

# Graph comparison meets Alexandrov

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## Abstract

Graph comparison is a certain type of condition on metric space encoded by a finite graph. We show that any nontrivial graph comparison implies one of two Alexandrov's comparisons. The proof gives a complete description of graphs with trivial graph comparisons.

The notion of graph comparison was introduced in [7]. It was studied further in [3–6, 10, 11]. Let us mention some of the results.

- ◊ Graph comparisons for the tripod and four-cycle capture nonnegative and nonpositive curvature in the sense of Alexandrov; see below.
- ◊ Graph comparison for certain trees is used to formulate a stronger version of the so-called *Lang-Schroeder-Sturm inequality* [2, 5, 9].
- ◊ The all-tree comparison gives a metric description of target spaces of submetrics from subsets of Hilbert space [7].
- ◊ For a certain tree, graph comparison has tight relation with the so-called *MTW condition* that was introduced by Xi-Nan Ma, Neil Trudinger, and Xu-Jia Wang [7, 8].
- ◊ Octahedron comparison holds in products of trees [6].

We will show that any nontrivial graph comparison implies one of two Alexandrov's comparisons.

Let us start with the definition. Suppose  $\Gamma$  is a graph with vertices  $v_1, \dots, v_n$ . We write  $v_i \sim v_j$  (or  $v_i \approx v_j$ ) if  $v_i$  is adjacent (respectively nonadjacent) to  $v_j$ .

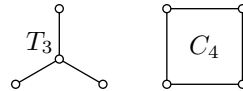
A metric space  $X$  meets the  $\Gamma$ -comparison if for any  $n$  points in  $X$  labeled by vertices of  $\Gamma$  there is a model configuration  $\tilde{v}_1, \dots, \tilde{v}_n$  in the Hilbert space  $\mathbb{H}$  such that

$$\begin{aligned} v_i \sim v_j &\implies |\tilde{v}_i - \tilde{v}_j|_{\mathbb{H}} \leq |v_i - v_j|_X, \\ v_i \approx v_j &\implies |\tilde{v}_i - \tilde{v}_j|_{\mathbb{H}} \geq |v_i - v_j|_X; \end{aligned}$$

here  $|\cdot|_X$  denotes distance in the metric space  $X$ . (Note that  $v_i$  may refer to a vertex in  $\Gamma$  and to the corresponding point in  $X$ .)

Denote by  $T_3$  and  $C_4$  and tripod and four-cycle shown on the diagram. The  $C_4$ -comparison is equivalent to nonnegative curvature, and  $T_3$ -comparison is equivalent to the nonpositive curvature in the sense of

Alexandrov [7]. These definitions are usually applied to length spaces, but they can be applied to general metric spaces; the latter convention is used in [1].

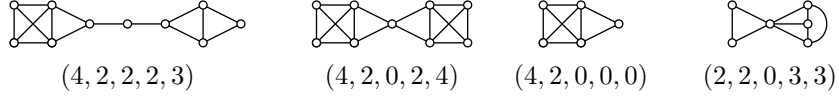


**Theorem.** *Let  $\Gamma$  be an arbitrary finite graph. Then either  $\Gamma$ -comparison holds in any metric space, or it implies  $C_4$ - or  $T_3$ -comparison.*

The following corollary describes all graphs  $\Gamma$  with trivial  $\Gamma$ -comparison; it follows from the proof of the theorem.

**Corollary.** *Let  $\Gamma$  be a finite connected graph. Suppose that  $\Gamma$ -comparison is trivial; that is, it holds in any metric space. Then  $\Gamma$  can be constructed from a path  $P_\ell$  of length  $\ell \geq 0$  and two complete graphs  $K_{m_1}$ ,  $K_{m_2}$  by attaching  $k_1$  vertices of  $K_{m_1}$  to the left end of  $P_\ell$  and  $k_2$  vertices of  $K_{m_2}$  to the right end of  $P_\ell$ .*

Note that the graph  $\Gamma$  in the corollary is described by five integers  $(m_1, k_1, \ell, k_2, m_2)$  such that  $\ell \geq 0$ ,  $m_i \geq k_i \geq 0$ , and  $k_i > 0$  if  $m_i > 0$  for each  $i$ . Examples of such graphs and their 5-arrays are shown below.



**Proof of the theorem.** Suppose  $\Gamma$  has connected components  $\Gamma_1, \dots, \Gamma_k$ . Observe that  $\Gamma$ -comparison holds in a metric space  $X$  if and only if so does every  $\Gamma_i$ -comparison. Therefore we can assume that  $\Gamma$  is connected.

Suppose  $\Gamma$  is a graph with vertices  $v_1, \dots, v_n$  as before. Let  $e$  be an edge in  $\Gamma$ ; we can assume that it connects  $v_1$  to  $v_2$ . Remove  $v_1$  and  $v_2$  from  $\Gamma$  and add a new vertex  $w$  such that for any other vertex  $u$  we have

- ◊ if  $u \sim v_1$  and  $u \sim v_2$ , then  $u \sim w$ ;
- ◊ if  $u \nsim v_1$  and  $u \nsim v_2$ , then  $u \nsim w$ ;
- ◊ in the remaining cases we can choose arbitrarily  $u \sim w$  or  $u \nsim w$ .

Denote the obtained graph by  $\Gamma'$ .

Applying the definition of  $\Gamma$ -comparison assuming that  $v_1 = v_2$  in  $X$ , we get the following.

**Claim.** *If  $\Gamma$ -comparison holds in a metric space  $X$ , then so does  $\Gamma'$ -comparison.*

The operation that produces  $\Gamma'$  from  $\Gamma$  will be called *edge shrinking*. If a graph  $\Delta$  can be obtained from  $\Gamma$  applying edge shrinking several times, then we will write  $\Delta \prec \Gamma$ .

Note that the claim implies the following two statements:

- ◊ If  $\Delta$  is an induced subgraph of a connected finite graph  $\Gamma$ , then  $\Delta \prec \Gamma$ .
- ◊ If  $\Delta \prec \Gamma$ , then  $\Gamma$ -comparison implies  $\Delta$ -comparison.

Taking all the above into account, we get the following reformulation of the theorem.

**Reformulation.** *For any finite connected graph  $\Gamma$*

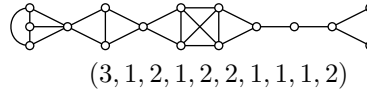
- (a)  *$\Gamma$ -comparison holds in any metric space, or*
- (b)  *$C_4 \prec \Gamma$ , or*

(c)  $T_3 \prec \Gamma$ .

A connected graph will be called *multipath* if it has an integer function  $\ell$  on the set of the vertex set such that  $v$  is adjacent to  $w$  if and only if

$$|\ell(v) - \ell(w)| \leq 1.$$

The value  $\ell(w)$  will be called the *level* of the vertex  $w$ . Multipath is completely described by a sequence of integers that give the number of vertexes on each level. An example of a multipath with its sequence is shown on the diagram.



**Lemma.** Let  $\Gamma$  be a connected finite graph such that  $C_4 \not\prec \Gamma$  and  $T_3 \not\prec \Gamma$ . Then  $\Gamma$  is a multipath.

*Proof of the lemma.* Let us denote by  $| - |_\Gamma$  the path metric on the vertex set of  $\Gamma$ ; it is equal to the number of edges in a shortest path connecting two vertices. Note that it is sufficient to show that

$$(*) \quad |u - w|_\Gamma \geq |u - v|_\Gamma \geq |v - w|_\Gamma \geq 2 \implies |u - w|_\Gamma = |u - v|_\Gamma + |v - w|_\Gamma$$

for any three vertices  $u$ ,  $v$ , and  $w$  in  $\Gamma$ .

Suppose  $(*)$  does not hold for  $u$ ,  $v$ , and  $w$ . Let us pass to a minimal connected induced subgraph  $\Delta \ni u, v, w$  of  $\Gamma$  such that  $(*)$  still does not hold in  $\Delta$ . Note



that  $\Delta$  is either a cycle or it has three paths from a vertex, say  $o$ , to each of  $u$ ,  $v$ , and  $w$  such that each of these paths do not visit the remaining vertices in the triple  $u, v, w$ . In these cases, we have  $C_4 \prec \Delta$  and  $T_3 \prec \Delta$  respectively. By the observation above  $\Delta \prec \Gamma$  — the lemma is proved.  $\square$

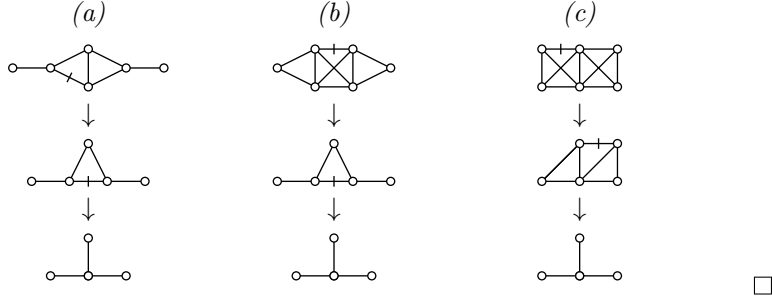
**Proposition.** Let  $\Gamma$  be a multipath with sequence  $(k_0, \dots, k_m)$ . Suppose  $C_4 \not\prec \Gamma$  and  $T_3 \not\prec \Gamma$ . Then

- (a) If  $m \geq 4$ , then  $k_2 = \dots = k_{m-2} = 1$ .
- (b) If  $m = 3$ , then  $k_1 = 1$  or  $k_2 = 1$ .
- (c) If  $m = 2$ , then  $k_0 = 1$ ,  $k_1 = 1$ , or  $k_2 = 1$ .

*Proof of the proposition.* Assuming the contrary in each case we get

- (a). If  $m \geq 4$ , then multipath  $(1, 1, 2, 1, 1)$  is an induced subgraph of  $\Gamma$ .
- (b). If  $m = 3$ , then multipath  $(1, 2, 2, 1)$  is an induced subgraph of  $\Gamma$ .
- (c). If  $m = 2$ , then multipath  $(2, 2, 2)$  is an induced subgraph of  $\Gamma$ .

In each case, we arrive at a contradiction by applying edge shrinking to the marked edges as shown on the diagram.



It remains to show that  $\Gamma$ -comparison holds in any metric space for every multipath  $\Gamma$  described in 6. This is done by prescribing the coordinates for the needed model configuration on the real line.

Each edge of  $\Gamma$  comes with weight — the distance between the endpoints in  $X$ . Define the distance  $\|v - w\|_\Gamma$  as the minimal total weight of paths connecting  $v$  to  $w$  in  $\Gamma$ . Note that

$$\|v - w\|_\Gamma \geq |v - w|_X$$

for any  $v$  and  $w$ .

If  $m \leq 1$  then  $\Gamma$  is a complete graph. In this case,  $\Gamma$ -comparison is trivial. It remains to consider cases  $m \geq 2$ .

Let us choose a special vertex  $w$  that is unique on its level and not too far from the middle of  $\Gamma$ . Namely, if  $m \geq 4$ , then choose  $w$  on the second level; by the proposition it is unique on its level. If  $m = 3$ , then by the proposition we can assume that  $k_2 = 1$ ; in this case choose  $w$  on the second level. Finally, if  $m = 2$ , let  $w$  be any vertex that is unique on its level; it exists by the proposition.

For every vertex  $v_i$ , let

$$\tilde{v}_i = \pm \|w - v_i\|_\Gamma,$$

where the sign is plus if  $v_i$  has a higher level than  $w$  and minus otherwise. By the triangle inequality, the obtained configuration  $\tilde{v}_1, \dots, \tilde{v}_n$  meets the condition of  $\Gamma$ -comparison.  $\square$

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