Tree comparison

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Abstract

We introduce a new type of metric comparison which is closely related to the Alexandrov comparison and continuity of optimal transport between regular measures.

1 Introduction

We will denote by $|a-b|_X$ the distance between points a and b in the metric space X.

Tree comparison. Fix a tree T with n vertexes.

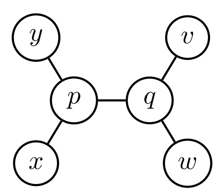
Let $(a_1,\dots a_n)$ be a point array in a metric space X labeled by the vertexes of T. We say that $(a_1,\dots a_n)$ satisfies the *T-tree comparison* if there is a point array $(\tilde a_1,\dots,\tilde a_n)$ in the Hilbert space $\mathbb{H}$ such that

$$|\tilde a_i-\tilde a_j|_{\mathbb{H}}\geqslant|a_i-a_j|_X$$

for any i and j and the equality holds if a_i and a_j are adjacent in T.

We say that a metric space X satisfies the *T-tree comparison* if every n-points arrays in X satisfies the T-tree comparison.

Instead of the Hilbert space $\mathbb{H}$ we use infinite dimensional sphere or infinite dimensional hyperbolic space. In this case we will arrive to spherical and hyperbolic tree comparisons.



Encoding of trees. To encode the labeled tree on the diagram, we will use notation $p/xy(q/vw)$. It means that we choose p as the root; p has two children leafs to x, y and one child q with two children leafs v and w. Taking another root for the same tree, we get different encodings, for eaxample $q/vw(p/xy)$ or $x/(p/y(q/vw))$.

If we do not need the labeling of vertexes, it is sufficient to write the number of leafs in the brackets; this way we can write $2(2)$ instead of $p/xy(q/vw)$ since the root (p) has 2 leafs (x and y) and yet another child (q) has 2 leafs (v and w). The same tree can be written as $(1(2))$ meaning that the root x has no leafs, p has 1 leaf y and one child q with 2 leafs v and w. Every vertex which is not the root and not a leaf corresponds to a pair of brackets in this notation.

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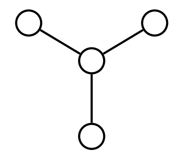
Using the described notation, we could say that a metric space *satisfies the 2(2)-tree comparison*, meaning that it satisfies the tree comparison on the diagram. We could also say *"applying the $p/xy(q/vw)$-tree comparison..."* meaning that we apply the comparison for these 6 points and the tree on the diagram.

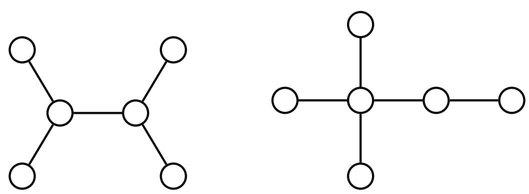
Monopolar trees. A vertex of a tree of degree at least two will be called *pole*.



Recall that *Alexandrov space* is defined as complete length space with curvature bounded below in the sense of Alexandrov; the latter is equivalent to the 3-tree comparison; that is, the comparison for the tripod-tree on the diagram.

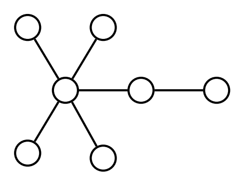
Using the introduced notation, a theorem in [1] can be restated the following way: *If length-metric space satisfies* 3*-tree comparison, then it also satisfies n-tree comparison for every positive integer n; in other words it satisfies all monopolar tree comparisons.*

Bipolar trees. The simplest bipolar trees are 3(1) and 2(2) shown on the diagram.



The following theorem staates that these two comparisons also follow form Alexandrov's comparison.

1.1. Theorem. *Any Alexandrov space with nonnegative curvature satisfies 3(1)-tree and 2(2)-tree comparisons.*



The next bipolar tree comparison is for the tree 4(1). It turns out to be related to the so called *continuity of transport property*, briefly CTP. A compact Riemannian manifold M is CTP if for any two regular measures with density functions bounded away from zero and infinity, the optimal transport can be described by a continuous map.

A necessary condition for CTP was given by Xi-Nan Ma, Neil Trudinger and Xu-Jia Wang in [7]. A key step in the understanding this condition was made by Grégoire Loeper in [6]. The manifolds satisfying this condition will be called *cost-convex*. We define it in the next section.

1.2. Theorem. *If a Riemannian manifold satisfies 4(1)-tree comparison then it is cost-convex; moreover it satisfies all bipolar tree comparisons.*

It is straightforward to check that the spherical 4(1)-tree comparison implies the strict cost-convexity, which in turns implies CTP.

All tree comparisons. Finally we consider spaces satisfying *all tree comparisons.*

Recall that a map $f\colon W \to X$ between metric spaces is called *submetry* if for any $w \in W$ and $r \geqslant 0$, we have

$$f[B(w,r)_W] = B(f(w),r)_X,$$

where $B(w,r)_W$ denotes the ball with center w and radius r in the space W. In other words submetry is a map which is 1-Lipschitz and 1-co-Lipschitz at the same time. Note that by the definition, any submetry is onto.

1.3. Theorem. *A separable metric space X satisfies all tree comparison if and only if X is isometric to a target space of submetry defined of a subset of the Hilbert space.*

The following proposition provides a source of examples of spaces satisfying all tree comparisons. For example, since $\mathbb{S}^n = \mathrm{SO}(n)/\mathrm{SO}(n-1)$, any round sphere has this property.

1.4. Proposition. *Suppose G is a compact Lie group with bi-invariant metric, so the action $G \times G \curvearrowright G$ defined by $(h_1,h_2)\cdot g = h_1\cdot g\cdot h_2^{-1}$ is isometric. Then for any closed subgroup $H < G \times G$, the bi-quotient space $G/\!\!/H$ satisfies all tree comparisons.*

2 On cost-convex manifolds

In this section we review necessary material from ???.

Cost-convex functions. Let M be a Riemannian manifold. Consider the *cost function* $\mathrm{cost}\colon M \times M \to \mathbb{R}$ defined by

$$\mathrm{cost}(x,y) = \tfrac{1}{2}\cdot|x-y|^2_M.$$

A function $f\colon M \to (-\infty,\infty]$ is called *cost-convex* if there is a nonempty subset of pairs $\mathcal{I} \subset M \times \mathbb{R}$ such that

$$f(x) = \sup\left\{\, r - \mathrm{cost}(p,x) \mid (p,r) \in \mathcal{I} \,\right\}.$$

If M has nonnegative curvature then any cost-convex function f is (-1)-convex; that is, $f'' \geqslant -1$, which means that the inequality

$$(f\circ\gamma)'' \geqslant -1 \qquad \text{❶}$$

holds in the barrier sense for any unit speed geodesic γ. On the other hand, the inequality ❶ does not imply that f is cost-convex.

Subgradient. Let $f\colon M \to (-\infty,\infty]$ be a semiconvex function defined on Riemannian manifold M. Assume $f(p)$ is finite. In this case the differential

$$d_pf\colon \mathrm{T}_p \to (-\infty,\infty]$$

is defined; it is a convex positive homogeneous function defined on the tangent space T_p.

A tangent vector $v \in \mathrm{T}_p M$ is a *subgradient* of f at p, briefly $v \in \underline{\nabla}_p f$ if

$$\langle v, w \rangle \leqslant d_p f(w)$$

for any $w \in \mathrm{T}_p$. Note that the set $\underline{\nabla}_p f$ is a convex subset of T_p.

The subset of tangent vectors $v \in \mathrm{T}_p$ such that there is a minimizing geodesic $[p, q]$ in the direction of v with length $|v|$ will be denoted as $\overline{\mathrm{TIL}}_p$. For p, q and v as above, we write $q = \exp_p v$.

2.1. Definition. *A Riemannian manifold will be called* cost-convex *if for any point p and any cost-convex function f which is finite at p we have $\underline{\nabla}_p f \subset \overline{\mathrm{TIL}}_p$ and for $q = \exp_p v$ the inequality*

$$\mathrm{cost}(q, p) - \mathrm{cost}(q, x) \geqslant f(x) - f(p)$$

holds for any $x \in M$.

It is easy to see that any cost-convex maniflold is nonnegatively curved. According to ???, if M is CTP then it is cost-convex. The converse is unknown, likely it holds, but there is a slightly stronger version of this condition which implies CTP.

CTIL and MTW. Let us formulate other two conditions on Riemannian manifolds which are together equivalent to the cost-convexity. These two conditions will be used in the proof.

Let M be a Riemannian manifold. The *tangent injectivity locus* at the point $p \in M$ (briefly TIL_p) is defined as the maximal open subset in the tangent space T_p such that for any $v \in \mathrm{TIL}_p$ the geodesic path $\gamma(t) = \exp_p(v \cdot t)$, $t \in [0, 1]$ is a minimizing. If the tangent injectivity locus at any point $p \in M$ is convex we say that M satisfies *convexity of tangent injectivity locus* or briefly M is CTIL.

Note that $\overline{\mathrm{TIL}}_p$ defiend above is closure of TIL_p in T_p. Therefore, M is CTIL if and only if the set $\overline{\mathrm{TIL}}_p$ is convex for any point $p \in M$.

The second condition is called MTW for Ma–Trudinger–Wang. We will use its reformulation close to the one given by Cédric Villani [8, 2.6]; it can be proved the same way.

Assume $u, v \in \mathrm{T}_p$ and $w = \frac{1}{2} \cdot (u + v)$ and $x = \exp_p u$, $y = \exp_p v$ and $q = \exp_p w$. If the three geodesic paths $[p, x]$, $[p, y]$ and $[p, q]$ described by the paths $t \mapsto \exp_p(t \cdot u)$, $t \mapsto \exp_p(t \cdot v)$, $t \mapsto \exp_p(t \cdot w)$ for $t \in [0, 1]$ are minimizing, then $[p, q]$ is called *median* of the hinge $[p\,{}^{x}_{y}]$. Note that in a CTIL Riemannian manifold, any hinge has a median.

2.2. MTW condition. *Assume M be a CTIL Riemannian manifold. Then M is MTW if and only if for a median $[p, q]$ of any hinge $[p\,{}^{x}_{y}]$ one of the following inequalities*

$$\left[\begin{aligned} &|p - q|^2_M - |z - q|^2_M \leqslant |p - x|^2_M - |z - x|^2_M, \\ &|p - q|^2_M - |z - q|^2_M \leqslant |p - y|^2_M - |z - y|^2_M. \end{aligned}\right.$$

holds for any $z \in M$.

3 On Kirszbraun's rigidity

In the proof we will use the rigidity case of the generalized Kirszbraun theorem proved by Urs Lang and Viktor Schroeder in [5], see also [1].

3.1. Kirszbraun rigidity theorem. *Let A be an Alexandrov space with non-negative curvature.*

Assume that for two point arrays $p, x_1, \dots, x_n \in A$ and $\tilde q, \tilde x_1, \dots, \tilde x_n \in \mathbb{H}$ we have that

$$|\tilde q - \tilde x_i| \geqslant |p - x_i|$$

for any i,

$$|\tilde x_i - \tilde x_j| \leqslant |x_i - x_i|$$

for any pair (i, j) and $\tilde q$ lies in the interior of the convex hull $\tilde K$ of $\tilde x_1, \dots, \tilde x_n$.

Then equalities hold in all the inequalities above. Moreover there is an distance preserving map $f\colon \tilde K \to A$ such that $f(\tilde x_i) = x_i$ and $f(\tilde q) = p$.

Proof. By the generalized Kirszbraun theorem, there is a short map $f\colon A \to \mathbb{H}$ such that $f(x_i) = \tilde x_i$. Set $\tilde p = f(p)$. By assumptions

$$|\tilde q - \tilde x_i| \geqslant |\tilde p - \tilde x_i|.$$

Since $\tilde q$ lies in the interior of K, $\tilde q = \tilde p$. It follows that the equality

$$|\tilde q - \tilde x_i| = |p - x_i|.$$

holds for each i.

???

Consider the tangent vectors $v_i \in \mathrm{T}_p$ such that $\exp_p v_i = x_i$ for each i. Note that these vectors are uniquely defined, all the vectors lie in an isometric copy of a Euclidean space and

$$|v_i - v_j|_{\mathrm{T}_p} = |x_i - x_j|_A.$$

In particular, the convex hull of $\{v_1, \dots, v_n\}$ in T_p is isometric to $\tilde K$, so we can keep notation $\tilde K$ for this convex hull.

Consider the gradient exponent $\mathrm{gexp}_p\colon \mathrm{T}_p \to A$, see [2]. It is a short map, $\mathrm{gexp}_p 0 = p$ and $\mathrm{gexp}_p v_i = x_i$ for each i. It remains to show that the restriction $\mathrm{gexp}_p|_{\tilde K}$ is distance-preserving.

Extend the sequence $v_1, \dots v_n$ to an infinite sequence of vectors $v_i \in \tilde K$ which is dense in $\tilde K$. Set $x_i = \mathrm{gexp}_p v_i$ for each i.

Note that it is sufficient to show that the map $v_i \mapsto x_i$ is distance preserving.

From above,

$$|v_i - v_j|_{\mathrm{T}_p} = |x_i - x_j|_A$$

if $i, j \leqslant n$; it provides a base for induction. Assume

$$|v_i - v_j|_{\mathrm{T}_p} = |x_i - x_j|_A$$

for all pairs $i, j \leqslant k-1$. Since the gradient exponent is short,

$$|v_i - v_k|_{\mathrm{T}_p} \geqslant |x_i - x_k|_A$$

for each $i \leqslant k$. From the first part of theorem we have

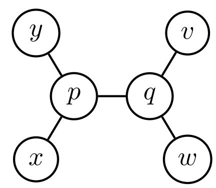
$$|v_i - v_k|_{\mathrm{T}_p} = |x_i - x_k|_A.$$

It proves the induction step and hence the second statement follows. $\square$

4 Pivotal trees.

Let X be a metric space. A point array $(a_1, \dots, a_n)$ in X together with a choice of a graph with n vertexes labeled by $(a_1, \dots, a_n)$ and a choice of geodesic $[a_i, a_j]$ for every adjacent pair (a_i, a_j) is called *geodesic graph.*

For geodesic trees we will use the same notation as for labeled combinatoric tree in square brackets; for example $[p, xy(q, vw)]$ will denote the geodesic tree with with combinatorics as on the diagram.



Fix a geodesic tree $T = [p_1/x_1 \dots x_k(p_2/x_{k+1} \dots x_n)]$; that is, T has two poles p_1, p_2 and each of the remaining vertexes are adjacent either to p_1 or p_2 — the vertexes $x_1, \dots, x_k$ are connected to p_1 and $x_{k+1}, \dots, x_n$ to p_2.

Assume X is a nonnegatively curved Alexandrov space; in particular the angle is defined for any geodesic hinge.

A geodesic tree $\tilde{T} = [\tilde{p}_1/\tilde{x}_1 \dots \tilde{x}_k(\tilde{p}_2/\tilde{x}_{k+1} \dots \tilde{x}_n)]$ in the Hilbert space $\mathbb{H}$ will be called *pivotal tree* for T if

(i) $|\tilde{p}_1 - \tilde{p}_2|_{\mathbb{H}} = |p_1 - p_2|_X$,

(ii) $|\tilde{p}_i - \tilde{x}_j|_{\mathbb{H}} = |p_i - p_j|_X$ for any edge $[p_i, x_j]$ in T and

(iii) $\measuredangle[\tilde{p}_j\,{}^{\tilde{x}_k}_{\tilde{p}_i}]_{\mathbb{H}} = \measuredangle[\tilde{p}_j\,{}^{\tilde{x}_k}_{\tilde{p}_i}]_X$ for any hinge $[p_j\,{}^{x_k}_{p_i}] = ([p_j, x_k], [p_j, p_i])$ in T.

4.1. Rigidity lemma. *Let X be a nonnegatively curved Alexandrov space and $T = [p_1/x_1 \dots x_k(p_2/x_{k+1} \dots x_n)]$ be geodesic tree in X Suppose $\tilde{T} = [\tilde{p}_1/\tilde{x}_1 \dots \tilde{x}_k(\tilde{p}_2/\tilde{x}_{k+1} \dots \tilde{x}_n)]$ is a pivotal tree for T. Assume that*

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}} \leqslant |x_i - x_j|_X \qquad ❷$$

for any pair (i, j) and the convex hull $\tilde{K}$ of $\{\tilde{x}_1, \dots \tilde{x}_n\}$ intersects the line thru $\tilde{p}_1$ and $\tilde{p}_2$. Then the equality holds in ❷ for each pair (i, j).

Proof. Let $\tilde{z}$ be a point on the line $(\tilde{p}_1, \tilde{p}_2)$. Assume that $\tilde{z}$ lies on the half-line from $\tilde{p}_1$ to $\tilde{p}_2$; otherwise swap the labels of $\tilde{p}_1$ and $\tilde{p}_2$.

Denote by ζ the direction of geodesic $[p_1, p_2]$ at p_1. Set

$$z = \operatorname{gexp}_{p_1}(|\tilde{z} - \tilde{p}_1| \cdot \zeta),$$

where $\operatorname{gexp}_{p_1}$ denotes the gradient exponent at p_1; see [2]. By comparison, we have

$$|x_i - z|_X \leqslant |\tilde{x}_i - \tilde{z}|_{\mathbb{R}^2}$$

for any i.

It remains to apply Kirszbraun rigidity theorem (3.1). □

Recall that X is a nonnegatively curved Alexandrov space.

Assume $[\tilde{p}_1/\tilde{x}_1 \dots \tilde{x}_k(\tilde{p}_2/\tilde{x}_{k+1} \dots \tilde{x}_n)]$ is a pivotal tree in $\mathbb{H}$ for the geodesic tree $[p_1/x_1 \dots x_k(p_2/x_{k+1} \dots x_n)]$ in X. Note that by angle comparison, for any i and j we have

$$|\tilde{x}_i - \tilde{p}_j|_{\mathbb{H}} \geqslant |x_i - p_j|_X.$$

It follows that the configuration $\tilde{p}_1,\ \tilde{p}_2,\ \tilde{x}_1, \dots \tilde{x}_n \in \mathbb{H}$ satisfies the tree comparison (see Section 1) if

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}} \geqslant |x_i - x_j|_X \qquad ❸$$

for all pairs (i, j).

Denote by $\tilde{\xi}_i$ the direction of the half-plane containing $\tilde{x}_i$ with the boundary line $(\tilde{p}_1, \tilde{p}_2)$. The direction $\tilde{\xi}_i$ lies in the unit sphere normal to the line $(\tilde{p}_1, \tilde{p}_2)$; we may assume that the dimension of the sphere is $n - 1$.

Note that up to a motion of $\mathbb{H}$, a pivotal configuration is completely described by the angles $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j)$. Moreover, the distance $|\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}}$ is determined by $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j)$ and the function $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j) \mapsto |\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}}$ is nondereasing.

Let us denote by $\alpha_{i,j}$ the minimal angle $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j)$ in a pivotal configuration such that ❸ holds. Note that the inequality ❸ is equivalent to

$$\measuredangle(\xi_i, \xi_j) \geqslant \alpha_{i,j}.$$

4.2. Corollary. *For any geodesic bipolar tree in a nonnegatively curved Alexandrov space the following conditions hold:*

(a) For any pair i and j, we have

$$\alpha_{i,j} \leqslant \pi.$$

(b) For any triple i, j and k, we have

$$\alpha_{i,j} + \alpha_{j,k} + \alpha_{k,i} \leqslant 2{\cdot}\pi.$$

In other words, if X is a nonnegatively curved Alexandrov space then

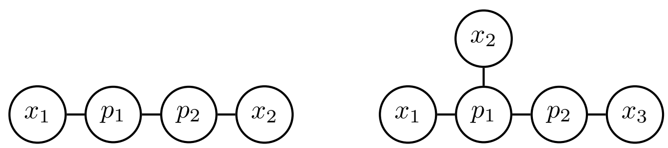
(a) For any broken geodesic line $[p_1/x_1(p_2/x_2)]$ in X there is a pivotal tree $[\tilde{p}_1/\tilde{x}_1(\tilde{p}_2/\tilde{x}_2)]$ such that

$$|\tilde{x}_1 - \tilde{x}_2|_{\mathbb{H}} \geqslant |x_1 - x_2|_X.$$

(b) For any $[p_1/x_1x_2(p_2/x_3)]$ in X, there is a pivotal tree $[\tilde p_1/\tilde x_1\tilde x_2(\tilde p_2/\tilde x_3)]$ such that

$$|\tilde x_i-\tilde x_j|_{\mathbb{H}}\geqslant|x_i-x_j|_X.$$

for all i and j.



Proof; (a). Consider the pivotal tree $[\tilde p_1/\tilde x_1(\tilde p_2/\tilde x_2)]$ (which is a polygonal path) with $\measuredangle(\tilde\xi_1,\tilde\xi_2)=\pi$. Note that the points $\tilde p_1,\tilde x_1,\tilde p_2,\tilde x_2$ are coplanar and the points $\tilde x_1$ and $\tilde x_2$ lie on the opposite sides from the line $(\tilde p_1,\tilde p_2)$. It remains to apply the rigidity lemma.

(b). By *(a)*, we can assume that

$$\alpha_{1,3}+\alpha_{2,3}>\pi. \tag{❹}$$

Consider the pivotal tree $[\tilde p_1/\tilde x_1\tilde x_2(\tilde p_2/\tilde x_3)]$ which lies in a 3-dimesional subspace in such a way that the points $\tilde x_1$ and $\tilde x_2$ lie on the opposite sides from the plane containing $\tilde p_1,\tilde p_2,\tilde x_3$, and

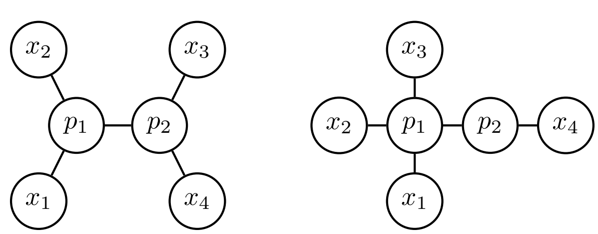
$$\measuredangle(\tilde\xi_1,\tilde\xi_3)=\alpha_{1,3},\qquad\qquad \measuredangle(\tilde\xi_2,\tilde\xi_3)=\alpha_{2,3}.$$

By ❹, the convex hull $\tilde K$ of $\{\tilde x_1,\tilde x_2,\tilde x_3\}$ intersects the line $(\tilde p_1,\tilde p_2)$. It remains to apply the rigidity lemma. □

Note that *(a)* and *(b)* implies that nonnegatively curved Alexandrov space satisfies 1(1)-tree and 2(1)-tree comparisons. However, 1(1)-tree comparison follows directly from the triangle inequality.

5 2(2)-tree and 3(1)-tree comparisons

The following theorem generalizes Theorem 1.1. It says in particular, that the comparisons for the following two trees holds in nonnegatively Alexandrov spaces.



5.1. Theorem. *Let X be an nonnegatively curved Alexandrov space. Then for any geodesic 2(2)-tree (or 3(1)-tree) there is a pivotal tree satisfying the corresponding tree comparison.*

In particular, any nonnegatively curved Alexandrov space satisfies the 2(2)-tree comparison as well as 3(1)-tree comparison.

The proofs in the two cases are nearly identical; they differ only by the choice made in the first line.

Proof. Fix a geodesic tree $[p_1/x_1x_2(p_2/x_3x_4)]$ or $[p_1/x_1x_2x_3(p_2/x_4)]$. Define the values $\{\alpha_{i,j}\}$ for each pair i, j as in the previous section.

Recall that p_1 and p_2 are the poles of the tree and each of remaining vertexes x_1, x_2, x_3, x_4 are connected to one of the poles.

Fix a smooth monotonic function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) = 0$ if $x \geqslant 0$ and $\varphi(x) > 0$ if $x < 0$. Consider a configuration of 4 points $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4$ in $\mathbb{S}^3$ which minimize the *energy*

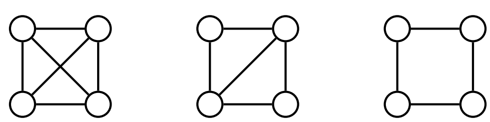
$$E(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4) = \sum_{i<j} \varphi(\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j) - \alpha_{i,j}).$$

Consider the geodesic graph Γ with 4 vertexes $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4$ in $\mathbb{S}^3$, where $\tilde{\xi}_i$ is adjacent to $\tilde{\xi}_j$ if $\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j) < \alpha_{i,j}$. If the comparison does not hold then Γ is not empty.

Note that a vertex of Γ can not lie in an open hemisphere with all its adjacent vertexes. Indeed, if it would be the case then we could move this vertex increasing the distances to all its adjacent vertexes. Along this move the energy decreases which is not possible.

Note that by Corollary 4.2, degree of any vertex is at least 2. Indeed existence of a vertex of degree 1 contradicts 4.2a and existence of a vertex of degree 0 contradicts 4.2b.

Therefore the graph Γ is isomorphic to one the following three graphs.



The 6-edege case (that is, the complete graph with 4 vertexes) can not appear by the rigidity lemma (see 4.1).

To do the remaining two cases, note that since the energy is minimal, the angle between the edges at every vertex of degree 2 of Γ has to be π. That is, the pair of edges at such vertex forms a geodesic.

Consider the 5-edge graph on the diagram. By the observation above the both triangles in the graph run along one equator. The latter contradicts Corollary 4.2b.

For the 4-edge graph (that is, for 4-cycle) by the same observation we have 4 points lie on the equator; moving the even pair to the north pole and odd pair to the south pole will decrease the energy, a contradiction. □

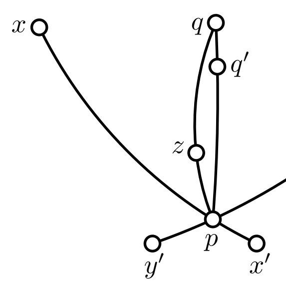
6 4(1)-tree comparison

In this section we will prove Theorem 1.2. Note that according to ???, it is sufficient to show that if a Riemannian manifold M satisfies 4(1)-tree comparison then it is CTIL and MTW. These two conditions will be proved separately.

Proof of CTIL. Assume the contrary; that is, there is $p \in M$ and $u, v \in \mathrm{TIL}_p$ such that $w = \frac{1}{2}\cdot(u+v) \notin \mathrm{TIL}_p$.

Let τ be the maximal value such that the geodesic $\gamma(t) = \exp_p(w\cdot t)$ is a length-minimizing on $[0,\tau]$. Set $w' = \tau\cdot w$. Note that $\tau < 1$ and $w' \in \partial\mathrm{TIL}_p$.

Set $q = \exp_p w'$. By general position argument, we can assume that there are at least two minimizing geodesics connecting p to q; see [4]. That is, there is $w'' \in \partial\mathrm{TIL}_p$ such that $w'' \neq w'$ and $\exp_p w' = \exp_p w''$.



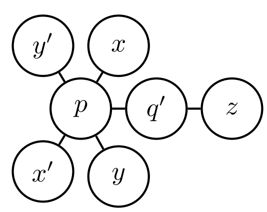
Fix small positive real numbers δ, ε and ζ. Consider the points

$$\begin{aligned} q' = q'(\varepsilon) &= \exp_p(1-\varepsilon)\cdot w', & z = z(\zeta) &= \exp_p(\zeta\cdot w''),\\ x &= \exp_p u, & x' = x'(\delta) &= \exp_p(-\delta\cdot u),\\ y &= \exp_p v, & y' = y'(\delta) &= \exp_p(-\delta\cdot v). \end{aligned}$$

We will show that for some choice of δ, ε and ζ the $p/xx'yy'(q'/z)$-tree comparison does not hold.

Assume contrary, that is, given any small positive numbers δ, ε and ζ, there is a point array $\tilde p$, $\tilde x$, $\tilde x'(\delta)$, $\tilde y$, $\tilde y'(\delta)$, $\tilde q'(\varepsilon)$, $\tilde z(\zeta) \in \mathbb{H}$ as in the definition of T-tree comparison; that is, the distances between the points in this array are at least as big as the distances of corresponding points in M and the equality holds for the pair adjacent in $p/xx'yy'(q'/z)$.

Since δ is small, we can assume that p lies on a necessary unique minimizing geodesic $[x,x']_M$. Hence



$$|x - x'|_M = |x - p|_M + |p - x'|_M.$$

By comparison

$$\begin{aligned} |\tilde x - \tilde x'|_{\mathbb{H}} &\geqslant |x - x'|_M,\\ |\tilde x - \tilde p|_{\mathbb{H}} &= |x - p|_M,\\ |\tilde x' - \tilde p|_{\mathbb{H}} &= |x' - p|_M. \end{aligned}$$

By triangle inequality,

$$|\tilde{x}-\tilde{x}'|_{\mathbb{H}} = |\tilde{x}-\tilde{p}|_{\mathbb{H}} + |\tilde{x}'-\tilde{p}|_{\mathbb{H}};$$

that is, $\tilde{p}\in[\tilde{x},\tilde{x}']_{\mathbb{H}}$. The same way we see that $\tilde{p}\in[\tilde{y},\tilde{y}']_{\mathbb{H}}$.

Fix ε and ζ. Note that as $\delta\to 0$ we have

$$\begin{aligned}
&\tilde{x}'\to\tilde{p}, && \tilde{y}'\to\tilde{p}.\\
&\measuredangle[\tilde{p}\,{}^{\tilde{x}'}_{\tilde{y}}]\to\measuredangle[p\,{}^{x'}_{y}], && \measuredangle[\tilde{p}\,{}^{\tilde{y}'}_{\tilde{x}}]\to\measuredangle[p\,{}^{y'}_{x}],\\
&\measuredangle[\tilde{p}\,{}^{\tilde{x}'}_{\tilde{q}'}]\to\measuredangle[p\,{}^{x'}_{q'}], && \measuredangle[\tilde{p}\,{}^{\tilde{y}'}_{\tilde{q}'}]\to\measuredangle[p\,{}^{y'}_{q'}],
\end{aligned}$$

It follows that

$$\measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{y}}]\to\measuredangle[p\,{}^{x}_{y}],\qquad \measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{q}'}]\to\measuredangle[p\,{}^{x}_{q'}],\qquad \measuredangle[\tilde{p}\,{}^{\tilde{y}}_{\tilde{q}'}]\to\measuredangle[p\,{}^{y}_{q'}].$$

Therefore, passing to a partial limit as $\delta\to 0$, we get a configuration of 5 points $\tilde{p},\tilde{x},\tilde{y},\tilde{q}'=\tilde{q}'(\varepsilon),\tilde{z}=\tilde{z}(\zeta)$ such that

$$\measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{y}}]=\measuredangle[p\,{}^{x}_{y}],\qquad \measuredangle[\tilde{p}\,{}^{\tilde{y}}_{\tilde{q}'}]=\measuredangle[p\,{}^{y}_{q'}],\qquad \measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{q}'}]=\measuredangle[p\,{}^{x}_{q'}].$$

In other words, the map sending 4 points $0,u,v,w'\in\mathrm{T}_p$ to $\tilde{p},\tilde{x},\tilde{y},\tilde{q}\in\mathbb{H}$ correspondingly is distance preserving.

Note that $q'\to q$ as $\varepsilon\to 0$. Therefore, in the limit, we get a configuration $\tilde{p}$, $\tilde{x}$, $\tilde{y}$, $\tilde{q}'$, $\tilde{z}=\tilde{z}(\zeta)$ such that in addition we have

$$\begin{aligned}
|\tilde{q}'-\tilde{z}| &= |q-z|, & |\tilde{p}-\tilde{z}| &\geqslant |p-z|,\\
|\tilde{x}-\tilde{z}| &\geqslant |x-p|, & |\tilde{y}-\tilde{z}| &\geqslant |y-z|
\end{aligned}$$

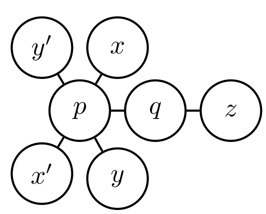
Since $w''\neq w'$, for small values ζ the last three inequalities imply

$$|\tilde{q}'-\tilde{z}|>|q-z|,$$

a contradiction.

Proof of MTW. Fix a hinge $[p\,{}^{x}_{y}]$ in M; from above M is CTIL. Therefore $[p\,{}^{x}_{y}]$ has a median; denote it by $[p,q]$. For $\delta>0$, define $x'=x'(\delta)$ and $y'=y'(\delta)$ as above.

Without loss of generality we can assume that $x,y\in\exp_p(\mathrm{TIL}_p)$. If δ is small, the latter implies that p lies on unique minimizing geodesics $[x,x']$ and $[y,y']$.



Consider the limit case of $p/xx'yy'(q/z)$-comparison as $\delta\to 0$. It gives a configuration of 5 points $\tilde{p}$, $\tilde{q}$, $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ such that

$$\measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{y}}]=\measuredangle[p\,{}^{x}_{y}],$$

$\tilde q$ is the midpoint of $[\tilde x,\tilde y]$. In particular,

$$2{\cdot}|\tilde z-\tilde q|^2_{\mathbb H}+|\tilde q-\tilde x|^2_{\mathbb H}+|\tilde q-\tilde y|^2_{\mathbb H}=|\tilde z-\tilde x|^2_{\mathbb H}+|\tilde z-\tilde y|^2_{\mathbb H},$$
$$2{\cdot}|\tilde p-\tilde q|^2_{\mathbb H}+|\tilde q-\tilde x|^2_{\mathbb H}+|\tilde q-\tilde y|^2_{\mathbb H}=|\tilde p-\tilde x|^2_{\mathbb H}+|\tilde p-\tilde y|^2_{\mathbb H},$$

By the comparison,

$$\begin{aligned}
|\tilde z-\tilde x|_{\mathbb H}&\geqslant|z-x|_M, & |\tilde z-\tilde y|_{\mathbb H}&\geqslant|z-y|_M,\\
|\tilde p-\tilde x|_{\mathbb H}&\geqslant|p-x|_M, & |\tilde p-\tilde y|_{\mathbb H}&\geqslant|p-y|_M,\\
|\tilde q-\tilde x|_{\mathbb H}&=|q-x|_M, & |\tilde q-\tilde y|_{\mathbb H}&=|q-y|_M,\\
|\tilde q-\tilde z|_{\mathbb H}&=|q-z|_M, & |\tilde q-\tilde p|_{\mathbb H}&=|q-p|_M,
\end{aligned}$$

Therefore

$$2{\cdot}|z-q|^2_M+|q-x|^2_M+|q-y|^2_M\geqslant|z-x|^2_M+|z-y|^2_M,$$
$$2{\cdot}|p-q|^2_M+|q-x|^2_M+|q-y|^2_M\leqslant|p-x|^2_M+|p-y|^2_M.$$

Hence the condition in 2.2 follows. □

7 Pull-back convexity

7.1. Proposition. *If a Riemannian manifold M satisfies 4(1)-tree comparison then*

(a) The set TIL_p is convex in the tangent space T_p.

(b) The restriction

$$f_{q,p}=\tfrac12\cdot\mathrm{dist}^2_q\circ\exp_p|_{\mathrm{TIL}_p}$$

is 1-concave.

Converse also holds. Moreover the conditions (a) and (b) implies that M satisfies all bipolar trees comparisons; that is the $m(n)$-tree comparison for any m and n.

Recall that for a continuous function f defined on metric space we write $f''\leqslant\lambda$ if for any unit-speed geodesic γ the function

$$t\mapsto f\circ\gamma(t)-\lambda{\cdot}\tfrac{t^2}{2}$$

is a concave real-to-real function.

Proof. It is sufficient to prove in the case $n=4$. Assume M satisfies 4(1)-tree comparison.

Since 4(1)-tree comparison implies 3-tree comparison, M has nonnegative sectional curvature.

Fix $u,v\in\mathrm{TIL}_p$ and $w\in[u,v]$. It is sufficient to show that if g is the function on $[u,v]$ on the segment which is uniquely defined by conditions: $g''=1$, $g(u)=f(u)$ and $g(w)=f(w)$ we will have that $g(v)\geqslant f(v)$.

Fix small $\varepsilon > 0$ and set

$$\begin{aligned} x &= \exp_p u, & y &= \exp_p v, & z &= \exp_p w, \\ x' &= \exp_p(-\varepsilon{\cdot}u), & y' &= \exp_p(-\varepsilon{\cdot}v) \end{aligned}$$

Let us apply $p/xyx'y'(z/q)$ comparison and pass to the limit as $\varepsilon \to 0$. We obtain points $\tilde{p}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{q} \in \mathbb{H}$, satisfying corresponding comparisons and such that

$$\measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{y}}] = \measuredangle[p\,{}^{x}_{y}], \qquad \measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{z}}] = \measuredangle[p\,{}^{x}_{z}], \qquad \measuredangle[\tilde{p}\,{}^{\tilde{z}}_{\tilde{y}}] = \measuredangle[p\,{}^{z}_{y}].$$

In particular, from above and Toponogov comparison, we have

$$\begin{aligned} |\tilde{x}-\tilde{y}|_{\mathbb{H}} &= |u-v|_{\mathrm{T}_p}, & |\tilde{z}-\tilde{y}|_{\mathbb{H}} &= |w-v|_{\mathrm{T}_p}, & |\tilde{x}-\tilde{z}|_{\mathbb{H}} &= |u-w|_{\mathrm{T}_p}, \\ |\tilde{q}-\tilde{z}|_{\mathbb{H}} &= |q-z|_{M}, & |\tilde{q}-\tilde{x}|_{\mathbb{H}} &\geqslant |q-x|_{M}, & |\tilde{q}-\tilde{y}|_{\mathbb{H}} &\geqslant |q-y|_{M}. \end{aligned}$$

Let $q^* \in \mathbb{H}$ be the point such that

$$|q^*-\tilde{z}| = |q-z|, \qquad |q^*-\tilde{x}| = |q-x|$$

Let φ be an isometry between segments in T_pM and $\mathbb{H}$ such that $\varphi(u) = \tilde{x}$, $\varphi(v) = \tilde{y}$ and $\varphi(w) = \tilde{z}$.

Then we can take $g := \mathrm{dist}^2_{q^*} \circ \varphi$. We have, that $g'' = 1$ and

$$g(u) = |q^*-\tilde{x}|^2 = |q-x|^2 = f(u), \quad g(w) = |q^*-\tilde{z}|^2 = |q-z|^2 = f(w)$$

$$g(v) = |q^*-\tilde{y}|^2 = |q-y|^2 \geqslant f(u).$$

Converse. Now assume M satisfies the conditions (a) and (b).

Fix points p and q in M; set $\tilde{q} = \log_p q \in \mathrm{T}_p$. Note that condition (b) implies that

$$f \leqslant \tilde{f}, \tag{5}$$

where

$$f = \tfrac{1}{2}\cdot \mathrm{dist}^2_q \circ \exp_p|_{\mathrm{TIL}_p} \quad \text{and} \quad \tilde{f} = \tfrac{1}{2}\cdot \mathrm{dist}^2_{\tilde{q}}.$$

Further note that the inequality ❺ is equivalent to the Toponogov comparison for all hinges $[p\,{}^{x}_{q}]$ in M. It follows that M has nonnegative sectional curvature.

Now fix a geodesic tree $[p/x_1 \dots x_n(q/y_1 \dots y_m)]$ in M. Set

$$\tilde{p} = 0 = \log_p p, \quad \tilde{q} = \log_p q, \quad \text{and} \quad \tilde{x}_i = \log_p x_i$$

for each i.

Since sectional curvature of M is nonnegative, the restriction $\exp_p|_{\mathrm{TIL}_p}$ is short. In particular, the linear map $\mathrm{T}_q \to \mathrm{T}_p$ defined by $\nabla_q h \mapsto \nabla_{\tilde{q}}(h \circ \exp_p)$ is short. Hence there is a linear map $z\colon \mathrm{T}_q \to \mathrm{T}_p$ such that, the map $\zeta\colon \mathrm{T}_q \to \mathrm{T}_p \oplus \mathrm{T}_p$ defined by

$$\zeta\colon \nabla_q h \mapsto \nabla_{\tilde{q}}(h \circ \exp_p) \oplus z(\nabla_q h)$$

is distance preserving.

Further set

$$h_i = \tfrac{1}{2}\cdot \operatorname{dist}_{y_i}^2 \qquad f_i = h_i \circ \exp_p|_{\mathrm{TIL}_p} \qquad \tilde y_i = \tilde q - \zeta(\nabla_q h_i).$$

By construction

$$|\tilde y_i - \tilde q|_{\mathrm{T}_p\oplus \mathrm{T}_p} = |y_i - q|_M.$$

At the point $\tilde q$ the restriction functions $\tilde f_i = \tfrac{1}{2}\cdot \operatorname{dist}_{\tilde w}^2|_{\mathrm{T}_p}$ and the function f_i have the same value and gradient. Since $f_i'' \leqslant 1$ and $\tilde f_i'' = 1$, we get $\tilde f_i \geqslant f_i$. The latter implies

$$|\tilde y_i - \tilde p|_{\mathrm{T}_p\oplus \mathrm{T}_p} \geqslant |y_i - p|_M \quad \text{and} \quad |\tilde y_i - \tilde x_j|_{\mathrm{T}_p\oplus \mathrm{T}_p} \geqslant |y_i - x_j|_M.$$

for any i and j.

Since $\mathrm{T}_p\oplus \mathrm{T}_p$ admits an isometric embedding into the Hilbert space, we get the needed configuration. □

The proof above is generalizable to complete length spaces therefore we get the following corollary.

7.2. Corollary. *If a complete length space X satisfies $4(1)$-tree comparison then it satisfies any bipolar tree comparisons.*

8 All tree comparisons

Proof of Theorem 1.3. The "if" part is left as an exercise; let us prove the "only if" part.

Fix a point array $a_1, \dots, a_n$ in X. Consider the complete graph K_n with $\{1, \dots, n\}$ as the set of vertexes.

Let $\hat K_n \to K_n$ be the universal covering of the complete graph K_n. Denote by $\hat V$ the set of vertexes of $\hat K_n$; given a vertex $\hat v \in \hat V$ denote by v the corresponding vertex of K_n.

Applying the tree comparison for finite subtrees in $\hat K_n$ and passing to a partial limit, we get the following:

($*$) There is a map $f\colon \hat V \to \mathbb{H}$ such that

$$|f(\hat v) - f(\hat w)|_{\mathbb{H}} \geqslant |a_v - a_w|_X$$

for any two vertexes $\hat v, \hat w \in \hat V$ and the equality holds if $(\hat v, \hat w)$ is an edge in $\hat K_n$.

This finish the proof if X is finite.

Since X is separable, it contains a countable everywhere dense set $\{a_1, a_2, \dots\}$. Applying the statement above for $X_n = \{a_1, \dots a_n\}$, we get a submetry from $Y_n = f_n(\hat V_n) \subset \mathbb{H}$ to X_n.

It remains to pass to the ulralimit Y of the subspaces Y_n. Clearly Y admits an isometric embedding into $\mathbb{H}$ and it admits submetry on $Y \to X$. Hence the statement follows. □

8.1. Proposition. *Suppose G be a compct Lie group with bi-invariant metric, so the action $G\times G \curvearrowright G$ defined by $(h_1,h_2)\cdot g = h_1\cdot g\cdot h_2^{-1}$ is isometric. Then for any closed subgroup $H < G\times G$, the bi-quotient space $G/\!\!/H$ satisfies multipolar comparison.*

We present a proof suggested by Alexander Lytchak, it is simplified vesrion of the construction of Chuu-Lian Terng and Gudlaugur Thorbergsson given in [9, Section 4].

Proof of Proposition 1.4. Denote by G^n the direct product of n copies of G. Consider the map $\varphi_n\colon G^n \to G/\!\!/H$ defined by

$$\varphi_n\colon (\alpha_1,\dots,\alpha_n)\mapsto [\alpha_1\cdots\alpha_n]_H,$$

where $[x]_H$ denotes the H-orbit of x in G.

Note that φ_n is a quotient map for the action of $H\times G^{n-1}$ on G^n defined by

$$(\beta_0,\dots,\beta_n)\cdot(\alpha_1,\dots,\alpha_n) = (\gamma_1\cdot\alpha_1\cdot\beta_1^{-1},\beta_1\cdot\alpha_2\cdot\beta_2^{-1},\dots,\beta_{n-1}\cdot\alpha_n\cdot\beta_n^{-1}),$$

where $\beta_i\in G$ and $(\beta_0,\beta_n)\in H<G\times G$.

Denote by ρ_n the product metric on G^n rescaled with factor $\sqrt{n}$. Note that the quotient $(G^n,\rho_n)/(H\times G^{n-1})$ is isometric to $G/\!\!/H = (G,\rho_1)/\!\!/H$.

As $n\to\infty$ the curvature of (G^n,ρ_n) converges to zero and its injectivity radius goes to infinity. Therefore passing to the ultra-limit of G^n as $n\to\infty$ we get the Hilbert space. It remains to observe that the limit action has the required property. □

9 Final remarks

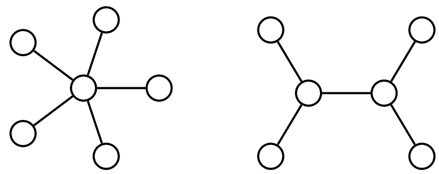
The following problem discussed in [1, 7.1] was one of the original motivations to study the tree comparison.

9.1. Problem. *Which finite metric spaces admit isometric embeddings into some Alexandrov spaces with nonnegative curvature.*

The problem is still open. According to [1, 4.1], the $(n-1)$-tree comparison provides a necessary condition for the problem n-point metric spaces. This condition is sufficient for the 4-point metric spaces. It might be still sufficient for 5-point metric spaces, but not for 6-point metric spaces.

The corresponding example of 6-point metric space was constructed by Sergei Ivanov, see [1]. Theorem 5.1, provides a source for such examples — any 6-point metric space which satisfy all 5-tree comparisons, but does not satisfy 2(2)-tree comparison provide an example. This class of examples includes the example of Sergei Ivanov — in the notations of [1, 7.1] it does not satisfies the comparison for the tree $y/az(q/xb)$.

We expect that it also provides a sufficient condition; that is, the 5-tree and 2(2)-tree comparisons (these are the comparisons for the trees on the diagram) are sufficient for 6-point metric spaces to be embeddable in a nonnegatively curved Alexandrov spaces.



Here an other candidate for a sufficient condition.

9.2. Question. *Assume F is a finite metric space which satisfies all tree comparisons. Is it true that F is isometric to a subset of Alexandrov space with nonpositive curvature?*

Note that even for finite metric space the all tree comparison has to be checked for infinite set of trees since one point of the space may be used as a label for several vertexes in the tree.

There is a chance that for 5-point and 6-point metric spaces, this condition is also necessary. Since there nonnegatively curved Riemannian manifolds which are not cost-convex, Theorem 1.2 implies that this condition can not be necessary for 7-point metric spaces.

For any metric space X with an isometric group action $G \curvearrowright X$ with closed orbits the quotient map $X \to X/G$ is a submetry. In particular, by Theorem 1.3, if $G \curvearrowright \mathbb{H}$ is an isometric action with closed orbits on the Hilbert space, then the quotient space $\mathbb{H}/G$ satisfies all tree comparisons.

9.3. Question. *Assume X is a metric space satisfying all tree comparisons. Is it always possible to construct an isometric group action with closed orbits on the Hilbert space $G \curvearrowright \mathbb{H}$ such that X is isometric to a subset in $\mathbb{H}/G$?*

Assume M is nonnegatively curved Riemannian manifold and $G \curvearrowright M$ is an isometric action with closed orbits. Then the quotient space $A = M/G$ is an Alexandrov space. It is easy to see that all singular points of A form an extremal set.

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