Graph comparison meets Alexandrov

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Abstract

Graph comparison is a certain type of condition on metric space encoded by a finite graph. We show that any nontrivial graph comparison implies one of two Alexandrov's comparisons. The proof gives a complete description of graphs with trivial graph comparisons.

The notion of graph comparison was introduced in [7]. It was studied further in [3–6, 10, 11]. Let us mention some of the results.

- ◇ Graph comparison captures nonnegative and nonpositive curvature in the sense of Alexandrov.
- ◇ Graph comparison for certain trees is used to formulate a stronger version of the so-called *Lang–Schroeder–Sturm inequality* [2, 9].
- ◇ The all-tree comparison gives a metric description of target spaces of submersions from subets of Hilbert space.
- ◇ For a certain tree, graph comparison has tight relation with the so-called *MTW condition* that was introduced by Xi-Nan Ma, Neil Trudinger, and Xu-Jia Wang [8].
- ◇ Octahedron comparison holds in products of trees.

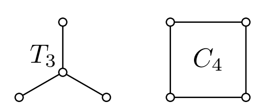
We will show that any nontrivial graph comparison implies one of two Alexandrov's comparisons.

Let us start with the definition. Suppose Γ is a graph with vertices $v_1, \dots, v_n$. We write $v_i \sim v_j$ (or $v_i \nsim v_j$) if v_i is adjacent (respectively nonadjacent) to v_j.

A metric space X meets the Γ-*comparison* if for any n points in X labeled by vertices of Γ there is a model configuration $\tilde v_1, \dots, \tilde v_n$ in the Hilbert space $\mathbb{H}$ such that

$$\begin{aligned} v_i \sim v_j \quad &\Longrightarrow \quad |\tilde v_i - \tilde v_j|_{\mathbb{H}} \leqslant |v_i - v_j|_X, \\ v_i \nsim v_j \quad &\Longrightarrow \quad |\tilde v_i - \tilde v_j|_{\mathbb{H}} \geqslant |v_i - v_j|_X; \end{aligned}$$

here $|\ -\ |_X$ denotes distance in the metric space X.



Denote by T_3 and C_4 and tripod and four-cycle shown on the diagram. The C_4-comparison is equivalent to nonnegative curvature, and T_3-comparison is equivalent to the nonpositive curvature in the sense of Alexandrov [7]. These definitions are usually applied to length spaces, but they can be applied to general metric spaces; the latter convention is used in [1].

Theorem. *Let Γ be an arbitrary finite graph. Then either Γ-comparison holds in any metric space, or it implies C_4- or T_3-comparison.*

The proposition below gives describes all graphs Γ with trivial Γ-comparison.

Proof. Suppose Γ has connected components $\Gamma_1, \dots, \Gamma_k$. Observe that Γ-comparison holds in a metric space X if and only if so does every Γ_i-comparison. Therefore we can assume that Γ is connected.

Suppose Γ is a graph with vertices $v_1, \dots, v_n$ as before. Let e be an edge in Γ; we can assume that it connects v_1 to v_2. Remove v_1 and v_2 from Γ and add a new vertex w such that for any other vertex u we have

- ⋄ if $u \sim v_1$ and $u \sim v_2$, then $u \sim w$;
- ⋄ if $u \nsim v_1$ and $u \nsim v_2$, then $u \nsim w$;
- ⋄ in the remaining cases we can choose arbitrarily $u \sim w$ or $u \nsim w$.

Denote the obtained graph by Γ'.

Applying the definition of Γ-comparison with $v_1 = v_2$, we get the following.

Claim. *If Γ-comparison holds in a metric space X, then so does Γ'-comparison.*

The operation that produces Γ' from Γ will be called *edge shrinking*. If a graph Δ can be obtained from Γ applying edge shrinking several times, then we will write $\Delta \prec \Gamma$.

Note that the claim implies the following two statements:

- ⋄ If Δ is an induced subgraph of a connected finite graph Γ, then $\Delta \prec \Gamma$.
- ⋄ If $\Delta \prec \Gamma$, then Γ-comparison implies Δ-comparison.

Taking all the above into account, we get the following reformulation of the theorem.

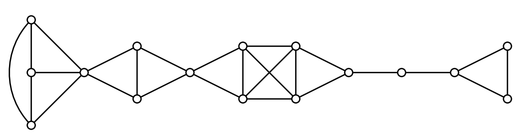
Reformulation. *For any finite connected graph Γ*

(a) Γ-comparison holds in any metric space, or
(b) $C_4 \prec \Gamma$, or
(c) $T_3 \prec \Gamma$.

A connected graph will be called *multipath* if it has an integer function ℓ on the set of the vertex set such that v is adjacent to w if and only if

$$|\ell(v) - \ell(w)| \leqslant 1.$$

The value $\ell(w)$ will be called the *level* of the vertex w. Multipath is completely described by a sequence of integers that give the number of vertexes on each level. For example, the multipath shown on the diagram can be described by



the sequence $(3, 1, 2, 1, 2, 2, 1, 1, 1, 2)$.

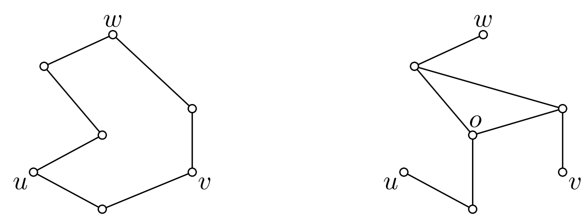
Lemma. *Let Γ be a connected finite graph such that $C_4 \not\prec \Gamma$ and $T_3 \not\prec \Gamma$. Then Γ is a multipath.*

Proof of the lemma. Let us denote by $| - |_\Gamma$ the path metric on the vertex set of Γ; it is equal to the number of edges in a shortest path connecting two vertices. Note that it is sufficient to show that

$$(*)\ \ |u-w|_\Gamma \geqslant |u-v|_\Gamma \geqslant |v-w|_\Gamma \geqslant 2 \quad \Longrightarrow \quad |u-w|_\Gamma = |u-v|_\Gamma + |v-w|_\Gamma$$

for any three vertices u, v, and w in Γ.

Suppose $(*)$ does not hold for u, v, and w. Let us pass to a minimal connected induced subgraph $\Delta \ni u, v, w$ of Γ such that $(*)$ still does not hold in Δ. Note



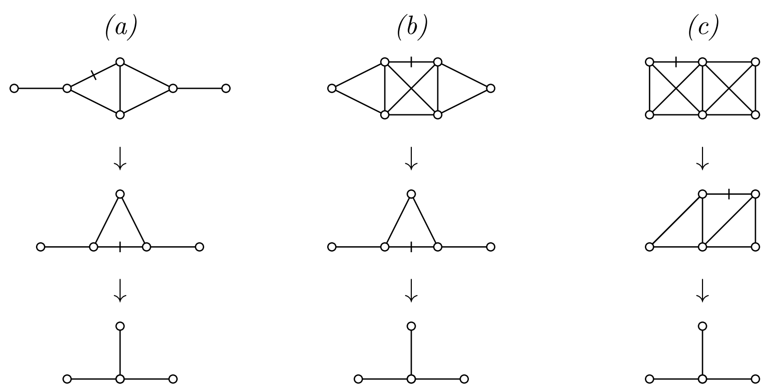
that Δ is either a cycle or it has three paths from a vertex, say o, to each of u, v, and w such that each of these paths do not visit the remaining vertices in the triple u, v, w. In these cases, we have $C_4 \prec \Delta$ and $T_3 \prec \Delta$ respectively. By the observation above $\Delta \prec \Gamma$ — the lemma is proved. ☐

Proposition. *Let Γ be a multipath with sequence $(k_1, \dots, k_m)$. Suppose $C_4 \not\prec \Gamma$ and $T_3 \not\prec \Gamma$. Then*

(a) If $m \geqslant 5$, then $k_i = 1$ for any $3 \leqslant i \leqslant m-2$.

(b) If $m = 4$, then $k_2 = 1$ or $k_3 = 1$.

(c) If $m = 3$, then $k_1 = 1$, $k_2 = 1$, or $k_3 = 1$.



Proof of the proposition. Assuming the contrary in each case we get

(a). If $m \geqslant 5$, then the multipath $(1, 1, 2, 1, 1)$ is an induced subgraph of Γ.

(b). If $m = 4$, then the multipath $(1, 2, 2, 1)$ is an induced subgraph of Γ.

(c). If $m = 3$, then the multipath $(2, 2, 2)$ is an induced subgraph of Γ.

In each case, we arrive at a contradiction by applying edge shrinking to the marked edges as shown on the diagram. $\square$

It remains to show that Γ-comparison holds in any metric space for every multipath Γ described in 5. This is done by prescribing the coordinates for the needed model configuration on the real line.

Each edge of Γ comes with weight — the distance between the endpoints in X. Define the distance $\|v-w\|_\Gamma$ as the minimal total weight of paths connecting v to w in Γ. Note that

$$\|v - w\|_\Gamma \geqslant |v - w|_X$$

for any v and w.

If $m \leqslant 2$ then Γ is a complete graph. In this case, Γ-comparison is trivial. It remains to consider cases $m \geqslant 3$.

Let us choose a special vertex v in Γ that is unique on its level. Namely, if $m \geqslant 5$, then v is on the third level. If $m = 4$, then we can assume that $k_2 = 1$; in this case choose v on the second level. Finally, if $m = 3$, let v be any vertex that is unique on its level; it exists by the proposition.

For every vertex v_i, let

$$\tilde{v}_i = \pm\|v - v_i\|_\Gamma,$$

where the sign is plus if v_i has a higher level than v and minus otherwise. By the triangle inequality, the obtained configuration $\tilde{v}_1, \ldots, \tilde{v}_n$ meets the condition of Γ-comparison. $\square$

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