Proof of 10.1.

If part. Assume $\mathcal{L}$ is geodesic. Consider $x^1, x^2, x^3 \in \mathcal{L}$ such that the model triangle $[\tilde{x}^1\tilde{x}^2\tilde{x}^3] = \tilde{\triangle}^{\kappa}(x^1x^2x^3)$ is defined. Choose $p \in]x^1x^2[$. Applying the short map extension property with $V_3 = \{x^1, x^2, x^3\}$, $V_4 = \{x^1, x^2, x^3, p\}$ and the map $f : x^i \mapsto \tilde{x}^i$, we obtain the point-on-side comparison (8.14*b*).

In case $\mathcal{L}$ is not geodesic, pass to its ultrapower $\mathcal{L}^\omega$. Note that the short map extension property survives for $\mathcal{L}^\omega$ and recall that $\mathcal{L}^\omega$ is geodesic (see 4.8). Thus, from above, $\mathcal{L}^\omega$ is a complete length CBB(κ) space. By Proposition 8.4, $\mathcal{L}$ is a complete length CBB(κ) space.

Only-if part. Assume the contrary: $\mathcal{L}$ is complete and CBB(κ), and $x^1, x^2, x^3, p \in \mathcal{L}$ and $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3 \in \mathbb{M}^2(\kappa)$ are such that $|\tilde{x}^i - \tilde{x}^j| \leqslant |x^i - x^j|$ for all i, j but there is no point $\tilde{p} \in \mathbb{M}^2(\kappa)$ such that $|\tilde{p} - \tilde{x}^i| \leqslant |p - x^i|$ for all i.

Note that in this case all comparison triangles $\tilde{\triangle}^{\kappa}(px^ix^j)$ are defined. This is always true if $\kappa \leqslant 0$. If $\kappa > 0$, and say $\tilde{\triangle}^{\kappa}(px^1x^2)$ is undefined, then

$$\begin{aligned}|p - x^1| + |p - x^2| &\geqslant 2 \cdot \varpi\kappa - |x^1 - x^2| \\ &\geqslant 2 \cdot \varpi\kappa - |\tilde{x}^1 - \tilde{x}^2| \\ &\geqslant |\tilde{x}^1 - \tilde{x}^3| + |\tilde{x}^2 - \tilde{x}^3|.\end{aligned}$$

Then the last inequality must be an equality. Thus we may extend by taking $\tilde{p}$ on $[\tilde{x}^1\tilde{x}^3]$ or $[\tilde{x}^2\tilde{x}^3]$. For each $i \in \{1, 2, 3\}$, consider a point $\tilde{p}^i \in \mathbb{M}^2(\kappa)$ such that $|\tilde{p}^i - \tilde{x}^i|$ is minimal among points satisfying $|\tilde{p}^i - \tilde{x}^j| \leqslant |p - x^j|$ for all $j \neq i$. Clearly, every $\tilde{p}^i$ is inside the triangle $[\tilde{x}^1\tilde{x}^2\tilde{x}^3]$ (that is, in $\mathrm{Conv}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$), and $|\tilde{p}^i - \tilde{x}^i| > |p - x^i|$ for each i. Since the function $x \mapsto \tilde{\measuredangle}^\kappa\{x; a, b\}$ is increasing, it follows that

(i) $|\tilde{p}^i - \tilde{x}^j| = |p - x^j|$ for $i \neq j$;

(ii) no pair of triangles from $[\tilde{p}^1\tilde{x}^2\tilde{x}^3]$, $[\tilde{p}^2\tilde{x}^3\tilde{x}^1]$, $[\tilde{p}^3\tilde{x}^1\tilde{x}^2]$ overlap in $[\tilde{x}^1\tilde{x}^2\tilde{x}^3]$.

As follows from ~~the~~ overlap lemma (10.2), in this case

$$\measuredangle\left[\tilde{p}^1\,{}^{\tilde{x}^2}_{\tilde{x}^3}\right] + \measuredangle\left[\tilde{p}^2\,{}^{\tilde{x}^3}_{\tilde{x}^1}\right] + \measuredangle\left[\tilde{p}^3\,{}^{\tilde{x}^1}_{\tilde{x}^2}\right] > 2 \cdot \pi.$$

Since $|\tilde{x}^i - \tilde{x}^j| \leqslant |x^i - x^j|$ we have

$$\measuredangle\left[\tilde{p}^k\,{}^{\tilde{x}^i}_{\tilde{x}^j}\right] \leqslant \tilde{\measuredangle}^\kappa\left(p\,{}^{x^i}_{x^j}\right)$$

if (i, j, k) is a permutation of $(1, 2, 3)$. Therefore

$$\tilde{\measuredangle}^\kappa\left(p\,{}^{x^1}_{x^2}\right) + \tilde{\measuredangle}^\kappa\left(p\,{}^{x^2}_{x^3}\right) + \tilde{\measuredangle}^\kappa\left(p\,{}^{x^3}_{x^1}\right) > 2 \cdot \pi,$$

contradicting the CBB(κ) comparison (8.2). □

10.3. Theorem. *Assume any pair of points at distance $< \varpi\kappa$ in the metric space $\mathcal{U}$ are joined by a unique geodesic. Then $\mathcal{U}$ is* CAT(κ) *if and only if for any 3-point* *set V_3 with perimeter $< 2\cdot\varpi\kappa$ and any 4-point set $V_4 \supset V_3$ in $\mathbb{M}^2(\kappa)$, any short map $f\colon V_3 \to \mathcal{U}$ can be extended to a short map $F\colon V_4 \to \mathcal{U}$.*

Note that the only-if part of Theorem 10.3 does not follow directly from Kirszbraun's theorem, since the desired extension is in $\mathcal{U}$—not its completion.

10.4. Lemma. *Let $x^1, x^2, x^3, y^1, y^2, y^3 \in \mathbb{M}(\kappa)$ be points such that $|x^i - x^j| \geqslant |y^i - y^j|$ for all i, j. Then there is a short map $\Phi\colon \mathbb{M}(\kappa) \to \mathbb{M}(\kappa)$ such that $\Phi(x^i) = y^i$ for all i; moreover, one can choose Φ so that*

$$\Im\Phi \subset \operatorname{Conv}(y^1, y^2, y^3).$$

We only give an idea of the proof of this lemma; alternatively, it can be obtained as a corollary of Kirszbraun's theorem (10.14)

Idea of the proof. The map Φ can be constructed as a composition of an isometry of $\mathbb{M}(\kappa)$ and the following folding map: given a halfspace H in $\mathbb{M}(\kappa)$, consider the map $\mathbb{M}(\kappa) \to H$ that is the identity on H and reflects all points outside of H into H. This map is a path isometry; in particular, it is short.

The last part of the lemma can be proved by composing this map with folding maps along the sides of triangle $[y^1y^2y^3]$, and passing to a partial limit. □

Proof of 10.3.

If part. The point-on-side comparison (9.14*b*) follows by taking $V_3 = \{\tilde x, \tilde y, \tilde p\}$ and $V_4 = \{\tilde x, \tilde y, \tilde p, \tilde z\}$ where $z \in\,]xy[$. It is only necessary to observe that $F(\tilde z) = z$ by uniqueness of $[xy]$.

Only-if part. Let $V_3 = \{\tilde x^1, \tilde x^2, \tilde x^3\}$ and $V_4 = \{\tilde x^1, \tilde x^2, \tilde x^3, \tilde p\}$.

Set $y^i = f(\tilde x^i)$ for all i. We need to find a point $q \in \mathcal{U}$ such that $|y^i - q| \leqslant |\tilde x^i - \tilde p|$ for all i.

Let D be the convex set in $\mathbb{M}^2(\kappa)$ bounded by the model triangle $[\tilde y^1\tilde y^2\tilde y^3] = \tilde\triangle^{\kappa} y^1y^2y^3$; that is, $D = \operatorname{Conv}(\tilde y^1, \tilde y^2, \tilde y^3)$.

Note that $|\tilde y^i - \tilde y^j| = |y^i - y^j| \leqslant |\tilde x^i - \tilde x^j|$ for all i, j. Applying Lemma 10.4, we get a short map $\Phi\colon \mathbb{M}(\kappa) \to D$ such that $\Phi\colon \tilde x^i \mapsto \tilde y^i$.

Further, by ~~the~~ majorization theorem (9.56), there is a short map $F\colon D \to \mathcal{U}$ such that $\tilde y^i \mapsto y^i$ for all i.

Thus one can take $q = F\circ\Phi(\tilde p)$. □

10.5. Exercise. *Assume $\mathcal{X}$ is a complete length space that satisfies the following condition: any 4-point subset admits a distance-preserving map to the Euclidean 3-space.*

Prove that $\mathcal{X}$ is isometric to a closed convex subset of a Hilbert space.

10.6. Exercise. *Let $\mathcal{F}_s$ be the metric on the 5-point set $\{p, q, x, y, z\}$ for which $|p-q| = s$ and all the remaining distances are equal 1. For which values s does the space $\mathcal{F}_s$ admit a distance-preserving map into*

(a) *a complete length* CAT(0) *space?*

(b) *a complete length* CBB(0) *space?*

The following exercise describes the first known definition of spaces with curvature bounded below; it was given by Abraham Wald [**157**].

10.7. Exercise. *Let $\mathcal{L}$ be a metric space and $\varkappa \leqslant 0$. Prove that $\mathcal{L}$ is* CBB($\varkappa$) *if and only if any quadruple of points p, q, r, $s \in \mathcal{L}$ admits a distance-preserving embedding into $\mathbb{M}^2(\mathrm{K})$ for some $\mathrm{K} \geqslant \varkappa$.*

Is the same true for $\varkappa > 0$; what is the difference?

B. $(1+n)$-point comparison

The following theorem gives a more sensitive analog of the CBB($\varkappa$) comparison (8.2).

10.8. $(1+n)$-point comparison. *Let $\mathcal{L}$ be a complete length* CBB($\varkappa$) *space. Then for any array $(p, x^1, \dots, x^n)$ of points in $\mathcal{L}$ there is a model array $(\tilde{p}, \tilde{x}^1, \dots, \tilde{x}^n)$ in $\mathbb{M}^n(\varkappa)$ such that*

(a) $|\tilde{p} - \tilde{x}^i| = |p - x^i|$ *for all i.*

(b) $|\tilde{x}^i - \tilde{x}^j| \geqslant |x^i - x^j|$ *for all i, j.*

Proof. It is sufficient to show that given $\varepsilon > 0$ there is an array $(\tilde{p}, \tilde{x}^1, \dots, \tilde{x}^n)$ in $\mathbb{M}^n(\varkappa)$ such that

$$|\tilde{x}^i - \tilde{x}^j| \geqslant |x^i - x^j| \quad \text{and} \quad |\tilde{p} - \tilde{x}^i| \lesseqgtr |p - x^i|| \pm \varepsilon.$$

Then one can pass to a limit array for $\varepsilon \to 0+$.

According to 8.11, the set $\mathrm{Str}(x^1, \dots, x^n)$ is dense in $\mathcal{L}$. Thus there is a point $p' \in \mathrm{Str}(\tilde{x}^1, \dots, \tilde{x}^n)$ such that $|p' - p| \leqslant \varepsilon$. According to Corollary 13.40, $\mathrm{T}_{p'}$ contains a subcone E isometric to a Euclidean space and containing all vectors $\log[p'x^i]$. Passing to a subspace if necessary, we may assume that $\dim E \leqslant n$.

Mark a point $\tilde{p} \in \mathbb{M}^n(\varkappa)$ and choose a distance-preserving map $\iota\colon E \to \mathrm{T}_{\tilde{p}}\,\mathbb{M}^n(\varkappa)$. Let

$$\tilde{x}^i = \exp_{\tilde{p}} \circ \iota(\log[p'x^i]).$$

Thus $|\tilde{p} - \tilde{x}^i| = |p' - x^i|$. Since $|p - p'| \leqslant \varepsilon$, we get

$$|\tilde{p} - \tilde{x}^i| \lesseqgtr |p - x^i| \pm \varepsilon.$$

From the hinge comparison (8.14c) we have

$$\tilde{\measuredangle}^\varkappa\left(\tilde{p}\,{}^{\tilde{x}^i}_{\tilde{x}^j}\right) = \measuredangle\left[\tilde{p}\,{}^{\tilde{x}^i}_{\tilde{x}^j}\right] = \measuredangle\left[p'\,{}^{x^i}_{x^j}\right] \geqslant \tilde{\measuredangle}^\varkappa\left(p'\,{}^{x^i}_{x^j}\right),$$

and thus

$$|\tilde{x}^i - \tilde{x}^j| \geqslant |x^i - x^j|.$$

□

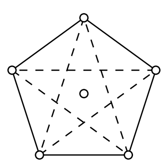
10.9. Exercise. *Let $(p, x_1, \dots, x_n)$ be a point array in a* CBB(0) *space. Consider the $n{\times}n$-matrix M with components*

$$m_{i,j} = \tfrac{1}{2} \cdot (|x_i - p|^2 + |x_j - p|^2 - |x_i - x_j|^2).$$

Show that

$$\mathbf{s} \cdot M \cdot \mathbf{s}^\top \geqslant 0 \qquad (1)$$

for any vector $\mathbf{s} = (s_1, \dots, s_n)$ with nonnegative components.



The above exercise describes the so-called *Lang–Schroeder–Sturm inequality*; it was discovered by Urs Lang and Viktor Schroeder [**98**] and rediscovered by Karl-Theodor Sturm [**151**]. It turns out to be weaker than $(1{+}n)$-point comparison. An example can be constructed by perturbing the 6-point metric isometric to a regular pentagon with its center, making its sides slightly longer and diagonals slightly shorter [**102**]. In particular, this inequality in general metric spaces (not necessarily length spaces) does not imply the inequality in the following exercise.

10.10. Exercise. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space. Show that for any points p, x_1, x_2, x_3, x_4, x_5 in $\mathcal{L}$ we have*

$$\tilde\measuredangle^\kappa\left(p\,{}^{x_1}_{x_5}\right) + \tilde\measuredangle^\kappa\left(p\,{}^{x_2}_{x_1}\right) + \tilde\measuredangle^\kappa\left(p\,{}^{x_3}_{x_2}\right) + \tilde\measuredangle^\kappa\left(p\,{}^{x_4}_{x_3}\right) + \tilde\measuredangle^\kappa\left(p\,{}^{x_5}_{x_4}\right) \leqslant 4 \cdot \pi,$$

assuming that the left-hand side is defined.

10.11. Exercise. *Give an example of a metric on a finite set that satisfies the comparison inequality*

$$\tilde\measuredangle^0\left(p\,{}^{x_1}_{x_2}\right) + \tilde\measuredangle^0\left(p\,{}^{x_2}_{x_3}\right) + \tilde\measuredangle^0\left(p\,{}^{x_3}_{x_1}\right) \leqslant 2 \cdot \pi$$

for any quadruple of points (p, x_1, x_2, x_3), but is not isometric to a subset of an Alexandrov space with curvature $\geqslant 0$.

10.12. Exercise. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space. Assume that a point array $(a^0, a^1, \dots, a^k)$ in $\mathcal{L}$ is κ*-strutting (*Definition* 15.1) *for a point $p \in \mathcal{L}$. Show that there are points $\tilde{p}$, $\tilde{a}^0, \dots, \tilde{a}^m$ in $\mathbb{M}^{m+1}(\kappa)$ such that*

$$|\tilde{p} - \tilde{a}^i| = |p - a^i| \quad \text{and} \quad |\tilde{a}^i - \tilde{a}^j| = |a^i - a^j|$$

for all i and j.

C. Helly's theorem

10.13. Helly's theorem. *Let $\mathcal{U}$ be a complete length* CAT(0) *space and $\{K_\alpha\}_{\alpha\in\mathcal{A}}$ be an arbitrary collection of closed bounded convex subsets in $\mathcal{U}$.*

If

$$\bigcap_{\alpha\in\mathcal{A}} K_\alpha = \emptyset,$$

then there is a finite index array $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathcal{A}$ such that

$$\bigcap_i K_{\alpha_i} = \emptyset.$$

Remarks.

(i) In general, none of the K_α may be compact; otherwise the statement is trivial.

(ii) If $\mathcal{U}$ is a Hilbert space (not necessarily separable), then Helly's theorem is equivalent to the following statement: if a convex bounded set is closed in the ordinary topology then it is compact in the weak topology. One can define *weak topology* in an arbitrary metric space by taking exteriors of closed ball as prebase. Then Helly's theorem implies the analogous statement for complete length CAT(0) spaces (compare to [**118**]).

We present the proof of Urs Lang and Viktor Schroeder [**98**].

Proof of 10.13. Assume the contrary. Then for any finite set $F \subset \mathcal{A}$,

$$K_F := \bigcap_{\alpha\in F} K_\alpha \neq \emptyset.$$

We will construct a point z such that $z \in K_\alpha$ for each α. Thus we will arrive at a contradiction since

$$\bigcap_{\alpha\in\mathcal{A}} K_\alpha = \emptyset.$$

Choose a point $p \in \mathcal{U}$, and let $r = \sup\{\operatorname{dist}_{K_F} p\}$ where F runs over all finite subsets of $\mathcal{A}$. Let p_F^* be the closest point on K_F to p; according to ~~the~~ closest-point projection lemma (9.73), p_F^* exists and is unique.

Take a nested sequence of finite subsets $F_1 \subset F_2 \subset \ldots$ of $\mathcal{A}$, such that $\operatorname{dist}_{K_{F_n}} p \to r$.

Let us show that the sequence $p_{F_n}^*$ is Cauchy. If not, then for fixed $\varepsilon > 0$, we can choose two subsequences y'_n and y''_n of $p_{F_n}^*$ such that $|y'_n - y''_n| \geqslant \varepsilon$. Let z_n be the midpoint of $[y'_n y''_n]$. From the point-on-side comparison (~~8.14b),~~ there is $\delta > 0$ such that

$$|p - z_n| \leqslant \max\{|p - y'_n|, |p - y''_n|\} - \delta.$$

Thus

$$\overline{\lim_{n\to\infty}} |p - z_n| < r.$$

On the other hand, from convexity, each K_{F_n} contains all z_k with sufficiently large k, a contradiction.

Thus, $p^*_{F_n}$ converges and we can take $z = \lim_n p^*_{F_n}$. Clearly

$$|p - z| = r.$$

Repeat the above arguments for the sequence $F'_n = F_n \cup \{\alpha\}$. As a result, we get another point z' such that $|p - z| = |p - z'| = r$ and $z, z' \in K_{F_n}$ for all n. Thus, if $z \neq z'$ the midpoint $\hat{z}$ of $[zz']$ would belong to all K_{F_n}, and from comparison, we would have $|p - \hat{z}| < r$, a contradiction.

Thus, $z' = z$; in particular $z \in K_\alpha$ for each $\alpha \in \mathcal{A}$. □

D. Kirszbraun's theorem

A slightly weaker version of the following theorem was proved by Urs Lang and Viktor Schroeder [**98**].

10.14. Kirszbraun's theorem. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space, $\mathcal{U}$ be a complete length* CAT(κ) *space, $Q \subset \mathcal{L}$ be arbitrary subset and $f\colon Q \to \mathcal{U}$ be a short map. Assume that there is $z \in \mathcal{U}$ such that $f(Q) \subset \mathrm{B}[z, \frac{\varpi\kappa}{2}]_{\mathcal{U}}$. Then $f\colon Q \to \mathcal{U}$ can be extended to a short map $F\colon \mathcal{L} \to \mathcal{U}$ (that is, there is a short map $F\colon \mathcal{L} \to \mathcal{U}$ such that $F|_Q = f$).*

The condition $f(Q) \subset \mathrm{B}[z, \frac{\varpi\kappa}{2}]$ trivially holds for any $\kappa \leqslant 0$ since in this case $\varpi\kappa = \infty$. The following example shows that this condition is needed for $\kappa > 0$.

Conjecture 10.22 (if true) gives an equivalent condition for the existence of a short extension; it states that the following example is the only obstacle.

10.15. Example. *Let $\mathbb{S}^m_+$ be a closed m-dimensional unit hemisphere. Denote its boundary, which is isometric to $\mathbb{S}^{m-1}$, by $\partial\mathbb{S}^m_+$. Clearly, $\mathbb{S}^m_+$ is* CBB(1) *and $\partial\mathbb{S}^m_+$ is* CAT(1)*, but the identity map $\partial\mathbb{S}^m_+ \to \partial\mathbb{S}^m_+$ cannot be extended to a short map $\mathbb{S}^m_+ \to \partial\mathbb{S}^m_+$ (there is no place for the pole).*

There is also a direct generalization of this example to a hemisphere in a Hilbert space of arbitrary cardinal dimension.

First we prove this theorem in the case $\kappa \leqslant 0$ (10.17). In the proof of the more complicated case $\kappa > 0$, we use the case $\kappa = 0$. The following lemma is the main ingredient in the proof.

10.16. Finite+one lemma. *Let $\kappa \leqslant 0$, $\mathcal{L}$ be a complete length* CBB(κ) *space, and $\mathcal{U}$ be a complete length* CAT(κ) *space. Suppose $x^1, x^2, \dots, x^n$ in $\mathcal{L}$ and $y^1, y^2, \dots, y^n$ in $\mathcal{U}$ are such that $|x^i - x^j| \geqslant |y^i - y^j|$ for all i, j.*

Then for any $p \in \mathcal{L}$, there is $q \in \mathcal{U}$ such that $|y^i - q| \leqslant |x^i - p|$ for each i.

Proof. It is sufficient to prove the lemma only for $\varkappa = 0$ and -1. The proofs of these two cases are identical, only the formulas differ. In the proof, we assume $\varkappa = 0$ and provide the formulas for $\varkappa = -1$ in the footnotes.

From the $(1+n)$-point comparison (10.8), there is a model configuration $\tilde{p}, \tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$ in $\mathbb{M}^n(\varkappa)$ such that $|\tilde{p} - \tilde{x}^i| = |p - x^i|$ and $|\tilde{x}^i - \tilde{x}^j| \geqslant |x^i - x^j|$ for all i, j. It follows that we can assume that $\mathcal{L} = \mathbb{M}^n(\varkappa)$.

For each i, consider functions $f^i\colon \mathcal{U} \to \mathbb{R}$ and $\tilde{f}^i\colon \mathbb{M}^n(\varkappa) \to \mathbb{R}$ defined as follows:[1]

$$(A)^0 \qquad f^i = \tfrac{1}{2} \cdot \mathrm{dist}^2_{y^i}, \quad \tilde{f}^i = \tfrac{1}{2} \cdot \mathrm{dist}^2_{\tilde{x}^i}.$$

Consider the function arrays

$$\mathbf{f} = (f^1, f^2, \dots, f^n)\colon \mathcal{U} \to \mathbb{R}^n \quad \text{and} \quad \tilde{\mathbf{f}} = (\tilde{f}^1, \tilde{f}^2, \dots, \tilde{f}^n)\colon \mathbb{M}^n(\varkappa) \to \mathbb{R}^n.$$

Define

$$\begin{aligned} \operatorname{Up} \mathbf{f}(\mathcal{U}) &= \left\{ \mathbf{v} \in \mathbb{R}^{k+1} : \exists\, \mathbf{w} \in \mathbf{f}(\mathcal{U}) \text{ such that } \mathbf{v} \succcurlyeq \mathbf{w} \right\}, \\ \operatorname{Min} \mathbf{f}(\mathcal{U}) &= \{ \mathbf{v} \in \mathbf{f}(\mathcal{U}) : \text{if } \mathbf{v} \succcurlyeq \mathbf{w} \in \mathbf{f}(\mathcal{U}) \text{ then } \mathbf{w} = \mathbf{v} \}. \end{aligned}$$

(See Definition 14.1.) Note it is sufficient to prove that $\tilde{\mathbf{f}}(\tilde{p}) \in \operatorname{Up} \mathbf{f}(\mathcal{U})$. Clearly, $(f^i)'' \geqslant 1$. Thus by Theorem 14.3*a*, the set $\operatorname{Up} \mathbf{f}(\mathcal{U}) \subset \mathbb{R}^n$ is convex.

Arguing by contradiction, let us assume that $\tilde{\mathbf{f}}(\tilde{p}) \notin \operatorname{Up} \mathbf{f}(\mathcal{U})$.

Then there exists a supporting hyperplane $\alpha_1 \cdot x_1 + \dots \alpha_n \cdot x_n = c$ to $\operatorname{Up} \mathbf{f}(\mathcal{U})$, separating it from $\tilde{\mathbf{f}}(\tilde{p})$. According to Lemma 14.6*b*, $\alpha_i \geqslant 0$ for each i. So we may assume that $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \Delta^{n-1}$ (that is, $\alpha_i \geqslant 0$ for each i and $\sum \alpha_i = 1$ and

$$\sum_i \alpha_i \cdot \tilde{f}^i(\tilde{p})) < \inf \left\{ \sum_i \alpha_i \cdot f^i(q) : q \in \mathcal{U} \right\}.$$

The latter contradicts the following claim.

(1) *Given $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Delta^{n-1}$, let*

$$\begin{aligned} h &= \sum_i \alpha_i \cdot f^i & h&\colon \mathcal{U} \to \mathbb{R} & z &= \operatorname{MinPoint} h \in \mathcal{U} \\ \tilde{h} &= \sum_i \alpha_i \cdot \tilde{f}^i & \tilde{h}&\colon \mathbb{M}^n(\varkappa) \to \mathbb{R} & \tilde{z} &= \operatorname{MinPoint} \tilde{h} \in \mathbb{M}^n(\varkappa). \end{aligned}$$

Then $h(z) \leqslant \tilde{h}(\tilde{z})$.

[1] In case $\varkappa = -1$,

$$(A)^- \qquad f^i = \cosh \circ \mathrm{dist}_{y^i}, \qquad \tilde{f}^i = \cosh \circ \mathrm{dist}_{\tilde{x}^i}.$$

Proof of the claim. Note that $\mathbf{d}_z h \geqslant 0$. Thus, for each i, we have[2]

$$\begin{aligned}
0 &\leqslant (\mathbf{d}_z h)(\uparrow_{[zy^i]})\\
&= -\sum_j \alpha_j \cdot |z-y^j| \cdot \cos\measuredangle\left[z\,{}^{y^i}_{y^j}\right]\\
(B)^0 \qquad &\leqslant -\sum_j \alpha_j \cdot |z-y^j| \cdot \cos\tilde\measuredangle^0\left(z\,{}^{y^i}_{y^j}\right)\\
&= -\tfrac{1}{2\cdot|z-y^i|}\cdot\sum_j \alpha_j\cdot\left[|z-y^i|^2+|z-y^j|^2-|y^i-y^j|^2\right].
\end{aligned}$$

In particular[3],

$$(C)^0 \qquad \sum_i \alpha_i\cdot\left[\sum_j \alpha_j\cdot\left[|z-y^i|^2+|z-y^j|^2-|y^i-y^j|^2\right]\right]\leqslant 0,$$

or[4]

$$(D)^0 \qquad 2\cdot h(z)\leqslant\sum_{i,j}\alpha_i\cdot\alpha_j\cdot|y^i-y^j|^2.$$

Note that if $\mathcal{U}\overset{iso}{=\!=}\mathbb{M}^n(\kappa)$, then all inequalities in (B,C,D) are sharp. Thus the same argument as above, repeated for $\tilde x^1,\tilde x^2,\dots,\tilde x^n$ in $\mathbb{M}^n(\kappa)$, gives[5]

$$(E)^0 \qquad 2\cdot\tilde h(\tilde z)=\sum_{i,j}\alpha_i\cdot\alpha_j\cdot|\tilde x^i-\tilde x^j|^2.$$

Note that

$$|\tilde x^i-\tilde x^j|\geqslant|x^i-x^j|\geqslant|y^i-y^j|$$

for all i, j. Thus, (D) and (E) imply the claim. □

[2] In case $\kappa=-1$, the same calculations give

$$(B)^- \qquad 0\leqslant\dots\leqslant-\tfrac{1}{\sinh|z-y^i|}\cdot\sum_j\alpha_j\cdot\left[\cosh|z-y^i|\cdot\cosh|z-y^j|-\cosh|y^i-y^j|\right].$$

[3] In case $\kappa=-1$, the same calculations give

$$(C)^- \qquad \sum_i\alpha_i\cdot\left[\sum_j\alpha_j\cdot\left[\cosh|z-y^i|\cdot\cosh|z-y^j|-\cosh|y^i-y^j|\right]\right]\leqslant 0.$$

[4] In case $\kappa=-1$,

$$(D)^- \qquad (h(z))^2\leqslant\sum_{i,j}\alpha_i\cdot\alpha_j\cdot\cosh|y^i-y^j|.$$

[5] In case $\kappa=-1$,

$$(E)^- \qquad (\tilde h(\tilde z))^2=\sum_{i,j}\alpha_i\cdot\alpha_j\cdot\cosh|\tilde x^i-\tilde x^j|.$$

10.17. Kirszbraun's theorem for nonpositive bound. *Let $\varkappa \leqslant 0$, $\mathcal{L}$ be a complete length* CBB($\varkappa$) *space, $\mathcal{U}$ be a complete length* CAT($\varkappa$) *space, $Q \subset \mathcal{L}$ be arbitrary subset and $f\colon Q \to \mathcal{U}$ be a short map. Then there is a short extension $F\colon \mathcal{L} \to \mathcal{U}$ of f; that is, there is a short map $F\colon \mathcal{L} \to \mathcal{U}$ such that $F|_Q = f$.*

Remark. If $\mathcal{U}$ is proper, then we do not need Helly's theorem (10.13); compactness of closed balls in $\mathcal{U}$ is sufficient in this case.

Proof of 10.17. By Zorn's lemma, we can assume that $Q \subset \mathcal{L}$ is a maximal set; that is, $f\colon Q \to \mathcal{U}$ does not admit a short extension to any larger set $Q' \supset Q$.

Let us argue by contradiction. Assume that $Q \neq \mathcal{L}$; choose $p \in \mathcal{L} \setminus Q$. Then

$$\bigcap_{x \in Q} \overline{\mathrm{B}}[f(x), |p-x|] = \emptyset.$$

Since $\varkappa \leqslant 0$, the balls are convex; thus, by Helly's theorem (10.13), one can choose points $x^1, x^2, \dots, x^n$ in Q such that

$$\bigcap_{i} \overline{\mathrm{B}}[y^i, |x^i - p|] = \emptyset, \tag{2}$$

where $y^i = f(x^i)$. Finally note that 2 contradicts the finite+one lemma (10.16). $\square$

Proof of Kirszbraun's theorem (10.14). The case $\varkappa \leqslant 0$ is already proved in 10.17. Thus it remains to prove the theorem only in case $\varkappa > 0$. After rescaling we can assume that $\varkappa = 1$ and therefore $\varpi\varkappa = \pi$.

Since $\overline{\mathrm{B}}[z, \pi/2]_{\mathcal{U}}$ is a complete length CAT($\varkappa$) space, we can assume $\mathcal{U} = \overline{\mathrm{B}}[z, \pi/2]_{\mathcal{U}}$. In particular, $\operatorname{diam} \mathcal{U} \leqslant \pi$.

Further, any two points $x, y \in \mathcal{U}$ such that $|x-y| < \pi$ are joined by a unique geodesic; if $|x-y| = \pi$, then the concatenation of $[xz]$ and $[zy]$ as a geodesic from x to y. Hence $\mathcal{U}$ is geodesic.

We may also assume that $\operatorname{diam} \mathcal{L} \leqslant \pi$. Otherwise $\mathcal{L}$ is one-dimensional (see 8.44); in this case the result follows since $\mathcal{U}$ is geodesic.

Assume the theorem is false. Then there is a set $Q \subset \mathcal{L}$, a short map $f\colon Q \to \mathcal{U}$, and $p \in \mathcal{L} \setminus Q$ such that

$$\bigcap_{x \in Q} \overline{\mathrm{B}}[f(x), |x-p|] = \emptyset. \tag{3}$$

We will apply 10.17 for $\varkappa = 0$ to the Euclidean cones $\mathring{\mathcal{L}} = \operatorname{Cone} \mathcal{L}$ and $\mathring{\mathcal{U}} = \operatorname{Cone} \mathcal{U}$. Note that

- $\mathring{\mathcal{U}}$ is a complete length CAT(0) space (see 11.7*a*),
- since $\operatorname{diam} \mathcal{L} \leqslant \pi$ we have $\mathring{\mathcal{L}}$ is CBB(0) (see 11.6*a*).

Further, we will view the spaces $\mathcal{L}$ and $\mathcal{U}$ as unit spheres in $\mathring{\mathcal{L}}$ and $\mathring{\mathcal{U}}$ respectively. In the cones $\mathring{\mathcal{L}}$ and $\mathring{\mathcal{U}}$ we will use "$|{*}|$" for distance to the tip, denoted by 0, "$\cdot$" for cone multiplication, "$\measuredangle(x,y)$" for $\measuredangle\left[0\,{}^{x}_{y}\right]$, and "$\langle x,y\rangle$" for $|x|\cdot|y|\cdot\cos\measuredangle\left[0\,{}^{x}_{y}\right]$. In particular,

- $|x-y|_{\mathcal{L}}=\measuredangle(x,y)$ for any $x,y\in\mathcal{L}$,
- $|x-y|_{\mathcal{U}}=\measuredangle(x,y)$ for any $x,y\in\mathcal{U}$,
- for any $y\in\mathcal{U}$, we have

$$\measuredangle(z,y)\leqslant\tfrac{\pi}{2}. \tag{4}$$

Let $\mathring{Q}=\operatorname{Cone}Q\subset\mathring{\mathcal{L}}$ and let $\mathring{f}\colon\mathring{Q}\to\mathring{\mathcal{U}}$ be the natural cone extension of f; that is, $y=f(x)\Rightarrow t\cdot y=\mathring{f}(t\cdot x)$ for $t\geqslant 0$. Clearly $\mathring{f}$ is short.

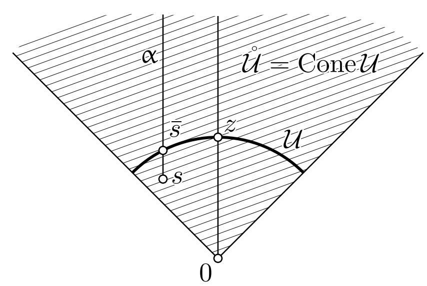
Applying 10.17 for $\mathring{f}$, we get a short extension map $\mathring{F}\colon\mathring{\mathcal{L}}\to\mathring{\mathcal{U}}$. Let $s=\mathring{F}(p)$. Then

$$|s-\mathring{f}(w)|\leqslant|p-w| \tag{5}$$

for any $w\in\mathring{Q}$. In particular, $|s|\leqslant 1$. Applying 5 for $w=t\cdot x$ and $t\to\infty$ we have

$$\langle f(x),s\rangle\geqslant\cos\measuredangle(p,x) \tag{6}$$

for any $x\in Q$.



By comparison, the geodesics $\mathrm{geod}_{[s\;t\cdot z]}$ converge as $t\to\infty$ and the limit is a half-line; denote it by $\alpha\colon[0,\infty)\to\mathring{\mathcal{U}}$. From 4, the function $t\mapsto\langle f(x),\alpha(t)\rangle$ is nondecreasing. From 6, for the necessarily unique point $\bar s$ on the half-line α such that $|\bar s|=1$, we also have

$$\langle f(x),\bar s\rangle\geqslant\cos\measuredangle(p,x)\quad\text{or, equivalently,}\quad\measuredangle(\bar s,f(x))\leqslant\measuredangle(p,f(x))$$

for any $x\in Q$, in contradiction to 3. □

10.18. Exercise. *Let $\mathcal{U}$ be* CAT(0). *Assume there are two point arrays, $(x^0,x^1,\dots,x^k)$ in $\mathcal{U}$ and $(\tilde x^0,\tilde x^1,\dots,\tilde x^k)$ in $\mathbb{E}^m$, such that $|x^i-x^j|_{\mathcal{U}}=|\tilde x^i-\tilde x^j|_{\mathbb{E}^m}$ for each i, j, and for any point $z_0\in\mathcal{U}$ there is $i>0$ such that $|z_0-x_i|\geqslant|x_0-x_i|$.*

Prove that there is a subset $Q\subset\mathcal{L}$ isometric to a convex set in $\mathbb{E}^m$ and containing all the points x^i.

10.19. Exercise. *Let $(x^0,x^1,\dots,x^k)$ in $\mathcal{L}$ and $(\tilde x^0,\tilde x^1,\dots,\tilde x^k)$ in $\mathbb{E}^m$ be two point arrays in complete length* CBB(0) *space $\mathcal{L}$. Assume that $|x^i-x^j|_{\mathcal{L}}=|\tilde x^i-\tilde x^j|_{\mathbb{E}^m}$ for each i, j and $\tilde x^0$ lies in the interior of* $\operatorname{Conv}(\tilde x^1,\dots,\tilde x^k)$.

Prove that there is a subset $Q\subset\mathcal{L}$ isometric to a convex set in $\mathbb{E}^m$ and containing all the points x^i.

The following statement we call $(2n+2)$-*point comparison*.

10.20. Exercise. *Let $\mathcal{U}$ be a complete length* $\mathrm{CAT}(\kappa)$ *space. Consider $x, y \in \mathcal{U}$ and an array $((p^1, q^1), (p^2, q^2), \dots, (p^n, q^n))$ of pairs of points in $\mathcal{U}$, such that there is a model configuration $\tilde{x}$, $\tilde{y}$ and array of pairs $((\tilde{p}^1, \tilde{q}^1), (\tilde{p}^2, \tilde{q}^2), \dots, (\tilde{p}^n, \tilde{q}^n))$ in $\mathbb{M}^3(\kappa)$ with the following properties:*

(*a*) $[\tilde{x}\tilde{p}^1\tilde{q}^1] = \tilde{\triangle}^{\kappa} x p^1 q^1$ *and* $[\tilde{y}\tilde{p}^n\tilde{q}^n] = \tilde{\triangle}^{\kappa} y p^n q^n$,

(*b*) *the simplex $\tilde{p}^i\tilde{p}^{i+1}\tilde{q}^i\tilde{q}^{i+1}$ is a model simplex of $p^i p^{i+1} q^i q^{i+1}$ for all i; that is,*

$$|\tilde{p}^i - \tilde{q}^i| = |p^i - q^i|,$$

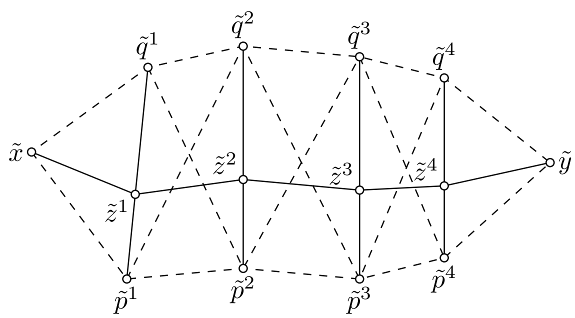
$$|\tilde{p}^i - \tilde{p}^{i+1}| = |p^i - p^{i+1}|,$$

$$|\tilde{q}^i - \tilde{q}^{i+1}| = |q^i - q^{i+1}|,$$

$$|\tilde{p}^i - \tilde{q}^{i+1}| = |p^i - q^{i+1}|,$$

and

$$|\tilde{p}^{i+1} - \tilde{q}^i| = |p^{i+1} - q^i|.$$



Then for any choice of n points $\tilde{z}^i \in [\tilde{p}^i\tilde{q}^i]$, we have

$$|\tilde{x} - \tilde{z}^1| + |\tilde{z}^1 - \tilde{z}^2| + \dots + |\tilde{z}^{n-1} - \tilde{z}^n| + |\tilde{z}^n - \tilde{y}| \geqslant |x - y|.$$

E. Remarks

The following problem and its relatives were mentioned by Michael Gromov [**71**, 1.19].

10.21. Open problem. *Find a necessary and sufficient condition for a finite metric space to admit distance-preserving embeddings into*

(*a*) *some length* $\mathrm{CBB}(\kappa)$ *space,*

(*b*) *some length* $\mathrm{CAT}(\kappa)$ *space.*

A metric on a finite set $\{a^1, a^2, \dots, a^n\}$, can be described by the matrix with components

$$s^{ij} = |a^i - a^j|^2,$$

which we will call the *associated matrix*. The set of associated matrices of all metrics that admit a distance-preserving map into a CBB(0) or a CAT(0) space form a convex cone. The latter follows since the rescalings and products of CBB(0) (or CAT(0)) spaces are CBB(0) (or CAT(0) respectively). This convexity gives a bit of hope that the cone admits an explicit description.

For the 5-point CAT(0) case, the (2+2)-comparison is a necessary and sufficient condition. This was proved by Tetsu Toyoda [**155**]; another proof was found by Nina Lebedeva and the third author [**104**]. For the 5-point CBB(0) case, the (1+4)-comparison is a necessary and sufficient condition; it was proved by Nina Lebedeva and the third author [**103**]. Starting from the 6-point case, only some necessary and some sufficient conditions are known; for more on the subject see [**8**, **102**, **104**].

The following conjecture (if true) would give the right generality for Kirszbraun's theorem (10.14). It states that ~~the~~ example 10.15 is the only obstacle to extending short maps.

10.22. Conjecture. *Assume $\mathcal{L}$ is a complete length* CBB(1) *space, $\mathcal{U}$ is a complete length* CAT(1) *space, $Q \subset \mathcal{L}$ is a proper subset, and $f\colon Q \to \mathcal{U}$ is a short map that does not admit a short extension to any bigger set $Q' \supset Q$. Then:*

(*a*) *Q is isometric to a sphere in a Hilbert space (of finite or cardinal dimension). Moreover, there is a point $p \in \mathcal{L}$ such that $|p - q| = \frac{\pi}{2}$ for any $q \in Q$.*

(*b*) *The map $f\colon Q \to \mathcal{U}$ is a distance-preserving map and there is no point $p' \in \mathcal{U}$ such that $|p' - q'| = \frac{\pi}{2}$ for any $q' \in f(Q)$.*

Curvature-free analogs. Let us present a collection of exercises on curvature-free analogs of Kirszbraun's theorem. It is worthwhile to know these results despite they are far from Alexandrov geometry.

10.23. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces, $A \subset \mathcal{X}$, and $f\colon A \to \mathcal{Y}$ be a short map. Assume $\mathcal{Y}$ is compact and for any finite set $F \subset \mathcal{X}$ there is a short map $F \to \mathcal{Y}$ that agrees with f on $F \cap A$. Then there is a short map $\mathcal{X} \to \mathcal{Y}$ that agrees with f on A.*

10.24. Exercise. *We say that a metric space $\mathcal{X}$ is* injective *if for an arbitrary metric space $\mathcal{Z}$ and a subset $Q \subset \mathcal{Z}$, any short map $Q \to \mathcal{X}$ can be extended as a short map $\mathcal{Z} \to \mathcal{X}$.*

(*a*) *Prove that any metric space $\mathcal{X}$ admits a distance-preserving embedding into an injective metric space.*

(*b*) *Use this to construct an analog of convex hull in the category of metric spaces; this is called the* injective hull.

Chapter 11

Warped products

The warped product is a construction that produces a new metric space, denoted by $\mathcal{B} \times_f \mathcal{F}$, from two metric spaces base $\mathcal{B}$ and fiber $\mathcal{F}$, and a function $f\colon \mathcal{B} \to \mathbb{R}_{\geqslant 0}$.

Many important constructions such as direct product, cone, spherical suspension, and join can be defined using warped products.

A. Definition

First we define the warped product for length spaces and then we expand the definition to allow for arbitrary fiber $\mathcal{F}$.

Let $\mathcal{B}$ and $\mathcal{F}$ be length spaces and $f\colon \mathcal{B} \to [0,\infty)$ be a continuous function.

For any path $\gamma\colon [0,1] \to \mathcal{B} \times \mathcal{F}$, we write $\gamma = (\gamma_{\mathcal{B}}, \gamma_{\mathcal{F}})$ where $\gamma_{\mathcal{B}}$ is the projection of γ to $\mathcal{B}$, and $\gamma_{\mathcal{F}}$ is the projection to $\mathcal{F}$. If $\gamma_{\mathcal{B}}$ and $\gamma_{\mathcal{F}}$ are Lipschitz, set

$$\text{(1)} \qquad \operatorname{length}_f \gamma \coloneqq \int\limits_0^1 \sqrt{v_{\mathcal{B}}^2 + (f\circ\gamma_{\mathcal{B}})^2 \cdot v_{\mathcal{F}}^2} \cdot dt,$$

where $\int$ is Lebesgue integral, and $v_{\mathcal{B}}$ and $v_{\mathcal{F}}$ are the speeds of $\gamma_{\mathcal{B}}$ and $\gamma_{\mathcal{F}}$ respectively. (Note that $\operatorname{length}_f \gamma \geqslant \operatorname{length} \gamma_{\mathcal{B}}$.)

Consider the pseudometric on $\mathcal{B} \times_f \mathcal{F}$ defined by

$$|x-y| \coloneqq \inf\left\{\operatorname{length}_f \gamma : \gamma(0) = x, \gamma(1) = y\right\}$$

where the exact lower bound is taken for all Lipschitz paths $\gamma\colon [0,1] \to \mathcal{B}\times\mathcal{F}$. The corresponding metric space is called the *warped product with base $\mathcal{B}$, fiber $\mathcal{F}$ and warping function f*; it will be denoted by $\mathcal{B} \times_f \mathcal{F}$.

The points in $\mathcal{B} \times_f \mathcal{F}$ can be described by corresponding pairs $(p, \phi) \in \mathcal{B} \times \mathcal{F}$. Note that if $f(p) = 0$ for $p \in \mathcal{B}$, then $(p, \phi) = (p, \psi)$ for any $\phi, \psi \in \mathcal{F}$.

We do not claim that every Lipschitz curve in $\mathcal{B} \times_f \mathcal{F}$ may be reparametrized as the image of a Lipschitz curve in $\mathcal{B} \times \mathcal{F}$; in fact this is not true.

11.1. Proposition. *The warped product $\mathcal{B} \times_f \mathcal{F}$ satisfies:*

(*a*) *The projection $(p, \phi_0) \mapsto p$ is a submetry. Moreover, its restriction to any horizontal leaf $\mathcal{B} \times \{\phi_0\}$ is an isometry to $\mathcal{B}$.*

(*b*) *If $f(p_0) \neq 0$, the projection $(p_0, \phi) \mapsto \phi$ of the vertical leaf $\{p_0\} \times \mathcal{F}$, with its length metric, is a homothety onto $\mathcal{F}$ with multiplier $\frac{1}{f(p_0)}$.*

(*c*) *If f achieves its (local) minimum at p_0, then the inclusion of the vertical leaf $\{p_0\} \times \mathcal{F}$ in $\mathcal{B} \times_f \mathcal{F}$ is (locally) distance-preserving.*

Proof. Claim (*b*) follows from the f-length formula 1.

Also, by 1, the projection of $\mathcal{B} \times_f \mathcal{F}$ onto $\mathcal{B} \times \{\phi_0\}$ given by $(p, \phi) \mapsto (p, \phi_0)$ is length-nonincreasing; hence (*a*).

Suppose p_0 is a local minimum point of f. Then the projection $(p, \phi) \mapsto (p_0, \phi)$ of a neighborhood of the vertical leaf $\{p_0\} \times \mathcal{F}$ to $\{p_0\} \times \mathcal{F}$ is length-nonincreasing.

If p_0 is a global minimum point of f, then the same holds for the projection of whole space. Hence (*c*). $\square$

Note that any horizontal leaf is weakly convex, but does not have to be convex even if $\mathcal{B} \times_f \mathcal{F}$ is a geodesic space. The latter follows since vanishing of the warping function f allows geodesics to bifurcate into distinct horizontal leaves. For instance, if there is a geodesic with the ends in the zero set

$$Z = \left\{ (p, \phi) \in \mathcal{B} \times_f \mathcal{F}) : f(p) = 0 \right\},$$

then there is a geodesic with the same endpoints in each horizontal leaf.

11.2. Proposition. *Suppose $\mathcal{B}$ and $\mathcal{F}$ are length spaces and $f : \mathcal{B} \to [0, \infty)$ is a continuous function. Then the warped product $\mathcal{B} \times_f \mathcal{F}$ is a length space.*

Proof. It is sufficient to show that for any $\alpha : [0,1] \to \mathcal{B} \times_f \mathcal{F}$ there is a path $\beta : [0,1] \to \mathcal{B} \times \mathcal{F}$ with the same endpoints such that

$$\operatorname{length} \alpha \geqslant \operatorname{length}_f \beta.$$

If $f \circ \alpha_{\mathcal{B}}(t) > 0$ for any t, then the vertical projection $\alpha_{\mathcal{F}}$ is defined. In this case, let $\beta(t) = (\alpha_{\mathcal{B}}(t), \alpha_{\mathcal{F}}(t)) \in \mathcal{B} \times \mathcal{F}$. Clearly

$$\operatorname{length} \alpha = \operatorname{length}_f \beta.$$

If $f \circ \alpha_{\mathcal{B}}(t_0) = 0$ for some t_0, let β be the concatenation of three curves in $\mathcal{B} \times \mathcal{F}$, namely: (1) the horizontal curve $(\alpha_{\mathcal{B}}(t), \phi)$ for $t \leqslant t_0$, (2) a vertical path in form (s, ϕ) to (s, ψ) and (3) the horizontal curve $(\alpha_{\mathcal{B}}(t), \psi)$ for $t \geqslant t_0$. By 1, the f-length of the middle path in the concatenation is vanishing; therefore the f-length of α cannot be smaller than length of $\alpha_{\mathcal{B}}$, that is,

$$\operatorname{length}_f \alpha \geqslant \operatorname{length} \alpha_{\mathcal{B}} = \operatorname{length}_f \beta.$$

The statement follows. □

The following theorem states that distance in a warped product is fiber-independent, in the sense that distances may be calculated by substituting for $\mathcal{F}$ a different length space:

11.3. Fiber-independence theorem. *Consider length spaces $\mathcal{B}$, $\mathcal{F}$ and $\check{\mathcal{F}}$, and a locally Lipschitz function $f\colon \mathcal{B} \to \mathbb{R}_{\geqslant 0}$. Assume $p, q \in \mathcal{B}$, $\phi, \psi \in \mathcal{F}$ and $\check{\phi}, \check{\psi} \in \check{\mathcal{F}}$. Then*

$$|\phi - \psi|_{\mathcal{F}} \geqslant |\check{\phi} - \check{\psi}|_{\check{\mathcal{F}}}$$
$$\Downarrow$$
$$|(p, \phi) - (q, \psi)|_{\mathcal{B} \times_f \mathcal{F}} \geqslant |(p, \check{\phi}) - (q, \check{\psi})|_{\mathcal{B} \times_f \check{\mathcal{F}}}.$$

In particular,

$$|(p, \phi) - (q, \psi)|_{\mathcal{B} \times_f \mathcal{F}} = |(p, 0) - (q, \ell)|_{\mathcal{B} \times_f \mathbb{R}},$$

where $\ell = |\phi - \psi|_{\mathcal{F}}$.

Proof. Let γ be a path in $(\mathcal{B} \times \mathcal{F})$.

Since $|\phi - \psi|_{\mathcal{F}} \geqslant |\check{\phi} - \check{\psi}|_{\check{\mathcal{F}}}$, there is a Lipschitz path $\gamma_{\check{\mathcal{F}}}$ from $\check{\phi}$ to $\check{\psi}$ in $\check{\mathcal{F}}$ such that

$$(\operatorname{speed} \gamma_{\mathcal{F}})(t) \geqslant (\operatorname{speed} \gamma_{\check{\mathcal{F}}})(t)$$

for almost all $t \in [0, 1]$. Consider the path $\check{\gamma} = (\gamma_{\mathcal{B}}, \gamma_{\check{\mathcal{F}}})$ from $(p, \check{\phi})$ to $(q, \check{\psi})$ in $\mathcal{B} \times_f \check{\mathcal{F}}$. Clearly

$$\operatorname{length}_f \gamma \geqslant \operatorname{length}_f \check{\gamma}.$$ □

11.4. Exercise. *Let $\mathcal{B}$ and $\mathcal{F}$ be length spaces and $f, g\colon \mathcal{B} \to \mathbb{R}_{\geqslant 0}$ be two locally Lipschitz nonnegative functions. Assume $f(b) \leqslant g(b)$ for any $b \in \mathcal{B}$. Show that $\mathcal{B} \times_f \mathcal{F} \leqslant \mathcal{B} \times_g \mathcal{F}$; that is, there is a distance-noncontracting map $\mathcal{B} \times_f \mathcal{F} \to \mathcal{B} \times_g \mathcal{F}$.*

B. Extended definitions

The fiber-independence theorem implies that

$$|(p, \phi) - (q, \psi)|_{\mathcal{B} \times_f \mathcal{F}} = |(p, 0) - (q, |\phi - \psi|_{\mathcal{F}})|_{\mathcal{B} \times_f \mathbb{R}}$$

for any $(p, \phi), (q, \psi) \in \mathcal{B} \times \mathcal{F}$. In particular, if $\iota\colon A \to \check{A}$ is an isometry between two subsets $A \subset \mathcal{F}$ and $\check{A} \subset \check{\mathcal{F}}$ in length spaces $\mathcal{F}$ and $\check{\mathcal{F}}$, and $\mathcal{B}$ is a length space, then for any warping function $f\colon \mathcal{B}\to\mathbb{R}_{\geqslant 0}$, the map ι induces an isometry between the sets $\mathcal{B}\times_f A\subset\mathcal{B}\times_f\mathcal{F}$ and $\mathcal{B}\times_f\check{A}\subset\mathcal{B}\times_f\check{\mathcal{F}}$.

This observation allows us to define the warped product $\mathcal{B}\times_f\mathcal{F}$ where the fiber $\mathcal{F}$ does not carry its length metric. Indeed we can use Kuratowsky embedding to realize $\mathcal{F}$ as a subspace in a length space, say $\mathcal{F}'$. Therefore we can take the warped product $\mathcal{B}\times_f\mathcal{F}'$ and identify $\mathcal{B}\times_f\mathcal{F}$ with its subspace consisting of all pairs (b,ϕ) such that $\phi\in\mathcal{F}$. According to the Fiber-independence theorem 11.3, the resulting space does not depend on the choice of $\mathcal{F}'$.

C. Examples

Direct product. The simplest example is the *direct product* $\mathcal{B}\times\mathcal{F}$, which can also be written as the warped product $\mathcal{B}\times_1\mathcal{F}$. That is, for $p,q\in\mathcal{B}$ and $\phi,\psi\in\mathcal{F}$, the latter metric simplifies to

$$|(p,\phi)-(q,\psi)|=\sqrt{|p-q|^2+|\phi-\psi|^2}.$$

Cones. The *Euclidean cone* $\operatorname{Cone}\mathcal{F}$ over a metric space $\mathcal{F}$ can be written as the warped product $[0,\infty)\times_{\mathrm{id}}\mathcal{F}$. That is, for $s,t\in[0,\infty)$ and $\phi,\psi\in\mathcal{F}$, the metric is given by the cosine rule

$$|(s,\phi)-(t,\psi)|=\sqrt{s^2+t^2-2\cdot s\cdot t\cdot\cos\alpha},$$

where $\alpha=\min\{\pi,|\phi-\psi|\}$. (See Section 6E.)

Instead of the Euclidean cosine rule, we may use the cosine rule in $\mathbb{M}^2(\kappa)$:

$$|(s,\phi)-(t,\psi)|=\tilde{\gamma}^{\kappa}\{\alpha;s,t\}.$$

This way we get *κ-cones* over $\mathcal{F}$, denoted by $\operatorname{Cone}^\kappa\mathcal{F}=[0,\infty)\times_{\operatorname{sn}^\kappa}\mathcal{F}$ for $\kappa\leqslant 0$ and $\operatorname{Cone}^\kappa\mathcal{F}=[0,\varpi\kappa]\times_{\operatorname{sn}^\kappa}\mathcal{F}$ for $\kappa>0$.

The 1-cone $\operatorname{Cone}^1\mathcal{F}$ is also called the *spherical suspension* over $\mathcal{F}$; it is also denoted by $\operatorname{Susp}\mathcal{F}$. That is,

$$\operatorname{Susp}\mathcal{F}=[0,\pi]\times_{\sin}\mathcal{F}.$$

11.5. Exercise. *Let $\mathcal{F}$ be a length space and $A\subset\mathcal{F}$. Show that* $\operatorname{Cone}^\kappa A$ *is convex in* $\operatorname{Cone}^\kappa\mathcal{F}$ *if and only if A is π-convex in $\mathcal{F}$.*

Doubling. The doubling space $\mathcal{W}$ of a metric space $\mathcal{V}$ on a closed subset $A\subset\mathcal{V}$ can be also defined as a special type of warped product. Consider the fiber $\mathbb{S}^0$ consisting of two points with distance 2 from each other. Then

$$\mathcal{W}\overset{iso}{=\!=}\mathcal{V}\times_{\operatorname{dist}_A}\mathbb{S}^0;$$

that is, $\mathcal{W}$ is isometric to the warped product with base $\mathcal{V}$, fiber $\mathbb{S}^0$ and warping function dist_A.

D. 1-dimensional base

The following theorems provide conditions for the spaces and functions in a warped product with 1-dimensional base to have curvature bounds. These theorems are originally due to Valerii Berestovskii [**23**]. They are baby cases of the characterization of curvature bounds in warped products given in [**6**, **7**].

11.6. Theorem.

(*a*) *If* $\mathcal{L}$ *is a complete length* CBB(1) *space and* diam $\mathcal{L} \leqslant \pi$, *then*

$$\begin{aligned} \operatorname{Susp}\mathcal{L} &= [0,\pi]\times_{\sin}\mathcal{L} && \textit{is}\quad \mathrm{CBB}(1),\\ \operatorname{Cone}\mathcal{L} &= [0,\infty)\times_{\mathrm{id}}\mathcal{L} && \textit{is}\quad \mathrm{CBB}(0),\\ \operatorname{Cone}^{-1}\mathcal{L} &= [0,\infty)\times_{\sinh}\mathcal{L} && \textit{is}\quad \mathrm{CBB}(-1). \end{aligned}$$

Moreover, the converse also holds in each of the three cases.

(*b*) *If* $\mathcal{L}$ *is a complete length* CBB(0) *space, then*

$$\begin{aligned} &\mathbb{R}\times\mathcal{L} && \textit{is a complete length } \mathrm{CBB}(0)\textit{ space},\\ &\mathbb{R}\times_{\exp}\mathcal{L} && \textit{is a complete length } \mathrm{CBB}(-1)\textit{ space}. \end{aligned}$$

Moreover, the converse also holds in each of the two cases.

(*c*) *If* $\mathcal{L}$ *is a complete length* CBB(−1) *space, if and only if the warped product* $\mathbb{R}\times_{\cosh}\mathcal{L}$ *is a complete length* CBB(−1) *space.*

11.7. Theorem. *Let* $\mathcal{L}$ *be a metric space.*

(*a*) *If* $\mathcal{L}$ *is* CAT(1), *then*

$$\begin{aligned} \operatorname{Susp}\mathcal{L} &= [0,\pi]\times_{\sin}\mathcal{L} && \textit{is}\quad \mathrm{CAT}(1),\\ \operatorname{Cone}\mathcal{L} &= [0,\infty)\times_{\mathrm{id}}\mathcal{L} && \textit{is}\quad \mathrm{CAT}(0),\\ \operatorname{Cone}^{-1}\mathcal{L} &= [0,\infty)\times_{\sinh}\mathcal{L} && \textit{is}\quad \mathrm{CAT}(-1). \end{aligned}$$

Moreover, the converse also holds in each of the three cases.

(*b*) *If* $\mathcal{L}$ *is a complete length* CAT(0) *space, then* $\mathbb{R}\times\mathcal{L}$ *is* CAT(0) *and* $\mathbb{R}\times_{\exp}\mathcal{L}$ *is* CAT(−1). *Moreover, the converse also holds in each of the two cases.*

(*c*) *If* $\mathcal{L}$ *is* CAT(−1) *if and only if* $\mathbb{R}\times_{\cosh}\mathcal{L}$ *is* CAT(−1).

In the proof of the above two theorems we will use the following proposition.

11.8. Proposition.

(*a*)

$$\operatorname{Susp}\mathbb{S}^{m-1} = [0,\pi]\times_{\sin}\mathbb{S}^{m-1} \overset{iso}{=\!=} \mathbb{S}^m,$$

$$\operatorname{Cone}\mathbb{S}^{m-1} = [0,\infty)\times_{\mathrm{id}}\mathbb{S}^{m-1} \overset{iso}{=\!=} \mathbb{E}^m,$$

$$\operatorname{Cone}^{-1}\mathbb{S}^{m-1} = [0,\infty)\times_{\sinh}\mathbb{S}^{m-1} \overset{iso}{=\!=} \mathbb{M}^m(-1).$$

(*b*)

$$\mathbb{R}\times\mathbb{E}^{m-1} \overset{iso}{=\!=} \mathbb{E}^m,$$

$$\mathbb{R}\times_{\exp}\mathbb{E}^{m-1} \overset{iso}{=\!=} \mathbb{M}^m(-1).$$

(*c*)

$$\mathbb{R}\times_{\cosh}\mathbb{M}^{m-1}(-1) \overset{iso}{=\!=} \mathbb{M}^m(-1).$$

The proof is left to the reader.

Proof of 11.6 and 11.7. Each proof is based on the fiber-independence theorem 11.3 and the corresponding statement in Proposition 11.8.

Let us prove the last statement in (*a*); the remaining statements of this theorem are similar. Choose an arbitrary quadruple of points

$$(s,\phi),(t^1,\phi^1),(t^2,\phi^2),(t^3,\phi^3)\in[0,\infty)\times_{\sinh}\mathcal{L}.$$

Since $\operatorname{diam}\mathcal{L}\leqslant\pi$, the (1+3)-point comparison (10.8) provides a quadruple of points $\psi,\psi^1,\psi^2,\psi^3\in\mathbb{S}^3$ such that

$$|\psi-\psi^i|_{\mathbb{S}^3} = |\phi-\phi^i|_{\mathcal{L}}$$

and

$$|\psi^i-\psi^j|_{\mathbb{S}^3} \geqslant |\phi^i-\phi^j|_{\mathcal{L}}$$

for all i and j.

According to Proposition 11.8*a*,

$$\operatorname{Cone}^{-1}\mathbb{S}^3 = [0,\infty)\times_{\sinh}\mathbb{S}^3 \overset{iso}{=\!=} \mathbb{M}^4(-1).$$

Consider the quadruple of points

$$(s,\psi),(t^1,\psi^1),(t^2,\psi^2),(t^3,\psi^3)\in\operatorname{Cone}^{-1}\mathbb{S}^3=\mathbb{M}^4(-1).$$

By the fiber-independence theorem 11.3,

$$|(s,\psi)-(t^i,\psi^i)|_{[0,\infty)\times_{\sinh}\mathbb{S}^3} = |(s,\phi)-(t^i,\phi^i)|_{[0,\infty)\times_{\sinh}\mathcal{L}}$$

and

$$|(t^i,\psi^i)-(t^j,\psi^j)|_{[0,\infty)\times_{\sinh}\mathbb{S}^3} \geqslant |(t^i,\phi^i)-(t^j,\phi^j)|_{[0,\infty)\times_{\sinh}\mathcal{L}}$$

for all i and j. Since four points of $\mathbb{M}^4(-1)$ lie in an isometric copy of $\mathbb{M}^3(-1)$, it remains to apply Exercise 8.3.

Now let us prove the converse to (*a*). Choose a quadruple $\phi, \phi^1, \phi^2, \phi^3 \in \mathcal{L}$ with all distances smaller than $\frac{\pi}{2}$. Choose small $s > 0$ and let t_i be the hypotenuse in a hyperbolic right triangle with angle $|\phi - \phi^i|_{\mathcal{L}}$ and the adjacent side s. Observe that the $\mathbb{M}(-1)$ model angles of the quadruple (s, ϕ), (t^1, ϕ^1), (t^2, ϕ^2), (t^3, ϕ^3) in $[0, \infty) \times_{\sinh} \mathcal{L}$ at (s, ϕ) are the same as the $\mathbb{M}(1)$ model angles of the quadruple $\phi, \phi^1, \phi^2, \phi^3 \in \mathcal{L}$ at ϕ. Whence CBB(1) comparison holds for $\phi, \phi^1, \phi^2, \phi^3$; in particular, $\mathcal{L}$ is locally CBB(1). It remains to apply the globalization theorem (8.31).

The proof of 11.7 is nearly identical, but one has to apply (2+2)-comparison (9.5). $\square$

11.9. Exercise. *The* spherical join $\mathcal{U} \star \mathcal{V}$ *of two metric spaces $\mathcal{U}$ and $\mathcal{V}$ is defined as the unit sphere equipped with the angle metric in the product of Euclidean cones* Cone $\mathcal{U} \times$ Cone $\mathcal{V}$.

Assume $\mathcal{U}$ and $\mathcal{V}$ are nonempty spaces.

(*a*) *Show that $\mathcal{U} \star \mathcal{V}$ is* CAT(1) *if and only if $\mathcal{U}$ and $\mathcal{V}$ are* CAT(1)*.*

(*b*) *Suppose $\mathcal{U}$ and $\mathcal{V}$ have* diam $\leqslant \pi$. *Show that $\mathcal{U} \star \mathcal{V}$ is* CBB(1) *if and only if $\mathcal{U}$ and $\mathcal{V}$ are* CBB(1)*.*

E. Remarks

Let us formulate general results on curvature bounds of warped products proved by the first author and Richard Bishop [**7**].

11.10. Theorem. *Let $\mathcal{B}$ be a complete finite-dimensional* CBB(κ) *length space, and $f\colon \mathcal{B} \to \mathbb{R}_{\geqslant 0}$ be a locally Lipschitz function. Denote by $Z \subset \mathcal{B}$ the zero set of the restriction of f to the boundary $\partial\mathcal{B}$ of $\mathcal{B}$.*

Suppose that $\mathcal{W}$ is doubling of $\mathcal{B}$ along the closure of $\partial\mathcal{B} \setminus Z$, and $\bar f\colon \mathcal{W} \to \mathbb{R}_{\geqslant 0}$ is the natural extension of f. Assume that $\mathcal{W}$ is CBB(κ) *and $\bar f'' + \kappa \cdot \bar f \leqslant 0$.*

Suppose $\mathcal{F}$ is a complete finite-dimensional CBB(κ') *space. Then the warped product $\mathcal{B} \times_f \mathcal{F}$ is* CBB(κ) *in the following two cases:*

(*a*) *If $Z = \emptyset$ and $\kappa' \geqslant \kappa \cdot f^2(b)$ for any $b \in \mathcal{B}$.*

(*b*) *If $Z \neq \emptyset$ and $|d_z f|^2 \leqslant \kappa'$ for any $z \in Z$.*

We mention that in the setting of this theorem, f necessarily vanishes only at boundary points if f is not identically 0.

11.11. Theorem. *Let $\mathcal{B}$ be a complete* CAT(κ) *length space, and the function $f\colon \mathcal{B} \to \mathbb{R}_{\geqslant 0}$ satisfy $f'' + \kappa \cdot f \geqslant 0$, where f is Lipschitz on bounded sets or B is locally compact. Denote by $Z \subset \mathcal{B}$ the zero set of f. Suppose $\mathcal{F}$ is a complete* CAT(κ') *space. Then the warped product $\mathcal{B} \times_f \mathcal{F}$ is* CAT(κ) *in the following two cases:*

(a) *If* $Z = \emptyset$ *and* $\kappa' \leqslant \kappa \cdot f^2(b)$ *for any* $b \in \mathcal{B}$.

(b) *If* $Z \neq \emptyset$ *and* $[(d_z f) \uparrow_{[zb]}]^2 \geqslant \kappa'$ *for any minimizing geodesic* $[zb]$ *from* Z *to a point* $b \in \mathcal{B}$ *and* $\kappa' \leqslant \kappa \cdot f^2(b)$ *for any* $b \in \mathcal{B}$ *such that* $\operatorname{dist}_Z b \geqslant \frac{\varpi\kappa}{2}$.

Chapter 12

Polyhedral spaces

A. Definitions

12.1. Definition. *A length space $\mathcal{P}$ is called a* piecewise $\mathbb{M}(\kappa)$ *if it admits a finite triangulation τ such that an arbitrary simplex σ in τ is isometric to a simplex in the model space $\mathbb{M}^{\dim\sigma}(\kappa)$.*

By triangulation *of a piecewise $\mathbb{M}(\kappa)$ space we will understand a triangulation as in the definition. If we do not wish to specify κ, we will say that $\mathcal{P}$ is a* polyhedral space.

By rescaling we can assume that $\kappa = 1, 0,$ or -1.

(a) *Piecewise $\mathbb{M}(1)$ spaces will also be called* spherical polyhedral spaces;

(b) *Piecewise $\mathbb{M}(0)$ spaces will also be called* Euclidean polyhedral spaces;

(c) *Piecewise $\mathbb{M}(-1)$ spaces will also be called* hyperbolic polyhedral spaces. ■

Note that according to the above definition, all polyhedral spaces are compact. However, most of the statements below admit straightforward generalizations to *locally polyhedral space*; that is, complete length spaces, any point of which admits a closed neighborhood isometric to a polyhedral space. The latter class of spaces includes in particular infinite covers of polyhedral spaces.

The dimension of a polyhedral space $\mathcal{P}$ is defined as the maximal dimension of a simplex in one (and therefore any) triangulation of $\mathcal{P}$.

Links. Let $\mathcal{P}$ be a polyhedral space and σ be a simplex in its triangulation τ.

The simplexes that contain σ form an abstract simplicial complex called the *link* of σ, denoted by Link_σ. If $m = \dim\sigma$, then the set of vertexes of Link_σ is formed by the $(m+1)$-simplexes that contain σ; the set of its edges are formed by the $(m+2)$-simplexes that contain σ, and so on.

The link Link_σ can be identified with the subcomplex of τ formed by all the simplexes σ' such that $\sigma\cap\sigma'=\emptyset$ but both σ and σ' are faces of a simplex of τ.

The points in Link_σ can be identified with the normal directions to σ at a point in the interior of σ. The angle metric between directions makes Link_σ into a spherical polyhedral space. We will always consider the link with this metric.

Tangent space and space of directions. Let τ be a triangulation of a polyhedral space $\mathcal{P}$. If a point $p\in\mathcal{P}$ lies in the interior of a k-simplex σ of τ then the tangent space $\mathrm{T}_p\mathcal{P}$ is naturally isometric to

$$\mathbb{E}^k\times(\mathrm{Cone}\,\mathrm{Link}_\sigma).$$

Equivalently, the space of directions Σ_p can be isometrically identified with the k-th spherical suspension over Link_σ; that is,

$$\Sigma_p\stackrel{iso}{=\!=}\mathrm{Susp}^k(\mathrm{Link}_\sigma).$$

If $\mathcal{P}$ is an m-dimensional polyhedral space, then for any $p\in\mathcal{P}$ the space of directions Σ_p is a spherical polyhedral space of dimension at most $m-1$.

In particular, for any point p in the interior of a simplex σ, the isometry class of Link_σ and $k=\dim\sigma$ determine the isometry class of Σ_p and the other way around.

A small neighborhood of p is isometric to a neighborhood of the tip of the κ-cone over Σ_p. In fact, if this property holds at any point of a compact length space $\mathcal{P}$ then $\mathcal{P}$ is a piecewise $\mathbb{M}(\kappa)$ space [**101**].

B. Curvature bounds

Recall that ℓ-simply connected spaces are defined in 9.66.

The following theorem provides a combinatorial description of polyhedral spaces with curvature bounded above.

12.2. Theorem. *Let $\mathcal{P}$ be a piecewise $\mathbb{M}(\kappa)$ space and τ be a triangulation of $\mathcal{P}$. Then*

(*a*) *$\mathcal{P}$ is locally* CAT(κ) *if and only if any connected component of the link of any simplex σ in τ is $(2\cdot\pi)$-simply connected. Equivalently, if and only if any closed local geodesic in* Link_σ *has length at least $2\cdot\pi$.*

(*b*) *$\mathcal{P}$ is a complete length* CAT(κ) *space if and only if $\mathcal{P}$ is $(2\cdot\varpi\kappa)$-simply connected and any connected component of the link of any simplex σ in τ is $(2\cdot\pi)$-simply connected.*

Proof. We will prove only the if part; the only-if part is evident by the generalized Hadamard–Cartan theorem (9.67) and Theorem 11.6.

Let us apply induction on $\dim \mathcal{P}$. The *base* case $\dim \mathcal{P} = 0$ is evident.

Induction Step. Assume that the theorem is proved in the case $\dim \mathcal{P} < m$. Suppose $\dim \mathcal{P} = m$.

Fix a point $p \in \mathcal{P}$. A neighborhood of p is isometric to a neighborhood of the tip in the $\varkappa$-cone over Σ_p. By Theorem 11.6*a*, it is sufficient to show that

(1) Σ_p *is* CAT(1).

Note that Σ_p is a spherical polyhedral space and its links are isometric to links of $\mathcal{P}$. By the induction hypothesis, Σ_p is locally CAT(1). Applying the generalized Hadamard–Cartan theorem (9.67), we get 1.

Part (*b*) follows from the generalized Hadamard–Cartan theorem. □

12.3. Exercise. *Show that any metric tree is* CAT($\varkappa$) *for any* $\varkappa$.

12.4. Exercise. *Show that if in a Euclidean polyhedral space* $\mathcal{P}$ *any two points can be connected by a unique geodesic, then* $\mathcal{P}$ *is* CAT(0).

The following theorem provides a combinatorial description of polyhedral spaces with curvature bounded below.

12.5. Theorem. *Let* $\mathcal{P}$ *be a piecewise* $\mathbb{M}(\varkappa)$ *space and* τ *be a triangulation of* $\mathcal{P}$. *Then* $\mathcal{P}$ *is* CBB($\varkappa$) *if and only if the following conditions hold.*

(*a*) τ *is* pure*; that is, any simplex in* τ *is a face of some simplex of dimension exactly* m.

(*b*) *The link of any simplex of dimension* $m-1$ *is formed by a single point or two points.*

(*c*) *Any link of any simplex of dimension* $m-2$ *has diameter at most* π.

(*d*) *The link of any simplex of dimension* $\leqslant m-2$ *is connected.*

Remarks. Condition (*d*) can be reformulated in the following way:

(*d*)′ Any path $\gamma\colon [0,1] \to \mathcal{P}$ can be approximated by paths γ_n that cross only simplexes of dimension m and $m-1$.

Further, modulo the other conditions, condition (*c*) is equivalent to the following:

(*c*)′ The link of any simplex of dimension $m-2$ is isometric to a circle of length $\leqslant 2\cdot\pi$ or a closed real interval of length $\leqslant \pi$.

Proof. We will prove the if part. The only-if part is similar and is left to the reader.

We apply induction on m. The *base* case $m = 1$ follows from ~~the~~ assumption (b).

Step. Assume that the theorem is proved for polyhedral spaces of dimesnion less than m. Suppose $\dim \mathcal{P} = m$.

According to the globalization theorem (8.31), it is sufficient to show that $\mathcal{P}$ is locally $\mathrm{CBB}(\kappa)$.

Fix $p \in \mathcal{P}$. Note that a spherical neighborhood of p is isometric to a spherical neighborhood of the tip of the tangent κ-cone

$$\mathrm{T}_p^\kappa = \mathrm{Cone}^\kappa(\Sigma_p).$$

Hence it is sufficient to show that

(2) $\quad$ T_p^κ *is* $\mathrm{CBB}(\kappa)$ *for any* $p \in \mathcal{P}$.

By Theorem 11.6a, the latter is equivalent to

(3) $\quad$ $\operatorname{diam} \Sigma_p \leqslant \pi$ *and* Σ_p *is* $\mathrm{CBB}(1)$.

If $m = 2$, then 3 follows from (b).

To prove the case $m \geqslant 3$, note that Σ_p is an $(m-1)$-dimensional spherical polyhedral space and all the conditions of the theorem hold for Σ_p. It remains to apply the induction hypothesis. □

12.6. Exercise. *Assume $\mathcal{P}$ is a piecewise $\mathbb{M}(\kappa)$ space and $\dim \mathcal{P} \geqslant 2$. Show that*

(a) *if $\mathcal{P}$ is $\mathrm{CBB}(\kappa')$, then $\kappa' \leqslant \kappa$ and $\mathcal{P}$ is $\mathrm{CBB}(\kappa)$,*

(b) *if $\mathcal{P}$ is $\mathrm{CAT}(\kappa')$, then $\kappa' \geqslant \kappa$ and $\mathcal{P}$ is $\mathrm{CAT}(\kappa)$.*

C. Flag complexes

12.7. Definition. *A simplicial complex $\mathcal{S}$ is* flag *if whenever $\{v_0, \dots, v_k\}$ is a set of distinct vertexes of $\mathcal{S}$ that are pairwise joined by edges, then the vertexes $v_0, \dots, v_k$ span a k-simplex in $\mathcal{S}$.*

If the above condition is satisfied for $k = 2$, then we say $\mathcal{S}$ satisfies the no-triangle condition.

Note that every flag complex is determined by its 1-skeleton.

12.8. Proposition. *A simplicial complex $\mathcal{S}$ is flag if and only if $\mathcal{S}$, as well as all the links of all its simplexes, satisfy the no-triangle condition.*

From the definition of flag complex we get the following:

12.9. Observation. *Any link of a flag complex is flag.*

Proof of Proposition 12.8. By Observation 12.9, the no-triangle condition holds for any flag complex and all its links.

Now assume a complex $\mathcal{S}$ and all its links satisfy the no-triangle condition. It follows that $\mathcal{S}$ includes a 2-simplex for each triangle. Applying the same observation for each edge we get that $\mathcal{S}$ includes a 3-simplex for any complete graph with 4 vertexes. Repeating this observation for triangles, 4-simplexes, 5-simplexes and so on we get that $\mathcal{S}$ is flag. □

Right-angled triangulation. A triangulation of a spherical polyhedral space is called *right-angled* if each simplex of the triangulation is isometric to a spherical simplex all of whose angles are right. Similarly, we say that a simplicial complex is equipped with a *right-angled spherical metric* if it is a length metric and each simplex is isometric to a spherical simplex all of whose angles are right.

Spherical polyhedral CAT(1) spaces glued from right-angled simplexes admit the following characterization discovered by Michael Gromov [**76**, p. 122].

12.10. Flag condition. *Assume that a spherical polyhedral space $\mathcal{P}$ admits a right-angled triangulation τ. Then $\mathcal{P}$ is* CAT(1) *if and only if τ is flag.*

Proof.

Only-if part. Assume there are three vertexes v_1, v_2 and v_3 of τ that are pairwise joined by edges but do not span a simplex. Note that in this case

$$\measuredangle\left[v_1\,{}^{v_2}_{v_3}\right] = \measuredangle\left[v_2\,{}^{v_3}_{v_1}\right] = \measuredangle\left[v_3\,{}^{v_1}_{v_1}\right] = \pi.$$

Equivalently,

(1) *The concatenation of the geodesics $[v_1v_2]$, $[v_2v_3]$ and $[v_3v_1]$ forms a closed local geodesic in $\mathcal{P}$.*

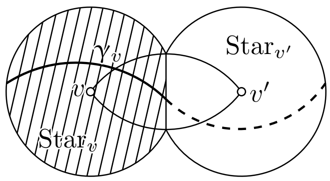
Now assume that $\mathcal{P}$ is CAT(1). Then by 11.6*a*, $\operatorname{Link}_\sigma \mathcal{P}$ is a compact length CAT(1) space for every simplex σ in τ.

Each of these links is a right-angled spherical complex and by Theorem 12.2, none of these links can contain a geodesic circle of length less than $2\cdot\pi$.

Therefore Proposition 12.8 and 1 imply the only-if part.

If part. By Observation 12.9 and Theorem 12.2, it is sufficient to show that any closed local geodesic γ in a flag complex $\mathcal{S}$ with right-angled metric has length at least $2\cdot\pi$.

Fix a flag complex $\mathcal{S}$. Recall that the *star* of a vertex v (briefly $\overline{\operatorname{Star}}_v$) is formed by all the simplexes containing v. Similarly, Star_v, the open star of v, is the union of all simplexes containing v with faces opposite v removed.



Choose a simplex σ that contains a point of γ. Let v be a vertex of σ. Set $f(t) = \cos|v - \gamma(t)|$. Note that

$$f''(t) + f(t) = 0$$

if $f(t) > 0$. Since the zeroes of sine are π apart, γ spends time π on every visit to Star_v.

After leaving Star_v, the local geodesic γ must enter another simplex, say σ', which has a vertex v' not joined to v by an edge.

Since τ is flag, the open stars Star_v and $\mathrm{Star}_{v'}$ do not overlap. The same argument as above shows that γ spends time π on every visit to $\mathrm{Star}_{v'}$. Therefore the total length of γ is at least $2 \cdot \pi$. □

12.11. Exercise. *Show that the barycentric subdivision of any simplicial complex is flag. Conclude that any finite simplicial complex is homeomorphic to a compact length* CAT(1) *space.*

12.12. Exercise. *Let p be a point in a product of metric trees. Show that a closed geodesic in the space of directions Σ_p has length either $2 \cdot \pi$ or at least $3 \cdot \pi$.*

12.13. Exercise. *Assume that a spherical polyhedral space $\mathcal{P}$ admits a triangulation τ such that all edgelengths of all simplexes in τ are at least $\frac{\pi}{2}$ Show that $\mathcal{P}$ is* CAT(1) *if τ is flag.*

12.14. Exercise. *Let $\phi_1, \phi_2, \dots, \phi_k \colon \mathcal{U} \to \mathcal{U}$ be commuting short retractions of a complete length* CAT(0) *space; that is,*

- *$\phi_i \circ \phi_i = \phi_i$ for each i;*
- *$\phi_i \circ \phi_j = \phi_j \circ \phi_i$ for any i and j;*
- *$|\phi_i(x) - \phi_i(y)|_{\mathcal{U}} \leqslant |x - y|_{\mathcal{U}}$ for each i and any $x, y \in \mathcal{U}$.*

Set $A_i = \mathfrak{I}\phi_i$ for all i. Note that each A_i is a weakly convex set.

Assume Γ is a finite graph (*without loops and multiple edges*) *with edges labeled by* 1, 2, …, *n. Denote by $\mathcal{U}^\Gamma$ the space obtained by taking a copy of $\mathcal{U}$ for each vertex of Γ and gluing two such copies along A_i if the corresponding vertexes are joined by an edge labeled by i.*

Show that $\mathcal{U}^\Gamma$ is CAT(0)

The space of trees. The following construction is given by Louis Billera, Susan Holmes, and Karen Vogtmann [**27**].

Let $\mathcal{T}_n$ be the set of all metric trees with n end-vertexes labeled by $a_1, \dots, a_n$. To describe one tree in $\mathcal{T}_n$ we may fix a topological tree τ with end vertexes a_1, …, a_n and all the other vertexes of degree 3, and prescribe the lengths of $2 \cdot n - 3$ edges. If the length of an edge is 0, we assume that edge degenerates; such a tree can be also described using a different topological tree τ'. The subset of $\mathcal{T}_n$

corresponding to the given topological tree τ can be identified with a convex closed cone in $\mathbb{R}^{2\cdot n-3}$. Equip each such subset with the metric induced from $\mathbb{R}^{2\cdot n-3}$ and consider the length metric on $\mathcal{T}_n$ induced by these metrics.

12.15. Exercise. *Show that $\mathcal{T}_n$ with the described metric is* CAT(0)*.*

Cubical complexes. The definition of a cubical complex mostly repeats the definition of a simplicial complex, with simplexes replaced by cubes.

Formally, a *cubical complex* is defined as a subcomplex of the unit cube in Euclidean space of large dimension; that is, a collection of faces of the cube, that with each face contains all its sub-faces. Each cube face in this collection will be called a *cube* of the cubical complex.

Note that according to this definition, any cubical complex is finite, that is, contains a finite number of cubes.

The union of all the cubes in a cubical complex $\mathcal{Q}$ will be called its *underlying space*; it will be denoted by $\mathcal{Q}$ or by $\underline{\mathcal{Q}}$ if we need to emphasize that we are talking about a topological space, not a complex. A homeomorphism from $\underline{\mathcal{Q}}$ to a topological space $\mathcal{X}$ is called a *cubulation of* $\mathcal{X}$.

The underlying space of a cubical complex $\mathcal{Q}$ will be always considered with the length metric induced from $\mathbb{R}^N$. In particular, with this metric, each cube of $\mathcal{Q}$ is isometric to the unit cube of the same dimension.

It is straightforward to construct a triangulation of $\underline{\mathcal{Q}}$ such that each simplex is isometric to a Euclidean simplex. In particular, $\underline{\mathcal{Q}}$ is a Euclidean polyhedral space.

The link of each cube in a cubical complex admits a natural right-angled triangulation; each simplex corresponds to an adjusted cube.

12.16. Exercise. *Show that a cubical complex $\mathcal{Q}$ is locally* CAT(0) *if and only if the link of each vertex in $\mathcal{Q}$ is flag.*

D. Remarks

The condition on polyhedral CAT(κ) spaces given in Theorem 12.2 might look easy to use, but in fact, it is hard to check even in simple cases. For example, the description of those coverings of $\mathbb{S}^3$ that branch at three great circles and are CAT(1) requires quite a bit of work; an answer is given by Ruth Charney and Michael Davis [**49**].

Analogs of the flag condition for spherical Coxeter simplexes could resolve the following problem.

12.17. Braid space problem. *Consider $\mathbb{C}^n$ with coordinates $z_1, \dots, z_n$. Let us remove from $\mathbb{C}^n$ the complex hyperplanes $z_i = z_j$ for all $i \neq j$, pass to the universal cover, and consider the completion $\mathcal{B}_n$ of the obtained space.*

Is it true that $\mathcal{B}_n$ is CAT(0) *for any n?*

The above question has an affirmative answer for $n \leqslant 3$ and is open for all $n \geqslant 4$ [**49**, **122**].

Recall that by the Hadamard–Cartan theorem (9.65), any complete length CAT(0) space is contractible. Therefore any complete length locally CAT(0) space is *aspherical*; that is, has contractible universal cover.

This observation can be used together with Exercise 12.16 to construct examples of exotic aspherical spaces; for example, compact topological manifolds with universal cover not homeomorphic to a Euclidean space. A survey on the subject is given by Michael Davis [**58**]; a more elementary introduction to the subject is given by the authors [**10**, Chapter 3].

The flag condition also leads to the so-called *hyperbolization* procedure, a flexible tool for constructing aspherical spaces; a good survey on the subject is given by Ruth Charney and Michael Davis [**50**].

The CAT(0) property of a cube complex admits interesting (and useful) geometric descriptions if one replaces the ℓ^2-metric with a natural ℓ^1- or ℓ^∞-metric on each cube. The following statement was proved by Brian Bowditch [**32**].

12.18. Theorem. *The following three conditions are equivalent.*

(*a*) *A cube complex Q equipped with ℓ^2-metric is* CAT(0).

(*b*) *A cube complex Q equipped with ℓ^∞-metric is* injective*; that is, for any metric space $\mathcal{X}$ with a subset A, any short map $A \to (Q, \ell^\infty)$ can be extended to a short map $\mathcal{X} \to (Q, \ell^\infty)$.*

(*c*) *A cube complex Q equipped with ℓ^1-metric is* median*; that is, for any three points x, y, z there is a unique point m (called the* median *of x, y, and z) that lies on some geodesics $[xy]$, $[xz]$ and $[yz]$.*

Part 3

Structure and tools

Chapter 13

First order differentiation

A. Ultratangent space

The following theorem is often used together with the observation that the ultralimit of any sequence of length spaces is geodesic (see 4.9).

13.1. Theorem. *(a) If $\mathcal{L}$ is a complete length* CBB(κ) *space and $p \in \mathcal{L}$, then T_p^ω is* CBB(0).

(b) If $\mathcal{U}$ is a complete length CAT(κ) *space and $p \in \mathcal{U}$, then T_p^ω is* CAT(0).

The proofs of both parts are nearly identical.

Proof.
(*a*). Since $\mathcal{L}$ is a complete length CBB(κ) space, then its blowup $n \cdot \mathcal{L}$ (see Section 6H) is a complete length CBB(κ/n^2) space. By Proposition 8.4, the ω-blowup $\omega \cdot \mathcal{L}$ is CBB(0) and so is T_p^ω as a metric component of $\omega \cdot \mathcal{L}$.

(*b*). Since $\mathcal{U}$ is a complete length CAT(κ) space, then its blowup $n \cdot \mathcal{U}$ is CAT(κ/n^2). By Proposition 9.7, $\omega \cdot \mathcal{U}$ is CAT(0) and so is T_p^ω as a metric component of $\omega \cdot \mathcal{U}$. □

Recall that the tangent space T_p can be considered as a subset of T_p^ω (see 6.17). Therefore we have the following:

13.2. Corollary. *(a) If $\mathcal{L}$ is a complete length* CBB(κ) *space and $p \in \mathcal{L}$, then T_p is* CBB(0). *Moreover, T_p satisfies the* $(1 + n)$*-point comparison* (10.8).

(*b*) *If $\mathcal{U}$ is a complete length* CAT(κ) *space and $p \in \mathcal{U}$, then* T_p *is* CAT(0). *Moreover,* T_p *satisfies the* $(2n+2)$*-comparison* (10.20).

13.3. Proposition. *Assume $\mathcal{Z}$ is a complete length* CBB *or* CAT *space and $f : \mathcal{Z} \multimap \mathbb{R}$ is a semiconcave locally Lipscitz subfunction. Then for any $p \in \operatorname{Dom} f$, the ultradifferential $\mathbf{d}_p^\omega : \mathrm{T}_p^\omega \to \mathbb{R}$ is a concave function.*

Proof. Fix a geodesic $[x^\omega y^\omega]$ in T_p^ω.

It is sufficient to show that for any subarc $[\bar{x}^\omega \bar{y}^\omega]$ of $[x^\omega y^\omega]$ that does not contain the ends there is a sequence of geodesics $[\bar{x}^n \bar{y}^n]$ in $n \cdot \mathcal{Z}$ converging to $[\bar{x}^\omega \bar{y}^\omega]$.

Choose any sequences $\bar{x}^n, \bar{y}^n \in n \cdot \mathcal{Z}$ such that $\bar{x}^n \to \bar{x}^\omega$, $\bar{y}^n \to \bar{y}^\omega$ as $n \to \omega$. We can assume that there is a geodesic $[\bar{x}^n \bar{y}^n]_{n \cdot \mathcal{Z}}$ for any n; see 8.11 and 9.8. Note that $[\bar{x}^n \bar{y}^n]$ converges to $[\bar{x}^\omega \bar{y}^\omega]$ as $n \to \omega$. The latter holds trivially in the CAT case, and the CBB case follows from 8.38. □

B. Length property of tangent space

13.4. Theorem. *Let $\mathcal{U}$ be a complete length* CAT(κ) *space and $p \in \mathcal{U}$. Then $\mathrm{T}_p \mathcal{U}$ is a length space.*

This theorem together with 13.2 imply the following.

13.5. Corollary. *For any point p in a complete length* CAT(κ) *space, the tangent space* T_p *is a complete length* CAT(0) *space.*

Proof of Theorem 13.4. Since $\mathrm{T}_p = \operatorname{Cone} \Sigma_p$, it is sufficient to show that for any hinge $[p\,{}^x_y]$ such that $\measuredangle[p\,{}^x_y] < \pi$ and any $\varepsilon > 0$, there is $z \in \mathcal{U}$ such that

$$\measuredangle[p\,{}^x_z] < \tfrac{1}{2} \cdot \measuredangle[p\,{}^x_y] + \varepsilon \quad \text{and} \quad \measuredangle[p\,{}^y_z] < \tfrac{1}{2} \cdot \measuredangle[p\,{}^x_y] + \varepsilon. \tag{1}$$

Fix a small $\delta > 0$. Let $\bar{x} \in\,]px]$ and $\bar{y} \in\,]py]$ denote the points such that $|p - \bar{x}| = |p - \bar{y}| = \delta$. Let z denote the midpoint between $\bar{x}$ and $\bar{y}$.

Since δ is small, we can assume that

$$\tilde{\measuredangle}^\kappa(p\,{}^{\bar{x}}_{\bar{y}}) < \measuredangle[p\,{}^x_y] + \varepsilon.$$

By Alexandrov's lemma (6.3), we have

$$\tilde{\measuredangle}^\kappa(p\,{}^{\bar{x}}_z) + \tilde{\measuredangle}^\kappa(p\,{}^{\bar{y}}_z) < \tilde{\measuredangle}^\kappa(p\,{}^{\bar{x}}_{\bar{y}}).$$

By construction,

$$\tilde{\measuredangle}^\kappa(p\,{}^{\bar{x}}_z) = \tilde{\measuredangle}^\kappa(p\,{}^{\bar{y}}_z).$$

Applying the angle comparison (9.14*c*), we get 1. □

The following example was constructed by Stephanie Halbeisen [**80**]. It shows that an analogous statement does not hold for CBB spaces. If the dimension is finite, such examples do not exist; for proper spaces the question is open, see 13.41.

13.6. Example. *There is a complete length* CBB *space* $\check{\mathcal{L}}$ *with a point* $p \in \check{\mathcal{L}}$ *such that the space of directions* $\Sigma_p\check{\mathcal{L}}$ *is not a* π*-length space, and therefore the tangent space* $\mathrm{T}_p\check{\mathcal{L}}$ *is not a length space.*

Construction. Let $\mathbb{H}$ be a Hilbert space formed by infinite sequences of real numbers $\mathbf{x} = (x_0, x_1, \dots)$ with the ℓ^2-norm $|\mathbf{x}|^2 = \sum_i (x_i)^2$. Fix $\varepsilon = 0.001$ and consider two functions $f, \check{f} : \mathbb{H} \to \mathbb{R}$:

$$f(\mathbf{x}) = |\mathbf{x}|,$$

$$\check{f}(\mathbf{x}) = \max\left\{|\mathbf{x}|, \max_{n \geqslant 1}\{(1+\varepsilon)\cdot x_n - \tfrac{1}{n}\}\right\}.$$

Both of these functions are convex and Lipschitz, therefore their graphs in $\mathbb{H}\times\mathbb{R}$ equipped with its length metric form infinite-dimensional Alexandrov spaces, say $\mathcal{L}$ and $\check{\mathcal{L}}$ (this is proved formally in 13.7).

Let p be the origin of $\mathbb{H}\times\mathbb{R}$. Note that $\check{\mathcal{L}} \cap \mathcal{L}$ is a starshaped subset of $\mathbb{H}$ with center at p. Further, $\check{\mathcal{L}} \setminus \mathcal{L}$ consists of a countable number of disjoint sets

$$\Omega_n = \left\{ (\mathbf{x}, \check{f}(\mathbf{x})) \in \check{\mathcal{L}} : (1+\varepsilon)\cdot x_n - \tfrac{1}{n} > |\mathbf{x}| \right\}.$$

Note that $|\Omega_n - p| > \frac{1}{n}$ for each n. It follows that for any geodesic $[pq]$ in $\check{\mathcal{L}}$, a small subinterval $[p\bar{q}] \subset [pq]$ is a straight line segment in $\mathbb{H}\times\mathbb{R}$, and also a geodesic in $\mathcal{L}$. Thus we can treat $\Sigma_p\mathcal{L}$ and $\Sigma_p\check{\mathcal{L}}$ as one set, with two angle metrics $\measuredangle$ and $\check{\measuredangle}$. Let us denote by $\measuredangle_{\mathbb{H}\times\mathbb{R}}$ the angle in $\mathbb{H}\times\mathbb{R}$.

The space $\mathcal{L}$ is isometric to the Euclidean cone over $\Sigma_p\mathcal{L}$ with vertex at p; $\Sigma_p\mathcal{L}$ is isometric to a sphere in Hilbert space with radius $\frac{1}{\sqrt{2}}$. In particular, $\measuredangle$ is the length metric of $\measuredangle_{\mathbb{H}\times\mathbb{R}}$ on $\Sigma_p\mathcal{L}$.

Therefore in order to show that $\check{\measuredangle}$ does not define a length metric on $\Sigma_p\mathcal{L}$, it is sufficient to construct a pair of directions (ξ_+, ξ_-) such that

$$\check{\measuredangle}(\xi_+, \xi_-) < \measuredangle(\xi_+, \xi_-).$$

Set $\mathbf{e}_0 = (1, 0, 0, \dots)$, $\mathbf{e}_1 = (0, 1, 0, \dots)$, $\dots \in \mathbb{H}$. Consider the following two half-lines in $\mathbb{H}\times\mathbb{R}$:

$$\gamma_+(t) = \tfrac{t}{\sqrt{2}}\cdot(\mathbf{e}_0, 1) \quad \text{and} \quad \gamma_-(t) = \tfrac{t}{\sqrt{2}}\cdot(-\mathbf{e}_0, 1), \qquad t \in [0, +\infty).$$

They form unit-speed geodesics in both $\mathcal{L}$ and $\check{\mathcal{L}}$. Let $\xi_\pm$ be the directions of $\gamma_\pm$ at p. Denote by σ_n the half-planes in $\mathbb{H}$ spanned by $\mathbf{e}_0$ and $\mathbf{e}_n$; that is, $\sigma_n = \{ x\cdot\mathbf{e}_0 + y\cdot\mathbf{e}_n : y \geqslant 0 \}$. Consider a sequence of 2-dimensional sectors $Q_n = \check{\mathcal{L}} \cap (\sigma_n \times \mathbb{R})$. For each n, the sector Q_n intersects Ω_n and is bounded by two geodesic half-lines $\gamma_\pm$. Note that $Q_n \xrightarrow[\text{GH}]{} Q$, where Q is a solid Euclidean angle in $\mathbb{E}^2$ with angle measure $\beta < \measuredangle(\xi_+, \xi_-) = \frac{\pi}{\sqrt{2}}$. Indeed, Q_n is path-isometric to the subset of $\mathbb{E}^3$ described by

$$y \geqslant 0 \quad \text{and} \quad z = \max\left\{\sqrt{x^2+y^2}, (1+\varepsilon)\cdot y - \tfrac{1}{n}\right\}$$

with length metric. Thus its limit Q is path-isometric to the subset of $\mathbb{E}^3$ described by

$$y \geqslant 0 \quad \text{and} \quad z = \max\left\{\sqrt{x^2+y^2}, (1+\varepsilon)\cdot y\right\}$$

with length metric. In particular, for any $t, \tau \geqslant 0$,

$$\begin{aligned} |\gamma_+(t) - \gamma_-(\tau)|_{\check{\mathcal{L}}} &\leqslant \lim_{n\to\infty} |\gamma_+(t) - \gamma_-(\tau)|_{Q_n} \\ &= \tilde{\gamma}^0\{\beta; t, \tau\}. \end{aligned}$$

That is, $\check{\measuredangle}(\xi_+, \xi_-) \leqslant \beta < \measuredangle(\xi_+, \xi_-)$. □

13.7. Lemma. *Let $\mathbb{H}$ be a Hilbert space, $f : \mathbb{H} \to \mathbb{R}$ be a convex Lipschitz function and $S \subset \mathbb{H} \times \mathbb{R}$ be the graph of f equipped with the length metric. Then S is* CBB(0).

Proof. Recall that for a subset $X \subset \mathbb{H}\times\mathbb{R}$, we will denote by $|{*}-{*}|_X$ the length metric on X.

By the theorem of Sergei Buyalo [**46**], sharpened by the authors in [**9**], any convex hypersurface in a Euclidean space, equipped with the length metric, is non-negatively curved. Thus it is sufficient to show that for any 4-point set $\{x_0, x_1, x_2, x_3\} \subset S$, there is a finite-dimensional subspace $E \subset \mathbb{H}\times\mathbb{R}$ such that $\{x_i\} \in E$ and $|x_i - x_j|_{S\cap E}$ is arbitrary close to $|x_i - x_j|_S$.

Clearly $|x_i - x_j|_{S\cap E} \geqslant |x_i - x_j|_S$; thus it is sufficient to show that for given $\varepsilon > 0$ one can choose E so that

$$|x_i - x_j|_{S\cap E} < |x_i - x_j|_S + \varepsilon. \tag{2}$$

For each pair (x_i, x_j), choose a polygonal line β_{ij} connecting x_i, x_j that lies under S (that is, outside of Conv S) in $\mathbb{H}\times\mathbb{R}$ and has length at most $|x_i - x_j|_S + \varepsilon$. Let E be the affine hull of all the vertexes in all β_{ij}. Thus

$$|x_i - x_j|_{S\cap E} \leqslant \operatorname{length}\beta_{ij}$$

and 2 follows. □

13.8. Exercise. *Construct a non-compact complete geodesic* CBB(0) *space that contains no half-lines.*

C. Rademacher theorem

At the end of this section we give an extension of the Rademacher theorem (see Section 3D) to CBB and CAT spaces (13.12); it was proved by Alexander Lytchak [**109**]. The following proposition is the 1-dimensional case of the extended Rademacher theorem.

Recall that differentiable curves are defined in 6.12.

13.9. Proposition. *Let $\alpha\colon \mathbb{I}\to\mathcal{Z}$ be a locally Lipschitz curve in a complete length space. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then α is differentiable almost everywhere.*

The following two lemmas provide sufficient conditions for existence of the one-sided derivative of a curve in CBB and CAT spaces. The proofs of both lemmas are similar.

13.10. Lemma. *Let $\alpha\colon \mathbb{I}\to\mathcal{L}$ be a 1-Lipschitz curve in a* CBB *space. Suppose that for some $t_0\in\mathbb{I}$ and any $\varepsilon>0$, there is a point p such that $|\alpha(t_0)-p|<\varepsilon$ and*

$$\varliminf_{t\to t_0+}\frac{\operatorname{dist}_p\circ\alpha(t)-\operatorname{dist}_p\circ\alpha(t_0)}{t-t_0}>1-\varepsilon.$$

Then the right derivative $\alpha^+(t_0)$ is defined and $|\alpha^+(t_0)|=1$.

Proof. Without loss of generality, we may assume that $t_0=0$. Set $x=\alpha(0)$. Fix a sequence of points $p_n\to x$ such that

$$\varliminf_{t\to 0+}\frac{|p_n-\alpha(t)|-|p_n-x|}{t}\to 1$$

as $n\to\infty$.

Observe that there are sequences $\delta_n\to 0+$ and $t_n\to 0+$ such that

$$\tilde\measuredangle^\kappa\left(x\,{}^{\alpha(s)}_{p_n}\right)>\pi-\delta_n\quad\text{and}\quad(1-\delta_n)\cdot s<|\alpha(s)-x|\leqslant s \tag{1}$$

for any $s\in(0,t_n]$.

For each n, choose $q_n\in\operatorname{Str}(x)$ sufficiently close $\alpha(t_n)$ that the inequality

$$\tilde\measuredangle^\kappa\left(x\,{}^{q_n}_{p_n}\right)>\pi-\delta_n$$

still holds (see Definition 8.10).

Set $\gamma_n=\operatorname{geod}_{[xq_n]}$. By comparison,

$$\begin{aligned}\tilde\measuredangle^\kappa\left(x\,{}^{\alpha(s)}_{\gamma_n(s)}\right)&\leqslant 2\cdot\pi-\tilde\measuredangle^\kappa\left(x\,{}^{p_n}_{\gamma_n(s)}\right)-\tilde\measuredangle^\kappa\left(x\,{}^{\alpha(s)}_{p_n}\right)\\&\leqslant 2\cdot\pi-\tilde\measuredangle^\kappa\left(x\,{}^{q_n}_{p_n}\right)-\tilde\measuredangle^\kappa\left(x\,{}^{\alpha(s)}_{p_n}\right)\\&<2\cdot\delta_n.\end{aligned}$$

Therefore 1 implies that

$$|\gamma_n(s) - \alpha(s)| < 10 \cdot \delta_n \cdot (s)$$

if s is a sufficiently small and positive. That is, $\alpha^+(0)$ is defined (see Definition 6.9). $\square$

13.11. Lemma. *Let $\alpha\colon \mathbb{I} \to \mathcal{U}$ be a 1-Lipschitz curve in a* CAT *space. Suppose that for some $t_0 \in \mathbb{I}$ and any $\varepsilon > 0$ there is a point q such that $|\alpha(t_0) - q| < \varepsilon$ and*

$$\varlimsup_{t\to t_0+} \frac{\operatorname{dist}_q \circ\alpha(t) - \operatorname{dist}_q \circ\alpha(t_0)}{t - t_0} < -1 + \varepsilon.$$

Then the right derivative $\alpha^+(t_0)$ is defined and $|\alpha^+(t_0)| = 1$.

Proof. Without loss of generatiy we may assume that $t_0 = 0$. Set $x = \alpha(0)$. Fix a sequence of points $q_n \to x$ such that

$$\varlimsup_{t\to 0+} \frac{|q_n - \alpha(t)| - |q_n - x|}{t} \to -1$$

as $n \to \infty$.

Observe that there are sequences $\delta_n \to 0+$ and $t_n \to 0+$ such that

$$\tilde\measuredangle^\kappa\left(x\,{}^{\alpha(s)}_{q_n}\right) < \delta_n \quad \text{and} \quad (1-\delta_n)\cdot s < |\alpha(s) - x| \leqslant s \tag{2}$$

for any $s \in (0, t_n]$.

Without loss of generality, we may assume that $|x - q_n| < \varpi\kappa$ for any n; in particular, the geodesic $\gamma_n = \operatorname{geod}_{[xq_n]}$ is uniquely defined.

By comparison,

$$\begin{aligned}\tilde\measuredangle^\kappa\left(x\,{}^{\alpha(s)}_{\gamma_n(s)}\right) &\leqslant \tilde\measuredangle^\kappa\left(x\,{}^{\alpha(s)}_{q_n}\right)\\ &< \delta_n.\end{aligned}$$

Therefore 2 implies that

$$|\gamma_n(s) - \alpha(s)| < 10 \cdot \delta_n \cdot s$$

if s is a sufficiently small and positive. That is, $\alpha^+(0)$ is defined (see Definition 6.9). $\square$

Proof of 13.9. By the standard Rademacher theorem, we may assume that α has an arc-length parametrization. In particular, α is 1-Lipschitz.

Recall that by Theorem 3.10,

$$\operatorname{speed}_s \alpha \overset{\text{a.e.}}{=\!=} 1. \tag{3}$$

Fix a countable dense set $T \subset \mathbb{I}$; given $t \in T$, let

$$h_t(s) = |\alpha(t) - \alpha(s)|.$$

Note that h_t is 1-Lipschitz for each $t \in T$. Therefore, by the standard Rademacher theorem and countability of T for almost all $s \in \mathbb{I}$, $h_t'(s)$ is defined for all $t \in T$.

Let

$$w^+(s) \coloneqq \varlimsup_{\substack{t \in T \\ t \to s-}} \{h_t'(s)\}.$$

Let us show that

$$w^+(s) \overset{a.e.}{=\!=} 1. \tag{4}$$

Note that once this is proved, Lemma 13.10 implies the proposition in the CBB case.

For a small $\varepsilon > 0$, denote by N_ε^+ the set of all points $s \in \mathbb{I}$ such that $w^+(s) < 1 - \varepsilon$. Note that the sets N_ε^+ are measurable.

Suppose N_ε^+ has positive measure. Let $s_0 \in N_\varepsilon^+$ be a Lebesgue point of α. We may assume that $\mathrm{speed}_{s_0} \alpha = 1$ and $h_t'(s_0)$ is defined for any $t \in T$. Suppose $t \in T$ is sufficiently close to s_0 and $t < s_0$. Since $\mathrm{speed}_{s_0} \alpha = 1$, we have

$$h_t(s_0) \geqslant (s_0 - t) \cdot (1 - \varepsilon^2). \tag{5}$$

Further, there is a set $A \subset [t, s_0]$ with measure at least $(1 - \varepsilon) \cdot |s_0 - t|$ such that

$$h_t'(s) < 1 - \varepsilon$$

for any $s \in A_n$. Since h_t is 1-Lipschitz, we have

$$\begin{aligned} h_t(s_0) &= \int_{[t,s_0] \setminus A} h_t'(s) \cdot \mathbf{d}s + \int_A h_t'(s) \cdot \mathbf{d}s \\ &\leqslant (s_0 - t) \cdot [\varepsilon + (1 - \varepsilon)^2]. \end{aligned}$$

The latter contradicts 5. Thus $w^+(s) \geqslant 1 - \varepsilon$ almost everywhere. Since $\varepsilon > 0$ is arbitrary, 4 follows.

In the same way we can show that

$$w^-(s) \overset{a.e.}{=\!=} -1, \tag{6}$$

where

$$w^-(s) \coloneqq \varliminf_{\substack{t \in T \\ t \to s+}} \{h_t'(s)\}.$$

Then Lemma 13.11 implies the proposition in the CAT case. □

13.12. Extended Rademacher theorem. *Let $f : \mathbb{E}^m \multimap \mathcal{Z}$ be a locally Lipschitz submap from a Euclidean space to a complete length space $\mathcal{Z}$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then the differential $\mathbf{d}_x f$ is defined at almost all points $x \in \mathrm{Dom}\, f$.*

Moreover the differential $\mathbf{d}_x f$ is linear *at almost all x in the following sense: the image $\mathfrak{I} f$ is a convex subcone of $\mathrm{T}_{f(x)} \mathcal{Z}$, and there is an isometry ι from $\mathfrak{I} f$ to a Euclidean space such that the composition $\iota \circ \mathbf{d}_x f$ is linear.*

The proof is a reduction to the 1-dimensional case (13.9) by standard arguments [**94**, **113**].

Proof. Without loss of generality, we may assume that $\operatorname{Dom} f$ is bounded and f is Lipschitz.

Fix a countable dense set of vectors $\{v_i\}$ in $\mathbb{E}^m$. Fix v_i and a point $p \in \operatorname{Dom} f$. By Proposition 13.9, the value $\mathbf{d}_x f(v_i)$ is defined at $x = p + t \cdot v_i$ for almost all t such that $x \in \operatorname{Dom} f$. It follows that $\mathbf{d}_x f(v_i)$ is defined for every i on a set A of full measure in $\operatorname{Dom} f$. Since the metric differential of f is defined almost everywhere (3.11), we have that $\mathbf{d}_x f(v)$ is defined for any v on a set B of full measure in $\operatorname{Dom} f$.

Applying the definitions of metric differential and differential (3.11) and 6.15, we obtain that the image of $\mathbf{d}_x f$ is a weakly convex set in $\mathrm{T}_{f(x)}$. It follows that $\Im \mathbf{d}_x f$ is CBB(0) or CAT(0) if the space $\mathcal{Z}$ is CBB or CAT respectively. It remains to apply Exercise 8.15 or 9.16 if the space $\mathcal{Z}$ is CBB or CAT respectively. $\square$

D. Differential

13.13. Exercise. *Let $\mathcal{U}$ be a complete length* CAT(κ) *space and $p, q \in \mathcal{U}$. Assume $|p - q| < \varpi\kappa$. Show that*

$$(\mathbf{d}_q \operatorname{dist}_p)(v) = -\langle \uparrow_{[qp]}, v \rangle.$$

13.14. Exercise. *Let $\mathcal{L}$ be a length* CBB(κ) *space and $p, q \in \mathcal{L}$ be distinct points. Assume $q \in \operatorname{Str}(p)$ or $p \in \operatorname{Str}(q)$. Show that*

$$(\mathbf{d}_q \operatorname{dist}_p)(v) = -\langle \uparrow_{[qp]}, v \rangle.$$

13.15. Lemma. *Let $\mathcal{U}$ be a complete length* CAT *space, $f \colon \mathcal{U} \multimap \mathbb{R}$ be a locally Lipschitz semiconcave subfunction, and $p \in \operatorname{Dom} f$. Then $\mathbf{d}_p f$ is a Lipschitz concave function on $\mathrm{T}_p\,\mathcal{U}$.*

Proof. Recall that the tangent space $\mathrm{T}_p = \mathrm{T}_p\,\mathcal{U}$ can be considered as a subspace of the ultratangent space T^ω_p (6.17). Since T^ω_p is CAT(0), 13.4 implies that T_p is a convex set in T^ω_p.

By 13.3, $\mathbf{d}^\omega_p f$ is a concave function on T^ω_p. It remains to apply that $\mathbf{d}_p f = (\mathbf{d}^\omega_p f)|_{\mathrm{T}_p}$ (6.16*c*). $\square$

As it is shown in Halbeisen's example (Section 13B), a CBB space might have tangent spaces that are not length spaces; thus concavity of the differential $\mathbf{d}_p f$ of a semiconcave function f is meaningless. Nevertheless, as the following lemma says, the differential $\mathbf{d}_p f$ of a semiconcave function always satisfies a weaker property similar to concavity (compare to [**137**, Prop. 136] and [**121**, 4.2]). In the finite dimensional case, $\mathbf{d}_p f$ is concave.

13.16. Lemma. *Let $\mathcal{Z}$ be a complete length space, $f : \mathcal{Z} \multimap \mathbb{R}$ be a locally Lipschitz semiconcave subfunction, and $p \in \operatorname{Dom} f$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then for any $u, v \in \mathrm{T}_p$, we have*

$$s \cdot \sqrt{|u|^2 + 2 \cdot \langle u, v\rangle + |v|^2} \geqslant (\mathbf{d}_p f)(u) + (\mathbf{d}_p f)(v),$$

where

$$s = \sup\left\{ (\mathbf{d}_p f)(\xi) : \xi \in \Sigma_p \right\}.$$

Proof. If $\mathcal{Z}$ is CAT, then the statement follows from 13.15. Indeed, let z be the midpoint of a geodesic $[uv]_{\mathrm{T}_p}$. Observe that

$$2 \cdot |z| = \sqrt{|u|^2 + 2 \cdot \langle u, v\rangle + |v|^2}.$$

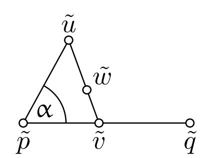
Since $\mathbf{d}_p f$ is concave, we have that

$$2 \cdot \mathbf{d}_p f(z) \geqslant \mathbf{d}_p f(u) + \mathbf{d}_p f(v).$$

It remains to choose $\xi \in \Sigma_p$ so that $\xi \cdot |z| = z$ and observe that $s \geqslant \mathbf{d}_p(\xi)$.

Now assume $\mathcal{Z}$ is CBB. We can assume that $\alpha = \measuredangle(u, v) > 0$, otherwise the statement is trivial. Moreover, we can assume that $\exp_p(t \cdot u)$ and $\exp_p(t \cdot v)$ are defined for all small $t > 0$; the latter follows since geodesic space of directions Σ_p' is dense in Σ_p.

Prepare a model configuration of five points: $\tilde{p}$, $\tilde{u}$, $\tilde{v}$, $\tilde{q}$, $\tilde{w} \in \mathbb{E}^2$ such that



- $\measuredangle\left[\tilde{p}\,{}^{\tilde{u}}_{\tilde{v}}\right] = \alpha$,
- $|\tilde{p} - \tilde{u}| = |u|$,
- $|\tilde{p} - \tilde{v}| = |v|$,
- $\tilde{q}$ lies on an extension of $[\tilde{p}\tilde{v}]$ so that $\tilde{v}$ is the midpoint of $[\tilde{p}\tilde{q}]$,
- $\tilde{w}$ is the midpoint between $\tilde{u}$ and $\tilde{v}$.

Note that

$$|\tilde{p} - \tilde{w}| = \tfrac{1}{2} \cdot \sqrt{|u|^2 + 2 \cdot \langle u, v\rangle + |v|^2}.$$

Assume that $\mathcal{Z}$ is geodesic.

For all small $t > 0$, construct points $u_t, v_t, q_t, w_t \in \mathcal{Z}$ as follows:

(a) $v_t = \exp_p(t \cdot v)$, $\quad q_t = \exp_p(2 \cdot t \cdot v)$

(b) $u_t = \exp_p(t \cdot u)$.

(c) w_t is the midpoint of $[u_t v_t]$.

Clearly $|p-u_t|=t\cdot|u|$, $|p-v_t|=t\cdot|v|$, $|p-q_t|=2\cdot t\cdot|v|$. Since $\measuredangle(u,v)$ is defined, we have $|u_t-v_t|=t\cdot|\tilde u-\tilde v|+o(t)$ and $|u_t-q_t|=t\cdot|\tilde u-\tilde q|+o(t)$ (see Theorem 8.14*c* and Section 8A).

From the point-on-side and hinge comparisons (8.14*b*) + 8.14*c*, we have

$$\tilde\measuredangle^\kappa\left(v_t\,{}^{p}_{w_t}\right)\geqslant\tilde\measuredangle^\kappa\left(v_t\,{}^{p}_{u_t}\right)\geqslant\measuredangle\left[\tilde v\,{}^{\tilde p}_{\tilde u}\right]+\tfrac{o(t)}{t}$$

and

$$\tilde\measuredangle^\kappa\left(v_t\,{}^{q_t}_{w_t}\right)\geqslant\tilde\measuredangle^\kappa\left(v_t\,{}^{q_t}_{u_t}\right)\geqslant\measuredangle\left[\tilde v\,{}^{\tilde q}_{\tilde u}\right]+\tfrac{o(t)}{t}.$$

Clearly, $\measuredangle\left[\tilde v\,{}^{\tilde p}_{\tilde u}\right]+\measuredangle\left[\tilde v\,{}^{\tilde q}_{\tilde u}\right]=\pi$. From the adjacent angle comparison (8.14*a*), $\tilde\measuredangle^\kappa\left(v_t\,{}^{p}_{u_t}\right)+\tilde\measuredangle^\kappa\left(v_t\,{}^{u_t}_{q_t}\right)\leqslant\pi$. Hence $\tilde\measuredangle^\kappa\left(v_t\,{}^{p}_{w_t}\right)\to\measuredangle\left[\tilde v\,{}^{\tilde p}_{\tilde w}\right]$ as $t\to 0+$ and thus

$$|p-w_t|=t\cdot|\tilde p-\tilde w|+o(t).$$

Since f is λ-concave, we have

$$\begin{aligned}2\cdot f(w_t)&\geqslant f(u_t)+f(v_t)+\tfrac{\lambda}{4}\cdot|u_t-v_t|^2\\&=2\cdot f(p)+t\cdot[(\mathbf{d}_pf)(u)+(\mathbf{d}_pf)(v)]+o(t).\end{aligned}$$

Applying λ-concavity of f, we have

$$(\mathbf{d}_pf)(\uparrow_{[pw_t]})\geqslant\frac{t\cdot[(\mathbf{d}_pf)(u)+(\mathbf{d}_pf)(v)]+o(t)}{2\cdot t\cdot|\tilde p-\tilde w|+o(t)}.\tag{1}$$

The lemma follows.

Finally, if $\mathcal{Z}$ is not geodesic one needs to make two adjustments in the above construction. Namely:

(i) For the geodesic $[u_tv_t]$ to be defined, in (b) one has to take $u_t\in\mathrm{Str}(v_t)$, $u_t\approx\exp_p(t\cdot u)$; more precisely,

$$|u_t-\exp_p(t\cdot u)|=o(t).$$

Thus instead of $|p-u_t|=t\cdot|u|$ we have

$$|p-u_t|=t\cdot|u|+o(t),$$

and this is sufficient for the rest of proof.

(ii) The direction $\uparrow_{[pw_t]}$ might be undefined. Thus in ~~the~~ estimate 1, instead of $\uparrow_{[pw_t]}$ one should take $\uparrow_{[pw_t']}$ for some point $w_t'\in\mathrm{Str}(p)$ near w_t (that is, $|w_t-w_t'|=o(t)$). $\square$

E. Definition of gradient

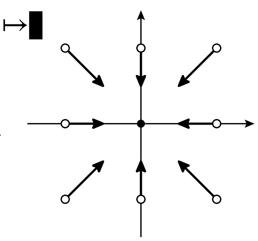
13.17. Definition of gradient. *Let $\mathcal{X}$ be a length space with defined angles and $f\colon \mathcal{X} \multimap \mathbb{R}$ be a subfunction. Suppose for a point $p \in \operatorname{Dom} f$ the differential $\mathbf{d}_p f\colon \mathrm{T}_p \to \mathbb{R}$ is defined.*

A tangent vector $\mathrm{g} \in \mathrm{T}_p$ is called a gradient of f at p *(briefly, $\mathrm{g} = \nabla_p f$) if*

(*a*) $(\mathbf{d}_p f)(w) \leqslant \langle \mathrm{g}, w\rangle$ *for any $w \in \mathrm{T}_p$, and*

(*b*) $(\mathbf{d}_p f)(\mathrm{g}) = \langle \mathrm{g}, \mathrm{g}\rangle$.

13.18. Example. *Consider the Euclidean plane with standard (x, y)-coordinates. Then the function $f\colon (x, y) \mapsto -|x| - |y|$ is concave; its gradient field is sketched on the figure.*



If a point does not lie on an axis, then its gradient has length $\sqrt{2}$ and takes one of four values $(\pm 1, \pm 1)$ depending on the quadrant of the point. At the origin the gradient vanish, and on the on the remaining parts of the x-axis and y-axis it is $(\pm 1, 0)$ and $(0, \pm 1)$ respectively.

13.19. Exercise. *Let $\mathcal{U}$ be a complete length* CAT(0) *space. Show that*

$$\nabla_p(-\operatorname{dist}_q) = \uparrow_{[pq]}$$

for any pair of distinct points $p, q \in \mathcal{U}$.

13.20. Existence and uniqueness of the gradient. *Let $\mathcal{Z}$ be a complete length space and $f\colon \mathcal{Z} \multimap \mathbb{R}$ be a locally Lipschitz and semiconcave subfunction. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then for any point $p \in \operatorname{Dom} f$, there is a unique gradient $\nabla_p f \in \mathrm{T}_p$.*

Proof.

Uniqueness. If $\mathrm{g}, \mathrm{g}' \in \mathrm{T}_p$ are two gradients of f, then

$$\langle \mathrm{g}, \mathrm{g}\rangle = (\mathbf{d}_p f)(\mathrm{g}) \leqslant \langle \mathrm{g}, \mathrm{g}'\rangle, \qquad \langle \mathrm{g}', \mathrm{g}'\rangle = (\mathbf{d}_p f)(\mathrm{g}') \leqslant \langle \mathrm{g}, \mathrm{g}'\rangle.$$

Therefore,

$$|\mathrm{g} - \mathrm{g}'|^2 = \langle \mathrm{g}, \mathrm{g}\rangle - 2\cdot\langle \mathrm{g}, \mathrm{g}'\rangle + \langle \mathrm{g}', \mathrm{g}'\rangle \leqslant 0.$$

It follows that $\mathrm{g} = \mathrm{g}'$.

Existence. Note first that if $\mathbf{d}_p f \leqslant 0$, then one can take $\nabla_p f = 0$.

Otherwise, if $s = \sup\left\{ (\mathbf{d}_p f)(\xi) : \xi \in \Sigma_p \right\} > 0$, it is sufficient to show that there is $\overline{\xi} \in \Sigma_p$ such that

(1) $$(\mathbf{d}_p f)\left(\overline{\xi}\right) = s.$$

Indeed, suppose $\bar{\xi}$ exists. Then applying Lemma 13.16 for $u = \bar{\xi}$, $v = \varepsilon \cdot w$ with $\varepsilon \to 0+$, we get

$$(\mathbf{d}_p f)(w) \leqslant \langle w, s \cdot \bar{\xi} \rangle$$

for any $w \in \mathrm{T}_p$; that is, $s \cdot \bar{\xi}$ is the gradient at p.

Take a sequence of directions $\xi_n \in \Sigma_p$, such that $(\mathbf{d}_p f)(\xi_n) \to s$. Applying Lemma 13.16 for $u = \xi_n$, $v = \xi_m$, we get

$$s \geqslant \frac{(\mathbf{d}_p f)(\xi_n) + (\mathbf{d}_p f)(\xi_m)}{\sqrt{2 + 2 \cdot \cos \measuredangle(\xi_n, \xi_m)}}.$$

Therefore $\measuredangle(\xi_n, \xi_m) \to 0$ as n, $m \to \infty$; that is, the sequence ξ_n is Cauchy. Clearly $\bar{\xi} = \lim_n \xi_n$ satisfies 1. □

13.21. Exercise. *Let p be a point in a complete length* CBB *space. Show that*

$$|\nabla_x \operatorname{dist}_p| = 1$$

for x in a dense G-delta subset.

F. Calculus of gradient

The next lemma states that the gradient points in the direction of maximal slope; moreover, if the slope in the given direction is almost maximal, then it is almost the direction of the gradient.

13.22. Lemma. *Let $\mathcal{Z}$ be a complete length space, $f : \mathcal{Z} \multimap \mathbb{R}$ be locally Lipschitz and semiconcave, and $p \in \operatorname{Dom} f$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT.

Assume $|\nabla_p f| > 0$; let $\bar{\xi} = \frac{1}{|\nabla_p f|} \cdot \nabla_p f$. Then:

(*a*) *If for some $v \in \mathrm{T}_p$, we have*

$$|v| \leqslant 1 + \varepsilon \quad \textit{and} \quad (\mathbf{d}_p f)(v) > |\nabla_p f| \cdot (1 - \varepsilon),$$

then

$$|\bar{\xi} - v| < 100 \cdot \sqrt{\varepsilon}.$$

(*b*) *If $v_n \in \mathrm{T}_p$ is a sequence of vectors such that*

$$\varlimsup_{n\to\infty} |v_n| \leqslant 1 \quad \textit{and} \quad \varliminf_{n\to\infty} (\mathbf{d}_p f)(v_n) \geqslant |\nabla_p f|,$$

then

$$\lim_{n\to\infty} v_n = \bar{\xi}.$$

(*c*) *$\bar{\xi}$ is the unique maximum direction for the restriction $\mathbf{d}_p f|_{\Sigma_p}$. In particular,*

$$|\nabla_p f| = \sup\{\, \mathbf{d}_p f : \xi \in \Sigma_p f \,\}.$$

Proof. According to the definition of gradient,

$$\begin{aligned}|\nabla_p f|\cdot(1-\varepsilon) &< (\mathbf{d}_p f)(v)\\ &\leqslant \langle v,\nabla_p f\rangle\\ &= |v|\cdot|\nabla_p f|\cdot\cos\measuredangle(\nabla_p f,v).\end{aligned}$$

Thus $|v|>1-\varepsilon$ and $\cos\measuredangle(\nabla_p f,v)>\frac{1-\varepsilon}{1+\varepsilon}$. Hence (*a*).

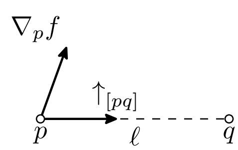
Statements (*b*) and (*c*) follow directly from (*a*)). □

As a corollary of the above lemma and Proposition 3.18 we obtain the following:

13.23. Chain rule. *Let $\mathcal{Z}$ be a complete length space, $f\colon \mathcal{Z}\multimap\mathbb{R}$ be a semiconcave function, and $\phi\colon\mathbb{R}\to\mathbb{R}$ be a nondecreasing semiconcave function. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then $\phi\circ f$ is semiconcave and $\nabla_x(\phi\circ f)=\phi^+(f(x))\cdot\nabla_x f$ for any $x\in\operatorname{Dom} f$.*

The following inequalities describe an important property of the "gradient vector field".

13.24. Lemma. *Let $\mathcal{Z}$ be a complete length space, $f\colon\mathcal{Z}\multimap\mathbb{R}$ satisfy $f''+\kappa\cdot f\leqslant\lambda$ for some $\kappa,\lambda\in\mathbb{R}$. Let $[pq]\subset\operatorname{Dom} f$. and $\ell=|p-q|$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then*



$$\langle\uparrow_{[pq]},\nabla_p f\rangle\geqslant\frac{f(q)-f(p)\cdot\operatorname{cs}^\kappa\ell-\lambda\cdot\operatorname{md}^\kappa\ell}{\operatorname{sn}^\kappa\ell}.$$

In particular,

(*a*) *if $\kappa=0$,*

$$\langle\uparrow_{[pq]},\nabla_p f\rangle\geqslant\Big(f(q)-f(p)-\tfrac{\lambda}{2}\cdot\ell^2\Big)/\ell;$$

(*b*) *if $\kappa=1$, $\lambda=0$ we have*

$$\langle\uparrow_{[pq]},\nabla_p f\rangle\geqslant(f(q)-f(p)\cdot\cos\ell)/\sin\ell;$$

(*c*) *if $\kappa=-1$, $\lambda=0$ we have*

$$\langle\uparrow_{[pq]},\nabla_p f\rangle\geqslant(f(q)-f(p)\cdot\cosh\ell)/\sinh\ell.$$

Proof of 13.24. Note that

$$\operatorname{geod}_{[pq]}(0)=p,\quad \operatorname{geod}_{[pq]}(\ell)=q,\quad (\operatorname{geod}_{[pq]})^+(0)=\uparrow_{[pq]}.$$

Thus,

$$\begin{aligned}\langle\uparrow_{[pq]},\nabla_p f\rangle &\geqslant d_p f(\uparrow_{[pq]})\\ &=(f\circ\mathrm{geod}_{[pq]})^+(0)\\ &\geqslant\frac{f(q)-f(p)\cdot\mathrm{cs}^\kappa\ell-\lambda\cdot\mathrm{md}^\kappa\ell}{\mathrm{sn}^\kappa\ell}.\end{aligned}\qquad\square$$

The following corollary states that the gradient vector field is monotonic in a sense similar to the definition of *monotone operators* [**133**].

13.25. λ-Monotonicity of gradient. *Let $\mathcal{Z}$ be a complete length space, $f\colon\mathcal{Z}\multimap\mathbb{R}$ be locally Lipschitz and λ-concave and $[pq]\subset\mathrm{Dom}\,f$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT*. Then*

$$\langle\uparrow_{[pq]},\nabla_p f\rangle+\langle\uparrow_{[qp]},\nabla_q f\rangle\geqslant-\lambda\cdot|p-q|.$$

Proof. Add two inequalities from 13.24*a*. $\square$

13.26. Lemma. *Let $\mathcal{Z}$ be a complete length space, $f,g\colon\mathcal{Z}\multimap\mathbb{R}$, and $p\in\mathrm{Dom}\,f\cap\mathrm{Dom}\,g$. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT.

Then

$$|\nabla_p f-\nabla_p g|^2_{\mathrm{T}_p}\leqslant s\cdot(|\nabla_p f|+|\nabla_p g|),$$

where

$$s=\sup\left\{\,|(\mathbf{d}_p f)(\xi)-(\mathbf{d}_p g)(\xi)|\;:\;\xi\in\Sigma_p\,\right\}.$$

In particular, if $f_n\colon\mathcal{Z}\multimap\mathbb{R}$ is a sequence of locally Lipschitz and semiconcave subfunctions, $p\in\mathrm{Dom}\,f_n$ for each n, and $\mathbf{d}_p f_n$ converges uniformly on Σ_p, then the sequence $\nabla_p f_n\in\mathrm{T}_p$ converges.

Proof. Clearly for any $v\in\mathrm{T}_p$, we have

$$|(\mathbf{d}_p f)(v)-(\mathbf{d}_p g)(v)|\leqslant s\cdot|v|.$$

From the definition of gradient (13.17) we have:

$$\begin{aligned}(\mathbf{d}_p f)(\nabla_p g)&\leqslant\langle\nabla_p f,\nabla_p g\rangle, & (\mathbf{d}_p g)(\nabla_p f)&\leqslant\langle\nabla_p f,\nabla_p g\rangle,\\ (\mathbf{d}_p f)(\nabla_p f)&=\langle\nabla_p f,\nabla_p f\rangle, & (\mathbf{d}_p g)(\nabla_p g)&=\langle\nabla_p g,\nabla_p g\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}|\nabla_p f-\nabla_p g|^2&=\langle\nabla_p f,\nabla_p f\rangle+\langle\nabla_p g,\nabla_p g\rangle-2\cdot\langle\nabla_p f,\nabla_p g\rangle\\ &\leqslant(\mathbf{d}_p f)(\nabla_p f)+(\mathbf{d}_p g)(\nabla_p g)-(\mathbf{d}_p f)(\nabla_p g)-(\mathbf{d}_p g)(\nabla_p f)\\ &\leqslant s\cdot(|\nabla_p f|+|\nabla_p g|).\end{aligned}\qquad\square$$

13.27. Exercise. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space, the function $f\colon\mathcal{L}\to\mathbb{R}$ be semiconcave and locally Lipschitz, and $\alpha\colon\mathbb{I}\to\mathcal{L}$ be a Lipschitz curve. Show that*

$$\langle\nabla_{\alpha(t)}f,\alpha^+(t)\rangle=(\mathbf{d}_{\alpha(t)}f)(\alpha^+(t))$$

for almost all $t \in \mathbb{I}$.

G. Semicontinuity of |gradient|

In this section we collect a few consequences of the following lemma.

13.28. Ultralimit of |gradient|. *Assume that*

- *$(\mathcal{Z}_n)$ is a sequence of complete length spaces and $(\mathcal{Z}_n, p_n) \to (\mathcal{Z}_\omega, p_\omega)$ as $n \to \omega$. Suppose that all $\mathcal{Z}_n$ are either* CBB *or* CAT.
- *$f_n\colon \mathcal{Z}_n \multimapinv \mathbb{R}$ and $f_\omega\colon \mathcal{Z}_\omega \multimapinv \mathbb{R}$ are locally Lipschitz and λ-concave, and $f_n \to f_\omega$ as $n \to \omega$.*
- *$x_n \in \operatorname{Dom} f_n$ and $x_n \to x_\omega \in \operatorname{Dom} f_\omega$ as $n \to \omega$.*

Then

$$|\nabla_{x_\omega} f_\omega| \leqslant \lim_{n\to\omega} |\nabla_{x_n} f_n|.$$

Remarks. The inequality might be strict. For example, consider $\mathcal{Z}_n = \mathbb{R}$, $f_n(x) = -|x|$ and $x_n \to 0+$.

From the convergence of gradient curves (proved ~~later~~ in 16.17), one can deduce the following slightly stronger statement.

13.29. Proposition. *Assume that*

- *$\mathcal{Z}_n$ is a sequence of complete length spaces and $(\mathcal{Z}_n, p_n) \to (\mathcal{Z}_\omega, p_\omega)$ as $n \to \omega$. Suppose that all $\mathcal{Z}_n$ are either* CBB *or* CAT.
- *$f_n\colon \mathcal{Z}_n \multimapinv \mathbb{R}$ and $f_\omega\colon \mathcal{Z}_\omega \multimapinv \mathbb{R}$ are locally Lipschitz and λ-concave and $f_n \to f_\omega$ as $n \to \omega$.*

Then

$$|\nabla_{x_\omega} f_\omega| = \inf\{\lim_{n\to\omega} |\nabla_{x_n} f_n|\},$$

where infimum is taken for all sequences $x_n \in \operatorname{Dom} f_n$ such that $x_n \to x_\omega \in \operatorname{Dom} f_\omega$ as $n \to \omega$.

Proof of 13.28. Fix an $\varepsilon > 0$ and choose $y_\omega \in \operatorname{Dom} f_\omega$ sufficiently close to x_ω that

$$|\nabla_{x_\omega} f_\omega| - \varepsilon < \frac{f_\omega(y_\omega) - f_\omega(x_\omega)}{|x_\omega - y_\omega|}.$$

Choose $y_n \in \mathcal{Z}_n$ such that $y_n \to y_\omega$ as $n \to \omega$. Since $|x_\omega - y_\omega|$ is sufficiently small, the λ-concavity of f_n implies that

$$|\nabla_{x_\omega} f_\omega| - 2\cdot\varepsilon < (\mathbf{d}_{x_n} f_n)(\uparrow_{[x_n y_n]})$$

for ω-almost all n. Hence

$$|\nabla_{x_\omega} f_\omega| - 2\cdot\varepsilon \leqslant \lim_{n\to\omega} |\nabla_{x_n} f_n|.$$

Since $\varepsilon > 0$ is arbitrary, the proposition follows. □

Note that the distance-preserving map $\iota\colon \mathcal{Z}\hookrightarrow\mathcal{Z}^\omega$ induces an embedding

$$\mathbf{d}_p\iota\colon\ \mathrm{T}_p\,\mathcal{Z}\hookrightarrow\mathrm{T}_p\,\mathcal{Z}^\omega.$$

Thus, we can (and will) consider $\mathrm{T}_p\,\mathcal{Z}$ as a subcone of $\mathrm{T}_p\,\mathcal{Z}^\omega$.

13.30. Corollary. *Let $\mathcal{Z}$ be a complete length space and $f\colon\mathcal{Z}\multimapinv\mathbb{R}$ be a locally Lipschitz semiconcave subfunction. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then*

$$\nabla_x f=\nabla_x f^\omega.$$

for any point $x\in\operatorname{Dom} f$.

Proof. Note that

$$\begin{array}{ccc}\mathcal{Z} & \subset & \mathcal{Z}^\omega\\ \cup & & \cup\\ \operatorname{Dom} f & \subset & \operatorname{Dom} f^\omega\end{array}.$$

Applying 13.28 for $\mathcal{Z}_n=\mathcal{Z}$ and $x_n=x$, we get that $|\nabla_x f|\geqslant|\nabla_x f^\omega|$.

On the other hand, $f=f^\omega|_{\mathcal{Z}}$, hence $\mathbf{d}_p f=\mathbf{d}_p f^\omega|_{\mathrm{T}_p\,\mathcal{Z}}$. Thus from 13.22c, $|\nabla_x f|\leqslant|\nabla_x f^\omega|$. Therefore

$$|\nabla_x f|=|\nabla_x f^\omega| \tag{1}$$

for any $x\in\mathcal{Z}$.

Further,

$$\begin{aligned}|\nabla_x f|^2&=(\mathbf{d}_x f)(\nabla_x f)\\&=\mathbf{d}_x f^\omega(\nabla_x f)\\&\leqslant\langle\nabla_x f^\omega,\nabla_x f\rangle\\&=|\nabla_x f^\omega|\cdot|\nabla_x f|\cdot\cos\measuredangle(\nabla_x f^\omega,\nabla_x f).\end{aligned}$$

Together with 1, this implies $\measuredangle(\nabla_x f^\omega,\nabla_x f)=0$ and the statement follows. □

13.31. Semicontinuity of |gradient|. *Let $\mathcal{Z}$ be a complete length space and $f\colon\mathcal{Z}\multimapinv\mathbb{R}$ be a locally Lipschitz semiconcave subfunction. Suppose that $\mathcal{Z}$ is either* CBB *or* CAT. *Then the function $x\mapsto|\nabla_x f|$ is lower-continuous; that is, for any sequence $x_n\to x\in\operatorname{Dom} f$, we have*

$$|\nabla_x f|\leqslant\varliminf_{n\to\infty}|\nabla_{x_n} f|.$$

Proof. According to 13.30, $|\nabla_x f|=|\nabla_x f^\omega|$. Applying 13.28 for $x_n\to x$, we obtain

$$\lim_{n\to\omega}|\nabla_{x_n} f|\geqslant|\nabla_x f^\omega|=|\nabla_x f|.$$

The same holds for an arbitrary subsequence of x_n—hence the result. □

H. Polar vectors

Here we give a corollary of Lemma 13.26. It will be used to prove basic properties of the tangent space.

13.32. Anti-sum lemma. *Let $\mathcal{L}$ be a complete length* CBB *space and $p \in \mathcal{L}$.*

Given two vectors $u, v \in \mathrm{T}_p$, there is a unique vector $w \in \mathrm{T}_p$ such that

$$\langle u, x\rangle + \langle v, x\rangle + \langle w, x\rangle \geqslant 0$$

for any $x \in \mathrm{T}_p$, and

$$\langle u, w\rangle + \langle v, w\rangle + \langle w, w\rangle = 0.$$

If T_p were a length space, then the lemma would follow from the existence of the gradient (13.20), applied to the function $\mathrm{T}_p \to \mathbb{R}$ defined by $x \mapsto -(\langle u, x\rangle + \langle v, x\rangle)$. However, the tangent space T_p might be not a length space; see Halbeisen's example 13.6.

Applying the above lemma for $u = v$, we have the following statement.

13.33. Existence of polar vector. *Let $\mathcal{L}$ be a complete length* CBB *space and $p \in \mathcal{L}$. Given a vector $u \in \mathrm{T}_p$, there is a unique vector $u^* \in \mathrm{T}_p$ such that $\langle u^*, u^*\rangle + \langle u, u^*\rangle = 0$ and u^* is* polar *to u; that is, $\langle u^*, x\rangle + \langle u, x\rangle \geqslant 0$ for any $x \in \mathrm{T}_p$.*

In particular, for any vector $u \in \mathrm{T}_p$ there is a polar vector $u^* \in \mathrm{T}_p$ such that $|u^*| \leqslant |u|$.

Milka's lemma provides a refinement of this statement; it states that in the finite-dimensional CBB space, any tangent vector u has a polar vector u^* such that $|u^*| = |u|$.

13.34. Example. *Let $\mathcal{L}$ be the upper half space of the Euclidean space $\mathbb{E}^n$; that is, $\mathcal{L} = \{(x_1, \dots, x_n) \in \mathbb{E}^n \mid x_n \geqslant 0\}$. It is a complete length* CBB(0) *space. For $p = 0$, the tangent space T_p can be canonically identified with $\mathcal{L}$. Then any vector $u = (v_1, \dots, v_n) \in \mathrm{T}_p$ a unique polar vector such that $|u^*| = |u|$ which is $u^* = (-v_1, \dots, -v_{n-1}, v_n)$. However, if $v_n \neq 0$, then u has other polar vectors, in particular $(-v_1, \dots, -v_{n-1}, 0)$.*

It is instructive to solve the following exercise before reading the proof of 13.32.

13.35. Exercise. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space and a, b, p be mutually distinct points in $\mathcal{L}$. Prove that*

$$(\mathbf{d}_p \operatorname{dist}_a)(\nabla_p \operatorname{dist}_b) \leqslant \cos \tilde{\measuredangle}^\kappa \left(p\,{}^a_b\right).$$

Proof of 13.32. Choose two sequences of points $a_n, b_n \in \operatorname{Str}(p)$ such that $\uparrow_{[pa_n]} \to u/|u|$ and $\uparrow_{[pb_n]} \to v/|v|$ as $n \to \infty$. Consider a sequence of functions

$$f_n = |u| \cdot \operatorname{dist}_{a_n} + |v| \cdot \operatorname{dist}_{b_n}.$$

According to Exercise 13.14,

$$(\mathbf{d}_p f_n)(x) = -|u| \cdot \langle \uparrow_{[pa_n]}, x\rangle - |v| \cdot \langle \uparrow_{[pb_n]}, x\rangle.$$

Thus we have the following uniform convergence for all $x \in \Sigma_p$:

$$(\mathbf{d}_p f_n)(x) \xrightarrow[n\to\infty]{} -\langle u, x\rangle - \langle v, x\rangle.$$

According to Lemma 13.26, the sequence $\nabla_p f_n$ converges. Let

$$w = \lim_{n\to\infty} \nabla_p f_n.$$

By the definition of gradient,

$$\begin{aligned}\langle w, w\rangle &= \lim_{n\to\infty} \langle \nabla_p f_n, \nabla_p f_n\rangle & \langle w, x\rangle &= \lim_{n\to\infty} \langle \nabla_p f_n, x\rangle\\ &= \lim_{n\to\infty} (\mathbf{d}_p f_n)(\nabla_p f_n) & &\geqslant \lim_{n\to\infty} (\mathbf{d}_p f_n)(x)\\ &= -\langle u, w\rangle - \langle v, w\rangle, & &= -\langle u, x\rangle - \langle v, x\rangle.\end{aligned}$$

□

I. Linear subspace of tangent space

13.36. Definition. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space, $p \in \mathcal{L}$ and u, $v \in \mathrm{T}_p$. We say that vectors u and v are* opposite *to each other; (briefly, $u + v = 0$) if $|u| = |v| = 0$ or $\measuredangle(u, v) = \pi$ and $|u| = |v|$.*

The subcone

$$\mathrm{Lin}_p = \left\{ v \in \mathrm{T}_p : \exists\, w \in \mathrm{T}_p \quad \textit{such that} \quad w + v = 0 \right\}$$

will be called the linear subcone *of* T_p.

The reason for the term "linear" will become evident in Theorem 13.39.

13.37. Proposition. *Let $\mathcal{L}$ be a complete length* CBB *space and $p \in \mathcal{L}$. Given two vectors u, $v \in \mathrm{T}_p$, the following statements are equivalent:*

(*a*) $u + v = 0$;

(*b*) $\langle u, x\rangle + \langle v, x\rangle = 0$ *for any* $x \in \mathrm{T}_p$;

(*c*) $\langle u, \xi\rangle + \langle v, \xi\rangle = 0$ *for any* $\xi \in \Sigma_p$.

Proof. The condition $u + v = 0$ is equivalent to

$$\langle u, u\rangle = -\langle u, v\rangle = \langle v, v\rangle;$$

thus (*b*) ⇒ (*a*). Since T_p is isometric to a subset of T_p^ω, the splitting theorem (16.22) applied to T_p^ω gives (*a*) ⇒ (*b*).

The equivalence (*b*)⇔(*c*) is trivial. □

13.38. Proposition. *Let $\mathcal{L}$ be a complete length* CBB *space and $p \in \mathcal{L}$. Then for any three vectors u, v, $w \in \mathrm{T}_p$, if $u + v = 0$ and $u + w = 0$ then $v = w$.*

Proof. By Proposition 13.37, both v and w satisfy the condition in corollary 13.33. Hence the result. $\square$

Let $u \in \mathrm{Lin}_p$; that is $u + v = 0$ for some $v \in \mathrm{T}_p$. Given $s < 0$, let

$$s \cdot u \coloneqq (-s) \cdot v.$$

This way we define multiplication of any vector in Lin_p by any real number (positive and negative). Proposition 13.38 implies that such multiplication is uniquely defined.

13.39. Theorem. *Let $\mathcal{L}$ be a complete length* $\mathrm{CBB}(\kappa)$ *space and $p \in \mathcal{L}$. Then* Lin_p *is a subcone of* T_p *isometric to a Hilbert space.*

Before proving the theorem, let us give a corollary.

13.40. Corollary. *Let $\mathcal{L}$ be a complete length* $\mathrm{CBB}(\kappa)$ *space and $p \in$* $\mathrm{Str}(x_1, \dots, x_n)$. *Then there is a subcone $E \subset \mathrm{T}_p$ that is isometric to a Euclidean space such that* $\log[px_i] \in E$ *for every i.*

Proof. By the definition of $\mathrm{Str}(x_1, \dots, x_n)$ (8.10), $\log[px_i] \in \mathrm{Lin}_p$ for each i. It remains to apply Theorem 13.39. $\square$

The main difficulty in the proof of Theorem 13.39 comes from the fact that in general T_p is not a length space; see Halbeisen's example (13.6). If the tangent space were a length space, the statement would follow directly from the splitting theorem (16.22). In fact the proof of Theorem 13.39 is very circuitous—we use the construction of the gradient, as well as the splitting theorem, namely its corollary (16.23). Thus in order to understand our proof one needs to read most of Chapter 16.

Proof of 13.39. First we show that Lin_p is a complete geodesic CBB(0) space.

Recall that T_p^ω is a complete geodesic CBB(0) space (see 4.9 and 13.1*a*) and Lin_p is a closed subset of T_p^ω. Thus, it is sufficient to show that the metric on Lin_p inherited from T_p^ω is a length metric.

Fix two vectors $x, y \in \mathrm{Lin}_p$. Let u and v be such that $u + \tfrac{1}{2} \cdot x = 0$ and $v + \tfrac{1}{2} \cdot y = 0$. Apply Lemma 13.32 to the vectors u and v; let $w \in \mathrm{T}_p$ denote the obtained tangent vector.

(1) *w is a midpoint of $[xy]$.*

Indeed, according to Lemma 13.32,

$$\begin{aligned}|w|^2 &= -\langle w, u\rangle - \langle w, v\rangle\\ &= \tfrac{1}{2} \cdot \langle w, x\rangle + \tfrac{1}{2} \cdot \langle w, y\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}
|x-w|^2+|w-y|^2 &= 2\cdot|w|^2+|x|^2+|y|^2-2\cdot\langle w,x\rangle-2\cdot\langle w,y\rangle\\
&= |x|^2+|y|^2-\langle w,x\rangle-\langle w,y\rangle\\
&\leqslant |x|^2+|y|^2+\langle u,x\rangle+\langle v,x\rangle+\langle u,y\rangle+\langle v,y\rangle\\
&= \tfrac12\cdot|x|^2+\tfrac12\cdot|y|^2-\langle x,y\rangle\\
&= \tfrac12\cdot|x-y|^2.
\end{aligned}$$

Thus $|x-w|=|w-y|=\frac12\cdot|x-y|$ and 1 follows. △

Note that for any $v\in\mathrm{Lin}_p$ there is a line ℓ that contains v and 0. Therefore by 16.23, Lin_p is isometric to a Hilbert space. □

J. Comments

13.41. Open question. *Let $\mathcal{L}$ be a proper length* CBB(κ) *space. Is it true that for any $p\in\mathcal{L}$, the tangent space* T_p *is a length space?*

Chapter 14

Dimension of CAT spaces

In this chapter we discuss constructions introduced by Bruce Kleiner [**95**].

The material of this chapter is used mostly for CAT spaces, but the results in section 14A find applications for finite-dimensional CBB spaces as well.

A. The case of complete geodesic spaces

The following construction gives a k-dimensional submanifold for a given "non-degenerate" array of $k+1$ strongly convex functions.

14.1. Definition. *For two real arrays* $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^{k+1}$, $\mathbf{v} = (v^0, v^1, \dots, v^k)$ *and* $\mathbf{w} = (w^0, w^1, \dots, w^k)$, *we will write* $\mathbf{v} \succcurlyeq \mathbf{w}$ *if* $v^i \geqslant w^i$ *for each* i.

Given a subset $Q \subset \mathbb{R}^{k+1}$, denote by $\operatorname{Up} Q$ the smallest upper set containing Q, and by $\operatorname{Min} Q$ the set of minimal elements of Q with respect to $\succcurlyeq$; that is,

$$\operatorname{Up} Q = \left\{ \mathbf{v} \in \mathbb{R}^{k+1} : \exists\, \mathbf{w} \in Q \text{ such that } \mathbf{v} \succcurlyeq \mathbf{w} \right\},$$
$$\operatorname{Min} Q = \{ \mathbf{v} \in Q : \text{if } \mathbf{v} \succcurlyeq \mathbf{w} \in Q \text{ then } \mathbf{w} = \mathbf{v} \}.$$

14.2. Definition. *Let* $\mathbf{f} = (f^0, f^1, \dots, f^k) \colon \mathcal{X} \to \mathbb{R}^{k+1}$ *be a function array on a metric space* $\mathcal{X}$. *The set*

$$\operatorname{Web} \mathbf{f} \coloneqq \mathbf{f}^{-1}\left[\operatorname{Min} \mathbf{f}(\mathcal{X})\right] \subset \mathcal{X}$$

will be called the web *of* $\mathbf{f}$.

Given an array $\mathbf{f} = (f^0, f^1, \dots, f^k)$, we denote by $\mathbf{f}^{-i}$ the subarray of $\mathbf{f}$ with f^i removed; that is,

$$\mathbf{f}^{-i} \coloneqq (f^0, \dots, f^{i-1}, f^{i+1}, \dots, f^k).$$

Clearly $\operatorname{Web}\mathbf{f}^{-i}\subset\operatorname{Web}\mathbf{f}$. Define the *inner web* of $\mathbf{f}$ as

$$\operatorname{InWeb}\mathbf{f}=\operatorname{Web}\mathbf{f}\Big\backslash\Big(\bigcup_i\operatorname{Web}\mathbf{f}^{-i}\Big).$$

We say that a function array is *nondegenerate* if $\operatorname{InWeb}\mathbf{f}\neq\emptyset$.

Example. If $\mathcal{X}$ is a geodesic space, then $\operatorname{Web}(\operatorname{dist}_x,\operatorname{dist}_y)$ is the union of all geodesics from x to y, and

$$\operatorname{InWeb}(\operatorname{dist}_x,\operatorname{dist}_y)=\operatorname{Web}(\operatorname{dist}_x,\operatorname{dist}_y)\setminus\{x,y\}.$$

Barycenters. Let us denote by $\Delta^k\subset\mathbb{R}^{k+1}$ the *standard k-simplex*; that is, $\mathbf{x}=(x^0,x^1,\dots,x^k)\in\Delta^k$ if $\sum_{i=0}^k x^i=1$ and $x^i\geqslant 0$ for all i.

Let $\mathcal{X}$ be a metric space and $\mathbf{f}=(f^0,f^1,\dots,f^k)\colon\mathcal{X}\to\mathbb{R}^{k+1}$ be a function array. Consider the map $\mathfrak{S}_{\mathbf{f}}\colon\Delta^k\to\mathcal{X}$ defined by

$$\mathfrak{S}_{\mathbf{f}}(\mathbf{x})=\operatorname{MinPoint}\sum_{i=0}^k x^i\cdot f^i,$$

where $\operatorname{MinPoint}f$ denotes a point of minimum of f. The map $\mathfrak{S}_{\mathbf{f}}$ will be called a *barycentric simplex* of $\mathbf{f}$. Note that for a general function array $\mathbf{f}$, the value $\mathfrak{S}_{\mathbf{f}}(\mathbf{x})$ might be undefined or nonuniquely defined.

It is clear from the definition that $\mathfrak{S}_{\mathbf{f}^{-i}}$ coincides with the restriction of $\mathfrak{S}_{\mathbf{f}}$ to the corresponding facet of Δ^k.

14.3. Theorem. *Let $\mathcal{X}$ be a complete geodesic space and $\mathbf{f}=(f^0,f^1,\dots,f^k)\colon\mathcal{X}\to\mathbb{R}^{k+1}$ be an array of strongly convex and locally Lipschitz functions. Then $\mathbf{f}$ defines a $C^{\frac12}$-embedding $\operatorname{Web}\mathbf{f}\hookrightarrow\mathbb{R}^{k+1}$.*

Moreover,

(*a*) *$W=\operatorname{Up}[\mathbf{f}(\mathcal{X})]$ is a convex closed subset of $\mathbb{R}^{k+1}$, and $S=\partial_{\mathbb{R}^{k+1}}W$ is a convex hypersurface in $\mathbb{R}^{k+1}$.*

(*b*)
$$\mathbf{f}(\operatorname{Web}\mathbf{f})=\operatorname{Min}W\subset S$$
and
$$\mathbf{f}(\operatorname{InWeb}\mathbf{f})=\operatorname{Interior}_S(\operatorname{Min}W).$$

(*c*) *The barycentric simplex $\mathfrak{S}_{\mathbf{f}}\colon\Delta^k\to\mathcal{X}$ is a uniquely defined Lipshitz map and $\mathfrak{I}\mathfrak{S}_{\mathbf{f}}=\operatorname{Web}\mathbf{f}$. In particular, $\operatorname{Web}\mathbf{f}$ is compact.*

(*d*) *Let us equip Δ^k with the metric induced by the ℓ^1-norm on $\mathbb{R}^{k+1}$. Then the Lipschitz constant of $\mathfrak{S}_{\mathbf{f}}\colon\Delta^k\to\mathcal{U}$ can be estimated in terms of positive lower bounds on $(f^i)''$ and Lipschitz constants of f^i in a neighborhood of $\operatorname{Web}\mathbf{f}$ for all i.*

In particular, by (a) and (b), InWeb $\mathbf{f}$ *is* $C^{\frac{1}{2}}$*-homeomorphic to an open set of* $\mathbb{R}^k$.

The proof is preceded by a few preliminary statements.

14.4. Lemma. *Suppose $\mathcal{X}$ is a complete geodesic space and $f\colon \mathcal{X} \to \mathbb{R}$ is a locally Lipschitz, strongly convex function. Then the minimum point of f is uniquely defined.*

Proof. Without loss of generality, we can assume that f is 1-convex. In particular, the following claim holds:

(1) *if z is a midpoint of the geodesic $[xy]$, then*

$$s \leqslant f(z) \leqslant \tfrac{1}{2}\cdot f(x) + \tfrac{1}{2}\cdot f(y) - \tfrac{1}{8}\cdot|x-y|^2,$$

where s is the infimum of f.

Uniqueness. Assume that x and y are distinct minimum points of f. From 1 we have

$$f(z) < f(x) = f(y),$$

a contradiction.

Existence. Fix a point $p \in \mathcal{X}$, and let $\ell \in \mathbb{R}$ be a Lipschitz constant of f in a neighborhood of p.

Choose a geodesic $[px]$; consider the function $\phi\colon t \mapsto f \circ \mathrm{geod}_{[px]}(t)$. Clearly ϕ is 1-convex and $\phi^+(0) \geqslant -\ell$. Setting $a = |p-x|$, we have

$$\begin{aligned} f(x) &= \phi(a) \\ &\geqslant f(p) - \ell\cdot a + \tfrac{1}{2}\cdot a^2 \\ &\geqslant f(p) - \tfrac{1}{2}\cdot \ell^2. \end{aligned}$$

In particular,

$$\begin{aligned} s &\coloneqq \inf\{\, f(x) : x \in \mathcal{X} \,\} \\ &\geqslant f(p) - \tfrac{1}{2}\cdot\ell^2. \end{aligned}$$

Choose a sequence of points $p_n \in \mathcal{X}$ such that $f(p_n) \to s$. Applying 1 for $x = p_n$, $y = p_m$, we see that p_n is a Cauchy sequence. Thus the sequence p_n converges to a minimum point of f. □

14.5. Definition. *Let Q be a closed subset of $\mathbb{R}^{k+1}$. A vector $\mathbf{x} = (x^0, x^1, \dots, x^k) \in \mathbb{R}^{k+1}$ is* subnormal *to Q at a point $\mathbf{v} \in Q$ if*

$$\langle \mathbf{x}, \mathbf{w} - \mathbf{v}\rangle \coloneqq \sum_i x^i \cdot (w^i - v^i) \geqslant 0$$

for any $\mathbf{w} \in Q$.

14.6. Lemma. *Let $\mathcal{X}$ be a complete geodesic space and $\mathbf{f} = (f^0, f^1, \dots, f^k) : \mathcal{X} \to \mathbb{R}^{k+1}$ be an array of strongly convex and locally Lipschitz functions. Let $W = \operatorname{Up}\mathbf{f}(\mathcal{X})$. Then:*

(*a*) *W is a closed convex set, bounded below with respect to $\succcurlyeq$.*

(*b*) *If $\mathbf{x}$ is a subnormal vector to W, then $\mathbf{x} \succcurlyeq \mathbf{0}$.*

(*c*) *$S = \partial_{\mathbb{R}^{k+1}} W$ is a complete convex hypersurface in $\mathbb{R}^{k+1}$.*

Proof. Denote by $\bar{W}$ the closure of W.

Convexity of all f^i implies that for any two points $p, q \in \mathcal{X}$ and $t \in [0,1]$ we have

$$(1-t)\cdot\mathbf{f}(p) + t\cdot\mathbf{f}(q) \succcurlyeq \mathbf{f}\circ\operatorname{path}_{[pq]}(t), \tag{2}$$

where $\operatorname{path}_{[pq]}$ denotes a geodesic path from p to q. Therefore W, as well as $\bar{W}$, are convex sets in $\mathbb{R}^{k+1}$.

Let

$$w^i = \min\{\, f^i(x) : x \in \mathcal{X} \,\}.$$

By Lemma 14.4, w^i is finite for each i. Evidently, $\mathbf{w} = (w^0, w^1, \dots, w^k)$ is a lower bound of $\bar{W}$ with respect to $\succcurlyeq$.

It is clear that W has nonempty interior, and $W \neq \mathbb{R}^{k+1}$ since W is bounded below. Therefore $S = \partial_{\mathbb{R}^{k+1}} W = \partial_{\mathbb{R}^{k+1}} \bar{W}$ is a complete convex hypersurface in $\mathbb{R}^{k+1}$.

Since $\bar{W}$ is closed and bounded below, we also have

$$\bar{W} = \operatorname{Up}[\operatorname{Min}\bar{W}]. \tag{3}$$

Choose an arbitrary $\mathbf{v} \in S$. Let $\mathbf{x} \in \mathbb{R}^{k+1}$ be a subnormal vector to $\bar{W}$ at $\mathbf{v}$. In particular, $\langle \mathbf{x}, \mathbf{y}\rangle \geqslant 0$ for any $\mathbf{y} \succcurlyeq \mathbf{0}$; that is, $\mathbf{x} \succcurlyeq \mathbf{0}$.

Further, according to Lemma 14.4, the function

$$p \mapsto \langle \mathbf{x}, \mathbf{f}(p)\rangle = \sum_i x^i \cdot f^i(p)$$

has a uniquely defined minimum point, say p. Clearly

$$\mathbf{v} \succcurlyeq \mathbf{f}(p) \quad\text{and}\quad \mathbf{f}(p) \in \operatorname{Min} W. \tag{4}$$

Note that for any $\mathbf{u} \in \bar{W}$ there is $\mathbf{v} \in S$ such that $\mathbf{u} \succcurlyeq \mathbf{v}$. Therefore 4 implies

$$\bar{W} \subset \operatorname{Up}[\operatorname{Min} W] \subset W.$$

Hence $\bar{W} = W$; that is, W is closed. $\square$

Proof of 14.3.

(a)+(b). Without loss of generality, we may assume that all f^i are 1-convex.

Given $\mathbf{v} = (v^0, v^1, \dots, v^k) \in \mathbb{R}^{k+1}$, consider the function $h_{\mathbf{v}} : \mathcal{X} \to \mathbb{R}$ defined by

$$h_{\mathbf{v}}(p) = \max_i \{f^i(p) - v^i\}.$$

Note that $h_{\mathbf{v}}$ is 1-convex. Let

$$\Phi(\mathbf{v}) \coloneqq \operatorname{MinPoint} h_{\mathbf{v}}.$$

According to Lemma 14.4, $\Phi(\mathbf{v})$ is uniquely defined.

From the definition of web (14.2) we have $\Phi \circ \mathbf{f}(p) = p$ for any $p \in \operatorname{Web}\mathbf{f}$; that is, Φ is a left inverse to the restriction $\mathbf{f}|_{\operatorname{Web}\mathbf{f}}$. In particular,

$$\operatorname{Web}\mathbf{f} = \mathfrak{I}\Phi. \tag{5}$$

Given $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{k+1}$, set $p = \Phi(\mathbf{v})$ and $q = \Phi(\mathbf{w})$. Since $h_{\mathbf{v}}$ and $h_{\mathbf{w}}$ are 1-convex, we have

$$h_{\mathbf{v}}(q) \geqslant h_{\mathbf{v}}(p) + \tfrac{1}{2} \cdot |p - q|^2, \qquad h_{\mathbf{w}}(p) \geqslant h_{\mathbf{w}}(q) + \tfrac{1}{2} \cdot |p - q|^2.$$

Therefore,

$$\begin{aligned} |p - q|^2 &\leqslant 2 \cdot \sup_{x \in \mathcal{X}} \{|h_{\mathbf{v}}(x) - h_{\mathbf{w}}(x)|\} \\ &\leqslant 2 \cdot \max_i \{|v^i - w^i|\}. \end{aligned}$$

In particular, Φ is $C^{\frac{1}{2}}$-continuous. Hence $\mathbf{f}|_{\operatorname{Web}\mathbf{f}}$ is a $C^{\frac{1}{2}}$-embedding.

As in Lemma 14.6, let $W = \operatorname{Up}\mathbf{f}(\mathcal{X})$ and $S = \partial_{\mathbb{R}^{k+1}} W$. Then S is a convex hypersurface in $\mathbb{R}^{k+1}$. Clearly $\mathbf{f}(\operatorname{Web}\mathbf{f}) = \operatorname{Min} W \subset S$. From the definition of inner web, we have $\mathbf{v} \in \mathbf{f}(\operatorname{InWeb}\mathbf{f})$ if and only if $\mathbf{v} \in S$ and for any i there is $\mathbf{w} = (w^0, w^1, \dots, w^k) \in W$ such that $w^j < v^j$ for all $j \neq i$. Thus $\mathbf{f}(\operatorname{InWeb}\mathbf{f})$ is open in S. That is, $\operatorname{InWeb}\mathbf{f}$ is $C^{\frac{1}{2}}$-homeomorphic to an open set in a convex hypersurface $S \subset \mathbb{R}^{k+1}$, and hence to an open set of $\mathbb{R}^k$, as claimed.

(c)+(d). Since f^i is 1-convex, for any $\mathbf{x} = (x^0, x^1, \dots, x^k) \in \Delta^k$ the convex combination

$$\left(\sum_i x^i \cdot f^i\right) : \mathcal{X} \to \mathbb{R}$$

is also 1-convex. Therefore, according to Lemma 14.4, the barycentric simplex $\mathfrak{S}_{\mathbf{f}}$ is uniquely defined on Δ^k.

For $\mathbf{x}, \mathbf{y} \in \Delta^k$, let

$$\begin{aligned} f_{\mathbf{x}} &= \sum_i x^i \cdot f^i, & f_{\mathbf{y}} &= \sum_i y^i \cdot f^i, \\ p &= \mathfrak{S}_{\mathbf{f}}(\mathbf{x}), & q &= \mathfrak{S}_{\mathbf{f}}(\mathbf{y}), \\ s &= |p - q|. \end{aligned}$$

Note the following:

- The function $\phi(t) = f_{\mathbf{x}} \circ \mathrm{geod}_{[pq]}(t)$ has minimum at 0. Therefore $\phi^+(0) \geqslant 0$.
- The function $\psi(t) = f_{\mathbf{y}} \circ \mathrm{geod}_{[pq]}(t)$ has minimum at s. Therefore $\psi^-(s) \geqslant 0$.

From 1-convexity of $f_{\mathbf{y}}$, we have $\psi^+(0) + \psi^-(s) + s \leqslant 0$.

Let ℓ be a Lipschitz constant for all f^i in a neighborhood $\Omega \ni p$. Then

$$\psi^+(0) \leqslant \phi^+(0) + \ell \cdot \|\mathbf{x} - \mathbf{y}\|_1,$$

where $\|\mathbf{x} - \mathbf{y}\|_1 = \sum_{i=0}^{k} |x^i - y^i|$. That is, given $\mathbf{x} \in \Delta^k$, there is a constant ℓ such that

$$\begin{aligned}|\mathfrak{S}_{\mathbf{f}}(\mathbf{x}) - \mathfrak{S}_{\mathbf{f}}(\mathbf{y})| &= s \\ &\leqslant \ell \cdot \|\mathbf{x} - \mathbf{y}\|_1\end{aligned}$$

for any $\mathbf{y} \in \Delta^k$. In particular, there is $\varepsilon > 0$ such that if $\|\mathbf{x}-\mathbf{y}\|_1 < \varepsilon$, $\|\mathbf{x}-\mathbf{z}\|_1 < \varepsilon$, then $\mathfrak{S}_{\mathbf{f}}(\mathbf{y}), \mathfrak{S}_{\mathbf{f}}(\mathbf{z}) \in \Omega$. Thus the same argument as above implies

$$|\mathfrak{S}_{\mathbf{f}}(\mathbf{y}) - \mathfrak{S}_{\mathbf{f}}(\mathbf{z})| \leqslant \ell \cdot \|\mathbf{y} - \mathbf{z}\|_1$$

for any $\mathbf{y}$ and $\mathbf{z}$ sufficiently close to $\mathbf{x}$; that is, $\mathfrak{S}_{\mathbf{f}}$ is locally Lipschitz. Since Δ^k is compact, $\mathfrak{S}_{\mathbf{f}}$ is Lipschitz.

Clearly $\mathfrak{S}_{\mathbf{f}}(\Delta^k) \subset \mathrm{Web}\,\mathbf{f}$. It remains to show that $\mathfrak{S}_{\mathbf{f}}(\Delta^k) \supset \mathrm{Web}\,\mathbf{f}$. According to Lemma 14.6, $W = \mathrm{Up}\,\mathbf{f}(\mathcal{X})$ is a closed convex set in $\mathbb{R}^{k+1}$. Let $p \in \mathrm{Web}\,\mathbf{f}$. Clearly $\mathbf{f}(p) \in \mathrm{Min}\,W \subset S = \partial_{\mathbb{R}^{k+1}} W$. Let $\mathbf{x}$ be a subnormal vector to W at $\mathbf{f}(p)$. According to Lemma 14.6, $\mathbf{x} \geqslant \mathbf{0}$. Without loss of generality, we may assume that $\sum_i x^i = 1$; that is, $\mathbf{x} \in \Delta^k$. By Lemma 14.4, p is the unique minimum point of $\sum_i x^i \cdot f^i$; that is, $p = \mathfrak{S}_{\mathbf{f}}(\mathbf{x})$. □

B. The case of CAT spaces

Let $\mathbf{a} = (a^0, a^1, \dots, a^k)$ be a point array in a metric space $\mathcal{U}$. Recall that $\mathrm{dist}_{\mathbf{a}}$ denotes the distance map

$$(\mathrm{dist}_{a^0}, \mathrm{dist}_{a^1}, \dots, \mathrm{dist}_{a^k}) \colon \mathcal{U} \to \mathbb{R}^{k+1},$$

which can be also regarded as a function array. The *radius* of the point array $\mathbf{a}$ is defined to be the radius of the set $\{a^0, a^1, \dots, a^k\}$; that is,

$$\mathrm{rad}\,\mathbf{a} = \inf\left\{ r > 0 : \exists z \in \mathcal{U} \text{ such that } a^i \in \mathrm{B}(z, r) \text{ for any } i \right\}.$$

Fix $\kappa \in \mathbb{R}$. Let $\mathbf{a} = (a^0, a^1, \dots, a^k)$ be a point array of radius $< \frac{\varpi\kappa}{2}$ in a metric space $\mathcal{U}$. Consider the function array $\mathbf{f} = (f^0, f^1, \dots, f^k)$ where

$$f^i(x) = \mathrm{md}^\kappa |a^i - x|.$$

Assuming the barycentric simplex $\mathfrak{S}_{\mathbf{f}}$ is defined, then $\mathfrak{S}_{\mathbf{f}}$ is called the *κ-barycentric simplex* for the point array $\mathbf{a}$; it will be denoted by $\mathfrak{S}^{\kappa}_{\mathbf{a}}$. The points a^0, a^1, ..., a^k are called *vertexes* of the κ-barycentric simplex. Note that once we say the κ-barycentric simplex is defined, we automatically assume that $\operatorname{rad}\mathbf{a} < \frac{\varpi\kappa}{2}$.

14.7. Theorem. *Let $\mathcal{U}$ be a complete length $\mathrm{CAT}(\kappa)$ space and $\mathbf{a} = (a^0, a^1, \dots a^k)$ be a point array with radius $< \frac{\varpi\kappa}{2}$. Then:*

(*a*) *The κ-barycentric simplex $\mathfrak{S}^{\kappa}_{\mathbf{a}}\colon \Delta^k \to \mathcal{U}$ is defined. Moreover, $\mathfrak{S}^{\kappa}_{\mathbf{a}}$ is a Lipschitz map, and if Δ^k is equipped with the ℓ^1-metric, then its Lipschitz constant can be estimated in terms of κ and the radius of $\mathbf{a}$ (in particular it does not depend on k).*

(*b*) $\mathrm{Web}(\operatorname{dist}_{\mathbf{a}}) = \mathfrak{I}\mathfrak{S}^{\kappa}_{\mathbf{a}}$. *Moreover, if a closed convex set $K \subset \mathcal{U}$ contains all a^i, then* $\mathrm{Web}(\operatorname{dist}_{\mathbf{a}}) \subset K$.

(*c*) *The restriction*[1] $\operatorname{dist}_{\mathbf{a}-0}|_{\mathrm{InWeb}(\operatorname{dist}_{\mathbf{a}})}$ *is an open $C^{\frac{1}{2}}$-embedding in $\mathbb{R}^k$. Thus there is an inverse of* $\operatorname{dist}_{\mathbf{a}-0}|_{\mathrm{InWeb}(\operatorname{dist}_{\mathbf{a}})}$, *say $\Phi\colon \mathbb{R}^k \multimap \mathcal{U}$.*

The subfunction $f = \operatorname{dist}_{a^0} \circ \Phi$ is semiconvex and locally Lipschitz. Moreover, if $\kappa \leqslant 0$, then f is convex.

In particular, $\mathrm{Web}(\operatorname{dist}_{\mathbf{a}})$ *is a compact set and* $\mathrm{InWeb}(\operatorname{dist}_{\mathbf{a}})$ *is $C^{\frac{1}{2}}$-homeomorphic to an open subset of $\mathbb{R}^k$.*

14.8. Definition. *The submap $\Phi\colon \mathbb{R}^k \multimap \mathcal{X}$ of Theorem 14.7c will be called the* $\operatorname{dist}_{\mathbf{a}}$-web embedding *with* brace $\operatorname{dist}_{a^0}$. *The terminology invokes Theorem 14.7c.*

14.9. Definition. *Let $\mathcal{U}$ be a complete length $\mathrm{CAT}(\kappa)$ space and $\mathbf{a} = (a^0, a^1, \dots a^k)$ be a point array with radius $< \frac{\varpi\kappa}{2}$. If* $\mathrm{InWeb}(\operatorname{dist}_{\mathbf{a}})$ *is nonempty, then the point array $\mathbf{a}$ is called* nondegenerate.

Lemma 14.11 will provide examples of nondegenerate point arrays, which can be used in 14.7c.

14.10. Corollary. *Let $\mathcal{U}$ be a complete length $\mathrm{CAT}(\kappa)$ space, $\mathbf{a} = (a^0, a^1, \dots a^m)$ be a nondegenerate point array of radius $< \frac{\varpi\kappa}{2}$ in $\mathcal{U}$ and $\sigma = \mathfrak{S}^{\kappa}_{\mathbf{a}}$ be the corresponding κ-baricentric simplex. Then for some $\mathbf{x} \in \Delta^m$, the differential $\mathbf{d}_{\mathbf{x}}\sigma$ is linear and the image $\mathfrak{I}\mathbf{d}_{\mathbf{x}}\sigma$ forms a subcone isometric to an m-dimensional Euclidean space in the tangent cone $\mathrm{T}_{\sigma(\mathbf{x})}$.*

Proof. Denote the distance map $\operatorname{dist}_{\mathbf{a}-0}$ by $\tau\colon \mathcal{U} \to \mathbb{R}^m$.

According to Theorem 14.7, σ is Lipschitz and the distance map τ gives an open embedding of $\mathrm{InWeb}(\operatorname{dist}_{\mathbf{a}}) = \sigma(\Delta^m) \setminus \sigma(\partial\Delta^m)$. Note that τ is Lipschitz. According to Rademacher's theorem (13.12), the differential $\mathbf{d}_{\mathbf{x}}(\tau\circ\sigma)$ is linear for almost all $\mathbf{x} \in \Delta^m$. Further, since $\mathrm{InWeb}(\operatorname{dist}_{\mathbf{a}}) \neq \emptyset$, the area formula [**92**] implies that $\mathbf{d}_{\mathbf{x}}(\tau\circ\sigma)$ is surjective on a set of positive masure of points $\mathbf{x} \in \Delta^m$.

[1] Recall that $\operatorname{dist}_{\mathbf{a}-0}$ denotes the array $(\operatorname{dist}_{a^1}, \dots, \operatorname{dist}_{a^k})$.

Note that $\mathbf{d}_{\mathbf{x}}(\tau\circ\sigma) = (\mathbf{d}_{\sigma(\mathbf{x})}\tau)\circ(\mathbf{d}_{\mathbf{x}}\sigma)$. Applying Rademacher's theorem again, we have linearity of $\mathbf{d}_{\mathbf{x}}\sigma$ for almost all $\mathbf{x}\in\Delta^m$; at these points $\mathfrak{I}\mathbf{d}_{\mathbf{x}}\sigma$ forms a subcone isometric to a Euclidean space in $\mathrm{T}_{\sigma(\mathbf{x})}$. Clearly the dimension of $\mathfrak{I}\mathbf{d}_{\mathbf{x}}(\tau\circ\sigma)$ is at least as big as the dimension of $\mathfrak{I}\mathbf{d}_{\mathbf{x}}\sigma$. Hence the result. □

Proof of 14.7. Fix $z\in\mathcal{U}$ and $r<\frac{\varpi\kappa}{2}$ such that $|z-a^i|<r$ for all i. Note that the set $K\cap\overline{\mathrm{B}}[z,r]$ is convex, closed, and contains all a^i. Applying the theorem on short retract (Exercise 9.75), we get the second part of (*b*).

The remaining statements are proved first in the case $\kappa\leqslant 0$, and then the remaining case $\kappa>0$ is reduced to the case $\kappa=0$.

Case $\kappa\leqslant 0$. Consider the function array $f^i=\mathrm{md}^\kappa\circ\mathrm{dist}_{a^i}$. From the definition of web (14.2), it is clear that $\mathrm{Web}(\mathrm{dist}_{\mathbf{a}})=\mathrm{Web}\,\mathbf{f}$. Further, by the definition of κ-barycentric simplex, $\mathfrak{S}^\kappa_{\mathbf{a}}=\mathfrak{S}_{\mathbf{f}}$.

All the functions f^i are strongly convex (see ~~9.25*b*~~). Therefore (*a*), (*b*), and the first statements in (*c*) follow from Theorem 14.3.

Case $\kappa>0$. Applying rescaling, we may assume $\kappa=1$, so $\varpi\kappa=\varpi 1=\pi$.

Let $\mathring{\mathcal{U}}=\mathrm{Cone}\,\mathcal{U}$. By 11.7*a*, $\mathring{\mathcal{U}}$ is CAT(0). Let us denote by ι the natural embedding of $\mathcal{U}$ as the unit sphere in $\mathring{\mathcal{U}}$, and by $\mathrm{proj}\colon\mathring{\mathcal{U}}\multimap\mathcal{U}$ the submap defined by $\mathrm{proj}(v)=\iota^{-1}(v/|v|)$ for all $v\neq 0$. Note that there is $z\in\mathcal{U}$ and $\varepsilon>0$ such that the set

$$K_\varepsilon=\left\{\,v\in\mathring{\mathcal{U}}\;:\;\langle\iota(z),v\rangle\geqslant\varepsilon\,\right\}$$

contains all $\iota(a^i)$. Then $0\notin K_\varepsilon$, and the set K_ε is closed and convex. The latter follows from Exercise 9.28, since $v\mapsto-\langle\iota(z),v\rangle$ is a Busemann function.

Denote by $\iota(\mathbf{a})$ the point array $(\iota(a^0),\iota(a^1),\dots,\iota(a^k))$ in $\mathring{\mathcal{U}}$. From the case $\kappa=0$, we get that $\mathfrak{I}\mathfrak{S}^0_{\iota(\mathbf{a})}\subset K_\varepsilon$. In particular, $\mathfrak{I}\mathfrak{S}^0_{\iota(\mathbf{a})}\not\ni 0$ and thus $\mathrm{proj}\circ\mathfrak{S}^0_{\iota(\mathbf{a})}$ is defined. Direct calculations show

$$\mathfrak{S}^1_{\mathbf{a}}=\mathrm{proj}\circ\mathfrak{S}^0_{\iota(\mathbf{a})}\quad\text{and}\quad\mathrm{Web}(\mathrm{dist}_{\mathbf{a}})=\mathrm{proj}[\mathrm{Web}(\mathrm{dist}_{\iota(\mathbf{a})})].$$

Thus the case $\kappa=1$ of the theorem is reduced to the case $\kappa=0$, which is proved already. □

14.11. Lemma. *Let* $\mathbf{a}=(a^0,a^1,\dots a^k)$ *be a point array of radius* $<\frac{\varpi\kappa}{2}$ *in a complete length* CAT(κ) *space* $\mathcal{U}$, *and* $B^i=\overline{\mathrm{B}}[a^i,r^i]$ *for an array of positive reals* $(r^0,r^1,\dots,r^k)$. *Assume that* $\bigcap_i B^i=\emptyset$, *but* $\bigcap_{i\neq j}B^i\neq\emptyset$ *for any* j. *Then* $\mathbf{a}$ *is nondegenerate.*

Proof. Without loss of generality, we may assume that $\mathcal{U}$ is geodesic and diam $\mathcal{U}<\varpi\kappa$. If not, choose $z\in\mathcal{U}$ and $r<\frac{\varpi\kappa}{2}$ so that $|z-a^i|\leqslant r$ for each i, and consider $\overline{\mathrm{B}}[z,r]$ instead of $\mathcal{U}$. The latter can be done since $\overline{\mathrm{B}}[z,r]$ is convex and closed, so $\overline{\mathrm{B}}[z,r]$ is a complete length CAT(κ) space and $\mathrm{Web}(\mathrm{dist}_{\mathbf{a}})\subset\overline{\mathrm{B}}[z,r]$; see 9.27 and 14.7*b*.

By Theorem 14.7, $\mathrm{Web}(\mathrm{dist}_{\mathbf{a}})$ is a compact set; therefore there is a point $p \in \mathrm{Web}(\mathrm{dist}_{\mathbf{a}})$ minimizing the function

$$f(x) = \max_i\{\mathrm{dist}_{B^i} x\} = \max\{0, |a^0 - x| - r^0, \dots, |a^k - x| - r^k\}.$$

By the definition of web (14.2), p is also the minimum point of f on $\mathcal{U}$. Let us prove the following claim:

(1) $\quad p \notin B^j$ *for any* j.

Indeed, assume the contrary; that is,

(2) $$p \in B^j$$

for some j. Then p is a point of local minimum for the function

$$h^j(x) = \max_{i \neq j}\{\mathrm{dist}_{B^i} x\}.$$

Hence

$$\max_{i \neq j}\{\measuredangle\left[p\,{}^{x}_{a^i}\right]\} \geqslant \tfrac{\pi}{2}$$

for any $x \in \mathcal{U}$. From the angle comparison (9.14c), it follows that p is a global minimum of h^j and hence

$$p \in \bigcap_{i \neq j} B^i.$$

The latter and 2 contradict $\bigcap_i B^i = \emptyset$. $\triangle$

From the definition of web, it also follows that

$$\mathrm{Web}(\mathrm{dist}_{\mathbf{a}^{-j}}) \subset \bigcup_{i \neq j} B^i.$$

Indeed, if $q \in \bigcap_{i \neq j} B^i$ and $q' \notin \bigcup_{i \neq j} B^i$, then $|a_i - q| < |a_i - q'|$ for any $i \neq j$ therefore $q' \notin \mathrm{Web}(\mathrm{dist}_{\mathbf{a}^{-j}})$. Therefore the claim implies that $p \notin \mathrm{Web}(\mathrm{dist}_{\mathbf{a}^{-j}})$ for each j; that is, $p \in \mathrm{InWeb}(\mathrm{dist}_{\mathbf{a}})$. $\square$

C. Dimension

See Chapter 7 for definitions of various dimension-like invariants of metric spaces.

We start with two examples.

The first example shows that the dimension of complete length CAT spaces is not local; that is, such spaces might have open sets with different linear dimensions.

Such an example can be constructed by gluing at one point two Euclidean spaces of different dimensions. According to Reshetnyak's gluing theorem (9.39), this construction gives a CAT(0) space.

The second example provides a complete length CAT space with topological dimension 1 and arbitrary large Hausdorff dimension. Thus for complete length CAT spaces, one should not expect any relations between topological and Hausdorff dimensions except for the one provided by Szpilrajn's theorem (7.5).

To construct the second type of example, note that the completion of any metric tree has topological dimension 1 and is CAT($\varkappa$) for any $\varkappa$. Start with a binary tree Γ, and a sequence $\varepsilon_n > 0$ such that $\sum_n \varepsilon_n < \infty$. Define the metric on Γ by prescribing the length of an edge from level n to level $n+1$ to be ε_n. For an appropriately chosen sequence ε_n, the completion of Γ will contain a Cantor set of arbitrarily large Hausdorff dimension.

The following is a version of a theorem proved by Bruce Kleiner [**95**], with an improvement made by Alexander Lytchak [**109**].

14.12. Theorem. *For any complete length* CAT($\varkappa$) *space $\mathcal{U}$, the following statements are equivalent:*

(a) $\operatorname{LinDim}\mathcal{U} \geqslant m$.

(b) *For some $z \in \mathcal{U}$ there is an array of $m+1$ balls $B^i = \mathrm{B}(a^i, r^i)$ with a^0, $a^1, \dots, a^m \in \mathrm{B}(z, \frac{\varpi\varkappa}{2})$ such that*
$$\bigcap_i B^i = \emptyset \quad \textit{and} \quad \bigcap_{i \neq j} B^i \neq \emptyset \quad \textit{for each } j.$$

(c) *There is a $C^{\frac{1}{2}}$-embedding $\Phi\colon\ \overline{\mathrm{B}}[1]_{\mathbb{E}^m} \hookrightarrow \mathcal{U}$; that is, Φ is bi-Hölder with exponent $\frac{1}{2}$.*

(d) *There is a closed separable set $K \subset \mathcal{U}$ such that*
$$\operatorname{TopDim} K \geqslant m.$$

Remarks. Theorem 14.15 gives a stronger version of part (c) in the finite-dimensional case. Namely, a complete length CAT space with linear dimension m admits a bi-Lipschitz embedding Φ of an open set of $\mathbb{R}^m$. Moreover, the Lipschitz constants of Φ can be made arbitrarily close to 1.

14.13. Corollary. *For any separable complete length* CAT *space $\mathcal{U}$, we have*
$$\operatorname{TopDim}\mathcal{U} = \operatorname{LinDim}\mathcal{U}.$$

Any simplicial complex can be equipped with a length metric such that each k-simplex is isometric to the standard simplex
$$\Delta^k = \left\{ (x_0, \dots, x_k) \in \mathbb{R}^{k+1} : x_i \geqslant 0, \quad x_0 + \dots + x_k = 1 \right\}$$

with the metric induced by the ℓ^1-norm on $\mathbb{R}^{k+1}$. This metric will be called the *ℓ^1-metric* on the simplicial complex.

14.14. Lemma. *Let $\mathcal{U}$ be a complete length* CAT(κ) *space and $\rho\colon \mathcal{U}\to\mathbb{R}$ be a continuous positive function. Then there is a simplicial complex $\mathcal{N}$ equipped with ℓ^1-metric, a locally Lipschitz map $\Phi\colon \mathcal{U}\to\mathcal{N}$, and a Lipschitz map $\Psi\colon \mathcal{N}\to\mathcal{U}$ such that:*

(*a*) *The displacement of the composition $\Psi\circ\Phi\colon \mathcal{U}\to\mathcal{U}$ is bounded by ρ; that is,*

$$|x-\Psi\circ\Phi(x)|<\rho(x)$$

for any $x\in\mathcal{U}$.

(*b*) *If* LinDim $\mathcal{U}\leqslant m$ *then the Ψ-image of any closed simplex in $\mathcal{N}$ coincides with the image of its m-skeleton.*

Proof. Without loss of generality, we may assume that for any x we have $\rho(x)<\rho_0$ for fixed $\rho_0<\frac{\varpi\kappa}{2}$.

By Stone's theorem, any metric space is paracompact. Thus, we can choose a locally finite covering $\{\,\Omega_\alpha : \alpha\in\mathcal{A}\,\}$ of $\mathcal{U}$ such that $\Omega_\alpha\subset \mathrm{B}(x,\frac13\cdot\rho(x))$ for any $x\in\Omega_\alpha$.

Denote by $\mathcal{N}$ the nerve of the covering $\{\Omega_\alpha\}$; that is, $\mathcal{N}$ is an abstract simplicial complex with vertex set $\mathcal{A}$, such that $\{\alpha^0,\alpha^1,\dots,\alpha^n\}\subset\mathcal{A}$ are vertexes of a simplex if and only if $\Omega_{\alpha^0}\cap\Omega_{\alpha^1}\cap\dots\cap\Omega_{\alpha^n}\neq\emptyset$.

Fix a Lipschitz partition of unity $\phi_\alpha\colon \mathcal{U}\to[0,1]$ subordinate to $\{\Omega_\alpha\}$. Consider the map $\Phi\colon \mathcal{U}\to\mathcal{N}$ such that the barycentric coordinate of $\Phi(p)$ is $\phi_\alpha(p)$. Note that Φ is locally Lipschitz. Clearly the Φ-preimage of any open simplex in $\mathcal{N}$ lies in Ω_α for some $\alpha\in\mathcal{A}$.

For each $\alpha\in\mathcal{A}$, choose $x_\alpha\in\Omega_\alpha$. Let us extend the map $\alpha\mapsto x_\alpha$ to a map $\Psi\colon\mathcal{N}\to\mathcal{U}$ that is κ-barycentric on each simplex. According to Theorem 14.7*a*, this extension exists, Ψ is Lipschitz, and its Lipschitz constant depends only on ρ_0 and κ.

(*a*). Fix $x\in\mathcal{U}$. Denote by Δ the minimal simplex that contains $\Phi(x)$, and let $\alpha^0,\alpha^1,\dots,\alpha^n$ be the vertexes of Δ. Note that α is a vertex of Δ if and only if $\phi_\alpha(x)>0$. Thus

$$|x-x_{\alpha^i}|<\tfrac13\cdot\rho(x)$$

for any i. Therefore

$$\operatorname{diam}\Psi(\Delta)\leqslant\max_{i,j}\{|x_{\alpha^i}-x_{\alpha^j}|\}<\tfrac23\cdot\rho(x).$$

In particular,

$$|x-\Psi\circ\Phi(x)|\leqslant|x-x_{\alpha^0}|+\operatorname{diam}\Psi(\Delta)<\rho(x).$$

(b). Assume the contrary; that is, $\Psi(\mathcal{N})$ is not included in the Ψ-image of the m-skeleton of $\mathcal{N}$. Then for some $k > m$, there is a k-simplex Δ^k in $\mathcal{N}$ such that the barycentric simplex $\sigma = \Psi|_{\Delta^k}$ is nondegenerate; that is,

$$W = \Psi(\Delta^k) \setminus \Psi(\partial\Delta^k) \neq \emptyset.$$

Applying Corollary 14.10 gives $\operatorname{LinDim} \mathcal{U} \geqslant k$, a contradiction. □

Proof of 14.12.

$(b) \Rightarrow (c) \Rightarrow (d)$. The implication $(b) \Rightarrow (c)$ follows from Lemma 14.11 and Theorem 14.7*c*, and $(c) \Rightarrow (d)$ is trivial.

$(d) \Rightarrow (a)$. According to Theorem 7.7, there is a continuous map $f\colon K \to \mathbb{R}^m$ with a stable value. By the Tietze extension theorem, it is possible to extend f to a continuous map $F\colon \mathcal{U} \to \mathbb{R}^m$.

Fix $\varepsilon > 0$. Since F is continuous, there is a continuous positive function ρ defined on $\mathcal{U}$ such that

$$|x - y| < \rho(x) \quad\Rightarrow\quad |F(x) - F(y)| < \tfrac{1}{3}\cdot\varepsilon.$$

Apply Lemma 14.14 to ρ. For the resulting simplicial complex $\mathcal{N}$ and the maps $\Phi\colon \mathcal{U} \to \mathcal{N}$, $\Psi\colon \mathcal{N} \to \mathcal{U}$, we have

$$|F \circ \Psi \circ \Phi(x) - F(x)| < \tfrac{1}{3}\cdot\varepsilon$$

for any $x \in \mathcal{U}$.

According to Lemma 3.5, there is a locally Lipschitz map $F_\varepsilon\colon \mathcal{U} \to \mathbb{R}^{m+1}$ such that $|F_\varepsilon(x) - F(x)| < \frac{1}{3}\cdot\varepsilon$ for any $x \in \mathcal{U}$.

Note that $\Phi(K)$ is contained in a countable subcomplex of $\mathcal{N}$, say $\mathcal{N}'$. Indeed, since K is separable, there is a countable dense collection of points $\{x_n\}$ in K. Denote by Δ_n the minimal simplex of $\mathcal{N}$ that contains $\Phi(x_n)$. Then $\Phi(K) \subset \bigcup_i \Delta_n$.

Arguing by contradiction, assume $\operatorname{LinDim} \mathcal{U} < m$. By 14.14*b*, the image $F_\varepsilon \circ \Psi \circ \Phi(K)$ lies in the F_ε-image of the $(m-1)$-skeleton of $\mathcal{N}'$; In particular, it can be covered by a countable collection of Lipschitz images of $(m-1)$-simplexes. Hence $\mathbf{0} \in \mathbb{R}^m$ is not a stable value of the restriction $F_\varepsilon \circ \Psi \circ \Phi|_K$. Since $\varepsilon > 0$ is arbitrary, then $\mathbf{0} \in \mathbb{R}^m$ is not a stable value of f—a contradiction.

$(a) \Rightarrow (b)$. Choose $q \in \mathcal{U}$ such that T_q contains a subcone E isometric to m-dimensional Euclidean space. Note that one can choose $\varepsilon > 0$ and a point arrray $(\dot a^0, \dot a^1, \dots, \dot a^m)$ in $E \subset \mathrm{T}_q$ such that $\bigcap_i \overline{\mathrm{B}}[\dot a^i, 1+\varepsilon] = \emptyset$ and $\bigcap_{i\neq j} \overline{\mathrm{B}}[\dot a^i, 1-\varepsilon] \neq \emptyset$ for each j.

For each i choose a geodesic γ^i from q that goes almost in the directions of $\dot a^i$. Choose small $\delta > 0$ and take the point a^i on γ^i at distance $\delta\cdot|\dot a^i|$ from q. We get a point array $(a^0, a^1, \dots, a^m)$ in $\mathcal{U}$ such that $\bigcap_i \overline{\mathrm{B}}[a^i, \delta] = \emptyset$ and

$\bigcap_{i\neq j}\overline{\mathrm{B}}[a^i,\delta]\neq\emptyset$ for each j. Since $\delta>0$ can be chosen arbitrarily small, (b) follows. □

D. Finite-dimensional spaces

Recall that a web embedding and its brace are defined in 14.8.

14.15. Theorem. *Suppose $\mathcal{U}$ is a complete length $\mathrm{CAT}(\kappa)$ space such that $\mathrm{LinDim}\,\mathcal{U}=m$, and $\mathbf{a}=(a^0,a^1,\dots,a^m)$ is a point array in $\mathcal{U}$ with radius $<\frac{\varpi\kappa}{2}$. Then the $\mathrm{dist}_{\mathbf{a}}$-web embedding $\Phi\colon\mathbb{R}^m\multimap\mathcal{U}$ with brace dist_{a^0} is locally Lipschitz.*

Note that if $\mathbf{a}$ is degenerate, that is, if $\mathrm{InWeb}(\mathrm{dist}_{\mathbf{a}})=\emptyset$, then the domain of Φ is empty, and hence the conclusion of the theorem trivially holds.

14.16. Lemma. *Let $\mathcal{U}$ be a complete length $\mathrm{CAT}(\kappa)$ space, and $\mathbf{a}=(a^0,a^1,\dots,a^k)$ be a point array with radius $<\frac{\varpi\kappa}{2}$. Then for any $p\in\mathrm{InWeb}(\mathrm{dist}_{\mathbf{a}})$, there is $\varepsilon>0$ such that if for some $q\in\mathrm{Web}(\mathrm{dist}_{\mathbf{a}})$ and $b\in\mathcal{U}$ we have*

$$(1)\qquad |p-q|<\varepsilon,\quad |p-b|<\varepsilon,\quad \textit{and}\quad \measuredangle\left[q\,{}^{b}_{a^i}\right]<\tfrac{\pi}{2}+\varepsilon$$

for each i, then the array $(b,a^0,a^1,\dots,a^m)$ is nondegenerate.

Proof. Without loss of generality, we may assume that $\mathcal{U}$ is geodesic and $\mathrm{diam}\,\mathcal{U}<\varpi\kappa$. If not, consider instead of $\mathcal{U}$, a ball $\overline{\mathrm{B}}[z,r]\subset\mathcal{U}$ for some $z\in\mathcal{U}$ and $r<\frac{\varpi\kappa}{2}$ such that $|z-a^i|\leqslant r$ for each i.

From the angle comparison (9.14c), it follows that $p\in\mathrm{InWeb}\,\mathbf{a}$ if and only if both of the following conditions hold:

1. $\max_i\{\measuredangle\left[p\,{}^{a^i}_{u}\right]\}\geqslant\frac{\pi}{2}$ for any $u\in\mathcal{U}$,
2. for each i there is $u^i\in\mathcal{U}$ such that $\measuredangle\left[p\,{}^{a^j}_{u^i}\right]<\frac{\pi}{2}$ for all $j\neq i$.

Due to the semicontinuity of angles (9.34), there is $\varepsilon>0$ such that for any $x\in\mathrm{B}(p,10\cdot\varepsilon)$ we have

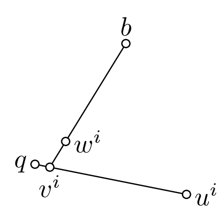
$$(2)\qquad \measuredangle\left[x\,{}^{a^j}_{u^i}\right]<\tfrac{\pi}{2}-10\cdot\varepsilon\quad\text{for all}\quad j\neq i.$$

Now assume that for sufficiently small $\varepsilon>0$ there are points $b\in\mathcal{U}$ and $q\in\mathrm{Web}(\mathrm{dist}_{\mathbf{a}})$ such that 1 holds. According to Theorem 14.7b, for all small $\varepsilon>0$ we have

$$\mathrm{rad}\{b,a^0,a^1,\dots,a^k\}<\tfrac{\varpi\kappa}{2}.$$

Fix a sufficiently small $\delta>0$ and let

$$v^i=\mathrm{geod}_{[qu^i]}(\tfrac13\cdot\delta)\quad\text{and}\quad w^i=\mathrm{geod}_{[v^ib]}(\tfrac23\cdot\delta).$$



Clearly

$$\begin{aligned}|b-w^i| &= |b-v^i| - \tfrac{2}{3}\cdot\delta\\ &\leqslant |b-q| - \tfrac{1}{3}\cdot\delta.\end{aligned}$$

Further, ~~the~~ inequalities 2 and 1 imply

$$\begin{aligned}|a^j-w^i| &< |a^j-v^i| + \tfrac{2}{3}\cdot\varepsilon\cdot\delta\\ &< |a^i-q| - \varepsilon\cdot\delta\\ &< |a^i-q|\end{aligned}$$

for all $i\neq j$.

Set $B^i = \overline{\mathrm{B}}[a^i, |a^i-q|]$ and $B^{m+1} = \overline{\mathrm{B}}[b, |a^i-q| - \tfrac{1}{3}\cdot\delta]$. Clearly

$$\bigcap_{i\neq m+1} B^i = \{q\},$$

$$\bigcap_{i\neq j} B^i \ni w^j \quad \text{for } j\neq m+1, \text{ and}$$

$$\bigcap_{i} B^i = \{q\}\cap B^{m+1} = \emptyset.$$

Lemma 14.11 finishes the proof. □

Proof of 14.15. Suppose Φ is not locally Lipshitz.; that is, there are sequences $\mathbf{y}_n, \mathbf{z}_n \to \mathbf{x}\in \operatorname{Dom}\Phi$ such that

$$\frac{|\Phi(\mathbf{y}_n)-\Phi(\mathbf{z}_n|)}{|\mathbf{y}_n-\mathbf{z}_n|}\to\infty \quad\text{as}\quad n\to\infty. \tag{3}$$

Set $p=\Phi(\mathbf{x})$, $q_n=\Phi(\mathbf{y}_n)$, and $b_n=\Phi(\mathbf{z}_n)$. By 14.8, $p, q_n, b_n\in \operatorname{InWeb}(\operatorname{dist}_{\mathbf{a}})$ and $q_n, b_n\to p$ as $n\to\infty$. Choose $\varepsilon>0$; note that 3 implies

$$\measuredangle\left[q_n\,{}^{a^i}_{b_n}\right] < \tfrac{\pi}{2}+\varepsilon$$

for all $i>0$ and all large n. Further, according to 14.8, the subfunction $(\operatorname{dist}_{\mathbf{a}^0})\circ\Phi$ is locally Lipschitz. Therefore we also have

$$\measuredangle\left[q_n\,{}^{a^0}_{b_n}\right] < \tfrac{\pi}{2}+\varepsilon$$

for all large n. According to Lemma 14.16, the point array $b_n, a^0, \dots, a^k$ for large n is nondegenerate.

Applying Corollary 14.10, we have a contradiction. □

E. Remarks

The following conjecture was formulated by Bruce Kleiner [**95**], see also [**74**, p. 133]. For separable spaces, it follows from Corollary 14.13.

14.17. Conjecture. *For any complete length* CAT *space* $\mathcal{U}$*, we have*

$$\operatorname{TopDim}\mathcal{U} = \operatorname{LinDim}\mathcal{U}.$$

Chapter 15

Dimension of CBB spaces

As the main dimension-like invariant, we will use the linear dimension LinDim; see Definition 7.9. In other words, by default *dimension* means *linear dimension.*

A. Struts and rank

Our definitions of strut and distance chart differ from the one in [**44**]; it is closer to Perelman's definitions [**124, 125**].

The term "strut" seems to have the closest meaning to the original Russian term used by Yuriy Burago, Grigory Perelman, and Michael Gromov [**44**]. In the official translation, it appears as "burst", and in the authors' translation it was "strainer". Neither seems intuitive, so we decided to switch to "strut".

15.1. Definition of struts. *Let $\mathcal{L}$ be a complete length* CBB *space. We say that a point array $(a^0, a^1, \dots, a^k)$ in $\mathcal{L}$ is κ-*strutting *for a point $p \in \mathcal{L}$ if $\tilde\measuredangle^\kappa\left(p\,{}^{a^i}_{a^j}\right) > \frac{\pi}{2}$ for all $i \neq j$.*

Recall that the packing number is defined in 2B. The following definition is motivated by the observation that $k = \operatorname{pack}_{\pi/2}(\mathbb{S}^{k-1}) - 1$ for any integer $k > 0$.

15.2. Definition. *Let $\mathcal{L}$ be a complete length* CBB *space and $p \in \mathcal{L}$. Let us define rank of $\mathcal{L}$ at p as*

$$\operatorname{rank}_p = \operatorname{rank}_p \mathcal{L} := \operatorname{pack}_{\pi/2} \Sigma_p - 1.$$

Thus rank takes values in $\mathbb{Z}_{\geqslant 0} \cup \{\infty\}$.

15.3. Proposition. *Let $\mathcal{L}$ be a complete length* CBB(κ) *space and $p \in \mathcal{L}$. Then the following conditions are equivalent:*

(*a*) $\operatorname{rank}_p \geqslant k$,

(*b*) *there is a point array $(a^0, a^1, \dots, a^k)$ that is κ-strutting at p.*

Proof of 15.3.
(*b*) ⇒ (*a*). For each i, choose a point $\acute{a}^i \in \operatorname{Str}(p)$ sufficiently close to a^i (so $[p\acute{a}^i]$ exists for each i). One can choose $\acute{a}^i$ so that we still have $\tilde{\measuredangle}^\kappa\left(p\,{}^{\acute{a}^i}_{\acute{a}^j}\right) > \frac{\pi}{2}$ for all $i \neq j$.

From hinge comparison (8.14*c*),

$$\measuredangle(\uparrow_{[p\acute{a}^i]}, \uparrow_{[p\acute{a}^j]}) \geqslant \tilde{\measuredangle}^\kappa\left(p\,{}^{\acute{a}^i}_{\acute{a}^j}\right) > \tfrac{\pi}{2}$$

for all $i \neq j$. In particular, $\operatorname{pack}_{\pi/2} \Sigma_p \geqslant k + 1$.

(*a*) ⇒ (*b*). Assume $(\xi^0, \xi^1, \dots, \xi^k)$ is an array of directions in Σ_p, such that $\measuredangle(\xi^i, \xi^j) > \frac{\pi}{2}$ if $i \neq j$.

Without loss of generality, we may assume that each direction ξ^i is geodesic; that is, for each i there is a geodesic γ^i in $\mathcal{L}$ such that $\gamma^i(0) = p$ and $\xi^i = (\gamma^i)^+(0)$. From the definition of angle, it follows that for sufficiently small $\varepsilon > 0$ the array of points $a^i = \gamma^i(\varepsilon)$ satisfies (*b*). □

15.4. Corollary. *Let $\mathcal{L}$ be a complete length* CBB *space and $k \in \mathbb{Z}_{\geqslant 0}$. Then the set of all points in $\mathcal{L}$ with rank $\geqslant k$ is open.*

Proof. Given an array of points $\mathbf{a} = (a^0, \dots, a^k)$ in $\mathcal{L}$, consider the set $\Omega_{\mathbf{a}}$ of all points $p \in \mathcal{L}$ such that array $\mathbf{a}$ is κ-strutting for a point p. Clearly $\Omega_{\mathbf{a}}$ is open.

According to Proposition 15.3, the set of points in $\mathcal{L}$ with rank $\geqslant k$ can be presented as

$$\bigcup_{\mathbf{a}} \Omega_{\mathbf{a}},$$

where the union is taken over all k-arrays $\mathbf{a}$ of points in $\mathcal{L}$. Hence the result. □

B. Right-inverse theorem

Suppose that $\mathbf{a} = (a^1, \dots, a^k)$ is a point array in a metric space $\mathcal{L}$. Recall that the map $\operatorname{dist}_{\mathbf{a}} : \mathcal{L} \to \mathbb{R}^n$ is defined by

$$\operatorname{dist}_{\mathbf{a}} p \coloneqq (|a^1 - p|, \dots, |a^n - p|).$$

15.5. Right-inverse theorem. *Suppose $\mathcal{L}$ is a complete length* CBB(κ) *space, $p, b \in \mathcal{L}$, and $\mathbf{a} = (a^1, \dots, a^k)$ is a point array in $\mathcal{L}$.*

Assume that $(b, a^1, a^2, \dots, a^k)$ is κ-strutting for p. Then the distance map $\operatorname{dist}_{\mathbf{a}} : \mathcal{L} \to \mathbb{R}^k$ *has a right inverse defined in a neighborhood of* $\operatorname{dist}_{\mathbf{a}} p \in \mathbb{R}^k$;

that is, there is a submap $\Phi\colon \mathbb{R}^k \multimap \mathcal{L}$ *such that* $\operatorname{Dom}\Phi \ni \operatorname{dist}_{\mathbf{a}} p$ *and* $\operatorname{dist}_{\mathbf{a}}[\Phi(\mathbf{x})] = \mathbf{x}$ *for any* $\mathbf{x} \in \operatorname{Dom}\Phi$. *Moreover,*

(*a*) *The map* Φ *can be chosen to be* $C^{\frac{1}{2}}$*-continuous (that is, Hölder continuous with exponent* $\frac{1}{2}$*) and such that*

$$\Phi(\operatorname{dist}_{\mathbf{a}} p) = p.$$

(*b*) *The distance map* $\operatorname{dist}_{\mathbf{a}}\colon \mathcal{L} \to \mathbb{R}^k$ *is locally co-Lipschitz (in particular, open) in a neighborhood of p.*

Part (*b*) of the theorem is closely related to [**44**, Theorem 5.4] by Yuriy Burago, Grigory Perelman, and Michael Gromov, but the proof presented here is different. Yet another proof can be built on [**110**, Proposition 4.3] by Alexander Lytchak.

Proof. Fix $\varepsilon, r, \lambda > 0$ such that the following conditions hold:

(i) Each distance function $\operatorname{dist}_{a^i}$ and dist_b is $\frac{\lambda}{2}$-concave in $\mathrm{B}(p,r)$.

(ii) For any $q \in \mathrm{B}(p,r)$, we have $\tilde\measuredangle^\kappa\left(q\,{}^{a^i}_{a^j}\right) > \frac{\pi}{2} + \varepsilon$ for all $i \neq j$ and $\tilde\measuredangle^\kappa\left(q\,{}^{b}_{a^i}\right) > \frac{\pi}{2} + \varepsilon$ for all i. In addition, $\varepsilon < \frac{1}{10}$.

Given $\mathbf{x} = (x^1, x^2, \dots, x^k) \in \mathbb{R}^k$, consider the function $f_{\mathbf{x}}\colon \mathcal{L} \to \mathbb{R}$ defined by

$$f_{\mathbf{x}} = \min_i\{h^i_{\mathbf{x}}\} + \varepsilon \cdot \operatorname{dist}_b,$$

where $h^i_{\mathbf{x}}(q) = \min\{0, |a^i - q| - x^i\}$. Note that for any $\mathbf{x} \in \mathbb{R}^k$, the function $f_{\mathbf{x}}$ is $(1+\varepsilon)$-Lipschitz and λ-concave in $\mathrm{B}(p,r)$. Denote by $\alpha_{\mathbf{x}}(t)$ the $f_{\mathbf{x}}$-gradient curve (see Chapter 16) that starts at p.

(1) *If for some* $\mathbf{x} \in \mathbb{R}^k$ *and* $t_0 \leqslant \frac{r}{2}$ *we have* $|\operatorname{dist}_{\mathbf{a}} p - \mathbf{x}| \leqslant \frac{\varepsilon^2}{10} \cdot t_0$, *then* $\operatorname{dist}_{\mathbf{a}}[\alpha_{\mathbf{x}}(t_0)] = \mathbf{x}$.

First note that 1 follows if for any $q \in \mathrm{B}(p,r)$, we have

(i) $(\mathbf{d}_q \operatorname{dist}_{a^i})(\nabla_q f_{\mathbf{x}}) < -\frac{1}{10} \cdot \varepsilon^2$ if $|a^i - q| > x^i$ and

(ii) $(\mathbf{d}_q \operatorname{dist}_{a^i})(\nabla_q f_{\mathbf{x}}) > \frac{1}{10} \cdot \varepsilon^2$ if

$$|a^i - q| - x^i = \min_j\left\{|a^j - q| - x^j\right\} < 0.$$

Indeed, since $t_0 \leqslant \frac{r}{2}$, then $\alpha_{\mathbf{x}}(t) \in \mathrm{B}(p,r)$ for all $t \in [0, t_0]$. Consider the following real-to-real functions:

$$\phi(t) \coloneqq \max_i\left\{|a^i - \alpha_{\mathbf{x}}(t)| - x^i\right\}, \qquad \psi(t) \coloneqq \min_i\left\{|a^i - \alpha_{\mathbf{x}}(t)| - x^i\right\}. \tag{2}$$

Then from (i), we have $\phi^+ < -\frac{1}{10}\cdot\varepsilon^2$ if $\phi > 0$ and $t \in [0, t_0]$. Similarly, from (ii), we have $\psi^+ > \frac{1}{10}\cdot\varepsilon^2$ if $\psi < 0$ and $t \in [0, t_0]$. Since $|\operatorname{dist}_{\mathbf{a}} p - \mathbf{x}| \leqslant \frac{\varepsilon^2}{10}\cdot t_0$, it follows that $\phi(0) \leqslant \frac{\varepsilon^2}{10}\cdot t_0$ and $\psi(0) \geqslant -\frac{\varepsilon^2}{10}\cdot t_0$. Thus $\phi(t_0) \leqslant 0$ and $\psi(t_0) \geqslant 0$. On the other hand, from 2 we have $\phi(t_0) \geqslant \psi(t_0)$. That is, $\phi(t_0) = \psi(t_0) = 0$; hence 1 follows.

Thus, to prove 1, it remains to prove (i) and (ii). First let us prove it assuming that $\mathcal{L}$ is geodesic.

Note that

$$(\mathbf{d}_q \operatorname{dist}_b)(\uparrow_{[qa^i]}) \leqslant \cos\tilde{\measuredangle}^\kappa\left(q\,{}^{b}_{a^j}\right) < -\tfrac{\varepsilon}{2} \tag{3}$$

for all i, and

$$(\mathbf{d}_q \operatorname{dist}_{a^j})(\uparrow_{[qa^i]}) \leqslant \cos\tilde{\measuredangle}^\kappa\left(q\,{}^{a^i}_{a^j}\right) < -\tfrac{\varepsilon}{2} \tag{4}$$

for all $j \neq i$. Further, 4 implies

$$(\mathbf{d}_q h^j_{\mathbf{x}})(\uparrow_{[qa^i]}) \leqslant 0 \tag{5}$$

for all $i \neq j$. The assumption in (i) implies

$$\mathbf{d}_q f_{\mathbf{x}} = \min_{j\neq i}\{\mathbf{d}_q h^j_{\mathbf{x}}\} + \varepsilon\cdot(\mathbf{d}_q \operatorname{dist}_b).$$

Thus

$$\begin{aligned}
-(\mathbf{d}_q \operatorname{dist}_{a^i})(\nabla_q f_{\mathbf{x}}) &\geqslant \langle\uparrow_{[qa^i]}, \nabla_q f_{\mathbf{x}}\rangle \\
&\geqslant (\mathbf{d}_q f_{\mathbf{x}})(\uparrow_{[qa^i]}) \\
&= \min_{i\neq j}\{(\mathbf{d}_q h^i_{\mathbf{x}})(\uparrow_{[qa^i]})\} + \varepsilon\cdot(\mathbf{d}_q \operatorname{dist}_b)(\uparrow_{[qa^i]}).
\end{aligned}$$

Therefore (i) follows from 3 and 5.

The assumption in (ii) implies that $f_{\mathbf{x}}(q) = h^i_{\mathbf{x}}(q) + \varepsilon\cdot\operatorname{dist}_b$ and

$$\mathbf{d}_q f_{\mathbf{x}} \leqslant \mathbf{d}_q \operatorname{dist}_{a^i} + \varepsilon\cdot(\mathbf{d}_p \operatorname{dist}_b).$$

Therefore,

$$\begin{aligned}
(\mathbf{d}_q \operatorname{dist}_{a^i})(\nabla_q f_{\mathbf{x}}) &\geqslant \mathbf{d}_q f_{\mathbf{x}}(\nabla_q f_{\mathbf{x}}) \\
&\geqslant \left[(\mathbf{d}_q f_{\mathbf{x}})(\uparrow_{[qb]})\right]^2 \\
&\geqslant \left[\min_i\{\cos\tilde{\measuredangle}^\kappa\left(q\,{}^{b}_{a^i}\right)\} - \varepsilon^2\right]^2.
\end{aligned}$$

Thus (ii) follows from 3, since $\varepsilon < \frac{1}{10}$.

Therefore 1 holds if $\mathcal{L}$ is geodesic. If $\mathcal{L}$ is not geodesic, perform the above estimate in $\mathcal{L}^\omega$, the ultrapower of $\mathcal{L}$. (Recall that according to 4.9, $\mathcal{L}^\omega$ is geodesic.) This completes the proof of 1. $\triangle$

Set $t_0(\mathbf{x}) = \frac{10}{\varepsilon^2} \cdot |\operatorname{dist}_{\mathbf{a}} p - \mathbf{x}|$, giving equality in 1. Define the submap Φ by

$$\Phi : \mathbf{x} \mapsto \alpha_{\mathbf{x}} \circ t_0(\mathbf{x}), \quad \operatorname{Dom}\Phi = \mathrm{B}(\operatorname{dist}_{\mathbf{a}} p, \tfrac{\varepsilon^2 \cdot r}{20}) \subset \mathbb{R}^k.$$

It follows from 1 that $\operatorname{dist}_{\mathbf{a}} [\Phi(\mathbf{x})] = \mathbf{x}$ for any $\mathbf{x} \in \operatorname{Dom}\Phi$.

Clearly $t_0(p) = 0$; thus $\Phi(\operatorname{dist}_{\mathbf{a}} p) = p$. Further, by construction of $f_{\mathbf{x}}$,

$$|f_{\mathbf{x}}(q) - f_{\mathbf{y}}(q)| \leqslant |\mathbf{x} - \mathbf{y}|,$$

for any $q \in \mathcal{L}$. Therefore, according to Lemma 16.13, Φ is $C^{\frac{1}{2}}$-continuous. Thus (a).

Further, note that

$$|p - \Phi(\mathbf{x})| \leqslant (1 + \varepsilon) \cdot t_0(\mathbf{x}) \leqslant \tfrac{11}{\varepsilon^2} \cdot |\operatorname{dist}_{\mathbf{a}} p - \mathbf{x}| \tag{6}$$

holds.

The above construction may be repeated for any $p' \in \mathrm{B}(p, \frac{r}{4})$, $\varepsilon' = \varepsilon$, and $r' = \frac{r}{2}$. ~~The~~ inequality 6 for the resulting map Φ' implies that for any $p', q \in \mathrm{B}(p, \frac{r}{4})$ there is $q' \in \mathcal{L}$ such that $\Phi'(q) = \Phi'(q')$ and

$$|p' - q'| \leqslant \tfrac{11}{\varepsilon^2} \cdot |\operatorname{dist}_{\mathbf{a}} p' - \mathbf{x}|.$$

That is, the distance map $\operatorname{dist}_{\mathbf{a}}$ is locally $\frac{11}{\varepsilon^2}$-co-Lipschitz in $\mathrm{B}(p, \frac{r}{4})$. □

C. Dimension theorem

The following theorem is the main result of this section.

15.6. Theorem. *Let $\mathcal{L}$ be a complete length* $\mathrm{CBB}(\kappa)$ *space, $q \in \mathcal{L}$, $R > 0$ and $m \in \mathbb{Z}_{\geqslant 0}$. Then the following statements are equivalent:*

(*A*) $\operatorname{LinDim}\mathcal{L} \geqslant m$.

(*B*) *There is a point $p \in \mathcal{L}$ that admits a κ-strutting array $(b, a^1, \dots, a^m) \in \mathcal{L}^{m+1}$.*

(*C*) *Let* Euk^m *be the set of all points $p \in \mathcal{L}$ such that there is a distance-preserving embedding $\mathbb{E}^m \hookrightarrow \mathrm{T}_p$ that preserves the cone structure (see Section 6E). Then* Euk^m *contains a dense G-delta set in $\mathcal{L}$.*

(*D*) *There is a $C^{\frac{1}{2}}$-embedding*

$$\overline{\mathrm{B}}[1]_{\mathbb{E}^m} \hookrightarrow \mathrm{B}(q, R);$$

that is, a bi-Hölder embedding with exponent $\frac{1}{2}$.

(*E*)

$$\operatorname{pack}_\varepsilon \mathrm{B}(q, R) > \frac{c}{\varepsilon^m}$$

for fixed $c > 0$ and any $\varepsilon > 0$.

In particular:

(i) *If* $\operatorname{LinDim}\mathcal{L} = \infty$*, then all the statements* (C)*,* (D)*, and* (E) *are satisfied for all* $m \in \mathbb{Z}_{\geqslant 0}$.

(ii) *If the statement* (D) *or* (E) *is satisfied for some choice of* $q \in \mathcal{L}$ *and* $R > 0$*, then it also is satisfied for any other choice of* q *and* R.

For finite-dimensional spaces, Theorem 15.13 gives a stronger version of the theorem above.

The above theorem with the exception of statement (D) was proved by Conrad Plaut [**136**]. At that time, it was not known whether

$$\operatorname{LinDim}\mathcal{L} = \infty \quad\Rightarrow\quad \operatorname{TopDim}\mathcal{L} = \infty$$

for any complete length CBB(κ) space $\mathcal{L}$. The latter implication was proved by Grigory Perelman and the third author [**123**]; it was done by combining an idea of Conrad Plaut with the technique of gradient flow. Part (D) is somewhat stronger.

To prove Theorem 15.6 we will need the following three propositions.

15.7. Proposition. *Let* p *be a point in a a complete length* CBB(κ) *space* $\mathcal{L}$*. Assume there is a distance-preserving embedding* $\iota\colon \mathbb{E}^m \hookrightarrow \mathrm{T}_p\mathcal{L}$ *that preserves the cone structure. Then either*

(*a*) $\Im\iota = \mathrm{T}_p\mathcal{L}$*, or*

(*b*) *there is a point* p' *arbitrarily close to* p *such that there is a distance-preserving embedding* $\iota'\colon \mathbb{E}^{m+1} \hookrightarrow \mathrm{T}_{p'}\mathcal{L}$ *that preserves the cone structure.*

Proof. Assume $\iota(\mathbb{E}^m)$ is a proper subset of $\mathrm{T}_p\mathcal{L}$. Equivalently, there is a direction $\xi \in \Sigma_p \setminus \iota(\mathbb{S}^{m-1})$, where $\mathbb{S}^{m-1} \subset \mathbb{E}^m$ is the unit sphere.

Fix $\varepsilon > 0$ so that $\measuredangle(\xi,\sigma) > \varepsilon$ for any $\sigma \in \iota(\mathbb{S}^{m-1})$. Choose a maximal ε-packing in $\iota(\mathbb{S}^{m-1})$; that is, an array $(\zeta^1,\zeta^2,\dots,\zeta^n)$ of directions in $\iota(\mathbb{S}^{m-1})$ such that $n = \operatorname{pack}_\varepsilon \mathbb{S}^{m-1}$ and $\measuredangle(\zeta^i,\zeta^j) > \varepsilon$ for any $i \neq j$.

Choose an array $(x, z^1, z^2, \dots, z^n)$ of points in $\mathcal{L}$ such that $\uparrow_{[px]}\approx \xi$, $\uparrow_{[pz^i]}\approx \zeta^i$; here we write "$\approx$" for "sufficiently close". We can choose this array so that $\tilde\measuredangle^\kappa\left(p\,{}^{x}_{z^i}\right) > \varepsilon$ for all i and $\tilde\measuredangle^\kappa\left(p\,{}^{z^i}_{z^j}\right) > \varepsilon$ for all $i \neq j$. Applying Corollary 13.40, there is a point p' arbitrarily close to p such that all directions $\uparrow_{[p'x]}$, $\uparrow_{[p'z^1]}$, $\uparrow_{[p'z^2]},\dots,\uparrow_{[p'z^n]}$ belong to an isometric copy of $\mathbb{S}^{k-1}$ in $\Sigma_{p'}$. In addition, we may assume that $\tilde\measuredangle^\kappa\left(p'\,{}^{x}_{z^i}\right) > \varepsilon$ and $\tilde\measuredangle^\kappa\left(p'\,{}^{z^i}_{z^j}\right) > \varepsilon$. From the hinge comparison (8.14*c*), $\measuredangle(\uparrow_{[p'x]},\uparrow_{[p'z^i]}) > \varepsilon$ and $\measuredangle(\uparrow_{[p'z^i]},\uparrow_{[p'z^j]}) > \varepsilon$; that is,

$$\operatorname{pack}_\varepsilon \mathbb{S}^{k-1} \geqslant n+1 > \operatorname{pack}_\varepsilon \mathbb{S}^{m-1}.$$

Hence $k > m$. $\square$