Tree comparison

Nina Lebedeva, Anton Petrunin and Vladimir Zolotov

Abstract

We introduce a new type of comparison for metric spaces which is closely related to the Alexandrov comparison and the condition introduced by Ma Trudinger and Wang which guarantees continuity of optimal transport.

1 Introduction

Tree comparison. We will denote by $|a-b|_X$ the distance between points a and b in the metric space X.

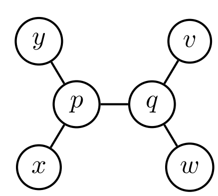
Let $(a_1, \dots a_n)$ be a point array in a metric space X and T be a tree with the vertexes labeled by $\{a_1, \dots, a_n\}$. We say that $(a_1, \dots a_n)$ satisfies the *T-tree comparison* if there is a point array $(\tilde a_1, \dots, \tilde a_n)$ in the Hilbert space $\mathbb{H}$ such that

$$|\tilde a_i - \tilde a_j|_{\mathbb{H}} \geqslant |a_i - a_j|_X$$

for any i and j and the equality holds if a_i and a_j are adjacent in T.

We say that a metric space X satisfies the *T-tree comparison* if every n-points arrays in X satisfies the T-tree comparison.

Encoding of trees. To encode the labeled tree on the diagram, we will use notation $p/xy(q/vw)$. It means that we choose p as the root; p has two children leafs to x, y and one child q with two children leafs v and w. Taking another root for the same tree, we get different encodings, for eaxample $q/vw(p/xy)$ or $x/(p/y(q/vw))$.



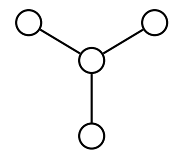
If we do not need the labeling of vertexes, it is sufficient to write the number of leafs in the brackets; this way we can write 2(2) instead of $p/xy(q/vw)$ since the root (p) has two leafs (x and y) and yet another child (q) has two leafs (v and w). The same tree can be written as (1(2)) meaning that the root x has no leafs, p has one leaf y and one child q with two leafs v and w. Every vertex which is not the root and not a leaf corresponds to a pair of bracketsin this notation.

Using the described notation, we could say that a metric space *satisfies the 2(2)-tree comparison*, meaning that it satisfies the tree comparison on the

... was partially supported ...

diagram. We could also say *"applying the $p/xy(q/vw)$-tree comparison we get..."* meaning that we apply the comparison for these 6 points and the tree on the diagram.

A vertex of a tree is called *pole* that is they have one vertex (*pole*) adjacent to all other vertexes.



Monopolar comparisons. We define *Alexandrov space* as complete length space with curvature bounded below in the sense of Alexandrov; the latter is equivalent to the 3-tree comparison for the tripod-tree on the diagram.

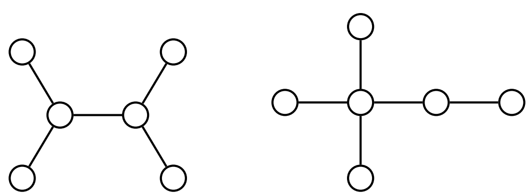
Using the introduced notation, Theorem 4.1. in [1] can be restated the following way: *If length-metric space satisfies* 3*-tree comparison, then it also satisfies n-tree comparison for every positive integer n.*

In this example the trees are *monopolar*; that is, they have one pole.

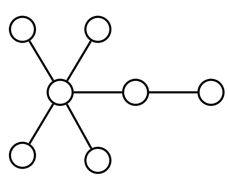
Dipolar comparison. A tree will be called *dipoar* if it has exactly two poles; that is, two vertexes of degree at least two.

Equivalently, dipolar trees could be defined as the trees with diameter 3 or as the trees which admit an encoding $m(n)$ for positive integers m and n; in this tree m vertexes are adjacent to the first pole which is the root and n vertexes are adjacent to the second pole and the two poles are adjacent.

In Section 4 we will show that the comparison for the any Alexandrov space with nonnegative curvature satisfies the comparisons for the trees 2(2) and 3(1) — the first two trees an the following diagram.



CTP, MTW and CTIL. In Section 5, we will show that the comparison for the tree 4(1) (see the diagram below) implies the curvature condition introduced by Ma–Trudinger–Wang (briefly MTW condition) and convexity of tangent injectivity loci (briefly CTIL condition).



These two conditions appear in the study of continuity of optimal transport between regular measures with positive continuous density functions. The continuity implies both MTW and CTIL conditions and slightly stronger version of these two conditions imply the continuity; see [3], [8] and the references there in.

Note that dipolar comparison provides a uniform way to treat combined CTIL+MTW condition; this partially answers the question of Cédric Villani in ???.

Note that 4(1)-tree comparison (as well as any T-tree comparison) is stable under passing to the limit. Namely, if a sequence of metric spaces X_n satisfies 4(1)-comparison and converges in the sense of Gromov–Hausdorff to the space then its limit X_∞ also satisfies 4(1)-comparison.

We expect that for Riemannian manifolds, the 4(1)-tree comparison is equivalent to CTP and to MTW+CTIL (see question ???). If this is indeed the case then from above it will follow that the properties CTP and MTW+CTIL are stable as well.

All tree comparisons. In Section 6 we show that if the space satisfies all tree comparisons then it is a target space of submetry from a subset in Hilbert space.

We also show that all bi-quotients of compact Lie groups with bi-invariant metrics satisfy all tree comparison.

2 Preliminaries

Cost-convex functions. Recall that complete length space with curvature bounded below in the sense of Alexandrov will be called *Alexandrov space*.

Let A be an Alexandrov space. Consider the *cost function* $\mathrm{cost}\colon A\times A\to\mathbb{R}$ defined by

$$\mathrm{cost}(x,y)=\tfrac{1}{2}\cdot|x-y|_A^2.$$

A function $f\colon A\to(-\infty,\infty]$ is called cost-convex if there is a nonempty subset of pairs $\mathcal{I}\subset A\times\mathbb{R}$ such that

$$f(x)=\sup\left\{\,r-\mathrm{cost}(p,x)\mid(p,r)\in\mathcal{I}\,\right\}.$$

If A has nonnegative curvature then any cost-convex function f is (-1)-convex, that is, it satisfies the inequality

$$f''\geqslant-1 \qquad ❶$$

in the barrier sense (see [2]). On the other hand, the inequality ❶ does not imply that f is cost-convex.

Subgradient. Let $f\colon A\to(-\infty,\infty]$ be a semiconvex function defined on Alexandrov space A. Assume $f(p)$ is finite. In this case the differential

$$d_pf\colon\mathrm{T}_p\to(-\infty,\infty]$$

is defined; it is a convex positive homogeneous function defined on the tangent cone T_p.

A tangent vector $v\in \mathrm{T}_pA$ is a *subgradient* of f at p, briefly $v\in \underline{\nabla}_p f$ if

$$\langle v,w\rangle \leqslant d_pf(w)$$

for any $w\in \mathrm{T}_p$. Note that the set $\underline{\nabla}_p f$ is a convex subset of T_p.

The sunset of tangent vectors $v\in\mathrm{T}_p$ such that there is a minimizing geodesic $[p,q]$ in the direction of v with length $|v|$ will be denoted as $\overline{\mathrm{TIL}}_p$. For p,q and v as above, we write $q=\exp_p v$.

An Alexandrov space will be called *cost-convex* if for any cost-convex function f any subgradient $v\in\underline{\nabla}_p f$ is geodesic and for $q=\exp_p v$ the inequality

$$\mathrm{cost}(q,p)-\mathrm{cost}(q,x)\geqslant f(x)-f(p)$$

holds for any $x\in A$.

2.1. Observation. *If A is a cost-convex Alexandrov space, then $\overline{\mathrm{TIL}}_p$ is a convex subset of T_p for any $p\in A$.*

Proof. Fix $p\in A$ and consider the cost-concave function

$$f=\inf\left\{\,\mathrm{cost}(q,x)-\mathrm{cost}(q,p)\,\middle|\,q\in A\,\right\}.$$

Note that any $\overline{\mathrm{TIL}}_p\subset\underline{\nabla}_p f$. Hence the statement follows. □

Riemannian manifold M satisfies *convexity of tangent injectivity locus*, or briefly M is CTIL, if the set $\overline{\mathrm{TIL}}_p$ is convex for any point $p\in M$. This property was considered in ??? as a necessary condition for the *continuity of transport property*, briefly CTP.

The observation above implies that a cost-convex complete Riemannian manifold are CTIL. Therefore, as it follows from [6], cost-convexity of complete Riemannian manifold is equivalent to MTW+CTIL condition. Here MTW is stays for an other necessary condition for CTP introduced by Xi-Nan Ma, Neil Trudinger and Xu-Jia Wang, Xu-Jia in [7].

Likely cost-convexity is also sufficient for CTP.

Kirszbraun theorem. In the proof we will use the rigidity case of the generalized Kirszbraun theorem proved by Urs Lang and Viktor Schroeder in [5], see also [1].

2.2. Kirszbraun rigidity theorem. *Let A be a complete* CBB[0] *length space.*

Assume that for two point arrays $p,x_1,\dots,x_n\in A$ and $\tilde q,\tilde x_1,\dots,\tilde x_n\in\mathbb{H}$ we have that

$$|\tilde q-\tilde x_i|\geqslant|p-x_i|$$

for any i,

$$|\tilde x_i-\tilde x_j|\leqslant|x_i-x_i|$$

for any pair (i,j) and $\tilde q$ lies in the interior of the convex hull $\tilde K$ of $\tilde x_1,\dots,\tilde x_n$.

Then equalities hold in all the inequalities above. Moreover there is an distance preserving map $f\colon\tilde K\to A$ such that $f(\tilde x_i)=x_i$ and $f(\tilde q)=p$.

Proof. By the generalized Kirszbraun theorem, there is a short map $f\colon A \to \mathbb{H}$ such that $f(x_i) = \tilde{x}_i$. Set $\tilde{p} = f(p)$. By assumptions

$$|\tilde{q} - \tilde{x}_i| \geqslant |\tilde{p} - \tilde{x}_i|.$$

Since $\tilde{q}$ lies in the interior of K, $\tilde{q} = \tilde{p}$. It follows that the equality

$$|\tilde{q} - \tilde{x}_i| = |p - x_i|.$$

holds for each i.

Consider the tangent vectors $v_i \in \mathrm{T}_p$ such that $\exp_p v_i = x_i$ for each i. Note that these vectors are uniquely defined, all the vectors lie in an isometric copy of a Euclidean space and

$$|v_i - v_j|_{\mathrm{T}_p} = |x_i - x_j|_A.$$

In particular, the convex hull of $\{v_1, \dots, v_n\}$ in T_p is isometric to $\tilde{K}$, so we can keep notation $\tilde{K}$ for this convex hull.

Consider the gradient exponent $\mathrm{gexp}_p\colon \mathrm{T}_p \to A$; it is a short map such that $\mathrm{gexp}_p 0 = p$ and $\mathrm{gexp}_p v_i = x_i$ for each i. It remains to show that the restriction $\mathrm{gexp}_p|_{\tilde{K}}$ is distance-preserving.

Extend the sequence $v_1, \dots v_n$ to an infinite sequence of vectors $v_i \in \tilde{K}$ which is dense in $\tilde{K}$. Set $x_i = \mathrm{gexp}_p v_i$ for each i.

Note that it is sufficient to show that the map $v_i \mapsto x_i$ is distance preserving. From above,

$$|v_i - v_j|_{\mathrm{T}_p} = |x_i - x_j|_A$$

if $i, j \leqslant n$; it provides a base for induction. Assume

$$|v_i - v_j|_{\mathrm{T}_p} = |x_i - x_j|_A$$

for all pairs $i, j \leqslant k - 1$. Since the gradient exponent is short,

$$|v_i - v_k|_{\mathrm{T}_p} \geqslant |x_i - x_k|_A$$

for each $i \leqslant k$. From the first part of theorem we have

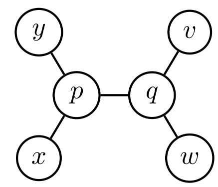
$$|v_i - v_k|_{\mathrm{T}_p} = |x_i - x_k|_A.$$

Hence the second statement follows. □

3 Pivotal trees.

Let X be a metric space. A point array $(a_1, \dots, a_n)$ in X together with a choice of a tree with n vertexes labeled by $(a_1, \dots, a_n)$ and a choice of geodesic $[a_i, a_j]$ for every adjacent pair (a_i, a_j) is called *geodesic tree.*

For geodesic trees we will use the same notation as for labeled combinatoric tree in square brackets; for example $[p, xy(q, vw)]$ will denote the geodesic tree with with combinatorics as on the diagram.



Fix a geodesic tree $T = [p_1/x_1 \dots x_k(p_2/x_{k+1} \dots x_n)]$; that is, the tree T has two poles p_1, p_2 and each of the remaining vertexes are adjacent either to p_1 or p_2 — the vertexes $x_1, \dots, x_k$ are connected to p_1 and $x_{k+1}, \dots, x_n$ to p_2.

Assume X is a nonnegatively curved Alexandrov space; in particular the angle is defined for any geodesic hinge.

A geodesic tree $\tilde T = [\tilde p_1/\tilde x_1 \dots \tilde x_k(\tilde p_2/\tilde x_{k+1} \dots \tilde x_n)]$ in the Hilbert space $\mathbb{H}$ will be called *pivotal tree* for $T = [p_1/x_1 \dots x_k(p_2/x_{k+1} \dots x_n)]$ if

(i) $|\tilde p_1 - \tilde p_2|_{\mathbb{H}} = |p_1 - p_2|_X$,

(ii) $|\tilde p_i - \tilde x_j|_{\mathbb{H}} = |p_i - p_j|_X$ for any edge $[p_i, x_j]$ in T and

(iii) $\measuredangle[\tilde p_j\,{}^{\tilde x_k}_{\tilde p_i}]_{\mathbb{H}} = \measuredangle[\tilde p_j\,{}^{x_k}_{\tilde p_i}]_X$ for any hinge $[p_j\,{}^{x_k}_{\tilde p_i}] = ([p_j, x_k], [p_j, p_i])$ in T.

3.1. Rigidity lemma. *Let X be a nonnegatively curved Alexandrov space and $T = [p_1/x_1 \dots x_k(p_2/x_{k+1} \dots x_n)]$ be geodesic tree in X Suppose $\tilde T = [\tilde p_1/\tilde x_1 \dots \tilde x_k(\tilde p_2/\tilde x_{k+1} \dots \tilde x_n)]$ is a pivotal tree for T. Assume that*

$$|\tilde x_i - \tilde x_j|_{\mathbb{H}} \leqslant |x_i - x_j|_X \qquad \text{❷}$$

for any pair (i, j) and the convex hull $\tilde K$ of $\{\tilde x_1, \dots \tilde x_n\}$ intersects the line thru $\tilde p_1$ and $\tilde p_2$. Then the equality holds in ❷ for each pair (i, j).

Proof. Assume that a point $\tilde z$ on the line $(\tilde p_1, \tilde p_2)$ is given. We can assume that $\tilde z$ lies on the half-line from $\tilde p_1$ to $\tilde p_2$; otherwise swap the labels of $\tilde p_1$ and $\tilde p_2$.

Denote by ζ the direction of geodesic $[p_1, p_2]$ at p_1. Set

$$z = \mathrm{gexp}_{p_1}(|\tilde z - \tilde p_1|\cdot\zeta),$$

where gexp_{p_1} denotes the gradient exponent at p_1; see [2]. By comparison, we have

$$|x_i - z|_X \leqslant |\tilde x_i - \tilde z|_{\mathbb{R}^2}$$

for any i.

It remains to apply Kirszbraun rigidity theorem (2.2). $\square$

Recall that X is a nonnegatively curved Alexandrov space.

Assume $[\tilde p_1, \tilde x_1 \dots \tilde x_k(\tilde p_2, \tilde x_{k+1} \dots \tilde x_n)]$ is a pivotal tree in $\mathbb{H}$ for the geodesic tree $[p_1, x_1 \dots x_k(p_2, x_{k+1} \dots x_n)]$ in X. Note that by angle comparison, for any i and j we have

$$|\tilde x_i - \tilde p_j|_{\mathbb{H}} \geqslant |x_i - p_j|_X.$$

It follows that the configuration $\tilde p_1$, $\tilde p_2$, $\tilde x_1, \dots \tilde x_n \in \mathbb{H}$ satisfies the tree comparison (see Section 1) if

$$|\tilde x_i - \tilde x_j|_{\mathbb{H}} \geqslant |x_i - x_j|_X \qquad \text{❸}$$

for all pairs (i, j).

Denote by $\tilde{\xi}_i$ the direction of the half-plane containing $\tilde{x}_i$ with the boundary line $(\tilde{p}_1, \tilde{p}_2)$. The direction $\tilde{\xi}_i$ lies in the unit sphere normal to the line $(\tilde{p}_1, \tilde{p}_2)$; we may assume that the dimension of the sphere is $n - 1$.

Note that up to a motion of $\mathbb{H}$, a pivotal configuration is completely described by the angles $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j)$. Moreover, the distance $|\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}}$ is determined by $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j)$ and the function $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j) \mapsto |\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}}$ is nondereasing.

Let us denote by $\alpha_{i,j}$ the minimal angle $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j)$ in a pivotal configuration such that ❸ holds, so the inequality ❸ is equivalent to

$$\measuredangle(\xi_i, \xi_j) \geqslant \alpha_{i,j}.$$

3.2. Corollary. *For any geodesic dipolar tree in a nonnegatively curved Alexandrov space the following conditions hold:*

(a) For any pair i and j, we have

$$\alpha_{i,j} \leqslant \pi.$$

(b) For any triple i, j and k, we have

$$\alpha_{i,j} + \alpha_{j,k} + \alpha_{k,i} \leqslant 2{\cdot}\pi.$$

In other words, if X is a nonnegatively curved Alexandrov space then

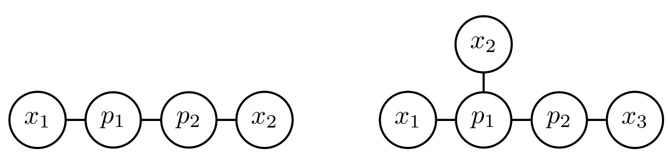
(a) For any broken geodesic line $[p_1/x_1(p_2/x_2)]$ in X there is a pivotal tree $[\tilde{p}_1, \tilde{x}_1(\tilde{p}_2, \tilde{x}_2)]$ such that

$$|\tilde{x}_1 - \tilde{x}_2|_{\mathbb{H}} \geqslant |x_1 - x_2|_X.$$

(b) For any geodesic tree $[p_1/x_1x_2(p_2/x_3)]$ in X there is a pivotal tree $[\tilde{p}_1/\tilde{x}_1\tilde{x}_2(\tilde{p}_2/\tilde{x}_3)]$ such that

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}} \geqslant |x_i - x_j|_X.$$

for all i and j.



Proof; (a). Consider the pivotal tree $[\tilde{p}_1/\tilde{x}_1(\tilde{p}_2/\tilde{x}_2)]$ (which is a polygonal path) with $\measuredangle(\tilde{\xi}_1, \tilde{\xi}_2) = \pi$. Note that the points $\tilde{p}_1, \tilde{x}_1, \tilde{p}_2, \tilde{x}_2$ are coplanar and the points $\tilde{x}_1$ and $\tilde{x}_2$ lie on the opposite sides from the line $(\tilde{p}_1, \tilde{p}_2)$. It remains to apply the rigidity lemma.

(b). By *(a)*, we can assume that

$$\alpha_{1,3} + \alpha_{2,3} > \pi. \qquad ❹$$

Consider the pivotal tree $[\tilde p_1/\tilde x_1 x_2(\tilde p_2/\tilde x_3)]$ which lies in a 3-dimesional subspace in such a way that the points $\tilde x_1$ and $\tilde x_2$ lie on the opposite sides from the plane containing $\tilde p_1, \tilde p_2, \tilde x_3$, and

$$\measuredangle(\tilde\xi_1, \tilde\xi_3) = \alpha_{1,3}, \qquad\qquad \measuredangle(\tilde\xi_2, \tilde\xi_3) = \alpha_{2,3}.$$

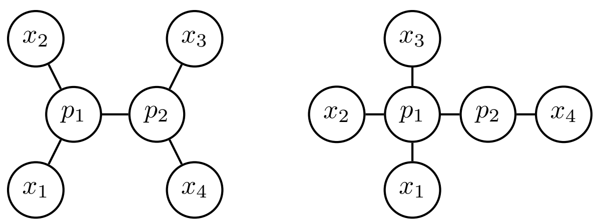
By ❹, the convex hull $\tilde K$ in the rigidity lemma intersects the line $(\tilde p_1, \tilde p_2)$. It remains to apply the lemma. □

Note that *(a)* and *(b)* implies that X satisfies the comparison for the dipolar trees 1(1) and 2(1). However, 1(1)-tree comparison follows from the triangle inequality.

4 Six point comparison

4.1. Theorem. *Let X be an nonnegatively curved Alexandrov space. Then for any geodesic 2(2)-tree and any 3(1)-tree (see the diagram) there is a pivotal tree satisfying the tree comparison.*

In particular, any nonnegatively curved Alexandrov space satisfies the 2(2)-tree comparison as well as 3(1)-tree comparison.



The proofs in the two cases are identical.

Proof. Fix a geodesic tree $[p_1/x_1x_2(p_2/x_3x_4)]$ or $[p_1/x_1x_2x_3(p_2/x_4)]$.

Recall that p_1 and p_2 are the poles of the tree and each of remaining vertexes x_1, x_2, x_3, x_4 are connected to one of the poles.

Define the values $\{\alpha_{i,j}\}$ for each pair i, j as in the previous section.

Fix a smooth monotonic function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) = 0$ if $x \geqslant 0$ and $\varphi(x) > 0$ if $x < 0$. Consider a configuration of 4 points $\tilde\xi_1, \tilde\xi_2, \tilde\xi_3, \tilde\xi_4$ in $\mathbb{S}^3$ which minimize the *energy*

$$E(\tilde\xi_1, \tilde\xi_2, \tilde\xi_3, \tilde\xi_4) = \sum_{i<j} \varphi(\measuredangle(\tilde\xi_i, \tilde\xi_j) - \alpha_{i,j}).$$

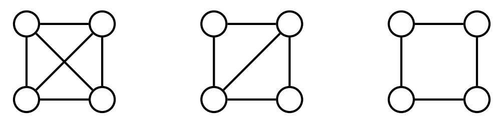
Consider the geodesic graph Γ with 4 vertexes $\tilde\xi_1, \tilde\xi_2, \tilde\xi_3, \tilde\xi_4$ in $\mathbb{S}^3$, where $\tilde\xi_i$ is adjacent to $\tilde\xi_j$ if $\measuredangle(\tilde\xi_i, \tilde\xi_j) < \alpha_{i,j}$. If the comparison does not hold then Γ is not empty.

Note that for any vertex $\tilde\xi_i$ can not lie in an open hemisphere with all its adjacent vertexes. Indeed, if it would be the case then we could move this $\tilde\xi_i$

increasing the distances to all its adjacent vertexes. Along this move the energy decreases which is not possible.

Note that by Corollary 3.2, degree of any vertex is at least 2. Indeed existence of a vertex of degree 1 contradicts 3.2a and existence of a vertex of degree 0 contradicts 3.2b.

Therefore the graph Γ is isomorphic to one the following three graphs.



The 6-edege case (that is, the complete graph with 4 vertexes) can not appear by the rigidity lemma (see 3.1).

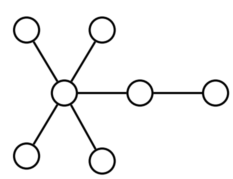
To do the remaining two cases, note that since the energy is minimal, the angle between the edges at every vertex of degree 2 of Γ has to be π. That is, the pair of edges at such vertex forms a geodesic.

Consider the 5-edge graph on the diagram. By the observation above the both triangles in the graph run along one equator. The latter contradicts Corollary 3.2b.

For the 4-edge graph, by the same observation we have 4 points lie on the equator; moving the even pair to the north pole will decrease the energy, a contradiction. $\square$

5 Seven point comparison

In this section we will consider 4(1)-tree comparison. This is the simplest tree comparison which does not follow from Alexandrov comparison.



Let M be a Riemannian manifold. The *tangent injectivity locus* at the point $p \in M$ (briefly TIL_p) is defined as the maximal open subset in the tangent space T_p such that for any $v \in \mathrm{TIL}_p$ the geodesic path $\gamma(t) = \exp_p(v{\cdot}t)$, $t \in [0,1]$ is a minimizing. If the tangent injectivity locus at any point $p \in M$ is convex we say that M satisfies *convexity of tangent injectivity locus* or briefly M is CTIL.

Xi-Nan Ma, Neil Trudinger and Xu-Jia Wang, Xu-Jia introduced a global differential geometric condition which is now called MTW, see [7]. The conditions CTIL and MTW are necessary for the regularity of optimal transport on Riemannian manifold M. Moreover, a slightly stronger version of these conditions gives the converse.

5.1. Proposition. *If a Riemannian manifold M satisfies the* 4(1)*-tree comparison then*

(a) M is CTIL;

(b) M is MTW.

In the proof we will use a reformulation of MTW condition given by Cédric Villani [8, 2.6]. More precisely, we will use the following reformulation of which can be proved the same way.

Assume $u, v \in \mathrm{T}_p$ and $w = \frac{1}{2}\cdot(u+v)$ and $x = \exp_p u$, $y = \exp_p v$ and $q = \exp_p w$. If the three geodesic paths $[p,x]$, $[p,y]$ and $[p,q]$ described by the paths $t \mapsto \exp_p(t\cdot u)$, $t \mapsto \exp_p(t\cdot v)$, $t \mapsto \exp_p(t\cdot w)$ for $t \in [0,1]$ are minimizing, then $[p,q]$ is called *median* of the hinge $[p\,{}^x_y]$. Note that in a CTIL Riemannian manifold, any hinge has a median.

5.2. MTW condition. *Assume M be a CTIL Riemannian manifold. Then M is MTW if and only if for a median $[p,q]$ of any hinge $[p\,{}^x_y]$ one of the following inequalities*

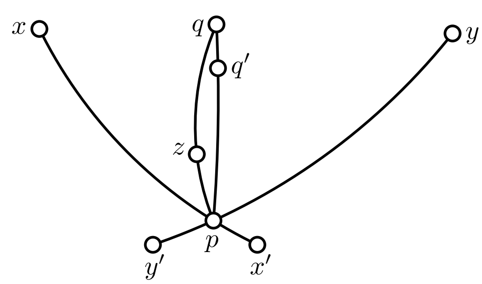
$$\left[\begin{aligned} &|p-q|^2_M - |z-q|^2_M \leqslant |p-x|^2_M - |z-x|^2_M,\\ &|p-q|^2_M - |z-q|^2_M \leqslant |p-y|^2_M - |z-y|^2_M. \end{aligned}\right.$$

holds for any $z \in M$.

Proof; (a). Assume the contrary; that is, there is $p \in M$ and $u, v \in \mathrm{TIL}_p$ such that $w = \frac{1}{2}\cdot(u+v) \notin \mathrm{TIL}_p$.

Let τ be the maximal value such that the geodesic $\gamma(t) = \exp_p(w\cdot t)$ is a length-minimizing on $[0,\tau]$. Set $w' = \tau\cdot w$. Note that $\tau < 1$ and $w' \in \partial\mathrm{TIL}_p$.

Set $q = \exp_p w'$. By general position argument, we can assume that there are at least two minimizing geodesics connecting p to q; see [4]. That is, there is $w'' \in \partial\mathrm{TIL}_p$ such that $w'' \neq w'$ and $\exp_p w' = \exp_p w''$.



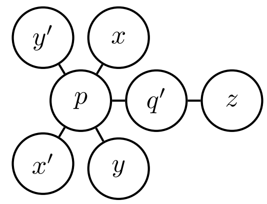
Fix small positive real numbers δ, ε and ζ. Consider the points

$$\begin{aligned} q' &= q'(\varepsilon) = \exp_p(1-\varepsilon)\cdot w', & z &= z(\zeta) = \exp_p(\zeta\cdot w''),\\ x &= \exp_p u, & x' &= x'(\delta) = \exp_p(-\delta\cdot u),\\ y &= \exp_p v, & y' &= y'(\delta) = \exp_p(-\delta\cdot v). \end{aligned}$$

We will show that for some choice of δ, ε and ζ the array p, x, x', y, y', q', z does not satisfy the T-tree comparison with the labeling as on the diagram.

Assume that given positive numbers δ, ε and ζ, there is a point array $\tilde p$, $\tilde x$, $\tilde x'(\delta)$, $\tilde y$, $\tilde y'(\delta)$, $\tilde q'(\varepsilon)$, $\tilde z(\zeta) \in \mathbb{H}$ as in the definition of T-tree comparison; that is, the distances between the points in this array are at least as big as the distances of corresponding points in M and the equality holds for the pair adjacent in T.

Since δ is small, we can assume that p lies on a necessary unique minimizing geodesic $[x, x']_M$. Hence



$$|x - x'|_M = |x - p|_M + |p - x'|_M.$$

By comparison

$$\begin{aligned}
|\tilde x - \tilde x'|_{\mathbb H} &\geqslant |x - x'|_M,\\
|\tilde x - \tilde p|_{\mathbb H} &= |x - p|_M,\\
|\tilde x' - \tilde p|_{\mathbb H} &= |x' - p|_M.
\end{aligned}$$

By triangle inequality,

$$|\tilde x - \tilde x'|_{\mathbb H} = |\tilde x - \tilde p|_{\mathbb H} + |\tilde x' - \tilde p|_{\mathbb H};$$

that is, $\tilde p \in [\tilde x, \tilde x']_{\mathbb H}$. The same way we see that $\tilde p \in [\tilde y, \tilde y']_{\mathbb H}$.

Fix ε and ζ. Note that as $\delta \to 0$ we have

$$\begin{aligned}
\tilde x' &\to \tilde p, & \tilde y' &\to \tilde p.\\
\measuredangle[\tilde p\,{}^{\tilde x'}_{\tilde y}] &\to \measuredangle[p\,{}^{x'}_{y}], & \measuredangle[\tilde p\,{}^{\tilde y'}_{\tilde x}] &\to \measuredangle[p\,{}^{y'}_{x}],\\
\measuredangle[\tilde p\,{}^{\tilde x'}_{\tilde q'}] &\to \measuredangle[p\,{}^{x'}_{q'}], & \measuredangle[\tilde p\,{}^{\tilde y'}_{\tilde q'}] &\to \measuredangle[p\,{}^{y'}_{q'}],
\end{aligned}$$

It follows that

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] \to \measuredangle[p\,{}^{x}_{y}], \qquad \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde q'}] \to \measuredangle[p\,{}^{x}_{q'}], \qquad \measuredangle[\tilde p\,{}^{\tilde y}_{\tilde q'}] \to \measuredangle[p\,{}^{y}_{q'}].$$

Therefore, passing to a partial limit as $\delta \to 0$, we get a configuration of 5 points $\tilde p, \tilde x, \tilde y, \tilde q' = \tilde q'(\varepsilon), \tilde z = \tilde z(\zeta)$ such that

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] = \measuredangle[p\,{}^{x}_{y}], \qquad \measuredangle[\tilde p\,{}^{\tilde y}_{\tilde q'}] = \measuredangle[p\,{}^{y}_{q'}], \qquad \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde q'}] = \measuredangle[p\,{}^{x}_{q'}].$$

In other words, the map sending 4 points $0, u, v, w' \in \mathrm{T}_p$ to $\tilde p, \tilde x, \tilde y, \tilde q \in \mathbb H$ correspondingly is distance preserving.

Note that $q' \to q$ as $\varepsilon \to 0$. Therefore, in the limit, we get a configuration $\tilde p$, $\tilde x$, $\tilde y$, $\tilde q'$, $\tilde z = \tilde z(\zeta)$ such that in addition we have

$$\begin{aligned}
|\tilde q' - \tilde z| &= |q - z|, & |\tilde p - \tilde z| &\geqslant |p - z|,\\
|\tilde x - \tilde z| &\geqslant |x - p|, & |\tilde y - \tilde z| &\geqslant |y - z|
\end{aligned}$$

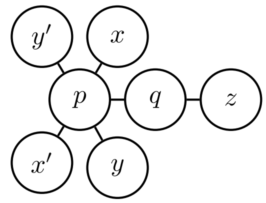
Since $w'' \neq w'$, for small values ζ the last three inequalities imply

$$|\tilde q' - \tilde z| > |q - z|,$$

a contradiction.

(b). Fix a hinge $[p\,{}^{x}_{y}]$ in M. By *(a)*, M is CTIL. Therefor $[p\,{}^{x}_{y}]$ has a median; denote it by $[p, q]$. For $\delta > 0$, define $x' = x'(\delta)$ and $y' = y'(\delta)$ as in *(a)*.

Without loss of generality we can assume that $x, y \in \exp_p(\mathrm{TIL}_p)$. If δ is small, the latter implies that p lies on unique minimizing geodesics $[x, x']$ and $[y, y']$.



Consider the limit case of $p/xx'yy'(q/z)$-comparison as $\delta \to 0$. It gives a configuration of 5 points $\tilde p$, $\tilde q$, $\tilde x$, $\tilde y$ and $\tilde z$ such that

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] = \measuredangle[p\,{}^{x}_{y}],$$

$\tilde q$ is the midpoint of $[\tilde x, \tilde y]$. In particular,

$$\begin{aligned}
2{\cdot}|\tilde z - \tilde q|^2_{\mathbb H} + |\tilde q - \tilde x|^2_{\mathbb H} + |\tilde q - \tilde y|^2_{\mathbb H} &= |\tilde z - \tilde x|^2_{\mathbb H} + |\tilde z - \tilde y|^2_{\mathbb H},\\
2{\cdot}|\tilde p - \tilde q|^2_{\mathbb H} + |\tilde q - \tilde x|^2_{\mathbb H} + |\tilde q - \tilde y|^2_{\mathbb H} &= |\tilde p - \tilde x|^2_{\mathbb H} + |\tilde p - \tilde y|^2_{\mathbb H},
\end{aligned}$$

By the comparison,

$$\begin{aligned}
|\tilde z - \tilde x|_{\mathbb H} &\geqslant |z - x|_M, & |\tilde z - \tilde y|_{\mathbb H} &\geqslant |z - y|_M,\\
|\tilde p - \tilde x|_{\mathbb H} &\geqslant |p - x|_M, & |\tilde p - \tilde y|_{\mathbb H} &\geqslant |p - y|_M,\\
|\tilde q - \tilde x|_{\mathbb H} &= |q - x|_M, & |\tilde q - \tilde y|_{\mathbb H} &= |q - y|_M,\\
|\tilde q - \tilde z|_{\mathbb H} &= |q - z|_M, & |\tilde q - \tilde p|_{\mathbb H} &= |q - p|_M,
\end{aligned}$$

Therefore

$$\begin{aligned}
2{\cdot}|z - q|^2_M + |q - x|^2_M + |q - y|^2_M &\geqslant |z - x|^2_M + |z - y|^2_M,\\
2{\cdot}|p - q|^2_M + |q - x|^2_M + |q - y|^2_M &\leqslant |p - x|^2_M + |p - y|^2_M.
\end{aligned}$$

Hence the condition in 5.2 follows. □

6 Multipolar comparison

Recall that a map $f\colon W \to X$ between metric spaces is called *submetry* if for any $w \in W$ and $r \geqslant 0$, we have

$$f[B(w, r)_W] = B(f(w), r)_X,$$

where $B(w, r)_W$ denotes the ball with center w and radius r in the space W. In other words submetry is a map which is 1-Lipschitz and 1-co-Lipschitz at the same time. Note that any submetry is onto.

6.1. Theorem. *A separable metric space X satisfies all tree comparison if and only if X is isometric to a target space of submetry defined of a subset of the Hilbert space.*

Proof. The "if" part is left as an exercise; let us prove the "only if" part.

Fix a point array $a_1, \dots, a_n$ in X. Consider the complete graph K_n with $\{1, \dots, n\}$ as the set of vertexes.

Let $\tilde{K}_n \to K_n$ be the universal covering of the complete graph K_n. Denote by $\tilde{V}$ the set of vertexes of $\tilde{K}_n$; given a vetex $\tilde{v} \in \tilde{V}$ denote by v the corresponding vertex of K_n.

By multipolar comparison, we have the following:

(*) There is a map $f\colon \tilde{V} \to \mathbb{H}$ such that

$$|f(\tilde{v}) - f(\tilde{w})|_{\mathbb{H}} \geqslant |a_v - a_w|_X$$

for any two vertexes $\tilde{v}, \tilde{w} \in \tilde{V}$ and the equality holds if $(\tilde{v}, \tilde{w})$ is an edge in $\tilde{K}_n$.

This finish the proof if X is finite.

Since X is separable, it contains a countable everywhere dense set $\{a_1, a_2, \dots\}$. Applying the statement above for $X_n = \{a_1, \dots a_n\}$, we get a submetry from $Y_n = f_n(\tilde{V}_n) \subset \mathbb{H}$ to X_n.

It remains to pass to the ulralimit Y of the subspaces Y_n. Clearly Y admits an isometric embedding into $\mathbb{H}$ and a submetry on X. Hence the statement follows. $\square$

6.2. Proposition. *Suppose G be a compct Lie group with bi-invariant metric, so the action $G \times G \curvearrowright G$ defined by $(h_1, h_2)\cdot g = h_1 \cdot g \cdot h_2^{-1}$ is isometric. Then for any closed subgroup $H < G \times G$, the bi-quotient space $G /\!\!/ H$ satisfies multipolar comparison.*

As a result we have many examples of spaces satisfying all tree comparison; for example, since $\mathbb{S}^n = \mathrm{SO}(n)/\mathrm{SO}(n-1)$, any round sphere satisfies multipolar comparison.

We present a proof suggested by Alexander Lytchak, it is simplified vesrion of the construction of Chuu-Lian Terng and Gudlaugur Thorbergsson given in [9, Section 4].

Proof. Denote by G^n the direct product of n copies of G. Consider the map $\varphi_n\colon G^n \to G$ defined by

$$\varphi_n\colon (\alpha_1, \dots, \alpha_n) \mapsto \alpha_1 \cdots \alpha_n.$$

Note that φ_n is a quotient map for the $H \times G^{n-1}$-action on G^n defined by

$$(\beta_0, \dots, \beta_n)\cdot(\alpha_1, \dots, \alpha_n) = (\gamma_1 \cdot \alpha_1 \cdot \beta_1^{-1}, \beta_1 \cdot \alpha_2 \cdot \beta_2^{-1}, \dots, \beta_{n-1} \cdot \alpha_n \cdot \beta_n^{-1}),$$

where $\beta_i \in G$ and $(\beta_0, \beta_n) \in H < G \times G$.

Denote by ρ_n the product metric on G^n rescaled with factor $\sqrt{n}$. Note that the quotient $(G^n, \rho_n)/(H \times G^{n-1})$ is isometric to $G /\!\!/ H = (G, \rho_1) /\!\!/ H$.

As $n \to \infty$ the curvature of (G^n, ρ_n) converges to zero and its injectivity radius goes to infinity. Therefore passing to the ultra-limit of G^n as $n \to \infty$ we get the Hilbert space. It remains to observe that the limit action has the required property. $\square$

7 Final remarks

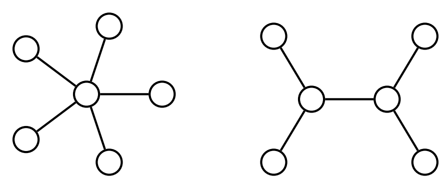
The following problem discussed in [1, 7.1] was one of the original motivations to study the tree comparison.

7.1. Problem. *Which finite metric spaces admit isometric embeddings into Alexandrov spaces with nonnegative curvature.*

The problem is still open. According to [1, 4.1], the $(n-1)$-tree comparison provides a necessary condition for the problem n-point metric spaces. This condition is sufficient for the 4-point metric spaces. This condition might be still sufficient for 5-point metric spaces. For 6-point metric spaces it is not sufficient.

The corresponding example of 6-point metric space was constructed by Sergei Ivanov, see [1]. Theorem 4.1, provides a source for such examples — any 6-point metric space which satisfy all 5-tree comparisons, but does not satisfy 2(2)-tree comparison provide an example. This class of examples includes the example of Sergei Ivanov — in the notations of [1, 7.1] it does not satisfies the comparison for the tree $y/az(q/xb)$.

We expect that the 5-tree and 2(2)-tree comparisons (these are the comparisons for the trees on the diagram) are sufficient for 6-point metric spaces.



Here an other candidate for a sufficient condition.

7.2. Question. *Assume F is a finite metric space which satisfies all tree comparisons. Is it true that F is isometric to a subset of Alexandrov space with nonpositive curvature?*

Note that even for finite metric space the all tree comparison has to be checked for infinite set of trees since one point of the space may may be used as a label for several vertexes in the tree.

There is a chance that for 5-point and 6-point metric spaces, this condition is also necessary. According to Proposition 5.1, this is not longer true for 7-point metric spaces.

For any metric space X with an isometric group action $G \curvearrowright X$ with closed orbits the quotient map $X \to X/G$ is a submetry. In particular, by Theorem 6.1, if $G \curvearrowright \mathbb{H}$ is an isometric action with closed orbits on the Hilbert space, then the quotient space $\mathbb{H}/G$ satisfies all tree comparisons.

7.3. Question. *Assume X is a metric space satisfying all tree comparisons. Is it always possible to construct an isometric group action with closed orbits on the Hilbert space $G \curvearrowright \mathbb{H}$ such that X is isometric to a subset in $\mathbb{H}/G$?*

Assume M is nonnegatively curved Riemannian manifold and $G \curvearrowright M$ is an isometric action with closed orbits. Then the quotient space $A = M/G$ is an Alexandrov space. It is easy to see that all singular points of A form an extremal set.

7.4. Question. *Is it true that the same holds for all Alexandrov spaces which satisfy 4(1)-tree comparison (or all dipolar tree comparisons)?*

If instead of the Hilbert space $\mathbb{H}$ we use infinite dimensional sphere or infinite dimensional hyperbolic space we will arrive to spherical and hyperbolic tree comparisons. Theorem 4.1 admits a straightforward generalization to these analogs of tree comparisons.

The spherical 4(1)-tree comparison implies the strict version of the condition MTW+CTIL and therefore implies CTP.

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