Invitation to comparison geometry

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Chapter 1

Metric spaces

Metric is a function that returns a real value $\operatorname{dist}(x, y)$ for any pair x, y in a given nonempty set $\mathcal{X}$ and satisfies the following axioms for any triple x, y, z:

(a) Positiveness:
$$\operatorname{dist}(x, y) \geqslant 0.$$

(b) $x = y$ if and only if
$$\operatorname{dist}(x, y) = 0.$$

(c) Symmetry:
$$\operatorname{dist}(x, y) = \operatorname{dist}(y, x).$$

(d) Triangle inequality:
$$\operatorname{dist}(x, z) \leqslant \operatorname{dist}(x, y) + \operatorname{dist}(y, z).$$

A set with a metric is called *metric space* and the elements of the set are called *points.*

Shortcut notation. Usually we consider only one metric on a set, therefore we can denote the metric space and its underlying set by the same letter, say $\mathcal{X}$. In this case we also use a shortcut notations $|x-y|$ or $|x-y|_{\mathcal{X}}$ for the *distance* $\operatorname{dist}(x, y)$ *from x to y in $\mathcal{X}$.* For example, the triangle inequality can be written as
$$|x-z|_{\mathcal{X}} \leqslant |x-y|_{\mathcal{X}} + |y-z|_{\mathcal{X}}.$$

Examples. Euclidean space and plane as well as real line will be the most important examples of metric spaces for us. In these examples the introduced notation $|x-y|$ for the distance from x to y has perfect sense as a norm of the vector $x-y$. However, in general metric space the expression $x-y$ has no sense, but anyway we use expression $|x-y|$ for the distance.

If we say plane or space we mean Eucledean plane or space. However the plane (as well as the space) admits many other metrics, for example the the so called Manhattan metric from the following exercise.

1.1. Exercise. *Consider the function*

$$\mathrm{dist}(p,q)=|x_p-x_q|+|y_p-y_q|,$$

where $p=(x_p,y_p)$ and $q=(x_q,y_q)$ are points in the coordinate plane $\mathbb{R}^2$. Show that dist *is a metric on $\mathbb{R}^2$.*

Another example of a bizarre metric space let us mention *discrete space* — arbitrary nonempty set $\mathcal{X}$ with the metric defined as $|x-y|_{\mathcal{X}}=0$ if $x=y$ and $|x-y|_{\mathcal{X}}=1$ otherwise.

Subspaces. Any subset of a metric space is also a metric space, by restricting the original metric to the subset; the obtained metric space is called a *subspace*. In particular, all subsets of Euclidean space are metric spaces.

Balls. Given a point p in a metric space $\mathcal{X}$ and a real number $R\geqslant 0$, the set of points x on the distance less then R (or at most R) from p is called open (or correspondingly closed) ball of radius R with center at p. The open ball is denoted as

$$B(p,R)\quad\text{or}\quad B(p,R)_{\mathcal{X}},$$

the second notation is used if we need to emphasize that the ball lies in the metric space $\mathcal{X}$. Analogously, the closed ball is denoted as

$$\bar{B}[p,R]\quad\text{or}\quad \bar{B}[p,R]_{\mathcal{X}}.$$

1.2. Exercise. *Let $\mathcal{X}$ be a metric space.*

(a) Show that if $\bar{B}[p,2]\subset\bar{B}[q,1]$ for some points $p,q\in\mathcal{X}$, then $\bar{B}[p,2]=\bar{B}[q,1]$.

(b) Construct a metric space $\mathcal{X}$ with two points p and q such that $B(p,\frac{3}{2})\subset B(q,1)$ and the inclusions is strict.

Calculus

In this section we will extend standard notions from calculus to the metric spaces.

1.3. Definition. *Let $\mathcal{X}$ be a metric space. A sequence of points $x_1, x_2, \dots$ in $\mathcal{X}$ is called* convergent *if there is $x_\infty \in \mathcal{X}$ such that $|x_\infty - x_n| \to 0$ as $n \to \infty$. That is, for every $\varepsilon > 0$, there is a natural number N such that for all $n \geqslant N$, we have*

$$|x_\infty - x_n| < \varepsilon.$$

In this case we say that the sequence (x_n) converges *to x_∞, or x_∞ is the* limit *of the sequence (x_n). Notationally, we write $x_n \to x_\infty$ as $n \to \infty$ or $x_\infty = \lim_{n\to\infty} x_n$.*

1.4. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called continuous if for any convergent sequence $x_n \to x_\infty$ in $\mathcal{X}$, we have $f(x_n) \to f(x_\infty)$ in $\mathcal{Y}$.*

Equivalently, $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$, there is $\delta > 0$ such that

$$|x - x'|_{\mathcal{X}} < \delta \quad \textit{implies} \quad |f(x) - f(x')|_{\mathcal{Y}} < \varepsilon.$$

1.5. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is* distance non-expanding map*; that is,*

$$|f(x) - f(x')|_{\mathcal{Y}} \leqslant |x - x'|_{\mathcal{X}}$$

for any $x, x' \in \mathcal{X}$. Show that f is continuous.

1.6. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A continous bijection $f\colon \mathcal{X} \to \mathcal{Y}$ is called a* homeomorphism *if its inverse $f^{-1}\colon \mathcal{Y} \to \mathcal{X}$ is also continuous.*

If there exists a homeomorphism $f\colon \mathcal{X} \to \mathcal{Y}$, we say that $\mathcal{X}$ is homeomorphic *to $\mathcal{Y}$, or $\mathcal{X}$ and $\mathcal{Y}$ are* homeomorphic*.*

If a metric space $\mathcal{X}$ is homeomorphic to a known space, for example plane, sphere, disc, circle and so on, we may also say that $\mathcal{X}$ is a *topological* plane, sphere, disc, circle and so on.

1.7. Definition. *A subset A of a metric space $\mathcal{X}$ is called* closed *if whenever a sequence (x_n) of points from A converges in $\mathcal{X}$, we have that $\lim_{n\to\infty} x_n \in A$.*

A set $\Omega \subset \mathcal{X}$ is called open *if the complement $\mathcal{X} \setminus \Omega$ is a closed set. Equivalently, $\Omega \subset \mathcal{X}$ is open if for any $z \in \Omega$, there is $\varepsilon > 0$ such that if $|x - z| < \varepsilon$, then $x \in \Omega$.*

An open set Ω that contains a given point p is called *neighborhood of* p.

1.8. Exercise. *Let Q be a subset of a metric space $\mathcal{X}$. Show that A is closed if and only if its complement $\Omega = \mathcal{X} \backslash Q$ is open.*

Chapter 2

Curves

Paths. Let $\mathcal{X}$ be a metric space. A continuous map $f\colon[0,1]\to\mathcal{X}$ is called a *path*. If $p=f(0)$ and $q=f(1)$, then we say that f *connects* p *to* q.

If any two points in $\mathcal{X}$ can be connected by a path then $\mathcal{X}$ is called *path connected*. Similarly, a subset $A\subset\mathcal{X}$ is called *path connected* if any two points $p,q\in A$ can be connected by a path that runs in A; equivalently, the subspace A is path connected.

Simple curves.

2.1. Definition. *A path connected subset γ in a metric space is called a* simple curve *if it is* locally *homeomorphic to a real interval; that is, any point $p\in\gamma$ has a neighborhood $U\ni p$ such that the intersection $U\cap\gamma$ is homeomorphic to a real interval.*

It turns out that any curve γ admits a homeomorphism from a real interval or a circle; that is, there is a continuous bijection $G\to\gamma$ with continuous inverse; here (and further) G denotes a circle or real interval. We omit a proof of this statement, but it is not hard.

The homeomorphism $G\to\gamma$ as above is called *parametrization* of γ. The parametrization completely defines the curve. Often will use the same letter for curve and its parametrization, so we can say curve γ has parametrization $\gamma\colon G\to\mathcal{X}$. Note however that any curve admits many different parametrization.

2.2. Exercise. *Find a continuous injective map $\gamma\colon[0,1)\to\mathbb{R}^2$ such that its image is not a simple curve.*

Hint: The image of γ should have a shape of digit 9.

If G is a circle, then the curve $\gamma\colon G\to\mathcal{X}$ is called *closed*. If G is a real interval, then we may say that γ is an *arc*.

Parameterized curves. A *parameterized curve* is defined as a continuous map $\gamma\colon G \to \mathcal{X}$. For a parameterized curve we do not assume that γ is injective; in other words the parameterized curve might have self-intersections.

Loops and periodic parametrization. A closed simple curve can be described as an image of a parameterized curve $\gamma\colon [0,1] \to \mathcal{X}$ such that $p = \gamma(0) = \gamma(1)$; such curves are called *loops*; the point p in this case is called *base* of the loop.

As you will see further it is more natural to present it as a *periodic* parameterized cure $\gamma\colon \mathbb{R} \to \mathcal{X}$; that is, there is a constant ℓ such that $\gamma(t+\ell) = \gamma(t)$ for any t. For example the unit circle in the plane can be described by periodic parametrization $(\cos t, \sin t)$.

2.3. Advanced exercise. *Let $\alpha\colon [0,1] \to \mathcal{X}$ be a path from p to q. Assume $p \neq q$. Show that there is a simple path connecting from p to q in $\mathcal{X}$.*

Smooth curves

A curve in the Euclidean space or plane, called *space* or *plane curve* correspondingly.

A space curve can be described by its coordinate functions

$$\gamma(t) = (x(t), y(t), z(t)).$$

Plane curves can be considered as a partial case of space curves with $z(t) \equiv 0$.

If each of the coordinate functions $x(t), y(t), z(t)$ of the space curve γ is a smooth (that is it has derivatives of all orders everywhere in its domain) then the parameterized curve is called *smooth.*

If the *velocity vector*

$$\gamma'(t) = (x'(t), y'(t), z'(t))$$

does not vanish at all points, then the parameterized curve γ is called *regular.*

A simple space curve is called *smooth and regular* if it admits a smooth and regular parametrization correspondingly. Regular smooth curves are among the main objects in differential geometry; the term *smooth curve* often used for *smooth regular curve.*

2.4. Exercise. *The function*

$$f(t) = \begin{cases} 0 & \textit{if } t \leqslant 0, \\ \frac{t}{e^{1/t}} & \textit{if } t > 0. \end{cases}$$

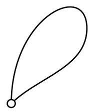
is smooth.[1]

Show that $\gamma(t) = (f(t), f(-t))$ gives a smooth parametrization of the curve S formed by the union of two half-axis in the plane.

Show that any smooth parametrization of S has vanishing velocity vector at the origin. Conclude that the curve S regular and smooth.

2.5. Exercise. *Describe the set of real numbers a such that the plane curve $\gamma_a(t) = (t + a\cdot \sin t, a\cdot \cos t)$, $t \in \mathbb{R}$ is*

(a) regular;
(b) simple.



Any smooth regular closed curve can be described by a smooth regular loop. But in general the closed curve that described by a smooth regular loop might fail to be smooth and regular — it might be not smooth the the base of the loop; an example shown on the diagram.

Implicitly defined curves

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function; that is, all its partial derivatives defined in its domain of definition. Consider the set S of solution of equation $f(x, y) = 0$ in the plane.

Assume S is path connected. According to implicit function theorem (A.2), the set S is a smooth regular simple curve if 0 is a *regular value* of f. In this case it means that the gradient ∇f does not vanish at any point $p \in S$. In other words, if $f(p) = 0$, then $\frac{\partial f}{\partial x}(p) \ne 0$ or $\frac{\partial f}{\partial y}(p) \ne 0$.

Similarly, assume f, h is a pair of smooth functions defined in $\mathbb{R}^3$. The system of equations $f(x, y, z) = h(x, y, z) = 0$ defines a regular smooth space curve if the set of solutions is path connected and 0 is a regular value of the map $(f, h)\colon \mathbb{R}^3 \to \mathbb{R}^2$. In this case it means that the gradients ∇f and ∇h are linearly independent at any point $p \in S$. In other words, if $f(p) = 0$, then at the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}$$

for the map $(x, y, z) \mapsto (f(x, y, z), h(x, y, z))$ has rank 2 at p.

The described way to define a curve is called *implicit*; if a curve is defined by its parametrization, we say that it is *explicitly defined.*

[1] The existence of all derivatives $f^{(n)}(x)$ at $x \ne 0$ is evident and direct calculations show that $f^{(n)}(0) = 0$ for any n.

While implicit function theorem guarantees the existence of regular smooth parametrizations, it does not mean that such parametrization can be found explicitly in a closed form. When it comes to calculations, usually it is easier to work directly with implicit presentation.

2.6. Exercise. *Describe the set of real numbers a such that the system of equations*

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ x^2 + a{\cdot}x + y^2 &= 0 \end{aligned}$$

describes a smooth regular curve.

Chapter 3

Length

Recall that a sequence

$$a = t_0 < t_1 < \dots < t_k = b.$$

is called a *partition* of the interval $[a, b]$.

3.1. Definition. *The* length *of a* $\alpha\colon [a, b] \to \mathcal{X}$ *is defined as*

$$\begin{aligned}\operatorname{length}\alpha = \sup\{|\alpha(t_0) - \alpha(t_1)| + |\alpha(t_1) - \alpha(t_2)| + \dots \\ \dots + |\alpha(t_{k-1}) - \alpha(t_k)|\}.\end{aligned}$$

where the exact upper bound is taken over all partitions

$$a = t_0 < t_1 < \dots < t_k = b.$$

The length of α *is a nonnegative real number or infinity; the curve* α *is called* rectifiable *if its length is finite.*

If α is a space curve, then the above definition says that it length is the exact upper bound of the lengths of polygonal lines $p_0 \dots p_k$ *inscribed* in the curve, where $p_i = \alpha(t_i)$ for a partition $a = t_0 < t_1 < \dots < t_k = b$.

3.2. Exercise. *Assume* $\alpha\colon [a, b] \to \mathbb{R}^2$ *is a smooth curve. Show that*

(a) $\operatorname{length}\alpha \geqslant \int_a^b |\alpha'(t)|\cdot dt$,

(b) $\operatorname{length}\alpha \leqslant \int_a^b |\alpha'(t)|\cdot dt$.

Conclude that

$$\operatorname{length}\alpha = \int\limits_a^b |\alpha'(t)|\cdot dt.$$

Hints: For (a), apply the fundamental theorem of calculus for each segment in a given partition. For (b) consider a partition such that α' is nearly constant on each of its segments.

3.3. Exercise. *Construct a nonrectifiable simple curve* $\alpha\colon [0,1] \to \to \mathbb{R}^2$.

Convex curves

A closed simple plane curve is called *convex* if it bounds a convex region.

3.4. Proposition. *Assume a convex figure A bounded by a curve α lies inside a figure B bounded by a curve β. Then*

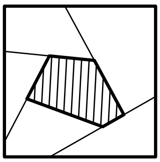
$$\operatorname{length}\alpha \leqslant \operatorname{length}\beta.$$

Note that it is sufficient to show that for any polygon P inscribed in α there is a polygon Q inscribed in β with $\operatorname{perim} P \leqslant \operatorname{perim} Q$, where $\operatorname{perim} P$ denotes the perimeter of P.

Therefore it is sufficient to prove the following lemma.

3.5. Lemma. *Let P and Q be polygons. Assume P is convex and $Q \supset P$. Then*

$$\operatorname{perim} P \leqslant \operatorname{perim} Q.$$



Proof. Note that by the triangle inequality, the inequality

$$\operatorname{perim} P \leqslant \operatorname{perim} Q$$

holds if P can be obtained from Q by cutting it along a chord; that is, a line segment with ends on the boundary of Q that lies in Q.

Note that there is an increasing sequence of polygons

$$P = P_0 \subset P_1 \subset \dots \subset P_n = Q$$

such that P_{i-1} obtained from P_i by cutting along a chord. Therefore

$$\operatorname{perim} P = \operatorname{perim} P_0 \leqslant \operatorname{perim} P_1 \leqslant \dots$$
$$\dots \leqslant \operatorname{perim} P_n = \operatorname{perim} Q$$

and the lemma follows. □

3.6. Corollary. *Any convex closed curve is rectifiable.*

Proof. Any closed curve is bounded; that is, it lies in a sufficiently large square.

By Proposition 3.4, the length of the curve can not exceed the perimeter of the square, hence the result. □

Semicontinuity of length

Recall that the lower limit of a sequence of real numbers (x_n) is denoted by

$$\varliminf_{n\to\infty} x_n.$$

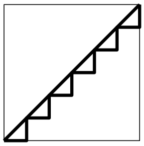
It is defined as the lowest partial limit; that is, the lowest possible limit of a subsequence of (x_n). The lower limit is defined for any sequence of real numbers and it lies in the exteded real line $[-\infty,\infty]$

3.7. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\alpha_n\colon[a,b]\to\mathbb{R}^2$ *converges pointwise to a curve* $\alpha_\infty\colon[a,b]\to\mathbb{R}^2$*; that is,* $\alpha_n(t)\to\alpha_\infty(t)$ *for any fixed* $t\in[a,b]$ *as* $n\to\infty$*. Then*

❶ $$\varliminf_{n\to\infty} \operatorname{length}\alpha_n \geqslant \operatorname{length}\alpha_\infty.$$

Note that the inequality ❶ might be strict. For example the diagonal α_∞ of the unit square



can be approximated by a sequence of stairs-like polygonal curves α_n with sides parallel to the sides of the square (α_6 is on the picture). In this case

$$\operatorname{length}\alpha_\infty=\sqrt{2} \quad\text{and}\quad \operatorname{length}\alpha_n=2$$

for any n.

Proof. Fix a partition $a=t_0<t_1<\dots<t_k=b$. Set

$$\begin{aligned}\Sigma_n &:= |\alpha_n(t_0)-\alpha_n(t_1)|+\dots+|\alpha_n(t_{k-1})-\alpha_n(t_k)|.\\ \Sigma_\infty &:= |\alpha_\infty(t_0)-\alpha_\infty(t_1)|+\dots+|\alpha_\infty(t_{k-1})-\alpha_\infty(t_k)|.\end{aligned}$$

Note that $\Sigma_n\to\Sigma_\infty$ as $n\to\infty$ and $\Sigma_n\leqslant\operatorname{length}\alpha_n$ for each n. Hence

❷ $$\varliminf_{n\to\infty} \operatorname{length}\alpha_n \geqslant \Sigma_\infty.$$

If α_∞ is rectifiable, we can assume that

$$\operatorname{length}\alpha_\infty < |\alpha_\infty(t_0) - \alpha_\infty(t_1)| + \dots + |\alpha_\infty(t_{k-1}) - \alpha_\infty(t_k)| + \varepsilon.$$

for any given $\varepsilon > 0$. By ❷ it follows that

$$\varliminf_{n\to\infty} \operatorname{length}\alpha_n > \operatorname{length}\alpha_\infty - \varepsilon$$

for any $\varepsilon > 0$; whence ❶ follows.

It remains to consider the case when α_∞ is not rectifiable; that is $\operatorname{length}\alpha_\infty = \infty$. In this case we can choose a partition so that $\Sigma_\infty > L$ for any real number L. By ❷ it follows that

$$\varliminf_{n\to\infty} \operatorname{length}\alpha_n > L$$

for any L; whence $\varliminf_{n\to\infty} = \infty$ and ❶ follows. □

Length metric

Let $\mathcal{X}$ be a metric space. Given two points x, y in $\mathcal{X}$, denote by $d(x, y)$ the exact lower bound for lengths of all paths connecting x to y; if there is no such path we assume that $d(x, y) = \infty$. Note that function d satisfies all the axioms of metric except it might take infinite value.

Therefore if any two points in $\mathcal{X}$ can be connected by a rectifiable curve then d defines a new metric on it; in this case d is called *induced length metric.*

Evidently $d(x, y) \geqslant |x - y|$ for any pair of points $x, y \in \mathcal{X}$. If the equality holds for any pair, then then the metric is called *length metric* and the space is called *length-metric space.*

Note that the Euclidean space is a length-metric space. Most of the time we consider length-metric spaces. A subspaces A of length-metric space $\mathcal{X}$ might be not a lenght-metric space; the induced length distance between points x and y in the subspace A will be denoted as $|x - y|_A$; that is $|x - y|_A$ is the exact lower bound for the length of paths in A.

3.8. Exercise. *Let $A \subset \mathbb{R}^3$ be a closed subset. Show that A is convex if and only if*

$$|x - y|_A = |x - y|_{\mathbb{R}^3}.$$

3.9. Exercise. *Let us denote by $\mathbb{S}^2$ the unit sphere in the space; that is,*

$$\mathbb{S}^2 = \left\{ (x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 = 1 \right\}.$$

Show that

$$|u - v|_{\mathbb{S}^2} = \measuredangle(u, v) := \arccos\langle u, v\rangle.$$

Hint: Use the following map $f\colon (r, \theta, \varphi) \mapsto (r, \theta, 0)$ in spherical coordinates. Note that f a distance nonexpanding map that maps $\mathbb{R}^3$ to a half-plane and $\mathbb{S}^2$ to one of its meridians.

Appendix A

Preliminaries

In this chapter we state and discuss results from different branches of mathematics which were used further in the book. The reader is not expected to know proofs of these statements, but it is better to check that his intuition agrees with each.

Multivariable calculus

A map $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^k$ can be thought as array of functions

$$f_1, \dots, f_k\colon \mathbb{R}^n \to \mathbb{R}.$$

The map $\boldsymbol{f}$ is called *smooth* if each function f_i is smooth; that is, all partial derivatives of f_i are defined in the domain of definition of $\boldsymbol{f}$.

Inverse function theorem gives a sufficient condition for a smooth function to be invertible in a neighborhood of a given point p in its domain. The condition is formulated in terms of partial derivative of f_i at p.

Implicit function theorem is a close relative to inverse function theorem; in fact it can be obtained as its corollary. It is used for instance when we need to pass from parametric to implicit description of curves and surface.

Both theorems reduce the existence of a map satisfying certain equation to a question in linear algebra. We use these two theorems only for $n \leqslant 3$.

These two theorems are discussed in any course of multivariable calculus, the classical book of Walter Rudin [1] is one of my favorites.

1.1. Inverse function theorem. *Let* $\boldsymbol{f} = (f_1, \dots, f_n)\colon \mathbb{R}^n \to \mathbb{R}^n$

be a smooth map. Assume that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is invertible at some point p *in the domain of definition of* $\boldsymbol{f}$. *Then there is a smooth function* $\boldsymbol{h}\colon \mathbb{R}^m \to \mathbb{R}^n$ *defined is a neighborhood* Ω_q *of* $q = \boldsymbol{f}(p)$ *that is* local inverse of $\boldsymbol{f}$ at p*; that is, there are neighborhoods* $\Omega_p \ni p$ *such that* $\boldsymbol{f}$ *defines a bijection* $\Omega_p \to \Omega_q$ *and* $\boldsymbol{f}(x) = y$ *if and only if* $x = \boldsymbol{h}(y)$ *for any* $x \in \Omega_p$ *and any* $y \in \Omega_q$.

1.2. Implicit function theorem. *Let* $\boldsymbol{f} = (f_1, \dots, f_n)\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ *be a smooth map,* $m, n \geqslant 1$. *Let us consider* $\mathbb{R}^{n+m}$ *as a product space* $\mathbb{R}^n \times \mathbb{R}^m$ *with coodinates* $x_1, \dots, x_n, y_1, \dots, y_m$. *Consider the following matrix*

$$M = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

formed by first n *columns of the Jacobian matrix. Assume* M *is invertible at some point* p *in the domain of definition of* $\boldsymbol{f}$ *and* $\boldsymbol{f}(p) = 0$. *Then there is a neighborhood* $\Omega_p \ni p$ *and smooth function* $\boldsymbol{h}\colon \mathbb{R}^m \to \mathbb{R}^n$ *defined is a neighborhood* $\Omega_0 \ni 0$ *that for any* $(x_1, \dots, x_n, y_1, \dots y_m) \in \Omega_p$ *the equality*

$$\boldsymbol{f}(x_1, \dots, x_n, y_1, \dots y_m) = 0$$

holds if and only if

$$(x_1, \dots x_n) = \boldsymbol{h}(y_1, \dots y_m).$$

If the assumption in the theorem holds for any point p such that $\boldsymbol{f}(p) = 0$, then we say that 0 is a regular value of $\boldsymbol{f}$. *Sard's theorem* states that most of the values of smooth map are regular; in particular generic smooth function satisfies the assumption of the theorem.

Rademacher's theorem

Recall that a function f is called Lipschitz if there is a constant L such that

$$|f(x) - f(y)| \leqslant L{\cdot}|x - y|$$

for values x and y in the domain of definition of f. This definition works for maps between metric spaces, but we will use it for real-to-real functions only.

1.3. Rademacher's theorem. *Let $f\colon [a,b] \to \mathbb{R}$ be a Lipschitz function. then derivative $f'(x)$ is defined for alomst all $x \in [a,b]$. Moreover the derivative is bounded and Lebesgue-integrable and satisfies the fundamental theorem of calculus; that is, the following identity*

$$f(b) - f(a) = \int_a^b f'(x)\cdot dx,$$

holds if the integral understood in the sense of Lebesgue.

Fundamental theorem of ODE

Picard theorem or the fundamental theorem of ordinary differential equations; it guarantees existance and uniqueness of a solution of an initial value problem for a system of ordinary differential equations

$$\begin{cases} x_1' = f_1(x_1, \dots, x_n), \\ \quad\dots \\ x_n' = f_n(x_1, \dots, x_n), \end{cases}$$

The array functions $(f_1, \dots, f_n)$ can be considered as one vector-valued function $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ and the array $(x_1, \dots, x_n)$ can be considered as a vector $\boldsymbol{x} \in \mathbb{R}^n$. Therefore the system can be rewritten as one vector equation

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}).$$

We use only the following partial case of this theorem.

1.4. Theorem. *Suppose $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. Then for any initial data $\boldsymbol{x}(0) = \boldsymbol{u}$ the differential equation*

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$$

has a unique solution $\boldsymbol{x}(t)$ defined at some open interval containing zero. Moreover the function $(\boldsymbol{u}, t) \mapsto \boldsymbol{x}(t)$ is smooth.

Appendix B

Homework assignments

HWA01, due Monday January 14. Exercises: 1.2, 2.4, 2.5, 2.6, 3.8.

Bibliography

[1] Walter Rudin. *Principles of mathematical analysis*. 1976.