15.8. Proposition. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space. Then for any two points p, $\bar p \in \mathcal{L}$ and any $R, \bar R > 0$, there is a constant $\delta = \delta(\kappa, R, \bar R, |p - \bar p|) > 0$ such that*

$$\operatorname{pack}_{\delta\cdot\varepsilon} \mathrm{B}(\bar p, \bar R) \geqslant \operatorname{pack}_\varepsilon \mathrm{B}(p, R).$$

Proof. According to 8.33, we can assume that $\kappa \leqslant 0$.

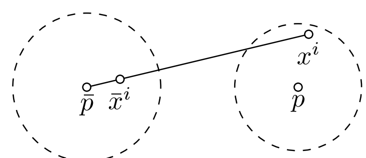
Let $n = \operatorname{pack}_\varepsilon \mathrm{B}(p, R)$ and $\{x^1, \dots, x^n\}$ be a maximal ε-packing in $\mathrm{B}(p, R)$; that is, $|x^i - x^j| > \varepsilon$ for all $i \neq j$. Without loss of generality, we may assume the x^i are in $\operatorname{Str}(\bar p)$. Thus, for each i there is a unique geodesic $[\bar p x^i]$ (see 8.11). Choose a factor $1 > s > 0$ so that $\bar R > s\cdot(|p - \bar p| + R)$. For each i, take $\bar x^i \in [\bar p x^i]$ so that $|\bar p - \bar x^i| = s \cdot (|p - x^i|)$. From 8.17*a*,

$$\tilde\measuredangle^\kappa\left(\bar p\,{}^{\bar x^i}_{\bar x^j}\right) \geqslant \tilde\measuredangle^\kappa\left(\bar p\,{}^{x^i}_{x^j}\right).$$

The cosine law gives a constant $\delta = \delta(\kappa, R, \bar R, |p - \bar p|) > 0$ such that

$$|\bar x^i - \bar x^j| > \delta \cdot (|x^i - x^j|) > \delta \cdot \varepsilon$$

for all $i \neq j$. Hence the statement follows. ☐



15.9. Proposition. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space, $r < \varpi\kappa$ and $p \in \mathcal{L}$. Assume that*

(1) $$\operatorname{pack}_\varepsilon \mathrm{B}(p, r) > \operatorname{pack}_\varepsilon \overline{\mathrm{B}}[r]_{\mathbb{M}^m(\kappa)}$$

for $\varepsilon > 0$. Then there is a G-delta set $A \subset \mathcal{L}$ that is dense in a neighborhood of p and such that $\dim \operatorname{Lin}_q > m$ *for any $q \in A$.*

Proof. Choose a maximal ε-packing in $\mathrm{B}(p, r)$, that is, an array $(x^1, x^2, \dots, x^n)$ of points in $\mathrm{B}(p, r)$ such that $n = \operatorname{pack}_\varepsilon \mathrm{B}(p, r)$ and $|x^i - x^j| > \varepsilon$ for any $i \neq j$. Choose a neighborhood $\Omega \ni p$ such that $|q - x^i| < r$ for any $q \in \Omega$ and all i. Let

$$A = \Omega \cap \operatorname{Str}(x^1, x^2, \dots, x^n).$$

According to Theorem 8.11, A is a G-delta set that is dense in Ω.

Assume $k = \dim \operatorname{Lin}_q \leqslant m$ for $q \in A$. Consider an array $(v^1, v^2, \dots, v^n)$ of vectors in Lin_q, where $v^i = \log[q x^i]$. Clearly

$$|v^i| = |q - x^i| < r,$$

and from the hinge comparison (8.14*c*) we have

$$\tilde\gamma^\kappa\left[0\,{}^{v^i}_{v^j}\right] \geqslant |x^i - x^j| > \varepsilon.$$

Note that the ball $\mathrm{B}(0,r)_{\mathrm{Lin}_q}$ equipped with the metric $\rho(v,w) = \tilde{\gamma}^\kappa\,[0\,{}^{v}_{w}]$ is isometric to $\overline{\mathrm{B}}[r]_{\mathbb{M}^k(\kappa)}$. Thus

$$\operatorname{pack}_\varepsilon \overline{\mathrm{B}}[r]_{\mathbb{M}^k(\kappa)} \geqslant \operatorname{pack}_\varepsilon \mathrm{B}(p,r),$$

which contradicts $k \leqslant m$ and 1. $\square$

The proof of Theorem 15.6 is essentially done in 15.7, 15.8, 15.9, 15.10, 15.5; now we assemble the proof from these parts.

Proof of 15.6. We will prove the implications

$$(C) \Rightarrow (A) \Rightarrow (B) \Rightarrow (E) \Rightarrow (C) \Rightarrow (D) \Rightarrow (E).$$

The implication $(C) \Rightarrow (A)$ is trivial.

$(A) \Rightarrow (B)$. Choose a point $p \in \mathcal{L}$ such that $\dim \mathrm{Lin}_p \geqslant m$. Clearly one can choose an array $(\xi^0, \xi^1, \dots, \xi^m)$ of directions in Lin_p such that $\measuredangle(\xi^i, \xi^j) > \frac{\pi}{2}$ for all $i \neq j$. Choose an array $(x^0, x^1, \dots, x^m)$ of points in $\mathcal{L}$ such that each $\uparrow_{[px^i]}$ is sufficiently close to ξ^i; in particular, we have $\measuredangle\left[p\,{}^{x^i}_{x^j}\right] > \frac{\pi}{2}$. Choose points $a^i \in\,]px^i]$ sufficiently close to p. This can be done so that each $\tilde{\measuredangle}^\kappa\left(p\,{}^{a^i}_{a^j}\right)$ is arbitrarily close to $\measuredangle\left[p\,{}^{a^i}_{a^j}\right]$, in particular $\tilde{\measuredangle}^\kappa\left(p\,{}^{a^i}_{a^j}\right) > \frac{\pi}{2}$. Finally, set $b = a^0$.

$(B) \Rightarrow (E)$. Let $p \in \mathcal{L}$ be a point that admits a κ-strutting array $(b, a^1, \dots, a^m)$ of points in $\mathcal{L}$. The right-inverse theorem (15.5b) implies that the distance map $\mathrm{dist}_{\mathbf{a}} : \mathcal{L} \to \mathbb{R}^m$,

$$\mathrm{dist}_{\mathbf{a}} : x \mapsto (|a^1 - x|, |a^2 - x|, \dots, |a^n - x|),$$

is open in a neighborhood of p. Since the distance map $\mathrm{dist}_{\mathbf{a}}$ is Lipschitz, for any $r > 0$, there is $c > 0$ such that

$$\operatorname{pack}_\varepsilon \mathrm{B}(p,r) > \frac{c}{\varepsilon^m}$$

for any $\varepsilon > 0$. Applying 15.8, we get a similar inequality for any other ball in $\mathcal{L}$; that is, for any $q \in \mathcal{L}$ and $R > 0$, there is $c' > 0$ such that

$$\operatorname{pack}_\varepsilon \mathrm{B}(q,R) > \frac{c'}{\varepsilon^m}.$$

$(E) \Rightarrow (C)$. Note that for any $q' \in \mathcal{L}$ and $R' > |q - q'| + R$ we have

$$\begin{aligned}\operatorname{pack}_\varepsilon \mathrm{B}(q',R') &\geqslant \operatorname{pack}_\varepsilon \mathrm{B}(q,R)\\ &\geqslant \frac{c}{\varepsilon^m}\\ &> \operatorname{pack}_\varepsilon \overline{\mathrm{B}}[R']_{\mathbb{M}^{m-1}(\kappa)}\end{aligned}$$

for all sufficiently small $\varepsilon > 0$. Applying 15.9, Euk^m contains a G-delta set that is dense in a neighborhood of any point $q' \in \mathcal{L}$.

$(C) \Rightarrow (D)$. Since Euk^m contains a dense G-delta set in $\mathcal{L}$, we can choose $p \in \mathrm{B}(q, R)$ with a distance-preserving cone embedding $\iota\colon \mathbb{E}^m \hookrightarrow \mathrm{T}_p$.

Repeating the construction in $(A) \Rightarrow (B)$, we get a $\varkappa$-strutting array $(p, a^1, \dots, a^m)$ for p.

Applying the right-inverse theorem (15.5), we obtain a $C^{\frac{1}{2}}$-submap

$$\Phi\colon \mathbb{R}^m \multimap \mathrm{B}(q, R)$$

that is a right inverse for $\mathrm{dist}_{\mathbf{a}} : \mathcal{L} \to \mathbb{R}^m$ and such that $\Phi(\mathrm{dist}_{\mathbf{a}} p) = p$. In particular, Φ is a $C^{\frac{1}{2}}$-embedding of $\mathrm{Dom}\,\Phi$.

$(D) \Rightarrow (E)$. This proof is valid for general metric spaces; it is based on general relations between topological dimension, Hausdorff measure and $\mathrm{pack}_\varepsilon$.

Let $W \subset \mathrm{B}(q, R)$ be the image of Φ. Since $\mathrm{TopDim}\, W = m$, Szpilrajn's theorem (7.5) implies that

$$\mathrm{HausMes}_m W > 0.$$

Given $\varepsilon > 0$, consider a maximal ε-packing of W, that is, an array $(x^1, x^2, \dots, x^n)$ of points in W such that $n = \mathrm{pack}_\varepsilon W$ and $|x^i - x^j| > \varepsilon$ for all $i \neq j$. Note that W is covered by balls $\mathrm{B}(x^i, 2 \cdot \varepsilon)$.

By the definition of Hausdorff measure,

$$\mathrm{pack}_\varepsilon W \geqslant \frac{c}{\varepsilon^m} \cdot \mathrm{HausMes}_m W$$

for a fixed constant $c > 0$ and all small $\varepsilon > 0$. Hence (E) follows. $\square$

D. Inverse function theorem

15.10. Inverse function theorem. *Let $\mathcal{L}$ be an m-dimensional complete length* $\mathrm{CBB}(\varkappa)$ *space and $p, b, a^1, a^2, \dots, a^m \in \mathcal{L}$.*

Assume that the point array $\mathbf{a} = (b, a^1, \dots, a^m)$ is $\varkappa$-strutting for p. Then there are $R > 0$ and $\varepsilon > 0$ such that:

(a) *For all $i \neq j$ and any $q \in \mathrm{B}(p, R)$ we have*

$$\tilde\measuredangle^\varkappa\left(q\,{}^{a^i}_{a^j}\right) > \tfrac{\pi}{2} + \varepsilon \quad \textit{and} \quad \tilde\measuredangle^\varkappa\left(q\,{}^{b}_{a^i}\right) > \tfrac{\pi}{2}.$$

(b) *The restriction of the distance map*

$$\mathrm{dist}_{\mathbf{a}} : x \mapsto (|a^1 - x|, \dots, |a^m - x|)$$

to the ball $\mathrm{B}(p, R)$ is an open $[\varepsilon, \sqrt{m}]$-bi-Lipschitz embedding $\mathrm{B}(p, R) \hookrightarrow \mathbb{R}^m$.

(c) *The value R depends only on $\varkappa$, $|p - a^i|$, $|a^i - a^j|$, and $|b - a^i|$ for all i and j.*

15.11. Definition. *Suppose $\mathcal{L}$ is an m-dimensional complete length* CBB(κ) *space. If a point array $(b, a^1, a^2, \dots, a^m)$ and the value R satisfy the conditions in Theorem* 15.10*, then the restriction $\mathbf{x} = \mathrm{dist}_{\mathbf{a}}|_{\mathrm{B}(p,R)}$ is called a* distance chart*, the restrictions $x^i = \mathrm{dist}_{a^i}|_{\mathrm{B}(p,R)}$ are called* coordinates*, and the restriction $y = \mathrm{dist}_b|_{\mathrm{B}(p,R)}$ is called the* strut *of the distance chart.*

15.12. Lemma. *Let p be a point in an m-dimensional complete length* CBB(κ) *space $\mathcal{L}$. Assume for the directions $\xi, \zeta^1, \zeta^2, \dots, \zeta^k \in \Sigma_p$ the following conditions hold:*

(a) $\measuredangle(\xi, \zeta^i) > \frac{\pi}{2} - \varepsilon$ *for all i,*

(b) $\measuredangle(\zeta^i, \zeta^j) > \frac{\pi}{2} + \varepsilon$ *for all $i \neq j$. Then $k \leqslant m$.*

Proof. Without loss of generality, we can assume that all $\xi, \zeta^1, \zeta^2, \dots, \zeta^k$ are geodesic directions; let $\xi = \uparrow_{[px]}$ and $\zeta^i = \uparrow_{[pz^i]}$ for all i. Fix a small $r > 0$, and let $\bar{x} \in]px]$ and $\bar{z}^i \in]pz^i]$ be points such that

$$|p - \bar{x}| = |p - \bar{z}^1| = \dots = |p - \bar{z}^k| = r.$$

From the definition of angle, if r is sufficiently small we have

- $\tilde{\measuredangle}^\kappa\left(p\,{}^{\bar{x}}_{\bar{z}^i}\right) > \frac{\pi}{2} - \varepsilon$ for all i, and $\tilde{\measuredangle}^\kappa\left(p\,{}^{\bar{z}^i}_{\bar{z}^j}\right) > \frac{\pi}{2} + \varepsilon$ for all $i \neq j$.

Choose a point $p' \in \mathrm{Str}(\bar{x}, \bar{z}^1, \bar{z}^2, \dots, \bar{z}^k)$ sufficiently close to p that the above conditions still hold for p'; that is,

(1) $\quad \tilde{\measuredangle}^\kappa\left(p'\,{}^{\bar{x}}_{\bar{z}^i}\right) > \frac{\pi}{2} - \varepsilon$ *for all i, and* $\tilde{\measuredangle}^\kappa\left(p'\,{}^{\bar{z}^i}_{\bar{z}^j}\right) > \frac{\pi}{2} + \varepsilon$ *for all $i \neq j$.*

Set $\acute{\xi} = \uparrow_{[p'\bar{x}]}$ and $\acute{\zeta}^i = \uparrow_{[p'\bar{z}^i]}$ for each i. By the hinge comparison (8.14c),

(2) $\quad \measuredangle(\acute{\xi}, \acute{\zeta}^i) > \frac{\pi}{2} - \varepsilon$ *for all i, and* $\measuredangle(\acute{\zeta}^i, \acute{\zeta}^j) > \frac{\pi}{2} + \varepsilon$ *for all $i \neq j$.*

According to Corollary 13.40, all directions $\acute{\xi}, \acute{\zeta}^1, \acute{\zeta}^2, \dots, \acute{\zeta}^k$ lie in an isometric copy of the standard n-sphere in $\Sigma_{p'}$. Clearly $n \leqslant m - 1$. Thus it remains to prove the following claim, which is a partial case of the lemma.

(3) $\quad$ *If $\xi, \zeta^1, \zeta^2, \dots, \zeta^k \in \mathbb{S}^{m-1}$, $|\xi - \zeta^i| > \frac{\pi}{2} - \varepsilon$ for all i, and $|\zeta^i - \zeta^j| > \frac{\pi}{2} + \varepsilon$ for all $i \neq j$, then $k \leqslant m$.*

For each i, let $\bar{\zeta}^i$ be the closest point to ζ^i in $\Xi = \mathbb{S}^{m-1} \setminus \mathrm{B}(\xi, \frac{\pi}{2}) \overset{iso}{=\!=} \mathbb{S}^{m-1}_+$ (if $\zeta \in \Xi$, then $\bar{\zeta}^i = \zeta^i$). By straightforward calculations, we have

$$|\bar{\zeta}^i - \bar{\zeta}^j| \geqslant |\zeta^i - \zeta^j| - \varepsilon > \tfrac{\pi}{2}.$$

Thus it is sufficient to show the ~~following~~ claim:

(4) $\quad \mathrm{pack}_{\frac{\pi}{2}} \mathbb{S}^{m-1}_+ = m.$

Clearly, $\operatorname{pack}_{\frac{\pi}{2}} \mathbb{S}^{m-1}_{+} \geqslant m$.

The opposite inequality is proved by induction on m. The base case $m = 1$ is obvious. Assume $(\bar\zeta^1, \bar\zeta^2, \dots, \bar\zeta^k)$ is an array of points in $\mathbb{S}^{m-1}_{+}$ with $|\bar\zeta^i - \bar\zeta^j| > \frac{\pi}{2}$. Without loss of generality, we can also assume that $\bar\zeta^k \in \partial\mathbb{S}^{m-1}_{+}$. For each $i < k$, let $\check\zeta^i = \uparrow_{[\bar\zeta^k\bar\zeta^i]} \in \Sigma_{\bar\zeta^k}\mathbb{S}^{m-1}_{+} \stackrel{iso}{=\!=} \mathbb{S}^{m-2}_{+}$. By the hinge comparison (8.14*c*), $\measuredangle(\check\zeta^i, \check\zeta^j) > \frac{\pi}{2}$ for all $i < j < k$. Thus from the induction hypothesis we have $k - 1 \leqslant m - 1$. □

Proof of 15.10.

(*a*). Fix $\varepsilon > 0$ such that $\tilde\measuredangle^\kappa\left(p\,{}^{a^i}_{a^j}\right) > \frac{\pi}{2} + \varepsilon$ and $\tilde\measuredangle^\kappa\left(p\,{}^{b}_{a^i}\right) > \frac{\pi}{2} + \varepsilon$ for all $i \neq j$. Choose $R > 0$ sufficiently small that $\tilde\measuredangle^\kappa\left(q\,{}^{a^i}_{a^j}\right) > \frac{\pi}{2} + \varepsilon$ and $\tilde\measuredangle^\kappa\left(q\,{}^{b}_{a^i}\right) > \frac{\pi}{2} + \varepsilon$ for all $i \neq j$ and any $q \in \mathrm{B}(p, R)$. Clearly, (*a*) holds for $\mathrm{B}(p, R)$.

(*b*). Note that the distance map $\operatorname{dist}_{\mathbf{a}}$ is Lipschitz and its restriction $\operatorname{dist}_{\mathbf{a}}|_{\mathrm{B}(p,R)}$ is open; the latter follows from the right-inverse theorem (15.5*b*). Thus to prove (*b*), it is sufficient to show that

$$\max_i \left\{ \big||a^i - x| - |a^i - y|\big| \right\} > \tfrac{\varepsilon}{2} \cdot |x - y| \tag{5}$$

for any $x, y \in \mathrm{B}(p, R)$.

According to Lemma 15.12,

$$\measuredangle\left[x\,{}^{y}_{b}\right] \leqslant \tfrac{\pi}{2} - \varepsilon \quad \text{or} \quad \measuredangle\left[x\,{}^{y}_{a^i}\right] \leqslant \tfrac{\pi}{2} - \varepsilon \quad \text{for some } i.$$

In the latter case, since $|x - y| < 2 \cdot R$ and R is small, the hinge comparison (8.14*c*) implies

$$|a^i - x| - |a^i - y| > \tfrac{\varepsilon}{2} \cdot |x - y| \quad \text{for some } i. \tag{6}$$

If $\measuredangle\left[x\,{}^{y}_{b}\right] \leqslant \frac{\pi}{2} - \varepsilon$, then switching x and y, we get

$$|a^j - y| - |a^j - x| > \tfrac{\varepsilon}{2} \cdot |x - y| \quad \text{for some } j. \tag{7}$$

Then 6 and 7 imply 5.

Finally, part (*c*) follows since the angle $\tilde\measuredangle^\kappa\left(q\,{}^{a^i}_{a^j}\right)$ depends continuously on κ, $|q - a^i|$, $|q - a^j|$ and $|a^i - a^j|$. □

E. Finite-dimensional spaces

The next theorem is a refinement of 15.6 for the finite-dimensional case; it was essentially proved by Yuriy Burago, Grigory Perelman, and Michael Gromov [**44**].

15.13. Theorem. *Suppose $\mathcal{L}$ is a complete length* CBB(κ) *space, m is a non-negative integer, $0 < R \leqslant \varpi\kappa$, and $q \in \mathcal{L}$. Then the following statements are equivalent:*

(*a*) $\operatorname{LinDim}\mathcal{L} = m$.

(*b*) *m is the maximal integer such that there is a point $p \in \mathcal{L}$ that admits a $\varkappa$-strutting array $(b, a^1, \dots, a^m)$.*

(*c*) $\mathrm{T}_p \stackrel{iso}{=\!=} \mathbb{E}^m$ *for any point p in a dense G-delta set of $\mathcal{L}$.*

(*d*) *There is an open bi-Lipschitz embedding*

$$\overline{\mathrm{B}}[1]_{\mathbb{E}^m} \hookrightarrow \mathrm{B}(q, R) \subset \mathcal{L}.$$

(*e*) *For any $\varepsilon > 0$,*

$$\operatorname{pack}_\varepsilon \overline{\mathrm{B}}[R]_{\mathbb{M}^m(\varkappa)} \geqslant \operatorname{pack}_\varepsilon \mathrm{B}(q, R).$$

moreover, there is $c = c(q, R) > 0$ such that

$$\operatorname{pack}_\varepsilon \mathrm{B}(q, R) > \frac{c}{\varepsilon^m}.$$

Using theorems 15.6 and 15.13, one can show that linear dimension is equal to many different types of dimension, such as *small* and *big inductive dimension* and *upper* and *lower box-counting dimension* (also known as *Minkowski dimension*), *homological dimension* and so on.

The next two corollaries follow from (*e*).

15.14. Corollary. *Any finite-dimensional complete length* CBB *space is proper and geodesic.*

15.15. Corollary. *Let $(\mathcal{L}_n)$ be a sequence of length* CBB$(\varkappa)$ *spaces and $\mathcal{L}_n \to \mathcal{L}_\omega$ as $n \to \omega$. Assume* $\operatorname{LinDim} L_n \leqslant m$ *for all n. Then* $\operatorname{LinDim} L_\omega \leqslant m$.

15.16. Corollary. *Let $\mathcal{L}$ be a complete length* CBB$(\varkappa)$ *space. Then for any open $\Omega \subset \mathcal{L}$, we have*

$$\operatorname{LinDim}\mathcal{L} = \operatorname{LinDim}\Omega = \operatorname{TopDim}\Omega = \operatorname{HausDim}\Omega,$$

where TopDim *and* HausDim *denote topological* (7.2) *and Hausdorff dimension* (7.1) *respectively.*

In particular, $\mathcal{L}$ is dimension-homogeneous; that is, all open sets have the same linear dimension.

Proof of 15.16. The equality

$$\operatorname{LinDim}\mathcal{L} = \operatorname{LinDim}\Omega$$

follows from 15.6*A*&(*C*).

If $\operatorname{LinDim}\mathcal{L} = \infty$, then applying 15.6*D* for $\mathrm{B}(q, R) \subset \Omega$, we find that there is a compact subset $K \subset \Omega$ having an arbitrarily large $\operatorname{TopDim} K$. Therefore

$$\operatorname{TopDim}\Omega = \infty.$$

By Szpilrajn's theorem (7.5), $\operatorname{HausDim} K \geqslant \operatorname{TopDim} K$. Thus we also have

$$\operatorname{HausDim} \Omega = \infty.$$

If $\operatorname{LinDim} \mathcal{L} = m < \infty$, then the first inequality in 15.13*e* implies that

$$\operatorname{HausDim} \mathrm{B}(q, R) \leqslant m.$$

According to Corollary 15.14, $\mathcal{L}$ is proper and in particular has countable base. Thus applying Szpilrajn's theorem again, we have

$$\operatorname{TopDim} \Omega \leqslant \operatorname{HausDim} \Omega \leqslant m.$$

Finally, 15.13*d* implies that $m \leqslant \operatorname{TopDim} \Omega$. □

Proof of 15.13. The equivalence $(a)\Leftrightarrow(b)$ follows from 15.6.

$(a) \Rightarrow (c)$. If $\operatorname{LinDim} \mathcal{L} = m$, then by Theorem 15.6, Euk^m contains a dense G-delta set in $\mathcal{L}$. From 15.7, it follows that T_p is isometric to $\mathbb{E}^m$ for any $p \in \mathrm{Euk}^m$.

$(c) \Rightarrow (d)$. This is proved in exactly the same way as implication $(C) \Rightarrow (D)$ of theorem 15.6, but applying the existence of a distance chart (15.10) instead of the right-inverse theorem (15.5).

$(d) \Rightarrow (e)$. From (d), it follows that there is a point $p \in \mathrm{B}(q, R)$ and $r > 0$ such that $\mathrm{B}(p, r) \subset \mathcal{L}$ is bi-Lipschitz homeomorphic to a bounded open set of $\mathbb{E}^m$. Thus there is $c > 0$ such that

$$\operatorname{pack}_\varepsilon \mathrm{B}(p, r) > \frac{c}{\varepsilon^m}. \tag{1}$$

Applying 15.8 shows that inequality 1, with different constants, holds for any other ball, in particular for $\mathrm{B}(q, R)$.

Applying 15.9 gives the first inequality in (e).

$(e) \Rightarrow (a)$. From Theorem 15.6, we have $\operatorname{LinDim} \mathcal{L} \geqslant m$. Applying Theorem 15.6 again, if $\operatorname{LinDim} \mathcal{L} \geqslant m + 1$ then for some $c > 0$ and any $\varepsilon > 0$,

$$\operatorname{pack}_\varepsilon \mathrm{B}(q, R) \geqslant \frac{c}{\varepsilon^{m+1}}.$$

But

$$\frac{c'}{\varepsilon^m} \geqslant \operatorname{pack}_\varepsilon \mathrm{B}(q, R)$$

for any $\varepsilon > 0$, a contradiction. □

15.17. Exercise. *Suppose $\mathcal{L}$ is a complete length* CBB *space and $\Sigma_p \mathcal{L}$ is compact for any $p \in \mathcal{L}$. Prove that $\mathcal{L}$ is finite-dimensional.*

F. One-dimensional spaces

15.18. Theorem. *Let $\mathcal{L}$ be a one-dimensional complete length* $\mathrm{CBB}(\kappa)$ *space. Then $\mathcal{L}$ is isometric to a connected complete Riemannian one-dimensional manifold with possibly non-empty boundary.*

Proof. Clearly $\mathcal{L}$ is connected. It remains to show the following:

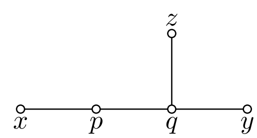
(1) *For any point $p \in \mathcal{L}$ there is $\varepsilon > 0$ such that $\mathrm{B}(p,\varepsilon)$ is isometric to either $[0,\varepsilon)$ or $(-\varepsilon,\varepsilon)$.*

First let us show:

(2) *If $p \in\ ]xy[$ for $x,\, y \in \mathcal{L}$ and $\varepsilon < \min\{|p-x|, |p-y|\}$, then $\mathrm{B}(p,\varepsilon) \subset\ ]xy[$. In particular, $\mathrm{B}(p,\varepsilon) \overset{iso}{=\!=} (-\varepsilon,\varepsilon)$.*

Assume the contrary; that is, there is

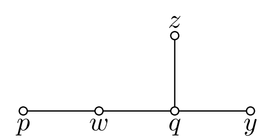
$$z \in \mathrm{B}(p,\varepsilon) \setminus\ ]xy[.$$



Consider a geodesic $[pz]$, and let $q \in [pz] \cap [xy]$ be the point that maximizes the distance $|p-q|$. At q, we have three distinct directions: to x, y, and z. Moreover, $\measuredangle\big[q\,{}^{x}_{y}\big] = \pi$. Thus, according to Proposition 15.7, $\mathrm{LinDim}\,\mathcal{L} > 1$, a contradiction. △

Now we assume no geodesic includes p as a non-endpoint. Since $\mathrm{LinDim}\,\mathcal{L} = 1$ there is a point $y \neq p$.

Fix a positive value $\varepsilon < |p-y|$. Let us show:



(3) $\mathrm{B}(p,\varepsilon) \subset [py]$; *in particular,* $\mathrm{B}(p,\varepsilon) \overset{iso}{=\!=} [0,\varepsilon)$.

Assume the contrary; let $z \in \mathrm{B}(p,\varepsilon) \setminus [py]$.

Choose a point $w \in\ ]py[$ such that

$$|p-w| + |p-z| < \varepsilon.$$

Consider geodesic $[wz]$, and let $q \in [py] \cap [wz]$ be the point that maximizes the distance $|w-q|$. Since no geodesic includes p as a non-endpoint, we have $p \neq q$. As above, $\measuredangle\big[q\,{}^{p}_{y}\big] = \pi$ and $\uparrow_{[qz]}$ is distinct from $\uparrow_{[qp]}$ and $\uparrow_{[qp]}$. Thus, according to Proposition 15.7, $\mathrm{LinDim}\,\mathcal{L} > 1$, a contradiction. △

Clearly $2 + 3 \Rightarrow 1$; hence the result. □

Chapter 16

Gradient flow

Gradient flow could be considered ~~as~~ a nonsmooth version of first-order ordinary differential equations. It provides a universal tool in Alexandrov geometry with most significant applications to CBB spaces.

The theory of gradient flows of semiconvex functions on Hilbert spaces (which are of course both CBB(0) and CAT(0)) is classical, see for example [**33**].

The technique of gradient flows in the context of comparison geometry takes its roots in *Sharafutdinov's retraction*, introduced by Vladimir Sharafutdinov [**147**]. It has been used widely in comparison geometry since then. In CBB spaces, it was first used by Grigory Perelman and the third author [**123**, **129**]. A bit later, independently Jürgen Jost and Uwe Mayer [**88**, **114**] used the gradient flow in CAT spaces. Later, Alexander Lytchak unified and generalized these two approaches to a wide class of metric spaces [**110**]. It was developed yet further by Shin-ichi Ohta [**121**] and by Giuseppe Sevaré [**146**]. It is based on the more analytic approach suitable for the study of synthetic spaces with lower Ricci bounds was developed by Luigi Ambrosio, Nicola Gigli and Giuseppe Sevaré in a general metric and metric measure setting [**19**].

The following exercise is a stripped-down version of Sharfutdinov's retraction; it gives the idea behind gradient flow.

16.1. Exercise. *Assume that a one-parameter family of convex sets $K_t \subset \mathbb{E}^m$ is nested; that is, $K_{t_1} \supset K_{t_2}$ if $t_1 \leqslant t_2$. Show that there is a family of short maps $\phi_t : \mathbb{E}^m \to K_t$ such that $\phi_t|_{K_t} = \mathrm{id}$ for any t and $\phi_{t_2} \circ \phi_{t_1} = \phi_{t_2}$ if $t_1 \leqslant t_2$.*

A. Gradient-like curves

Gradient-like curves will be used later in the construction of gradient curves. The latter are a special reparametrization of gradient-like curves.

16.2. Definition. *Let $\mathcal{Z}$ be a complete length space and $f\colon \mathcal{Z} \multimap \mathbb{R}$ be locally Lipschitz semiconcave subfunction. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT.

*A Lipschitz curve $\hat\alpha\colon [s_{\min}, s_{\max}) \to \operatorname{Dom} f$ will be called an f-*gradient-like curve *if*

$$\hat\alpha^+ = \tfrac{1}{|\nabla_{\hat\alpha} f|}\cdot \nabla_{\hat\alpha} f;$$

that is, for any $s \in [s_{\min}, s_{\max})$, the right derivative $\hat\alpha^+(s)$ is defined and

$$\hat\alpha^+(s) = \tfrac{1}{|\nabla_{\hat\alpha(s)} f|}\cdot \nabla_{\hat\alpha(s)} f.$$

Note that this definition implies that $|\nabla_p f| > 0$ for any point p on $\hat\alpha$.

The following theorem gives a seemingly weaker condition that is equivalent to the definition of gradient-like curve.

16.3. Theorem. *Suppose $\mathcal{Z}$ is a complete length space, $f\colon \mathcal{Z} \multimap \mathbb{R}$ is a locally Lipschitz semiconcave subfunction, and $|\nabla_p f| > 0$ for any $p \in \operatorname{Dom} f$. Assume that $\mathcal{Z}$ is either* CBB *or* CAT.

A curve $\hat\alpha\colon [s_{\min}, s_{\max}) \to \operatorname{Dom} f$ is an f-gradient-like curve if and only if it is 1*-Lipschitz and*

$$\varliminf_{s\to s_0+} \frac{f\circ\hat\alpha(s) - f\circ\hat\alpha(s_0)}{s-s_0} \geqslant |\nabla_{\hat\alpha(s_0)} f| \tag{1}$$

for almost all $s_0 \in [s_{\min}, s_{\max})$.

Proof. The only-if part follows directly from the definition. To prove the if part, note that for any $s_0 \in [s_{\min}, s_{\max})$ we have

$$\begin{aligned}\varliminf_{s\to s_0+} \frac{f\circ\hat\alpha(s) - f\circ\hat\alpha(s_0)}{s-s_0} &\geqslant \varliminf_{s\to s_0+} \frac{1}{s-s_0}\cdot \int\limits_{s_0}^{s} |\nabla_{\hat\alpha(\varsigma)} f|\cdot \mathbf{d}\varsigma \\ &\geqslant |\nabla_{\hat\alpha(s_0)} f|;\end{aligned}$$

the first inequality follows from 1 and the second from lower semicontinuity of the function $x \mapsto |\nabla_x f|$, see 13.31. From 13.22, we have

$$\hat\alpha^+(s_0) = \tfrac{1}{|\nabla_{\hat\alpha(s_0)} f|}\cdot \nabla_{\hat\alpha(s_0)} f.$$

Hence the result. □

Recall that second-order differential inequalities are understood in a barrier sense; see Section 3E.

16.4. Theorem. *Let $\mathcal{Z}$ be a complete length space and $f\colon \mathcal{Z} \multimap \mathbb{R}$ be locally Lipschitz and λ-concave. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Assume $\hat\alpha\colon [0, s_{\max}) \to \operatorname{Dom} f$ is an f-gradient-like curve. Then*

$$(f \circ \hat\alpha)'' \leqslant \lambda$$

everywhere on $[0, s_{\max})$.

Closely related statements were proved independently by Uwe Mayer [**114**, 2.36] and Shin-ichi Ohta [**121**, 5.7].

Before the proof, let us formulate and prove a corollary.

16.5. Corollary. *Let $\mathcal{Z}$ be a complete length space, $f\colon \mathcal{Z} \multimap \mathbb{R}$ be a locally Lipschitz and semiconcave function, and $\hat\alpha\colon [0, s_{\max}) \to \operatorname{Dom} f$ be an f-gradient-like curve. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then the function $s \mapsto |\nabla_{\hat\alpha(s)} f|$ is right-continuous; that is, for any $s_0 \in [0, s_{\max})$ we have*

$$|\nabla_{\hat\alpha(s_0)} f| = \lim_{s \to s_0+} |\nabla_{\hat\alpha(s)} f|.$$

Proof. Applying 16.4 locally, we have that $f \circ \hat\alpha(s)$ is semiconcave. The statement follows since

$$(f \circ \hat\alpha)^+(s) = (\mathbf{d}_p f)\left(\tfrac{1}{|\nabla_{\hat\alpha(s)} f|} \cdot \nabla_{\hat\alpha(s)} f\right) = |\nabla_{\hat\alpha(s)} f|. \qquad \square$$

Proof of 16.4. For any $s > s_0$,

$$\begin{aligned}
(f \circ \hat\alpha)^+(s_0) &= |\nabla_{\hat\alpha(s_0)} f| \\
&\geqslant (d_{\hat\alpha(s_0)} f)(\uparrow_{[\hat\alpha(s_0)\hat\alpha(s)]}) \\
&\geqslant \frac{f \circ \hat\alpha(s) - f \circ \hat\alpha(s_0)}{|\hat\alpha(s) - \hat\alpha(s_0)|} - \tfrac{\lambda}{2} \cdot |\hat\alpha(s) - \hat\alpha(s_0)|.
\end{aligned}$$

Let $\lambda_+ = \max\{0, \lambda\}$. Since $s - s_0 \geqslant |\hat\alpha(s) - \hat\alpha(s_0)|$, for any $s > s_0$ we have

$$(f \circ \hat\alpha)^+(s_0) \geqslant \frac{f \circ \hat\alpha(s) - f \circ \hat\alpha(s_0)}{s - s_0} - \tfrac{\lambda_+}{2} \cdot (s - s_0). \tag{2}$$

Thus $f \circ \hat\alpha$ is λ_+-concave. That finishes the proof for $\lambda \geqslant 0$. For $\lambda < 0$ we get only that $f \circ \hat\alpha$ is 0-concave.

Note that $|\hat\alpha(s) - \hat\alpha(s_0)| = s - s_0 - o(s - s_0)$. Thus

$$(f \circ \hat\alpha)^+(s_0) \geqslant \frac{f \circ \hat\alpha(s) - f \circ \hat\alpha(s_0)}{s - s_0} - \tfrac{\lambda}{2} \cdot (s - s_0) + o(s - s_0). \tag{3}$$

Together, 2 and 3 imply that $f \circ \hat\alpha$ is λ-concave. $\square$

16.6. Proposition. *Let $\mathcal{L}$ be a complete length* $\mathrm{CBB}(\kappa)$ *space, $p, q \in \mathcal{L}$. Assume $\hat\alpha\colon [s_{\min}, s_{\max}) \to \mathcal{L}$ is a dist_p-gradient-like curve such that $\hat\alpha(s) \to z \in \,]pq[$ as $s \to s_{\max}+$. Then $\hat\alpha$ is a unit-speed geodesic that lies in $[pq]$.*

Proof. Clearly,

$$\tfrac{d^+}{dt}|q-\hat\alpha(t)| \geqslant -1. \tag{4}$$

On the other hand,

$$\begin{aligned}\tfrac{d^+}{dt}|p-\hat\alpha(t)| &\geqslant (\mathbf{d}_{\hat\alpha(t)}\operatorname{dist}_p)(\uparrow_{[\hat\alpha(t)q]}) \\ &\geqslant -\cos\tilde\measuredangle^\kappa\left(\hat\alpha(t)\,{}^p_q\right).\end{aligned} \tag{5}$$

Inequalities 4 and 5 imply that the function $t\mapsto\tilde\measuredangle^\kappa\left(q\,{}^{\hat\alpha(t)}_{p}\right)$ is nondecreasing. Hence the result. $\square$

B. Gradient curves

In this section we define gradient curves and tie them tightly to gradient-like curves which were introduced in Section 16A.

16.7. Definition. *Let $\mathcal{Z}$ be a complete length space and $f\colon\mathcal{Z}\multimap\mathbb{R}$ be a locally Lipschitz and semiconcave subfunction. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT*.*

*A locally Lipschitz curve $\alpha\colon[t_{\min},t_{\max})\to\operatorname{Dom}f$ will be called an f-*gradient curve *if*

$$\alpha^+=\nabla_\alpha f;$$

that is, for any $t\in[t_{\min},t_{\max})$, $\alpha^+(t)$ is defined and $\alpha^+(t)=\nabla_{\alpha(t)}f$.

The following exercise describes a global geometric property of a gradient curve without direct reference to its function. It uses the notion of *self-contracting curves* introduced by Aris Daniilidis, Olivier Ley, Stéphane Sabourau [**56**].

16.8. Exercise. *Let $\mathcal{Z}$ be a complete length space, $f\colon\mathcal{Z}\to\mathbb{R}$ a concave locally Lipschitz function, and $\alpha\colon\mathbb{I}\to\mathcal{Z}$ an f-gradient curve. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT*.*

Show that α is self-contracting*; that is,*

$$t_1\leqslant t_2\leqslant t_3 \quad\Longrightarrow\quad |\alpha(t_1)-\alpha(t_3)|_{\mathcal{Z}}\geqslant|\alpha(t_2)-\alpha(t_3)|_{\mathcal{Z}}.$$

The next lemma states that gradient and gradient-like curves are special reparametrizations of each other.

16.9. Lemma. *Let $\mathcal{Z}$ be a complete length space and $f\colon\mathcal{Z}\multimap\mathbb{R}$ be a locally Lipschitz semiconcave subfunction such that $|\nabla_p f|>0$ for any $p\in\operatorname{Dom}f$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT*.*

Assume that $\alpha\colon [0, t_{\max}) \to \operatorname{Dom} f$ *is a locally Lipschitz curve and* $\hat\alpha\colon [0, s_{\max}) \to \operatorname{Dom} f$ *is its reparametrization by arc-length, so* $\alpha = \hat\alpha\circ\varsigma$ *for a homeomorphism* $\varsigma\colon [0, t_{\max}) \to [0, s_{\max})$. *Then*

$$\alpha^+ = \nabla_\alpha f$$
$$\Updownarrow$$
$$\hat\alpha^+ = \frac{1}{|\nabla_{\hat\alpha} f|}\cdot\nabla_{\hat\alpha} f \quad \textit{and} \quad \varsigma^{-1}(s) = \int\limits_0^s \frac{\mathbf{d}\mathfrak{s}}{(f\circ\hat\alpha)'(\mathfrak{s})}.$$

Proof. ($\Downarrow$). According to 3.10,

$$\begin{aligned}\varsigma'(t) &\overset{\text{a.e.}}{=\!=} |\alpha^+(t)| \\ &= |\nabla_{\alpha(t)} f|.\end{aligned} \tag{1}$$

Note that

$$\begin{aligned}(f\circ\alpha)'(t) &\overset{\text{a.e.}}{=\!=} (f\circ\alpha)^+(t) \\ &= |\nabla_{\alpha(t)} f|^2.\end{aligned}$$

Setting $s = \varsigma(t)$, we have

$$\begin{aligned}(f\circ\hat\alpha)'(s) &\overset{\text{a.e.}}{=\!=} \frac{(f\circ\alpha)'(t)}{\varsigma'(t)} \\ &\overset{\text{a.e.}}{=\!=} |\nabla_{\alpha(t)} f| \\ &= |\nabla_{\hat\alpha(s)} f|.\end{aligned}$$

From 16.3, it follows that $\hat\alpha(t)$ is an f-gradient-like curve; that is,

$$\hat\alpha^+ = \frac{1}{|\nabla_{\hat\alpha} f|}\cdot\nabla_{\hat\alpha} f.$$

In particular, $(f\circ\hat\alpha)^+(s) = |\nabla_{\hat\alpha^+(s)} f|$, and by 1,

$$\begin{aligned}\varsigma^{-1}(s) &= \int\limits_0^s \frac{\mathbf{d}\mathfrak{s}}{|\nabla_{\hat\alpha(\mathfrak{s})} f|} \\ &= \int\limits_0^s \frac{\mathbf{d}\mathfrak{s}}{(f\circ\hat\alpha)'(\mathfrak{s})}.\end{aligned}$$

(⇑). Clearly,

$$\begin{aligned}\varsigma(t) &= \int\limits_0^t (f\circ\hat\alpha)^+(\varsigma(\mathfrak{t}))\cdot \mathbf{d}\mathfrak{t}\\ &= \int\limits_0^t |\nabla_{\alpha(\mathfrak{t})} f|\cdot \mathbf{d}\mathfrak{t}.\end{aligned}$$

According to 16.5, the function $s\mapsto|\nabla_{\hat\alpha(s)}f|$ is right-continuous. Therefore so is the function $t\mapsto|\nabla_{\hat\alpha\circ\varsigma(t)}f|=|\nabla_{\alpha(t)}f|$. Hence, for any $t_0\in[0,t_{\max})$ we have

$$\begin{aligned}\varsigma^+(t_0) &= \lim_{t\to t_0+}\frac{1}{t-t_0}\cdot\int\limits_{t_0}^t |\nabla_{\alpha(\mathfrak{t})} f|\cdot \mathbf{d}\mathfrak{t}\\ &= |\nabla_{\alpha(t_0)} f|.\end{aligned}$$

Thus, we have

$$\begin{aligned}\alpha^+(t_0) &= \varsigma^+(t_0)\cdot\hat\alpha^+(\varsigma(t_0))\\ &= \nabla_{\alpha(t_0)} f.\end{aligned}$$

□

16.10. Exercise. *Let $\mathcal{Z}$ be a complete length space, and $f\colon\mathcal{Z}\to\mathbb{R}$ be a semiconcave locally Lipschitz function. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Assume $\alpha\colon\mathbb{I}\to\mathcal{Z}$ is a Lipschitz curve such that*

$$\begin{aligned}\alpha^+(t)&\leqslant|\nabla_{\alpha(t)}f|,\\ (f\circ\alpha)^+(t)&\geqslant|\nabla_{\alpha(t)}f|^2\end{aligned}$$

for almost all t. Show that α is an f-gradient curve.

16.11. Exercise. *Let $\mathcal{Z}$ be a complete length space and $f\colon\mathcal{Z}\to\mathbb{R}$ be a concave locally Lipschitz function. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Show that $\alpha\colon\mathbb{R}\to\mathcal{Z}$ is an f-gradient curve if and only if*

$$|x-\alpha(t_1)|^2_{\mathcal{Z}}-|x-\alpha(t_0)|^2_{\mathcal{Z}}\leqslant 2\cdot(t_1-t_0)\cdot(f\circ\alpha(t_1)-f(x))$$

for any $t_1>t_0$ and $x\in\mathcal{Z}$.

C. Distance estimates

16.12. First distance estimate. *Let $\mathcal{Z}$ be a complete length space, and $f\colon\mathcal{Z}\to\mathbb{R}$ be a locally Lipschitz λ-concave function. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Let $\alpha,\beta\colon[0,t_{\max})\to\mathcal{Z}$ be two f-gradient curves. Then*

$$|\alpha(t)-\beta(t)|\leqslant e^{\lambda\cdot t}\cdot|\alpha(0)-\beta(0)|$$

for any t.

Moreover, the statement holds for a locally Lipschitz λ-concave subfunction $f\colon \mathcal{Z} \multimap \mathbb{R}$ if there is a geodesic $[\alpha(t)\,\beta(t)]$ in $\operatorname{Dom} f$ *for any t.*

Proof. If $\mathcal{Z}$ is not geodesic, then pass to its ultrapower $\mathcal{Z}^\omega$.

Fix a choice of geodesic $[\alpha(t)\,\beta(t)]$ for each t.

Setting $\ell(t) = |\alpha(t) - \beta(t)|$, from the first variation inequality (6.7) and the estimate in 13.25 we get

$$\ell^+(t) \leqslant -\langle \uparrow_{[\alpha(t)\beta(t)]}, \nabla_{\alpha(t)} f\rangle - \langle \uparrow_{[\beta(t)\alpha(t)]}, \nabla_{\beta(t)} f\rangle \leqslant \lambda \cdot \ell(t).$$

Here one has to apply the first variation inequality for distance to the midpoint m of $[\alpha(t)\,\beta(t)]$, and apply the triangle inequality. Hence the result. □

16.13. Second distance estimate. *Let $\mathcal{Z}$ be a complete length space, $\varepsilon > 0$, and f, $g\colon \mathcal{Z} \to \mathbb{R}$ be two λ-concave locally Lipschitz functions such that $|f - g| < \varepsilon$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT*. Assume $\alpha, \beta\colon [0, t_{\max}) \to \mathcal{Z}$ are respectively f- and g-gradient curves. Let $\ell\colon t \mapsto |\alpha(t) - \beta(t)|$. Then*

$$\ell^+ \leqslant \lambda \cdot \ell + \tfrac{2\cdot\varepsilon}{\ell}.$$

In particular, if $\alpha(0) = \beta(0)$ and $t_{\max} < \infty$, then

$$|\alpha(t) - \beta(t)| \leqslant c \cdot \sqrt{\varepsilon \cdot t}$$

for a constant $c = c(t_{\max}, \lambda)$.

Moreover, the same conclusion holds for locally Lipschitz λ-concave subfunctions f, $g\colon \mathcal{Z} \multimap \mathbb{R}$ if for any $t \in [0, t_{\max})$ there is a geodesic $[\alpha(t)\,\beta(t)]$ in $\operatorname{Dom} f \cap \operatorname{Dom} g$.

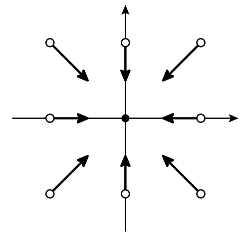
Proof. Set $\ell = \ell(t) = |\alpha(t) - \beta(t)|$. Fix t, and let $p = \alpha(t)$ and $q = \beta(t)$. From the first variation formula and 13.24,

$$\begin{aligned}
\ell^+ &\leqslant -\langle \uparrow_{[pq]}, \nabla_p f\rangle - \langle \uparrow_{[qp]}, \nabla_q g\rangle \\
&\leqslant -\Big(f(q) - f(p) - \lambda \cdot \tfrac{\ell^2}{2}\Big)/\ell - \Big(g(p) - g(q) - \lambda \cdot \tfrac{\ell^2}{2}\Big)/\ell \\
&\leqslant \lambda \cdot \ell + \tfrac{2\cdot\varepsilon}{\ell}.
\end{aligned}$$

By integrating, we get the second statement. □

D. Existence and uniqueness

In general, the "past" of gradient curves can not be determined by the "present". For example, consider the concave function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = -|x|$. The two curves $\alpha(t) = \min\{0, t\}$ with $\beta(t) = 0$ are f-gradient with $\alpha(t) = \beta(t) = 0$ for all $t \geqslant 0$; however $\alpha(t) \neq \beta(t)$ for all $t < 0$. Another example can be given as follows.



16.14. Example. *Let f be as in 13.18; that is, $f\colon (x,y)\mapsto -|x|-|y|$ be the concave function on the (x,y)-plane; its gradient field is sketched on the figure.*

Let α be an f-gradient curve that starts at $p=(x,y)$ for $x>y>0$. Then

$$\alpha(t)=\begin{cases}(x-t,y-t) & \text{for } 0\leqslant t\leqslant x-y,\\ (x-t,0) & \text{for } x-y\leqslant t\leqslant x,\\ (0,0) & \text{for } x\leqslant t.\end{cases}$$

In particular, gradient curves can merge even in the region where $|\nabla f|\neq 0$. Hence their past cannot be uniquely determined from their present.

The next theorem shows that the future gradient curve is determined by its present.

16.15. Picard's theorem. *Let $\mathcal{Z}$ be a complete length space, $f\colon \mathcal{Z} \multimap \mathbb{R}$ be a semiconcave subfunction. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Assume $\alpha,\beta\colon [0,t_{\max})\to$ $\operatorname{Dom} f$ are two f-gradient curves such that $\alpha(0)=\beta(0)$. Then $\alpha(t)=\beta(t)$ for any $t\in[0,t_{\max})$.*

Proof. Follows from the first distance estimate (16.12). □

16.16. Local existence. *Let $\mathcal{Z}$ be a complete length space and $f\colon \mathcal{Z}\multimap\mathbb{R}$ be locally Lipschitz λ-concave subfunction. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then for any $p\in\operatorname{Dom} f$,*

(a) *if $|\nabla_p f|>0$, then for some $\varepsilon>0$, there is an f-gradient-like curve $\hat\alpha\colon[0,\varepsilon)\to\mathcal{Z}$ that starts at p (that is, $\hat\alpha(0)=p$);*

(b) *for some $\delta>0$, there is an f-gradient curve $\alpha\colon[0,\delta)\to\mathcal{Z}$ that starts at p (that is $\alpha(0)=p$).*

This theorem was proved by Grigory Perelman and the third author [**123**]; we present a simplified proof given by Alexander Lytchak [**110**].

Proof. If $|\nabla_p f|=0$, then the constant curve $\alpha(t)=p$ is f-gradient.

Otherwise, choose $\varepsilon>0$ such that $\mathrm{B}(p,\varepsilon)\subset\operatorname{Dom} f$, the restriction $f|_{\mathrm{B}(p,\varepsilon)}$ is Lipschitz, and $|\nabla_x f|>\varepsilon$ for all $x\in\mathrm{B}(p,\varepsilon)$; the latter is possible due to semicontinuity of |gradient| (13.31).

The curves $\hat\alpha$ and α will be constructed in the following three steps. First we construct an f^ω-gradient-like curve $\hat\alpha_\omega\colon[0,\varepsilon)\to\mathcal{Z}^\omega$ as an ω-limit of a certain sequence of broken geodesics in $\mathcal{Z}$. Second, we parametrize $\hat\alpha_\omega$ as in 16.9, to obtain an f^ω-gradient curve α_ω in $\mathcal{Z}^\omega$. Third, applying Picard's theorem (16.15) together with Lemma 4.5, we obtain that α_ω lies in $\mathcal{Z}\subset\mathcal{Z}^\omega$ and therefore one can take $\alpha=\alpha_\omega$ and $\hat\alpha=\hat\alpha_\omega$.

Note that if $\mathcal{Z}$ is proper, then $\mathcal{Z}$ is a metric component of $\mathcal{Z}^\omega$ and $f = f^\omega|_{\mathcal{Z}}$. Thus, in this case, the third step is not necessary.

Step 1. Given $n \in \mathbb{N}$, by an open-closed argument, we can construct a unit-speed curve $\hat{\alpha}_n \colon [0, \varepsilon] \to \mathcal{Z}$ starting at p, with a partition of $[0, \varepsilon)$ into a countable number of half-open intervals $[\varsigma_i, \bar{\varsigma}_i)$ such that for each i we have

(i) $\hat{\alpha}_n([\varsigma_i, \bar{\varsigma}_i])$ is a geodesic and $\bar{\varsigma}_i - \varsigma_i < \frac{1}{n}$,

(ii) $f \circ \hat{\alpha}_n(\bar{\varsigma}_i) - f \circ \hat{\alpha}_n(\varsigma_i) > (\bar{\varsigma}_i - \varsigma_i) \cdot (|\nabla_{\hat{\alpha}_n(\varsigma_i)} f| - \frac{1}{n})$.

Passing to a subsequence of $\hat{\alpha}_n$ such that $f \circ \hat{\alpha}_n$ uniformly converges, let

$$h(s) = \lim_{n \to \infty} f \circ \hat{\alpha}_n(s).$$

Let $\hat{\alpha}_\omega = \lim_{n \to \omega} \hat{\alpha}_n$; it is a curve in $\mathcal{Z}^\omega$ that starts at $p \in \mathcal{Z} \subset \mathcal{Z}^\omega$.

Clearly $\hat{\alpha}_\omega$ is 1-Lipschitz. From (ii) and 13.28, we have

$$(f^\omega \circ \hat{\alpha}_\omega)^+(\varsigma) \geqslant |\nabla_{\hat{\alpha}_\omega(\varsigma)} f^\omega|.$$

According to 16.3, $\hat{\alpha}_\omega \colon [0, \varepsilon) \to \mathcal{Z}^\omega$ is an f^ω-gradient-like curve.

Step 2. Clearly $h(s) = f^\omega \circ \alpha_\omega$. Therefore, according to 16.4, h is λ-concave. Thus we can define a homeomorphism $\varsigma \colon [0, \delta] \to [0, \varepsilon]$ by

$$\varsigma^{-1}(s) = \int_0^s \frac{\mathbf{d}\underset{\smile}{s}}{h'(\underset{\smile}{s})}, \tag{1}$$

According to 16.9, $\alpha(t) = \hat{\alpha} \circ \varsigma(t)$ is an f^ω-gradient curve in $\mathcal{Z}^\omega$.

Step 3. Clearly, $\nabla_p f = \nabla_p f^\omega$ for any $p \in \mathcal{Z} \subset \mathcal{Z}^\omega$; more formally, if $\iota \colon \mathcal{Z} \hookrightarrow \mathcal{Z}^\omega$ is the natural embedding, then $(\mathbf{d}_p \iota)(\nabla_p f) = \nabla_p f^\omega$. Thus it is sufficient to show that α_ω lies in $\mathcal{Z}$. Assume the contrary; then according to 4.5, there is a subsequence $\hat{\alpha}_{n_k}$ such that

$$\hat{\alpha}_\omega \neq \hat{\alpha}'_\omega := \lim_{k \to \omega} \hat{\alpha}_{n_k}.$$

Clearly $h(s) = f^\omega \circ \hat{\alpha}_\omega = f^\omega \circ \hat{\alpha}'_\omega$. Thus for $\varsigma \colon [0, \delta] \to [0, \varepsilon]$ defined by 1, we have that both curves $\hat{\alpha}_\omega \circ \varsigma$ and $\hat{\alpha}'_\omega \circ \varsigma$ are f^ω-gradient. From Picard's theorem (16.15), we have $\hat{\alpha}_\omega \circ \varsigma = \hat{\alpha}'_\omega \circ \varsigma$. Therefore $\hat{\alpha}_\omega = \hat{\alpha}'_\omega$, a contradiction. □

E. Convergence

16.17. Ultralimit of gradient curves. *Assume*

- *$\mathcal{Z}_n$ is a sequence of complete spaces, $\mathcal{Z}_n \to \mathcal{Z}_\omega$ as $n \to \omega$, and $p_n \to p_\omega$ for a sequence of points $p_n \in \mathcal{Z}_n$,*
- *all spaces $\mathcal{Z}_n$ are either* CBB(κ) *or* CAT(κ)*,*

- *$f_n : \mathcal{Z}_n \multimap \mathbb{R}$ are ℓ-Lipschitz and λ-concave, $f_n \to f_\omega$ as $n \to \omega$, and $p_\omega \in \operatorname{Dom} f_\omega$.*

Then:

(*a*) *f_ω is λ-concave.*

(*b*) *If $|\nabla_{p_\omega} f_\omega| > 0$, then there is $\varepsilon > 0$ such that, the f_n-gradient-like curves $\hat\alpha_n : [0,\varepsilon) \to \mathcal{Z}_n$ are defined for ω-almost all n. Moreover, a curve $\hat\alpha_\omega : [0,\varepsilon) \to \mathcal{Z}_\omega$ is a gradient-like curve that starts at p_ω if and only if $\hat\alpha_n(s) \to \hat\alpha_\omega(s)$ as $n \to \omega$ for all $s \in [0,\varepsilon)$.*

(*c*) *For some $\delta > 0$, the f_n-gradient curves $\alpha_n : [0,\delta) \to \mathcal{Z}_n$ are defined for ω-almost all n. Moreover, a curve $\alpha_\omega : [0,\delta) \to \mathcal{Z}_\omega$ is a gradient curve that starts at p_ω if and only if $\alpha_n(t) \to \alpha_\omega(t)$ as $n \to \omega$ for all $t \in [0,\delta)$.*

Note that according to Exercise 4.17, part (*a*) does not hold for general metric spaces. The idea of the proof is the same as in the proof of local existence (16.16).

Proof of 16.17.

(*a*). Fix a geodesic $\gamma_\omega : \mathbb{I} \to \operatorname{Dom} f_\omega$; we need to show that the function

$$t \mapsto f_\omega \circ \gamma_\omega(t) - \tfrac{\lambda}{2} \cdot t^2 \tag{1}$$

is concave.

Since the f_n are ℓ-Lipschitz, so is f_ω. Therefore it is sufficient to prove concavity in the interior of $\mathbb{I}$. In particular, we can assume that γ_ω is sufficiently short and can be extended behind its ends p_ω and q_ω as a minimizing geodesic. If $\mathcal{Z}$ is CBB, then by Theorem 8.11, γ_ω is the unique geodesic connecting p_ω to q_ω. The same holds true if $\mathcal{Z}$ is CAT by the uniqueness of geodesics (9.8).

Construct two sequences of points $p_n, q_n \in \mathcal{Z}_n$ such that $p_n \to p_\omega$ and $q_n \to q_\omega$ as $n \to \omega$. Applying either 8.11 or 9.8, we can assume that for each n there is a geodesic γ_n from p_n to q_n in $\mathcal{Z}_n$.

Since f_n is λ-concave, the function

$$t \mapsto f_n \circ \gamma_n(t) - \tfrac{\lambda}{2} \cdot t^2$$

is concave.

The ω-limit of the sequence γ_n is a geodesic in $\mathcal{Z}_\omega$ from p_ω to q_ω. By uniqueness of such geodesics, we have that $\gamma_n \to \gamma_\omega$ as $n \to \omega$. Passing to the limit, we have 1.

If part of (*b*). Take $\varepsilon > 0$ so small that $\mathrm{B}(p_\omega, \varepsilon) \subset \operatorname{Dom} f_\omega$ and $|\nabla_{x_\omega} f_\omega| > 0$ for any $x_\omega \in \mathrm{B}(p_\omega, \varepsilon)$ (this is possible by 13.31).

Clearly $\hat\alpha_\omega$ is 1-Lipschitz. From 13.28, we get

$$(f_\omega \circ \hat\alpha_\omega)^+(s) \geqslant |\nabla_{\hat\alpha_\omega(s)} f^\omega|.$$

According to 16.3, $\hat{\alpha}_\omega\colon [0,\varepsilon) \to \mathcal{Z}^\omega$ is an f_ω-gradient-like curve.

If part of (c). Assume first that $|\nabla_{p_\omega} f_\omega| > 0$, so we can apply the if part of (b). Let $h_n = f_n \circ \hat{\alpha}_n\colon [0,\varepsilon) \to \mathbb{R}$ and $h_\omega = f_\omega \circ \hat{\alpha}_\omega$. From 16.4, the h_n are λ-concave, and clearly $h_n \to h_\omega$ as $n \to \omega$. Let us define reparametrizations

$$\varsigma_n^{-1}(s) = \int_0^s \frac{\mathbf{d}\varsigma}{h_n'(\varsigma)}, \qquad \varsigma_\omega^{-1}(s) = \int_0^s \frac{\mathbf{d}\varsigma}{h_\omega'(\varsigma)}.$$

The λ-convexity of the h_n implies that $\sigma_n \to \sigma_\omega$ as $n \to \omega$. By 16.9, $\alpha_n = \hat{\alpha}_n \circ \varsigma_n$. Applying the if part of (b) together with Lemma 16.9, we get that $\alpha_\omega = \hat{\alpha}_\omega \circ \varsigma_\omega$ is gradient curve.

The remaining case $|\nabla_{p_\omega} f_\omega| = 0$ can be reduced to the one above using the following trick. Consider the sequence of spaces $\mathcal{Z}_n^\times = \mathcal{Z}_n \times \mathbb{R}$, with the sequence of subfunctions $f_n^\times\colon \mathcal{Z}_n^\times \to \mathbb{R}$ defined by

$$f_n^\times(p,t) = f_n(p) + t.$$

Applying either 11.7b or 11.6b, we have that $\mathcal{Z}_n^\times$ is a CBB(κ_-) space for $\kappa_- = \min\{\kappa, 0\}$, or CAT($\kappa_+$) space for $\kappa_+ = \max\{\kappa, 0\}$. Note that the $f_n^\times$ are λ_+-concave for $\lambda_+ = \max\{\lambda, 0\}$. Now let $\mathcal{Z}_\omega^\times = \mathcal{Z}_\omega \times \mathbb{R}$, and $f_\omega^\times(p,t) = f_\omega(p) + t$.

Clearly $\mathcal{Z}_n^\times \to \mathcal{Z}_\omega^\times$, $f_n^\times \to f_\omega^\times$ as $n \to \omega$, and $|\nabla_x f_\omega^\times| > 0$ for any $x \in \operatorname{Dom} f_\omega^\times$. Thus for the sequence $f_n^\times\colon \mathcal{Z}_n^\times \multimap \mathbb{R}$, we can apply the if part of (b). It remains to note that the curve $\alpha_\omega^\times(t) = (\alpha_\omega(t), t)$ is an $f_\omega^\times$-gradient curve in $\mathcal{Z}_\omega^\times$ if and only if $\alpha_\omega(t)$ is an f_ω-gradient curve.

Only-if part of (c) *and* (b). The only-if part of (c) follows from the if part of (c) and Picard's theorem (16.15). Applying Lemma 16.9, we get the only-if part of (b). □

From local existence (16.16) and the distance estimates (16.12), we obtain the following.

16.18. Global existence. *Let $f\colon \mathcal{Z} \multimap \mathbb{R}$ be a locally Lipschitz and λ-concave subfunction on a complete length space $\mathcal{Z}$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then for any $p \in \operatorname{Dom} f$, there is $t_{\max} \in (0,\infty]$ such that there is an f-gradient curve $\alpha\colon [0,t_{\max}) \to \mathcal{Z}$ with $\alpha(0) = p$. Moreover, for any sequence $t_n \to t_{\max}-$, the sequence $\alpha(t_n)$ does not have a limit point in* Dom f.

The following theorem guarantees the existence of gradient curves for all times for the special type of semiconcave functions that play important role in the theory. It follows from 16.18, 16.4 and 16.9.

16.19. Theorem. *Let $\mathcal{Z}$ be a complete length space and $f\colon \mathcal{Z} \to \mathbb{R}$ satisfies*

$$f'' + \kappa \cdot f \leqslant \lambda$$

for real constants κ and λ. Suppose that $\mathcal{Z}$ is either CBB *or* CAT. *Then f has* complete gradient*; that is, for any $x \in \mathcal{Z}$ there is a f-gradient curve $\alpha\colon [0,\infty) \to \mathcal{Z}$ that starts at x.*

F. Gradient flow

In this section we define gradient flow for semiconcave subfunctions and reformulate theorems obtained earlier in this chapter using this new terminology.

Let $\mathcal{Z}$ be a complete length space and $f\colon \mathcal{Z} \multimap \mathbb{R}$ be a locally Lipschitz semiconcave subfunction. Suppose that $\mathcal{Z}$ is either CBB or CAT. For any $t \geqslant 0$, we write $\mathrm{Flow}_f^t(x) = y$ if there is an f-gradient curve α such that $\alpha(0) = x$ and $\alpha(t) = y$. The partially defined map Flow_f^t from $\mathcal{Z}$ to itself is called the *f-gradient flow* for time t.

From 16.13, it follows that for any $t \geqslant 0$, the domain of definition of Flow_f^t is an open subset of $\mathcal{Z}$; that is, Flow_f^t is a submap. Moreover, if f is defined on all of $\mathcal{Z}$ and $f'' + \mathrm{K}\cdot f \leqslant \lambda$ for constants $\mathrm{K}, \lambda \in \mathbb{R}$, then according to 16.19, $\mathrm{Flow}_f^t(x)$ is defined for all pairs $(x,t) \in \mathcal{Z} \times \mathbb{R}_{\geqslant 0}$.

Clearly $\mathrm{Flow}_f^{t_1+t_2} = \mathrm{Flow}_f^{t_1} \circ \mathrm{Flow}_f^{t_2}$; in other words, gradient flow is given by an action of the semigroup $(\mathbb{R}_{\geqslant 0}, +)$.

From the first distance estimate (16.12), we have the following:

16.20. Proposition. *Let $\mathcal{Z}$ be a complete length* CBB *or* CAT *space and $f\colon \mathcal{Z} \to \mathbb{R}$ be a semiconcave function. Then the map $x \mapsto \mathrm{Flow}_f^t(x)$ is locally Lipschitz.*

Moreover, if f is λ-concave, then Flow_f^t is $e^{\lambda\cdot t}$-Lipschitz.

The next proposition states that gradient flow is stable under Gromov–Hausdorff convergence. The proposition follows directly from the proposition on ultralimit of gradient curves 16.17.

16.21. Proposition. *Supose $\mathcal{Z}_\infty, \mathcal{Z}_1, \mathcal{Z}_2, \dots$ are complete length* $\mathrm{CBB}(\kappa)$ *space, $\mathcal{Z}_n \xrightarrow{\tau} \mathcal{Z}_\infty$, and $f_n\colon \mathcal{Z}_n \to \mathbb{R}$ is a sequence of λ-concave functions that converges to $f_\infty\colon \mathcal{Z}_\infty \to \mathbb{R}$. Then $\mathrm{Flow}_{f_n}^t\colon \mathcal{Z}_n \to \mathcal{Z}_n$ converges to $\mathrm{Flow}_{f_\infty}^t\colon \mathcal{Z}_\infty \to \mathcal{Z}_\infty$.*

G. Line splitting theorem

Let $\mathcal{X}$ be a metric space and $A, B \subset \mathcal{X}$. We say that $\mathcal{X}$ is a *direct sum* of A and B, briefly
$$\mathcal{X} = A \oplus B,$$
if there are projections $\mathrm{proj}_A\colon \mathcal{X} \to A$ and $\mathrm{proj}_B\colon \mathcal{X} \to B$ such that
$$|x-y|^2 = |\mathrm{proj}_A(x) - \mathrm{proj}_A(y)|^2 + |\mathrm{proj}_B(x) - \mathrm{proj}_B(y)|^2$$
for any two points $x, y \in \mathcal{X}$.

Note that if

$$\mathcal{X} = A \oplus B,$$

then

- A intersects B at a single point,
- both sets A and B are convex sets in $\mathcal{X}$.

Recall that a line in a metric space is a both-sided infinite geodesic; thus it minimizes the length on each segment.

16.22. Line splitting theorem. *Let $\mathcal{L}$ be a complete length* CBB(0) *space and γ be a line in $\mathcal{L}$. Then*

$$\mathcal{L} = \mathcal{L}' \oplus \gamma(\mathbb{R})$$

for a subset $\mathcal{L}' \subset \mathcal{L}$.

For smooth 2-dimensional surfaces, this theorem was proved by Stefan Cohn-Vossen [**55**]. For Riemannian manifolds of higher dimensions it was proved by Victor Toponogov [**154**]. Then it was generalized by Anatoliy Milka [**117**] to Alexandrov spaces; nearly the same proof is used in [**37**, 1.5].

Further generalizations of the splitting theorem for Riemannian manifolds with nonnegative Ricci curvature were obtained by Jeff Cheeger and Detlef Gromoll [**53**]. This was further generalized by Jeff Cheeger and Toby Colding for limits of Riemannian manifolds with almost nonnegative Ricci curvature [**51**] and to their synthetic generalizations, so-called *RCD spaces*, by Nicola Gigli [**68**, **69**]. Jost-Hinrich Eschenburg obtained an analogous result for Lorentzian manifolds [**64**], that is, pseudo-Riemannian manifolds of signature $(1, n)$.

We present a proof that uses gradient flow for Busemann functions. It is close in spirit to the proof given in [**53**].

Before going into the proof, let us state a few corollaries of the theorem.

16.23. Corollary. *Let $\mathcal{L}$ be a complete length* CBB(0) *space. Then there is an isometric splitting*

$$\mathcal{L} = \mathcal{L}' \oplus H,$$

where $H \subset \mathcal{L}$ is a subset isometric to a Hilbert space, and $\mathcal{L}' \subset \mathcal{L}$ is a convex subset that contains no line.

16.24. Corollary. *Let $\mathcal{K}$ be a finite-dimensional complete length* CBB(0) *cone and, $v_+, v_- \in \mathcal{K}$ be a pair of opposite vectors (that is, $v_+ + v_- = 0$, see Definiton* 13.36*). Then there is an isometry $\iota\colon K \to K' \times \mathbb{R}$ such that $\iota : v_\pm \mapsto (0', \pm|v_\pm|)$, where K' is a complete length* CBB(0) *space having a cone structure with tip* $0'$.

16.25. Corollary. *Let $\mathcal{L}$ be an m-dimensional complete length* CBB(1) *space, $2 \leqslant m < \infty$, and* $\operatorname{rad} \mathcal{L} = \pi$. *Then*

$$\mathcal{L} \overset{iso}{=\!=} \mathbb{S}^m.$$

The following lemma is closely relevant to the first distance estimate (16.12); its proof goes along the same lines.

16.26. Lemma. *Let $\mathcal{L}$ be a complete length* CBB(0) *space. Suppose $f : \mathcal{L} \to \mathbb{R}$ be a concave 1-Lipschitz function. Consider two f-gradient curves α and β. Then for any $t, s \geqslant 0$ we have*

$$|\alpha(s) - \beta(t)|^2 \leqslant |p - q|^2 + 2 \cdot (f(p) - f(q)) \cdot (s - t) + (s - t)^2,$$

where $p = \alpha(0)$ and $q = \beta(0)$.

Proof. If $\mathcal{L}$ is not geodesic, then pass to its ultrapower $\mathcal{L}^\omega$.

Since f is 1-Lipschitz, $|\nabla f| \leqslant 1$. Therefore

$$f \circ \beta(t) \leqslant f(q) + t$$

for any $t \geqslant 0$.

Set $\ell(t) = |p - \beta(t)|$. Applying 13.24*a* and the first variation inequality (6.7), we get

$$\begin{aligned}\ell^2(t)^+ &\leqslant 2 \cdot (f \circ \beta(t) - f(p)) \\ &\leqslant 2 \cdot (f(q) + t - f(p)).\end{aligned}$$

Therefore

$$\ell^2(t) - \ell^2(0) \leqslant 2 \cdot (f(q) - f(p)) \cdot t + t^2.$$

It proves the needed inequality in case $s = 0$. Combining it with the first distance estimate (16.12), we get the result in case $s \leqslant t$. The case $s \geqslant t$ follows by switching the roles of s and t. □

Proof of 16.22. Consider two Busemann functions, bus_+ and bus_-, asociated with half-lines $\gamma : [0, \infty) \to \mathcal{L}$ and $\gamma : (-\infty, 0] \to \mathcal{L}$ respectively; that is,

$$\operatorname{bus}_\pm(x) = \lim_{t \to \infty} |\gamma(\pm t) - x| - t.$$

According to Exercise 8.25, both functions $\operatorname{bus}_\pm$ are concave.

Fix $x \in \mathcal{L}$. Note that since γ is a line, we have

$$\operatorname{bus}_+(x) + \operatorname{bus}_-(x) \geqslant 0.$$

On the other hand, by 8.23*b*, $f(t) = \operatorname{dist}_x^2 \circ \gamma(t)$ is 2-concave. In particular, $f(t) \leqslant t^2 + at + b$ for some constants $a, b \in \mathbb{R}$. Passing to the limit as $t \to \pm\infty$, we have

$$\operatorname{bus}_+(x) + \operatorname{bus}_-(x) \leqslant 0.$$

Hence

$$\operatorname{bus}_+(x) + \operatorname{bus}_-(x) = 0$$

for any $x \in \mathcal{L}$. In particular, the functions $\operatorname{bus}_\pm$ are *affine*; that is, they are convex and concave at the same time.

It follows that for any x,

$$\begin{aligned}|\nabla_x \operatorname{bus}_\pm| &= \sup\{\, \mathbf{d}_x \operatorname{bus}_\pm(\xi) : \xi \in \Sigma_x \,\} \\ &= \sup\{\, -\mathbf{d}_x \operatorname{bus}_\mp(\xi) : \xi \in \Sigma_x \,\} \cancel{\equiv} \\ &\equiv 1.\end{aligned}$$

By Exercise 16.10, a 1-Lipschitz curve α such that $\operatorname{bus}_\pm(\alpha(t)) = t + c$ is a $\operatorname{bus}_\pm$-gradient curve. In particular, $\alpha(t)$ is a bus_+-gradient curve if and only if $\alpha(-t)$ is a bus_--gradient curve. It follows that for any $t > 0$, the $\operatorname{bus}_\pm$-gradient flows commute; that is,

$$\operatorname{Flow}^t_{\operatorname{bus}_+} \circ \operatorname{Flow}^t_{\operatorname{bus}_-} = \operatorname{id}_{\mathcal{L}}.$$

Setting

$$\operatorname{Flow}^t = \begin{cases} \operatorname{Flow}^t_{\operatorname{bus}_+} & \text{if } t \geqslant 0 \\ \operatorname{Flow}^t_{\operatorname{bus}_-} & \text{if } t \leqslant 0 \end{cases}$$

defines an $\mathbb{R}$-action on $\mathcal{L}$.

Consider the level set $\mathcal{L}' = \operatorname{bus}_+^{-1}(0) = \operatorname{bus}_-^{-1}(0)$; it is a closed convex subset of $\mathcal{L}$, and therefore forms an Alexandrov space. Consider the map $h\colon \mathcal{L}'\times\mathbb{R} \to \mathcal{L}$ defined by $h\colon (x,t) \mapsto \operatorname{Flow}^t(x)$. Note that h is onto. Applying Lemma 16.26 for $\operatorname{Flow}^t_{\operatorname{bus}_+}$ and $\operatorname{Flow}^t_{\operatorname{bus}_-}$ shows that h is short and non-contracting at the same time; that is, h is an isometry. □

H. Radial curves

The radial curves are specially reparametrized gradient curves for distance functions. This parametrization makes them behave like unit-speed geodesics in a natural comparison sense (16I).

16.27. Definition. *Assume $\mathcal{L}$ is a complete length* CBB *space, $\kappa \in \mathbb{R}$, and $p \in \mathcal{L}$. A curve*

$$\sigma\colon [s_{\min}, s_{\max}) \to \mathcal{L}$$

*is called a (p,κ)-*radial curve *if*

$$s_{\min} = |p - \sigma(s_{\min})| \in (0, \tfrac{\varpi\kappa}{2})$$

and σ satisfies the differential equation

$$\sigma^+(s) = \frac{\operatorname{tg}^\kappa |p - \sigma(s)|}{\operatorname{tg}^\kappa s} \cdot \nabla_{\sigma(s)} \operatorname{dist}_p \tag{1}$$

for any $s \in [s_{\min}, s_{\max})$, where $\operatorname{tg}^\kappa x := \frac{\operatorname{sn}^\kappa x}{\operatorname{cs}^\kappa x}$.

If $x = \sigma(s_{\min})$, we say that σ starts *at x.*

Note that according to the definition, $s_{\max} \leqslant \frac{\varpi\kappa}{2}$.

In the the next section, we will see that (p,κ)-radial curves work best for CBB(κ) spaces.

16.28. Definition. *Let $\mathcal{L}$ be a complete length* CBB *space and $p \in \mathcal{L}$. A unit-speed geodesic $\gamma\colon \mathbb{I} \to \mathcal{L}$ is called a p*-radial geodesic *if $|p - \gamma(s)| \equiv s$.*

The proofs of the following two propositions follow directly from the definitions.

16.29. Proposition. *Let $\mathcal{L}$ be a complete length* CBB *space and $p \in \mathcal{L}$. Assume $\frac{\varpi\kappa}{2} \geqslant s_{\max}$. Then any p-radial geodesic $\gamma\colon [s_{\min}, s_{\max}) \to \mathcal{L}$ is a (p,κ)-radial curve.*

16.30. Proposition. *Suppose $\mathcal{L}$ is a complete length* CBB *space, $p \in \mathcal{L}$. Then for any (p,κ)-radial curve $\sigma\colon [s_{\min}, s_{\max}) \to \mathcal{L}$ and $s \in [s_{\min}, s_{\max})$, we have $|p - \sigma(s)| \leqslant s$, and therefore, σ is 1-Lipschitz.*

Moreover, if for some s_0 we have $|p - \sigma(s_0)| = s_0$, then the restriction $\sigma|_{[s_{\min}, s_0]}$ is a p-radial geodesic.

16.31. Existence and uniqueness. *Let $\mathcal{L}$ be a complete length* CBB *space, $\kappa \in \mathbb{R}$, $p \in \mathcal{L}$, and $x \in \mathcal{L}$. Assume $0 < |p - x| < \frac{\varpi\kappa}{2}$. Then there is a unique (p,κ)-radial curve $\sigma\colon [|p - x|, \frac{\varpi\kappa}{2}) \to \mathcal{L}$ that starts at x; that is, $\sigma(|p - x|) = x$.*

Proof.

Existence. Let us define *integral tangent*

$$\operatorname{itg}^\kappa\colon [0, \tfrac{\varpi\kappa}{2}) \to \mathbb{R}, \qquad \operatorname{itg}^\kappa(t) = \int\limits_0^t \operatorname{tg}^\kappa \mathfrak{t} \cdot \mathbf{d}\mathfrak{t}.$$

Clearly $\operatorname{itg}^\kappa$ is smooth and increasing. From 3.18 it follows that the composition

$$f = \operatorname{itg}^\kappa \circ \operatorname{dist}_p$$

is semiconcave in $\mathrm{B}(p, \frac{\varpi\kappa}{2})$.

According to 16.16, there is an f-gradient curve $\alpha\colon [0, t_{\max}) \to \mathcal{L}$ defined on the maximal interval such that $\alpha(0) = x$.

Now consider the solution $\tau(t)$ for the initial value problem $\tau' = \operatorname{tg}^\kappa \tau$, $\tau(0) = r$. Note that $\tau(t)$ is also a gradient curve for the function $\operatorname{itg}^\kappa$ defined on $[0, \frac{\varpi\kappa}{2})$. Direct calculations show that the composition $\alpha \circ \tau^{-1}$ is a (p,κ)-radial curve.

Uniqueness. Assume σ^1, σ^2 are two (p,κ)-radial curves that start at x. Then the compositions $\sigma^i \circ \tau$ both give f-gradient curves. By Picard's theorem (16.15), we have $\sigma^1 \circ \tau \equiv \sigma^2 \circ \tau$. Therefore $\sigma^1(s) = \sigma^2(s)$ for any $s \geqslant r$ such that both sides are defined. ☐

I. Radial comparisons

In this section we show that radial curves behave in a comparison sense like unit-speed geodesics. Recall that notation $\tilde{\measuredangle}^\kappa\{a;b,c\}$ is introduced in 1A.

16.32. Radial monotonicity. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space and p, q be distinct points in $\mathcal{L}$. Let $\sigma\colon [s_{\min}, \frac{\varpi\kappa}{2}) \to \mathcal{L}$ be a (p,κ)-radial curve. Then the function*

$$\psi\colon\ s \mapsto \tilde{\measuredangle}^\kappa\{|q-\sigma(s)|;|p-q|,s\}$$

is nonincreasing in its domain of definition. Moreover, if $\psi(s)$ is undefined, then $|q-\sigma(s)| < s - |p-q|$.

If one extends the definition of $\tilde{\measuredangle}^\kappa\{a;b,c\}$ by stating $\tilde{\measuredangle}^\kappa\{a;b,c\} = 0$ if $a < |b-c|$ and $\tilde{\measuredangle}^\kappa\{a;b,c\} = \pi$ if $a > b+c$. Then the radial monotonicity implies that ψ is a nonincreasing function defined in $[s_{\min}, \frac{\varpi\kappa}{2})$.

Radial monotonicity implies the following by straightforward calculations.

16.33. Corollary. *Let $\kappa \leqslant 0$, $\mathcal{L}$ be a complete* CBB(κ) *space, and $p, q \in \mathcal{L}$. Let $\sigma\colon [s_{\min}, \infty) \to \mathcal{L}$ be a (p,κ)-radial curve. Then for any $w \geqslant 1$, the function*

$$s \mapsto \tilde{\measuredangle}^\kappa\{|q-\sigma(s)|;|p-q|, w\cdot s\}$$

is nonincreasing in its domain of definition.

16.34. Radial comparison. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space and $p \in \mathcal{L}$. Let $\rho\colon [r_{\min}, \frac{\varpi\kappa}{2}) \to \mathcal{L}$ and $\sigma\colon [s_{\min}, \frac{\varpi\kappa}{2}) \to \mathcal{L}$ be two (p,κ)-radial curves. Let*

$$\phi_{\min} = \tilde{\measuredangle}^\kappa\left(p\,{}^{\rho(r_{\min})}_{\sigma(s_{\min})}\right).$$

Then for any $r \in [r_{\min}, \frac{\varpi\kappa}{2})$ and $s \in [s_{\min}, \frac{\varpi\kappa}{2})$, we have

$$\tilde{\measuredangle}^\kappa\{|\rho(r)-\sigma(s)|;r,s\} \leqslant \phi_{\min}, \tag{1}$$

if the left-hand side is defined. Moreover,

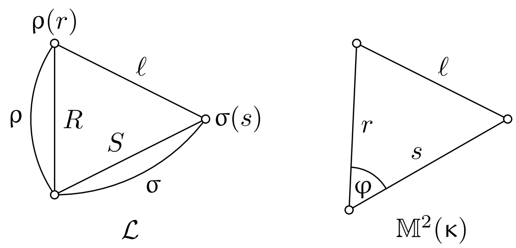
$$|\rho(r)-\sigma(s)| \leqslant \tilde{\gamma}^\kappa\{\phi_{\min};r,s\}. \tag{2}$$

for all r, s.

We prove Theorems 16.32 and 16.34 simultaneously. The proof is an application of 13.24 plus trigonometric manipulations. We give a proof first in the simplest case $\kappa = 0$, and then in the harder case $\kappa \neq 0$. The arguments for both cases are nearly the same, but the case $\kappa \neq 0$ requires an extra twist.

Proof of 16.32 and 16.34 in case $\varkappa = 0$. Set

$$
\begin{aligned}
R &= R(r) = |p - \rho(r)|,\\
S &= S(s) = |p - \sigma(s)|,\\
\ell &= \ell(r,s) = |\rho(r) - \sigma(s)|,\\
\phi &= \phi(r,s) = \tilde{\measuredangle}^0\{\ell(r,s); r, s\}.
\end{aligned}
$$



Therefore it will be sufficient to prove the following inequalities:

$(*)^0_\phi$ $$\tfrac{\partial^+}{\partial r}\phi(s_{\min}, r) \leqslant 0, \qquad \tfrac{\partial^+}{\partial s}\phi(s, r_{\min}) \leqslant 0$$

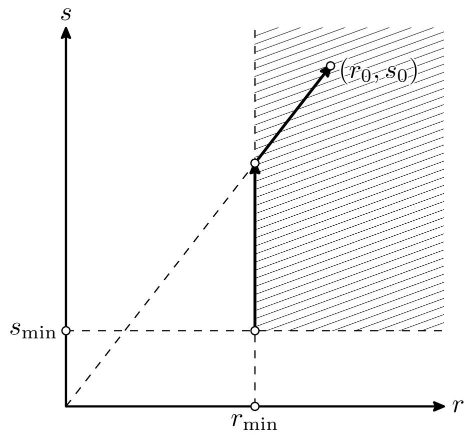
$(**)^0_\phi$ $$s \cdot \tfrac{\partial^+}{\partial s}\phi + r \cdot \tfrac{\partial^+}{\partial r}\phi \leqslant 0.$$

Indeed, from $(*)^0_\phi$, we get that ψ is locally nonincreasing. If ψ is defined in $[s_{\min}, \frac{\varpi\kappa}{2})$, then it implies the radial monotonicity. Otherwise, note that ψ is continuous, and it is defined on a closed subset of $[s_{\min}, \frac{\varpi\kappa}{2})$. Suppose $[s_{\min}, s_{\max}]$ is a maximal interval where ψ is defined. Note that $\phi(s_{\max}) = 0$, therefore $|q - \sigma(s_{\max})| = |s_{\max} - |p-q||$. Note that $|q - \sigma(s_n)| < |s_n - |p-q||$ for some sequence $s_n \to s_{\max}+$. The triangle inequality and 16.30 imply that $|q - \sigma(s_n)| \geqslant -s_n + |p-q|$. Therefore $|q - \sigma(s_n)| < s_n - |p-q|$. By the triangle inequality $|q - \sigma(s)| < s - |p-q|$ for any $s > s_n$. Hence $\phi(s)$ is undefined for all $s > s_{\max}$, and the radial monotonicity follows.

Similarly, 1 follows from $(*)^0_\phi$ and $(**)^0_\phi$. Indeed, one can connect $(s_{\min}, r_{\min})$ and (s_0, r_0) in $[s_{\min}, \infty) \times [r_{\min}, \infty)$ by a concatenation of a coordinate segment and a segment defined by $r/s = r_0/s_0$ as in the figure. By $(*)^0_\phi$ and $(**)^0_\phi$, we have that ϕ does not increase while the pair (r,s) moves along this concatenation with nondecreasing r and s. Thus

$$\phi(r_0, s_0) \leqslant \phi(r_{\min}, s_{\min}) = \phi_{\min}.$$

Finally, if $\phi(r_0, s_0)$ is defined, then 1 implies 2; otherwise, $\ell(r_0, s_0) < |r_0 - s_0|$, and 2 trivially holds.



Let us rewrite the inequalities $(*)^0_\phi$ and $(**)^0_\phi$ in an equivalent form:

$$(*)^0_\ell \qquad \begin{aligned} \tfrac{\partial^+}{\partial s}\ell(s, r_{\min}) &\leqslant \cos\tilde\measuredangle^0\{r_{\min}; s, \ell\}, \\ \tfrac{\partial^+}{\partial r}\ell(s_{\min}, r) &\leqslant \cos\tilde\measuredangle^0\{s_{\min}; r, \ell\}, \end{aligned}$$

$$(**)^0_\ell \qquad s\cdot\tfrac{\partial^+}{\partial s}\ell + r\cdot\tfrac{\partial^+}{\partial r}\ell \leqslant s\cdot\cos\tilde\measuredangle^0\{r; s, \ell\} + r\cdot\cos\tilde\measuredangle^0\{s; r, \ell\} = \ell.$$

Let

$$(A)^0 \qquad f = \tfrac{1}{2}\cdot\operatorname{dist}^2_p.$$

Clearly f is 1-concave, and

$$(B)^0 \qquad \rho^+(r) = \tfrac{1}{r}\cdot\nabla_{\rho(r)}f \quad\text{and}\quad \sigma^+(s) = \tfrac{1}{s}\cdot\nabla_{\sigma(s)}f.$$

Thus from 13.24, we have

$$(C)^0 \qquad \tfrac{\partial^+}{\partial r}\ell = -\tfrac{1}{r}\cdot\langle\nabla_{\rho(r)}f, \uparrow_{[\rho(r)\sigma(s)]}\rangle \leqslant \frac{\ell^2 + R^2 - S^2}{2\cdot\ell\cdot r}.$$

Since $R(r) \leqslant r$ and $S(s_{\min}) = s_{\min}$, we have

$$(D)^0 \qquad \begin{aligned} \tfrac{\partial^+}{\partial r}\ell(r, s_{\min}) &\leqslant \frac{\ell^2 + r^2 - s^2_{\min}}{2\cdot\ell\cdot r} \\ &= \cos\tilde\measuredangle^0\{s_{\min}; r, \ell\}, \end{aligned}$$

which is the first inequality in $(*)^0_\ell$. By switching ρ and σ we obtain the second inequality in $(*)^0_\ell$. Further, adding $(C)^0$ and its mirror-inequality for $\tfrac{\partial^+}{\partial s}\ell$, we have

$$(E)^0 \qquad r\cdot\tfrac{\partial^+}{\partial r}\ell + s\cdot\tfrac{\partial^+}{\partial s}\ell \leqslant \frac{\ell^2 + R^2 - S^2}{2\cdot\ell} + \frac{\ell^2 + S^2 - R^2}{2\cdot\ell} = \ell,$$

which is $(**)^0_\ell$. $\square$

Remark. Alternatively, we could redefine $\cos\psi$ and $\cos\phi$ via the cosine law even if the corresponding comparison triangles do not exist. That is, set $(\cos\tilde{\measuredangle}^0)\{\ell; r, s\} := \frac{r^2+s^2-\ell^2}{2rs}$; it is always defined but might have absolute value bigger than 1. Then inequality $(*)^0_\ell$ implies that $\cos\psi$ is nondecreasing on $[s_{\min}, \frac{\varpi\kappa}{2})$. This in particular shows that if $\cos\psi(s_0) > 1$ for some s_0, then $\cos\psi(s) > 1$ for $s \geqslant s_0$. Or, equivalently, if $|q-\sigma(s_0)| < s_0 - |p-q|$, then $|q-\sigma(s)| < s - |p-q|$ for $s \geqslant s_0$.

Similarly, inequalities $(*)^0_\ell$ and $(**)^0_\ell$ imply that $\cos\phi(r, s)$ is nondecreasing while the point (r, s) moves along this concatenation in the figure with nondecreasing r and s. Likewise, if at some point along this concatenation the triangle inequality fails then it also fails for all later times.

Proof of 16.32 and 16.34 in case $\kappa \neq 0$. As before, let

$$\begin{aligned} R &= R(r) = |p-\rho(r)|, & \ell &= \ell(r,s) = |\rho(r) - \sigma(s)|,\\ S &= S(s) = |p-\sigma(s)|, & \phi &= \phi(r,s) = \tilde{\measuredangle}^\kappa\{\ell(r,s); r, s\}. \end{aligned}$$

It suffices to prove the following three inequalities:

$$(*)^\pm_\phi \qquad \tfrac{\partial^+}{\partial r}\phi(s_{\min}, r) \leqslant 0, \qquad \tfrac{\partial^+}{\partial s}\phi(s, r_{\min}) \leqslant 0,$$

$$(**)^\pm_\phi \qquad \operatorname{sn}^\kappa s\cdot\operatorname{cs}^\kappa S\cdot\tfrac{\partial^+}{\partial s}\phi + \operatorname{sn}^\kappa r\cdot\operatorname{cs}^\kappa R\cdot\tfrac{\partial^+}{\partial r}\phi \leqslant 0..$$

Then radial monotonicity follows from $(*)^\pm_\phi$ the same way as in the $\kappa = 0$ case. The radial comparison follows from $(*)^0_\phi$ and $(**)^\pm_\phi$. Indeed, the functions $s \mapsto \operatorname{sn}^\kappa s\cdot\operatorname{cs}^\kappa S$ and $r \mapsto \operatorname{sn}^\kappa r\cdot\operatorname{cs}^\kappa R$ are Lipschitz. Thus there is a solution for the differential equation

$$(r', s') = (\operatorname{sn}^\kappa s\cdot\operatorname{cs}^\kappa S, \operatorname{sn}^\kappa r\cdot\operatorname{cs}^\kappa R)$$

with any initial data $(r_0, s_0) \in \left[r_{\min}, \frac{\varpi\kappa}{2}\right)\times\left[s_{\min}, \frac{\varpi\kappa}{2}\right)$. (Unlike the case $\kappa = 0$, the solution cannot be written explicitly.) Since $\operatorname{sn}^\kappa s\cdot\operatorname{cs}^\kappa S, \operatorname{sn}^\kappa r\cdot\operatorname{cs}^\kappa R > 0$, this solution $t \mapsto (r(t), s(t))$ must meet one of the coordinate rays $\{r_{\min}\}\times\left[s_{\min}, \frac{\varpi\kappa}{2}\right)$ or $\left[r_{\min}, \frac{\varpi\kappa}{2}\right)\times\{s_{\min}\}$. That is, one can connect the pair $(s_{\min}, r_{\min})$ to (s_0, r_0) by a concatenation of a coordinate segment (vertical or horizontal) and part of the solution $(r(t), s(t))$. According to $(*)^\pm_\phi$ and $(**)^\pm_\phi$, the value of ϕ does not increase while the pair (r, s) moves along this concatenation in direction of increasing r and s. So, the same argument as in the $\kappa = 0$ case, implies 1 and 2.

As before, we rewrite the inequalities $(*)^\pm_\phi$ and $(**)^\pm_\phi$ in terms of ℓ:

$$(*)^\pm_\ell \qquad \begin{aligned} &\tfrac{\partial^+}{\partial s}\ell(s, r_{\min}) \leqslant \cos\tilde{\measuredangle}^\kappa\{r_{\min}; s, \ell\},\\ &\tfrac{\partial^+}{\partial r}\ell(s_{\min}, r) \leqslant \cos\tilde{\measuredangle}^\kappa\{s_{\min}; r, \ell\}, \end{aligned}$$

$$(**)^\pm_\ell \qquad \begin{aligned} &\operatorname{sn}^\kappa s\cdot\operatorname{cs}^\kappa S\cdot\tfrac{\partial^+}{\partial s}\ell + \operatorname{sn}^\kappa r\cdot\operatorname{cs}^\kappa R\cdot\tfrac{\partial^+}{\partial r}\ell\\ &\qquad\leqslant \operatorname{sn}^\kappa s\cdot\operatorname{cs}^\kappa S\cdot\cos\tilde{\measuredangle}^\kappa\{r; s, \ell\} + \operatorname{sn}^\kappa r\cdot\operatorname{cs}^\kappa R\cdot\cos\tilde{\measuredangle}^\kappa\{s; r, \ell\}. \end{aligned}$$

Let

$$(A)^{\pm} \qquad f = -\tfrac{1}{\varkappa}\cdot \operatorname{cs}^{\varkappa}\circ \operatorname{dist}_p = \operatorname{md}^{\varkappa}\circ \operatorname{dist}_p - \tfrac{1}{\varkappa}.$$

Clearly $f'' + \varkappa\cdot f \leqslant 0$ and

$$(B)^{\pm} \qquad \begin{aligned} \rho^+(r) &= \frac{1}{\operatorname{tg}^{\varkappa} r\cdot \operatorname{cs}^{\varkappa} R}\cdot \nabla_{\rho(r)}f,\\ \sigma^+(s) &= \frac{1}{\operatorname{tg}^{\varkappa} s\cdot \operatorname{cs}^{\varkappa} S}\cdot \nabla_{\sigma(s)}f.\end{aligned}$$

Thus from 13.24, we have

$$(C)^{\pm} \qquad \begin{aligned} \tfrac{\partial^+}{\partial r}\ell &= -\frac{1}{\operatorname{tg}^{\varkappa} r\cdot \operatorname{cs}^{\varkappa} R}\cdot\langle \nabla_{\rho(r)}f, \uparrow_{[\rho(r)\sigma(s)]}\rangle\\ &\leqslant \frac{1}{\operatorname{tg}^{\varkappa} r\cdot \operatorname{cs}^{\varkappa} R}\cdot \frac{\operatorname{cs}^{\varkappa} S - \operatorname{cs}^{\varkappa} R\cdot \operatorname{cs}^{\varkappa}\ell}{\varkappa\cdot \operatorname{sn}^{\varkappa}\ell}\\ &= \frac{\frac{\operatorname{cs}^{\varkappa} S}{\operatorname{cs}^{\varkappa} R} - \operatorname{cs}^{\varkappa}\ell}{\varkappa\cdot \operatorname{tg}^{\varkappa} r\cdot \operatorname{sn}^{\varkappa}\ell}.\end{aligned}$$

Note that for all $\varkappa \neq 0$, the function $x \mapsto \frac{1}{\varkappa\cdot\operatorname{cs}^{\varkappa} x}$ is increasing. Thus, since $R(r) \leqslant r$ and $S(s_{\min}) = s_{\min}$, we have

$$(D)^{\pm} \qquad \begin{aligned} \tfrac{\partial^+}{\partial r}\ell(r, s_{\min}) &\leqslant \frac{\frac{\operatorname{cs}^{\varkappa} s_{\min}}{\operatorname{cs}^{\varkappa} r} - \operatorname{cs}^{\varkappa}\ell}{\varkappa\cdot \operatorname{tg}^{\varkappa} r\cdot \operatorname{sn}^{\varkappa}\ell}\\ &= \frac{\operatorname{cs}^{\varkappa} s_{\min} - \operatorname{cs}^{\varkappa}\ell\cdot \operatorname{cs}^{\varkappa} r}{\varkappa\cdot \operatorname{sn}^{\varkappa} r\cdot \operatorname{sn}^{\varkappa}\ell}\\ &= \cos\tilde\measuredangle^{\varkappa}\{s_{\min}; r, \ell\},\end{aligned}$$

which is the first inequality in $(*)^{\pm}_{\ell}$ for $\varkappa \neq 0$. By switching ρ and σ we obtain the second inequality in $(*)^{\pm}_{\ell}$. Further, adding $(C)^{\pm}$ and its mirror-inequality for $\frac{\partial^+}{\partial s}\ell$, we have

$$(E)^{\pm} \qquad \begin{aligned} &\operatorname{sn}^{\varkappa} r\cdot \operatorname{cs}^{\varkappa} R\cdot \tfrac{\partial^+}{\partial r}\ell + \operatorname{sn}^{\varkappa} s\cdot \operatorname{cs}^{\varkappa} S\cdot \tfrac{\partial^+}{\partial s}\ell\\ &\quad\leqslant \frac{\operatorname{cs}^{\varkappa} S\cdot \operatorname{cs}^{\varkappa} r - \operatorname{cs}^{\varkappa}\ell\cdot \operatorname{cs}^{\varkappa} R\cdot \operatorname{cs}^{\varkappa} r}{\varkappa\cdot \operatorname{sn}^{\varkappa}\ell} + \frac{\operatorname{cs}^{\varkappa} R\cdot \operatorname{cs}^{\varkappa} s - \operatorname{cs}^{\varkappa}\ell\cdot \operatorname{cs}^{\varkappa} S\cdot \operatorname{cs}^{\varkappa} s}{\varkappa\cdot \operatorname{sn}^{\varkappa}\ell}\\ &\quad= \operatorname{sn}^{\varkappa} r\cdot \operatorname{cs}^{\varkappa} R\cdot \frac{\operatorname{cs}^{\varkappa} s - \operatorname{cs}^{\varkappa}\ell\cdot \operatorname{cs}^{\varkappa} r}{\varkappa\cdot \operatorname{sn}^{\varkappa} r\cdot \operatorname{sn}^{\varkappa}\ell} + \operatorname{sn}^{\varkappa} s\cdot \operatorname{cs}^{\varkappa} S\cdot \frac{\operatorname{cs}^{\varkappa} r - \operatorname{cs}^{\varkappa}\ell\cdot \operatorname{cs}^{\varkappa} s}{\varkappa\cdot \operatorname{sn}^{\varkappa} s\cdot \operatorname{sn}^{\varkappa}\ell}\\ &\quad= \operatorname{sn}^{\varkappa} r\cdot \operatorname{cs}^{\varkappa} R\cdot \cos\tilde\measuredangle^{\varkappa}\{r; s, \ell\} + \operatorname{sn}^{\varkappa} s\cdot \operatorname{cs}^{\varkappa} S\cdot \cos\tilde\measuredangle^{\varkappa}\{s; r, \ell\},\end{aligned}$$

which is $(**)^{\pm}_{\ell}$. □

16.35. Exercise. *Suppose $\mathcal{L}$ is a complete length* CBB($\varkappa$) *space and $x, y, z \in \mathcal{L}$. Assume $\tilde\measuredangle^{\varkappa}(z\,{}^{x}_{y}) = \pi$. Show that there is a geodesic $[xy]$ that contains z. In particular, x can be connected to y by a minimizing geodesic.*

J. Gradient exponent

Let $\mathcal{L}$ be a complete length CBB($\varkappa$) space, $p \in \mathcal{L}$, and $\xi \in \Sigma_p$. Consider a sequence of points $x_n \in \mathcal{L}$ such that $\uparrow_{[px_n]} \to \xi$. Let $r_n = |p - x_n|$, and let $\sigma_n \colon [r_n, \frac{\varpi\varkappa}{2}) \to \mathcal{L}$ be the $(p,\varkappa)$-radial curve that starts at x_n.

By the radial comparison (16.34), the curves $\sigma_n \colon [r_n, \frac{\varpi\varkappa}{2}) \to \mathcal{L}$ converge to a curve $\sigma_\xi \colon (0, \frac{\varpi\varkappa}{2}) \to \mathcal{L}$, and this limit is independent of the choice of the sequence x_n. Let $\sigma_\xi(0) = p$, and if $\varkappa > 0$ define

$$\sigma_\xi(\tfrac{\varpi\varkappa}{2}) = \lim_{t \to \frac{\varpi\varkappa}{2}} \sigma_\xi(t).$$

The resulting curve σ_ξ will be called the $(p,\varkappa)$-*radial curve* in direction ξ.

The *gradient exponential map* $\mathrm{gexp}_p^\varkappa \colon \overline{\mathrm{B}}[0, \frac{\varpi\varkappa}{2}]_{\mathrm{T}_p} \to \mathcal{L}$ is defined by

$$\mathrm{gexp}_p^\varkappa \colon r \cdot \xi \mapsto \sigma_\xi(r).$$

Here are properties of radial curves reformulated in terms of the gradient exponential map:

16.36. Theorem. *Let $\mathcal{L}$ be a complete length* CBB($\varkappa$) *space. Then:*

(a) *If $p, q \in \mathcal{L}$ are points such that $|p - q| \leqslant \frac{\varpi\varkappa}{2}$, then for any geodesic $[pq]$ in $\mathcal{L}$ we have*

$$\mathrm{gexp}_p^\varkappa(\log[pq]) = q.$$

(b) *For any $v, w \in \overline{\mathrm{B}}[0, \frac{\varpi\varkappa}{2}]_{\mathrm{T}_p}$,*

$$|\mathrm{gexp}_p^\varkappa v - \mathrm{gexp}_p^\varkappa w| \leqslant \tilde{\mathrm{Y}}^\varkappa \left[0\,{}^{v}_{w}\right].$$

In other words, if we denote by $\mathcal{T}_p^\varkappa$ the set $\overline{\mathrm{B}}[0, \frac{\varpi\varkappa}{2}]_{\mathrm{T}_p}$ equipped with the metric $|v - w|_{\mathcal{T}_p^\varkappa} = \tilde{\mathrm{Y}}^\varkappa \left[0\,{}^{v}_{w}\right]$, then

$$\mathrm{gexp}_p^\varkappa \colon \mathcal{T}_p^\varkappa \to \mathcal{L}$$

is a short map.

(c) *Suppose $p, q \in \mathcal{L}$ and $|p - q| \leqslant \frac{\varpi\varkappa}{2}$. If $v \in \mathrm{T}_p$, $|v| \leqslant 1$, and*

$$\sigma(t) = \mathrm{gexp}_p^\varkappa(t \cdot v),$$

then the function

$$s \mapsto \tilde{\measuredangle}^\varkappa(\sigma|_0^s, q) \coloneqq \tilde{\measuredangle}^\varkappa\{|q - \sigma(s)|; |q - \sigma(0)|, s\}$$

is nonincreasing in its domain of definition.

Proof. Follows directly from the construction of $\mathrm{gexp}_p^\varkappa$ and the radial comparison (16.34). ☐

Applying the theorem above together with 15.13*c*, we obtain the following.

16.37. Corollary. *Let p be a point in an m-dimensional complete length* CBB(κ) *space $\mathcal{L}$, $m < \infty$, and $0 < R \leqslant \frac{\varpi\kappa}{2}$. Then there is a short map $f\colon\ \overline{\mathrm{B}}[R]_{\mathbb{M}^m(\kappa)} \to \mathcal{L}$ such that $\Im f = \overline{\mathrm{B}}[p, R] \subset \mathcal{L}$.*

16.38. Exercise. *Let $\mathcal{L} \subset \mathbb{E}^2$ be the Euclidean halfplane. Clearly $\mathcal{L}$ is a two-dimensional complete length* CBB(0) *space. Given a point $x \in \mathbb{E}^2$, denote by* $\mathrm{proj}(x)$ *the closest point to x on $\mathcal{L}$.*

Apply the radial comparison (16.34) *to show that for any interior point $p \in \mathcal{L}$ and any $v \in \mathbb{R}^2$ we have*

$$\mathrm{gexp}_p\, v = \mathrm{proj}(p + v).$$

16.39. Exercise. *Suppose x, p, and q are points in a complete length* CBB(κ) *space, and $x \in [pq[$. Show that there is a unique vector $v \in \mathrm{T}_p$ such that $\mathrm{gexp}_p\, v = x$.*

K. Remarks

Gradient flow on Riemannian manifolds. The gradient flow for general semiconcave functions on smooth Riemannian manifolds can be introduced with much less effort. To do this note that the distance estimates proved in the Section 16C can be proved in the same way for gradient curves of smooth semiconcave subfunctions. By the Greene–Wu lemma [**70**], given a λ-concave function f, a compact set $K \subset \operatorname{Dom} f$, and $\varepsilon > 0$ there is a smooth $(\lambda - \varepsilon)$-concave function that is ε-close to f on K. Hence one can apply smoothing and pass to the limit as $\varepsilon \to 0$. Note that by the second distance estimate (16.13), the limit curve obtained does not depend on the smoothing.

Gradient curves of a family of functions. Gradient flow can be extended to a family of functions. This type of flow was studied by Chanyoung Jun [**90**, **91**], by Lucas Ferreira and Julio Valencia-Guevara [**65**], and by Alexander Mielke, Riccarda Rossi, and Giuseppe Savaré [**116**]. We will follow the simplified and generalized approach given by Alexander Lytchak and the third author [**106**], where an application related to this type of flow is given. The original motivation of Chanyoung Jun came from the study of pursuit-evasion problems. Another application of this type of flow comes from the fact that the optimal transport plan, or equivalently geodesics in the Wasserstein metric, can be described as gradient flow for a family of semiconcave functions. This observation was used by the third author to prove that Alexandrov spaces with nonnegative curvature have nonnegative Ricci curvature in the sense of Lott–Villani–Sturm [**128**].

Suppose that $\mathcal{Z}$ is either CBB or CAT. Let f_t be a family of functions defined on open subsets $\operatorname{Dom} f_t$ of $\mathcal{Z}$. More precisely, we assume that the parameter t lies in a real interval $\mathbb{I}$ and

$$\Omega = \{\,(x,t) \in \mathcal{Z} \times \mathbb{I} \;:\; x \in \operatorname{Dom} f_t\,\}$$

is an open subset in $\mathcal{Z} \times \mathbb{I}$.

A family of functions f_t is called *Lipschitz* if the function $(x,t) \mapsto f_t(x)$ is L-Lipschitz for some constant L.

A family of functions f_t will be called *semiconcave* if the function $x \mapsto f_t(x)$ is λ-concave for each t. A family f_t is called *locally semiconcave* if for each $(p_0,t_0) \in \Omega$ there is a neighborhood Ω' and $\lambda \in \mathbb{R}$ such that the restriction of f_t to Ω' is semiconcave.

One cannot expect that a direct generalization of Definition 16.7 holds for every family of functions f_t; that is, gradient curves of a family f_t cannot be defined as curves satisfying the equation $\alpha^+ = \nabla_\alpha f$.

For example, consider a 1-Lipschitz curve α in the real line. It is reasonable to assume that α is an f_t-gradient curve for the family $f_t(x) = -|x - \alpha(t)|$. (Indeed α can be realized as a limit of gradient curves for a family of functions obtained by smoothing f_t.) On the other hand, $\alpha^+(t)$ might be undefined, and even if it is defined, in general $\alpha^+(t) \neq 0$, while $\nabla_{\alpha(t)} f_t \equiv 0$.

Instead we define an *f_t-gradient curve* as a Lipschitz curve α that satisfies the following inequality for any point p, time t, and small $\varepsilon > 0$:

$$\operatorname{dist}_p \circ \alpha(t+\varepsilon) \leqslant \operatorname{dist}_p \circ \alpha(t) - \varepsilon \cdot \mathbf{d}_{\alpha(t)} f_t(\uparrow_{[\alpha(t)p]}) + o(\varepsilon). \tag{1}$$

If there is no geodesic $[\alpha(t)\,p]$, then we impose no condition.

If $\alpha^+(t) = \nabla_{\alpha(t)} f_t$ for all t, then 1 holds by the definition of gradient (13.17). On the other hand, the example above shows that the converse does not hold; that is, 1 generalizes Definition 16.7. The defining inequality 1 is closely related to the so-called *evolution variational inequality* [**19**, Theorem 4.0.4(iii)].

16.40. Distance estimate. *Let f_t and h_t be two Lipschitz families of λ-concave functions on a complete length space $\mathcal{Z}$, and $s \geqslant 0$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Assume f_t and h_t have common domain $\Omega \subset \mathcal{Z}\times\mathbb{R}$, and $|f_t(x) - h_t(x)| \leqslant s$ for any $(x,t) \in \Omega$. Assume $t \mapsto \alpha(t)$ and $t \mapsto \beta(t)$ are f_t- and h_t-gradient curves, respectively, defined on a common interval $t \in [a,b)$, and let $\ell(t) = |\alpha(t) - \beta(t)|_{\mathcal{Z}}$. If for all t, a minimizing geodesic $[\alpha(t)\,\beta(t)]$ lies in $\{\,x \in \mathcal{Z} \,:\, (x,t) \in \Omega\,\}$, then*

$$\ell'(t) \leqslant \lambda \cdot \ell(t) + 2 \cdot s/\ell(t),$$

whenever the left-hand side is defined. Moreover,

$$\ell(t)^2 + \tfrac{2\cdot s}{\lambda} \leqslant \left(\ell(a)^2 + \tfrac{2\cdot s}{\lambda}\right) \cdot e^{2\cdot\lambda\cdot(t-a)}.$$

In particular, these inequalities hold for any $t \in \mathbb{I}$ if $\Omega \supset B(p, 2\cdot r) \times \mathbb{I}$ and $\alpha(t), \beta(t) \in B(p, r)$ for any $t \in \mathbb{I}$.

Note that if $f_t = h_t$ then $s = 0$; in this case the second inequality can be written as

$$\ell(t) \leqslant \ell(a) \cdot e^{\lambda\cdot(t-a)}. \tag{2}$$

In particular, it implies uniqueness of the future of gradient curves with given initial data. This inequality also makes it possible to estimate the distance between two gradient curves for close functions. In particular, it implies convergence for f_t^n-gradient curves if a sequence of ℓ-Lipschitz and λ-concave families f_t^n converges uniformly as $n \to \infty$.

Proof of 16.40. Fix a time moment t and set $f = f_t$ and $h = h_t$. Let m be the midpoint of the geodesic $[\alpha(t)\beta(t)]$. Let $\gamma\colon [0, \ell(t)] \to \mathcal{Z}$ be an arclength parametrization of $[\alpha(t)\beta(t)]$. Note that $\mathbf{d}_{\alpha(t)} f(\uparrow_{[\alpha(t)m]})$ is the right derivative of $f \circ \gamma$ at 0 and $-\mathbf{d}_{\alpha(t)} h(\uparrow_{[\beta(t)m]})$ is the left derivative of $h \circ \gamma$ at $\ell(t)$. Since f and h are λ-concave,

$$\begin{aligned} f\circ\beta(t) &\leqslant f\circ\alpha(t) + \ell(t)\cdot \mathbf{d}_{\alpha(t)} f(\uparrow_{[\alpha(t)m]}) + \tfrac{1}{2}\cdot\lambda\cdot\ell(t)^2,\\ h\circ\alpha(t) &\leqslant h\circ\beta(t) + \ell(t)\cdot \mathbf{d}_{\alpha(t)} h(\uparrow_{[\beta(t)m]}) + \tfrac{1}{2}\cdot\lambda\cdot\ell(t)^2. \end{aligned}$$

Adding these inequalities and taking into account $|f(x) - h(x)| < s$ for any x, we conclude that

$$\mathbf{d}_{\alpha(t)} f(\uparrow_{[\alpha(t)m]}) + \mathbf{d}_{\alpha(t)} h(\uparrow_{[\beta(t)m]}) \geqslant \lambda\cdot\ell(t) + 2\cdot s/\ell(t).$$

Applying the triangle inequality and 1 at m, we obtain

$$\begin{aligned} \ell(t+\varepsilon) &= |\alpha(t+\varepsilon) - \beta(t+\varepsilon)|\\ &\leqslant |\alpha(t+\varepsilon) - m| + |\beta(t+\varepsilon) - m|\\ &\leqslant |\alpha(t) - m| - \varepsilon\cdot \mathbf{d}_{\alpha(t)} f(\uparrow_{[\alpha(t)m]})\\ &\quad + |\beta(t+\varepsilon) - m| - \varepsilon\cdot \mathbf{d}_{\beta(t)} h(\uparrow_{[\beta(t)m]}) + o(\varepsilon)\\ &= \ell(t) - \varepsilon\cdot(\lambda\cdot\ell(t) + 2\cdot s/\ell(t)) + o(\varepsilon) \end{aligned}$$

for $\varepsilon > 0$. The first inequality follows.

Since α and β are Lipschitz, $t \mapsto \ell(t)$ is a Lipschitz function. By Rademacher's theorem, its derivative ℓ' is defined almost everywhere and satisfies the fundamental theorem of calculus. Therefore the first inequality implies the second. ☐

16.41. Proposition. *Suppose $\mathcal{Z}$ is a complete length space that is either* CBB *or* CAT. *Let f_t be a family of λ-concave functions for $t \in [a, b)$, where* Dom $f_t \supset B(z, 2\cdot r)$ *for fixed $z \in \mathcal{Z}$, $r > 0$ and any t.*

Let $\alpha\colon [a,b) \to B(z,r)$ be Lipschitz. Then α is an f_t-gradient curve if and only if

$$\text{(3)}\quad \operatorname{dist}_p \circ\alpha(t+\varepsilon) \leqslant \operatorname{dist}_p \circ\alpha(t) - \varepsilon\cdot\left[\frac{f_t(p)-f_t\circ\alpha(t)}{|p-\alpha(t)|} - \tfrac{\lambda}{2}\cdot|p-\alpha(t)|\right] + o(\varepsilon)$$

for any $t \in [a,b)$ and $p \in B(z,r)\setminus\{\alpha(t)\}$.

Proof. Note that the geodesics $[\alpha(t)p]$ lie in $\operatorname{Dom} f_t$ for any t.

Since f_t is λ-concave, we have

$$\mathbf{d}_{\alpha(t)} f_t(\uparrow_{[\alpha(t)p]}) \geqslant \frac{f(p)-f\circ\alpha(t)}{|p-\alpha(t)|} - \tfrac{\lambda}{2}\cdot|p-\alpha(t)|.$$

Hence the only-if part follows.

Given $p \in \mathcal{Z}$ and t, consider a point $\bar p \in [\alpha(t)p]$. Applying 3 for $\bar p$, and the triangle inequality, we have

$$\operatorname{dist}_p \circ\alpha(t+\varepsilon) \leqslant \operatorname{dist}_p \circ\alpha(t) - \varepsilon\cdot\left[\frac{f(\bar p)-f\circ\alpha(t)}{|\bar p-\alpha(t)|} - \tfrac{\lambda}{2}\cdot|\bar p-\alpha(t)|\right] + o(\varepsilon).$$

By taking $\bar p$ close to $\alpha(t)$, the value $\frac{f(\bar p)-f\circ\alpha(t)}{|\bar p-\alpha(t)|} - \frac{\lambda}{2}\cdot|\bar p-\alpha(t)|$ can be made arbitrarily close to $\mathbf{d}_{\alpha(t)} f_t(\uparrow_{[\alpha(t)p]})$. Therefore, given $\delta > 0$, the inequality

$$\operatorname{dist}_p \circ\alpha(t+\varepsilon) \leqslant \operatorname{dist}_p \circ\alpha(t) - \varepsilon\cdot\mathbf{d}_{\alpha(t)} f_t(\uparrow_{[\alpha(t)p]}) + \varepsilon\cdot\delta$$

holds for all sufficiently small $\varepsilon > 0$. Therefore 1 holds. □

Now we are ready to formulate and prove global existence of gradient curves for time-dependent families—an analog of 16.18.

16.42. Theorem. *Suppose $\mathcal{Z}$ is a complete length space that is either* CBB *or* CAT. *Let $\{f_t\}$ be a family of functions defined on an open set*

$$\Omega = \{\,(x,t)\in\mathcal{Z}\times\mathbb{R} : x\in\operatorname{Dom} f_t\,\}.$$

Suppose that f_t is Lipschitz and locally semiconcave. Then for any time a and initial point $p \in \operatorname{Dom} f_a$, there is a unique f_t-gradient curve $t\mapsto\alpha(t)$ defined on a maximal semiopen interval $[a,b)$. Moreover, if $b<\infty$ then $(\alpha(t),t)$ escapes from any closed set $K\subset\Omega$.

Proof. Let L be a Lipschitz constant of f_t. Fix $b > a$ sufficiently small that $\operatorname{Dom} f_t \supset B(p,\varepsilon\cdot L)$ for any $t\in[a,b)$. Consider a sequence $a = t_0 < t_1\cdots < t_n = b$, and a piecewise constant family of functions on $B(p,\varepsilon\cdot L)$ defined by $\hat f_t = f_{t_i}$ if $t_i \leqslant t < t_{i+1}$.

Note that $\hat f_t$ is time-independent on each interval $[t_i,t_{i+1})$. By 16.18 applied recursively on each interval $[t_i,t_{i+1})$, the proposition holds for $\hat f_t$. That is, there is a unique $\hat f_t$-gradient curve $\hat\alpha$ that starts at p and is defined on the interval $[a,b)$.

The distance estimates (16.40) show that as the partition gets finer, the gradient curves $\hat\alpha$ form a Cauchy sequence; denote its limit by α. Then

$$\begin{aligned}\operatorname{dist}_p \circ \hat\alpha(t+\varepsilon) &\leqslant \operatorname{dist}_p \circ \hat\alpha(t) - \varepsilon \cdot \left[\frac{\hat f_t(p) - \hat f_t \circ \hat\alpha(t)}{|p-\alpha(t)|} - \tfrac{\lambda}{2}\cdot|p-\hat\alpha(t)|\right] + o(\varepsilon)\\ &\leqslant \operatorname{dist}_p \circ \hat\alpha(t)\\ &\qquad - \varepsilon \cdot \left[\frac{f_t(p) - f_t \circ \hat\alpha(t) - 2\cdot s}{|p-\alpha(t)|} - \tfrac{\lambda}{2}\cdot|p-\hat\alpha(t)|\right] + o(\varepsilon),\end{aligned}$$

where

$$s = \sup_{t,x}\{|f_t(x) - \hat f_t(x)|\}.$$

Since $s \to 0$ as $\hat\alpha \to \alpha$, then 3 holds for α; that is, α is an f_t-gradient curve.

This proves short time existence. Applying this argument recursively, we can find a gradient curve defined on a maximal interval $[a,b)$. Uniqueness of this curve follows from the distance estimate 2.

Note that α is L-Lipschitz. In particular, if $b<\infty$ then $\alpha(t)\to p'$ as $t\to b$. If $(p',b)\in\Omega$ then we can repeat the procedure; otherwise α escapes from any closed set in Ω. □

Gradient curves for non-Lipschitz functions. In this book, we only consider gradient curves for locally Lipschitz semiconcave subfunctions; this turns out to be sufficient for all our needs. However, instead of Lipschitz semiconcave subfunctions, it is more natural to consider upper semicontinuous semiconcave functions with target in $[-\infty,\infty)$, and to assume in addition that the functions take finite values at a dense set in the domain of definition. Suppose that $\mathcal{Z}$ is a complete length space that is either CBB or CAT. The set of such subfunctions on $\mathcal{Z}$ will be denoted by $\mathrm{LCC}(\mathcal{Z})$ (for **l**ower semi-**c**ontinous and semi-**c**oncave).

In this section we describe the adjustments needed to construct gradient curves for the subfunctions in $\mathrm{LCC}(\mathcal{Z})$.

When $\mathcal{Z} = H$ is a Hilbert space the theory we develop is equivalent to the classical theory of gradient flows on Hilbert space mentioned earlier [**33**].

Further examples of such functions include the entropy and other closely related functionals on the Wasserstein space over a CBB(0) space. Another important example is given by the Cheeger energy on metric measure spaces, its gradient flow leads to the notion of the heat flow on such spaces. The gradient flow for these functions plays an important role in the theory of optimal transport, see [**156**] and references there-in.

Differential. First we need to extend the definition of differential (6.15) to LCC subfunctions.

Let $\mathcal{Z}$ be a complete length space and $f \in \mathrm{LCC}(\mathcal{Z})$. Suppose that $\mathcal{Z}$ is either CBB or CAT. Given a point $p \in \operatorname{Dom} f$ and a geodesic direction $\xi = \uparrow_{[pq]}$, let $\hat{\mathbf{d}}_p f(\xi) = (f \circ \mathrm{geod}_{[pq]})^+(0)$. Since f is semiconcave, the value $\hat{\mathbf{d}}_p f(\xi)$ is defined if $f \circ \mathrm{geod}_{[pq]}(t)$ is finite at all sufficiently small values $t > 0$, but $\hat{\mathbf{d}}_p f(\xi)$ may take value ∞. Note that $\hat{\mathbf{d}}_p f$ is defined on a dense subset of Σ_p.

Let

$$\mathbf{d}_p f(\zeta) = \varlimsup_{\xi \to \zeta} \hat{\mathbf{d}}_p f(\xi),$$

and $\mathbf{d}_p f(v) = |v| \cdot \mathbf{d}_p f(\xi)$ if $v = |v| \cdot \xi$ for some $\xi \in \Sigma_p$.

In other words, we define differential as the smallest upper semi-continuous positive-homogeneous function $\mathbf{d}_p f \colon \mathrm{T}_p \to \mathbb{R}$ such that if $\hat{\mathbf{d}}_p f(\xi)$ is defined, then $\mathbf{d}_p f(\xi) \geqslant \hat{\mathbf{d}}_p f(\xi)$.

Existence and uniqueness of the gradient. Note that in the proof of 13.20, we used the Lipschitz condition just once, to show that

$$s = \sup\{(\mathbf{d}_p f)(\xi) : \xi \in \Sigma_p\} = \varlimsup_{x \to p} \frac{f(x) - f(p)}{|x - p|} < \infty.$$

The value s above will be denoted by $|\nabla|_p f$. Note that if the gradient $\nabla_p f$ is defined then $|\nabla|_p f = |\nabla_p f|$, and otherwise $|\nabla|_p f = \infty$.

Summarizing the discussion above, we have the following.

13.20′ Existence and uniqueness of the gradient. *Assume $\mathcal{Z}$ is a complete space and $f \in \mathrm{LCC}(\mathcal{Z})$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then for any point $p \in \operatorname{Dom} f$, either there is a unique gradient $\nabla_p f \in \mathrm{T}_p$ or $|\nabla|_p f = \infty$.*

Further, in all the results of Section 13F we may assume only that both f and the gradient of f are defined at the points under consideration. The proofs are the same.

Sections 13G–16D require almost no changes. Mainly, where appropriate one needs to change $|\nabla_p f|$ to $|\nabla|_p f$ and/or assume that the gradient is defined at the points of interest. Also, 1 in Theorem 16.3 is taken as the definition of gradient-like curve. Then the theorem states that any gradient-like curve $\alpha\colon \mathbb{I} \to \mathcal{Z}$ satisfies Definition 16.2 at $t \in \mathbb{I}$ if $\nabla_{\hat{\alpha}(s)} f$ is defined. Further, Definition 16.7, should be changed to the following:

16.7′. Definition. *Let $\mathcal{Z}$ be a complete length space and $f \in \mathrm{LCC}(\mathcal{Z})$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT.

*A curve $\alpha\colon [t_{\min}, t_{\max}) \to \operatorname{Dom} f$ will be called an f-*gradient curve *if*

$$\alpha^+(t) = \nabla_{\alpha(t)} f$$

when $\nabla_{\alpha(t)} f$ *is defined and*

$$(f\circ\alpha)^+(t) = \infty$$

otherwise.

In the proof of local existence (16.16), condition (ii) should be changed to the following condition:

(ii)′ $f\circ\hat{\alpha}_n(\bar{\varsigma}_i) - f\circ\hat{\alpha}_n(\varsigma_i) > (\bar{\varsigma}_i - \varsigma_i)\cdot\max\{n, |\nabla|_{\hat{\alpha}_n(\varsigma_i)} f - \frac{1}{n})\}$.

Any gradient curve $\alpha[0,\ell)\to\mathcal{Z}$ for a subfunction $f\in\mathrm{LCC}(\mathcal{Z})$ satisfies the equation

$$\alpha^+(t) = \nabla_{\alpha(t)} f$$

at all values t, with the possible exception of $t=0$. In particular, the gradient of f is defined at all points of any f-gradient curve, with the possible exception of the initial point.

16.43. Example. *Let* $\mathcal{X} = L^2(\mathbb{R}^n)$*. Let* $F\colon\ \mathcal{X}\to\mathbb{R}$ *be given by* $F(f) = -\int|\nabla f|^2 d\mathcal{L}$*. Then* F *is a concave but not locally Lipschitz functional on* $\mathcal{X}$ *and its finite precisely at functions* $f\in W^{1,2}(\mathbb{R}^n)\subset L^2(\mathbb{R}^n)$*. Integration by parts shows that for any smooth* $f\in W^{1,2}(\mathbb{R}^n)$ *it holds that* $\nabla_f F = \Delta f$*. The gradient flow of* F *is given by the heat flow* f_t *starting at* f *and* f_t *is smooth for all positive* t*.*

Slower radial curves. Let $\varkappa\geqslant 0$. Assume that for some function ψ, the curves defined by the equation

$$\sigma^+(s) = \psi(s, |p-\sigma(s)|)\cdot\nabla_{\sigma(s)}\operatorname{dist}_p$$

satisfy radial comparison 16.34. Then in fact the $\sigma(s)$ are radial curves; that is,

$$\psi(s,|p-\sigma(s)|) = \frac{\operatorname{tg}^\varkappa|p-\sigma(s)|)}{\operatorname{tg}^\varkappa s},$$

see exercise 16.38.

In case $\varkappa<0$, such a function ψ is not unique. In particular, one can take curves defined by the simpler equation

$$\sigma^+(s) = \frac{\operatorname{sn}^\varkappa|p-\sigma(s)|}{\operatorname{sn}^\varkappa s}\cdot\nabla_{\sigma(s)}\operatorname{dist}_p\ = \frac{1}{\operatorname{sn}^\varkappa s}\cdot\nabla_{\sigma(s)}(\operatorname{md}^\varkappa\circ\operatorname{dist}_p).$$

Among all curves of that type, the radial curves for curvature $\varkappa$ as defined in 16.27 maximize the growth of $|p-\sigma(s)|$.

Appendix A

Semisolutions

1.3. Suppose α is a closed spherical curve. By Crofton's formula, the length of α is $\pi \cdot n_\alpha$, where n_α denotes the average number of crossings of α with equators.

Since α is closed, almost all equators cross it at an even number of points (we assume that ∞ is an even number). If length $\alpha < 2 \cdot \pi$ then $n_\alpha < 2$. Therefore there is an equator that does not cross α—hence the result.

2.9. (*a*). Note that any Cauchy sequence x_n in $(\mathcal{X}, \|{*}-{*}\|)$ is also Cauchy in $\mathcal{X}$. Since $\mathcal{X}$ is complete, x_n converges; denote its limit by x_∞.

Passing to a subsequence, we may assume that $\|x_{n-1} - x_n\| < \frac{1}{2^n}$. It follows that there is a 1-Lipschitz curve $\alpha \colon [0,1] \to (\mathcal{X}, \|{*}-{*}\|)$ such that $x_n = \alpha(\frac{1}{2^n})$ and $x_\infty = \alpha(0)$. In particular, $\|x_n - x_\infty\| \to 0$ and $n \to \infty$.

(*b*). Fix two points $x, y \in \mathcal{X}$ such that $\ell = \|x - y\| < \infty$. Let α_n be a sequence of paths from x to y such that length$(\alpha_n) \to \ell$ as $n \to \infty$. Without loss of generality, we may assume that each α_n is $(\ell + 1)$-Lipschitz.

Since $\mathcal{X}$ is compact, there is a partial limit α_∞ of α_n as $n \to \infty$. By semicontinuity of length, length $\alpha_\infty \leqslant \ell$; that is; α is a shortest path in $\mathcal{X}$.

Source. Part (*a*) appears as a Corollary in [**84**]; see also [**126**, Lemma 2.3].

2.10. The following example was suggested by Fedor Nazarov [**120**].

Consider the unit ball (B, ρ_0) in the space c_0 of all sequences converging to zero equipped with the sup-norm.

Consider another metric ρ_1 which is different from ρ_0 by the conformal factor

$$\phi(\mathbf{x}) = 2 + \tfrac{1}{2} \cdot x_1 + \tfrac{1}{4} \cdot x_2 + \tfrac{1}{8} \cdot x_3 + \dots,$$

where $\mathbf{x} = (x_1, x_2 \dots) \in B$. That is, if $\mathbf{x}(t)$, $t \in [0, \ell]$, is a curve parametrized by ρ_0-length then its ρ_1-length is

$$\operatorname{length}_{\rho_1} \mathbf{x} = \int_0^\ell \phi \circ \mathbf{x}.$$

Note that the metric ρ_1 is bi-Lipschitz equivalent to ρ_0.

Assume $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are two curves parametrized by ρ_0-length that differ only in the m-th coordinate; denote them by $x_m(t)$ and $x'_m(t)$ respectively. Note that if $x'_m(t) \leqslant x_m(t)$ for any t and the function $x'_m(t)$ is locally 1-Lipschitz at all t such that $x'_m(t) < x_m(t)$, then

$$\operatorname{length}_{\rho_1} \mathbf{x}' \leqslant \operatorname{length}_{\rho_1} \mathbf{x}.$$

Moreover this inequality is strict if $x'_m(t) < x_m(t)$ for some t.

Fix a curve $\mathbf{x}(t)$, $t \in [0, \ell]$, parametrized by ρ_0-length. We can choose m large so that $x_m(t)$ is sufficiently close to 0 for any t. In particular, for some values t, we have $y_m(t) < x_m(t)$, where

$$y_m(t) = (1 - \tfrac{t}{\ell}) \cdot x_m(0) + \tfrac{t}{\ell} \cdot x_m(\ell) - \tfrac{1}{100} \cdot \min\{t, \ell - t\}.$$

Consider the curve $\mathbf{x}'(t)$ as above with

$$x'_m(t) = \min\{x_m(t), y_m(t)\}.$$

Note that $\mathbf{x}'(t)$ and $\mathbf{x}(t)$ have the same endpoints, and by the above

$$\operatorname{length}_{\rho_1} \mathbf{x}' < \operatorname{length}_{\rho_1} \mathbf{x}.$$

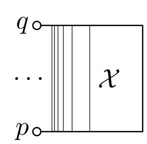
That is, for any curve $\mathbf{x}(t)$ in (B, ρ_1), we can find a shorter curve $\mathbf{x}'(t)$ with the same endpoints. In particular, (B, ρ_1) has no geodesics.

2.11. Choose a sequence of positive numbers $\varepsilon_n \to 0$ and an ε_n-net N_n of K for each n. Assume N_0 is a one-point set, so $\varepsilon_0 > \operatorname{diam} K$. Connect each point $x \in N_{k+1}$ to a point $y \in N_k$ by a curve of length at most ε_k.

Consider the union K' of all these curves with K; observe that K' is compact and path-connected.

Source: This problem was suggested by Eugene Bilokopytov [**28**].

2.16. Consider the following subset of $\mathbb{R}^2$ equipped with the induced length metric



$$\mathcal{X} = \big((0,1] \times \{0,1\}\big) \cup \big(\{1, \tfrac12, \tfrac13, \dots\} \times [0,1]\big).$$

Note that $\mathcal{X}$ is locally compact and geodesic.

Its completion $\bar{\mathcal{X}}$ is isometric to the closure of $\mathcal{X}$ equipped with the induced length metric; $\bar{\mathcal{X}}$ is obtained from $\mathcal{X}$ by adding two points $p = (0,0)$ and $q = (0,1)$.

The point p admits no compact neighborhood in $\bar{\mathcal{X}}$ and there is no geodesic connecting p to q in $\bar{\mathcal{X}}$.

Source: This example is taken from [**34**].

2.22. Let $\mathcal{X}$ be a compact metric space. Let us identify $\mathcal{X}$ with its image in $\mathrm{Bnd}(\mathcal{X},\mathbb{R})$ under the Kuratowsky embedding (Section 2G). Denote by $\mathcal{K}$ the *linear* convex hull of $\mathcal{X}$ in the space of bounded functions on $\mathcal{X}$; that is, $x \in \mathcal{K}$ if and only if x cannot be separated from $\mathcal{X}$ by a hyperplane.

Since $\mathcal{X}$ is compact, so is $\mathcal{K}$. It remains to observe that $\mathcal{K}$ is a length space since it is convex.

Remark. Alternatively, one can use the embedding of $\mathcal{X}$ into its injective hull; see [**86**].

4.2. Let $F = \{\, n \in \mathbb{N} : f(n) = n \,\}$; we need to show that $\omega(F) = 1$.

Consider an oriented graph Γ with vertex set $\mathbb{N}\setminus F$ such that m is connected to n if $f(m) = n$. Show that each connected component of Γ has at most one cycle. Use it to subdivide vertices of Γ into three sets S_1, S_2, and S_3 such that $f(S_i) \cap S_i = \emptyset$ for each i.

Conclude that $\omega(S_1) = \omega(S_2) = \omega(S_3) = 0$ and hence

$$\omega(F) = \omega(\mathbb{N} \setminus (S_1 \cup S_2 \cup S_3)) = 1.$$

Source: The presented proof was given by Robert Solovay [**149**], but the key statement is due to Miroslav Katětov [**93**].

4.6. Choose a nonprincipal ultrafilter ω and set $L(\mathbf{s}) = s_\omega$. It remains to observe that L is linear.

Remark. By this exercise, ω corresponds to a vector in $(\ell^\infty)^* \setminus \ell^1$.

4.7. Use 4.2.

4.10. ; (*a*). Show that there is $\delta > 0$ such that sides of any geodesic triangle in $\mathbb{M}^2(1)$ intersect a disk of radius δ. Observe that $\mathbb{M}^2(n) = \frac{1}{\sqrt{n}} \cdot \mathbb{M}^2(1)$, and use it to show that any geodesic triangle in $\mathcal{T}$ is a tripod.

(*b*). Observe and use that $\mathbb{M}^2(n)$ are homogeneous.

(*c*). Choose $p_1 \in \mathbb{M}^2(1)$, denote by p_n the corresponding point in $\mathbb{M}^2(n) = \frac{1}{\sqrt{n}} \cdot \mathbb{M}^2(1)$ Suppose $p_n \to p_\omega$ as $n \to \omega$; we can assume that $p_\omega \in \mathcal{T}$. By (*b*), it is sufficient to show that p_ω has a continuum degree.

Choose distinct geodesics $\gamma_1, \gamma_2 : [0,\infty) \to \mathbb{M}^2(1)$ that start at a point p_1. Show that the limits of γ_1 and γ_2 run in the different connected components of $\mathcal{T} \setminus \{p_\omega\}$. Since there is a continuum of distinct geodesics starting at p, we get that the degree of p_ω is at least continuum.

On the other hand, the set of sequences of points $q_n \in \mathbb{M}^2(n)$ has cardinality continuum. In particular, the set of points in $\mathcal{T}$ has cardinality at most continuum. It follows that the degree of any vertex is at most continuum.

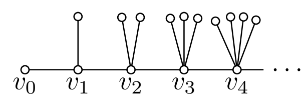
Remark. The properties (b) and (c) describe the tree $\mathcal{T}$ up to isometry [**61**]. In particular, $\mathcal{T}$ does not depend on the choice of the ultrafilter.

4.13. Show and use that the spaces $\mathcal{X}^\omega$ and $(\mathcal{X}^\omega)^\omega$ have discrete metrics and both have cardinality of the continuum.

4.14. Choose a bijection $\iota\colon \mathbb{N} \to \mathbb{N}\times\mathbb{N}$. Given a set $S \subset \mathbb{N}$, consider the sequence $S_1, S_2, \dots$ of subsets in $\mathbb{N}$ defined by $m \in S_n$ if $(m,n) = \iota(k)$ for some $k \in S$. Set $\omega_1(S) = 1$ if and only if $\omega(S_n) = 1$ for ω-almost all n. It remains to check that ω_1 meets the conditions of the exercise.

Comment. It turns out that $\omega_1 \neq \omega$ for any ι; see the post of Andreas Blass [**30**].

4.15. Consider the infinite metric $\mathcal{T}$ tree with unit edges shown on the diagram. Observe that $\mathcal{T}$ is proper.



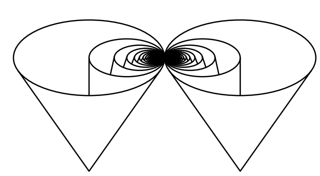
Consider the vertex $v_\omega = \lim_{n\to\omega} v_n$ in the ultrapower $\mathcal{T}^\omega$. Observe that ω has an infinite degree. Conclude that $\mathcal{T}^\omega$ is not locally compact.

4.17. Let $\mathcal{X}_n$ be the square $\{(x,y) \in \mathbb{R}^2, |x| \leqslant 1, |y| \leqslant 1\}$ with the metric induced by the ℓ^n-norm and let $f_n(x,y) = x$ for all n. Observe that $\mathcal{X}_\omega$ is the square with the metric induced by the ℓ^∞-norm where the limit function $f_\omega(x,y) = x$ is not concave.

5.13. ~~;~~ (a)~~.~~ Suppose $\mathcal{X}_n \xrightarrow[\text{GH}]{} \mathcal{X}$ and $\mathcal{X}_n$ are simply connected length metric spaces. It is sufficient to show that any nontrivial covering map $f : \tilde{\mathcal{X}} \to \mathcal{X}$ corresponds to a nontrivial covering map $f_n : \tilde{\mathcal{X}}_n \to \mathcal{X}_n$ for large n.

The latter can be constructed by covering $\mathcal{X}_n$ by small balls that lie close to sets in $\mathcal{X}$ evenly covered by f, preparing a few copies of these sets and gluing them in the same way as the inverse images of the evenly covered sets in $\mathcal{X}$ are glued to obtain $\tilde{\mathcal{X}}$.

(b)~~.~~ Let $\mathcal{V}$ be a cone over Hawaiian earrings. Consider the *doubled cone* $\mathcal{W}$—two copies of $\mathcal{V}$ with their base points glued (see the diagram).



The space $\mathcal{W}$ can be equipped with a length metric (for example, the induced length metric from the shown embedding).

Show that $\mathcal{V}$ is simply connected, but $\mathcal{W}$ is not; the latter is a good exercise in topology.

If we delete from the earrings all small circles, then the obtained double cone becomes simply connected and it remains close to $\mathcal{W}$. That is, $\mathcal{W}$ is a Gromov–Hausdorff limit of simply connected spaces.

Remark. Note that from part (b), the limit does not admit a nontrivial covering. So, if we define the fundamental group as the inverse image of groups of deck transformations for all the coverings of the given space, then one may say that a Gromov–Hausdorff limit of simply connected length spaces is simply connected.

5.14. (a). Suppose that a metric on $\mathbb{S}^2$ is close to the disk $\mathbb{D}^2$. Note that $\mathbb{S}^2$ contains a circle γ that is close to the boundary curve of $\mathbb{D}^2$. By the Jordan curve theorem, γ divides $\mathbb{S}^2$ into two disks, say D_1 and D_2.

By 5.13a, the Gromov–Hausdorff limits of D_1 and D_2 have to contain the whole $\mathbb{D}^2$, otherwise the limit would admit a nontrivial covering.

Consider points $p_1 \in D_1$ and $p_2 \in D_2$ that are close to the center of $\mathbb{D}^2$. If n is large, the distance $|p_1 - p_2|_n$ has to be very small. On the other hand, any curve from p_1 to p_2 must cross γ, so it has length about 2—a contradiction.

(b). Make holes in the unit 3-disc, that do not change its topology and do not change its length metric much and pass to its doubling in the boundary.

Source: The exercise is taken from [**37**].

5.15. Modify proof of 5.12, or apply 5.16b.

6.6. If $\measuredangle\left[p\,{}^x_z\right] + \measuredangle\left[p\,{}^y_z\right] < \pi$, then by the triangle inequality for angles (6.5) we have $\measuredangle\left[p\,{}^x_y\right] < \pi$. The latter implies that $[xy]$ fails to be minimizing near p.

6.10. By the definition of a right derivative, there is a geodesic γ such that both limits

$$\varlimsup_{\varepsilon\to0+}\frac{|\alpha(\varepsilon)-\gamma(\varepsilon)|_{\mathcal{X}}}{\varepsilon}\quad\text{and}\quad\varlimsup_{\varepsilon\to0+}\frac{|\beta(\varepsilon)-\gamma(\varepsilon)|_{\mathcal{X}}}{\varepsilon}$$

are arbitrarily small. By the triangle inequality, we get

$$\varlimsup_{\varepsilon\to0+}\frac{|\alpha(\varepsilon)-\beta(\varepsilon)|_{\mathcal{X}}}{\varepsilon}=0.$$

6.13. Follows directly from the definition.

6.14. Observe that

$$\operatorname{speed}_t\alpha = |\alpha^+(t)| = |\alpha^-(t)|.$$

Apply Theorem 3.10 to show that

$$|\alpha^+(t)-\alpha^-(t)|_{\mathrm{T}_{\alpha(t)}} = 2\cdot\operatorname{speed}_t\alpha.$$

7.10. Choose two non-Euclidean norms $\|{*}\|_{\mathcal{X}}$ and $\|{*}\|_{\mathcal{Y}}$ on $\mathbb{R}^{10}$ such that the sum $\|{*}\|_{\mathcal{X}} + \|{*}\|_{\mathcal{Y}}$ is Euclidean. See [**145**] for more details.

8.3. Assume $|p - x^i| = |q - y^i|$ for each i. Observe and use that

$$|x^i - x^j| \leqslant |y^i - y^j| \quad \iff \quad \tilde\measuredangle^\kappa\left(p\,{}^{x^i}_{x^j}\right) \leqslant \tilde\measuredangle^\kappa\left(q\,{}^{y^i}_{y^j}\right).$$

8.3. Apply the four-point comparison (8.1).

8.8. Modify the induced length metric on the unit sphere in an infinite-dimensional Hilbert space in small neighborhoods of a countable collection of points. To prove that the obtained space is CBB(0), you may need to use the technique from Halbeisen's example (13.6).

8.12. Mimic the proof of Theorem 8.11.

8.13. On the plane, any nonnegatively curved metric having an everywhere dense set of singular points will do the job, where by singular point we mean a point having total angle around it strictly smaller than $2 \cdot \pi$.

Indeed, if x_i is a singular point, then there is $0 < \varepsilon_i < 1/20$ such that no geodesic with ends outside of $\mathrm{B}(x_i, r)$ can meet the ball $\mathrm{B}(x_i, \varepsilon_i \cdot r)$. The set

$$\Omega_n = \bigcup_i \mathrm{B}(x_i, \tfrac{\varepsilon_i}{n})$$

is open and everywhere dense. Note that Ω_n may intersect a geodesic of length $1/n$ only within $\frac{1}{10n}$ of its endpoints. The intersection of the Ω_n is a G-delta dense set that does not intersect the interior of any geodesic.

8.15. Note that rescaling does not change the space. Therefore if the space is CBB(κ) then it is CBB($\lambda \cdot \kappa$) for any $\lambda > 0$. Passing to the limit as $\lambda \to 0$, we may assume that the space is CBB(0).

The point-on-side comparison (8.14*b*) for $p = v$, $x = w$, $y = -w$ and $z = 0$ implies that

$$\|v + w\|^2 + \|v - w\|^2 \leqslant 2 \cdot \|v\|^2 + 2 \cdot \|w\|^2.$$

Applying the comparison for $p = v + w$, $x = w - v$, $y = v - w$ and $z = 0$ gives the opposite inequality. That is, the parallelogram identity

$$\|v + w\|^2 + \|v - w\|^2 = 2 \cdot \|v\|^2 + 2 \cdot \|w\|^2$$

holds for any vectors v and w. Whence the statement follows.

8.16. Apply the hinge comparison (8.14*c*).

8.18. Without loss of generality, we may assume that the points x, v, w, y appear on the geodesic $[xy]$ in that order. By the point-on-side comparison (8.14b) we have

$$\begin{aligned}\tilde\measuredangle^\kappa(x\,{}^{y}_{p})\leqslant\tilde\measuredangle^\kappa(x\,{}^{w}_{p})\leqslant\tilde\measuredangle^\kappa(x\,{}^{v}_{p}),\\ \tilde\measuredangle^\kappa(y\,{}^{w}_{p})\geqslant\tilde\measuredangle^\kappa(y\,{}^{v}_{p})\geqslant\tilde\measuredangle^\kappa(y\,{}^{x}_{p}).\end{aligned}$$

Therefore

$$\begin{aligned}\tilde\measuredangle^\kappa(x\,{}^{y}_{p})<\tilde\measuredangle^\kappa(x\,{}^{w}_{p})\quad&\Longrightarrow\quad\tilde\measuredangle^\kappa(x\,{}^{y}_{p})<\tilde\measuredangle^\kappa(x\,{}^{v}_{p}),\\ \tilde\measuredangle^\kappa(y\,{}^{x}_{p})<\tilde\measuredangle^\kappa(y\,{}^{w}_{p})\quad&\Longleftarrow\quad\tilde\measuredangle^\kappa(y\,{}^{x}_{p})<\tilde\measuredangle^\kappa(y\,{}^{v}_{p}).\end{aligned}$$

By Alexandrov's lemma (6.3), we have

$$\begin{aligned}\tilde\measuredangle^\kappa(x\,{}^{y}_{p})<\tilde\measuredangle^\kappa(x\,{}^{v}_{p})\quad&\Longleftrightarrow\quad\tilde\measuredangle^\kappa(y\,{}^{x}_{p})<\tilde\measuredangle^\kappa(y\,{}^{v}_{p}),\\ \tilde\measuredangle^\kappa(x\,{}^{y}_{p})<\tilde\measuredangle^\kappa(x\,{}^{w}_{p})\quad&\Longleftrightarrow\quad\tilde\measuredangle^\kappa(y\,{}^{x}_{p})<\tilde\measuredangle^\kappa(y\,{}^{w}_{p}).\end{aligned}$$

Hence the statement follows.

8.19. See the construction of Urysohn's space [**71**, $3.11\tfrac{3}{2}{}_{+}$] or [**132**].

8.20. Read [**100**].

8.21. Apply the angle-sidelength monotonicity (8.17) twice.

8.22. The first part follows from the angle-sidelength monotonicity (8.17). An example for the second part can be found among metrics on $\mathbb{R}^2$ induced by a norm. (Compare to Exercise 8.15.)

Remark. This exercise is inspired by Busemann's definition [**45**].

8.25 and 9.28. ; (a). By the function comparison definitions of CBB(0) space (8.23b), for any $p\in\mathcal{L}$ and $\varepsilon>0$ the function dist_p is ε-concave everywhere sufficiently far from p. Applying the definition of Busemann function we get the result.

The CAT(0) case is analogous; we have to apply (9.25b) and use ε-convexity.

(b). By the definition of Busemann function (see 6.1),

$$\begin{aligned}\exp(\operatorname{bus}_\gamma)&=\lim_{t\to\infty}\exp(\operatorname{dist}_{\gamma(t)}-t)\\&=\lim_{t\to\infty}\left[\exp(\operatorname{dist}_{\gamma(t)}-t)+\exp(-\operatorname{dist}_{\gamma(t)}-t)\right]\\&=\lim_{t\to\infty}\left(2\cdot\exp(-t)\cdot\cosh\circ\operatorname{dist}_{\gamma(t)}\right).\end{aligned}$$

By the function comparison definitions of CAT(κ) space (9.25b) or CBB(κ) space (8.23b), for any $p\in\mathcal{U}$ the function $f=\cosh\circ\operatorname{dist}_p$ satisfies $f''+\kappa\cdot f\geqslant 1$ (respectively $f''+\kappa\cdot f\leqslant 1$). The result follows.

8.32. Read [**127**].

8.46. If $\operatorname{diam}(\mathcal{L}/G) > \frac{\pi}{2}$, then for some $x \in \mathcal{L}$ we have

$$\sup\{\operatorname{dist}_{G\cdot x}(y) : y \in \mathcal{L}\} > \tfrac{\pi}{2}.$$

Use comparison to show that there is a unique point y^* that lies at maximal distance from the orbit $G \cdot x$. Observe that y^* is a fixed point.

8.47. Assume there are 4 such points x_1, x_2, x_3, x_4. Since the space $\mathcal{L}$ is CBB(1), it is also CBB(0). By the angle comparison, the sum of the angles in any geodesic triangle in an CBB(0) space is $\geqslant \pi$. Therefore the average of the $\measuredangle\left[x_i\,{}^{x_j}_{x_k}\right]$ is larger than $\frac{\pi}{3}$. On the other hand, since each x_i has space of directions $\leqslant \frac{1}{2}\cdot\mathbb{S}^n$ and the perimeter of any triangle in $\frac{1}{2}\cdot\mathbb{S}^n$ is at most π, the average of $\measuredangle\left[x_i\,{}^{x_j}_{x_k}\right]$ is at most $\frac{\pi}{3}$—a contradiction.
Source: Based on the main idea in [**83**].

9.4. Suppose that

$$\tilde\measuredangle^\kappa\left(x^0\,{}^{x^1}_{x^2}\right) + \tilde\measuredangle^\kappa\left(x^0\,{}^{x^2}_{x^3}\right) < \tilde\measuredangle^\kappa\left(x^0\,{}^{x^1}_{x^3}\right).$$

Show that

$$\tilde\measuredangle^\kappa\left(x^2\,{}^{x^0}_{x^1}\right) + \tilde\measuredangle^\kappa\left(x^2\,{}^{x^1}_{x^3}\right) + \tilde\measuredangle^\kappa\left(x^2\,{}^{x^3}_{x^0}\right) > 2\cdot\pi.$$

Conclude that one can take $p = x^2$.

9.5. Modify the configuration in 9.2.

9.6. Read [**144**]; the original proof [**25**] is harder to follow.

An example for the second part of the problem can be found among 4-point metric spaces. It is sufficient to take four vertices of a generic convex quadrangle and increase one of its diagonals slightly; it will still satisfy the inequality for all relabeling but will fail to meet 9.2.

9.9. Suppose that a geodesic $[px]$ is not extendable beyond x. We may assume that $|p - x| < \varpi\kappa$; otherwise move p along the geodesic toward x.

By the uniqueness of geodesics (9.8), any point y in a neighborhood $\Omega \ni x$ is connected to p by a unique geodesic path; denote it by γ_y. Note that $h_t(y) = \gamma_y(t)$ defines a homotopy, called the *geodesic homotopy*, between the identity map of Ω and the constant map with value p.

Since $[px]$ is not extendable, $x \notin h_t(\Omega)$ for any $t < 1$. In particular, the local homology groups vanish at x—a contradiction.

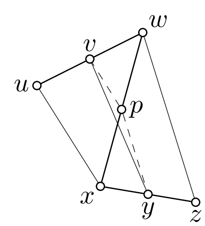
9.10. Choose a sequence of directions ξ_n at p; by $\gamma_n : \mathbb{R} \to \mathcal{U}$ the corresponding local geodesics. Since the space $\mathcal{U}$ is locally compact, we may pass to a converging subsequence of (γ_n); it's limit is a local geodesic by Corollary 9.22. Denote the limit by γ_∞ and its direction by ξ_∞. By comparison, ξ_∞ is a limit of (ξ_n).

9.16. Follow the solution in the 8.15, reversing all the inequalities.

9.17. It is sufficient to show that if v and y are midpoints of geodesics $[uw]$ and $[xz]$ in $\mathcal{U}$, then

$$|v - y| \leqslant \tfrac{1}{2} \cdot (|u - x| + |w - z|).$$

Denote by p the midpoint of $[uz]$. Applying the angle-sidelength monotonicity (9.15) twice, we have



$$|v - p| \leqslant \tfrac{1}{2} \cdot |w - z|.$$

Similarly we have

$$|y - p| \leqslant \tfrac{1}{2} \cdot |u - x|.$$

It remains to add these two inequalities and apply the triangle inequality.

Remark. This inequality also follows directly from the majorization theorem (9.56).

9.20. The only-if part is evident. Use 9.14 to show that $(c) \Rightarrow (b) \Rightarrow (a)$. By 9.14, condition (a) implies that the natural map is distance-preserving on the sides $[\tilde{x}\tilde{y}]$ and $[\tilde{x}\tilde{z}]$. Applying it again, we have that condition (a) holds for all permutations of the labels x, y, z. Whence the natural map is distance-preserving on all three sides.

Remark. These conditions imply that the natural map can be extended to a distance-preserving map to the solid model triangle. In fact the image of the line-of-sight map (9.32) is isometric to the model triangle.

9.28. See the solution of Exercise 8.25.

9.33. (a) Suppose that $x_n \to x_\infty$, $y_n \to y_\infty$ as $n \to \infty$, but $[x_n y_n]$ does not converge to $[x_\infty y_\infty]$. Since the space is proper, we can pass to a subsequence such that $[x_n y_n]$ converges to another geodesic. That is, we have at least two geodesics between x_∞ and y_∞.

(b) Let Δ_n be a sequence of solid spherical triangles with angle $\frac{\pi}{4}$ and adjacent sides $\pi - \frac{1}{n}$. Let us glue each Δ_n to $[0, \pi]$ along an isometry of one of the longer sides. It remains to show that the obtained space $\mathcal{X}$ is a needed example.

Source: The example (b) is taken from [**34**, Chapter I, Exercise 3.14].

9.41. Subdivide Q into a a half-plane A bounded by the extension of γ_1 and the remaining solid angle B; it has angle measure measure $\pi-\alpha$. First glue B along γ_2, and then glue A. Each time apply the Reshetnyak gluing theorem (9.39), to show that the obltained space is CAT(0).

9.42. Suppose that A is not convex. Then there is a geodesic $[xy]$ with ends in A that does not lie in A completely. Note that $[xy]$ can be lifted to two different geodesics with the same ends in the doubling, and apply uniqueness of geodesics (9.13).

9.43. Since K is π-convex, it is CAT(1). By 11.7, the spherical suspension $\mathrm{Susp}\,K$ is CAT(1) as well. Let us glue $\mathrm{Susp}\,K$ to $\mathcal{U}$ along K; according to the Reshetnyak gluing theorem, the resulting space, say $\mathcal{U}'$, is CAT(1).

Consider the geodesic path $\gamma\colon\ [0,1]$ from p to a pole of the suspension in $\mathcal{U}'$. Set $K_t=\mathcal{U}\cap\overline{\mathrm{B}}[\gamma(t),\tfrac{\pi}{2}]$. By 9.27, K_t is π-convex for any t; monotonicity and continuity of the family should be evident.

Source: This construction was used in [**106**]. Applying it together with Sharafutdinov retraction leads to another solution of Exercise 9.75.

9.44. Apply the Reshetnyak gluing theorem, or its reformulation 9.40.

9.59. By 9.56, there is a majorization $F\colon\ D\to\mathcal{U}$ of the polygonal line β. Show and use that D is a convex plane polygon and its external angles cannot exceed the corresponding external angle of β.

9.60. This exercise generalizes the so-called Fáry–Milnor theorem. An elementary proof is given in the first author and Richard Bishop [**4**]; another proof is given by Stephan Stadler [**150**].

9.61. *(Easier way.)* Let $(t,s)\mapsto\gamma_t(s)$ be the line-of-sight map for α with respect to $\alpha(0)$, and $(t,s)\mapsto\tilde\gamma_t(s)$ be the line-of-sight map for $\tilde\alpha$ with respect to $\tilde\alpha(0)$. Consider the map $F\colon\ \mathrm{Conv}\,\tilde\alpha\to\mathcal{U}$ such that $F\colon\ \tilde\gamma_t(s)\mapsto\gamma_t(s)$.

Show that F majorizes α and conclude that F is distance-preserving.

(Harder way.) Prove and apply the following statement together with the Majorization theorem.

- Let α and β be two convex curves in $\mathbb{M}^2(\kappa)$. Assume

$$\operatorname{length}\alpha=\operatorname{length}\beta<2\cdot\varpi\kappa$$

  and there is a short bijecction $f\colon\ \alpha\to\beta$. Then f is an isometry.

9.62. Suppose that points p, x, q, y appear on the curve in that cyclic order. Assume that the geodesics $[pq]$ and $[xy]$ do not intersect. Use the argument in the proof of the majorization theorem (9.56) to show that in this case there are nonequivalent majorization maps.

Now we can assume that pairs of geodesics $[pq]$ and $[xy]$ intersect for all choices of points p, x, q, y on the curve in that cyclic order. Show that in this case the convex hull K of the curve is isometric to a convex figure.

Note that the composition of a majorization map and closest point projection to K is a majorization. Show and use that the boundary of a convex figure in the plane admits a unique majorization up to equivalence.

Remark. A typical rectifiable closed curve in a CAT(0) space can be majorized by more than one convex figure. There are two exceptions: (1) the majorization map is distance-preserving, and (2) the curve is geodesic triangle. It is expected that there are no other exceptions; this question was asked by Richard Bishop in a private conversation.

9.64. Show that quadrangle $[x^1x^2x^3x^4]$ is majorized by the solid quadrangle $[\tilde{x}^1\tilde{x}^2\tilde{x}^3\tilde{x}^4]$. Further show that the majorization is isometric; argue as in 8.18.

9.70. If ℓ and m do not intersect, then the double cover $\mathcal{X}$ is not simply connected. In particular, by the Hadamard–Cartan theorem, $\mathcal{X}$ is not CAT(0).

If ℓ and m intersect then $\mathcal{X}$ is a cone over a double cover Σ of $\mathbb{S}^2$ branching at two pairs (x, y) and (v, w) of antipodal points. Suppose $|x - v|_{\mathbb{S}^2} = \ell < \frac{\pi}{2}$. Note that the inverse image of $[xv]_{\mathbb{S}^2}$ is a closed geodesic of length $4 \cdot \ell < 2 \cdot \pi$. Therefore, by the generalized Hadamard–Cartan theorem, Σ is not CAT(1). Hence $\mathcal{X}$ is not CAT(0) by Theorem 11.7 on curvature of cones.

9.71. Let us do the second part first. Assume A has nonempty interior. Note that the space $\tilde{\mathcal{U}}$ is simply connected and locally isometric to the doubling $\mathcal{W}$ of $\mathcal{U}$ in A; that is, any point in $\tilde{\mathcal{U}}$ has a neighborhood that is isometric to a neighborhood of a point in $\mathcal{W}$.

By the Reshetnyak gluing theorem (9.39), $\mathcal{W}$ is CAT(0). Therefore $\tilde{\mathcal{U}}$ is locally CAT(0); it remains to apply the Hadamard–Cartan theorem (9.65).

Let us come back to the general case. The above argument can be applied to a closed ε-neighborhood of A. After that we need to pass to a limit as $\varepsilon \to 0$.

The first part of the problem follows since a geodesic is a convex set.

9.74. Let $p \mapsto \bar{p}$ denote the closest-point projection to K. We need to show that $|\bar{p} - \bar{q}| \leqslant |p - q|$ for any $p, q \in \mathcal{U}$.

Assume $p \neq \bar{p} \neq \bar{q} \neq q$. Note that in this case $\measuredangle\left[\bar{p}\,{}^{p}_{\bar{q}}\right] \geqslant \frac{\pi}{2}$ and $\measuredangle\left[\bar{q}\,{}^{q}_{\bar{p}}\right] \geqslant \frac{\pi}{2}$. Otherwise a point on the geodesic $[\bar{p}\bar{q}]$ would be closer to p or to q than $\bar{p}$ or $\bar{q}$, respectively. The latter is impossible since K is convex and therefore $[\bar{p}\bar{q}] \subset K$.

Applying the arm lemma (9.63), we get the statement.

The cases $p = \bar{p} \neq \bar{q} \neq q$ and $p \neq \bar{p} \neq \bar{q} = q$ can be done similarly. The rest of the cases are trivial.

9.75. A more transparent, but less elementary solution via gradient flow is given by Alexander Lytchak and the third author [**106**].

Without loss of generality, we may assume that $p \in K$.

If $\operatorname{dist}_K x \geqslant \pi$, then set $\Psi(x) = p$.

Otherwise, if $\operatorname{dist}_K x < \pi$, by the closest-point projection lemma (9.73), there is a unique point $x^* \in K$ that minimizes distance to x; that is, $|x^* - x| = \operatorname{dist}_K x$. Let us define ℓ_x, ϕ_x and ψ_x using the following identities:

$$\begin{aligned}\ell_x &= |p - x^*|,\\ \phi_x &= \tfrac{\pi}{2} - |x^* - x|,\\ \sin\psi_x &= \sin\phi_x \cdot \sin\ell_x, \quad 0 \leqslant \psi_x \leqslant \tfrac{\pi}{2}.\end{aligned}$$

Let

$$\Psi(x) = \operatorname{geod}_{[px^*]}(\psi_x).$$

Note that Ψ is a retraction to K; that is, $\Psi(x) \in K$ for any $x \in \mathcal{U}$ and $\Psi(a) = a$ for any $a \in K$.

Let us show that Ψ is short. Given $x, y \in \mathrm{B}(K, \frac{\pi}{2})$, let

$$\boxed{\begin{aligned} x' &= \Psi(x) & \qquad y' &= \Psi(y)\\ r &= |x - y| & r' &= |x' - y'|\\ d &= |x^* - y^*| & \alpha &= \tilde{\measuredangle}^1\left(p\,{}^{x^*}_{y^*}\right).\end{aligned}}$$

Note that

$$\cos r \leqslant \cos\phi_x \cdot \cos\phi_y - \cos d \cdot \sin\phi_x \cdot \sin\phi_y. \tag{1}$$

Indeed, if $x, y \notin K$, then $\measuredangle\left[x^*\,{}^{x}_{y*}\right], \measuredangle\left[y^*\,{}^{y}_{x*}\right] \geqslant \frac{\pi}{2}$ and ~~the~~ inequality 1 follows from the arm lemma (9.63). If $x \in K$ and $y \notin K$, we obtain 1 by the angle comparison (9.14c) since $\measuredangle\left[y^*\,{}^{y}_{x*}\right] \geqslant \frac{\pi}{2}$. In the same way, 1 is proved if $x \notin K$ and $y \in K$. Finally, if $x, y \in K$, then $\phi_x = \phi_y = \frac{\pi}{2}$ and $r = d$; that is, the inequality trivially holds.

Further note that

$$\cos\alpha = \frac{\cos d - \cos\ell_x \cdot \cos\ell_y}{\sin\ell_x \cdot \sin\ell_y}.$$

Applying the angle-sidelength monotonicity (9.15), we have

$$\begin{aligned}\cos r' &\geqslant \cos\psi_x\cdot\cos\psi_y - \cos\alpha\cdot\sin\psi_x\cdot\sin\psi_y\\ &= \cos\psi_x\cdot\cos\psi_y - (\cos d - \cos\ell_x\cdot\cos\ell_y)\cdot\sin\phi_x\cdot\sin\phi_y\\ &\geqslant \cos\psi_x\cdot\cos\psi_y - \cos d\cdot\sin\phi_x\cdot\sin\phi_y.\end{aligned}$$

Note that $\psi_x \leqslant \phi_x$ and $\psi_y \leqslant \phi_y$; in particular,

$$\cos\phi_x\cdot\cos\phi_y \leqslant \cos\psi_x\cdot\cos\psi_y.$$

Hence

$$\cos r' \geqslant \cos r;$$

that is, the restriction $\Psi|_{\mathrm{B}(K,\frac{\pi}{2})}$ is short. Clearly Ψ is continuous. Since the complement of $\mathrm{B}(K,\frac{\pi}{2})$ is mapped to p, Ψ is short; that is,

$$r' \leqslant r \tag{2}$$

for any $x, y \in \mathcal{U}$.

If we have equality in 2 then

$$\cos\ell_x\cdot\cos\ell_y\cdot\sin\phi_x\cdot\sin\phi_y = 0.$$

If $K \subset \mathrm{B}(p,\frac{\pi}{2})$, then $\ell_x, \ell_y < \frac{\pi}{2}$, which implies that $x \in K$ or $y \in K$. Without loss of generality, we may assume that $x \in K$.

It remains to show that if $y \notin K$ then the inequality 2 is strict. If $\operatorname{dist}_K y \geqslant \frac{\pi}{2}$, then 2 holds since the left-hand side is $< \frac{\pi}{2}$ while the right-hand side is $\geqslant \frac{\pi}{2}$. If $\operatorname{dist}_K y < \frac{\pi}{2}$, then $\phi_y > 0$. Clearly $\psi_y < \phi_y$, hence ~~the~~ inequality 2 is strict.

Below you will find a geometric way to think about the given construction; it is close to the construction in the proof of Kirszbraun's theorem (10.14).

Geometric interpretation of the map Ψ. Let $\mathring{\mathcal{U}} = \operatorname{Cone}\mathcal{U}$, and denote by $\mathring{K}$ the subcone of $\mathring{\mathcal{U}}$ spanned by K. The space $\mathcal{U}$ can be naturally identified with the unit sphere in $\mathring{\mathcal{U}}$, that is, the set

$$\left\{ z \in \mathring{\mathcal{U}} : |z| = 1 \right\}.$$

According to 11.7, $\mathring{\mathcal{U}}$ is CAT(0). Note that $\mathring{K}$ forms a convex closed subset of $\mathring{\mathcal{U}}$. According to 9.73, for any point x there is a unique point $\hat{x} \in \mathring{K}$ that minimizes the distance to x, that is, $|\hat{x} - x| = \operatorname{dist}_K x$. (If $|\hat{x}| \neq 0$, then in the notation above we have $x^* = \frac{1}{|\hat{x}|}\cdot\hat{x}$.)

Consider the half-line $t \mapsto t\cdot p$ in $\mathring{\mathcal{U}}$. By comparison, for given $s \in \mathring{\mathcal{U}}$ the geodesics $\operatorname{geod}_{[s\, t\cdot p]}$ converge as $t \to \infty$ to a half-line, say $\alpha_s\colon [0,\infty) \to \mathring{\mathcal{U}}$.

Note that if $|x| = 1$, then $|\hat{x}| \leqslant 1$. By assumption, for any $a \in K$ the function $t \mapsto |\alpha_a(t)|$ is monotonically increasing. Therefore there is a unique value $t_x \geqslant 0$ such that $|\alpha_{\hat{x}}(t_x)| = 1$. Define $\Psi\colon\ \mathcal{U} \to K$ by

$$\Psi(x) = \alpha_{\hat{x}}(t_x).$$

9.80. Prove that the angle comparison (9.14c) holds.

9.81. Mimic the proof of the Hadamard–Cartan theorem.

10.5. Note that it is sufficient to show that any finite set of points $x^1, \dots, x^n \in \mathcal{X}$ lies in an isometric copy of a Euclidean polyhedron.

Observe that $\mathcal{X}$ is CBB(0) and CAT(0) at the same time. Show that there is a unique point p that minimizes the sum $|p - x^1| + \dots + |p - x^n|$. Note that the vectors $v^i = \log[px^i]$ lie in a linear subspace of T_p. Moreover if K is the convex hull of v_i, then the origin of T_p lies in the interior of K relative to its affine hull. Finally observe that the exponential map is defined on all of K and is distance-preserving. The statement follows since the exponential map sends $v^i \mapsto x^i$ for each i.

10.6. The answers are $s \leqslant \sqrt{3}$ and $s \leqslant 2$ respectively.

Let us start with the CAT(0) case. The upper bound $s \leqslant \sqrt{3}$ follows from (2+2)-point comparison. The Euclidean space works as an example if s is smaller than the large diagonal of the double pyramid with unit side (that is, if $s \leqslant 2 \cdot \sqrt{2/3}$). Otherwise it can be embedded into a product of the real line with a two-dimensional cone.

For the CBB(0) case, the needed space can be constructed by doubling a polyhedron $K \subset \mathbb{E}^3$ in its boundary. The obtained space is CBB(0) by 12.5; the same follows from Perelman's doubling theorem [**125**]. We assume that the points correspond to vertices of a regular tetrahedron with 3 vertices on the boundary of K and one in its interior; this point corresponds to a pair of points in the doubling at distance s from each other.

Remark. The CAT(0) case also follows from [**104**, **155**].

10.7. Choose a quadruple of points p, q, r, s. Suppose that it admits a distance-preserving embedding into some $\mathbb{M}^2(\mathrm{K})$ for some $\mathrm{K} \geqslant \kappa$. Then

$$\tilde{\measuredangle}^{\mathrm{K}}\left(p\,{}^{q}_{r}\right) + \tilde{\measuredangle}^{\mathrm{K}}\left(p\,{}^{r}_{s}\right) + \tilde{\measuredangle}^{\mathrm{K}}\left(p\,{}^{a}_{q}\right) \leqslant 2 \cdot \pi.$$

Applying monotonicity of the function $\kappa \mapsto \tilde{\measuredangle}^{\kappa}\left(p\,{}^{q}_{r}\right)$ (1.1d) shows that

$$\tilde{\measuredangle}^{\kappa}\left(p\,{}^{q}_{r}\right) + \tilde{\measuredangle}^{\kappa}\left(p\,{}^{r}_{s}\right) + \tilde{\measuredangle}^{\kappa}\left(p\,{}^{s}_{q}\right) \leqslant 2 \cdot \pi.$$

Since the quadruple p, q, r, s is arbitrary, the if part follows.

Now let us prove the only-if part. Denote by σ the exact upper bound on values $\mathrm{K} \geqslant \kappa$ such that all model triangles with the vertices p, q, r, s are defined.

Recall that $\tilde\measuredangle^{\mathrm{K}+}\left(p\,{}^{q}_{r}\right)$ denotes extended angle (8.49). Observe that if

$$\tilde\measuredangle^{\mathrm{K}+}\left(p\,{}^{q}_{r}\right)+\tilde\measuredangle^{\mathrm{K}+}\left(p\,{}^{r}_{s}\right)+\tilde\measuredangle^{\mathrm{K}+}\left(p\,{}^{s}_{q}\right)=2\cdot\pi \tag{3}$$

for some $\sigma\geqslant\mathrm{K}\geqslant\kappa$, then the quadruple admits a distance-preserving embedding into $\mathbb{M}^2(\mathrm{K})$.

Observe that the left-hand side of 3 is continuous in K. Since $\mathcal{L}$ is CBB(κ), for $\mathrm{K}=\kappa$ the left-hand side cannot exceed $2\cdot\pi$. Therefore it remains smaller than $2\cdot\pi$ for all $\sigma\geqslant\mathrm{K}\geqslant\kappa$; moreover the same holds for all permutations of the labels p,q,r,s.

Note that we can assume the perimeter of the triple q,r,s is $2\cdot\varpi\sigma$, and use this and the overlap lemma (10.2) to arrive at a contradiction.

According to our definition, the real line is CBB(κ) for any $\kappa\in\mathbb{R}$, but it does not satisfy the property for $\kappa>0$. The condition $\kappa\leqslant0$ was used just once to ensure that the κ-model triangles with the vertices p,q,r,s are defined. One can assume instead that perimeters of all triangles in $\mathcal{L}$ are at most $2\cdot\varpi\kappa$. This condition holds for all complete length CBB(κ) spaces of dimension at least 2; see 8.44.

10.9. Let $\tilde p,\tilde x_1,\dots,\tilde x_n$ be the array in $\mathbb{E}^n$ provided by the $(1+n)$-point comparison (10.8). We may assume that $\tilde p$ is the origin of $\mathbb{E}^n$.

Consider an $n{\times}n$-matrix $\tilde M$ with components

$$\tilde m_{i,j}=\tfrac12\cdot(|\tilde x_i-\tilde p|^2+|\tilde x_j-\tilde p|^2-|\tilde x_i-\tilde x_j|^2).$$

Note that $\tilde m_{i,j}=\langle\tilde x_i,\tilde x_j\rangle$. It follows that $\tilde M=A\cdot A^\top$ for an $n{\times}n$-matrix A that defines a linear transformation sending the standard basis to the array $\tilde x_1,\dots,\tilde x_n$. Therefore

$$\mathbf{s}\cdot\tilde M\cdot\mathbf{s}^\top=|A^\top\cdot\mathbf{s}^\top|^2\geqslant0$$

for any vector $\mathbf{s}$. Further show and use that

$$\mathbf{s}\cdot M\cdot\mathbf{s}^\top\geqslant\mathbf{s}\cdot\tilde M\cdot\mathbf{s}^\top$$

for any vector $\mathbf{s}=(s_1,\dots,s_n)$ with nonnegative components.

10.10. Apply the (5+1)-point comparison (10.8).

10.11. It is sufficient to construct a metric on the set of points $\{p,x^1,x^2,x^3,x^4\}$ that does not satisfy (1+4)-point comparison but does satisfy all (1+3)-point comparisons. To do this, set x^i to be the vertices of a regular tetrahedron in $\mathbb{E}^3$. Suppose p is its center and reduce the distances $|p-x^i|$ slightly.

Remark. There are examples of 6-point metric spaces that satisfy all (1+5)-point comparisons, but do not admit embedding into a complete length CBB(0) space [**102**].

10.12. By the $(1+n)$-point comparison (10.8), there is a point array $\tilde p$, $\tilde a^0, \dots, \tilde a^m \in \mathbb{M}^{m+1}(\kappa)$ such that

$$|\tilde p - \tilde a^i| = |p - a^i| \quad \text{and} \quad |\tilde a^i - \tilde a^j| \geqslant |a^i - a^j|$$

for all i and j.

For each i, set $\tilde\xi^i = \uparrow_{[\tilde p \tilde a^i]} \in \mathbb{S}^m = \Sigma_{\tilde p}(\mathbb{M}^{m+1}(\kappa))$. Note that

$$|\tilde\xi^i - \tilde\xi^j|_{\mathbb{S}^m} \geqslant \tilde\measuredangle^\kappa\left(p\,{}^{a^i}_{a^j}\right) > \tfrac{\pi}{2}.$$

Consider two matrices S and $\tilde S$ with components $s_{i,j} = \langle \tilde\xi^i, \xi^j\rangle$ and $\tilde s_{i,j} = \cos[\tilde\measuredangle^\kappa\left(p\,{}^{a^i}_{a^j}\right)]$. By construction, $S \geqslant 0$; that is $\mathbf{v}\cdot S\cdot \mathbf{v}^\top \geqslant 0$ for any vector $\mathbf{v}$.

Observe that it is sufficient to show that $\tilde S \geqslant 0$. The latter follows since $s_{i,j} \leqslant \tilde s_{i,j} \leqslant 0$ if $i \ne j$ and $s_{i,j} = \tilde s_{i,j} = 1$ if $i = j$.

10.18. Set $\tilde Q = \mathrm{Conv}\{\tilde x^0, \tilde x^1, \dots, \tilde x^k\}$. By Kirszbraun's theorem, the map $\tilde x^i \mapsto x^i$ can be extended to a short map $F\colon \tilde Q \to \mathcal{L}$; it remains to show that the map F is distance-preserving.

Consider the logarithm map $G\colon x \mapsto \log[x_0 x]$; note that G is short. Observe that the composition $G\circ F$ is distance-preserving. Therefore F is distance-preserving; in particular we can take $Q = F(\tilde Q)$.

10.19. Consider vectors $v^i = \log[x^0 x^i] \in \mathrm{T}_{x^0}$. Show that all the v^i lie in a linear subspace of T_{x^0} and that $x^i \mapsto v^i$ is distance-preserving. It follows that we can identify the convex hull K of the v^i with the convex hull of the $\tilde x^i$.

Note that the gradient exponential map gexp_{x_0} maps v^i to x^i. By assumption,

$$|v^i - v^j| = |x^i - x^j| \tag{4}$$

for all i and j. By 16.36, gexp_{x_0} is a short map. By 4, gexp_{x_0} cannot be strictly short at a pair of points in K. That is, gexp_{x_0} is distance-preserving on K.

10.20. Apply 10.17 for each of the following maps

- $f_0\colon \tilde x \mapsto x$, $\tilde p^1 \mapsto p^1$, $\tilde q^1 \mapsto q^1$;
- $f_i\colon \tilde p^i \mapsto p^i$, $\tilde p^{i+1} \mapsto p^{i+1}$, $\tilde q^i \mapsto q^i$, $\tilde q^{i+1} \mapsto q^{i+1}$ for $1 \leqslant i < n$;
- $f_n\colon \tilde y \mapsto y$, $\tilde p^n \mapsto p^n$, $\tilde q^n \mapsto q^n$.

Denote by F_i the short extension of f_i. Observe and use that $F_{i-1}(\tilde z_i) = F_i(\tilde z_i)$ for each i.

10.23. Consider the space $\mathcal{Y}^{\mathcal{X}}$ of all maps $\mathcal{X} \to \mathcal{Y}$ equipped with the product topology.

Denote by $\mathfrak{S}_F$ the set of maps $h \in \mathcal{Y}^{\mathcal{X}}$ such that the restriction $h|_F$ is short and agrees with f in $F \cap A$. Note that the sets $\mathfrak{S}_F \subset \mathcal{Y}^{\mathcal{X}}$ are closed and any finite intersection of these sets is nonempty.

According to Tikhonov's theorem, $\mathcal{Y}^{\mathcal{X}}$ is compact. By the finite intersection property, the intersection $\bigcap_F \mathfrak{S}_F$ for all finite sets $F \subset X$ is nonempty. Hence the statement follows.

Source: This statement appears in [**126**]; it is an analogous of the finite+one lemma (10.16).

10.24. The Kuratowsky embedding is a distance-preserving map of $\mathcal{X}$ into the space of bounded functions $\mathcal{X}$ equipped with the metric induced by the sup-norm (Section 2G). It remains to show that the latter space is injective.

The second part of the exercise is a classical result of John Isbell [**86**] which was rediscovered several times after him; for more on the subject see lecture notes of the third author [**132**].

11.4. It is sufficient to show that the natural map $\mathcal{B} \times_g \mathcal{F} \to \mathcal{B} \times_f \mathcal{F}$ is short. The latter follows from the fiber-independence theorem (11.3).

11.5. Show and use that any geodesic path in $\mathrm{Cone}^\kappa \mathcal{F}$ projects to a reparametrized geodesic in $\mathcal{F}$ of length less than π.

11.9. By 11.6*a*, the space $\mathcal{U}$, $\mathcal{V}$, or $\mathcal{U} \star \mathcal{V}$ is CBB(1) if and only if $\mathrm{Cone}\,\mathcal{U}$, $\mathrm{Cone}\,\mathcal{V}$, or $\mathrm{Cone}(\mathcal{U} \star \mathcal{V}) = \mathrm{Cone}\,\mathcal{U} \times \mathrm{Cone}\,\mathcal{V}$ is CBB(0) respectively.

By 11.7*a*, the space $\mathcal{U}$, $\mathcal{V}$, or $\mathcal{U} \star \mathcal{V}$ is CAT(1) if and only if $\mathrm{Cone}\,\mathcal{U}$, $\mathrm{Cone}\,\mathcal{V}$, or $\mathrm{Cone}(\mathcal{U} \star \mathcal{V}) = \mathrm{Cone}\,\mathcal{U} \times \mathrm{Cone}\,\mathcal{V}$ is CAT(0) respectively.

It remains to show that the product of two spaces is CBB(0) or CAT(0) if and only if each space is CBB(0) or CAT(0) respectively.

12.3. Apply Reshetnyak gluing theorem (9.39) several times.

12.4. Assume $\mathcal{P}$ is not CAT(0). Then by 12.2, a link Σ of some simplex contains a closed local geodesic α with length $4 \cdot \ell < 2 \cdot \pi$. We can assume that Σ has minimal possible dimension; then by 12.2, Σ is locally CAT(1).

Divide α into two equal arcs α_1 and α_2.

Assume α_1 and α_2 are length-minimizing, and parametrize them by $[-\ell, \ell]$. Fix a small $\delta > 0$ and consider the two curves in $\mathrm{Cone}\,\Sigma$ given in polar coordinates by

$$\gamma_i(t) = \left(\alpha_i\left(\arctan \tfrac{t}{\delta}\right), \sqrt{\delta^2 + t^2}\right).$$

Show that the curves γ_1 and γ_2 are geodesics in $\operatorname{Cone}\Sigma$ having common endpoints.

Observe that a small neighborhood of the tip of $\operatorname{Cone}\Sigma$ admits a distance-preserving embedding into $\mathcal{P}$. Hence we can construct two geodesics γ_1 and γ_2 in $\mathcal{P}$ with common endpoints.

It remains to consider the case where α_1 (and therefore α_2) is not length-minimizing.

Pass to a maximal length-minimizing arc $\bar\alpha_1$ of α_1. Since Σ is locally CAT(1), by the no-conjugate-point theorem (9.46) there is another geodesic $\bar\alpha_2$ in Σ_p that shares endpoints with $\bar\alpha_1$. It remains to repeat the above construction for the pair $\bar\alpha_1, \bar\alpha_2$.

Remark. By 9.8 the converse holds as well. This problem was suggested by Dmitri Burago.

12.6. Apply 13.1, 12.5, and 12.2.

12.11. Observe and use that (1) in the barycentric subdivision every vertex corresponds to a simplex of the original triangulation, and (2) a simplex of the subdivision corresponds to a decreasing sequence of simplexes in the original triangulation.

Remark. The second statement, *any finite simplicial complex is homeomorphic to a compact length* CAT(1) *space*, is due to Valerii Berestovskii [**23**].

12.13. Use induction on the dimension to prove that if in a spherical simplex $\triangle$ every edge is at least $\frac{\pi}{2}$, then all dihedral angles of $\triangle$ are at least $\frac{\pi}{2}$.

The rest of the proof goes along the same lines as the proof of the flag condition (12.10). The only difference is that a geodesic may spend time *at least* π on each visit to Star_v.

Remark. It is not sufficient to assume only that all the dihedral angles of the simplexes are at least $\frac{\pi}{2}$. Indeed, the two-dimensional sphere with the interior of a small rhombus removed is a spherical polyhedral space glued from four triangles with angles at least $\frac{\pi}{2}$. On the other hand, the boundary of the rhombus is a closed local geodesic in this space and has length less than $2\cdot\pi$. Therefore the space cannot be CAT(1).

12.14. Observe that if we glue two copies of spaces along A_i, then the copies of A_j for some $j\ne i$ form a convex subset in the glued space. Use this and the Reshetnyak gluing theorem (9.39) n times, once for each label of the edges.

12.15. The space $\mathcal{T}_n$ has a natural cone structure whose vertex is the completely degenerate tree—all its edges have zero length.

Note that the space Σ over which the cone is taken comes naturally with a triangulation by right-angled spherical simplexes. Each simplex corresponds to the combinatorics of a possibly degenerate tree.

Note that the link of any simplex of this triangulation satisfies the no-triangle condition. Indeed, fix a simplex $\triangle$ of the complex; suppose it is described by a possibly degenerate topological tree t. A triangle in the link of $\triangle$ can be described by three ways to resolve a degeneracy of t by adding one edge, where (1) any pair of these resolutions can be done simultaneously, but (2) all three cannot be done simultaneously. Direct inspection shows that this is impossible.

By Proposition 12.8, our complex is flag. It remains to apply the flag condition (12.10) and 11.6*a*.

12.16. Apply the flag condition (12.10) and Theorem 11.7*a*.

13.8. Consider a cube in the ℓ^2-space defined by $|x_i| \leqslant 1$.

13.13 and 13.14. Apply the strong angle lemmas (9.35 and 8.43).

13.19. Apply 13.13.

13.21. Apply 8.11.

13.27. Since α is Lipschitz, so is $f \circ \alpha$. By the standard Rademacher theorem, the derivative $(f \circ \alpha)'$ is defined almost everywhere. In particular,

$$(\mathbf{d}_{\alpha(t)} f)(\alpha^+(t)) + (\mathbf{d}_{\alpha(t)} f)(\alpha^-(t)) \stackrel{a.e.}{=\!=} 0.$$

Further, by the extended Rademacher theorem (more precisely its 1-dimensional case; see Proposition 13.9), we have

$$\alpha^+(t) + \alpha^-(t) \stackrel{a.e.}{=\!=} 0.$$

In particular,

$$\langle \nabla_{\alpha(t)} f, \alpha^+(t) \rangle + \langle \nabla_{\alpha(t)} f, \alpha^-(t) \rangle \stackrel{a.e.}{=\!=} 0.$$

Finally, by the definition of gradient, we have

$$\langle \nabla_{\alpha(t)} f, \alpha^\pm(t) \rangle \geqslant (\mathbf{d}_{\alpha(t)} f)(\alpha^\pm(t)).$$

Hence the result follows.

13.35. Let us pass to the ultrapower $\mathcal{L}^\omega \supset \mathcal{L}$. Argue as in 13.14 to show that there is a geodesic $[pb]_{\mathcal{L}^\omega}$ such that

$$\mathbf{d}_p \operatorname{dist}_b(\uparrow_{[pa]}) = -\langle \uparrow_{[pb]}, \uparrow_{[pa]} \rangle.$$

It follows that

$$\begin{aligned}
(\mathbf{d}_p \operatorname{dist}_a)(\nabla_p \operatorname{dist}_b) &\leqslant -\langle \uparrow_{[pa]}, \nabla_p \operatorname{dist}_b \rangle \\
&\leqslant -\mathbf{d}_p \operatorname{dist}_b(\uparrow_{[pa]}) \\
&= \langle \uparrow_{[pb]}, \uparrow_{[pa]} \rangle \\
&= \cos \measuredangle \left[p\, {}^a_b\right]_{\mathcal{L}^\omega} \\
&\leqslant \cos \tilde{\measuredangle}^\kappa \left(p\, {}^a_b\right).
\end{aligned}$$

15.17. Suppose that $\mathcal{L}$ is infinite-dimensional. Denote by $\Omega_m \subset \mathcal{L}$ the set of all points p with $\operatorname{rank}_p \geqslant m$. Evidently $\Omega_1 \supset \Omega_2 \supset \dots$, and Ω_m is open for each m.

By 15.6*C*, each Ω_m is dense in $\mathcal{L}$. Hence there is a G-delta dense set of points $p \in \mathcal{L}$ such that $\operatorname{rank}_p = \infty$. It follows that Σ_p is not compact.

Source: It was suggested by Alexander Lytchak.

16.1. Choose a finite sequence $t_0 < \dots < t_n$. Denote by Φ_{t_i} the composition of the closest-point projections to $K_{t_0}, \dots, K_{t_i}$. Pass to a limit of the Φ_{t_i} as the sequence becomes denser in the parameter interval. Show and use that the limit ϕ_t does not depend on the choice of the sequences.

16.8. Let $\ell(t) = |\alpha(t) - \alpha(t_3)|$. Note that

$$\ell'(t) \leqslant -\langle \nabla_{\alpha(t)} f, \uparrow_{[\alpha(t)\alpha(t_3)]} \rangle.$$

Observe that the function $t \mapsto f \circ \alpha(t)$ is nondecreasing; in particular, $f(\alpha(t_1)) \leqslant f(\alpha(t_2)) \leqslant f(\alpha(t_3))$. Therefore

$$\langle \nabla_{\alpha(t)} f, \uparrow_{[\alpha(t)\alpha(t_3)]} \rangle \geqslant \mathbf{d}_{\alpha(t)} f(\uparrow_{[\alpha(t)\alpha(t_3)]}) \geqslant 0$$

for any $t \in [t_1, t_2]$. Therefore $\ell' \leqslant 0$ for any $t \in [t_1, t_2]$. Hence the statement.

16.10. Without loss of generality, we may assume that $(f \circ \alpha)'(t) > 0$ for any t. Let $\hat{\alpha}$ be the arclength reparametrization of α. Note that

$$(f \circ \hat{\alpha})'(s) \geqslant |\nabla_{\hat{\alpha}(s)} f|$$

almost everywhere. Therefore, by Theorem 16.3, $\hat{\alpha}$ is a gradient-like curve. It remains to apply Lemma 16.9.

16.11. Use 16.10 to prove the only-if part.

To prove the if part, set $h(z) = \frac{1}{2} \cdot |x - z|^2$. If α is an f-gradient curve, then

$$\begin{aligned}(h \circ \alpha)^+ &\geqslant |\alpha(t) - x| \cdot \langle \uparrow_{[\alpha(t)x]}, \nabla_{\alpha(t)} f\rangle \\ &\geqslant |\alpha(t) - x| \cdot \mathbf{d}_{\alpha(t)} f(\uparrow_{[\alpha(t)x]}) \\ &\geqslant f(x) - f \circ \alpha(t).\end{aligned}$$

It remains to integrate the inequality and observe that $f \circ \alpha$ is nondecreasing.

16.35. Consider $(x, \varkappa)$- and $(z, \varkappa)$-radial curves that start at y and observe that they form a geodesic from x to z. (Compare to Exercise 10.19.)

16.38. Set $q = p + v$ and $q' = \mathrm{gexp}_p v$. By radial comparison, $|q' - x| \leqslant |q - x|$ for any $x \in \mathcal{L}$. If $q \in \mathcal{L}$, this implies that $q = q'$. Otherwise note that q' lies on the boundary line of $\mathcal{L}$, and $\mathrm{proj}(q)$ is the only point on this line that satisfies the inequality.

16.39. By the angle comparison, $|\nabla_x \mathrm{dist}_p| \geqslant -\cos \tilde\measuredangle^\varkappa \left(x\,{}^p_q\right)$.

Choose a $(p, \varkappa)$-radial curve α that starts at p. Observe that

$$(\mathrm{dist}_p \circ \alpha)^+(t) \geqslant -|\alpha^+(t)| \cdot \cos \tilde\measuredangle^\varkappa \left(\alpha(t)\,{}^p_q\right)$$

and

$$(\mathrm{dist}_q \circ \alpha)^+(t) \geqslant -|\alpha^+(t)|.$$

Therefore $t \mapsto \tilde\measuredangle^\varkappa \left(q\,{}^{\alpha(t)}_p\right)$ is nondecreasing, hence the result.

Bibliography

[1] Stephanie B. Alexander, I. David Berg, and Richard L. Bishop, *Geometric curvature bounds in Riemannian manifolds with boundary*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 703–716, DOI 10.2307/2154294. MR1113693

[2] Stephanie B. Alexander and Richard L. Bishop, *Gauss equation and injectivity radii for subspaces in spaces of curvature bounded above*, Geom. Dedicata **117** (2006), 65–84, DOI 10.1007/s10711-005-9011-6. MR2231159

[3] ______, *Comparison theorems for curves of bounded geodesic curvature in metric spaces of curvature bounded above*, Differential Geom. Appl. **6** (1996), no. 1, 67–86, DOI 10.1016/0926-2245(96)00008-3. MR1384880

[4] ______, *The Fary-Milnor theorem in Hadamard manifolds*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3427–3436, DOI 10.1090/S0002-9939-98-04423-2. MR1459103

[5] ______, *The Hadamard-Cartan theorem in locally convex metric spaces*, Enseign. Math. (2) **36** (1990), no. 3-4, 309–320. MR1096422

[6] ______, *Warped products admitting a curvature bound*, Adv. Math. **303** (2016), 88–122, DOI 10.1016/j.aim.2016.07.005. MR3552521

[7] S. B. Alexander and R. L. Bishop, *Curvature bounds for warped products of metric spaces*, Geom. Funct. Anal. **14** (2004), no. 6, 1143–1181, DOI 10.1007/s00039-004-0487-2. MR2135163

[8] Stephanie Alexander, Vitali Kapovitch, and Anton Petrunin, *Alexandrov meets Kirszbraun*, Proceedings of the Gökova Geometry-Topology Conference 2010, Int. Press, Somerville, MA, 2011, pp. 88–109. MR2931882

[9] ______, *An optimal lower curvature bound for convex hypersurfaces in Riemannian manifolds*, Illinois J. Math. **52** (2008), no. 3, 1031–1033. MR2546022

[10] ______, *An invitation to Alexandrov geometry: CAT(0) spaces*, SpringerBriefs in Mathematics, Springer, Cham, 2019, DOI 10.1007/978-3-030-05312-3. MR3930625

[11] A. Alexandroff, *Existence of a convex polyhedron and of a convex surface with a given metric* (Russian, with English summary), Rec. Math. [Mat. Sbornik] N.S. **11/53** (1942), 15–65. MR0013540

[12] ______, *The inner geometry of an arbitrary convex surface*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **32** (1941), 467–470. MR0009146

[13] A. D. Alexandroff, *On the gluing of convex surfaces*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **54** (1946), 99–101. MR0019948

[14] A. D. Aleksandrov, *Vnutrennyaya Geometriya Vypuklyh Poverhnosteĭ* (Russian), OGIZ, Moscow-Leningrad, 1948. MR29518

[15] A. D. Aleksandrov and V. A. Zalgaller, *Intrinsic geometry of surfaces*, Translations of Mathematical Monographs, Vol. 15, American Mathematical Society, Providence, RI, 1967. Translated from the Russian by J. M. Danskin. MR216434

[16] A. Alexandroff, *The inner geometry of an arbitrary convex surface*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **32** (1941), 467–470. MR9146

[17] A. D. Alexandrow, *Über eine Verallgemeinerung der Riemannschen Geometrie* (German), Schr. Forschungsinst. Math. **1** (1957), 33–84. MR87119

[18] A. D. Aleksandrov, *A theorem on triangles in a metric space and some of its applications* (Russian), Trudy Mat. Inst. Steklov. **38** (1951), 5–23. MR0049584

[19] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré, *Gradient flows in metric spaces and in the space of probability measures*, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008. MR2401600

[20] Werner Ballmann, *Singular spaces of nonpositive curvature*, Sur les groupes hyperboliques d'après Mikhael Gromov (Bern, 1988), Progr. Math., vol. 83, Birkhäuser Boston, Boston, MA, 1990, pp. 189–201, DOI 10.1007/978-1-4684-9167-8_10. MR1086658

[21] ———, *Lectures on spaces of nonpositive curvature*, DMV Seminar, vol. 25, Birkhäuser Verlag, Basel, 1995. With an appendix by Misha Brin, DOI 10.1007/978-3-0348-9240-7. MR1377265

[22] W. Ballmann, *On the geometry of metric spaces*, MPIM, 2004.

[23] V. N. Berestovskĭ, *Borsuk's problem on metrization of a polyhedron* (Russian), Dokl. Akad. Nauk SSSR **268** (1983), no. 2, 273–277. MR0688238

[24] V. N. Berestovskij and I. G. Nikolaev, *Multidimensional generalized Riemannian spaces*, Geometry, IV, Encyclopaedia Math. Sci., vol. 70, Springer, Berlin, 1993, pp. 165–243, 245–250, DOI 10.1007/978-3-662-02897-1_2. MR1263965

[25] I. D. Berg and I. G. Nikolaev, *Quasilinearization and curvature of Aleksandrov spaces*, Geom. Dedicata **133** (2008), 195–218, DOI 10.1007/s10711-008-9243-3. MR2390077

[26] A. S. Besicovitch and P. A. P. Moran, *The measure of product and cylinder sets*, J. London Math. Soc. **20** (1945), 110–120, DOI 10.1112/jlms/s1-20.2.110. MR16448

[27] Louis J. Billera, Susan P. Holmes, and Karen Vogtmann, *Geometry of the space of phylogenetic trees*, Adv. in Appl. Math. **27** (2001), no. 4, 733–767, DOI 10.1006/aama.2001.0759. MR1867931

[28] E. Bilokopytov, *Is it possible to connect every compact set?*, MathOverflow (May 5, 2020).

[29] Wilhelm Blaschke, *Kreis und Kugel* (German), Chelsea Publishing Co., New York, 1949. MR76364

[30] A. Blass, *Selective ultrafilter and bijective mapping*, MathOverflow.

[31] B. H. Bowditch, *Notes on locally* CAT(1) *spaces*, Geometric group theory (Columbus, OH, 1992), Ohio State Univ. Math. Res. Inst. Publ., vol. 3, de Gruyter, Berlin, 1995, pp. 1–48. MR1355101

[32] Brian H. Bowditch, *Median and injective metric spaces*, Math. Proc. Cambridge Philos. Soc. **168** (2020), no. 1, 43–55, DOI 10.1017/s0305004118000555. MR4043820

[33] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert* (French), North-Holland Mathematics Studies, No. 5, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. Notas de Matemática [Mathematical Notes], No. 50. MR348562

[34] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999, DOI 10.1007/978-3-662-12494-9. MR1744486

[35] Elia Bruè, Andrea Mondino, and Daniele Semola, *The metric measure boundary of spaces with Ricci curvature bounded below*, Geom. Funct. Anal. **33** (2023), no. 3, 593–636, DOI 10.1007/s00039-023-00626-x. MR4597638

[36] Dmitri Burago, *Hard balls gas and Alexandrov spaces of curvature bounded above*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), Doc. Math. **Extra Vol. II** (1998), 289–298. MR1648079

[37] Dmitri Burago, Yuri Burago, and Sergei Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001, DOI 10.1090/gsm/033. MR1835418

[38] D. Burago, S. Ferleger, and A. Kononenko, *A geometric approach to semi-dispersing billiards*, Ergodic Theory Dynam. Systems **18** (1998), no. 2, 303–319, DOI 10.1017/S014338579811564X. MR1619558

[39] ———, *Uniform estimates on the number of collisions in semi-dispersing billiards*, Ann. of Math. (2) **147** (1998), no. 3, 695–708, DOI 10.2307/120962. MR1637663

[40] ———, *Topological entropy of semi-dispersing billiards*, Ergodic Theory Dynam. Systems **18** (1998), no. 4, 791–805, DOI 10.1017/S0143385798108246. MR1645377

[41] ———, *Unfoldings and global bounds on the number of collisions for generalized semi-dispersing billiards*, Asian J. Math. **2** (1998), no. 1, 141–152, DOI 10.4310/AJM.1998.v2.n1.a4. MR1656555

[42] ———, *A geometric approach to semi-dispersing billiards*, Hard ball systems and the Lorentz gas, Encyclopaedia Math. Sci., vol. 101, Springer, Berlin, 2000, pp. 9–27, DOI 10.1007/978-3-662-04062-1_2. MR1805324

[43] D. Burago, S. Ferleger, B. Kleiner, and A. Kononenko, *Gluing copies of a* 3*-dimensional polyhedron to obtain a closed nonpositively curved pseudomanifold*, Proc. Amer. Math. Soc. **129** (2001), no. 5, 1493–1498, DOI 10.1090/S0002-9939-01-05554-X. MR1707510

[44] Yu. Burago, M. Gromov, and G. Perel′man, *A. D. Aleksandrov spaces with curvatures bounded below* (Russian, with Russian summary), Uspekhi Mat. Nauk **47** (1992), no. 2(284), 3–51, 222, DOI 10.1070/RM1992v047n02ABEH000877; English transl., Russian Math. Surveys **47** (1992), no. 2, 1–58. MR1185284

[45] Herbert Busemann, *Spaces with non-positive curvature*, Acta Math. **80** (1948), 259–310, DOI 10.1007/BF02393651. MR29531

[46] S. V. Bujalo, *Shortest paths on convex hypersurfaces of a Riemannian space* (Russian, with English summary), Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **66** (1976), 114–132, 207. Studies in topology, II. MR643664

[47] É. Cartan, *Leçons sur la Géométrie des Espaces de Riemann* (French), Gauthier-Villars, Paris, 1946. 2d ed. MR20842

[48] Nicola Cavallucci, *The* L^2*-completion of the space of Riemannian metrics is* CAT(0)*: a shorter proof*, Proc. Amer. Math. Soc. **151** (2023), no. 7, 3183–3187, DOI 10.1090/proc/16360. MR4579388

[49] Ruth Charney and Michael Davis, *Singular metrics of nonpositive curvature on branched covers of Riemannian manifolds*, Amer. J. Math. **115** (1993), no. 5, 929–1009, DOI 10.2307/2375063. MR1246182

[50] Ruth M. Charney and Michael W. Davis, *Strict hyperbolization*, Topology **34** (1995), no. 2, 329–350, DOI 10.1016/0040-9383(94)00027-I. MR1318879

[51] Jeff Cheeger and Tobias H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), no. 1, 189–237, DOI 10.2307/2118589. MR1405949

[52] Jeff Cheeger, *Comparison and Finiteness Theorems for Riemannian Manifolds*, ProQuest LLC, Ann Arbor, MI, 1967. Thesis (Ph.D.)–Princeton University. MR2616706

[53] Jeff Cheeger and Detlef Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971/72), 119–128. MR303460

[54] Brian Clarke, *Geodesics, distance, and the* CAT(0) *property for the manifold of Riemannian metrics*, Math. Z. **273** (2013), no. 1-2, 55–93, DOI 10.1007/s00209-012-0996-x. MR3010152

[55] S. Cohn-Vossen. *Totalkrümmung und geodätische Linien auf einfachzusammenhängenden offenen vollständigen Flächenstücken*, German. *Mat. Sb.* **1** (1936), no. 2, 43.

[56] Aris Daniilidis, Olivier Ley, and Stéphane Sabourau, *Asymptotic behaviour of self-contracted planar curves and gradient orbits of convex functions* (English, with English and French summaries), J. Math. Pures Appl. (9) **94** (2010), no. 2, 183–199, DOI 10.1016/j.matpur.2010.03.007. MR2665417

[57] Tamás Darvas, *The Mabuchi completion of the space of Kähler potentials*, Amer. J. Math. **139** (2017), no. 5, 1275–1313, DOI 10.1353/ajm.2017.0032. MR3702499

[58] Michael W. Davis, *Exotic aspherical manifolds*, Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001), ICTP Lect. Notes, vol. 9, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002, pp. 371–404. MR1937019

[59] ______, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2) **117** (1983), no. 2, 293–324, DOI 10.2307/2007079. MR690848

[60] ______, *Exotic aspherical manifolds*, Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001), ICTP Lect. Notes, vol. 9, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002, pp. 371–404. MR1937019

[61] Anna Dyubina and Iosif Polterovich, *Explicit constructions of universal $\mathbb{R}$-trees and asymptotic geometry of hyperbolic spaces*, Bull. London Math. Soc. **33** (2001), no. 6, 727–734, DOI 10.1112/S002460930100844X. MR1853785

[62] David A. Edwards, *The structure of superspace*, Studies in topology (Proc. Conf., Univ. North Carolina, Charlotte, N.C., 1974; dedicated to Math. Sect. Polish Acad. Sci.), Academic Press, New York-London, 1975, pp. 121–133. MR401069

[63] Ryszard Engelking, *Dimension theory*, North-Holland Mathematical Library, vol. 19, North-Holland Publishing Co., Amsterdam-Oxford-New York; PWN—Polish Scientific Publishers, Warsaw, 1978. Translated from the Polish and revised by the author. MR482697

[64] J.-H. Eschenburg, *The splitting theorem for space-times with strong energy condition*, J. Differential Geom. **27** (1988), no. 3, 477–491. MR940115

[65] Lucas C. F. Ferreira and Julio C. Valencia-Guevara, *Gradient flows of time-dependent functionals in metric spaces and applications to PDEs*, Monatsh. Math. **185** (2018), no. 2, 231–268, DOI 10.1007/s00605-017-1037-y. MR3748335

[66] Zdeněk Frolík, *Concerning topological convergence of sets* (English, with Russian summary), Czechoslovak Math. J. **10(85)** (1960), 168–180. MR116303

[67] Jian Ge and Nan Li, *Gluing of multiple Alexandrov spaces*, Adv. Math. **399** (2022), Paper No. 108248, 26, DOI 10.1016/j.aim.2022.108248. MR4383015

[68] Nicola Gigli, *An overview of the proof of the splitting theorem in spaces with non-negative Ricci curvature*, Anal. Geom. Metr. Spaces **2** (2014), no. 1, 169–213, DOI 10.2478/agms-2014-0006. MR3210895

[69] ______, *The splitting theorem in non-smooth context*, Preprint, `arXiv:1302.5555`, 2013.

[70] R. E. Greene and H. Wu, *On the subharmonicity and plurisubharmonicity of geodesically convex functions*, Indiana Univ. Math. J. **22** (1972/73), 641–653, DOI 10.1512/iumj.1973.22.22052. MR422686

[71] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Reprint of the 2001 English edition, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007. Based on the 1981 French original; With appendices by M. Katz, P. Pansu and S. Semmes; Translated from the French by Sean Michael Bates. MR2307192

[72] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–73. MR623534

[73] M. Gromov, *Almost flat manifolds*, J. Differential Geometry **13** (1978), no. 2, 231–241. MR540942

[74] ______, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR1253544

[75] Michael Gromov, *Curvature, diameter and Betti numbers*, Comment. Math. Helv. **56** (1981), no. 2, 179–195, DOI 10.1007/BF02566208. MR630949

[76] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263, DOI 10.1007/978-1-4613-9586-7_3. MR919829

[77] Karsten Grove and Peter Petersen V, *Bounding homotopy types by geometry*, Ann. of Math. (2) **128** (1988), no. 1, 195–206, DOI 10.2307/1971439. MR951512

[78] Karsten Grove and Burkhard Wilking, *A knot characterization and 1-connected nonnegatively curved 4-manifolds with circle symmetry*, Geom. Topol. **18** (2014), no. 5, 3091–3110, DOI 10.2140/gt.2014.18.3091. MR3285230

[79] J. Hadamard, *Sur la forme des lignes géodésiques à l'infini et sur les géodésiques des surfaces réglées du second ordre* (French), Bull. Soc. Math. France **26** (1898), 195–216. MR1504321

[80] Stephanie Halbeisen, *On tangent cones of Alexandrov spaces with curvature bounded below*, Manuscripta Math. **103** (2000), no. 2, 169–182, DOI 10.1007/s002290070018. MR1796313

[81] Felix Hausdorff, *Grundzüge der Mengenlehre* (German), Chelsea Publishing Co., New York, 1949. MR31025

[82] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T. Tyson, *Sobolev spaces on metric measure spaces: An approach based on upper gradients*, New Mathematical Monographs, vol. 27, Cambridge University Press, Cambridge, 2015, DOI 10.1017/CBO9781316135914. MR3363168

[83] Wu-Yi Hsiang and Bruce Kleiner, *On the topology of positively curved* 4*-manifolds with symmetry*, J. Differential Geom. **29** (1989), no. 3, 615–621. MR992332

[84] Thakyin Hu and W. A. Kirk, *Local contractions in metric spaces*, Proc. Amer. Math. Soc. **68** (1978), no. 1, 121–124, DOI 10.2307/2040922. MR464180

[85] Witold Hurewicz and Henry Wallman, *Dimension Theory*, Princeton Mathematical Series, vol. 4, Princeton University Press, Princeton, NJ, 1941. MR6493

[86] J. R. Isbell, *Six theorems about injective metric spaces*, Comment. Math. Helv. **39** (1964), 65–76, DOI 10.1007/BF02566944. MR182949

[87] S. Ivanov, *To what extent is convexity a local property?*, MathOverflow (June 9, 2012).

[88] Jürgen Jost, *Nonlinear Dirichlet forms*, New directions in Dirichlet forms, AMS/IP Stud. Adv. Math., vol. 8, Amer. Math. Soc., Providence, RI, 1998, pp. 1–47, DOI 10.1090/amsip/008/01. MR1652278

[89] ———, *Nonpositive curvature: geometric and analytic aspects*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1997, DOI 10.1007/978-3-0348-8918-6. MR1451625

[90] C. Jun, *Pursuit-evasion and time-dependent gradient flow in singular spaces*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2012.

[91] ———, *Time-dependent gradient curves on CAT(0) spaces*, Preprint, `arXiv:1801.04583`, 2018.

[92] M. B. Karmanova, *Metricheskaya teorema Rademakhera i formula ploshchadi dlya lipshitsevykh otobrazheniĭ so znacheniyami v metricheskom prostranstve*, Russian. Vestn. NGU. Ser. matem., mekh., inform. **6** (2006), no. 4, 50–69.

[93] M. Katětov, *On universal metric spaces*, General topology and its relations to modern analysis and algebra, VI (Prague, 1986), Res. Exp. Math., vol. 16, Heldermann, Berlin, 1988, pp. 323–330. MR952617

[94] Bernd Kirchheim, *Rectifiable metric spaces: local structure and regularity of the Hausdorff measure*, Proc. Amer. Math. Soc. **121** (1994), no. 1, 113–123, DOI 10.2307/2160371. MR1189747

[95] Bruce Kleiner, *The local structure of length spaces with curvature bounded above*, Math. Z. **231** (1999), no. 3, 409–456, DOI 10.1007/PL00004738. MR1704987

[96] Bruce Kleiner and Bernhard Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Inst. Hautes Études Sci. Publ. Math. **86** (1997), 115–197 (1998). MR1608566

[97] N. N. Kosovskiĭ, *Gluing of Riemannian manifolds of curvature* $\geq \kappa$ (Russian, with Russian summary), Algebra i Analiz **14** (2002), no. 3, 140–157; English transl., St. Petersburg Math. J. **14** (2003), no. 3, 467–478. MR1921991

[98] U. Lang and V. Schroeder, *Kirszbraun's theorem and metric spaces of bounded curvature*, Geom. Funct. Anal. **7** (1997), no. 3, 535–560, DOI 10.1007/s000390050018. MR1466337

[99] Urs Lang and Viktor Schroeder, *On Toponogov's comparison theorem for Alexandrov spaces*, Enseign. Math. **59** (2013), no. 3-4, 325–336, DOI 10.4171/LEM/59-3-6. MR3189039

[100] Nina Lebedeva and Anton Petrunin, *Curvature bounded below: a definition a la Berg-Nikolaev*, Electron. Res. Announc. Math. Sci. **17** (2010), 122–124, DOI 10.3934/era.2010.17.122. MR2735031

[101] ———, *Local characterization of polyhedral spaces*, Geom. Dedicata **179** (2015), 161–168, DOI 10.1007/s10711-015-0072-x. MR3424662

[102] Nina Lebedeva, Anton Petrunin, and Vladimir Zolotov, *Bipolar comparison*, Geom. Funct. Anal. **29** (2019), no. 1, 258–282, DOI 10.1007/s00039-019-00481-9. MR3925110

[103] N. Lebedeva and A. Petrunin, *5-point Toponogov theorem*, Preprint, `arXiv:2202.13049`, 2022.

[104] Nina Lebedeva and Anton Petrunin, *5-point* CAT(0) *spaces after Tetsu Toyoda*, Anal. Geom. Metr. Spaces **9** (2021), no. 1, 160–166, DOI 10.1515/agms-2020-0126. MR4298102

[105] J. Liberman, *Geodesic lines on convex surfaces*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **32** (1941), 310–313. MR0010994

[106] Alexander Lytchak and Anton Petrunin, *Short retractions of CAT(1) spaces*, Proc. Amer. Math. Soc. **149** (2021), no. 3, 1247–1257, DOI 10.1090/proc/15268. MR4211878

[107] Alexander Lytchak and Stephan Stadler, *Conformal deformations of* CAT(0) *spaces*, Math. Ann. **373** (2019), no. 1-2, 155–163, DOI 10.1007/s00208-018-1703-y. MR3968869

[108] A. Lytchak and S. Stadler, *Curvature bounds of subsets in dimension two*, 2021.

[109] A. Lytchak, *Differentiation in metric spaces*, Algebra i Analiz **16** (2004), no. 6, 128–161, DOI 10.1090/S1061-0022-05-00888-5; English transl., St. Petersburg Math. J. **16** (2005), no. 6, 1017–1041. MR2117451

[110] ______, *Open map theorem for metric spaces*, Algebra i Analiz **17** (2005), no. 3, 139–159, DOI 10.1090/S1061-0022-06-00916-2; English transl., St. Petersburg Math. J. **17** (2006), no. 3, 477–491. MR2167848

[111] Alexander Lytchak, *On the geometry of subsets of positive reach*, Manuscripta Math. **115** (2004), no. 2, 199–205, DOI 10.1007/s00229-004-0491-8. MR2098470

[112] Hans von Mangoldt, *Ueber diejenigen Punkte auf positiv gekrümmten Flächen, welche die Eigenschaft haben, dass die von ihnen ausgehenden geodätischen Linien nie aufhören, kürzeste Linien zu sein* (German), J. Reine Angew. Math. **91** (1881), 23–53, DOI 10.1515/crll.1881.91.23. MR1579879

[113] G. A. Margulis and G. D. Mostow, *Some remarks on the definition of tangent cones in a Carnot-Carathéodory space*, J. Anal. Math. **80** (2000), 299–317, DOI 10.1007/BF02791539. MR1771529

[114] Uwe F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom. **6** (1998), no. 2, 199–253, DOI 10.4310/CAG.1998.v6.n2.a1. MR1651416

[115] Karl Menger, *Untersuchungen über allgemeine Metrik* (German), Math. Ann. **100** (1928), no. 1, 75–163, DOI 10.1007/BF01448840. MR1512479

[116] Alexander Mielke, Riccarda Rossi, and Giuseppe Savaré, *Nonsmooth analysis of doubly nonlinear evolution equations*, Calc. Var. Partial Differential Equations **46** (2013), no. 1-2, 253–310, DOI 10.1007/s00526-011-0482-z. MR3016509

[117] A. D. Milka, *Metric structure of a certain class of spaces that contain straight lines* (Russian), Ukrain. Geometr. Sb. **4** (1967), 43–48. MR0256327

[118] Nicolas Monod, *Superrigidity for irreducible lattices and geometric splitting*, J. Amer. Math. Soc. **19** (2006), no. 4, 781–814, DOI 10.1090/S0894-0347-06-00525-X. MR2219304

[119] Jun-iti Nagata, *Modern dimension theory*, Revised edition, Sigma Series in Pure Mathematics, vol. 2, Heldermann Verlag, Berlin, 1983. MR715431

[120] F. Nazarov, *Intrinsic metric with no geodesics* (February 18, 2010).

[121] Shin-ichi Ohta, *Gradient flows on Wasserstein spaces over compact Alexandrov spaces*, Amer. J. Math. **131** (2009), no. 2, 475–516, DOI 10.1353/ajm.0.0048. MR2503990

[122] D. Panov and A. Petrunin, *Ramification conjecture and Hirzebruch's property of line arrangements*, Compos. Math. **152** (2016), no. 12, 2443–2460, DOI 10.1112/S0010437X16007648. MR3594282

[123] G. Perelman and A. Petrunin, *Quasigeodesics and Gradient curves in Alexandrov spaces.*

[124] G. Ya. Perelʹman, *Elements of Morse theory on Aleksandrov spaces* (Russian, with Russian summary), Algebra i Analiz **5** (1993), no. 1, 232–241; English transl., St. Petersburg Math. J. **5** (1994), no. 1, 205–213. MR1220498

[125] G. Perelman. *Alexandrov spaces with curvature bounded from below II.* Preprint LOMI, 1991. 35pp. `http://www.math.psu.edu/petrunin/`.

[126] Anton Petrunin and Stephan Stadler, *Metric-minimizing surfaces revisited*, Geom. Topol. **23** (2019), no. 6, 3111–3139, DOI 10.2140/gt.2019.23.3111. MR4039186

[127] Anton Petrunin, *A globalization for non-complete but geodesic spaces*, Math. Ann. **366** (2016), no. 1-2, 387–393, DOI 10.1007/s00208-015-1295-8. MR3552243

[128] ______, *Alexandrov meets Lott-Villani-Sturm*, Münster J. Math. **4** (2011), 53–64. MR2869253

[129] ______, *Quasigeodesics in multidimensional Alexandrov spaces*, ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–University of Illinois at Urbana-Champaign. MR2693118

[130] ______, *Semiconcave functions in Alexandrov's geometry*, Surveys in differential geometry. Vol. XI, Surv. Differ. Geom., vol. 11, Int. Press, Somerville, MA, 2007, pp. 137–201, DOI 10.4310/SDG.2006.v11.n1.a6. MR2408266

[131] ______, *Applications of quasigeodesics and gradient curves*, Comparison geometry (Berkeley, CA, 1993), Math. Sci. Res. Inst. Publ., vol. 30, Cambridge Univ. Press, Cambridge, 1997, pp. 203–219. MR1452875

[132] A. Petrunin, *Pure metric geometry: introductory lectures*, Preprint, `arXiv:2007.09846`, 2020.

[133] Robert R. Phelps, *Convex functions, monotone operators and differentiability*, 2nd ed., Lecture Notes in Mathematics, vol. 1364, Springer-Verlag, Berlin, 1993. MR1238715

[134] P. Pizzetti, *Paragone fra due triangoli a lati uguali.* Atti della Reale Accademia dei Lincei, Rendiconti (5). Classe di Scienze Fisiche, Matematiche e Naturali, **16** (1907), no. 1, 6–11.

[135] C. Plaut, *Toponogov's theorem for metric spaces II*, MPIM Preprints (1990).

[136] Conrad Plaut, *Spaces of Wald curvature bounded below*, J. Geom. Anal. **6** (1996), no. 1, 113–134, DOI 10.1007/BF02921569. MR1402389

[137] ______, *Metric spaces of curvature* $\geq k$, Handbook of geometric topology, North-Holland, Amsterdam, 2002, pp. 819–898. MR1886682

[138] L. Pontrjagin, *Sur une hypothèse fondamentale de la théorie de la dimension*, French. C. R. Acad. Sci., Paris, **190** (1930), 1105–1107.

[139] Yu. G. Reshetnyak, *Two-dimensional manifolds of bounded curvature* (Russian), Geometry, 4 (Russian), Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989, pp. 7–189, 273–277, 279. MR1099202

[140] Ju. G. Rešetnjak, *Non-expansive maps in a space of curvature no greater than K* (Russian), Sibirsk. Mat. Ž. **9** (1968), 918–927. MR244922

[141] Willi Rinow, *Die innere Geometrie der metrischen Räume*, Die Grundlehren der mathematischen Wissenschaften, Band 105, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961. MR123969

[142] Walter Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–419. MR80902

[143] I. Kh. Sabitov, *Around the proof of the Legendre-Cauchy lemma on convex polygons* (Russian, with Russian summary), Sibirsk. Mat. Zh. **45** (2004), no. 4, 892–919, DOI 10.1023/B:SIMJ.0000035837.80962.0a; English transl., Siberian Math. J. **45** (2004), no. 4, 740–762. MR2091654

[144] Takashi Sato, *An alternative proof of Berg and Nikolaev's characterization of* CAT(0)*-spaces via quadrilateral inequality*, Arch. Math. (Basel) **93** (2009), no. 5, 487–490, DOI 10.1007/s00013-009-0057-9. MR2563595

[145] Thomas Foertsch and Viktor Schroeder, *Minkowski versus Euclidean rank for products of metric spaces*, Adv. Geom. **2** (2002), no. 2, 123–131, DOI 10.1515/advg.2002.002. MR1895343

[146] Giuseppe Savaré, *Gradient flows and diffusion semigroups in metric spaces under lower curvature bounds* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **345** (2007), no. 3, 151–154, DOI 10.1016/j.crma.2007.06.018. MR2344814

[147] V. A. Šarafutdinov, *The Pogorelov-Klingenberg theorem for manifolds that are homeomorphic to* $\mathbf{R}^n$ (Russian), Sibirsk. Mat. Ž. **18** (1977), no. 4, 915–925, 958. MR0487896

[148] Katsuhiro Shiohama, *An introduction to the geometry of Alexandrov spaces*, Lecture Notes Series, vol. 8, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1993. MR1320267

[149] R. M. Solovay, *Maps preserving measures*, 2011.

[150] Stephan Stadler, *The structure of minimal surfaces in CAT(0) spaces*, J. Eur. Math. Soc. (JEMS) **23** (2021), no. 11, 3521–3554, DOI 10.4171/jems/1075. MR4310811

[151] K. T. Sturm, *Metric spaces of lower bounded curvature*, Exposition. Math. **17** (1999), no. 1, 35–47. MR1687468

[152] K.-T. Sturm, *The space of spaces: curvature bounds and gradient flows on the space of metric measure spaces*, Preprint, `arXiv:1208.0434`, 2012.

[153] Karl-Theodor Sturm, *Probability measures on metric spaces of nonpositive curvature*, Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), Contemp. Math., vol. 338, Amer. Math. Soc., Providence, RI, 2003, pp. 357–390, DOI 10.1090/conm/338/06080. MR2039961

[154] V. A. Toponogov, *Riemann spaces with curvature bounded below* (Russian), Uspehi Mat. Nauk **14** (1959), no. 1(85), 87–130. MR0103510

[155] Tetsu Toyoda, *An intrinsic characterization of five points in a* CAT(0) *space*, Anal. Geom. Metr. Spaces **8** (2020), no. 1, 114–165, DOI 10.1515/agms-2020-0111. MR4141370

[156] Cédric Villani, *Optimal transport*: *Old and new*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009, DOI 10.1007/978-3-540-71050-9. MR2459454

[157] A. Wald, *Begründung eiiner Koordinatenlosen Differentialgeometrie der Flächen*, German. Ergebnisse eines mathematischen Kolloquium, **6** (1935), 24–46.

[158] R. A. Wijsman, *Convergence of sequences of convex sets, cones and functions. II*, Trans. Amer. Math. Soc. **123** (1966), 32–45, DOI 10.2307/1994611. MR196599

[159] Sumio Yamada, *On the geometry of Weil-Petersson completion of Teichmüller spaces*, Math. Res. Lett. **11** (2004), no. 2-3, 327–344, DOI 10.4310/MRL.2004.v11.n3.a5. MR2067477

[160] Takao Yamaguchi, *Collapsing and pinching under a lower curvature bound*, Ann. of Math. (2) **133** (1991), no. 2, 317–357, DOI 10.2307/2944340. MR1097241

Index