Part 2

Fundamentals

Chapter 8

Fundamentals of curvature bounded below

A. Four-point comparison

Recall (Section 6B) that the model angle $\tilde\measuredangle^\kappa\left(p\,{}^x_y\right)$ is defined if

$$|p-x|+|p-y|+|x-y|<\varpi\kappa;$$

here $\varpi\kappa$ denotes the diameter of model space $\mathbb{M}^2(\kappa)$.

8.1. Four-point comparison. *A quadruple of points p, x^1, x^2, x^3 in a metric space satisfies* CBB(κ) comparison *if*

$$\tilde\measuredangle^\kappa\left(p\,{}^{x^1}_{x^2}\right)+\tilde\measuredangle^\kappa\left(p\,{}^{x^2}_{x^3}\right)+\tilde\measuredangle^\kappa\left(p\,{}^{x^3}_{x^1}\right)\leqslant 2\cdot\pi. \tag{1}$$

or at least one of the model angles $\tilde\measuredangle^\kappa\left(p\,{}^{x^i}_{x^j}\right)$ is not defined.

8.2. Definition. *Let $\mathcal{L}$ be a metric space.*

(*a*) *$\mathcal{L}$ is* CBB(κ) *if any quadruple in $\mathcal{L}$ satisfies* CBB(κ) *comparison.*

(*b*) *$\mathcal{L}$ is* locally CBB(κ) *if any point $q\in\mathcal{L}$ admits a neighborhood $\Omega\ni q$ such that any quadruple in Ω satisfies* CBB(κ) *comparison.*

(*c*) *$\mathcal{L}$ is a* CBB *space if $\mathcal{L}$ is* CBB(κ) *for some $\kappa\in\mathbb{R}$.*

Remarks.

- CBB(κ) length spaces are often called *spaces with curvature $\geqslant\kappa$ in the sense of Alexandrov*. These spaces will usually be denoted by $\mathcal{L}$, for $\mathcal{L}$ower curvature bound.

- In the definition of CBB(κ), when $\kappa > 0$ most authors assume in addition that the diameter is at most the model diameter $\varpi\kappa$. For a complete length space, the latter means that it is not isometric to one of the exceptional spaces, see 8.44. We do not make this assumption. In particular, we consider the real line to have curvature $\geqslant 1$.
- If $\kappa < \mathrm{K}$, then any complete length CBB(K) space is CBB(κ). Moreover directly from the definition it follows that if $\mathrm{K} \leqslant 0$, then any CBB(K) space is CBB(κ). However, in the case $\mathrm{K} > 0$ the latter statement does not hold and the former statement is not trivial; it will be proved in 8.33.

8.3. Exercise. *Let $\mathcal{L}$ be a metric space and $\kappa \leqslant 0$. Show that $\mathcal{L}$ is* CBB(κ) *if for any quadruple of points p, $x^1, x^2, x^3 \in \mathcal{L}$ there is a quadruple of points q, $y^1, y^2, y^3 \in \mathbb{M}^2(\kappa)$ such that*

$$|p - x^i| = |q - y^i| \quad \textit{and} \quad |x^i - x^j| \leqslant |y^i - y^j|$$

for all i and j.

The exercise above is a special case of $(1 + n)$-point comparison (10.8).

Recall that ω denotes a selective ultrafilter on $\mathbb{N}$, which is fixed once and for all. The following proposition follows directly from the definition of CBB(κ) comparison and the definitions of ω-limit and ω-power given in Section 4B.

8.4. Proposition. *Let $\mathcal{L}_n$ be a* CBB(κ_n) *space for each n. Assume $\mathcal{L}_n \to \mathcal{L}_\omega$ and $\kappa_n \to \kappa_\omega$ as $n \to \omega$. Then $\mathcal{L}_\omega$ is* CBB(κ_ω).

Moreover, a metric space $\mathcal{L}$ is CBB(κ) *if and only if so is its ultrapower $\mathcal{L}^\omega$.*

8.5. Theorem. *Let $\mathcal{L}$ be a* CBB(κ) *space, $\mathcal{M}$ be a metric space, and $\sigma\colon \mathcal{L} \to \mathcal{M}$ be a submetry. Assume p, x^1, x^2, x^3 is a quadruple of points in $\mathcal{M}$ such that $|p - x^i| < \frac{\varpi\kappa}{2}$ for any i. Then the quadruple satisfies* CBB(κ) *comparison.*

In particular,

(*a*) *The space $\mathcal{M}$ is locally* CBB(κ). *Moreover, any open ball of radius $\frac{\varpi\kappa}{4}$ in $\mathcal{M}$ is* CBB(κ).

(*b*) *If $\kappa \leqslant 0$, then $\mathcal{M}$ is* CBB(κ).

Corollary 8.34 gives a stronger statement; it states that if $\mathcal{L}$ is a complete length space, then $\mathcal{M}$ is always CBB(κ). The theorem above together with Proposition 3.8 imply the following:

8.6. Corollary. *Assume that the group G acts isometrically on a* CBB(κ) *space $\mathcal{L}$ and has closed orbits. Then the quotient space $\mathcal{L}/G$ is locally* CBB(κ).

8.7. Example. *If $G < \mathrm{O}(n+1)$ is a closed subgroup, then $\mathbb{S}^n/G$ is* CBB(1) *and $\mathbb{R}^{n+1}/G$ is* CBB(0).

Proof of 8.5. Fix a quadruple of points $p, x^1, x^2, x^3 \in \mathcal{M}$ such that $|p - x^i| < \frac{\varpi\kappa}{2}$ for any i. Choose an arbitrary $\hat{p} \in \mathcal{L}$ such that $\sigma(\hat{p}) = p$.

Since σ is submetry, we can choose the points $\hat{x}^1, \hat{x}^2, \hat{x}^3 \in \mathcal{L}$ such that $\sigma(\hat{x}_i) = x_i$ and

$$|p - x^i|_{\mathcal{M}} \lessgtr |\hat{p} - \hat{x}^i|_{\mathcal{L}} \pm \delta$$

for all i and any fixed $\delta > 0$.

Note that

$$|x^i - x^j|_{\mathcal{M}} \leqslant |\hat{x}^i - \hat{x}^j|_{\mathcal{L}} \leqslant |p - x^i|_{\mathcal{M}} + |p - x^j|_{\mathcal{M}} + 2 \cdot \delta$$

for all i and j.

Since $|p - x^i| < \frac{\varpi\kappa}{2}$, we can choose $\delta > 0$ above so that the angles $\tilde{\measuredangle}^\kappa\left(\hat{p}\,{}^{\hat{x}^i}_{\hat{x}^j}\right)$ are defined. Moreover, given $\varepsilon > 0$, the value δ can be chosen in such a way that the inequality

$$\tilde{\measuredangle}^\kappa\left(p\,{}^{x^i}_{x^j}\right) < \tilde{\measuredangle}^\kappa\left(\hat{p}\,{}^{\hat{x}^i}_{\hat{x}^j}\right) + \varepsilon \tag{2}$$

holds for all i and j.

By CBB(κ) comparison in $\mathcal{L}$, we have

$$\tilde{\measuredangle}^\kappa\left(\hat{p}\,{}^{\hat{x}^1}_{\hat{x}^2}\right) + \tilde{\measuredangle}^\kappa\left(\hat{p}\,{}^{\hat{x}^2}_{\hat{x}^3}\right) + \tilde{\measuredangle}^\kappa\left(\hat{p}\,{}^{\hat{x}^3}_{\hat{x}^1}\right) \leqslant 2 \cdot \pi.$$

Applying 2, we get

$$\tilde{\measuredangle}^\kappa\left(p\,{}^{x^1}_{x^2}\right) + \tilde{\measuredangle}^\kappa\left(p\,{}^{x^2}_{x^3}\right) + \tilde{\measuredangle}^\kappa\left(p\,{}^{x^3}_{x^1}\right) < 2 \cdot \pi + 3 \cdot \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we have

$$\tilde{\measuredangle}^\kappa\left(p\,{}^{x^1}_{x^2}\right) + \tilde{\measuredangle}^\kappa\left(p\,{}^{x^2}_{x^3}\right) + \tilde{\measuredangle}^\kappa\left(p\,{}^{x^3}_{x^1}\right) \leqslant 2 \cdot \pi;$$

that is, the CBB(κ) comparison holds for this quadruple in $\mathcal{M}$. □

B. Geodesics

Recall that general complete length spaces might have no geodesics; see Exercise 2.10.

8.8. Exercise. *Construct a complete length* CBB(0) *space that is not geodesic.*

We are going to show that all complete length CBB spaces have plenty of geodesics in the following sense. Recall that a subset of a topological space is called G-delta if it is a countable intersection of open sets.

8.9. Definition. *A metric space $\mathcal{X}$ is called* G-delta geodesic *if for any point $p \in \mathcal{X}$ there is a dense G-delta set $W_p \subset \mathcal{X}$ such that for any $q \in W_p$ there is a geodesic $[pq]$.*

A metric space $\mathcal{X}$ is called locally G-delta geodesic *if for any point $p \in \mathcal{X}$ there is a G-delta set $W_p \subset \mathcal{X}$ such that W_p is dense in a neighborhood of p and for any $q \in W_p$ there is a geodesic $[pq]$.*

8.10. Definition. *Let $\mathcal{X}$ be a metric space and $p \in \mathcal{X}$. A point $q \in \mathcal{X}$ is called p-*straight *(briefly, $q \in \operatorname{Str}(p)$) if*

$$\varlimsup_{r\to q} \frac{|p-r|-|p-q|}{|q-r|} = 1.$$

For an array of points $x^1, x^2, \dots, x^k$, we use the notation

$$\operatorname{Str}(x^1, x^2, \dots, x^k) = \bigcap_i \operatorname{Str}(x^i).$$

8.11. Theorem. *Let $\mathcal{L}$ be a complete length* CBB *space and $p \in \mathcal{L}$. Then the set $\operatorname{Str}(p)$ is a dense G-delta set. Moreover, for any $q \in \operatorname{Str}(p)$ there is a unique geodesic $[pq]$.*

In particular, $\mathcal{L}$ is G-delta geodesic.

This theorem was proved by by Conrad Plaut [**137**, Th. 27].

Proof. Given a positive integer n, consider the set Ω_n of all points $q \in \mathcal{L}$ such that

$$(1 - \tfrac{1}{n}) \cdot |q-r| < |p-r| - |p-q| < \tfrac{1}{n}$$

for some $r \in \mathcal{L}$. Clearly Ω_n is open; let us show that Ω_n is dense in $\mathcal{L}$.

Assuming the contrary, there is a point $x \in \mathcal{L}$ such that

$$\mathrm{B}(x,\varepsilon) \cap \Omega_n = \emptyset$$

for $\varepsilon > 0$. Since $\mathcal{L}$ is a length space, for any $\delta > 0$, there exists a point $y \in \mathcal{L}$ such that

$$|x-y| < \tfrac{\varepsilon}{2} + \delta \quad \text{and} \quad |p-y| < |p-x| - \tfrac{\varepsilon}{2} + \delta.$$

If ε and δ are sufficiently small, then

$$(1 - \tfrac{1}{n}) \cdot |y-x| < |p-x| - |p-y| < \tfrac{1}{n};$$

that is, $y \in \Omega_n$, a contradiction.

Note that $\operatorname{Str}(p) = \bigcap_n \Omega_n$; therefore, $\operatorname{Str}(p)$ is a dense G-delta set.

Assuming $q \in \operatorname{Str}(p)$, let us show that there is a unique geodesic connecting p and q. Note that it is sufficient to show that for all sufficiently small $t > 0$ there is a unique point z such that

$$t = |q-z| = |p-q| - |p-z|. \tag{1}$$

First let us show uniqueness. Assume z and z' both satisfy 1. Take a sequence $r_n \to q$ such that

$$\frac{|p-r_n|-|p-q|}{|q-r_n|} \to 1.$$

By the triangle inequality,

$$|z-r| - |z-q|, \quad |z'-r| - |z'-q| \geqslant |p-r| - |p-q|;$$

thus, as $n \to \infty$,

$$\frac{|z - r_n| - |z - q|}{|q - r_n|}, \quad \frac{|z' - r_n| - |z' - q|}{|q - r_n|} \to 1.$$

Therefore $\tilde{\measuredangle}^\kappa(q\,{}^{z}_{r_n}) \to \pi$ and $\tilde{\measuredangle}^\kappa(q\,{}^{z'}_{r_n}) \to \pi$. (Here we use that t is small, otherwise if $\kappa > 0$ the angles might be undefined.)

From CBB(κ) comparison (8.2), $\tilde{\measuredangle}^\kappa(q\,{}^{z}_{z'}) = 0$ and thus $z = z'$.

The proof of existence is similar. Choose a sequence r_n as above. Since $\mathcal{L}$ is a complete length space, there is a sequence $z_k \in \mathcal{L}$ such that $|q - z_k| \to t$ and $|p - q| - |p - z_k| \to t$ as $k \to \infty$. Then

$$\lim_{n\to\infty} \lim_{k\to\infty} \tilde{\measuredangle}^\kappa(q\,{}^{z_k}_{r_n}) = \pi.$$

Thus, for any $\varepsilon > 0$ and sufficiently large n, k, we have $\tilde{\measuredangle}^\kappa(q\,{}^{z_k}_{r_n}) > \pi - \varepsilon$. From CBB($\kappa$) comparison (8.2), for all large k and j, we have $\tilde{\measuredangle}^\kappa(q\,{}^{z_k}_{z_j}) < 2 \cdot \varepsilon$ and thus

$$|z_k - z_j| < \varepsilon \cdot c(\kappa, t);$$

that is, z_n is a Cauchy sequence, and its limit z satisfies 1. $\square$

8.12. Exercise. *Let $\mathcal{L}$ be a complete length* CBB *space and $A \subset \mathcal{L}$ be a closed subset. Show that there is a dense G-delta set $W \subset \mathcal{L}$ such that for any $q \in W$, there is a unique geodesic $[pq]$ with $p \in A$ that realizes the distance from q to A; that is, $|p - q| = \operatorname{dist}_A q$.*

8.13. Exercise. *Construct a complete length* CBB *space $\mathcal{L}$ with an everywhere dense G-delta set A such that $A \cap \,]xy[\, = \emptyset$ for any geodesic $[xy]$ in $\mathcal{L}$.*

C. More comparisons

The following theorem makes it easier to use Euclidean intuition in the Alexandrov setting.

8.14. Theorem. *If $\mathcal{L}$ is a* CBB(κ) *space, then the following conditions hold for all $p, x, y \in \mathcal{L}$, provided the model triangle $\tilde{\triangle}^\kappa(pxy)$ is defined.*

(*a*) *(adjacent angle comparison) for any geodesic $[xy]$ and $z \in \,]xy[$, $z \neq p$ we have*

$$\tilde{\measuredangle}^\kappa(z\,{}^{p}_{x}) + \tilde{\measuredangle}^\kappa(z\,{}^{p}_{y}) \leqslant \pi.$$

(*b*) *(point-on-side comparison) for any geodesic $[xy]$ and $z \in \,]xy[$, we have*

$$\tilde{\measuredangle}^\kappa(x\,{}^{p}_{y}) \leqslant \tilde{\measuredangle}^\kappa(x\,{}^{p}_{z});$$

or, equivalently,

$$|\tilde{p} - \tilde{z}| \leqslant |p - z|,$$

where $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}^\kappa(pxy)$, $\tilde{z} \in \,]\tilde{x}\tilde{y}[$, $|\tilde{x} - \tilde{z}| = |x - z|$.

(*c*) (*hinge comparison*) *for any hinge* $[x\,{}^p_y]$*, the angle* $\measuredangle[x\,{}^p_y]$ *is defined and*

$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle^\kappa(x\,{}^p_y),$$

or equivalently

$$\tilde\gamma^\kappa[x\,{}^p_y] \geqslant |p-y|.$$

Moreover,

$$\measuredangle[z\,{}^p_y] + \measuredangle[z\,{}^p_x] \leqslant \pi$$

for any two adjacent hinges $[z\,{}^p_y]$ *and* $[z\,{}^p_x]$*.*

Moreover, in each case, the converse holds if $\mathcal{L}$ *is G-delta geodesic. That is, if one of the conditions (a), (b), or (c) holds in a G-delta geodesic space* $\mathcal{L}$*, then* $\mathcal{L}$ *is* CBB(κ)*.*

A slightly stronger form of (c) is given in 8.29. See also Problem 8.51.

Proof. Since $z \in\,]xy[$, we have $\tilde\measuredangle^\kappa(z\,{}^x_y) = \pi$. Thus, CBB($\kappa$) comparison

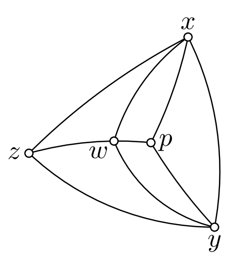
$$\tilde\measuredangle^\kappa(z\,{}^x_y) + \tilde\measuredangle^\kappa(z\,{}^p_x) + \tilde\measuredangle^\kappa(z\,{}^p_y) \leqslant 2\cdot\pi$$

implies

$$\tilde\measuredangle^\kappa(z\,{}^p_x) + \tilde\measuredangle^\kappa(z\,{}^p_y) \leqslant \pi.$$

(a)⇔(b). Follows from Alexandrov's lemma (6.3).

(a) + (b) ⇒ (c). From (b) we get that for $\bar p \in\,]xp]$ and $\bar y \in\,]xy]$, the function $(|x-\bar p|, |x-\bar y|) \mapsto \tilde\measuredangle^\kappa(x\,{}^{\bar p}_{\bar y})$ is nonincreasing in each argument. In particular, $\measuredangle[x\,{}^p_y] = \sup\{\tilde\measuredangle^\kappa(x\,{}^{\bar p}_{\bar y})\}$ is defined and is at least $\tilde\measuredangle^\kappa(x\,{}^p_y)$.



From above and (a), it follows that

$$\measuredangle[z\,{}^p_y] + \measuredangle[z\,{}^p_x] \leqslant \pi.$$

Converse. Assume first that $\mathcal{L}$ is geodesic. Consider a point $w \in\,]pz[$ close to p. From (c), it follows that

$$\measuredangle[w\,{}^x_z] + \measuredangle[w\,{}^x_p] \leqslant \pi \quad \text{and} \quad \measuredangle[w\,{}^y_z] + \measuredangle[w\,{}^y_p] \leqslant \pi.$$

Since $\measuredangle[w\,{}^x_y] \leqslant \measuredangle[w\,{}^x_p] + \measuredangle[w\,{}^y_p]$ (see 6.5), we get

$$\measuredangle[w\,{}^x_z] + \measuredangle[w\,{}^y_z] + \measuredangle[w\,{}^x_y] \leqslant 2\cdot\pi.$$

Applying the first inequality in (c),

$$\tilde\measuredangle^\kappa(w\,{}^x_z) + \tilde\measuredangle^\kappa(w\,{}^y_z) + \tilde\measuredangle^\kappa(w\,{}^x_y) \leqslant 2\cdot\pi.$$

Passing to the limits $w \to p$, we have

$$\tilde\measuredangle^\kappa(p\,{}^x_z) + \tilde\measuredangle^\kappa(p\,{}^y_z) + \tilde\measuredangle^\kappa(p\,{}^x_y) \leqslant 2\cdot\pi.$$

If $\mathcal{L}$ is only G-delta geodesic, we can apply the above arguments to sequences of points $p_n, w_n \to p$, $x_n \to x$, $y_n \to y$ such that $[p_n z]$ exists, $w_n \in\,]zp_n[$ and $[x_n w_n]$, $[y_n w_n]$ exist, and then pass to the limit as $n \to \infty$. □

8.15. Exercise. *Let $\mathcal{L}$ be $\mathbb{R}^m$ with a metric defined by a norm. Show that $\mathcal{L}$ is a complete length* CBB *space if and only if $\mathcal{L} \overset{iso}{=\!=} \mathbb{E}^m$.*

8.16. Exercise. *Assume $\mathcal{L}$ is a complete length* CBB *space, and $[px]$, $[py]$ be two geodesics in the same geodesic direction $\xi \in \Sigma'_p$. Show that*

$$[px] \subset [py] \quad \text{or} \quad [px] \supset [py].$$

8.17. Angle-sidelength monotonicity. *Let p, x, y be points in a complete length* CBB(κ) *space $\mathcal{L}$. Suppose that the model triangle $\tilde{\triangle}^{\kappa}(pxy)$ is defined and there is a geodesic $[xy]$. Then for $\bar{y} \in\,]xy]$ the function*

$$|x - \bar{y}| \mapsto \tilde{\measuredangle}^{\kappa}\left(x\,{}^{p}_{\bar{y}}\right)$$

is nonincreasing.

In particular, if a geodesic $[xp]$ exists and $\bar{p} \in\,]xp]$, then

(*a*) *the function*

$$(|x - \bar{y}|, |x - \bar{p}|) \mapsto \tilde{\measuredangle}^{\kappa}\left(x\,{}^{\bar{p}}_{\bar{y}}\right)$$

is nonincresing in each argument,

(*b*) *The angle $\measuredangle\left[x\,{}^{p}_{y}\right]$ is defined and*

$$\measuredangle\left[x\,{}^{p}_{y}\right] = \sup\left\{\, \tilde{\measuredangle}^{\kappa}\left(x\,{}^{\bar{p}}_{\bar{y}}\right) \,:\, \bar{p} \in\,]xp],\ \bar{y} \in\,]xy] \,\right\}.$$

The proof is contained in the first part of $(a) + (b) \Rightarrow (c)$ of the proof above.

8.18. Exercise. *Let $\mathcal{L}$ be a* CBB(κ) *space, p, x, $y \in \mathcal{L}$ and v, $w \in\,]xy[$. Prove that*

$$\tilde{\measuredangle}^{\kappa}\left(x\,{}^{y}_{p}\right) = \tilde{\measuredangle}^{\kappa}\left(x\,{}^{v}_{p}\right) \iff \tilde{\measuredangle}^{\kappa}\left(x\,{}^{y}_{p}\right) = \tilde{\measuredangle}^{\kappa}\left(x\,{}^{w}_{p}\right).$$

8.19. Advanced exercise. *Construct a geodesic space $\mathcal{X}$ that is not* CBB(0)*, but meets the following condition: for any 3 points p, x, $y \in \mathcal{X}$, there is a geodesic $[xy]$ such that for any $z \in\,]xy[$*

$$\tilde{\measuredangle}^{0}\left(z\,{}^{p}_{x}\right) + \tilde{\measuredangle}^{0}\left(z\,{}^{p}_{y}\right) \leqslant \pi.$$

8.20. Advanced exercise. *Let $\mathcal{L}$ be a complete length space such that for any quadruple p, x, y, $z \in \mathcal{L}$ the following inequality holds,*

$$(1) \qquad |p - x|^2 + |p - y|^2 + |p - z|^2 \geqslant \tfrac{1}{3} \cdot \left[|x - y|^2 + |y - z|^2 + |z - x|^2\right].$$

Prove that $\mathcal{L}$ is CBB(0).

Construct a 4-point metric space $\mathcal{X}$ that satisfies inequality 1 for any relabeling of its points by p, x, y, z, such that $\mathcal{X}$ is not CBB(0).

Assume that for a given triangle $[x^1x^2x^3]$ in a metric space, its κ-model triangle $[\tilde{x}^1\tilde{x}^2\tilde{x}^3] = \tilde{\triangle}^{\kappa}(x^1x^2x^3)$ is defined. We say the triangle $[x^1x^2x^3]$ is κ-*thick* if the natural map (see definition 9.19) $[\tilde{x}^1\tilde{x}^2\tilde{x}^3] \to [x^1x^2x^3]$ is distance non-contracting.

8.21. Exercise. *Prove that any triangle with perimeter $< \varpi\kappa$ in a* CBB(κ) *space is* κ-thick.

8.22. Exercise. (*a*) *Show that any* CBB(0) *space $\mathcal{L}$ satisfies the following condition: for any three points p, q, $r \in \mathcal{L}$, if $\bar{q}$ and $\bar{r}$ are midpoints of geodesics $[pq]$ and $[pr]$ respectively, then $2 \cdot |\bar{q} - \bar{r}| \geqslant |q - r|$.*

(*b*) *Show that there is a metric on $\mathbb{R}^2$ defined by a norm that satisfies the above condition, but is not* CBB(0).

Remarks. Monotonicity of the model angle with respect to adjacent side lengths (8.17) was named the *convexity property* by Alexandrov.

D. Function comparison

In this section we will translate the angle comparison definitions (Theorem 8.14) to a concavity-like property of the distance functions as defined in Section 3F. This is a conceptual step—we reformulate a global geometric condition into an infinitesimal condition on distance functions.

8.23. Theorem. *Let $\mathcal{L}$ be a complete length space. Then the following statements are equivalent:*

(*a*) *$\mathcal{L}$ is* CBB(κ).

(*b*) (*function comparison*) *$\mathcal{L}$ is G-delta geodesic and for any $p \in \mathcal{L}$, the function $f = \operatorname{md}^\kappa \circ \operatorname{dist}_p$ satisfies the differential inequality*

$$f'' \leqslant 1 - \kappa \cdot f.$$

in $\mathrm{B}(p, \varpi\kappa)$.

8.24. Corollary. *A complete G-delta geodesic space $\mathcal{L}$ is* CBB(0) *if and only if for any $p \in \mathcal{L}$, the function $\operatorname{dist}_p^2 : \mathcal{L} \to \mathbb{R}$ is 2-concave.*

Proof of 8.23. Let $[xy]$ be a geodesic in $\mathrm{B}(p, \varpi\kappa)$ and $\ell = |x - y|$. Consider the model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}^\kappa(pxy)$. Set

$$\tilde{r}(t) = |\tilde{p} - \operatorname{geod}_{[\tilde{x}\tilde{y}]}(t)|, \qquad r(t) = |p - \operatorname{geod}_{[xy]}(t)|.$$

Clearly $\tilde{r}(0) = r(0)$ and $\tilde{r}(\ell) = r(\ell)$. Set $\tilde{f} = \operatorname{md}^\kappa \circ \tilde{r}$ and $f = \operatorname{md}^\kappa \circ r$. From 1.1*a* we get that $\tilde{f}'' = 1 - \kappa \cdot \tilde{f}$.

Note that the point-on-side comparison (8.14*b*) for point p and geodesic $[xy]$ is equivalent to $\tilde{r} \leqslant r$. Since md^κ is increasing on $[0, \varpi\kappa)$, $\tilde{r} \leqslant r$ is equivalent to $\tilde{f} \leqslant f$. The latter is Jensen's inequality (3.14*c*) for the function $t \mapsto \operatorname{md}^\kappa |p - \operatorname{geod}_{[xy]}(t)|$ on the interval $[0, \ell]$. Hence the result. □

Recall that Busemann functions are defined in Proposition 6.1.

8.25. Exercise. *Let $\mathcal{L}$ be a complete length* $\mathrm{CBB}(\kappa)$ *space and* $\mathrm{bus}_\gamma\colon \mathcal{L}\to\mathbb{R}$ *be the Busemann function for a half-line* $\gamma\colon [0,\infty)\to\mathcal{L}$*.*

(*a*) *If* $\kappa=0$*, then the Busemann function* bus_γ *is concave.*

(*b*) *If* $\kappa=-1$*, then the function* $f=\exp\circ\,\mathrm{bus}_\gamma$ *satisfies*

$$f''-f\leqslant 0.$$

Exercise 9.28 is an analogous statement for upper curvature bound.

E. Development

In this section we reformulate the function comparison using a more geometric language based on the definition of development given below.

This definition appears in [**17**] and an earlier form of it can be found in [**105**]. The definition is somewhat lengthy, but it defines a useful comparison object for a curve. Often it is easier to write proofs in terms of function comparison but think in terms of developments.

8.26. Lemma-definition. *Let* $\kappa\in\mathbb{R}$*,* $\mathcal{X}$ *be a metric space,* $\gamma\colon\mathbb{I}\to\mathcal{X}$ *be a* 1*-Lipschitz curve,* $p\in\mathcal{X}$*, and* $\tilde p\in\mathbb{M}^2(\kappa)$*. Assume* $0<|p-\gamma(t)|<\varpi\kappa$ *for all* $t\in\mathbb{I}$*. Then there exists a unique up to rotation curve* $\tilde\gamma\colon\mathbb{I}\to\mathbb{M}^2(\kappa)$*, parametrized by arc-length, such that* $|\tilde p-\tilde\gamma(t)|=|p-\gamma(t)|$ *for all* t *and the direction of* $[\tilde p\tilde\gamma(t)]$ *monotonically turns around* $\tilde p$ *counterclockwise as* t *increases.*

If p*,* $\tilde p$*,* γ*, and* $\tilde\gamma$ *are as above, then* $\tilde\gamma$ *is called the* κ-development *of* γ *with respect to* p*; the point* $\tilde p$ *is called the* basepoint *of the development. When we say that the* κ*-development of* γ *with respect to* p *is defined, we always assume that* $0<|p-\gamma(t)|<\varpi\kappa$ *for all* $t\in\mathbb{I}$*.*

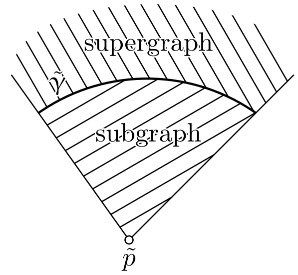
Proof. Consider the functions $\rho,\theta\colon\mathbb{I}\to\mathbb{R}$ defined as

$$\rho(t)=|p-\gamma(t)|,\qquad \theta(t)=\int\limits_{t_0}^{t}\frac{\sqrt{1-(\rho')^2}}{\mathrm{sn}^\kappa\rho},$$

where $t_0\in\mathbb{I}$ is a fixed number and $\int$ denotes Lebesgue integral. Since γ is 1-Lipshitz, so is $\rho(t)$, and thus the function θ is defined and nondecreasing.

It is straightforward to check that (ρ,θ) uniquely describe $\tilde\gamma$ in polar coordinates on $\mathbb{M}^2(\kappa)$ with center at $\tilde p$. ☐

We need the following analogs of sub- and super-graphs and convex/concave functions, adapted to polar coordinates in $\mathbb{M}^2(\kappa)$.



8.27. Definition. *Let $\tilde\gamma\colon \mathbb{I}\to\mathbb{M}^2(\kappa)$ be a curve and $\tilde p\in\mathbb{M}^2(\kappa)$ be such that there is a unique geodesic $[\tilde p\,\tilde\gamma(t)]$ for any $t\in\mathbb{I}$ and the direction of $[\tilde p\,\tilde\gamma(t)]$ turns monotonically as t grows.*

The set formed by all geodesics from $\tilde p$ to the points on $\tilde\gamma$ is called the subgraph *of $\tilde\gamma$ with respect to $\tilde p$.*

The set of all points $\tilde x\in\mathbb{M}^2(\kappa)$ such that a geodesic $[\tilde p\tilde x]$ intersects $\tilde\gamma$ is called the supergraph *of $\tilde\gamma$ with respect to $\tilde p$.*

The curve $\tilde\gamma$ is called convex (concave) with respect to $\tilde p$ *if the subgraph (supergraph) of $\tilde\gamma$ with respect to $\tilde p$ is convex.*

The curve $\tilde\gamma$ is called locally convex (concave) with respect to $\tilde p$ *if for any interior value t_0 in $\mathbb{I}$ there is a subsegment $(a,b)\subset\mathbb{I}$, $(a,b)\ni t_0$, such that the restriction $\tilde\gamma|_{(a,b)}$ is convex (concave) with respect to $\tilde p$.*

Note that if $\kappa>0$, then the supergraph of a curve is the subgraph with respect to the opposite point.

For developments, all the notions above will be considered with respect to their basepoints. In particular, if $\tilde\gamma$ is a development, we will say it is (*locally*) *convex* if it is (locally) convex with respect to its basepoint.

8.28. Development comparison. *A complete G-delta geodesic space $\mathcal{L}$ is* CBB(κ) *if and only if for any point $p\in\mathcal{L}$ and any geodesic γ in* $\mathrm{B}(p,\varpi\kappa)\setminus\{p\}$, *its κ-development with respect to p is convex.*

A simpler proof of the only-if part can be built on the adjacent angle comparison ~~(8.14*a*)~~. We use a longer proof since it also implies the short hinge lemma (8.29).

Proof.

Only-if part. Let $\gamma\colon [0,T]\to \mathrm{B}(p,\varpi\kappa)\setminus\{p\}$ be a unit-speed geodesic in $\mathcal{L}$.

Consider a fine partition

$$0=t_0<t_1<\dots<t_n=T.$$

Set $x_i=\gamma(t_i)$ and choose a point

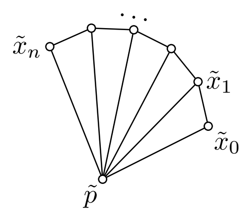
$$p'\in\mathrm{Str}(x_0,x_1,\dots,x_n)$$

sufficiently close to p; recall that geodesics $[p'x_i]$ exist for all i (see 8.11).

Let us construct a chain of model triangles $[\tilde p'\tilde x_{i-1}\tilde x_i]=\tilde\triangle^{\kappa}(p'x_{i-1}x_i)$ in such a way that direction $[\tilde p'\tilde x_i]$ turns counterclockwise as i grows. By the hinge comparison (8.14*c*), we have

$$
\begin{aligned}
\measuredangle\left[\tilde x_i\,{}^{\tilde x_{i-1}}_{\tilde p'}\right] + \measuredangle\left[\tilde x_i\,{}^{\tilde x_{i+1}}_{\tilde p'}\right] &= \tilde\measuredangle^\kappa\left(x_i\,{}^{x_{i-1}}_{p'}\right) + \tilde\measuredangle^\kappa\left(x_i\,{}^{x_{i+1}}_{p'}\right) \\
&\leqslant \measuredangle\left[x_i\,{}^{x_{i-1}}_{p'}\right] + \measuredangle\left[x_i\,{}^{x_{i+1}}_{p'}\right] \\
&\leqslant \pi.
\end{aligned}
\tag{1}
$$

Further, since γ is a unit-speed geodesic, we have



$$
\sum_{i=1}^{n} |x_{i-1} - x_i| \leqslant |p' - x_0| + |p' - x_n|. \tag{2}
$$

Since $p' \notin \gamma$, the development comparison implies that $\tilde p'$ does not lie on the polygonal line $\tilde x_0 \dots \tilde x_n$.

If $\kappa \leqslant 0$, then 2 implies that

$$
\theta \coloneqq \sum_{i=1}^{n} \measuredangle\left[\tilde p'\,{}^{\tilde x_i}_{\tilde x_{i-1}}\right] \leqslant \pi. \tag{3}
$$

In the case $\kappa > 0$, the proof of 3 requires more work. Applying rescaling, we can assume that $\kappa = 1$. Since γ lies in $B_\pi(p')$, the point-on-side comparison implies that the antipodal point of $\tilde p'$ does not lie on the polygonal line $\tilde x_0 \dots \tilde x_n$.

Consider the space L glued from the solid model triangles $[\tilde p' \tilde x_0 \tilde x_1]$, $\dots$, $[\tilde p' \tilde x_{n-1} \tilde x_n]$ along the corresponding sides. Note that θ is the total angle of L at $\tilde p'$. We can assume that L has nonempty interior. Otherwise all the triangles are degenerate, and therefore $\theta = 0$; the latter holds since $\tilde p'$ is not on the polygonal line $\tilde x_0 \dots \tilde x_n$.

Consider a minimizing geodesic $[\tilde x_0 \tilde x_n]_L$. By 2 we may assume that $\tilde p' \notin [\tilde x_0 \tilde x_n]_L$. Further if the geodesic $[\tilde x_0 \tilde x_n]_L$ contains one of the points $\tilde x_1, \dots, \tilde x_{n-1}$, then it coincides with the polygonal line $\tilde x_0 \dots \tilde x_n$. (In particular, we have equality in 1 for each i.) In this case, the sum in the left-hand side of 2 must be at most π; otherwise $[\tilde x_0 \tilde x_n]_L$ is not minimizing. Therefore 3 follows. In the remaining case $[\tilde x_0 \tilde x_n]_L$ meets the boundary of L only at its ends. In this case, $|\tilde x_0 - \tilde x_n|_L \leqslant \pi$; otherwise $[\tilde x_0 \tilde x_n]_L$ is not minimizing. Whence 3 follows.

Inequalities 1 and 3 imply that the polygon $[\tilde p' \tilde x_0 \tilde x_1 \dots \tilde x_n]$ is convex.

Let us take finer and finer partitions and pass to the limit of the polygon $\tilde p' \tilde x_0 \tilde x_1 \dots \tilde x_n$ as $p' \to p$. We obtain a convex curvilinear triangle formed by a curve $\tilde\gamma\colon [0, T] \to \mathbb{M}^2(\kappa)$—the limit of polygonal line $\tilde x_0 \tilde x_1 \dots \tilde x_n$ and two geodesics $[\tilde p'\, \tilde\gamma(0)]$, $[\tilde p'\, \tilde\gamma(T)]$. Since $[\tilde p' \tilde x_0 \tilde x_1 \dots \tilde x_n]$ is convex, the natural parametrization of $\tilde x_0 \tilde x_1 \dots \tilde x_n$ converges to the natural parametrization of $\tilde\gamma$. Thus $\tilde\gamma$ is the κ-development of γ with respect to p. This proves the only-if part of (8.28).

If part. Assuming convexity of the development, we will prove the point-on-side comparison (8.14*b*). We can assume that $p \notin [xy]$; otherwise the statement is trivial.

Set $T=|x-y|$ and $\gamma(t)=\mathrm{geod}_{[xy]}(t)$; note that γ is a geodesic in $\mathrm{B}(p,\varpi\kappa)\setminus\{p\}$. Let $\tilde\gamma\colon [0,T]\to\mathbb{M}^2(\kappa)$ be the κ-development with base $\tilde p$ of γ with respect to p. Take a partition $0=t_0<t_1<\dots<t_n=T$, and set

$$\tilde y_i=\tilde\gamma(t_i)\quad\text{and}\quad \tau_i=|\tilde y_0-\tilde y_1|+|\tilde y_1-\tilde y_2|+\dots+|\tilde y_{i-1}-\tilde y_i|.$$

Since $\tilde\gamma$ is convex, for a fine partition we have that the polygonal line $\tilde y_0\tilde y_1\dots\tilde y_n$ is also convex. Applying Alexandrov's lemma (6.3) inductively to pairs of model triangles

$$\tilde\triangle^{\kappa}\{\tau_{i-1},|\tilde p-\tilde y_0|,|\tilde p-\tilde y_{i-1}|\}$$

and

$$\tilde\triangle^{\kappa}\{|\tilde y_{i-1}-\tilde y_i|,|\tilde p-\tilde y_{i-1}|,|\tilde p-\tilde y_i|\}$$

we obtain that the sequence $\tilde\measuredangle^{\kappa}\{|\tilde p-\tilde y_i|;|\tilde p-\tilde y_0|,\tau_i\}$ is non increasing.

For finer and finer partitions we have

$$\max_i\{|\tau_i-t_i|\}\to 0.$$

Thus, the point-on-side comparison (8.14*b*) follows. □

Note that in the proof of if part we could use a slightly weaker version of the hinge comparison (8.14*c*). Namely, we proved the following lemma, which will be needed later in the proof of the globalization theorem (8.31).

8.29. Short hinge lemma. *Let $\mathcal{L}$ be a complete G-delta geodesic space such that for any hinge $[x\,{}^p_y]$ in $\mathcal{L}$ the angle $\measuredangle[x\,{}^p_y]$ is defined, and*

$$\measuredangle[x\,{}^p_y]+\measuredangle[x\,{}^p_z]\leqslant\pi$$

for any two adjacent hinges.

Assume that for any hinge $[x\,{}^p_y]$ in $\mathcal{L}$ we have

$$|p-x|+|x-y|<\varpi\kappa\quad\Rightarrow\quad\measuredangle[x\,{}^p_y]\geqslant\tilde\measuredangle^{\kappa}(x\,{}^p_y).$$

Then $\mathcal{L}$ is CBB(κ).

F. Local definitions and globalization

In this section we discuss locally CBB(κ) spaces. In particular, we prove the *globalization theorem*: equivalence of local and global definitions for complete length spaces.

The following theorem summarizes equivalent definitions of locally CBB(κ) spaces.

8.30. Theorem. *Let $\mathcal{X}$ be a complete length space and $p\in\mathcal{X}$. Then the following conditions are equivalent:*

1) *(local* CBB(κ) *comparison) there is* $R_1 > 0$ *such that the comparison*

$$\tilde{\measuredangle}^\kappa\left(q\,{}^{x^1}_{x^2}\right) + \tilde{\measuredangle}^\kappa\left(q\,{}^{x^2}_{x^3}\right) + \tilde{\measuredangle}^\kappa\left(q\,{}^{x^3}_{x^1}\right) \leqslant 2\cdot\pi$$

holds for any q, x^1, x^2, $x^3 \in \mathrm{B}(p, R_1)$.

2) *(local Kirszbraun property) there is* $R_2 > 0$ *such that for any 3-point subset* F_3 *and any 4-point subset* $F_4 \supset F_3$ *in* $\mathrm{B}(p, R_2)$, *any short map* $f : F_3 \to \mathbb{M}^2(\kappa)$ *can be extended to a short map* $\bar f : F_4 \to \mathbb{M}^2(\kappa)$ *(so* $f = \bar f|_{F_3}$*).*

3) *(local function comparison) there is* $R_3 > 0$ *such that* $\mathrm{B}(p, R_3)$ *is G-delta geodesic and for any* $q \in \mathrm{B}(p, R_3)$, *the function* $f = \mathrm{md}^\kappa \circ \mathrm{dist}_q$ *satisfies* $f'' \leqslant 1 - \kappa\cdot f$ *in* $\mathrm{B}(p, R_3)$.

4) *(local adjacent angle comparison) there is* $R_4 > 0$ *such that* $\mathrm{B}(p, R_4)$ *is G-delta geodesic, and if* q *and a geodesic* $[xy]$ *lie in* $\mathrm{B}(p, R_4)$ *and* $z \in \,]xy[$, *then*

$$\tilde{\measuredangle}^\kappa\left(z\,{}^q_x\right) + \tilde{\measuredangle}^\kappa\left(z\,{}^q_y\right) \leqslant \pi.$$

5) *(local point-on-side comparison) there is* $R_5 > 0$ *such that* $\mathrm{B}(p, R_5)$ *is G-delta geodesic and if* q *and a geodesic* $[xy]$ *lie in* $\mathrm{B}(p, R_5)$ *and* $z \in \,]xy[$, *then*

$$\tilde{\measuredangle}^\kappa\left(x\,{}^q_y\right) \leqslant \tilde{\measuredangle}^\kappa\left(x\,{}^q_z\right);$$

or, equivalently,

$$|\tilde p - \tilde z| \leqslant |p - z|,$$

where $[\tilde p\tilde x\tilde y] = \tilde\triangle^\kappa(pxy)$, $\tilde z \in \,]\tilde x\tilde y[$, $|\tilde x - \tilde z| = |x - z|$.

6) *(local hinge comparison) there is* $R_6 > 0$ *such that* $\mathrm{B}(p, R_6)$ *is G-delta geodesic and if* $x \in \mathrm{B}(p, R_6)$, *then for any hinge* $\left[x\,{}^q_y\right]$, *the angle* $\measuredangle\left[x\,{}^q_y\right]$ *is defined, and*

$$\measuredangle\left[x\,{}^q_y\right] + \measuredangle\left[x\,{}^q_z\right] \leqslant \pi$$

for any two adjacent hinges. Moreover, if a hinge $\left[x\,{}^q_y\right]$ *lies in* $\mathrm{B}(p, R_6)$, *then*

$$\measuredangle\left[x\,{}^q_y\right] \geqslant \tilde{\measuredangle}^\kappa\left(x\,{}^q_y\right),$$

or, equivalently,

$$\tilde{\mathrm{Y}}^\kappa\left[x\,{}^q_y\right] \geqslant |q - y|.$$

7) *(local development comparison) there is* $R_7 > 0$ *such that* $\mathrm{B}(p, R_7)$ *is G-delta geodesic, and if a geodesic* γ *lies in* $\mathrm{B}(p, R_7)$ *and* $q \in \mathrm{B}(p, R_7) \setminus \gamma$, *then the* κ*-development* $\tilde\gamma$ *with respect to* q *is convex.*

Moreover, for each pair $i, j \in \{1, 2, \dots, 7\}$ *we can assume that*

$$R_i > \tfrac{1}{9}\cdot R_j.$$

The proofs of each of these equivalences repeat the proofs of the corresponding global equivalences in localized form; see the proofs of Theorems 8.14, 8.23, 8.26, 10.1.

8.31. Globalization theorem. *Any complete length locally* CBB(κ) *space is* CBB(κ).

In the two-dimensional case this theorem was proved by Paolo Pizzetti [**134**]; later it was reproved independently by Alexandr Alexandrov [**17**]. Victor Toponogov [**154**] proved it for Riemannian manifolds of all dimensions. In the above generality, the theorem first appears in the paper of Michael Gromov, Yuriy Burago, and Grigory Perelman [**44**]; simplifications and modifications were given by Conrad Plaut [**136**], Katsuhiro Shiohama [**148**], and in the book of Dmitry Burago, Yuriy Burago, and Sergei Ivanov [**37**]. A generalization for non-complete but geodesic spaces was obtained by the third author [**127**]; namely it solves the following exercise:

8.32. Advanced exercise. *Show that any locally* CBB(κ) *geodesic space is* CBB(κ).

The following corollary of the globalization theorem says that the expression "space with curvature $\geqslant \kappa$" makes sense.

8.33. Corollary. *Let $\mathcal{L}$ be a complete length space. Then $\mathcal{L}$ is* CBB(K) *if and only if $\mathcal{L}$ is* CBB(κ) *for any $\kappa <$ K.*

Proof. Note that if K $\leqslant 0$, this statement follows directly from the definition of an Alexandrov space (8.2) and monotonicity of the function $\kappa \mapsto \tilde\measuredangle^\kappa\left(x\,{}^y_z\right)$ (1.1*d*).

The *if* part also follows directly from the definition.

For K > 0, the angle $\tilde\measuredangle^{\mathrm{K}}\left(x\,{}^y_z\right)$ might be undefined while $\tilde\measuredangle^\kappa\left(x\,{}^y_z\right)$ is defined. However, $\tilde\measuredangle^{\mathrm{K}}\left(x\,{}^y_z\right)$ is defined if x, y, and z are sufficiently close to each other. Thus, if K $> \kappa$, then any CBB(K) space is locally CBB(κ). It remains to apply the globalization theorem. □

8.34. Corollary. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space. Assume that a space $\mathcal{M}$ is the target space of a submetry from $\mathcal{L}$. Then $\mathcal{M}$ is a complete length space* CBB(κ) *space.*

In particular, if $G \curvearrowright \mathcal{L}$ is an isometric group action with closed orbits, then the quotient space $\mathcal{L}/G$ is a complete length CBB(κ) *space.*

Proof. This follows from the globalization theorem and Theorem 8.5. □

Our proof of ~~the~~ globalization theorem (8.31) is based on presentations in [**136**] and [**37**]; this proof was rediscovered independently by Urs Lang and Viktor Schroeder [**99**]. We will need the short hinge lemma (8.29) together with the following two lemmas. The following lemma says that if comparison holds for all small hinges, then it holds for slightly bigger hinges near the given point.

8.35. Key lemma. *Let $\kappa \in \mathbb{R}$, $0 < \ell \leqslant \varpi\kappa$, $\mathcal{X}$ be a complete geodesic space and $p \in \mathcal{X}$ be a point such that $\mathrm{B}(p, 2 \cdot \ell)$ is locally $\mathrm{CBB}(\kappa)$.*

Assume that for any point $q \in \mathrm{B}(p, \ell)$ the comparison

$$\measuredangle\left[x\,{}^{y}_{q}\right] \geqslant \tilde\measuredangle^{\kappa}\left(x\,{}^{y}_{q}\right)$$

holds for any hinge $\left[x\,{}^{y}_{q}\right]$ with $|x - y| + |x - q| < \tfrac{2}{3} \cdot \ell$. Then the comparison

$$\measuredangle\left[x\,{}^{p}_{q}\right] \geqslant \tilde\measuredangle^{\kappa}\left(x\,{}^{p}_{q}\right)$$

holds for any hinge $\left[x\,{}^{p}_{q}\right]$ with $|x - p| + |x - q| < \ell$.

Proof. It is sufficient to prove the inequality

$$\tilde\gamma^{\kappa}\left[x\,{}^{p}_{q}\right] \geqslant |p - q| \tag{1}$$

for any hinge $\left[x\,{}^{p}_{q}\right]$ with $|x - p| + |x - q| < \ell$.

Fix q. Consider a hinge $\left[x\,{}^{p}_{q}\right]$ such that

$$\tfrac{2}{3} \cdot \ell \leqslant |p - x| + |x - q| < \ell.$$

First we construct a new smaller hinge $\left[x'\,{}^{p}_{q}\right]$ with

$$|p - x| + |x - q| \geqslant |p - x'| + |x' - q|, \tag{2}$$

such that

$$\tilde\gamma^{\kappa}\left[x\,{}^{p}_{q}\right] \geqslant \tilde\gamma^{\kappa}\left[x'\,{}^{p}_{q}\right]. \tag{3}$$

Assume $|x - q| \geqslant |x - p|$; otherwise switch the roles of p and q in the following construction. Take $x' \in [xq]$ such that

$$|p - x| + 3 \cdot |x - x'| = \tfrac{2}{3} \cdot \ell. \tag{4}$$

Choose a geodesic $[x'p]$ and consider the hinge $\left[x'\,{}^{p}_{q}\right]$ formed by $[x'p]$ and $[x'q] \subset [xq]$. (In fact by 8.38 the condition $[x'q] \subset [xq]$ always holds.) Then 2 follows from the triangle inequality.

Further, note that we have $x, x' \in \mathrm{B}(p, \ell) \cap \mathrm{B}(q, \ell)$ and moreover

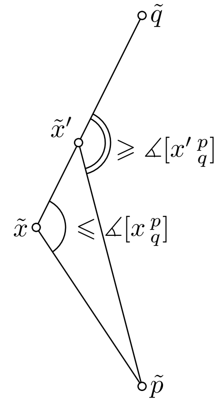
$$|p - x| + |x - x'| < \tfrac{2}{3} \cdot \ell, \qquad |p - x'| + |x' - x| < \tfrac{2}{3} \cdot \ell.$$

In particular,

$$\measuredangle\left[x\,{}^{p}_{x'}\right] \geqslant \tilde\measuredangle^{\kappa}\left(x\,{}^{p}_{x'}\right) \quad \text{and} \quad \measuredangle\left[x'\,{}^{p}_{x}\right] \geqslant \tilde\measuredangle^{\kappa}\left(x'\,{}^{p}_{x}\right). \tag{5}$$

Now, let $[\tilde x\tilde x'\tilde p]=\tilde\triangle^{\kappa}(xx'p)$. Take $\tilde q$ on the extension of $[\tilde x\tilde x']$ beyond x' such that $|\tilde x-\tilde q|=|x-q|$ (and therefore $|\tilde x'-\tilde q|=|x'-q|$). From 5,

$$\measuredangle\left[x\,{}^{p}_{q}\right]=\measuredangle\left[x\,{}^{p}_{x'}\right]\geqslant\tilde\measuredangle^{\kappa}\left(x\,{}^{p}_{x'}\right)\quad\Rightarrow\quad\tilde{\curlyvee}^{\kappa}\left[x\,{}^{q}_{p}\right]\geqslant|\tilde p-\tilde q|.$$

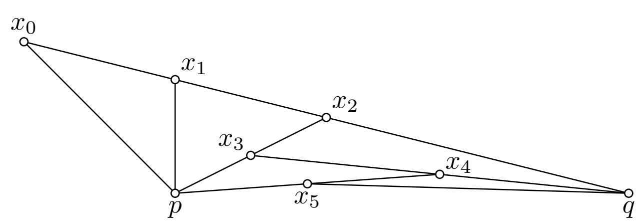


Hence

$$\begin{aligned}\measuredangle\left[\tilde x'\,{}^{\tilde p}_{\tilde q}\right]&=\pi-\tilde\measuredangle^{\kappa}\left(x'\,{}^{p}_{x}\right)\\&\geqslant\pi-\measuredangle\left[x'\,{}^{p}_{x}\right]\\&=\measuredangle\left[x'\,{}^{p}_{q}\right],\end{aligned}$$

and 3 follows.

Let us continue the proof. Set $x_0=x$. Let us apply inductively the above construction to get a sequence of hinges $\left[x_n\,{}^{p}_{q}\right]$ with $x_{n+1}=x_n'$. From 3, we have that the sequence $s_n=\tilde{\curlyvee}^{\kappa}\left[x_n\,{}^{p}_{q}\right]$ is nonincreasing.



The sequence might terminate at x_n only if $|p-x_n|+|x_n-q|<\frac{2}{3}\cdot\ell$. In this case, by the assumptions of the lemma, $\tilde{\curlyvee}^{\kappa}\left[x_n\,{}^{p}_{q}\right]\geqslant|p-q|$. Since the sequence s_n is nonincreasing, inequality 1 follows.

Otherwise, the sequence $r_n=|p-x_n|+|x_n-q|$ is nonincreasing and $r_n\geqslant\frac{2}{3}\cdot\ell$ for all n. Note that by construction, the distances $|x_n-x_{n+1}|$, $|x_n-p|$, and $|x_n-q|$ are bounded away from zero for all large n. Indeed, since on each step we move x_n toward to the point p or q that is further away, the distances $|x_n-p|$ and $|x_n-q|$ become about the same. Namely, by 4, we have that $|p-x_n|-|x_n-q|\leqslant\frac{2}{9}\cdot\ell$ for all large n. Since $|p-x_n|+|x_n-q|\geqslant\frac{2}{3}\cdot\ell$, we have $|x_n-p|\geqslant\frac{\ell}{100}$ and $|x_n-q|\geqslant\frac{\ell}{100}$. Further, since $r_n\geqslant\frac{2}{3}\cdot\ell$, 4 implies that $|x_n-x_{n+1}|>\frac{\ell}{100}$.

Since the sequence r_n is nonincreasing, it converges. In particular, $r_n-r_{n+1}\to0$ as $n\to\infty$. It follows that $\tilde\measuredangle^{\kappa}\left(x_n\,{}^{p_n}_{x_{n+1}}\right)\to\pi$, where $p_n=p$ if $x_{n+1}\in[x_nq]$, and otherwise $p_n=q$. Since $\measuredangle\left[x_n\,{}^{p_n}_{x_{n+1}}\right]\geqslant\tilde\measuredangle^{\kappa}\left(x_n\,{}^{p_n}_{x_{n+1}}\right)$, we have $\measuredangle\left[x_n\,{}^{p_n}_{x_{n+1}}\right]\to\pi$ as $n\to\infty$.

It follows that

$$r_n-s_n=|p-x_n|+|x_n-q|-\tilde{\curlyvee}^{\kappa}\left[x_n\,{}^{p}_{q}\right]\to0.$$

(Here we used that $\ell \leqslant \varpi\kappa$.) Together with the triangle inequality

$$|p - x_n| + |x_n - q| \geqslant |p - q|$$

this yields

$$\lim_{n\to\infty} \tilde{\gamma}^{\kappa}\left[x_n\,{}^{p}_{q}\right] \geqslant |p - q|.$$

Applying monotonicity of the sequence $s_n = \tilde{\gamma}^{\kappa}\left[x_n\,{}^{p}_{q}\right]$, we obtain 1. □

The final part of the proof above resembles the *cat's cradle construction* introduced by the first author and Richard Bishop [**5**].

The following lemma works in all complete spaces; it will be used as a substitute for the existence of a minimum point of a continuous function on a compact space.

8.36. Lemma on almost minimum. *Let $\mathcal{X}$ be a complete metric space. Suppose $r\colon \mathcal{X} \to \mathbb{R}$ is a function, $p \in \mathcal{X}$, and $\varepsilon > 0$. Assume that the function r is strictly positive in $\overline{\mathrm{B}}[p, \frac{1}{\varepsilon^2} \cdot r(p)]$ and $\underline{\lim}_n\, r(x_n) > 0$ for any convergent sequence $x_n \to x \in \overline{\mathrm{B}}[p, \frac{1}{\varepsilon^2} \cdot r(p)]$.*

Then there is a point $p^* \in \overline{\mathrm{B}}[p, \frac{1}{\varepsilon^2} \cdot r(p)]$ such that

(a) $r(p^*) \leqslant r(p)$ and

(b) $r(x) > (1 - \varepsilon) \cdot r(p^*)$ for any $x \in \overline{\mathrm{B}}[p^*, \frac{1}{\varepsilon} \cdot r(p^*)]$.

Proof. Assume the statement is wrong. Then for any $x \in \mathrm{B}(p, \frac{1}{\varepsilon^2} \cdot r(p))$ with $r(x) \leqslant r(p)$, there is a point $x' \in \mathcal{X}$ such that

$$|x - x'| < \tfrac{1}{\varepsilon} \cdot r(x) \quad \text{and} \quad r(x') \leqslant (1 - \varepsilon) \cdot r(x).$$

Take $x_0 = p$ and consider a sequence of points x_n such that $x_{n+1} = x'_n$. Clearly

$$|x_{n+1} - x_n| \leqslant \tfrac{r(p)}{\varepsilon} \cdot (1 - \varepsilon)^n \quad \text{and} \quad r(x_n) \leqslant r(p) \cdot (1 - \varepsilon)^n.$$

In particular, $|p - x_n| < \frac{1}{\varepsilon^2} \cdot r(p)$. Therefore the sequence x_n is Cauchy, $x_n \to x \in \overline{\mathrm{B}}[p, \frac{1}{\varepsilon^2} \cdot r(p)]$ and $\lim_n r(x_n) = 0$, a contradiction. □

Proof of the globalization theorem (8.31). Exactly the same argument as in the proof of Theorem 8.11 shows that $\mathcal{L}$ is G-delta geodesic. By Theorem 8.30-6, for any hinge $\left[x\,{}^{p}_{y}\right]$ in $\mathcal{L}$ the angle $\measuredangle\left[x\,{}^{p}_{y}\right]$ is defined and

$$\measuredangle\left[x\,{}^{p}_{y}\right] + \measuredangle\left[x\,{}^{p}_{z}\right] \leqslant \pi$$

for any two adjacent hinges.

Let us denote by $\mathrm{ComRad}(p, \mathcal{L})$ (which stands for *comparison radius* of $\mathcal{L}$ at p) the maximal value (possibly ∞) such that the comparison

$$\measuredangle\left[x\,{}^{p}_{y}\right] \geqslant \tilde{\measuredangle}^{\kappa}\left(x\,{}^{p}_{y}\right)$$

holds for any hinge $[x\,{}^{p}_{y}]$ with $|p-x|+|x-y|<\operatorname{ComRad}(p,\mathcal{L})$.

As follows from 8.30-3, $\operatorname{ComRad}(p,\mathcal{L})>0$ for any $p\in\mathcal{L}$ and

$$\varliminf_{n\to\infty}\operatorname{ComRad}(p_n,\mathcal{L})>0$$

for any converging sequence of points $p_n\to p$. That makes it possible to apply the lemma on almost minimum (8.36) to the function $p\mapsto\operatorname{ComRad}(p,\mathcal{L})$.

According to ~~the~~ short hinge lemma (8.29), it is sufficient to show that

(6) $$s_0=\inf_{p\in\mathcal{L}}\operatorname{ComRad}(p,\mathcal{L})\geqslant\varpi\kappa\quad\text{for any}\quad p\in\mathcal{L}.$$

We argue by contradiction, assuming that 6 does not hold.

The rest of the proof is easier for geodesic spaces and easier still for compact spaces. Thus we give three different arguments for each of these cases.

Compact case. Assume $\mathcal{L}$ is compact.

By Theorem 8.30-3, $s_0>0$. Take a point $p^*\in\mathcal{L}$ such that $r^*=\operatorname{ComRad}(p^*,\mathcal{L})$ is sufficiently close to s_0 (p^* such that $s_0\leqslant r^*<\min\{\varpi\kappa,\frac32\cdot s_0\}$ will do). Then ~~the~~ key lemma (8.35) applied to p^* and ℓ slightly bigger than r^* (say, such that $r^*<\ell<\min\{\varpi\kappa,\frac32\cdot s_0\}$) implies that

$$\measuredangle\left[x\,{}^{p^*}_{q}\right]\geqslant\tilde\measuredangle^\kappa\left(x\,{}^{p^*}_{q}\right)$$

for any hinge $\left[x\,{}^{p^*}_{q}\right]$ such that $|p^*-x|+|x-q|<\ell$. Thus $r^*\geqslant\ell$, a contradiction.

Geodesic case. Assume $\mathcal{L}$ is geodesic.

Fix a small $\varepsilon>0$ ($\varepsilon=0.0001$ will do). Apply ~~the~~ lemma on almost minimum ~~(8.36)~~ to find a point $p^*\in\mathcal{L}$ such that

$$r^*=\operatorname{ComRad}(p^*,\mathcal{L})<\varpi\kappa$$

and

(7) $$\operatorname{ComRad}(q,\mathcal{L})>(1-\varepsilon)\cdot r^*$$

for any $q\in\overline{\mathrm{B}}[p^*,\frac1\varepsilon\cdot r^*]$.

Applying ~~the~~ key lemma (8.35) for p^* and ℓ slightly bigger than r^* leads to a contradiction.

General case. Let us construct $p^*\in\mathcal{L}$ as in the previous case. Since $\mathcal{L}$ is not geodesic, we cannot apply the key lemma directly. Instead, let us pass to the ultrapower $\mathcal{L}^\omega$, which is a geodesic space (see 4.9).

In Theorem 8.30, inequality 7 implies that condition 8.30-1 holds for some fixed $R_1=\frac{r^*}{100}>0$ at any point $q\in\overline{\mathrm{B}}[p^*,\frac{1}{2\cdot\varepsilon}\cdot r^*]\subset\mathcal{L}$. Therefore a similar statement is true in the ultrapower $\mathcal{L}^\omega$; that is, for any point $q_\omega\in\overline{\mathrm{B}}[p^*,\frac{1}{2\cdot\varepsilon}\cdot r^*]\subset\mathcal{L}^\omega$, condition 8.30-1 holds for, say, $R_1=\frac{r^*}{101}$.

Note that $r^* \geqslant \operatorname{ComRad}(p^*, \mathcal{L}^\omega)$. Therefore we can apply the lemma on almost minimum at the point p^* to the function $x \mapsto \operatorname{ComRad}(x, \mathcal{L}^\omega)$ and $\varepsilon' = \sqrt{\varepsilon} = 0.01$.

For the resulting point $p^{**} \in \mathcal{L}^\omega$, we have

$$r^{**} = \operatorname{ComRad}(p^{**}, \mathcal{L}) < \varpi\kappa, \quad \text{and} \quad \operatorname{ComRad}(q_\omega, \mathcal{L}^\omega) > (1-\varepsilon') \cdot r^{**}$$

for any $q_\omega \in \overline{\mathrm{B}}[p^{**}, \tfrac{1}{\varepsilon'} \cdot r^{**}]$. Thus applying ~~the~~ key lemma (8.35) for p^{**} and for ℓ slightly bigger than r^{**} leads to a contradiction. □

G. Properties of geodesics and angles

Remark. All proofs in this section can be easily modified to use only the local definition of CBB spaces without use of ~~the~~ globalization theorem (8.31).

8.37. Geodesics do not split. *In a* CBB *space, geodesics do not bifurcate.*

More precisely, let $\mathcal{L}$ be a CBB *space and* $[px]$, $[py]$ *be two geodesics. Then:*

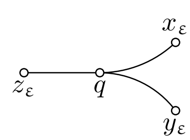
(*a*) *If there is* $\varepsilon > 0$ *such that* $\operatorname{geod}_{[px]}(t) = \operatorname{geod}_{[py]}(t)$ *for all* $t \in [0, \varepsilon)$*, then* $[px] \subset [py]$ *or* $[py] \subset [px]$.

(*b*) *If* $\measuredangle\left[p\,{}^x_y\right] = 0$*, then* $[px] \subset [py]$ *or* $[py] \subset [px]$.

8.38. Corollary. *Let $\mathcal{L}$ be a* CBB *space. Then the restriction of any geodesic in $\mathcal{L}$ to a proper segment is the unique minimal geodesic joining its endpoints.*

In case $\kappa \leqslant 0$, the proof is easier, since the model triangles are always defined. To deal with $\kappa > 0$ we have to argue locally.

Proof of 8.37.



(*a*). Let $t_{\max}$ be the maximal value such that $\operatorname{geod}_{[px]}(t) = \operatorname{geod}_{[py]}(t)$ for all $t \in [0, t_{\max})$. Since geodesics are continuous, $\operatorname{geod}_{[px]}(t_{\max}) = \operatorname{geod}_{[py]}(t_{\max})$. Let

$$q = \operatorname{geod}_{[px]}(t_{\max}) = \operatorname{geod}_{[py]}(t_{\max}).$$

We must show that $t_{\max} = \min\{|p-x|, |p-y|\}$.

If that is not true, choose a sufficiently small $\varepsilon > 0$ such that the points

$$x_\varepsilon = \operatorname{geod}_{[px]}(t_{\max} + \varepsilon) \quad \text{and} \quad y_\varepsilon = \operatorname{geod}_{[py]}(t_{\max} + \varepsilon)$$

are distinct. Let

$$z_\varepsilon = \operatorname{geod}_{[px]}(t_{\max} - \varepsilon) = \operatorname{geod}_{[py]}(t_{\max} - \varepsilon).$$

Clearly, $\tilde\measuredangle^\kappa\left(q\,{}^{z_\varepsilon}_{x_\varepsilon}\right) = \tilde\measuredangle^\kappa\left(q\,{}^{z_\varepsilon}_{y_\varepsilon}\right) = \pi$. Thus from the CBB($\kappa$) comparison (8.2), $\tilde\measuredangle^\kappa\left(q\,{}^{x_\varepsilon}_{y_\varepsilon}\right) = 0$ and thus $x_\varepsilon = y_\varepsilon$, a contradiction.

(*b*). From hinge comparison 8.14*c*,

$$\measuredangle\left[p\,{}^{x}_{y}\right]=0 \quad\Rightarrow\quad \tilde\measuredangle^{\kappa}\left(p\,{}^{\mathrm{geod}_{[px]}(t)}_{\mathrm{geod}_{[py]}(t)}\right)=0$$

and thus $\mathrm{geod}_{[px]}(t)=\mathrm{geod}_{[py]}(t)$ for all small t. Therefore we can apply (*a*). □

8.39. Adjacent angle lemma. *Let $\mathcal{L}$ be a* CBB *space. Assume that two hinges $\left[z\,{}^{x}_{p}\right]$ and $\left[z\,{}^{y}_{p}\right]$ in $\mathcal{L}$ are adjacent. Then*

$$\measuredangle\left[z\,{}^{p}_{y}\right]+\measuredangle\left[z\,{}^{p}_{x}\right]=\pi.$$

Proof. From the hinge comparison (8.14*c*) we have that both angles $\measuredangle\left[z\,{}^{p}_{y}\right]$ and $\measuredangle\left[z\,{}^{p}_{x}\right]$ are defined and

$$\measuredangle\left[z\,{}^{p}_{y}\right]+\measuredangle\left[z\,{}^{p}_{x}\right]\leqslant\pi.$$

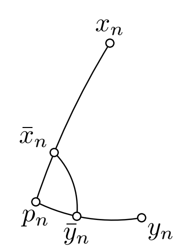
Clearly $\measuredangle\left[z\,{}^{x}_{y}\right]=\pi$. Thus the result follows from the triangle inequality for angles (6.5). □

8.40. Angle semicontinuity. *Suppose $\mathcal{L}_n$ is a sequence of* CBB(κ) *spaces and $\mathcal{L}_n\to\mathcal{L}_\omega$ as $n\to\omega$. Assume that a sequence of hinges $\left[p_n\,{}^{x_n}_{y_n}\right]$ in $\mathcal{L}_n$ converges to a hinge $\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right]$ in $\mathcal{L}_\omega$. Then*

$$\measuredangle\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right]\leqslant\lim_{n\to\omega}\measuredangle\left[p_n\,{}^{x_n}_{y_n}\right].$$

Proof. From 8.17,

$$\measuredangle\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right]=\sup\left\{\tilde\measuredangle^{\kappa}\left(p_\omega\,{}^{\bar x_\omega}_{\bar y_\omega}\right):\bar x_\omega\in\,]p_\omega x_\omega],\ \bar y_\omega\in\,]p_\omega x_\omega]\right\}.$$



For fixed $\bar x_\omega\in\,]p_\omega x_\omega]$ and $\bar y_\omega\in\,]p_\omega x_\omega]$, choose $\bar x_n\in\,]px_n]$ and $\bar y_n\in\,]py_n]$ so that $\bar x_n\to\bar x_\omega$ and $\bar y_n\to\bar y_\omega$ as $n\to\omega$. Clearly

$$\tilde\measuredangle^{\kappa}\left(p_n\,{}^{\bar x_n}_{\bar y_n}\right)\to\tilde\measuredangle^{\kappa}\left(p_\omega\,{}^{\bar x_\omega}_{\bar y_\omega}\right)$$

as $n\to\omega$.

From the hinge comparison (8.14*c*), $\measuredangle\left[p_n\,{}^{x_n}_{y_n}\right]\geqslant\tilde\measuredangle^{\kappa}\left(p_n\,{}^{\bar x_n}_{\bar y_n}\right)$. Hence the result. □

8.41. Angle continuity. *Let $\mathcal{L}_1,\mathcal{L}_2,\dots$ be a sequence of complete length* CBB(κ) *spaces, and $\mathcal{L}_n\to\mathcal{L}_\omega$ as $n\to\omega$. Assume that sequences of points p_n, x_n, y_n in $\mathcal{L}_n$ converge to points $p_\omega, x_\omega, y_\omega$ in $\mathcal{L}_\omega$ as $n\to\omega$, and the following two conditions hold:*

(*a*) $p_\omega\in\mathrm{Str}(x_\omega)$,

(*b*) $p_\omega\in\mathrm{Str}(y_\omega)$ *or* $y_\omega\in\mathrm{Str}(p_\omega)$.

Then

$$\measuredangle\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right] = \lim_{n\to\omega} \measuredangle\left[p_n\,{}^{x_n}_{y_n}\right].$$

Proof. By Corollary 8.33, we may assume that $\varkappa \leqslant 0$.

By Plaut's theorem (8.11), the hinge $\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right]$ is uniquely defined. Therefore the hinges $\left[p_n\,{}^{x_n}_{y_n}\right]$ converge to $\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right]$ as $n \to \omega$. Hence by the angle semicontinuity (8.40), we have

$$\measuredangle\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right] \leqslant \lim_{n\to\omega} \measuredangle\left[p_n\,{}^{x_n}_{y_n}\right].$$

It remains to show that

$$\measuredangle\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right] \geqslant \lim_{n\to\omega} \measuredangle\left[p_n\,{}^{x_n}_{y_n}\right]. \tag{1}$$

Fix $\varepsilon > 0$. Since $p_\omega \in \mathrm{Str}(x_\omega)$, there is a point $q_\omega \in \mathcal{L}_\omega$ such that

$$\tilde\measuredangle^\varkappa\left(p_\omega\,{}^{x_\omega}_{q_\omega}\right) > \pi - \varepsilon.$$

The hinge comparison (8.14*c*) implies that

$$\measuredangle\left[p_\omega\,{}^{x_\omega}_{q_\omega}\right] > \pi - \varepsilon. \tag{2}$$

By the triangle inequality for angles (6.5),

$$\begin{aligned} \measuredangle\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right] &\geqslant \measuredangle\left[p_\omega\,{}^{x_\omega}_{q_\omega}\right] - \measuredangle\left[p_\omega\,{}^{y_\omega}_{q_\omega}\right] \\ &> \pi - \varepsilon - \measuredangle\left[p_\omega\,{}^{y_\omega}_{q_\omega}\right]. \end{aligned} \tag{3}$$

Note that we can assume in addition that $q_\omega \in \mathrm{Str}(p_\omega)$. Choose $q_n \in \mathcal{L}_n$ such that $q_n \to q_\omega$ as $n \to \omega$. Note that by angle semicontinuity we again have

$$\begin{aligned} \measuredangle\left[p_\omega\,{}^{x_\omega}_{q_\omega}\right] &\leqslant \lim_{n\to\omega} \measuredangle\left[p_n\,{}^{x_n}_{q_n}\right], \\ \measuredangle\left[p_\omega\,{}^{y_\omega}_{q_\omega}\right] &\leqslant \lim_{n\to\omega} \measuredangle\left[p_n\,{}^{y_n}_{q_n}\right]. \end{aligned} \tag{4}$$

By the CBB($\varkappa$) comparison (8.2) and 8.17*b*,

$$\measuredangle\left[p_n\,{}^{x_n}_{y_n}\right] + \measuredangle\left[p_n\,{}^{y_n}_{q_n}\right] + \measuredangle\left[p_n\,{}^{x_n}_{q_n}\right] \leqslant 2\cdot\pi$$

for all n. Together with 4, 2 and 3, this implies

$$\begin{aligned} \lim_{n\to\omega} \measuredangle\left[p_n\,{}^{x_n}_{y_n}\right] &\leqslant 2\cdot\pi - \lim_{n\to\omega} \measuredangle\left[p_n\,{}^{x_n}_{q_n}\right] - \lim_{n\to\omega} \measuredangle\left[p_n\,{}^{y_n}_{q_n}\right] \\ &\leqslant 2\cdot\pi - \measuredangle\left[p_\omega\,{}^{x_\omega}_{q_\omega}\right] - \measuredangle\left[p_\omega\,{}^{y_\omega}_{q_\omega}\right] \\ &< \measuredangle\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right] + 2\cdot\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, 1 follows. □

8.42. First variation formula. *Let $\mathcal{L}$ be a complete length* CBB *space. For any point q and any geodesic $[px]$ in $\mathcal{L}$ with $p \neq q$, we have*

$$|q-\mathrm{geod}_{[px]}(t)| = |q-p| - t\cdot\cos\phi + o(t), \tag{5}$$

where ϕ is the infimum of angles between $[px]$ and all geodesics from p to q in the ultrapower $\mathcal{L}^\omega$.

Remark. If $\mathcal{L}$ is a proper space, then $\mathcal{L}^\omega = \mathcal{L}$; see Section 4B. Therefore the infimum ϕ is achieved on a particular geodesic from p to q.

As a corollary we obtain the following classical result:

8.43. Strong angle lemma. *Let $\mathcal{L}$ be a complete length* CBB *space ~~and~~ $p\neq q\in\mathcal{L}$ be such that there is unique geodesic from p to q in the ultrapower $\mathcal{L}^\omega$. Then for any hinge $[p\,{}^q_x]$ we have*

$$\measuredangle[p\,{}^q_x] = \lim_{\substack{\bar x\to p\\ \bar x\in\,]px]}} \tilde\measuredangle^\kappa(p\,{}^q_{\bar x}) \tag{6}$$

for any $\kappa\in\mathbb{R}$ such that $|p-q|<\varpi\kappa$.

In particular, 6 holds if $p\in\mathrm{Str}(q)$ as well as if $q\in\mathrm{Str}(p)$.

Remark.

- The above lemma is essentially due to Alexandrov. The right-hand side in 6 is called the *strong angle* of the hinge $[p\,{}^q_x]$. Note that in a general metric space the angle and the strong angle of the same hinge might differ.
- As follows from Corollary 4.12, if there is a unique geodesic $[pq]$ in the ultrapower $\mathcal{L}^\omega$, then $[pq]$ lies in $\mathcal{L}$.

Proof of 8.43. The first statement follows directly from the first variation formula (8.42) and the definition of model angle (see Section 6C).

The second statement follows from Plaut's theorem (8.11) applied to $\mathcal{L}^\omega$. (Note that according to Proposition 8.4, $\mathcal{L}^\omega$ is a complete length CBB space.) $\square$

Proof of 8.42. By Corollary 8.33, we can assume that $\kappa\leqslant 0$. The inequality

$$|q-\mathrm{geod}_{[px]}(t)| \leqslant |q-p| - t\cdot\cos\phi + o(t)$$

follows from the first variation inequality (6.7). Thus, it is sufficient to show that

$$|q-\mathrm{geod}_{[px]}(t)| \geqslant |q-p| - t\cdot\cos\phi + o(t).$$

Assume the contrary. Then there is $\varepsilon > 0$ such that $\phi+\varepsilon < \pi$, and for a sequence $t_n \to 0+$ we have

$$|q - \mathrm{geod}_{[px]}(t_n)| < |q-p| - t_n \cdot \cos(\phi - \varepsilon). \tag{7}$$

Let $x_n = \mathrm{geod}_{[px]}(t_n)$. Clearly

$$\tilde\measuredangle^\kappa\left(x_n\,{}^p_q\right) > \pi - \phi + \tfrac{\varepsilon}{2}$$

for all large n.

Assume $\mathcal{L}$ is geodesic. Choose a sequence of geodesics $[x_nq]$. Let $[x_nq] \to [pq]_{\mathcal{L}^\omega}$ as $n \to \omega$ (in general $[pq]$ might lie in $\mathcal{L}^\omega$). Applying both parts of hinge comparison (8.14*c*), we have $\measuredangle\left[x_n\,{}^q_x\right] < \phi - \frac{\varepsilon}{2}$ for all large n. According to 8.40, the angle between $[pq]$ and $[px]$ is at most $\phi - \frac{\varepsilon}{2}$, a contradiction.

Finally, if $\mathcal{L}$ is not geodesic, choose a sequence $q_n \in \mathrm{Str}(x_n)$, such that $q_n \to q$ and the inequality

$$\tilde\measuredangle^\kappa\left(x_n\,{}^{p}_{q_n}\right) > \pi - \phi + \tfrac{\varepsilon}{2}$$

still holds. Then the same argument as above shows that $[x_nq_n]$ ω-converges to a geodesic $[pq]_{\mathcal{L}^\omega}$ from p to q having angle at most $\phi - \frac{\varepsilon}{2}$ with $[px]$. $\square$

H. On positive lower bound

In this section we consider CBB(κ) spaces for $\kappa > 0$. Applying rescaling we can assume that $\kappa = 1$.

The following theorem states that if one ignores a few exceptional spaces, then the diameter of a space with positive lower curvature bound is bounded. Many authors (but not us) exclude these spaces in the definition of Alexandrov space with positive lower curvature bound.

8.44. On diameter of a space. *Let $\mathcal{L}$ be a complete length* CBB(1) *space. Then either*

(*a*) $\operatorname{diam}\mathcal{L} \leqslant \pi$;

(*b*) ~~*or*~~ *$\mathcal{L}$ is isometric to one of the following exceptional spaces:*

- (a) *real line* $\mathbb{R}$,
- (b) *a half-line* $\mathbb{R}_{\geqslant 0}$,
- (c) *a closed interval* $[0,a] \in \mathbb{R}$, $a > \pi$,
- (d) *a circle* $\mathbb{S}^1_a$ *of length* $a > 2\cdot\pi$.

Proof. Assume that $\mathcal{L}$ is a geodesic space and $\operatorname{diam}\mathcal{L} > \pi$. Choose $x, y \in \mathcal{L}$ so that $|x-y| = \pi+\varepsilon$, $0 < \varepsilon < \frac{\pi}{4}$. By moving y slightly, we can also assume that the geodesic $[xy]$ is unique; to prove this, use either Plaut's theorem (8.11) or the fact that geodesics do not split (8.37). Let z be the midpoint of the geodesic $[xy]$.

Consider the function $f = \operatorname{dist}_x + \operatorname{dist}_y$. As follows from Lemma 8.45, f is concave in $\mathrm{B}(z, \frac{\varepsilon}{4})$. Let $p \in \mathrm{B}(z, \frac{\varepsilon}{4})$. Choose a geodesic $[zp]$, and let $h(t) = f \circ \operatorname{geod}_{[zp]}(t)$ and $\ell = |z - p|$. Clearly h is concave. From the adjacent angle lemma (8.39), we have $h^+(0) = 0$. Therefore h is nonincreasing which means that

$$|x - p| + |y - p| = h(\ell) \leqslant h(0) = |x - y|.$$

Since the geodesic $[xy]$ is unique this means that $p \in [xy]$, and hence $\mathrm{B}(z, \frac{\varepsilon}{4})$ only contains points of $[xy]$.

Since in CBB spaces, geodesics do not bifurcate (8.37*a*), it follows that all of $\mathcal{L}$ coincides with the maximal extension of $[xy]$ as a local geodesic γ (which might not be minimizing). In other words, $\mathcal{L}$ is isometric to a 1-dimensional Riemannian manifold with possibly nonempty boundary. From this, it is easy to see that $\mathcal{L}$ falls into one of the exceptional spaces described in the theorem.

Lastly, if $\mathcal{L}$ is not geodesic and $\operatorname{diam} \mathcal{L} > \pi$, then the above argument applied to $\mathcal{L}^\omega$ yields that each metric component of $\mathcal{L}^\omega$ is isometric to one of the exceptional spaces. As all of those spaces are proper, $\mathcal{L}$ is a metric component in $\mathcal{L}^\omega$. ☐

8.45. Lemma. *Let $\mathcal{L}$ be a complete length* CBB(1) *space ~~and~~ $p \in \mathcal{L}$. Then* $\operatorname{dist}_p : \mathcal{L} \to \mathbb{R}$ *is concave in* $\mathrm{B}(p, \pi) \setminus \mathrm{B}(p, \frac{\pi}{2})$.

In particular, if $\operatorname{diam} \mathcal{L} \leqslant \pi$*, then the complements* $\mathcal{L} \setminus \mathrm{B}(p, r)$ *and* $\mathcal{L} \setminus \overline{\mathrm{B}}[p, r]$ *are convex for any* $r > \frac{\pi}{2}$.

Proof. This is a consequence of 8.23*b*. ☐

8.46. Exercise. *Let $\mathcal{L}$ be a complete length* CBB(1) *space that is not exceptional (that is,* $\operatorname{diam} \mathcal{L} \leqslant \pi$*). Assume that a group G acts on $\mathcal{L}$ by isometries, has closed orbits, and*

$$\operatorname{diam}(\mathcal{L}/G) > \tfrac{\pi}{2}.$$

Show that the action of G has a fixed point in $\mathcal{L}$.

8.47. Advanced exercise. *Let $\mathcal{L}$ be a complete length* CBB(1) *space Show that $\mathcal{L}$ contains at most 3 points with space of directions $\leqslant \frac{1}{2} \cdot \mathbb{S}^n$ (see Definition* 5.9*).*

8.48. On perimeter of a triple. *Suppose $\mathcal{L}$ is a complete length* CBB(1) *space and* $\operatorname{diam} \mathcal{L} \leqslant \pi$*. Then the perimeter of any triple of points $p, q, r \in \mathcal{L}$ is at most $2 \cdot \pi$.*

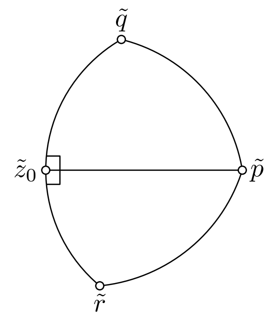
Proof. Arguing by contradiction, suppose

$$|p - q| + |q - r| + |r - p| > 2 \cdot \pi \tag{1}$$

for $p, q, r \in \mathcal{L}$. Rescaling the space slightly, we can assume that $\operatorname{diam} \mathcal{L} < \pi$, but ~~the~~ inequality 1 still holds. By Corollary 8.33, after rescaling $\mathcal{L}$ is still CBB(1).

Since $\mathcal{L}$ is G-delta geodesic (8.11), it is sufficient to consider the case when there is a geodesic $[qr]$.

First note that since $\operatorname{diam} \mathcal{L} < \pi$, by 8.23*b* the function



$$y(t) = \operatorname{md}^1 |p - \operatorname{geod}_{[qr]}(t)|$$

satisfies the differential inequality $y'' \leqslant 1 - y$.

Take $z_0 \in [qr]$ so that the restriction $\operatorname{dist}_p|_{[qr]}$ attains its maximum at z_0, and set $t_0 = |q - z_0|$ so $z_0 = \operatorname{geod}_{[qr]}(t_0)$. Consider the following model configuration: two geodesics $[\tilde p \tilde z_0]$, $[\tilde q \tilde r]$ in $\mathbb{S}^2$ such that

$$|\tilde p - \tilde z_0| = |p - z_0|, \quad |\tilde q - \tilde r| = |q - r|,$$
$$|\tilde z_0 - \tilde q| = |z_0 - q|, \quad |\tilde z_0 - \tilde r| = |z_0 - q|$$

and

$$\measuredangle\left[\tilde z_0\,{}^{\tilde q}_{\tilde p}\right] = \measuredangle\left[\tilde z_0\,{}^{\tilde r}_{\tilde p}\right] = \tfrac{\pi}{2}.$$

Clearly, $\bar y(t) = \operatorname{md}^1 |\tilde p - \operatorname{geod}_{[\tilde q \tilde r]}(t)|$ satisfies $\bar y'' = 1 - \bar y$ and $\bar y'(t_0) = 0$, $\bar y(t_0) = y(t_0)$. Since z_0 is a maximum point, $y(t) \leqslant y(t_0) + o(t - t_0)$; thus, $\bar y(t)$ is a barrier for $y(t) = \operatorname{md}^1 |p - \operatorname{geod}_{[qr]}(t)|$ at t_0 by 3.14*d*. From the barrier inequality 3.14*d*, we get

$$|\tilde p - \operatorname{geod}_{[\tilde q \tilde r]}(t)| \geqslant |p - \operatorname{geod}_{[qr]}(t)|,$$

and hence $|\tilde p - \tilde q| \geqslant |p - q|$ and $|\tilde p - \tilde r| \geqslant |p - r|$.

Therefore $|p - q| + |q - r| + |r - p|$ cannot exceed the perimeter of the spherical triangle $[\tilde p \tilde q \tilde r]$. In particular,

$$|p - q| + |q - r| + |r - p| \leqslant 2 \cdot \pi$$

—a contraction. □

Let $\kappa > 0$. Consider the following extension $\tilde\measuredangle^{\kappa+}(*\,{}^*_*)$ of the model angle function $\tilde\measuredangle^{\kappa}(*\,{}^*_*)$. This definition works well for CBB spaces; for CAT spaces there is a similar but definition. Some authors define the comparison angle to be $\tilde\measuredangle^{\kappa+}(*\,{}^*_*)$.

8.49. Definition of extended angle. *Suppose p, q, r are points in a metric space, and $p \ne q$, $p \ne r$. Let*

$$\tilde\measuredangle^{\kappa+}\left(p\,{}^q_r\right) = \sup\left\{\, \tilde\measuredangle^{\mathrm K}\left(p\,{}^q_r\right) : \mathrm K \leqslant \kappa \,\right\}.$$

The value $\tilde\measuredangle^{\kappa+}\left(p\,{}^q_r\right)$ is called the extended model angle *of the triple p, q, r.*

8.50. Extended angle comparison. *Let $\kappa > 0$ and $\mathcal{L}$ be a complete length* $\mathrm{CBB}(\kappa)$ *space. Then for any hinge $\left[p\,{}^q_r\right]$ we have $\measuredangle\left[p\,{}^q_r\right] \geqslant \tilde\measuredangle^{\kappa+}\left(p\,{}^q_r\right)$.*

Moreover, the extended model angle $\tilde\measuredangle^{\kappa+}\left(p\,{}^q_r\right)$ can be calculated using the following rule:

(*a*) $\tilde\measuredangle^{\kappa+}\left(p\,{}^q_r\right) = \tilde\measuredangle^{\kappa}\left(p\,{}^q_r\right)$ *if* $\tilde\measuredangle^{\kappa}\left(p\,{}^q_r\right)$ *is defined;*

(*b*) $\tilde\measuredangle^{\kappa+}\left(p\,{}^q_r\right) = \tilde\measuredangle^{\kappa+}\left(p\,{}^r_q\right) = 0$ *if* $|p-q| + |q-r| = |p-r|$*;*

(*c*) $\tilde\measuredangle^{\kappa+}\left(p\,{}^q_r\right) = \pi$ *if none of the above is applicable.*

Proof. From Corollary 8.33, $\mathrm{K} < \kappa$ implies that any complete length CBB(K) space is CBB(κ); thus the extended angle comparison follows from the definition.

The rule for calculating extended angle is an easy consequence of its definition. □

I. Remarks

The question whether the first part of 8.14*c* suffices to conclude that $\mathcal{L}$ is CBB(κ) is a long-standing open problem (possibly dating back to Alexandrov); it was first stated in [**37**, footnote in 4.1.5].

8.51. Open question. *Let* $\mathcal{L}$ *be a complete geodesic space (you can also assume that* $\mathcal{L}$ *is homeomorphic to* $\mathbb{S}^2$ *or* $\mathbb{R}^2$*) such that for any hinge* $\left[x\,{}^p_y\right]$ *in* $\mathcal{L}$*, the angle* $\measuredangle\left[x\,{}^p_y\right]$ *is defined and*

$$\measuredangle\left[x\,{}^p_y\right] \geqslant \tilde\measuredangle^0\left(x\,{}^p_y\right).$$

Is it true that $\mathcal{L}$ *is* CBB(0)*?*

Examples and constructions. Let us list important sources of examples of CBB spaces. We do not provide all the proofs and some proofs are deferred to later chapters.

Complete Riemannian manifolds with sectional curvature at least κ, their *Gromov–Hausdorff limits*, as well as their *ultralimits*, are CBB(κ). This statement follows from 8.4, 5.16 and the Toponogov comparison which is a partial case of ~~the~~ globalization theorem (8.31). For example, if M is a Riemannian manifold of nonnegative sectional curvature, then the limit of its rescalings $\frac{1}{n}\cdot M$ as $n \to \infty$ is CBB(0); this is the so-called *asymptotic cone* of M.

Most of applications to Riemannian geometry are based on the described sources and the following corollary of Gromov's selection theorem (5.6) and the Bishop–Gromov inequality.

8.52. Gromov compactness theorem. *Let* M_n *be a sequence of Riemannian manifolds with sectional curvature at least* κ*. Then, for any choice of marked points* $p_n \in M_n$*, a subsequence of* M_n *admits a Gromov–Hausdorff convergence such that the corresponding subsequence of* p_n *converges.*

By Corollary 8.34, the *target of submetry* from CBB(κ) is CBB(κ). In particular, if M is a Riemannian manifold with sectional curvature at least κ and G is a closed subgroup of isometries on M, then the *quotient space* M/G is CBB(κ).

Yet another source is given by *convex hypersurfaces*. Namely, suppose M is a complete Riemannian manifold of sectional curvature at least $\varkappa$. Let N be a closed (as a subset) hypersurface in M. Then N with the induced inner metric is CBB($\varkappa$). In particular, any closed convex hypersurface in $\mathbb{R}^n$ is CBB(0).

The smooth case of this statement follows from the Gauss formula. This has been generalized by Sergei Buyalo [**46**] to the nonsmooth case and sharpened by the authors [**9**].

Let us mention that an analogous statement about convex hypersurfaces in CBB($\varkappa$) space is completely open. Namely, it is unknown if *the boundary of a complete finite-dimensional* CBB($\varkappa$) *length space has to be* CBB($\varkappa$).

Further, CBB spaces behave nicely with respect to several natural constructions. For example, the product of CBB(0) spaces is a CBB(0). Also, the Euclidean cone over CBB(1) space is a CBB(0). These are the first examples of the so-called *warped products* that are discussed in Chapter 11; a general statement is given in 11.10. Perelman's doubling theorem can be considered as a partial case; it states that if $\mathcal{L}$ is a finite-dimensional CBB($\varkappa$) length space with nonempty boundary, then its *doubling across the boundary* is CBB($\varkappa$) as well.

More conceptually, *Wasserstein space* of order 2 over CBB(0) space is CBB(0). Also, there is a natural metric on the *space of metric-measure spaces* that makes it CBB(0) space; it was constructed by Karl-Theodor Sturm [**152**].

Among less important examples, let us mention *polyhedral spaces*, an if-and-only-if condition is given in 12.5. Also, in addition to Perelman's doubling theorem, there are several versions of gluing theorems [**67**, **97**, **131**]; they give conditions that guarantee that gluing of two (or more) spaces is CBB($\varkappa$).

Chapter 9

Fundamentals of curvature bounded above

A. Four-point comparison.

9.1. Four-point comparison. *A quadruple of points* p^1, p^2, x^1, x^2 *in a metric space satisfies* CAT(κ) comparison *if*

(*a*) $\tilde\measuredangle^\kappa\left(p^1\,{}^{x^1}_{x^2}\right) \leqslant \tilde\measuredangle^\kappa\left(p^1\,{}^{p^2}_{x^1}\right) + \tilde\measuredangle^\kappa\left(p^1\,{}^{p^2}_{x^2}\right)$, *or*

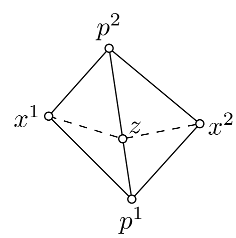
(*b*) $\tilde\measuredangle^\kappa\left(p^2\,{}^{x^1}_{x^2}\right) \leqslant \tilde\measuredangle^\kappa\left(p^2\,{}^{p^1}_{x^1}\right) + \tilde\measuredangle^\kappa\left(p^2\,{}^{p^1}_{x^2}\right)$, *or*

(*c*) *one of the six model angles*

$$\begin{aligned}&\tilde\measuredangle^\kappa\left(p^1\,{}^{x^1}_{x^2}\right), \quad \tilde\measuredangle^\kappa\left(p^1\,{}^{p^2}_{x^1}\right), \quad \tilde\measuredangle^\kappa\left(p^1\,{}^{p^2}_{x^2}\right),\\ &\tilde\measuredangle^\kappa\left(p^2\,{}^{x^1}_{x^2}\right), \quad \tilde\measuredangle^\kappa\left(p^2\,{}^{p^1}_{x^1}\right), \quad \tilde\measuredangle^\kappa\left(p^2\,{}^{p^1}_{x^2}\right)\end{aligned}$$

is undefined.

Here is a more intuitive formulation.



9.2. Reformulation. *Let* $\mathcal{X}$ *be a metric space. A quadruple* $p^1, p^2, x^1, x^2 \in \mathcal{X}$ *satisfies* CAT(κ) *comparison if one of the following holds:*

(*a*) *One of the triples* (p^1, p^2, x^1) *or* (p^1, p^2, x^2) *has perimeter* $> 2 \cdot \varpi\kappa$.

(*b*) *If* $[\tilde p^1\tilde p^2\tilde x^1]=\tilde\triangle^{\kappa}(p^1p^2x^1)$ *and* $[\tilde p\tilde p^2\tilde x^2]=\tilde\triangle^{\kappa}p^1p^2x^2$, *then*

$$|\tilde x^1-\tilde z|+|\tilde z-\tilde x^2|\geqslant|x^1-x^2|,$$

for any $\tilde z\in[\tilde p^1\tilde p^2]$.

9.3. Definition. *Let* $\mathcal{U}$ *be a metric space.*

(*a*) $\mathcal{U}$ *is* CAT(κ) *if any quadruple in* $\mathcal{X}$ *satisfies* CAT(κ) *comparison.*

(*b*) $\mathcal{U}$ *is* locally CAT(κ) *if any point* $q\in\mathcal{U}$ *admits a neighborhood* $\Omega\ni q$ *such that any quadruple in* Ω *satisfies* CAT(κ) *comparison.*

(*c*) $\mathcal{U}$ *is a* CAT *space if* $\mathcal{U}$ *is* CAT(κ) *for some* $\kappa\in\mathbb{R}$.

The condition $\mathcal{U}$ is CAT(κ) should be understood as "$\mathcal{U}$ has global curvature $\leqslant\kappa$". In Proposition 9.18, it will be shown that this formulation makes sense; in particular, if $\kappa\leqslant K$, then any CAT(κ) space is CAT(K).

This terminology was introduced by Michael Gromov; CAT stands for Élie Cartan, Alexandr Alexandrov, and Victor Toponogov. Originally these spaces were called $\mathfrak{R}_\kappa$ *domains*; this is Alexandrov's terminology and is still in use.

9.4. Exercise. *Let* $\mathcal{U}$ *be a metric space. Show that* $\mathcal{U}$ *is* CAT(κ) *if and only if every quadruple of points in* $\mathcal{U}$ *admits a labeling by* (p,x^1,x^2,x^3) *such that the three angles* $\tilde\measuredangle^\kappa\left(p\,{}^{x^1}_{x^2}\right)$, $\tilde\measuredangle^\kappa\left(p\,{}^{x^2}_{x^3}\right)$ *and* $\tilde\measuredangle^\kappa\left(p\,{}^{x^1}_{x^3}\right)$ *satisfy all three triangle inequalities or one of these angles is undefined.*

9.5. Exercise. *Show that* $\mathcal{U}$ *is* CAT(κ) *if and only if for any quadruple of points* p^1, p^2, x^1, x^2 *in* $\mathcal{U}$ *such that* $|p^1-p^2|,|x^1-x^2|\leqslant\varpi\kappa$, *there is a quadruple* q^1, q^2, y^1, y^2 *in* $\mathbb{M}^m(\kappa)$ *such that*

$$|q^1-q^2|=|p^1-p^2|,\qquad|y^1-y^2|=|x^1-x^2|,\qquad|q^i-y^j|\leqslant|p^i-x^j|$$

for any i *and* j.

9.6. Advanced exercise. *Let* $\mathcal{U}$ *be a complete length space such that for any quadruple* $p,q,x,y\in\mathcal{L}$ *the following inequality holds*

$$|p-q|^2+|x-y|^2\leqslant|p-x|^2+|p-y|^2+|q-x|^2+|q-y|^2. \tag{1}$$

Prove that $\mathcal{U}$ *is* CAT(0).

Construct a 4-point metric space $\mathcal{X}$ *that satisfies inequality 1 for any relabeling of its points by* p, q, x, y, *and such that* $\mathcal{X}$ *is not* CAT(0).

The next proposition follows directly from Definition 9.3 and the definitions of ultralimit and ultrapower; see Section 4B for the related definitions. Recall that ω denotes a fixed selective ultrafilter on $\mathbb{N}$.

9.7. Proposition. *Let* $\mathcal{U}_n$ *be a* CAT(κ_n) *space for each* $n\in\mathbb{N}$. *Assume* $\mathcal{U}_n\to\mathcal{U}_\omega$ *and* $\kappa_n\to\kappa_\omega$ *as* $n\to\omega$. *Then* $\mathcal{U}_\omega$ *is* CAT(κ_ω).

Moreover, a metric space $\mathcal{U}$ *is* CAT(κ) *if and only if so is its ultrapower* $\mathcal{U}^\omega$.

B. Geodesics

9.8. Uniqueness of geodesics. *In a complete length* CAT(κ) *space, pairs of points at distance* $< \varpi\kappa$ *are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

Proof. Fix a complete length CAT(κ) space $\mathcal{U}$. Fix two points $p^1, p^2 \in \mathcal{U}$ such that

$$|p^1 - p^2|_{\mathcal{U}} < \varpi\kappa.$$

Choose a sequence of approximate midpoints z_n between p^1 and p^2; that is,

(1) $$|p^1 - z_n|, |p^2 - z_n| \to \tfrac{1}{2} \cdot |p^1 - p^2| \quad \text{as} \quad n \to \infty.$$

By the law of cosines, $\tilde\measuredangle^\kappa\left(p^1\,{}^{z_n}_{p^2}\right)$ and $\tilde\measuredangle^\kappa\left(p^2\,{}^{z_n}_{p^1}\right)$ are arbitrarily small when n is sufficiently large.

Let us apply CAT(κ) comparison (9.1) to the quadruple p^1, p^2, z_n, z_k with large n and k. We conclude that $\tilde\measuredangle^\kappa\left(p\,{}^{z_n}_{z_k}\right)$ is arbitrarily small when n, k are sufficiently large and p is either p^1 or p^2. By 1 and the law of cosines, the sequence z_n converges.

Since $\mathcal{U}$ is complete, the sequence z_n converges to a midpoint between p^1 and p^2. By Lemma 2.13 we obtain the existence of a geodesic $[p^1 p^2]$.

Now suppose $p^1_n \to p^1$, $p^2_n \to p^2$ as $n \to \infty$. Let z_n be the midpoint of a geodesic $[p^1_n p^2_n]$ and z be the midpoint of a geodesic $[p^1 p^2]$.

It suffices to show that

(2) $$|z_n - z| \to 0 \quad \text{as} \quad n \to \infty.$$

By the triangle inequality, the z_n are approximate midpoints between p^1 and p^2. Apply the CAT(κ) comparison (9.1) to the quadruple p^1, p^2, z_n, z. For $p = p^1$ or $p = p^2$, we see that $\tilde\measuredangle^\kappa\left(p\,{}^{z_n}_{z}\right)$ is arbitrarily small when n is sufficiently large. By the law of cosines, 2 follows. □

9.9. Exercise. *Let $\mathcal{U}$ be a complete length* CAT *space. Assume $\mathcal{U}$ is a topological manifold. Show that any geodesic in $\mathcal{U}$ can be extended as a two-side infinite local geodesic.*

Moreover the same holds for any locally geodesic locally CAT *space $\mathcal{U}$ with nontrivial local homology groups at any point; the latter holds in particular if $\mathcal{U}$ is a homological manifold.*

9.10. Exercise. *Assume $\mathcal{U}$ is a locally compact geodesic* CAT *space with extendable geodesics; that is, any geodesic in $\mathcal{U}$ can be extended to a both-sided infinite local geodesic.*

Show that the space of geodesic directions Σ'_p is complete for any $p \in \mathcal{U}$.

By the uniqueness of geodesics (9.8), we have the following.

9.11. Corollary. *Any complete length* $\mathrm{CAT}(\kappa)$ *space is* ϖ^κ*-geodesic.*

9.12. Proposition. *The completion* $\bar{\mathcal{U}}$ *of any geodesic* $\mathrm{CAT}(\kappa)$ *space* $\mathcal{U}$ *is a complete length* $\mathrm{CAT}(\kappa)$ *space.*

Moreover, $\mathcal{U}$ *is a geodesic* $\mathrm{CAT}(\kappa)$ *space if and only if there is a complete length* $\mathrm{CAT}(\kappa)$ *space* $\bar{\mathcal{U}}$ *that contains a* ϖ^κ*-convex dense set isometric to* $\mathcal{U}$*.*

Proof. By Theorem 9.8, in order to show that $\bar{\mathcal{U}}$ is $\mathrm{CAT}(\kappa)$, it is sufficient to verify that the completion of a length space is a length space; this is straightforward.

For the second part, note that the completion $\bar{\mathcal{U}}$ contains the original space $\mathcal{U}$ as a dense ϖ^κ-convex subset, and the metric on $\mathcal{U}$ coincides with the induced length metric from $\bar{\mathcal{U}}$. □

Here is a corollary from Proposition 9.12 and Theorem 9.8.

9.13. Corollary. *Let* $\mathcal{U}$ *be a* ϖ^κ*-geodesic* $\mathrm{CAT}(\kappa)$ *space. Then pairs of points in* $\mathcal{U}$ *at distance less than* ϖ^κ *are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

Moreover for any pair of points $p, q \in \mathcal{U}$ *and any value*

$$\ell > \sup\left\{\frac{\operatorname{sn}^\kappa r}{\operatorname{sn}^\kappa|p-q|} : 0 \leqslant r \leqslant |p-q|\right\}$$

there are neighborhoods $\Omega_p \ni p$ *and* $\Omega_q \ni q$ *such that the map*

$$(x, y, t) \mapsto \operatorname{path}_{[xy]}(t)$$

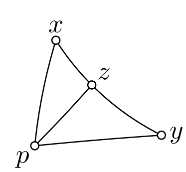
is ℓ*-Lipschitz in* $\Omega_p \times \Omega_q \times [0,1]$.

Proof. By Proposition 9.12, any geodesic $\mathrm{CAT}(\kappa)$ space is isometric to a convex dense subset of a complete length $\mathrm{CAT}(\kappa)$ space. It remains to apply Theorem 9.8. □

C. More comparisons

Here we give a few reformulations of Definition 9.3.

9.14. Theorem. *If* $\mathcal{U}$ *is a* $\mathrm{CAT}(\kappa)$ *space, then the following conditions hold for all triples* $p, x, y \in \mathcal{U}$ *of perimeter* $< 2\cdot\varpi^\kappa$*:*



(a) *(adjacent angle comparison) for any geodesic* $[xy]$ *and* $z \in\,]xy[$*, we have*

$$\tilde\measuredangle^\kappa\left(z\,{}^{p}_{x}\right) + \tilde\measuredangle^\kappa\left(z\,{}^{p}_{y}\right) \geqslant \pi.$$

(b) *(point-on-side comparison) for any geodesic* $[xy]$ *and* $z \in]xy[$*, we have*

$$\tilde\measuredangle^\kappa(x\,{}^p_y) \geqslant \tilde\measuredangle^\kappa(x\,{}^p_z),$$

or equivalently,

$$|\tilde p - \tilde z| \geqslant |p - z|,$$

where $[\tilde p\tilde x\tilde y] = \tilde\triangle^\kappa(pxy)$*,* $\tilde z \in]\tilde x\tilde y[$*,* $|\tilde x - \tilde z| = |x - z|$*.*

(c) *(hinge comparison) for any hinge* $[x\,{}^p_y]$*, the angle* $\measuredangle[x\,{}^p_y]$ *exists and*

$$\measuredangle[x\,{}^p_y] \leqslant \tilde\measuredangle^\kappa(x\,{}^p_y),$$

or equivalently,

$$\tilde\gamma^\kappa[x\,{}^p_y] \leqslant |p - y|.$$

Moreover, if $\mathcal{U}$ *is* $\varpi\kappa$*-geodesic, then the converse holds in each case.*

Remark. In the following proof, the part $(c) \Rightarrow (a)$ only requires that the CAT(κ) comparison (9.1) hold for any quadruple, and does not require the existence of geodesics at distance $< \varpi\kappa$. The same is true of the parts $(a)\Leftrightarrow(b)$ and $(b) \Rightarrow (c)$. Thus the conditions (a), (b) and (c) are valid for any metric space (not necessarily a length space) that satisfies CAT(κ) comparison (9.1). The converse does not hold; for example, all these conditions are vacuously true in a totally disconnected space, while CAT(κ) comparison is not.

Proof.

(a). Since the perimeter of p, x, y is $< 2\cdot\varpi\kappa$, so is the perimeter of any subtriple of p, z, x, y by the triangle inequality. By Alexandrov's lemma (6.3),

$$\tilde\measuredangle^\kappa(p\,{}^z_x) + \tilde\measuredangle^\kappa(p\,{}^z_y) < \tilde\measuredangle^\kappa(p\,{}^x_y) \quad \text{or} \quad \tilde\measuredangle^\kappa(z\,{}^p_x) + \tilde\measuredangle^\kappa(z\,{}^p_y) = \pi.$$

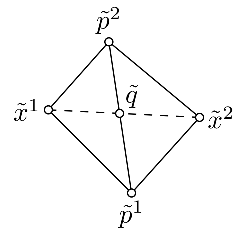
In the former case, the CAT(κ) comparison (9.1) applied to the quadruple p, z, x, y implies

$$\tilde\measuredangle^\kappa(z\,{}^p_x) + \tilde\measuredangle^\kappa(z\,{}^p_y) \geqslant \tilde\measuredangle^\kappa(z\,{}^x_y) = \pi.$$

$(a)\Leftrightarrow(b)$. Follows from Alexandrov's lemma (6.3).

$(b) \Rightarrow (c)$. By (b), for $\bar p \in]xp]$ and $\bar y \in]xy]$ the function $(|x - \bar p|, |x - \bar y|) \mapsto \tilde\measuredangle^\kappa(x\,{}^{\bar p}_{\bar y})$ is nondecreasing in each argument. In particular, $\measuredangle[x\,{}^p_y] = \inf \tilde\measuredangle^\kappa(x\,{}^{\bar p}_{\bar y})$. Thus $\measuredangle[x\,{}^p_y]$ exists and is at most $\tilde\measuredangle^\kappa(x\,{}^p_y)$.

Converse. Assume $\mathcal{U}$ is $\varpi\kappa$-geodesic. Let us first show that in this case $(c) \Rightarrow (a)$.



Indeed, by (c) and the triangle inequality for angles (6.5),

$$\tilde\measuredangle^\kappa\left(z\,{}^p_x\right)+\tilde\measuredangle^\kappa\left(z\,{}^p_y\right)\geqslant\measuredangle\left[z\,{}^p_x\right]+\measuredangle\left[z\,{}^p_y\right]\geqslant\pi.$$

It remains to prove the converse for (b).

Given a quadruple p^1, p^2, x^1, x^2 whose subtriples have perimeter $< 2\cdot\varpi\kappa$, we must verify the $\mathrm{CAT}(\kappa)$ comparison (9.1). In $\mathbb{M}^2(\kappa)$, construct the model triangles $[\tilde p^1\tilde p^2\tilde x^1]=\tilde\triangle^\kappa(p^1p^2x^1)$ and $[\tilde p^1\tilde p^2\tilde x^2]=\tilde\triangle^\kappa(p^1p^2x^2)$, lying on either side of a common segment $[\tilde p^1\tilde p^2]$. We may suppose

$$\tilde\measuredangle^\kappa\left(p^1\,{}^{p^2}_{x^1}\right)+\tilde\measuredangle^\kappa\left(p^1\,{}^{p^2}_{x^2}\right)\leqslant\pi\quad\text{and}\quad\tilde\measuredangle^\kappa\left(p^2\,{}^{p^1}_{x^1}\right)+\tilde\measuredangle^\kappa\left(p^2\,{}^{p^1}_{x^2}\right)\leqslant\pi,$$

since otherwise $\mathrm{CAT}(\kappa)$ comparison holds trivially. Then $[\tilde p^1\tilde p^2]$ and $[\tilde x^1\tilde x^2]$ intersect, say at $\tilde q$.

By assumption, there is a geodesic $[p^1p^2]$. Choose $q\in[p^1p^2]$ corresponding to $\tilde q$; that is, $|p^1-q|=|\tilde p^1-\tilde q|$. Then

$$|x^1-x^2|\leqslant|x^1-q|+|q-x^2|\leqslant|\tilde x^1-\tilde q|+|\tilde q-\tilde x^2|=|\tilde x^1-\tilde x^2|,$$

where the second inequality follows from (b). By monotonicity of the function $a\mapsto\tilde\measuredangle^\kappa\{a;b,c\}$ (1.1c),

$$\tilde\measuredangle^\kappa\left(p^1\,{}^{x^1}_{x^2}\right)\leqslant\measuredangle\left[\tilde p^1\,{}^{\tilde x^1}_{\tilde x^2}\right]=\tilde\measuredangle^\kappa\left(p^1\,{}^{p^2}_{x^1}\right)+\tilde\measuredangle^\kappa\left(p^1\,{}^{p^2}_{x^2}\right).$$

$\square$

Let us display a corollary of the proof of 9.14, namely, monotonicity of the model angle with respect to adjacent sidelengths.

9.15. Angle-sidelength monotonicity. *Suppose $\mathcal{U}$ is a $\varpi\kappa$-geodesic* $\mathrm{CAT}(\kappa)$ *space, and p, x, $y\in\mathcal{U}$ have perimeter $<2\cdot\varpi\kappa$. Then for $\bar y\in\,]xy]$, the function*

$$|x-\bar y|\mapsto\tilde\measuredangle^\kappa\left(x\,{}^p_{\bar y}\right)$$

is nondecreasing.

In particular, if $\bar p\in\,]xp]$, then

(a) *the function*

$$(|x-\bar y|,|x-\bar p|)\mapsto\tilde\measuredangle^\kappa\left(x\,{}^{\bar p}_{\bar y}\right)$$

is nondecreasing in each argument,

(b) $\measuredangle\left[x\,{}^p_y\right]=\inf\left\{\,\tilde\measuredangle^\kappa\left(x\,{}^{\bar p}_{\bar y}\right)\;:\;\bar p\in\,]xp],\;\bar y\in\,]xy]\,\right\}.$

9.16. Exercise. *Let $\mathcal{U}$ be $\mathbb{R}^m$ with the metric defined by a norm. Show that $\mathcal{U}$ is a complete length* CAT *space if and only if $\mathcal{U}\overset{\mathrm{iso}}{=\!=}\mathbb{E}^m$.*

9.17. Exercise. *Assume $\mathcal{U}$ is a geodesic* CAT(0) *space. Show that for any two geodesic paths $\gamma, \sigma\colon [0,1] \to \mathcal{U}$ the function*

$$t \mapsto |\gamma(t) - \sigma(t)|$$

is convex.

9.18. Proposition. *Assume $\kappa < \mathrm{K}$. Then any complete length* CAT(κ) *space is* CAT(K).

Moreover a space $\mathcal{U}$ is CAT(κ) *if $\mathcal{U}$ is* CAT(K) *for all $\mathrm{K} > \kappa$.*

Proof. The first statement follows from Corollary 9.11, the adjacent-angles comparison (9.14*a*) and the monotonicity of the function $\kappa \mapsto \tilde{\measuredangle}^\kappa\left(x\,{}^y_z\right)$ (1.1*d*).

The second statement follows since the function $\kappa \mapsto \tilde{\measuredangle}^\kappa\left(x\,{}^y_z\right)$ is continuous. $\square$

D. Thin triangles

In this section we define thin triangles and use them to characterize CAT spaces. Inheritance for thin triangles with respect to decomposition is the main result of this section. It will lead to two fundamental constructions: Alexandrov's patchwork globalization (9.30) and Reshetnyak gluing (9.39).

9.19. Definition of κ-thin triangles. *Let $[x^1x^2x^3]$ be a triangle of perimeter $< 2\cdot\varpi\kappa$ in a metric space and $[\tilde{x}^1\tilde{x}^2\tilde{x}^3] = \tilde{\triangle}^\kappa(x^1x^2x^3)$. Consider the* natural map *$[\tilde{x}^1\tilde{x}^2\tilde{x}^3] \to [x^1x^2x^3]$ that sends a point $\tilde{z} \in [\tilde{x}^i\tilde{x}^j]$ to the corresponding point $z \in [x^ix^j]$ (that is, such that $|\tilde{x}^i - \tilde{z}| = |x^i - z|$ and therefore $|\tilde{x}^j - \tilde{z}| = |x^j - z|$).*

We say the triangle $[x^1x^2x^3]$ is κ-thin *if the natural map $[\tilde{x}^1\tilde{x}^2\tilde{x}^3] \to [x^1x^2x^3]$ is short.*

9.20. Exercise. *Let $\mathcal{U}$ be a $\varpi\kappa$-geodesic* CAT(κ) *space. Let $[xyz]$ be a triangle in $\mathcal{U}$ and $[\tilde{x}\tilde{y}\tilde{z}]$ be its model triangle in $\mathbb{M}^2(\kappa)$. Prove that the natural map $f\colon [\tilde{x}\tilde{y}\tilde{z}] \to [xyz]$ is distance-preserving if and only if one of the following conditions hold:*

(*a*) $\measuredangle\left[x\,{}^y_z\right] = \tilde{\measuredangle}^\kappa\left(x\,{}^y_z\right)$,

(*b*) $|x - w| = |\tilde{x} - \tilde{w}|$ *for some* $\tilde{w} \in]\tilde{y}\tilde{z}[$ *and* $w = f(\tilde{w})$,

(*c*) $|v - w| = |\tilde{v} - \tilde{w}|$ *for some* $\tilde{v} \in]\tilde{x}\tilde{y}[$, $\tilde{w} \in]\tilde{x}\tilde{z}[$ *and* $v = f(\tilde{v})$, $w = f(\tilde{w})$.

9.21. Proposition. *Let $\mathcal{U}$ be a $\varpi\kappa$-geodesic space. Then $\mathcal{U}$ is* CAT(κ) *if and only if every triangle of perimeter $< 2\cdot\varpi\kappa$ in $\mathcal{U}$ is κ-thin.*

Proof. The if part follows from the point-on-side comparison (9.14*b*). The only-if part follows from angle-sidelength monotonicity (9.15*a*). $\square$

9.22. Corollary. *Suppose $\mathcal{U}$ is a $\varpi\kappa$-geodesic* CAT(κ) *space. Then any local geodesic in $\mathcal{U}$ of length $< \varpi\kappa$ is length-minimizing.*

Proof. Suppose $\gamma\colon [0,\ell] \to \mathcal{U}$ is a local geodesic that is not minimizing, with $\ell < \varpi\kappa$. Choose a to be the maximal value such that γ is minimizing on $[0,a]$. Further choose $b > a$ so that γ is minimizing on $[a,b]$.

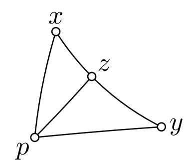
Since triangle $[\gamma(0)\gamma(a)\gamma(b)]$ is κ-thin, we have

$$|\gamma(a-\varepsilon) - \gamma(a+\varepsilon)| < 2\cdot\varepsilon$$

for all small $\varepsilon > 0$, a contradiction. $\square$

Now let us formulate the main result of this section. The inheritance lemma states that in any metric space, a triangle is κ-thin if it decomposes into κ-thin triangles. In contrast, κ-thickness of triangles (8.21) is not inherited in this way.

9.23. Inheritance lemma. *In a metric space, consider a triangle $[pxy]$ that* decomposes *into two triangles $[pxz]$ and $[pyz]$; that is, $[pxz]$ and $[pyz]$ have common side $[pz]$, and the sides $[xz]$ and $[zy]$ together form the side $[xy]$ of $[pxy]$.*



If the triangle $[pxy]$ has perimeter $< 2\cdot\varpi\kappa$ and both triangles $[pxz]$ and $[pyz]$ are κ-thin, then triangle $[pxy]$ is κ-thin.

The following model-space lemma is extracted from Lemma 2 in [**140**].

9.24. Lemma. *Let $[\tilde p\tilde x\tilde y]$ be a triangle in $\mathbb{M}^2(\kappa)$ and $\tilde z \in [\tilde x\tilde y]$. Consider the solid triangle $\tilde D =$* Conv$[\tilde p\tilde x\tilde y]$. *Construct points $\dot p, \dot x, \dot z, \dot y \in \mathbb{M}^2(\kappa)$ such that*

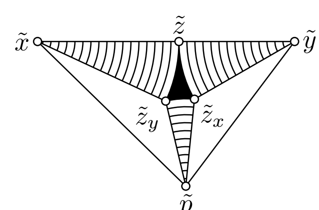
$$\begin{aligned} |\dot p - \dot x| &= |\tilde p - \tilde x|, & |\dot p - \dot y| &= |\tilde p - \tilde y|, & |\dot p - \dot z| &\leqslant |\tilde p - \tilde z|,\\ |\dot x - \dot z| &= |\tilde x - \tilde z|, & |\dot y - \dot z| &= |\tilde y - \tilde z|, \end{aligned}$$

where points $\dot x$ and $\dot y$ lie on either side of $[\dot p\dot z]$. Set

$$\dot D = \mathrm{Conv}[\dot p\dot x\dot z] \cup \mathrm{Conv}[\dot p\dot y\dot z].$$

Then there is a short map $F\colon \tilde D \to \dot D$ that maps $\tilde p$, $\tilde x$, $\tilde y$ and $\tilde z$ to $\dot p$, $\dot x$, $\dot y$ and $\dot z$ respectively.

Proof. By Alexandrov's lemma (6.3), there are nonoverlapping triangles $[\tilde p\tilde x\tilde z_y] \overset{iso}{=\!=} [\dot p\dot x\dot z]$ and $[\tilde p\tilde y\tilde z_x] \overset{iso}{=\!=} [\dot p\dot y\dot z]$ inside triangle $[\tilde p\tilde x\tilde y]$.



Connect points in each pair $(\tilde z, \tilde z_x)$, $(\tilde z_x, \tilde z_y)$ and $(\tilde z_y, \tilde z)$ with arcs of circles centered at $\tilde y$, $\tilde p$, and $\tilde x$ respectively. Define F as follows.

- Map Conv$[\tilde p\tilde x\tilde z_y]$ isometrically onto Conv$[\dot p\dot x\dot y]$; similarly map Conv$[\tilde p\tilde y\tilde z_x]$ onto Conv$[\dot p\dot y\dot z]$.

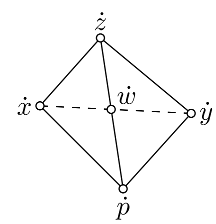
- If w is in one of the three circular sectors, say at distance r from the center of the circle, let $F(w)$ be the point on $[\dot p\dot z]$, $[\dot x\dot z]$, or $[\dot y\dot z]$ whose distance from the left-hand endpoint of the segment is r.
- Finally, if w lies in the remaining curvilinear triangle $\tilde z\tilde z_x\tilde z_y$, set $F(w) = \dot z$.

By construction, F meets the conditions of the lemma. $\square$

Proof of 9.23. Construct model triangles $[\dot p\dot x\dot z] = \tilde\triangle^{\kappa}(pxz)$ and $[\dot p\dot y\dot z] = \tilde\triangle^{\kappa}(pyz)$ so that $\dot x$ and $\dot y$ lie on opposite sides of $[\dot p\dot z]$.

Suppose

$$\tilde\measuredangle^{\kappa}\left(z\,{}^{p}_{x}\right) + \tilde\measuredangle^{\kappa}\left(z\,{}^{p}_{y}\right) < \pi.$$



Then for some point $\dot w \in [\dot p\dot z]$, we have

$$|\dot x - \dot w| + |\dot w - \dot y| < |\dot x - \dot z| + |\dot z - \dot y| = |x - y|.$$

Let $w \in [pz]$ correspond to $\dot w$; that is, $|z - w| = |\dot z - \dot w|$. Since $[pxz]$ and $[pyz]$ are κ-thin, we have

$$|x - w| + |w - y| < |x - y|,$$

contradicting the triangle inequality.

Thus

$$\tilde\measuredangle^{\kappa}\left(z\,{}^{p}_{x}\right) + \tilde\measuredangle^{\kappa}\left(z\,{}^{p}_{y}\right) \geqslant \pi.$$

By Alexandrov's lemma (6.3), this is equivalent to

$$\tilde\measuredangle^{\kappa}\left(x\,{}^{p}_{z}\right) \leqslant \tilde\measuredangle^{\kappa}\left(x\,{}^{p}_{y}\right). \tag{1}$$

Let $[\tilde p\tilde x\tilde y] = \tilde\triangle^{\kappa}(pxy)$ and $\tilde z \in [\tilde x\tilde y]$ correspond to z; that is, $|x - z| = |\tilde x - \tilde z|$. Inequality 1 is equivalent to $|p - z| \leqslant |\tilde p - \tilde z|$. Hence Lemma 9.24 applies. Therefore there is a short map F that sends $[\tilde p\tilde x\tilde y]$ to $\dot D = \mathrm{Conv}[\dot p\dot x\dot z] \cup \mathrm{Conv}[\dot p\dot y\dot z]$ in such a way that $\tilde p \mapsto \dot p$, $\tilde x \mapsto \dot x$, $\tilde z \mapsto \dot z$ and $\tilde y \mapsto \dot y$.

By assumption, the natural maps $[\dot p\dot x\dot z] \to [pxz]$ and $[\dot p\dot y\dot z] \to [pyz]$ are short. By composition, the natural map from $[\tilde p\tilde x\tilde y]$ to $[pyz]$ is short, as claimed. $\square$

E. Function comparison

In this section we give analytic and geometric ways of viewing the point-on-side comparison (9.14*b*) as a convexity condition.

First we obtain a corresponding differential inequality for the distance function in $\mathcal{U}$; see Section 3F for the definition.

9.25. Theorem. *Suppose $\mathcal{U}$ is a $\varpi\kappa$-geodesic space. Then the following are equivalent:*

(a) *$\mathcal{U}$ is* $\mathrm{CAT}(\kappa)$*,*

(b) *for any $p \in \mathcal{U}$, the function $f = \mathrm{md}^\kappa \circ \mathrm{dist}_p$ satisfies*

$$f'' + \kappa \cdot f \geqslant 1$$

in $\mathrm{B}(p, \varpi\kappa)$*.*

9.26. Corollary. *A geodesic space $\mathcal{U}$ is* $\mathrm{CAT}(0)$ *if and only if for any $p \in \mathcal{U}$, the function* $\mathrm{dist}_p^2 \colon \mathcal{U} \to \mathbb{R}$ *is 2-convex.*

Proof of 9.25. Fix a sufficiently short geodesic $[xy]$ in $\mathrm{B}(p, \varpi\kappa)$. We can assume that the model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}^\kappa(pxy)$ is defined. Let

$$\tilde{r}(t) = |\tilde{p} - \mathrm{geod}_{[\tilde{x}\tilde{y}]}(t)|, \qquad r(t) = |p - \mathrm{geod}_{[xy]}(t)|.$$

Let $\tilde{f} = \mathrm{md}^\kappa \circ \tilde{r}$ and $f = \mathrm{md}^\kappa \circ r$. By 1.1*a*, we have $\tilde{f}'' = 1 - \kappa \cdot \tilde{f}$. Clearly $\tilde{f}(t)$ and $f(t)$ agree at $t = 0$ and $t = |x - y|$. The point-on-side comparison (9.14*b*) is the condition $r(t) \leqslant \tilde{r}(t)$ for all $t \in [0, |x - y|]$. Since md^κ is increasing on $[0, \varpi\kappa)$, then $r \leqslant \tilde{r}$ and $f \leqslant \tilde{f}$ are equivalent. Thus the claim follows by Jensen's inequality (3.14*c*). □

9.27. Corollary. *Suppose $\mathcal{U}$ is a $\varpi\kappa$-geodesic* $\mathrm{CAT}(\kappa)$ *space. Then any ball (closed*[redacted] *or open) of radius $R < \frac{\varpi\kappa}{2}$ in $\mathcal{U}$ is convex.*

Moreover, any open ball of radius $\frac{\varpi\kappa}{2}$ is convex and any closed ball of radius $\frac{\varpi\kappa}{2}$ is $\varpi\kappa$-convex.

Proof. Suppose $p \in \mathcal{U}$, $R \leqslant \varpi\kappa/2$, and two points x and y lie in $\overline{\mathrm{B}}[p, R]$ or $\mathrm{B}(p, R)$. By the triangle inequality, if $|x - y| < \varpi\kappa$, then any geodesic $[xy]$ lies in $\mathrm{B}(p, \varpi\kappa)$.

By the function comparison (9.25), the geodesic $[xy]$ lies in $\overline{\mathrm{B}}[p, R]$ or $\mathrm{B}(p, R)$[redacted] respectively.

Thus any ball (closed or open) of radius $R < \frac{\varpi\kappa}{2}$ is $\varpi\kappa$-convex. This implies convexity unless there is a pair of points in the ball at distance at least $\varpi\kappa$. By the triangle inequality, the latter is possible only for the closed ball of radius $\frac{\varpi\kappa}{2}$. □

Recall that Busemann functions are defined in Proposition 6.1. The following exercise is analogous to Exercise 8.25.

9.28. Exercise. *Let $\mathcal{U}$ be a complete length* $\mathrm{CAT}(\kappa)$ *space and* $\mathrm{bus}_\gamma \colon \mathcal{U} \to \mathbb{R}$ *be the Busemann function for a half-line $\gamma \colon [0, \infty) \to \mathcal{L}$.*

(a) *If $\kappa = 0$, then the Busemann function* bus_γ *is convex.*

(*b*) *If* $\kappa = -1$, *then the function* $f = \exp \circ \mathrm{bus}_\gamma$ *satisfies*

$$f'' - f \geqslant 0.$$

F. Development

Geometrically, the development construction (8.26) translates distance comparison into a local convexity statement for subsets of $\mathbb{M}^2(\kappa)$. Recall that a curve in $\mathbb{M}^2(\kappa)$ is (*locally*) *concave* with respect to p if (locally) its supergraph with respect to p is a convex subset of $\mathbb{M}^2(\kappa)$; see Definition 8.27.

9.29. Development criterion. *For a* $\varpi\kappa$*-geodesic space* $\mathcal{U}$*, the following statements hold:*

(*a*) *For any* $p \in \mathcal{U}$ *and any geodesic* $\gamma\colon [0, T] \to \mathrm{B}(p, \varpi\kappa)$*, suppose the* κ*-development* $\tilde\gamma$ *in* $\mathbb{M}^2(\kappa)$ *of* γ *with respect to* p *is locally concave. Then* $\mathcal{U}$ *is* $\mathrm{CAT}(\kappa)$*.*

(*b*) *If* $\mathcal{U}$ *is* $\mathrm{CAT}(\kappa)$*, then for any geodesic* $\gamma\colon [0, T] \to \mathcal{U}$ *and* $p \in \mathcal{U}$ *such that the triangle* $[p\gamma(0)\gamma(T)]$ *has perimeter* $< 2\cdot\varpi\kappa$*, the* κ*-development* $\tilde\gamma$ *in* $\mathbb{M}^2(\kappa)$ *of* γ *with respect to* p *is concave.*

Proof.

(*a*). Let $\gamma = \mathrm{geod}_{[xy]}$ and $T = |x - y|$. Let $\tilde\gamma\colon [0, T] \to \mathbb{M}^2(\kappa)$ be the concave κ-development based at $\tilde p$ of γ with respect to p. Let us show that the function

$$t \mapsto \tilde\measuredangle^\kappa\left(x\,{}^{p}_{\gamma(t)}\right) \tag{1}$$

is nondecreasing.

For a partition $0 = t^0 < t^1 < \dots < t^n = T$, let

$$\tilde y^i = \tilde\gamma(t^i) \quad\text{and}\quad \tau^i = |\tilde y^0 - \tilde y^1| + |\tilde y^1 - \tilde y^2| + \dots + |\tilde y^{i-1} - \tilde y^i|.$$

Since $\tilde\gamma$ is locally concave, for a sufficiently fine partition the polygonal line $\tilde y^0\tilde y^1\dots\tilde y^n$ is locally concave with respect to $\tilde p$. Alexandrov's lemma (6.3), applied inductively to pairs of triangles $\tilde\triangle^\kappa\{\tau^{i-1}, |p-\tilde y^0|, |p-\tilde y^{i-1}|\}$ and $\tilde\triangle^\kappa\{|\tilde y^{i-1}-\tilde y^i|, |p-\tilde y^{i-1}|, |p-\tilde y^i|\}$, shows that the sequence $\tilde\measuredangle^\kappa\{|\tilde p-\tilde y^i|; |\tilde p-\tilde y^0|, \tau^i\}$ is nondecreasing.

Taking finer partitions and passing to the limit, we get

$$\max_i\{|\tau^i - t^i|\} \to 0.$$

Therefore 1 and the point-on-side comparison (9.14*b*) follows.

(b)~~.~~ Consider a partition $0 = t^0 < t^1 < \dots < t^n = T$, and let $x^i = \gamma(t^i)$. Construct a chain of model triangles $[\tilde p \tilde x^{i-1} \tilde x^i] = \tilde\triangle^{\kappa}(p x^{i-1} x^i)$ with the direction of $[\tilde p \tilde x^i]$ turning counterclockwise as i grows. By the angle comparison (9.14c),

$$\measuredangle\left[\tilde x^i\,{}^{\tilde x^{i-1}}_{\tilde p}\right] + \measuredangle\left[\tilde x^i\,{}^{\tilde x^{i+1}}_{\tilde p}\right] \geqslant \pi. \tag{2}$$

Since γ is a geodesic,

$$\operatorname{length}\gamma = \sum_{i=1}^{n} |x^{i-1} - x^i| \leqslant |p - x^0| + |p - x^n|. \tag{3}$$

By repeated application of Alexandrov's lemma (6.3), and inequality 3,

$$\sum_{i=1}^{n} \measuredangle\left[\tilde p\,{}^{\tilde x^{i-1}}_{\tilde x^i}\right] \leqslant \tilde\measuredangle^{\kappa}\left(p\,{}^{x^0}_{x^n}\right) \leqslant \pi.$$

Then by 2, the polygonal line $\tilde p \tilde x^0 \tilde x^1 \dots \tilde x^n$ are concave with respect to $\tilde p$.

Note that under finer partitions, the polygonal line $\tilde x^0 \tilde x^1 \dots \tilde x^n$ approach the development of γ with respect to p. Since the polygonal lines are convex, their lengths converge to the length of γ. Hence the result. □

G. Patchwork globalization

If $\mathcal{U}$ is a CAT(κ) space, then it is locally CAT(κ). The converse does not hold even for complete length space. For example, $\mathbb{S}^1$ is locally isometric to $\mathbb{R}$, and so is locally CAT(0), but it is easy to find a quadruple of points in $\mathbb{S}^1$ that violates CAT(0) comparison.

The following theorem was essentially proved by Alexandr Alexandrov [**17**, Satz 9]; it gives a global condition on geodesics that is necessary and sufficient for a locally CAT(κ) space to be globally CAT(κ). The proof uses thin-triangle decompositions and ~~the~~ inheritance lemma (9.23).

9.30. Patchwork globalization theorem. *For any complete length space $\mathcal{U}$, the following two statements are equivalent:*

- (a) *$\mathcal{U}$ is* CAT(κ).
- (b) *$\mathcal{U}$ is locally* CAT(κ)*; moreover, pairs of points in $\mathcal{U}$ at distance $< \varpi\kappa$ are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

Note that the implication $(a) \Rightarrow (b)$ follows from Theorem 9.8.

9.31. Corollary. *Let $\mathcal{U}$ be a complete length space and $\Omega \subset \mathcal{U}$ be an open locally* CAT(κ) *subset. Then for any point $p \in \Omega$ there is $R > 0$ such that $\overline{\mathrm{B}}[p, R]$ is a convex subset of $\mathcal{U}$ and $\overline{\mathrm{B}}[p, R]$ is* CAT(κ).

Proof. Fix $R > 0$ such that CAT(κ) comparison holds in $\mathrm{B}(p, R)$.

We may assume that $\mathrm{B}(p, R) \subset \Omega$ and $R < \varpi\mathrm{K}$. The same argument as in the proof of the theorem on uniqueness of geodesics (9.8) shows that any two points in $\overline{\mathrm{B}}[p, \frac{R}{2}]$ can be joined by a unique geodesic that depends continuously on the endpoints.

The same argument as in the proof of Corollary 9.27 shows that $\overline{\mathrm{B}}[p, \frac{R}{2}]$ is a convex set. Then $(b) \Rightarrow (a)$ of the patchwork globalization theorem implies that $\overline{\mathrm{B}}[p, \frac{R}{2}]$ is CAT(κ). □

The proof of patchwork globalization uses the following construction:

9.32. Definition (Line-of-sight map). *Let p be a point and α be a curve of finite length in a length space $\mathcal{U}$. Let $\bar{\alpha} : [0, 1] \to \mathcal{U}$ be the constant-speed parametrization of α. If $\gamma_t : [0, 1] \to \mathcal{U}$ is a geodesic path from p to $\bar{\alpha}(t)$, we say that the map $[0, 1] \times [0, 1] \to \mathcal{U}$ defined by*

$$(t, s) \mapsto \gamma_t(s)$$

is a line-of-sight map *for α with respect to p.*

Note that a line-of-sight map is closely related to geodesic homotopy (Section 9M).

Proof of 9.30. It only remains to prove $(b) \Rightarrow (a)$.

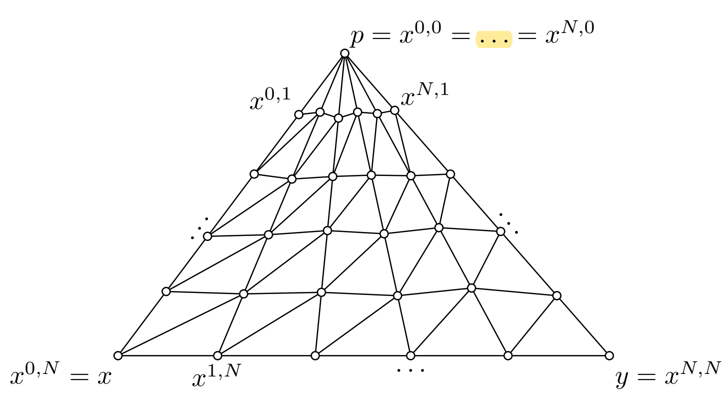
Let $[pxy]$ be a triangle of perimeter $< 2 \cdot \varpi\kappa$ in $\mathcal{U}$. According to 9.21 and 9.18, it is sufficient to show the triangle $[pxy]$ is κ-thin.

Since pairs of points at distance $< \varpi\kappa$ are joined by unique geodesics and these geodesics depend continuously on their endpoint pairs, there is a unique and continuous line-of-sight map for $[xy]$ with respect to p.

For a partition

$$0 = t^0 < t^1 < \dots < t^N = 1,$$

let $x^{i,j} = \gamma_{t^i}(t^j)$. Since the line-of-sight map is continuous, we may assume



each triangle $[x^{i,j}x^{i,j+1}x^{i+1,j+1}]$ and $[x^{i,j}x^{i+1,j}x^{i+1,j+1}]$ is κ-thin (see Proposition 9.21).

Now we show that the κ-thin property propagates to $[pxy]$, by repeated application of ~~the~~ inheritance lemma (9.23):

- First, for fixed i, sequentially applying the lemma shows that the triangles $[xx^{i,1}x^{i+1,2}]$, $[xx^{i,2}x^{i+1,2}]$, $[xx^{i,2}x^{i+1,3}]$, and so on are κ-thin.

In particular, for each i, the long triangle $[xx^{i,N}x^{i+1,N}]$ is κ-thin.

- Applying the lemma again shows that the triangles $[xx^{0,N}x^{2,N}]$, $[xx^{0,N}x^{3,N}]$, and so on are κ-thin.

In particular, $[pxy] = [px^{0,N}x^{N,N}]$ is κ-thin. □

The following exercise implies that if the space is proper, then one can drop the condition on continuous dependence of geodesics in the formulation of patchwork globalization.

9.33. Exercise. (*a*) *Suppose pairs of points in a geodesic space $\mathcal{U}$ are joined by unique geodesics. Show that if $\mathcal{U}$ is proper, then these geodesics depend continuously on their endpoint pairs.*

(*b*) *Construct an example of a complete geodesic space $\mathcal{U}$ such that pairs of points in $\mathcal{U}$ are joined by unique geodesics, but these geodesics do not depend continuously on their endpoint pairs.*

H. Angles

Recall that ω denotes a selective nonprincipal ultrafilter on $\mathbb{N}$, see Section 4B.

9.34. Angle semicontinuity. *Suppose $\mathcal{U}_1, \mathcal{U}_2, \dots$ is a sequence of $\varpi\kappa$-geodesic* CAT(κ) *spaces and $\mathcal{U}_n \to \mathcal{U}_\omega$ as $n \to \omega$. Assume that a sequence of hinges $\left[p_n\,{}^{x_n}_{y_n}\right]$ in $\mathcal{U}_n$ converges to a hinge $\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right]$ in $\mathcal{U}_\omega$ as $n \to \omega$. Then*

$$\measuredangle\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right] \geqslant \lim_{n\to\omega} \measuredangle\left[p_n\,{}^{x_n}_{y_n}\right].$$

Proof. By the angle-sidelength monotonicity (9.15),

$$\measuredangle\left[p_\omega\,{}^{x_\omega}_{y_\omega}\right] = \inf\left\{\,\tilde\measuredangle^\kappa\left(p_\omega\,{}^{\bar x_\omega}_{\bar y_\omega}\right) : \bar x_\omega \in \,]p_\omega x_\omega],\ \bar y_\omega \in \,]p_\omega y_\omega]\,\right\}.$$

For fixed $\bar x_\omega \in \,]p_\omega x_\omega]$ and $\bar y_\omega \in \,]p_\omega x_\omega]$, choose $\bar x_n \in \,]px_n]$ and $\bar y_n \in \,]py_n]$ so that $\bar x_n \to \bar x_\omega$ and $\bar y_n \to \bar y_\omega$ as $n \to \omega$. Clearly

$$\tilde\measuredangle^\kappa\left(p_n\,{}^{\bar x_n}_{\bar y_n}\right) \to \tilde\measuredangle^\kappa\left(p_\omega\,{}^{\bar x_\omega}_{\bar y_\omega}\right)$$

as $n \to \omega$.

By the angle comparison (9.14*c*), $\measuredangle\left[p_n\,{}^{x_n}_{y_n}\right] \leqslant \tilde\measuredangle^\kappa\left(p_n\,{}^{\bar x_n}_{\bar y_n}\right)$. Hence the result. □

Now we verify that the first variation formula holds in the CAT setting. Compare it to the first variation inequality (6.7) which holds for general metric spaces and to the strong angle lemma (8.42) for CBB spaces.

9.35. Strong angle lemma. *Let $\mathcal{U}$ be a $\varpi\kappa$-geodesic* CAT(κ) *space. Then for any hinge $[p\,{}^q_y]$ in $\mathcal{U}$, we have*

(1)
$$\measuredangle[p\,{}^q_y] = \lim_{\bar y \to p}\{\,\tilde\measuredangle^\kappa(p\,{}^q_{\bar y}) : \bar y \in\,]py]\,\}$$

for any $\kappa \in \mathbb{R}$ such that $|p-q| < \varpi\kappa$.

Proof. By angle-sidelength monotonicity (9.15), the right-hand side is defined and bigger than or equal to the left-hand side.

By Lemma 6.4, we may take $\kappa = 0$ in 1. By the cosine law and the first variation inequality (6.7), the right-hand side is less than or equal to the left-hand side. □

9.36. First variation. *Let $\mathcal{U}$ be a $\varpi\kappa$-geodesic* CAT(κ) *space. For any nontrivial geodesic $[py]$ in $\mathcal{U}$ and point $q \ne p$ such that $|p-q| < \varpi\kappa$, we have*

$$|q - \mathrm{geod}_{[py]}(t)| = |q-p| - t\cdot\cos\measuredangle[p\,{}^q_y] + o(t).$$

Proof. The first variation equation is equivalent to ~~the~~ strong angle lemma (9.35), as follows from the cosine law. □

9.37. Both-endpoints first variation. *Let $\mathcal{U}$ be a $\varpi\kappa$-geodesic* CAT(κ) *space. Then for any nontrivial geodesics $[py]$ and $[qz]$ in $\mathcal{U}$ such that $p \ne q$ and $|p-q| < \varpi\kappa$, we have*

$$|\mathrm{geod}_{[py]}(t) - \mathrm{geod}_{[qz]}(\tau)| = |q-p| - t\cdot\cos\measuredangle[p\,{}^q_y] - \tau\cdot\cos\measuredangle[q\,{}^p_z] + o(t+\tau).$$

Proof. By 9.14*c*,

$$\begin{aligned}
&|\mathrm{geod}_{[py]}(t) - \mathrm{geod}_{[qz]}(\tau)|\\
&\quad\geqslant |q - \mathrm{geod}_{[py]}(t)| - \tau\cdot\cos\measuredangle\left[q\,{}^{\mathrm{geod}_{[py]}(t)}_{z}\right] + o(\tau)\\
&\quad\geqslant |q-p| - t\cdot\cos\measuredangle[p\,{}^q_y] + o(t) - \tau\cdot\cos\measuredangle\left[q\,{}^{\mathrm{geod}_{[py]}(t)}_{z}\right] + o(\tau)\\
&\quad= |q-p| - t\cdot\cos\measuredangle[p\,{}^q_y] - \tau\cdot\cos\measuredangle[q\,{}^p_z] + o(t+\tau).
\end{aligned}$$

Here the final equality follows from

(2)
$$\lim_{t\to 0}\measuredangle\left[q\,{}^{\mathrm{geod}_{[py]}(t)}_{z}\right] = \measuredangle[q\,{}^p_z].$$

The angle semicontinuity (9.34) implies "$\leqslant$" in 2, and "$\geqslant$" holds by the triangle inequality for angles, since angle comparison (9.14*c*) gives

$$\lim_{t\to 0}\measuredangle\left[q\,{}^{p}_{\mathrm{geod}_{[py]}(t)}\right] = 0.$$

The opposite inequality follows from 9.36 and the triangle inequality

$$|\mathrm{geod}_{[py]}(t) - \mathrm{geod}_{[qz]}(\tau)| \leqslant |\mathrm{geod}_{[py]}(t) - m| + |m - \mathrm{geod}_{[qz]}(\tau)|,$$

where m is the midpoint of $[pq]$. □

We have given elementary proofs of the first-variation statements 9.35, 9.36 and 9.37. Note however that the no-conjugate-point theorem 9.46 not only provides proofs of these statements but also extends the statements from geodesics in $\mathrm{CAT}(\kappa)$ spaces to local geodesics in locally $\mathrm{CAT}(\kappa)$ spaces as follows:

9.38. First variation for local geodesics. *Let $\gamma_t\colon [0,1] \to \mathcal{U}$ be a continuous family of local geodesics in a locally $\mathrm{CAT}(\kappa)$. Set $\alpha(t) = \gamma_t(0)$ and $\beta(t) = \gamma_t(1)$. Suppose that γ_0 is unit-speed and $\alpha^+(0)$ and $\beta^+(0)$ are defined. Then*

$$\operatorname{length}\gamma_t = \operatorname{length}\gamma_0 - (\langle\alpha^+(0), \gamma_0^+(0)\rangle + \langle\beta^+(0), \gamma_0^-(1)\rangle)\cdot t + o(t).$$

I. Reshetnyak gluing theorem

The following theorem was proved by Yuriy Reshetnyak [**140**], assuming $\mathcal{U}^1$, $\mathcal{U}^2$ are proper and complete. In the following form, the theorem appears in the book of Martin Bridson and André Haefliger [**34**].

9.39. Reshetnyak gluing theorem. *Suppose $\mathcal{U}^1$, $\mathcal{U}^2$ are $\varpi\kappa$-geodesic spaces with isometric complete $\varpi\kappa$-convex sets $A^i \subset \mathcal{U}^i$. Let $\iota\colon A^1 \to A^2$ be an isometry. Let $\mathcal{W} = \mathcal{U}^1 \sqcup_\iota \mathcal{U}^2$; that is, $\mathcal{W}$ is the gluing of $\mathcal{U}^1$ and $\mathcal{U}^2$ along ι (see Section 2E).*

Then:

(*a*) *Both canonical mappings $\jmath_i\colon \mathcal{U}^i \to \mathcal{W}$ are distance-preserving and the images $\jmath_i(\mathcal{U}^i)$ are $\varpi\kappa$-convex subsets in $\mathcal{W}$.*

(*b*) *If $\mathcal{U}^1$, $\mathcal{U}^2$ are $\mathrm{CAT}(\kappa)$, then so is $\mathcal{W}$.*

Proof. Part (*a*) follows directly from $\varpi\kappa$-convexity of the A^i.

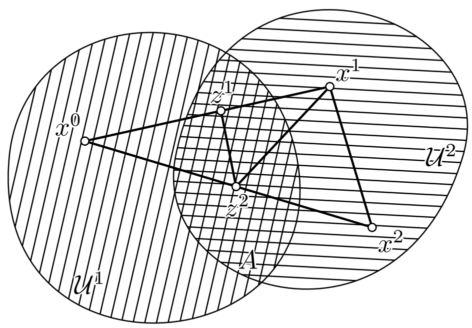
(*b*). According to (*a*), we can identify $\mathcal{U}^i$ with its image $\jmath_i(\mathcal{U}^i)$ in $\mathcal{W}$; in this way, the subsets $A^i \subset \mathcal{U}^i$ will be identified and denoted further by A. Thus $A = \mathcal{U}^1 \cap \mathcal{U}^2 \subset \mathcal{W}$, and A is $\varpi\kappa$-convex in $\mathcal{W}$.

Part (*b*) can be reformulated as follows:

9.40. Reformulation of 9.39*b*. *Let $\mathcal{W}$ be a length space having two $\varpi\kappa$-convex subsets $\mathcal{U}^1, \mathcal{U}^2 \subset \mathcal{W}$ such that $\mathcal{W} = \mathcal{U}^1 \cup \mathcal{U}^2$. Assume the subset $A = \mathcal{U}^1 \cap \mathcal{U}^2$ is complete and $\varpi\kappa$-convex in $\mathcal{W}$, and $\mathcal{U}^1$, $\mathcal{U}^2$ are $\mathrm{CAT}(\kappa)$ spaces. Then $\mathcal{W}$ is a $\mathrm{CAT}(\kappa)$ space.*

(1) *If $\mathcal{W}$ is $\varpi\kappa$-geodesic, then $\mathcal{W}$ is $\mathrm{CAT}(\kappa)$.*

Indeed, according to 9.21, it is sufficient to show that any triangle $[x^0x^1x^2]$ of perimeter $< 2\cdot\varpi\kappa$ in $\mathcal{W}$ is κ-thin. This is obviously true if all three points x^0, x^1, x^2 lie in a single $\mathcal{U}^i$. Thus, without loss of generality, we may assume that $x^0\in\mathcal{U}^1$ and $x^1, x^2\in\mathcal{U}^2$.



Choose points $z^1, z^2\in A=\mathcal{U}^1\cap\mathcal{U}^2$ lying respectively on the sides $[x^0x^1]$, $[x^0x^2]$. Note that all distances between any pair of points from x^0, x^1, x^2, z^1, z^2 are less than $\varpi\kappa$. Therefore

- triangle $[x^0z^1z^2]$ lies in $\mathcal{U}^1$,
- both triangles $[x^1z^1z^2]$ and $[x^1z^2x^2]$ lie in $\mathcal{U}^2$.

In particular, each triangle $[x^0z^1z^2]$, $[x^1z^1z^2]$, $[x^1z^2x^2]$ is κ-thin.

Applying ~~the~~ inheritance lemma for thin triangles (~~9.23~~) twice, we get that $[x^0x^1z^2]$ and consequently $[x^0x^1x^2]$ is κ-thin. $\triangle$

(2) $\quad$ *$\mathcal{W}$ is* CAT(κ) *if* $\kappa\leqslant 0$.

By 1 it suffices to prove that $\mathcal{W}$ is geodesic.

For $p^1\in\mathcal{U}^1$, $p^2\in\mathcal{U}^2$, we may choose a sequence $z_n\in A$ such that $|p^1-z_n|+|p^2-z_n|$ converges to $|p^1-p^2|$, and $|p^1-z_n|$ and $|p^2-z_n|$ converge. Since A is complete, it suffices to show z_n is a Cauchy sequence. In that case, the limit point z of z_n satisfies $|p^1-z|+|p^2-z|=|p^1-p^2|$, so the geodesics $[p^1z]$ in $\mathcal{U}^1$ and $[p^2z]$ in $\mathcal{U}^2$ together give a geodesic $[p^1p^2]$ in $\mathcal{U}$.

Suppose z_n is not a Cauchy sequence. Then there are subsequences x_n and y_n of z_n satisfying $\lim|x_n-y_n|>0$. Let m_n be the midpoint of $[x_ny_n]$. Since $|p^1-m_n|+|p^2-m_n|\geqslant|p^1-p^2|$, and $|p^1-x_n|+|p^2-x_n|$ and $|p^1-y_n|+|p^2-y_n|$ converge to $|p^1-p^2|$, then for any $\epsilon>0$, we may assume (taking subsequences and possibly relabeling p^1 and p^2)

$$|p^1-m_n|\geqslant|p^1-x_n|-\epsilon,\qquad |p^1-m_n|\geqslant|p^1-y_n|-\epsilon.$$

Since triangle $[p^1 x_n y_n]$ is thin, the analogous inequalities hold for the Euclidean model triangle $[\tilde p^1 \tilde x_n \tilde y_n]$. Then there is a nondegenerate limit triangle $[pxy]$ in the Euclidean plane satisfying $|p-x| = |p-y| \leqslant |p-m|$ where m is the midpoint of $[xy]$. This contradiction proves the claim. $\vartriangle$

Finally suppose $\varkappa > 0$; by rescaling, take $\varkappa = 1$. Consider the Euclidean cones $\mathrm{Cone}\,\mathcal{U}^i$ (see Section 6E). By Theorem 11.7*a*, $\mathrm{Cone}\,\mathcal{U}^i$ is a CAT(0) space for $i = 1, 2$.

Geodesics contained in the complement of the tip of $\mathrm{Cone}\,\mathcal{U}^i$ project to geodesics of length $< \pi$ in $\mathcal{U}^i$. It follows that $\mathrm{Cone}\,A$ is convex in $\mathrm{Cone}\,\mathcal{U}^1$ and $\mathrm{Cone}\,\mathcal{U}^2$. By the cone distance formula, $\mathrm{Cone}\,A$ is complete since A is complete.

Gluing along $\mathrm{Cone}\,A$ and applying 1 and 2 for $\varkappa = 0$, we find that $\mathrm{Cone}\,\mathcal{W}$ is a CAT(0) space. By Theorem 11.7*a*, $\mathcal{W}$ is a CAT(1) space. $\square$

9.41. Exercise. *Let Q be the nonconvex subset of the plane bounded by two half-lines γ_1 and γ_2 with a common starting point and angle α between them. Assume $\mathcal{U}$ is a complete length* CAT(0) *space and γ_1', γ_2' are two half-lines in $\mathcal{U}$ with a common starting point and angle α between them. Show that the space glued from Q and $\mathcal{U}$ along the corresponding half-lines is a* CAT(0) *space.*

9.42. Exercise. *Suppose $\mathcal{U}$ is a complete length* CAT(0) *space and $A \subset \mathcal{U}$ is a closed subset. Assume that the doubling of $\mathcal{U}$ in A is* CAT(0)*. Show that A is a convex set of $\mathcal{U}$.*

9.43. Exercise. *Let $\mathcal{U}$ be a complete length* CAT(1) *space and $K \subset \mathcal{U}$ be a closed π-convex set. Assume $K \subset \overline{\mathrm{B}}[p, \tfrac{\pi}{2}]$ for $p \in K$. Show that there is a decreasing continuous one-parameter family of closed convex sets K_t for $t \in [0,1]$ such that $K_0 = \overline{\mathrm{B}}[p, \tfrac{\pi}{2}]$ and $K_1 = K$.*

(*Decreasing means with respect to inclusion; that is $K_{t_0} \supset K_{t_1}$ if $t_0 \leqslant t_1$. Continuous means with respect to Hausdorff distance; that is $K_t \underset{H}{\to} K_{t_0}$ as $t \to t_0$.*)

9.44. Exercise. *Let A and B be two closed convex sets in a complete length* CAT(0) *space. Assume $A \cap B \neq \emptyset$. Show that the union $A \cup B$ equipped with induced length metric is* CAT(0)*.*

J. Space of geodesics

In this section we prove a no-conjugate-point theorem for spaces with upper curvature bounds and derive from it a number of statements about local geodesics. These statements will be used to prove ~~the~~ Hadamard–Cartan theorem (9.65) and ~~the~~ lifting globalization theorem (9.50), in much the same way as the exponential map is used in Riemannian geometry.

9.45. Proposition. *Let $\mathcal{U}$ be a locally* $\mathrm{CAT}(\kappa)$ *space. Let $\gamma_n\colon [0,1]\to\mathcal{U}$ be a sequence of local geodesic paths converging to a path $\gamma_\infty\colon [0,1]\to\mathcal{U}$. Then γ_∞ is a local geodesic path. Moreover*

$$\operatorname{length}\gamma_n \to \operatorname{length}\gamma_\infty$$

as $n\to\infty$.

Proof. Fix $t\in[0,1]$. By Corollary 9.31, we may choose R satisfying $0<R<\varpi\mathrm{K}$, and such that the ball $\mathcal{B}=\mathrm{B}(\gamma_\infty(t),R)$ is a convex subset of $\mathcal{U}$ and forms a $\mathrm{CAT}(\kappa)$ space.

A local geodesic segment with length less than $R/2$ that intersects $\mathrm{B}(\gamma_\infty(t),R/2)$ cannot leave $\mathcal{B}$, and hence is minimizing by Corollary 9.22. In particular, for all sufficiently large n, if subsegment of γ_n has length less than $R/2$ and contains $\gamma_n(t)$, then it is a geodesic.

Since $\mathcal{B}$ is $\mathrm{CAT}(\kappa)$, geodesic segments in $\mathcal{B}$ depend uniquely and continuously on their endpoint pairs by Theorem 9.8. Thus there is a subinterval $\mathbb{I}$ of $[0,1]$ that contains a neighborhood of t in $[0,1]$ and such that $\gamma_n|_{\mathbb{I}}$ is minimizing for all large n. It follows that the restriction $\gamma_\infty|_{\mathbb{I}}$ is a geodesic, and therefore γ_∞ is a local geodesic. $\square$

The following theorem was proved by the first author and Richard Bishop [**5**]. In analogy with Riemannian geometry, the main statement of the following theorem could be restated as: *In a space of curvature $\leqslant\kappa$, two points cannot be conjugate along a local geodesic of length $<\varpi\kappa$.*

9.46. No-conjugate-point theorem. *Suppose $\mathcal{U}$ is a locally complete, length, locally* $\mathrm{CAT}(\kappa)$ *space. Let $\gamma\colon[0,1]\to\mathcal{U}$ be a local geodesic path with length $<\varpi\kappa$. Then for some neighborhoods $\Omega^0\ni\gamma(0)$ and $\Omega^1\ni\gamma(1)$, there is a unique continuous map from the direct product $\Omega^0\times\Omega^1\times[0,1]$ to $\mathcal{U}$,*

$$(x,y,t)\mapsto\gamma_{xy}(t),$$

such that $\gamma_{xy}\colon[0,1]\to\mathcal{U}$ is a local geodesic path with $\gamma_{xy}(0)=x$ and $\gamma_{xy}(1)=y$ for each $(x,y)\in\Omega^0\times\Omega^1$, and the family γ_{xy} contains γ. Moreover, we can assume that the map

$$(x,y,t)\mapsto\gamma_{xy}(t)\colon\ \Omega^0\times\Omega^1\times[0,1]\to\mathcal{U}$$

is ℓ-Lipschitz for any $\ell>\max\left\{\frac{\operatorname{sn}^\kappa r}{\operatorname{sn}^\kappa \ell}\ :\ 0\leqslant r\leqslant\ell\right\}$.

9.47. Patchwork along a geodesic. *Let $\mathcal{U}$ be a locally complete, length, locally* $\mathrm{CAT}(\kappa)$ *space, and $\alpha\colon[a,b]\to\mathcal{U}$ be a local geodesic.*

Then there is a complete length $\mathrm{CAT}(\kappa)$ *space $\mathcal{N}$ with an open set $\hat\Omega\subset\mathcal{N}$, a local geodesic $\hat\alpha\colon[a,b]\to\hat\Omega$, and an open locally distance-preserving map $\Phi\colon\hat\Omega\nrightarrow\mathcal{U}$ such that $\Phi\circ\hat\alpha=\alpha$.*

Moreover if α is simple, then one can assume in addition that Φ is an open embedding; thus $\hat{\Omega}$ is locally isometric to a neighborhood of $\Omega = \Phi(\hat{\Omega})$ of α.

This lemma and its proof were suggested by Alexander Lytchak. The proof proceeds by piecing together CAT(κ) neighborhoods of points on a curve to construct a new CAT(κ) space. Exercise 9.80 is inspired by the original idea of the proof of the no-conjugate-point theorem (9.46) given in [**5**].

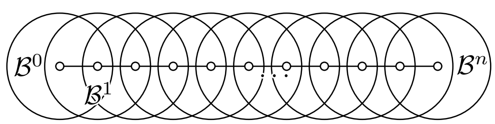
Proof. According to Corollary 9.31, we can choose $r > 0$ such that for any $t \in [a, b]$ the closed ball $\overline{\mathrm{B}}[\alpha(t), r]$ is a convex set that forms a complete length CAT(κ) space.

Choose balls $\mathcal{B}_i = \overline{\mathrm{B}}[\alpha(t_i), r]$ for some partition $a = t_0 < t_1 < \dots < t_n = b$ in such a way that

$$\operatorname{Interior} \mathcal{B}_i \supset \alpha([t_{i-1}, t_i])$$

for all $i > 0$. We can assume in addition that $\mathcal{B}_{i-1} \cap \mathcal{B}_{i+1} \subset \mathcal{B}_i$ if $0 < i < n$.

Consider the disjoint union $\bigsqcup_i \mathcal{B}_i = \{ (i, x) : x \in \mathcal{B}_i \}$ with the minimal equivalence relation $\sim$ such that $(i, x) \sim (i - 1, x)$ for all $i > 0$. Let $\mathcal{N}$ be the space obtained by gluing the $\mathcal{B}_i$ along $\sim$. Note that $A_i = \mathcal{B}_i \cap \mathcal{B}_{i-1}$ is convex in $\mathcal{B}_i$ and in $\mathcal{B}_{i-1}$. Applying ~~the~~ Reshetnyak gluing theorem (9.39) several times, we conclude that $\mathcal{N}$ is a complete length CAT(κ) space.



For $t \in [t_{i-1}, t_i]$, let $\hat{\alpha}(t)$ be the equivalence class of $(i, \alpha(t))$ in $\mathcal{N}$. Let $\hat{\Omega}$ be the ε-neighborhood of $\hat{\alpha}$ in $\mathcal{N}$, where $\varepsilon > 0$ is chosen so that $\mathrm{B}(\alpha(t), \varepsilon) \subset \mathcal{B}_i$ for all $t \in [t_{i-1}, t_i]$.

Define $\Phi : \hat{\Omega} \to \mathcal{U}$ by sending the equivalence class of (i, x) to x. It is straightforward to check that $\Phi : \mathcal{N} \to \mathcal{U}$, $\hat{\alpha} : [a, b] \to \mathcal{N}$ and $\hat{\Omega} \subset \mathcal{N}$ satisfy the conclusion of the main part of the lemma.

To prove the final statement in the lemma, we only have to choose $\varepsilon > 0$ so that in addition, $|\alpha(\tau) - \alpha(\tau')| > 2 \cdot \varepsilon$ if $\tau \leqslant t_{i-1}$ and $t_i \leqslant \tau'$ for some i. $\square$

Proof of 9.46. Apply patchwork along γ (9.47). $\square$

~~The~~ No-conjugate-point theorem (9.46) allows us to move a local geodesic so that its endpoints follow given trajectories. The following corollary describes how this process might terminate.

9.48. Corollary. *Let $\mathcal{U}$ be a locally complete, length, locally* CAT(κ) *space. Suppose $\gamma : [0, 1] \to \mathcal{U}$ is a local geodesic with length $< \varpi\kappa$. Let $\alpha^i : [0, 1] \to \mathcal{U}$, for $i = 0, 1$, be curves starting at $\gamma(0)$ and $\gamma(1)$ respectively.*

Then there is a uniquely determined pair consisting of an interval $\mathbb{I}$ satisfying $0 \in \mathbb{I} \subset [0,1]$, and a continuous family of local geodesics $\gamma_t : [0,1] \to \mathcal{U}$ for $t \in \mathbb{I}$, such that

(*a*) $\gamma_0 = \gamma$, $\gamma_t(0) = \alpha^0(t)$, $\gamma_t(1) = \alpha^1(t)$, *and γ_t has length $< \varpi\kappa$,*

(*b*) *if $\mathbb{I} \neq [0,1]$, then $\mathbb{I} = [0,a)$, where either γ_t converges uniformly to a local geodesic γ_a of length $\varpi\kappa$, or for some fixed $s \in [0,1]$ the curve $\gamma_t(s) : [0,a) \to \mathcal{U}$ is a Lipschitz curve with no limit as $t \to a-$.*

Proof. Uniqueness follows from Theorem 9.46.

Let $\mathbb{I}$ be the maximal interval for which there is a family γ_t satisfying condition (a). By Theorem 9.46, such an interval exists and is open in $[0,1]$. Suppose $\mathbb{I} \neq [0,1]$. Then $\mathbb{I} = [0,a)$ for $0 < a \leqslant 1$.

For each fixed $s \in [0,1]$, define the curve $\alpha_s : [0,a) \to \mathcal{U}$ by $\alpha_s(t) = \gamma_t(s)$. By Theorem 9.46, each α_s is locally Lipschitz.

If α_s for some value of s does not converge as $t \to a-$, then condition (b) holds. If each α_s converges as $t \to a-$, then γ_t converges as $t \to a-$, say to γ_a. By Proposition 9.45, γ_a is a local geodesic and

$$\operatorname{length}\gamma_t \to \operatorname{length}\gamma_a \leqslant \varpi\kappa.$$

By maximality of $\mathbb{I}$, length $\gamma_a = \varpi\kappa$ and so condition (b) again holds. $\square$

9.49. Corollary. *Let $\mathcal{U}$ be a complete locally* $\mathrm{CAT}(\kappa)$ *length space, and $\alpha : [0,1] \to \mathcal{U}$ be a path of length $< \varpi\kappa$ that starts at p and ends at q. Then:*

(*a*) *There is a unique homotopy of local geodesic paths $\gamma_t : [0,1] \to \mathcal{U}$ such that $\gamma_0(t) = \gamma_t(0) = p$ and $\gamma_t(1) = \alpha(t)$ for any t.*

(*b*) *For any $t \in [0,1]$,*

$$\operatorname{length}\gamma_t \leqslant \operatorname{length}(\alpha|_{[0,t]}),$$

and equality holds for given t if and only if the restriction $\alpha|_{[0,t]}$ is a reparametrization of γ_t.

Moreover, instead of completeness of $\mathcal{U}$, one can assume that the subspace

$$W = \{\, x \in \mathcal{U} : |x-p| + |x-q| \leqslant \ell \,\}$$

is complete.

Proof. By Corollary 9.48, taking $\alpha^0(t) = p$ and $\alpha^1(t) = \alpha(t)$ for all $t \in [0,1]$, there is an interval $\mathbb{I}$ such that (a) holds for all $t \in \mathbb{I}$, and either $\mathbb{I} = [0,1]$ or $\mathbb{I} = [0,a)$ for $a \leqslant 1$.

By patchwork along a curve (9.47), the values of t for which condition (b) holds form an open subset of $\mathbb{I}$ containing 0; clearly this subset is also closed in $\mathbb{I}$. Therefore (b) holds on all of $\mathbb{I}$.

Corollary 9.48 implies that $\mathbb{I} = [0, 1]$. Indeed if $\mathbb{I} = [0, a)$, then either $\operatorname{length}\gamma_t \to \varpi\kappa$ as $t \to a-$, or for some fixed $s \in [0, 1]$ the Lipschitz curve $\gamma_t(s)$ has no limit as $t \to a-$. Since $\operatorname{length}\alpha < \varpi\kappa$, 9.48 implies that neither of these is possible. □

K. Lifting globalization

~~The~~ Hadamard–Cartan theorem (9.65) states that the universal metric cover of a complete locally CAT(0) space is CAT(0). The lifting globalization theorem gives an appropriate generalization of the above statement to arbitrary curvature bounds; it could be also described as a global version of Gauss's lemma.

9.50. Lifting globalization theorem. *Suppose $\mathcal{U}$ is a complete length locally* CAT(κ) *space and $p \in \mathcal{U}$. Then there is a complete* CAT(κ) *length space $\mathcal{B}$, with a point $\hat p$ such that there is a locally distance-preserving map $\Phi\colon \mathcal{B} \to \mathcal{U}$ such that $\Phi(\hat p) = p$ and the following lifting property holds: for any path $\alpha\colon [0, 1] \to \mathcal{U}$ with $\alpha(0) = p$ and* $\operatorname{length}\alpha < \varpi\kappa/2$, *there is a unique path $\hat\alpha\colon [0, 1] \to \mathcal{B}$ such that $\hat\alpha(0) = \hat p$ and $\Phi \circ \hat\alpha \equiv \alpha$.*

Note that the lifting property implies that $\Phi(\mathcal{B}) \supset \mathrm{B}(p, \varpi\kappa/2)$ and by completeness $\Phi(\mathcal{B}) \supset \overline{\mathrm{B}}[p, \varpi\kappa/2]$. Also, since $\mathcal{B}$ is CAT(κ), the closed ball $\overline{\mathrm{B}}[\hat p, \frac{\varpi\kappa}{2}]_{\mathcal{B}}$ is a weakly convex set in $\mathcal{B}$ (see 9.27); in particular $\overline{\mathrm{B}}[\hat p, \frac{\varpi\kappa}{2}]_{\mathcal{B}}$ is a complete length CAT(κ) space. Therefore we can assume in addition that $|\hat p - \hat x| \leqslant \varpi\kappa/2$ for any $\hat x \in \mathcal{B}$; or equivalently

$$\overline{\mathrm{B}}[\hat p, \tfrac{\varpi\kappa}{2}]_{\mathcal{B}} = \mathcal{B}.$$

Before proving the theorem we state and prove its corollary.

9.51. Corollary. *Suppose $\mathcal{U}$ is a complete length locally* CAT(κ) *space. Then for any $p \in \mathcal{U}$ there is $\rho_p > 0$ such that $\overline{\mathrm{B}}[p, \rho_p]$ is a complete length* CAT(κ) *space.*

Moreover, we can assume that $\rho_p < \frac{\varpi\kappa}{2}$ for any p and the function $p \mapsto \rho_p$ is 1-Lipschitz.

Proof. Assume $\Phi\colon \mathcal{B} \to \mathcal{U}$ and $\hat p \in \mathcal{B}$ are provided by ~~the~~ lifting globalization theorem (9.50).

Since Φ is local isometry, we can choose $r > 0$ so that the restriction of Φ to $\overline{\mathrm{B}}[\hat p, r]$ is distance-preserving. By the lifting globalization, the image $\Phi(\overline{\mathrm{B}}[\hat p, r])$ coincides with the ball $\overline{\mathrm{B}}[p, r]$. This proves the first part of the theorem.

To prove the second part, let us choose ρ_p to be the maximal value $\leqslant \frac{\varpi\kappa}{2}$ such that $\overline{\mathrm{B}}[p, \rho_p]$ is a complete length CAT(κ) space. By Corollary 9.27, the ball

$$\overline{\mathrm{B}}[q, \rho_p - |p - q|]$$

is weakly convex in $\overline{\mathrm{B}}[p,\rho_p]$. Therefore

$$\overline{\mathrm{B}}[q,\rho_p-|p-q|]$$

is a complete length $\mathrm{CAT}(\kappa)$ space for any $q\in\mathrm{B}(p,\rho_p)$. In particular, $\rho_q\geqslant\rho_p-|p-q|$ for any $p,q\in\mathcal{U}$. Hence the second statement follows. □

The proof of the lifting globalization theorem relies heavily on the properties of the space of local geodesic paths discussed in Section 9J. The following lemma is a key step in the proof; it was proved by the first author and Richard Bishop [**3**].

9.52. Radial lemma. *Let $\mathcal{U}$ be a length locally $\mathrm{CAT}(\kappa)$ space, and suppose $p\in\mathcal{U}$, $R\leqslant\varpi\kappa$. Assume the ball $\overline{\mathrm{B}}[p,\bar R]$ is complete for any $\bar R<R$, and there is a unique geodesic path, $\mathrm{path}_{[px]}$, from p to any point $x\in\mathrm{B}(p,R)$ that depends continuously on x. Then $\mathrm{B}(p,\frac{R}{2})$ is a $\varpi\kappa$-geodesic $\mathrm{CAT}(\kappa)$ space.*

Proof. Without loss of generality, we may assume $\mathcal{U}=\mathrm{B}(p,R)$.

Set $f=\mathrm{md}^\kappa\circ\mathrm{dist}_p$. Let us show that

(1) $$f''+\kappa\cdot f\geqslant 1.$$

Fix $z\in\mathcal{U}$. We will apply ~~the~~ no-conjugate-point theorem (9.46) for the unique geodesic path γ from p to z. The notations Ω^0, Ω^1, γ_{xy}, $\mathcal{N}$, $\hat x$, $\hat y$ will be as in the formulation of ~~the~~ lifting globalization theorem (9.50); in particular, $z\in\Omega^1$.

By assumption, $\gamma_{py}=\mathrm{path}_{[py]}$ for any $y\in\Omega^1$. Consequently, $f(y)=\mathrm{md}^\kappa|\hat p-\hat y|_{\mathcal{N}}$. Applying the function comparison (9.25) in $\mathcal{N}$, we have that $f''+\kappa\cdot f\geqslant 1$ in Ω^1; whence 1 follows. □

Fix $r<\frac{R}{2}$. Proving the following claim takes most of the remaining proof:

(2) *$\overline{\mathrm{B}}[p,r]$ is a convex set in $\mathcal{U}$.*

Choose arbitrary x, $z\in\overline{\mathrm{B}}[p,r]$. First note that 1 implies the following claim.

(3) *If $\gamma\colon[0,1]\to\mathcal{U}$ is a local geodesic path from x to z and* $\mathrm{length}\,\gamma<$ [redacted] $\varpi\kappa$*, then* $\mathrm{length}\,\gamma\leqslant 2\cdot r$ *and γ lies completely in $\overline{\mathrm{B}}[p,r]$.*

Note that $|x-z|<\varpi\kappa$. Thus to prove Claim 2, it is sufficient to show that there is a geodesic path from x to z. Note that by assumption $\overline{\mathrm{B}}[p,2\cdot r]$ is complete. Therefore Corollary 9.49 implies the following:

(4) *Given a path $\alpha\colon [0,1] \to \mathcal{U}$ from x to z with* $\operatorname{length}\alpha < 2\cdot r$, *there is a local geodesic path γ from x to z such that*

$$\operatorname{length}\gamma \leqslant \operatorname{length}\alpha.$$

Further, let us prove the following:

(5) *There is a unique local geodesic path γ_{xz} in $\overline{\mathrm{B}}[p,r]$ from x to z.*

Denote by Δ_{xz} the set of all local geodesic paths in $\overline{\mathrm{B}}[p,r]$ from x to z. By Corollary 9.48, there is a bijection $\Delta_{xz} \to \Delta_{pp}$. According to 1, Δ_{pp} contains only the constant path. Claim 5 follows.

Note that claims 3, 4 and 5 imply that γ_{xz} is minimizing; hence Claim 2.

Further, Claim 3 and ~~the~~ no-conjugate-point theorem (9.46) together imply that the map $(x,z) \mapsto \gamma_{xz}$ is continuous.

By ~~the~~ patchwork globalization theorem (9.30), $\overline{\mathrm{B}}[p,r]$ is a $\varpi\kappa$-geodesic CAT(κ) space.

Since

$$\mathrm{B}(p,R) = \bigcup_{r<R} \overline{\mathrm{B}}[p,r],$$

then $\mathrm{B}(p,R)$ is convex in $\mathcal{U}$ and CAT(κ) comparison holds for any quadruple in $\mathrm{B}(p,R)$. Therefore $\mathrm{B}(p,\varpi\kappa/2)$ is CAT(κ). △

In the following proof, we construct a space $\mathfrak{G}_p$ of local geodesic paths that start at p. The space $\mathfrak{G}_p$ comes with a marked point $\hat{p}$ and the endpoint map $\Phi\colon \mathfrak{G}_p \to \mathcal{U}$. One can think of the map Φ as an analog of the exponential map $\exp_p$ in ~~the~~ Riemannian geometry; in this case, the space $\mathfrak{G}_p$ corresponds to the ball of radius $\varpi\kappa$ in the tangent space at p, equipped with the metric pulled back by $\exp_p$.

We are going to set $\mathcal{B} = \mathrm{B}(\hat{p},\varpi\kappa/2) \subset \mathfrak{G}_p$, and use ~~the~~ radial lemma (9.52) to prove that $\mathcal{B}$ is a $\varpi\kappa$-geodesic CAT(κ) space.

Proof of 9.50. Suppose $\hat{\gamma}$ is a homotopy of local geodesic paths that start at p. Thus the map

$$\hat{\gamma}\colon (t,\tau) \mapsto \hat{\gamma}_t(\tau)\colon [0,1]\times[0,1] \to \mathcal{U}$$

is continuous, and the following holds for each t:

- $\hat{\gamma}_t(0) = p$,
- $\hat{\gamma}_t\colon [0,1] \to \mathcal{U}$ is a local geodesic path in $\mathcal{U}$.

Denote by $\theta(\hat\gamma)$ the length traced by the ends of $\hat\gamma_t$; that is, $\theta(\hat\gamma)$ is the length of the path $t \mapsto \hat\gamma_t(1)$.

Let $\mathfrak{G}_p$ be the set of all local geodesic paths with length $< \varpi\kappa$ in $\mathcal{U}$ that start at p. Denote by $\hat p \in \mathfrak{G}_p$ the constant path $\hat p(t) \equiv p$. Given $\alpha, \beta \in \mathfrak{G}_p$, define

$$|\alpha - \beta|_{\mathfrak{G}_p} = \inf_{\hat\gamma}\{\theta(\hat\gamma)\},$$

with the exact lower bound taken along all homotopies $\hat\gamma\colon [0,1]\times[0,1] \to \mathcal{U}$ such that $\hat\gamma_0 = \alpha$, $\hat\gamma_1 = \beta$ and $\hat\gamma_t \in \mathfrak{G}_p$ for all $t \in [0,1]$.

By ~~the~~ no-conjugate-point theorem (9.46), we have $|\alpha-\beta|_{\mathfrak{G}_p} > 0$ for distinct α and β; that is,

(6) $\quad |{*} - {*}|_{\mathfrak{G}_p}$ *is a metric on* $\mathfrak{G}_p$.

Further, again from ~~the~~ no-conjugate-point theorem (9.46), we have

(7) $\quad$ *The map*

$$\Phi\colon \xi \mapsto \xi(1)\colon \mathfrak{G}_p \to \mathcal{U}$$

is a local isometry. In particular, $\mathfrak{G}_p$ *is locally* $\mathrm{CAT}(\kappa)$.

Let $\alpha\colon [0,1] \to \mathcal{U}$ be a path, length $\alpha < \varpi\kappa$, and $\alpha(0) = p$. The homotopy constructed in Corollary 9.49 can be regarded as a path in $\mathfrak{G}_p$, say $\hat\alpha\colon [0,1] \to \mathfrak{G}_p$, such that $\hat\alpha(0) = \hat p$ and $\Phi\circ\hat\alpha = \alpha$; in particular $\hat\alpha_t(1) \equiv \alpha(t)$ for any t. By 7,

$$\operatorname{length}(\hat\alpha)_{\mathfrak{G}_p} = \operatorname{length}(\alpha)_{\mathcal{U}}.$$

Moreover, it follows that α is a local geodesic path of $\mathcal{U}$ if and only if $\hat\alpha$ is a local geodesic path of $\mathfrak{G}_p$.

Further, from Corollary 9.49, for any $\xi \in \mathfrak{G}_p$ and path $\hat\alpha\colon [0,1] \to \mathfrak{G}_p$ from $\hat p$ to ξ, we have

$$\begin{aligned}\operatorname{length}\hat\alpha &= \operatorname{length}\Phi\circ\hat\alpha \\ &\geqslant \operatorname{length}\xi \\ &= \operatorname{length}\hat\xi\end{aligned}$$

where equality holds only if $\hat\alpha$ is a reparametrization of $\hat\xi$. In particular,

(8) $$|\hat p - \xi|_{\mathfrak{G}_p} = \operatorname{length}\xi$$

and $\hat\xi\colon [0,1] \to \mathfrak{G}_p$ is the unique geodesic path from $\hat p$ to ξ. Clearly, the map $\xi \mapsto \hat\xi$ is continuous.

By 8 and Proposition 9.45,

(9) *For any $\bar R < \varpi\kappa$, the closed ball $\overline{\mathrm{B}}[\hat p, \bar R]$ in $\mathfrak{G}_p$ is complete.*

Take $\mathrm{B}(\hat p, \varpi\kappa/2)$ and Φ constructed above. According to ~~the~~ radial lemma (9.52), $\mathrm{B}(\hat p, \varpi\kappa/2)$ is a $\varpi\kappa$-geodesic $\mathrm{CAT}(\kappa)$ space. The map Φ extends to its completion $\mathcal{B} = \overline{\mathrm{B}}[\hat p, \varpi\kappa/2]$. All the remaining statements are already proved. $\square$

L. Reshetnyak majorization

9.53. Definition. *Let $\mathcal{X}$ be a metric space, $\tilde\alpha$ be a simple closed curve of finite length in $\mathbb{M}^2(\kappa)$, and $D \subset \mathbb{M}^2(\kappa)$ be a closed region bounded by $\tilde\alpha$. A length-nonincreasing map $F\colon D \to \mathcal{X}$ is called* majorizing *if it is length-preserving on $\tilde\alpha$.*

In this case, we say that D majorizes *the curve $\alpha = F\circ\tilde\alpha$ under the map F.*

The following proposition is a consequence of the definition.

9.54. Proposition. *Let α be a closed curve in a metric space $\mathcal{X}$. Suppose $D \subset \mathbb{M}^2(\kappa)$ majorizes α under $F\colon D \to \mathcal{X}$. Then any geodesic subarc of α is the image under F of a subarc of $\partial_{\mathbb{M}^2(\kappa)} D$ that is geodesic in the length metric of D.*

In particular, if D is convex, then the corresponding subarc is a geodesic in $\mathbb{M}^2(\kappa)$.

Proof. For a geodesic subarc $\gamma\colon [a,b] \to \mathcal{X}$ of $\alpha = F\circ\tilde\alpha$, set

$$\tilde r = |\tilde\gamma(a) - \tilde\gamma(b)|_D, \qquad \tilde\gamma = (F|_{\partial D})^{-1}\circ\gamma,$$
$$s = \operatorname{length}\gamma, \qquad \tilde s = \operatorname{length}\tilde\gamma.$$

Then

$$\tilde r \geqslant r = s = \tilde s \geqslant \tilde r.$$

Therefore $\tilde s = \tilde r$. $\square$

9.55. Corollary. *Let $[pxy]$ be a triangle of perimeter $< 2\cdot\varpi\kappa$ in a metric space $\mathcal{X}$. Assume a convex region $D \subset \mathbb{M}^2(\kappa)$ majorizes $[pxy]$. Then $D = \mathrm{Conv}[\tilde p\tilde x\tilde y]$ for a model triangle $[\tilde p\tilde x\tilde y] = \tilde\triangle^{\kappa}(pxy)$, and the majorizing map sends $\tilde p$, $\tilde x$ and $\tilde y$ respectively to p, x and y.*

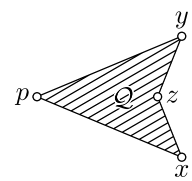
Now we come to the main theorem of this section.

9.56. Majorization theorem. *Any closed curve α with length smaller than $2\cdot\varpi\kappa$ in a $\varpi\kappa$-geodesic $\mathrm{CAT}(\kappa)$ space is majorized by a convex region in $\mathbb{M}^2(\kappa)$.*

This theorem was proved by Yuriy Reshetnyak [**140**]; our proof uses a trick that we learned from the lectures of Werner Ballmann [**21**]. Another proof can be built on Kirszbraun's theorem (10.14), but it works only for complete spaces.

The case when α is a triangle, say $[pxy]$, is the base and is nontrivial. In this case, by Corollary 9.55, the majorizing convex region has to be isometric to $\mathrm{Conv}[\tilde p\tilde x\tilde y]$, where $[\tilde p\tilde x\tilde y] = \tilde\triangle^{\kappa}(pxy)$. There is a majorizing map for $[pxy]$ whose image W is the image of the line-of-sight map (definition 9.32) for $[xy]$ from p, but as one can see from the following example, the line-of-sight map is not majorizing in general.

Example. Let Ω be a solid quadrangle $[pxzy]$ in $\mathbb{E}^2$ formed by two congruent triangles, which is non-convex at z (as in the picture). Equip Ω with the length metric. Then Ω is CAT(0) by Reshetnyak gluing (9.39). For triangle $[pxy]_\Omega$ in Ω and its model triangle $[\tilde p\tilde x\tilde y]$ in $\mathbb{E}^2$, we have



$$|\tilde x - \tilde y| = |x - y|_\Omega = |x - z| + |z - y|.$$

Then the map F defined by matching line-of-sight parameters satisfies $F(\tilde x) = x$ and $|x - F(\tilde w)| > |\tilde x - \tilde w|$ if $\tilde w$ is near the midpoint $\tilde z$ of $[\tilde x\tilde y]$ and lies on $[\tilde p\tilde z]$. Indeed, by ~~the~~ first variation formula (8.42), for $\varepsilon = 1 - s$ we have

$$|\tilde x - \tilde w| = |\tilde x - \tilde\gamma_{\frac12}(s)| = |x - z| + o(\varepsilon)$$

and

$$|x - F(\tilde w)| = |x - \gamma_{\frac12}(s)| = |x - z| - \varepsilon\cdot\cos\measuredangle\left[z\,{}^{p}_{x}\right] + o(\varepsilon).$$

Thus F is not majorizing.

In the following proofs, $x^1 \dots x^n$ ($n \geqslant 3$) denotes a polygonal line $x^1, \dots, x^n$, and $[x^1 \dots x^n]$ denotes the corresponding (closed) polygon. For a subset R of the ambient metric space, we denote by $[x^1 \dots x^n]_R$ a polygon in the length metric of R.

Our first lemma gives a model space construction based on repeated application of Lemma 9.24 from the proof of the inheritance. Recall that convex and concave curves with respect to a point are defined in 8.27.

9.57. Lemma. *In $\mathbb{M}^2(\kappa)$, let β be a curve from x to y that is concave with respect to p. Let D be the subgraph of β with respect to p. Assume*

$$\mathrm{length}\,\beta + |p - x| + |p - y| < 2\cdot\varpi\kappa.$$

(*a*) *Then β forms a geodesic $[xy]_D$ in D and therefore β, $[px]$ and $[py]$ form a triangle $[pxy]_D$ in the length metric of D.*

(b) *Let* $[\tilde p\tilde x\tilde y]$ *be the model triangle for* $[pxy]_D$. *Then there is a short map*

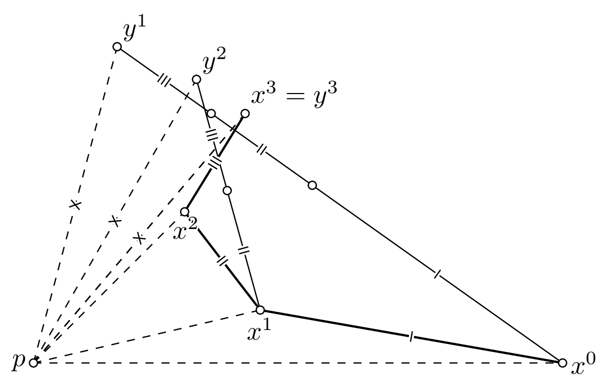
$$G\colon\ \mathrm{Conv}[\tilde p\tilde x\tilde y]\to D$$

such that $\tilde p\mapsto p$, $\tilde x\mapsto x$, $\tilde y\mapsto y$, *and* G *is length-preserving on each side of* $[\tilde p\tilde x\tilde y]$. *In particular,* $\mathrm{Conv}[\tilde p\tilde x\tilde y]$ *majorizes triangle* $[pxy]_D$ *in* D *under* G.

Proof. We prove the lemma for a polygonal line β; the general case then follows by approximation. Namely, since β is concave it can be approximated by polygonal lines that are concave with respect to p, with their lengths converging to length β. Passing to a partial limit we will obtain the needed map G.

Suppose $\beta = x^0x^1\dots x^n$ is a polygonal line with $x^0 = x$ and $x^n = y$. Consider a sequence of polygonal lines $\beta_i = x^0x^1\dots x^{i-1}y_i$ such that $|p-y_i| = |p-y|$ and β_i has same length as β; that is,

$$|x^{i-1}-y_i| = |x^{i-1}-x^i| + |x^i-x^{i+1}| + \dots + |x^{n-1}-x^n|.$$



Clearly $\beta_n = \beta$. Sequentially applying Alexandrov's lemma (6.3) shows that each of the polygonal lines $\beta_{n-1}, \beta_{n-2}, \dots, \beta_1$ is concave with respect to p. Let D_i be the subgraph of β_i with respect to p. Applying Lemma 9.24 gives a short map $G_i\colon D_i\to D_{i+1}$ that maps $y_i\mapsto y_{i+1}$ and does not move p and x (in fact, G_i is the identity everywhere except on $\mathrm{Conv}[px^{i-1}y_i]$). Thus the composition

$$G_{n-1}\circ\dots\circ G_1\colon D_1\to D_n$$

is short. The result follows since $D_1 \overset{iso}{=\!=} \mathrm{Conv}[\tilde p\tilde x\tilde y]$. □

9.58. Lemma. *Let* $[pxy]$ *be a triangle of perimeter* $< 2\cdot\varpi\kappa$ *in a* $\varpi\kappa$*-geodesic* $\mathrm{CAT}(\kappa)$ *space* $\mathcal U$. *In* $\mathbb M^2(\kappa)$, *let* $\tilde\gamma$ *be the* κ*-development of* $[xy]$ *with respect to* p, *where* $\tilde\gamma$ *has basepoint* $\tilde p$ *and subgraph* D. *Consider the map* $H\colon D\to\mathcal U$ *that sends the point with parameter* (t,s) *under the line-of-sight map for* $\tilde\gamma$ *with respect to* $\tilde p$, *to the point with the same parameter under the line-of-sight map* f *for* $[xy]$

with respect to p. Then H is length-nonincreasing. In particular, D majorizes triangle $[pxy]$.

Proof. Let $\gamma = \mathrm{geod}_{[xy]}$ and $T = |x - y|$. As in the proof of ~~the~~ development criterion (9.29), take a partition

$$0 = t^0 < t^1 < \dots < t^n = T,$$

and set $x^i = \gamma(t^i)$. Construct a chain of model triangles $[\tilde{p}\tilde{x}^{i-1}\tilde{x}^i] = \tilde{\triangle}^{\kappa}(px^{i-1}x^i)$, with $\tilde{x}^0 = \tilde{x}$ and the direction of $[\tilde{p}\tilde{x}^i]$ turning counterclockwise as i grows. Let D_n be the subgraph with respect to $\tilde{p}$ of the polygonal line $\tilde{x}^0 \dots \tilde{x}^n$.

Let δ_n be the maximum radius of a circle inscribed in any of the triangles $[\tilde{p}\tilde{x}^{i-1}\tilde{x}^i]$.

Now we construct a map $H_n\colon D_n \to \mathcal{U}$ that increases distances by at most $2 \cdot \delta_n$. Suppose $w \in D_n$. Then w lies on or inside some triangle $[\tilde{p}\tilde{x}^{i-1}\tilde{x}^i]$. Define $H_n(w)$ by first mapping w to a nearest point on $[\tilde{p}\tilde{x}^{i-1}\tilde{x}^i]$ (choosing one if there are several), followed by the natural map to the triangle $[px^{i-1}x^i]$.

Since triangles in $\mathcal{U}$ are κ-thin (9.21), the restriction of H_n to each triangle $[\tilde{p}\tilde{x}^{i-1}\tilde{x}^i]$ is short. Then the triangle inequality implies that the restriction of H_n to

$$U_n = \bigcup_{1\leqslant i\leqslant n} [\tilde{p}\tilde{x}^{i-1}\tilde{x}^i]$$

is short with respect to the length metric on D_n. Since nearest-point projection from D_n to U_n increases the D_n-distance between two points by at most $2 \cdot \delta_n$, the map H_n also increases the D_n-distance by at most $2 \cdot \delta_n$.

Consider converging sequences $v_n \to v$ and $w_n \to w$ such that $v_n, w_n \in D_n$ and therefore $v, w \in D$. Note that

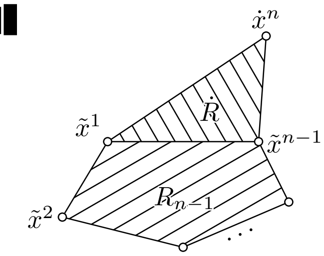
$$|H_n(v_n) - H_n(w_n)| \leqslant |v_n - w_n|_{D_n} + 2 \cdot \delta_n, \tag{1}$$

for each n. Since $\delta_n \to 0$ and geodesics in $\mathcal{U}$ vary continuously with their endpoints (9.30), we have $H_n(v_n) \to H(v)$ and $H_n(w_n) \to H(w)$. Therefore the left-hand side in 1 converges to $|H(v){-}H(w)|$ and the right-hand side converges to $|v - w|_D$, it follows that H is short. $\square$

Proof of 9.56. We begin by proving the theorem in case α is polygonal.

First suppose α is a triangle, say $[pxy]$. By assumption, the perimeter of $[pxy]$ is less than $2 \cdot \varpi\kappa$. This is the base case for the induction.

Let $\tilde{\gamma}$ be the κ-development of $[xy]$ with respect to p, where $\tilde{\gamma}$ has base-point $\tilde{p}$ and subgraph D. By ~~the~~ development criterion (9.29), $\tilde{\gamma}$ is concave. By Lemma 9.57, there is a short map $G\colon\ \mathrm{Conv}\,\tilde{\triangle}^{\kappa}(pxy) \to D$. Further, by Lemma 9.58, D majorizes $[pxy]$ under a majorizing map $H\colon D \to \mathcal{U}$. Clearly $H \circ G$ is a majorizing map for $[pxy]$.



Now we claim that any closed n-gon $[x^1x^2\dots x^n]$[redacted] of perimeter less than $2\cdot\varpi\kappa$ in a CAT(κ) space is majorized by a convex polygonal region

$$R_n = \mathrm{Conv}[\tilde{x}^1\tilde{x}^2\dots\tilde{x}^n]$$

under a map F_n such that $F_n\colon\ \tilde{x}^i\mapsto x^i$ for each i.

Assume the statement is true for $(n-1)$-gons, $n\geqslant 4$. Then $[x^1x^2\dots x^{n-1}]$ is majorized by a convex polygonal region

$$R_{n-1} = \mathrm{Conv}[\tilde{x}^1\tilde{x}^2,\dots\tilde{x}^{n-1}],$$

in $\mathbb{M}^2(\kappa)$ under a map F_{n-1} satisfying $F_{n-1}(\tilde{x}^i) = x^i$ for all i. Take $\dot{x}^n\in\mathbb{M}^2(\kappa)$ such that $[\tilde{x}^1\tilde{x}^{n-1}\dot{x}^n] = \tilde{\triangle}^{\kappa}(x^1x^{n-1}x^n)$ and this triangle lies on the other side of $[\tilde{x}^1\tilde{x}^{n-1}]$ from R_{n-1}. Let $\dot{R} = \mathrm{Conv}[\tilde{x}^1\tilde{x}^{n-1}\dot{x}^n]$, and $\dot{F}\colon\ \dot{R}\to\mathcal{U}$ be a majorizing map for $[x^1x^{n-1}x^n]$ as provided above.

Set $R = R_{n-1}\cup\dot{R}$, where R carries its length metric. Since F_n and F agree on $[\tilde{x}^1\tilde{x}^{n-1}]$, we may define $F\colon\ R\to\mathcal{U}$ by

$$F(x) = \begin{cases} F_{n-1}(x), & x\in R_{n-1},\\ \dot{F}(x), & x\in\dot{R}.\end{cases}$$

Then F is length-nonincreasing and is a majorizing map for $[x^1x^2\dots x^n]$ (as in Definition 9.53).

If R is a convex subset of $\mathbb{M}^2(\kappa)$, we are done.

If R is not convex, the total internal angle of R at $\tilde{x}^1$ or $\tilde{x}^{n-1}$ or both is $>\pi$. By relabeling we may suppose this holds for $\tilde{x}^{n-1}$.

The region R is obtained by gluing R_{n-1} to $\dot{R}$ by $[x^1x^{n-1}]$. Thus, by Reshetnyak gluing (9.39), R carrying its length metric is a CAT(κ)-space. Moreover $[\tilde{x}^{n-2}\tilde{x}^{n-1}]\cup[\tilde{x}^{n-1}\dot{x}^n]$ is a geodesic of R. Thus $[\tilde{x}^1\tilde{x}^2\dots\tilde{x}^{n-2}\dot{x}^n]_R$ is a closed $(n-1)$-gon in R, to which the induction hypothesis applies. The resulting short map from a convex region in $\mathbb{M}^2(\kappa)$ to R, followed by F, is the desired majorizing map.

Note that in fact we have proved the following:

(2) *Let F_{n-1} be a majorizing map for the polygon $[x^1x^2\dots x^{n-1}]$, and $\dot{F}$ be a majorizing map for the triangle $[x^1x^{n-1}x^n]$. Then there is a majorizing map F_n for the polygon $[x^1x^2\dots x^n]$ such that*

$$\mathfrak{I}F_{n+1} = \mathfrak{I}F_n\cup\mathfrak{I}\dot{F}.$$

We now use this claim to prove the theorem for general curves.

Assume $\alpha\colon [0,\ell]\to\mathcal{U}$ is an arbitrary closed curve with natural parameter. Choose a sequence of partitions $0=t_n^0<t_n^2<\dots<t_n^n=\ell$ so that:

- The set $\{t_{n+1}^i\}_{i=0}^{n+1}$ is obtained from the set $\{t_n^i\}_{i=0}^{n}$ by adding one element.
- For a sequence $\varepsilon_n\to 0+$, we have $t_n^i-t_n^{i-1}<\varepsilon_n$ for all i.

Inscribe in α a sequence of polygons P_n with vertexes $\alpha(t_n^i)$. Apply the claim above, to get a sequence of majorizing maps $F_n\colon R_n\to\mathcal{U}$. Note that for all $m>n$ we have

- $\Im F_m$ lies in an ε_n-neighborhood of $\Im F_n$,
- $\Im F_m\setminus\Im F_n$ lies in an ε_n-neighborhood of α.

It follows that the set

$$K=\alpha\cup\left(\bigcup_n \Im F_n\right)$$

is compact. Therefore the sequence (F_n) has a partial limit as $n\to\infty$; say F. Clearly F is a majorizing map for α. □

If $p_1\dots p_n$ is a polygon, then values $\theta_i=\pi-\measuredangle\left[p_i\,{}^{p_{i-1}}_{p_{i+1}}\right]$ for all $i\pmod n$ are called *external angles* of the polygon. The following exercise is a generalization of Fenchel's theorem.

9.59. Exercise. *Show that the sum of external angles of any polygon in a complete length* CAT(0) *space cannot be smaller than* $2\cdot\pi$.

9.60. Very advanced exercise. *Suppose that a simple polygon β in a complete length* CAT(0) *space does not bound an embedded disc. Show that the sum of external angles of β cannot be smaller than* $4\cdot\pi$.

Give an example of such a polygon β with the sum of external angles exactly $4\cdot\pi$.

The following exercise is the rigidity case of the majorization theorem.

9.61. Exercise. *Let $\mathcal{U}$ be a $\varpi\kappa$-geodesic* CAT(κ) *space and $\alpha\colon[0,\ell]\to\mathcal{U}$ be a closed curve with arclength parametrization. Assume that $\ell<2\cdot\varpi\kappa$ and there is a closed convex curve $\tilde\alpha\colon[0,\ell]\to\mathbb{M}^2(\kappa)$ such that*

$$|\alpha(t_0)-\alpha(t_1)|_{\mathcal{U}}=|\tilde\alpha(t_0)-\tilde\alpha(t_1)|_{\mathbb{M}^2(\kappa)}$$

for any t_0 and t_1. Then there is a distance-preserving map $F\colon \operatorname{Conv}\tilde\alpha\to\mathcal{U}$ such that $F\colon\tilde\alpha(t)\mapsto\alpha(t)$ for any t.

9.62. Exercise. *Two majorizations $F\colon D\to\mathcal{U}$ and $F'\colon D'\to\mathcal{U}$ will be called* equivalent *if $F'=F\circ\iota$ for an isometry $\iota\colon D\to D'$.*

Show that a closed rectifiable curve in a CAT(0) *space has an isometric majorization map if and only if the majorization map is unique up to equivalence.*

The following lemma states, in particular, that in a CAT(κ) space, a sharp triangle comparison implies the presence of an isometric copy of the convex hull of the model triangle. The latter statement was proved by Alexandr Alexandrov [**17**].

9.63. Arm lemma. *Let $\mathcal{U}$ be a $\varpi\kappa$-geodesic* CAT(κ) *space, and $P = [x^0 x^1 \dots x^{n+1}]$ be a polygon of length $< 2\cdot\varpi\kappa$ in $\mathcal{U}$. Suppose $\tilde{P} = [\tilde{x}^0\tilde{x}^1 \dots \tilde{x}^{n+1}]$ is a convex polygon in $\mathbb{M}^2(\kappa)$ such that*

$$(3) \qquad |\tilde{x}^i - \tilde{x}^{i-1}|_{\mathbb{M}^2(\kappa)} = |x^i - x^{i-1}|_{\mathcal{U}} \quad \textit{and} \quad \measuredangle\left[x^i\,{}^{x^{i-1}}_{x^{i+1}}\right] \geqslant \measuredangle\left[\tilde{x}^i\,{}^{\tilde{x}^{i-1}}_{\tilde{x}^{i+1}}\right]$$

for all i. Then

(*a*) $|\tilde{x}^0 - \tilde{x}^{n+1}|_{\mathbb{M}^2(\kappa)} \leqslant |x^0 - x^{n+1}|_{\mathcal{U}}$.

(*b*) *Equality holds in* (*a*) *if and only if the map $\tilde{x}^i \mapsto x^i$ can be extended to a distance-preserving map of* $\mathrm{Conv}(\tilde{x}^0, \tilde{x}^1 \dots \tilde{x}^{n+1})$ *onto* $\mathrm{Conv}(x^0, x^1 \dots x^{n+1})$.

Proof.

(*a*). By majorization (9.56), P is majorized by a convex region $\tilde{D}$ in $\mathbb{M}^2(\kappa)$. By Proposition 9.54 and the definition of angle, $\tilde{D}$ is bounded by a convex polygon $\tilde{P}_R = [\tilde{y}^0\tilde{y}^1 \dots \tilde{y}^{n+1}]$ that satisfies

$$|\tilde{y}^i - \tilde{y}^{i\pm1}|_{\mathbb{M}^2(\kappa)} = |x^i - x^{i\pm1}|_{\mathcal{U}}, \qquad |\tilde{y}^0 - \tilde{y}^{n+1}|_{\mathbb{M}^2(\kappa)} = |x^0 - x^{n+1}|_{\mathcal{U}},$$
$$\measuredangle\left[\tilde{y}^i\,{}^{\tilde{y}^{i-1}}_{\tilde{y}^{i+1}}\right] \geqslant \measuredangle\left[x^i\,{}^{x^{i-1}}_{x^{i+1}}\right] \geqslant \measuredangle\left[\tilde{x}^i\,{}^{\tilde{x}^{i-1}}_{\tilde{x}^{i+1}}\right]$$

for $1 \leqslant i \leqslant n$; the last inequality follows from 3.

The classical arm lemma [**143**] gives $|\tilde{x}^0 - \tilde{x}^{n+1}| \leqslant |\tilde{y}^0 - \tilde{y}^{n+1}|$. Since $|\tilde{y}^0 - \tilde{y}^{n+1}| = |x^0 - x^{n+1}|$, part (*a*) follows.

(*b*). Suppose equality holds in (*a*). Then angles at the j-th vertex of $\tilde{P}$, P, and $\tilde{P}_R$ are equal for $1 \leqslant j \leqslant n$, and we may take $\tilde{P} = \tilde{P}_R$.

Let $F\colon \tilde{D} \to \mathcal{U}$ be the majorizing map for P, where $\tilde{D}$ is the convex region bounded by $\tilde{P}$, and $F|_{\tilde{P}}$ is length-preserving.

(4) *Let $\tilde{x}, \tilde{y}, \tilde{z}$ be three vertexes of $\tilde{P}$, and x, y, z be the corresponding vertexes of P. If $|\tilde{x} - \tilde{y}| = |x - y|$, $|\tilde{y} - \tilde{z}| = |x - z|$ and $\measuredangle\left[\tilde{y}\,{}^{\tilde{x}}_{\tilde{z}}\right] = \measuredangle\left[y\,{}^{x}_{z}\right]$, then $F|_{\mathrm{Conv}(\tilde{x},\tilde{y},\tilde{z})}$ is distance-preserving.*

Indeed, since F is majorizing, F restricts to distance-preserving maps from $[\tilde{x}\tilde{y}]$ to $[xy]$ and $[\tilde{y}\tilde{z}]$ to $[yz]$. Suppose $\tilde{p} \in [\tilde{x}\tilde{y}]$ and $\tilde{q} \in [\tilde{y}\tilde{z}]$. Then

$$(5) \qquad |\tilde{p} - \tilde{q}|_{\mathbb{M}^2(\kappa)} = |F(\tilde{p}) - F(\tilde{q})|_{\mathcal{U}}.$$

This inequality holds in one direction by majorization, and in the other direction by the angle comparison (9.14*c*). By the first variation formula (9.37), it follows that each pair of corresponding angles of triangles $[\tilde{x}\tilde{y}\tilde{z}]$ and $[xyz]$ are equal. But then 5 holds for p, q on any two sides of these triangles, so F is distance-preserving on every geodesic of $\mathrm{Conv}(\tilde p, \tilde x, \tilde y)$. Hence the claim. $\triangle$

(6) *Suppose $F|_{\mathrm{Conv}(\tilde x^0,\tilde x^1,\dots,\tilde x^k)}$ is distance-preserving for $2 \leqslant k \leqslant n-1$. Then the restriction $F|_{\mathrm{Conv}(\tilde x^0,\tilde x^1,\dots,\tilde x^{k+1})}$ is distance-preserving.*

To verify this claim, let

$$\tilde p = [\tilde x^{k-1}\tilde x^{k+1}] \cap [\tilde x^k \tilde x^0] \quad \text{and} \quad p = F(p).$$

Note that the following maps are distance-preserving:

(i) $F|_{\mathrm{Conv}(\tilde x^{k-1},\tilde x^k,\tilde x^{k+1})}$,

(ii) $F|_{\mathrm{Conv}(\tilde x^{k+1},\tilde x^{k-1},\tilde x^0)}$,

(iii) $F|_{\mathrm{Conv}(\tilde x^0,\tilde x^k,\tilde x^{k+1})}$.

Indeed, (i) follows from 4. Therefore $|\tilde x^{k-1} - \tilde x^{k+1}| = |x^{k-1} - x^{k+1}|$, and so F restricts to a distance-preserving map from $[\tilde x^{k-1}\tilde x^{k+1}]$ onto $[x^{k-1}x^{k+1}]$. With the induction hypothesis in 6, it follows that $p = [x^{k-1}x^{k+1}] \cap [x^k x^0]$, hence

(7) $$\measuredangle\left[\tilde x^{k-1}\,{}^{\tilde x^{k+1}}_{\tilde x^0}\right] = \measuredangle\left[x^{k-1}\,{}^{x^{k+1}}_{x^0}\right].$$

Then (ii) follows from 7 and 4. Since $|\tilde x^k - \tilde x^0| = |x^k - x^0|$, (iii) follows from 7 and (i).

Let $\tilde\gamma$ be a geodesic of $\mathrm{Conv}(\tilde x^0, \tilde x^0, \tilde x^1 \dots \tilde x^{k+1})$. Then length $\tilde\gamma < \varpi\kappa$. If $\tilde\gamma$ does not contain the point $\tilde p$, then by the induction hypothesis in 6 and (i) + (ii) + (iii), we get that $\gamma = F \circ \tilde\gamma$ is a local geodesic of length $< \varpi\kappa$. By 9.22, γ is a geodesic.

By continuity, $F \circ \tilde\gamma$ is a geodesic for all $\tilde\gamma$; so 6 follows.

The base of the induction is provided by 4. It finishes the proof of part (b). $\square$

9.64. Exercise. *Let $\mathcal U$ be a complete length* CAT(0) *space and suppose for 4 points $x^1, x^2, x^3, x^4 \in \mathcal U$ there is a convex quadrangle $[\tilde x^1\tilde x^2\tilde x^3\tilde x^4]$ in $\mathbb E^2$ such that*

$$|x^i - x^j|_{\mathcal U} = |\tilde x^i - \tilde x^j|_{\mathbb E^2}$$

for all i and j. Show that $\mathcal U$ contains an isometric copy of solid quadrangle *$[\tilde x^1\tilde x^2\tilde x^3\tilde x^4]$; that is, the convex hull of $\tilde x^1, \tilde x^2, \tilde x^3, \tilde x^4$ in $\mathbb E^2$.*

M. Hadamard–Cartan theorem

The development of Alexandrov geometry was greatly influenced by the Hadamard–Cartan theorem. Its original formulation states that if M is a complete Riemannian manifold with nonpositive sectional curvature, then the exponential map at any point $p \in M$ is a covering; in particular it implies that the universal cover of M is diffeomorphic to the Euclidean space of the same dimension.

In this generality, the theorem appeared in the lectures of Élie Cartan [**47**]. For surfaces in the Euclidean space, the theorem was proved by Hans von Mangoldt [**112**], and a few years later independently by Jacques Hadamard [**79**].

Formulations for metric spaces of different generality were proved by Herbert Busemann [**45**], Willi Rinow [**141**], and Michael Gromov [**76**, p. 119]. A detailed proof of Gromov's statement when $\mathcal{U}$ is proper was given by Werner Ballmann [**20**], using Birkhoff's curve-shortening. A proof in the non-proper geodesic case was given by the first author and Richard Bishop [**5**]. This proof applies more generally, to *convex spaces* (see Exercise 9.80). It was pointed out by Bruce Kleiner [**21**] and independently by Martin Bridson and André Haefliger [**34**] that this proof extends to length spaces, as well as geodesic spaces, giving the following statement:

9.65. Hadamard–Cartan theorem. *Let $\kappa \leqslant 0$, and $\mathcal{U}$ be a complete, simply connected length locally* CAT(κ) *space. Then $\mathcal{U}$ is* CAT(κ).

Proof. Since $\varpi\kappa = \infty$, Theorem 9.50 implies that there is a CAT(κ) space $\mathcal{B}$ and a *metric covering* $\Phi\colon \mathcal{B} \to \mathcal{U}$; that is, Φ is a length-preserving covering map.

Since $\mathcal{U}$ is simply connected, $\Phi\colon \mathcal{B} \to \mathcal{U}$ is an isometry—hence the result. □

To formulate the generalized Hadamard–Cartan theorem, we need the following definition.

9.66. Definition. *Given $\ell \in (0, \infty]$, a metric space $\mathcal{X}$ is called ℓ-simply connected if it is connected and any closed curve of length $< \ell$ is null-homotopic in the class of curves of length $< \ell$ in $\mathcal{X}$.*

Note that there is a subtle difference between simply connected and ∞-simply connected spaces; the first states that any closed curve is null-homotopic while the second means that any rectifiable curve is null-homotopic in the class of rectifiable curves. However, as follows from Proposition 9.69, for locally CAT(κ) spaces these two definitions are equivalent. This fact makes it possible to deduce the Hadamard–Cartan theorem directly from the generalized Hadamard–Cartan theorem.

9.67. Generalized Hadamard–Cartan theorem. *A complete length space $\mathcal{U}$ is* CAT(κ) *if and only if $\mathcal{U}$ is $2 \cdot \varpi\kappa$-simply connected and $\mathcal{U}$ is locally* CAT(κ).

For proper spaces, the generalized Hadamard–Cartan theorem was proved by Brian Bowditch [**31**]. In the proof we need the following lemma.

9.68. Lemma. *Let $\mathcal{U}$ be a complete length locally* CAT(κ) *space,* $\varepsilon > 0$*, and* $\gamma_1, \gamma_2 \colon \mathbb{S}^1 \to \mathcal{U}$ *be two closed curves. Assume*

(a) $\operatorname{length}\gamma_i < 2 \cdot \varpi\kappa - 4 \cdot \varepsilon$ *for* $i = 1, 2$*;*

(b) $|\gamma_1(x) - \gamma_2(x)| < \varepsilon$ *for any* $x \in \mathbb{S}^1$*, and the geodesic* $[\gamma_1(x)\gamma_2(x)]$ *is uniquely defined and depends continuously on* x*;*

(c) γ_1 *is majorized by a convex region in* $\mathbb{M}^2(\kappa)$*.*

Then γ_2 *is majorized by a convex region in* $\mathbb{M}^2(\kappa)$*.*

Proof. Let D be a convex region in $\mathbb{M}^2(\kappa)$ that majorizes γ_1 under the map $F \colon D \to \mathcal{U}$ (see Definition 9.53). Denote by $\tilde{\gamma}_1$ the curve bounding D such that $F \circ \tilde{\gamma}_1 = \gamma_1$. Since

$$\begin{aligned}\operatorname{length}\tilde{\gamma}_1 &= \operatorname{length}\gamma_1\\ &< 2 \cdot \varpi\kappa - 4 \cdot \varepsilon,\end{aligned}$$

there is a point $\tilde{p} \in D$ such that $|\tilde{p} - \tilde{\gamma}(x)|_{\mathbb{M}^2(\kappa)} < \frac{\varpi\kappa}{2} - \varepsilon$ for any $x \in \mathbb{S}^1$. Denote by α_x the concatenation of the paths $F \circ \operatorname{path}_{[p\tilde{\gamma}_1(x)]_{\mathbb{M}^2(\kappa)}}$ and $\operatorname{path}_{[\gamma_1(x)\gamma_2(x)]}$ in $\mathcal{U}$. Note that α_x depends continuously on x, and

$$\operatorname{length}\alpha_x < \tfrac{\varpi\kappa}{2} \quad \text{and} \quad \alpha_x(1) = \gamma_2(x)$$

hold for any x.

Let us apply ~~the~~ lifting globalization theorem (9.50) for $p = F(\tilde{p})$. We obtain a $\varpi\kappa$-geodesic CAT(κ) space $\mathcal{B}$ and a locally distance-preserving map $\Phi \colon \mathcal{B} \to \mathcal{U}$ with $\Phi(\hat{p}) = p$ for some $\hat{p} \in \mathcal{B}$, and with the lifting property for the curves starting at p with length $< \varpi\kappa/2$. Applying the lifting property for α_x, we get existence of a curve $\hat{\gamma}_2 \colon \mathbb{S}^1 \to \mathcal{B}$ such that

$$\gamma_2 = \Phi \circ \hat{\gamma}_2.$$

Since $\mathcal{B}$ is a geodesic CAT(κ) space, we can apply ~~the~~ majorization theorem (9.56) for $\hat{\gamma}_2$. The composition of the obtained majorization with Φ is a majorization of γ_2. □

Proof of Theorem 9.67. The only-if part follows from the Reshetnyak majorization theorem (9.56).

Let γ_t, $t \in [0, 1]$ be a null-homotopy of curves in $\mathcal{U}$; that is, $\gamma_0(x) = p$ for some $p \in \mathcal{U}$ and any $x \in \mathbb{S}^1$. Assume further that $\operatorname{length}\gamma_t < 2 \cdot \varpi\kappa$ for any t. To prove the if part, it is sufficient to show that γ_1 is majorized by a convex region in $\mathbb{M}^2(\kappa)$ if $\mathcal{U}$ is locally CAT(κ).

By semicontinuity of length (2.6), we can choose $\varepsilon > 0$ sufficiently small that

$$\operatorname{length}\gamma_t < 2 \cdot \varpi\kappa - 4 \cdot \varepsilon$$

for all t.

By Corollary 9.51, we may assume in addition that $\mathrm{B}(\gamma_t(x),\varepsilon)$ is $\mathrm{CAT}(\kappa)$ for any t and x.

Choose a partition $0 = t_0 < t_1 < \dots < t_n = 1$ so that $|\gamma_{t_i}(x) - \gamma_{t_{i-1}}(x)| < \varepsilon$ for any i and x. According to 9.8, for any i, the geodesic $[\gamma_{t_i}(x)\gamma_{t_{i-1}}(x)]$ depends continuously on x.

Note that $\gamma_0 = \gamma_{t_0}$ is majorized by a convex region in $\mathbb{M}^2(\kappa)$. Applying the lemma n times, we see that the same holds for $\gamma_1 = \gamma_{t_n}$. $\square$

9.69. Proposition. *Let $\mathcal{U}$ be a complete length locally* $\mathrm{CAT}(\kappa)$ *space. Then $\mathcal{U}$ is simply connected if and only if it is ∞-simply connected.*

Proof.

If part. It is sufficient to show that any closed curve in $\mathcal{U}$ is homotopic to a polygon.

Let γ_0 be a closed curve in $\mathcal{U}$. According to Corollary 9.51, there is $\varepsilon > 0$ such that $\mathrm{B}(\gamma(x),\varepsilon)$ is $\mathrm{CAT}(\kappa)$ for any x.

Choose a polygon γ_1 such that $|\gamma_0(x) - \gamma_1(x)| < \varepsilon$ for any x. By 9.8, $\mathrm{path}_{[\gamma_0(x)\gamma_1(x)]}$ is uniquely defined and depends continuously on x.

Hence $\gamma_t(x) = \mathrm{path}_{[\gamma_0(x)\gamma_1(x)]}(t)$ gives a homotopy from γ_0 to γ_1.

Only-if part. The proof is similar.

Assume γ_t is a homotopy between two rectifiable curves γ_0 and γ_1. Fix $\varepsilon > 0$ so that the ball $\mathrm{B}(\gamma_t(x),\varepsilon)$ is $\mathrm{CAT}(\kappa)$ for any t and x. Choose a partition $0 = t_0 < t_1 < \dots < t_n = 1$ so that

$$|\gamma_{t_{i-1}}(x) - \gamma_{t_i}(x)| < \tfrac{\varepsilon}{10}$$

for any i and x. Set $\hat\gamma_{t_0} = \gamma_0$, $\hat\gamma_{t_n} = \gamma_{t_n}$. For each $0 < i < n$, approximate γ_{t_i} by a polygon $\hat\gamma_i$.

Construct the *geodesic homotopy* from $\hat\gamma_{t_{i-1}}$ to $\hat\gamma_{t_i}$; that is, set

$$\hat\gamma_t = \mathrm{path}_{[\hat\gamma_{t_{i-1}}(x)\hat\gamma_{t_i}(x)]}(t)$$

for $t \in [t_{i-1}, t_i]$. Since ε is sufficiently small, by 9.13, we get that

$$\operatorname{length}\hat\gamma_t < 10 \cdot (\operatorname{length}\hat\gamma_{t_{i-1}} + \operatorname{length}\hat\gamma_{t_i})$$

for any $t \in [t_{i-1}, t_i]$. In particular, $\hat\gamma_t$ is rectifiable for all t.

Joining the obtained homotopies for all i, we obtain a homotopy from γ_0 to γ_1 in the class of rectifiable curves. $\square$

9.70. Exercise. *Let $\mathcal{X}$ be a double cover of $\mathbb{E}^3$ that branches along two distinct lines ℓ and m. Show that $\mathcal{X}$ is* $\mathrm{CAT}(0)$ *if and only if ℓ intersects m at a right angle.*

9.71. Exercise. *Let $\mathcal{U}$ be a complete length* $\mathrm{CAT}(0)$ *space. Assume $\tilde{\mathcal{U}} \to \mathcal{U}$ is a metric covering branching along a geodesic. Show that $\tilde{\mathcal{U}}$ is* $\mathrm{CAT}(0)$.

More generally, assume $A \subset \mathcal{U}$ is a closed convex subset and $f : \mathcal{X} \to \mathcal{U}\backslash A$ is a metric cover. Denote by $\bar{\mathcal{X}}$ the completion of $\mathcal{X}$, and $\bar{f} : \bar{\mathcal{X}} \to \mathcal{U}$ the continuous extension of f. Let $\tilde{\mathcal{U}}$ be the space glued from $\bar{\mathcal{X}}$ and A by identifying x and $\bar{f}(x)$ if $\bar{f}(x) \in A$. Show that $\tilde{\mathcal{U}}$ is CAT(0).

N. Convex sets

Recall that according to Corollary 9.27, any ball (closed or open) of radius $R < \frac{\varpi\kappa}{2}$ in a $\varpi\kappa$-geodesic CAT(κ) space is convex. From the uniqueness of geodesics in CAT(κ) spaces (9.8) we get the following:

9.72. Observation. *Any weakly $\varpi\kappa$-convex set in a complete length* CAT(κ) *space is $\varpi\kappa$-convex.*

9.73. Closest-point projection lemma. *Let p be a point in a complete length* CAT(κ) *space $\mathcal{U}$, and $K \subset \mathcal{U}$ be a closed $\varpi\kappa$-convex set. If $\operatorname{dist}_K p < \frac{\varpi\kappa}{2}$, then there is a unique point $p^* \in K$ that minimizes the distance to p; that is, $|p^* - p| = \operatorname{dist}_K p$.*

Proof. Fix r properly between $\operatorname{dist}_K p$ and $\frac{\varpi\kappa}{2}$. By the function comparison (9.25), the function $f = \operatorname{md}^\kappa \circ \operatorname{dist}_p$ is strongly convex in $\bar{\mathrm{B}}[p, r]$.

The lemma follows from Lemma 14.4 applied to the subspace $K' = K \cap \bar{\mathrm{B}}[p, r]$ and the restriction $f|_{K'}$. □

9.74. Exercise. *Let $\mathcal{U}$ be a complete length* CAT(0) *space and $K \subset \mathcal{U}$ be a closed convex set. Show that the closest-point projection $\mathcal{U} \to K$ is short.*

9.75. Advanced exercise. *Let $\mathcal{U}$ be a complete length* CAT(1) *space and $K \subset \mathcal{U}$ be a closed π-convex set. Assume $K \subset \bar{\mathrm{B}}[p, \frac{\pi}{2}]$ for $p \in K$. Show that there is a short retraction of $\mathcal{U}$ to K.*

9.76. Proposition. *Let $\mathcal{U}$ be a $\varpi\kappa$-geodesic* CAT(κ) *space and $K \subset \mathcal{U}$ be a closed $\varpi\kappa$-convex set. Let*

$$f = \operatorname{sn}^\kappa \circ \operatorname{dist}_K .$$

Then

$$f'' + \kappa \cdot f \geqslant 0$$

holds in $\mathrm{B}(K, \frac{\varpi\kappa}{2})$.

Proof. It is sufficient to show that Jensen's inequality (3.14*c*) holds on a sufficiently short geodesic $[pq]$ in $\mathrm{B}(K, \frac{\varpi\kappa}{2})$. We may assume that

(1) $$|p - q| + \operatorname{dist}_K p + \operatorname{dist}_K q < \varpi\kappa.$$

For each $x \in [pq]$, we need to find a value $h(x) \in \mathbb{R}$ such that

$$h(p) = f(p), \qquad h(q) = f(q), \qquad h(x) \leqslant f(x)$$

for any $x \in [pq]$, and

(2) $$h'' + \kappa \cdot h \geqslant 0$$

along $[pq]$.

Denote by p^* and q^* the closest-point projections of p and q on K; they are provided by 9.73. From 1 and the triangle inequality, we have

$$|p^* - q^*| < \varpi\kappa.$$

Since K is $\varpi\kappa$-convex, $K \supset [p^*q^*]$; in particular

$$\operatorname{dist}_K x \leqslant \operatorname{dist}_{[p^*q^*]} x$$

for any $x \in \mathcal{U}$.

There is a majorizing map $F : D \to \mathcal{U}$ for quadrangle $[pp^*q^*q]$, as in Definition 9.53 and ~~the~~ Reshetnyak majorization theorem (9.56). By Proposition 9.54, the figure D is a solid convex quadrangle $[\tilde{p}\tilde{p}^*\tilde{q}^*\tilde{q}]$ in $\mathbb{M}^2(\kappa)$ such that

$$|\tilde{p} - \tilde{p}^*|_{\mathbb{M}^2(\kappa)} = |p - p^*|_{\mathcal{U}} \qquad |\tilde{p} - \tilde{q}|_{\mathbb{M}^2(\kappa)} = |p - q|_{\mathcal{U}}$$
$$|\tilde{q} - \tilde{q}^*|_{\mathbb{M}^2(\kappa)} = |q - q^*|_{\mathcal{U}} \qquad |\tilde{p}^* - \tilde{q}^*|_{\mathbb{M}^2(\kappa)} = |p^* - q^*|_{\mathcal{U}}.$$

Given $x \in [pq]$, denote by $\tilde{x}$ the corresponding point on $[\tilde{p}\tilde{q}]$. Then

$$\operatorname{dist}_{[pq]} x \leqslant \operatorname{dist}_{[\tilde{p}\tilde{q}]} \tilde{x}.$$

Set

$$h(x) = \operatorname{sn}^\kappa \circ \operatorname{dist}_{[\tilde{p}\tilde{q}]} \tilde{x}.$$

By straightforward calculations, 2 holds and hence the statement follows. □

9.77. Corollary. *Let $\mathcal{U}$ be a complete length* CAT(κ) *space and $K \subset \mathcal{U}$ be a closed locally convex set. Then there is an open set $\Omega \supset K$ such that the function $f = \operatorname{sn}^\kappa \circ \operatorname{dist}_K$ satisfies*

$$f'' + \kappa \cdot f \geqslant 0$$

in Ω.

Proof. Fix $p \in K$. By Corollary 9.27, $\overline{\mathrm{B}}[p, r]$ is convex for all small $r > 0$.

Since K is locally convex, there is $r_p > 0$ such that the intersection $K' = K \cap \mathrm{B}(p, r_p)$ is convex.

Note that

$$\operatorname{dist}_K x = \operatorname{dist}_{K'} x$$

for any $x \in \mathrm{B}(p, \tfrac{r_p}{2})$. Therefore the statement holds for

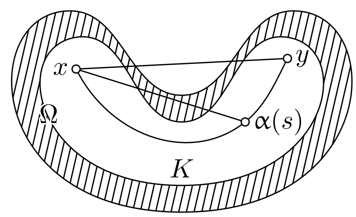
$$\Omega = \bigcup_{p \in K} \mathrm{B}(p, \tfrac{r_p}{2}).$$

□

9.78. Theorem. *Assume $\mathcal{U}$ is a complete length* CAT(κ) *space and $K \subset \mathcal{U}$ is a closed connected locally convex set. Assume $|x - y| < \varpi\kappa$ for any $x, y \in K$. Then K is convex.*

In particular, if $\kappa \leqslant 0$, then any closed connected locally convex set in $\mathcal{U}$ is convex.

The following proof is due to Sergei Ivanov [**87**].



Proof. Since K is locally convex, it is locally path-connected. Since K is connected and locally path connected it is path-connected.

Fix two points $x, y \in K$. Let us connect x to y by a path $\alpha\colon [0,1] \to K$. Since $|x - \alpha(s)| < \varpi\kappa$ for any s, Theorem 9.8 implies that the geodesic $[x\alpha(s)]$ is uniquely defined and depends continuously on s.

Let $\Omega \supset K$ be the open set provided by Corollary 9.77. If $[xy] = [x\alpha(1)]$ does not completely lie in K, then there is a value $s \in [0,1]$ such that $[x\alpha(s)]$ lies in Ω but does not completely lie in K. By Corollary 9.77, the function $f = \mathrm{sn}^\kappa \circ \mathrm{dist}_K \mathcal{U}$ satisfies the differential inequality

$$f'' + \kappa \cdot f \geqslant 0 \tag{3}$$

along $[x\alpha(s)]$.

Since

$$|x - \alpha(s)| < \varpi\kappa, \qquad f(x) = f(\alpha(s)) = 0,$$

then the barrier inequality (3.14*b*) implies that $f(z) \leqslant 0$ for $z \in [x\alpha(s)]$; that is $[x\alpha(s)] \subset K$, a contradiction. $\square$

O. Remarks

The following question was known in folklore in the 80's, but it seems that in print it was first mentioned by Michael Gromov [**74**, 6.B$_1$(f)]. We do not see any reason why it should be true, but we also cannot construct a counterexample.

9.79. Open question. *Let $\mathcal{U}$ be a complete length* CAT(0) *space and $K \subset \mathcal{U}$ be a compact set. Is it true that K lies in a convex compact set $\bar{K} \subset \mathcal{U}$?*

The question can easily be reduced to the case when K is finite; so far it is not even known if any three points in a complete length CAT(0) space lie in a compact convex set.

We expect that 9.47 can be extended to all curves, not necessarily local geodesics, but the proof does not admit a straightforward generalization.

About convex spaces. A *convex space* $\mathcal{X}$ is a geodesic space such that the function $t \mapsto |\gamma(t) - \sigma(t)|$ is convex for any two geodesic paths $\gamma, \sigma : [0,1] \to \mathcal{X}$. A *locally convex space* is a length space in which every point has a neighborhood that is a convex space in the restricted metric.

9.80. Exercise. *Assume $\mathcal{X}$ is a convex space such that the angle of any hinge is defined. Show that $\mathcal{X}$ is* CAT(0)*.*

The following exercise gives an analog of the Hadamard–Cartan theorem for locally convex spaces; see also [**5**].

9.81. Exercise. *Show that a complete, simply connected, locally convex space is a convex space.*

Examples and constructions. Let us list important sources of examples of CAT spaces. This should be beneficial to the reader despite that we do not provide all the proofs and some proofs are deferred to later chapters.

Riemannian manifolds with sectional curvature at most κ are locally CAT(κ). This statement follows from the Rauch comparison, and it is one of the main motivations for CAT(κ) comparison. *Hilbert spaces* are another motivating example of CAT(0) spaces.

The question of when a Riemannian *manifold-with-boundary* is locally CAT(κ) has been completely answered by the first author, David Berg, and Richard Bishop [**1**]. If the sectional curvatures of the interior and the outward sectional curvatures of the boundary do not exceed κ then it is locally CAT(κ) (where an outward sectional curvature is one that corresponds to a tangent 2-plane all of whose normal curvature vectors point outward). In particular, if a Riemannian manifold and its boundary have sectional curvature at most κ, then it is locally CAT(κ).

Subsets of *positive reach* in Riemannian manifolds were studied in this context by the first author, Richard Bishop, and Alexander Lytchak [~~**2**,~~ **111**]. In particular, any compact subset of positive reach in a Riemannian manifold is CAT; as usual, we assume that it is equipped with the induced length metric.

It was shown by Alexander Lytchak and Stephan Stadler [**108**] that any *simply-connected subset* of a contractible two-dimensional CAT(κ) length space (equipped with induced length metric) is CAT(κ). In higher dimensions things are more complicated, even for a Euclidean ambient space; see [**10**, Chapter 4] for related questions (open and solved).

By the Gauss formula smooth saddle surfaces in manifolds with sectional curvature at most κ are locally CAT(κ). The nonsmooth analog of this statement is wide open; this is the so-called Shefel conjecture [**10**, Chapter 4]. However, the following weaker statement was proved by the third author and Stephan Stadler [**126**]: *metric-minimizing surfaces* in CAT(κ) space are locally CAT(κ);

metric minimizing means that it is impossible to decrease its length metric by a small deformation.

Note that any *metric tree* (see Section 4C) is CAT($-\infty$); that is, a metric tree is CAT(κ) for *any* $\kappa \in \mathbb{R}$. In particular, any tree (in the graph-theoretical sense) with a length metric such that every edge is isometric to a line segment is CAT($-\infty$).

Suppose $A^1, \dots, A^n$ is an array of convex closed sets in the Euclidean space $\mathbb{E}^m$. Let us prepare $n+1$ copy of $\mathbb{E}^m$ and glue successive pairs of spaces along $A^1, \dots, A^n$. The resulting space is called *puff pastry*; by the Reshetnyak gluing theorem it is CAT(0). (This observation was used in one of the most beautiful applications of the Reshetnyak gluing theorem given by Dmitri Burago, Serge Ferleger, and Alexey Kononenko [**38**–**41**]. They use it to study billiards; a short survey on the subject was written by Dmitri Burago [**36**]; see also our book [**10**].)

An if-and-only-if condition on *polyhedral spaces* is given in 12.2. It implies the so-called Gromov~~'s~~ flag condition 12.10 which provides in particular a flexible way to construct CAT(0) *cube complexes*. (Several applications are mentioned in Section 12D.)

By 9.7 and 5.16, the *ultralimits*, as well as *Gromov–Hausdorff limits* of CAT(κ) spaces are CAT(κ). In particular, if $\mathcal{U}$ is a CAT(0) space then its *asymptotic cone* defined as the ultralimit of its rescalings $\frac{1}{n} \cdot \mathcal{U}$ as $n \to \omega$ is again CAT(0). Unlike in the case of CBB(0) spaces, the use of ultralimits is necessary even if $\mathcal{U}$ is a manifold due to the lack of compactness theorem. Asymptotic cones of CAT(0) spaces and their generalizations play an important role in geometric group theory.

Conformal deformations of CAT spaces were studied by Alexander Lytchak and Stephan Stadler [**107**]. In particular, if $\mathcal{U}$ is a CAT(0) space and $f\colon \mathcal{U} \to \mathbb{R}$ is continuous, convex and bounded below, then the conformally equivalent space with conformal factor e^f is CAT(0).

Further, CAT spaces behave nicely with respect to some natural constructions. For example, the product of CAT(0) spaces is again CAT(0). Also, the Euclidean cone over a CAT(1) space is CAT(0). These are the first examples of the so-called *warped products* that are discussed in Chapter 11; a general statement is given in 11.11. Also, as it was observed by Karl-Theodor Sturm [**153**, Prop. 3.10], the *space of L^2-maps* from a measure space to a complete CAT(0) length space is CAT(0).

Among more conceptual examples, let us mention a result of Brian Clarke [**54**]: the *space of Riemannian metrics* on a compact, orientable smooth manifold with respect to the L^2-distance is CAT(0); a shorter proof of this statement was given by Nicola Cavallucci [**48**]. By a result of Tamás Darvas [**57**], the *space* *of Kähler potentials* on a compact Kähler manifold is CAT(0). The *Teichmüller space* with the Weil–Petersson metric is CAT(0); the latter was shown by Sumio Yamada [**159**].

Chapter 10

Kirszbraun revisited

This chapter is based on our paper [**8**] and an earlier paper of Urs Lang and Viktor Schroeder [**98**].

A. Short map extension definitions

10.1. Theorem. *A complete length space $\mathcal{L}$ is* CBB(κ) *if and only if for any 3-point set V_3 and any 4-point set $V_4 \supset V_3$ in $\mathcal{L}$, any short map $f\colon V_3 \to \mathbb{M}^2(\kappa)$ can be extended to a short map $F\colon V_4 \to \mathbb{M}^2(\kappa)$ (so $f = F|_{V_3}$).*

The only-if part of Theorem 10.1 can be obtained as a corollary of Kirszbraun's theorem (10.14). We present another, more elementary proof; using the following analog of Alexandrov's lemma (6.3).

We say that two triangles with a common vertex *do not overlap* if their convex hulls intersect only at the common vertex.

10.2. Overlap lemma. *Let $[\tilde{x}^1\tilde{x}^2\tilde{x}^3]$ be a triangle in $\mathbb{M}^2(\kappa)$. Let $\tilde{p}^1$, $\tilde{p}^2$, $\tilde{p}^3$ be points in $\mathbb{M}^2(\kappa)$ such that, for any permutation $\{i,j,k\}$ of $\{1,2,3\}$, we have*

(i) $|\tilde{p}^i - \tilde{x}^k| = |\tilde{p}^j - \tilde{x}^k|$,

(ii) *$\tilde{p}^i$ and $\tilde{x}^i$ lie in the same closed halfspace determined by $[\tilde{x}^j\tilde{x}^k]$,*

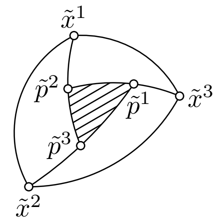
If no pair of triangles $[\tilde{p}^i\tilde{x}^j\tilde{x}^k]$ overlap, then

$$\measuredangle\tilde{p}^1 + \measuredangle\tilde{p}^2 + \measuredangle\tilde{p}^3 > 2\cdot\pi,$$

where $\measuredangle\tilde{p}^i := \measuredangle\left[\tilde{p}^i\,{}^{\tilde{x}^k}_{\tilde{x}^j}\right]$ for a permutation $\{i,j,k\}$ of $\{1,2,3\}$.

Remarks. If $\varkappa \leqslant 0$, then the overlap lemma can be proved without using condition (i). This follows since the sum of external angles for the hexagon $[\tilde p^1\tilde x^2\tilde p^3\tilde x^1\tilde p^2\tilde x^3]$ and its area is $2\cdot\pi-\varkappa\cdot a$, where a denotes the area of the hexagon.

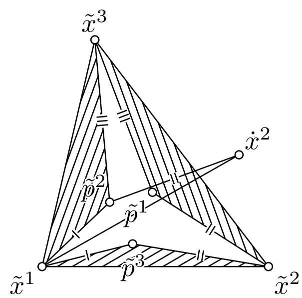
The diagram shows that condition (i) is essential in case $\varkappa>0$.

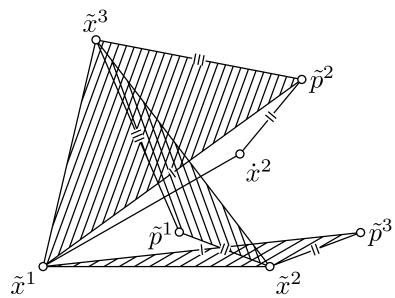


Proof. Rotate the triangle $[\tilde p^3\tilde x^1\tilde x^2]$ around $\tilde x^1$ to make $[\tilde x^1\tilde p^3]$ coincide with $[\tilde x^1\tilde p^2]$. Let $\dot x^2$ denote the image of $\tilde x^2$ after rotation. Note that

$$\measuredangle\left[\tilde x^1\,\begin{smallmatrix}\tilde x^3\\ \dot x^2\end{smallmatrix}\right]=\min\{\measuredangle\left[\tilde x^1\,\begin{smallmatrix}\tilde x^2\\ \tilde p^3\end{smallmatrix}\right]+\measuredangle\left[\tilde x^1\,\begin{smallmatrix}\tilde p^2\\ \tilde x^3\end{smallmatrix}\right],\ 2\cdot\pi-(\measuredangle\left[\tilde x^1\,\begin{smallmatrix}\tilde x^2\\ \tilde p^3\end{smallmatrix}\right]+\measuredangle\left[\tilde x^1\,\begin{smallmatrix}\tilde p^2\\ \tilde x^3\end{smallmatrix}\right])\}.$$

By (ii), the triangles $[\tilde p^3\tilde x^1\tilde x^2]$ and $[\tilde p^2\tilde x^3\tilde x^1]$ do not overlap if and only if





(1) $$\measuredangle\left[\tilde x^1\,\begin{smallmatrix}\tilde x^3\\ \tilde x^2\end{smallmatrix}\right]>\measuredangle\left[\tilde x^1\,\begin{smallmatrix}\tilde x^3\\ \dot x^2\end{smallmatrix}\right];$$

and

(2) $$2\cdot\pi>\measuredangle\left[\tilde x^1\,\begin{smallmatrix}\tilde x^2\\ \tilde p^3\end{smallmatrix}\right]+\measuredangle\left[\tilde x^1\,\begin{smallmatrix}\tilde p^2\\ \tilde x^3\end{smallmatrix}\right]+\measuredangle\left[\tilde x^1\,\begin{smallmatrix}\tilde x^2\\ \tilde x^3\end{smallmatrix}\right]$$

~~The~~ condition 1 holds if and only if $|\tilde x^2-\tilde x^3|>|\dot x^2-\tilde x^3|$, which in turn holds if and only if

(3) $$\begin{aligned}\measuredangle\tilde p^1&>\measuredangle\left[\tilde p^2\,\begin{smallmatrix}\tilde x^3\\ \dot x^2\end{smallmatrix}\right]\\ &=\min\{\measuredangle\tilde p^3+\measuredangle\tilde p^2,2\cdot\pi-(\measuredangle\tilde p^3+\measuredangle\tilde p^2)\}.\end{aligned}$$

The inequality follows since the corresponding hinges have the same pairs of sidelengths. (The two pictures show that both possibilities for the minimum can occur.)

Now assume $\measuredangle\tilde p^1+\measuredangle\tilde p^2+\measuredangle\tilde p^3\leqslant 2\cdot\pi$. Then 3 implies

$$\measuredangle\tilde p^i>\measuredangle\tilde p^j+\measuredangle\tilde p^k.$$

Since no pair of triangles overlap, the same holds for any permutation (i,j,k) of $(1,2,3)$. Therefore

$$\measuredangle\tilde p^1+\measuredangle\tilde p^2+\measuredangle\tilde p^3>2\cdot(\measuredangle\tilde p^1+\measuredangle\tilde p^2+\measuredangle\tilde p^3),$$

a contradiction. □