Graph comparison meets Alexandrov

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Abstract

Graph comparison is a certain type of condition on metric space encoded by a finite graph. We show that any nontrivial graph comparison implies one of two Alexandrov's comparisons. The proof gives a complete description of graphs with trivial graph comparisons.

The notion of graph comparison was introduced in [7]. It was studied further in [3–6, 10, 11]. Let us mention some of the results.

- ⋄ Graph comparison captures nonnegative and nonpositive curvature in the sense of Alexandrov.
- ⋄ Graph comparison for certain trees is used to formulate a stronger version of the so-called *Lang–Schroeder–Sturm inequality* [2, 9].
- ⋄ The all-tree comparison gives a metric description of target spaces of submersions from subets of Hilbert space.
- ⋄ For a certain tree, graph comparison has tight relation with the so-called *MTW condition* that was introduced by Xi-Nan Ma, Neil Trudinger, and Xu-Jia Wang [8].
- ⋄ Octahedron comparison holds in products of trees.

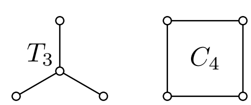
We will show that any nontrivial graph comparison implies one of two Alexandrov's comparisons.

Let us start with the definition. Suppose Γ is a graph with vertices $v_1, \dots, v_n$. We write $v_i \sim v_j$ (or $v_i \nsim v_j$) if v_i is adjacent (respectively nonadjacent) to v_j.

A metric space X meets the Γ-*comparison* if for any n points in X labeled by vertices of Γ there is a model configuration $\tilde v_1, \dots, \tilde v_n$ in the Hilbert space $\mathbb{H}$ such that

$$\begin{aligned} v_i \sim v_j \quad &\Longrightarrow \quad |\tilde v_i - \tilde v_j|_{\mathbb{H}} \leqslant |v_i - v_j|_X, \\ v_i \nsim v_j \quad &\Longrightarrow \quad |\tilde v_i - \tilde v_j|_{\mathbb{H}} \geqslant |v_i - v_j|_X; \end{aligned}$$

here $|-|_X$ denotes distance in the metric space X. (Note that v_i may refer to a vertex in Γ and to the corresponding point in X.)



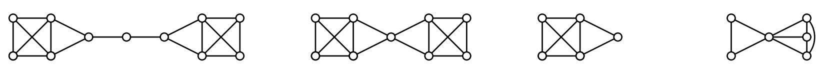
Denote by T_3 and C_4 and tripod and four-cycle shown on the diagram. The C_4-comparison is equivalent to nonnegative curvature, and T_3-comparison is equivalent to the nonpositive curvature in the sense of Alexandrov [7]. These definitions are usually applied to length spaces, but they can be applied to general metric spaces; the latter convention is used in [1].

Theorem. *Let Γ be an arbitrary finite graph. Then either Γ-comparison holds in any metric space, or it implies C_4- or T_3-comparison.*

The following corollary describes all graphs Γ with trivial Γ-comparison; it follows from the proof of the theorem.

Corollary. *Let Γ be a finite connected graph. Suppose that Γ-comparison is trivial; that is, it holds in any metric space. Then Γ can be constructed from a path P_ℓ of length $\ell \geqslant 0$ and two complete graphs K_{m_1}, K_{m_2} by attaching k_1 vertices of K_{m_1} to the left end of P_ℓ and k_2 vertices of K_{m_2} to the right end of P_ℓ.*

Note that the graph Γ in the corollary is described by a 5-array of integers $(m_1, k_1, \ell, k_2, m_2)$ such that $m_i \geqslant k_i$, $\ell \geqslant 0$, and $k_i > 0$ if $n_i > 0$. Examples of graphs with their 5-arrays are given on the diagram.



$(4,2,2,2,4)$ $(4,2,0,2,4)$ $(4,2,0,0,0)$ $(2,2,0,3,3)$

Proof

Suppose Γ has connected components $\Gamma_1, \dots, \Gamma_k$. Observe that Γ-comparison holds in a metric space X if and only if so does every Γ_i-comparison. Therefore we can assume that Γ is connected.

Suppose Γ is a graph with vertices $v_1, \dots, v_n$ as before. Let e be an edge in Γ; we can assume that it connects v_1 to v_2. Remove v_1 and v_2 from Γ and add a new vertex w such that for any other vertex u we have

◇ if $u \sim v_1$ and $u \sim v_2$, then $u \sim w$;
◇ if $u \nsim v_1$ and $u \nsim v_2$, then $u \nsim w$;
◇ in the remaining cases we can choose arbitrarily $u \sim w$ or $u \nsim w$.

Denote the obtained graph by Γ'.

Applying the definition of Γ-comparison assuming that $v_1 = v_2$ in X, we get the following.

Claim. *If Γ-comparison holds in a metric space X, then so does Γ'-comparison.*

The operation that produces Γ' from Γ will be called *edge shrinking*. If a graph Δ can be obtained from Γ applying edge shrinking several times, then we will write $\Delta \prec \Gamma$.

Note that the claim implies the following two statements:

◇ If Δ is an induced subgraph of a connected finite graph Γ, then $\Delta \prec \Gamma$.
◇ If $\Delta \prec \Gamma$, then Γ-comparison implies Δ-comparison.

Taking all the above into account, we get the following reformulation of the theorem.

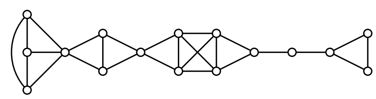
Reformulation. *For any finite connected graph Γ*
(a) Γ-comparison holds in any metric space, or

(b) $C_4 \prec \Gamma$*, or*
(c) $T_3 \prec \Gamma$.

A connected graph will be called *multipath* if it has an integer function ℓ on the set of the vertex set such that v is adjacent to w if and only if

$$|\ell(v) - \ell(w)| \leqslant 1.$$

The value $\ell(w)$ will be called the *level* of the vertex w. Multipath is completely described by a sequence of integers that give the number of vertexes on each level. For example, on the diagram you see multipath $(3, 1, 2, 1, 2, 2, 1, 1, 1, 2)$.



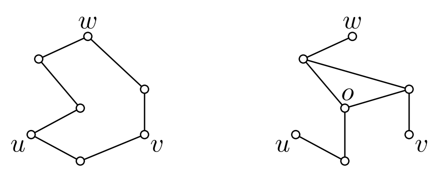
Lemma. *Let Γ be a connected finite graph such that $C_4 \not\prec \Gamma$ and $T_3 \not\prec \Gamma$. Then Γ is a multipath.*

Proof of the lemma. Let us denote by $|\ -\ |_\Gamma$ the path metric on the vertex set of Γ; it is equal to the number of edges in a shortest path connecting two vertices. Note that it is sufficient to show that

$$(*)\ \ |u-w|_\Gamma \geqslant |u-v|_\Gamma \geqslant |v-w|_\Gamma \geqslant 2 \quad \Longrightarrow \quad |u-w|_\Gamma = |u-v|_\Gamma + |v-w|_\Gamma$$

for any three vertices u, v, and w in Γ.

Suppose $(*)$ does not hold for u, v, and w. Let us pass to a minimal connected induced subgraph $\Delta \ni u, v, w$ of Γ such that $(*)$ still does not hold in Δ. Note



that Δ is either a cycle or it has three paths from a vertex, say o, to each of u, v, and w such that each of these paths do not visit the remaining vertices in the triple u, v, w. In these cases, we have $C_4 \prec \Delta$ and $T_3 \prec \Delta$ respectively. By the observation above $\Delta \prec \Gamma$ — the lemma is proved. $\square$

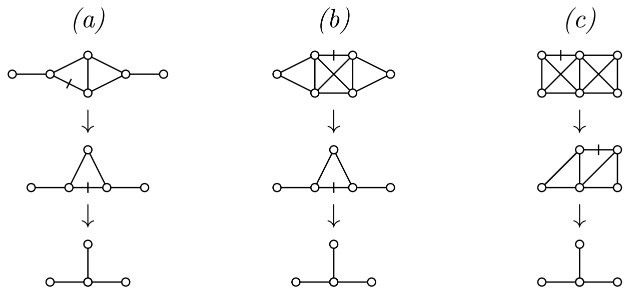
Proposition. *Let Γ be a multipath with sequence $(k_0, \dots, k_m)$. Suppose $C_4 \not\prec \Gamma$ and $T_3 \not\prec \Gamma$. Then*
(a) If $m \geqslant 4$, then $k_2 = \dots = k_{m-2} = 1$.
(b) If $m = 3$, then $k_1 = 1$ or $k_2 = 1$.
(c) If $m = 2$, then $k_0 = 1$, $k_1 = 1$, or $k_2 = 1$.

Proof of the proposition. Assuming the contrary in each case we get

(a). If $m \geqslant 4$, then multipath $(1, 1, 2, 1, 1)$ is an induced subgraph of Γ.

(b). If $m = 3$, then multipath $(1, 2, 2, 1)$ is an induced subgraph of Γ.

(c). If $m = 2$, then multipath $(2, 2, 2)$ is an induced subgraph of Γ.



In each case, we arrive at a contradiction by applying edge shrinking to the marked edges as shown on the diagram. □

It remains to show that Γ-comparison holds in any metric space for every multipath Γ described in 6. This is done by prescribing the coordinates for the needed model configuration on the real line.

Each edge of Γ comes with weight — the distance between the endpoints in X. Define the distance $\|v - w\|_\Gamma$ as the minimal total weight of paths connecting v to w in Γ. Note that

$$\|v - w\|_\Gamma \geqslant |v - w|_X$$

for any v and w.

If $m \leqslant 1$ then Γ is a complete graph. In this case, Γ-comparison is trivial. It remains to consider cases $m \geqslant 2$.

Let us choose a special vertex w that is unique on its level and not too far from the middle of Γ. Namely, if $m \geqslant 4$, then choose w on the second level; by the proposition it is unique on its level. If $m = 3$, then by the proposition we can assume that $k_2 = 1$; in this case choose w on the second level. Finally, if $m = 2$, let w be any vertex that is unique on its level; it exists by the proposition.

For every vertex v_i, let

$$\tilde{v}_i = \pm\|w - v_i\|_\Gamma,$$

where the sign is plus if v_i has a higher level than w and minus otherwise. By the triangle inequality, the obtained configuration $\tilde{v}_1, \ldots, \tilde{v}_n$ meets the condition of Γ-comparison. □

Acknowledgments. The first author was partially supported by the Russian Foundation for Basic Research grant 20-01-00070; the second author was partially supported by the National Science Foundation grant DMS-2005279.

References

[1] S. Alexander, V. Kapovitch, and A. Petrunin. *Alexandrov geometry: foundations*. 2022. arXiv: 1903.08539 [math.DG].

[2] U. Lang and V. Schroeder. "Kirszbraun's theorem and metric spaces of bounded curvature". *Geom. Funct. Anal.* 7.3 (1997), 535–560.

[3] N. Lebedeva. "On open flat sets in spaces with bipolar comparison". *Geom. Dedicata* 203 (2019), 347–351.

[4] N. Lebedeva and A. Petrunin. "5-point CAT(0) spaces after Tetsu Toyoda". *Anal. Geom. Metr. Spaces* 9.1 (2021), 160–166.

[5] N. Lebedeva and A. Petrunin. *5-point Toponogov theorem*. 2022. arXiv: 2202.13049 [math.DG].

[6] N. Lebedeva and A. Petrunin. *Trees meet octahedron comparison*.

[7] N. Lebedeva, A. Petrunin, and V. Zolotov. "Bipolar comparison". *Geom. Funct. Anal.* 29.1 (2019), 258–282.

[8] X.-N. Ma, N. Trudinger, and X.-J. Wang. "Regularity of potential functions of the optimal transportation problem". *Arch. Ration. Mech. Anal.* 177.2 (2005), 151–183.

[9] K. T. Sturm. "Metric spaces of lower bounded curvature". *Exposition. Math.* 17.1 (1999), 35–47.

[10] T. Toyoda. "An intrinsic characterization of five points in a CAT(0) space". *Anal. Geom. Metr. Spaces* 8.1 (2020), 114–165.

[11] T. Toyoda. *A non-geodesic analogue of Reshetnyak's majorization theorem*. 2019. arXiv: 1907.09067 [math.MG].

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