Chapter 1

Isoperimetric inequality

For any plane figure F with perimeter ℓ, its area a satisfies the following inequality:

$$4\pi\cdot a \leqslant \ell^2. \quad ❶$$

Moreover the equality holds iff F is congruent to a round disk.

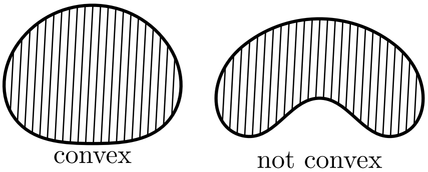
This is the so-called *isoperimetric inequality* on the plane. Let us restate it without formulas, using the comparison language.

1.1. Isoperimetric inequality. *The area of plane figure bounded by a closed curve of length ℓ can not exceed the area of round disk with the same circumference ℓ. Moreover the equality holds only if the figure is congruent to the disk.*

The comparison reformulation has some advantages — it is more intuitive and it is also easier to generalize.

1.2. Exercise. *Come up with a formulation of the isoperimetric inequality on the unit sphere. Try to reformulate it as an algebraic inequality similar to ❶.*

Recall that a plane figure F is called *convex* if for every pair of points $x, y \in F$, the line segment $[x, y]$ that joins the pair of points lies also in F.



The following exercise reduces the isoperimetric inequality to the case of convex figures:

1.3. Exercise. *Assume F is a plane figure bounded by a closed curve of length ℓ. Show that there is a convex figure $F' \supset F$ bounded by a closed curve of length at most ℓ.*

The following problem is named after Dido, the legendary founder and first queen of Carthage.

1.4. Dido's problem. *The figure of maximal area bounded by a straight line and a curve of given length with endpoints on that line is a half-disk.*

1.5. Exercise. *Show that Dido's problem follows from the isoperimetric inequality and the other way around.*

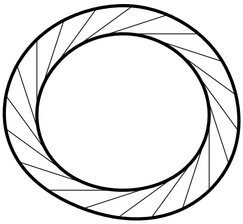
1.6. Exercise. *Use the isoperimetric inequality in the plane to show that among all the polygons with the given sides, the convex polygons inscribed in a circle have maximal area.*

1.7. Exercise. *Find the minimal length of a curve that divides the unit square in a given ratio α.*

1.1 Lawlor's proof

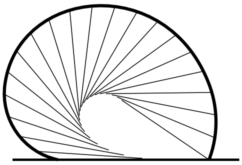
Here we present a sketch of the proof of Dido's problem based of the idea of Gary Lawlor in [1]. Before getting into the proof, try to solve the following exercise.

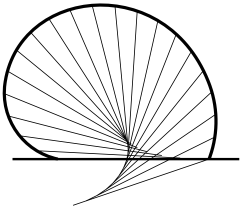
1.8. Exercise. *An old man walks along a trail around a convex meadows and pulls a brick tied to a rope of unit length (the rope is always strained). After walking around he noticed that the brick is at the same position as at the beginning. Show that the area between the trail and the path of the brick equals the area of the unit disk.*

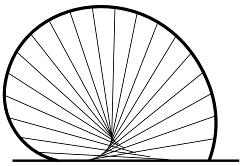


Sketch of the proof. Let F be a convex figure bounded by a line and a curve $\gamma(t)$ of length ℓ; we can assume that γ is a unit speed curve so the set of parameters is $[0, \ell]$.

Imagine that we are walking along the curve with a stick of length r so that the other end of the stick drags on the flow. Assume that initially at $t = 0$ the stick points in the direction of $\gamma(\ell)$ — the other end of γ.







Note that if r is small then most of the time we drag the stick behind. Therefore at the end of the walk the stick will have made more than half turn and will point to the same side of the figure.

Let R be the radius of the half-circle $\tilde{\gamma}(t)$ of length ℓ. Assume we walk along $\tilde{\gamma}$ with a stick of length R the same way as described above. Note that the other end does not move (it always lies in the center) and the direction of the stick changes with rate $\frac{1}{R}$. Note further that for γ this rate is at most $\frac{1}{R}$. Therefore after walking along γ, the stick of length R will rotate at most as much as if we would walk along $\tilde{\gamma}$.

It follows that there is a positive value $r \leqslant R$ such that after walking along γ with a stick of length r, it will rotate exactly half turn, so at the end it will point towards $\gamma(0)$.

Imagine that the stick of length r is covered with red paint and it paints the area below it. If we move with velocity v then the angular velocity (radians per second) of the stick is at most $\frac{v}{r}$ and the equality holds if we move perpendicularly to the stick. Therefore we paint at a rate of at most $\frac{1}{2}v{\cdot}r$.

If we do the same for the half-disk of radius R and a stick of length R with blue paint, then we paint the area of the disk without overlap with the rate $\frac{1}{2}v{\cdot}R$. Since $r \leqslant R$, the total red-painted area can not exceed the blue-painted area, that is, D.

It remains to show that all F is red-painted. Fix a point $p \in F$. Notice that at the beginning the point p lies on the left from the stick and at the end it lies on the right form it. Therefore there will be a moment t_0 when the side changes from left to right. At this time the point must be on the line containing the stick. Moreover, if it lies on the extension then the side changes from right to left. Therefore p has to lie under the stick; that is, p is painted.

$\square$

1.9. Exercise. *Find the places with cheating in the proof above and try to fix them.*

1.10. Exercise. *Read and understand the original proof of Gary Lawlor in [1].*

Chapter 2

Length

The material of this and the following chapters overlaps largely with [2, Chapter 5].

2.1 Length of curve

2.1. Definition. *Consider a* plane curve $\alpha\colon [a,b] \to \mathbb{R}^2$*; that is α is a continuous mapping from the real interval $[a,b]$ to the Euclidean plane $\mathbb{R}^2$.*

If $\alpha(a) = p$ and $\alpha(b) = q$, we say that α is a curve from p to q*.*

A curve $\alpha\colon [a,b] \to \mathbb{R}^2$ is called closed *if $\alpha(a) = \alpha(b)$.*

A curve α called simple *if it is described by an injective map; that is $\alpha(t) = \alpha(t')$ if and only if $t = t'$. However, a closed curve $\alpha\colon [a,b] \to \mathbb{R}^2$ is called simple if it is injective everywhere except the ends; that is, if $\alpha(t) = \alpha(t')$ for $t < t'$ then $t = a$ and $t' = b$.*

Recall that a sequence

$$a = t_0 < t_1 < \dots < t_k = b.$$

is called *partition* of interval $[a,b]$.

2.2. Definition. *Let $\alpha\colon [a,b] \to \mathbb{R}^2$ be a curve. The* length *of α is defined as*

$$\begin{aligned}\operatorname{length}\alpha = \sup\{|\alpha(t_0) - \alpha(t_1)| + |\alpha(t_1) - \alpha(t_2)| + \dots \\ \dots + |\alpha(t_{k-1}) - \alpha(t_k)|\}.\end{aligned}$$

where the exact upper bound is taken over all partitions

$$a = t_0 < t_1 < \dots < t_k = b.$$

Note that length $\alpha \in [0, \infty]$*; the curve α is called* rectifiable *if its length is finite.*

Informally, one could say that the length of curve is the exact upper bound of lengths of polygonal lines *inscribed* in the curve.

2.3. Exercise. *Assume $\alpha\colon [a,b] \to \mathbb{R}^2$ is smooth curve, in particular the velocity vector $\alpha'(t)$ is defined and depends continuously on t. Show that*

$$\operatorname{length}\alpha = \int_a^b |\alpha'(t)|\cdot dt.$$

2.4. Exercise. *Construct a nonrectifiable curve $\alpha\colon [0,1] \to \mathbb{R}^2$.*

A closed simple plane curve is called *convex* if it bounds a convex figure.

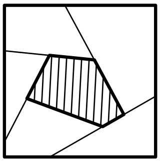
2.5. Proposition. *Assume a convex figure A bounded by a curve α lies in a figure B bounded by a curve β. Then*

$$\operatorname{length}\alpha \leqslant \operatorname{length}\beta.$$

Note that it is sufficient to show that for any polygon P inscribed in α there is a polygon Q inscribed in β such that $\operatorname{perim} P \leqslant \operatorname{perim} Q$, where $\operatorname{perim} P$ denotes the perimeter of P.

Therefore it is sufficient to prove the following lemma.

2.6. Lemma. *Let P and Q be polygons. Assume P is convex and $Q \supset P$. Then* $\operatorname{perim} P \leqslant \operatorname{perim} Q$.



Proof. Note that by triangle inequality, the inequality

$$\operatorname{perim} P \leqslant \operatorname{perim} Q$$

holds if P can be obtained from Q by cutting it along a chord; that is, a line segment with ends on the boundary of Q that lies in Q.

Note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \cdots \subset P_n = Q$$

such that P_{i-1} obtained from P_i by cutting along a chord. Therefore

$$\begin{aligned}\operatorname{perim} P = \operatorname{perim} P_0 \leqslant \operatorname{perim} P_1 \leqslant \dots\\ \dots \leqslant \operatorname{perim} P_n = \operatorname{perim} Q\end{aligned}$$

and the lemma follows. □

2.7. Corollary. *Any convex closed curve is rectifiable.*

Proof. Fix a curve $\alpha\colon [a,b]\to\mathbb{R}^2$. Note that α is bounded; indeed

Any closed curve is bounded; that is, it lies in a sufficiently large square.

By Proposition 2.5, the length of the curve can not exceed the perimeter of the square, hence the result. □

AFTER THIS LINE READ AT YOUR OWN RISK!!!

2.2 Semicontinuity of length

Recall that lower limit of a sequence of real numbers (x_n) is denoted by

$$\varliminf_{n\to\infty} x_n.$$

It is defined as the lowest partial limit; that is a the lowest possible limit of a subsequence of (x_n). The lower limit is defined for any sequence of real numbers and it takes value in the exteded real line $[-\infty,\infty]$

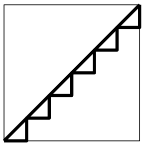
2.8. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\alpha_n\colon[a,b]\to\mathbb{R}^2$ converges pointwise to a curve $\alpha_\infty\colon[a,b]\to\mathbb{R}^2$; that is, $\alpha_n(t)\to\alpha_\infty(t)$ for any fixed $t\in[a,b]$ and $n\to\infty$. Then

❶ $$\varliminf_{n\to\infty} \operatorname{length}\alpha_n \geqslant \operatorname{length}\alpha_\infty.$$

Note that the inequality ❶ might be strict. For example the diagonal of unit square α_∞

can be approximated by a sequence of stairs-like polygonal curves α_n with sides parallel to the sides of the square (α_6 is on the picture). In this case



$$\operatorname{length}\alpha_\infty=\sqrt2 \quad\text{and}\quad \operatorname{length}\alpha_n=2$$

for any n.

Proof. Fix $\varepsilon>0$ and choose a sequence $a=t_0<t_1<\dots<t_k=b$ such that

$$\operatorname{length}\alpha_\infty<|\alpha_\infty(t_0)-\alpha_\infty(t_1)|+\dots+|\alpha_\infty(t_{k-1})-\alpha_\infty(t_k)|+\varepsilon.$$

Set

$$\Sigma_n := |\alpha_n(t_0) - \alpha_n(t_1)| + \cdots + |\alpha_n(t_{k-1}) - \alpha_n(t_k)|.$$
$$\Sigma_\infty := |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + \cdots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|.$$

Note that $\Sigma_n \to \Sigma_\infty$ as $n \to \infty$ and $\Sigma_n \leqslant \operatorname{length} \alpha_n$ for each n. Hence

$$\varliminf_{n\to\infty} \operatorname{length} \alpha_n \geqslant \operatorname{length} \alpha_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get ❶. □

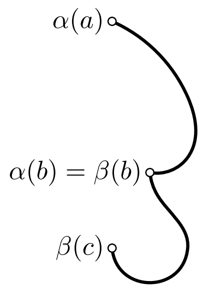
2.3 Axioms of length

Concatenation. Assume $\alpha\colon [a,b] \to \mathbb{R}^2$ and $\beta\colon [b,c] \to \mathbb{R}^2$ are two curves such that $\alpha(b) = \beta(b)$. Then one can combine these two curves in one $\gamma\colon [a,c] \to \mathbb{R}^2$ by assuming that $\gamma(t) = \alpha(t)$ for $t \leqslant b$ and $\gamma(t) \geqslant \beta(t)$ for $t \geqslant b$. The obtained curve γ is called the *concatenation* of α and β which can be written as $\gamma = \alpha * \beta$.

Note that

$$\operatorname{length}(\alpha * \beta) = \operatorname{length} \alpha + \operatorname{length} \beta$$

for any two curves α and β such that the concatenation $\alpha * \beta$ is defined.



Reparametrization. Assume $\alpha\colon [a,b] \to \mathbb{R}^2$ is a curve and $\tau\colon [c,d] \to [a,b]$ is a continuous strictly monotonic onto map. Consider the curve $\alpha'\colon [c,d] \to \mathbb{R}^2$ defined by $\alpha' = \alpha \circ \tau$. The curves α' is called *reparametrization* of α.

Note that

$$\operatorname{length} \alpha' = \operatorname{length} \alpha$$

if α' is a reparametrization of α.

2.9. Proposition. *Let ℓ be a functional that returns a value in $[0,\infty]$ for any curve $\alpha\colon [a,b] \to \mathbb{R}$.*

Assume it satisfies the following properties:

(i) (Normalization) If $\alpha\colon [a,b] \to \mathbb{R}^2$ is a linear curve,[1] then

$$\ell(\alpha) = |\alpha(a) - \alpha(b)|.$$

(ii) (Additivity) If the concatenation $\alpha * \beta$ is defined, then

$$\ell(\alpha * \beta) = \ell(\alpha) + \ell(\beta).$$

[1]That is $\alpha = w + v{\cdot}t$ for some vectors w and v.

(iii) *(Motion invariance) The functional ℓ is invariant with respect to the motions of the plane; that is, if m is a motion then*

$$\ell(m\circ\alpha)=\ell(\alpha)$$

for any curve α.

(iv) *(Reparametrization invariance) If α' is a reparametrization of a curve α then*

$$\ell(\alpha')=\ell(\alpha).$$

(In fact linear reparametrizations will be sufficient.)

(v) *(Semi-continuity) If a sequence of curves $\alpha_n\colon[a,b]\to\mathbb{R}^2$ converges to a curve pointwise to a curve $\alpha_\infty\colon[a,b]\to\mathbb{R}^2$, then*

$$\varliminf_{n\to\infty}\ell(\alpha_n)\geqslant\ell(\alpha_\infty).$$

Then

❷ $$\ell(\alpha)=\operatorname{length}\alpha$$

for any plane curve α.

Proof. Note that from normalization and additivity, the identity

❸ $$\ell(\beta)=\operatorname{length}\beta$$

holds for any polyhonal line β that is linear on each edge.

Note that the following two inequalities

❹ $$\ell(\alpha)\leqslant\operatorname{length}\alpha$$

❺ $$\ell(\alpha)\geqslant\operatorname{length}\alpha$$

imply ❷; we will prove them separately.

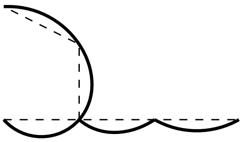
Fix a curve $\alpha\colon[a,b]\to\mathbb{R}^2$ and a partition $a=t_0<t_1<\dots<$ $<t_k=b$. Consider the curve $\beta\colon[a,b]\to\mathbb{R}^2$ defined as a linear curve from $\alpha(t_i)$ to $\alpha(t_{i+1})$ on each segment $t\in[t_i,t_j]$. By the definition of length,

$$\operatorname{length}\beta\leqslant\operatorname{length}\alpha.$$

Since the map $\alpha\colon[a,b]\to\mathbb{R}^2$ is continuous, one can find a sequence of partitions of $[a,b]$ such that the corresponding curves β_n converge to α pointwise. Applying semi-continuity of ℓ, ❸ and the definition of length, we get that

$$\begin{aligned}\ell(\alpha)&\leqslant\varliminf_{n\to\infty}\ell(\beta_n)=\\&=\varliminf_{n\to\infty}\operatorname{length}\beta_n\leqslant\\&\leqslant\operatorname{length}\alpha.\end{aligned}$$

Hence ❹ follows.



Note that a curve $\alpha\colon [a,b] \to \mathbb{R}^2$ with a partition $a = t_0 < t_1 < \ldots < t_k = b$ can be considered as a concatenation

$$\alpha = \alpha_1 * \alpha_2 * \dots * \alpha_k$$

where α_i is the restriction of α to $[t_{i-1}, t_i]$.

Note that there is a sequence of motions m_i of the plane so that

$$m_i \circ \alpha(t_i) = m_{i+1} \circ \alpha(t_i)$$

for any i and the points

$$m_1 \circ \alpha(t_0), m_1 \circ \alpha(t_1), \dots m_k \circ \alpha(t_k)$$

appear on a line in the same order. For the concatenation

$$\gamma = (m_1 \circ \alpha_1) * (m_2 \circ \alpha_2) * \dots * (m_k \circ \alpha_k)$$

we have

$$\ell(\gamma) = \ell(\alpha).$$

Note that one can find a sequence of partitions of $[a,b]$ such that reparametrizationsof γ_n converge to linear curve γ'_∞; denote these reparametrizations by γ'_n. We can assume in addition that length $\gamma'_\infty =$ $=$ length α; indeed since γ'_∞ is linear,

$$\begin{aligned}\operatorname{length}\gamma'_\infty &= |\gamma'(a) - \gamma'(b)| = \\ &= \lim_{n\to\infty} \Sigma_n = \\ &= \operatorname{length}\alpha.\end{aligned}$$

where Σ_n is the sum in the definition of length for the n-th partition. Hence it is sufficient to choose a sequence of partitions such that $\Sigma_n \to$ $\to$ length α — by the definition of length this is possible.

Applying additivity, invariance of ℓ with respact to motions and reparametizations, we get that

$$\begin{aligned}\ell(\alpha) &= \lim_{n\to\infty} \ell(\gamma_n) = \\ &= \lim_{n\to\infty} \ell(\gamma'_n) \geqslant \\ &\geqslant \ell(\gamma'_\infty) = \\ &= \operatorname{length}\alpha.\end{aligned}$$

Hence ❺ follows. □

2.10. Exercise. *Construct a functional ℓ that is different from length and satisfies all the conditions in Proposition 2.9 except the semi-continuity.*

2.4 Crofton formula

Let α be a plane curve and u is a unit vector. Denote by α_u the orthogonal projection of α to a line ℓ in the direction of u; that is, $\alpha_u(t) \in \ell$ and $\alpha(t) - \alpha_u(t) \perp \ell$ for any t.

2.11. Crofton formula. *The length of any plane curve α is proportional to the average of lengths of its projection α_u for all unit vectors u. Moreover for any plane curve α we have*

$$\operatorname{length}\alpha = \tfrac{\pi}{2}\cdot\overline{\operatorname{length}\alpha_u},$$

where $\overline{\operatorname{length}\alpha_u}$ *denotes the average value of* $\operatorname{length}\alpha_u$.

Proof. First let us show that the formula

❻ $$\operatorname{length}\alpha = k\cdot\overline{\operatorname{length}\alpha_u},$$

holds for some fixed coefficient k. It will follow once we show that both sides of formula satisfies the length axioms in 2.9.

The normalization can be acheaved by adjusting k.

The semi-continuity of the right hand side follows since $\operatorname{length}\alpha_u$ is semi-continuous and therefore the average has to be semi-continuous.

It is straightforward to check the remaining properties.

It remains to find k. Let us apply the formula ❻ to the unit circle. The circle has length $2\cdot\pi$ and its projection to any line has length 4 — it is a segment of length 2 traveled back and forth. Evidently the average value is also 4. Therefore

$$2\cdot\pi = k\cdot 4$$

and therefore $k = \frac{\pi}{2}$. □

Reformulation via number of intersections. Given a pair a unit vector u and a real number ρ, consider the line of vectors w on the plane described by the equation

$$\langle u, w\rangle = \rho,$$

where $\langle u, w\rangle$ denotes the scalar product. Any line on the plane admits exactly two such presentations with pair (u, ρ) and $(-u, -\rho)$. A pair (u, ρ) describes uniquely an *oriented* line — that is a line with chosen unit normal vector.

Fix a unit vector u_0 and denote by $u(\varphi)$ its rotation by angle φ. Denote by $\ell(\varphi, \rho)$ the oriented line for the pair $(u(\varphi), \rho)$. To describe any line, we need a pair $(\varphi, \rho) \in (-\pi, \pi] \times \mathbb{R}$.

For a curve α set $n_\alpha(\varphi,\rho)$ to be the number of parameter values t such that $\alpha(t)$ lies on the line $\ell(\varphi,\rho)$. The value $n_\alpha(\varphi,\rho)$ has to be nonnegative integer or ∞. Note that if α is simple curve then $n_\alpha(\ell)$ is the number of intersections of α with ℓ.

2.12. An other Crofton formula. *For any curve α,*

$$\operatorname{length}\alpha = \tfrac{1}{4}\cdot \iint\limits_{(-\pi,\pi]\times\mathbb{R}} n_\alpha(\rho,\varphi)\cdot d\rho\cdot d\varphi.$$

the integral is to be understood in the sense of Lebesgue.

By definition of average value,

$$\overline{\operatorname{length}\alpha_u} = \frac{1}{2\cdot\pi}\cdot\int\limits_{-\pi}^{\pi}\operatorname{length}\alpha_{u(\varphi)}\cdot d\varphi.$$

Therefore proof of this reformulation of the Crofton follows from the following observation.

2.13. Observation. *If $u = u(\varphi)$, then*

$$\operatorname{length}\alpha_u = \int\limits_{\mathbb{R}} n_\alpha(\rho,\varphi)\cdot d\rho;$$

The proof is straightforward for those who understand Lebesgue integral.

Variations. The same argument can be used to derive other formulas of the same type. For example.

Recall that big circle in a sphere is the intersection of sphere with a plane passing thru its center. For example equator as well as any meridian are a big circles.

2.14. Spherical Crofton formula. *The length of any curve α in the unit sphere is π times the average number of its crossings with big circles.*

More presciently, given a unit vector u, denote by $n_\alpha(u)$ the number of crossings of α and the equator with pole at u. Then

$$\operatorname{length}\alpha = \overline{n_\alpha(u)}.$$

Equivalently,

$$\operatorname{length}\alpha = \overline{\operatorname{length}\alpha_u},$$

where α_u denotes the curve obtianed by cloasest point projection of α on the equator with pole at u.

2.15. Exercise. *Come up with a Crofton formulas for curves in the Euclidean space via projections to lines and to planes. Find the coefficients in these formulas.*

2.5 Applications

Alternative proof of Proposition 2.5. Note that

$$\operatorname{length}\beta_u \geqslant \operatorname{length}\alpha_u$$

for any unit vector u. Indeed α_u runs back and forth along a line segment and β_u has to run at least as much.

It follows that

$$\overline{\operatorname{length}\beta_u} \geqslant \overline{\operatorname{length}\alpha_u}.$$

It remains to apply Crofton formula. □

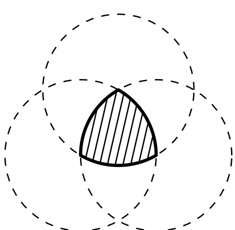
Recall that diameter of a plane figure F is defined as the least upper bound on the distances between pairs of its points; that is,

$$\operatorname{diam} F = \sup\{\,|x-y| \;:\; x,y\in F\,\}.$$

The equilateral triangle with side 1 gives an example of convex figure of diameter 1 that can not be covered by a round disc of diameter 1.

2.16. Exercise. *Assume F is a convex figure of diameter 1 and D is the round disc of diameter 1. Show that*

$$\operatorname{perim} F \leqslant \operatorname{perim} D.$$



A convex figure F has constant width a if the orthogonal projection of F to any line has length a. There are many non-circular shapes of constant width. A nontrivial example is the Reuleaux triangle shown on the picture; it is the intersection of three round disks, each having its center on the boundary of the other two. The following exercise is the so called Barbier's theorem.

2.17. Exercise. *Show that figures has constant width a have the same perimeter (which is equal to $\pi{\cdot}a$ — the perimeter of the round disc of diameter a).*

2.18. Exercise. *Let γ be a closed curve in the unit sphere of length smaller than $2{\cdot}\pi$. Show that γ lies in a hemisphere.*

2.19. Exercise. *Let α be a closed curve of length π. Show that it lies between a pair of parallel lines on distance 1 from each other.*

2.20. Exercise. *A spaceship flies around nonrotating planet or unit radius and came back to the original position; it was able to make a picture of every point on the surface of the planet.*

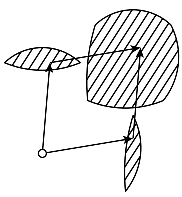
Try to use the Crofton formulas to get a lower bound on the length of its trajectory (does not need to be exact, but should be bigger than $2{\cdot}\pi$).

What do you think could be the shortest trajectory?

Hausdorff distance $d_H(F, G)$ between closed bounded sets F and G in the plane is defined as the exact lower bound on $\varepsilon > 0$ such that ε-neightborhood of F contains G and ε-neightborhood of G contains F.

2.21. Exercise. *Assume F and G be two closed convex figures on the plane such that $d_H(F, G) < \varepsilon$. Show that*

$$|\operatorname{perim} F - \operatorname{perim} G| < 2{\cdot}\pi{\cdot}\varepsilon.$$



The set C is called *Minkowski sum* of two sets A and B (briefly $C = A{+}B$) if C is formed by adding each vector in A to each vector in B; that is,

$$C = \{\, a + b \;:\; a \in A,\ b \in B \,\}.$$

Note that if A and B are convex then so is $C = A + B$.

Indeed, A is convex if and only if for any pair of points $a_0, a_1 \in A$ and any $t \in [0, 1]$, the point $a_t = (1 - t){\cdot}a_0 + t{\cdot}a_1$ belongs to A. Similarly, B is convex if and only if for any pair of points $b_0, b_1 \in B$ and any $t \in [0, 1]$, the point $b_t = (1 - t){\cdot}b_0 + t{\cdot}b_1$ belongs to A.

Fix a pair of points $c_0, c_1 \in C$; by the definition of Minkowski sum, there are two pairs of points $a_0, a_1 \in A$ and $b_0, b_1 \in B$ such that $c_0 = a_0 + b_0$ and $c_1 = a_1 + b_1$. Then

$$\begin{aligned} c_t &= (1 - t){\cdot}c_0 + t{\cdot}c_1 = \\ &= (1 - t){\cdot}(a_0 + b_0) + t{\cdot}(a_1 + b_1) = \\ &= [(1 - t){\cdot}a_0 + t{\cdot}a_1] + [(1 - t){\cdot}b_0 + t{\cdot}b_1] = \\ &= a_t + b_t. \end{aligned}$$

That is, $c_t \in C$ for any $t \in [0,1]$, hence the result.

2.22. Exercise. *Show that*

$$\operatorname{perim}(A+B) = \operatorname{perim} A + \operatorname{perim} B$$

for any pair of convex figures in the plane.

2.23. Exercise. *Let γ be a curve that lies in a convex figure F in the plane. Assume that*

$$2 \cdot \operatorname{length} \gamma \geqslant n \cdot \operatorname{perim} F$$

for some integer n. Show that there is a line ℓ that has γ at least n distinct points of intersection with γ.

Chapter 3

Curvature of curves

3.1 Total curvature

Here we introduce the so called *total curvature of curve.* In general term *curvature* is used for something that measures how a geometric object deviates from being a straight; total curvature is not an exception — as you will see if the total curvature of a curve is vanishing then the curve runs along a straight line.

Let $\alpha\colon [a,b] \to \mathbb{R}^3$ be a *smooth regular* curve — smooth means that the velocity vector $\alpha'(t)$ is defined and continuous with respect to t and regular means that $\alpha'(t) \neq 0$ for any t. If the curve α is closed then we assume in addition that $\alpha'(a) = \alpha'(b)$.

Denote by $\tau(t)$ the unit vector in the direction of $\alpha'(t)$; that is, $\tau(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$. The $\tau\colon [a,b] \to \mathbb{S}^2$ is an other curve which is called *tangent indicatrix* of α. The length of τ is called *total curvature of* α; that is,

$$\operatorname{TotCurv}\alpha := \operatorname{length}\tau.$$

3.1. Exercise. *Show that*

$$\operatorname{TotCurv}\alpha \geqslant 2{\cdot}\pi$$

for any smooth closed regular curve α.

Moreover, the equality holds if and only if α *is a closed and convex curve that lies in a plane.*

The above exercise is the so called Fenchel's theorem.

3.2 On inscribed broken lines

The total curvature of a polygonal line is defined as the sum of its external angles. More precisely, assume $p_0, \dots, p_n$ are the vertexes of a polygonal line then. The external angle at the vertex p_i is defined as $\alpha_i = \pi - \measuredangle p_{i-1}p_ip_{i+1}$. The the total curvature of the polygonal line $p_0 \dots p_n$ is defined as the sum

$$\alpha_1 + \dots + \alpha_{n-1}.$$

If the polygonal line $p_0 \dots p_n$ is closed; that is $p_0 = p_n$ you add one more angle

$$\alpha_0 + \alpha_1 + \dots + \alpha_{n-1},$$

where $\alpha_0 = \pi - \measuredangle p_np_0p_1$.

3.2. Exercise. *Let a, b, c, d and x be distinct points in $\mathbb{R}^3$. Show that*

$$\mathrm{TotCurv}\, abcd \geqslant \mathrm{TotCurv}\, abxcd.$$

3.3. Exercise. *Use Exercise ex:monotonic-tc to prove an analog of Fenchel's theorem (Exercise 3.1) for closed polygonal lines.*

Let $\alpha\colon [a, b] \to \mathbb{R}^3$ be a curve and $a = t_0 < \dots < t_n = b$ a partition. Set $p_i = \alpha(t_i)$. Then the polygonal line $p_0 \dots p_n$ is called inscribed in α.

We gave two definitions of total curvature: the first, via tangent indicatrix — it works for smooth regular curves; the second, via external angles — it works for polygonal lines. The following definition tells that these two definitions agree.

3.4. Theorem. *Total curvature of a smooth regular curve equals to the exact upper bound on the total curvature of inscribed polygonal lines; if the original curve is closed then the inscribed polygonal line is assumed to be closed as well.*

Appendix A

Semisolutions

Exercise 1.5. First let us show that Dido's problem follows from the isoperimetric inequality.

Assume F is a figure bounded by a straight line and a curve of length ℓ whose endpoints belong to that line. Let F' be the reflection of F in the line. Note that the union $G = F \cup F'$ is a figure bounded by a closed curve of length $2{\cdot}\ell$.

Applying the isoperimetric inequality, we get that the area of G can not exceed the area of round disc with the same circumference $2{\cdot}\ell$ and the equality holds only if the figure is congruent to the disc. Since F and F' are congruent, Dido's problem follows.

Now let us show that the isoperimetric inequality follows from the Dido's problem.

Assume G is a convex figure bounded by a closed curve of length $2{\cdot}\ell$. Cut G by a line that splits the perimeter in two equal parts — ℓ each. Denote by F and F' the two parts. Applying the Dido's problem for each part, we get that that are of each does not exceed the area of half-disc bounded by a half-circle. The two half-disc could be arranged into a round disc of circumference ℓ, hence the isoperimetric inequality follows.

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