercise.** Let  $Q$  be a subset of a metric space  $\mathcal{X}$ . Show that  $A$  is closed if and only if its complement  $\Omega = \mathcal{X} \setminus Q$  is open.

# Chapter 2

## Curves

**Paths.** Let  $\mathcal{X}$  be a metric space. A continuous map  $f: [0, 1] \rightarrow \mathcal{X}$  is called a *path*. If  $p = f(0)$  and  $q = f(1)$ , then we say that  $f$  *connects*  $p$  to  $q$ .

If any two points in  $\mathcal{X}$  can be connected by a path then  $\mathcal{X}$  is called *path connected*. Similarly, a subset  $A \subset \mathcal{X}$  is called *path connected* if any two points  $p, q \in A$  can be connected by a path that runs in  $A$ ; equivalently, the subspace  $A$  is path connected.

**Simple curves.**

**2.1. Definition.** A path connected subset  $\gamma$  in a metric space is called a *simple curve* if it is locally homeomorphic to a real interval; that is, any point  $p \in \gamma$  has a neighborhood  $U \ni p$  such that the intersection  $U \cap \gamma$  is homeomorphic to a real interval.

It turns out that any curve  $\gamma$  admits a homeomorphism from a real interval or a circle; that is, there is a continuous bijection  $G \rightarrow \gamma$  with continuous inverse; here (and further)  $G$  denotes a circle or real interval. We omit a proof of this statement, but it is not hard.

The homeomorphism  $G \rightarrow \gamma$  as above is called *parametrization* of  $\gamma$ . The parametrization completely defines the curve. Often will use the same letter for curve and its parametrization, so we can say curve  $\gamma$  has parametrization  $\gamma: G \rightarrow \mathcal{X}$ . Note however that any curve admits many different parametrization.

**2.2. Exercise.** Find a continuous injective map  $\gamma: [0, 1] \rightarrow \mathbb{R}^2$  such that its image is not a simple curve.

*Hint:* The image of  $\gamma$  should have a shape of digit 9.

If  $G$  is a circle, then the curve  $\gamma: G \rightarrow \mathcal{X}$  is called *closed*. If  $G$  is a real interval, then we may say that  $\gamma$  is an *arc*.

**Parameterized curves.** A *parameterized curve* is defined as a continuous map  $\gamma: G \rightarrow \mathcal{X}$ . For a parameterized curve we do not assume that  $\gamma$  is injective; in other words the parameterized curve might have self-intersections.

**2.3. Advanced exercise.** Let  $\alpha: [0, 1] \rightarrow \mathcal{X}$  be a path from  $p$  to  $q$ . Assume  $p \neq q$ . Show that there is a simple path connecting from  $p$  to  $q$  in  $\mathcal{X}$ .

## Smooth curves

A curve in the Euclidean space or plane, called *space* or *plane curve* correspondingly.

A space curve can be described by its coordinate functions

$$\gamma(t) = (x(t), y(t), z(t)).$$

Plane curves can be considered as a partial case of space curves with  $z(t) \equiv 0$ .

If each of the coordinate functions  $x(t), y(t), z(t)$  of the space curve  $\gamma$  is a smooth (that is, it has derivatives of all orders everywhere in its domain) then the parameterized curve is called *smooth*.

If the *velocity vector*

$$\gamma'(t) = (x'(t), y'(t), z'(t))$$

does not vanish at all points, then the parameterized curve  $\gamma$  is called *regular*.

A simple space curve is called *smooth and regular* if it admits a smooth and regular parametrization correspondingly. Regular smooth curves are among the main objects in differential geometry; the term *smooth curve* often used for *smooth regular curve*.

**2.4. Exercise.** The function

$$f(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \frac{t}{e^{1/t}} & \text{if } t > 0. \end{cases}$$

is smooth.<sup>1</sup>

Show that  $\gamma(t) = (f(t), f(-t))$  gives a smooth parametrization of the curve  $S$  formed by the union of two half-axis in the plane.

---

<sup>1</sup>The existence of all derivatives  $f^{(n)}(x)$  at  $x \neq 0$  is evident and direct calculations show that  $f^{(n)}(0) = 0$  for any  $n$ .

Show that any smooth parametrization of  $S$  has vanishing velocity vector at the origin. Conclude that the curve  $S$  is not regular and smooth.

**2.5. Exercise.** Describe the set of real numbers  $a$  such that the plane curve  $\gamma_a(t) = (t + a \cdot \sin t, a \cdot \cos t)$ ,  $t \in \mathbb{R}$  is

- (a) regular;
- (b) simple.

**Loops and periodic parametrization.** A closed simple curve can be described as an image of a parameterized curve  $\gamma: [0, 1] \rightarrow \mathcal{X}$  such that  $p = \gamma(0) = \gamma(1)$ ; such curves are called *loops*; the point  $p$  in this case is called *base* of the loop.

However, it is more natural to present it as a *periodic* parameterized curve  $\gamma: \mathbb{R} \rightarrow \mathcal{X}$ ; that is, there is a constant  $\ell$  such that  $\gamma(t + \ell) = \gamma(t)$  for any  $t$ . For example the unit circle in the plane can be described by  $2\pi$ -periodic parametrization  $\gamma(t) = (\cos t, \sin t)$ .

Any smooth regular closed curve can be described by a smooth regular loop. But in general the closed curve that described by a smooth regular loop might fail to be smooth and regular — it might fail to be smooth at its base; an example shown on the diagram.



## Implicitly defined curves

Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a smooth function; that is, all its partial derivatives defined in its domain of definition. Consider the set  $S$  of solution of equation  $f(x, y) = 0$  in the plane.

Assume  $S$  is path connected. According to implicit function theorem (A.2), the set  $S$  is a smooth regular simple curve if 0 is a *regular value* of  $f$ . In this case it means that the gradient  $\nabla f$  does not vanish at any point  $p \in S$ . In other words, if  $f(p) = 0$ , then  $\frac{\partial f}{\partial x}(p) \neq 0$  or  $\frac{\partial f}{\partial y}(p) \neq 0$ .

Similarly, assume  $f, h$  is a pair of smooth functions defined in  $\mathbb{R}^3$ . The system of equations  $f(x, y, z) = h(x, y, z) = 0$  defines a regular smooth space curve if the set of solutions is path connected and 0 is a regular value of the map  $F: (x, y, z) \mapsto (f(x, y, z), h(x, y, z))$ . In this case it means that the gradients  $\nabla f$  and  $\nabla h$  are linearly independent at any point  $p \in S$ . In other words, if  $f(p) = 0$ , then at the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix}$$



for the map  $F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  has rank 2 at  $p$ .

The described way to define a curve is called *implicit*; if a curve is defined by its parametrization, we say that it is *explicitly defined*. While implicit function theorem guarantees the existence of regular smooth parametrizations, do not expect it to be in a closed form. When it comes to calculations, usually it is easier to work directly with implicit presentation.

**2.6. Exercise.** Describe the set of real numbers  $a$  such that the system of equations

$$\begin{aligned}x^2 + y^2 + z^2 &= 1 \\ x^2 + a \cdot x + y^2 &= 0\end{aligned}$$

describes a smooth regular curve.

# Chapter 3

## Length

Recall that a sequence

$$a = t_0 < t_1 < \cdots < t_k = b.$$

is called a *partition* of the interval  $[a, b]$ .

**3.1. Definition.** Let  $\alpha: [a, b] \rightarrow \mathcal{X}$  be a curve in a metric space. The length of a  $\alpha$  is defined as

$$\text{length } \alpha = \sup\{|\alpha(t_0) - \alpha(t_1)| + |\alpha(t_1) - \alpha(t_2)| + \cdots + |\alpha(t_{k-1}) - \alpha(t_k)|\},$$

where the exact upper bound is taken over all partitions

$$a = t_0 < t_1 < \cdots < t_k = b.$$

The length of  $\alpha$  is a nonnegative real number or infinity; the curve  $\alpha$  is called *rectifiable* if its length is finite.

The length of a closed curve is defined as the length of a corresponding loop. If a curve is defined on a open or closed-open interval then its length is defined as the exact upper bound for lengths of all its closed arcs.

If  $\alpha$  is a space curve, then the above definition says that its length is the exact upper bound of the lengths of polygonal lines  $p_0 \dots p_k$  inscribed in the curve, where  $p_i = \alpha(t_i)$  for a partition  $a = t_0 < t_1 < \cdots < t_k = b$ . If  $\alpha$  is closed then  $p_0 = p_k$  and therefore the inscribed polygonal line is also closed.

**3.2. Exercise.** Assume  $\alpha: [a, b] \rightarrow \mathbb{R}^2$  is a smooth curve. Show that  
(a)  $\text{length } \alpha \geq \int_a^b |\alpha'(t)| \cdot dt$ ,

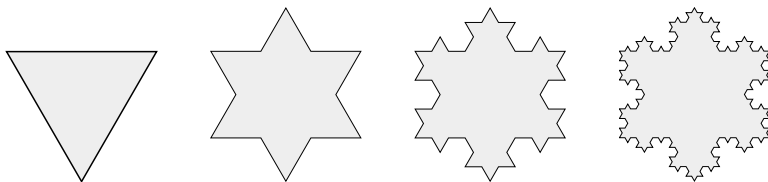
(b)  $\text{length } \alpha \leq \int_a^b |\alpha'(t)| \cdot dt$ .  
 Conclude that

$$\text{length } \alpha = \int_a^b |\alpha'(t)| \cdot dt.$$

*Hints:* For (a), apply the fundamental theorem of calculus for each segment in a given partition. For (b) consider a partition such that the velocity vector  $\alpha'(t)$  is nearly constant on each of its segments.

**Nonrectifiable curves.** A classical example of a nonrectifiable curve is the so called *Koch snowflake*; it is a fractal curve that can be constructed the following way:

Start with an equilateral triangle, divide each of its side into three segments of equal length and add an equilateral triangle with base at the middle segment. Repeat this construction recursively to the obtained polygons. Few first iterations of the construction are shown



on the diagram. The Koch snowflake is the boundary of the union of all the polygons.

### 3.3. Exercise.

- (a) Show that Koch snowflake is a closed simple curve; that is, it admits a homeomorphism to a circle.
- (b) Show that Koch snowflake is not rectifiable.

## Arc length parametrization

We say that a parametrized curve  $\gamma$  has an *arc length parametrization*<sup>1</sup> if for any two values of parameters  $t_1 < t_2$ , the value  $t_2 - t_1$  is the length of  $\gamma|_{[t_1, t_2]}$ ; that is, the closed arc of  $\gamma$  from  $t_1$  to  $t_2$ .

Note that a smooth space curve  $\gamma(t) = (x(t), y(t), z(t))$  has arc length parametrization if and only if it has unit velocity vector at all times; that is

$$|\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = 1;$$

---

<sup>1</sup>which is also called *natural parametrization*

by that reason arc length parametrization of smooth curves with also called *unit-speed curves*. Note that smooth unit-speed curves are automatically regular.

Any rectifiable curve can be parameterized by arc length. For a parametrized smooth curve  $\gamma$ , the arc length parameter  $s$  can be written as an integral

$$s(t) = \int_{t_0}^t |\gamma'(\tau)| \cdot d\tau.$$

Note that  $s(t)$  is a smooth increasing function. Further by fundamental theorem of calculus,  $s'(t) = |\gamma'(t)|$ . Therefore if  $\gamma$  is regular, then  $s'(t) \neq 0$  for any parameter value  $t$ . By inverse function theorem (A.1) the inverse function  $s^{-1}(t)$  is also smooth. Therefore after the reparametrization of  $\gamma$  by arclength  $s$  it remains smooth and regular.

We will be interested in the properties of curves that are invariant under a reparametrization. Therefore we can always assume that the given smooth regular curve comes with a arc length parametrization. A good property of arc length parametrizations is that it is almost canonical — these parametrizations differ only by a sign and additive constant. On the other hand, often it is impossible to find an arc length parametrization in a closed form which makes it hard to use it calculations; usually it is more convenient to use the original parametrization.

**3.4. Exercise.** Reparametrize the helix  $\gamma_a(t) = (a \cdot \cos t, a \cdot \sin t, t)$  by arc length.

## Convex curves

A closed simple plane curve is called *convex* if it bounds a convex region.

**3.5. Proposition.** Assume a convex curve  $\alpha$  lies inside the domain bounded by a closed simple plane curve  $\beta$ . Then

$$\text{length } \alpha \leq \text{length } \beta.$$

Note that it is sufficient to show that for any polygon  $P$  inscribed in  $\alpha$  there is a polygon  $Q$  inscribed in  $\beta$  with  $\text{perim } P \leq \text{perim } Q$ , where  $\text{perim } P$  denotes the perimeter of  $P$ .

Therefore it is sufficient to prove the following lemma.

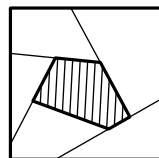
**3.6. Lemma.** *Let  $P$  and  $Q$  be polygons. Assume  $P$  is convex and  $Q \supset P$ . Then*

$$\text{perim } P \leq \text{perim } Q.$$

*Proof.* Note that by the triangle inequality, the inequality

$$\text{perim } P \leq \text{perim } Q$$

holds if  $P$  can be obtained from  $Q$  by cutting it along a chord; that is, a line segment with ends on the boundary of  $Q$  that lies in  $Q$ .



Note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \cdots \subset P_n = Q$$

such that  $P_{i-1}$  obtained from  $P_i$  by cutting along a chord. Therefore

$$\begin{aligned} \text{perim } P = \text{perim } P_0 &\leq \text{perim } P_1 \leq \cdots \\ &\leq \text{perim } P_n = \text{perim } Q \end{aligned}$$

and the lemma follows.  $\square$

**3.7. Corollary.** *Any convex closed plane curve is rectifiable.*

*Proof.* Any closed curve is bounded; that is, it lies in a sufficiently large square. Indeed the curve can be described as an image of a loop  $\alpha: [0, 1] \rightarrow \mathbb{R}^2$ ,  $\alpha(t) = (x(t), y(t))$ . The coordinate functions  $x(t)$  and  $y(t)$  are continuous functions defined on  $[0, 1]$ . Therefore the absolute values of both of these functions are bounded by some constant  $C$ . That is  $\alpha$  lies in the square defined by the inequalities  $|x| \leq C$  and  $|y| \leq C$ .

By Proposition 3.5, the length of the curve can not exceed the perimeter of the square  $8 \cdot C$ , whence the result.  $\square$

Recall that convex hull of a set  $X$  is the smallest convex set that contains  $X$ ; in other words convex hull is the intersection of all convex sets containing  $X$ .

**3.8. Exercise.** *Let  $\alpha$  be a closed simple plane curve. Denote by  $K$  the convex hull of  $\alpha$ ; let  $\beta$  be the boundary curve of  $X$ . Show that*

$$\text{length } \alpha \geq \text{length } \beta.$$

*Try to show that the statement holds for arbitrary closed plane curve  $\alpha$ , assuming that  $X$  has nonempty interior.*

## Semicontinuity of length

Recall that the lower limit of a sequence of real numbers  $(x_n)$  is denoted by

$$\varliminf_{n \rightarrow \infty} x_n.$$

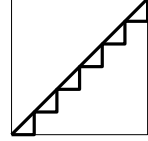
It is defined as the lowest partial limit; that is, the lowest possible limit of a subsequence of  $(x_n)$ . The lower limit is defined for any sequence of real numbers and it lies in the extended real line  $[-\infty, \infty]$

**3.9. Theorem.** *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

*More precisely, assume that a sequence of curves  $\alpha_n: [a, b] \rightarrow \mathcal{X}$  in a metric space  $\mathcal{X}$  converges pointwise to a curve  $\alpha_\infty: [a, b] \rightarrow \mathcal{X}$ ; that is,  $\alpha_n(t) \rightarrow \alpha_\infty(t)$  for any fixed  $t \in [a, b]$  as  $n \rightarrow \infty$ . Then*

$$\textcircled{1} \quad \varliminf_{n \rightarrow \infty} \text{length } \alpha_n \geq \text{length } \alpha_\infty.$$

Note that the inequality  $\textcircled{1}$  might be strict. For example the diagonal  $\alpha_\infty$  of the unit square can be approximated by a sequence of stairs-like polygonal curves  $\alpha_n$  with sides parallel to the sides of the square ( $\alpha_6$  is on the picture). In this case



$$\text{length } \alpha_\infty = \sqrt{2} \quad \text{and} \quad \text{length } \alpha_n = 2$$

for any  $n$ .

*Proof.* Fix a partition  $a = t_0 < t_1 < \cdots < t_k = b$ . Set

$$\Sigma_n := |\alpha_n(t_0) - \alpha_n(t_1)| + \cdots + |\alpha_n(t_{k-1}) - \alpha_n(t_k)|.$$

$$\Sigma_\infty := |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + \cdots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|.$$

Note that  $\Sigma_n \rightarrow \Sigma_\infty$  as  $n \rightarrow \infty$  and  $\Sigma_n \leq \text{length } \alpha_n$  for each  $n$ . Hence

$$\textcircled{2} \quad \varliminf_{n \rightarrow \infty} \text{length } \alpha_n \geq \Sigma_\infty.$$

If  $\alpha_\infty$  is rectifiable, we can assume that

$$\text{length } \alpha_\infty < \Sigma_\infty + \varepsilon.$$

for any given  $\varepsilon > 0$ . By  $\textcircled{2}$  it follows that

$$\varliminf_{n \rightarrow \infty} \text{length } \alpha_n > \text{length } \alpha_\infty - \varepsilon$$

for any  $\varepsilon > 0$ ; whence ❶ follows.

It remains to consider the case when  $\alpha_\infty$  is not rectifiable; that is length  $\alpha_\infty = \infty$ . In this case we can choose a partition so that  $\Sigma_\infty > L$  for any real number  $L$ . By ❷ it follows that

$$\varliminf_{n \rightarrow \infty} \text{length } \alpha_n > L$$

for any  $L$ ; whence  $\varliminf_{n \rightarrow \infty} = \infty$  and ❶ follows.  $\square$

## Length metric

Let  $\mathcal{X}$  be a metric space. Given two points  $x, y$  in  $\mathcal{X}$ , denote by  $d(x, y)$  the exact lower bound for lengths of all paths connecting  $x$  to  $y$ ; if there is no such path we assume that  $d(x, y) = \infty$ .

Note that function  $d$  satisfies all the axioms of metric except it might take infinite value. Therefore if any two points in  $\mathcal{X}$  can be connected by a rectifiable curve, then  $d$  defines a new metric on  $\mathcal{X}$ ; in this case  $d$  is called *induced length metric*.

Evidently  $d(x, y) \geq |x - y|$  for any pair of points  $x, y \in \mathcal{X}$ . If the equality holds for any pair, then the metric is called *length metric* and the space is called *length-metric space*.

Most of the time we consider length-metric spaces. In particular the Euclidean space is a length-metric space. A subspace  $A$  of length-metric space  $\mathcal{X}$  might be not a length-metric space; the induced length distance between points  $x$  and  $y$  in the subspace  $A$  will be denoted as  $|x - y|_A$ ; that is  $|x - y|_A$  is the exact lower bound for the length of paths in  $A$ .

**3.10. Exercise.** Let  $A \subset \mathbb{R}^3$  be a closed subset. Show that  $A$  is convex if and only if

$$|x - y|_A = |x - y|_{\mathbb{R}^3}.$$

**3.11. Exercise.** Let us denote by  $\mathbb{S}^1$  the unit circle in the plane; that is,

$$\mathbb{S}^1 = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}.$$

Show that

$$|u - v|_{\mathbb{S}^1} = \angle(u, v) := \arccos \langle u, v \rangle.$$

## Spherical curves

**3.12. Exercise.** Let us denote by  $\mathbb{S}^2$  the unit sphere in the space; that is,

$$\mathbb{S}^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}.$$

Show that

$$|u - v|_{\mathbb{S}^2} = \angle(u, v) := \arccos \langle u, v \rangle.$$

*Hint:* Use Exercise 3.11 and the following map  $f: (r, \theta, \varphi) \mapsto (r, \theta, 0)$  in spherical coordinates. Note that  $f$  is distance nonexpanding and it maps  $\mathbb{R}^3$  to a half-plane and  $\mathbb{S}^2$  to one of its meridians.

**3.13. Hemisphere lemma.** Any closed curve of length  $< 2 \cdot \pi$  in  $\mathbb{S}^2$  lies in an open hemisphere.

*Proof.* Let  $\alpha$  be a closed curve in  $\mathbb{S}^2$  of length  $2 \cdot \ell$ .

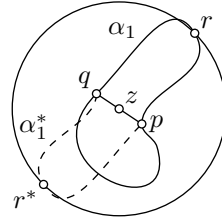
Assume  $\ell < \pi$ .

Let us divide  $\alpha$  into two arcs  $\alpha_1$  and  $\alpha_2$  of length  $\ell$ , with endpoints  $p$  and  $q$ . According to Exercise 3.12,  $\angle(p, q) \leq \ell < \pi$ . Denote by  $z$  be the midpoint between  $p$  and  $q$  in  $\mathbb{S}^2$ ; that is  $z$  is the midpoint of an equator arc from  $p$  to  $q$ . We claim that  $\alpha$  lies in the open north hemisphere with north pole at  $z$ . If not,  $\alpha$  intersects the equator in a point, say  $r$ . Without loss of generality we may assume that  $r$  lies on  $\alpha_1$ .

Rotate the arc  $\alpha_1$  by angle  $\pi$  around the line thru  $z$  and the center of the sphere. The obtained arc  $\alpha_1^*$  together with  $\alpha_1$  forms a closed curve of length  $2 \cdot \ell$  that passes thru  $r$  and its antipodal point  $r^*$ . Therefore

$$\frac{1}{2} \cdot \text{length } \alpha = \ell \geq \angle(r, r^*) = \pi,$$

a contradiction. □



The north hemisphere corresponds to the disc and the south hemisphere — its complement.

**3.14. Exercise.** Consider the following problem, find a flaw in the given wrong solution. Come up with a correct argument.

**Problem.** Suppose that a closed plane curve  $\alpha$  has length at most 4. Show that  $\alpha$  lies in a unit disc.



*Wrong solution.* Note that it is sufficient to show that diameter of  $\alpha$  is at most 2; that is the distance between any two pairs of points  $p$  and  $q$  of  $\alpha$  can not exceed 2.

The length of  $\alpha$  can not be smaller than the closed inscribed polygonal line which goes from  $p$  to  $q$  and back to  $p$ . Therefore

$$2 \cdot |p - q| \leq \text{length } \alpha \leq 4.$$

□

# Appendix A

## Preliminaries

In this chapter we state and discuss results from different branches of mathematics which were used further in the book. The reader is not expected to know proofs of these statements, but it is better to check that his intuition agrees with each.

### Multivariable calculus

A map  $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^k$  can be thought as array of functions

$$f_1, \dots, f_k: \mathbb{R}^n \rightarrow \mathbb{R}.$$

The map  $\mathbf{f}$  is called *smooth* if each function  $f_i$  is smooth; that is, all partial derivatives of  $f_i$  are defined in the domain of definition of  $\mathbf{f}$ .

Inverse function theorem gives a sufficient condition for a smooth function to be invertible in a neighborhood of a given point  $p$  in its domain. The condition is formulated in terms of partial derivative of  $f_i$  at  $p$ .

Implicit function theorem is a close relative to inverse function theorem; in fact it can be obtained as its corollary. It is used for instance when we need to pass from parametric to implicit description of curves and surface.

Both theorems reduce the existence of a map satisfying certain equation to a question in linear algebra. We use these two theorems only for  $n \leq 3$ .

These two theorems are discussed in any course of multivariable calculus, the classical book of Walter Rudin [1] is one of my favorites.

**1.1. Inverse function theorem.** *Let  $\mathbf{f} = (f_1, \dots, f_n): \mathbb{R}^n \rightarrow \mathbb{R}^n$*

be a smooth map. Assume that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is invertible at some point  $p$  in the domain of definition of  $\mathbf{f}$ . Then there is a smooth function  $\mathbf{h}: \mathbb{R}^m \rightarrow \mathbb{R}^n$  defined in a neighborhood  $\Omega_q$  of  $q = \mathbf{f}(p)$  that is local inverse of  $\mathbf{f}$  at  $p$ ; that is, there are neighborhoods  $\Omega_p \ni p$  such that  $\mathbf{f}$  defines a bijection  $\Omega_p \rightarrow \Omega_q$  and  $\mathbf{f}(x) = y$  if and only if  $x = \mathbf{h}(y)$  for any  $x \in \Omega_p$  and any  $y \in \Omega_q$ .

**1.2. Implicit function theorem.** Let  $\mathbf{f} = (f_1, \dots, f_n): \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  be a smooth map,  $m, n \geq 1$ . Let us consider  $\mathbb{R}^{n+m}$  as a product space  $\mathbb{R}^n \times \mathbb{R}^m$  with coordinates  $x_1, \dots, x_n, y_1, \dots, y_m$ . Consider the following matrix

$$M = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

formed by first  $n$  columns of the Jacobian matrix. Assume  $M$  is invertible at some point  $p$  in the domain of definition of  $\mathbf{f}$  and  $\mathbf{f}(p) = 0$ . Then there is a neighborhood  $\Omega_p \ni p$  and smooth function  $\mathbf{h}: \mathbb{R}^m \rightarrow \mathbb{R}^n$  defined in a neighborhood  $\Omega_0 \ni 0$  that for any  $(x_1, \dots, x_n, y_1, \dots, y_m) \in \Omega_p$  the equality

$$\mathbf{f}(x_1, \dots, x_n, y_1, \dots, y_m) = 0$$

holds if and only if

$$(x_1, \dots, x_n) = \mathbf{h}(y_1, \dots, y_m).$$

If the assumption in the theorem holds for any point  $p$  such that  $\mathbf{f}(p) = 0$ , then we say that  $0$  is a regular value of  $\mathbf{f}$ . Sard's theorem states that most of the values of smooth map are regular; in particular generic smooth function satisfies the assumption of the theorem.

## Rademacher's theorem

Recall that a function  $f$  is called Lipschitz if there is a constant  $L$  such that

$$|f(x) - f(y)| \leq L \cdot |x - y|$$

for values  $x$  and  $y$  in the domain of definition of  $f$ . This definition works for maps between metric spaces, but we will use it for real-to-real functions only.

**1.3. Rademacher's theorem.** *Let  $f: [a, b] \rightarrow \mathbb{R}$  be a Lipschitz function. then derivative  $f'(x)$  is defined for almost all  $x \in [a, b]$ . Moreover the derivative is bounded and Lebesgue-integrable and satisfies the fundamental theorem of calculus; that is, the following identity*

$$f(b) - f(a) = \int_a^b f'(x) \cdot dx,$$

*holds if the integral understood in the sense of Lebesgue.*

## Fundamental theorem of ODE

Picard theorem or the fundamental theorem of ordinary differential equations; it guarantees existence and uniqueness of a solution of an initial value problem for a system of ordinary differential equations

$$\begin{cases} x'_1 = f_1(x_1, \dots, x_n), \\ \dots \\ x'_n = f_n(x_1, \dots, x_n), \end{cases}$$

The array functions  $(f_1, \dots, f_n)$  can be considered as one vector-valued function  $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  and the array  $(x_1, \dots, x_n)$  can be considered as a vector  $\mathbf{x} \in \mathbb{R}^n$ . Therefore the system can be rewritten as one vector equation

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

We use only the following partial case of this theorem.

**1.4. Theorem.** *Suppose  $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a smooth function. Then for any initial data  $\mathbf{x}(0) = \mathbf{u}$  the differential equation*

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})$$

*has a unique solution  $\mathbf{x}(t)$  defined at some open interval containing zero. Moreover the function  $(\mathbf{u}, t) \mapsto \mathbf{x}(t)$  is smooth.*

# Appendix B

## Homework assignments

**HWA01.** Exercises: 1.2, 2.4, 2.5, 2.6, 3.10.

**HWA02.** Exercises: 3.3(b), 3.4, 3.8, 3.11, 3.14.

# Bibliography

- [1] Walter Rudin. *Principles of mathematical analysis*. 1976.