Extra pearls in graph theory

Anton Petrunin

I used these topics together with the book "Pearls in graph theory" [12] to teach an undergraduate course in graph theory at the Pennsylvania State University. I tried to keep clarity and simplicity on the same level.

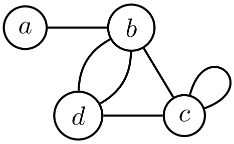
Hope that someone will find it useful for something.

Contents

Chapter 1

Introduction

Terminology



The diagram on the right may describe regular flights of an airline. It has six flights which serve four airports labeled by a, b, c and d.

For this and similar type of data, mathematicians use the notion of *pseudograph*.

Formally, a pseudograph is a finite nonempty set of *vertexes* (in our example a vertex is an airport) and finite collection of *edges*, each edge connects two vertexes (in the example above, edge is a regular flight). A pair of vertexes can be connected by few edges, such edges are called *parallel* (in our example, it might mean that the airline makes few flight a day between these airports). Also an edge can connect a vertex to itself, such edge is called a *loop* (we might think of it as a sightseeing flight).

Thus, from mathematical point of view, the diagram above describes an example of pseudograph with vertexes a, b, c, d and six edges; among them one loop at c and a pair of parallel edges between b and d.

The number of edges comming from one vertex is called its *degree*, the loops are counted twice. In the example above, the degrees of a, b, c and d are $1, 4, 4$ и 3 correspondingly.

A vertex with zero degree is called *isolated* and a vertex of degree one is called *end vertex*.

A pseudograph without loops is also called *multigraph*. A multigraph without parallel edges is also called *graph*. Most of the time we will work with graphs.

If x and y are vertexes of a pseudograph G, we say x is *adjacent* to y if there is an edge between x and y. We say that a vertex x is *incident* with an edge e if x is an end vertex of e.

Wolf, goat and cabbage

Usually we visualize the vertexes of a graph by points and its edges are represented by a line connecting two vertexes.

However, the vertexes and edges of the graph might have a very different nature. As an example, let us consider the following classical problem.

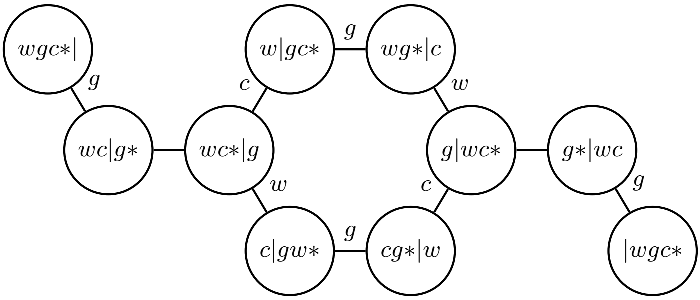
1.1. Problem. *A farmer purchased a wolf, a goat and a cabbage; he needs to cross a river. He has a boat, but he could carry only himself and a single one of his purchases: the wolf, the goat, or the cabbage.*

If left unattended together, the wolf would eat the goat, or the goat would eat the cabbage.

The farmer has to carry himself and his purchases to the far bank of the river, leaving each purchase intact. How did he do it?

Solution. Let us denote the farmer by $*$, the river by a vertical line $|$ the wolf by w, the goat by g and the cabbage by c. For example $wc|g*$ means that wolf and cabbage are on the left bank of the river and the goat with the farmer are on the right bank.

The starting position is $wgc*|$; that is, everyone is one the left bank. The following graph describes all possible positions which can be achieved; each edge is labeled by the transported purchase.



This graph shows that the farmer can achieve $|wgc*$ by legal moves. It solves the problem, and also shows that there are exactly two different solutions; assuming that the farmer does not want to repeat the same position twice. □

Often graph comes with an extra structure, for example labeling of edges and/or vertexes as in the example above.

Here is a small variation of an other classical problem.

1.2. Problem. *Missionaries and cannibals must cross a river using a boat which can carry at most two people, under the constraint that, for both banks, if there are missionaries present on the bank, they cannot be outnumbered by cannibals; the missionaries will be eaten otherwise. The boat cannot cross the river by itself with no people on board.*

Let us introduce a notation to describe positions of missionaries, cannibals and the boat on the banks. The river will be denoted by vertical line $|$; let $*$ denotes the boat, we will write the number of cannibals on each side of $|$ and the number of missionaries by subscript. For example $4^*_2|0_2$ means that on the left bank we have four cannibals, two missionaries and the boat, and on the right bank there is no cannibals and two missionaries.

1.3. Exercise. *Assume four missionaries and four cannibals need to cross the river; in other words the beginning stage is $4^*_4|0_0$. Draw a graph for all possible positions which can be achieved.*

Conclude that all of them can not cross the river.

Chapter 2

Ramsey numbers continued

Lower bounds

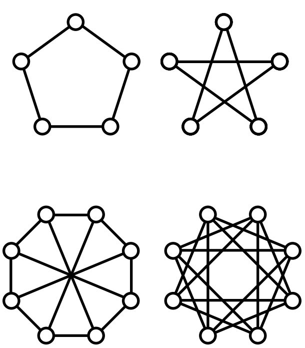
Recall that Ramsey number $r(m,n)$ is a least positive integer for which every blue-red coloring of edges in the complete graph $K_{r(m,n)}$ contains a blue K_m or a red K_n.

Equivalently, for any decomposition of $K_{r(m,n)}$ into two subgraphs G and H either G contains a copy of K_m or H contains a copy of K_n.

Therefore, in order to show that

$$r(m,n) \geqslant s+1,$$

it is sufficient to decompose K_s into two subgraphs with no isomorphic copy of K_m in the first one and no isomorphic copy of K_n in the second one.



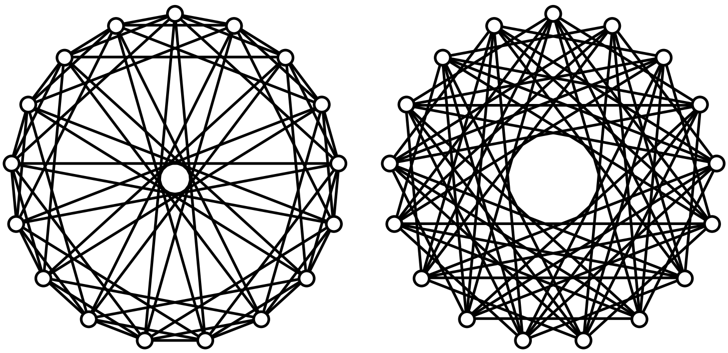
For example, the subgraphs in the decomposition of K_5 on the diagram has no monochromatic triangles; the latter implies that $r(3,3) \geqslant 6$. We showed already that for any decomposition of K_6 into two subgraphs, one of the subgraphs has a triangle; that is, $r(3,3) = 6$.

Similarly, to show that $r(3,4) \geqslant 9$, we need to construct a decomposition of K_8 in to two subgraphs G and H such that G contains no triangle K_3 and H contains no K_4. In fact any decomposition of K_9

into two subgraphs, first subgraph contains a triangle or the second contains a K_4. That is, $r(3,4) = 9$; see [12, p. 82–83].

Similarly, to show that $r(4,4) \geqslant 18$, we need to construct a decomposition of K_{17} in to two subgraphs with no K_4. (In fact, $r(4,4) = 18$, but we are not going to prove it.)

The corresponding decomposition is given on the following diagram.



The constructed decomposition is rationally symmetric; the first subgraph contains the chords of angle lengths 1, 2, 4, and 8 and the second to all the cords of angle lengths 3, 5, 6 and 7.

2.1. Exercise. *Show that*

(a) In the decomposition of K_8 above, the left graph contains no triangle and the right graph contains no K_4.

(b) In the decomposition of K_{17} above, both graph contain no K_4.

Hint: In each cases, fix one vertex v and draw the subgraph induced by the vertexes connected to v. (If uncertain, see the definition of *induced subgraph.*)

For larger values m and n the problem of finding the exact lower bound for $r(m,n)$ is quickly becomes too hard. Even getting a reasonable estimate is challenging. In the next section we will show how to obtain such estimate using probability.

Probabilistic method

The probabilistic method makes possible to prove the existence of graphs with certain properties without constructing them explicitly. The idea is to show that if one randomly chooses a graph or its coloring from a specified class, then probability that the result is of the needed property is more than zero. The latter implies that a graph with needed property exists.

Despite that this method of proof uses probability, the final conclusion is determined for certain, without any possible error.

Recall that $\binom{n}{m}$ denotes the *binomial coefficient*; that is, m and n are integers, $n \geqslant 0$ and

$$\binom{n}{m} = \frac{n!}{m! \cdot (n-m)!}$$

if $0 \leqslant m \leqslant n$ and $\binom{n}{m} = 0$ otherwise.

The number $\binom{n}{m}$ plays an important role in combinatorics — it gives the number of ways that m objects can be chosen from among n different objects.

2.2. Theorem. *Assume that the inequality*

$$\binom{N}{n} < 2^{\binom{n}{2}-1}$$

holds for a pair of positive integers N and n. Then $r(n,n) > N$.

Proof. We need to show that the complete graph K_N admits a coloring of edges in red and blue such that it has no monochromatic subgraph isomorphic to K_n.

Let us color the edges randomly — color each edge independently with probability $\frac{1}{2}$ in red and otherwise in blue.

Fix a set S of n vertexes. Define the variable $X(S)$ to be 1 if every edge the vertexes in S has the same color, and 0 otherwise. Note that the number of monochromatic n-subgraphs in K_N is the sum of $X(S)$ over all possible n-vertex subsets S.

Note that the expected value of $X(S)$ is simply the probability that all of the $\binom{n}{2} = \frac{n \cdot (n-1)}{2}$ edges in S are the same color. The probability that all the edges with the ends in S are blue is $1/2^{\binom{n}{2}}$ and with the same probability all edges are red. Since these two possibilities exclude each other the expected value of $X(S)$ is $2/2^{\binom{n}{2}}$.

This holds for any n-vertex subset S of the vertexes of K_N. The total number of such subsets is $\binom{N}{n}$. Therefore the expected value for the sum of $X(S)$ over all S is

$$X = 2 \cdot \binom{N}{n} / 2^{\binom{n}{2}}.$$

Assume that $X < 1$. Note that at least in one coloring suppose to have at most X complete monochromatic n-subgraphs. Since this number has to be an integer, at least one coloring must have no monochromatic K_n.

Therefore if $\binom{N}{n} < 2^{\binom{n}{2}-1}$, then there is a coloring K_N without monochromatic n-subgraphs. Hence the statement follows. □

The following corollary implies that the function $n \mapsto r(n,n)$ grows at lest exponentially.

2.3. Corollary. *$r(n,n) > \frac{1}{8}\cdot 2^{\frac{n}{2}}$ for all positive integers $n \geqslant 2$.*

Proof. Set $N = \lfloor \frac{1}{8}\cdot 2^{\frac{n}{2}} \rfloor$; that is, N is the largest integer $\leqslant \frac{1}{8}\cdot 2^{\frac{n}{2}}$.

Note that
$$2^{\binom{n}{2}-1} > (2^{\frac{n-3}{2}})^n \geqslant N^n.$$
and
$$\binom{N}{n} = \frac{N\cdot(N-1)\cdots(N-n+1)}{n!} < N^n.$$

Therefore
$$\binom{N}{n} < 2^{\binom{n}{2}-1}.$$

By Theorem 2.2, we get that $r(n,n) > N$. □

In the following exercise, mimic the proof of Theorem 2.2, crude estimates will do the job.

2.4. Exercise. *By random coloring we will understand a coloring edges of a given graph in red and blue such that each edge is colored independently in red or blue with equal chances.*

Assume the edges of the complete graph K_{100} is colored randomly. Find expected number of monochromatic Hamiltonian cycles in K_{100}. (You may use factorial in the answer.)

Counting proof

In this section we will repeat the proof of Theorem 2.2 using a different language, without use of probability. We do this to affirm that probabilistic method provides real proof, without any possible error.

In principle, any probabilistic proof admits such translation, but in most cases, the translation is less intuitive.

Proof of 2.2. The graph K_N has $\binom{N}{2}$ edges. Each edge can be colored in blue or red therefore the total number of different colorings is
$$\Omega = 2^{\binom{N}{2}}.$$

Fix a subgraph isomorphic to K_n in K_N. Note that this graph is red in $\Omega/2^{\binom{n}{2}}$ different colorings and yet in $\Omega/2^{\binom{n}{2}}$ different colorings this subgraph is blue.

There are $\binom{N}{n}$ different subgraphs isomorphic to K_n in K_N. Therefore the total number of monochromatic K_n's in all the colorings is

$$M = \tbinom{N}{n}\cdot\Omega\cdot 2/2^{\binom{n}{2}}.$$

If $M < \Omega$, then by the pigeonhole principle, there is a coloring with no monochromatic K_n. Hence the result. □

Remarks

The probabilistic method was introduced by Paul Erdős. It finds applications in many areas of mathematics; not only in graph theory.

Note that probabilistic method is not nonconstructive — often when the existence of a certain graph is probed by probabilistic method, it is still uncontrollably hard to describe a concrete example.

More involved examples of proofs based on the probabilistic method deal with *typical properties* of random graphs.

To describe the concept, let us consider the following *random process* which generates graph G_p with p vertexes.

Fix a positive integer p. Consider a graph G_p with the vertexes labeled by $1, \dots, p$, where every edge in G_p exists with probability $\frac{1}{2}$.

Note that the described process depends only on p and as a result we can get any graph on p vertexes with the same probability $1/2^{\binom{p}{2}}$.

Fix a property of a graph (for example connectedness) and let α_p be the probability that G_p has this property. We say that the property is *typical* if $\alpha_p \to 1$ as $p \to \infty$.

2.5. Exercise. *Show that random graphs are typically have diameter 2. That is, the probability that G_p is has diameter 2 converges to 1 as $p \to \infty$.*

Hint: Find the probability that two given vertexes lie on the distance > 2 from each other in G_p; find the average number of such pairs in G_p; make a conclusion.

Note that from the exercise above, it follows that in the described random process the random graphs are *typically connected.*

The following theorem gives a deeper illustration for probabilistic method with use of typical properties, a proof can be found in [1, Chapter 44].

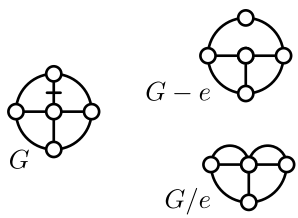
2.6. Theorem. *Given a positive integer g and k there is a graph G with girth at least g and chromatic number at least k.*

Chapter 3

Deletion and contraction

Definitions

Let e be an edge in the pseudograph G. Denote by $G-e$ the pseudograph obtained from G by deleting e, and by G/e the pseudograph obtained from G by contraction the edge e to a point; see the diagram.



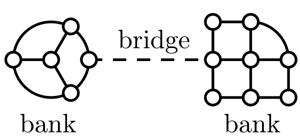
Assume G is a graph; that is, G has no loops and no parallel edges. Then so is $G-e$, but G/e might have parallel edges but no loops; that is, G/e is a multigraph.

If G is a multigraph then so is $G-e$. If the edge e is parallel to f in G, then f in G/e becomes a loop; that is, G/e is a pseudograph in general.

Number of spanning trees

Recall that $s(G)$ denotes the number of spanning trees in the pseudograph G.

An edge e in a connected graph G is called *bridge*, if deletion of this edge makes it disconnected; that is, the remaining graph has two connected components which are called *banks*.



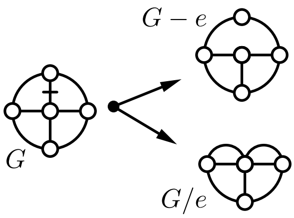
3.1. Exercise. *Assume that the graph G contains a bridge between* *banks H_1 and H_2. Show that*

$$s(G) = s(H_1) \cdot s(H_2).$$

3.2. Deletion-plus-contraction formula. *Let e be an edge in the pseudograph G. Assume e is not a loop, then the following identity holds*

➊ $$s(G) = s(G - e) + s(G/e).$$

Often it is convenient to write the identity ➊ using a diagram as on the picture; the edge e is marked on the diagram.



Proof. Note that the spanning trees of G can be subdivided into two groups — (1) those which contain the edge e and (2) those which do not. For the trees in the first group, contraction of e to a point gives a spanning tree in G/e, while the trees in the second group are also spanning trees in $G - e$.

Moreover, both of the described correspondences are one-to-one. Hence the formula follows. □

Note that a spanning tree can not have loops. Therefore if we remove all loops from the pseudograph, then the number of spanning trees remains unchanged. In other words, for any loop e the following identity holds

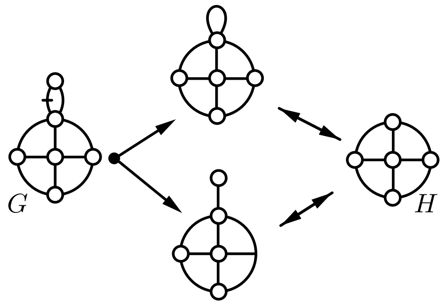
$$s(G) = s(G - e).$$

From the deletion-plus-contraction formula we can deduce few other useful identities. For example, assume that the graph G has an end vertex w (that is, $\deg w = 1$). If we remove the vertex w and its edge from G, then in obtained graph $G - w$ the number of spanning trees remains unchanged; that is,

➋ $$s(G) = s(G - w).$$

Indeed, denote by e the only edge incident to w. Note that the graph $G - e$ is not connected, since the vertex w is isolated. Therefore $s(G - e) = 0$. On the other hand $G/e = G - w$ therefore ➊ implies ➋.

On the diagrams, we will use two-sided arrow "↔" for the graphs with equal number of the spanning trees. For example, from the discussed identities we can draw the diagram, which in particular implies the following identity:



$$s(G) = 2\cdot s(H).$$

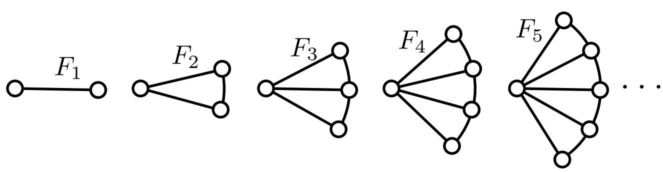
The deletion-plus-contraction formula gives an algorithm to calculate the value $s(G)$ for given pseudograph G. Indeed, for any edge e, both graphs $G-e$ and G/e have smaller number of edges. That is, the deletion-plus-contraction formula reduces the problem of finding number of the trees to simpler graphs; applying this formula few times we can reduce the question to a collection of graphs with evident answer for each. In the next section we will show how it works.

Fans and their relatives

Recall that Fibonacci numbers f_n are defined using the recursive identity $f_{n+1} = f_n + f_{n-1}$ with $f_1 = f_2 = 1$. The sequence of Fibonacci numbers starts as

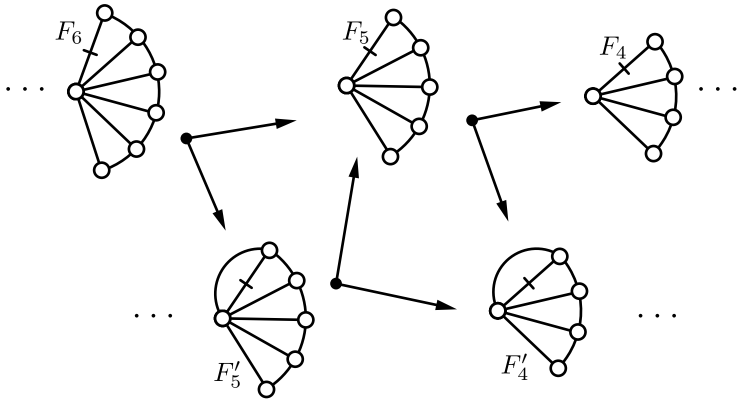
$$1, 1, 2, 3, 5, 8, 13, \ldots$$

The graphs of the following type are called *fans*; a fan with $n+1$ vertex will be denoted by F_n.



3.3. Theorem. $s(F_n) = f_{2\cdot n}$.

Proof. Applying the delection-plus-contraction formula, we can draw the following infinite diagram. In addition to the fans F_n we use its variations F'_n, which differ from F_n by an extra parallel edge.



Set $a_n = s(F_n)$ and $a'_n = s(F'_n)$. From the diagram we get the following two recursive relations:

$$\begin{aligned} a_{n+1} &= a'_n + a_n, \\ a'_n &= a_n + a'_{n-1}. \end{aligned}$$

That is, in the sequence

$$a_1, a'_1, a_2, a'_2, a_3 \dots$$

every number starting from a_2 is sum of previous two.

Further note that F_1 has two vertexes connected by unique edge, and F'_1 has two vertexes connected by a pair of parallel edges. Hence $a_1 = 1 = f_2$ and $a'_1 = 2 = f_3$ and therefore

$$a_n = f_{2\cdot n}$$

for any n. □

Comments. We can deduce a recursive relation for a_n, without using a'_n:

$$\begin{aligned} a_{n+1} &= a'_n + a_n = \\ &= 2\cdot a_n + a'_{n-1} = \\ &= 3\cdot a_n - a_{n-1}. \end{aligned}$$

Sequences defined by the *linear recursion* as above are called *constant-recursive sequences*. The general term of such sequence can be expressed by a closed formula — read [16] if you wonder how. In our case it is

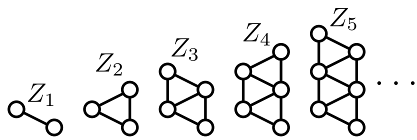
$$a_n = \tfrac{1}{\sqrt{5}} \cdot \Big((\tfrac{3+\sqrt{5}}{2})^n - (\tfrac{3-\sqrt{5}}{2})^n \Big).$$

Since a_n is integer and $0 < \frac{1}{\sqrt{5}}\cdot(\frac{3-\sqrt{5}}{2})^n < 1$ for any $n \geqslant 1$ a shorter formula can be written

$$a_n = \left\lfloor \frac{1}{\sqrt{5}}\cdot(\frac{3+\sqrt{5}}{2})^n \right\rfloor,$$

where $\lfloor x \rfloor$ denotes floor of x; that is, $\lfloor x \rfloor$ is the maximal integer such that $\lfloor x \rfloor \leqslant x$.

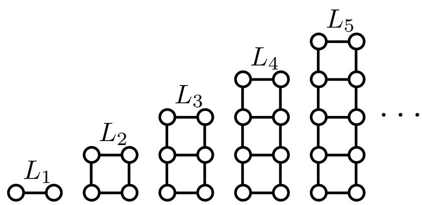
3.4. Exercise. *Consider the sequence of zig-zag graphs Z_n of the following type:*



Show that $s(Z_n) = f_{2\cdot n}$ for any n.

Hint: Use the induction on n and/or mimic the proof of Theorem 3.3.

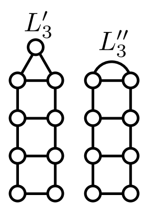
3.5. Exercise. *Let us denote by b_n the number of spanning trees in the n-*step ladder *L_n; that is, in the graph of the following type:*



Apply the method we used for fans F_n to show that the sequence b_n satisfies the following linear recursive relation

$$b_{n+1} = 4\cdot b_n - b_{n-1}.$$

Hint: To construct the recursive relation, in addition to the ladders L_n you will need two of its analogs L'_n and L''_n shown on the diagram.

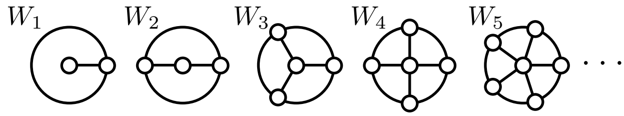


Note that $b_1 = 1$ и $b_2 = 4$; applying the exercise we could calculate first numbers of the sequence (b_n):

$$1, 4, 15, 56, 209, 780, 2911, \dots$$

The following exercise is analogous, but more complicated.

3.6. Advanced exercise. *Recall that a wheel W_n is the graph of following type:*



Show that the sequence $c_n = s(W_n)$ *satisfies the following recursive relation*

$$c_{n+1} = 4{\cdot}c_n - 4{\cdot}c_{n-1} + c_{n-2}.$$

Using the exercise above and applying induction one can show that

$$c_n = f_{2\cdot n+1} + f_{2\cdot n-1} - 2 = l_{2\cdot n} - 2$$

for any n; the numbers $l_n = f_{n+1} + f_{n-1}$ are called *Lucas numbers*; they pop up in combinatorics as often as Fibonacci numbers.

Remarks

The *delection-plus-contraction* formula together with Kirchhoff's rules were used in the solution of the so called *squaring the square problem.* The history of this problem and its solution are discussed in a book of Martin Gradner [8, Chapter 17].

The proof of recurrent relation above is given by Mohammad Hassan Shirdareh Haghighi and Khodakhast Bibak in [21]; this problem is also discussed in a book of Ronald Graham, Donald Knuth and Oren Patashnik [15] which is a classical book.

Chapter 4

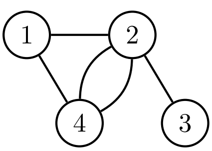
Matrix theorem

Adjacency matrix

Let us describe a way to encode the given multigraph G with p vertexes by an $p{\times}p$ matrix. First, enumerate the vertexes of the multigraph by numbers from 1 to p; such multigraph will be called *labeled*. Consider the matrix $A = A_G$ with the component $a_{i,j}$ equal to the number of edges from i-th vertex to the j-th vertex of G.

This matrix A is called *adjacency matrix* of G. Note that A is *symmetric*; that is, $a_{i,j} = a_{j,i}$ for any pair i, j. Also, the diagonal components of A vanish; that is, $a_{i,i} = 0$ for any i.

For example, for the labeled multigraph G shown on the diagram, we get the following adjacency matrix



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}.$$

4.1. Exercise. *Let A be the* adjacency matrix *of a labeled multigraph. Show that the components $b_{i,j}$ of the n-th power A^n is the number of walks of length n in the graph from vertex i to vertex j.*

Hint: Use induction on n.

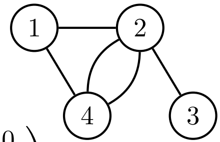
Kirchhoff minor

In this section we construct a special matrix, called *Kirchhoff minor*, associated with a pseudograph and discuss its basic properties. This matrix will be used in the next section in a formula for the number of spanning trees in a pseudograph G. Since the loops do not change the number of spanning trees, we can remove all of them. In other words we can (and will) always assume that G is a multigraph.

Fix a multigraph G and consider its adjacency matrix $A = A_G$; it is a $p{\times}p$ symmetric matrix with zeros on the diagonal.

1. Revert the signs of the components of A and exchange the zeros on the diagonal by the degrees of the corresponding vertexes. (The matrix A' is called *Kirchhoff matrix*, *Laplacian matrix* or *admittance matrix* of the graph G.)
2. Delete from A' the last column and the last row; the obtained matrix $M = M_G$ will be called *Kirchhoff minor* of the labeled pseudograph G.

For example, the labeled multigraph G on the diagram has the following Kirchhoff matrix and Kirchhoff minor:



$$A' = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -1 & -2 \\ 0 & -1 & 1 & 0 \\ -1 & -2 & 0 & 3 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

4.2. Exercise. *Show that in any Kirchhoff matrix A' the sum of components in each row or column vanishes. Conclude that*

$$\det A' = 0.$$

4.3. Exercise. *Draw a labeled pseudograph with following Kirchhoff minor:*

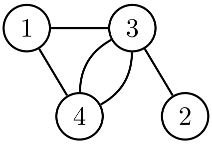
$$\begin{pmatrix} 4 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & 0 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 4 \end{pmatrix}$$

4.4. Exercise. *Show that the sum of all components in every column of Kirchhoff minor is nonnegative.*

Moreover the sum of all components in i-th column vanish if and only if i-th vertex is not adjacent to the last vertex.

Relabeling. Let us understand what happens with Kirchhoff minor and its determinant as we swap two labels distinct from the last one.

For example, if we swap the labels 2 and 3 in the graph above, we get an other labeling shown on the diagram. Then the corresponding Kirchhoff minor will be



$$M' = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 4 \end{pmatrix}$$

which is obtained from M by swapping columns 2 and 3 following by swapping rows 2 and 3.

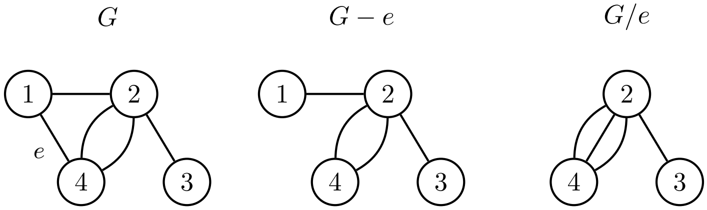
Note that swapping a pair of columns or rows changes the sign of determinant. Therefore swapping one pair of rows and one pair of columns does not change the determinant. The same holds in general; let us summarize it:

4.5. Observation. *Assume G is a labeled graph with p vertexes and M_G is its Kirchhoff minor. If we swap two labels $i, j < p$ then corresponding Kirchhoff minor M'_G can be obtained from M_G by swapping columns i and j following by swapping rows i and j. In particular,*

$$\det M'_G = \det M_G.$$

Deletion and contraction. Next let us understand what happens with Kirchhoff minor if we delete or contract an edge in the labeled multigraph. (If after contraction of an edge we get loops, we remove it; this way we obtain a multigraph.).

Assume edge e connects first and last vertex of labeled multigraph G as in the following example:



Note that deleting e only reduce the corner component of M_G by one, while contracting it removes first row and column. That is, since

$$M_G = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

we have

$$M_{G-e} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad M_{G/e} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}.$$

Again, the same holds in general, let us summarize it in the following observation.

4.6. Observation. *Assume e is an edge of labeled multigraph G between the first and last vertex then and M_G is the Kirchhoff minor of G. Then*

(a) the Kirchhoff minor M_{G-e} of $G-e$ can be obtained from M_G by subtracting 1 from the corner element with index (1,1);

(b) the Kirchhoff minor $M_{G/e}$ of G/e can be obtained from M_G by removing the first row and the first column in M_G.

In particular by cofactor expansion of determinant we get that

$$\det M_G = \det M_{G-e} + \det M_{G/e}$$

Note that the last formula reminds deletion-plus-contraction formula. This is the key observation in the proof of the matrix theorem; see the next section.

Matrix theorem

4.7. Matrix theorem. *Let M be the Kirchhoff minor of labeled multigraph G with at least two vertexes. Then*

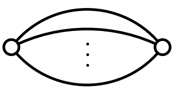
$$❶ \qquad s(G) = \det M,$$

where $s(G)$ denotes the number of spanning trees in G.

Proof. As usual we denote by p and q the number of vertexes and edges in G. We will use induction on the number $p+q$; we will always assume that $p \geqslant 2$.

The base case will include two cases $p = 2$ and $q = 0$.

Assume $p = 2$; that is, G has two vertexes and q parallel edges connecting them. Clearly $s(G) = q$. Further note that $M_G = (q)$; that is, the Kirchhoff minor M_G is a $1{\times}1$ matrix with single component q. In particular $\det M_G = q$ and therefore the formula ❶ holds if $p = 2$. It proves the statement in the first base case $p = 2$.



Denote by d the degree of the last vertex in G.

Assume $d = 0$. Then G is not connected and therefore $s(G) = 0$. On the other hand, the sum in each row of M_G vanish (compare to Exercise 4.4). Hence the sum of all columns in M_G vanish; in particular, the columns in M_G are linearly dependent and hence $\det M_G = 0$. In particular if $q = 0$ then $d = 0$ hence in this case formula ➊ holds if $q = 0$. It proves the statement in the second base case $q = 0$.

It remains to consider the case $d > 0$. In this case we may assume that the first and last vertexes of G are adjacent, otherwise permute pair of labels 1 and some $j < p$ and apply Observation 4.5. Denote by e the edge between the first and last vertex.

Note that the total number of vertexes and edges in the pseudographs $G - e$ and G/e are smaller than $p + q$; hence by induction hypothesis the matrix theorem holds for these two pseudographs.

Applying deletion-plus-contraction formula, the induction hypothesis and Observation 4.6, we get that

$$\begin{aligned} s(G) &= s(G - e) + s(G/e) = \\ &= \det M_{G-e} + \det M_{G/e} = \\ &= \det M_G. \end{aligned}$$

Which proves the induction step. □

4.8. Exercise. *Fix a labeling for each of the following graphs, find its Kirchhoff minor and use matrix theorem to find the number of spanning trees.*

(a) $s(K_{3,3})$*;*

(b) $s(W_6)$*;*

(c) $s(Q_3)$*.*

(You can use `http://matrix.reshish.com/determinant.php`*, or any other matrix calculator.)*

Calculation of determinants

In this section we recall key properties of determinant which will be used in the next section.

Let M be an $n{\times}n$-matrix; that is, a table $n{\times}n$, filled with numbers which are called *components of the matrix.* The determinat $\det M$ is a polynomial of the n^2 components of M, which satisfies the following conditions:

1. The unit matrix has determinant 1; that is,

$$\det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = 1.$$

2. If we multiply each component of one of the rows of the matrix M multiply by a number λ, then for the obtained matrix M', we have

$$\det M' = \lambda \cdot \det M.$$

3. If one of the rows in the matrix M add (or subtract) term-by-term to an other row, then the obtained matrix M' has the same determinant

$$\det M' = \det M.$$

These three conditions define determinant in a unique way. We will not give a proof of the statement, it is not evident and not complicated (soon or later you will have to learn it, if it is not done already).

4.9. Exercise. *Show that the following property follows from the properties above.*

4. If we permute two rows in the matrix M then the obtained matrix M' will have determinant of opposite sign; that is,

$$\det M' = -\det M.$$

The determinant of $n \times n$-matrix can be written explicitly as a sum of $n!$ terms. For example,

$$a_1 \cdot b_2 \cdot c_3 + a_2 \cdot b_3 \cdot c_1 + a_3 \cdot b_1 \cdot c_2 - a_3 \cdot b_2 \cdot c_1 - a_2 \cdot b_1 \cdot c_3 - a_1 \cdot b_3 \cdot c_2$$

is the determinant of the matrix

$$M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

However, the properties described above give a more convinient and faster way to calculate the determinant, especially for larger values n.

Let us show it on one example which will be needed in the next section:

$$
\begin{aligned}
\det\begin{pmatrix} 4 & -1 & -1 & -1\\ -1 & 4 & -1 & -1\\ -1 & -1 & 4 & -1\\ -1 & -1 & -1 & 4\end{pmatrix} &= \det\begin{pmatrix} 1 & 1 & 1 & 1\\ -1 & 4 & -1 & -1\\ -1 & -1 & 4 & -1\\ -1 & -1 & -1 & 4\end{pmatrix} =\\
&= \det\begin{pmatrix} 1 & 1 & 1 & 1\\ 0 & 5 & 0 & 0\\ 0 & 0 & 5 & 0\\ 0 & 0 & 0 & 5\end{pmatrix} =\\
&= 5^3\cdot \det\begin{pmatrix} 1 & 1 & 1 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} =\\
&= 5^3\cdot \det\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} =\\
&= 5^3.
\end{aligned}
$$

Let us describe what we used on each line above:

1. property 3 three times — we add to the first row each of the remaining rows;
2. property 3 three times — we add first row to the each of the remaining three rows;
3. property 2 three times;
4. property 3 three times — we subtract from the first row the remaining three rows;
5. property 1.

Cayley formula

Recall that the *complete graph* is the graph where each pair of vertexes is connected by an edge; complete graph with p vertexes is denoted by K_p.

Note that every vertex of K_p has degree $p - 1$. Therefore the Kirchhoff minor $M = M_{K_p}$ in the matrix formula ❶ for K_p is the following $(p-1)\times(p-1)$-matrix:

$$M = \begin{pmatrix} p-1 & -1 & \cdots & -1 \\ -1 & p-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p-1 \end{pmatrix}.$$

The argument given in the end of the previous section admits a direct generalization:

$$\begin{aligned} \det \begin{pmatrix} p-1 & -1 & \cdots & -1 \\ -1 & p-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p-1 \end{pmatrix} &= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & p-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p-1 \end{pmatrix} = \\ &= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p \end{pmatrix} = \\ &= p^{p-2}. \end{aligned}$$

That is,

$$\det M = p^{\,p-2}.$$

Therefore, applying the matrix theorem, we get the following:

4.10. Cayley formula. *The number of spanning trees in the complete graph K_p is p^{p-2}; that is,*

$$s(K_p) = p^{p-2}.$$

4.11. Exercise. *Use the matrix theorem to show that*

$$s(K_{m,n}) = m^{n-1}{\cdot}n^{m-1}.$$

Remarks

There is strong connection between counting spanning trees of the given graph and calculations of currents in an electric chain.

Assume that the graph G describes an electic chain; each edge has resistance one Ohm and battery is connected to the vertexes a and b.

Assume that the total current between these vertexes is one Ampere and we need to calculate the current thru edge e.

Fix an orientation of e. Note that the spanning trees of G can be subdivided into the following three groups: (1) those where the edged e appears on the (necessary unique) path from a to b with positive orientation, (2) those where the edge e appears on the path from a to b with negative orientation, (3) those which the edge e do not appear on the path from a to b. Denote by s_+, s_- and s_0 the number of the trees in each group. Clearly

$$s(G) = s_+ + s_- + s_0.$$

The current I_e along e can be calculated using the following formula:

$$I_e = \frac{s_+ - s_-}{s(G)} \cdot I.$$

This statement can be proved by checking Kirchhoff's rules for the currents calculated by this formula.

There are many other applications of Kirchhoff's rules to graph theory For example, in [18], they were used to prove the Eulers's formula

$$p - q + r = 2,$$

where p, q и r denotes the number of vertexes, edges and regions of in a plane drawing of graph.

Few interesting proofs of Cayley formula are given in [1, Chapter 30]; the most popular proof using the Prüfer's code is given in [12].

Chapter 5

Polynomials

Counting problems often lead to specially organized data. Sometimes it is convenient to *pack* this data in a polynomial. If this *packing* is to be done in a smart way, then the algebraic structure of polynomial reflects the original combinatoric structure.

Chromatic polynomial

Denote by $P_G(x)$ the number of different colorings of the graph G in x colors such that the ends of each edge get different colors.

5.1. Exercise. *Assume that a graph G has exactly two connected components H_1 and H_2. Show that*

$$P_G(x) = P_{H_1}(x) \cdot P_{H_2}(x)$$

for any x.

5.2. Exercise. *Show that for any integer $n \geqslant 3$,*

$$P_{W_n}(x+1) = (x+1) \cdot P_{C_n}(x),$$

where W_n denotes the wheel with n spokes and C_n is the cycle of length n.

5.3. Deletion-minus-contraction formula. *For any edge e in the pseudograph G we have*

$$P_G(x) = P_{G-e}(x) - P_{G/e}(x).$$

Proof. The valid colorings of $G - e$ can be divided into two groups: (1) those where the ends of the edge e get different colors — these remain to be valid colorings of G and (2) those where the ends of e get the same color — each of such colorings corresponds to unique coloring of G/e. Hence

$$P_{G-e}(x) = P_G(x) + P_{G/e}(x),$$

which is equivalent to the deletion-minus-contraction formula. □

Note that if the pseudograph G has loops then $P_G(x) = 0$ for any x. Indeed in a valid coloring the ends of loop should get different colors, which is impossible.

The latter can be also proved using the deletion-minus-contraction formula. Indeed, if e is a loop in G, then $G/e = G - e$; therefore $P_{G-e}(x) = P_{G/e}(x)$ and

$$P_G(x) = P_{G-e}(x) - P_{G/e}(x) = 0.$$

Similarly, removing a parallel edge from a pseudograph G does not change the value $P_G(x)$ for any x. Indeed, if e is an edge of G which has a parallel edge f then in G/e the edge f becomes a loop. Therefore $P_{G/e}(x) = 0$ for any x and by deletion-minus-contraction formula we get that

$$P_G(x) = P_{G-e}(x).$$

The same identity can be seen directly — any admissible coloring of $G - e$ is also admissible in G — since the ends of f get different colors, so does e.

Summarizing above discussion: the problem of finding $P_G(x)$ for a pseudograph G can be reduced to the case when G is a graph — if G has a parallel edge, removing it does not change $P_G(x)$ and if G has a loop then $P_G(x) = 0$ for all x.

Recall that polynomial P of x is an expression of the following type

$$P(x) = a_0 + a_1 \cdot x + \dots + a_n \cdot x^n,$$

with constants $a_0, \dots, a_n$, which are called *coefficients* of the polynomial. If $a_n \neq 0$, it is called *leading coefficient* of P; in this case n is the degree of P. If the leading coefficient is 1 then the polynomial is called *monic.*

5.4. Theorem. *Let G be a pseudograph with p vertexes. Then $P_G(x)$ is a polynomial with integer coefficients.*

Moreover, if G has a loop then $P_G(x) \equiv 0$; otherwise $P_G(x)$ is monic and has degree p.

Based on this result we can call $P_G(x)$ the *chromatic polynomial* of the graph G. The deletion-minus-contraction formula will play the central role in the proof.

Proof. As usual, denote by p and q the number of vertexes and edges in G. To prove the first part, we will use the induction on q.

As the base case, consider the null graph N_p; that is, the graph with p vertexes and no edges. Since N_p has no edges, any coloring of N_p is admissible. We have x choices for each of n vertexes therefore

$$P_{N_p}(x) = x^p.$$

In particular, the function $x \mapsto P_{N_p}(x)$ is given by monic polynomial of degree p with integer coefficients.

Assume that the first statement holds for all pseudographs with at most $q-1$ edges. Fix a pseudograph G with q edges. Applying the deletion-minus-contraction formula for some edge e in G, we get that

$$❶ \qquad P_G(x) = P_{G-e}(x) - P_{G/e}(x).$$

Note that the pseudographs $G-e$ and G/e have $q-1$ edges. By induction hypothesis, $P_{G-e}(x)$ and $P_{G/e}(x)$ are polynomials with integer coefficients. Hence ❶ implies the same for $P_G(x)$.

First note that if G has a loop then $P_G(x) = 0$ as G has no valid colorings. To prove the remaining statement, we also use the induction.

Assume that the statement holds for any multigraph G with at most $q-1$ edges and at most p vertexes.

Fix a multigraph G with p vertexes and q edges. Note that $G-e$ is a multigraph with p vertexes and $q-1$ edges. By the assumption, its chromatic polynomial P_{G-e} is monic of degree p.

Further the pseudograph G/e has $p-1$ vertexes, and its chromatic polynomial $P_{G/e}$ is either vanish or has degree $p-1$. In both cases the difference $P_{G-e} - P_{G/e}$ is a monic polynomial of degree p. It remains to apply the deletion-minus-contraction formula ❶. □

5.5. Advanced exercise. *Let G be a graph with p vertexes and q edges. Show that the coefficient in front of x^{p-1} of its chromatic polynomial $P_G(x)$ equals to $(-q)$.*

Hint: Apply induction on q and use the deletion-minus-contraction formula the same ways as in the proof of the theorem.

5.6. Exercise. *Use induction and deletion-minus-contraction formula to show that*

(a) $P_T(x) = x\cdot(x-1)^q$ *for any tree T with q edges;*
(b) $P_{C_p}(x) = (x-1)^p + (-1)^p\cdot(x-1)$ *for the cycle C_p of length p.*

5.7. Exercise. *Show that graph G is a tree if and only if*

$$P_G(x) = x\cdot(x-1)^{p-1}$$

for some positive integer p.

5.8. Exercise. *Show that*

$$P_{K_p}(x) = x\cdot(x-1)\cdots(x-p+1).$$

Remark. Note that for any graph G with p vertexes we have

$$P_{K_p}(x) \leqslant P_G(x) \leqslant P_{N_p}(x)$$

for any x. Since both polynomials

$$P_{K_p}(x) = x\cdot(x-1)\cdots(x-p+1), \qquad \text{and} \qquad P_{N_p}(x) = x^p,$$

are monic of degree p, it follows that so is P_G.

Hence Exercise 5.8 leads to an alternative way to prove the second statement in Theorem 5.4.

5.9. Exercise. *Construct a pair of nonisomorphic graphs with equal chromatic polynomials.*

Matching polynomial

Recall that a *matching* in a graph is a set of edges without common vertexes.

Given an integer $n \geqslant 0$, denote by $m_n = m_n(G)$ the number of matchings with n edges in the graph G.

Note that for a graph G with p vertexes and q edges, we have $m_0(G) = 1$, $m_1(G) = q$, and if $2\cdot n > p$, then $m_n(G) = 0$. The maximal integer k such that $m_k(G) \neq 0$ is called *matching number* of G. The expression

$$M_G(x) = m_0 + m_1\cdot x + \cdots + m_k\cdot x^k$$

is called *matching polynomial* of G.

Matching polynomial $M_G(x)$ gives a convenient way to work with all the numbers $m_n(G)$ simultaneously. The degree of $M_G(x)$ is the matching number of G and the total number of matching in G is its value at 1:

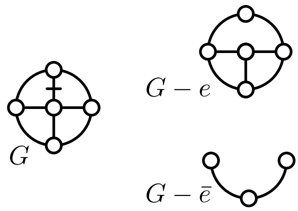
$$M_G(1) = m_0 + m_1 + \cdots + m_k.$$

5.10. Exercise. *Show that the values*

$$\tfrac{1}{2}\cdot[M_G(1)+M_G(-1)] \quad \textit{and} \quad \tfrac{1}{2}\cdot[M_G(1)-M_G(-1)]$$

equals to the number of matchings with even and odd number of edges correspondingly.

Assume e is an edge in a graph G. Recall that the graph $G-e$ is obtained from G by deleting e. Let us denote by $G-\bar{e}$ the graph obtained from G by deleting the vertexes of e with all their edges; that is, if e connects two vertexes v and w then



$$G-\bar{e} = G-\{v,w\}.$$

The following exercise is analogous to the deletion-contraction formulas 5.3 and 3.2.

5.11. Exercise. *Let G be a graph*

(a) *Show that*
$$M_G(x) = M_{G-e}(x) + x\cdot M_{G-\bar{e}}(x)$$
for any edge e in G

(b) *Use part (a) to show that the matching polynomials of complete graphs satisfy the following recursive relation:*
$$M_{K_{p+1}}(x) = M_{K_p}(x) + n\cdot x\cdot M_{K_{p-1}}(x).$$

(c) *Use (b) to calculate $M_{K_6}(x)$.*

Spanning-tree polynomial

Consider a connected graph G with the vertexes labeled by $x_1,\dots,x_p$; let us treat each label x_i as an independent variable.

For each spanning tree T in G consider the monomial

$$x_1^{d_1-1}\cdots x_p^{d_p-1},$$

where d_i denotes the degree of i-th vertex in T.

The tree T has $p-1$ edges and therefore $d_1+\dots+d_p = 2\cdot(p-1)$. It follows that the sum of degrees of the monomial is $p-2$; that is, our monomial has degree $p-2$.

The sum of these monomials is a polynomial of degree $p-2$ of p variables $x_1,\dots,x_p$. This polynomial will be called *spanning-tree polynomial* and it will be denoted by $S_G(x_1,\dots,x_p)$.

Note that the total number of spanning trees in G equals to the value $S_G(1,\dots,1)$. The following exercise shows that, the polynomial S_G keeps lot more information about spanning trees in G.

5.12. Exercise. *Let G be a graph with vertexes labeled by $x_1,\dots,x_p$ and $S_G(x_1,\dots,x_p)$ be its spanning tree polynomial. Show the following*

(a) *The value $S_G(0,1,\dots,1)$ can be interpreted as the number of spanning trees with the leaf x_1.*

(b) *The coefficient of S_G in front of $x_1\cdots x_{p-2}$ equals to the number of paths of length p connecting x_{p-1} and x_p.*

(c) *The partial derivative*

$$\tfrac{\partial}{\partial x_1}S_G(0,1,\dots,1)$$

is the numbers of spannig trees in G for which have x_1 have degree 2.

(d) *The values*

$$\tfrac{1}{2}\cdot\left(S_G(1,1,\dots,1)\pm S_G(-1,1,\dots,1)\right)$$

is the numbers of spanig trees in G for which have x_1 has odd or even degree correspondingly.

The following theorem generalizes the Cayley formula (4.10).

5.13. Theorem. *Assume that the vertexes of K_n are labeled by $x_1,\dots,x_n$. Then*

$$S_{K_p}(x_1,\dots,x_p)=(x_1+\dots+x_p)^{p-2}.$$

In particular, $s(K_p)=p^{p-2}$; that is the number of spanning trees in K_p is p^{p-2}.

Proof. Let us apply induction on p; the base case $p=1$ is evident.

Assume that the statement holds for $p-1$; that is,

❷
$$S_{K_{p-1}}(x_1,\dots,x_{p-1})=(x_1+\dots+x_{p-1})^{p-3}$$

We need to show that

$$S_{K_p}(x_1,\dots,x_p)=(x_1+\dots+x_p)^{p-2}.$$

Note that S_{K_p} is a homogeneous symmetric polynomial of degree $p-2$; that is, each monomial in S_{K_p} has degree $p-2$ and permutation of values $x_1,\dots x_p$ does not change the value $S_G(x_1,\dots,x_p)$.

Therefore it suffices to prove that all monomials in S_{K_p} without x_p sum up to $(x_1+\dots+x_{p-1})^{p-2}$.

Each of these monomials correspond to a spanning tree T with $d_p = 1$; that is, the vertex x_p is a leaf of T. In other words, T is obtained from tree T' on the vertexes $x_1, \dots, x_{p-1}$ by adding an edge from x_n to x_i with $i < p$. So the monomial for T equals to the monomial for T' times x_i.

Summing up for all such trees T' and all x_i we get

$$S_{K_{p-1}}(x_1, \dots, x_{p-1}) \cdot (x_1 + \dots + x_{p-1}).$$

By ❷, the latter equals to

$$(x_1 + \dots + x_{p-1})^{p-2}$$

which proves the required statement.

To prove the last statement, it remains to note that

$$s(G) = S_G(1, \dots, 1) = (1 + \dots + 1)^{p-2} = p^{p-2}.$$ □

5.14. Exercise. *Assume that the vertexes of the left part of $K_{m,n}$ are labeled by $x_1, \dots, x_m$ and the vertexes in the right part are labeled by $y_1, \dots, y_n$. Show that*

$$S_{K_n}(x_1, \dots, x_m, y_1, \dots, y_n) = (x_1 + \dots + x_m)^{n-1} \cdot (y_1 + \dots + y_n)^{m-1}.$$

Conclude that $s(K_{m,n}) = m^{n-1} \cdot n^{m-1}$.

Hint: Modify the proof of Theorem 5.13.

Remarks

A very good expository paper on chromatic polynomials is writen by Ronald Read; see [20]. Matching polynomials are discussed in a paper of Christopher Godsil and Ivan Gutman; see [9].

Our discussion of spanning-tree polynomials is based of a modification of Fedor Petrov [19] of the original proof of Arthur Cayley [3].

Chapter 6

Generating functions

For this chapter, the reader has to be familiar with power series.

Exponential generating functions

The power series

$$A(x) = a_0 + a_1{\cdot}x + \tfrac{1}{2}{\cdot}a_2{\cdot}x^2 + \dots + \tfrac{1}{n!}{\cdot}a_n{\cdot}x^n + \dots$$

is called *exponential generating function* of the sequence $a_0, a_1, \dots$.

If the series $A(x)$ converges in some neighborhood of zero, then it defines a function which remembers all information of the sequence a_n. The latter follows since

❶ $$A^{(n)}(0) = a_n;$$

that is, the n-th derivative of $A(x)$ at 0 equals to a_n.

However, without assuming the convergence, we can treat $A(x)$ as a formal power series. We are about to describe how to do addition, multiplying, taking derivative and so on.

Sum and product. Consider two exponential generating functions

$$\begin{aligned} A(x) &= a_0 + a_1{\cdot}x + \tfrac{1}{2}{\cdot}a_2{\cdot}x^2 + \tfrac{1}{6}{\cdot}a_3{\cdot}x^3 + \dots \\ B(x) &= b_0 + b_1{\cdot}x + \tfrac{1}{2}{\cdot}b_2{\cdot}x^2 + \tfrac{1}{6}{\cdot}b_3{\cdot}x^3 + \dots \end{aligned}$$

We will write

$$S(x) = A(x) + B(x), \quad P(x) = A(x){\cdot}B(x)$$

if the power series $S(x)$ and $P(x)$ are obtained from $A(x)$ and $B(x)$ by opening the parentheses these formulas and combining similar terms.

It is straightforward to check that $S(x)$ is the exponential generating function for the sequence

$$\begin{aligned} s_0 &= a_0 + b_0, \\ s_1 &= a_1 + b_1, \\ &\dots \\ s_n &= a_n + b_n, \\ &\dots \end{aligned}$$

The product $P(x)$ is also exponential generating function for the sequence

$$\begin{aligned} p_0 &= a_0{\cdot}b_0, \\ p_1 &= a_0{\cdot}b_1 + a_1{\cdot}b_0, \\ p_2 &= a_0{\cdot}b_2 + 2{\cdot}a_1{\cdot}b_1 + a_2{\cdot}b_0, \end{aligned}$$

❷ $$p_3 = a_0{\cdot}b_3 + 3{\cdot}a_1{\cdot}b_2 + 3{\cdot}a_2{\cdot}b_1 + a_3{\cdot}b_0,$$

$$\begin{aligned} &\dots \\ p_n &= \sum_{i=0}^{n} \tbinom{i}{n}{\cdot}a_i{\cdot}b_{n-i}. \end{aligned}$$

6.1. Exercise. *Assume $A(x)$ exponential generating function of the sequence $a_0, a_1, \dots$. Show that $B(x) = x{\cdot}A(x)$ corresponds to the sequence $b_n = n{\cdot}a_{n-1}$.*

Composition. Once we define addition and multiplication of power series we can also plug in one power series in an other. For example, if $a_0 = 0$ the expression

$$E(x) = e^{A(x)}$$

is an other power series which is obtained by pluging $A(x)$ instead of x in the power series of exponent:

$$e^x = 1 + x + \tfrac{1}{2}{\cdot}x^2 + \tfrac{1}{6}{\cdot}x^3 + \dots$$

It is harder to express the sequence (e_n) corresponding to $E(x)$ in terms of a_n, but it is easy to find first few terms. Since we assume $a_0 = 0$, we have

$$\begin{aligned} e_0 &= 1, \\ e_1 &= a_1, \\ e_2 &= a_2 + 2{\cdot}a_1^2, \\ e_3 &= a_3 + 6{\cdot}a_1{\cdot}a_2, \\ &\dots \end{aligned}$$

Derivative. The derivative of $A(x)$ is defined as the following formal power series

$$A'(x) = a_1 + a_2{\cdot}x + \tfrac{1}{2}{\cdot}a_3{\cdot}x^2 + \dots + \tfrac{1}{n!}{\cdot}a_{n+1}{\cdot}x^n + \dots$$

Note that $A'(x)$ coincides with the ordinary derivative of $A(x)$ if the latter converges.

Note that $A'(x)$ is the exponential generating function of the sequence

$$a_1, a_2, a_3, \dots$$

obtained from the original sequence

$$a_0, a_1, a_2, \dots$$

by deleting the zero-term and shifting the indexes by 1.

6.2. Exercise. *Let $A(x)$ be the exponential generating function of the sequence $a_0, a_1, a_2 \dots$. Describe the sequence b_n for which*

$$B(x) = x{\cdot}A'(x).$$

is the exponential generating function.

Calculus. If $A(x)$ converges and

$$E(x) = e^{A(x)},$$

then we have

$$\ln E(x) = A(x).$$

Also taking derivative of $E(x) = e^{A(x)}$ we get that

$$\begin{aligned} E'(x) &= e^{A(x)}{\cdot}A'(x) = \\ &= E(x){\cdot}A'(x). \end{aligned}$$

These identities have perfect meaning in terms of formal power series and they still hold without assuming the convergence. We will not prove it formally, but this is not hard.

Fibonacci numbers

Recall that Fibonacci numbers f_n are defined using the recursive identity $f_{n+1} = f_n + f_{n-1}$ with $f_0 = 0$, $f_1 = 1$.

6.3. Exercise. *Let $F(x)$ be the exponential generating function of Fibonacci numbers f_n.*

(a) *Show that it satisfies the following differential equation*

$$F''(x) = F(x) + F'(x).$$

(b) *Conclude that*

$$F(x) = \frac{1}{\sqrt{5}} \cdot \left(e^{\frac{1+\sqrt{5}}{2} \cdot x} - e^{\frac{1-\sqrt{5}}{2} \cdot x}\right).$$

(c) *Use the identity* ❶ *to derive*

$$f_n = \tfrac{1}{\sqrt{5}} \cdot \left((\tfrac{1+\sqrt{5}}{2})^n - (\tfrac{1+\sqrt{5}}{2})^n\right).$$

(This is the so called Binet's formula.)

Exponential formula

Fix a set of connected graphs $\mathcal{S}$. Denote by $c_n = c_n(\mathcal{S})$ the number of spanning subgraphs of K_n isomorphic to one of the graphs in $\mathcal{S}$. Let

$$C(x) = C_{\mathcal{S}}(x)$$

be the exponential generating function of the sequence c_n.

6.4. Theorem. *Let $\mathcal{S}$ be a set of connected graphs.*

(a) *Fix a positive integer k and denote by w_n the number of spanning subgraphs of K_n which have exactly k connected components and each connected component is isomorphic to one of the graphs in $\mathcal{S}$. Then*

$$W_k(x) = \tfrac{1}{k!} \cdot C_{\mathcal{S}}(x)^k,$$

where $W_k(x)$ is the exponential generating function of the sequence w_n.

(b) *Denote by a_n the number of* all *spanning subgraphs of K_n such that each connected component of it is from the class and let $A(x)$ be the corresponding exponential generating function. Then*

$$1 + A(x) = e^{C_{\mathcal{S}}(x)}.$$

Taking logarithm and derivative of the formula in (b), we get the following.

6.5. Corollary. *Assume $A(x)$ and $C(x)$ as in Theorem 6.4(b). Then*

$$\ln[1 + A(x)] = C(x) \quad \textit{and} \quad A'(x) = [1 + A(x)] \cdot C'(x).$$

The second formula in this corollary provides a recursive formula for the corresponding sequences which will be important latter.

Proof; (a). Denote by v_n the number of spanning subgraphs of K_n which have k ordered connected components and each connected component is isomorphic to one of the graphs in $\mathcal{S}$. Let $V_k(x)$ be the corresponding generating function.

Note that for each graph as above there are $k!$ ways to order its k components. Therefore $w_n = \frac{1}{k!}\cdot v_n$ for any n and

$$W_k(x) = \tfrac{1}{k!}\cdot V(x).$$

Hence it is sufficient to show that

❸ $$V_k(x) = C(x)^k.$$

To prove the latter identity, we apply induction on k and the multiplication formula ❷ for exponential generating functions. The base case $k = 1$ is evident.

Assuming that the identity ❸ holds for k; we need to show that

❹ $$V_{k+1} = V_k(x)\cdot C(x).$$

Assume that a spanning graph with ordered $k + 1$ connected components of K_n is given. Denote by m the number of vertexes in the first k components. There are $\binom{m}{n}$ ways to choose these vertexes among n vertexes of K_n and for each choice we have v_m ways to choose spanning subgraph with k components in it; the last component has $m - n$ vertexes and we have c_{n-m} ways to choose a subgraph from $\mathcal{S}$. All together we get that

$$\tbinom{m}{n}\cdot v_m\cdot c_n.$$

Summing it up for all m we get the multiplication formula ❷ for exponential generating functions. Hence ❹ follows.

(b). To count all graphs we need to add number of spanning graphs for all number of components; that is,

$$A(x) = W_1(x) + W_2(x) + \dots =$$

Applying to part (a), we can continue

$$\begin{aligned} &= C(x) + \tfrac{1}{2}\cdot C(x)^2 + \tfrac{1}{6}\cdot C(x)^3 + \dots = \\ &= e^{C(x)} - 1. \end{aligned}$$

The last equality follows since

$$e^x = 1 + x + \tfrac{1}{2}\cdot x^2 + \tfrac{1}{6}\cdot x^3 + \dots$$

Hence the result. □

Sample applications

The following calculations can be done without using Therem 6.4; this theorem only provides a general point of view to these problems.

Perfect matchings. Recall that a perfect matching is 1-factor of the graph. In other words, it is a set of isolated edges which cover all the vertexes. Note that if a graph admits a perfect matching then the number of its vertexes is even.

Recall that *double factorial* is the product of all the integers from 1 up to some non-negative integer n that have the same parity (odd or even) as n; the double factorial of n is denoted by $n!!$. For example,

$$9!! = 9{\cdot}7{\cdot}5{\cdot}3{\cdot}1 = 945 \quad \text{and} \quad 10!! = 10{\cdot}8{\cdot}6{\cdot}4{\cdot}2 = 3840.$$

6.6. Exercise. *Let a_n denotes the number of perfect matching in K_n. Show that*

(a) $a_2 = 1$;
(b) $a_n = 0$ *for odd* n;
(c) $a_{n+1} = n{\cdot}a_{n-1}$ *for any integer* $n \geqslant 2$.
(d) *Conclude that* $a_n = 0$ *and* $a_{n+1} = n!!$ *for odd* n.

Now we give a more complicated proof of Exercise 6.6(d).

6.7. Problem. *Use Theorem 6.4 to show that number of perfect matching in $K_{2\cdot n}$ is $(2{\cdot}n - 1)!!$.*

Soluion. Denote by a_n the number of perfect matching in K_n and let $A(x)$ be the corresponding exponential generating function.

Note that a perfect matching can be defined as a spanning subgraph such that each connected component is isomorphic to K_2. So we can apply the formula in Theorem 6.4 for the set $\mathcal{S}$ consisting of only one graph K_2.

Note that if K_n contains a spanning subgraph isomorphic to K_2, then $n = 2$. It follows that $c_2(\mathcal{S}) = 1$ and $c_n(\mathcal{S}) = 0$ for $n \neq 2$. Therefore

$$C(x) = C_{\mathcal{S}}(x) = \tfrac{1}{2}{\cdot}x^2.$$

By Theorem 6.4(b),

$$\begin{aligned}1 + A(x) &= e^{C(x)} = \\ &= e^{\frac{1}{2}\cdot x^2} = \\ &= 1 + \tfrac{1}{2}{\cdot}x + \tfrac{1}{2\cdot 2}{\cdot}x^2 + \tfrac{1}{6\cdot 4}{\cdot}x^3 + \dots + \tfrac{1}{n!\cdot 2^n}{\cdot}x^n + \dots\end{aligned}$$

That is,

$$\tfrac{1}{(2\cdot n-1)!}\cdot a_{2\cdot n-1} = 0 \quad \text{and} \quad \tfrac{1}{(2\cdot n)!}\cdot a_{2\cdot n} = \tfrac{1}{n!\cdot 2^n}$$

for any positive integer n. In particular,

$$\begin{aligned} a_{2\cdot n} &= \frac{(2\cdot n)!}{n!\cdot 2^n} = \\ &= \frac{1\cdot 2\cdots(2\cdot n)}{2\cdot 4\cdots(2\cdot n)} = \\ &= 1\cdot 3\cdots(2\cdot n-1) = \\ &= (2\cdot n-1)!! \end{aligned}$$

That is, $a_n = 0$ for odd n and $a_n = (n-1)!!$ for even n. □

Remark. Note that by Corollary 6.5, we also have

$$A'(x) = [1 + A(x)]\cdot x,$$

which is equivalent to the recursive identity

$$a_{n+1} = n\cdot a_{n-1}$$

in Exercise 6.6(c).

All matchings. Now let $\mathcal{S}$ is the set of two graphs K_1 or K_2. In this case $c_1(\mathcal{S}) = c_2(\mathcal{S}) = 1$, since K_1 and K_2 are spanning subgraph of itself. Further we have $c_n(\mathcal{S}) = 0$ for all $n \geqslant 3$ since K_n contains no spanning subgraphs isomorphic to K_1 or K_2.

Therefore the exponential generating function of the sequence $c_n(\mathcal{S})$ is a polynomial of degree 2

$$C(x) = x + \tfrac{1}{2}\cdot x^2.$$

Note that a matching in a graph G can be identified with a spanning subgraph with all connected components isomorphic to K_1 or K_2 — that is few isolated edges and few isolated vertexes. If we denote by a_n the number of all matchings and by $A(x)$ the corresponding exponential generating function then by Theorem 6.4(b), we get that

$$A(x) = e^{x+\frac{1}{2}\cdot x^2}.$$

Applying Corollary 6.5, we also have

$$A'(x) = [1 + A(x)]\cdot(1 + x).$$

The latter is equivalent to the following recursive formula for a_n:

❺ $$a_{n+1} = a_n + n{\cdot}a_{n-1}.$$

Since $a_1 = 1$ and $a_2 = 2$, we can easily find first few terms of this sequence:

$$1, 2, 4, 10, 26, \ldots$$

6.8. Exercise. *Prove formula ❺ directly — without using generating functions. Compare to Exercise 5.11(b).*

2-factors. Let $\mathcal{S}$ be the set of all cycles.

Note that 2-factor of graph can be defined as a spanning subgraph with components isomorphic to cycles. Denote by a_n and c_n the number of 2-factors and spanning cycles in K_n. Let $A(x)$ and $C(x)$ be the corresponding exponential generating functions.

6.9. Exercise.

(a) *Show that* $c_1 = c_2 = 0$ *and*

$$c_n = (n-1)!/2$$

for $n \geqslant 3$. *In particular*

$$c_1 = 0, c_2 = 0, c_3 = 1, c_4 = 3, c_5 = 12, c_6 = 60.$$

(b) *Use the identity*

$$A'(x) = [1 + A(x)]{\cdot}C'(x)$$

to find $a_1, \ldots, a_6$ *using the part (a).*

(c) *Count the number of 2-factors in* $K_1, \ldots, K_6$ *and compare with the result in the part (b).*

(d) *Use part (a) to conclude that*

$$C(x) = \tfrac{1}{2}\cdot \ln(1-x) - \tfrac{1}{2}{\cdot}x - \tfrac{1}{4}{\cdot}x^2$$

(e) *Use part (d) and Theorem 6.4(b) to show that*

$$A(x) = \sqrt{1-x}/e^{\frac{x}{2}+\frac{x^2}{4}}.$$

Counting spanning forests

Recall that a forest is a graph without cycles. Assume we want to count the number of spanning forests in K_n; denote by a_n its number and by c_n the number of connected spanning forests, that is, the number of spanning trees in K_n.

By Corollary 6.5, the following identity

$$A'(x) = [1 + A(x)]\cdot C'(x)$$

holds for the corresponding exponential generating functions.

According to Cayley theorem, $c_n = n^{n-2}$; therefore

$$c_1 = 1, c_2 = 1, c_3 = 3, c_4 = 16, \ldots$$

Applying the product formula ❷, we can use c_n to calculate a_n recurrently:

$$\begin{aligned}
a_1 &= c_1 = 1,\\
a_2 &= c_2 + a_1\cdot c_1 =\\
&= 1 + 1\cdot 1 = 2,\\
a_3 &= c_3 + 2\cdot a_1\cdot c_2 + a_2\cdot c_1 =\\
&= 3 + 2\cdot 1\cdot 1 + 2\cdot 1 = 7,\\
a_4 &= c_4 + 3\cdot a_1\cdot c_3 + 3\cdot a_2\cdot c_2 + a_3\cdot c_1 =\\
&= 16 + 3\cdot 1\cdot 3 + 3\cdot 2\cdot 1 + 7\cdot 1 = 38\\
&\ldots
\end{aligned}$$

It is instructive to check by hands there are exactly 38 spanning forests in K_4.

For the general term of a_n no simple formula is known, however the recursive formula above provides sufficiently fast way to calculate its terms.

Counting connected subgraphs

Let a_n be the number of all subgraphs of K_n and c_n is the number of connected subgraphs of K_n Assume $A(x)$ and $C(x)$ are the corresponding exponential generating functions. These two series diverge for all $x \neq 0$; nevertheless, the formula for formal power series in Theorem 6.4(b) still holds and by Corollary 6.5 we can write again

$$A'(x) = [1 + A(x)]\cdot C'(x).$$

Note that $a_n = 2^{\binom{n}{2}}$; indeed, to describe a subgraph of K_n we can choose any subset of $\binom{n}{2}$ edges of K_n, and a_n is the total number of $\binom{n}{2}$ these independent choices. In particular the first few terms of a_n are

$$a_1 = 1, a_2 = 2, a_3 = 8, a_4 = 64, \dots$$

Applying the product formula ❷, we can calculate the first few terms of c_n:

$$\begin{aligned}
c_1 &= a_1 = 1 \\
c_2 &= a_2 - a_1 \cdot c_1 = \\
&= 2 - 1 \cdot 1 = 1, \\
c_3 &= a_3 - 2 \cdot a_1 \cdot c_2 - a_2 \cdot c_1 = \\
&= 8 - 2 \cdot 1 \cdot 1 - 2 \cdot 1 = 4, \\
c_4 &= a_4 - 3 \cdot a_1 \cdot c_3 - 3 \cdot a_2 \cdot c_2 - 1 \cdot a_3 \cdot c_1 = \\
&= 64 - 3 \cdot 1 \cdot 4 - 3 \cdot 2 \cdot 1 - 1 \cdot 8 \cdot 1 = 38, \\
&\dots
\end{aligned}$$

Note that in the previous section we found a_n from c_n and now we go in the opposite direction. For the general term of c_n no closed formula is known, but the recursive formula is as good a closed formula.

Remarks

The method of Let us mention an other application of exponential generating functions.

Assume r_n denotes the number of rooted spanning trees in K_n, that is a tree with one marked vertex called its *root*. Then it is not hard to see that the exponential generating function of r_n satisfies the following identity

❻
$$R(x) = x \cdot e^{R(x)}.$$

By Lagrange inversion theorem, formula ❻ implies that $r_n = n^{n-1}$.

Since in any spanning tree of K_n we have n choices for the root, we have

$$r_n = n \cdot s(K_n).$$

This way we get yet an other proof of the Cayley formula (4.10)

$$s(K_n) = n^{n-2}.$$

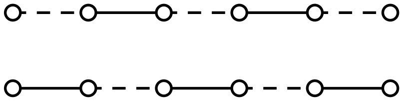
For more on the subject we recommend a classical book of Frank Harary and Edgar Palmer [11].

Chapter 7

Marriage theorem and its relatives

Let G be a bipartite graph (or briefly *bigraph*) and M is a matching in G. Recall that a path P in G is called *M-alternated* if the edges in P alternate between edges from M and edges not from M. If the path P connects two unmatched vertexes of G then it is called *M-augmenting.*

If there is an M-augmenting path P then the matching M can be improved by deleting from M the edges in P and adding the remaining edges of P. On the diagrams we denote the edges in M by solid lines and the remaining edges by dashed lines; the following diagram gives an example of the improvement.



This construction plays the central role in the Hungarian algorithm. In this chapter we will show more ways to use this construction and its analogs.

First note the following.

7.1. Observation. *Assume G be a bigraph and M is a maximal matching in G. Then G has no M-augmenting path.*

The following two exercises follow from the definitions given above; they will be used further in the sequel.

7.2. Exercise. *Let M be a matching in a bigraph G. Show that any M-augmenting path connects vertexes from the opposite parts of the bigraph.*

7.3. Exercise. *Let M be a maximal matching in a bigraph G. Assume two unmatched vertexes l and r lie in the opposite parts of G. Show that no pair of M-alternated paths starting from l and r can have a common vertex.*

Marriage theorem

Assume that G is a bigraph and S is a set of its vertexes. We say that a matching M of G *covers* S if any vertex in S is incident to an edge in M.

Given a set of vertexes W in a graph G, the set W' of all vertexes adjacent to at least one of vertexes in W will be called the *set of neighbors* of W. Note that if G is a bigraph and W lies in the left part then W' lies in the right part.

7.4. Marriage theorem. *Let G be a bigraph with the left and right parts L and R. Then G has a matching which covers L if and only if for any subset $W \subset L$ the set $W' \subset R$ of all neighbors of W contains at least as many vertexes as W; that is,*

$$|W'| \geqslant |W|.$$

Proof. Note that if there is a matching M covering L then for any set $W \subset L$ the set W' of its neighbors includes the vertexes matched with W. In particular,

$$|W'| \geqslant |W|;$$

it proves the "only if" part.

Consider a maximal matching M of G; to prove the "if" part it is sufficient to show that M covers L. Assume the contrary; that is, there is a vertex w in L which is not incident to any edge in M.

Consider the maximal set S of vertexes in G which are reachable from w by a M-alternated paths. Denote by W and W' the set of left and right vertexes in S correspondingly.

Since S is maximal, W' is the set of neighbors of W. According to Observation 7.1, the matching M provides a bijection between $W - w$ and W'. In particular,

$$|W| = |W'| + 1;$$

the latter contradicts the assumption. $\square$

7.5. Exercise. *Assume G is a r-regular bigraph; $r \geqslant 1$. Show that*

(a) G admits a 1-factor;

(b) *the edge chromatic number of G is r; in other words, G can be decomposed into 1-factors.*

Remark. If $r = 2^n$ for an integer $n \geqslant 1$, then G in the exercise above has an Euler's circuit. Note that the total number of edges in G is even, so we can deltete all odd edges from the circuit. The obtained graph G' is regular with degree 2^{n-1}. Repeating the described procedure recursively n times, we will end up at 1-factor of G.

There is a tricky way to make this idea working for arbitrary r, not necessary a power of 2; it is discovered by Noga Alon, see [2] and also [13].

7.6. Exercise. *Children from 25 countries, 10 kids from each, decided to stand in a rectangular formation with 25 rows of 10 children in each row. Show that you can always choose one child from each row so that all 25 of them will be from different countries.*

7.7. Exercise. *The sons of the king divided the kingdom between each other into 23 parts of equal area — one for each son. Later a new son was born. The king proposed a new subdivision into 24 equal parts and gave one of the parts to the newborn son.*

Show that each of 23 older sons can choose a part of land in the new subdivision which overlaps with his old part.

7.8. Exercise. *A table $n{\times}n$ filled with nonnegative numbers. Assume that the sum in each column and each row is 1. Show that one can choose n cells with positive numbers which do not share columns and rows.*

7.9. Advanced exercise. *In a group of people, for some fixed s and any k, any k girls like at least $k - s$ boys in total. Show that then all but s girls may get married on the boys they like.*

Vertex covers

A set S of vertexes in a graph is called *vertex cover* if any edge is incident to at least one of the vertexes in S.

7.10. Theorem. *In any bigraph, the number of edges in a maximal matching equals the number of vertexes in a minimal vertex cover.*

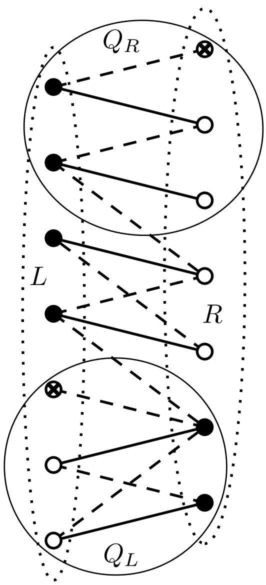
On the following diagram, a maximal matching is marked by solid lines; the remaining edges of the graph are marked by dashed lines; the vertexes of constructed cover are marked by in black and the remaining vertexes in white; the only unmatched vertexes are marked by a cross.

Proof. Fix a bigraph G; denote by L and R its left and right part. Let M be a maximal matching in G.

Assume S is a vertex cover. Then any edge m in M is incident to at least one vertex in S. Therefore

$$|S| \geqslant |M|;$$

that is, the number of vertexes in S is at least as large as the number of edges in any matching M. It remains to construct a vertex cover S such that $|S| = |M|$.



Denote by U_L and U_R the set of left and right unmatched vertexes. Denote by Q_L and Q_R the set of vertexes in G which can be reached by M-alternated paths starting from U_L and from U_R correspondingly.

Note that according to Exercise 7.3, Q_L and Q_R do not overlap.

Further note that if two vertexes are matched then they both lie in Q_L or in Q_R or neither. In particular, if m is an edge in M, then both of the vertexes of m lie in Q_L or in Q_R or neither.

For each edge m in M, include in S the right of m if it connects vertexes in Q_L and left vertex otherwise. Since S is constructed by taking exactly one vertex incident to each edge of M, we have

$$|S| = |M|.$$

It remains to prove that S is a cover; that is, at least one vertex of any edge e in G is in S. Consider the following three cases:

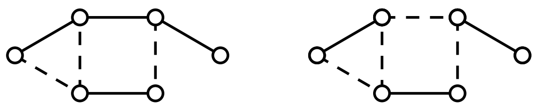
- ◇ if e connects a vertex in Q_L to a vertex in the complement of Q_L, then has both vertexes in S;
- ◇ if e connects vertexes in Q_L, then the right vertex of e is in S;
- ◇ if e connects vertexes outside Q_L, then the left vertex of e is in S. □

7.11. Exercise. *On the chess board few squares are marked. Show that the minimal number of ranks and files which cover all marked squares is the same as the maximal number of rooks on the marked squares which do not threaten each other.*

Edge cover

A collection of edges N in a graph is called *edge cover* if every vertex is incident with at least one of the edges in N.

On he following diagram two edge cover of the same graph are marked in solid lines. The second cover is minimal — there is no edge cover with smaller number of edges.



7.12. Exercise. *Show that a minimal edge cover of any graph contains no paths of length 3 and no triangle.*

7.13. Exercise. *Let G be a bigraph with p vertexes.*

Assume that a minimal edge cover of a connected bigraph G contains n edges and a maximal matching of G contains m edges. Show that

$$m + n = p.$$

Minimal cut

Recall that *directed graph* (or briefly *digraph*) is a graph, where the edges have a direction associated with them; that is, an edge in a digraph is defined as an *ordered* pair of vertexes.

7.14. Min-cut theorem. *Let s and t be two vertexes in a digraph G. Then the maximal number of oriented paths from s to t which do not have common edges equals to the minimal number of edges one can remove from G so that there will be no oriented path from s to t.*

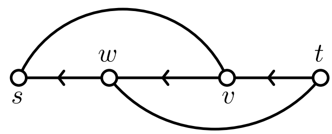
Proof. Denote by m the maximal number of oriented paths from s to t which do not have common edges and by n the minimal number of edges one can remove from G to make s disconnected from t.

Let $P_1, \dots, P_m$ be a maximal collection of paths from s to t which have no common edges. Note that to make s and t disconnected, we have to cut at least one edge from each path P_i; therefore $n \geqslant m$.

Consider the new orientation on G where each path P_i is oriented backwards — from t to s.

Consider the set S of the vertexes which are reachable from s by oriented paths for this new orientation.

Assume S contains t; that is, there is a path Q from s to t which can move along P_i only backwards. (Further the path Q will be used the same way as the augmenting path in the proof of marriage theorem.)



Assume Q overlaps with some of $P_1, \dots, P_m$. Without loss of generality, we can assume that Q first overlaps with P_1 — assume it meets P_1 at the vertex v and leaves it at the vertex w. Let us modify the paths Q and P_1 the following way: Instead of the path P_1 consider the path P_1' which goes along Q from s to v and after that goes along P_1 to t. Instead of the path Q consider the trail Q' which goes along P_1 from s to w and after that goes along Q to t.

If the constructed trail Q' is not a path (that is, if Q' visits some vertexes several times) then we can discard a maximal circuit from Q' to obtain a genuine path, which we will still denote by Q'.

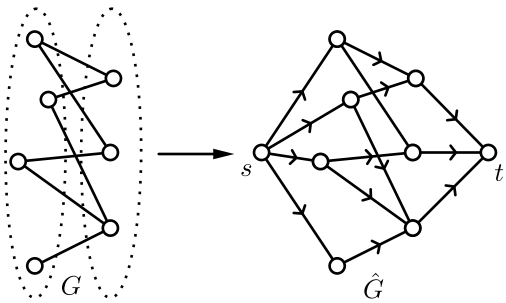
Note that the obtained collection of paths $Q', P_1', P_2 \dots, P_m$ satisfies the same conditions as the original collection, but it has smaller number of overlaps. Therefore repeating the described procedure several times, we get $m+1$ paths without overlaps — a contradiction.

It follows that $S \not\ni t$. In this case, all edges which connect S to the remaining vertexes of G are oriented toward to S. That is, every such edge which comes out of S in the original orientation belongs to one of the paths P_i.

Moreover for each path P_i there is only one such edge; in other words if a path P_i leaves S then it can not come back. Otherwise S could be made larger by moving backwards along P_i. Therefore cutting one edge in each paths P_i makes impossible to leave S. In particular we can cut m edges leving no oriented path from s and t disconnected; that is, $n \leqslant m$. $\square$

Remark. The described process has the following physical interpretation. Think of each path $P_1, \dots, P_m$ and Q as of watter pipelines from s to t. At each overlap of Q with the remaining paths the watter runs opposite direction, so we can cut the overlapping edges and connect the open ends of the pipes to each other while keeping the watter flow unchanged. As the result, we get $m+1$ pipes form s to t with no common edges and possibly some cycles which we can discard.

Assume G is a bigraph. Let us add to G two vertexes s and t so that s is connected to each vertex in the left part of G and t is connected to each vertex in the right part of G and orient the graph from left to right. Denote the obtained digraph by $\hat{G}$.



7.15. Advanced exercise. *Give an other proof of the marriage theorem for a bigraph G, applying the min-cut theorem to $\hat{G}$.*

Remarks

The marriage theorem was proved by Philip Hall in [10]; it has many applications in all branches of mathematics. The theorem on vertex cover was discovered by Dénes Kőnig [17] and independently by Jenő Egerváry [4]. The theorem on min-cut was proven by Peter Elias, Amiel Feinstein, and Claude Shannon [5], and independently also by Lester Ford and Delbert Fulkerson [7].

An extensive overview of the marriage theorem and its relatives is given by Alexander Evnin in [6], which I recommend to everyone who can read Russian.

Appendix A

Correction for 8.4.1

There is an inaccuracy in the proof of [12, Theorem 8.4.1] about stretchable planar graphs. Namely in the planar drawing of $G - h$, the region R might be unbounded.

To fix this inaccuracy, one needs to prove slightly stronger statement. Namely that any planar drawing of the maximal planar graph G can be stretched. That is, given a planar drawing of G there is a stretched drawing of G and a bijection between the bounded (necessary triangular) regions such that corresponding triangles have the same edges of G as the sides.

The remaining part of the proof works with no other changes.

Bibliography

[1] M. Aigner, G. M. Ziegler, *Proofs from The Book.* 2014.

[2] N. Alon, *A simple algorithm for edge-coloring bipartite multigraphs.* Inform. Process. Lett. 85 (2003), no. 6, 301–302.

[3] A. Cayley, *A theorem on trees.* Quart. J. Pure Appl. Math. 23 (1889) 376–378.

[4] E. Egerváry, *Matrixok Kombinatorius Tulajdonságairol* Matematikai és Fizikai Lapok. 38 (1931), 16–28.

[5] P. Elias, A. Feinstein, and C. Shannon. *A note on the maximum flow through a network.* IRE Transactions on Information Theory 2, no. 4 (1956), 117–119.

[6] А. Ю. Эвнин, *Вокруг теоремы Холла.* Матем. обр. 3(34) 2005, 2—23.

[7] L. Ford, D. Fulkerson. *Maximal flow through a network.* Canadian journal of Mathematics 8 (1956) no. 3, 399–404.

[8] M. Gardner, *The 2nd Scientific American Book of Mathematical Puzzles and Diversions.*

[9] C. D. Godsil, I. Gutman, *On the theory of the matching polynomial.* J. Graph Theory 5 (1981), no. 2, 137–144.

[10] P. Hall, *On Representatives of Subsets*, J. London Math. Soc., vol. 10 (1935), 26–30.

[11] F. Harary, E. Palmer, *Graphical enumeration.* 1973.

[12] N. Hartsfield and G. Ringel, *Pearls in graph theory: a comprehensive introduction.* 2013.

[13] G. Kalai, *The seventeen camels riddle, and Noga Alon's camel proof and algorithms.* `https://gilkalai.wordpress.com/` 2017/02/16/

[14] L. Lovász, László *Combinatorial problems and exercises.* Corrected reprint of the 1993 second edition. AMS Chelsea Publishing, Providence, RI, 2007. 642 pp. ISBN: 978-0-8218-4262-1

[15] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete mathematics. A foundation for computer science.* 1994.

[16] C. Jordan, *Calculus of finite differences.* 1939

[17] D. Kőnig, *Gráfok és mátrixok*, Matematikai és Fizikai Lapok, 38 (1931): 116–119.

[18] M. Levi, An Electrician's (or a plumber's) proof of Euler's polyhedral formula, *SIAM News* 50, no. 4, May 2017.

[19] F. Petrov, Generating function in graph theory, *MathOverflow* `https://mathoverflow.net/q/287767` (version: 2017-12-05)

[20] R. Read, An introduction to chromatic polynomials. *J. Combinatorial Theory* 4 1968 52–71.

[21] M. H. Shirdareh Haghighi, Kh. Bibak, Recursive relations for the number of spanning trees. *Appl. Math. Sci. (Ruse)* 3 (2009), no. 45–48, 2263–2269.