Tree comparison

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Abstract

We introduce a new type of comparison for metric spaces which is closely related to the Alexandrov comparison and the condition introduced by Ma Trudinger and Wang which guarantees continuity of optimal transport.

1 Introduction

We will denote by $|a - b|_X$ the distance between points a and b in the metric space X.

Let $(a_1, \ldots a_n)$ be a point array in a metric space X and T be a tree with the vertexes labeled by $\{a_1, \ldots, a_n\}$. We say that $(a_1, \ldots a_n)$ satisfies the *tree comparison* for T if there is a point array $(\tilde{a}_1, \ldots, \tilde{a}_n)$ in the Hilbert space $\mathbb{H}$ such that

$$|\tilde{a}_i - \tilde{a}_j|_{\mathbb{H}} \geqslant |a_i - a_j|_X$$

for any i and j and the equality holds if a_i and a_j are adjacent in T.

Fix a tree T with n vertexes as above. We say that a metric space X satisfies the tree comparison for T if every n-points arrays in X satisfies the tree comparison for T. We say that X satisfies *all tree comparisons* if X satisfies the tree comparison for all trees.

Notation for trees. To encode the labeled tree as on the diagram, we will use notation $p/xy(q/vw)$. It means that we choose p as the root; p has two children leafs to x, y and one child q with two children leafs v and w. Taking another root for the same tree, we may get different encodings, for eaxample $q/vw(p/xy)$ or $x/(p/y(q/vw))$.

If we do not need the labeling of vertexes, it is sufficient to write the number of leafs in the brackets; this way we can write $2(2)$ instead of $p/xy(q/vw)$ meaning that the root has two leafs x and y and its other child q has two leafs v and w. The same tree can be written as $(1(2))$ meaning that the root x has no leafs, p has one leaf y and one child q with two leafs v and w.

Using the described notation, we could say that a metric space *satisfies the comparison for the tree 2(2)*, or briefly *2(2)-tree comparison*, meaning that it
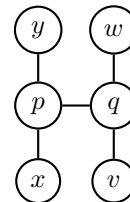
satisfies the tree comparison on the diagram. We could also say *by p/xy(q/vw)-comparison we have...* meaning that we apply the comparison for these 6 points and the tree on the diagram.

Alexandrov spaces. We define *Alexandrov space* as complete length space with curvature bounded below in the sense of Alexandrov; the latter is equivalent to the 3-tree comparison for the tripod-tree on the diagram.
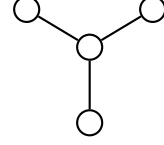

Using the introduced notation, [1, 4.1] can be restated the following way: *If length-metric space satisfies 3-tree comparison, then it also satisfies n-tree comparison for every positive integer n.*

In this example the trees are *monopolar*; that is they have one vertex (*pole*) adjacent to all other vertexes.

Dipolar comparison. A tree will be called *dipoar* if it has exactly two poles; that is, two vertexes of degree at least two. Equivalently, they could be defined as the trees with diameter 3. These are $m(n)$-trees for positive integers m and n; in this tree m vertexes are adjacent to the first pole, n vertexes are adjacent to the second pole and the tw poles are adjacent.

In Section 4 we will show that the comparison for the any Alexandrov space with nonnegative curvature satisfies the comparisons for the trees 2(2) and 3(1) — the first two trees an the following diagram.
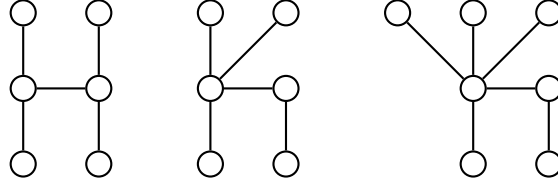
In Section 5, we will show that the comparison for the tree 4(1) (the third on the diagram) implies the curvature condition introduced by Ma–Trudinger–Wang (briefly MTW condition) and convexity of tangent injectivity loci (briefly CTIL condition). These two conditions appear in the study of contunuity of optimal transport betweens regular measures with positive continuous density functions. The continuity implies both MTW and CTIL conditions and slightly stronger version of these two conditions imply the continuity; see [3], [7] and the references there in.

Note that dipolar comparison provides a uniform way to treat combined CTIL+MTW condition; this partially answers the question of Cédric Villani in ???.

All tree comparisons. In Section 6 we show that if the space satisfies all tree comparisons then it is a target space of submetry from a subset in Hilbert space.

We also show that all bi-quotients of compact Lie groups with bi-invariant metrics satisfy all tree comparison.

2 Preliminaries

2.1. Kirszbraun rigidity theorem. *Let A be a complete* CBB[0] *length space.*
Assume that for two point arrays $p, x_1, \ldots, x_n \in A$ and $\tilde{q}, \tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{H}$ we have that

$$|\tilde{q} - \tilde{x}_i| \geqslant |p - x_i|$$

for any i,

$$|\tilde{x}_i - \tilde{x}_j| \leqslant |x_i - x_i|$$

for any pair (i, j) and $\tilde{q}$ lies in the interior of the convex hull K of $\tilde{x}_1, \ldots, \tilde{x}_n$.
Then equalities hold in all the inequalities above. Moreover there is an distance preserving map $f \colon K \to A$ such that $f(\tilde{x}_i) = x_i$ and $f(\tilde{q}) = p$.

Proof. By the generalized Kirszbraun theorem, there is a short map $f \colon A \to \mathbb{H}$ such that $f(x_i) = \tilde{x}_i$. Set $\tilde{p} = f(p)$. By assumptions

$$|\tilde{q} - \tilde{x}_i| \geqslant |\tilde{p} - \tilde{x}_i|.$$

Since $\tilde{q}$ lies in the interior of K, $\tilde{q} = \tilde{p}$. It follows that the equality

$$|\tilde{q} - \tilde{x}_i| = |p - x_i|.$$

holds for each i.

According to ???, there is a short map $\mathbb{H} \to \mathrm{T}_p$ such which admits a right inverse $\mathrm{T}_p \to \mathbb{H}$ such that ??? $\qquad\square$

Cost-concave functions. Let $f \colon A \to \mathbb{R}$ be a semiconcave function defined on Alexandrov space. A tangent vector $v \in \mathrm{T}_p A$ is called supporting vector of f at p, briefly $v \in \nabla_p^- f$ if

$$\langle v, x \rangle + d_p f(x) \leqslant 0$$

for any $x \in \mathrm{T}_p$.

It is straightforward to see that the set of supporting vectors is a convex subset of T_p.

Let A be an Alexandrov space. Consider the *cost function* $\mathrm{cost} \colon A \times A \to \mathbb{R}$ defined by

$$\mathrm{cost}(x, y) = \tfrac{1}{2} \cdot |x - y|_A^2.$$

A function $f \colon A \to \mathbb{R}$ is called cost-concave if thereis a subset of pairs $\mathcal{I} \subset A \times \mathbb{R}$ such that

$$f(x) = \inf \{ \, \mathrm{cost}(p, x) + r \mid (p, r) \in \mathcal{I} \, \}.$$

Note that any cost-concave function f is 1-concave, that is, it satisfies the inequality

$$f'' \leqslant 1$$

in the barrier sense (see [2]). On the other hand, inequality $f'' \leqslant 1$ does not imply that f is cost-concave.

Cost-convex spaces. Recall that a tangent vector $v \in \mathrm{T}_p$ is called geodesic if there is a minimizing geodesic $[p, q]$ in the direction of v with length $|v|$; in this case we write $q = \exp_p v$. The set of all geodesic vectors will be denoted by T'_p;

An Alexandrov space will be called *cost-convex* if for any cost-concave function f any supporting vector $v \in \nabla^-_p f$ is geodesic and for $q = \exp_p v$ the inequality
$$\mathrm{cost}(q, x) - \mathrm{cost}(q, p) \geqslant f(x) - f(p)$$
holds for any $x \in A$.

The following two observations implies that an Alexandrov space A is cost-convex if and only if A is CTIL and MTW as defiened below.

2.2. Observation. *If A is a cost-concave Alexandrov space then set of geodesic tangent vectors T'_p is convex for any $p \in A$.*

Proof. Fix $p \in A$ and consider the cost-concave function
$$f = \inf \left\{ \, \mathrm{cost}(q, x) - \mathrm{cost}(q, p) \, | \, q \in A \, \right\}.$$
Note that any $\mathrm{T}'_p \subset \nabla^-_p f$. Hence the statement follows. $\qquad\square$

Transport continuity. Consider the space $\mathcal{P}X$ of probability measures on X. Consider the metric on $\mathcal{P}X$ defined by
$$|\mu_0 - \mu_1|^2_{\mathcal{P}X} = \inf\{ \, (\mathrm{cost}\cdot\Pi)(X \times X) \, \},$$
where the gratest lower bound is taken for all probability measures Π on $X \times X$ such that $\Pi(A \times X) = \mu_0(A)$ and $\Pi(X \times A) = \mu_1(A)$ for any Borel set $A \subset X$.

Any measure Π satisfying the latter condition is called a *plan* and a measure Π for which the minimum is achieved is called *optimal plan*; it minimizes the cost for moving μ_0 into μ_1.

If we assume that X is ???, then the optimal transport exists for any pair of probability measures in $\mathcal{P}X$. If in addition X is a geodesic space, then so is $\mathcal{P}X$. That is, given two probability measures μ_0 and μ_1 there is one parameter family of measures μ_t, $t \in [0, 1]$ such that
$$|\mu_a - \mu_b|_{\mathcal{P}_2 X} = t\cdot|\mu_0 - \mu_1|_{\mathcal{P}_2 X}$$
for any $a, b \in [0, 1]$.

The family of measures μ_t described above is a geodesic path in $\mathcal{P}X$ connecting μ_0 and μ_1.

Assume that X is equipped with a backgraund metric vol called *volume*; that is X is a metric-measure space. Denote by $\mathcal{R}$ the subset of regular measures in $\mathcal{P}X$ with the distribution bounded away from zero and infinity. If the map $[0, 1] \times \mathcal{R} \times \mathcal{R} \to \mathcal{P}X$ defined by $(t, \mu_0, \mu_1) \mapsto \mu_t$ is uniquelly defined and continuous then we say that X satisfies transport continuity property, or briefly X is TCP.

The TCP Riemannian manifolds were studied by Xi-Nan Ma, Neil Trudinger and Xu-Jia Wang, Xu-Jia in [6]. They introduced a global differential geometric

condition which is now called MTW. CTIL is an other necessary condition was introduced in ???. The conditions CTIL+MTW are necessary for TCP. Moreover, a slightly stronger version of these conditions gives the converse.

3   Pivotal trees.

Let X be a metric space. A point array $(a_1, \ldots, a_n)$ in X together with a choice of a tree with n vertexes labeled by $(a_1, \ldots, a_n)$ and a choice of geodesic $[a_i, a_j]$ for every adjacent pair (a_i, a_j) is called *geodesic tree*.

For geodesic trees we will use the same notation as for labeled combinatoric tree in square brackets; for example $[p, xy(q, vw)]$ will denote the geodesic tree with with combinatorics as on the diagram.

Fix a geodeisc dipolar tree $T = [p_1/x_1 \ldots x_k(p_2/x_{k+1} \ldots x_n)]$; that is, the tree T has two poles p_1, p_2 and each of the remaining vertexes are adjacent either to p_1 or p_2; $x_1, \ldots, x_k$ are connected to p_1 and $x_{k+1}, \ldots, x_n$ to p_2.
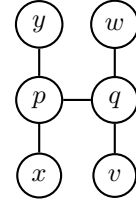
Assume X is a nonnegatively curved Alexandrov space; in particular the angle is defined for any geodesic hinge.

A geodesic tree $\tilde{T} = [\tilde{p}_1/\tilde{x}_1 \ldots \tilde{x}_k(\tilde{p}_2/\tilde{x}_{k+1} \ldots \tilde{x}_n)]$ in the Hilbert space $\mathbb{H}$ will be called *pivotal tree* for $T = [p_1/x_1 \ldots x_k(p_2/x_{k+1} \ldots x_n)]$ if

(i) $|\tilde{p}_1 - \tilde{p}_2|_{\mathbb{H}} = |p_1 - p_2|_X$,

(ii) $|\tilde{p}_i - \tilde{x}_j|_{\mathbb{H}} = |p_i - p_j|_X$ for any edge $[p_i, x_j]$ in T and

(iii) $\measuredangle[\tilde{p}_j\, {}^{\tilde{x}_k}_{\tilde{p}_i}]_{\mathbb{H}} = \measuredangle[\tilde{p}_j\, {}^{x_k}_{\tilde{p}_i}]_X$ for any hinge $[p_j\, {}^{x_k}_{p_i}]$ in T.

3.1. Rigidity lemma.   *Let X be a nonnegatively curved Alexandrov space and $T = [p_1/x_1 \ldots x_k(p_2/x_{k+1} \ldots x_n)]$ be geodesic tree in X Suppose $\tilde{T} = [\tilde{p}_1/\tilde{x}_1 \ldots \tilde{x}_k(\tilde{p}_2/\tilde{x}_{k+1} \ldots \tilde{x}_n)]$ is a pivotal tree for T. Assume that*

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}} \leqslant |x_i - x_j|_X \qquad\qquad ❶$$

for any pair (i, j) and the convex hull $\tilde{K}$ of $\{\tilde{x}_1, \ldots \tilde{x}_n\}$ intersects the line thru $\tilde{p}_1$ and $\tilde{p}_2$. Then the equality holds in ❶ for each pair (i, j).

Proof. Assume that a point $\tilde{z}$ on the line $(\tilde{p}_1, \tilde{p}_2)$ is given. We can assume that $\tilde{z}$ lies on the half-line from $\tilde{p}_1$ thru $\tilde{p}_2$; otherwise swap the labels of $\tilde{p}_1$ and $\tilde{p}_2$.

Denote by ζ the direction of geodesic $[p_1, p_2]$ at p_1. Set

$$z = \mathrm{gexp}_{p_1}(|\tilde{z} - \tilde{p}_1| \cdot \zeta),$$

where gexp_{p_1} denotes the gradient exponent at p_1; see [2]. By comparison, we have

$$|x_i - z|_X \leqslant |\tilde{x}_i - \tilde{z}|_{\mathbb{R}^2}$$

for any i.

It remains to apply Kirszbraun rigidity theorem (2.1).   □

5

Recall that X is a nonnegatively curved Alexandrov space. Note that by angle comparison, for any i and j we have

$$|\tilde{x}_i - \tilde{p}_j|_{\mathbb{H}} \geqslant |x_i - p_j|_X,$$

where $[\tilde{p}_1, \tilde{x}_1 \dots \tilde{x}_k(\tilde{p}_2, \tilde{x}_{k+1} \dots \tilde{x}_n)]$ is a pivotal tree for a geodesic tree with the vertexes $[p_1, x_1 \dots x_k(p_2, x_{k+1} \dots x_n)]$ in X.

It follows that the configuration $\tilde{p}_1,\ \tilde{p}_2,\ \tilde{x}_1, \dots \tilde{x}_n\ \in\ \mathbb{H}$ satisfies the tree comparison (see Section 1) if

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}} \geqslant |x_i - x_j|_X \qquad \qquad ❷$$

for all pairs (i, j).

Denote by $\tilde{\xi}_i$ the direction of the half-plane thru $\tilde{x}_i$ with the boundary line $(\tilde{p}_1, \tilde{p}_2)$. The direction $\tilde{\xi}_i$ lies in the unit sphere normal to the line $(\tilde{p}_1, \tilde{p}_2)$; we may assume that the dimension of the sphere is $n - 1$.

Note that up to a motion of $\mathbb{H}$, a pivotal configuration is completely described by the angles $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j)$. Moreover, the distance $|\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}}$ is determined by $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j)$ and the function $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j) \mapsto |\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}}$ is nondereasing.

Let us denote by $\alpha_{i,j}$ the minimal angle $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j)$ in a pivotal configuration such that ❷ holds, so the inequality ❷ is equivalent to

$$\measuredangle(\xi_i, \xi_j) \geqslant \alpha_{i,j}.$$

3.2. Corollary. *For any geodesic dipolar tree in a nonnegatively curved Alexandrov space the following conditions hold:*
 (a) For any pair i and j, we have

$$\alpha_{i,j} \leqslant \pi.$$

 (b) For any triple i, j and k, we have

$$\alpha_{i,j} + \alpha_{j,k} + \alpha_{k,i} \leqslant 2 \cdot \pi.$$

In other words, if X is a nonnegatively curved Alexandrov space then
 (a) For any broken geodesic line $[p_1/x_1(p_2/x_2)]$ in X there is a pivotal tree $[\tilde{p}_1, \tilde{x}_1(\tilde{p}_2, \tilde{x}_2)]$ such that

$$|\tilde{x}_1 - \tilde{x}_2|_{\mathbb{H}} \geqslant |x_1 - x_2|_X.$$

 (b) For any geodesic tree $[p_1/x_1 x_2(p_2/x_3)]$ in X there is a pivotal tree $[\tilde{p}_1/\tilde{x}_1 \tilde{x}_2(\tilde{p}_2/\tilde{x}_3)]$ such that

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}} \geqslant |x_i - x_j|_X.$$

 for all i and j.

Proof; (a). Consider the pivotal tree $[\tilde{p}_1/\tilde{x}_1(\tilde{p}_2/\tilde{x}_2)]$ (which is a polygonal path) with $\angle(\tilde{\xi}_1, \tilde{\xi}_2) = \pi$. Note that the points $\tilde{p}_1, \tilde{x}_1, \tilde{p}_2, \tilde{x}_2$ are coplanar and the points $\tilde{x}_1$ and $\tilde{x}_2$ lie on the opposite sides from the line $(\tilde{p}_1, \tilde{p}_2)$. It remains to apply the rigidity lemma.

(b). By *(a)*, we can assume that

$$\alpha_{1,3} + \alpha_{2,3} > \pi. \qquad\qquad ❸$$

Consider the pivotal tree $[\tilde{p}_1/\tilde{x}_1 x_2(\tilde{p}_2/\tilde{x}_3)]$ which lies in a 3-dimesional sub-space in such a way that the points $\tilde{x}_1$ and $\tilde{x}_2$ lie on the opposite sides from the plane $(\tilde{p}_1, \tilde{p}_2, \tilde{x}_3)$ and

$$\angle(\tilde{\xi}_1, \tilde{\xi}_3) = \alpha_{1,3}, \qquad\qquad \angle(\tilde{\xi}_2, \tilde{\xi}_3) = \alpha_{2,3}.$$

By ❸, the convex hull $\tilde{K}$ in the rigidity lemma intersects the line $(\tilde{p}_1, \tilde{p}_2)$. It remains to apply the lemma. $\qquad\square$

Note that *(a)* and *(b)* implies that X satisfies the comparison for the dipolar trees 1(1) and 2(1) shown on the digram. However, the tree comparison for (1(1)) follows from the triangle inequality.
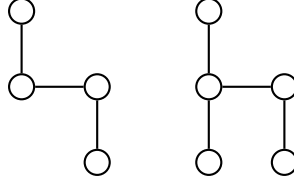
4   Six point comparison

4.1. Theorem. *Let X be an nonnegatively curved Alexandrov space. Then for any geodesic 2(2)-tree and any 3(1)-tree (see the diagram) there is a pivotal tree satisfying the tree comparison.*

In particular, any nonnegatively curved Alexandrov space satisfies the comparison for the bipolar trees 2(2) and 3(1).
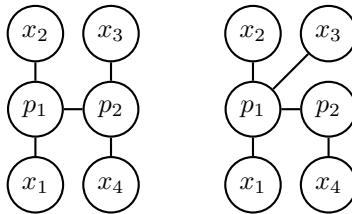
Proof. The proof in these two cases will be identical. Fix a geodesic tree $[p_1/x_1 x_2(p_2/x_3 x_4)]$ or $[p_1/x_1 x_2 x_3(p_2/x_4)]$.

Recall that p_1 and p_2 are the poles of the tree and each of remaining vertexes x_1, x_2, x_3, x_4 are connected to one of the poles.

Define the values $\{\alpha_{i,j}\}$ for each pair i, j as in the previous section.

Fix a smooth monotonic function $\varphi \colon \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) = 0$ if $x \geqslant 0$ and $\varphi(x) > 0$ if $x < 0$. Consider a configuration of 4 points $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4$ in $\mathbb{S}^3$ which minimize the *energy*

$$E(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4) = \sum_{i<j} \varphi(\angle(\tilde{\xi}_i, \tilde{\xi}_j) - \alpha_{i,j}).$$

Consider the geodesic graph Γ with 4 vertexes $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4$ in $\mathbb{S}^3$, where $\tilde{\xi}_i$ is adjacent to $\tilde{\xi}_j$ if $\angle(\tilde{\xi}_i, \tilde{\xi}_j) < \alpha_{i,j}$. If there the comparison does not hold then this graph is not empty.

Note that for any vertex $\tilde{\xi}_i$ can not lie in an open hemisphere with all its adjacent vertexes. Indeed, if it would be the case then we could move this $\tilde{\xi}_i$ increasing the distances to all its adjacent vertexes. Along this move the energy decreases which is not possible.

Applying the rigidity lemma, we conclude that Γ can not be complete; that is, the graph on the diagram.

Note that by Corollary 3.2, degree of any vertex is at least 2. Indeed if there is a vertex of degree 1, the statement follows from *(a)* and if there is a vertex of degree 2 then the statement follows from *(b)*.
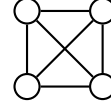
Since energy is minimal, the angle between the edges at every vertex of degree 2 has to be π; that is, the pair of edges at such vertex continue each other.

Threre are only three graphs with 4 vertexes and all the degrees at least two: they have 6, 5 and 4 edges corrspondingly. The complete graph with 6 edges is already forbidden above. For the 5-edge graph on the diagram, by the observation above the two triangles in the graph forms an equator. The latter contradicts Corollary 3.2*b*.
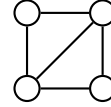
For the 4-edge graph, by the same observation we have 4 points lie on the equator; moving one pair to north pole and the other to south pole will increase all the sides, a contradiction. $\square$

5 Seven point comparison

Let M be a Riemannian manifold. The *tangent injectivity locus* at the point $p \in M$ (briefly TIL_p) is defined as the maximal open subset in the tangent space T_p such that for any $v \in \mathrm{TIL}_p$ the geodesic path $\gamma(t) = \exp_p(v \cdot t)$, $t \in [0, 1]$ is a minimizing. If the tangent injectivity locus at any point $p \in M$ is convex we say that M satisfies *convexity of tangent injectivity locus* or briefly M is CTIL.
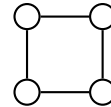
Xi-Nan Ma, Neil Trudinger and Xu-Jia Wang, Xu-Jia introduced a global differential geometric condition which is now called MTW, see [6]. The conditions CTIL and MTW are necessary for the regularity of optimal transport on

Riemannian manifold M. Moreover, a slightly stronger version of these conditions gives the converse.

5.1. Proposition. *Let T be the tree as on the diagram. If a Riemannian manifold M satisfies the T-tree comparison then*

   (a) *M is CTIL;*
   (b) *M is MTW.*
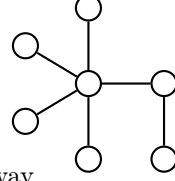
In the proof we will use a reformulation of MTW condition given by Cédric Villani [7, 2.6]. More precisely, we will use the following reformulation of which can be proved the same way.

Assume $u, v \in \mathrm{T}_p$ and $w = \frac{1}{2} \cdot (u + v)$ and $x = \exp_p u$, $y = \exp_p v$ and $q = \exp_p w$. If the three geodesic paths $[p, x]$, $[p, y]$ and $[p, q]$ described by the paths $t \mapsto \exp_p(t \cdot u)$, $t \mapsto \exp_p(t \cdot v)$, $t \mapsto \exp_p(t \cdot w)$ for $t \in [0, 1]$ are minimizing, then $[p, q]$ is called *median* of the hinge $[p\,{}^x_y]$. Note that in a CTIL Riemannian manifold, any hinge has a median.

5.2. MTW condition. *Assume M be a CTIL Riemannian manifold. Then M is MTW if and only if for a median $[p, q]$ of any hinge $[p\,{}^x_y]$ one of the following inequalities*

$$\left[\begin{array}{l} |p - q|^2_M - |z - q|^2_M \leqslant |p - x|^2_M - |z - x|^2_M, \\ |p - q|^2_M - |z - q|^2_M \leqslant |p - y|^2_M - |z - y|^2_M. \end{array} \right.$$

holds for any $z \in M$.

Proof; (a). Assume the contrary; that is, there is $p \in M$ and $u, v \in \mathrm{TIL}_p$ such that $w = \frac{1}{2} \cdot (u + v) \notin \mathrm{TIL}_p$.

Let τ be the maximal value such that the geodesic $\gamma(t) = \exp_p(w \cdot t)$ is a length-minimizing on $[0, \tau]$. Set $w' = \tau \cdot w$. Note that $\tau < 1$ and $w' \in \partial \mathrm{TIL}_p$.

Set $q = \exp_p w'$. By general position argument, we can assume that there are at least two minimizing geodesics connecting p to q; see [4]. That is, there is $w'' \in \partial \mathrm{TIL}_p$ such that $w'' \neq w'$ and $\exp_p w' = \exp_p w''$.
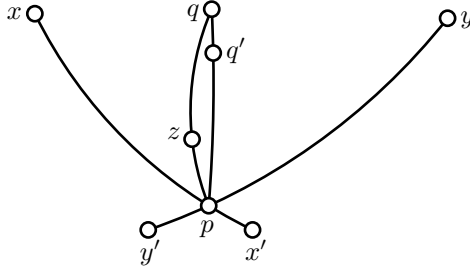
Fix small positive real numbers δ, ε and ζ. Consider the points

$$\begin{aligned} q' &= q'(\varepsilon) = \exp_p(1 - \varepsilon) \cdot w', & z &= z(\zeta) = \exp_p(\zeta \cdot w''), \\ x &= \exp_p u, & x' &= x'(\delta) = \exp_p(-\delta \cdot u), \\ y &= \exp_p v, & y' &= y'(\delta) = \exp_p(-\delta \cdot v). \end{aligned}$$

We will show that for some choice of δ, ε and ζ the array p, x, x', y, y', q', z does not satisfy the T-tree comparison with the labeling as on the diagram below.

Assume that given positive numbers δ, ε and ζ, there is a point array $\tilde{p}, \tilde{x}, \tilde{x}'(\delta), \tilde{y}, \tilde{y}'(\delta), \tilde{q}'(\varepsilon), \tilde{z}(\zeta) \in \mathbb{H}$ as in the definition of T-tree comparison; that is, the distances between the points in this array are at least as big as the distances of corresponding points in M and the equality holds for the pair adjacent in T.

Since δ is small, we can assume that p lies on a necessary unique minimizing geodesic $[x, x']_M$. Hence
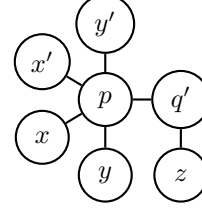
$$|x - x'|_M = |x - p|_M + |p - x'|_M.$$

By comparison

$$|\tilde{x} - \tilde{x}'|_{\mathbb{H}} \geqslant |x - x'|_M,$$
$$|\tilde{x} - \tilde{p}|_{\mathbb{H}} = |x - p|_M,$$
$$|\tilde{x}' - \tilde{p}|_{\mathbb{H}} = |x' - p|_M.$$

By triangle inequality,

$$|\tilde{x} - \tilde{x}'|_{\mathbb{H}} = |\tilde{x} - \tilde{p}|_{\mathbb{H}} + |\tilde{x}' - \tilde{p}|_{\mathbb{H}};$$

that is, $\tilde{p} \in [\tilde{x}, \tilde{x}']_{\mathbb{H}}$. The same way we see that $\tilde{p} \in [\tilde{y}, \tilde{y}']_{\mathbb{H}}$.

Fix ε and ζ. Note that as $\delta \to 0$ we have

$$\tilde{x}' \to \tilde{p}, \qquad\qquad\qquad \tilde{y}' \to \tilde{p}.$$
$$\measuredangle[\tilde{p}\,{}^{\tilde{x}'}_{\tilde{y}}] \to \measuredangle[p\,{}^{x'}_{y}], \qquad\qquad \measuredangle[\tilde{p}\,{}^{\tilde{y}'}_{\tilde{x}}] \to \measuredangle[p\,{}^{y'}_{x}],$$
$$\measuredangle[\tilde{p}\,{}^{\tilde{x}'}_{\tilde{q}'}] \to \measuredangle[p\,{}^{x'}_{q'}], \qquad\qquad \measuredangle[\tilde{p}\,{}^{\tilde{y}'}_{\tilde{q}'}] \to \measuredangle[p\,{}^{y'}_{q'}],$$

It follows that

$$\measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{y}}] \to \measuredangle[p\,{}^{x}_{y}], \qquad \measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{q}'}] \to \measuredangle[p\,{}^{x}_{q'}], \qquad \measuredangle[\tilde{p}\,{}^{\tilde{y}}_{\tilde{q}'}] \to \measuredangle[p\,{}^{y}_{q'}].$$

Therefore, passing to a partial limit as $\delta \to 0$, we get a configuration of 5 points $\tilde{p}, \tilde{x}, \tilde{y}, \tilde{q}' = \tilde{q}'(\varepsilon), \tilde{z} = \tilde{z}(\zeta)$ such that

$$\measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{y}}] = \measuredangle[p\,{}^{x}_{y}], \qquad \measuredangle[\tilde{p}\,{}^{\tilde{y}}_{\tilde{q}'}] = \measuredangle[p\,{}^{y}_{q'}], \qquad \measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{q}'}] = \measuredangle[p\,{}^{x}_{q'}].$$

In other words, the map sending 4 points $0, u, v, w' \in \mathrm{T}_p$ to $\tilde{p}, \tilde{x}, \tilde{y}, \tilde{q} \in \mathbb{H}$ correspondingly is distance preserving.

Note that $q' \to q$ as $\varepsilon \to 0$. Therefore, in the limit, we get a configuration $\tilde{p}, \tilde{x}, \tilde{y}, \tilde{q}', \tilde{z} = \tilde{z}(\zeta)$ such that in addition we have

$$|\tilde{q}' - \tilde{z}| = |q - z|, \qquad\qquad |\tilde{p} - \tilde{z}| \geqslant |p - z|,$$
$$|\tilde{x} - \tilde{z}| \geqslant |x - p|, \qquad\qquad |\tilde{y} - \tilde{z}| \geqslant |y - z|$$

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Since $w'' \neq w'$, for small values ζ the last three inequalities imply

$$|\tilde{q}' - \tilde{z}| > |q - z|,$$

a contradiction.

(b). Fix a hinge $[p \, {}^{x}_{y}]$ in M. By *(a)*, M is CTIL. Therefor $[p \, {}^{x}_{y}]$ has a median; denote it by $[p, q]$. For $\delta > 0$, define $x' = x'(\delta)$ and $y' = y'(\delta)$ as above.

Without loss of generality we can assume that $x, y \in$ $\in \exp_p(\mathrm{TIL}_p)$. If δ is small, the latter implies that p lies on unique minimizing geodesics $[x, x']$ and $[y, y']$.

Consider a limit case T-tree comparison as $\delta \to 0$; we get a configuration of 5 points $\tilde{p}$, $\tilde{q}$, $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ such that

$$\angle[\tilde{p} \, {}^{\tilde{x}}_{\tilde{y}}] = \angle[p \, {}^{x}_{y}],$$
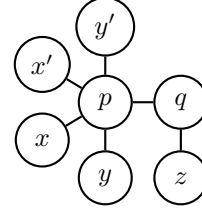
$\tilde{q}$ is the midpoint of $[\tilde{x}, \tilde{y}]$. In particular,

$$2 {\cdot} |\tilde{z} - \tilde{q}|^2_{\mathbb{H}} + |\tilde{q} - \tilde{x}|^2_{\mathbb{H}} + |\tilde{q} - \tilde{y}|^2_{\mathbb{H}} = |\tilde{z} - \tilde{x}|^2_{\mathbb{H}} + |\tilde{z} - \tilde{y}|^2_{\mathbb{H}},$$
$$2 {\cdot} |\tilde{p} - \tilde{q}|^2_{\mathbb{H}} + |\tilde{q} - \tilde{x}|^2_{\mathbb{H}} + |\tilde{q} - \tilde{y}|^2_{\mathbb{H}} = |\tilde{p} - \tilde{x}|^2_{\mathbb{H}} + |\tilde{p} - \tilde{y}|^2_{\mathbb{H}},$$

By the comparison,

$$|\tilde{z} - \tilde{x}|_{\mathbb{H}} \geqslant |z - x|_M, \qquad\qquad |\tilde{z} - \tilde{y}|_{\mathbb{H}} \geqslant |z - y|_M,$$
$$|\tilde{p} - \tilde{x}|_{\mathbb{H}} \geqslant |p - x|_M, \qquad\qquad |\tilde{p} - \tilde{y}|_{\mathbb{H}} \geqslant |p - y|_M,$$
$$|\tilde{q} - \tilde{x}|_{\mathbb{H}} = |q - x|_M, \qquad\qquad |\tilde{q} - \tilde{y}|_{\mathbb{H}} = |q - y|_M,$$
$$|\tilde{q} - \tilde{z}|_{\mathbb{H}} = |q - z|_M, \qquad\qquad |\tilde{q} - \tilde{p}|_{\mathbb{H}} = |q - p|_M,$$

Therefore

$$2 {\cdot} |z - q|^2_M + |q - x|^2_M + |q - y|^2_M \geqslant |z - x|^2_M + |z - y|^2_M,$$
$$2 {\cdot} |p - q|^2_M + |q - x|^2_M + |q - y|^2_M \leqslant |p - x|^2_M + |p - y|^2_M.$$

Hence the condition in 5.2 follows. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6 Polypolar comparison

Recall that a map $f \colon W \to X$ between metric spaces is called *submetry* if for any $w \in W$ and $r \geqslant 0$, we have

$$f[B(w, r)_W] = B(f(w), r)_X,$$

where $B(w, r)_W$ denotes the ball with center w and radius r in the space W. In other words submetry is a map which is 1-Lipschitz and 1-co-Lipschitz at the same time. Note that any submetry is onto.

6.1. Theorem. *A separable metric space X satisfies all tree comparison if and only if X is isometric to a target space of submetry defined of a subset of the Hilbert space.*

Proof. The "if" part is left as an exercise; let us prove the "only if" part.

Fix a point array $a_1, \ldots, a_n$ in X. Consider the complete graph K_n with $\{1, \ldots, n\}$ as the set of vertexes.

Let $\tilde{K}_n \to K_n$ be the universal covering of the complete graph K_n. Denote by $\tilde{V}$ the set of vertexes of $\tilde{K}_n$; given a vetex $\tilde{v} \in \tilde{V}$ denote by v the corresponding vertex of K_n.

By multipolar comparison, we have the following:

($*$) There is a map $f \colon \tilde{V} \to \mathbb{H}$ such that

$$|f(\tilde{v}) - f(\tilde{w})|_{\mathbb{H}} \geqslant |a_v - a_w|_X$$

for any two vertexes $\tilde{v}, \tilde{w} \in \tilde{V}$ and the equality holds if $(\tilde{v}, \tilde{w})$ is an edge in $\tilde{K}_n$.

Since X is separable, it contains a countable everywhere dense set $\{a_1, a_2, \ldots\}$. Applying the statement above for $X_n = \{a_1, \ldots a_n\}$, we get an isometric action $\Gamma_n \curvearrowright \mathbb{H}$ and invariant sets $Y_n = f(\tilde{V}_n) \subset \mathbb{H}$ such that X_n is isometric to Y_n/Γ_n.

It remains to fix an ultra filter ω on $\mathbb{N}$ and pass to the ω-limit action on $\mathbb{H}$. $\qquad\square$

6.2. Proposition. *Suppose G be a compct Lie group with bi-invariant metric, so the action $G \times G \curvearrowright G$ defined by $(h_1, h_2) \cdot g = h_1 \cdot g \cdot h_2^{-1}$ is isometric. Then for any closed subgroup $H < G \times G$, the bi-quotient space $G /\!\!/ H$ satisfies multipolar comparison.*

As a result we have many examples of spaces satisfying all tree comparison; for example, since $\mathbb{S}^n = \mathrm{SO}(n)/\mathrm{SO}(n-1)$, any round sphere satisfies multipolar comparison.

We present a proof suggested by Alexander Lytchak, it is simplified vesrion of the construction of Chuu-Lian Terng and Gudlaugur Thorbergsson given in [8, Section 4].

Proof. Denote by G^n the direct product of n copies of G. Consider the map $\varphi_n \colon G^n \to G$ defined by

$$\varphi_n \colon (\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1 \cdots \alpha_n.$$

Note that φ_n is a quotient map for the $H \times G^{n-1}$-action on G^n defined by

$$(\beta_0, \ldots, \beta_n) \cdot (\alpha_1, \ldots, \alpha_n) = (\gamma_1 \cdot \alpha_1 \cdot \beta_1^{-1}, \beta_1 \cdot \alpha_2 \cdot \beta_2^{-1}, \ldots, \beta_{n-1} \cdot \alpha_n \cdot \beta_n^{-1}),$$

where $\beta_i \in G$ and $(\beta_0, \beta_n) \in H < G \times G$.

Denote by ρ_n the product metric on G^n rescaled with factor $\sqrt{n}$. Note that the quotient $(G^n, \rho_n)/(H \times G^{n-1})$ is isometric to $G /\!\!/ H = (G, \rho_1) /\!\!/ H$.

As $n \to \infty$ the curvature of (G^n, ρ_n) converges to zero and its injectivity radius goes to infinity. Therefore passing to the ultra-limit of G^n as $n \to \infty$ we get the Hilbert space. It remains to observe that the limit action has the required property. $\qquad\square$

7   Final remarks

The following problem discussed in [1, 7.1] was one of the original motivations to study the tree comparison.

7.1. Problem.   *Which finite metric spaces admit isometric embeddings into Alexandrov spaces with curvature $\geqslant \kappa$.*

The problem is still open. According to [1, 4.1], the n-tree comparison provides a necessary condition for the problem. This condition is sufficient for the 4-point metric spaces and possibly for 5-point metric spaces, but not sufficient for 6-point metric spaces. The corresponding example of 6-point metric space was constructed by Sergei Ivanov, see [1].

Theorem 4.1, provides a source for such examples — it is sufficient to construct construct 6-pint metric space which satisfy all 5-tree comparisons, but does not satisfy 2(2)-tree comparison. This class of examples includes the example of Sergei Ivanov; it does not satisfies the comparison for $y/az(q/xb)$ in the notations of [1, 7.1].

We expect that the 5-tree and 2(2)-tree comparisons (see the trees on the diagram) are sufficient for 6-point metric spaces.
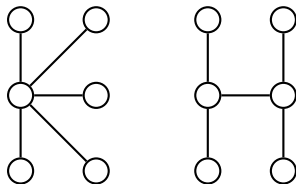
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