Tree comparison

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Abstract

We introduce a natural comparison for metric spaces which is closely related the condition which guarantee continuity of optimal transport on smooth manifolds.

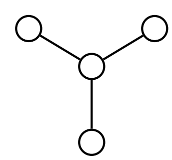
1 Introduction

Let $(a_1, \dots a_n)$ be a point array in a metric space X and T be a tree with the vertexes labeled by $\{a_1, \dots, a_n\}$. We say that $(a_1, \dots a_n)$ satisfies T-tree comparison if there is a point array $(\tilde a_1, \dots, \tilde a_n)$ in the Hilbert space $\mathbb{H}$ such that

$$|\tilde a_i - \tilde a_j| \geqslant |a_i - a_j|$$

for any i and j and the equality holds if for every edge (i, j) in T.

We say that a metric space X satisfies T-tree comparison if every n-points arrays in X satisfies the T-tree comparison. We say that X satisfies all tree comparison If X satisfies T-tree comparison for all trees T then we say that



Monopolar comparison. The (3+1)-comparison described in [1] is S_3-tree comparison, where S_3 denotes the star-like tree as on the diagram. Similarly, the $(n+1)$-comparison described in [1] as S_n-tree comparison for the star-like tree S_n with one vertex of degree n and n end-vertexes. In these examples the trees are *monopolar*; that is they have one vertex (*pole*) adjacent to all other vertexes.

In [1, 4.1] it is proved that any Alexandrov space with nonnegative curvature satisfies the so called $(n+1)$-comparison. Using the tree comparison, the later statement can be reformulated the following way. If a length-metric space satisfies S_3-comparison, then it also satisfies S_n-comparison for all integer n.

Dipolar comparison. A tree will be called *dipoar* if it has exactly two poles; that is, two vertexes of degree at least two.

In Section 4 we will show that the comparison for the any Alexandrov space with nonnegative curvature satisfies the tree comparisons for the first two trees an the following diagram.

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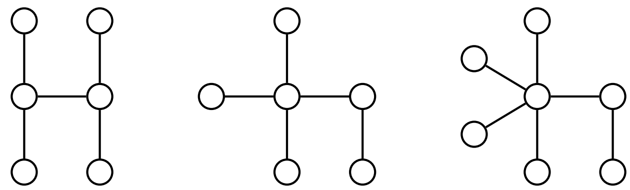


In Section 5, we will show that the comparison for the third tree on the diagram implies the curvature condition introduced by Ma–Trudinger–Wang (briefly MTW condition) and convexity of tangent injectivity loci (briefly CTIL condition). These two conditions appear in the study of contunuity of optimal transport betweens regular measures with positive continuous density functions. The continuity implies both MTW and CTIL conditions and slightly stronger version of these two conditions imply the continuity; see [3], [7] and the references there in.

Note that dipolar comparison provides a uniform way to treat combined CTIL+MTW condition; this partially answers the question of Cédric Villani in ???.

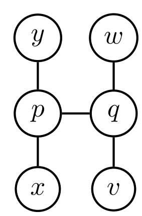
Polypolar comparison. In Section 6 we show that if the space satisfies all tree comparisons then it is a target space of submetry from a subset in Hilbert space.

We also show that all bi-quotients of compact Lie groups with bi-invariant metrics satisfy all tree comparison.

2 Preliminaries

Proposed notation for trees. We will encode the trees using brackets.

For example $(()()(()()))$ encodes the tree with 6 vertexes on the diagram. One vertex for each pair of brakes; we assume that a pair vertexes is adjacent if the corresponding pairs of brackets nested directly one into another.



We will use shortcut

$$n = \underbrace{()\dots()}_{n \text{ times}};$$

so we can write $(2(2))$ instead of $(()()(()()))$.

To encode the labelig of vertexes as on the diagram, we will use notation $p, xy(q, vw)$; here p and q are connected to the first element in each group followed by comma. Taking another root for the tree, we can write it as $q, vw(p, x, y)$ or as $x, (p, y(q, vw))$.

2.1. Kirszbraun rigidity theorem. *Let A be a complete* CBB[0] *length space.*

Assume that for two point arrays $p, x_1, \dots, x_n \in A$ and $\tilde q, \tilde x_1, \dots, \tilde x_n \in \mathbb{H}$ we have that

$$|\tilde q - \tilde x_i| \geqslant |p - x_i|$$

for any i,

$$|\tilde{x}_i - \tilde{x}_j| \leqslant |x_i - x_i|$$

for any pair (i, j) and $\tilde{q}$ lies in the interior of the convex hull K of $\tilde{x}_1, \dots, \tilde{x}_n$.

Then equalities hold in all the inequalities above. Moreover there is an distance preserving map $f\colon K \to A$ such that $f(\tilde{x}_i) = x_i$ and $f(\tilde{q}) = p$.

Proof. By the generalized Kirszbraun theorem, there is a short map $f\colon A \to \mathbb{H}$ such that $f(x_i) = \tilde{x}_i$. Set $\tilde{p} = f(p)$. By assumptions

$$|\tilde{q} - \tilde{x}_i| \geqslant |\tilde{p} - \tilde{x}_i|.$$

Since $\tilde{q}$ lies in the interior of K, $\tilde{q} = \tilde{p}$. It follows that the equality

$$|\tilde{q} - \tilde{x}_i| = |p - x_i|.$$

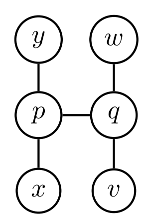
holds for each i.

According to ???, there is a short map $\mathbb{H} \to \mathrm{T}_p$ such which admits a right inverse $\mathrm{T}_p \to \mathbb{H}$ such that ...

□

3 Pivotal configurations.

Let X be a metric space. A point array $(a_1, \dots, a_n)$ in X together with a choice of a tree with the vertexes labeled by $(a_1, \dots, a_n)$ and a choice of geodesic $[a_i, a_j]$ for every adjacent pair (a_i, a_j) is called *geodesic tree.*



For geodesic trees we will use the same notation as for labeled combinatoric tree in square brackets; for example $[p, xy(q, vw)]$ will denote the geodesic tree with with combinatorics as on the diagram.

Fix a geodeisc dipolar tree $T = [p_1, x_1 \dots x_k(p_2, x_{k+1} \dots x_n)]$; that is, the tree T has two poles p_1, p_2 and each of the remaining vertexes are adjacent either to p_1 or p_2 — $x_1, \dots, x_k$ are connected to p_1 and $x_{k+1}, \dots, x_n$ to p_2.

Assume X is a complete nonempty curved length space; in particular the angle is defined for any geodesic hinge.

A geodesic tree $\tilde{T} = [\tilde{p}_1, \tilde{x}_1 \dots \tilde{x}_k(\tilde{p}_2, \tilde{x}_{k+1} \dots \tilde{x}_n)]$ in $\mathbb{H}$ will be called *pivotal tree* for $T = [p_1, x_1 \dots x_k(p_2, x_{k+1} \dots x_n)]$ if

(i) $|\tilde{p}_1 - \tilde{p}_2| = |p_1 - p_2|$,
(ii) $|\tilde{p}_i - \tilde{x}_j| = |p_i - p_j|$ for any edge $[p_i, x_j]$ in T and
(iii) $\measuredangle[\tilde{p}_j\,{}^{\tilde{x}_k}_{\tilde{p}_i}]_{\mathbb{H}} = \measuredangle[\tilde{p}_j\,{}^{\tilde{x}_k}_{\tilde{p}_i}]_A$ for any hinge $[p_j\,{}^{x_k}_{\tilde{p}_i}]$ in T.

3.1. Rigidity lemma. *Let A be a complete* CBB[0] *length space. Suppose $[\tilde{p}_1, \tilde{x}_1 \dots \tilde{x}_k(\tilde{p}_2, \tilde{x}_{k+1} \dots \tilde{x}_n)]$ is a pivotal tree for a geodesic tree with the vertexes $[p_1, x_1 \dots x_k(p_2, x_{k+1} \dots x_n)]$ in A. Assume that*

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{H}} \leqslant |x_i - x_j|_A \tag{❶}$$

for any pair (i,j) and the convex hull $\tilde K$ of $\{\tilde x_1,\dots\tilde x_n\}$ intersects the line $\tilde p_1\tilde p_2$. Then the equality holds in ❶ *for each pair (i,j).*

Proof. Assume that a point $\tilde z$ on the line $(\tilde p_1,\tilde p_2)$ is given. We can assume that $\tilde z$ lies on the half-line $[\tilde p_1,\tilde p_2)$; otherwise swap the labels of $\tilde p_1$ and $\tilde p_2$.

Denote by ζ the direction of geodesic $[p_1,p_2]$ at p_1. Set $z=\mathrm{gexp}_{p_1}(|\tilde z-\tilde p_1|\cdot\zeta)$, where gexp_{p_1} denotes the gradient exponent at p_1; see [2]. By comparison, we have

$$|x_i-z|_A\leqslant|\tilde x_i-\tilde z|_{\mathbb{R}^2}$$

for any i.

It remains to apply Kirszbraun rigidity theorem (2.1). □

Suppose $[\tilde p_1,\tilde x_1\dots\tilde x_k(\tilde p_2,\tilde x_{k+1}\dots\tilde x_n)]$ is a pivotal tree for a geodesic tree with the vertexes $[p_1,x_1\dots x_k(p_2,x_{k+1}\dots x_n)]$ in A. Note that by angle comparison

$$|\tilde x_i-\tilde p_j|_{\mathbb{H}}\geqslant|x_i-p_j|_A$$

for any i and j. It follows that the configuration $\tilde p_1$, $\tilde p_2$, $\tilde x_1,\dots\tilde x_n\in\mathbb{H}$ satisfies the tree comparison (see Section 1) if

$$|\tilde x_i-\tilde x_j|_{\mathbb{H}}\geqslant|x_i-x_j|_A \qquad ❷$$

for all pairs (i,j).

Denote by $\tilde\xi_i$ the direction of the half-plane thru $\tilde x_i$ with the boundary line $(\tilde p_1,\tilde p_2)$. We may assume that all $\tilde\xi_i$ belong to a unit sphere, of dimension at most $n-1$.

Note that up to a motion of $\mathbb{H}^n$, a pivotal configuration is completely described by the angles $\measuredangle(\tilde\xi_i,\tilde\xi_j)$. Moreover, the distance $|\tilde x_i-\tilde x_j|_{HH}$ is determined by $\measuredangle(\tilde\xi_i,\tilde\xi_j)$.

Let us denote by $\alpha_{i,j}$ the minimal angle $\measuredangle(\tilde\xi_i,\tilde\xi_j)$ in a pivotal configuration such that ❷ holds, so the inequality ❷ is equivalent to

$$\measuredangle(\xi_i,\xi_j)\geqslant\alpha_{i,j}.$$

3.2. Corollary. *For any geodesic dipolar tree in a complete* CBB[0] *length space the following conditions hold:*

(a) For any pair i and j, we have

$$\alpha_{i,j}\leqslant\pi.$$

(b) For any triple i, j and k, we have

$$\alpha_{i,j}+\alpha_{j,k}+\alpha_{k,i}\leqslant2\cdot\pi.$$

In other words, if A is a nonnegatively curved complete length space then

(a) *For any broken geodesic line* $[p_1, x_1(p_2, x_2)]$ *in* A *there is a pivotal tree* $[\tilde p_1, \tilde x_1(\tilde p_2, \tilde x_2)]$ *such that*

$$|\tilde x_1 - \tilde x_2|_{\mathbb{H}} \geqslant |x_1 - x_2|_A.$$

(b) *For any geodesic tree* $[p_1, x_1x_2(p_2, x_3)]$ *in* A *there is a pivotal tree* $[\tilde p_1, \tilde x_1\tilde x_2(\tilde p_2, \tilde x_3)]$ *such that*

$$|\tilde x_i - \tilde x_j|_{\mathbb{H}} \geqslant |x_i - x_j|_A.$$

for all i *and* j.

Proof; (a). Consider the pivotal tree $[\tilde p_1, \tilde x_1(\tilde p_2, \tilde x_2)]$ with $\measuredangle(\tilde\xi_1, \tilde\xi_2) = \pi$. In particular the points $\tilde x_1$ and $\tilde x_2$ lie on the opposite sides from the line $(\tilde p_1, \tilde p_2)$. It remains to apply the rigidity lemma.

(b). By *(a)*, we can assume that

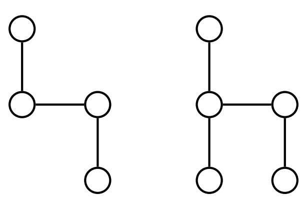
$$\alpha_{1,3} + \alpha_{2,3} > \pi. \qquad ❸$$

Consider the pivotal tree $[\tilde p_1, \tilde x_1 x_2(\tilde p_2, \tilde x_3)]$ which lies in 3-dimesional space in such a way that the points $\tilde x_1$ and $\tilde x_2$ lie on the opposite sides from the plane $(\tilde p_1, \tilde p_2, \tilde x_3)$ and

$$\measuredangle(\tilde\xi_1, \tilde\xi_3) = \alpha_{1,3}, \qquad \measuredangle(\tilde\xi_2, \tilde\xi_3) = \alpha_{2,3}.$$

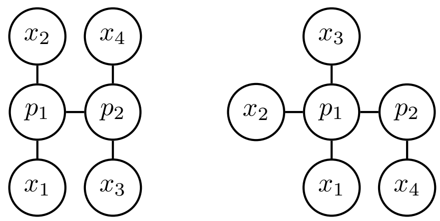
Te inequality ❸ implies that the convex hull K in the rigidity lemma intersects the line $(\tilde p_1, \tilde p_2)$. It remains to apply the lemma. □

Note that *(a)* and *(b)* implies that A satisfies tree comparison for the two trees (1(1)) and (2(1)) on the digram. However, the tree comparison for the first tree (1(1)) follows from the triangle inequality.



4 Six point comparison

4.1. Theorem. *Let* A *be an complete nonnegatively curved lenght space. Then for any tree* $[p_1, x_1x_2x_3(p_2, x_4)]$ *or* $[p_1, x_1x_2(p_2, x_3x_4)]$ *(see the diagram) there is a pivotal tree satisfying the tree comparison.*



In particular, any complete nonnegatively curved lenght space satisfies the comparison for 2(2) and 3(1) bipolar trees.

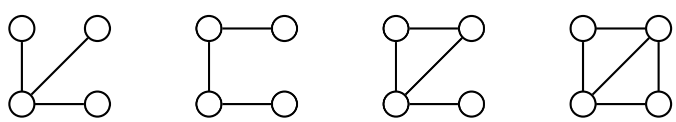
Proof. Let us fix a geodesic tree $[p_1, x_1x_2x_3(p_2, x_4)]$ or $[p_1, x_1x_2(p_2, x_3x_4)]$. The rest of the proof in these two cases will be identical. Recall that p_1 and p_2 are the poles of the tree and each of remaining vertexes $\xi_1, \xi_2, \xi_3, \xi_4$ are connected to one of the poles.

Define the values $\{\alpha_{i,j}\}$ for each pair i, j as in the previous section. The following algorithm, produces a metric graph with the vertexes denoted by $\xi_1, \xi_2, \xi_3, \xi_4$.

1. List the values $\{\alpha_{i,j}\}$ in the non-increasing order.
2. If $\alpha_{i,j}$ the first value in the list, connect vertexes ξ_i and ξ_j by an edge of length $\alpha_{i,j}$.
3. Do the same for the second value in the list.
4. Starting from the third step, we attach a new edge corresponding to the next value $\alpha_{i,j}$ only if the already constructed edges in the graph will remain to be the shortest path between their vertexes; otherwise go to the next value in the list.
5. Repeat until the end of the list.

Denote the obtained metric graph by Γ. Note that Γ is connected; therefore Γ have to be isomorphic to one of 6 graphs shown below.

In the following 4 cases, Corollary 3.2, there is a geodesic graph $\tilde{\Gamma}$ in $\mathbb{S}^2$ isometric to Γ; denote its vertexes by $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4 \in \mathbb{S}^2$. Let $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$, $\tilde{x}_4$ be the vertexes of the corresponding pivotal tree in $\mathbb{H}$; that is, ξ_i is the normal direction of the half-plane thru $\tilde{x}_i$ with $(\tilde{p}_1, \tilde{p}_2)$ as the boundary line.

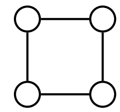


Note that in this case

$$\measuredangle(\tilde{\xi}_i, \tilde{\xi}_j) \geqslant \alpha_{i,j}$$

for any pair (i, j). Indeed, if ξ_i is adjacent to ξ_j in Γ then the equality holds; otherwise the inequality follows from the triangle inequality in $\mathbb{S}^2$.

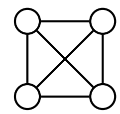
Assume Γ is a cycle of length 4. If there is an isometic geodesic graph $\tilde\Gamma$ in $\mathbb{S}^2$ then the statement can be proved along the same lines. If there is no such graph $\tilde\Gamma$ then ???



It remains to consider two cases — when Γ is the cycle of length 4 and when it is the complete graph (see the diagrams blow).

Assume Γ is the complete graph, in other words, the inequalities

$$\alpha_{i,k} \leqslant \alpha_{i,j} + \alpha_{j,k}$$



hold for all triples (i, j, k).

Remove the edge (ξ_3, ξ_4) from Γ; denote the obtained graph by Γ'. Note that there is geodesic graph $\tilde\Gamma'$ in $\mathbb{S}^2$ which is isometric to Γ'; denote by $\tilde\xi_1, \tilde\xi_2, \tilde\xi_3, \tilde\xi_4$ the corresponding vertexes in $\tilde\Gamma'$. We can (and will) assume that the vertexes $\tilde\xi_3$ and $\tilde\xi_4$ lie on the opposite side from the equator containing $[\tilde\xi_1, \tilde\xi_2]$. Let $\tilde p_1$, $\tilde p_2$, $\tilde x_1$, $\tilde x_2$, $\tilde x_3$, $\tilde x_4 \in \mathbb{H}$ be the corresponding pivotal configuration; that is, $\tilde\xi_i$ is the normal direction of the half-plane $\tilde p_1 \tilde p_2 \tilde x_i$ to the line $\tilde p_1 \tilde p_2$.

Note that by construction, we have

$$|\tilde x_i - \tilde x_j|_{\mathbb{H}} \geqslant |x_i - x_j|_A$$

for each pair $i < j$ except $(3, 4)$.

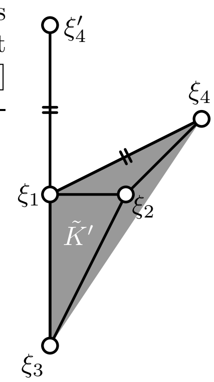
Denote by $\tilde K$ the convex hull of $\tilde x_1, \tilde x_2, \tilde x_3, \tilde x_4$ in $\mathbb{H}$ and by $\tilde K'$ the convex hull of $\tilde\xi_1, \tilde\xi_2, \tilde\xi_3, \tilde\xi_4$ in $\mathbb{S}^2$.

Assume the interior of $\tilde K$ intersects the line $\tilde p_1 \tilde p_2$; or equivalently $\tilde K' = \mathbb{S}^2$. Then by the rigidity lemma, we have

$$|\tilde x_3 - \tilde x_4|_{\mathbb{H}} \geqslant |x_3 - x_4|_A.$$

In particular, the array $\tilde p_1$, $\tilde p_2$, $\tilde x_1$, $\tilde x_2$, $\tilde x_3$, $\tilde x_4 \in \mathbb{H}$ satisfies the tree comparison.

In the remaining case $\tilde K' \neq \mathbb{S}^2$, the boundary $\partial_{\mathbb{S}^2} \tilde K'$ is nonempty; moreover it contains at least 3 of the vertexes $\tilde\xi_1, \tilde\xi_2, \tilde\xi_3, \tilde\xi_4$. Without loss of generality we may assume that $\tilde\xi_1 \in \partial_{\mathbb{S}^2} K$. Denote by $\tilde\xi_4'$ the point on the extension of $[\tilde\xi_3, \tilde\xi_1]$ behind $\tilde\xi_1$ such that $|\tilde\xi_1 - \tilde\xi_4'|_{\mathbb{S}^2} = |\tilde\xi_1 - \tilde\xi_4|_{\mathbb{S}^2}$. Since the increasing of angle increase the opposite side, we have



$$\begin{aligned} |\tilde\xi_2 - \tilde\xi_4'|_{\mathbb{S}^2} &\geqslant |\tilde\xi_2 - \tilde\xi_4|_{\mathbb{S}^2} = \\ &= \alpha_{2,4}. \end{aligned}$$

Note that

$$|\tilde\xi_3 - \tilde\xi_4'|_{\mathbb{S}^2} = \min\{\, \alpha_{1,3} + \alpha_{1,4},\ \pi - (\alpha_{1,3} + \alpha_{1,4})\,\}$$

Since Γ is complete, we have

$$\alpha_{1,3} + \alpha_{1,4} \geqslant \alpha_{3,4}$$

and by Corollary 3.2,

$$\alpha_{1,3} + \alpha_{1,4} + \alpha_{3,4} \leqslant 2{\cdot}\pi.$$

Therefore

$$|\tilde\xi_3-\tilde\xi_4'|_{\mathbb{S}^2}\geqslant\alpha_{3,4}.$$

It follows that the pivotal configuration with normal directions $\tilde\xi_1,\tilde\xi_2,\tilde\xi_3,\tilde\xi_4'$ satisfies the definition of tree comparison. $\square$

5 Seven point comparison

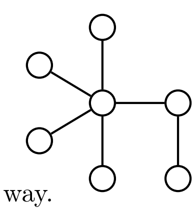
Let M be a Riemannian manifold. The *tangent injectivity locus* at the point $p\in M$ (briefly TIL_p) is defined as the maximal open subset in the tangent space T_p such that for any $v\in\mathrm{TIL}_p$ the geodesic path $\gamma(t)=\exp_p(v\cdot t)$, $t\in[0,1]$ is a minimizing. If the tangent injectivity locus at any point $p\in M$ is convex we say that M satisfies *convexity of tangent injectivity locus* or briefly M is CTIL.

Xi-Nan Ma, Neil Trudinger and Xu-Jia Wang, Xu-Jia introduced a global differential geometric condition which is now called MTW, see [6]. The conditions CTIL and MTW are necessary for the regularity of optimal transport on Riemannian manifold M. Moreover, a slightly stronger version of these conditions gives the converse.

5.1. Proposition. *Let T be the tree as on the diagram. If a Riemannian manifold M satisfies the T-tree comparison then*

(a) M is CTIL;

(b) M is MTW.



In the proof we will use a reformulation of MTW condition given by Cédric Villani [7, 2.6]. More precisely, we will use the following reformulation of which can be proved the same way.

Assume $u,v\in\mathrm{T}_p$ and $w=\tfrac12\cdot(u+v)$ and $x=\exp_p u$, $y=\exp_p v$ and $q=\exp_p w$. If the three geodesic paths $[p,x]$, $[p,y]$ and $[p,q]$ described by the paths $t\mapsto\exp_p(t\cdot u)$, $t\mapsto\exp_p(t\cdot v)$, $t\mapsto\exp_p(t\cdot w)$ for $t\in[0,1]$ are minimizing, then $[p,q]$ is called *median* of the hinge $[p\,{}^x_y]$. Note that in a CTIL Riemannian manifold, any hinge has a median.

5.2. MTW condition. *Assume M be a CTIL Riemannian manifold. Then M is MTW if and only if for a median $[p,q]$ of any hinge $[p\,{}^x_y]$ one of the following inequalities*

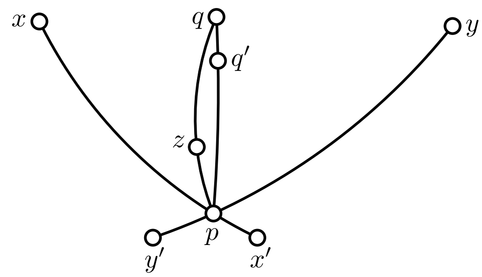
$$\left[\begin{aligned}&|p-q|^2_M-|z-q|^2_M\leqslant|p-x|^2_M-|z-x|^2_M,\\&|p-q|^2_M-|z-q|^2_M\leqslant|p-y|^2_M-|z-y|^2_M.\end{aligned}\right.$$

holds for any $z\in M$.

Proof; (a). Assume the contrary; that is, there is $p\in M$ and $u,v\in\mathrm{TIL}_p$ such that $w=\tfrac12\cdot(u+v)\notin\mathrm{TIL}_p$.

Let τ be the maximal value such that the geodesic $\gamma(t)=\exp_p(w\cdot t)$ is a length-minimizing on $[0,\tau]$. Set $w'=\tau\cdot w$. Note that $\tau<1$ and $w'\in\partial\mathrm{TIL}_p$.

Set $q=\exp_p w'$. By general position argument, we can assume that there are at least two minimizing geodesics connecting p to q; see [4]. That is, there is $w''\in\partial\mathrm{TIL}_p$ such that $w''\neq w'$ and $\exp_p w'=\exp_p w''$.

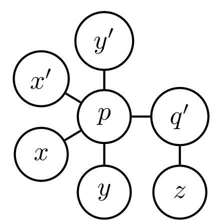


Fix small positive real numbers δ, ε and ζ. Consider the points

$$\begin{aligned} q' = q'(\varepsilon) &= \exp_p(1-\varepsilon)\cdot w', & z = z(\zeta) &= \exp_p(\zeta\cdot w''), \\ x &= \exp_p u, & x' = x'(\delta) &= \exp_p(-\delta\cdot u), \\ y &= \exp_p v, & y' = y'(\delta) &= \exp_p(-\delta\cdot v). \end{aligned}$$

We will show that for some choice of δ, ε and ζ the array p, x, x', y, y', q', z does not satisfy the T-tree comparison with the labeling as on the diagram below.

Assume that given positive numbers δ, ε and ζ, there is a point array $\tilde p$, $\tilde x$, $\tilde x'(\delta)$, $\tilde y$, $\tilde y'(\delta)$, $\tilde q'(\varepsilon)$, $\tilde z(\zeta) \in \mathbb{H}$ as in the definition of T-tree comparison; that is, the distances between the points in this array are at least as big as the distances of corresponding points in M and the equality holds for the pair adjacent in T.



Since δ is small, we can assume that p lies on a necessary unique minimizing geodesic $[x, x']_M$. Hence

$$|x - x'|_M = |x - p|_M + |p - x'|_M.$$

By comparison

$$\begin{aligned} |\tilde x - \tilde x'|_{\mathbb{H}} &\geqslant |x - x'|_M, \\ |\tilde x - \tilde p|_{\mathbb{H}} &= |x - p|_M, \\ |\tilde x' - \tilde p|_{\mathbb{H}} &= |x' - p|_M. \end{aligned}$$

By triangle inequality,

$$|\tilde x - \tilde x'|_{\mathbb{H}} = |\tilde x - \tilde p|_{\mathbb{H}} + |\tilde x' - \tilde p|_{\mathbb{H}};$$

that is, $\tilde p \in [\tilde x, \tilde x']_{\mathbb{H}}$. The same way we see that $\tilde p \in [\tilde y, \tilde y']_{\mathbb{H}}$.

Fix ε and ζ. Note that as $\delta \to 0$ we have

$$\begin{aligned} \tilde x' &\to \tilde p, & \tilde y' &\to \tilde p. \\ \measuredangle[\tilde p\,{}^{\tilde x'}_{\tilde y}] &\to \measuredangle[p\,{}^{x'}_{y}], & \measuredangle[\tilde p\,{}^{\tilde y'}_{\tilde x}] &\to \measuredangle[p\,{}^{y'}_{x}], \\ \measuredangle[\tilde p\,{}^{\tilde x'}_{\tilde q'}] &\to \measuredangle[p\,{}^{x'}_{q'}], & \measuredangle[\tilde p\,{}^{\tilde y'}_{\tilde q'}] &\to \measuredangle[p\,{}^{y'}_{q'}], \end{aligned}$$

It follows that

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] \to \measuredangle[p\,{}^{x}_{y}], \qquad \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde q'}] \to \measuredangle[p\,{}^{x}_{q'}], \qquad \measuredangle[\tilde p\,{}^{\tilde y}_{\tilde q'}] \to \measuredangle[p\,{}^{y}_{q'}].$$

Therefore, passing to a partial limit as $\delta \to 0$, we get a configuration of 5 points $\tilde p, \tilde x, \tilde y, \tilde q' = \tilde q'(\varepsilon), \tilde z = \tilde z(\zeta)$ such that

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] = \measuredangle[p\,{}^{x}_{y}], \qquad \measuredangle[\tilde p\,{}^{\tilde y}_{\tilde q'}] = \measuredangle[p\,{}^{y}_{q'}], \qquad \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde q'}] = \measuredangle[p\,{}^{x}_{q'}].$$

In other words, the map sending 4 points $0, u, v, w' \in \mathrm{T}_p$ to $\tilde p, \tilde x, \tilde y, \tilde q \in \mathbb{H}$ correspondingly is distance preserving.

Note that $q' \to q$ as $\varepsilon \to 0$. Therefore, in the limit, we get a configuration $\tilde p$, $\tilde x$, $\tilde y$, $\tilde q'$, $\tilde z = \tilde z(\zeta)$ such that in addition we have

$$\begin{aligned} |\tilde q' - \tilde z| &= |q - z|, & |\tilde p - \tilde z| &\geqslant |p - z|, \\ |\tilde x - \tilde z| &\geqslant |x - p|, & |\tilde y - \tilde z| &\geqslant |y - z| \end{aligned}$$

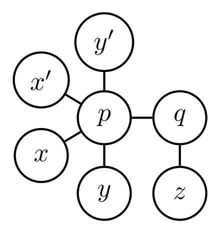
Since $w'' \neq w'$, for small values ζ the last three inequalities imply

$$|\tilde q' - \tilde z| > |q - z|,$$

a contradiction.

(b). Fix a hinge $[p\,{}^{x}_{y}]$ in M. By *(a)*, M is CTIL. Therefor $[p\,{}^{x}_{y}]$ has a median; denote it by $[p, q]$. For $\delta > 0$, define $x' = x'(\delta)$ and $y' = y'(\delta)$ as above.

Without loss of generality we can assume that $x, y \in \exp_p(\mathrm{TIL}_p)$. If δ is small, the latter implies that p lies on unique minimizing geodesics $[x, x']$ and $[y, y']$.



Consider a limit case T-tree comparison as $\delta \to 0$; we get a configuration of 5 points $\tilde p$, $\tilde q$, $\tilde x$, $\tilde y$ and $\tilde z$ such that

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] = \measuredangle[p\,{}^{x}_{y}],$$

$\tilde q$ is the midpoint of $[\tilde x, \tilde y]$. In particular,

$$\begin{aligned} 2\cdot|\tilde z - \tilde q|^2_{\mathbb{H}} + |\tilde q - \tilde x|^2_{\mathbb{H}} + |\tilde q - \tilde y|^2_{\mathbb{H}} &= |\tilde z - \tilde x|^2_{\mathbb{H}} + |\tilde z - \tilde y|^2_{\mathbb{H}}, \\ 2\cdot|\tilde p - \tilde q|^2_{\mathbb{H}} + |\tilde q - \tilde x|^2_{\mathbb{H}} + |\tilde q - \tilde y|^2_{\mathbb{H}} &= |\tilde p - \tilde x|^2_{\mathbb{H}} + |\tilde p - \tilde y|^2_{\mathbb{H}}, \end{aligned}$$

By the comparison,

$$\begin{aligned} |\tilde z - \tilde x|_{\mathbb{H}} &\geqslant |z - x|_M, & |\tilde z - \tilde y|_{\mathbb{H}} &\geqslant |z - y|_M, \\ |\tilde p - \tilde x|_{\mathbb{H}} &\geqslant |p - x|_M, & |\tilde p - \tilde y|_{\mathbb{H}} &\geqslant |p - y|_M, \\ |\tilde q - \tilde x|_{\mathbb{H}} &= |q - x|_M, & |\tilde q - \tilde y|_{\mathbb{H}} &= |q - y|_M, \\ |\tilde q - \tilde z|_{\mathbb{H}} &= |q - z|_M, & |\tilde q - \tilde p|_{\mathbb{H}} &= |q - p|_M, \end{aligned}$$

Therefore

$$\begin{aligned} 2\cdot|z - q|^2_M + |q - x|^2_M + |q - y|^2_M &\geqslant |z - x|^2_M + |z - y|^2_M, \\ 2\cdot|p - q|^2_M + |q - x|^2_M + |q - y|^2_M &\leqslant |p - x|^2_M + |p - y|^2_M. \end{aligned}$$

Hence the condition in 5.2 follows. $\square$

6 Polypolar comparison

Recall that a map $f\colon W \to X$ between metric spaces is called *submetry* if for any $w \in W$ and $r \geqslant 0$, we have

$$f[B(w,r)_W] = B(f(w),r)_X,$$

where $B(w,r)_W$ denotes the ball with center w and radius r in the space W. In other words submetry is a map which is 1-Lipschitz and 1-co-Lipschitz at the same time. Note that any submetry is onto.

6.1. Theorem. *A separable metric space X satisfies all tree comparison if and only if X is isometric to a target space of submetry defined of a subset of the Hilbert space.*

Proof. The "if" part is left as an exercise; let us prove the "only if" part.

Fix a point array $a_1,\dots,a_n$ in X. Consider the complete graph K_n with $\{1,\dots,n\}$ as the set of vertexes.

Let $\tilde K_n \to K_n$ be the universal covering of the complete graph K_n. Denote by $\tilde V$ the set of vertexes of $\tilde K_n$; given a vetex $\tilde v \in \tilde V$ denote by v the corresponding vertex of K_n.

By multipolar comparison, we have the following:

($*$) There is a map $f\colon \tilde V \to \mathbb{H}$ such that

$$|f(\tilde v) - f(\tilde w)|_{\mathbb{H}} \geqslant |a_v - a_w|_X$$

for any two vertexes $\tilde v, \tilde w \in \tilde V$ and the equality holds if $(\tilde v, \tilde w)$ is an edge in $\tilde K_n$.

Since X is separable, it contains a countable everywhere dense set $\{a_1, a_2, \dots\}$. Applying the statement above for $X_n = \{a_1, \dots a_n\}$, we get an isometric action $\Gamma_n \curvearrowright \mathbb{H}$ and invariant sets $Y_n = f(\tilde V_n) \subset \mathbb{H}$ such that X_n is isometric to Y_n/Γ_n.

It remains to fix an ultra filter ω on $\mathbb{N}$ and pass to the ω-limit action on $\mathbb{H}$. ☐

6.2. Proposition. *Suppose G be a compct Lie group with bi-invariant metric, so the action $G\times G \curvearrowright G$ defined by $(h_1,h_2)\cdot g = h_1\cdot g h_2^{-1}$ is isometric. Then for any closed subgroup $H < G\times G$, the bi-quotient space $G/\!\!/H$ satisfies multipolar comparison.*

As a result we have many examples of spaces satisfying all tree comparison; for example, since $\mathbb{S}^n = \mathrm{SO}(n)/\mathrm{SO}(n-1)$, any round sphere satisfies multipolar comparison.

We present a proof suggested by Alexander Lytchak, it is simplified vesrion of the construction of Chuu-Lian Terng and Gudlaugur Thorbergsson given in [8, Section 4].

Proof. Denote by G^n the direct product of n copies of G. Consider the map $\varphi_n\colon G^n \to G$ defined by

$$\varphi_n\colon (\alpha_1,\dots,\alpha_n) \mapsto \alpha_1\cdots\alpha_n.$$

Note that φ_n is a quotient map for the $H \times G^{n-1}$-action on G^n defined by

$$(\beta_0, \dots, \beta_n)\cdot(\alpha_1, \dots, \alpha_n) = (\gamma_1\cdot\alpha_1\cdot\beta_1^{-1}, \beta_1\cdot\alpha_2\cdot\beta_2^{-1}, \dots, \beta_{n-1}\cdot\alpha_n\cdot\beta_n^{-1}),$$

where $\beta_i \in G$ and $(\beta_0, \beta_n) \in H < G \times G$.

Denote by ρ_n the product metric on G^n rescaled with factor $\sqrt{n}$. Note that the quotient $(G^n, \rho_n)/(H \times G^{n-1})$ is isometric to $G/\!\!/H = (G, \rho_1)/\!\!/H$.

As $n \to \infty$ the curvature of (G^n, ρ_n) converges to zero and its injectivity radius goes to infinity. Therefore passing to the ultra-limit of G^n as $n \to \infty$ we get the Hilbert space. It remains to observe that the limit action has the required property. $\square$

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