Tree comparison

(preliminary version)

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Abstract

We introduce a new type of metric comparison which is closely related to the Alexandrov comparison and continuity of optimal transport between regular measures.

1 Introduction

We will denote by $|a-b|_X$ the distance between points a and b in the metric space X.

Tree comparison. Fix a tree T with n vertexes.

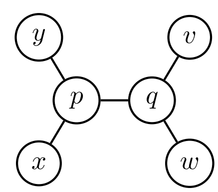
Let $(a_1,\dots a_n)$ be a point array in a metric space X labeled by the vertexes of T. We say that $(a_1,\dots a_n)$ satisfies the *T-tree comparison* if there is a point array $(\tilde a_1,\dots,\tilde a_n)$ in the Hilbert space $\mathbb{H}$ such that

$$|\tilde a_i-\tilde a_j|_{\mathbb{H}}\geqslant|a_i-a_j|_X$$

for any i and j and the equality holds if a_i and a_j are adjacent in T.

We say that a metric space X satisfies the *T-tree comparison* if every n-points array in X satisfies the T-tree comparison.

Instead of the Hilbert space $\mathbb{H}$ we may use infinite dimensional sphere or infinite dimensional hyperbolic space. In this case we will it defines *spherical* and *hyperbolic* tree comparisons.



Encoding of trees. To encode the labeled tree on the diagram, we will use notation $p/xy(q/vw)$. It means that we choose p as the root; p has two children leafs to x, y and one child q with two children leafs v and w. Taking another root for the same tree, we get different encodings, for eaxample $q/vw(p/xy)$ or $x/(p/y(q/vw))$.

If we do not need the labeling of vertexes, it is sufficient to write the number of leafs in the brackets; this way we can write $2(2)$ instead of $p/xy(q/vw)$ since the root (p) has 2 leafs (x and y) and yet another child (q) which has 2 leafs (v and w). The same tree can be encoded as $(1(2))$ meaning that the root x has

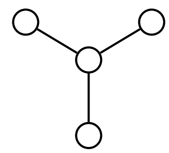
... was partially supported ...

no leafs, p has 1 leaf y and one child q with 2 leafs v and w. Every vertex which is not the root and not a leaf corresponds to a pair of brackets in this notation.

Using the described notation, we could say that a metric space *satisfies the 2(2)-tree comparison*, meaning that it satisfies the tree comparison on the diagram. We could also say *"applying the tree comparison for $p/xy(q/vw)$ ..."* meaning that we apply the comparison for these 6 points labeled as on the diagram.

Monopolar trees. A vertex of a tree of degree at least two will be called *pole*.

Recall that *Alexandrov space* with nonnegative curvature is defined as complete length space with curvature bounded below in the sense of Alexandrov; the latter is equivalent to the 3-tree comparison; that is, the comparison for the tripod-tree on the diagram.



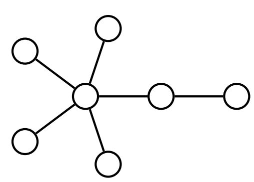
Using the introduced notation, a theorem in [1] can be restated the following way: *If a complete length-metric space satisfies* 3*-tree comparison, then it also satisfies n-tree comparison for every positive integer n; in other words it satisfies all monopolar tree comparisons.*

Bipolar trees. Consider the bipolar trees 3(1) and 2(2) shown on the diagram.

The following theorem is proved in Section 6; it states that the corresponding comparisons also follow form Alexandrov's comparison.

1.1. Theorem. *Any Alexandrov space with nonnegative curvature satisfies 3(1)-tree and 2(2)-tree comparisons.*

The 4(1)-tree comparison turns out to be related to the so called *transport continuity property*, briefly TCP. A compact Riemannian manifold M is TCP if for any two regular measures with density functions bounded away from zero and infinity the generalized solution of Monge–Ampère equation provided by optimal transport is a genuine solution.



A necessary condition for TCP was given by Xi-Nan Ma, Neil Trudinger and Xu-Jia Wang in [8]. A key step in the understanding this condition was made by Grégoire Loeper in [7]. The manifolds satisfying this condition will be called *cost-convex*; using terminology of [10], cost-convexity is CTIL+MTW. Cost-convexity is defined and discussed in Section 3.

1.2. Theorem. *If a complete Riemannian manifold satisfies 4(1)-tree comparison, then it is cost-convex; moreover it satisfies all bipolar tree comparisons.*

The two parts of the theorem are proved in sections 7 and 8. The proof of the second part works for all complete length spaces (see Corollary 8.2). In this poof, we describe 4(1)-tree comparison using convexity of certain function on the tangent space (see Proposition 8.1); we also give an analogous characterization of cost-convex manifolds (see Proposition 8.3).

It is straightforward to check that the spherical 4(1)-tree comparison implies the strict cost-convexity, which in turns implies TCP; see [3].

The following question remains open.

1.3. Question. *Is it true that Rieamnnian manifold is cost-convex if and only if it satisfies 4(1)-tree comparison.*

From the theorem above we get that for complete Riemannian manifold satisfying 4(1)-tree comparison also satisfies 3(2)-tree and 3(3)-tree comparisons. In both cases, the converse is not known.

All tree comparisons. Finally we consider spaces satisfying *all tree comparisons.*

Recall that a map $f\colon W \to X$ between metric spaces is called *submetry* if for any $w \in W$ and $r \geqslant 0$, we have

$$f[B(w,r)_W] = B(f(w),r)_X,$$

where $B(w,r)_W$ denotes the ball with center w and radius r in the space W. In other words submetry is a map which is 1-Lipschitz and 1-co-Lipschitz at the same time. Note that by the definition, any submetry is onto.

1.4. Theorem. *A separable metric space X satisfies all tree comparisons if and only if X is isometric to a target space of submetry defined on a subset of the Hilbert space.*

The following proposition provides a source of examples of spaces satisfying all tree comparisons. For example, since $\mathbb{S}^n = \mathrm{SO}(n)/\mathrm{SO}(n-1)$, any round sphere has this property.

1.5. Proposition. *Suppose G is a compact Lie group with bi-invariant metric, so the action $G \times G \curvearrowright G$ defined by $(h_1, h_2)\cdot g = h_1 \cdot g \cdot h_2^{-1}$ is isometric. Then for any closed subgroup $H < G \times G$, the bi-quotient space $G/\!\!/H$ satisfies all tree comparisons*

From proposition above and Theorem 1.2, it follows that the bi-quotient space $G/\!\!/H$ is cost-convex (or equivalently CTIL+MTW), see Section 3. This makes it closely related to the result of Young-Heon Kim and Robert McCann in [5].

2 Digressions

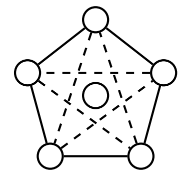
On matrix inequality. The comparison for monopolar trees has an algebraic corollary which was used Urs Lang and Viktor Schroeder in [6], see also [9].

Namely, given a point array $p, x_1, \ldots, x_n$ in a metric space X consider the matrix M with the components

$$m_{i,j} = \tfrac{1}{2}\cdot(|x_i - p|^2 + |x_j - p|^2 - |x_i - x_j|^2).$$

If the tree comparison for $p/x_1, \ldots, x_n$ holds, then

$$\boldsymbol{s}\cdot M\cdot\boldsymbol{s}^\top \geqslant 0$$



for any vector $\boldsymbol{s} = (s_1, \ldots, s_n)$ with nonnegative components.

However, this condition is not sufficient for the tree comparison for $p/x_1x_2x_3x_4x_5$. (In general, is not easy to describe tree comparisons using a system of inequalities.)

An example can be constructed by perturbing the configuration on the plane as on the diagram — if the diameter of diagram is 1, then increasing the distances between the pairs of points connected by dashed lines by $\varepsilon = 10^{-9}$ and decreasing the distances between the pairs of points connected by sold lines by $\delta = 10^{-6}$ does the job. The obtained metric 6-point metric space satisfies the matrix inequality with center at each point, but does not satisfy the tree comparison with pole at the central point.

On graph comparison. Analogously to the tree comparison one can define graph comparison for any graph by stating that there is model configuration such that the distance between pairs of adjacent points is at most as big and nonadjacent is at least as big.

2.1. Exercise. *Show that if a graph is a tree, then the graph comparison defined above is equivalent to the tree comparison defined at the beginning of the paper.*

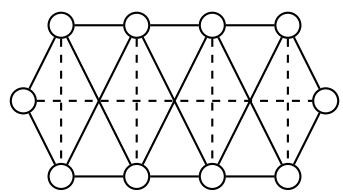
Note that nonnegative and nonpositive curvature can be defined using the comparison for following two graphs on 4 vertexes:

By Reshetnyak majorization theorem, the nonpositive curvature could be also defined using the comparison for cycle; for example the 6-cycle, the first graph the following diagram.

The comparison for the octahedron graph (the second graph on the diagram) implies that the space is nonnegatively curved. The latter follows since in this graph, a 4-cycle appears as an induced subgraph. On the other hand, this comparison might be stronger and it might be interesting to understand.

(The tree comparison holds for factors spaces of Hilbert space — this was the main motivation to consider this type of comparison. Unfortunately we do not have analogous examples of nonnegatively curved spaces which would guide us to the right definition.)

On colored graph comparison. It is also possible to use graph with $(\mp)$-colored edges and define comparison by model configuration such that the distances between vertexes adjacent by a $(-)$-edge does not get larger and by $(+)$-edge does not get smaller. For example, the $(2{\cdot}n+2)$-comparison (which holds in CAT[0] length spaces, see [1]) can be considered as a comparison for the following colored graph, where $(-)$-edges are marked by solid lines and $(+)$-edges by dashed lines.



3 Cost-convexity

In this section we review necessary material we learned from [10]; see also the references there in.

Cost-convex functions. Let M be a complete Riemannian manifold. Consider the *cost function* $\mathrm{cost}\colon M\times M\to\mathbb{R}$ defined by

$$\mathrm{cost}(x,y)=\tfrac{1}{2}\cdot|x-y|^2_M.$$

A function $f\colon M\to(-\infty,\infty]$ is called *cost-convex* if for any $p\in M$ such that $f(p)<\infty$ there is $q\in M$ such that

$$f(x)+\mathrm{cost}(q,x)\geqslant f(p)+\mathrm{cost}(q,p)$$

for any $x\in M$.

If M has nonnegative sectional curvature, then any cost-convex function f is (-1)-convex; that is, $f''\geqslant-1$, which means that the inequality

$$(f\circ\gamma)''\geqslant-1 \qquad ❶$$

holds in the barrier sense for any unit speed geodesic γ. On the other hand, $f''\geqslant-1$ does not imply that f is cost-convex.

Subgradient. Let $f\colon M\to(-\infty,\infty]$ be a semiconvex function defined on the complete Riemannian manifold M. Assume $f(p)$ is finite. In this case the differential

$$d_pf\colon \mathrm{T}_p\to(-\infty,\infty]$$

is defined; it is a convex positive homogeneous function defined on the tangent space T_p.

A tangent vector $v\in\mathrm{T}_pM$ is a *subgradient* of f at p, briefly $v\in\underline{\nabla}_pf$ if

$$\langle v,w\rangle\leqslant d_pf(w)$$

for any $w\in\mathrm{T}_p$. Note that the set $\underline{\nabla}_pf$ is a convex subset of T_p.

The subset of tangent vectors $v\in\mathrm{T}_p$ such that there is a minimizing geodesic $[p\,q]$ in the direction of v with length $|v|$ will be denoted as $\overline{\mathrm{TIL}}_p$. For p,q and v as above, we write $q=\exp_pv$.

3.1. Definition. *A complete Riemannian manifold will be called* cost-convex *if for any point p and any cost-convex function f which is finite at p we have $\underline{\nabla}_pf\subset\overline{\mathrm{TIL}}_p$ and for $q=\exp_pv$ the inequality*

$$f(x)+\mathrm{cost}(q,x)\geqslant f(p)+\mathrm{cost}(q,p)$$

holds for any $x\in M$.

Note that any cost-convex manifold has nonnegative sectional curvature. According to [3] and Proposition 3.3, if M is TCP, then it is cost-convex. The converse is unknown, likely it holds; there is a slightly stronger version of cost-convexity which implies TCP, see [3].

CTIL and MTW. Let us formulate other two conditions on Riemannian manifolds which are together equivalent to the cost-convexity. These two conditions will be used in the proof.

Let M be a complete Riemannian manifold. The *tangent injectivity locus* at the point $p\in M$ (briefly TIL_p) is defined as the maximal open subset in the tangent space T_p such that for any $v\in\mathrm{TIL}_p$ the geodesic path $\gamma(t)=\exp_p(v{\cdot}t)$, $t\in[0,1]$ is minimizing. If the tangent injectivity locus at any point $p\in M$ is convex we say that M satisfies *convexity of tangent injectivity locus* or briefly M is CTIL.

Note that the set $\overline{\mathrm{TIL}}_p$ defined above is closure of TIL_p in T_p. Therefore, M is CTIL if and only if the set $\overline{\mathrm{TIL}}_p$ is convex for any point $p\in M$.

If M is cost-convex, then it is CTIL. Indeed, fix a point $p\in M$ and consider the cost-convex function

$$f(x)=\sup\{\,\mathrm{cost}(q,p)-\mathrm{cost}(q,x)\mid q\in M\,\}$$

It remains to note that $\underline{\nabla}_pf=\overline{\mathrm{TIL}}_p$.

The second condition is called MTW for Ma–Trudinger–Wang. We will use its reformulation close to the one given by Cédric Villani [10, 2.6]; it can be proved the same way.

Assume $u, v \in \mathrm{T}_p$ and $w = \frac{1}{2}\cdot(u + v)$ and $x = \exp_p u$, $y = \exp_p v$ and $q = \exp_p w$. If the three geodesic paths $[p\,x]$, $[p\,y]$ and $[p\,q]$ described by the paths $t \mapsto \exp_p(t\cdot u)$, $t \mapsto \exp_p(t\cdot v)$, $t \mapsto \exp_p(t\cdot w)$ for $t \in [0, 1]$ are minimizing, then $[p\,q]$ is called *median* of the hinge $[p\,{}^x_y]$. Note that in a CTIL Riemannian manifold, any hinge has a median.

3.2. MTW condition. *Assume M be a CTIL Riemannian manifold. Then M is MTW if and only if for a median $[p\,q]$ of any hinge $[p\,{}^x_y]$ one of the following inequalities*

$$\left[\begin{array}{l} |p-q|^2_M - |z-q|^2_M \leqslant |p-x|^2_M - |z-x|^2_M, \\ |p-q|^2_M - |z-q|^2_M \leqslant |p-y|^2_M - |z-y|^2_M. \end{array}\right.$$

holds for any $z \in M$.

The following statement was proved by Grégoire Loeper in [7], see also [10, Proposition 2.5].

3.3. Proposition. *A complete Riemannian manifold is cost-convex if and only if it is CTIL and MTW.*

4 Kirszbraun's rigidity

In the proof we will use the rigidity case of the generalized Kirszbraun theorem proved by Urs Lang and Viktor Schroeder in [6], see also [1].

4.1. Kirszbraun rigidity theorem. *Let A be an Alexandrov space with non-negative curvature.*

Assume that for two point arrays $p, x_1, \dots, x_n \in A$ and $\tilde q, \tilde x_1, \dots, \tilde x_n \in \mathbb{H}$ we have that

$$|\tilde q - \tilde x_i| \geqslant |p - x_i|$$

for any i,

$$|\tilde x_i - \tilde x_j| \leqslant |x_i - x_i|$$

for any pair (i, j) and $\tilde q$ lies in the interior of the convex hull $\tilde K$ of $\tilde x_1, \dots, \tilde x_n$.

Then equalities hold in all the inequalities above. Moreover there is an distance preserving map $f\colon \tilde K \to A$ such that $f(\tilde x_i) = x_i$ and $f(\tilde q) = p$.

Proof. By the generalized Kirszbraun theorem, there is a short map $f\colon A \to \mathbb{H}$ such that $f(x_i) = \tilde x_i$. Set $\tilde p = f(p)$. By assumptions

$$|\tilde q - \tilde x_i| \geqslant |\tilde p - \tilde x_i|.$$

Since $\tilde q$ lies in the interior of K, $\tilde q = \tilde p$. It follows that the equality

$$|\tilde q - \tilde x_i| = |p - x_i|.$$

holds for each i.

The remaining part will be reduced to the case when A is a Euclidean space; which is left as an exercise.

According to Plaut's theorem (see [1]) there is a dense G-delta set $S \subset A$ of points s such that minimizing geodesics $[s\,x_i]$ are uniquely defined and their directions at s lie in a Euclidean subspace E_s of the tangent cone $\mathrm{T}_s A$.

Fix $s \in S$ and set $v_i = \log_s x_i$. Recall that the gradient exponent

$$\operatorname{gexp}_s \colon E_s \to A$$

is a short map and $\operatorname{gexp}_s \colon v_i \mapsto x_i$ for each i (see [1]). Note that $|v_i|_{E_s} = |s - x_i|_A$ for any i. Therefore by triangle inequality

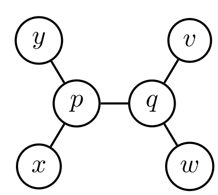
$$|v_i|_{E_s} \geqslant |p - x_i|_A - |p - s|_A.$$

Passing to the limit of the point array v_i and gexp_s as $s \to p$ we get (1) a collection of points in a Euclidean space $v_1, \dots, v_n \in E$ such that $|v_i|_E = |x_i - p|_A$ and (2) a short map $g \colon E \to A$ such that $g \colon v_i \to x_i$ and $g(0) = p$.

The composition $f \circ g \colon E \to \mathbb{H}$ is short and by classical Kirszbraun rigidity it has to be distance preserving on the convex hull $\tilde K'$ of v_i. Hence $\tilde K'$ is isometric to $\tilde K$ and the restriction $g|_{\tilde K'}$ is distance preserving. Hence the result. $\square$

5 Pivotal trees.

Let X be a metric space. A point array $(a_1, \dots, a_n)$ in X together with a choice of a graph with n vertexes labeled by $(a_1, \dots, a_n)$ and a choice of geodesic $[a_i\,a_j]$ for every adjacent pair (a_i, a_j) is called *geodesic graph.*



For geodesic trees we will use the same notation as for labeled combinatoric tree in square brackets; for example $[p/xy(q/vw)]$ will denote the geodesic tree with with combinatorics as on the diagram.

Fix a geodesic tree $T = [p_1/x_1 \dots x_k(p_2/x_{k+1} \dots x_n)]$; that is, T has two poles p_1, p_2 and each of the remaining vertexes are adjacent either to p_1 or p_2 — the vertexes $x_1, \dots, x_k$ are connected to p_1 and $x_{k+1}, \dots, x_n$ to p_2.

Assume X is a nonnegatively curved Alexandrov space; in particular the angle measure $\measuredangle[p\,{}^{x}_{z}]$ is defined for any geodesic hinge $[p\,{}^{x}_{y}] = ([p\,x], [p\,y])$.

A geodesic tree $\tilde T = [\tilde p_1/\tilde x_1 \dots \tilde x_k(\tilde p_2/\tilde x_{k+1} \dots \tilde x_n)]$ in the Hilbert space $\mathbb{H}$ will be called *pivotal tree* for T if

(i) $|\tilde p_1 - \tilde p_2|_{\mathbb{H}} = |p_1 - p_2|_X$,
(ii) $|\tilde p_i - \tilde x_j|_{\mathbb{H}} = |p_i - p_j|_X$ for any edge $[p_i\,x_j]$ in T and
(iii) $\measuredangle[\tilde p_j\,{}^{\tilde x_k}_{\tilde p_i}]_{\mathbb{H}} = \measuredangle[\tilde p_j\,{}^{\tilde x_k}_{\tilde p_i}]_X$ for any hinge $[p_j\,{}^{x_k}_{\tilde p_i}]$ in T.

5.1. Rigidity lemma. *Let X be a nonnegatively curved Alexandrov space and $T = [p_1/x_1 \dots x_k(p_2/x_{k+1} \dots x_n)]$ be geodesic tree in X Suppose $\tilde T = [\tilde p_1/\tilde x_1 \dots \tilde x_k(\tilde p_2/\tilde x_{k+1} \dots \tilde x_n)]$ is a pivotal tree for T. Assume that*

$$|\tilde x_i - \tilde x_j|_{\mathbb{H}} \leqslant |x_i - x_j|_X \tag{❷}$$

for any pair (i,j) and the convex hull $\tilde K$ of $\{\tilde x_1,\dots\tilde x_n\}$ intersects the line $(\tilde p_1,\tilde p_2)$. Then the equality holds in ❷ for each pair (i,j).

Proof. Let $\tilde z$ be a point on the line $(\tilde p_1,\tilde p_2)$. Assume that $\tilde z$ lies on the half-line from $\tilde p_1$ to $\tilde p_2$; otherwise swap the labels of $\tilde p_1$ and $\tilde p_2$.

Denote by ζ the direction of geodesic $[p_1\,p_2]$ at p_1. Set

$$z=\operatorname{gexp}_{p_1}(|\tilde z-\tilde p_1|\cdot\zeta),$$

where $\operatorname{gexp}_{p_1}$ denotes the gradient exponent at p_1; see [2]. By comparison, we have

$$|x_i-z|_X\leqslant|\tilde x_i-\tilde z|_{\mathbb R^2}$$

for any i.

It remains to apply Kirszbraun rigidity theorem (4.1). □

As above, we assume that X is a nonnegatively curved Alexandrov space and $[\tilde p_1/\tilde x_1\dots\tilde x_k(\tilde p_2/\tilde x_{k+1}\dots\tilde x_n)]$ is a pivotal tree in $\mathbb H$ for the geodesic tree $[p_1/x_1\dots x_k(p_2/x_{k+1}\dots x_n)]$ in X.

Note that by angle comparison, for any i and j we have

$$|\tilde x_i-\tilde p_j|_{\mathbb H}\geqslant|x_i-p_j|_X.$$

It follows that the configuration $\tilde p_1$, $\tilde p_2$, $\tilde x_1,\dots,\tilde x_n\in\mathbb H$ satisfies the tree comparison (see Section 1) if

❸ $$|\tilde x_i-\tilde x_j|_{\mathbb H}\geqslant|x_i-x_j|_X$$

for all pairs (i,j).

Denote by $\tilde\xi_i$ the direction of the half-plane containing $\tilde x_i$ with the boundary line $(\tilde p_1,\tilde p_2)$. The direction $\tilde\xi_i$ lies in the unit sphere normal to the line $(\tilde p_1,\tilde p_2)$; we may assume that the dimension of the sphere is $n-1$.

Note that up to a motion of $\mathbb H$, a pivotal configuration is completely described by the angles $\measuredangle(\tilde\xi_i,\tilde\xi_j)$. Moreover, the distance $|\tilde x_i-\tilde x_j|_{\mathbb H}$ is determined by $\measuredangle(\tilde\xi_i,\tilde\xi_j)$ and the function $\measuredangle(\tilde\xi_i,\tilde\xi_j)\mapsto|\tilde x_i-\tilde x_j|_{\mathbb H}$ is nondereasing.

Let us denote by $\alpha_{i,j}$ the minimal angle $\measuredangle(\tilde\xi_i,\tilde\xi_j)$ in a pivotal configuration such that ❸ holds. Note that the inequality ❸ is equivalent to

$$\measuredangle(\xi_i,\xi_j)\geqslant\alpha_{i,j}.$$

5.2. Corollary. *For any geodesic bipolar tree in a nonnegatively curved Alexandrov space the following conditions hold:*

(a) For any pair i and j, we have

$$\alpha_{i,j}\leqslant\pi.$$

(b) For any triple i, j and k, we have

$$\alpha_{i,j}+\alpha_{j,k}+\alpha_{k,i}\leqslant2\cdot\pi.$$

In other words, if X is a nonnegatively curved Alexandrov space, then

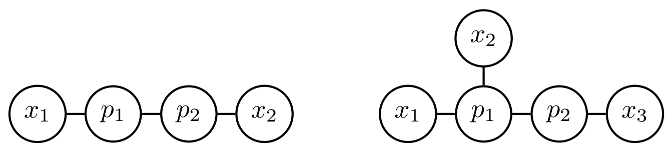
(a) *For any broken geodesic line $[p_1/x_1(p_2/x_2)]$ in X there is a pivotal tree $[\tilde p_1/\tilde x_1(\tilde p_2/\tilde x_2)]$ such that*

$$|\tilde x_1-\tilde x_2|_{\mathbb H}\geqslant|x_1-x_2|_X.$$

(b) *For any $[p_1/x_1x_2(p_2/x_3)]$ in X, there is a pivotal tree $[\tilde p_1/\tilde x_1\tilde x_2(\tilde p_2/\tilde x_3)]$ such that*

$$|\tilde x_i-\tilde x_j|_{\mathbb H}\geqslant|x_i-x_j|_X.$$

for all i and j.



Proof; (a). Consider the pivotal tree $[\tilde p_1/\tilde x_1(\tilde p_2/\tilde x_2)]$ (which is a polygonal path) with $\measuredangle(\tilde\xi_1,\tilde\xi_2)=\pi$. Note that the points $\tilde p_1,\tilde x_1,\tilde p_2,\tilde x_2$ are coplanar and the points $\tilde x_1$ and $\tilde x_2$ lie on the opposite sides from the line $(\tilde p_1,\tilde p_2)$. It remains to apply the rigidity lemma.

(b). By *(a)*, we can assume that

❹ $$\alpha_{1,3}+\alpha_{2,3}>\pi.$$

Consider the pivotal tree $[\tilde p_1/\tilde x_1\tilde x_2(\tilde p_2/\tilde x_3)]$ which lies in a 3-dimesional subspace in such a way that the points $\tilde x_1$ and $\tilde x_2$ lie on the opposite sides from the plane containing $\tilde p_1,\tilde p_2,\tilde x_3$, and

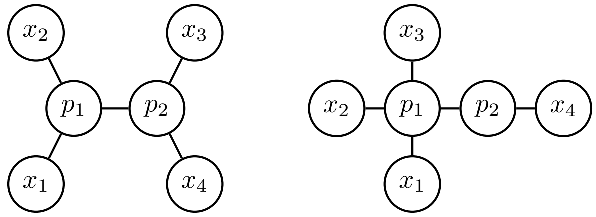
$$\measuredangle(\tilde\xi_1,\tilde\xi_3)=\alpha_{1,3},\qquad\qquad \measuredangle(\tilde\xi_2,\tilde\xi_3)=\alpha_{2,3}.$$

By ❹, the convex hull $\tilde K$ of $\{\tilde x_1,\tilde x_2,\tilde x_3\}$ intersects the line $(\tilde p_1,\tilde p_2)$. It remains to apply the rigidity lemma. □

Note that *(a)* and *(b)* imply that nonnegatively curved Alexandrov space satisfies 1(1)-tree and 2(1)-tree comparisons. However, 1(1)-tree comparison follows directly from the triangle inequality.

6 2(2)-tree and 3(1)-tree comparisons

The following theorem generalizes Theorem 1.1. It says in particular, that the comparisons for the following two trees holds in nonnegatively Alexandrov spaces.



6.1. Theorem. *Let X be an nonnegatively curved Alexandrov space. Then for any geodesic 2(2)-tree (or 3(1)-tree) there is a pivotal tree satisfying the corresponding tree comparison.*

In particular, any nonnegatively curved Alexandrov space satisfies the 2(2)-tree comparison as well as 3(1)-tree comparison.

The proofs in the two cases are nearly identical; they differ only by the choice made in the first line.

Proof. Fix a geodesic tree $[p_1/x_1x_2(p_2/x_3x_4)]$ or $[p_1/x_1x_2x_3(p_2/x_4)]$. Define the values $\{\alpha_{i,j}\}$ for each pair i, j as in the previous section.

Fix a smooth monotonic function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) = 0$ if $x \geqslant 0$ and $\varphi(x) > 0$ if $x < 0$. Consider a configuration of 4 points $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4$ in $\mathbb{S}^3$ which minimize the *energy*

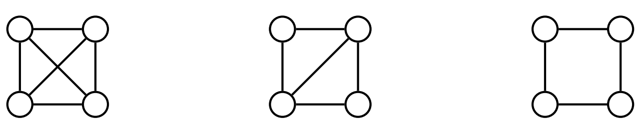
$$E(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4) = \sum_{i<j} \varphi(\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j) - \alpha_{i,j}).$$

Consider the geodesic graph Γ with 4 vertexes $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4$ in $\mathbb{S}^3$, where $\tilde{\xi}_i$ is adjacent to $\tilde{\xi}_j$ if $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j) < \alpha_{i,j}$. If the comparison does not hold, then Γ is not empty.

Note that a vertex of Γ can not lie in an open hemisphere with all its adjacent vertexes. Indeed, if it would be the case, then we could move this vertex increasing the distances to all its adjacent vertexes. Along this move the energy decreases which is not possible.

Note that by Corollary 5.2, degree of any vertex is at least 2. Indeed existence of a vertex of degree 1 contradicts 5.2a and existence of a vertex of degree 0 contradicts 5.2b.

Therefore the graph Γ is isomorphic to one the following three graphs.



The 6-edege case (that is, the complete graph with 4 vertexes) can not appear by the rigidity lemma (see 5.1).

To do the remaining two cases, note that since the energy is minimal, the angle between the edges at every vertex of degree 2 of Γ has to be π. That is, the concatenation of two edges at such vertex is a geodesic.

Consider the 5-edge graph on the diagram. By the observation above the both triangles in the graph run along one equator. The latter contradicts Corollary 5.2*b*.

For the 4-edge graph (that is, for 4-cycle) by the same observation we have 4 points lie on the equator; moving the even pair to the north pole and odd pair to the south pole will decrease the energy, a contradiction. □

7 4(1)-tree comparison

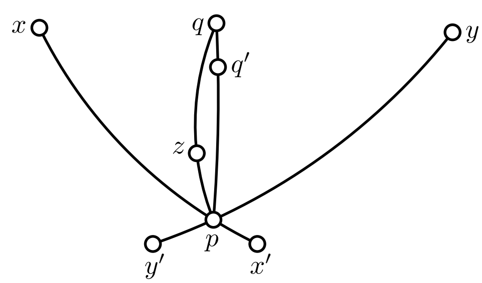
In this section we will prove the first part of Theorem 1.2. Note that according to Proposition 3.3, it is sufficient to show that if a Riemannian manifold M satisfies 4(1)-tree comparison, then it is CTIL and MTW. These two conditions will be proved separately.

7.1. Proposition. *Any complete cost-convex Riemannian manifold is CTIL.*

Proof. Let M be a cost-convex Riemannian manifold. Assume there is $p \in M$ and $u, v \in \mathrm{TIL}_p$ such that $w = \frac{1}{2}\cdot(u+v) \notin \mathrm{TIL}_p$.

Let τ be the maximal value such that the geodesic $\gamma(t) = \exp_p(w\cdot t)$ is a length-minimizing on $[0,\tau]$. Set $w' = \tau\cdot w$. Note that $\tau < 1$ and $w' \in \partial\mathrm{TIL}_p$.

Set $q = \exp_p w'$. By general position argument, we can assume that there are at least two minimizing geodesics connecting p to q; see [4]. That is, there is $w'' \in \partial\mathrm{TIL}_p$ such that $w'' \neq w'$ and $\exp_p w' = \exp_p w''$.



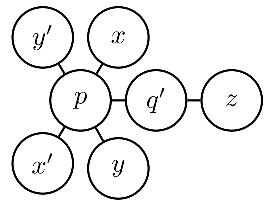
Fix small positive real numbers δ, ε and ζ. Consider the following points

$$
\begin{aligned}
q' &= q'(\varepsilon) = \exp_p(1-\varepsilon)\cdot w', & z &= z(\zeta) = \exp_p(\zeta\cdot w''),\\
x &= \exp_p u, & x' &= x'(\delta) = \exp_p(-\delta\cdot u),\\
y &= \exp_p v, & y' &= y'(\delta) = \exp_p(-\delta\cdot v).
\end{aligned}
$$

We will show that for some choice of δ, ε and ζ the tree comparison for $p/xx'yy'(q'/z)$ does not hold.

Assume contrary, that is, given any small positive numbers δ, ε and ζ, there is a point array $\tilde p$, $\tilde x$, $\tilde x'(\delta)$, $\tilde y$, $\tilde y'(\delta)$, $\tilde q'(\varepsilon)$, $\tilde z(\zeta) \in \mathbb{H}$ as in the definition of T-tree comparison.

Since δ is small, we can assume that p lies on a necessary unique minimizing geodesic $[x\,x']_M$. Hence



$$|x - x'|_M = |x - p|_M + |p - x'|_M.$$

By comparison

$$\begin{aligned}
|\tilde x - \tilde x'|_{\mathbb{H}} &\geqslant |x - x'|_M,\\
|\tilde x - \tilde p|_{\mathbb{H}} &= |x - p|_M,\\
|\tilde x' - \tilde p|_{\mathbb{H}} &= |x' - p|_M.
\end{aligned}$$

By triangle inequality,

$$|\tilde x - \tilde x'|_{\mathbb{H}} = |\tilde x - \tilde p|_{\mathbb{H}} + |\tilde x' - \tilde p|_{\mathbb{H}};$$

that is, $\tilde p \in [\tilde x\,\tilde x']_{\mathbb{H}}$. The same way we see that $\tilde p \in [\tilde y\,\tilde y']_{\mathbb{H}}$.

Fix ε and ζ. Note that as $\delta \to 0$ we have

$$\begin{aligned}
&\tilde x' \to \tilde p, && \tilde y' \to \tilde p.\\
&\measuredangle[\tilde p\,{}^{\tilde x'}_{\tilde y}] \to \measuredangle[p\,{}^{x'}_{y}], && \measuredangle[\tilde p\,{}^{\tilde y'}_{\tilde x}] \to \measuredangle[p\,{}^{y'}_{x}],\\
&\measuredangle[\tilde p\,{}^{\tilde x'}_{\tilde q'}] \to \measuredangle[p\,{}^{x'}_{q'}], && \measuredangle[\tilde p\,{}^{\tilde y'}_{\tilde q'}] \to \measuredangle[p\,{}^{y'}_{q'}],
\end{aligned}$$

It follows that

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] \to \measuredangle[p\,{}^{x}_{y}], \qquad \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde q'}] \to \measuredangle[p\,{}^{x}_{q'}], \qquad \measuredangle[\tilde p\,{}^{\tilde y}_{\tilde q'}] \to \measuredangle[p\,{}^{y}_{q'}].$$

Therefore, passing to a partial limit as $\delta \to 0$, we get a configuration of 5 points $\tilde p, \tilde x, \tilde y, \tilde q' = \tilde q'(\varepsilon), \tilde z = \tilde z(\zeta)$ such that

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] = \measuredangle[p\,{}^{x}_{y}], \qquad \measuredangle[\tilde p\,{}^{\tilde y}_{\tilde q'}] = \measuredangle[p\,{}^{y}_{q'}], \qquad \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde q'}] = \measuredangle[p\,{}^{x}_{q'}].$$

In other words, the map sending the points $0, u, v, w' \in \mathrm{T}_p$ to $\tilde p, \tilde x, \tilde y, \tilde q \in \mathbb{H}$ correspondingly is distance preserving.

Note that $q' \to q$ as $\varepsilon \to 0$. Therefore, in the limit, we get a configuration $\tilde p$, $\tilde x$, $\tilde y$, $\tilde q'$, $\tilde z = \tilde z(\zeta)$ such that in addition we have

$$\begin{aligned}
&|\tilde q' - \tilde z| = |q - z|, && |\tilde p - \tilde z| \geqslant |p - z|,\\
&|\tilde x - \tilde z| \geqslant |x - p|, && |\tilde y - \tilde z| \geqslant |y - z|
\end{aligned}$$

Since $w'' \neq w'$, for small values ζ the last three inequalities imply

$$|\tilde q' - \tilde z| > |q - z|,$$

a contradiction.

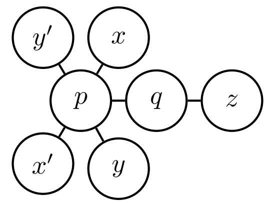
7.2. Proposition. *Any cost-convex CTIL Riemannian manifold is MTW.*

Proof. Let M be a cost-convex CTIL Riemannian manifold.

Fix a hinge $[p\,{}^{x}_{y}]$ in M. Since M is CTIL, the hinge $[p\,{}^{x}_{y}]$ has a median; denote it by $[p\,q]$.

For $\delta > 0$, define $x' = x'(\delta)$ and $y' = y'(\delta)$ as above.

Without loss of generality we can assume that $x, y \in \exp_p(\mathrm{TIL}_p)$. If δ is small, the latter implies that p lies on unique minimizing geodesics $[x\,x']$ and $[y\,y']$.



Consider the limit case of the comparison configuration for $p/xx'yy'(q/z)$ as $\delta \to 0$. It gives a configuration of 5 points $\tilde p$, $\tilde q$, $\tilde x$, $\tilde y$ and $\tilde z$ such that

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] = \measuredangle[p\,{}^{x}_{y}],$$

$\tilde q$ is the midpoint of $[\tilde x\,\tilde y]$. In particular,

$$\begin{aligned}
2{\cdot}|\tilde z - \tilde q|^2_{\mathbb{H}} + |\tilde q - \tilde x|^2_{\mathbb{H}} + |\tilde q - \tilde y|^2_{\mathbb{H}} &= |\tilde z - \tilde x|^2_{\mathbb{H}} + |\tilde z - \tilde y|^2_{\mathbb{H}},\\
2{\cdot}|\tilde p - \tilde q|^2_{\mathbb{H}} + |\tilde q - \tilde x|^2_{\mathbb{H}} + |\tilde q - \tilde y|^2_{\mathbb{H}} &= |\tilde p - \tilde x|^2_{\mathbb{H}} + |\tilde p - \tilde y|^2_{\mathbb{H}},
\end{aligned}$$

By the comparison,

$$\begin{aligned}
|\tilde z - \tilde x|_{\mathbb{H}} &\geqslant |z - x|_M, & |\tilde z - \tilde y|_{\mathbb{H}} &\geqslant |z - y|_M,\\
|\tilde p - \tilde x|_{\mathbb{H}} &\geqslant |p - x|_M, & |\tilde p - \tilde y|_{\mathbb{H}} &\geqslant |p - y|_M,\\
|\tilde q - \tilde x|_{\mathbb{H}} &= |q - x|_M, & |\tilde q - \tilde y|_{\mathbb{H}} &= |q - y|_M,\\
|\tilde q - \tilde z|_{\mathbb{H}} &= |q - z|_M, & |\tilde q - \tilde p|_{\mathbb{H}} &= |q - p|_M,
\end{aligned}$$

Therefore

$$\begin{aligned}
2{\cdot}|z - q|^2_M + |q - x|^2_M + |q - y|^2_M &\geqslant |z - x|^2_M + |z - y|^2_M,\\
2{\cdot}|p - q|^2_M + |q - x|^2_M + |q - y|^2_M &\leqslant |p - x|^2_M + |p - y|^2_M.
\end{aligned}$$

Hence the condition in 3.2 follows. □

8 Pull-back convexity

In this section we reformulate 4(1)-tree comparison using convexity of certain function on tangent space and use it to prove the second part of Theorem 1.2.

Recall that for a function f defined on metric space we write $f'' \leqslant \lambda$ if for any unit-speed geodesic γ the function

$$t \mapsto f \circ \gamma(t) - \lambda{\cdot}\tfrac{t^2}{2}$$

is a concave real-to-real function.

8.1. Proposition. *If a complete Riemannian manifold M satisfies 4(1)-tree comparison, then it is CTIL and for any $p, q \in M$, we have $f'' \leqslant 1$, where f is the function $f\colon \mathrm{TIL}_p \to \mathbb{R}$ defined by*

$$f(v) = \tfrac{1}{2}\cdot \mathrm{dist}_q^2 \circ \exp_p(v).$$

The converse also holds; moreover, if $f''\leqslant 1$ for any p,q in a CTIL Riemannian manifold M, then M satisfies all bipolar tree comparisons; that is, the $m(n)$-tree comparison holds for any m and n.

Proof. Note that 4(1)-tree comparison implies 3-tree comparison. Hence M has nonnegative sectional curvature.

Fix $u,v\in\mathrm{TIL}_p$ and $w\in[u\,v]$. It is sufficient to show that there is a function $g\colon \mathrm{T}_p\to\mathbb{R}$ such that

$$g''=1,\quad g(w)=f(w),\quad g(u)\geqslant f(u)\quad\text{and}\quad g(v)\geqslant f(v).$$

Fix small $\varepsilon>0$ and set

$$\begin{aligned}x&=\exp_p u, & y&=\exp_p v, & z&=\exp_p w,\\ x'&=\exp_p(-\varepsilon\cdot u), & y'&=\exp_p(-\varepsilon\cdot v).\end{aligned}$$

Apply the $p/xyx'y'(z/q)$ comparison and pass to the limit as $\varepsilon\to 0$. We obtain a configuration of points $\tilde p,\tilde x,\tilde y,\tilde z,\tilde q\in\mathbb{H}$, satisfying corresponding comparisons and in addition

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}]=\measuredangle[p\,{}^{x}_{y}],\qquad \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde z}]=\measuredangle[p\,{}^{x}_{z}],\qquad \measuredangle[\tilde p\,{}^{\tilde z}_{\tilde y}]=\measuredangle[p\,{}^{z}_{y}].$$

In particular, from above and Toponogov comparison, we have

$$\begin{aligned}|\tilde x-\tilde y|_{\mathbb{H}}&=|u-v|_{\mathrm{T}_p}, & |\tilde z-\tilde y|_{\mathbb{H}}&=|w-v|_{\mathrm{T}_p}, & |\tilde x-\tilde z|_{\mathbb{H}}&=|u-w|_{\mathrm{T}_p},\\ |\tilde q-\tilde z|_{\mathbb{H}}&=|q-z|_{M}, & |\tilde q-\tilde x|_{\mathbb{H}}&\geqslant|q-x|_{M}, & |\tilde q-\tilde y|_{\mathbb{H}}&\geqslant|q-y|_{M}.\end{aligned}$$

In particular, there is a distance-preserving map $\mathrm{T}_p\to\mathbb{H}$ such that $u\mapsto\tilde x$, $v\mapsto\tilde y$, $w\mapsto\tilde z$ and $0\mapsto\tilde p$. Further, we identify T_p and a subset of $\mathbb{H}$ using this map.

Consider the function $g(s):=\tfrac12\cdot|s-\tilde q|^2_{\mathrm{T}_p}$. Note that $g''=1$ and

$$\begin{aligned}g(w)&=\tfrac12\cdot|\tilde q-\tilde z|^2_{\mathrm{T}_p}=\tfrac12\cdot|q-z|^2_M=f(w),\\ g(u)&=\tfrac12\cdot|\tilde q-\tilde x|^2_{\mathrm{T}_p}\geqslant\tfrac12\cdot|q-x|^2_M=f(u),\\ g(v)&=\tfrac12\cdot|\tilde q-\tilde y|^2_{\mathrm{T}_p}\geqslant\tfrac12\cdot|q-y|^2_M=f(u).\end{aligned}$$

Hence the statement.

Converse. Fix points p and q in M; set $\tilde q=\log_p q\in\mathrm{T}_p$. Note that

❺ $$f\leqslant\tilde f,$$

where

$$\tilde f(v)=\tfrac12\cdot|v-\tilde q|^2_{\mathrm{T}_p}.$$

Further note that the inequality ❺ is equivalent to the Toponogov comparison for all hinges $[p\,{}^{x}_{q}]$ in M. It follows that M has nonnegative sectional curvature.

Fix a bipolar geodesic tree $[p/x_1\dots x_n(q/y_1\dots y_m)]$ in M. Set

$$\tilde p=0=\log_p p,\quad \tilde q=\log_p q,\quad\text{and}\quad \tilde x_i=\log_p x_i$$

for each i.

Consider the linear map $\psi_1\colon \mathrm{T}_q\to\mathrm{T}_p$ such that for any smooth function h

$$\psi_1\colon \nabla_q h\mapsto \nabla_{\tilde q}(h\circ\exp_p).$$

Since sectional curvature of M is nonnegative, the restriction $\exp_p|_{\mathrm{TIL}_p}$ is short and therefore so is ψ_1.

In particular there is a linear map $\psi_2\colon\mathrm{T}_q\to\mathrm{T}_p$ such that, the map $\iota\colon\mathrm{T}_q\to$ $\to\mathrm{T}_p\oplus\mathrm{T}_p$ defined by

$$\iota\colon v\mapsto\psi_1(v)\oplus\psi_2(v)$$

is distance preserving.

Further set

$$h_i=\tfrac12\cdot\mathrm{dist}^2_{y_i},\quad g_i=h_i\circ\exp_p|_{\mathrm{TIL}_p},\quad \tilde y_i=\tilde q-\iota(\nabla_q h_i).$$

By construction

$$|\tilde y_i-\tilde q|_{\mathrm{T}_p\oplus\mathrm{T}_p}=|y_i-q|_M.$$

At the point $\tilde q$ the restriction functions $\tilde g_i=\frac12\cdot\mathrm{dist}^2_{\tilde y_i}|_{\mathrm{T}_p\oplus 0}$ and the function g_i have the same value and gradient. Since $g_i''\leqslant 1$ and $\tilde g_i''=1$, we get $\tilde g_i\geqslant g_i$. The latter implies

$$|\tilde y_i-\tilde p|_{\mathrm{T}_p\oplus\mathrm{T}_p}\geqslant|y_i-p|_M\quad\text{and}\quad|\tilde y_i-\tilde x_j|_{\mathrm{T}_p\oplus\mathrm{T}_p}\geqslant|y_i-x_j|_M.$$

for any i and j.

Since there is an isometric embedding $\mathrm{T}_p\oplus\mathrm{T}_p\hookrightarrow\mathbb{H}$, we get the needed configuration. ☐

Note that we almost did not use smoothness of manifold M. In fact the Plaut's theorem makes possible to extend the proof above to complete length spaces; that is, we get the following corollary from the proof.

8.2. Corollary. *If a complete length space X satisfies $4(1)$-tree comparison, then it satisfies any bipolar tree comparisons.*

For the cost-convex manifolds we have somewhat weaker statement.

8.3. Proposition. *A complete Riemannian manifold M is cost-convex if and only if it is CTIL and for any $p,q\in M$ the function $h\colon\mathrm{TIL}_p\to\mathbb{R}$ defined by*

$$h(v)=\tfrac12\cdot|v|^2-\tfrac12\cdot\mathrm{dist}^2_q\circ\exp_p(v)$$

has convex sub level sets.

Proof; "only if" part. If M is cost-convex, then Proposition 7.2 it is CTIL.

Fix $p\in M$ and $u,v\in\mathrm{TIL}_p$ and set $x=\exp_p u$ and $y=\exp_p v$.

Consider the cost-convex function

$$f(s) = \min\{\mathrm{cost}(x,p) - \mathrm{cost}(x,s), \mathrm{cost}(y,p) - \mathrm{cost}(y,s)\}.$$

Note that the subgradient of $\underline{\nabla}_p f$ is the line segment $[u\,v]$. From the definition of cost-convex manifold (3.1) it follows that if u and v lie in sub level set of h, then so is the segment $[u\,v]$.

"If" part. Follows from the propositions 7.2 and 7.2. $\square$

9 All tree comparisons

Proof of Theorem 1.4. The "if" part is left as an exercise; let us prove the "only if" part.

Fix a point array $a_1, \dots, a_n$ in X. Consider the complete graph K_n with the vertexes labeled by $a_1, \dots, a_n$.

Let $\hat{K}_n \to K_n$ be the universal covering of the complete graph K_n. Denote by $\hat{V}$ the set of vertexes of $\hat{K}_n$; given a vertex $\hat{v} \in \hat{V}$ denote by v the corresponding vertex of K_n.

Applying the tree comparison for finite subtrees in $\hat{K}_n$ and passing to a partial limit, we get the following:

($*$) There is a map $f\colon \hat{V} \to \mathbb{H}$ such that

$$|f(\hat{v}) - f(\hat{w})|_{\mathbb{H}} \geqslant |v - w|_X$$

for any two vertexes $\hat{v}, \hat{w} \in \hat{V}$ and the equality holds if $(\hat{v}, \hat{w})$ is an edge in $\hat{K}_n$.

It finishes the proof if X is finite.

Since X is separable, it contains a countable everywhere dense set $\{a_1, a_2, \dots\}$. Applying the statement above for $X_n = \{a_1, \dots a_n\}$, we get a submetry from $Y_n = f_n(\hat{V}_n) \subset \mathbb{H}$ to X_n.

It remains to pass to the ulralimit Y of the subspaces Y_n. Clearly Y admits an isometric embedding into $\mathbb{H}$ and it admits submetry on $Y \to X$. Hence the statement follows. $\square$

The following proof was suggested by Alexander Lytchak, it is simplified version of the construction of Chuu-Lian Terng and Gudlaugur Thorbergsson given in [11, Section 4].

Proof of Proposition 1.5. Denote by G^n the direct product of n copies of G. Consider the map $\varphi_n\colon G^n \to G/\!\!/H$ defined by

$$\varphi_n\colon (\alpha_1, \dots, \alpha_n) \mapsto [\alpha_1 \cdots \alpha_n]_H,$$

where $[x]_H$ denotes the H-orbit of x in G.

Note that φ_n is a quotient map for the action of $H \times G^{n-1}$ on G^n defined by

$$(\beta_0, \dots, \beta_n)\cdot(\alpha_1, \dots, \alpha_n) = (\gamma_1\cdot\alpha_1\cdot\beta_1^{-1}, \beta_1\cdot\alpha_2\cdot\beta_2^{-1}, \dots, \beta_{n-1}\cdot\alpha_n\cdot\beta_n^{-1}),$$

where $\beta_i \in G$ and $(\beta_0, \beta_n) \in H < G \times G$.

Denote by ρ_n the product metric on G^n rescaled with factor $\sqrt{n}$. Note that the quotient $(G^n, \rho_n)/(H \times G^{n-1})$ is isometric to $G/\!\!/H = (G, \rho_1)/\!\!/H$.

As $n \to \infty$ the curvature of (G^n, ρ_n) converges to zero and its injectivity radius goes to infinity. Therefore passing to the ultra-limit of G^n as $n \to \infty$ we get the Hilbert space. It remains to observe that the limit action has the required property. $\square$

10 Final remarks

The following problem discussed in [1, 7.1] was one of the original motivations to study the tree comparison.

10.1. Problem. *Which finite metric spaces admit isometric embeddings into some Alexandrov spaces with nonnegative curvature.*

The problem is still open. According to [1, 4.1], the $(n-1)$-tree comparison provides a necessary condition for the problem n-point metric spaces. This condition is sufficient for the 4-point metric spaces. It might be still sufficient for 5-point metric spaces, but not for 6-point metric spaces.

The corresponding example of 6-point metric space was constructed by Sergei Ivanov, see [1]. Theorem 6.1, provides a source for such examples — any 6-point metric space which satisfy all 5-tree comparisons, but does not satisfy 2(2)-tree comparison provide an example. This class of examples includes the example of Sergei Ivanov — in the notations of [1, 7.1] it does not satisfies the comparison for the tree $y/az(q/xb)$.

By Theorem 6.1 and a theorem in [1], 5-tree and 2(2)-tree comparisons provide a necessary condition for 6-point metric spaces. We expect that these conditions are sufficient

Here an other candidate for a sufficient condition.

10.2. Question. *Assume F is a finite metric space which satisfies all tree comparisons. Is it true that F is isometric to a subset of Alexandrov space with nonpositive curvature?*

Note that even for finite metric space the all tree comparison has to be checked for infinite set of trees since one point of the space may be used as a label for several vertexes in the tree.

There is a chance that for 5-point and 6-point metric spaces, this condition is also necessary. Since there are nonnegatively curved Riemannian manifolds which are not cost-convex, Theorem 1.2 implies that this condition can not be necessary for 7-point metric spaces.

For any metric space X with an isometric group action $G \curvearrowright X$ with closed orbits the quotient map $X \to X/G$ is a submetry. In particular, by Theorem 1.4, if $G \curvearrowright \mathbb{H}$ is an isometric action with closed orbits on the Hilbert space, then the quotient space $\mathbb{H}/G$ satisfies all tree comparisons.

10.3. Question. *Assume X is a metric space satisfying all tree comparisons. Is it always possible to construct an isometric group action with closed orbits on the Hilbert space $G \curvearrowright \mathbb{H}$ such that X is isometric to a subset in $\mathbb{H}/G$?*

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