Chapter 1

Isoperimetric inequality

AFTER THIS LINE READ AT YOUR OWN RISK!!!

For any plane figure F with perimeter ℓ, its area a satisfies the following inequality:

❶ $$4\pi\cdot a \leqslant \ell^2.$$

Moreover the equality holds if F is congruent to a round disc.

This is so-called *isoperimetric inequality* on the plane. Let us reformulate it without formulas, using the comparison language.

1.1. Isoperimetric inequality. *The area of plane figure bounded by a closed curve of length ℓ can not exceed the area of round disc with the same circumference ℓ. Moreover the equality holds only if the figure is congruent to the disc.*

The comparison reformulation has some advantages — it is more intuitive and it is also easier to generalize.

1.2. Exercise. *Come up with a formulation of isoperimetric inequality on the units sphere. Try to reformulate it as an algebraic inequality similar to* ❶*.*

Recall that a plane figure F is called *convex* if for every pair of points $x, y \in F$, the line segment $[x, y]$ that joins the pair of points lies also in F.

convex   not convex
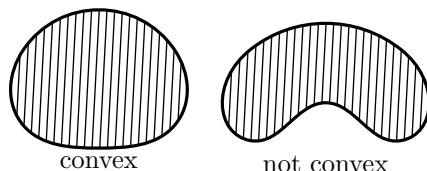
The following exercise reduces the isoperimetric inequality to the case of convex figures:

1.3. Exercise. *Assume F is a plane figure bounded by a closed curve of length ℓ. Show that there is a convex figure $F' \supset F$ bounded by a closed curve of length at most ℓ.*

The following problem is named after Dido, the legendary founder and first queen of Carthage.

1.4. Dido's problem. *A figure of the maximal area bounded by a straight line and a curve of given length with endpoints on that line is a half-disc.*

1.5. Exercise. *Show that Dido's problem follows from the isoperimetric inequality and the other way around.*

1.6. Exercise. *Use the isoperimetric inequality in the plane to show that inscribed convex polygons have maximal area among all the polygons with the given sides.*

1.7. Exercise. *Find the minimal length of curve that divides the unit square in the given ratio α.*

1.1   Lawlor's proof

Here we present a sketch of the proof of Dido's problem based of the idea of Gary Lawlor in [1]. Before getting into the proof, try to solve the following exercise.

1.8. Exercise. *An old man walks along a trail around a convex meadows and pulls a brick on a rope of unit length (the rope is always strained). After walking around he noticed that the brick is at the same position as at the beginning. Show that the area between the trail and the path of the brick equals to the area of the unit disc.*

Sketch of the proof. Let F be a convex figure bounded by a line and a curve $\gamma(t)$ of length ℓ; we can assume that γ is a unit speed curve so the set of parameters is $[0, \ell]$.

Imagine that we are walking along the curve with a stick of length r so that the other end of the stick drags on the flow. Assume that at initially $t = 0$ the stick points in the direction of $\gamma(\ell)$ — the other end of γ.

Note that is r is small then most of the time we drag the stick behind. Therefore at the end of walk the stick will make more than half turn and will point to the same side of the figure.

Let R be the radius of half-circle $\tilde{\gamma}(t)$ of length ℓ. Assume we walk along $\tilde{\gamma}$ with a stick as or length R the same way as described above. Note that the other end does not move (it always lies in the center) and the direction of stick changes with rate $\frac{1}{R}$. Note further for γ this rate would be at most $\frac{1}{R}$. Therefore after walking along γ, the stick of length R will rotate at most as much as if we would walk along $\tilde{\gamma}$.

It follows that there is a positive value $r \leqslant$ $\leqslant R$ such that after walking along γ, a stick of length r will rotate exactly half turn, so it will point in the direction of $\gamma(0)$.

Imagine that the stick of length r is covered with red paint and it paints the area below it. If we move with velocity v then angular velocity (radians per second) of the stick is at most $\frac{v}{r}$ the equality hold if we move perpendicularly to the stick. Therefore we paint at the rate at most most $\frac{1}{2}v{\cdot}r$.

If we do the same for the half-disc of radius R and stick of length R with blue paint, then we paint area of disc without overlap with the rate $\frac{1}{2}v{\cdot}R$. Since $r \leqslant R$, the total red-painted area can not exceed the blue-painted area, that is, D.

It remains to show that all F is red-painted. Fix a point $p \in F$. Notice that at the beginning the point p lies on the left from the stick and at the end it lies on the right form it. Therefore it will be a moment of time t_0 when the sides change from left to right. At this time the point must be on the line containing the stick. Moreover if it lies on the extension then the sides change from right to left. Therefore p have to lie under the stick; that is, p is painted. $\square$

1.9. Exercise. *Find the places with cheating in the proof above and try to fix them.*

1.10. Exercise. *Read and understand the original proof of Gary Lawlor in [1].*
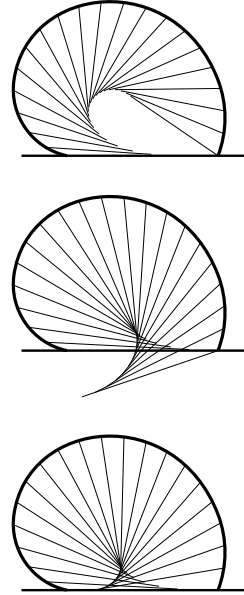
Chapter 2

Length of curves

2.1. Definition. *A* plane curve *is a continuous mapping* $\alpha\colon [a,b] \to$ $\to \mathbb{R}^2$, *where* $\mathbb{R}^2$ *denotes the Euclidean plane.*

If $\alpha(a) = p$ *and* $\alpha(b) = q$, *we say that* α *is a* curve from p to q.

A curve $\alpha\colon [a,b] \to \mathbb{R}^2$ *is called* closed *if* $\alpha(a) = \alpha(b)$.

A curve α *called* simple *if it is described by an injective map; that is* $\alpha(t) = \alpha(t')$ *if and only if* $t = t'$. *However, a closed curve* $\alpha\colon [a,b] \to$ $\mathbb{R}^2$ *is called simple if it is injective everywhere except the ends; that is, if* $\alpha(t) = \alpha(t')$ *for* $t < t'$ *then* $t = a$ *and* $t' = b$.

2.2. Definition. *Let* $\alpha\colon [a,b] \to \mathbb{R}^2$ *be a curve. The* length *of* α *is defined as*

$$\operatorname{length} \alpha = \sup\{|\alpha(t_0) - \alpha(t_1)| + |\alpha(t_1) - \alpha(t_2)| + \dots$$
$$\dots + |\alpha(t_{k-1}) - \alpha(t_k)|\}.$$

where the exact upper bound is taken over all k *and all sequences*

$$a = t_0 < t_1 < \dots < t_k = b.$$

(Such sequences called partitions *of* $[a,b]$*.)*

A curve is called rectifiable *if its length is finite.*

In other words length of curve is the exact upper bound of lengths of polygonal lines *inscribed* in the curve.

2.3. Exercise. *Assume* $\alpha\colon [a,b] \to \mathbb{R}^2$ *is smooth curve, in particular the velocity vector* $\alpha'(t)$ *is defined and depends continuously on* t. *Show that*

$$\operatorname{length} \alpha = \int_a^b |\alpha'(t)| \cdot dt.$$

2.4. Exercise. *Construct a nonrectifiable curve* $\alpha\colon [0,1] \to \mathbb{R}^2$.

A closed simple plane curve is called *convex* if it bounds a convex figure.

2.5. Proposition. *Assume a convex figure A bounded by a curve α lies in a figure B bounded by a curve β. Then*

$$\mathrm{length}\,\alpha \leqslant \mathrm{length}\,\beta.$$

Note that it is sufficient to show that for any polygon P inscribed in α there is a polygon Q inscribed in β such that $\mathrm{perim}\,P \leqslant \mathrm{perim}\,Q$, where $\mathrm{perim}\,P$ denotes the perimeter of P.

Therefore it is sufficient to prove the following lemma.

2.6. Lemma. *Let P and Q be polygons. Assume P is convex and $Q \supset P$. Then* $\mathrm{perim}\,P \leqslant \mathrm{perim}\,Q$.

Proof. Note that by triangle inequality, the inequality

$$\mathrm{perim}\,P \leqslant \mathrm{perim}\,Q$$

holds if P can be obtained from Q by cutting it along a chord; that is, the boundary curve of P is formed by a part of boundary curve of Q and one line segment lying in Q.

Note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \cdots \subset P_n = Q$$

such that P_{i-1} obtained from P_i by cutting along a chord. Therefore

$$\mathrm{perim}\,P = \mathrm{perim}\,P_0 \leqslant \mathrm{perim}\,P_1 \leqslant \ldots$$
$$\ldots \leqslant \mathrm{perim}\,P_n = \mathrm{perim}\,Q$$

and the lemma follows. $\qquad\square$

2.7. Corollary. *Any convex closed curve is rectifiable.*

Proof. Any closed curve is bounded; that is, it lies in a sufficiently large square. By Proposition 1.15, the length of the curve can not exceed the perimeter of the square, hence the result. $\qquad\square$
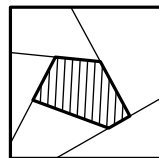
## 2.1 Semicontinuity of length

2.8. Theorem. *Length is a lower semi-continuous with respect to point-wise convergence of curves.*

More precisely, assume that a sequence of curves $\alpha_n\colon [a,b] \to \mathbb{R}^2$ *converges point-wise to a curve* $\alpha_\infty\colon [a,b] \to \mathbb{R}^2$*; that is,* $\alpha_n(t) \to$ $\to \alpha_\infty(t)$ *for any fixed* $t \in [a,b]$ *and* $n \to \infty$*. Then*

❶
$$\varliminf_{n\to\infty} \operatorname{length} \alpha_n \geqslant \operatorname{length} \alpha_\infty,$$

where $\varliminf_{n\to\infty} a_n$ *denotes the lower limit; that is the lowest limit of subsequences of* a_n*.*

Note that the inequality ❷ might be strict. For example the diagonal of unit square α_∞ can be approximated by a sequence of stairs-like polygonal curves α_n with sides parallel to the sides of the square, α_6 is on the picture. In this case
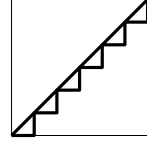
$$\operatorname{length} \alpha_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length} \alpha_n = 2$$

for all n.

Proof. Fix $\varepsilon > 0$ and choose a sequence $a = t_0 < t_1 < \cdots < t_k = b$ such that

$$\operatorname{length} \alpha_\infty - (|\alpha_\infty(t_0) - \alpha_\infty(t_1)| + |\alpha_\infty(t_1) - \alpha_\infty(t_2)| + \ldots$$
$$\cdots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|) < \varepsilon$$

Set

$$\Sigma_n := |\alpha_n(t_0) - \alpha_n(t_1)| + |\alpha_n(t_1) - \alpha_n(t_2)| + \ldots$$
$$\cdots + |\alpha_n(t_{k-1}) - \alpha_n(t_k)|.$$
$$\Sigma_\infty := |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + |\alpha_\infty(t_1) - \alpha_\infty(t_2)| + \ldots$$
$$\cdots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)|.$$

Note that $\Sigma_n \to \Sigma_\infty$ as $n \to \infty$ and $\Sigma_n \leqslant \operatorname{length} \alpha_n$ for each n. Hence

$$\varliminf_{n\to\infty} \operatorname{length} \alpha_n \geqslant \operatorname{length} \alpha_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get ❷. □

2.2 Axioms of length

Assume $\alpha\colon [a,b] \to \mathbb{R}^2$ and $\beta\colon [b,c] \to \mathbb{R}^2$ are two curves such that $\alpha(b) = \beta(b)$. Then one can combine these two curves in one $\gamma\colon [a,c] \to \to \mathbb{R}^2$ the assuming that $\gamma(t) = \alpha(t)$ for $t \leqslant b$ and $\gamma(t) \geqslant \beta(t)$ for $t \geqslant b$. The obtained curve γ is called the *concatenation* of α and β which can be written as $\gamma = \alpha * \beta$. Note that if the concatenation $\alpha * \beta$ is defined, then

$$\text{length}(\alpha * \beta) = \text{length}\,\alpha + \text{length}\,\beta.$$

Assume $\alpha\colon [a,b] \to \mathbb{R}^2$ is a curve and $\tau\colon [c,d] \to [a,b]$ is a continuous strictly monotonic onto map. Consider the curve $\alpha'\colon [c,d] \to \mathbb{R}^2$ defined by $\alpha' = \alpha \circ \tau$. The curves α' is called *reparametrization* of α. Note that

$$\text{length}\,\alpha' = \text{length}\,\alpha$$

if α' is a reparametrization of α.

2.9. Proposition. *Let ℓ be a functional that returns a value in $[0,\infty]$ for any curve $\alpha\colon [a,b] \to \mathbb{R}$. Assume it satisfies the following properties*

  (i) (Normalization) If $\alpha\colon [a,b] \to \mathbb{R}^2$ is a linear curve,[1] then

$$\ell(\alpha) = |\alpha(a) - \alpha(b)|.$$

  (ii) (Additivity) If the concatenation $\alpha * \beta$ is defined, then

$$\ell(\alpha * \beta) = \ell(\alpha) + \ell(\beta).$$

  (iii) (Motion invariance) The functional ℓ is invariant with respect to the motions of the plane; that is, if m is a motion then

$$\ell(m \circ \alpha) = \ell(\alpha)$$

  for any curve α.
  (iv) (Reparametrization invariance) If α' is a reparametrization of a curve α then

$$\ell(\alpha') = \ell(\alpha).$$

  (In fact linear reparametrizations will be sufficient.)
  (v) (Semi-continuity) If a sequence of curves $\alpha_n\colon [a,b] \to \mathbb{R}^2$ converges to a curve pointwise to a curve $\alpha_\infty\colon [a,b] \to \mathbb{R}^2$, then

$$\lim_{n \to \infty} \ell(\alpha_n) \geqslant \ell(\alpha_\infty).$$

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[1] That is $\alpha = w + v{\cdot}t$ for some vectors w and v.

Then $\ell(\alpha) = \text{length}\, \alpha$ for any plane curve α.

Proof. Note that from normalization and additivity, the identity

❷ $$\ell(\beta) = \text{length}\, \beta$$

holds for any polyhonal line β that is linear on each edge.

Fix a curve $\alpha\colon [a,b] \to \mathbb{R}^2$ and a partition $a = t_0 < t_1 < \ldots < < t_k = b$. Consider the curve $\beta\colon [a,b] \to \mathbb{R}^2$ defined as a linear curve from $\alpha(t_i)$ to $\alpha(t_{i+1})$ on each segment $t \in [t_i, t_j]$. Note that

$$\text{length}\, \beta = |\alpha(t_0) - \alpha(t_1)| + \cdots + |\alpha(t_{k-1}) - \alpha(t_k)|.$$

Since the map $\alpha\colon [a,b] \to \mathbb{R}^2$ is continuous, one can find a sequence of partitions of $[a,b]$ such that the corresponding curves β_n converge to α pointwise. Applying semi-continuity of ℓ, ❸ and the definition of length, we get that

$$\ell(\alpha) \leqslant \varliminf_{n\to\infty} \ell(\beta_n) =$$
$$= \varliminf_{n\to\infty} \text{length}\, \beta_n \leqslant$$
$$\leqslant \text{length}\, \alpha.$$

It remains to prove the opposite inequality. Note that a curve $\alpha\colon [a,b] \to \mathbb{R}^2$ with a partition $a = t_0 < t_1 < \ldots < t_k = b$ can be considered as a concatenation

$$\alpha = \alpha_1 * \alpha_2 * \cdots * \alpha_k$$
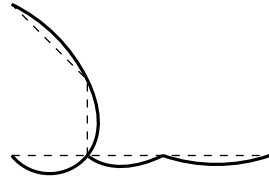
where α_i is the restriction of α to $[t_{i-1}, t_i]$.

Note that there is a sequence of motions m_i of the plane so that $m_i \circ \alpha(t_i) = m_{i+1} \circ \alpha(t_i)$ for any i and the points

$$m_1 \circ \alpha(t_0), m_1 \circ \alpha(t_1), \ldots m_k \circ \alpha(t_k)$$

appear on a line in the same order. Note that for the concatenation

$$\gamma = (m_1 \circ \alpha_1) * (m_2 \circ \alpha_2) * \cdots * (m_k \circ \alpha_k)$$

we have

$$\ell(\gamma) = \ell(\alpha).$$

Note that one can find a sequence of partitions of $[a,b]$ such that reparametrizations γ_n' of the corresponding curves γ_n converge to linear curve γ_∞'. We can assume in addition that length $\gamma_\infty' = \text{length}\, \alpha$;

indeed since γ'_∞ is linear,

$$\text{length}\,\gamma'_\infty = |\gamma'(a) - \gamma'(b)| =$$
$$= \lim_{n\to\infty} \Sigma_n =$$
$$= \text{length}\,\alpha.$$

where Σ_n is the sum in the definition of length for the n-th partition.

Applying invariance of ℓ with respact to reparametizations, we get that

$$\ell(\alpha) = \lim_{n\to\infty} \ell(\gamma_n) =$$
$$= \lim_{n\to\infty} \ell(\gamma'_n) \geqslant$$
$$\geqslant \ell(\gamma'_\infty) =$$
$$= \text{length}\,\alpha. \qquad \square$$

2.3   Crofton formula

Let α be a plane curve and u is a unit vector. Denote by α_u the orthogonal projection of α to a line ℓ in the direction of u; that is, $\alpha_u(t) \in \ell$ and $\alpha(t) - \alpha_u(t) \perp \ell$ for any t.

2.10. Crofton formula.   *The length of any plane curve α is proportional to the average of lengths of its projection α_u for all unit vectors u. Moreover for any plane curve α we have*

$$\text{length}\,\alpha = \tfrac{\pi}{2}\cdot\overline{\text{length}\,\alpha_u},$$

where $\overline{\text{length}\,\alpha_u}$ denotes the average value of $\text{length}\,\alpha_u$.

Proof. First let us show that the formula

❸ $$\text{length}\,\alpha = k\cdot\overline{\text{length}\,\alpha_u},$$

holds for some fixed coefficient k. It will follow once we show that both sides of formula satisfies the length axioms in 1.19.

The semi-continuity of the right hand side follows since $\text{length}\,\alpha_u$ is semi-continuous and therefore the average has to be semi-continuous. It is straightforward to check the remaining properties exept the normalization; the normalization needs a coefficient which we denote by k.

It remains to find k. Let us apply ❹ to the unit circle. The circle has length $2\cdot\pi$ and its projection to any line has length 4 — it is a segment of length 2 traveled back and forth. Therefore

$$2\cdot\pi = k\cdot 4;$$

hence $k = \frac{\pi}{2}$. $\square$

2.11. Exercise.   *Come up with a Crofton formula for curves in the unit sphere.*

Let us use Crofton formula to give an other proof of Proposition 1.15.

Alternative proof of Proposition 1.15. Note that

$$\text{length}\, \beta_u \geqslant \text{length}\, \alpha_u$$

for any unit vector u. Indeed α_u runs back and forth along a line segment and β_u has to run at least as much.

It follows that
$$\overline{\text{length}\, \beta_u} \geqslant \overline{\text{length}\, \alpha_u}.$$

It remains to apply Crofton formula. $\square$

Appendix A

Semisolutions

Exercise 1.5. First let us show that Dido's problem follows from the isoperimetric inequality.

Assume F is a figure bounded by a straight line and a curve of length ℓ whose endpoints belong to that line. Let F' be the reflection of F in the line. Note that the union $G = F \cup F'$ is a figure bounded by a closed curve of length $2{\cdot}\ell$.

Applying the isoperimetric inequality, we get that the area of G can not exceed the area of round disc with the same circumference $2{\cdot}\ell$ and the equality holds only if the figure is congruent to the disc. Since F and F' are congruent, Dido's problem follows.

Now let us show that the isoperimetric inequality follows from the Dido's problem.

Assume G is a convex figure bounded by a closed curve of length $2{\cdot}\ell$. Cut G by a line that splits the perimeter in two equal parts — ℓ each. Denote by F and F' the two parts. Applying the Dido's problem for each part, we get that that are of each does not exceed the area of half-disc bounded by a half-circle. The two half-disc could be arranged into a round disc of circumference ℓ, hence the isoperimetric inequality follows.

Bibliography

[1]   Gary Lawlor. "A new area-maximization proof for the circle". *The Mathematical Intelligencer* 20.1 (1998), pp. 29–31.