All-set-homogeneous spaces

Nina Lebedeva and Anton Petrunin

Abstract

We give a classification of a certain subclass of all-set-homogeneous length spaces; a metric space is all-set-homogeneous if any of its partial isometries can be extended to a genuine isometry.

1 Main result

A metric space M is said to be *all-set-homogeneous* if for any subset $A \subset M$ any distance-preserving map $A \to M$ can be extended to an isometry $M \to M$.

Given a metric space M and a positive integer n, consider all pseudometrics induced on n points $x_1, \dots, x_n \in M$. Any such metric is completely described by $N = \frac{n\cdot(n-1)}{2}$ real numbers $|x_i - x_j|$ for $i < j$, so it can be encoded by a point in $\mathbb{R}^N$. The set of all these points $F_n(M) \subset \mathbb{R}^N$ will be called n^{th} *fingerprint* of M.

Let us denote by $\mathbb{H}^m$ the hyperbolic space, $\mathbb{E}^m$ — Euclidean space, $\mathbb{S}^m$ — unit sphere, $\mathbb{R}\mathrm{P}^m$ — projective space, all with standard metrics; here m stands for the dimension of the space.

Given a metric space M, we denote by $\lambda\cdot M$ the rescaled M with factor $\lambda > 0$.

Theorem. *Let M be a complete all-set-homogeneous length space. Suppose that all fingerprints of M are closed. Then M is isometric to $\mathbb{E}^m$, $\lambda\cdot\mathbb{H}^m$, $\lambda\cdot\mathbb{S}^m$, or $\lambda\cdot\mathbb{R}\mathrm{P}^m$ for some $\lambda > 0$ and nonnegative integer m.*

The question about classifying all-set-homogeneous space was posted by Joseph O'Rourke [2]. The locally compact case of the theorem is proven by Jacques Tits [3]. In this case, the condition on fingerprints is not needed. Moreover, the three-point-homogeneity is sufficient; that is, any distance preserving map defined on a subset with at most 3 points can be extended to an isometry. Our result shows that one can trade local compactness to a better type of homogeneity and closed fingerprints.

Proof. If M is locally compact, then the statement follows from the result of Jacques Tits [3].

Assume M is not locally compact. Then there is an infinite sequence of points $x_1, x_2, \dots$ such that $\varepsilon < |x_i - x_j| < 1$ for some $\varepsilon > 0$. Applying the Ramsey theorem, we get that for arbitrary positive integer n and $\delta > 0$ there is a sequence $x_1, x_2, \dots, x_n$ such that $|x_i - x_j| \lesseqgtr r \pm \delta$ where $\varepsilon \leqslant r \leqslant 1$. Since the fingerprints are closed, there is an arbitrarily long sequence $x_1, x_2, \dots, x_n$ such that $|x_i - x_j| = r$ for some fixed $r > 0$.

Choose a maximal (with respect to inclusion) set of points $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ such that $|x_\alpha - x_\beta| = r$ for any $\alpha \neq \beta$. Since M is all-set-homogeneous, we can assume that $\mathcal{A}$ is infinite. In particular, there is an injective map $f\colon \mathcal{A}\to\mathcal{A}$ such that $f(\mathcal{A})$ is a proper subset of $\mathcal{A}$.

Note that the map $x_\alpha\mapsto x_{f(\alpha)}$ is distance preserving. Since $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is maximal, for any $y\notin\{x_\alpha\}_{\alpha\in\mathcal{A}}$ we have that $|y-x_\alpha|_M\ne r$ for some α. It follows that a distance preserving map $M\to M$ that agrees with $x_\alpha\mapsto x_{f(\alpha)}$ cannot have in its image a point x_α for $\alpha\in\mathcal{A}\setminus f(\mathcal{A})$. In particular, no isometry $M\to M$ agrees with the map $x_\alpha\mapsto x_{f(\alpha)}$ — a contradiction. □

2 Example

Recall that for any cardinality $n\geqslant 2$ there is the so-called *universal* $\mathbb{R}$*-tree of valence* n; let us denote it by $\mathbb{T}_n$.

- ◇ The space $\mathbb{T}_n$ is a complete $\mathbb{R}$-tree; in particular, $\mathbb{T}_n$ is geodesic.
- ◇ $\mathbb{T}_n$ is homogeneous; that is, the group of isometries acts transitively on $\mathbb{T}_n$.
- ◇ The space $\mathbb{T}_n$ is n-universal; that is, $\mathbb{T}_n$ includes a copy of any $\mathbb{R}$-tree of maximal valence at most n

An explicit construction of $\mathbb{T}_n$ is given by Anna Dyubina (now Erschler) and Iosif Polterovich [1]. Their proof of the universality of $\mathbb{T}_n$ admits a straightforward modification that proves the following claim.

Claim. *If n is finite, then $\mathbb{T}_n$ is all-set-homogeneous.*

The claim implies that the condition on fingerprints in the theorem is necessary.

Proof. Choose a subset $A\subset\mathbb{T}_n$. Let us extend any distance preserving map $f\colon A\to\mathbb{T}_n$ to a distance preserving map $\mathbb{T}_n\to\mathbb{T}_n$.

Applying the Zorn lemma, we can assume that A is maximal; that is, the domain of f cannot be extended by a single point. Note that in this case, A is a closed convex set in $\mathbb{T}_n$; in particular, A is an $\mathbb{R}$-tree with maximal valence at most n. Suppose $A\ne\mathbb{T}_n$, choose $a\in A$ and $b\notin A$. Let $c\in A$ be the last point on the geodesic $[ab]_{\mathbb{T}_n}$. Note that the valence of c in A is smaller than n.

Let $c'=f(c)$; since n is finite at least one of connected components $\mathbb{T}_n\setminus\{c'\}$ does not intersect $A'=f(A)$. Choose a point b' in this connected component on distance $|c-b|$ from c'. Observe that f can be extended by $b\mapsto b'$ — a contradiction.

It remains to show that $f(\mathbb{T}_n)=\mathbb{T}_n$ for any distance-preserving map $f\colon\mathbb{T}_n\to\to\mathbb{T}_n$. Assume the contrary; that is, $B=f(\mathbb{T}_n)$ is a proper subset on $\mathbb{T}_n$; choose $a\notin B$. Note that B is a closed convex set in $\mathbb{T}_n$. Let $b\in\partial B$ be the point that minimizes the distance to a. Observe that the valence of b in B is smaller than n — a contradiction. □

3 Remarks

Let us list examples for related classification problems. We would be interested to see other examples or a proof that there are no more.

First of all, we do not see other examples of complete all-set-homogeneous length spaces except those listed in the theorem and the claim.

The definition of all-set homogeneity can be restricted to the distance-preserving map with *small* domains; for example, *finite* or *compact* domains. In these cases, we say that the space is *finite-set-homogeneous* or *compact-set-homogeneous* respectively.

Examples of complete separable compact-set-homogeneous length spaces with closed fingerprints include the spaces listed in the theorem, plus the Urysohn space $\mathbb{U}$ and its version $\mathbb{U}_d$ of diameter d (it is isometric to a sphere of radius $\frac{d}{2}$ in $\mathbb{U}$). (Without separability condition, we get in addition the $\mathbb{R}$-trees from the claim.)

For finite-set-homogeneous we get in addition infinite-dimensional analogs of the spaces in the theorem; in particular the Hilbert space.

Acknowledgments. The second author wants to thank Rostislav Matveyev for an interesting discussion on the Rubinstein Street.

References

[1] A. Dyubina and I. Polterovich. "Explicit constructions of universal $\mathbb{R}$-trees and asymptotic geometry of hyperbolic spaces". *Bull. London Math. Soc.* 33.6 (2001), 727–734.

[2] J. O'Rourke. *Which metric spaces have this superposition property?* MathOverflow. eprint: `https://mathoverflow.net/q/118008`.

[3] J. Tits. "Sur certaines classes d'espaces homogènes de groupes de Lie". *Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8°* 29.3 (1955), 268.