# Bipolar comparison

(preliminary version)

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**Abstract**

We define the so called tree comparison — a type of metric comparison which generalize the Alexandrov comparison and defined in a similar fashion. This type of comparison turns out to have strong connections to continuity of optimal transport between regular measures on a Riemannian manifold, in particular to the so called MTW condition introduced by Xi-Nan Ma, Neil Trudinger and Xu-Jia Wang.

## 1 Introduction

We will denote by $|a-b|_X$ the distance between points $a$ and $b$ in the metric space $X$.

**Tree comparison.** Fix a tree $T$ with $n$ vertexes.

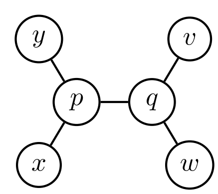
Let $(a_1, \dots a_n)$ be a point array in a metric space $X$ labeled by the vertexes of $T$. We say that $(a_1, \dots a_n)$ satisfies the $T$*-tree comparison* if there is a point array $(\tilde a_1, \dots, \tilde a_n)$ in the Hilbert space $\mathbb{H}$ such that

$$|\tilde a_i - \tilde a_j|_{\mathbb{H}} \geqslant |a_i - a_j|_X$$

for any $i$ and $j$ and the equality holds if $a_i$ and $a_j$ are adjacent in $T$.

We say that a metric space $X$ satisfies the $T$*-tree comparison* if every $n$-points array in $X$ satisfies the $T$-tree comparison.

Instead of the Hilbert space $\mathbb{H}$ we take use infinite dimensional sphere or infinite dimensional hyperbolic space. In this case we will it defines *spherical* and *hyperbolic* tree comparisons.




**Encoding of trees.** To encode the labeled tree on the diagram, we will use notation $p/xy(q/vw)$. It means that we choose $p$ as the root; $p$ has two children leafs $x$, $y$ and one child $q$ with two children leafs $v$ and $w$. Taking another root for the same tree, we get different encodings, for eaxample $q/vw(p/xy)$ or $x/(p/y(q/vw))$.

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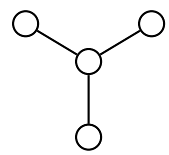
If we do not need the labeling of vertexes, it is sufficient to write the number of leafs in the brackets; this way we can write 2(2) instead of $p/xy(q/vw)$ since the root ($p$) has 2 leafs ($x$ and $y$) and yet another child ($q$) which has 2 leafs ($v$ and $w$). The same tree can be encoded as (1(2)) meaning that the root $x$ has no leafs, $p$ has 1 leaf $y$ and one child $q$ with 2 leafs $v$ and $w$. Note that every vertex that is not the root and not a leaf corresponds to a pair of brackets in this notation.

Using the described notation, we could say that a metric space *satisfies the 2(2)-tree comparison*, meaning that it satisfies the tree comparison on the diagram. We could also say *"applying the tree comparison for $p/xy(q/vw)$ ..."* meaning that we apply the comparison for these 6 points in a metric space labeled as on the diagram.

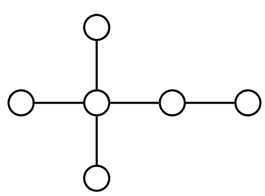
**Monopolar trees.** A vertex of a tree of degree at least two will be called *pole*.

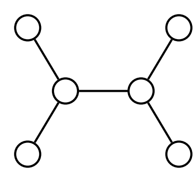
Recall that *Alexandrov space* with nonnegative curvature is defined as complete length space with *nonnegative curvature in the sense of Alexandrov*; the latter is equivalent to the 3-tree comparison; that is, the comparison for the tripod-tree on the diagram.



Using the introduced notation, the theorem on $(n+1)$-point comparison in [1] can be restated the following way: *If a complete length-metric space satisfies* 3*-tree comparison, then it also satisfies $n$-tree comparison for every positive integer $n$; in other words it satisfies all monopolar tree comparisons.*

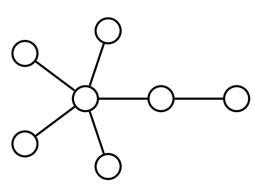
**Bipolar trees.** The following theorem is proved in sections 3 and 4; it describes the comparisons for the bipolar trees 3(1) and 2(2) shown on the diagram.





**1.1. Theorem.** *A complete Riemannian manifold has nonnegative sectional curvature if and only if it satisfies 3(1)-tree or 2(2)-tree comparison.*

The following theorem gives a description of Riemannian manifolds satisfying 4(1)-tree comparison (the 4(1)-tree is shown on the diagram). In order to formulate it we need to define CTIL Riemannian manifolds (CTIL stands for *convexity of tangent injectivity locus*).



Let $M$ be a Riemannian manifold and $p \in M$. The subset of tangent vectors $v \in \mathrm{T}_p$ such that there is a minimizing geodesic $[p\,q]$ in the direction of $v$ with length $|v|$ will be denoted as $\overline{\mathrm{TIL}}_p$. The interior of $\overline{\mathrm{TIL}}_p$ is denoted by $\mathrm{TIL}_p$; it is called *tangent injectivity locus* at $p$. If at $\mathrm{TIL}_p$ is convex for any $p \in M$, then $M$ is called CTIL.

For a function $f$ defined on an open convex set of Euclidean space we write $f'' \leqslant \lambda$ if for any unit-speed geodesic $\gamma$ the function

$$t \mapsto f\circ\gamma(t) - \tfrac{\lambda}{2}\cdot t^2$$

is a concave real-to-real function. (The same definition can be applied in arbitrary metric space.)

**1.2. Theorem.** *If a complete Riemannian manifold $M$ satisfies $4(1)$-tree comparison, then it is CTIL and the following two equivalent condition hold:*

*(i)* *For any $p, q \in M$, we have $f'' \leqslant 1$, where $f$ is the function $f\colon \mathrm{TIL}_p \to \mathbb{R}$ defined by*

$$f(v) = \tfrac{1}{2}\cdot \mathrm{dist}_q^2 \circ \exp_p(v).$$

*(ii)* *For any point $p \in M$ and any three tangent vectors $W \in \mathrm{TIL}_p$, $X, Y \in \mathrm{T}_p$, we have*

$$\frac{\partial^4}{\partial^2 s\,\partial^2 t}\left|\exp_p(s\cdot X) - \exp_p(W + t\cdot Y)\right|_M^2 \leqslant 0 \tag{❶}$$

*at $t = s = 0$.*

*The converse also holds; if one of the conditions holds in a CTIL Riemannian manifold $M$, then $M$ satisfies all bipolar tree comparisons; that is, the $m(n)$-tree comparison holds for any $m$ and $n$.*

The part $(i)$ is proved in Section 5 and the equivalence of the conditions $(i)$ and $(ii)$ is proved in Section 6.

Note that theorems 1.1 and 1.2 do not provide descriptions of manifolds only for the following two bipolar trees trees: 3(2) and 3(3).

A Riemannian manifold is called MTW if inequality ❶ holds for $X \perp Y$. The condition $(ii)$ in the theorem is stronger since it does not require orthogonality.[1]

The MTW condition was introduced by Xi-Nan Ma, Neil Trudinger and Xu-Jia Wang in [10] in connection to the so called *transport continuity property*, briefly TCP. An important step in the understanding MTW was made by Grégoire Loeper in [9].

A compact Riemannian manifold $M$ is called TCP if for any two regular measures with density functions bounded away from zero and infinity the generalized solution of Monge–Ampère equation provided by optimal transport is a genuine (continuous) solution. MTW turns out to be a necessary condition for TCP.

In [4], Alessio Figalli, Ludovic Rifford and Cédric Villani showed that CTIL and a strict version of MTW provides a sufficient condition for TCP. From the proof of Theorem 1.2 it is evident that spherical 4(1)-tree comparison implies the strict version of MTW; in particular, we get the following:

**1.3. Corollary.** *Any Riemannian manifold satisfying spherical 4(1)-tree comparison is TCP.*

[1] Likely the condition $(ii)$ is strictly stronger, but it is not known; also it is not known if MTW implies CTIL [18].

**All tree comparisons.** Recall that a map $f\colon W\to X$ between metric spaces is called *submetry* if for any $w\in W$ and $r\geqslant 0$, we have

$$f[B(w,r)_W]=B(f(w),r)_X,$$

where $B(w,r)_W$ denotes the open ball with center $w$ and radius $r$ in the space $W$. In other words submetry is a map that is 1-Lipschitz and 1-co-Lipschitz at the same time. Note that by the definition, any submetry is onto.

**1.4. Exercise.** *Let $W\to X$ be a submetry and $T$ be a tree. Assume $W$ satisfies the $T$-tree comparison. Show that the same holds for $X$.*

Directly from the definition of tree comparison, it follows that Hilbert space satisfies all tree comparisons. According to the exercise, the same holds for the target spaces of submetries defined of Hilbert space or its subsets. The following theorem gives a converse of the latter statement.

**1.5. Theorem.** *A separable metric space $X$ satisfies all tree comparisons if and only if $X$ is isometric to a target space of submetry defined on a subset of the Hilbert space.*

The following proposition provides a source of examples of spaces satisfying all tree comparisons. For example, since $\mathbb{S}^n=\mathrm{SO}(n)/\mathrm{SO}(n-1)$, any round sphere has this property.

**1.6. Proposition.** *Suppose $G$ is a compact Lie group with bi-invariant metric, so the action $G\times G\curvearrowright G$ defined by $(h_1,h_2)\cdot g=h_1\cdot g\cdot h_2^{-1}$ is isometric. Then for any closed subgroup $H<G\times G$, the bi-quotient space $G/\!\!/H$ satisfies all tree comparisons*

The theorem and propsition are proved in Section 7.

From proposition above and Theorem 1.2, it follows that the bi-quotient space $G/\!\!/H$ is CTIL and MTW. This makes it closely related to the result of Young-Heon Kim and Robert McCann [6].

## 2 Kirszbraun's rigidity

In the proof we will use the rigidity case of the generalized Kirszbraun theorem proved by Urs Lang and Viktor Schroeder in [7], see also [1].

**2.1. Kirszbraun rigidity theorem.** *Let $M$ be a complete Riemannian manifold with nonnegative curvature sectional curvature.*

*Assume that for two point arrays $p,x_1,\dots,x_n\in M$ and $\tilde q,\tilde x_1,\dots,\tilde x_n\in\mathbb{H}$ we have that*

$$|\tilde q-\tilde x_i|\geqslant|p-x_i|$$

*for any $i$,*

$$|\tilde x_i-\tilde x_j|\leqslant|x_i-x_i|$$

*for any pair $(i,j)$ and $\tilde q$ lies in the interior of the convex hull $\tilde K$ of $\tilde x_1,\dots,\tilde x_n$.*

*Then equalities hold in all the inequalities above. Moreover there is an distance preserving map $f\colon \tilde K \to M$ such that $f(\tilde x_i) = x_i$ and $f(\tilde q) = p$.*

*Proof.* By the generalized Kirszbraun theorem, there is a short map $f\colon M \to \mathbb{H}$ such that $f(x_i) = \tilde x_i$. Set $\tilde p = f(p)$. By assumptions

$$|\tilde q - \tilde x_i| \geqslant |\tilde p - \tilde x_i|.$$

Since $\tilde q$ lies in the interior of $K$, we have that $\tilde q = \tilde p$ and the equality

$$|\tilde q - \tilde x_i| = |p - x_i|.$$

holds for each $i$.

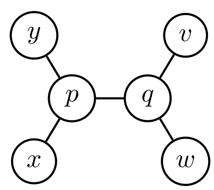
Set $v_i = \log_p x_i$; that is $t \mapsto \exp_p t$ for $t \in [0,1]$ is a minimizing geodesic from $p$ to $x_i$ or, equivalently, $|v_i| = |p - x_i|$ and $\exp_p v_i = x_i$. Recall that the gradient exponent $\mathrm{gexp}_p\colon \mathrm{T}_p \to M$ is a short map and $\mathrm{gexp}_p\colon v_i \mapsto x_i$ for each $i$ (see [1]).

The composition $f \circ \mathrm{gexp}_p\colon \mathrm{T}_p \to \mathbb{H}$ is short and by classical Kirszbraun rigidity it has to be distance preserving on the convex hull $\tilde K'$ of $v_i$. Hence $\tilde K'$ is isometric to $\tilde K$ and the restriction $g|_{\tilde K'}$ is distance preserving. Hence the result. ☐

# 3 Pivotal trees

Assume $M$ is a complete Riemannian manifold. A point array $(a_1, \dots, a_n)$ in $M$ together with a choice of a graph with $n$ vertexes labeled by $(a_1, \dots, a_n)$ and a choice of geodesic $[a_i\, a_j]$ for every adjacent pair $(a_i, a_j)$ is called *geodesic graph.*

For geodesic trees we will use the same notation as for labeled combinatoric tree in square brackets; for example $[p/xy(q/vw)]$ will denote the geodesic tree with with combinatorics as on the diagram.




Fix a geodesic tree $T = [p_1/x_1 \dots x_k(p_2/x_{k+1} \dots x_n)]$; that is, $T$ has two poles $p_1$, $p_2$ and each of the remaining vertexes are adjacent either to $p_1$ or $p_2$ — the vertexes $x_1, \dots, x_k$ are connected to $p_1$ and $x_{k+1}, \dots, x_n$ to $p_2$.

A geodesic tree $\tilde T = [\tilde p_1/\tilde x_1 \dots \tilde x_k(\tilde p_2/\tilde x_{k+1} \dots \tilde x_n)]$ in the Hilbert space $\mathbb{H}$ will be called *pivotal tree* for $T$ if

(i) $|\tilde p_1 - \tilde p_2|_{\mathbb{H}} = |p_1 - p_2|_M$,

(ii) $|\tilde p_i - \tilde x_j|_{\mathbb{H}} = |p_i - p_j|_M$ for any edge $[p_i\, x_j]$ in $T$ and

(iii) $\measuredangle[\tilde p_j\, {}^{\tilde x_k}_{\tilde p_i}]_{\mathbb{H}} = \measuredangle[\tilde p_j\, {}^{\tilde x_k}_{\tilde p_i}]_M$ for any hinge $[p_j\, {}^{x_k}_{\tilde p_i}]$ in $T$.

**3.1. Rigidity lemma.** *Let $M$ be a complete Riemannian manifold with nonnegative sectional curvature and $T = [p_1/x_1 \dots x_k(p_2/x_{k+1} \dots x_n)]$ be geodesic tree in $M$. Suppose $\tilde T = [\tilde p_1/\tilde x_1 \dots \tilde x_k(\tilde p_2/\tilde x_{k+1} \dots \tilde x_n)]$ is a pivotal tree for $T$. Assume that*

❶ $$|\tilde x_i - \tilde x_j|_{\mathbb{H}} \leqslant |x_i - x_j|_M$$

*for any pair $(i,j)$ and the convex hull $\tilde K$ of $\{\tilde x_1,\dots\tilde x_n\}$ intersects the line $(\tilde p_1,\tilde p_2)$. Then the equality holds in ❶ for each pair $(i,j)$.*

*Proof.* Let $\tilde z$ be a point on the line $(\tilde p_1,\tilde p_2)$. Assume that $\tilde z$ lies on the half-line from $\tilde p_1$ to $\tilde p_2$; otherwise swap the labels of $\tilde p_1$ and $\tilde p_2$.

Denote by $\zeta$ the direction of geodesic $[p_1\,p_2]$ at $p_1$. Set

$$z=\exp_{p_1}(|\tilde z-\tilde p_1|\cdot\zeta).$$

By comparison, we have

$$|x_i-z|_M\leqslant|\tilde x_i-\tilde z|_{\mathbb{R}^2}$$

for any $i$.

It remains to apply Kirszbraun rigidity theorem (2.1). □

As above, we assume that $M$ is a complete Riemannian manifold with nonnegative sectional curvature and $[\tilde p_1/\tilde x_1\dots\tilde x_k(\tilde p_2/\tilde x_{k+1}\dots\tilde x_n)]$ is a pivotal tree in $\mathbb{H}$ for the geodesic tree $[p_1/x_1\dots x_k(p_2/x_{k+1}\dots x_n)]$ in $M$.

Note that by angle comparison, for any $i$ and $j$ we have

$$|\tilde x_i-\tilde p_j|_{\mathbb{H}}\geqslant|x_i-p_j|_M.$$

It follows that the configuration $\tilde p_1$, $\tilde p_2$, $\tilde x_1,\dots,\tilde x_n\in\mathbb{H}$ satisfies the tree comparison (see Section 1) if

❷ $$|\tilde x_i-\tilde x_j|_{\mathbb{H}}\geqslant|x_i-x_j|_M$$

for all pairs $(i,j)$.

Denote by $\tilde\xi_i$ the direction of the half-plane containing $\tilde x_i$ with the boundary line $(\tilde p_1,\tilde p_2)$. The direction $\tilde\xi_i$ lies in the unit sphere normal to the line $(\tilde p_1,\tilde p_2)$; we may assume that the dimension of the sphere is $n-1$.

Note that up to a motion of $\mathbb{H}$, a pivotal configuration is completely described by the angles $\measuredangle(\tilde\xi_i,\tilde\xi_j)$. Moreover, the distance $|\tilde x_i-\tilde x_j|_{\mathbb{H}}$ is determined by $\measuredangle(\tilde\xi_i,\tilde\xi_j)$ and the function $\measuredangle(\tilde\xi_i,\tilde\xi_j)\mapsto|\tilde x_i-\tilde x_j|_{\mathbb{H}}$ is nondereasing.

Let us denote by $\alpha_{i,j}$ the minimal angle $\measuredangle(\tilde\xi_i,\tilde\xi_j)$ in a pivotal configuration such that ❷ holds. Note that the inequality ❷ is equivalent to

$$\measuredangle(\xi_i,\xi_j)\geqslant\alpha_{i,j}.$$

**3.2. Corollary.** *For any geodesic bipolar tree in a complete Riemannian manifold $M$ with nonnegative sectional curvature the following conditions hold:*

*(a) For any pair $i$ and $j$, we have*

$$\alpha_{i,j}\leqslant\pi.$$

*(b) For any triple $i$, $j$ and $k$, we have*

$$\alpha_{i,j}+\alpha_{j,k}+\alpha_{k,i}\leqslant2\cdot\pi.$$

*In other words:*

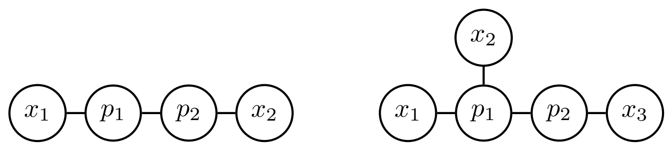
(a) *For any 1(1) geodesic tree (which is a broken geodesic line)* $[p_1/x_1(p_2/x_2)]$ *in* $M$ *there is a pivotal tree* $[\tilde p_1/\tilde x_1(\tilde p_2/\tilde x_2)]$ *such that*

$$|\tilde x_1-\tilde x_2|_{\mathbb H}\geqslant|x_1-x_2|_M.$$

(b) *For any 2(1) geodesic tree* $[p_1/x_1x_2(p_2/x_3)]$ *in* $M$*, there is a pivotal tree* $[\tilde p_1/\tilde x_1\tilde x_2(\tilde p_2/\tilde x_3)]$ *such that*

$$|\tilde x_i-\tilde x_j|_{\mathbb H}\geqslant|x_i-x_j|_M.$$

*for all* $i$ *and* $j$*.*




*Proof; (a).* Consider the pivotal tree $[\tilde p_1/\tilde x_1(\tilde p_2/\tilde x_2)]$ (which is a polygonal path) with $\measuredangle(\tilde\xi_1,\tilde\xi_2)=\pi$. Note that the points $\tilde p_1,\tilde x_1,\tilde p_2,\tilde x_2$ are coplanar and the points $\tilde x_1$ and $\tilde x_2$ lie on the opposite sides from the line $(\tilde p_1,\tilde p_2)$. It remains to apply the rigidity lemma.

*(b).* By $(a)$, we can assume that

❸ $$\alpha_{1,3}+\alpha_{2,3}>\pi.$$

Consider the pivotal tree $[\tilde p_1/\tilde x_1\tilde x_2(\tilde p_2/\tilde x_3)]$ which lies in a 3-dimesional subspace in such a way that the points $\tilde x_1$ and $\tilde x_2$ lie on the opposite sides from the plane containing $\tilde p_1,\tilde p_2,\tilde x_3$, and

$$\measuredangle(\tilde\xi_1,\tilde\xi_3)=\alpha_{1,3},\qquad\qquad \measuredangle(\tilde\xi_2,\tilde\xi_3)=\alpha_{2,3}.$$

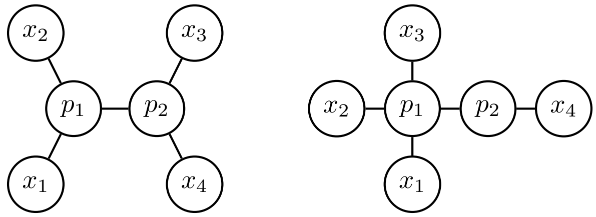
By ❸, the convex hull $\tilde K$ of $\{\tilde x_1,\tilde x_2,\tilde x_3\}$ intersects the line $(\tilde p_1,\tilde p_2)$. It remains to apply the rigidity lemma. □

Note that $(a)$ and $(b)$ imply that nonnegatively curved Alexandrov space satisfies 1(1)-tree and 2(1)-tree comparisons. However, 1(1)-tree comparison follows directly from the triangle inequality.

# 4 2(2) and 3(1)

Note that both 2(2)-tree and 3(1)-tree comparisons imply Alexandrov comparison; indeed the tripod (that is 3-tree) is an subtree of both trees 2(2) and 3(1) and the 3-tree comparison is equivalent Alexandrov comparison, see the introduction. Hence the if part of Theorem 1.1 follows.

The following proposition is a slightly stronger version of the only-if part — namely we will show that the required model configuration in Theorem 1.1 can be found among pivotal trees.




**4.1. Proposition.** *Let $M$ be a complete Riemannian manifold with nonnegative sectional curvature. Then for any geodesic 2(2)-tree (or 3(1)-tree) there is a pivotal tree satisfying the corresponding tree comparison.*

*In particular, $M$ satisfies the 2(2)-tree comparison as well as 3(1)-tree comparison.*

The proofs in the two cases are nearly identical; they differ only by the choice made in the first line.

*Proof.* Fix a geodesic tree $[p_1/x_1x_2(p_2/x_3x_4)]$ or $[p_1/x_1x_2x_3(p_2/x_4)]$. Define the values $\{\alpha_{i,j}\}$ for each pair $i,j$ as in the previous section.

Fix a smooth monotonic function $\varphi\colon \mathbb{R}\to\mathbb{R}$ such that $\varphi(x)=0$ if $x\geqslant 0$ and $\varphi(x)>0$ if $x<0$. Consider a configuration of 4 points $\tilde\xi_1,\tilde\xi_2,\tilde\xi_3,\tilde\xi_4$ in $\mathbb{S}^3$ that minimize the *energy*

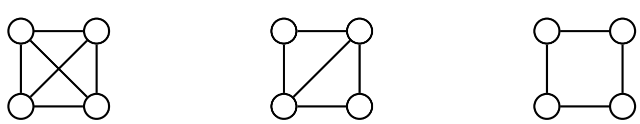
$$E(\tilde\xi_1,\tilde\xi_2,\tilde\xi_3,\tilde\xi_4)=\sum_{i<j}\varphi(\measuredangle(\tilde\xi_i,\tilde\xi_j)-\alpha_{i,j}).$$

Consider the geodesic graph $\Gamma$ with 4 vertexes $\tilde\xi_1,\tilde\xi_2,\tilde\xi_3,\tilde\xi_4$ in $\mathbb{S}^3$, where $\tilde\xi_i$ is adjacent to $\tilde\xi_j$ if $\measuredangle(\tilde\xi_i,\tilde\xi_j)<\alpha_{i,j}$. If the comparison does not hold, then $\Gamma$ is not empty.

Note that a vertex of $\Gamma$ can not lie in an open hemisphere with all its adjacent vertexes. Indeed, if it would be the case, then we could move this vertex increasing the distances to all its adjacent vertexes. Along this move the energy decreases — a contradiction.

Note that by Corollary 3.2, degree of any vertex is at least 2. Indeed existence of a vertex of degree 1 contradicts 3.2*a* and existence of a vertex of degree 0 contradicts 3.2*b*.

Therefore the graph $\Gamma$ is isomorphic to one the following three graphs.



The 6-edege case (that is, the complete graph with 4 vertexes) can not appear by the rigidity lemma (see 3.1).

To do the remaining two cases, note that since the energy is minimal, the angle between the edges at every vertex of degree 2 of $\Gamma$ has to be $\pi$. That is, the concatenation of two edges at such vertex is a geodesic.

Consider the 5-edge graph on the diagram. By the observation above the both triangles in the graph run along one equator. The latter contradicts Corollary 3.2*b*.

For the 4-edge graph (that is, for 4-cycle) by the same observation we have 4 points lie on the equator; moving the even pair to the north pole and odd pair to the south pole will decrease the energy, a contradiction. □

# 5 Pull-back convexity

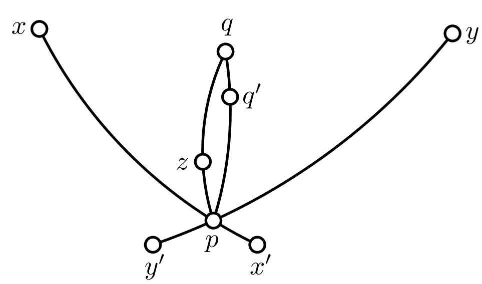
The part i of Theorem 1.2 follows from the three propositions in this section.

**5.1. Proposition.** *If a complete Riemannian manifold satisfies 4(1)-tree comparison the it is CTIL.*

*Proof.* Assume that a Riemannian manifold $M$ satisfies 4(1)-tree comparison. Assume there is $p \in M$ and $u, v \in \mathrm{TIL}_p$ such that $w = \frac{1}{2}\cdot(u+v) \notin \mathrm{TIL}_p$. It is sufficient to show that $\gamma(t) = \exp_p(w\cdot t)$ is a length-minimizing on $[0,1]$.

Assume the contrary, that is, $\tau < 1$ is the maximal value such that the geodesic $\gamma(t) = \exp_p(w\cdot t)$ is a length-minimizing on $[0,\tau]$. Set $w' = \tau\cdot w$. Note that $w' \in \partial\mathrm{TIL}_p$.

Set $q = \exp_p w'$. By general position argument, we can assume that there are at least two minimizing geodesics connecting $p$ to $q$; see [5]. That is, there is $w'' \in \partial\mathrm{TIL}_p$ such that $w'' \neq w'$ and $\exp_p w' = \exp_p w''$.



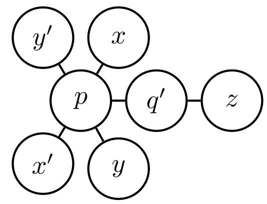

Fix small small positive real numbers $\delta, \varepsilon$ and $\zeta$. Consider the following points

$$\begin{aligned} q' = q'(\varepsilon) &= \exp_p(1-\varepsilon)\cdot w', & z = z(\zeta) &= \exp_p(\zeta\cdot w''), \\ x &= \exp_p u, & x' = x'(\delta) &= \exp_p(-\delta\cdot u), \\ y &= \exp_p v, & y' = y'(\delta) &= \exp_p(-\delta\cdot v). \end{aligned}$$

We will show that for some choice of $\delta, \varepsilon$ and $\zeta$ the tree comparison for $p/xx'yy'(q'/z)$ does not hold.

Assume the contrary, that is, given any positive numbers $\delta, \varepsilon$ and $\zeta$, there is a point array $\tilde p$, $\tilde x$, $\tilde x'(\delta)$, $\tilde y$, $\tilde y'(\delta)$, $\tilde q'(\varepsilon)$, $\tilde z(\zeta) \in \mathbb{H}$ as in the definition of $T$-tree comparison.

If $\delta$ is small, we can assume that $p$ lies on a necessary unique minimizing geodesic $[x\,x']_M$. Hence




$$|x-x'|_M=|x-p|_M+|p-x'|_M.$$

By comparison

$$\begin{aligned}|\tilde x-\tilde x'|_{\mathbb H}&\geqslant|x-x'|_M,\\ |\tilde x-\tilde p|_{\mathbb H}&=|x-p|_M,\\ |\tilde x'-\tilde p|_{\mathbb H}&=|x'-p|_M.\end{aligned}$$

By triangle inequality,

$$|\tilde x-\tilde x'|_{\mathbb H}=|\tilde x-\tilde p|_{\mathbb H}+|\tilde x'-\tilde p|_{\mathbb H};$$

that is, $\tilde p\in[\tilde x\,\tilde x']_{\mathbb H}$. The same way we see that $\tilde p\in[\tilde y\,\tilde y']_{\mathbb H}$.

Fix $\varepsilon$ and $\zeta$. Note that as $\delta\to 0$ we have

$$\begin{aligned}&\tilde x'\to\tilde p, &&\tilde y'\to\tilde p.\\ &\measuredangle[\tilde p\,{}^{\tilde x'}_{\tilde y}]\to\measuredangle[p\,{}^{x'}_{y}], &&\measuredangle[\tilde p\,{}^{\tilde y'}_{\tilde x}]\to\measuredangle[p\,{}^{y'}_{x}],\\ &\measuredangle[\tilde p\,{}^{\tilde x'}_{\tilde q'}]\to\measuredangle[p\,{}^{x'}_{q'}], &&\measuredangle[\tilde p\,{}^{\tilde y'}_{\tilde q'}]\to\measuredangle[p\,{}^{y'}_{q'}],\end{aligned}$$

It follows that

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}]\to\measuredangle[p\,{}^{x}_{y}],\qquad \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde q'}]\to\measuredangle[p\,{}^{x}_{q'}],\qquad \measuredangle[\tilde p\,{}^{\tilde y}_{\tilde q'}]\to\measuredangle[p\,{}^{y}_{q'}].$$

Therefore, passing to a partial limit as $\delta\to 0$, we get a configuration of 5 points $\tilde p,\tilde x,\tilde y,\tilde q'=\tilde q'(\varepsilon),\tilde z=\tilde z(\zeta)$ such that

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}]=\measuredangle[p\,{}^{x}_{y}],\qquad \measuredangle[\tilde p\,{}^{\tilde y}_{\tilde q'}]=\measuredangle[p\,{}^{y}_{q'}],\qquad \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde q'}]=\measuredangle[p\,{}^{x}_{q'}].$$

In other words, the map sending the points $0,u,v,w'\in\mathrm{T}_p$ to $\tilde p,\tilde x,\tilde y,\tilde q\in\mathbb H$ correspondingly is distance preserving.

Note that $q'\to q$ as $\varepsilon\to 0$. Therefore, in the limit, we get a configuration $\tilde p$, $\tilde x$, $\tilde y$, $\tilde q'$, $\tilde z=\tilde z(\zeta)$ such that in addition we have

$$\begin{aligned}&|\tilde q'-\tilde z|=|q-z|, &&|\tilde p-\tilde z|\geqslant|p-z|,\\ &|\tilde x-\tilde z|\geqslant|x-p|, &&|\tilde y-\tilde z|\geqslant|y-z|\end{aligned}$$

Since $w''\neq w'$, for small values $\zeta$ the last three inequalities imply

$$|\tilde q'-\tilde z|>|q-z|,$$

a contradiction.

**5.2. Proposition.** *If a complete CTIL Riemannian manifold $M$ satisfies 4(1)-tree comparison, then for any $p,q\in M$, we have $f''\leqslant 1$, where $f$ is the function $f\colon\mathrm{TIL}_p\to\mathbb R$ defined by*

$$f(v)=\tfrac12\cdot\mathrm{dist}_q^2\circ\exp_p(v).$$

*Proof.* Note that 4(1)-tree comparison implies 3-tree comparison. Hence $M$ has nonnegative sectional curvature.

Fix $u, v \in \mathrm{TIL}_p$ and $w \in [u\,v]$. It is sufficient to show that there is a function $g\colon \mathrm{T}_p \to \mathbb{R}$ such that

$$g'' = 1, \quad g(w) = f(w), \quad g(u) \geqslant f(u) \quad \text{and} \quad g(v) \geqslant f(v).$$

Fix small $\varepsilon > 0$ and set

$$\begin{aligned} x &= \exp_p u, & y &= \exp_p v, & z &= \exp_p w, \\ x' &= \exp_p(-\varepsilon\cdot u), & y' &= \exp_p(-\varepsilon\cdot v). \end{aligned}$$

Apply the $p/xyx'y'(z/q)$ comparison and pass to the limit as $\varepsilon \to 0$. We obtain a configuration of points $\tilde p, \tilde x, \tilde y, \tilde z, \tilde q \in \mathbb{H}$, satisfying corresponding comparisons and in addition

$$\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}] = \measuredangle[p\,{}^{x}_{y}], \qquad \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde z}] = \measuredangle[p\,{}^{x}_{z}], \qquad \measuredangle[\tilde p\,{}^{\tilde z}_{\tilde y}] = \measuredangle[p\,{}^{z}_{y}].$$

In particular, from above and Toponogov comparison, we have

$$\begin{aligned} |\tilde x - \tilde y|_{\mathbb{H}} &= |u - v|_{\mathrm{T}_p}, & |\tilde z - \tilde y|_{\mathbb{H}} &= |w - v|_{\mathrm{T}_p}, & |\tilde x - \tilde z|_{\mathbb{H}} &= |u - w|_{\mathrm{T}_p}, \\ |\tilde q - \tilde z|_{\mathbb{H}} &= |q - z|_M, & |\tilde q - \tilde x|_{\mathbb{H}} &\geqslant |q - x|_M, & |\tilde q - \tilde y|_{\mathbb{H}} &\geqslant |q - y|_M. \end{aligned}$$

In particular, there is a distance-preserving map $\mathrm{T}_p \to \mathbb{H}$ such that $u \mapsto \tilde x$, $v \mapsto \tilde y$, $w \mapsto \tilde z$ and $0 \mapsto \tilde p$. Further, we identify $\mathrm{T}_p$ and a subset of $\mathbb{H}$ using this map.

Consider the function $g(s) := \tfrac12\cdot|s - \tilde q|^2_{\mathrm{T}_p}$. Note that $g'' = 1$ and

$$\begin{aligned} g(w) &= \tfrac12\cdot|\tilde q - \tilde z|^2_{\mathrm{T}_p} = \tfrac12\cdot|q - z|^2_M = f(w), \\ g(u) &= \tfrac12\cdot|\tilde q - \tilde x|^2_{\mathrm{T}_p} \geqslant \tfrac12\cdot|q - x|^2_M = f(u), \\ g(v) &= \tfrac12\cdot|\tilde q - \tilde y|^2_{\mathrm{T}_p} \geqslant \tfrac12\cdot|q - y|^2_M = f(u). \end{aligned}$$

Hence the first statement follows.

**5.3. Proposition.** *Assume $M$ is a complete CTIL Riemannian manifold such that for any $p, q \in M$, we have $f'' \leqslant 1$, where $f$ is the function $f\colon \mathrm{TIL}_p \to \mathbb{R}$ defined by*

$$f(v) = \tfrac12\cdot \mathrm{dist}^2_q \circ \exp_p(v),$$

*Then $M$ satisfies all bipolar tree comparisons.*

*Proof.* Fix points $p$ and $q$ in $M$; set $\tilde q = \log_p q \in \mathrm{T}_p$ and $\tilde f(v) = \tfrac12\cdot|v - \tilde q|^2_{\mathrm{T}_p}$. Note that

$$f \leqslant \tilde f. \qquad ❶$$

Further note that the inequality ❶ is equivalent to the Toponogov comparison for all hinges $[p\,{}^{x}_{q}]$ in $M$. It follows that $M$ has nonnegative sectional curvature.

Fix a bipolar geodesic tree $[p/x_1 \dots x_n(q/y_1 \dots y_m)]$ in $M$. Set

$$\tilde p = 0 = \log_p p, \quad \tilde q = \log_p q, \quad \text{and} \quad \tilde x_i = \log_p x_i$$

for each $i$.

Consider the linear map $\psi_1 \colon \mathrm{T}_q \to \mathrm{T}_p$ such that for any smooth function $h$

$$\psi_1 \colon \nabla_q h \mapsto \nabla_{\tilde q}(h \circ \exp_p).$$

Since sectional curvature of $M$ is nonnegative, the restriction $\exp_p|_{\mathrm{TIL}_p}$ is short and therefore so is $\psi_1$.

In particular there is a linear map $\psi_2 \colon \mathrm{T}_q \to \mathrm{T}_p$ such that, the map $\iota \colon \mathrm{T}_q \to \to \mathrm{T}_p \oplus \mathrm{T}_p$ defined by

$$\iota \colon v \mapsto \psi_1(v) \oplus \psi_2(v)$$

is distance preserving.

Further set

$$h_i = \tfrac{1}{2} \cdot \mathrm{dist}^2_{y_i}, \quad g_i = h_i \circ \exp_p|_{\mathrm{TIL}_p}, \quad \tilde y_i = \tilde q - \iota(\nabla_q h_i).$$

By construction

$$|\tilde y_i - \tilde q|_{\mathrm{T}_p \oplus \mathrm{T}_p} = |y_i - q|_M.$$

At the point $\tilde q$ the restriction functions $\tilde g_i = \frac{1}{2} \cdot \mathrm{dist}^2_{\tilde y_i}|_{\mathrm{T}_p \oplus 0}$ and the function $g_i$ have the same value and gradient. Since $g_i'' \leqslant 1$ and $\tilde g_i'' = 1$, we get $\tilde g_i \geqslant g_i$. The latter implies

$$|\tilde y_i - \tilde p|_{\mathrm{T}_p \oplus \mathrm{T}_p} \geqslant |y_i - p|_M \quad \text{and} \quad |\tilde y_i - \tilde x_j|_{\mathrm{T}_p \oplus \mathrm{T}_p} \geqslant |y_i - x_j|_M.$$

for any $i$ and $j$.

Since there is an isometric embedding $\mathrm{T}_p \oplus \mathrm{T}_p \hookrightarrow \mathbb{H}$, we get the needed configuration. $\square$

# 6 MTW

The Proposition 6.1 below provides the equivalence of properties (*i*) and (*ii*) which finished the proof Theorem 1.2. The equivalence is proved by calculations along the same lines as in [20, Chapter 12].

Let us introduce notations and use them to reformulate the property (*ii*).

**Tangent vectors.** Let $M$ be a Riemannian manifold, $p \in M$. Denote by $\mathrm{IL}_p$ the *inner locus* of $p$; it can be defined as the $\exp_p$-image of $\mathrm{TIL}_p$ or, equivalently, as the complement $M \backslash \mathrm{CL}_p$, where $\mathrm{CL}_p$ denotes the cut locus of $p$. Note that $q \in \mathrm{IL}_p$ if and only if $p \in \mathrm{IL}_p$.

Assume $q \in \mathrm{IL}_p$; that is, $q = \exp_p W$ for some $W \in \mathrm{TIL}_p M$. Given a vector $Y \in T_q$, consider the unique vector $Y_p \in \mathrm{T}_p$ such that

$$Y = (d_W \exp_p) Y_p.$$

Note that $p = \exp_q(-W_q)$ if $p$, $q$ and $W$ are as above.

Given $x \in \mathrm{IL}_p$ such that $x = \exp_p X$ for some $X \in \mathrm{TIL}_p$, set

$$\tilde{Y}_p(x) = (d_X \exp_p) Y_p;$$

this way we defined a vector field $\tilde{Y}_p$ in $\mathrm{IL}_p$.

Note that in the vector field $\tilde{Y}_p$ is constant in the normal coordinates at $p$; in particular

❶ $$\nabla_X \tilde{Y}_p = 0$$

for any $X \in \mathrm{T}_p$. Further, note that

❷ $$Y \tilde{Y}_p f = Y \tilde{Y}_q f + (\nabla_Y \tilde{Y}_p) f$$

for $Y \in \mathrm{T}_q$ and any smooth function $f$. Indeed applying ❶, we get that

$$\begin{aligned}(Y\tilde{Y}_p - Y\tilde{Y}_q) f &= (\tilde{Y}_q \tilde{Y}_p - \tilde{Y}_p \tilde{Y}_q) f(q) = \\ &= (\nabla_Y \tilde{Y}_p - \nabla_Y \tilde{Y}_q) f = \nabla_Y \tilde{Y}_p f.\end{aligned}$$

**Column notation.** Given two points $p$ and $q$ in a Riemannian manifold $M$, let us define the cost function $(p, q) \mapsto \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ as

$$\begin{bmatrix} p \\ q \end{bmatrix} = \tfrac{1}{2} \cdot |p - q|^2_M$$

We will need to differentiate the cost function by both argument. In order to avoid possible confusion, we will write the vector next to the differentiated argument. For example

$$\begin{matrix} X \\ Y \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

is the second mixed derivative of the cost function at the pair $(p, q)$, once by the first argument ($p$) along the vector $X \in \mathrm{T}_p$ and once by the second argument ($q$) along the vector field $Y \in \mathrm{T}_q$. We may also write a vector field instead of the vectors.

Using the introduced notations, we can reformulate the property ($ii$) in Theorem 1.2 as

*(ii)′ If $X \in \mathrm{T}_p$, $Y \in \mathrm{T}_q$ and $q \in \mathrm{IL}_p$, then*

$$\begin{matrix} X\tilde{X}_p \\ Y\tilde{Y}_p \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} \leqslant 0.$$

The left hand side of the last inequality, multiplied by $(-\frac{3}{2})$ is the so called $\mathfrak{S}$-curvature; it is denoted by $\mathfrak{S}(X, Y)$, see [20, equation 12.21]; if $p = q$ then $\mathfrak{S}(X, Y)$ coincides with the curvature $\langle \mathrm{Rm}(X, Y)Y, X\rangle$, see [20, 12.30]. In particular, if the condition ($ii$) holds then the manifold has nonnegative sectional curvature.

Assume $q = \exp_p W$ for some $W \in \mathrm{TIL}_p$. Then

❸ $$X \begin{bmatrix} p \\ q \end{bmatrix} = -\langle X, W \rangle;$$

❹ $$\begin{matrix} X \\ Y \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} = -\langle X, Y_p \rangle = -\langle X_q, Y \rangle$$

and

❺ $$\begin{matrix} X \\ Y\tilde{Y}_p \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} = 0.$$

Indeed, ❸ is equivalent to the first variation formula. Taking the derivative of ❸ in the normal coordinates at $p$ we get ❹.

$$\begin{matrix} X \\ Y \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} = -\langle X, Y_p \rangle.$$

Since $q \in \mathrm{IL}_p$ if and only if $p \in \mathrm{IL}_q$ we can swap $p$ and $q$ and get the second identity in ❹. Finally, the value $\langle X, Y_p \rangle$ does not depend on $q$; therefore the derivative along the second argument ❹ has to vanish; hence ❺ follows.

Let us use the identities to show that

❻ $$\begin{matrix} X\tilde{X}_p \\ Y\tilde{Y}_p \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{matrix} X\tilde{X}_p \\ Y\tilde{Y}_p \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} \quad \text{or, equivalently} \quad \mathfrak{S}(X, Y) = \mathfrak{S}(Y, X).$$

This identity will not be used in the sequel, but it might help the reader to adapt to the column notation.

Applying ❷, we get that

$$\begin{aligned} \begin{matrix} X\tilde{X}_p \\ Y\tilde{Y}_p \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} &= \begin{matrix} X\tilde{X}_p \\ Y\tilde{Y}_q \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{matrix} X\tilde{X}_p \\ \nabla_Y\tilde{Y}_q \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} = \\ &= \begin{matrix} X\tilde{X}_p \\ Y\tilde{Y}_q \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{matrix} X\tilde{X}_q \\ \nabla_Y\tilde{Y}_q \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} - \begin{matrix} \nabla_X\tilde{X}_p \\ \nabla_Y\tilde{Y}_q \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} \end{aligned}$$

By ❺,

$$\begin{matrix} X\tilde{X}_q \\ \nabla_Y\tilde{Y}_q \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} = 0.$$

Therefore

$$\begin{matrix} X\tilde{X}_p \\ Y\tilde{Y}_p \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{matrix} X\tilde{X}_p \\ Y\tilde{Y}_q \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix} - \begin{matrix} \nabla_X\tilde{X}_p \\ \nabla_Y\tilde{Y}_q \end{matrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

The right hand side is symmetric in $p$ and $q$; hence ❻ follows.

**6.1. Proposition.** *Let $M$ be a CTIL Riemannian manifold. Then the following conditions are equivalent:*

*(a) For any $p\in M$, $q\in \mathrm{IL}_p$, $X\in \mathrm{T}_p$ and $Y\in\mathrm{T}_q$ we have*

$$\begin{array}{r} X\tilde X_p \\ Y\tilde Y_p \end{array}\begin{bmatrix} p \\ q \end{bmatrix} \leqslant 0.$$

*(b) For any $p_0,p_1\in M$, the function $h\colon \mathrm{TIL}_{p_0}\to\mathbb{R}$ defined by*

$$h(X)=\left[\substack{p_1\\ \exp_{p_0} X}\right]-\left[\substack{p_0\\ \exp_{p_0} X}\right]$$

*is concave;*

*(c) For any $p_0,p_1\in M$, the function $f\colon \mathrm{TIL}_{p_0}\to\mathbb{R}$ defined by*

$$f(X)=\left[\substack{p_1\\ \exp_{p_0} X}\right]$$

*is 1-concave.*

*Proof.* Note that $s(X)=\tfrac12\cdot|X|^2=\left[\substack{p\\ \exp_q X}\right]$, in particular the function $s$ is 1-affine (that is 1-concave and 1-convex at the same time).

Evidently $f=h+s$, therefore $(b)\iff(c)$.

*(a) $\Rightarrow$ (b).* Note that the function $h$ is semiconcave. Therefore $h$ is concave if

$$\tfrac{d^2}{dt^2}h(U+t\cdot V)\leqslant 0$$

at $t=0$ for almost all vectors $U\in\mathrm{TIL}_p$ and $V\in\mathrm{T}_p$.

Using the column notation, we can rewrite the inequality in the following equivalent form:

❼
$$\begin{array}{r} \\ Y\tilde Y_{p_0} \end{array}\left(\begin{bmatrix} p_1 \\ q \end{bmatrix}-\begin{bmatrix} p_0 \\ q \end{bmatrix}\right)\leqslant 0$$

for any $p_0,p_1,q$ and $Y\in\mathrm{T}_q$; from above it is sufficient to prove ❼ for almost all $q$; in particular, we can assume that $p_0,p_1\in\mathrm{IL}_q$.

Let $W,X\in\mathrm{TIL}_q$ be such that $p_0=\exp_q W$, $p_1=\exp_q(W+X)$. Since $M$ is CTIL, $W+t\cdot X\in\mathrm{TIL}_q$ for any $t\in[0,1]$; set $p_t=\exp_q(W+t\cdot X)$.

Let us use the identity $f(1)-f(0)-f'(0)=\int_0^1 f''(t)\cdot(1-t)\cdot dt$, for the function

$$f(t)=\begin{array}{r} \\ Y\tilde Y_q \end{array}\begin{bmatrix} p_t \\ q \end{bmatrix};$$

Note that

$$f'(0)=\begin{array}{r} X \\ Y\tilde Y_q \end{array}\begin{bmatrix} p_0 \\ q \end{bmatrix}\quad\text{and}\quad f''(t)=\begin{array}{r} \tilde X_q\tilde X_q \\ Y\tilde Y_q \end{array}\begin{bmatrix} p_t \\ q \end{bmatrix},$$

therefore

$$\begin{array}{r} \\ Y\tilde Y_q \end{array}\begin{bmatrix} p_1 \\ q \end{bmatrix}-\begin{array}{r} \\ Y\tilde Y_q \end{array}\begin{bmatrix} p_0 \\ q \end{bmatrix}-\begin{array}{r} X \\ Y\tilde Y_q \end{array}\begin{bmatrix} p_0 \\ q \end{bmatrix}=\int\limits_0^1 \begin{array}{r} \tilde X_q\tilde X_q \\ Y\tilde Y_q \end{array}\begin{bmatrix} p_t \\ q \end{bmatrix}\cdot dt.$$

By $(a)$, the term under the integral is nonpositive; therefore

$$
{}_{Y\tilde{Y}_q}\left(\begin{bmatrix}p_1\\q\end{bmatrix}-\begin{bmatrix}p_0\\q\end{bmatrix}\right)\leqslant {}^{\quad X}_{Y\tilde{Y}_q}\begin{bmatrix}p_0\\q\end{bmatrix}.
$$

By ❷, we can rewrite the last inequality the following way:

❽
$$
{}_{Y\tilde{Y}_{p_0}}\left(\begin{bmatrix}p_1\\q\end{bmatrix}-\begin{bmatrix}p_0\\q\end{bmatrix}\right)\leqslant {}_{\nabla_Y\tilde{Y}_{p_0}}\left(\begin{bmatrix}p_1\\q\end{bmatrix}-\begin{bmatrix}p_0\\q\end{bmatrix}\right)+{}^{\quad X}_{Y\tilde{Y}_q}\begin{bmatrix}p_0\\q\end{bmatrix}.
$$

Applying ❸, we get that

$$
\begin{aligned}
{}_{\nabla_Y\tilde{Y}_{p_0}}\left(\begin{bmatrix}p_1\\q\end{bmatrix}-\begin{bmatrix}p_0\\q\end{bmatrix}\right)&=-\langle\nabla_Y\tilde{Y}_{p_0},W+X\rangle+\langle\nabla_Y\tilde{Y}_{p_0},W\rangle=\\
&=-\langle\nabla_Y\tilde{Y}_{p_0},X\rangle.
\end{aligned}
$$

Further, applying ❺ and ❹, we get that

$$
\begin{aligned}
{}^{\quad X}_{Y\tilde{Y}_q}\begin{bmatrix}p_0\\q\end{bmatrix}&={}^{\quad X}_{Y\tilde{Y}_{p_0}}\begin{bmatrix}p_0\\q\end{bmatrix}-{}^{\qquad X}_{\nabla_Y\tilde{Y}_{p_0}}\begin{bmatrix}p_0\\q\end{bmatrix}=\\
&=0+\langle\nabla_Y\tilde{Y}_{p_0},X\rangle.
\end{aligned}
$$

It follows that the right hand side in ❽ vanishes; hence ❼ follows.

*(b) ⇒ (a).* Let $p_t$, $q$, $W$, $X$ and $Y$ be as above; set

$$
h_t(Z)=\begin{bmatrix}p_t\\\exp_{p_0}Z\end{bmatrix}-\begin{bmatrix}p_0\\\exp_{p_0}Z\end{bmatrix}.
$$

Note that $h_0\equiv 0$; in particular

$$
Y\tilde{Y}_{p_0}h_0=0
$$

for any $Y\in\mathrm{T}_q$.

By $(b)$,

$$
Y\tilde{Y}_{p_0}h_t\leqslant 0.
$$

It follows that

$$
\frac{d^2}{dt^2}(Y\tilde{Y}_{p_0}h_t)\leqslant 0
$$

at $t=0$. Finally note that

$$
{}^{X\tilde{X}_{p_0}}_{Y\tilde{Y}_{p_0}}\begin{bmatrix}p_0\\q\end{bmatrix}=\frac{d^2}{dt^2}(Y\tilde{Y}_{p_0}h_t);
$$

hence the part $(b)$ follows. □

## 7 All tree comparisons

*Proof of Theorem 1.5.* The "if" part is left as an exercise; let us prove the "only if" part.

Fix a point array $a_1, \dots, a_n$ in $X$. Consider the complete graph $K_n$ with the vertexes labeled by $a_1, \dots, a_n$.

Let $\hat K_n \to K_n$ be the universal covering. Denote by $\hat V$ the set of vertexes of $\hat K_n$; given a vertex $\hat v \in \hat V$ denote by $v$ the corresponding vertex of $K_n$.

Applying the tree comparison for finite subtrees in $\hat K_n$ and passing to a partial limit, we get the following:

($*$) There is a map $f\colon \hat V \to \mathbb{H}$ such that

$$|f(\hat v) - f(\hat w)|_{\mathbb{H}} \geqslant |v - w|_X$$

for any two vertexes $\hat v, \hat w \in \hat V$ and the equality holds if $(\hat v, \hat w)$ is an edge in $\hat K_n$.

It finishes the proof if $X$ is finite.

Since $X$ is separable, it has a countable everywhere dense set $\{a_1, a_2, \dots\}$. Applying the statement above for $X_n = \{a_1, \dots a_n\}$, we get a submetry from $Y_n = f_n(\hat V_n) \subset \mathbb{H}$ to $X_n$.

It remains to pass to the ulralimit $Y$ of the subspaces $Y_n$. Clearly $Y$ admits an isometric embedding into $\mathbb{H}$ and it admits submetry on $Y \to X$. Hence the statement follows. □

The following proof was suggested by Alexander Lytchak, it is simplified version of the construction of Chuu-Lian Terng and Gudlaugur Thorbergsson given in [17, Section 4].

*Proof of Proposition 1.6.* Denote by $G^n$ the direct product of $n$ copies of $G$. Consider the map $\varphi_n\colon G^n \to G /\!\!/ H$ defined by

$$\varphi_n\colon (\alpha_1, \dots, \alpha_n) \mapsto [\alpha_1 \cdots \alpha_n]_H,$$

where $[x]_H$ denotes the $H$-orbit of $x$ in $G$.

Note that $\varphi_n$ is a quotient map for the action of $H \times G^{n-1}$ on $G^n$ defined by

$$(\beta_0, \dots, \beta_n) \cdot (\alpha_1, \dots, \alpha_n) = (\gamma_1 \cdot \alpha_1 \cdot \beta_1^{-1}, \beta_1 \cdot \alpha_2 \cdot \beta_2^{-1}, \dots, \beta_{n-1} \cdot \alpha_n \cdot \beta_n^{-1}),$$

where $\beta_i \in G$ and $(\beta_0, \beta_n) \in H < G \times G$.

Denote by $\rho_n$ the product metric on $G^n$ rescaled with factor $\sqrt{n}$. Note that the quotient $(G^n, \rho_n)/(H \times G^{n-1})$ is isometric to $G /\!\!/ H = (G, \rho_1) /\!\!/ H$.

As $n \to \infty$ the curvature of $(G^n, \rho_n)$ converges to zero and its injectivity radius goes to infinity. Therefore passing to the ultra-limit of $G^n$ as $n \to \infty$ we get the Hilbert space. It remains to observe that the limit action has the required property. □

## 8 Remarks

**On graph comparison.** Analogously to the tree comparison one can define *graph comparison* for any graph by stating that there is a model configuration in $\mathbb{H}$ such that

- ⋄ the distance between each pair of adjacent points is at most as big and
- ⋄ the distance between each pair of nonadjacent is at least as big.

**8.1. Exercise.** *Show that if a graph is a tree, then the graph comparison defined above is equivalent to the tree comparison defined at the beginning of the paper.*

Note that nonnegative and nonpositive curvature can be defined using the comparison for following two graphs on 4 vertexes:

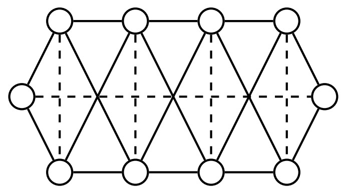
If a graph $G$ has two induced subgraphs that isomorphic to each of these two graphs, then the corresponding graph comparison implies that the curvature vanish in the sense of Alexandrov. In particular, any complete length spaces satisfying $G$-graph comparison is isometric to a convex set in a Hilbert space.

By Reshetnyak majorization theorem, the nonpositive curvature could be also defined using the comparison for cycle; for example the 6-cycle — the first graph the following diagram.

The comparison for the octahedron graph (the second graph on the diagram) implies that the space is nonpositively curved. The latter follows since in this graph, a 4-cycle appears as an induced subgraph. On the other hand, this comparison might be stronger and it might be interesting to understand.

The quotients of Hilbert space provide motivating examples to consider the tree comparison (see Theorem 1.5). Unfortunately we do not have such guiding examples in nonpositive curvature spaces — we can only fumble around for something without light.

**On colored graph comparison.** It is also possible to use graph with $(\mp)$-colored edges and define comparison by model configuration such that the distances between vertexes adjacent by a $(-)$-edge does not get larger and by $(+)$-edge does not get smaller (no condition on the remaining pairs).



For example, the $(2{\cdot}n{+}2)$-comparison (which holds in CAT[0] length spaces, see [1]) can be considered as a comparison for the colored graph above, where $(-)$-edges are marked by solid lines and $(+)$-edges by dashed lines.

**Finite subsets of Alexandrov spaces.** The following problem discussed in [1, 7.1] was one of the original motivations to study the tree comparison: *Which finite metric spaces admit isometric embeddings into some Alexandrov spaces with nonnegative curvature.*

This problem is still open (as well as its analog for CAT(0) spaces). According to [1, 4.1], the $(n-1)$-tree comparison provides a necessary condition for the problem $n$-point metric spaces. An other well known necessary condition, the so called *matrix inequality* turns out to be weaker than $(n-1)$-tree comparison; see the discussion below. This condition is sufficient for the 4-point metric spaces. It might be still sufficient for 5-point metric spaces, but not for 6-point metric spaces.

The corresponding example of 6-point metric space was constructed by Sergei Ivanov, see [1]. Theorem 4.1, provides a source for such examples — any 6-point metric space that satisfy all 5-tree comparisons, but does not satisfy 2(2)-tree comparison provides an example. This class of examples includes the example of Sergei Ivanov — in the notations of [1, 7.1] it does not satisfies the comparison for the tree $y/az(q/xb)$.

By Theorem 4.1 and a theorem in [1], 5-tree and 2(2)-tree comparisons provide a necessary condition for 6-point metric spaces. We expect that these conditions are sufficient

Here an other candidate for a sufficient condition.

**8.2. Question.** *Assume $F$ is a finite metric space that satisfies all tree comparisons. Is it true that $F$ is isometric to a subset of an Alexandrov space with nonnegative curvature?*

Note that even for finite metric space the all tree comparison has to be checked for an infinite set of trees since one point of the space may be used as a label for several vertexes in the tree.

There is a chance that for 5-point and 6-point metric spaces, the condition in Question 8.2 is also necessary. However, since there are nonnegatively curved Riemannian manifolds that do not satisfy 4(1)-tree comparison, Theorem 1.2 implies that this condition can not be necessary for 7-point metric spaces.

For any metric space $X$ with an isometric group action $G \curvearrowright X$ with closed orbits the quotient map $X \to X/G$ is a submetry. In particular, by Theorem 1.5, if $G \curvearrowright \mathbb{H}$ is an isometric action with closed orbits on the Hilbert space, then the quotient space $\mathbb{H}/G$ satisfies all tree comparisons.

**8.3. Question.** *Assume $X$ is a metric space satisfying all tree comparisons. Is it always possible to construct an isometric group action with closed orbits on the Hilbert space $G \curvearrowright \mathbb{H}$ such that $X$ is isometric to a subset in $\mathbb{H}/G$?*

**On matrix inequality.** The comparison for monopolar trees has an algebraic corollary which was used Urs Lang and Viktor Schroeder in [7], a similar inequality was used by Karl-Theodor Sturm in [16].

Namely, given a point array $p, x_1, \dots, x_n$ in a metric space $X$ consider the $n{\times}n$-matrix $M$ with the components

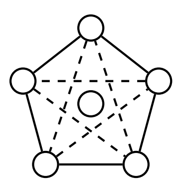
$$m_{i,j} = \tfrac{1}{2}\cdot(|x_i - p|^2 + |x_j - p|^2 - |x_i - x_j|^2).$$

If the tree comparison for $p/x_1, \dots, x_n$ holds, then

❶ $$\boldsymbol{s}\cdot M\cdot\boldsymbol{s}^\top \geqslant 0$$

for any vector $\boldsymbol{s} = (s_1, \dots, s_n)$ with nonnegative components.

The converse does not hold; that is, for some point array $p, x_1, \dots, x_n$ in a metric space the inequality ❶ might hold, while the tree comparison for $p/x_1x_2x_3x_4x_5$ does not. (We do not know an explicit way to describe tree comparisons using a system of inequalities.)



An example can be constructed by perturbing the configuration on the plane as on the diagram — if the diameter of diagram is 1, then increasing the distances between the pairs of points connected by dashed lines by $\varepsilon = 10^{-9}$ and decreasing the distances between the pairs of points connected by sold lines by $\delta = 10^{-6}$ does the job. The obtained metric 6-point metric space satisfies the matrix inequality with center at each point, but does not satisfy the tree comparison with the pole at the central point.

Many necessary conditions on finite subsets of nonnegatively curved Alexandrov spaces are known; in addition to the comparisons discussed above, let us mention authors results in [8] and [13] and the Markov type inequality proved by Shin-ichi Ohta in [12]; see also the survey by Assaf Naor [11] on the Ribe program.

**On tree comparisons in length spaces.** Note that if a tree $T$ is not a path, then it contains a tripod as subtree. Therefore $T$-tree comparison implies Alexandrov comparison, in particular any complete length-metric space satisfying $T$-tree comparison is a nonnegatively curved Alexandrov space.

It is straightforward to generalize Theorem 1.1 to Alexandrov spaces; that is, *a complete length space $L$ satisfies 3(1) or 2(2)-tree comparison if and only if $L$ is a nonnegatively curved Alexandrov space.* We expect that Theorem 1.2 (after appropriate reformulation) can be also generalized to Alexandrov spaces — the only obstacle we see is the proof of Proposition 5.1. Such a generalization would characterize length-metric spaces satisfying most of 4(1)-tree comparison (as well as most of bipolar tree comparisons).

The Alexandrov spaces which satisfy 4(1) seem to have properties which remind the quotients of Riemannian manifolds by isometric group actions. For example we expect that if a finite dimensional Alexandrov space $A$ satisfies 4(1)-tree, then the tangent space at any point $p \in A$ is a product of a Euclidean space $\mathbb{E}^k$ and a cone $K$ over space $\Sigma$ of diameter at most $\frac{\pi}{2}$ (the space $\Sigma$ might be empty, in this case $K$ is a one-point space). In particular it implies that the set of all metric singularities of $A$ is an extremal subset, see [14]. (For big branchy trees, the properties of spaces with the tree comparison should remind the quotients of Hilbert space even more.)

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