What is Alexandrov geometry?

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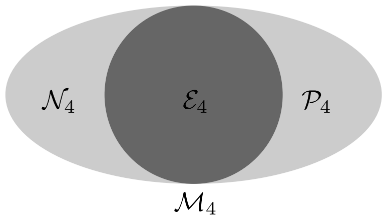
Manifesto

Consider the space $\mathcal{M}_4$ of all isometry classes of 4-point metric space. Each element in $\mathcal{M}_4$ can be described by 6 numbers — the distances between all 6 pairs of its points, say $\ell_{i,j}$ for $1 \leqslant i < j \leqslant 4$ modulo permutations of $(1, 2, 3, 4)$. These 6 numbers is subject to 12 triangle inequalities; that is,

$$\ell_{i,j} + \ell_{j,k} \geqslant \ell_{i,k}$$

holds for all i, j and k; here we assume that $\ell_{j,i} = \ell_{i,j}$ and $\ell_{i,i} = 0$.

Consider the subset $\mathcal{E}_4 \subset \mathcal{M}_4$ of all isometry classes of metric spaces which admit an isometric embedding into Euclidean space. The complement $\mathcal{M}_4 \backslash \mathcal{E}_4$ has two connected components.



0.0.1. Exercise. *Prove the latter statement.*

One of the component will be denoted by $\mathcal{P}_4$ and the other by $\mathcal{N}_4$. Here $\mathcal{P}$ and $\mathcal{N}$ stay for *positive* and *negative curvature* because spheres have no quadruples of type $\mathcal{N}_4$ and hyperbolic plane has no quadruples of type $\mathcal{P}_4$.

A metric space with an intrinsic metric which has no quadruples of points of type $\mathcal{P}_4$ or $\mathcal{N}_4$ are called Alexandrov space with non-positive or non-negative curvature correspondingly.

Here is an exercise, solving which would force the reader to rebuild considerable part of the theory.

0.0.2. Advanced exercise. *Assume $\mathcal{X}$ is a complete metric space with intrinsic metric which contains only quadruples of type $\mathcal{E}_4$. Show that $\mathcal{X}$ is isometric to the convex set in a Hilbert space.*

In fact it would be helpful to think about this exercise for a couple of days before you proceed on reading.

In the definition above, instead of Euclidean space, one can take the hyperbolic plane. In this case one arrives to the definition of spaces with curvature bounded above or below by -1.

To define spaces with curvature bounded above or below by 1, one has to take unit 3-sphere instead, and allow to check only the quadruples of points such that each of the 4 triangles has perimeter at most $2\cdot\pi$. The later condition could be thought as a part of *spherical triangle inequality.*

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Chapter 1

Preliminaries

In this chapter we fix some conventions and remind the main definitions. For more background in metric geometry, reader advised to read the book of Burgos and Ivanov [31].

1.1 Metric spaces

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|xy|$ or $|xy|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

The function $|x*| \colon y \mapsto |xy|$ is called *distance function* from x.

- ⋄ The *diameter* of metric space $\mathcal{X}$ is defined as

$$\operatorname{diam} \mathcal{X} = \sup\{\, |xy|_{\mathcal{X}} \mid x, y \in \mathcal{X} \,\}.$$

- ⋄ Given $R \in [0, \infty]$ and $x \in \mathcal{X}$, the sets

$$\begin{aligned} \mathrm{B}(x,R) &= \{y \in \mathcal{X} \mid |xy| < R\}, \\ \overline{\mathrm{B}}[x,R] &= \{y \in \mathcal{X} \mid |xy| \leqslant R\} \end{aligned}$$

are called respectively the *open* and the *closed balls* of radius R with center x. Again, if we need to emphasize that these balls are taken in the metric space $\mathcal{X}$, we write

$$\mathrm{B}(x,R)_{\mathcal{X}} \quad \text{and} \quad \overline{\mathrm{B}}[x,R]_{\mathcal{X}}$$

correspondingly.

A metric space $\mathcal{X}$ is called *proper* if all closed bounded sets in $\mathcal{X}$ are compact. This condition is equivalent to each of the following statements:

1. For some (and therefore any) point $p \in \mathcal{X}$ and any $R < \infty$, the closed ball $\overline{\mathrm{B}}[p, R] \subset \mathcal{X}$ is compact.
2. The function $|p*| \colon \mathcal{X} \to \mathbb{R}$ is proper for some (and therefore any) point $p \in \mathcal{X}$; that is, for any compact set $K \subset \mathbb{R}$, its inverse image $\{\, x \in \mathcal{X} \mid |px|_{\mathcal{X}} \in K \,\}$ is compact.

1.2 Constructions

Product space. Given two metric spaces $\mathcal{U}$ and $\mathcal{V}$, the *product space* $\mathcal{U} \times \mathcal{V}$ is defined as the set of all pairs (u, v) in the set $\mathcal{U} \times \mathcal{V}$ with the metric defined by formulas

$$|(u^1, v^1)\ (u^2, v^2)|_{\mathcal{U}\times\mathcal{V}} = \sqrt{|u^1 u^2|^2_{\mathcal{U}} + |v^1 v^2|^2_{\mathcal{V}}}.$$

Given a metric spaces $\mathcal{U}$, we say that $\mathcal{V}$ is Euclidean cone over $\mathcal{U}$ if the underlying set of $\mathcal{V}$ is formed

Cone. The *cone* $\mathcal{V} = \mathrm{Cone}\,\mathcal{U}$ over metric space $\mathcal{U}$ is defined as the metric space with underlying set formed by the equivalence classes on $[0, \infty) \times \mathcal{U}$ with the minimal equivalence relation "$\sim$" such that $(0, p) \sim (0, q)$ for any points $p, q \in \mathcal{U}$ and the metric given by cosine rule

$$|(p, s)\ (q, t)|_{\mathcal{V}} = \sqrt{s^2 + t^2 - 2{\cdot}s{\cdot}t{\cdot}\cos\alpha},$$

where $\alpha = \max\{\pi, |pq|_{\mathcal{U}}\}$.

The point in the cone $\mathcal{V}$ formed by the equivalence class of $0 \times \mathcal{U}$ is called *tip of the cone* and denoted by 0 or $0_{\mathcal{V}}$. The distance $|0v|_{\mathcal{V}}$ is called norm of v and denoted as $|v|$ or $|v|_{\mathcal{V}}$.

Suspension. The *suspension* $\mathcal{V} = \mathrm{Susp}\,\mathcal{U}$ over metric space $\mathcal{U}$ is defined as the metric space with underlying set formed by the equivalence classes on $[0, \infty] \times \mathcal{U}$ with the equivalence relation "$\sim$" defined by $(0, p) \sim (0, q)$ and $(\pi, p) \sim (\pi, q)$ for any points $p, q \in \mathcal{U}$ and the metric given by spherical cosine rule

$$\cos|(p, s)\ (q, t)|_{\mathrm{Cone}\,\mathcal{U}} = \cos s{\cdot}\cot t + \sin s{\cdot}\sin t{\cdot}\cos\alpha,$$

where $\alpha = \max\{\pi, |pq|_{\mathcal{U}}\}$.

The points in the cone $\mathcal{V}$ formed by the equivalence class of $0 \times \mathcal{U}$ and $\pi \times \mathcal{U}$ is called *north south pole* of the suspension.

1.2.1. Exercise. *Let $\mathcal{U}$ be a metric space. Show that the spaces*

$$\mathbb{R} \times \mathrm{Cone}\,\mathcal{U} \quad \textit{and} \quad \mathrm{Cone}[\mathrm{Susp}\,\mathcal{U}]$$

are isometric.

1.3 Geodesics, triangles and hinges

Geodesics. Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ be a real interval. A globally isometric map $\gamma\colon \mathbb{I}\to\mathcal{X}$ is called a *unit-speed geodesic*[1]; in other words, $\gamma\colon \mathbb{I}\to\mathcal{X}$ is a unit-speed geodesic if

$$|\gamma(s)\ \gamma(t)|_{\mathcal{X}} = |s-t|$$

for any pair $s,t\in\mathbb{I}$.

A unit-speed geodesic $\gamma\colon \mathbb{R}_{\geqslant 0}\to\mathcal{X}$ is called a *ray*.

A unit-speed geodesic $\gamma\colon \mathbb{R}\to\mathcal{X}$ is called a *line*.

1.3.1. Proposition. *Suppose $\mathcal{X}$ is a metric space and $\gamma\colon [0,\infty)\to\mathcal{X}$ is a ray. Then the* Busemann function $\operatorname{bus}_\gamma\colon \mathcal{X}\to\mathbb{R}$

❶ $$\operatorname{bus}_\gamma(x) = \lim_{t\to\infty} |\gamma(t)\ x| - t$$

is defined and 1*-Lipschitz.*

Proof. As follows from the triangle inequality, the function

$$t\mapsto |\gamma(t)\ x| - t$$

is nonincreasing in t. Clearly $|\gamma(t)\ x| - t \geqslant -|\gamma(0)\ x|$. Thus the limit in ❶ is defined. □

A unit-speed geodesic between p and q in $\mathcal{X}$ will be denoted by $\operatorname{geod}_{[pq]}$. We assume $\operatorname{geod}_{[pq]}$ is parametrized starting at p; that is, $\operatorname{geod}_{[pq]}(0)=p$ and $\operatorname{geod}_{[pq]}(|pq|)=q$. The image of $\operatorname{geod}_{[pq]}$ will be denoted by $[pq]$ and called a *geodesic*. The term *geodesic* will also be used for a linear reparametrization of a unit-speed geodesic; when a confusion is possible we call the latter a *constant-speed geodesic*. With slight abuse of notation, we will use $[pq]$ also for the class of all linear reparametrizations of $\operatorname{geod}_{[pq]}$.

We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$. Also we use the following short-cut notation:

$$]pq[\ = [pq]\backslash\{p,q\}, \qquad]pq] = [pq]\backslash\{p\}, \qquad [pq[\ = [pq]\backslash\{q\}.$$

In general, a geodesic between p and q need not exist and if it exists, it need not to be unique. However, once we write $\operatorname{geod}_{[pq]}$ or $[pq]$ we mean that we made a choice of geodesic.

A metric space is called *geodesic* if any pair of its points can be jointed by a geodesic.

[1] Various authors call it differently: *shortest path*, *minimizing geodesic*.

A constant-speed geodesic $\gamma\colon [0,1] \to \mathcal{X}$ is called a *geodesic path*. Given a geodesic $[pq]$, we denote by $\mathrm{path}_{[pq]}$ the corresponding geodesic path; that is,

$$\mathrm{path}_{[pq]}(t) \equiv \mathrm{geod}_{[pq]}(t{\cdot}|pq|).$$

A curve $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a *local geodesic*, if for any $t \in \mathbb{I}$ there is a neighborhood $U \ni t$ in $\mathbb{I}$ such that the restriction $\gamma|_U$ is a constant-speed geodesic. If $\mathbb{I} = [0,1]$, then γ is called a *local geodesic path*.

Triangles. For a triple of points $p, q, r \in \mathcal{X}$, a choice of triple of geodesics $([qr], [rp], [pq])$ will be called a *triangle* and we will use short notation $[pqr] = ([qr], [rp], [pq])$. Again given a triple $p, q, r \in \mathcal{X}$ it may be no triangle $[pqr]$ simply because one of the pairs of these points can not be joined by a geodesic, and also it maybe many different triangles with these vertexes, any of which can be denoted by $[pqr]$. Once we write $[pqr]$, it means we made a choice of such a triangle, that is, a choice of each $[qr], [rp]$ and $[pq]$. The value $|pq| + |qr| + |rp|$ will be called *perimeter of triangle* $[pqr]$.

Hinges. Let $p, x, y \in \mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair geodesics $([px], [py])$ will be called *hinge* and briefly, it will be denoted by $[p\,{}^x_y] = ([px], [py])$.

Convex sets. Let $\mathcal{X}$ be a metric space. A set $A \subset \mathcal{X}$ is called *convex* if for every two points $p, q \in A$, every geodesic $[pq]$ of $\mathcal{X}$ lies in A.

A set $A \subset \mathcal{X}$ is called *locally convex* if every point $a \in A$ admits an open neighborhood $\Omega \ni a$ such that for every two points $p, q \in A \cap \Omega$ every geodesic $[pq] \subset \Omega$ lies in A.

Note that any open set is locally convex by definition.

1.4 Length spaces

Recall that a curve is a continuous map from a real interval to a space.

1.4.1. Definition. *Let $\mathcal{X}$ be a metric space and $\alpha\colon \mathbb{I} \to \mathcal{X}$ is a curve. We define the* length *of α as*

$$\mathrm{length}\,\alpha \stackrel{def}{=\!=} \sup_{t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n} \sum_i |\alpha(t_i)\ \alpha_{i+1}|$$

It is easy to see that if $\tau\colon\ [c,d] \to [a,b]$ is monotonic onto function then $\mathrm{length}\,\alpha = \mathrm{length}(\alpha \circ \tau)$.

Given two points x and y in a metric space $\mathcal{X}$ consider the value

$$\|xy\| = \inf_\alpha \{\mathrm{length}\,\alpha\},$$

where infimum is taken for all paths α from x to y.

If the value $\|xy\|$ is finite for any pair of points x and y then $\|{*}{*}\|$ defines a metric on $\mathcal{X}$; it will be called the induced *length-metric* on $\mathcal{X}$.

In this book, most of the time we consider length spaces. If $\mathcal{X}$ is length space, and $A \subset \mathcal{X}$. The set A comes with the inherited metric from $\mathcal{X}$ which might be not a length-metric. The corresponding length-metric on A will be denoted as $|{*}{*}|_A$.

1.4.2. Definition. *If $\|xy\| = |xy|$ for any pair of points $x, y \in \mathcal{X}$ then $\mathcal{X}$ is called a* length space.

In other words, a metric space $\mathcal{X}$ is a *length space* if for any $\varepsilon > 0$ and any two points $x, y \in \mathcal{X}$ there is a path $\alpha\colon [0,1] \to \mathcal{X}$ connecting[2] x to y such that

$$\operatorname{length}\alpha < |xy| + \varepsilon.$$

Note that any geodesic space is a length space; as you see from the following example, the contrary does not hold.

1.4.3. Example. *Let $\mathcal{X}$ be obtained by gluing a countable collection of disjoint intervals I_i of length $1 + 1/i$ where for each I_i one end is glued to $p = \{0\}$ and the other to $q = \{1\}$. Then $\mathcal{X}$ carries a natural complete length metric with respect to which $|pq| = 1$ but there is no geodesic connecting p to q.*

1.4.4. Exercise. *Give an example of a complete length space for which no pair of distinct points can be joined by a geodesic.*

1.4.5. Definition. *Let $\mathcal{X}$ be a metric space and $x, y \in \mathcal{X}$.*

(i) A point $z \in \mathcal{X}$ is called a midpoint *of x and y if*

$$|xz| = |yz| = \tfrac{1}{2}\cdot|xy|.$$

(ii) Assume $\varepsilon \geqslant 0$. A point $z \in \mathcal{X}$ is called ε-midpoint *of x and y if*

$$|xz|, |yz| < \tfrac{1}{2}\cdot|xy| + \varepsilon.$$

Note that a 0-midpoint is the same as a midpoint.

1.4.6. Lemma. *Let $\mathcal{X}$ be a complete metric space.*

a) Assume that for any pair of points $x, y \in \mathcal{X}$ and any $\varepsilon > 0$ there is a ε-midpoint z. Then $\mathcal{X}$ is a length space.

b) Assume that for any pair of points $x, y \in \mathcal{X}$ such that $|xy| < R$ there is a midpoint z. Then $\mathcal{X}$ is a geodesic space.

[2] That is, such that $\alpha(0) = x$ and $\alpha(1) = y$.

Proof. Let $x, y \in \mathcal{X}$ be a pair of points such that $|xy| < R$.

Set $\varepsilon_n = \frac{\varepsilon}{2^{2\cdot n}}$.

Set $\alpha(0) = x$ and $\alpha(1) = y$.

Set $\alpha(\frac{1}{2})$ to be an ε_1-midpoint of $\alpha(0)$ and $\alpha(1)$. Further, set $\alpha(\frac{1}{4})$ and $\alpha(\frac{3}{4})$ to be ε_2-midpoints for the pairs $(\alpha(0), \alpha(\frac{1}{2})$ and $(\alpha(\frac{1}{2}), \alpha(1)$ respectively. Applying the above procedure recursively, on the n-th step we define $\alpha(\frac{\hbar}{2^n})$ for every odd integer $\hbar$ such that $0 < \frac{\hbar}{2^n} < 1$, as an ε_n-midpoint of the already defined $\alpha(\frac{\hbar-1}{2^n})$ and $\alpha(\frac{\hbar+1}{2^n})$.

In this way we define $\alpha(t)$ for $t \in W$, where W denotes the set of dyadic rationals in $[0,1]$. For any $t \in [0,1]$ consider a sequence of $t_n \in W$ such that $t_n \to t$ as $n \to \infty$. Note that the sequence $\alpha(t_n)$ converges; define $\alpha(t)$ as its limit. It is easy to see that $\alpha(t)$ does not depend on the choice of the sequence t_n and $\alpha\colon [0,1] \to \mathcal{X}$ is a path from x to y. Moreover,

❶ $$\begin{aligned} \operatorname{length}\alpha &\leqslant |xy| + \sum_{n=1}^{\infty} 2^{n-1}\cdot\varepsilon_n \leqslant \\ &\leqslant |xy| + \tfrac{\varepsilon}{2}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get (a).

To prove (b), one should repeat the same argument taking midpoints instead of ε_n-midpoints. In this case ❶ holds for $\varepsilon_n = \varepsilon = 0$. □

Since in a compact space a sequence of $1/n$-midpoints z_n contains a convergent subsequence, Lemma 1.4.6 immediately implies

1.4.7. Proposition. *A proper length space is geodesic.*

1.4.8. Hopf–Rinow theorem. *Any complete, locally compact length space is proper.*

Proof. Let $\mathcal{X}$ be a locally compact length space. Given $x \in \mathcal{X}$, denote by $\rho(x)$ the supremum of all $R > 0$ such that the closed ball $\overline{\mathrm{B}}[x, R]$ is compact. Since $\mathcal{X}$ is locally compact

❷ $$\rho(x) > 0 \quad \text{for any} \quad x \in \mathcal{X}.$$

It is sufficient to show that $\rho(x) = \infty$ for some (and therefore any) point $x \in \mathcal{X}$.

Assume the contrary; that is, $\rho(x) < \infty$.

❸ $B = \overline{\mathrm{B}}[x, \rho(x)]$ *is compact for any* x.

Indeed, $\mathcal{X}$ is a length space; therefore for any $\varepsilon > 0$, the set $\overline{\mathrm{B}}[x, \rho(x) - \varepsilon]$ forms a compact ε-net in B. Since B is closed and hence complete, it has to be compact. △

❹ $|\rho(x) - \rho(y)| \leqslant |xy|_{\mathcal{X}}$, *in particular* $\rho\colon \mathcal{X} \to \mathbb{R}$ *is a continuous function.*

Indeed, assume the contrary; that is, $\rho(x) + |x - y| < \rho(y)$ for some $x, y \in \mathcal{X}$. Then $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$ is a closed subset of $\overline{\mathrm{B}}[y, \rho(y)]$ for some $\varepsilon > 0$. Then compactness of $\overline{\mathrm{B}}[y, \rho(y)]$ implies compactness of $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$, a contradiction. △

Set $\varepsilon = \min_{y \in B}\{\rho(y)\}$; the minimum is defined since B is compact. From ❷, we have $\varepsilon > 0$.

Choose a finite $\frac{\varepsilon}{10}$-net $\{a_1, a_2, \dots, a_n\}$ in B. The union W of the closed balls $\overline{\mathrm{B}}[a_i, \varepsilon]$ is compact. Clearly $\overline{\mathrm{B}}[x, \rho(x) + \frac{\varepsilon}{10}] \subset W$. Therefore $\overline{\mathrm{B}}[x, \rho(x) + \frac{\varepsilon}{10}]$ is compact; a contradiction. □

1.4.9. Exercise. *Construct a geodesic space which is locally compact, but whose completion is neither geodesic nor locally compact.*

1.5 Model angles and triangles.

Let $\mathcal{X}$ be a metric space, $p, q, r \in \mathcal{X}$. Let us define its *model triangle* $[\tilde{p}\tilde{q}\tilde{r}]$ (briefly, $[\tilde{p}\tilde{q}\tilde{r}] = \tilde{\triangle}(pqr)_{\mathbb{E}^2}$) to be a triangle in the plane $\mathbb{E}^2$ such that

$$|\tilde{p}\tilde{q}| = |pq|, \quad |\tilde{q}\tilde{r}| = |qr|, \quad |\tilde{r}\tilde{p}| = |rp|.$$

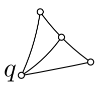
The same way we can define a *hyperbolic* and *spherical model triangles* $\tilde{\triangle}(pqr)_{\mathbb{H}^2}$, $\tilde{\triangle}(pqr)_{\mathbb{S}^2}$ in the hyperbolic plane $\mathbb{H}^2$ and sphere $\mathbb{S}^2$. In the latter case the model triangle is said to be defined if in addition

$$|pq| + |qr| + |rp| < 2{\cdot}\pi.$$

In this case it also exists and unique up to isometry of $\mathbb{S}^2$.

If $[\tilde{p}\tilde{q}\tilde{r}] = \tilde{\triangle}(pqr)_{\mathbb{E}^2}$ and $|pq|, |pr| > 0$, the angle measure of $[\tilde{p}\tilde{q}\tilde{r}]$ at $\tilde{p}$ will be called *model angle* of triple p, q, r and it will be denoted by $\tilde{\measuredangle}(p\,{}^{q}_{r})_{\mathbb{E}^2}$. The same way we define $\tilde{\measuredangle}(p\,{}^{q}_{r})_{\mathbb{H}^2}$ and $\tilde{\measuredangle}(p\,{}^{q}_{r})_{\mathbb{S}^2}$; in the latter case we assume in addition that the model triangle $\tilde{\triangle}(pqr)_{\mathbb{S}^2}$ is defined.

1.5.1. Alexandrov's lemma. *Let p, q, r, z be distinct points in a metric space such that $z \in\,]pr[$. Then the following expressions have the same sign:*



a) $\tilde{\measuredangle}(p\,{}^{q}_{z}) - \tilde{\measuredangle}(p\,{}^{q}_{r})$,

b) $\tilde{\measuredangle}(z\,{}^{q}_{p}) + \tilde{\measuredangle}(z\,{}^{q}_{r}) - \pi$.

Moreover,

$$\measuredangle(q\,{}^{p}_{r}) \geqslant \measuredangle(q\,{}^{p}_{z}) + \measuredangle(q\,{}^{z}_{r}),$$

with equality if and only if the expressions in (a) and (b) vanish.

The same holds for hyperbolic and spherical model angles, but in the latter case one has to assume in addition that

$$|p\,q| + |q\,r| + |r\,p| < 2{\cdot}\pi.$$

Proof. By the triangle inequality,

$$|p\,q| + |q\,z| + |z\,p| \leqslant |p\,q| + |q\,r| + |r\,p| < 2{\cdot}\varpi^{\kappa}.$$

Therefore the model triangle $[\tilde p\tilde q\tilde z] = \tilde\triangle(pqz)_{\mathbb{E}^2}$ is defined. Take a point $\tilde r$ on the extension of $[\tilde p\tilde z]$ beyond $\tilde z$ so that $|\tilde p\,\tilde r| = |p\,r|$ (and therefore $|\tilde p\,\tilde z| = |p\,z|$).

Since the increasing the opposite side in the plane triangle increase the corresponding angle, the following expressions have the same sign:

(i) $\measuredangle[\tilde p\,{}^{\tilde q}_{\tilde r}] - \measuredangle(p\,{}^{q}_{r})$;

(ii) $|\tilde p\,\tilde r| - |p\,r|$;

(iii) $\measuredangle[\tilde z\,{}^{\tilde q}_{\tilde r}] - \measuredangle(z\,{}^{q}_{r})$.

Since

$$\measuredangle[\tilde p\,{}^{\tilde q}_{\tilde r}] = \measuredangle[\tilde p\,{}^{\tilde q}_{\tilde z}] = \measuredangle(p\,{}^{q}_{z})$$

and

$$\measuredangle[\tilde z\,{}^{\tilde q}_{\tilde r}] = \pi - \measuredangle[\tilde z\,{}^{\tilde p}_{\tilde q}] = \pi - \measuredangle(z\,{}^{p}_{q}),$$

the first statement follows.

For the second statement, construct $[\tilde q\tilde z\tilde r'] = \tilde\triangle(qzr)_{\mathbb{E}^2}$ on the opposite side of $[\tilde q\tilde z]$ from $[\tilde p\tilde q\tilde z]$. Since

$$|\tilde p\,\tilde r'| \leqslant |\tilde p\,\tilde z| + |\tilde z\,\tilde r'| = |p\,z| + |z\,r| = |p\,r|,$$

then

$$\begin{aligned}\measuredangle(q\,{}^{p}_{z}) + \measuredangle(q\,{}^{z}_{r}) &= \measuredangle[\tilde q\,{}^{\tilde p}_{\tilde z}] + \measuredangle[\tilde q\,{}^{\tilde z}_{\tilde r'}] = \\ &= \measuredangle[\tilde q\,{}^{\tilde p}_{\tilde r'}] \leqslant \\ &\leqslant \measuredangle(q\,{}^{p}_{r}).\end{aligned}$$

Equality holds if and only if $|\tilde p\,\tilde r'| = |p\,r|$, as required. $\square$

1.6 Angles and the first variation.

Given a hinge $[p\,{}^x_y]$, we define its *angle* as follows:

$$❶ \qquad \measuredangle[p\,{}^x_y] \xlongequal{def} \lim_{\bar x,\bar y\to p} \tilde\measuredangle(p\,{}^{\bar x}_{\bar y})_{\mathbb{E}^2},$$

where $\bar x \in\,]px]$ and $\bar y \in\,]py]$.

1.6.1. Lemma. *For any three points p, x, y in a metric space the following inequalities*

$$❷ \qquad \begin{aligned} |\tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2} - \tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2}| &\leqslant |px|\cdot|py|,\\ |\tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2} - \tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2}| &\leqslant |px|\cdot|py|, \end{aligned}$$

hold whenever the left hand side is defined.

The lemma above implies that the definition of angle ❶ one can use $\tilde\measuredangle(p\,{}^{\bar x}_{\bar y})_{\mathbb{S}^2}$ or $\tilde\measuredangle(p\,{}^{\bar x}_{\bar y})_{\mathbb{H}^2}$ instead of $\tilde\measuredangle(p\,{}^{\bar x}_{\bar y})_{\mathbb{E}^2}$. In particular, may use Euclidean plane, so that the angle can be calculated from the cosine law:

$$\cos\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} = \frac{|px|^2 + |py|^2 - |xy|^2}{2\cdot|px|\cdot|py|}.$$

Proof. Note that

$$\tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2} \leqslant \tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} \leqslant \tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2}.$$

Therefore

$$\begin{aligned} 0 \leqslant \tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2} - \tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2} &\leqslant \tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2} + \tilde\measuredangle(x\,{}^p_y)_{\mathbb{S}^2} + \tilde\measuredangle(y\,{}^p_x)_{\mathbb{S}^2} -\\ &\quad - \tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2} - \tilde\measuredangle(x\,{}^p_y)_{\mathbb{H}^2} - \tilde\measuredangle(y\,{}^p_x)_{\mathbb{H}^2} =\\ &= \operatorname{area}\tilde\triangle(pxy)_{\mathbb{S}^2} + \operatorname{area}\tilde\triangle(pxy)_{\mathbb{H}^2}. \end{aligned}$$

Thus, ❷ follows since

$$\begin{aligned} 0 &\leqslant \operatorname{area}\tilde\triangle(pxy)_{\mathbb{H}^2} \leqslant\\ &\leqslant \operatorname{area}\tilde\triangle(pxy)_{\mathbb{S}^2} \leqslant\\ &\leqslant |px|\cdot|py|. \end{aligned}$$

□

1.6.2. Triangle inequality for angles. *Let $[px^1]$, $[px^2]$ and $[px^3]$ be three geodesics in a metric space. If all of the angles $\alpha^{ij} = \measuredangle[p\,{}^{x^i}_{x^j}]$ are defined then they satisfy the triangle inequality:*

$$\alpha^{13} \leqslant \alpha^{12} + \alpha^{23}.$$

Proof. Since $\alpha^{13} \leqslant \pi$, we can assume that $\alpha^{12} + \alpha^{23} < \pi$. Set $\gamma^i = \mathrm{geod}_{[px^i]}$. Given any $\varepsilon > 0$, for all sufficiently small $t, \tau, s \in \mathbb{R}_+$ we have

$$\begin{aligned}|\gamma^1(t)\ \gamma^3(\tau)| \leqslant &|\gamma^1(t)\ \gamma^2(s)| + |\gamma^2(s)\ \gamma^3(\tau)| < \\ &< \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha^{12} + \varepsilon)} + \\ &\quad + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\alpha^{23} + \varepsilon)} \leqslant\end{aligned}$$

(Below we define $s(t, \tau)$ so that for $s = s(t, \tau)$, this chain of inequalities continues the following way.)

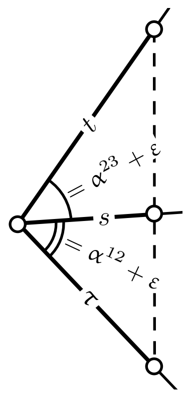
$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha^{12} + \alpha^{23} + 2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\alpha^{13} \leqslant \alpha^{12} + \alpha^{23} + 2{\cdot}\varepsilon.$$

Hence the result.

To define $s(t, \tau)$, consider three rays $\tilde\gamma^1$, $\tilde\gamma^2$, $\tilde\gamma^3$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde\gamma^1, \tilde\gamma^2) = \alpha^{12} + \varepsilon$, $\measuredangle(\tilde\gamma^2, \tilde\gamma^3) = \alpha^{23} + \varepsilon$ and $\measuredangle(\tilde\gamma^1, \tilde\gamma^3) = \alpha^{12} + \alpha^{23} + 2{\cdot}\varepsilon$. We parametrize each ray by length from the starting point. Given two positive numbers $t, \tau \in \mathbb{R}_+$, let $s = s(t, \tau)$ be the number such that $\tilde\gamma^2(s) \in [\tilde\gamma^1(t)\ \tilde\gamma^3(\tau)]$. Clearly $s \leqslant \max\{t, \tau\}$, so t, τ, s may be taken sufficiently small. ☐



1.6.3. Exercise. *Prove that the sum of adjacent angles is at least* π.

More precisely: let $\mathcal{X}$ *be a complete length space and* $p, x, y, z \in \mathcal{X}$. *If* $p \in]xy[$, *then*

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \geqslant \pi$$

whenever each angle on the left-hand side is defined.

1.6.4. First variation inequality. *Assume for hinge* $[q\,{}^p_x]$ *the angle* $\alpha = \measuredangle[q\,{}^p_x]$ *is defined then*

$$|p\,\mathrm{geod}_{[qx]}(t)| \leqslant |q\,p| - t{\cdot}\cos\alpha + o(t).$$

Proof. Take sufficiently small $\varepsilon > 0$. For all sufficiently small $t > 0$, we have that

$$\begin{aligned}|\mathrm{geod}_{[qp]}(t/\varepsilon)\ \mathrm{geod}_{[qx]}(t)| &\leqslant \tfrac{t}{\varepsilon}{\cdot}\sqrt{1 + \varepsilon^2 - 2{\cdot}\varepsilon{\cdot}\cos\alpha} + o(t) \leqslant \\ &\leqslant \tfrac{t}{\varepsilon} - t{\cdot}\cos\alpha + t{\cdot}\varepsilon.\end{aligned}$$

Applying triangle inequality, we get

$$\begin{aligned}|p\,\mathrm{geod}_{[qx]}(t)| &\leqslant |p\,\mathrm{geod}_{[qp]}(t/\varepsilon)| + |\mathrm{geod}_{[qp]}(t/\varepsilon)\ \mathrm{geod}_{[qx]}(t)| \leqslant \\ &\leqslant |pq| - t\cdot\cos\alpha + t\cdot\varepsilon\end{aligned}$$

for any $\varepsilon > 0$ and all sufficiently small t. Hence the result. □

1.7 Space of directions and tangent space

Let $\mathcal{X}$ be a metric space with defined angles. Fix a point $p \in \mathcal{X}$.

Consider the set $\mathfrak{S}_p$ of all nontrivial unit-speed geodesics which start at p. By 1.6.2 the triangle inequality holds for $\measuredangle$ on $\mathfrak{S}_p$; that is, $(\mathfrak{S}_p, \measuredangle)$ forms a pseudometric space; that is the angle satisfies all the consitions of metric except the angle between distinct geodesics might vanish.

The metric space corresponding to $(\mathfrak{S}_p, \measuredangle)$ is called *space of geodesic directions* at p and denoted as Σ'_p or $\Sigma'_p\mathcal{X}$. The elements of Σ'_p are called *geodesic directions* at p. Each geodesic direction is formed by an equivalence class of geodesics starting from p for the equivalence relation

$$[px] \sim [py] \quad\Longleftrightarrow\quad \measuredangle[p\,{}^{x}_{y}] = 0.$$

1.7.1. Exercise. *Assume $\mathcal{U}$ is a* CAT *proper length space with extendable geodesics; that is for any geodesic in $\mathcal{U}$ is an arc in a both-side infinite local geodesic.*

Show that the space of geodesic directions Σ'_p is complete for any $p \in \mathcal{U}$.

The completion of Σ'_p is called *space of directions* at p and is denoted as Σ_p or $\Sigma_p\mathcal{X}$. The elements of Σ_p are called *directions* at p.

The Euclidean cone $\mathrm{Cone}\,\Sigma_p$ over the space of directions Σ_p is called tangent space at p and denoted as T_p or $\mathrm{T}_p\mathcal{X}$.

The tangent space T_p could be also defined directly, without introducing the space of direction. To do so consider the set $\mathfrak{T}_p$ of all geodesics starting at p, with arbitrary speed. Given $\alpha, \beta \in \mathfrak{T}_p$, set

❶ $$|\alpha\,\beta|_{\mathfrak{T}_p} = \lim_{\varepsilon\to 0}\frac{|\alpha(\varepsilon)\ \beta(\varepsilon)|_{\mathcal{X}}}{\varepsilon}$$

Since the angles in $\mathcal{X}$ are defined, ❶ defines a pseudometric on $\mathfrak{T}_p$.

The corresponding metric space admits a natuaral isometric identification with the cone $\mathrm{T}'_p = \mathrm{Cone}\,\Sigma'_p$. The elements of T'_p are formed by the equivalence classes for the realtion

$$\alpha \sim \beta \quad\Longleftrightarrow\quad |\alpha(t)\ \beta(t)|_{\mathcal{X}} = o(t).$$

The completion of T'_p is therefore natuaral isometric to T_p.

The elements of T_p will be called tangent vector at p, despite that T_p is only cone — not a vector space. The elements of T'_p will be called geodesic tangent vector at p.

1.8 Hemisphere lemma

1.8.1. Hemisphere lemma. *For $\kappa > 0$, any closed path of length $< 2\cdot\varpi^\kappa$ (respectively, $\leqslant 2\cdot\varpi^\kappa$) in $\mathbb{M}^2[\kappa]$ lies in an open (respectively, closed) hemisphere.*

Proof. By rescaling, we may assume that $\kappa = 1$ and thus $\varpi^\kappa = \pi$ and $\mathbb{M}^2[\kappa] = \mathbb{S}^2$. Let α be a closed curve in $\mathbb{S}^2$ of length $2\cdot\ell$.

Assume $\ell < \pi$. Let $\breve\alpha$ be a subarc of α of length ℓ, with endpoints p and q. Since $|pq| \leqslant \ell < \pi$, there is a unique geodesic $[pq]$ in $\mathbb{S}^2$. Let z be the midpoint of $[pq]$. We claim that α lies in the open hemisphere centered at z. If not, α intersects the boundary great circle in a point say r. Without loss of generality we may assume that $r \in \breve\alpha$. The arc $\breve\alpha$ together with its reflection in z form a closed curve of length $2\cdot\ell$ which passes through r and its antipodal point r'. Thus $\ell = \operatorname{length}\breve\alpha \geqslant$ $\geqslant |r\,r'| = \pi$, a contradiction.

If $\ell = \pi$, then either α is a local geodesic, and hence a great circle, or α may be strictly shortened by substituting a geodesic arc for a subarc of α whose endpoints p^1, p^2 are arbitrarily close to some point p on α. In the latter case, α lies in a closed hemisphere obtained as a limit of closures of open hemispheres containing the shortened curves as p^1, p^2 approach p. $\square$

1.8.2. Exercise. *Build a proof of Hemisphere lemma 1.8.1 based on Crofton's formula.*

Part I

Curvature bounded above

Chapter 2

Gluing theorem and billiards

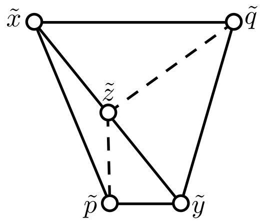
In this chapter we define CAT spaces and give the first application to the billiards.

2.1 4-point condition

Given a quadrouple of points p, q, x, y in a metric space $\mathcal{X}$, consider two model triangles in the plane $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{E}^2}$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)_{\mathbb{E}^2}$ with common side $[\tilde x\tilde y]$.

If the inequality

$$|pq|_{\mathcal{X}} \leqslant |\tilde p\tilde z|_{\mathbb{E}^2} + |\tilde z\tilde q|_{\mathbb{E}^2}$$



holds for any point $\tilde z \in [\tilde x\tilde y]$ then we say that the quadruple p, q, x, y *satisfies* CAT[0] *comparison.*

If one does the same for spherical model triangles $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{S}^2}$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)_{\mathbb{S}^2}$ then he arrives to the definition of CAT[1] comparison. If one of the spherical model triangles is undefined[1] then it is assumed that CAT[1] comparison automatically holds for this quadruple.

We can do the same for the model plane with curvature κ; that is a sphere if $\kappa > 0$, Euclidean plane if $\kappa = 0$ and Lobachevsky plane if

def of sec $\equiv \kappa$?

[1] that is, if

$$|px| + |py| + |xy| \geqslant 2\cdot\pi \quad \text{or} \quad |qx| + |qy| + |xy| \geqslant 2\cdot\pi.$$

$\kappa < 0$. In this case we ~~will~~ arrive ~~to a~~ the definition of CAT[κ] comparison. However, in these notes we will consider mostly CAT[0] comparison the reader and occasionally CAT[1] comparison; so, if ~~you see~~ whenever CAT[κ], ~~you~~ is mentioned, can assume that κ is 0 or 1.

If all quadruples ~~of~~ in ~~the~~ a metric space $\mathcal{X}$ satisfy CAT[κ] comparison then we say that the space $\mathcal{X}$ is CAT[κ].

Note that in order to check CAT[κ] comparison, it is sufficient to know 6 distances between all the pairs of points in the quadruple. The latter observation implies the following.

2.1.1. Proposition. *The Gromov–Hausdorff limit of a sequence of* CAT[κ] *spaces is* CAT[κ].

In fact, it does not ~~even~~ matter which definition of convergence for metric spaces ~~you use~~ one uses ~~as far as~~ so long as any quadruple of points in the limit space can be arbitrary well approximated by ~~the~~ quadruples in the sequence of metric spaces.

2.1.2. Exercise. *Let $\mathcal{V}$ be a metric space and $\mathcal{U} = \operatorname{Cone}\mathcal{V}$. Show that $\mathcal{U}$ is a* CAT[0] *if and only if $\mathcal{V}$ is* CAT[1].

Analogously, if $\mathcal{U} = \operatorname{Susp}\mathcal{V}$ then $\mathcal{U}$ is a CAT[1] *if and only if $\mathcal{V}$ is* CAT[1].

The following exercise is bit simpler, but can be proved in essentially the same way.

2.1.3. Exercise. *Assume $\mathcal{U}$ and $\mathcal{V}$ are* CAT[0] *spaces. Show that the product space $\mathcal{U} \times \mathcal{V}$ is* CAT[0].

2.2 Thin triangles

The inheritance lemma 2.2.6 proved below plays ~~the~~ a central role in the theory; it will lead to two fundamental constructions: patchwork globalization (3.3.2) and Reshetnyak gluing (2.3.1) which in turn are used to prove the globalization theorem (3.3.1).

Recall that a *triangle* $[x^1\,x^2\,x^3]$ in a space $\mathcal{X}$ is a triple of minimizing geodesics $[x^1x^2]$, $[x^2x^3]$ and $[x^3x^1]$. Consider ~~the~~ a model triangle $[\tilde{x}^1\,\tilde{x}^2\,\tilde{x}^3] = \tilde{\triangle}(x^1x^2x^3)_{\mathbb{E}^2}$ in the Euclidean plane. The *natural map* $[\tilde{x}^1\,\tilde{x}^2\,\tilde{x}^3] \to [x^1\,x^2\,x^3]$ sends a point $\tilde{z} \in [\tilde{x}^i\tilde{x}^j]$ to the corresponding point $z \in [x^ix^j]$; that is, z is the point such that $|\tilde{x}^i\,\tilde{z}| = |x^i\,z|$ and therefore $|\tilde{x}^j\,\tilde{z}| = |x^j\,z|$.

The ~~same way~~ natural map can be defined similarly for ~~by~~ the spherical model triangle $\tilde{\triangle}(x^1x^2x^3)_{\mathbb{S}^2}$. and more generally for the comparison triangle in $\mathbb{S}^2_\kappa$

2.2.1. Definition of thin triangles. *A triangle* $[x^1\,x^2\,x^3]$ *in* ~~*the*~~ a metric *space* $\mathcal{X}$ *is called* thin *if the natural map* $\tilde{\triangle}(x^1x^2x^3)_{\mathbb{E}^2} \to [x^1\,x^2\,x^3]$ *is short.* (i.e. distance non-increasing)

Analogously a *triangle* $[x^1\,x^2\,x^3]$ *in* ~~*the space*~~ $\mathcal{X}$ *is called* spherically thin *if the natural map from the spherical model triangle* $\tilde{\triangle}(x^1x^2x^3)_{\mathbb{S}^2}$ *to* $[x^1\,x^2\,x^3]$ *is short.*

2.2.2. Exercise. *Show that a geodesic space is* CAT[0] *(correspondingly,* CAT[1]*) if and only if all its triangles are thin (*~~*or*~~ *correspondingly, spherically thin).*

The cone and suspension are defined in Section 1.2.

2.2.3. Uniqueness of geodesics. *In a* CAT[0] *proper length space, pairs of points are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

Analogously, in a CAT[1] *proper length space, pairs of points on distance less than* π *are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.* (define π-geodesics?)

Proof. Given 4 points p^1, p^2, q^1, q^2 in a CAT[0] proper length space $\mathcal{U}$, consider two triangles $[p^1\,q^1\,p^2]$ and $[p^2\,q^2\,q^1]$. Since both these triangles are thin, we get

$$|\mathrm{path}_{[p^1q^1]}(t)\ \mathrm{path}_{[p^2q^1]}(t)|_{\mathcal{U}} \leqslant (1-t)\cdot|p^1\,p^2|_{\mathcal{U}},$$
$$|\mathrm{path}_{[p^2q^1]}(t)\ \mathrm{path}_{[p^2q^2]}(t)|_{\mathcal{U}} \leqslant t\cdot|q^1\,q^2|_{\mathcal{U}}.$$

It follows that

$$|\mathrm{path}_{[p^1q^1]}(t)\ \mathrm{path}_{[p^2q^2]}(t)|_{\mathcal{U}} \leqslant \max\{|p^1\,p^2|_{\mathcal{U}}, |q^1\,q^2|_{\mathcal{U}}\}.$$

Hence continuity and uniqueness in the CAT[0] case follow~~s~~. The CAT[1] case is done in essentially the same way. □

2.2.4. Corollary. *Any* CAT[0] ~~*proper length*~~ change to geodesic? ← more general. *space is contractible.*

Analogously, any CAT[1] ~~*proper length*~~ geodesic *space with diameter* $< \cancel{\frac{\pi}{2}}$ π *is contractible.*

geodesic space is more general then proper.

Proof. Let $\mathcal{U}$ be a CAT[0] proper length space. Fix a point $p \in \mathcal{U}$ in the space.

For each point x consider the geodesic path $\gamma_x\colon [0,1] \to \mathcal{U}$ from p to x. Consider the map $h_t\colon x \mapsto \gamma_x(t)$ for each t. By uniqueness of geodesics (2.2.3) the map $(t,x) \mapsto h_t(x)$ is continuous.

It remains to note that $h_1(x) = x$ and $h_0(x) = p$ for any x.

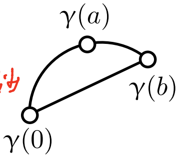
The proof ~~of~~ in the CAT[1] case is identical. □

2.2.5. Proposition. *Suppose $\mathcal{U}$ is a* CAT[0] *proper length space. Then any local geodesic in $\mathcal{U}$ is a minimizing geodesic.*

Analogously, if $\mathcal{U}$ is a CAT[1] *proper length space then any local geodesic with length $<\pi$ in $\mathcal{U}$ is a minimizing geodesic.*

Proof. Suppose $\gamma\colon[0,\ell]\to\mathcal{U}$ is a local geodesic that is not minimizing. Choose a to be the maximal value such that γ is minimizing on $[0,a]$. Further choose $b>a$ so that γ is minimizing on $[a,b]$.

Since the triangle $[\gamma(0)\,\gamma(a)\,\gamma(b)]$ is thin, we have

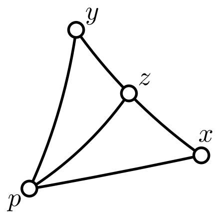


$$|\gamma(a-\varepsilon)\ \gamma(a+\varepsilon)|<2{\cdot}\varepsilon$$

for all small $\varepsilon>0$, a contradiction. to local minimality of γ

The spherical case is done the same way. □

Now let us formulate the in main result of this section.



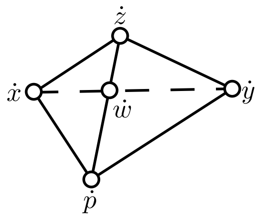
2.2.6. Inheritance lemma. *Assume that a triangle $[pxy]$ is* decomposed *into two triangles $[pxz]$ and $[pyz]$; that is, $[pxz]$ and $[pyz]$ have common side $[pz]$, and the sides $[xz]$ and $[zy]$ together form the side $[xy]$ of $[pxy]$.*

If both triangles $[pxz]$ and $[pyz]$ are thin, then the *triangle $[pxy]$ is* also *thin.*

Analogously, if $[pxy]$ has perimeter $<2{\cdot}\varpi^\kappa$ π *and both triangles $[pxz]$ and $[pyz]$ are spherically thin, then triangle $[pxy]$ is spherically thin.*

Proof. Construct model triangles $[\dot p\dot x\dot z]=\tilde\triangle(pxz)_{\mathbb{E}^2}$ and $[\dot p\dot y\dot z]=\tilde\triangle(pyz)_{\mathbb{E}^2}$ so that $\dot x$ and $\dot y$ lie on opposite sides of $[\dot p\dot z]$.

Let us show that



❶ $$\measuredangle(z\,{}^p_x)+\measuredangle(z\,{}^p_y)\geqslant\pi.$$

Suppose contrary, that is

$$\measuredangle(z\,{}^p_x)+\measuredangle(z\,{}^p_y)<\pi.$$

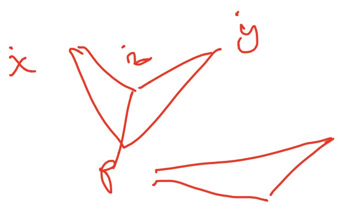
Then for some point $\dot w\in[\dot p\dot z]$, we have

$$|\dot x\,\dot w|+|\dot w\,\dot y|<|\dot x\,\dot z|+|\dot z\,\dot y|=|x\,y|.$$

Let $w\in[pz]$ correspond to $\dot w$; that is $|z\,w|=|\dot z\,\dot w|$. Since $[pxz]$ and $[pyz]$ are thin, we have

$$|x\,w|+|w\,y|<|x\,y|,$$

contradicting the triangle inequality.



no need for convex hulls here

Now consider the set

not defined

$$\dot{D} = \mathrm{Conv}[\dot{p}\dot{x}\dot{z}] \cup \mathrm{Conv}[\dot{p}\dot{y}\dot{z}].$$

By ❶, there is a short map F that sends $[\tilde{p}\tilde{x}\tilde{y}]$ to $\dot{D}$ in such a way that

$$\tilde{p} \mapsto \dot{p}, \qquad \tilde{x} \mapsto \dot{x}, \qquad \tilde{z} \mapsto \dot{z}, \qquad \tilde{y} \mapsto \dot{y}.$$

?

2.2.7. Exercise. *Prove the last statement.*

Should we refer to Alexandrov's Lemma instead of leaving this as exercise? it has to be proved anyway.

By assumption, the natural maps $[\dot{p}\dot{x}\dot{z}] \to [pxz]$ and $[\dot{p}\dot{y}\dot{z}] \to [pyz]$ are short. By composition, the natural map from $[\tilde{p}\tilde{x}\tilde{y}]$ to $[pyz]$ is short, as claimed.

The spherical case is done along the same lines. □

Recall that a set A in a metric space $\mathcal{U}$ is called locally convex if for any point $p \in A$ there is a open neighborhood $\Omega \ni p$ such that any geodesic with the ends in A which lie in Ω lies in A.

2.2.8. Exercise. *Any ball in a* CAT[0] *proper length space forms a convex set.*

Analogousely, any ball in a CAT[1] *proper length space of radius* $R < \frac{\pi}{2}$ *forms a convex set.*

2.2.9. Exercise. *Show that in any* CAT[0] *proper length space any closed connected locally convex set is convex.*

2.2.10. Exercise. *Let $\mathcal{U}$ be a* CAT[0] *proper length space and $K \subset \mathcal{U}$ be a closed convex set. Show that*

a) *For each point $p \in \mathcal{U}$ there is unique point $p^* \in K$ which minimizes the distance $|pp^*|$.*

b) *The ~~defined~~ closest-point projection $p \mapsto p^*$ which is well-defined by a) is short.*

2.3 Reshetnyak's gluing theorem

Suppose $\mathcal{U}^1$ and $\mathcal{U}^2$ are proper length spaces with isometric closed convex sets $A^i \subset \mathcal{U}^i$ and $\iota\colon A^1 \to A^2$ be an isometry. Consider the space $\mathcal{W}$ of all equivalence classes on $\mathcal{U}^1 \sqcup \mathcal{U}^2$ for the equivalence relation such that $a \sim \iota(a)$ for any $a \in A^1$.

It is straightforward to see that $\mathcal{W}$ forms a proper length space if equipped with the following metric

$$\begin{aligned} |xy|_{\mathcal{W}} &= |xy|_{\mathcal{U}^i} \\ &\quad \text{if} \quad x, y \in \mathcal{U}^i, \quad \text{and} \\ |xy|_{\mathcal{W}} &= \min\{\, |xa|_{\mathcal{U}^1} + |y\,\iota(a)|_{\mathcal{U}^2} \;\big|\; a \in A^1 \,\} \\ &\quad \text{if} \quad x \in \mathcal{U}^1 \quad \text{and} \quad y \in \mathcal{U}^2. \end{aligned}$$

Abusing notation, we denoted by x and y the points in $\mathcal{U}^1 \sqcup \mathcal{U}^2$ and their equivalence classes in $\mathcal{U}^1 \sqcup \mathcal{U}^2/\sim$.

The obtained space $\mathcal{W}$ is called the *gluing* of of $\mathcal{U}^1$ and $\mathcal{U}^2$ along ι. If one applies this construction to two copies of one space $\mathcal{U}$ with a set $A \subset \mathcal{U}$ and identity map $\iota\colon A \to A$, then the obtained space is called *doubling* of $\mathcal{U}$ in A.

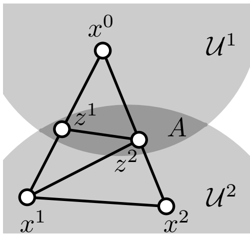
We can (and will) identify $\mathcal{U}^i$ with its image in $\mathcal{W}$; this way both subsets $A^i \subset \mathcal{U}^i$ will be identified and denoted further by A. Note that $A = \mathcal{U}^1 \cap \mathcal{U}^2 \subset \mathcal{W}$, therefore A is also a convex set in $\mathcal{W}$.

The following theorem was proved by Reshetnyak in [147]. Recall, that once you see CAT[κ], you may read CAT[0] or CAT[1].

2.3.1. Reshetnyak gluing. *Suppose $\mathcal{U}^1$ and $\mathcal{U}^2$ are* CAT[0] *proper length spaces with isometric closed convex sets $A^i \subset \mathcal{U}^i$, and $\iota\colon A^1 \to A^2$ is an isometry. Then the gluing of $\mathcal{U}^1$ and $\mathcal{U}^2$ along ι is* CAT[0] *proper length space.*

Proof. By construction of the gluing space, the statement can be reformulated the following way.

2.3.2. Reformulation of 2.3.1. *Let $\mathcal{W}$ be a proper length space which has two closed convex sets $\mathcal{U}^1, \mathcal{U}^2 \subset \mathcal{W}$ such that $\mathcal{U}^1 \cup \mathcal{U}^1 = \mathcal{W}$, $A = \mathcal{U}^1 \cap \mathcal{U}^2$ is complete, and $\mathcal{U}^1$, $\mathcal{U}^2$ are* CAT[0]. *Then $\mathcal{W}$ is a* CAT[0] *space.*



It suffices to show that any triangle $[x^0\,x^1\,x^2]$ in $\mathcal{W}$ is thin. This is obviously true if all three points x^0, x^1, x^2 lie in one of $\mathcal{U}^i$. Thus, without loss of generality, we may assume that $x^0 \in \mathcal{U}^1$ and $x^1, x^2 \in \mathcal{U}^2$.

Choose points $z^1, z^2 \in A$ which lie correspondingly on the sides $[x^0x^1], [x^0x^2]$. Note that

- ◇ triangle $[x^0\,z^1\,z^2]$ lies in $\mathcal{U}^1$,
- ◇ both triangles $[x^1\,z^1\,z^2]$ and $[x^1\,z^2\,x^2]$ lie in $\mathcal{U}^2$.

In particular each triangle $[x^0\,z^1\,z^2]$, $[x^1\,z^1\,z^2]$ and $[x^1\,z^2\,x^2]$ is thin.

Applying the inheritance lemma for thin triangles (2.2.6) twice we get that $[x^0\,x^1\,z^2]$ and consequently $[x^0\,x^1\,x^2]$ is thin. □

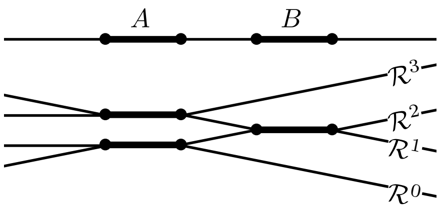
2.3.3. Exercise. *Let Q be the nonconvex subset of the plane bounded by two rays $\tilde\gamma^1$ and $\tilde\gamma^2$ with common starting point and angle α between them. Assume $\mathcal{U}$ is a* CAT[0] *proper length space and γ^1, γ^2 be two rays in $\mathcal{U}$ starting point and angle α between them. Show that the space glued from Q and $\mathcal{U}$ along the corresponding rays is a* CAT[0] *space.*

2.4 Reshetnyak's puff pastry

In this section we introduce *Reshetnyak's puff pastry*, a construction which will be used in the next section to prove Collision Theorem 2.5.3.

Let $\boldsymbol{A} = (A^1, \dots, A^N)$ be an array of convex closed sets in the Euclidean space $\mathbb{E}^m$. Consider an array of $N+1$ copies of $\mathbb{E}^m$. Assume that the space $\mathcal{R}$ is obtained by gluing successive pairs of spaces in the array along $A^1, \dots, A^N$ respectively.

The obtained space $\mathcal{R}$ will be called *Reshetnyak's puff pastry* for the array $\boldsymbol{A}$. The copies of $\mathbb{E}^m$ in the puff pastry $\mathcal{R}$ will be called *levels*; they will be denoted by $\mathcal{R}^0, \dots, \mathcal{R}^N$. The point in the $\mathfrak{k}$-th level $\mathcal{R}^{\mathfrak{k}}$ which corresponds to $x \in \mathbb{E}^m$ will be denoted by $x^{\mathfrak{k}}$.



Puff pastry for (A, B, A).

Given $x \in \mathbb{E}^m$, any point $x^{\mathfrak{k}} \in \mathcal{R}$ is called a *lifting* of x; we also can consider the liftings of maps to $\mathbb{E}^m$, in particular the liftings of subsets in $\mathbb{E}^m$.

Note that

- ◇ The intersection $A^1 \cap \dots \cap A^N$ admits a unique lifting in $\mathcal{R}$.
- ◇ Moreover, $x^i = x^j$ for some $i < j$ if and only if $x \in A^{i+1} \cap \cap \dots \cap A^j$.
- ◇ The restriction $\mathcal{R}^{\mathfrak{k}} \to \mathbb{E}^m$ of the natural projection $x^{\mathfrak{k}} \mapsto x$ is an isometry.

2.4.1. Observation. *Any Reshetnyak's puff pastry is a* CAT[0] *proper length space.*

Proof. Apply Reshetnyak's gluing theorem 2.3.1 recursively for the convex sets in the array. □

2.4.2. Proposition. *Assume $(A^1, \dots, A^N)$ and $(\check A^1, \dots, \check A^N)$ are two arrays of convex closed sets in $\mathbb{E}^m$ such that $A^{\mathfrak{k}} \subset \check A^{\mathfrak{k}}$ for each $\mathfrak{k}$. Let $\mathcal{R}$ and $\check{\mathcal{R}}$ be the corresponding Reshetnyak's puff pastries. Then the map $\mathcal{R} \to \check{\mathcal{R}}$ defined as $x^{\mathfrak{k}} \mapsto \check x^{\mathfrak{k}}$ is short.*

Moreover, if

$$|x^i y^j|_{\mathcal{R}} = |\check x^i \check y^j|_{\check{\mathcal{R}}}$$

for some $x, y \in \mathbb{E}^m$ and $i, j \in \{0, \dots, n\}$, then the geodesic $[\check x^i \check y^j]_{\check{\mathcal{R}}}$ is the image of geodesic $[x^i y^j]_{\mathcal{R}}$ under the map $x^i \mapsto \check x^i$.

Proof. The first statement in the proposition follows from the construction of Reshetnyak's puff pastries.

By Observation 2.4.1, $\mathcal{R}$ and $\check{\mathcal{R}}$ are CAT[0] proper length spaces, hence $[x^i y^j]_{\mathcal{R}}$ and $[\check{x}^i \check{y}^j]_{\check{\mathcal{R}}}$ are unique. Since the map $\mathcal{R} \to \check{\mathcal{R}}$ is short, the image of $[x^i y^j]_{\mathcal{R}}$ is a geodesic of $\check{\mathcal{R}}$ joining $\check{x}^i$ to $\check{y}^j$. Hence the second statement follows. □

2.4.3. Definition. *A Reshetnyak's puff pastry $\mathcal{R}$ is called* end-to-end convex *if the union of lower and upper levels forms a convex set in $\mathcal{R}$.*

If $\mathcal{R}$ is the Reshetnyak's puff pastry for an array of convex sets $\boldsymbol{A} = (A^1, \dots, A^N)$ then $\mathcal{R}$ is end-to-end convex if and only if the union of lower and upper levels $\mathcal{R}^0 \cup \mathcal{R}^N$ is isometric to the doubling of $\mathbb{E}^m$ in the nonempty intersection $A^1 \cap \dots \cap A^N$.

2.4.4. Observation. *Let $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ be arrays of convex bodies in $\mathbb{E}^m$. Assume the array $\boldsymbol{A}$ is obtained by inserting in $\check{\boldsymbol{A}}$, copies of the bodies which were already listed in $\check{\boldsymbol{A}}$.*

For example, if $\check{\boldsymbol{A}} = (A, C, B, C, A)$, by placing B in the second place and A in the fourth place, we can obtain $\boldsymbol{A} = (A, B, C, A, B, C, A)$.

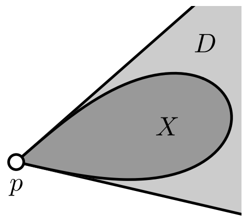
Denote by $\check{\mathcal{R}}$ and $\mathcal{R}$ the Reshetnyak's puff pastries for $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ respectively.

If $\check{\mathcal{R}}$ is end-to-end convex then so is $\mathcal{R}$.

Proof. Without loss of generality we can assume that $\boldsymbol{A}$ is obtained by inserting one element in $\check{\boldsymbol{A}}$, say at the place number $\hbar$.

Note that $\check{\mathcal{R}}$ is isometric to the puff pastry for $\boldsymbol{A}$ with $A^{\hbar}$ replaced by $\mathbb{E}^m$. It remains to apply Proposition 2.4.2. □

Let X be a convex set in a Euclidean space. By a *dihedral angle* we understand an intersection of two half-spaces; the intersection of corresponding hyperplanes is called the *edge* of the angle. We say that a dihedral angle D is supporting X at point $p \in X$ if D contains X and the edge of D contains p.



2.4.5. Lemma. *Let A and B be two convex sets in $\mathbb{E}^m$. Assume that any dihedral angle supporting $A \cap B$ at some point has measure at least α. Then the Reshetnyak's puff pastry for the array*

$$(\underbrace{A, B, A, \dots}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \text{ times}}).$$

is end-to-end convex.

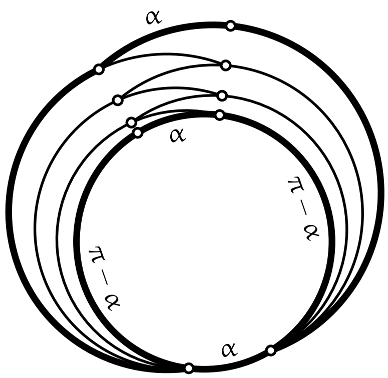
The proof of the lemma is based its partial case which we formulate as a sublemma.

2.4.6. Sublemma. *Let $\ddot{A}$ and $\ddot{B}$ be two half-planes in $\mathbb{E}^2$, where $\ddot{A} \cap \cap \ddot{B}$ is an angle with angle measure α. Then the Reshetnyak's puff pastry for the array*

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \dots}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \text{ times}})$$

is end-to-end convex.

Proof. Note that the puff pastry $\ddot{\mathcal{R}}$ is isometric to the cone over the space glued from the unit circles as shown on the diagram.



All the short arcs on the diagram have length α; the long arcs have length $\pi-\alpha$, so making a circuit along any path will take $2\cdot\pi$.

Observe that end-to-end convexity of $\ddot{\mathcal{R}}$ is equivalent to the fact that any geodesic shorter than π with the ends on inner and outer circles lies completely in these two circles.

The latter holds if the zigzag line has length at least π. This line is formed by $\lceil \frac{\pi}{\alpha} \rceil$ arcs with length α each. Hence the sublemma follows. □

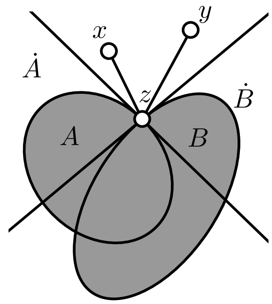
Proof of Lemma 2.4.5. Fix arbitrary $x, y \in \mathbb{E}^m$. Choose a point $z \in \in A \cap B$ for which the sum

$$|x\,z| + |y\,z|$$

is minimal. To show the end-to-end convexity of $\mathcal{R}$, it is sufficient to prove the following.

❶ *The geodesic $[x^0 y^N]_{\mathcal{R}}$ passes though $z^0 = z^N \in \mathcal{R}$.*

Without loss of generality we can assume that $z \in \partial A \cap \partial B$. Indeed, since the puff pastry for 1-array (B) is end-to-end convex, Proposition 2.4.2 together with Observation 2.4.4 imply ❶ in case z lies in the interior of A. The same way we can treat the case, z lies in the interior of B.



Further we will use the following exercise in convex geometry

2.4.7. Exercise. *There are half-spaces $\dot{A}$ and $\dot{B}$ such that $\dot{A} \supset A$ and $\dot{B} \supset B$ and*

$$|x\,z| + |y\,z|$$

takes minimal value for all $z \in \dot{A} \cap \dot{B}$.

Note that the angle measure of $\dot{A} \cap \dot{B}$ is at least α.

Note that $\mathbb{E}^m$ admits isometric splitting $\mathbb{E}^{m-2} \times \mathbb{E}^2$ such that

$$\dot{A} = \mathbb{E}^{m-2} \times \ddot{A}$$
$$\dot{B} = \mathbb{E}^{m-2} \times \ddot{B}$$

where $\ddot{A}$ and $\ddot{B}$ are half-planes in $\mathbb{E}^2$.

Let us replace each A by $\dot{A}$ and each B by $\dot{B}$ in the array, to get the array

$$(\underbrace{\dot{A}, \dot{B}, \dot{A}, \dots}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \text{ times}}).$$

The corresponding puff pastry $\dot{\mathcal{R}}$ splits as a product of $\mathbb{E}^{m-2}$ and a puff pastry, say $\ddot{\mathcal{R}}$, glued from the copies of the plane $\mathbb{E}^2$ for the array

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \dots}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \text{ times}}).$$

According to Sublemma 2.4.6 and Observation 2.4.4, $\ddot{\mathcal{R}}$ is end-to-end convex.

Since $\dot{\mathcal{R}} \overset{iso}{=\!=} \mathbb{E}^{m-2} \times \ddot{\mathcal{R}}$, the puff pastry $\dot{\mathcal{R}}$ is also end-to-end convex.

It follows that the geodesic $[\dot{x}^0 \dot{y}^N]_{\dot{\mathcal{R}}}$ passes through $\dot{z}^0 = \dot{z}^N \in \dot{\mathcal{R}}$. By Proposition 2.4.2, the image of $[\dot{x}^0 \dot{y}^N]_{\dot{\mathcal{R}}}$ under the map $\dot{x}^{\bar{k}} \mapsto x^{\bar{k}}$ is the geodesic $[x^0 y^N]_{\mathcal{R}}$. Hence Claim ❶ follows. □

The proof of the following proposition is based on Lemma 2.4.5, which is essentially the case $n = 2$.

We say that a closed convex set $A \subset \mathbb{E}^m$ has *ε-wide corners* for given $\varepsilon > 0$ if together with each point p the set A contains a small right circular cone with tip at p and aperture ε.

For example, a plane polygon has ε-wide corners if all its interior angles are at least ε.

We will consider finite collections of closed convex sets $A^1, \dots$ $\dots, A^n \subset \mathbb{E}^m$ such that for any subset $F \subset \{1, \dots, n\}$ the intersection

$\bigcap_{i\in F} A^i$ has ε-wide corners. In this case we may say briefly *all intersections of A^i have ε-wide corners.*

2.4.8. Proposition. *Given $\varepsilon > 0$ and a positive integer n there is an array of integers $\boldsymbol{j}_\varepsilon = (j_1, \dots, j_N)$ such that*

a) For each $\bar{k}$ we have $1 \leqslant j_{\bar{k}} \leqslant n$ and each number $1, \dots, n$ appears in $\boldsymbol{j}_\varepsilon$ at least once.

b) If $A^1, \dots, A^n$ is a collection of closed convex sets in $\mathbb{E}^m$ with common point and all their intersections have ε-wide corners then the puff pastry for the array $(A^{j_1}, \dots, A^{j_N})$ is end-to-end convex.

Moreover we can assume that $N \leqslant (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^n$.

Proof. The array $(j_1, \dots, j_N)$ is constructed recursively. For $n = 1$, the array is (1).

Now assume an array for n is already constructed. Let us exchange each occurrence of n by the alternating string

$$\underbrace{n, n+1, n, \dots}_{\lceil \frac{\pi}{\varepsilon} \rceil + 1 \text{ times}}.$$

Applying Lemma 2.4.5, we get that the obtained array meets the conditions of the proposition.

The upper bound on N follows directly from the construction and it can be improved easily. ☐

2.5 Billiards

Let $A^1, A^2, \dots A^n$ be a finite collection of closed convex sets in $\mathbb{E}^m$. Assume that for each i the boundary ∂A^i is a smooth hypersurface.

Consider the billiard table formed by the closure of the complement

$$T = \overline{\mathbb{E}^m \backslash \bigcup_i A^i}.$$

The sets A^i will be called *walls* of the table T and the billiards described above will be called *billiards with convex walls.*

A *billiard trajectory* on the table T is a unit-speed broken line γ which follows the standard law of billiards at the break points on ∂A^i — in particular, the angle of reflection is equal to the angle of incidence. The break points of the trajectory will be called *collisions.* We assume the trajectory meets only one wall at a time.

Recall that $A \subset \mathbb{E}^m$ has ε-wide corners if together with each point p the set A contains a small right circular cone with tip at p and aperture ε.

2.5.1. Exercise. *Assume that the walls of billiard table T are compact, convex and have a common interior point. Show that all the intersections of walls of T have ε-wide corners for some $\varepsilon > 0$.*

2.5.2. Exercise. *Assume that a billiard table T has centrally symmetric convex walls with common center. Show that all the intersections of the walls of T have ε-wide corners for some $\varepsilon > 0$.*

2.5.3. Collision Theorem. *Assume $T \subset \mathbb{E}^m$ is a billiard table with convex walls. Assume that the walls of T have common interior point and all their intersections have ε-wide corners. Then the number of collisions of any trajectory in T is bounded by a number N which depends only on the number n of walls and ε.*

As you will see from the proof, the value N can be found explicitly, say one can take

$$N = (\lceil \tfrac{\pi}{\varepsilon} \rceil + 1)^{n^2}.$$

The Collision Theorem was proved by Burago, Ferleger and Kononenko in [32]; we present their proof with minor improvements.

Let us formulate and prove a corollary of the Collision Theorem.

2.5.4. Corollary. *Consider n homogeneous hard balls moving freely and colliding elastically in empty space $\mathbb{R}^3$. Every ball moves along a straight line with constant speed until two balls collide, and then the new velocities of the two balls are determined by the laws of classical mechanics. We assume that only two balls can collide at the same time.*

Then the total number of collisions cannot exceed some number N which depends on the radii and masses of the balls. If the balls are identical then N depends only on n.

The proof below admits a straightforward generalization to all dimensions.

Proof. Denote by $a_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ the center of the i-th ball. Consider the corresponding point in $\mathbb{R}^{3\cdot N}$

$$\begin{aligned}\boldsymbol{a} &= (a_1, a_2, \ldots, a_n) = \\ &= (x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n).\end{aligned}$$

The i-th and j-th ball intersect if

$$|a_i - a_j| \leqslant R_i + R_j,$$

where R_i denoted the radius of the i-th ball. These inequalities define $\frac{n\cdot(n-1)}{2}$ cylinders

$$C_{i,j} = \left\{\, (a_1, a_2, \ldots, a_n) \in \mathbb{R}^{3\cdot n} \;\middle|\; |a_i - a_j| \leqslant R_i + R_j \,\right\}.$$

The closure of the complement

$$T = \overline{\mathbb{R}^{3\cdot n}\backslash \bigcup_{i<j} C_{i,j}}$$

is the configuration space of our system. Its points correspond to valid positions of the system of balls.

The evolution of the system of balls is described by the motion of the point $\boldsymbol{a} \in \mathbb{R}^{3\cdot n}$. It moves straight and at a constant speed until it hits one of the cylinders $C_{i,j}$; this event corresponds to a collision in the system of balls.

Consider the norm of $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^{3\cdot n}$ defined by

$$\|\boldsymbol{a}\| = \sqrt{M_1\cdot|a_1|^2 + \ldots + M_n\cdot|a_n|},$$

where $|a_i| = \sqrt{x_i^2 + y_i^2 + z_i^2}$ and M_i denotes the mass of the i-th ball. In the metric defined by $\|*\|$, the collisions follow the standard law of billiards: the angle of reflection is equal to the angle of incidence.

By construction, the number of collisions of hard balls that we need to estimate is the same as the number of collisions of the corresponding billiard trajectory on the table T with $C_{i,j}$ as the walls.

Note that each cylinder $C_{i,j}$ is a convex set; it has smooth boundary and it is centrally symmetric around the origin. By Exercise 2.5.2 all the intersections of the walls have ε-wide corners for some $\varepsilon > 0$ that depends on the radiuses R_i and the masses M_i. It remains to apply Collision Theorem 2.5.3. □

Now we present the proof of Collision Theorem (2.5.3) based on the results developed in the previous section.

Proof of Collision Theorem (2.5.3). Let us apply induction on n.

Base; $n = 1$. The number of collisions cannot exceed 1. Indeed, by the convexity of A^1, if the trajectory is reflected once in ∂A^1, then it cannot return to A^1.

Step. Assume γ is a trajectory which meets the walls in the order $A^{i_1}, \ldots, A^{i_N}$ for a large integer N.

Consider the array

$$\boldsymbol{A}_\gamma = (A^{i_1}, \ldots, A^{i_N}).$$

The induction hypothesis implies:

❶ *There is a positive integer M such that any M consecutive elements of $\boldsymbol{A}_\gamma$ contain each A^i at least once.*

Let $\mathcal{R}_\gamma$ be the Reshetnyak's puff pastry for $\boldsymbol{A}_\gamma$.

Consider the lift of γ to $\mathcal{R}_\gamma$ defined as $\bar\gamma(t) = \gamma^{\bar k}(t) \in \mathcal{R}_\gamma$ for any moment of time t between $\bar k$ and $(\bar k + 1)$-th collisions. Since γ follows the standard law of billiards at break points — in particular, the angle of reflection is equal to the angle of incidence — the lift $\bar\gamma$ is locally a geodesic in $\mathcal{R}_\gamma$. Since $\mathcal{R}_\gamma$ is a CAT[0] proper length space (see Observation 2.4.1), $\bar\gamma$ is a geodesic.

Since γ does not pass through $A^1 \cap \ldots \cap A^n$, the lift $\bar\gamma$ does not lie in $\mathcal{R}^0_\gamma \cup \mathcal{R}^N_\gamma$. In particular, $\mathcal{R}_\gamma$ is not end-to-end convex.

Let

$$\boldsymbol{B} = (A^{j_1}, \ldots, A^{j_K})$$

be the array provided by Proposition 2.4.8; so $\boldsymbol{B}$ contains each A^i at least once and the puff pastry $\mathcal{R}_{\boldsymbol{B}}$ for $\boldsymbol{B}$ is end-to-end convex. If N is sufficiently large, namely $N \geqslant K \cdot M$, then ❶ implies that $\boldsymbol{A}_\gamma$ can be obtained by inserting a finite number of A^i's in $\boldsymbol{B}$.

By Observation 2.4.4, $\mathcal{R}_\gamma$ is end-to-end convex, a contradiction. □

Chapter 3

Globalization and asphericity

In this chapter we introduce locally CAT[0] spaces and show that universal cover of locally CAT[0] proper length space forms a CAT[0] proper length space. The latter implies that any locally CAT$^{[}$[0]] proper length space is aspherical; that is its universal cover is contractible.

The globalization theorem leads to a *construction toy set*, described by Flag condition 3.5.5. Playing with this toy set we produce examples of exotic aspherical spaces.

3.1 Locally CAT spaces

We say that space $\mathcal{U}$ is *locally* CAT[0] (or *locally* CAT[1]) if a small closed ball centered at any point p in $\mathcal{U}$ forms a CAT[0] (or correspondingly CAT[1] space).

For example, $\mathbb{S}^1$ is locally isometric to $\mathbb{R}$, and so $\mathbb{S}^1$ is locally CAT[0]. On the other hand, $\mathbb{S}^1$ is not CAT[0] since the closed geodesic in $\mathbb{S}^1$ is not minimizing, so $\mathbb{S}^1$ does not satisfy Theorem 2.2.3.

This definition is equivalent to saying that each point $p \in \mathcal{U}$ admits an open neighborhood Ω such that any triangle in Ω is thin, or correspondingly spherically thin. The proof goes along the same lines as Exercise 2.2.8.

3.2 Space of local geodesics

In this section we will study behavior of local geodesics in the locally CAT spaces. The proved statements will be used in the proof of Globalization theorem (3.3.1).

3.2.1. Proposition. *Let $\mathcal{U}$ be a locally* CAT[0] *or locally* CAT[1] *proper length space.*

Assume $\gamma_n\colon [0,1] \to \mathcal{U}$ be a sequence of local geodesic paths converging to a path $\gamma_\infty\colon [0,1] \to \mathcal{U}$. Then γ_∞ is a local geodesic path. Moreover

$$\operatorname{length}\gamma_n \to \operatorname{length}\gamma_\infty$$

as $n \to \infty$.

Proof. Fix $t \in [0,1]$. Let $R > 0$ be the value such that $\overline{\mathrm{B}}[\gamma_\infty(t), R]$ forms a CAT[0] proper length space.

A local geodesic segment with length $< R/2$, and intersecting $\mathrm{B}(\gamma_\infty(t), R/2)$, cannot leave $\overline{\mathrm{B}}[\gamma_\infty(t), R]$ and hence is minimizing by Corollary 2.2.5. In particular, for all sufficiently large n, any subsegment of γ_n through $\gamma_n(t)$ with length $< R/2$ is a geodesic.

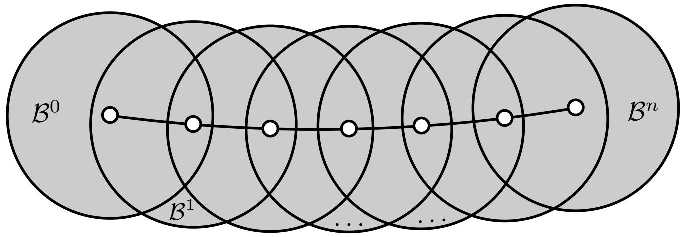
Since $\overline{\mathrm{B}}[\gamma_\infty(t), R]$ is a CAT[0] proper length space, by Theorem 2.2.3, geodesic segments in $\mathcal{B}$ depend uniquely and continuously on their endpoint pairs. Thus there is a subinterval $\mathbb{I}$ of $[0,1]$, which contains a neighborhood of t in $[0,1]$ and such that $\gamma_n|\mathbb{I}$ is minimizing for all large n. It follows that $\gamma_\infty|\mathbb{I}$ is a geodesic, and therefore γ_∞ is a local geodesic. □

The following lemma and its proof were suggested to us by Alexander Lytchak. This lemma will allow us to move a local geodesic path so that its endpoints follow given trajectories.

3.2.2. Patchwork along a curve. *Let $\mathcal{U}$ be a locally* CAT[κ] *proper length space, and $\gamma\colon [0,1] \to \mathcal{U}$ be a locally geodesic path.*

Then there is a CAT[κ] *proper length space $\mathcal{N}$, an open set $\hat{\Omega} \subset \mathcal{N}$, and a geodesic path $\hat{\gamma}\colon [0,1] \to \hat{\Omega}$, such that there is an open locally isometric immersion $\Phi\colon \hat{\Omega} \looparrowright \mathcal{U}$ such that $\Phi \circ \hat{\gamma} = \gamma$.*

Proof. Fix $r > 0$ so that for each $t \in [0,1]$, the closed ball $\overline{\mathrm{B}}[\gamma(t), r]$ forms a CAT[κ] proper length space.



Choose a partition $0 = t^0 < t^1 < \ldots < t^n = 1$ so that

$$\mathrm{B}(\gamma(t^i), r) \supset \gamma([t^{i-1}, t^i])$$

for all $n > i > 0$. Set $\mathcal{B}^i = \overline{\mathrm{B}}[\gamma(t^i), r]$.

Consider the disjoint union $\bigsqcup_i \mathcal{B}^i = \{ (i, x) \mid x \in \mathcal{B}^i \}$ with the minimal equivalence relation $\sim$ such that $(i, x) \sim (i-1, x)$ for all i. Let $\mathcal{N}$ be the space obtained by gluing the $\mathcal{B}^i$ along $\sim$.

Note that $A^i = \mathcal{B}^i \cap \mathcal{B}^{i-1}$ is convex in $\mathcal{B}^i$ and in $\mathcal{B}^{i-1}$. Applying the Reshetnyak gluing theorem (2.3.1) n times, we conclude that $\mathcal{N}$ is a CAT[κ] proper length space.

For $t \in [t^{i-1}, t^i]$, define $\hat\gamma(t)$ as the equivalence class of $(i, \gamma(t))$ in $\mathcal{N}$. Let $\hat\Omega$ be the ε-neighborhood of $\hat\gamma$ in $\mathcal{N}$, where $\varepsilon > 0$ is chosen so that $\mathrm{B}(\gamma(t), \varepsilon) \subset \mathcal{B}^i$ for all $t \in [t^{i-1}, t^i]$.

Define $\Phi\colon \hat\Omega \to \mathcal{U}$ by sending the equivalence class of (i, x) to x. It is straightforward to check that Φ, $\hat\gamma$ and $\hat\Omega \subset \mathcal{N}$ satisfy the conclusion of the main part of the lemma. $\square$

The following two corollaries follow from Patchwork along a curve (3.2.2) and Theorem on uniqueness of geodesics 2.2.3.

3.2.3. Corollary. *If $\mathcal{U}$ is a locally* CAT[0] *proper length space than for any pair of points $p, q \in \mathcal{U}$ the space of all locally geodesic paths from p to q is discrete; that is for any local geodesic path γ connecting p to q there is $\varepsilon > 0$ such that for any other geodesic path δ from p to q we have $|\gamma(t)\ \delta(t)|_{\mathcal{U}} > \varepsilon$ for some $t \in [0,1]$.*

Analogously, if $\mathcal{U}$ is a locally CAT[1] *proper length space than for any pair of points $p, q \in \mathcal{U}$ the space of all locally geodesic paths from p to q with length less than π is discrete.*

Recall that *path* is a curve parametrized by $[0,1]$. The spaces of paths in $\mathcal{U}$ comes with the natural metric

$$|\alpha\,\beta| = \sup\{\, |\alpha(t)\ \beta(t)|_{\mathcal{U}} \mid t \in [0,1] \,\}.$$

3.2.4. Corollary. *Let $\mathcal{U}$ is a locally* CAT[0] *proper length space than for any path α there is a choice of local geodesic path γ connecting the* *ends of* α *such that the map* $\alpha \mapsto \gamma$ *is continuous and if* α *is a local geodesic path then* $\gamma = \alpha$.

Analogously, if $\mathcal{U}$ *is a locally* CAT[1] *proper length space than for any path* α *with length less than* π *there is a choice of local geodesic path* γ *with length less than* π *connecting the ends of* α *such that the map* $\alpha \mapsto \gamma$ *is continuous and if* α *is a local geodesic path then* $\gamma = \alpha$.

Given a path $\alpha\colon [0,1] \to \mathcal{U}$, we denote by $\bar\alpha$ the same path traveled in the opposite direction; that is,

$$\bar\alpha(t) = \alpha(1-t).$$

Joint of two paths will be denoted with "$*$"; if two paths α and β connect the same pair of points then the joint $\bar\alpha * \beta$ forms a closed curve.

3.2.5. Exercise. *Assume* $\mathcal{U}$ *is a locally* CAT[1] *proper length space and the construction* $\alpha \mapsto \gamma_\alpha$ *provided by Corollary 3.2.4.*

Assume α *and* β *are two paths connecting the same pair of points in* $\mathcal{U}$, *each has lengths less than* π *and the joint* $\bar\alpha*\beta$ *is null-homotopic in the class of closed curves with length smaller than* $2\cdot\pi$ *then* $\gamma_\alpha = \gamma_\beta$.

3.3 Globalization

The original formulation of *Globalization theorem*, or *Hadamard–Cartan theorem* states that if M is a complete Riemannian manifold with sectional curvature $\leqslant 0$ then exponential map at any point $p \in M$ is a covering; in particular it implies that universal cover of M is diffeomorphic to the Euclidean space of the same dimension.

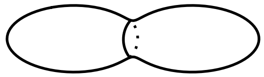
In this generality, theorem appeared in the lectures of Cartan, see [39]. For surfaces in the Euclidean plane, this theroem was proved by Hans von Mangoldt see [105] and few years later independently by Hadamard [70].

Formulations for metric spaces of different generality were proved by Busemann see [34], Rinow see [150], Gromov [64, p.119]. A detailed proof of Gromov's statement when $\mathcal{U}$ is proper was given by Alexander and Bishop in [21] and generalized further [6].

For CAT[1] spaces, the globalization theorem was proved by Bowditch in [28].

3.3.1. Globalization theorem. *Any simply connected locally* CAT[0] *proper length space is* CAT[0].

Analogously, assume $\mathcal{U}$ *is a locally* CAT[1] *proper length space such that any closed curve* $\gamma\colon \mathbb{S}^1 \to \mathcal{U}$ *with length smaller than* $2\cdot\pi$ *is null-homotopic in the class of closed curves length smaller than* $2\cdot\pi$. *Then* $\mathcal{U}$ *is a* CAT[1] *space.*



The surface of revolution showed on the diagram, is an example of simply connected space which is locally CAT[1] but not CAT[1]. To contract the marked curve one has to increase its length to $2{\cdot}\pi$ or more.

The proof of globalization theorem relies on the following theorem, which essentially is [15, Satz 9]. The proof use a thin-triangle decompositions, and the inheritance lemma (2.2.6).

3.3.2. Patchwork globalization theorem. *A locally* CAT[0] *proper length space $\mathcal{U}$ is* CAT[0] *if and only if the pairs of points in $\mathcal{U}$ are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

Analogously, a locally CAT[1] *proper length space $\mathcal{U}$ is* CAT[1] *if and only if the pairs of points in $\mathcal{U}$ at distance less π are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

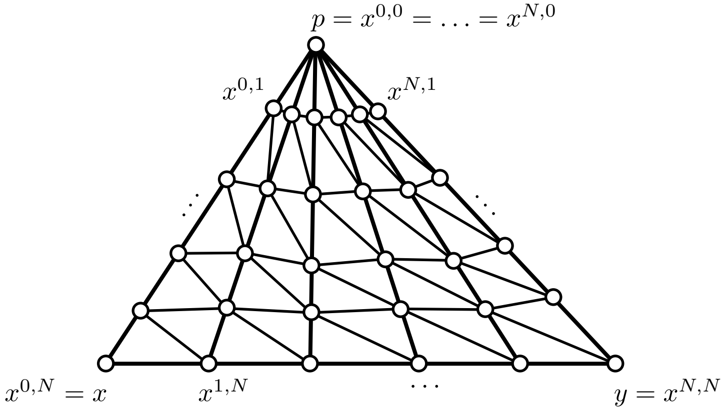
The proof of Patchwork globalization uses the following construction:

3.3.3. Line-of-sight map. *Let p be a point and α be a curve of finite length in a length space $\mathcal{X}$. Let $\bar\alpha : [0,1] \to \mathcal{U}$ be the constant-speed parameterization of α. If $\gamma_t \colon [0,1] \to \mathcal{U}$ is a geodesic from p to $\bar\alpha(t)$, we say*

$$[0,1] \times [0,1] \to \mathcal{U} \colon (t,s) \mapsto \gamma_t(s)$$

is a line-of-sight map from p to α .

Proof of Patchwork globalization theorem (3.3.2). Note that the implication the "only if" part is already proved in Theorem 2.2.3; it only remains to prove "if" part.



Fix a triangle $[pxy]$ in $\mathcal{U}$. We need to show that $[pxy]$ is thin.

By the assumptions the line-of-sight map (3.3.3)

$$(t, \tau) \mapsto \gamma_t(\tau)$$

for $[xy]$ from p is uniquely defied and continuous.

Fix a fine a partition

$$0 = t^0 < t^1 < \ldots < t^N = 1,$$

set $x^{i,j} = \gamma_{t^i}(t^j)$. Since the line-of-sight map is continuous and $\mathcal{U}$ is locally $\mathrm{CAT}[0]$, we may assume that each triangle

$$[x^{i,j}\, x^{i,j+1}\, x^{i+1,j+1}] \quad \text{and} \quad [x^{i,j}\, x^{i+1,j}\, x^{i+1,j+1}]$$

are thin.

Now we show that the thin property propagates to $[pxy]$ by repeated application of the inheritance lemma (2.2.6):

- ◇ First, for fixed i, sequentially applying the lemma shows that the triangles $[x\, x^{i,1}\, x^{i+1,2}]$, $[x\, x^{i,2}\, x^{i+1,2}]$, $[x\, x^{i,2}\, x^{i+1,3}]$, and so on are thin.

In particular, for each i, the long triangle $[x\, x^{i,N}\, x^{i+1,N}]$ is thin.

- ◇ Applying the lemma again shows that the triangles $[x\, x^{0,N}\, x^{2,N}]$, $[x\, x^{0,N}\, x^{3,N}]$, and so on are thin.

In particular, $[pxy] = [p\, x^{0,N}\, x^{N,N}]$ is thin. □

Proof of Globalization theorem; $\mathrm{CAT}[0]$ *case.* Given a path $\alpha\colon [0,1] \to \mathcal{U}$ denote by γ_α the local geodesic provided by Corollary 3.2.4. Since the map $\alpha \mapsto \gamma_\alpha$ is continuous, we have $\gamma_\alpha = \gamma_\beta$ for any pair of paths α and β homotopic rel. to the ends.

Since $\mathcal{U}$ is simply connected any pair of paths with common ends are homotopic. It follows that for any two points $p, q \in \mathcal{U}$ are joint by unique local geodesic.

It remains to apply the patchwork globalization theorem (3.3.2).

$\mathrm{CAT}[1]$ *case.* The proof goes along the same lines, but one one need to use Exercise 3.2.5. □

3.4 Polyhedral spaces

3.4.1. Definition. *A length space $\mathcal{P}$ is called* polyhedral space *if it admits a finite triangulation τ such that an every simplex in τ is isometric to a simplex in a Euclidean space of appropriate dimension.*

Analogousely, a length space $\mathcal{P}$ is called spherical polyhedral space *if it admits a finite triangulation τ such that an every simplex in τ is isometric to a simplex in a unit sphere of appropriate dimension.*

By a triangulation of a polyhedral space *we will always understand the triangulation as above.*

Note that according to the above definition, all polyhedral spaces are compact. However, most of the statements below admit straightforward generalizations to *locally polyhedral spaces*; that is, complete length spaces, any point of which admits a closed neighborhood isometric to a polyhedral space. The latter class of spaces includes in particular the infinite covers of polyhedral spaces.

The dimension of a polyhedral space $\mathcal{P}$ is defined as the maximal dimension of the simplex in one (and therefore any) triangulation of $\mathcal{P}$.

Links. Let $\mathcal{P}$ be a polyhedral space and σ be a simplex in a triangulation τ of $\mathcal{P}$.

The simplices which contain σ form an abstract simplicial complex called the *link* of σ, denoted by Link_σ. If m is the dimension of σ then the set of vertices of Link_σ is formed by the $(m+1)$-simplices which contain σ; the set of its edges are formed by the $(m+2)$-simplices which contain σ, and so on.

The link Link_σ can be identified with the subcomplex of τ formed by all the simplices σ' such that $\sigma \cap \sigma' = \varnothing$ but both σ and σ' are faces of the same simplex.

The points in Link_σ can be identified with the normal directions to σ at a point in its interior. The angle metric between directions makes Link_σ into a spherical polyhedral space. We will always consider the link with this metric.

Tangent space and space of directions. Let $\mathcal{P}$ be a polyhedral space and let τ be a triangulation of $\mathcal{P}$. If a point $p \in \mathcal{P}$ lies in the interior of a $\hbar$-simplex σ of τ then the tangent space $\mathrm{T}_p\mathcal{P}$ is naturally isometric to

$$\mathbb{E}^{\hbar} \times (\mathrm{Cone}\,\mathrm{Link}_\sigma).$$

Equivalently, the space of directions Σ_p can be isometrically identified with the $\hbar$-th spherical suspension over Link_σ; that is,

$$\Sigma_p \stackrel{iso}{=\!=} \mathrm{Susp}^{\hbar}(\mathrm{Link}_\sigma).$$

If $\mathcal{P}$ is an m-dimensional polyhedral space, then for any $p \in \mathcal{P}$ the space of directions Σ_p is a spherical polyhedral space of dimension at most $m-1$.

In particular, for any point p in σ, the isometry class of Link_σ together with $\hbar = \dim\sigma$ determines the isometry class of Σ_p and the other way around.

A small neighborhood of p is isometric to a neighborhood of the tip of the κ-cone over Σ_p. In fact, if this property holds at any point of a compact length space $\mathcal{P}$ then $\mathcal{P}$ is a polyhedral space see [99].

The following theorem provides a combinatorial description of polyhedral spaces with curvature bounded above.

3.4.2. Theorem. *Let $\mathcal{P}$ be a polyhedral space and τ be its triangulation. Then $\mathcal{P}$ is locally* $\mathrm{CAT}[0]$ *if and only if the link of any simplex in τ has no closed local geodesic shorter than $2{\cdot}\pi$.*

Analogously, if $\mathcal{P}$ be a spherical polyhedral space and τ be a its triangulation. Then $\mathcal{P}$ is $\mathrm{CAT}[1]$ *if and only if $\mathcal{P}$ and every link of any simplex in τ has no closed local geodesic shorter than $2{\cdot}\pi$.*

In the proof we will use the following corollary of Globalization theorem 3.3.1).

3.4.3. Corollary. *Any locally* $\mathrm{CAT}[0]$ *compact space which contains no closed geodesics is* $\mathrm{CAT}[0]$.

Analogously, any locally $\mathrm{CAT}[1]$ *compact space which contains no closed geodesics of length smaller than $2{\cdot}\pi$ is* $\mathrm{CAT}[1]$.

Proof. By Globalization theorem (3.3.1), we need to show that the space is simply connected. Assume contrary. Fix a nontrivial homotopy class of closed curves.

Denote by ℓ the exact lower bound for the lengths of curves in the class. Note that $\ell > 0$; otherwise there would be a closed noncontractable curve in a $\mathrm{CAT}[0]$ spherical neighborhood of some poin which contradicts Corollary 2.2.4.

Since the space is compact the class contains a length minimizing curve which has to be a closed geodesic.

The $\mathrm{CAT}[1]$ is analogous, one only has to consider the homotopy class of closed curves shorter than $2{\cdot}\pi$. □

3.4.4. Exercise. *In the assumptions of Corollary 3.4.3, prove that the space contains* geodesic circle*; that is a simple closed curve γ such that for any two points $p, q \in \gamma$ one the arcs of with endpoints p and q forms a minimizing geodesic.*

Proof of Theorem 3.4.2. We prove "only if" part and leave "if" part as an exercise.

We apply induction on $\dim\mathcal{P}$. The base case $\dim\mathcal{P} = 0$ is evident. Let us start with $\mathrm{CAT}[1]$ case.

Step. Assume that the theorem is proved in the case $\dim \mathcal{P} < m$. Suppose $\dim \mathcal{P} = m$.

Fix a point $p \in \mathcal{P}$. A neighborhood of p is isometric to the neighborhood of the north pole in the suspension over the Σ_p. By the second part of Exercise 2.1.2 it is sufficient to show that

❶ $$\Sigma_p \in \mathrm{CAT}[1].$$

Note that Σ_p is a spherical polyhedral space and its links are isometric to links of $\mathcal{P}$. By the induction hypothesis, Σ_p is CAT[1]. Applying Grolbalization (3.3.1), we get the statement.

The CAT[0] case is different only in the last step, where we need to use the first part of Exercise 2.1.2. □

3.4.5. Exercise. *Show that if in a polyhedral space $\mathcal{P}$ any two points can be connected by a unique geodesic then $\mathcal{P}$ is a* CAT[0] *space.*

3.5 Flag complexes

3.5.1. Definition. *A simplicial complex $\mathcal{S}$ is called* flag *if whenever $\{v^0, \dots, v^{\Bbbk}\}$ is a set of distinct vertices of $\mathcal{S}$ which are pairwise joined by edges, then the vertexes $v^0, \dots, v^{\Bbbk}$ span a $\Bbbk$-simplex in $\mathcal{S}$.*

If the above condition is satisfied only for $\Bbbk = 2$, then we say $\mathcal{S}$ satisfies the no-triangle condition*.*

Note that every flag complex is determined by its 1-skeleton.

3.5.2. Exercise. *Show that the barycentric subdivision of any simplicial complex is a flag complex.*

Conclude that any finite simplicial complex is homeomorphic to a CAT[1] *proper length space.*

3.5.3. Proposition. *A simplicial complex $\mathcal{S}$ is flag if and only if $\mathcal{S}$ as well as all the links of all its simplices satisfies the no-triangle condition.*

From the definition of flag complex we get the following.

3.5.4. Lemma. *Any link of a flag complex is flag.*

Proof of Proposition 3.5.3. By Lemma 3.5.4, the no-triangle condition holds for any flag complex and all its links.

Now assume a complex $\mathcal{S}$ and all its links satisfy the no-triangle condition. It follows that $\mathcal{S}$ includes a 2-simplex for each triangle. Applying the same observation for each edge we get that $\mathcal{S}$ includes a 3-simplex for any complete graph with 4 vertices. Repeating this observation for triangles, 4-simplexes, 5-simplices and so on we get that $\mathcal{S}$ is flag. □

All-right triangulation. A triangulation of a spherical polyhedral space is called an *all-right triangulation* if each simplex of the triangulation is isometric to a spherical simplex all of whose angles are right. Similarly, we say that a simplicial complex is equipped with an *all-right spherical metric* if it is a length metric and each simplex is isometric to a spherical simplex all of whose angles are right.

Spherical polyhedral CAT[1] spaces glued from of right-angled simplices admit the following characterization discovered by Gromov [64, p. 122].

3.5.5. Flag condition. *Assume that a spherical polyhedral space $\mathcal{P}$ admits an all-right triangulation τ. Then $\mathcal{P}$ is a* CAT[1] *space if and only if τ is flag.*

Proof; "only-if" part. Assume there are three vertices v^1, v^2 and v^3 of τ that are pairwise joined by edges but do not span a simplex. Note that in this case

$$\measuredangle[v^1\,{}^{v^2}_{v^3}] = \measuredangle[v^2\,{}^{v^3}_{v^1}] = \measuredangle[v^3\,{}^{v^1}_{v^1}] = \pi.$$

Equivalently,

❶ *The join of the geodesics $[v^1v^2]$, $[v^2v^3]$ and $[v^3v^1]$ forms a locally geodesic loop in $\mathcal{P}$.*

Now assume that $\mathcal{P}$ is a CAT[1] space. Then by Theorem 3.4.2, $\mathrm{Link}_\sigma \mathcal{P}$ is a CAT[1] space for every simplex σ in τ.

Each of these links is an all-right spherical complex and by Theorem 3.4.2, none of these links can contain a geodesic circle of length less than $2\cdot\pi$.

Therefore Proposition 3.5.3 and ❶ imply the "only-if" part.

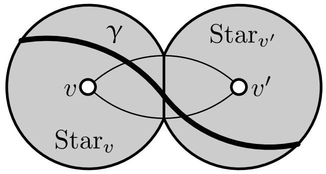
"If" part. By Lemma 3.5.4 and Theorem 3.4.2, it is sufficient to show that any closed local geodesic γ in a flag complex $\mathcal{S}$ with all-right metric has length at least $2\cdot\pi$.

Recall that the *closed star* of a vertex v (briefly $\overline{\mathrm{Star}}_v$) is formed by all the simplices containing v. Similarly, Star_v, the open star of v, is the union of all simplices containing v with faces opposite v removed.

Choose a simplex σ which contains a point of γ. Let v be a vertex of σ. Set $f(t) = \cos|v\gamma(t)|$. Note that

$$f''(t) + f(t) = 0$$

if $f(t) > 0$. Since the zeroes of f are π apart, γ spends time π on every visit to Star_v.



After leaving Star_v, the local geodesic γ has to enter another simplex, say σ', which has a vertex v' not joined to v by an edge.

Since τ is flag, we have that the stars Star_v and $\mathrm{Star}_{v'}$ do not overlap. The same argument as above shows that γ spends time π on every visit to $\mathrm{Star}_{v'}$. Therefore the total length of γ is at least $2{\cdot}\pi$. □

3.5.6. Exercise. *Assume that a spherical polyhedral space $\mathcal{P}$ admits a triangulation τ such that all dihedral angles of all simplexes are at least $\frac{\pi}{2}$. Show that $\mathcal{P}$ is a* CAT[1] *space if τ is flag.*

3.5.7. Exercise. *Let $\mathcal{U} \in \mathrm{CAT}[0]$ and $\varphi^1, \varphi^2, \dots, \varphi^k\colon \mathcal{U} \to \mathcal{U}$ be commuting short retractions; that is*

- ◇ $\varphi^i \circ \varphi^i = \varphi^i$ *for each* i*;*
- ◇ $\varphi^i \circ \varphi^j = \varphi^j \circ \varphi^i$ *for any* i *and* j*;*
- ◇ $|\varphi^i(x)\ \varphi^i(y)|_{\mathcal{U}} \leqslant |x\,y|_{\mathcal{U}}$ *for each* i *and any* $x, y \in \mathcal{U}$*.*

Set $A^i = \operatorname{Im} \varphi^i$ for all i; note that each A^i is a weakly convex set.

Assume Γ is a finite graph (without loops and multiple edges) with edges labeled by $1, 2, \dots, n$. Denote by $\mathcal{U}^\Gamma$ the space obtained by taking a copy of $\mathcal{U}$ for each vertex of Γ and gluing two such copies along A^i if the corresponding vertices are joint by an edge labeled by i.

Show that $\mathcal{U}^\Gamma$ is a CAT[0] *space*

The space of trees. The following construction is given by Billera, Holmes and Vogtmann in [26].

Let $\mathcal{T}_n$ be the set of all metric trees with n end-vertices labeled by $a^1, \dots, a^n$. To describe one tree in $\mathcal{T}_n$ we may fix a topological tree τ with end vertices $a^1, \dots, a^n$ and all the other verices of degree 3 and prescribe the lengths of $2{\cdot}n - 3$ edges. If the lengh of an edge is 0, we assume that edge degenerates; such a tree can be also decribed using a different topological tree τ'. The subset of $\mathcal{T}_n$ corresponding to the given topological tree τ can be identified with a convex closed cone in $\mathbb{R}^{2\cdot n-3}$. Equip each such subset with the metric induced from $\mathbb{R}^{2\cdot n-3}$ and consider the legth metric on $\mathcal{T}_n$ induced by these metrics.

3.5.8. Exercise. *Show that $\mathcal{T}_n$ with the described metric is a* CAT[0] *space.*

3.6 Cubical complexes

The definition of a cubical complex mostly repeats the definition of a simplicial complex, with simplices replaced by cubes.

Formally, a *cubical complex* is defined as a subcomplex of the unit cube in the Euclidean space of large dimension; that is, a collection of faces of cube such that together with each face it contains all its sub-faces. Each cube face in this collection will be called a *cube* of the cubical complex.

Note that according to this definition, any cubical complex is finite, that is, contains a finite number of cubes.

The union of all the cubes in a cubical complex $\mathcal{Q}$ will be called its *underlying space*; it will be denoted by $\mathcal{Q}$ or by $\underline{\mathcal{Q}}$ if we need to emphasize that we are talking about a set, not a complex. A homeomorphism from $\underline{\mathcal{Q}}$ to a topological space $\mathcal{X}$ is called a *cubulation of* $\mathcal{X}$.

The underlying space of a cubical complex $\mathcal{Q}$ will be always considered with the length metric induced from $\mathbb{R}^N$. In particular, with this metric, each cube of $\mathcal{Q}$ is isometric to the unit cube of the same dimension.

It is straightforward to construct a triangulation of $\underline{\mathcal{Q}}$ such that each simplex is isometric to a Euclidean simplex. In particular $\underline{\mathcal{Q}}$ is a Euclidean polyhedral space.

The link of each cube in a cubical complex admits a natural all-right triangulation; each simplex corresponds to an adjusted cube.

Cubical analog of a simplicial complex. Let $\mathcal{S}$ be a simplicial complex and $\{v_1,\dots,v_N\}$ be the set of its vertexes.

Consider $\mathbb{R}^N$ with the standard basis $\{e_1,\dots,e_N\}$. Denote by $\square^N$ the standard unit cube in $\mathbb{R}^N$; that is

$$\square^N=\left\{\,(x_1,\dots,x_N)\in\mathbb{R}^N\;\middle|\;0\leqslant x_i\leqslant 1 \text{ for each } i\,\right\}.$$

Given a $\mathchar'26\mkern-9mu k$-dimensional simplex $\langle v_{i_0},\dots,v_{i_{\mathchar'26\mkern-9mu k}}\rangle$ in $\mathcal{S}$, mark the $(\mathchar'26\mkern-9mu k+1)$-dimensional faces in $\square^N$ (there are $2^{N-\mathchar'26\mkern-9mu k}$ of them) which are parallel to the coordinate $(k+1)$-plane spanned by $e_{i_0},\dots,e_{i_{\mathchar'26\mkern-9mu k}}$.

Note that the set of all marked faces of $\square^N$ forms a cubical complex; it will be called the *cubical analog* of $\mathcal{S}$ and will be denoted as $\square_{\mathcal{S}}$.

Note that if a simplicial complex is connected then so is its cubical analog.

3.6.1. Proposition. *Let $\mathcal{S}$ be a connected simplicial complex and $\mathcal{Q}=\square_{\mathcal{S}}$ be its cubical analog. Then $\underline{\mathcal{Q}}$ is connected and the link of any vertex of $\mathcal{Q}$ is isometric to $\mathcal{S}$ equipped with the spherical right-angled metric.*

In particular, if $\mathcal{S}$ is a flag complex then $\mathcal{Q}$ is a locally CAT[0] *and therefore its universal cover $\tilde{\mathcal{Q}}$ is a* CAT[0] *space.*

Proof. The first part of the proposition follows from the construction above.

If $\mathcal{S}$ is flag, then by Flag condition (3.5.5) the link of any cube in $\mathcal{Q}$ is a CAT[1] space. Therefore, by cone construction (Exercise 2.1.2) $\mathcal{Q}$ is locally CAT[0] space. It remains to apply Globalization theorem 3.3.1. $\square$

From Proposition 3.6.1, it follows that the cubical analog of any flag complex is aspherical. The following exercise states that the converse also holds, see [47, 5.4].

3.6.2. Exercise. *A simplicial complex is flag if and only if its cubical analog is aspherical.*

3.7 Exotic aspherical manifolds

By Globalization theorem (3.3.1) any CAT[0] proper length space is contractible. Therefore any complete locally CAT[0] proper length space is *aspherical*; that is, they have contractible universal covers. This observation can be used to construct examples of aspherical spaces.

Let $\mathcal{X}$ be a proper topological space. Recall that $\mathcal{X}$ is called *simply connected at infinity* if for any compact set $K \subset \mathcal{X}$ there is a bigger compact set $K' \supset K$ such that $\mathcal{X}\backslash K'$ is path connected and any loop which lies in $\mathcal{X}\backslash K'$ is null-homotopic in $\mathcal{X}\backslash K$.

Recall that path connected spaces are not empty by definition. Therefore compact spaces are not simply connected at infinity.

The following statement was proved by Michael Davis in [46].

3.7.1. Proposition. *For any $m \geqslant 4$ there is a closed aspherical m-dimensional piecewise linear manifold whose universal cover is not simply connected at infinity.*

In particular, the universal cover of this manifold is not homeomorphic to the m-dimensional Euclidean space.

The proof requires the following lemma.

3.7.2. Lemma. *Let $\mathcal{S}$ be a flag complex, $\mathcal{Q} = \square_{\mathcal{S}}$ be its cubical analog and $\tilde{\mathcal{Q}}$ be the universal cover of $\mathcal{Q}$.*

Assume $\tilde{\mathcal{Q}}$ is simply connected at infinity. Then $\mathcal{S}$ is simply connected.

Proof of Lemma 3.7.2. Assume $\mathcal{S}$ is not simply connected. Choose a shortest noncontractible circle $\gamma\colon \mathbb{S}^1 \to \mathcal{S}$ formed by the edges of $\mathcal{S}$.

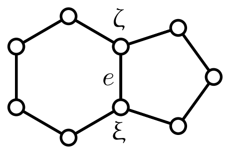
Note that γ forms a 1-dimensional subcompelex of $\mathcal{S}$ which is a closed local geodesic. Denote by G the subcomplex of $\mathcal{Q}$ which corresponds to γ.

Fix a vertex $v \in G$; let G_v be the connected component of G containing v. Let $\tilde{G}$ be the inverse image of G_v in $\tilde{\mathcal{Q}}$ for the universal cover $\tilde{\mathcal{Q}} \to \mathcal{Q}$. Fix a point $\tilde{v} \in \tilde{G}$ in the inverse image of v.

Note that

❶ *$\tilde{G}$ forms a convex set in $\tilde{\mathcal{Q}}$.*

Indeed, according to Proposition 3.6.1, $\tilde{\mathcal{Q}}$ is CAT[0]. By Exercise 2.2.9, it is sufficient to show that $\tilde{G}$ is locally convex in $\tilde{\mathcal{Q}}$, or equivalently G is locally convex in $\mathcal{Q}$.



The latter can only fail if γ passes through two vertices, say ξ to ζ in $\mathcal{S}$, which are joined by an edge not in γ; denote this edge by e.

Each edge of $\mathcal{S}$ has length $\frac{\pi}{2}$. Therefore each of two circles formed by e and an arc of γ from ξ to ζ is shorter that γ. Moreover, at least one of them is noncontractable since γ is not. That is, γ is not a shortest noncontractible circle — a contradiction. △

Further, note that $\tilde{G}$ is homeomorphic to the plane, since $\tilde{G}$ is a 2-dimensional manifold without boundary which by the above is CAT[0] and hence is contractible.

Denote by C_R the circle of radius R in $\tilde{G}$ centered at $\tilde{v}$. All C_R are homotopic to each other in $\tilde{G}\backslash\{\tilde{v}\}$ and therefore in $\tilde{\mathcal{Q}}\backslash\{\tilde{v}\}$.

Note that the map $\tilde{\mathcal{Q}}\backslash\{\tilde{v}\} \to \mathcal{S}$ defined by $x \mapsto \operatorname{dir}[\tilde{v}x]$ maps C_R to a circle homotopic to γ. Therefore C_R is not contractible in $\tilde{\mathcal{Q}}\backslash\{\tilde{v}\}$.

In particular, C_R is not contactable in $\tilde{\mathcal{Q}}\backslash K$ if $K \supset \tilde{v}$. If R is large, the circle C_R lies outside of any compact set K' in $\tilde{\mathcal{Q}}$. It follows that $\tilde{\mathcal{Q}}$ is not simply connected at infinity, a contradiction. □

3.7.3. Exercise. *Under the assumptions of the Lemma 3.7.2, for any vertex v in $\mathcal{S}$ the complement $\mathcal{S}\backslash\{v\}$ is simply connected.*

Proof of Proposition 3.7.1. Let Σ^{m-1} be an $(m-1)$-dmensional smooth homology sphere which is not simply connected and bounds a contractible smooth compact m-dimensional manifold $\mathcal{W}$.

For $m \geqslant 5$ the existence of such $(\mathcal{W}, \Sigma)$ follows from [84]. For $m = 4$ it follows from the construction in [108].

Pick any smooth triangulation τ of W and let $\mathcal{S}$ be the resulting subcomplex which triangulates Σ.

We can assume that $\mathcal{S}$ is flag; otherwise pass to the barycentric subdivision of τ and apply Exercise 3.5.2.

Let $\mathcal{Q} = \square_{\mathcal{S}}$ be the cubical analog of $\mathcal{S}$.

By Proposition 3.6.1, $\mathcal{Q}$ is a homology manifold. It follows that $\mathcal{Q}$ is a piecewise linear manifold with a finite number of singularities at its vertices.

Removing a small neighborhood of each vertex in $\mathcal{Q}$, we can obtain a piecewise linear manifold which boundary is formed by several copies of Σ. To each copy of Σ, glue a copy of $\mathcal{W}$ along its boundary. This results in a closed piecewise linear manifold $\mathcal{M}$ which is homotopically equivalent to $\mathcal{Q}$.

Finally, by the Lemma 3.7.2, the universal cover $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ is not simply connected at infinity.

The same holds for the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$. The later follows since homotopy equivalences $f\colon \mathcal{Q} \to \mathcal{M}$ and $g\colon \mathcal{M} \to \mathcal{Q}$ lift to proper maps $\tilde{f}\colon \tilde{\mathcal{Q}} \to \tilde{\mathcal{M}}$ and $\tilde{g}\colon \tilde{\mathcal{M}} \to \tilde{\mathcal{Q}}$. □

3.8 Comments

The construction used in the proof of Proposition 3.7.1 does many other interesting thins; some of them discussed in the survey [47] by Michael Davis.

The following Proposition was proved by Fredric Ancel, Michael Davis and Craig Guilbault in [20].

3.8.1. Proposition. *Given $m \geqslant 5$ there is a Euclidean polyhedral space $\mathcal{P}$ such that*

a) $\mathcal{P}$ is homeomorphic to a closed m-dimensional manifold.

b) $\operatorname{curv} \mathcal{P} \leqslant 0$

c) The universal cover of $\mathcal{P}$ is not simply connected at infinity.

Proof. Apply Exercise 3.7.3 to the barycentric subdivision of the simplicial complex $\mathcal{S}$ provided by Exercise 3.8.2. □

3.8.2. Exercise. *Given a positive integer $m \geqslant 5$ construct an $(m-1)$-dimensional simplicial complex $\mathcal{S}$ such that $\operatorname{Cone} \mathcal{S}$ is homeomorphic to $\mathbb{E}^m$ and $\pi_1(\mathcal{S}\backslash\{v\}) \neq 0$ for some vertex v in $\mathcal{S}$.*

It worth to note that any complete simply connected Riemannian manifold with nonpositive curvature is homeomorphic to the Euclidean space of the same dimension. In fact by Globalization theorem (3.3.1), the exponential map at one point of such manifold is a homeomorphism. In particular there is no Riemannian analog of the proposition.

Moreover, according to a theorem of Stone, see [161, 48], there is no piecewise linear analog of the proposition; that is the homeomorphism to a manifold in Proposition 3.8.1 can not be made piecewise linear.

A similar construction was used by Michael Davis, Tadeusz Januszkiewicz and Jean-François Lafont [49] to construct a closed smooth 4-dimensional manifold M with universal cover $\tilde{M}$ diffeomorphic to $\mathbb{R}^4$ such that (1) M admits a polyhedral metric which is locally CAT[0] but it does not admit a Riemannian metric with nonpositive sectional curvature.

The construction described in this chapter also lead to so called *hyperbolization*, a flexible tool to construct a aspherical spaces; [41] is a good survey in the subject.

Chapter 4

Subsets

In this chapter we give a partial answer to the following question. *Which subset of Euclidean space equipped with the induced length-metric form* CAT[0] *spaces?*

4.1 Motivating examples

Consider three subgraphs of different paraboloids

$$\begin{aligned} A &= \{\, (x,y,z) \in \mathbb{E}^3 \mid z \leqslant x^2 + y^2 \,\}, \\ B &= \{\, (x,y,z) \in \mathbb{E}^3 \mid z \leqslant -x^2 - y^2 \,\}, \\ C &= \{\, (x,y,z) \in \mathbb{E}^3 \mid z \leqslant x^2 - y^2 \,\}. \end{aligned}$$

Here is a question which motivates the rest of the chapter.

4.1.1. Question. *Which of the sets A, B and C, if equipped with the induced length metric, form a* CAT[0] *space and why?*

The answers is given below, but it is instructive to think about this question before reading further.

A. No, $A \notin \mathrm{CAT}[0]$.

The boundary ∂A is the paraboloid described by $z = x^2 + y^2$ in particular it bounds an open convex set in $\mathbb{E}^3$ which complement is A. The closest point projection of $A \to \partial A$ is short (Exercise 2.2.10). It follows that ∂A is a convex set in A equipped with the induced length metric.

Therefore if A forms a CAT[0] then so is ∂A. The latter is not true, ∂A is a smooth convex surface; by Gauss formula it has strictly positive curvature.

B. Yes, $B \in \mathrm{CAT}[0]$.

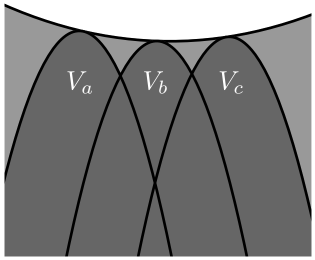
Evidently B is a convex closed set in $\mathbb{E}^3$. Therefore length metric on B coinsides with Euclidean metric and (2+2)-comparison holds.

C. Yes, $C \in \mathrm{CAT}[0]$, but the proof is not as easy as before. We give a sketch here; a complete proof of more general statement is given in Section 4.3.

Consider the one-parameter family of sets

$$V_t = \left\{ (x, y, z) \in \mathbb{E}^3 \;\middle|\; z \leqslant x^2 - y^2 - 2{\cdot}(x - t)^2 \right\}.$$

Each set V_t is closed, convex and



$$C = \bigcup_t V_t.$$

Further note that

❶ $$V_b \supset V_a \cap V_c$$

for any choice real numbers $a < b < c$.

The inclusion ❶ makes possible to apply recursively Reshetnyak gluing theorem 2.3.1 and show that any finite union of V_t forms a $\mathrm{CAT}[0]$ space. By approximation, the (2+2)-point comparison holds for any 4 points in C; that is, $C \in \mathrm{CAT}[0]$.

Remark. The set C is not convex, but it is two-convex as defined in the next section. As you will see two-convexity of subsets is closely related to the inheritance of upper curvature bound by a subset.

4.2 Two-convexity

The following definition is closely related to the one given by Gromov in [66, §$\tfrac{1}{2}$], see also [123].

4.2.1. Definition. *We say that a subset $K \subset \mathbb{E}^m$ is* two-convex *if the following condition holds for any plane $W \subset \mathbb{E}^m$. If γ is a closed curve in $W \cap K$ which is null-homotopic in K then it is null-homotopic in $W \cap K$.*

The following proposition follows immediately from the definition.

4.2.2. Proposition. *Any subset in $\mathbb{E}^2$ is 2-convex.*

The following proposition describes closed two-convex sets with smooth boundary.

4.2.3. Proposition. *Let $K \subset \mathbb{E}^m$ be a closed subset.*

Assume that the boundary of K forms a smooth hypersurface S. Consider the orthonormal vector field ν on S which points outside of K. Denote by $k_1 \leqslant \ldots \leqslant k_{m-1}$ the scalar fields of principal curvatures on S with respect to ν.

Then K is two-convex if and only if $k_2(p) \geqslant 0$ at any point $p \in S$. Moreover, if $k_2(p) < 0$ at some point p then K the Definition 4.2.1 fails for a curve γ forming a triangle.

Proof; "if" part. If $k_2(p) < 0$ for some $p \in S$, consider the plane W through p which spanned by the first two principle directions. Choose a small triangle in W which surrounds p and move it slightly in the direction of $\nu(p)$. We get a triangle $[xyz]$ which is null-homotopic in K, but the solid triangle $\Delta = \operatorname{Conv}\{x, y, x\}$ bounded by $[xyz]$ does not lie in K completely. Therefore K is not two-convex.

"Only-if" part. Recall that a smooth function $f\colon \mathbb{E}^m \to \mathbb{R}$ is called *strongly convex* if its Hessinan is positive definte at each point.

Fix a smooth strongly convex function $f\colon \mathbb{E}^m \to \mathbb{R}$, such that the restriction $f|_S$ is a Morse function. Note that generic smooth strongly convex function $f\colon \mathbb{E}^m \to \mathbb{R}$ has this property.

For a critical point p of $f|_S$ the outer normal vector $\nu(p)$ is parallel to the gradient $\nabla_p f$; we say that p is *positive critical point* if $\nu(p)$ and $\nabla_p f$ point in the same direction and *negative* otherwise. (If $\nabla_p f = 0$, that is p is the minimum point of f, then the sign is undefined.)

Since S is two-convex and f is convex, the negative critical points of $f|_S$ have index at most 1.

Given a real value s, set

$$K_s = \{\, x \in K \mid f(x) < s \,\}.$$

Assume $\varphi_0\colon \mathbb{D} \to K$ is a continuous map of the disc $\mathbb{D}$ such that $\varphi(\partial\mathbb{D}) \subset K_s$. By Morse Lemma, there is a homotopy $\varphi_t\colon \mathbb{D} \to K$ such that $\varphi_1(\mathbb{D}) \subset K_s$.

Consider a closed curve $\gamma\colon \mathbb{S}^1 \to K$ which is null-homotopic in K. Note that the distance function $f_0(x) = |\operatorname{Conv}\gamma\ x|_{\mathbb{E}^m}$ is convex. Therefore f_0 can be approximated by smooth strongly convex function f such that the restriction $f|_S$ is a Morse function. From above the there is a disc in K with boundary γ which lies arbitrary close to $\operatorname{Conv}\gamma$. Since K is closed, the statement follows. $\square$

4.2.4. Proposition. *Intersection of arbitrary collection of two-convex sets in $\mathbb{E}^m$ is two-convex.*

4.2.5. Proposition. *Interior of any two-convex set in $\mathbb{E}^m$ is a two-convex set.*

Proofs. The Proposition 4.2.4 is evident.

To prove Proposition 4.2.5, apply the Definition 4.2.1 for parallel translations of the curve γ by small vectors. □

4.2.6. Definition. *Given a subset $K \subset \mathbb{E}^m$ define its two-convex hull (briefly $\operatorname{Conv}_2 K$) as the intersection of all two-convex subsets containing K.*

Note that by Proposition 4.2.4, two-convex hull of any set is two-convex. Further, by Proposition 4.2.5, two-convex hull of open set is open.

4.3 Sets with smooth boundary

In this section we characterize the subsets in $\mathbb{E}^m$ with smooth boundary which form CAT[0] spaces.

4.3.1. Theorem. *Let K be a closed connected subset in $\mathbb{E}^m$ equipped with the induced length metric. Assume K is bounded by a smooth hypersurface. Then $K \in \mathrm{CAT}[0]$ if and only if K is two-convex.*

This theorem is a baby case of the main result in [2], it is discussed in the end of chapter.

Proof. Denote by S and by Ω the boundary and the interior of K. Since K is connected and S is smooth, Ω is also connected.

Denote by $k_1(p) \leqslant \ldots \leqslant k_{m-1}(p)$ the principle curvatures of S at $p \in S$ with respect to the normal vector pointing out of K. By Proposition 4.2.3, K is two-convex if and only if $k_2(p) \geqslant 0$ at any $p \in S$.

"Only-if" part. Assume K is not two-convex. Then by Proposition 4.2.3, there is a triangle $[xyz]$ in K which is null-homotopic in K, but the solid triangle $\Delta = \operatorname{Conv}\{x, y, z\}$ does not lie in K completely. Evidently Δ is not thin in K, that is K is not CAT[0].

"If" part. Since K is simply connected, by Globalization theorem 3.3.1, it is sufficient to show that $\operatorname{curv}_p K \leqslant 0$ at any point $p \in K$.

If $p \in \operatorname{Int} K$ then it admits a neighborhood isometric to a subset of $\mathbb{E}^m$; therefore $\operatorname{curv}_p K \leqslant 0$.

Fix $p \in S$. Assume that $k_2(p) > 0$. Fix sufficiently small $\varepsilon > 0$ and set $K' = K \cap \overline{\mathrm{B}}[p, \varepsilon]$. Let us show that

❶ *K' is a* CAT[0] *space.*

Consider the coordinate system with the origin at p and the principle directions and $\nu(p)$ as the coordinate directions. For small $\varepsilon > 0$, the set K' can be described as a subgraph

$$K' = \left\{ (x_1, \dots, x_m) \in \overline{\mathrm{B}}[p, \varepsilon] \;\middle|\; x_m \leqslant f(x_1, \dots, x_{m-1} \right\}.$$

Moreover, since ε is small and $k_2(p) > 0$, for any fixed $s \in [-\varepsilon, \varepsilon]$ the function

$$(x_2, \dots, x_{m-1}) \mapsto f(s, x_2, \dots, x_{m-1})$$

is concave for $|x_i| < 2{\cdot}\varepsilon$ for each i

Fix a negative real value $\lambda < k_1(p)$. Given $s \in (-\varepsilon, \varepsilon)$, consider the set

$$V_s = \left\{ (x_1, \dots, x_m) \in K' \;\middle|\; x_m \leqslant f(x_1, \dots, x_{m-1}) + \lambda{\cdot}(x_1 - s)^2 \right\}.$$

Note that the function

$$(x_1, \dots, x_{m-1}) \mapsto f(x_1, \dots, x_{m-1}) + \lambda{\cdot}(x_1 - s)^2$$

is concave near the origin. Since ε is small, we can assume that V_s forms a convex set in $\mathbb{E}^m$ for any s.

Further note that

$$K' = \bigcup_{s \in [-\varepsilon, \varepsilon]} V_s$$

and

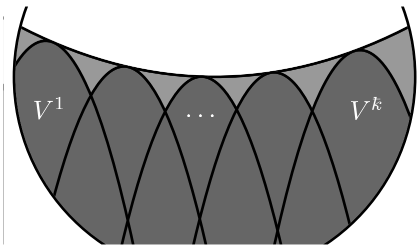
❷ $$V_b \supset V_a \cap V_c$$

for any choice real numbers $a < b < c$.

Given an array of values $s^1 < \dots < s^{\bar{k}}$ in $[-\varepsilon, \varepsilon]$, set $V^i = V_{s^i}$ and consider the unions

$$W^i = V^1 \cup \dots \cup V^i$$

equipped with the induced intrinsic metric.



Note that the array (s^n) can be chosen in such a way that $W^{\bar{k}}$ is arbitrary close to K'.

Therefore, in order to prove ❶, it is sufficient to show the following.

❸ *All W^i form* CAT[0] *spaces.*

This claim is proved by induction. Base: $W^1 = V^1$ is a CAT[0] space as a convex subset in $\mathbb{E}^m$.

Step: assume that $W^i \in \mathrm{CAT}[0]$. According to ❷

$$V^{i+1} \cap W^i = V^{i+1} \cap V^i.$$

Moreover, this set forms a convex set in $\mathbb{E}^m$ and therefore it is a convex set in W^i and in V^{i+1}. By Reshetnyak's gluing theorem, $W^{i+1} \in$ $\in \mathrm{CAT}[0]$. Hence the claim follows. △

Note that we proved the following

❹ *K is a* CAT[0] *space if K is* strongly two-convex*; that is $k_2(p) > 0$ at any point $p \in S$.*

It remains to show that $\mathrm{curv}_p K \leqslant 0$ in the case $k_2(p) = 0$.

Choose a coordinate system $(x_1, \dots, x_n)$ as above, so $(x_1, \dots, x_{m-1})$-coordinate hyperplane forms the tangent subspace to S at p.

Fix $\varepsilon > 0$ so that a neighborhood of p in S forms a graph

$$x_m = f(x_1, \dots, x_{m-1})$$

for a function f defined at the open ball B of radius ε centered at the origin in the $(x_1, \dots, x_{m-1})$-hyperplane. Fix a smooth positive strongly convex function $\varphi\colon B \to \mathbb{R}_+$ such that $\varphi(x) \to \infty$ as x approaches the boundary B. Note that for $\delta > 0$, the subgraph K_δ defined by the inequality

$$x_m \leqslant f(x_1, \dots, x_{m-1}) - \delta{\cdot}\varphi(x_1, \dots, x_{m-1})$$

is strongly two-convex. By ❹, K_δ is a CAT[0] space.

Finally as $\delta \to 0$, the closed ε-neighborhoods of p in K_δ converges to the closed ε-neighborhood of p in K. Therefore $\mathrm{curv}_p K \leqslant 0$. □

The following CBB-analog of the above theorem; its proof is simpler.

4.4 Open plane sets

In this section we will consider inheritance of upper curvature bound by subsets of Euclidean plane.

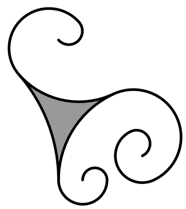
4.4.1. Theorem. *Let Ω be an open simply connected subset of $\mathbb{E}^2$. Equip Ω with induced length metric and denote by by K its completion. Show that K is a* CAT[0] *space.*

The assumption that the set Ω is not critical, instead one ca assume that the induced length metric takes finite values at all points of Ω. We sketch the proof from Bishop's note [27] and suggest to finish the details as an exercise.

Sketch of proof. It is sufficient to show that any triangle in K is thin, as defined in 2.2.1.

Note that K admits a length-preserving map to $\mathbb{E}^2$ which extends the embedding $\Omega \hookrightarrow \mathbb{E}^2$. Therefore the triangle $[xyz]$ can be mapped to the plane in a length-preserving way. Since Ω is simply connected, any open region, say Δ, surrounded by the image of $[xyz]$ completely lies in Ω.

Note that in each triangle $[xyz]$ in K the sides $[xy]$, $[yz]$ and $[zx]$ intersect each other along a geodesic starting at common vertex; possibly a one-point geodesic. In other words, every triangle in K look like one on the diagram.



Assuming contrary, there will be a diangle in K; that is two minimizing geodesics with common ends, but no other common points. The image of this diangle in the plane have to bound an open region of Ω and both sides of the region have to be concave, otherwise one could shorten the sides by pushing them into Ω. Evidently, there is no plane diangle with concave sides, a contradiction.

Note that it is enough to consider only injective triangle $[xyz]$; that is the sides $[xy]$, $[yz]$ and $[zx]$ intersect each other only at the common vertexes. If this is not the case, chopping the overlapping part of sides reduces to the injective case.

Again, the open region, say Δ, bounded by the image of $[xyz]$ has concave sides, otherwise one could shorten the sides by pushing them into Ω. It remains to solve the exercise below. $\square$

4.4.2. Exercise. *Assume Δ is a solid plane triangle with concave sides. Show that Δ equipped with induced intrinsic metric is thin.*

Here is a spherical analog of above lemma; which can be proved along the same lines. It will be used in the next section.

4.4.3. Exercise. *Let Θ be an open connected subset of the unit sphere $\mathbb{S}^2$ which does not contain a closed hemisphere. Equip Θ with induced length metric. Let $\tilde\Theta$ be a metric covering of Θ such that any closed curve in $\tilde\Theta$ of length smaller than $2{\cdot}\pi$ is contactable.*

Show that the completion of $\tilde\Theta$ is a $\mathrm{CAT}[1]$ *space.*

4.5 Open space sets

The material below is inspired by work of Samuel Shefel who seems to be very intrigued by survival of metric properties under affine transformation. To describe an instance of such phenomena, note that two-convexity survives under affine transformations of Euclidean space. Therefore, as a consequence of Theorem 4.3.1, the following holds.

4.5.1. Corollary. *Let K be closed connected subset of Euclidean space equipped with the induced length metric. Assume K is bounded by a smooth hypersurface and it forms* $\mathrm{CAT}[0]$ *space. Then any affine transformation of K also forms* $\mathrm{CAT}[0]$ *space.*

By Corollary 4.5.4, an analogous statement holds for sets bounded by Lipschitz surfaces in three-dimensional Euclidean space. In higher dimensions this is not longer true.

Here is the main theorem of the section.

4.5.2. Shefel's Theorem. *Let Ω be an open set $\mathbb{E}^3$. Equipp Ω with the induced length metric and denote by K the completion of universal metric cover of Ω. Then K is a* $\mathrm{CAT}[0]$ *space if and only if Ω is two-convex.*

The following exercise shows that analogous statement does not hold in higher dimensions.

4.5.3. Exercise. *Let Π_1, Π_2 be two planes in $\mathbb{E}^4$ intersecting at single point. Consider the double cover K of $\mathbb{E}^4$ branching along Π_1 and Π_2 and equip it with induced length metric.*

Show that K is a $\mathrm{CAT}[0]$ *space if and only if $\Pi_1 \perp \Pi_2$.*

Before coming to the proof of Shefel's theorem, let us formulate few corollaries. The following corollary look more like a generalization of Theorem 4.3.1 for three-dimensional Euclidean space.

4.5.4. Corollary. *Let K be a closed subset in $\mathbb{E}^3$ bounded by a Lipschitz hypersurface. Then K with the induced length metric is a* $\mathrm{CAT}[0]$ *space if and only if the interior of K is two-convex and simply connected.*

Proof. Set $\Omega = \operatorname{Int} K$. Since K is simply connected and bounded by a surface, Ω is also simply connected.

Apply Shefel's Theorem to Ω. Note that the completion of Ω equipped with the induced length metric is isometric to K with induced intrinsic metric. Hence the result follows. $\square$

The following corollary is the main statement in Shefel's original paper [156].

Let U be an open set in $\mathbb{R}^2$. Recall, that a continuous function $f\colon U\to\mathbb{R}$ is called *saddle* if for any linear function $\ell\colon\mathbb{R}^2\to\mathbb{R}$ the difference $f-\ell$ does not have local maxima and minima in U. Equivalently, open subgraph and epigraph of f

$$\left\{\,(x,y,z)\in\mathbb{E}^3\;\middle|\;z<f(x,y),\ (x,y)\in U\,\right\},$$
$$\left\{\,(x,y,z)\in\mathbb{E}^3\;\middle|\;z>f(x,y),\ (x,y)\in U\,\right\}$$

form a two-convex subsets in $\mathbb{E}^3$

4.5.5. Corollary. *Let $f\colon\mathbb{D}\to\mathbb{R}$ be a Lipschitz function which is saddle in the interior of the closed unit disc $\mathbb{D}$. Then the graph $z=f(x,y)$ in $\mathbb{E}^3$ equipped with induced length metric is a* CAT[0] *space.*

Proof. Since the function f is Lipschitz, its graph Γ with the induced length metric is bi-Lipschitz to the Euclidean metric on $\mathbb{D}$.

Consider the sequence of sets

$$K_n=\left\{\,(x,y,z)\in\mathbb{E}^3\;\middle|\;z\lesseqgtr f(x,y)\pm\tfrac{1}{n},\ (x,y)\in\mathbb{D}\,\right\}.$$

Note that each K_n is closed, simply connected and two-convex. Moreover the boundary of K_n forms a Lipschitz surface.

Equip K_n with induced length metric. By Corollary 4.5.4, K_n is a CAT[0] space. It remains to note that as $n\to\infty$ the sequence of space K_n converges to Γ in the sense of Gromov–Hausdorff. $\square$

Now we are back to the proof of Shefel's theorem 4.5.2.

Recall, that a subset P of $\mathbb{E}^m$ is called *polytope* if it can be presented as a union of finite number of simplices. Similarly, a *spherical polytope* is a union of finite number of simplices in $\mathbb{S}^m$.

Note any polytope admits a finite triangulation. Therefore any polytope equipped with induced intrinsic metric forms a Euclidean polyhedral space as defined in 3.4.1.

Let P be a polytope and Ω its interior, both considered with induced intrinsic metric. Typically the completion K of Ω is isometric to P. However in general we only have locally distance preserving map $K\to P$; it does not have to be onto and it may be not injective. An example can be guessed from the picture.

It is easy to see that K is a polyhedral spaces.

4.5.6. Lemma. *Shefel's Theorem 4.5.2 holds if the set Ω is formed by interior of a polytope.*

The statement might look obvious, but there is an underwater stone in the proof.

Proof. Denote by $\tilde{\Omega}$ the universal metric cover of Ω. Let $\tilde{K}$ and K be the completion of $\tilde{\Omega}$ and Ω correspondingly.

Note that K is a polyhedral space and the covering $\tilde{\Omega} \to \Omega$ extends to a covering map $\tilde{K} \to K$ which might be branching at some vertexes.

Fix a point $\tilde{p} \in \tilde{K} \backslash \tilde{\Omega}$; denote by p the image of $\tilde{p}$ in K. By Globalization theorem (3.3.1) it is sufficient to show that

❶ *A small neighborhood of $\tilde{p}$ in $\tilde{K}$ forms a* $\mathrm{CAT}[0]$ *space.*

Recall that $\Sigma_{\tilde{p}} = \Sigma_{\tilde{p}}\tilde{K}$ denotes the space of directions at $\tilde{p}$. Note that a small neighborhood of $\tilde{p}$ in $\tilde{K}$ isometric to an open set in the cone over $\Sigma_{\tilde{p}}\tilde{K}$. By Exercise 2.1.2, ❶ follows if

❷ $\Sigma_{\tilde{p}}$ *is a* $\mathrm{CAT}[1]$ *space.*

By rescaling, we can assume every face of K which does not contain p lies on the distance at least 2 from p. Denote by $\mathbb{S}^2$ the unit sphere centered at p, set $\Theta = \mathbb{S}^2 \cap \Omega$. Note that $\Sigma_p K$ is isomeric to the completion of Θ and $\Sigma_{\tilde{p}}\tilde{K}$ is a completion of a regular metric covering $\tilde{\Theta}$ of Θ which is induced by universal metric cover $\tilde{\Omega} \to \Omega$.

By Exercise 4.4.3, it remains to show that the following.

❸ *Any closed curve in $\tilde{\Theta}$ of length less than $2\cdot\pi$ is contactable.*

Consider a closed curve $\tilde{\gamma}$ of length $< 2\cdot\pi$ in $\tilde{\Theta}$. Its projection γ in $\Theta \subset \mathbb{S}^2$ has the same length. Therefore by Hemisphere lemma 1.8.1, γ lies in an open hemisphere. Then for a plane Π close to p, the central projection γ' of γ to Π is defined and lies in Ω. From two-convexity of Ω, the curve γ' is contractible in $\Pi \cap \Omega$. It follows that γ is contractible in Θ and therefore $\tilde{\gamma}$ is contractible in $\tilde{\Theta}$. □

The following proposition describes a construction which was essentially given by Shefel in [156]. It produce $\mathrm{Conv}_2\,\Omega$ for an open set $\Omega \subset \mathbb{E}^3$.

4.5.7. Proposition. *Let $\Pi_1, \Pi_2 \dots$ be an everywhere dense sequence of planes in $\mathbb{E}^3$. Given an open set Ω consider the recucevly defined sequence of open sets $\Omega = \Omega_0 \subset \Omega_1 \subset \dots$ such that Ω_n is the union of Ω_{n-1} and all the bounded compontnets of $\mathbb{E}^3 \backslash (\Pi_n \cup \Omega_{n-1})$. Then*

$$\mathrm{Conv}_2\,\Omega = \bigcup_n \Omega_n.$$

Proof. Set

$$\Omega' = \bigcup_n \Omega_n.$$

Note that Ω' is a union of open set, in particular it is open.

The inclusion $\mathrm{Conv}_2\,\Omega \supset \Omega'$ is evident.

It remains to show that Ω' is two-convex. Assume contrary; that is, there is a plane Π and a closed curve $\gamma\colon \mathbb{S}^1 \to \Pi\cap\Omega'$ which is null-homotopic in Ω', but not null-homotopic in $\Pi\cap\Omega'$.

By approximation we can assume that $\Pi = \Pi_n$ for a large enough n and that γ lies in Ω_{n-1}. The latter contradicts the n-th step in the construction. □

4.5.8. Key lemma. *The two-convex hull of the interior of polytope in $\mathbb{E}^3$ is an interior of a polytope.*

Proof. Fix a polytope P in $\mathbb{E}^3$. Set $\Omega = \mathrm{Int}\,P$, we can assume that Ω is dense in P Denote by $F_1,\dots,F_m$ the facets of P.

Set $\Omega' = \mathrm{Conv}_2\,\Omega$ and let P' be the closure of Ω'. Further, for each i, set $F_i' = F_i\backslash\Omega'$. In other words, F_i' is the subset of facet F_i which remains on the boundary of K'.

From the construction of two-convex hull (4.5.7)

❹ *F_i' is convex subset of F_i.*

Further, since Ω' is two-convex, we get the following.

❺ *Each connected component of the complement $F_i\backslash F_i'$ is convex.*

Indeed, assume a connected component A of $F_i\backslash F_i'$ fails to be convex. Then there is a suppoiting line ℓ to A touching A at a single point in the interior of F_i. Then one could rotate the plane of F_i slightly arounf ℓ and move it parallel to cut a hat from the complement of Ω. The latter means that Ω is not two-convex, a contradiction. △

From ❹ and ❺, we get that

❻ *F_i' is a convex polygon for each i.*

Consider the complement $\mathbb{E}^3\backslash\Omega$ equipped with the length metric. By construction of two-convex hull (4.5.7), the complement $L = \mathbb{E}^3\backslash(\Omega'\cup K)$ is locally convex; that is, any points of L admits a convex neighborhood.

Summarizing (1) Ω' is a 2-convex open set, (2) the boundary $\partial\Omega'$ contains a finite number of polygons F_i' and the remaing part is locally concave. It remains to show that (1) and (2) imply that $\partial\Omega'$ is piecewise linear.

4.5.9. Exercise. *Prove of the last statement.*

□

Proof of 4.5.2. Note that it is sufficient to show that (2+2)-comparison holds for any 4 points $x^1, x^2, x^3, x^4 \in \Omega$.

Fix $\varepsilon > 0$. Choose six broken lines connecting all the pairs of points x^1, x^2, x^3, x^4 such that length of each at most ε bigger than the distance between its ends in the length metric on Ω. Choose a polytope P in Ω such that the interior $\operatorname{Int} P$ is simply connected and it contains all these six broken lines.

Denote by Ω' the two-convex hull of the interior of P. According to Key Lemma (4.5.8) Ω' is an interior of a polytope.

Equip Ω' with the induced length metric. Consider the universal metric cover $\tilde{\Omega}'$ of Ω'. (The covering $\tilde{\Omega}' \to \Omega'$ might be nontrivial; despite that $\operatorname{Int} P$ is simply connected, its two-convex hull Ω' might be not simply connected.) Denote by $\tilde{K}'$ the completion of $\tilde{\Omega}'$.

By Lemma 4.5.6, $\tilde{K}' \in \mathrm{CAT}[0]$.

Since $\operatorname{Int} P$ is simply connected, the embedding $\operatorname{Int} P \hookrightarrow \Omega'$ admits a lifting $\iota\colon \operatorname{Int} P \hookrightarrow \tilde{K}'$. By construction, ι almost preserves the distances between the points x^1, x^2, x^3, x^4; namely

$$|\iota(x^i)\ \iota(x^j)|_L \gtrless |x^i\, x^j|_{\operatorname{Int} P} \pm \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and (2+2)-comparison holds in $\tilde{K}'$, we get that (2+2)-comparison holds in Ω for x^1, x^2, x^3, x^4.

The statement follows since the quadruple $x^1, x^2, x^3, x^4 \in \Omega$ is arbitrary. □

4.5.10. Exercise. *Assume $K \subset \mathbb{E}^m$ is a closed set bounded by a Lipschitz hypersurface. Equip K with the induced length metric. Show that if $K \in \mathrm{CAT}[0]$ then K is two-convex.*

The following exercise is analogous to Exercise 4.5.3. It provides a counterexample to the analog of Corollary 4.5.4 in higher dimensions.

4.5.11. Exercise. *Let $K = W \cap W'$, where*

$$W = \left\{\, (x, y, z, t) \in \mathbb{E}^4 \;\middle|\; z \geqslant -x^2 - y^2 \,\right\}$$

and $W' = \iota(W)$ for some motion $\iota\colon \mathbb{E}^4 \to \mathbb{E}^4$.

Show that K is always two-convex and one can choose ι so that K with the induced length metric is not $\mathrm{CAT}[0]$.

4.5.12. Exercise. *Let $\mathcal{U}$ be a $\mathrm{CAT}[0]$ proper length space. Assume $\tilde{\mathcal{U}} \to \mathcal{U}$ is a metric covering branching along a geodesic. Show that $\tilde{\mathcal{U}}$ is a $\mathrm{CAT}[0]$ space.*

4.6 Comments and open problems

Sets with smooth boundary. In [2] Alexander, Bishop and Berg gave the exact upper bound on Alexandrov's curvature for the Riemannian manifolds with boundary. Namely they show the following.

4.6.1. Theorem. *Let M be a Riemannian manifold with boundary ∂M. A direction tangent to the boundary will be called concave if there is a short geodesic in this direction which leaves the boundary and goes into interior of M. A sectional direction will be called concave if all the directions in it are concave.*

Then $\operatorname{curv} M \leqslant \kappa$*, where κ is the infimum of sectional curvatures in M and the curvatures of ∂M in the concave directions.*

4.6.2. Corollary. *Let M be a Riemannian manifold with boundary ∂M. Assume all the sectional curvatures of M and ∂M are bounded from above by κ. Then* $\operatorname{curv} M \leqslant \kappa$*.*

A surface S in $\mathbb{E}^3$ is called saddle if for any linear function $f\colon \mathcal{X} \to \mathbb{R}$, the restriction $f|S$ has no local maxima or maxima in the interior of S.

Nonsmooth sets.

4.6.3. Shefel's conjecture. *Any saddle hypersurface in $\mathbb{R}^3$ equipped with the length-metric has curvature $\leqslant 0$ at any point.*

From Corollary 4.5.4, it follows that if there is a counterexample then arbitrary neighborhood of some point of the surface can not be presented as graph in any coordinate system.

Further more, if a counterexample exists, then it forms a piece of boundary of a two-convex set Ω with the following property. There is a point $p \in \partial\Omega$ and $\varepsilon > 0$ such that for any $\delta > 0$ we have

$$\mathrm{B}(p, \varepsilon) \subset \operatorname{Conv}_2[\Omega \cup \mathrm{B}(p, \delta)].$$

None of such examples of Ω are known so far.

A surface S in a metric space. We say that S is saddle if for any convex function $f\colon \mathcal{X} \to \mathbb{R}$, the restriction $f|S$ has no local maxima in the interior of S.

4.6.4. Generalized Shefel's conjecture. *Any 2-dimensional saddle surface in a* $\mathrm{CAT}[\kappa]$ *equipped with the length-metric has curvature $\leqslant \kappa$ at any point.*

A subset K is called strongly two-convex if any null-homotopic circle $\gamma\colon \mathbb{S}^1 \to K$ is also null-homotopic in $K \cap \operatorname{Conv}[\gamma(\mathbb{S}^1)]$.

4.6.5. Question. *Is it true that any closed strongly two-convex set with dense interior bounded by a Lipschitz hypersurface in* $\mathbb{E}^m$ *forms a* CAT[0] *space?*

Metric minimizing surfaces. In [138] was shown that discs in CAT[0] spaces which do not admit a length-decreasing deformations with fixed boundary form CAT[0] spaces.

Any metric minimizing disc forms a saddle surface as defined above. There are saddle surfaces which are not metric minimizing globally and it is expected that there saddle surfaces which are not metric minimizing in arbitrary neighborhood of a fixed point. Constructing such examples could shed light on the Shefel's question.

Semisolutions

Exercise 0.0.1. Let $\mathcal{X}$ be a 4-point metric space.

Fix a tetrahedron $\triangle$ in $\mathbb{R}^3$. The verices of tetrahedron, say x_0, x_1, x_2, x_3 can be identified with the points of $\mathcal{X}$.

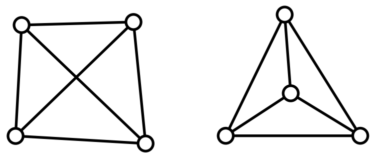
Note that there is unique qadratic form W on $\mathbb{R}^3$ such that

$$W(x_i - x_j) = |x_i x_j|^2_{\mathcal{X}}$$

for all i and j; here $|x_i x_j|_{\mathcal{X}}$ denotes the distance beween x_i and x_j in $\mathcal{X}$.

By triangle inequality $W(v) \geqslant 0$ for any vector v parallel to one of the faces of $\triangle$.

Note that $\mathcal{X}$ is isometric to a 4-point subset in the Euclidean space if and ony if $W(v) \geqslant 0$ for any vector v in $\mathbb{R}^3$.



Therefore if $\mathcal{X}$ is not of type $\mathcal{E}_4$ then $W(v) < 0$ for some vector v. From above, the vector v must be transveral to each of 4 faces of $\triangle$. Therefore if we project $\triangle$ along v to a plane transveral to v we see one of the folwing two pictures.

It is easy to see that the combinatorics of the picture does not depend on the choice of v. Hence $\mathcal{M}_4 \backslash \mathcal{E}_4$ not connected.

It remains to show that if the combinatorics of the pictures for two spaces is the same then one can continuousely deform one space into the other. This can be easily done deforming W and apply permutation of x_0, x_1, x_2, x_3 if necessay.

Exercise 0.0.2. The simplest proof we know require proof tangent cones for Alexandrov spaces.

Exercise 1.2.1.

Exercise 1.4.4. Given a metric graph Γ define $P_k(\Gamma)$ as follows. Let Γ^b be the barycentric subdivision of Γ with the natural metric. For any two adjacent vertices $p, q \in \Gamma^b$ substitute the edge $[pq]$ by a countable collection of intervals $\{I_i\}_{i\geqslant 1}$ of length $|pq| + \frac{|pq|}{2^k i}$ where one end of each I_i is glued to p and the other to q. Note that the resulting space $P_k(\Gamma)$ is again a metric graph with an inner metric.

Let $\mathcal{X}_0 = [0,1]$ and define $\mathcal{X}_k$ for $k \geqslant 1$ inductively as $\mathcal{X}_k = P_k(\mathcal{X}_{k-1})$.

Let $\mathcal{Y}_k$ be the set of vertices of $\mathcal{X}_k$ with the induced metric. By construction the inclusion $\mathcal{Y}_k \subset \mathcal{Y}_{k+1}$ is distance preserving.

Let $\mathcal{Y}_\infty = \cup_{k\geqslant 1}\mathcal{Y}_k$ with the obvious metric and let $\mathcal{Y} = \bar{\mathcal{Y}}_\infty$ be its metric completion. Then $\mathcal{Y}$ is a length space since it satisfies the almost midpoint property. But it is not hard to see that no two distinct points in $\mathcal{Y}$ can be connected by a shortest geodesic. □

AN OTHER SOLUTION from here http://mathoverflow.net/q/15720 if we want it, I can ask fedya.???

Well, the unit ball in c_0 is almost what you want (there is no unique shortest curve between points). All we need now is to enhance "bypasses" and to give disadvantage to "straight lines". This can easily be done by taking the distance element to be

$$(2 + \sum_n 2^{-n} x_n)^{-1} \|dx\|_\infty,$$

which is never less than the usual distance element in c_0 and never greater than 3 times it in the unit ball. Now, if we have any continuous finite length curve $x(t)$ from y to z parametrized by the arclength, we can easily shorten it by replacing the m-th position by the maximum of the actual value of $x_m(t)$ and

$$y_m + t(z_m - y_m)/d + \frac{1}{2}\min(t, d-t),$$

where d is the length of $x(t)$, which will work if m is large enough since $\max_t |x_m(t)| \to 0$ as $m \to \infty$ and both functions change slower than the distance along the original curve.

Exercise 1.4.9. The following example is taken from the book of Bridson and Haefliger [29].

Consider the following subset of $\mathbb{R}^2$:

$$\mathcal{X} = \big((0,1] \times \{0,1\}\big) \cup \big(\bigcup_{n\geqslant 1} \{1/n\} \times [0,1]\big)$$

Consider the induced inner metric on $\mathcal{X}$. It's obviously locally compact and geodesic. However, it's immediate to check that its metric completion

$$\bar{\mathcal{X}} = \big([0,1] \times \{0,1\}\big) \cup \big(\bigcup_{n\geqslant 1} \{1/n\} \times [0,1]\big)$$

is neither. $\square$

Exercise 1.6.3. Assume contrary; that is

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] < \pi$$

By triangle inequality for angles (1.6.2) we have

$$\measuredangle[p\,{}^x_y] < \pi$$

The latter cotradicts the triangle inequality for the triangle $[\bar{x}p\bar{y}]$, where the points $\bar{x} \in]px]$ and $\bar{y} \in]py]$ are sufficiently close to p.

Exercise 1.7.1. Fix sufficiently small $\varepsilon > 0$.

Consider a sequence of directions ξ_n of geodesics $[pq_n]$. Since the geodesics are extandable, we can assume that the distances $|pq_n|_{\mathcal{U}}$ are equal.

Since $\mathcal{U}$ is proper, the sequence q_n has partial limit, say q. It remains to note that the direction ξ of $[pq]$ is the limit of directions ξ_n, assuming the latter is defined.

Exercise 1.8.2. Let α be a closed curve in $\mathbb{S}^2$ of length $\leqslant 2{\cdot}\pi$. We wish to prove that it's contained in a hemisphere in $\mathbb{S}^2$. By approximation it's clearly enough to prove this for smooth curves of length $< 2{\cdot}\pi$ with transverse self-intersections.

Given $v \in \mathbb{S}^2$, denote by $v^\perp$ the equator in $\mathbb{S}^2$ with the pole at v. Further, $\#X$ will denote the number of points in the set X.

Obviously, if $\#(\alpha \cap v^\perp) = 0$ then α is contained in one of the hemispheres determined by $v^\perp$. Note that $\#(\alpha\cap v^\perp)$ is even for almost all v.

Therefore, if α does not lie in a hemisphere then $\#(\alpha \cap v^\perp) \geqslant 2$ for almost all $v \in \mathbb{S}^2$.

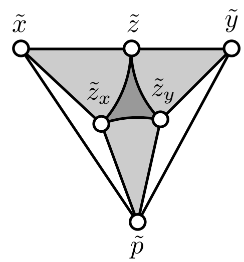
By Crofton's formula we have that

$$\begin{aligned}\operatorname{length}(\alpha) &= \frac{1}{4}\cdot\int\limits_{\mathbb{S}^2} \#(\alpha \cap v^\perp)\cdot \boldsymbol{d}_v \operatorname{area} \geqslant \\ &\geqslant 2{\cdot}\pi.\end{aligned}$$

Exercise 2.1.2.

Exercise 2.1.3.

Exercise 2.2.2.



Exercise 2.2.7. By Alexandrov's lemma (1.5.1), there are nonoverlapping triangles $[\tilde{p}\tilde{x}\tilde{z}_y] \overset{iso}{=\!=} [\dot{p}\dot{x}\dot{z}]$ and $[\tilde{p}\tilde{y}\tilde{z}_x] \overset{iso}{=\!=} [\dot{p}\dot{y}\dot{z}]$ inside triangle $[\tilde{p}\tilde{x}\tilde{y}]$.

Connect points in each pair $(\tilde{z}, \tilde{z}_x)$, $(\tilde{z}_x, \tilde{z}_y)$ and $(\tilde{z}_y, \tilde{z})$ with arcs of circles centered at $\tilde{y}$, $\tilde{p}$, and $\tilde{x}$ respectively. Define F as follows.

- ◇ Map $\mathrm{Conv}[\tilde{p}\tilde{x}\tilde{z}_y]$ isometrically onto $\mathrm{Conv}[\dot{p}\dot{x}\dot{y}]$; similarly map $\mathrm{Conv}[\tilde{p}\tilde{y}\tilde{z}_x]$ onto $\mathrm{Conv}[\dot{p}\dot{y}\dot{z}]$.
- ◇ If x is in one of the three circular sectors, say at distance r from center of the circle, let $F(x)$ be the point on the corresponding segment $[pz]$, $[xz]$ or $[yz]$ whose distance from the lefthand endpoint of the segment is r.
- ◇ Finally, if x lies in the remaining curvilinear triangle $\tilde{z}\tilde{z}_x\tilde{z}_y$, set $F(x) = z$.

By construction, F satisfies the conditions.

Exercise 2.2.8. It is sufficient to show that

$$|pz| \leqslant \max\{|px|, |py|\}$$

for any p and any $z \in [xy]$.

The statement follows since the triangle $[pxy]$ is thin and the above condition holds in the Euclidean plane.

In $\mathrm{CAT}[1]$ case, the proof is the same, but we need to assume in addition that $\max\{|px|, |py|\} \leqslant \frac{\pi}{2}$

Exercise 2.2.10. Since $\mathcal{U}$ is proper, the set $K \cap \overline{\mathrm{B}}[p, R]$ is compact for any $R < \infty$. Hence the existance of p^* follows.

Assume p^* is not uniquely defined; that is, two distinct points in K, say x and y, minimize the distance from p. Sicne K is convex the midpoint z of $[xy]$ lies in K.

Note that $|pz| < |px| = |py|$, a contradiction.

It remains to show that the map $p \mapsto p^*$ is short; that is

❶ $$|pq| \geqslant |p^* q^*|.$$

Note that

$$\measuredangle[p^*\,{}^{p}_{q^*}] \geqslant \tfrac{\pi}{2} \quad \text{and} \quad \measuredangle[q^*\,{}^{p}_{p^*}] \geqslant \tfrac{\pi}{2},$$

if the lefthand sides are defined.

Construct the model triangles $[\tilde{p}\,\tilde{p}^*\,\tilde{q}^*]$ and $[\tilde{p}\tilde{q}\tilde{q}^*]$ of $[p\,p^*\,q^*]$ and $[pqq^*]$ so that the points $\tilde{p}^*$ and $\tilde{q}$ lie on the opposite sides from $[\tilde{p}\tilde{q}^*]$.

By comparison and triangle inequality for angles, we get 1.6.2

$$\measuredangle[\tilde{p}^*\,{}^{\tilde{p}}_{\tilde{q}^*}] \geqslant \measuredangle[p^*\,{}^{p}_{q^*}] \geqslant \tfrac{\pi}{2} \quad \text{and} \quad \measuredangle[\tilde{q}^*\,{}^{\tilde{q}}_{\tilde{p}^*}] \geqslant \measuredangle[q^*\,{}^{q}_{p^*}] \geqslant \tfrac{\pi}{2}$$

assuming that the left hand sides are defined. Hence

$$|\tilde{p}\tilde{q}| \geqslant |\tilde{p}^*\,\tilde{q}^*|.$$

The latter is equivalent to ❶.

Exercise 2.3.3. Consider the angle A in the plane of measure $\pi - \alpha$. Note that A is $\mathrm{CAT}[0]$. Therefore by Reshetnyak gluing theorem 2.3.1, by gluing a side of A to γ_1 in $\mathcal{U}$ we obtain a $\mathrm{CAT}[0]$ space, say $\mathcal{U}'$.

Note that γ_2 together with the other side of A forms a both sides infinite geodesic, say γ in $\mathcal{U}'$. In particular, γ is a convex set isometric to $\mathbb{R}$.

Glue a half-plane along its boundary to γ. By Reshetnyak gluing theorem 2.3.1 the obtained space is CAT[0].

It remains to note that this space can be obtained directly by gluing $\mathcal{U}$ to with Q along γ_1 and γ_2.

Exercise 2.4.7. By approximation, it is sufficient to consider the case when A and B have smooth boundary. In the latter case the choice of $\dot{A}$ and $\dot{B}$ is unique; and a variation argument shows that it satisfies the condition.

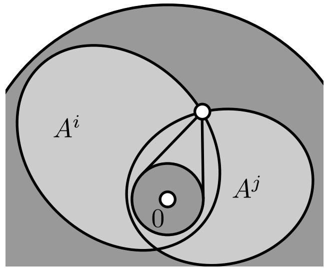
Exercise 2.5.1. Fix $r > 0$ and $R < \infty$ so that

$$\mathrm{B}(0, r) \subset A^i \subset \mathrm{B}(0, R)$$

for each wall A^i.

It remains to note that all the intersections of walls have ε-wide for

$$\varepsilon = 2\cdot \arcsin \tfrac{r}{R}.$$



Exercise 2.5.2. Note that there is $R < \infty$ such that if X is an intersection of arbitrary number of walls and then there is an isometry of X which moves any point $p \in X$ to a point in $\mathrm{B}(0, R)$.

The rest of the proof goes along the same lines as Exercise 2.5.1.

Exercise 3.2.5. Note that the existance of null-homotopy is equivalent to the following. There are two one-parameter family of paths α_τ and β_τ, $\tau \in [0, 1]$ such that

- ⋄ length α_τ, length $\beta_\tau < \pi$ for any τ.
- ⋄ $\alpha_\tau(0) = \beta_\tau(0)$ and $\alpha_\tau(1) = \beta_\tau(1)$ for any τ.
- ⋄ $\alpha_0(t) = \beta_0(t)$ for any t.
- ⋄ $\alpha_1(t) = \alpha(t)$ and $\beta_1(t) = \beta(t)$ for any t.

By Corollary 3.2.3, the construction in Corollary 3.2.4 produces the same result for α_τ and β_τ. Hence the result follows.

“If” part of Theorem 3.4.2.

Exercise 3.4.4.

Exercise 3.4.5.

Exercise 3.5.2. Checking the flag condition is straightforward once you know the following description of the barycentric subdevision.

Each vertexes v of the barycentric subdevision corresponds to a simplex $\triangle_v$ of the original triangulation. A set vertices form a simplex in the subdevision, if it can be ordered, say as $v_1, \dots, v_{k}$, such that the correspoding simplexes form a nested sequence

$$\triangle_{v_1} \subset \dots \subset \triangle_{v_k}.$$

Exercise 3.5.6. The proof goes along the same lines as the proof of Flag condition 3.5.5. The only difference that geodesic spends time at least π the star of each vertex.

Exercise 3.5.7. Start with a copy of $\mathcal{U}$ for each vertex of Γ and glue them recuresevely using the following operation and stages.

At the stage n glue only spaces along A^n.

In one step of the stage, choose two connected components of partially glued copies and glue them along the corresponding A^n's. Apply Reshetnyak gluing theorem to show that each connected component after the stage is CAT[0] assuming that all connected component were CAT[0] before the step.

Exercise 3.5.8. The space $\mathcal{T}_n$ has natural cone structure with the vertex formed by completely degenerate tree — all its edges have zero length. Note that the space Σ, over which the cone is taken comes naturally with triangulation with all-right spherical simplicies.

It remains to check that the complex is flag and apply Exercise 2.1.2.

Exercise 3.6.2.

Exercise 3.7.3. Solution goes along the same lines as the proof of Lemma 3.7.2. The only difference, the set G formed by squares with the vertexes at the midpoints of some edges of the cubical analog.

Exercise 3.8.2. In the proof we apply the following lemma from [50]; it follows from the disjoint discs property.

4.6.6. Lemma. *Let $\mathcal{S}$ be a simplicial complex which is an m-dimensional homology manifold for some $m \geqslant 5$. Assume all the vertices of $\mathcal{S}$ have simply connected links. Then $\mathcal{S}$ is a topological manifold.*

It is sufficient to construct a simplicial complex $\mathcal{S}$ such that

- ⋄ $\mathcal{S}$ is a closed $(m-1)$-dimensional homology manifold;
- ⋄ $\pi_1(\mathcal{S}\backslash\{v\}) \neq 0$ for some vertex v in $\mathcal{S}$;
- ⋄ $\mathcal{S} \sim \mathbb{S}^{m-1}$; that is, $\mathcal{S}$ is homotopy equivalent to $\mathbb{S}^{m-1}$.

Indeed, assume such $\mathcal{S}$ is constructed. Then the suspension $\mathcal{R} = \operatorname{Susp}\mathcal{S}$ is an m-dimensional homology manifold with a natural triangulation coming from $\mathcal{S}$. By Lemma 4.6.6, $\mathcal{R}$ is a topological manifold. According to generalized Poincaré conjecture, $\mathcal{R} \simeq \mathbb{S}^m$; that is $\mathcal{R}$ is homeomorphic to $\mathbb{S}^m$. Since $\operatorname{Cone}\mathcal{S} \simeq \mathcal{R}\backslash\{s\}$ where s denotes a south pole of the suspension and $\mathbb{E}^m \simeq \mathbb{S}^m\backslash\{p\}$ for any point $p \in \mathbb{S}^m$ we get

$$\operatorname{Cone}\mathcal{S} \simeq \mathbb{E}^m.$$

Let us construct $\mathcal{S}$. Fix an $(m-2)$-dimensional homology sphere Σ with a triangulation such that $\pi_1\Sigma \neq 0$. According to [84] an example of that type exists for any $m \geqslant 5$.

Remove from Σ one $(m-2)$-simplex. Denote the obtained complex by Σ'. Since $m \geqslant 5$, we have $\pi_1\Sigma = \pi_1\Sigma'$.

Consider the product $\Sigma' \times [0,1]$. Attach to it the cone over its boundary $\partial(\Sigma' \times [0,1])$. Denote by $\mathcal{S}$ the obtained simplicial complex and by v the tip of the attached cone.

Note that $\mathcal{S}$ is homotopy equivalent to the spherical suspension over Σ which is a simply connected homology sphere and hence is homotopy equivalent to $\mathbb{S}^{m-1}$. Hence $\mathcal{S} \sim \mathbb{S}^{m-1}$.

The complement $\mathcal{S}\backslash\{v\}$ is homotopy equivalent to Σ'. Therefore

$$\pi_1(\mathcal{S}\backslash\{v\}) = \pi_1\Sigma' = \pi_1\Sigma \neq 0.$$

That is, $\mathcal{S}$ satisfies the conditions above.

Exercise 4.4.2. By approximation, it is sufficient to consider the case of poygonalal sides.

The latter case can be done by induction on number of sides. The base case of triangle is evident.

To prove the step, apply Alexndrov's lemma (1.5.1) together with the costruction in Exercise 2.2.7.

Exercise 4.4.3.

Exercise 4.5.3. The space K forms a cone over branched covering Σ of $\mathbb{S}^3$ infinitely branching along two big circles.

If the planes are not orthogonal then the minimal distance between the circles is less than $\frac{\pi}{2}$. Assume that this distance is realized by a geodesic $[\xi\zeta]$. broken line made by four liftings of $[\xi\zeta]$ forms a closed geodesic in Σ. By Corollary 3.4.3, Σ is not CAT[1]. Therefore by Exercise 2.1.2, K is not CAT[0].

If the planes are orthogonal then the corresponding big circles in $\mathbb{S}^3$ are formes by subcomplexes of a flag triangulation of $\mathbb{S}^3$ with all-right simplicies. The branching cover is also flag. It remains to apply the flag condition 3.5.5.

Comments. In [40], Charney and Davis gave a complete answer to the analogous question for three planes. In particular they show that if a covering space of $\mathbb{E}^4$ branching at three planes through one origin is CAT[0] then these all are complex planes for some complex structure on $\mathbb{E}^4$.

Exercise 4.5.9. Consider the surface S formed by closure of the remaining part of the boundary. Note that the boundary ∂S of S is formed by some number of broken lines.

Assume S is not a piecewise linear. Show that there is a line segment $[pq]$ in $\mathbb{E}^3$ tangent to S at p which has no common points with S except p.

Since S is locally concave, there is a local inner supporting plane Π at p which contains the segment $[pq]$.

Note that $\Pi \cap S$ contains a segment $[xy]$ passing through p with the ends at ∂S.

Show that there is a line segmet of ∂S starting at x or y which lies in Π.

From the latter statement and local convexity of S, all points of $[pq]$ sufficiently close to p lie in S, a contradiction.

Exercise 4.5.10. Show that if K is not two-convex then there is a plane triangle $\triangle$ which sides lying completely in K, but its interior contains the points of the complement $\mathbb{E}^m \backslash K$.

It remains to note that $\triangle$ is not thin in K.

Exercise 4.5.11. Clearly the set W is two-convex. Therefore K is two-convex as an interesection of two convex sets.

The set K on big scale looks like the complement of the half-spaces of two 3-dimnsional spaces in $\mathbb{E}^4$. More precisely the limit, say K' of $\frac{1}{n}{\cdot}K$ is the completion of the complement of the half-spaces of two 3-dimensional spaces in $\mathbb{E}^4$ equipped with intrinsic metric.

In particular K' is a cone over space Σ which can be obtained as the completion of sphere $\mathbb{S}^3$ with removed two 2-dimensional hemispheres, say H_1 and H_2. The intersection of these hemispheres is tipically formed by a geodesic segment, say $[\xi\zeta]$.

Consider the hemispheres H_1 and H_2 so that $[\xi\zeta]$ is orthogonal to the boundary spheres and its length less than $\frac{\pi}{2}$. Then the projection of a closed geodesic in Σ to $\mathbb{S}^3$ is formed by a joint of 4 copies of $[\xi\zeta]$. In particular there is a closed geodesic in Σ shorter than $2{\cdot}\pi$. Hence Σ is not CAT[1] and therefore K' and consequently K are not CAT[0].

Exercise 4.5.12. Consider a ε-neighborhood A of the geodesic. Note that A_ε forms a convex set. By Reshetnyak gluing theorem the doubling $\mathcal{W}_\varepsilon$ of $\mathcal{U}$ in A_ε is a CAT[0] space.

Consider the other space $\mathcal{W}'_\varepsilon$ obtained by double covering of $\mathcal{U}\backslash A_\varepsilon$ and gluing back A_ε.

Note that $\mathcal{W}'_\varepsilon$ is locally isometric to $\mathcal{W}_\varepsilon$. That is for any point $p' \in \mathcal{W}'_\varepsilon$ there is a point $p \in \mathcal{W}_\varepsilon$ such that δ-neighborhood of p is isometric to δ-neighborhood of p for all small $\delta > 0$.

Further note that $\mathcal{W}'_\varepsilon$ is simply connected. By globalization theorem, $\mathcal{W}'_\varepsilon$ s CAT[0].

It remains to note that $\tilde{\mathcal{U}}$ can be obtained as a limit of $\mathcal{W}'_\varepsilon$ as $\varepsilon \to 0$ and apply Proposition 2.1.1.

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