All-set-homogeneous spaces

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Abstract

A metric space is said to be all-set-homogeneous if any of its partial isometries can be extended to a genuine isometry. We give a classification of a certain subclass of all-set-homogeneous length spaces.

1 Main result

A metric space M is said to be *all-set-homogeneous* if for any subset $A \subset M$ any distance-preserving map $A \to M$ can be extended to an isometry $M \to M$.

Examples of all-set-homogeneous spaces include all *classical spaces*; these are complete simply-connected Rieamnnian manifolds.

Nonclassical examples include the universal $\mathbb{R}$-trees of finite valence; these are discussed in the next section.

The following two results are closely related to our theorem; see also the survey by Semeon Bogatyi [3].

◇ *Any complete all-set-homogeneous geodesic space with locally unique nonbifurcating geodesics is classical;* it was proved by Garrett Birkhoff [2].

◇ *Any locally compact three-point-homogeneous geodesic space is classical.* This result was proved by Herbert Busemann [4]; it also follows from the more general result of Jacques Tits [8] about two-point-homogeneous spaces.

Given a metric space M and a positive integer n, consider all pseudometrics induced on n points $x_1, \dots, x_n \in M$. Any such metric is completely described by $N = \frac{n\cdot(n-1)}{2}$ distances $|x_i - x_j|_M$ for $i < j$, so it can be encoded by a point in $\mathbb{R}^N$. The set of all these points $F_n(M) \subset \mathbb{R}^N$ will be called n^{th} *fingerprint* of M.

Theorem. *Let M be a complete all-set-homogeneous length space. Suppose that all fingerprints of M are closed. Then M is classical.*

Proof. If M is locally compact, then the statement follows from the result of Jacques Tits [8].

Assume M is not locally compact. Then there is an infinite sequence of points $x_1, x_2, \dots$ such that $\varepsilon < |x_i - x_j| < 1$ for some $\varepsilon > 0$. Applying the Ramsey theorem, we get that for arbitrary positive integer n and $\delta > 0$ there is a sequence $x_1, x_2, \dots, x_n$ such that $|x_i - x_j| \lesseqgtr r \pm \delta$ where $\varepsilon \leqslant r \leqslant 1$. Since the fingerprints are closed, there is an arbitrarily long sequence $x_1, x_2, \dots, x_n$ such that $|x_i - x_j| = r$ for some fixed $r > 0$.

Choose a maximal (with respect to inclusion) set of points $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ such that $|x_\alpha - x_\beta| = r$ for any $\alpha \neq \beta$. Since M is all-set-homogeneous, we can assume that $\mathcal{A}$ is infinite. In particular, there is an injective map $f\colon \mathcal{A} \to \mathcal{A}$ such that $f(\mathcal{A})$ is a proper subset of $\mathcal{A}$.

Note that the map $x_\alpha \mapsto x_{f(\alpha)}$ is distance preserving. Since $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is maximal, for any $y \notin \{x_\alpha\}_{\alpha\in\mathcal{A}}$ we have that $|y - x_\alpha|_M \neq r$ for some α. It follows that a distance preserving map $M \to M$ that agrees with $x_\alpha \mapsto x_{f(\alpha)}$ cannot have in its image a point x_α for $\alpha \in \mathcal{A} \setminus f(\mathcal{A})$. In particular, no isometry $M \to M$ agrees with the map $x_\alpha \mapsto x_{f(\alpha)}$ — a contradiction. □

2 Example

For any cardinality $n \geqslant 2$ there is a uniquely defined up to isometry space $\mathbb{T}_n$ that satisfies the following properties:

◇ The space $\mathbb{T}_n$ is a complete $\mathbb{R}$-tree; in particular, $\mathbb{T}_n$ is geodesic.

◇ $\mathbb{T}_n$ is homogeneous; that is, the group of isometries acts transitively on $\mathbb{T}_n$.

◇ The space $\mathbb{T}_n$ is n-universal; that is, $\mathbb{T}_n$ includes an isometric copy of any $\mathbb{R}$-tree of maximal valence at most n.

The space $\mathbb{T}_n$ is called a *universal* $\mathbb{R}$*-tree of valence* n. An explicit construction of $\mathbb{T}_n$ is given by Anna Dyubina and Iosif Polterovich [5]. Their proof of the universality of $\mathbb{T}_n$ admits a straightforward modification that proves the following claim.

Claim. *If n is finite, then $\mathbb{T}_n$ is all-set-homogeneous.*

Note that the claim implies that the condition on fingerprints in the theorem is necessary. In fact, if $n \geqslant 3$, then the $(n+1)^{\text{th}}$ fingerprint of $\mathbb{T}_n$ is not closed — $\mathbb{T}_n$ does not contain $n+1$ points on distance 1 from each other, but it contains an arbitrarily large set with pairwise distances arbitrarily close to 1.

Proof. Let $f\colon A \to \mathbb{T}_n$ be a distance preserving map for some subset $A \subset \mathbb{T}_n$. Let us extend f to a distance preserving map $\mathbb{T}_n \to \mathbb{T}_n$.

Applying the Zorn lemma, we can assume that A is maximal; that is, the domain of f cannot be extended by a single point. Note that in this case, A is a closed convex set in $\mathbb{T}_n$; in particular, A is an $\mathbb{R}$-tree with maximal valence at most n.

Arguing by contradiction, suppose $A \neq \mathbb{T}_n$, choose $a \in A$ and $b \notin A$. Let $c \in A$ be the last point on the geodesic $[ab]_{\mathbb{T}_n}$. Note that the valence of c in A is smaller than n.

Let $c' = f(c)$; since n is finite, at least one of connected components of $\mathbb{T}_n \setminus \{c'\}$ does not intersect $A' = f(A)$. Choose a point b' in this component such that $|c' - b'|_{\mathbb{T}_n} = |c - b|_{\mathbb{T}_n}$. Observe that f can be extended by $b \mapsto b'$ — a contradiction.

It remains to show that $f(\mathbb{T}_n) = \mathbb{T}_n$ for any distance-preserving map $f\colon \mathbb{T}_n \to \to \mathbb{T}_n$. Assume the contrary; that is, $B = f(\mathbb{T}_n)$ is a proper subset on $\mathbb{T}_n$. Note that B is a closed convex set in $\mathbb{T}_n$. Choose $a \in B$ and $b \notin B$. Let $c \in B$ be the last point on the geodesic $[ab]_{\mathbb{T}_n}$. Observe that the valence of c in B is smaller than n — a contradiction. □

3 Remarks

Let us list examples for related classification problems. We would be interested to see other examples or a proof that there are no more.

First of all, we do not see other examples of complete all-set-homogeneous length spaces except those listed in the theorem and the claim.

Without length-metric assumption, examples include finite discrete spaces, Cantor sets with natural ultrametrics, and many more spaces.

The definition of all-set-homogeneous spaces can be restricted to the distance-preserving map with *small* domains; for example, *finite* or *compact* domains. In these cases, we say that the space is *finite-set-homogeneous* or *compact-set-homogeneous* respectively.

Examples of complete separable compact-set-homogeneous length spaces include the spaces listed in the theorem, plus the Urysohn spaces $\mathbb{U}$ and $\mathbb{U}_d$ (the space $\mathbb{U}_d$ is isometric to a sphere of radius $\frac{d}{2}$ in $\mathbb{U}$). Without the separability condition, we get in addition the $\mathbb{R}$-trees from the claim.

The finite-set-homogeneous spaces include, in addition, infinite-dimensional analogs of the spaces in the theorem; in particular the Hilbert space.

Let us also mention that finite-set homogeneity is closely related to the *metric version of Fraïssé limit* introduced by Itay Ben-Yaacov [1].

Acknowledgments. This note is inspired by the question of Joseph O'Rourke [7]. We want to thank James Hanson for his interesting and detailed comments to our question [6]. The second author wants to thank Rostislav Matveyev for an interesting discussion on Rubinstein Street.

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