A journey into Alexandrov geometry: curvature bounded below

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Contents

Preface

This book is similar to our "Invitation to Alexandrov geometry" written jointly with Stephanie Alexander [5]. We try to demonstrate the beauty and power of Alexandrov geometry by reaching interesting applications and theorems with minimal preparation. This time we do spaces with curvature bounded below in the sense of Alexandrov.

This subject is more technical; it takes more preparation, and we had to jump over several proofs. Namely, we do not prove the existence part in generalized Picard's theorem (5.3) and Perelman's theorem about conic neighborhoods (9.1); up to the last lecture, the rest is nearly rigorous. Several times, proofs of important statements are left as exercises, but all of them are solved at the end of the book; in all these cases, the statement is more important than its proof.

In Lecture 1, we discuss necessary preliminaries and fix notations.

Lecture 2 introduces the main object of our study — spaces with curvature bounded below in the sense of Alexandrov.

In Lecture 3, we formulate and prove the globalization theorem that local Alexandrov condition implies global. To simplify the presentation we consider only the compact case. This case is leading, it gives the main ideas of the proof but is less technical.

In Lecture 4, we develop calculus — tangent space and space of directions, differential, and gradient.

Lecture 5 introduces gradient flow, which will be further used as the main technical tool.

Lecture 6 proves the line splitting theorem. Furthermore, we introduce and study linear subspaces of tangent spaces.

In Lecture 7, we introduce linear dimension and volume. Further, we prove the Bishop–Gromov inequality and the right-inverse theorem, introduce distance charts, and show that all reasonable types of dimension are the same for Alexandrov spaces.

In Lecture 8 we show that a lower curvature bound survives under Gromov–Hausdorff limits and prove Gromov's selection theorem. Further, we present Perelman's construction of strictly concave functions and apply it with Gromov's selection theorem to prove the homotopy finiteness theorem. This proof illustrates one of the main sources of applications of Alexandrov geometry.

In Lecture 9, we introduce the boundary of finite-dimensional Alexandrov spaces and prove the doubling theorem.

In Lecture 10, we show that quotients of Alexandrov spaces by isometric group action are Alexandrov spaces. This gives another source of applications of Alexandrov geometry, several examples are given.

Finally, in Lecture 11 we briefly discuss convex surfaces in Euclidean space; this subject is the main precursor to the modern Alexandrov geometry.

Let us list available texts on Alexandrov spaces with curvature bounded below:

- ⋄ The first introduction to Alexandrov geometry of all dimensions is given in the original paper by Yuriy Burago, Michael Gromov, and Grigory Perelman [18] and its extension [79] written by Perelman.
- ⋄ A brief and reader-friendly introduction was written by Katsuhiro Shiohama [102, Sections 1–8].
- ⋄ Another reader-friendly introduction, written by Dmitri Burago, Yuriy Burago, and Sergei Ivanov [17, Chapter 10].
- ⋄ Survey by Conrad Plaut [93].
- ⋄ Survey by the second author [86].
- ⋄ Our book written jointly with Stephanie Alexander [6].

Exercises that are used later in the sequel are marked with an exclamation point: **Exercise!**

Acknowledgments. We would like to thank Stephanie Alexander and Nina Lebedeva for their help. The idea for this book emerged in collaboration with Stephanie Alexander. We had to write it without her but drew extensively from another book co-authored by the three of us [6]. Our notes were shaped in a series of lectures given by the authors on various occasions: at Penn State, including the MASS program; at the Summer School "Algebra and Geometry" in Yaroslavl; at SPbSU; and at the University of Toronto. We are grateful to these institutions for their hospitality and support.

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Lecture 1

Preliminaries

A Prerequisites

The main prerequisite: the reader should know and love elementary geometry, including convexity.

We also assume that the reader is familiar with the following topics in metric geometry:

◇ Compactness and proper metric spaces; recall that a metric space is *proper* if all its closed balls (of finite radius) are compact.

◇ Complete metric spaces and completion.

◇ Curves, semicontinuity of length, and rectifiability.

◇ Hausdorff and Gromov–Hausdorff convergence. These are discussed briefly in 1J–1N. The definitions are there, but some prior familiarity with these concepts will be very helpful

All these topics are treated in detail in the classic book of Dmitri Burago, Yurii Burago, and Sergei Ivanov [17]; see also a short book of second author [90]. Occasionally, we use the Baire category theorem and Rademacher's theorem, but these could be used as black boxes.

We use some topology. An introductory text in algebraic topology should be sufficient most of the time. For some examples, we apply more advanced results, but these are used as black boxes.

Since most of the applications come from Riemannian geometry, it is beneficial to be familiar with the Toponogov comparison theorem and related topics. The classical book by Jeff Cheeger and David Ebin [21] contains more than one needs.

B Notations

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

Given radius $r\in[0,\infty]$ and center $x\in\mathcal{X}$, the sets

$$\begin{aligned}\mathrm{B}(x,r)&=\{\,y\in\mathcal{X}\,:\,|x-y|<r\,\},\\ \overline{\mathrm{B}}[x,r]&=\{\,y\in\mathcal{X}\,:\,|x-y|\leqslant r\,\}\end{aligned}$$

are called, respectively, the open and the closed balls. The notations $\mathrm{B}(x,r)_{\mathcal{X}}$ and $\overline{\mathrm{B}}[x,r]_{\mathcal{X}}$ might be used if we need to emphasize that these balls are taken in the metric space $\mathcal{X}$.

The diameter of $\mathcal{X}$ is defined as

$$\operatorname{diam}\mathcal{X}=\sup\{\,|x-y|\,:\,x,y\in\mathcal{X}\,\}.$$

The radius of $\mathcal{X}$ is defined as

$$\operatorname{rad}\mathcal{X}=\inf\{\,R>0\,:\,\mathrm{B}(x,R)=\mathcal{X}\text{ for some }x\in\mathcal{X}\,\}.$$

We will denote by $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$ the n-dimensional sphere (with angle metric), the Euclidean space, and the Lobachevsky space respectively. More generally, $\mathbb{M}^n(\kappa)$ will denote the model n-space of curvature κ; that is,

- ◇ if $\kappa>0$, then $\mathbb{M}^n(\kappa)$ is the n-sphere of radius $\frac{1}{\sqrt{\kappa}}$, so $\mathbb{S}^n=\mathbb{M}^n(1)$
- ◇ $\mathbb{M}^n(0)=\mathbb{E}^n$,
- ◇ if $\kappa<0$, then $\mathbb{M}^n(\kappa)$ is the Lobachevsky n-space $\mathbb{H}^n$ rescaled by factor $\frac{1}{\sqrt{-\kappa}}$; in particular $\mathbb{M}^n(-1)=\mathbb{H}^n$.

C Nets and packing

Let S be a subset of a metric space $\mathcal{X}$. Recall that a set $Z\subset\mathcal{X}$ is called an ε-net of S if for any point $s\in S$, there is a point $z\in Z$ such that $|s-z|\leqslant\varepsilon$.

The ε-pack of $\mathcal{X}$ (or packing number) is the maximal number (possibly infinite) of points in $\mathcal{X}$ at distance $>\varepsilon$ from each other; it is denoted by $\operatorname{pack}_\varepsilon\mathcal{X}$. If $m=\operatorname{pack}_\varepsilon\mathcal{X}<\infty$, then a set $\{x_1,x_2,\dots,x_m\}$ in $\mathcal{X}$ such that $|x_i-x_j|>\varepsilon$ is called a maximal ε-packing in $\mathcal{X}$.

We will use the following characterizations of compact sets.

1.1. Exercise! *Let X be a closed subset in a complete metric space.*

(a) Show that X is compact if and only if it admits a finite ε-net for any $\varepsilon>0$.

(b) *Show that X is compact if and only if it admits a compact ε-net for any $\varepsilon > 0$.*

1.2. Exercise! *Show that any maximal ε-packing $x_1, \dots, x_n$ is an ε-net. Conclude that a complete metric space $\mathcal{X}$ is compact if and only if* $\operatorname{pack}_\varepsilon \mathcal{X} < \infty$ *for any $\varepsilon > 0$.*

D Length spaces

Let $\mathcal{X}$ be a metric space. If for any $\varepsilon > 0$ and any pair of points $x, y \in \mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length} \alpha < |x - y| + \varepsilon,$$

then $\mathcal{X}$ is called a length space and the metric on $\mathcal{X}$ is called a length metric.

1.3. Exercise. *Let $\mathcal{X}$ be a complete length space. Show that for any compact subset $K \subset \mathcal{X}$ there is a compact path-connected subset $K' \subset \mathcal{X}$ that contains K.*

Induced length metric. Directly from the definition, it follows that if $\alpha\colon [0, 1] \to \mathcal{X}$ is a path from x to y (that is, $\alpha(0) = x$ and $\alpha(1) = y$), then

$$\operatorname{length} \alpha \geqslant |x - y|.$$

Set

$$\|x - y\| = \inf\{ \operatorname{length} \alpha \},$$

where the greatest lower bound is taken for all paths from x to y. It is straightforward to check that $(x, y) \mapsto \|x - y\|$ is an ∞-metric; that is, $(x, y) \mapsto \|x - y\|$ is a metric in the extended positive reals $[0, \infty]$. The metric $\|{*} - {*}\|$ is called the induced length metric.

1.4. Exercise. *Suppose $(\mathcal{X}, |{*} - {*}|)$ is a complete metric space. Show that $(\mathcal{X}, \|{*} - {*}\|)$ is complete; that is, any Cauchy sequence of points in $(\mathcal{X}, \|{*} - {*}\|)$ converges in $(\mathcal{X}, \|{*} - {*}\|)$.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x - y|_A = \inf_\alpha \{ \operatorname{length} \alpha \},$$

where the greatest lower bound is taken for all paths α from x to y in A. In other words, $|{*} - {*}|_A$ denotes the induced length metric on the subspace A. (The notation $|{*} - {*}|_A$ conflicts with the previously defined notation for distance $|x - y|_{\mathcal{X}}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.)

E Geodesics

Let $\mathcal{X}$ be a metric space, and let $\mathbb{I}$ be a real interval. A distance-preserving map $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a geodesic[1]; in other words, $\gamma\colon \mathbb{I} \to \mathcal{X}$ is a geodesic if

$$|\gamma(s) - \gamma(t)| = |s - t|$$

for any pair $s, t \in \mathbb{I}$.

If $\gamma\colon [a, b] \to \mathcal{X}$ is a geodesic such that $p = \gamma(a)$, $q = \gamma(b)$, then we say that γ is a geodesic from p to q. In this case, the image of γ is denoted by $[pq]$, and, with abuse of notations, we also call it a geodesic. We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ lies in the space $\mathcal{X}$.

In general, a geodesic from p to q need not exist and if it exists, it need not be unique; for example, any meridian is a geodesic between poles on the sphere. However, once we write $[pq]$ we assume that we have chosen such a geodesic.

A geodesic path is a geodesic with constant-speed reparameterization by the unit interval $[0, 1]$.

A metric space is called geodesic if any pair of its points can be joined by a geodesic. A metric space $\mathcal{X}$ is called ℓ-geodesic if any two points $x, y \in \mathcal{X}$ such that $|x - y| < \ell$ can be connected by a geodesic. For instance, any geodesic space is ∞-geodesic.

Evidently, any geodesic space is a length space.

1.5. Exercise. *Show that any proper length space is geodesic.*

F Menger's lemma

1.6. Menger's lemma. *Let $\mathcal{X}$ be a complete metric space. Assume that for any pair of points $x, y \in \mathcal{X}$, there is a midpoint z. Then $\mathcal{X}$ is a geodesic space.*

This lemma is due to Karl Menger [71, Section 6].

Proof. Choose $x, y \in \mathcal{X}$; set $\gamma(0) = x$, and $\gamma(1) = y$.

$$x = \gamma(0) \quad \gamma(\tfrac{1}{4}) \quad \gamma(\tfrac{1}{2}) \quad \gamma(\tfrac{3}{4}) \quad \gamma(1) = y$$

Let $\gamma(\frac{1}{2})$ be a midpoint between $\gamma(0)$ and $\gamma(1)$. Further, let $\gamma(\frac{1}{4})$ and $\gamma(\frac{3}{4})$ be midpoints between the pairs $(\gamma(0), \gamma(\frac{1}{2}))$ and $(\gamma(\frac{1}{2}), \gamma(1))$

[1] Other texts may refer to geodesics in our sense of the word as *shortest path* or *minimizing geodesic*. Also, our meaning of the term *geodesic* is closely related, but different from what is used in Riemannian geometry.

respectively. Applying the above procedure recursively, on the n-th step we define $\gamma(\frac{k}{2^n})$, for every odd integer k such that $0 < \frac{k}{2^n} < 1$, as a midpoint of the already defined $\gamma(\frac{k-1}{2^n})$ and $\gamma(\frac{k+1}{2^n})$.

This way we define $\gamma(t)$ for all dyadic rationals t in $[0,1]$. Moreover, the map γ has Lipschitz constant $|x-y|$. Since $\mathcal{X}$ is complete, γ can be extended continuously to $[0,1]$. Moreover,

$$\operatorname{length}\gamma \leqslant |x-y|.$$

Therefore γ is a geodesic path from x to y. $\square$

1.7. Exercise. *Let $\mathcal{X}$ be a complete metric space. Assume that for any pair of points $x, y \in \mathcal{X}$, there is an almost midpoint; that is, given $\varepsilon > 0$, there is a point z such that*

$$|x-z| < \tfrac{1}{2}\cdot|x-y| + \varepsilon \quad \textit{and} \quad |y-z| < \tfrac{1}{2}\cdot|x-y| + \varepsilon.$$

Show that $\mathcal{X}$ is a length space.

G Triangles and model triangles

Triangles. Given a triple of points p, q, r in a metric space $\mathcal{X}$, a choice of geodesics $([qr], [rp], [pq])$ will be called a triangle and denoted by $[pqr] = [pqr]_{\mathcal{X}}$.

Given a triple of points $p, q, r \in \mathcal{X}$ there may be no triangle $[pqr]$ simply because one of the pairs of these points cannot be joined by a geodesic. Also, many different triangles with these vertices may exist, any of which can be denoted by $[pqr]$. If we write $[pqr]$, it means that we have chosen such a triangle.

Model triangles. Given three points p, q, r in a metric space $\mathcal{X}$, let us define the model triangle $[\tilde{p}\tilde{q}\tilde{r}]$ (briefly, $[\tilde{p}\tilde{q}\tilde{r}] = \tilde{\triangle}(pqr)_{\mathbb{E}^2}$) to be a triangle in the Euclidean plane $\mathbb{E}^2$ with the same sides; that is,

$$|\tilde{p}-\tilde{q}|_{\mathbb{E}^2} = |p-q|_{\mathcal{X}}, \quad |\tilde{q}-\tilde{r}|_{\mathbb{E}^2} = |q-r|_{\mathcal{X}}, \quad |\tilde{r}-\tilde{p}|_{\mathbb{E}^2} = |r-p|_{\mathcal{X}}.$$

In the same way, we can define the hyperbolic and the spherical model triangles $\tilde{\triangle}(pqr)_{\mathbb{H}^2}$, $\tilde{\triangle}(pqr)_{\mathbb{S}^2}$ in the Lobachevsky plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2$. In the latter case, the model triangle is said to be defined if in addition

$$|p-q| + |q-r| + |r-p| < 2\cdot\pi.$$

In this case, the model triangle again exists and is unique up to an isometry of $\mathbb{S}^2$.

Model angles. If $[\tilde p\tilde q\tilde r] = \tilde\triangle(pqr)_{\mathbb{E}^2}$ and $|p-q|, |p-r| > 0$, the angle measure of $[\tilde p\tilde q\tilde r]$ at $\tilde p$ will be called the *model angle* of the triple p, q, r and will be denoted by $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{E}^2}$.

For example, if $|p-q| = |q-r| = |r-p|$, then $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{E}^2} = \frac{\pi}{3}$ regardless of the existence and relative position of geodesics $[pq]$ and $[pr]$.

In the same way we define $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$; in particular, $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{H}^2}$ and $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{S}^2}$. We may use the notation $\tilde\measuredangle(p\,{}^q_r)$ if it is evident which of the model spaces is meant.

1.8. Exercise! *Show that for any triple of points p, q, and r, the function*

$$\kappa \mapsto \tilde\measuredangle(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$$

is nondecreasing in its domain of definition.

H Hinges and their angle measure

Hinges. Let $p, x, y \in \mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair of geodesics $([px], [py])$ will be called a *hinge* and will be denoted by $[p\,{}^x_y] = ([px], [py])$.

Angles. The angle measure of a hinge $[p\,{}^x_y]$ is defined as the following limit

$$\measuredangle[p\,{}^x_y] = \lim_{\bar x, \bar y \to p} \tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x \in\,]px]$ and $\bar y \in\,]py]$ (The angle is only defined if this limit exists).

If $\measuredangle[p\,{}^x_y]$ is defined, then

$$0 \leqslant \measuredangle[p\,{}^x_y] \leqslant \pi.$$

1.9. Exercise. *Suppose that in the above definition, one uses spherical or hyperbolic model angles instead of Euclidean ones. Show that it does not change the value $\measuredangle[p\,{}^x_y]$.*

1.10. Exercise. *Give an example of a hinge $[p\,{}^x_y]$ in a metric space with an undefined angle measure $\measuredangle[p\,{}^x_y]$.*

I Triangle inequality for angles

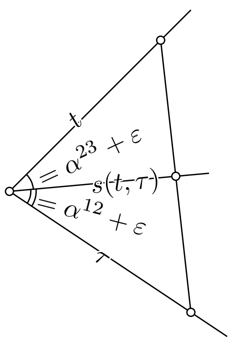
1.11. Proposition. *Consider three geodesics $[px_1]$, $[px_2]$, and $[px_3]$ in a metric space. Suppose all the angle measures $\alpha_{i,j} = \measuredangle[p\,{}^{x_i}_{x_j}]$ are* *defined. Then*

$$\alpha_{1,3} \leqslant \alpha_{1,2} + \alpha_{2,3}.$$

Proof. We can assume that $\alpha_{1,3} > 0$. Furthermore, since $\alpha_{1,3} \leqslant \pi$, we can assume that $\alpha_{1,2} + \alpha_{2,3} < \pi$.

Denote by γ_i the unit-speed parametrization of $[px_i]$ from p to x_i. Given small $\varepsilon > 0$, for all sufficiently small $t, \tau, s \in [0, \infty)$ we have

$$\begin{aligned}\sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha_{1,3} - \varepsilon)} &\leqslant |\gamma_1(t) - \gamma_3(\tau)| \leqslant \\ &\leqslant |\gamma_1(t) - \gamma_2(s)| + |\gamma_2(s) - \gamma_3(\tau)| < \\ &< \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha_{1,2} + \varepsilon)} + \\ &+ \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\alpha_{2,3} + \varepsilon)} \leqslant\end{aligned}$$



Below we define $s(t, \tau)$ so that for $s = s(t, \tau)$, this chain of inequalities can be continued as

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha_{1,2} + \alpha_{2,3} + 2{\cdot}\varepsilon)}.$$

Thus

$$\alpha_{1,3} \leqslant \alpha_{1,2} + \alpha_{2,3} + 3{\cdot}\varepsilon.$$

Hence the result follows.

To define $s(t, \tau)$, consider three half-lines $\tilde\gamma_1$, $\tilde\gamma_2$, $\tilde\gamma_3$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde\gamma_1, \tilde\gamma_2) = \alpha_{1,2} + \varepsilon$, $\measuredangle(\tilde\gamma_2, \tilde\gamma_3) = \alpha_{2,3} + \varepsilon$, and $\measuredangle(\tilde\gamma_1, \tilde\gamma_3) = \alpha_{1,2} + \alpha_{2,3} + 2{\cdot}\varepsilon$. We parametrize each half-line by the distance from the starting point. Given two positive numbers t, τ, let $s = s(t, \tau)$ be the number such that $\tilde\gamma_2(s) \in [\tilde\gamma_1(t)\ \tilde\gamma_3(\tau)]$. Clearly, $s \leqslant \max\{t, \tau\}$, so t, τ, s may be taken sufficiently small. ☐

1.12. Exercise! *Prove that the sum of adjacent angles is at least π.*

More precisely: suppose two hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ are a d j a c e n t; *that is, they share side $[pz]$, and the union of two sides $[px]$ and $[py]$ forms a geodesic $[xy]$. Show that*

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \geqslant \pi$$

whenever each angle on the left-hand side is defined.

Give an example showing that the inequality can be strict.

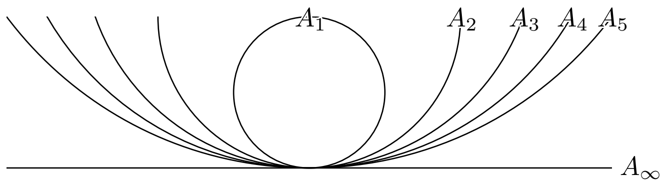
1.13. Exercise. *Let γ be the unit speed parametrization of $[qx]$ from q to x. Assume that the angle measure $\varphi = [q\,{}^p_x]$ is defined. Show that*

$$|p - \gamma(t)| \leqslant |q - p| - t{\cdot}\cos\varphi + o(t).$$

J Hausdorff convergence

1.14. Definition. *Let $A_1, A_2, \dots$ be a sequence of closed sets in a metric space $\mathcal{X}$. We say that the sequence A_n converges to a closed set A_∞ in the sense of Hausdorff if, for any $x \in \mathcal{X}$, we have $\operatorname{dist}_{A_n}(x) \to \operatorname{dist}_{A_\infty}(x)$ as $n \to \infty$.*

For example, suppose $\mathcal{X}$ is the Euclidean plane and A_n is the circle with radius n and center at the point $(0, n)$. Then A_n converges to the x-axis as $n \to \infty$.



Further, consider the sequence of one-point sets $B_n = \{(n, 0)\}$ in the Euclidean plane. It converges to the empty set; indeed, for any point x we have $\operatorname{dist}_{B_n}(x) \to \infty$ as $n \to \infty$ and $\operatorname{dist}_\varnothing(x) = \infty$ for any x.

The following exercise is an extension of the so-called Blaschke selection theorem to our version of Hausdorff convergence.

1.15. Exercise. *Show that any sequence of closed sets in a proper metric space has a convergent subsequence in the sense of Hausdorff.*

1.16. Exercise. *Construct a metric space with a sequence of points $\{x_0, x_1, \dots\}$ such that $|x_0 - x_n| = 1$ for any $n > 0$ and the two-point sets $\{x_0, x_n\}$ converge to one-point set $\{x_0\}$ in the sense of Hausdorff.*

K Hausdorff metric

1.17. Definition. *Let A and B be two nonempty compact subsets of a metric space $\mathcal{X}$. Then the Hausdorff distance between A and B is defined as*

$$|A - B|_{\operatorname{Haus}\mathcal{X}} := \sup_{x \in \mathcal{X}} \{\, |\operatorname{dist}_A(x) - \operatorname{dist}_B(x)| \,\}.$$

The following observation gives a useful reformulation of the definition:

1.18. Observation. *Suppose A and B are two compact subsets of a metric space $\mathcal{X}$. Then $|A-B|_{\mathrm{Haus}\,\mathcal{X}} < R$ if and only if B lies in an R-neighborhood of A, and A lies in an R-neighborhood of B.*

According to the following exercise, Hausdorff convergence of nonempty compact subsets is the convergence in Hausdorff metric.

1.19. Exercise. *Let $A_1, A_2, \dots,$ and A_∞ be compact nonempty sets in a proper metric space $\mathcal{X}$. Show that $|A_n - A_\infty|_{\mathrm{Haus}\,\mathcal{X}} \to 0$ as $n \to \infty$ if and only if $A_n \to A_\infty$ in the sense of Hausdorff.*

L Gromov–Hausdorff convergence

Let $\mathcal{X}_1, \mathcal{X}_2, \dots,$ and $\mathcal{X}_\infty$ be a sequence of proper metric spaces. Suppose that there is a metric on the disjoint union

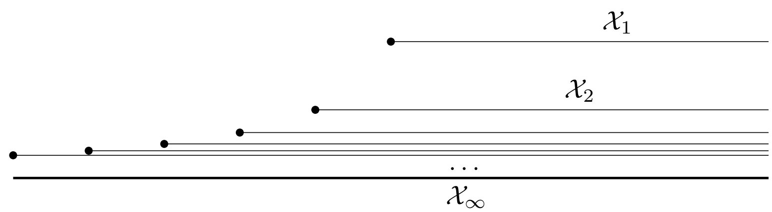
$$\boldsymbol{X} = \bigsqcup_{n\in\mathbb{N}\cup\{\infty\}} \mathcal{X}_n$$

that satisfies the following property:

1.20. Property. *The restriction of the metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric, the space $\boldsymbol{X}$ is proper, and $\mathcal{X}_n \to \mathcal{X}_\infty$ as subsets in $\boldsymbol{X}$ in the sense of Hausdorff.*

In this case we say that the metric on $\boldsymbol{X}$ *defines* a convergence $\mathcal{X}_n \to \mathcal{X}_\infty$ in the sense of Gromov–Hausdorff. The metric on $\bigsqcup \mathcal{X}_n$ makes it possible to talk about limits of sequences $x_n \in \mathcal{X}_n$ as $n \to \infty$, as well as weak limits of a sequence of Borel measures μ_n on $\mathcal{X}_n$ and so on.

The limit space is not uniquely defined by the sequence. For example, if each space $\mathcal{X}_n$ in the sequence is isometric to the half-line, then its limit might be isometric to the half-line or the whole line. The first convergence is evident and the second could be guessed from the picture.



Note that any sequence of spaces has an empty space as its limit in some Gromov–Hausdorff convergence. As we will see further, if the limit is nonempty and compact, then it is unique up to isometry.

Pointed convergence. The isometry class of the limit can be fixed by marking a point p_n in each space $\mathcal{X}_n$. We say that $(\mathcal{X}_n, p_n)$ converges to $(\mathcal{X}_\infty, p_\infty)$ if there is a metric on $\boldsymbol{X}$ as in 1.20 such that $p_n \to p_\infty$. This is called pointed Gromov–Hausdorff convergence. For example, the sequence $(\mathcal{X}_n, p_n) = ([0,\infty), 0)$ converges to $([0,\infty), 0)$, while $(\mathcal{X}_n, p_n) = ([0,\infty), n)$ converges to $(\mathbb{R}, 0)$ as $n \to \infty$.

M Gromov–Hausdorff metric

In this section we cook up a metric space out of all compact nonempty metric spaces that defines Gromov–Hausdorff convergence. We want to define the metric on the set of *isometry classes* of compact metric spaces. Further, the term *metric space* might also stand for its *isometry class.*

Loosely speaking, the distance between (isometry classes of) compact metric spaces will be defined as the maximal value such that *the distance between subspaces in a metric space is not greater than the Hausdorff distance between them.*

Observe that we *know* a real number x if we can give yes/no answer to the question $x < r$ for any r. Keep this in mind while reading the following definition.

1.21. Definition. *The Gromov–Hausdorff distance $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}}$ between compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is defined by the following relation.*

Given $r > 0$, we have $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} < r$ if and only if there exists a metric space $\mathcal{W}$ and subspaces $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{W}$ that are isometric to $\mathcal{X}$ and $\mathcal{Y}$, respectively, such that $|\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}} < r$. (Here $|\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}}$ denotes the Hausdorff distance between sets $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{W}$.)

These distances actually define a metric on (isometry classes of) compact metric spaces; for a proof we refer to [17, 90]. The obtained metric is called the Gromov–Hausdorff metric; the corresponding metric space will be denoted by GH. This means in particular that if $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} = 0$ for compact metric spaces X and Y, then they are isometric. Moreover, each of the references above provide the following statement.

1.22. Proposition. GH *is a complete metric space.*

The proofs of the mentioned statements are not quite straightforward. The following two exercises should help the reader who wants to reconstruct these proofs without reading the mentioned textbooks.

1.23. Exercise. *Let f be a distance noncontracting map from a compact metric space $\mathcal{K}$ to itself. Show that f is an isometry; that is, it is a distance-preserving bijection.*

1.24. Exercise. *Show that any surjective short map (that is, distance-nonexpanding) from a compact metric space to itself is an isometry.*

Two convergences. Notice that now we have two notions of convergence of metric spaces. One is defined in the previous section; it works for proper metric spaces and its limit might not be uniquely defined. The other is defined as convergence in GH; the limit in GH is unique, if it exists. Therefore, these convergences are different. On the other hand, if we restrict our attention only to nonempty compact spaces, then these two convergences are essentially the same. Let us sketch why.

Suppose $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \to 0$ as $n \to \infty$; that is $\mathcal{X}_n$ converges to $\mathcal{X}_\infty$ in GH. Then there is a metric on $\mathcal{V}_n = \mathcal{X}_n \sqcup \mathcal{X}_\infty$ such that the restriction of metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric, and $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{Haus}\,\mathcal{V}_n} < \varepsilon_n$ for some sequence $\varepsilon_n \to 0$. Gluing all $\mathcal{V}_n$ along $\mathcal{X}_\infty$, we get the required space $\boldsymbol{X}$, which defines convergence.

Once the convergence is fixed, we can talk about limits of sequences $x_n \in \mathcal{X}_n$ in $\mathcal{X}_\infty$ as $n \to \infty$, as well as weak limits of a sequence of Borel measures μ_n on $\mathcal{X}_n$ and so on — all these limits are defined as the corresponding limits in the ambient space $\boldsymbol{X}$.

Now suppose we have a Gromov–Hausdorff convergence $\mathcal{X}_n \to \mathcal{X}_\infty$ with the common space $\boldsymbol{X}$. Assume that all spaces $\mathcal{X}_1, \mathcal{X}_2, \dots,$ as well as $\mathcal{X}_\infty$ are compact, and $\mathcal{X}_\infty$ is nonempty. (The latter condition means that the convergence is nontrivial.) Observe that in this case $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{Haus}\,\boldsymbol{X}} \to 0$ as $n \to \infty$, and therefore $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \to 0$. In other words $\mathcal{X}_n$ converges to $\mathcal{X}_\infty$ in GH.

N Almost isometries

1.25. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called an ε-isometry if the following two conditions hold:*

(a) $f(\mathcal{X})$ is an ε-net in $\mathcal{Y}$; that is, for any $y \in \mathcal{Y}$ there is $x \in \mathcal{X}$ such that

$$|f(x) - y|_{\mathcal{Y}} \leqslant \varepsilon.$$

(b) $f(\mathcal{X})$ is ε-distance-preserving; that is,

$$|f(x) - f(x')|_{\mathcal{Y}} \lesssim |x - x'|_{\mathcal{X}} \pm \varepsilon$$

for any $x, x' \in \mathcal{X}$.

When dealing with Gromov–Hausdorff convergence the following lemma allows to bypass constructing explicit metrics on the disjoint unions of $\mathcal{X}_1, \mathcal{X}_2, \dots,$ and $\mathcal{X}_\infty$

1.26. Lemma. *Let $\mathcal{X}_1, \mathcal{X}_2, \dots,$ and $\mathcal{X}_\infty$ be compact metric spaces, and let $\varepsilon_n \to 0+$ as $n \to \infty$. Suppose that either*

(a) for each n there is an ε_n-isometry $f_n \colon \mathcal{X}_n \to \mathcal{X}_\infty$, or

(b) for each n there is an ε_n-isometry $h_n \colon \mathcal{X}_\infty \to \mathcal{X}_n$.

Then there is a Gromov–Hausdorff convergence $\mathcal{X}_n \to \mathcal{X}_\infty$.

Furthermore, a partial converse also holds.

(c) Suppose we have a Gromov–Hausdorff convergence $\mathcal{X}_n \to \mathcal{X}_\infty$ and $\mathcal{X}_\infty$ is nonempty and compact. Then there exist $\varepsilon_n \to 0+$ as $n \to \infty$ and ε_n-isometries $f_n \colon \mathcal{X}_n \to \mathcal{X}_\infty$ (and $h_n \colon \mathcal{X}_\infty \to \mathcal{X}_n$) such that $x_n \in \mathcal{X}_n$ converges to $x_\infty \in \mathcal{X}_\infty$ with respect to the convergence $\mathcal{X}_n \to \mathcal{X}_\infty$ if and only if $f_n(x_n) \to x_\infty$ (respectively, $|h_n(x_\infty) - x_n|_{\mathcal{X}_n} \to 0$) as $n \to \infty$.

Proof. To prove part *(a)* let us construct a common space $\boldsymbol{X}$ for the spaces $\mathcal{X}_1, \mathcal{X}_2, \dots,$ and $\mathcal{X}_\infty$ by taking the metric ρ on the disjoint union $\mathcal{X}_\infty \sqcup \mathcal{X}_1 \sqcup \mathcal{X}_2 \sqcup \dots$ that is defined the following way:

$$\begin{aligned}
|x_n - y_n|_{\boldsymbol{X}} &= |x_n - y_n|_{\mathcal{X}_n},\\
|x_\infty - y_\infty|_{\boldsymbol{X}} &= |x_\infty - y_\infty|_{\mathcal{X}_\infty},\\
|x_n - x_\infty|_{\boldsymbol{X}} &= \inf\left\{\, |x_n - y_n|_{\mathcal{X}_n} + \varepsilon_n + |x_\infty - f(y_n)|_{\mathcal{X}_\infty} \;:\; y_n \in \mathcal{X}_n \,\right\},\\
|x_n - x_m|_{\boldsymbol{X}} &= \inf\left\{\, |x_n - y_\infty|_{\boldsymbol{X}} + |x_m - y_\infty|_{\boldsymbol{X}} \;:\; y_\infty \in \mathcal{X}_\infty \,\right\},
\end{aligned}$$

where we assume that $x_\infty, y_\infty \in \mathcal{X}_\infty$, and $x_n, y_n \in \mathcal{X}_n$ for each n. It remains to observe that this indeed defines a metric on $\boldsymbol{X}$, and $\mathcal{X}_n \to \mathcal{X}_\infty$ in the sense of Hausdorff.

The proof of the second part is analogous; one only needs to change one line in the definition of the metric as follows:

$$|x_n - x_\infty|_{\boldsymbol{X}} = \inf\left\{\, |x_n - h(y_\infty)|_{\mathcal{X}_n} + \varepsilon_n + |x_\infty - y_\infty|_{\mathcal{X}_\infty} \;:\; y_\infty \in \mathcal{X}_\infty \,\right\}.$$

We leave part *(c)* as an exercise. $\square$

For two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $\mathcal{X} \leqslant \mathcal{Y} + \varepsilon$ if there is a map $f \colon \mathcal{X} \to \mathcal{Y}$ such that

$$|x - x'|_{\mathcal{X}} \leqslant |f(x) - f(x')|_{\mathcal{Y}} + \varepsilon$$

for any $x, x' \in \mathcal{X}$.

1.27. Exercise. *Let $\mathcal{X}_1, \mathcal{X}_2, \dots,$ and $\mathcal{X}_\infty$ be nonempty compact metric spaces. Show that there is a Gromov–Hausdorff convergence $\mathcal{X}_n \to \to \mathcal{X}_\infty$ if and only if for some sequence $\varepsilon_n \to 0$, we have*

$$\mathcal{X}_\infty \leqslant \mathcal{X}_n + \varepsilon_n \quad \textit{and} \quad \mathcal{X}_n \leqslant \mathcal{X}_\infty + \varepsilon_n.$$

Lemma 1.26 has a natural analogue for pointed convergence. For simplicity we only state part *(a)* of the lemma. Parts *(b)* and *(c)* can be rephrased similarly; in *(c)* we have to assume that the space is proper.

1.28. Lemma. *Let $(\mathcal{X}_1, p_1), (\mathcal{X}_2, p_2), \dots,$ let $(\mathcal{X}_\infty, p_\infty)$ be pointed metric spaces, and let $\varepsilon(n, R) \to 0+$ as $n \to \infty$ for any fixed $R > 0$. Suppose that for each n there is a map $f_n \colon \mathcal{X}_n \to \mathcal{X}_\infty$ such that*

(a) $f_n(p_n) \to p_\infty$

(b) $\big||f_n(x) - f_n(x')|_{\mathcal{X}_\infty} - |x - x'|_{\mathcal{X}_n}\big| \leqslant \varepsilon(n, R)$ *for any* $x, x' \in \in \mathrm{B}(p_n, R)$.

(c) For any $x \in \mathrm{B}(p_\infty, R)$ there is $x_n \in \mathrm{B}(p_n, R)$ such that $|x - - f_n(x_n)| \leqslant \varepsilon(n, R)$

Then there is a pointed Gromov–Hausdorff convergence $(\mathcal{X}_n, p_n) \to \to (\mathcal{X}_\infty, p_\infty)$.

The proofs of 1.28 and 1.26 are analogous; we leave the former to the reader.

O Remarks

In principle, our definition of Gromov–Hausdorff distance can be applied to complete metric spaces that are not necessarily compact. However, according to the following exercise, it only defines a semimetric; that is, zero Gromov–Hausdorff distance does not imply that the spaces are isometric. For that reason it is not in use.

1.29. Exercise.

(a) Construct two nonisometric proper (noncompact) metric spaces with vanishing Gromov–Hausdorff distance.

(b) Construct two nonisometric complete geodesic metric spaces of bounded diameter with vanishing Gromov–Hausdorff distance.

Lecture 2

Definitions

In this lecture we prove equivalence of several definitions of Alexandrov space.

A Four-point comparison

Recall that $\tilde\measuredangle(p\,{}^x_y)$ denotes the model angle; see page 10.

Let p, x, y, z be a quadruple of points in a metric space. If the inequality

$$❶ \qquad \tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^y_z)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^z_x)_{\mathbb{E}^2} \leqslant 2\cdot\pi$$

holds, then we say that the quadruple meets $\mathbb{E}^2$-comparison. If the left-hand side is undefined, then we assume that the comparison holds.

2.1. Exercise. *Suppose $\mathbb{E}^2$-comparison holds for quadruple p, x_1, x_2, x_3. Show that $\mathbb{E}^2$-comparison holds for quadruple q, y_1, y_2, y_3 if*

$$|q - y_i| \geqslant |p - x_i| \qquad \textit{and} \qquad |y_i - y_j| \leqslant |x_i - x_j|$$

for all i and j.

Instead of $\mathbb{E}^2$, we can use $\mathbb{S}^2$ or $\mathbb{H}^2$. This way we get the definition of $\mathbb{S}^2$- or $\mathbb{H}^2$-comparisons. Recall that $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2}$ and $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2}$ are defined if $p \neq x$, $p \neq y$, but for $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2}$ we require in addition that

$$|p - x| + |p - y| + |x - y| < 2\cdot\pi;$$

if this does not hold for one of the angles, then we assume that $\mathbb{S}^2$-comparison holds for this quadruple.

More generally, one may apply this definition to $\mathbb{M}^2(\kappa)$ and define $\mathbb{M}^2(\kappa)$-comparison for any real κ. However, if you see $\mathbb{M}^2(\kappa)$-comparison, it is safe to assume that $\kappa = -1$, 0, or 1; applying rescaling, the $\mathbb{M}^2(\kappa)$-comparison can be reduced to these three cases.

2.2. Definition. *A metric space $\mathcal{X}$ has curvature $\geqslant \kappa$ in the sense of Alexandrov if $\mathbb{M}^2(\kappa)$-comparison holds for any quadruple of points in $\mathcal{X}$.*

While this definition can be applied to any metric space, we will use it mostly for geodesic spaces that are complete (and often compact or proper). If a complete geodesic space has curvature $\geqslant \kappa$ in the sense of Alexandrov, then it will be called an ALEX(κ) space; here ALEX(κ) is an adjective. An $\mathcal{X}$ is ALEX(κ) for some κ, then we say that $\mathcal{X}$ is an Alexandrov space.

It is common practice in Alexandrov geometry to write proofs for nonnegative curvature and leave the general curvature bound as an exercise. These generalizations are usually straightforward. We will add notes when they are not. We will also often formulate statements just for $\kappa = 0$ even when they admit straightforward generalizations to arbitrary curvature bounds; see [6] for a more formal tratment.

2.3. Exercise. *Show that $\mathbb{E}^n$ is* ALEX(0).

2.4. Exercise. *Show that a metric space $\mathcal{X}$ has nonnegative curvature in the sense of Alexandrov if and only if for any quadruple of points $p, x_1, x_2, x_3 \in \mathcal{X}$ there is a quadruple of points $q, y_1, y_2, y_3 \in \mathbb{E}^3$ such that*

$$|p - x_i|_{\mathcal{X}} \geqslant |q - y_i|_{\mathbb{E}^2} \quad \text{and} \quad |x_i - x_j|_{\mathcal{X}} \leqslant |y_i - y_j|_{\mathbb{E}^2}$$

for all i and j.

B Alexandrov's lemma

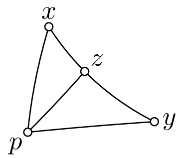
Recall that $[xy]$ denotes a geodesic from x to y; set

$$]xy] = [xy] \setminus \{x\}, \quad [xy[= [xy] \setminus \{y\}, \quad]xy[= [xy] \setminus \{x, y\}.$$

2.5. Alexandrov's lemma. *Let p, x, y, z be distinct points in a metric space such that $z \in\,]xy[$. Then the following expressions have the same sign:*

(a) $\tilde\measuredangle(x\,{}^p_y) - \tilde\measuredangle(x\,{}^p_z)$,

(b) $\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) - \pi$.



The same holds for the hyperbolic and spherical model angles, but in the latter case, one has to assume in addition that

$$|p-x|+|p-y|+|x-y|<2\cdot\pi.$$

In the proof we will apply the following statement from elementary geometry.

2.6. Observation. *Increasing the opposite side in a plane triangle increases the corresponding angle, and the other way around.*

Moreover, the same statement holds for spherical and hyperbolic triangles.

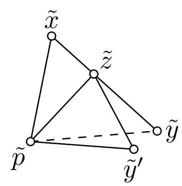
Proof. Consider the model triangle $[\tilde x\tilde p\tilde z]=\tilde\triangle(xpz)$. Take a point $\tilde y$ on the extension of $[\tilde x\tilde z]$ beyond $\tilde z$ so that $|\tilde x-\tilde y|=|x-y|$ (and therefore $|\tilde x-\tilde z|=|x-z|$).

By 2.6, the following expressions have the same sign:

(i) $\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}]-\tilde\measuredangle(x\,{}^{p}_{y})$,

(ii) $|\tilde p-\tilde y|-|p-y|$,

(iii) $\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}]-\tilde\measuredangle(z\,{}^{p}_{y})$.



Since

$$\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}]=\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde z}]=\tilde\measuredangle(x\,{}^{p}_{z})$$

and

$$\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}]=\pi-\measuredangle[\tilde z\,{}^{\tilde x}_{\tilde p}]=\pi-\tilde\measuredangle(z\,{}^{x}_{p}),$$

the statement follows.

The spherical and hyperbolic cases can be proved along the same lines. □

2.7. Exercise! *Assume p,x,y,z are as in Alexandrov's lemma (2.5). Show that*

$$\tilde\measuredangle(p\,{}^{x}_{y})\geqslant\tilde\measuredangle(p\,{}^{x}_{z})+\tilde\measuredangle(p\,{}^{z}_{y}),$$

with equality if and only if the expressions in (a) and (b) in Alexandrov's lemma vanish.

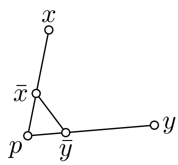
Note that

$$p\in\,]xy[\quad\Longrightarrow\quad\tilde\measuredangle(p\,{}^{x}_{y})=\pi.$$

Applying it with Alexandrov's lemma and $\mathbb{E}^2$-comparison, we get the following.

2.8. Claim. *If p,x,y,z are points in an* $\mathrm{A{\small LEX}}(0)$ *space. Suppose $z\in\,]xy[$, then*

$$\tilde\measuredangle(x\,{}^{y}_{p})\leqslant\tilde\measuredangle(x\,{}^{z}_{p}).$$



2.9. Exercise! *Let $[p\,{}^{x}_{y}]$ be a hinge in an* Alex(0) *space. Consider the function*

$$f\colon (|p-\bar{x}|, |p-\bar{y}|) \mapsto \tilde{\measuredangle}(p\,{}^{\bar{x}}_{\bar{y}}),$$

where $\bar{x} \in\,]px]$ and $\bar{y} \in\,]py]$. Show that f is nonincreasing in each argument.

The first statement in the last exercise is called a n g l e - s i d e l e n g t h m o n o t o n i c i t y. It implies the following.

2.10. Angle existence and comparison. *The angle measure of any hinge in an* Alex(0) *space is defined and is at least as large as the corresponding model angle; that is,*

$$\measuredangle[p\,{}^{x}_{y}] \geqslant \tilde{\measuredangle}(p\,{}^{x}_{y})$$

for any hinge $[p\,{}^{x}_{y}]$ in an Alex(0).

2.11. Exercise! *Let $[p\,{}^{x}_{y}]$ be a hinge in an* Alex(0) *space. Suppose $\measuredangle[p\,{}^{x}_{y}] = 0$; show that $[px] \subset [py]$ or $[py] \subset [px]$.*

Conclude that geodesics in Alex(0) *space cannot b i f u r c a t e; that is, if two geodesics $[px]$ and $[py]$ share a nontrivial arc with an end at p, then $[px] \subset [py]$ or $[py] \subset [px]$.*

2.12. Exercise. *Let $[xy]$ be a geodesic in an* Alex(0) *space. Suppose $z \in\,]xy[$. Show that there is a unique geodesic $[xz]$ and $[xz] \subset [xy]$.*

Recall that adjacent hinges are defined in 1.12.

2.13. Exercise! *Let $[p\,{}^{x}_{z}]$ and $[p\,{}^{y}_{z}]$ be adjacent hinges in an* Alex(0) *space. Show that*

$$\measuredangle[p\,{}^{x}_{z}] + \measuredangle[p\,{}^{y}_{z}] = \pi.$$

2.14. Exercise. *Let $\mathcal{A}$ be an* Alex(0) *space. Show that*

$$\tilde{\measuredangle}(x\,{}^{y}_{p}) = \tilde{\measuredangle}(x\,{}^{v}_{p}) \quad\Longleftrightarrow\quad \tilde{\measuredangle}(x\,{}^{y}_{p}) = \tilde{\measuredangle}(x\,{}^{w}_{p})$$

for any points p, x, y, v, w in $\mathcal{A}$ such that $v, w \in\,]xy[$.

2.15. Exercise (semicontinuity of angles). *Let $\mathcal{A}$ be an* Alex(0) *space. Suppose hinges $[x_n\,{}^{y_n}_{z_n}]$ in $\mathcal{A}$ converge to a hinge $[x_\infty\,{}^{y_\infty}_{z_\infty}]$; that is, geodesics $[x_n y_n]$ and $[x_n z_n]$ converge to the geodesics $[x_\infty y_\infty]$ and $[x_\infty z_\infty]$ in the sense of Hausdorff. Show that*

$$\varliminf_{n\to\infty} \measuredangle[x_n\,{}^{y_n}_{z_n}] \geqslant \measuredangle[x_\infty\,{}^{y_\infty}_{z_\infty}].$$

The last inequality might be strict; for example, on the surface of convex polyhedron, which is a Alex(0) space by 11.15.

C Hinge comparison

Let $[p\,^{x}_{y}]$ be a hinge in an $\mathrm{A{\scriptstyle LEX}}(0)$ space $\mathcal{A}$. By 2.9, the angle measure $\measuredangle[p\,^{x}_{y}]$ is defined and

$$\measuredangle[p\,^{x}_{y}] \geqslant \tilde\measuredangle(p\,^{x}_{y}).$$

Further, according to 2.13, we have

$$\measuredangle[p\,^{x}_{z}] + \measuredangle[p\,^{y}_{z}] = \pi$$

for adjacent hinges $[p\,^{x}_{z}]$ and $[p\,^{y}_{z}]$ in $\mathcal{A}$.

The following theorem provides a converse.

2.16. Theorem. *A complete geodesic space $\mathcal{A}$ is* $\mathrm{A{\scriptstyle LEX}}(0)$ *if the following conditions hold.*

(a) For any hinge $[x\,^{p}_{y}]$ in $\mathcal{A}$, the angle $\measuredangle[x\,^{p}_{y}]$ is defined and

$$\measuredangle[x\,^{p}_{y}] \geqslant \tilde\measuredangle(x\,^{p}_{y}).$$

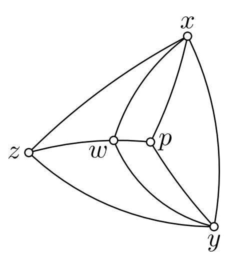
(b) For any two adjacent hinges $[p\,^{x}_{z}]$ and $[p\,^{y}_{z}]$ in $\mathcal{A}$, we have

$$\measuredangle[p\,^{x}_{z}] + \measuredangle[p\,^{y}_{z}] \leqslant \pi.$$

Proof. Consider a point $w \in\,]pz[$ close to p. From *(b)*, it follows that

$$\measuredangle[w\,^{x}_{z}] + \measuredangle[w\,^{x}_{p}] \leqslant \pi \quad \text{and} \quad \measuredangle[w\,^{y}_{z}] + \measuredangle[w\,^{y}_{p}] \leqslant \pi.$$

Since $\measuredangle[w\,^{x}_{y}] \leqslant \measuredangle[w\,^{x}_{p}] + \measuredangle[w\,^{y}_{p}]$ (see 1.11), we get



$$\measuredangle[w\,^{x}_{z}] + \measuredangle[w\,^{y}_{z}] + \measuredangle[w\,^{x}_{y}] \leqslant 2\cdot\pi.$$

Applying *(a)*,

$$\tilde\measuredangle(w\,^{x}_{z}) + \tilde\measuredangle(w\,^{y}_{z}) + \tilde\measuredangle(w\,^{x}_{y}) \leqslant 2\cdot\pi.$$

Passing to the limits as $w \to p$, we have

$$\tilde\measuredangle(p\,^{x}_{z}) + \tilde\measuredangle(p\,^{y}_{z}) + \tilde\measuredangle(p\,^{x}_{y}) \leqslant 2\cdot\pi.$$

□

D Equivalent conditions

The following theorem summarizes 2.8, 2.10, 2.13, and 2.16.

2.17. Theorem. *Let $\mathcal{A}$ be a complete geodesic space. Then the following conditions are equivalent.*

(a) $\mathcal{A}$ *is* $\mathrm{A\small{LEX}}(0)$.

(b) (adjacent angle comparison)

$$\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) \leqslant \pi$$

for any geodesic $[xy]$ *and point* $z \in\,]xy[$, $z \ne p$ *in* $\mathcal{A}$.

(c) (point-on-side comparison)

$$\tilde\measuredangle(x\,{}^p_y) \leqslant \tilde\measuredangle(x\,{}^p_z)$$

for any geodesic $[xy]$ *and* $z \in\,]xy[$ *in* $\mathcal{A}$.

(d) (hinge comparison) the angle $\measuredangle[x\,{}^p_y]$ *is defined for any hinge* $[x\,{}^p_y]$ *in* $\mathcal{A}$. *Moreover,*

$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle(x\,{}^p_y)$$

for any hinge $[x\,{}^p_y]$, *and*

$$\measuredangle[z\,{}^p_y] + \measuredangle[z\,{}^p_x] \leqslant \pi$$

for any adjacent hinges $[z\,{}^p_y]$ *and* $[z\,{}^p_x]$.

Moreover, the implications (a)⇒(b)⇒(c)⇒(d) hold in any space, not necessarily a geodesic one.

2.18. Advanced exercise. *Construct a complete geodesic space* $\mathcal{X}$ *that is not* $\mathrm{A\small{LEX}}(0)$, *but satisfies the following weaker version of the adjacent angle comparison 2.17b.*

For any three points $p, x, y \in \mathcal{X}$ *there is a geodesic* $[xy]$ *such that for any* $z \in\,]xy[$

$$\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) \leqslant \pi.$$

2.19. Exercise. *Let* $\mathcal{W}$ *be* $\mathbb{R}^n$ *with the metric induced by a norm. Show that if* $\mathcal{W}$ *is* $\mathrm{A\small{LEX}}(0)$, *then* $\mathcal{W}$ *is isometric to the Euclidean space* $\mathbb{E}^n$.

E Function comparison

Real-to-real functions. Choose $\lambda\in\mathbb{R}$. Let $s\colon\mathbb{I}\to\mathbb{R}$ be a locally Lipschitz function defined on an interval $\mathbb{I}$. The following statement are equivalent; if one (and therefore any) of them holds for s, then we say that s is λ-concave.

◇ We have inequality $s''\leqslant\lambda$, where the second derivative s'' is understood in the sense of distributions.

◇ The function $t\mapsto s(t)-\lambda\cdot\frac{t^2}{2}$ is concave.

◇ The Jensen inequality

$$s(a\cdot t_0+(1-a)\cdot t_1)\geqslant a\cdot s(t_0)+(1-a)\cdot s(t_1)+\tfrac{\lambda}{2}\cdot a\cdot(1-a)\cdot(t_1-t_0)^2$$

holds for any $t_0,t_1\in\mathbb{I}$ and $a\in[0,1]$.

◇ for any $t_0\in\mathbb{I}$ there is a quadratic polynomial $\ell=\frac{\lambda}{2}\cdot t^2+a\cdot t+b$ (it is called a barrier) that supports (locally) s at t_0 from above; that is, $\ell(t_0)=s(t_0)$ and $\ell(t)\geqslant s(t)$ for any t (in a neighborhood of t_0)

To prove equivalence, approximate f by smooth functions taking a convolutions $f_n=f*k_n$ for a suitable sequence of kernels k_n. Note that all the conditions are equivalent for f_n; passing to the limit we get the same for f.

2.20. Exercise. *Show that λ-concave functions are one-sided differentiable.*

The following exercise implies that if the function defined on an open interval, then the Lipschitz condition can be dropped from the definition of λ-concavity.

2.21. Exercise. *Suppose a real-to-real function f is defined on an open inerval and for some $\lambda\in\mathbb{R}$, it satisfies one the Jensen inequality stated above. Show that f is locally Lipscitz.*

Functions on metric spaces. A function on a metric space $\mathcal{A}$ will usually mean a *locally Lipschitz real-valued function defined on an open subset of $\mathcal{A}$*. The domain of a function f will be denoted by $\operatorname{Dom} f$.

We say that f is λ-concave (briefly $f''\leqslant\lambda$) if for any unit-speed geodesic $\gamma\colon\mathbb{I}\to\operatorname{Dom} f$ the real-to-real function $t\mapsto f\circ\gamma(t)$ is λ-concave.

The following proposition is simple but conceptual — it reduces a global comparison to an infinitesimal condition on distance functions.

2.22. Proposition. *A complete geodesic space $\mathcal{A}$ is* $\mathrm{ALEX}(0)$ *if and only if $f''\leqslant 1$ for any function f of the form*

$$f\colon x\mapsto\tfrac{1}{2}\cdot|p-x|^2.$$

Proof. Choose a unit-speed geodesic γ in $\mathcal{A}$ and two points $x = \gamma(t_0)$, $y = \gamma(t_1)$ for some $t_0 < t_1$. Consider the model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$. Let $\tilde{\gamma}\colon [t_0, t_1] \to \mathbb{E}^2$ be the unit-speed parametrization of $[\tilde{x}\tilde{y}]$ from $\tilde{x}$ to $\tilde{y}$.

Set

$$\tilde{r}(t) := |\tilde{p} - \tilde{\gamma}(t)|, \qquad\qquad r(t) := |p - \gamma(t)|.$$

Clearly, $\tilde{r}(t_0) = r(t_0)$ and $\tilde{r}(t_1) = r(t_1)$. Note that the point-on-side comparison (2.17*c*) says that the implication

$$❶ \qquad t_0 \leqslant t \leqslant t_1 \qquad \Longrightarrow \qquad \tilde{r}(t) \leqslant r(t)$$

holds for any γ and $t_0 < t_1$.

Observe that $(\tilde{r}^2)'' \equiv 2$. If $(r^2)'' \leqslant 2$, then $t \mapsto r^2(t) - \tilde{r}^2(t)$ is concave. Since $\tilde{r}(t_0) = r(t_0)$ and $\tilde{r}(t_1) = r(t_1)$, we get ❶, which proves the if part.

On the other hand, ❶ implies the Jensen inequality for the function $h\colon t \mapsto r(t)^2 - t^2$. Since the subinterval $[t_0, t_1]$ can be chosen arbitrarily, we conclude that $h'' \leqslant 0$, or, equivalently, $(\frac{1}{2}\cdot r^2)'' \leqslant 2$, and the only-if part follows. □

F Semiconcave functions

Recall that λ-concave functions were defined in Section 2E, and when we say *function* we usually mean a *locally Lipschitz function defined on an open domain.*

Let f be a locally Lipschitz real-valued function defined in an open subset $\operatorname{Dom} f$ of an Alexandrov space $\mathcal{A}$. Suppose φ is a continuous function defined in $\operatorname{Dom} f$. We will write $f'' \leqslant \varphi$ if for any point $x \in \operatorname{Dom} f$ and any $\varepsilon > 0$ there is a neighborhood $U \ni x$ such that the restriction $f|_U$ is $(\varphi(x) + \varepsilon)$-concave.

If $f'' \leqslant \varphi$ for some continuous function φ, then f is called semiconcave.

2.23. Exercise. *Let f be a distance function on an* ALEX(0) *space $\mathcal{A}$; that is, $f(x) \equiv |p - x|$ for some $p \in \mathcal{A}$. Show that $f'' \leqslant \frac{1}{f}$. In particular, f is semiconcave in $\mathcal{A} \setminus \{p\}$.*

Proposition 2.22 admits the following generalization. The is nearly the same, but the formulas are getting more complicated.

2.24. Proposition. *A complete geodesic space $\mathcal{A}$ is* ALEX(∓1) *if $f'' \leqslant \pm f$ for any function of the type $f = \cosh\circ\operatorname{dist}_p$ (respectively, $f = -\cos\circ\operatorname{dist}_p|_{\mathrm{B}(p,\pi)}$).*

The geometric meaning of these inequalities remains the same: *distance functions are more concave than distance functions in* $\mathbb{M}^2(\kappa)$.

G Remarks

Note that Alexandrov's lemma is a result in neutral geometry; it has the following useful variation; see [6, 10.2] or [3, 3.3].

2.25. Overlap lemma. *Let $\tilde x^1$, $\tilde x^2$, $\tilde x^3$, $\tilde p^1$, $\tilde p^2$, ans $\tilde p^3$ be points in $\mathbb{E}^2$, $\mathbb{S}^2$, or $\mathbb{H}^2$. Assume that, for any permutation $\{i,j,k\}$ of $\{1,2,3\}$, we have*

(i) $|\tilde p^i-\tilde x^k|=|\tilde p^j-\tilde x^k|$,

(ii) $\tilde p^i$ *and* $\tilde x^i$ *lie in the same closed half-space determined by* $[\tilde x^j\tilde x^k]$,

If no pair of triangles $[\tilde p^i\tilde x^j\tilde x^k]$ *overlap, then*

$$\measuredangle\tilde p^1+\measuredangle\tilde p^2+\measuredangle\tilde p^3>2\cdot\pi,$$

where $\measuredangle\tilde p^i:=\measuredangle[\tilde p^i\,{}^{\tilde x^k}_{\tilde x^j}]$ *for a permutation* $\{i,j,k\}$ *of* $\{1,2,3\}$.

The condition *(b)* in 2.16 might be superfluous. This is a long-standing open problem possibly dating back to Alexandrov [17, footnote in 4.1.5]. Let us state it formally.

2.26. Open question. *Let $\mathcal{A}$ be a complete geodesic space (you can also assume that $\mathcal{A}$ is homeomorphic to $\mathbb{S}^2$ or $\mathbb{R}^2$) such that for any hinge $[x\,{}^p_y]$ in $\mathcal{A}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and*

$$\measuredangle[x\,{}^p_y]\geqslant\tilde\measuredangle(x\,{}^p_y).$$

Is it true that $\mathcal{A}$ is an Alexandrov space?

Our 4-point comparison in Section 2A is closely related to the so-called CAT comparison, which defines an *upper* curvature bound in the sense of Alexandrov; this is the subject of our previous book [5].

In both comparisons we check certain conditions on the 6 distances between pairs of points in a 4-point set. Michael Gromov [37, Section 1.19_+] suggested considering other conditions of that type for n-point subsets; see [29, 38, 58–61, 63, 65, 89, 107] for the development of this idea.

One coul define Alxandrov space as a complete *length* space with curvature $\geqslant\kappa$. This condition is more natural and general, but many statements can be reduced to the geodesic case. In particular, suppose $\mathcal{A}$ is a complete length space with curvature $\geqslant\kappa$, then $\mathcal{A}$ *can be isometrically embedded into an* $\mathrm{ALEX}(\kappa)$ *space* — the ultrapower of $\mathcal{A}$; see [6, 4.11+8.4]. Also, by Plaut's theorem, any point p in $\mathcal{A}$ can be connected by geodesics to *most* of points in $\mathcal{A}$ [6, 8.11]; compare to 6.19c.

Lecture 3

Globalization

The globalization theorem states that a locally Alexandrov space is globally Alexandrov. We prove it for compact spaces and indicate the proof of the general case.

A Globalization

A complete geodesic metric space $\mathcal{A}$ is locally $\mathrm{A{\scriptsize LEX}}(0)$ if any point $p \in \mathcal{A}$ admits a neighborhood $U \ni p$ such that the $\mathbb{E}^2$-comparison holds for any quadruple of points in U.

It is straightforward to obtain a local version of Theorem 2.17 for locally $\mathrm{A{\scriptsize LEX}}(0)$ spaces; it gives a list of equivalent properties that hold in sufficiently small neighborhoods of any point. In particular, the analog of 2.17*d* implies that *the angle measure of any hinge in locally* $\mathrm{A{\scriptsize LEX}}(0)$ *space is well defined.*

3.1. Globalization theorem. *Any compact locally* $\mathrm{A{\scriptsize LEX}}(0)$ *space is* $\mathrm{A{\scriptsize LEX}}(0)$.

Proof modulo the key lemma (3.2). Note that condition 2.16*b* holds in $\mathcal{A}$ (the proof is the same). It remains to check 2.16*a*; that is,

$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle(x\,{}^p_y) \qquad ❶$$

for any hinge $[x\,{}^p_y]$ in $\mathcal{A}$.

Inequality ❶ holds for hinges in a small neighborhood of any point; this can be proved the same way as 2.10 and 2.13, applying the local version of the $\mathbb{E}^2$-comparison. Since $\mathcal{A}$ is compact, there is $\varepsilon > 0$ such that ❶ holds if $|x-p|+|p-y|<\varepsilon$. Applying the key lemma several times we get that ❶ holds for any given hinge. $\square$

3.2. Key lemma. *Let $\mathcal{A}$ be a locally* Alex(0) *space. Assume that the comparison*

$$\measuredangle[x\,^p_q] \geqslant \tilde{\measuredangle}(x\,^p_q)$$

holds for any hinge $[x\,^p_q]$ with $|x-y|+|x-q|<\tfrac{2}{3}\cdot\ell$. Then the comparison

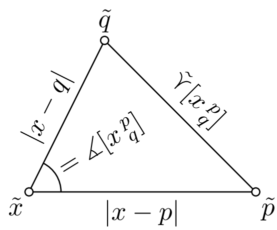
$$\measuredangle[x\,^p_q] \geqslant \tilde{\measuredangle}(x\,^p_q)$$

holds for any hinge $[x\,^p_q]$ with $|x-p|+|x-q|<\ell$.

Let $[x\,^p_q]$ be a hinge in $\mathcal{A}$. Denote by $\tilde{\Upsilon}[x\,^p_q]$ its model side; this is the opposite side in a flat triangle with the same angle and two adjacent sides as in $[x\,^p_q]$.

More precisely, consider the model hinge $[\tilde{x}\,^{\tilde{p}}_{\tilde{q}}]$ in $\mathbb{E}^2$ such that

$$\begin{aligned}
\measuredangle[\tilde{x}\,^{\tilde{p}}_{\tilde{q}}]_{\mathbb{E}^2} &= \measuredangle[x\,^p_q]_{\mathcal{A}},\\
|\tilde{x}-\tilde{p}|_{\mathbb{E}^2} &= |x-p|_{\mathcal{A}},\\
|\tilde{x}-\tilde{q}|_{\mathbb{E}^2} &= |x-q|_{\mathcal{A}};
\end{aligned}$$



then

$$\tilde{\Upsilon}[x\,^p_q]_{\mathcal{A}} := |\tilde{p}-\tilde{q}|_{\mathbb{E}^2}.$$

Note that

$$\tilde{\Upsilon}[x\,^p_q] \geqslant |p-q| \quad\iff\quad \measuredangle[x\,^p_q] \geqslant \tilde{\measuredangle}(x\,^p_q).$$

Proof. Consider a hinge $[x\,^p_q]$ such that

❷ $$\tfrac{2}{3}\cdot\ell \leqslant |p-x|+|x-q|<\ell.$$

It is sufficient to prove that

❸ $$\tilde{\Upsilon}[x\,^p_q] \geqslant |p-q|$$

First, let us describe a construction of a new hinge $[x'\,^p_q]$, which has many properties; in particular,

❹ $$|p-x|+|x-q| \geqslant |p-x'|+|x'-q|,$$

and

❺ $$\tilde{\Upsilon}[x\,^p_q] \geqslant \tilde{\Upsilon}[x'\,^p_q].$$

Construction. Assume $|x-q|\geqslant|x-p|$; otherwise, switch the roles of p and q. Take $x'\in[xq]$ such that

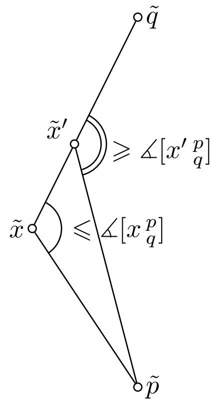
❻ $$|p-x|+3{\cdot}|x-x'|=\tfrac{2}{3}{\cdot}\ell.$$

Choose a geodesic $[x'p]$ and consider the hinge $[x'\,{}^p_q]$ formed by $[x'p]$ and $[x'q]\subset[xq]$. The triangle inequality implies ❹. Furthermore,

$$|p-x|+|x-x'|<\tfrac{2}{3}{\cdot}\ell,\qquad |p-x'|+|x'-x|<\tfrac{2}{3}{\cdot}\ell.$$

In particular, the assumption of the lemma implies that

❼ $$\measuredangle[x\,{}^{p}_{x'}]\geqslant\tilde\measuredangle(x\,{}^{p}_{x'})\quad\text{and}\quad\measuredangle[x'\,{}^{p}_{x}]\geqslant\tilde\measuredangle(x'\,{}^{p}_{x}).$$



Let $[\tilde x\tilde x'\tilde p]=\tilde\triangle(xx'p)$. Take $\tilde q$ on the extension of $[\tilde x\tilde x']$ beyond x' such that $|\tilde x-\tilde q|=|x-q|$ (and therefore $|\tilde x'-\tilde q|=|x'-q|$). By ❼,

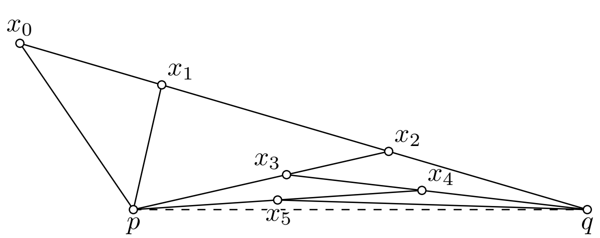
$$\measuredangle[x\,{}^p_q]=\measuredangle[x\,{}^{p}_{x'}]\geqslant\tilde\measuredangle(x\,{}^{p}_{x'})\quad\Rightarrow\quad\tilde\curlyvee[x\,{}^q_p]\geqslant|\tilde p-\tilde q|.$$

Hence

$$\begin{aligned}\measuredangle[\tilde x'\,{}^{\tilde p}_{\tilde q}]&=\pi-\tilde\measuredangle(x'\,{}^p_x)\geqslant\\&\geqslant\pi-\measuredangle[x'\,{}^p_x]=\\&=\measuredangle[x'\,{}^p_q],\end{aligned}$$

and ❺ follows.

Let us continue the proof. Set $x_0=x$. Let us apply inductively the above construction to get a sequence of hinges $[x_n\,{}^p_q]$ with $x_{n+1}=x'_n$. From ❺, we have that the sequence $s_n=\tilde\curlyvee[x_n\,{}^p_q]$ is nonincreasing.



The sequence x_n terminates at some n if $|p-x_n|+|x_n-q|<\tfrac{2}{3}{\cdot}\ell$. In this case, by the assumptions of the lemma, $\tilde\curlyvee[x_n\,{}^p_q]\geqslant|p-q|$. Since the sequence s_n is nonincreasing, inequality ❸ follows.

From now we assume that the process does not terminate. Let us prove the following claim.

❽ *The distances* $|x_n - x_{n+1}|$, $|x_n - p|$, *and* $|x_n - q|$ *are bounded away from zero for all large* n.

Set

$$\begin{aligned} a_n &= \min\{|p - x_n|, |q - x_n|\},\\ b_n &= \max\{|p - x_n|, |q - x_n|\},\\ r_n &= |p - x_n| + |x_n - q| = a_n + b_n. \end{aligned}$$

By the triangle inequality, r_n is a nonincreasing sequence, and since x_n does not terminate, we have $\tfrac{2}{3}\cdot\ell \leqslant r_n < \ell$ for all n.

By ❻, $|x_n - x_{n+1}| = \tfrac{1}{3}\cdot(\tfrac{2}{3}\cdot\ell - a_n)$. Since $a_n + b_n = r_n < l$ and $a_n \leqslant b_n$ it holds that $a_n \leqslant \tfrac{1}{2}\cdot\ell$. Hence

❾ $$\tfrac{2}{9}\cdot\ell \geqslant |x_n - x_{n+1}| \geqslant \tfrac{1}{18}\cdot\ell,$$

which proves the claim for $|x_n - x_{n+1}|$.

Note that

$$a_{n+1} = b_n - |x_n - x_{n+1}| \qquad \text{or} \qquad b_{n+1} = b_n - |x_n - x_{n+1}|$$

In the latter case,

$$\begin{aligned} a_{n+1} &= r_{n+1} - b_{n+1} =\\ &= (r_{n+1} - r_n) + (r_n - b_n) + |x_n - x_{n+1}|\\ &= -(r_n - r_{n+1}) + a_n + |x_n - x_{n+1}|. \end{aligned}$$

Since r_n does not increase, it must converge; so, $r_n - r_{n+1} \to 0$ as $n \to \infty$. Therefore, $r_n - r_{n+1} \leqslant \tfrac{1}{100}\cdot\ell$ for all large n. Observe that $b_n \geqslant \tfrac{1}{2}\cdot r_n$, and therefore $b_n \geqslant \tfrac{1}{3}\cdot\ell$. Taking ❾ into account, we get that in both cases $a_{n+1} \geqslant \tfrac{1}{100}\cdot\ell$ for all large n, which finishes the proof of the claim. △

Since $r_n - r_{n+1} \to 0$, ❽ implies that $\tilde\measuredangle(x_n\,{}^{p_n}_{x_{n+1}}) \to \pi$, where $p_n = p$ if $x_{n+1} \in [x_n q]$, and otherwise $p_n = q$. Since $\measuredangle[x_n\,{}^{p_n}_{x_{n+1}}] \geqslant \tilde\measuredangle(x_n\,{}^{p_n}_{x_{n+1}})$, we have $\measuredangle[x_n\,{}^{p_n}_{x_{n+1}}] \to \pi$ as $n \to \infty$.

It follows that

$$r_n - s_n = |p - x_n| + |x_n - q| - \tilde\Upsilon[x_n\,{}^{p}_{q}] \to 0.$$

By the triangle inequality $r_n \geqslant |p - q|$; therefore,

$$\lim_{n\to\infty} s_n \geqslant |p - q|.$$

Finally, the monotonicity of the sequence $s_n = \tilde\Upsilon[x_n\,{}^{p}_{q}]$ implies ❸. □

B On general case

3.3. Theorem. *Any locally* $\mathrm{A\scriptsize LEX}(\kappa)$ *space is* $\mathrm{A\scriptsize LEX}(\kappa)$*.*

This is a more general version of our globalization theorem 3.1. It holds for an arbitrary curvature bound κ and replaces compactness with completeness. A complete proof can be found in [6, 2F]; it is based on the idea from 3.1, but requires a few additional twists that we are about to discuss.

Trading compactness for completeness. Assume for a moment that $\kappa = 0$. To replace compactness by completeness, we need the following statement.

3.4. Exercise. *Let $\mathcal{X}$ be a complete metric space. Suppose $r\colon \mathcal{X} \to \mathbb{R}$ is a positive continuous function. Show that for any $\varepsilon > 0$, there exists a point $p \in \mathcal{X}$ such that*

$$r(x) > (1-\varepsilon)\cdot r(p)$$

for every $x \in \overline{\mathrm{B}}[p, \tfrac{1}{\varepsilon}\cdot r(p)]$.

Define $r(x)$ to be the maximal radius of a ball centered at x in which the comparison condition holds (this definition should be made precise, but the details are not important here). Note that if $r(x) = \infty$ for one (and therefore for every) point x, then the theorem follows immediately. To prove the theorem, we need to choose a point p satisfying the conclusion of the exercise and apply the key lemma.

On general curvature bounds. The case $\kappa \leqslant 0$ in 3.3 can be proved in the same way as the case $\kappa = 0$, but the case $\kappa > 0$ requires extra care. This is because model triangles may be undefined in $\mathbb{M}^2(\kappa)$ when $\kappa > 0$.[1]

Meaning of curvature bounded below. Recall that an $\mathrm{A\scriptsize LEX}(\kappa)$ space is defined as a complete geodesic space having "curvature $\geqslant \kappa$ in the sense of Alexandrov." This suggests that the following claim should hold.

3.5. Claim. *If* $\mathrm{K} > \kappa$*, then any* $\mathrm{A\scriptsize LEX}(\mathrm{K})$ *space is* $\mathrm{A\scriptsize LEX}(\kappa)$*.*

The proof of this statement uses 3.3.

Proof. By 1.8,

$$\tilde{\measuredangle}(x\,{}^{y}_{z})_{\mathbb{M}^2(\kappa)} \leqslant \tilde{\measuredangle}(x\,{}^{y}_{z})_{\mathbb{M}^2(\mathrm{K})}$$

[1] Recall that according to the definitions in Section 1G, if $\tilde{\measuredangle}(p\,{}^{q}_{r})_{\mathbb{M}^2(\kappa)}$, and hence $\tilde{\triangle}(pqr)_{\mathbb{M}^2(\kappa)}$, are defined for some $\kappa > 0$, then $|p-q| + |q-r| + |r-p| < 2\cdot\pi/\sqrt{\kappa}$.

whenever the right-hand side is defined. This completes the proof in the case $K \leqslant 0$. In general, however, the comparison holds only for sufficiently small triangles; in particular, any $\mathrm{ALEX}(K)$ space is locally $\mathrm{ALEX}(\kappa)$. Therefore, 3.3 completes the proof. $\square$

While the expression "curvature bounded below by κ" makes sense for geodesic spaces, it does not make much sense for general metric spaces. For example, let $\mathcal{X}$ be the set $\{p, x_1, x_2, x_3\}$ with the metric defined by $|p - x_i| = \pi$ and $|x_i - x_j| = 2{\cdot}\pi$ for all $i \neq j$. Since $\mathcal{X}$ has no defined spherical model angles, it formally has curvature $\geqslant 1$. On the other hand, $\mathcal{X}$ does not have curvature $\geqslant 0$, since

$$\tilde\measuredangle(p\,{}^{x_1}_{x_2})_{\mathbb{E}^2} = \tilde\measuredangle(p\,{}^{x_2}_{x_3})_{\mathbb{E}^2} = \tilde\measuredangle(p\,{}^{x_1}_{x_3})_{\mathbb{E}^2} = \pi.$$

3.6. Exercise! *Let p and q be points in an* $\mathrm{ALEX}(1)$ *space $\mathcal{A}$. Suppose $|p-q| > \pi$, and let m be the midpoint of $[pq]$. Show that for any hinge $[m\,{}^{x}_{p}]$, we have either $\measuredangle[m\,{}^{x}_{p}] = 0$ or $\measuredangle[m\,{}^{x}_{p}] = \pi$.*

Conclude that $\mathcal{A}$ is isometric to a line segment or a circle.

3.7. Exercise. *Suppose $\mathcal{A}$ is an* $\mathrm{ALEX}(1)$ *and* $\operatorname{diam}\mathcal{A} \leqslant \pi$*. Show that*

$$|x - y| + |y - z| + |z - x| \leqslant 2{\cdot}\pi$$

for any triple of points $x, y, z \in \mathcal{A}$.

C Remarks

The globalization theorem is also known as the *generalized Toponogov theorem.* Its two-dimensional case was proved by Paolo Pizzetti [91] and rediscovered by Alexandr Alexandrov [11]. Victor Toponogov [106] proved it for Riemannian manifolds of all dimensions. For Alexandrov spaces of all dimensions, the theorem first appears in the paper of Michael Gromov, Yuriy Burago, and Grigory Perelman [18]. They prove globalization *complete length spaces.* Another version for *noncomplete, but geodesic spaces* was proved by the second author [84].

We took the proof from our book [6], but reduced generality. This proof is based on simplifications obtained by Conrad Plaut [92] and Dmitry Burago, Yuriy Burago, and Sergei Ivanov [17]. The same proof was rediscovered independently by Urs Lang and Viktor Schroeder [57]. Another simplified argument was found by Katsuhiro Shiohama [102].

Lecture 4

Calculus

This lecture defines several notions related to the first-order derivatives in Alexandrov spaces, including space of directions, tangent space, differential, and gradient.

A Space of directions

Let $\mathcal{A}$ be an Alexandrov space. By 2.9, the angle measure of any hinge in $\mathcal{A}$ is defined. Given $p \in \mathcal{A}$, consider the set $\mathfrak{S}_p$ of all nontrivial geodesics starting at p. By 1.11, the triangle inequality holds for $\measuredangle$ on $\mathfrak{S}_p$, that is, $(\mathfrak{S}_p, \measuredangle)$ forms a semimetric space; that is, $\measuredangle$ behaves like a metric, but might vanish for distinct geodesics.

The metric space corresponding to $(\mathfrak{S}_p, \measuredangle)$ is called the space of geodesic directions at p, denoted by Σ'_p or $\Sigma'_p\mathcal{A}$. The elements of Σ'_p are called geodesic directions at p. Each geodesic direction is formed by an equivalence class of geodesics starting from p for the equivalence relation

$$[px] \sim [py] \qquad \Longleftrightarrow \qquad \measuredangle[p\,{}^x_y] = 0;$$

the direction of $[px]$ is denoted by $\uparrow_{[px]}$. By 2.11,

$$[px] \sim [py] \qquad \Longleftrightarrow \qquad [px] \subset [py] \quad \text{or} \quad [px] \supset [py].$$

The completion of Σ'_p is called the space of directions at p and is denoted by Σ_p or $\Sigma_p\mathcal{A}$. The elements of Σ_p are called directions at p.

4.1. Exercise. *Let $\mathcal{A}$ be an Alexandrov space. Assume that a sequence of geodesics $[px_n]$ converge to a geodesic $[px_\infty]$ in the sense of Hausdorff, and $x_\infty \neq p$. Suppose Σ_p is compact. Show that $\uparrow_{[px_n]} \to \to \uparrow_{[px_\infty]}$ as $n \to \infty$.*

B Tangent space

The Euclidean cone $\mathcal{V} = \operatorname{Cone}\mathcal{X}$ over a metric space $\mathcal{X}$ is defined as the metric space whose underlying set consists of equivalence classes in $[0,\infty)\times\mathcal{X}$ with the equivalence relation "$\sim$" given by $(0,p)\sim(0,q)$ for any points $p,q\in\mathcal{X}$, and whose metric is given by the cosine rule

$$|(s,p)-(t,q)|_{\mathcal{V}} = \sqrt{s^2+t^2-2{\cdot}s{\cdot}t{\cdot}\cos\theta},$$

where $\theta = \min\{\pi, |p-q|_{\mathcal{X}}\}$.

Note that

$$\operatorname{Cone}\mathbb{S}^n \overset{iso}{=\!=} \mathbb{E}^{n+1};$$

here "$\overset{iso}{=\!=}$" stands for "isometric to".

Now let us extend several notions from Euclidean space to Euclidean cones.

The point in $\mathcal{V}$ that corresponds $(t,x)\in[0,\infty)\times\mathcal{X}$ will be denoted by $t{\cdot}x$. The point in $\mathcal{V}$ formed by the equivalence class of $\{0\}\times\mathcal{X}$ is called the origin of the cone and is denoted by 0 or $0_{\mathcal{V}}$. For $v\in\mathcal{V}$ the distance $|0-v|_{\mathcal{V}}$ is called the norm of v and is denoted by $|v|$ or $|v|_{\mathcal{V}}$. The scalar product $\langle v,w\rangle$ of $v=s{\cdot}p$ and $w=t{\cdot}q$ is defined by

$$\langle v,w\rangle := |v|{\cdot}|w|{\cdot}\cos\theta$$

where $\theta=\min\{\pi,|p-q|_{\mathcal{X}}\}$. The value θ is undefined if $v=0$ or $w=0$; in these cases we set $\langle v,w\rangle := 0$.

4.2. Exercise. *Show that* $\operatorname{Cone}\mathcal{X}$ *is geodesic if and only if* $\mathcal{X}$ *is* π-*geodesic; that is, any two points* $x,y\in\mathcal{X}$ *such that* $|x-y|_{\mathcal{X}}<\pi$ *can be joined by a geodesic in* $\mathcal{X}$.

Tangent space. The Euclidean cone $\operatorname{Cone}\Sigma_p$ over the space of directions Σ_p is called the tangent space at p and is denoted by T_p or $\mathrm{T}_p\mathcal{A}$. The elements of $\mathrm{T}_p\mathcal{A}$ will be called tangent vectors at p (despite that T_p is not a vector space). The space of directions Σ_p can be (and will be) identified with the unit sphere in T_p; that is, with the set $\{\, v\in\mathrm{T}_p : |v|=1\,\}$.

4.3. Proposition. *Any tangent space to an Alexandrov space has nonnegative curvature in the sense of Alexandrov.*

Halbeisen's example [6, 13.6] shows that the tangent space T_p at some point of Alexandrov space might fail to be geodesic; in this case T_p is *not* $\mathrm{ALEX}(0)$.

Proof. Consider the tangent space $\mathrm{T}_p = \operatorname{Cone}\Sigma_p$ of an Alexandrov space $\mathcal{A}$ at a point p. We need to show that the $\mathbb{E}^2$-comparison holds for a given quadruple $v_0, v_1, v_2, v_3\in\mathrm{T}_p$.

Recall that the space of geodesic directions Σ'_p is dense in Σ_p. It follows that the subcone $\mathrm{T}'_p = \mathrm{Cone}\,\Sigma'_p$ is dense in T_p. Therefore, it is sufficient to consider the case $v_0, v_1, v_2, v_3 \in \mathrm{T}'_p$.

For each i, choose a geodesic γ_i from p in the direction of v_i; reparametrize each γ_i so that it has speed $|v_i|$. Since the angles are defined, we have

$$❶\qquad |\gamma_i(\varepsilon) - \gamma_j(\varepsilon)|_{\mathcal{A}} = \varepsilon\cdot|v_i - v_j|_{\mathrm{T}_p} + o(\varepsilon)$$

for $\varepsilon > 0$. The quadruple $\gamma_0(\varepsilon), \gamma_1(\varepsilon), \gamma_2(\varepsilon), \gamma_3(\varepsilon)$ meets the $\mathbb{M}^2(\kappa)$-comparison. After rescaling all the distances by $\frac{1}{\varepsilon}$, it becomes the $\mathbb{M}^2(\varepsilon^2\cdot\kappa)$-comparison. Passing to the limit as $\varepsilon \to 0$ and applying ❶, we get the $\mathbb{E}^2$-comparison for v_0, v_1, v_2, v_3. $\square$

4.4. Exercise! *Let p be a point in an Alexandrov space $\mathcal{A}$, and let $\lambda_n \to \infty$. Suppose Σ_p is compact. Show that there is a pointed Gromov–Hausdorff convergence $(\lambda_n\cdot\mathcal{A}, p) \to (\mathrm{T}_p, 0)$. Moreover, the convergence can be shoosen so that for any geodesic γ that starts at p, we have*

$$\iota_n \circ \gamma(t/\lambda_n) \to t\cdot\gamma^+(0),$$

where ι_n sends a point in $\mathcal{A}$ to the corresponding point in $\lambda_n\cdot\mathcal{A}$.

C Differential

Let f be a semiconcave function on an Alexandrov space $\mathcal{A}$, and $p \in \in \mathrm{Dom}\, f$. Choose a unit-speed geodesic γ that starts at p; let $\xi \in \Sigma_p$ be its direction. Define

$$(\boldsymbol{d}_p f)(\xi) := (f \circ \gamma)^+(0),$$

here $(f\circ\gamma)^+$ denotes the right derivative of $(f \circ \gamma)$; it is defined since f is semiconcave.

By the following exercise, $\boldsymbol{d}_p f$ is a Lipschitz function on Σ'_p. It follows that the function $\boldsymbol{d}_p f\colon \Sigma'_p \to \mathbb{R}$ can be uniquely extended to a Lipschitz function $\boldsymbol{d}_p f\colon \Sigma_p \to \mathbb{R}$. Further, we can extend it to the tangent space by setting

$$(\boldsymbol{d}_p f)(r\cdot\xi) := r\cdot(\boldsymbol{d}_p f)(\xi)$$

for any $r \geqslant 0$ and $\xi \in \Sigma_p$. The obtained function $\boldsymbol{d}_p f\colon \mathrm{T}_p \to \mathbb{R}$ is Lipschitz; it is called the differential of f at p.

4.5. Exercise. *Let f be a semiconcave function on an Alexandrov space. Suppose γ_1 and γ_2 are geodesics that start at $p \in \mathrm{Dom}\, f$; denote* *by θ the angle between γ_1 and γ_2 at p. Show that*

$$|(f\circ\gamma_1)^+(0)-(f\circ\gamma_2)^+(0)|\leqslant L\cdot\theta,$$

where L is the Lipschitz constant of f in a neighborhood of p.

4.6. Exercise (First variation formula)! *Let p and q be distinct points in an Alexandrov space $\mathcal{A}$.*

(a) Show that $\boldsymbol{d}_p\mathrm{dist}_q(v)\leqslant-\langle\uparrow_{[pq]},v\rangle$ for any $v\in\mathrm{T}_p$.

(b) Suppose $\mathcal{A}$ is proper. Let $\Uparrow_p^q$ be the set of all direction of geodesics from p to q. Show that

$$\boldsymbol{d}_p\mathrm{dist}_q(v)=-\max_{\xi\in\Uparrow_p^q}\langle\xi,v\rangle$$

for any $v\in\mathrm{T}_p$.

D Gradient

The following definition generalizes the gradient to semiconcave functions on Alexandrov space. This generalization is not trivial even for concave functions on Euclidean space; we suggest keeping this example in mind while reading further.

4.7. Definition. *Let f be a semiconcave function on an Alexandrov space. A tangent vector $g\in\mathrm{T}_p$ is called a gradient of f at p (briefly, $g=\nabla_p f$) if*

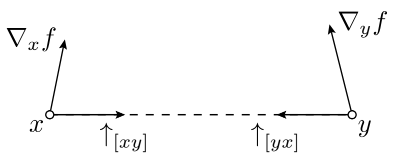
(a) $(\boldsymbol{d}_p f)(w)\leqslant\langle g,w\rangle$ for any $w\in\mathrm{T}_p$, and

(b) $(\boldsymbol{d}_p f)(g)=\langle g,g\rangle$.

The following exercise provides a property of gradients that will play a key role in the proof of the first distance estimate (5.6).

4.8. Exercise! *Let f be a λ-concave function on an Alexandrov space. Suppose that gradients $\nabla_x f$ and $\nabla_y f$ are defined. Show that*

$$\langle\uparrow_{[xy]},\nabla_x f\rangle+\langle\uparrow_{[yx]},\nabla_y f\rangle+\lambda\cdot|x-y|\geqslant 0.$$



4.9. Proposition. *Suppose that a semiconcave function f is defined in a neighborhood of a point p in an Alexandrov space. Then the gradient $\nabla_p f$ is uniquely defined.*

Moreover, if $\boldsymbol{d}_p f \leqslant 0$, then we have $\nabla_p f = 0$; otherwise, $\nabla_p f = s\cdot\bar\xi$, where $s = \boldsymbol{d}_p f(\bar\xi)$ and $\bar\xi \in \Sigma_p$ is the direction that maximize the value $\boldsymbol{d}_p f(\xi)$ for $\xi \in \Sigma_p$.

4.10. Key lemma. *Let f be a semiconcave function that is defined in a neighborhood of a point p in an Alexandrov space $\mathcal{A}$. Then for any $u, v \in \mathrm{T}_p$, we have*

$$s\cdot\sqrt{|u|^2 + 2\cdot\langle u, v\rangle + |v|^2} \geqslant (\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v),$$

where

$$s = \sup\{\, (\boldsymbol{d}_p f)(\xi) \;:\; \xi \in \Sigma_p \,\}.$$

If $\mathrm{T}_p \stackrel{iso}{=\!=} \mathbb{E}^m$ and $\boldsymbol{d}_p f$ is a concave function, then

$$2\cdot(\boldsymbol{d}_p f)(\tfrac{u+v}{2}) \geqslant (\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v).$$

The latter implies the statement since $|u+v| = \sqrt{|u|^2 + 2\cdot\langle u, v\rangle + |v|^2}$. In general, T_p is not geodesic (and not even a length space), so concavity of $\boldsymbol{d}_p f$ does not make sense. The key lemma however says that in a certain sense $\boldsymbol{d}_p f$ behaves as a concave function.

Solving the following exercise should help to find an approach to the key lemma.

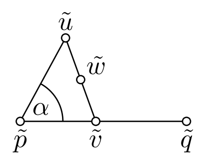
4.11. Exercise! *Let p and q be distinct points in an Alexandrov space $\mathcal{A}$. Suppose the geodesic $[pq]$ can be extended beyond q.*

Show that

$$\boldsymbol{d}_p \mathrm{dist}_q(v) = -\langle \uparrow_{[pq]}, v\rangle$$

for any $v \in \mathrm{T}_p$.

Proof of 4.10. We will assume that $\mathcal{A}$ is $\mathrm{ALEX}(0)$ and f is concave; the general case requires only minor modifications. We can assume that $v \neq 0$, $w \neq 0$, and $\alpha = \measuredangle(u, v) > 0$; otherwise, the statement is trivial.



Consider a model configuration of five points: $\tilde p$, $\tilde u$, $\tilde v$, $\tilde q$, $\tilde w \in \mathbb{E}^2$ such that

- $\measuredangle[\tilde p\,{}^{\tilde u}_{\tilde v}] = \alpha$,
- $|\tilde p - \tilde u| = |u|$,
- $|\tilde p - \tilde v| = |v|$,
- $\tilde q$ lies on an extension of $[\tilde p\tilde v]$ so that $\tilde v$ is the midpoint of $[\tilde p\tilde q]$,
- $\tilde w$ is the midpoint between $\tilde u$ and $\tilde v$.

Note that

❶
$$|\tilde p-\tilde w|=\tfrac12\cdot\sqrt{|u|^2+2\cdot\langle u,v\rangle+|v|^2}.$$

Since the geodesic space of directions Σ'_p is dense in Σ_p, we can assume that there are geodesics in the directions of u and v. Choose such geodesics γ_u and γ_v and assume that they are parametrized with speed $|u|$ and $|v|$ respectively. For all small $t>0$, consider points $u_t,v_t,q_t,w_t\in\mathcal{A}$ such that

◇ $v_t=\gamma_v(t)$, $\quad q_t=\gamma_v(2\cdot t)$

◇ $u_t=\gamma_u(t)$.

◇ w_t is the midpoint of $[u_tv_t]$.

Clearly

$$|p-u_t|=t\cdot|u|,\qquad |p-v_t|=t\cdot|v|,\qquad |p-q_t|=2\cdot t\cdot|v|.$$

Since $\measuredangle(u,v)$ is defined, we have

$$|u_t-v_t|=t\cdot|\tilde u-\tilde v|+o(t),\qquad |u_t-q_t|=t\cdot|\tilde u-\tilde q|+o(t).$$

From the point-on-side and hinge comparisons (2.17c+2.17d), we have

$$\tilde\measuredangle(v_t\,{}^{p}_{w_t})\geqslant\tilde\measuredangle(v_t\,{}^{p}_{u_t})\geqslant\measuredangle[\tilde v\,{}^{\tilde p}_{\tilde u}]+\tfrac{o(t)}{t}$$

and

$$\tilde\measuredangle(v_t\,{}^{q_t}_{w_t})\geqslant\tilde\measuredangle(v_t\,{}^{q_t}_{u_t})\geqslant\measuredangle[\tilde v\,{}^{\tilde q}_{\tilde u}]+\tfrac{o(t)}{t}.$$

Clearly, $\measuredangle[\tilde v\,{}^{\tilde p}_{\tilde u}]+\measuredangle[\tilde v\,{}^{\tilde q}_{\tilde u}]=\pi$. From the adjacent angle comparison (2.17b), $\tilde\measuredangle(v_t\,{}^{p}_{u_t})+\tilde\measuredangle(v_t\,{}^{u_t}_{q_t})\leqslant\pi$. Hence $\tilde\measuredangle(v_t\,{}^{p}_{w_t})\to\measuredangle[\tilde v\,{}^{\tilde p}_{\tilde w}]$ as $t\to0+$ and thus

$$|p-w_t|=t\cdot|\tilde p-\tilde w|+o(t).$$

Without loss of generality, we can assume that $f(p)=0$. Since f is concave, we have

$$\begin{aligned}2\cdot f(w_t)&\geqslant f(u_t)+f(v_t)=\\&=t\cdot[(\boldsymbol{d}_pf)(u)+(\boldsymbol{d}_pf)(v)]+o(t).\end{aligned}$$

Applying concavity of f, we have

$$\begin{aligned}(\boldsymbol{d}_pf)(\uparrow_{[pw_t]})&\geqslant\frac{f(w_t)}{|p-w_t|}\geqslant\\&\geqslant\frac{t\cdot[(\boldsymbol{d}_pf)(u)+(\boldsymbol{d}_pf)(v)]+o(t)}{2\cdot t\cdot|\tilde p-\tilde w|+o(t)}.\end{aligned}$$

By ❶, the key lemma follows. □

Proof of 4.9; uniqueness. If $g, g' \in \mathrm{T}_p$ are two gradients of f, then

$$\langle g, g\rangle = (\boldsymbol{d}_p f)(g) \leqslant \langle g, g'\rangle, \qquad \langle g', g'\rangle = (\boldsymbol{d}_p f)(g') \leqslant \langle g, g'\rangle.$$

Therefore,

$$|g - g'|^2 = \langle g, g\rangle - 2\cdot\langle g, g'\rangle + \langle g', g'\rangle \leqslant 0.$$

It follows that $g = g'$.

Existence. If $\boldsymbol{d}_p f \leqslant 0$, then one can take $\nabla_p f = 0$.

Suppose $s = \sup\{\,(\boldsymbol{d}_p f)(\xi) \,:\, \xi \in \Sigma_p\,\} > 0$, it is sufficient to show that there is $\overline{\xi} \in \Sigma_p$ such that

❷ $$(\boldsymbol{d}_p f)\left(\overline{\xi}\right) = s.$$

Indeed, suppose $\overline{\xi}$ exists. Applying 4.10 for $u = \overline{\xi}$, $v = \varepsilon\cdot w$ with $\varepsilon \to 0+$, we get

$$(\boldsymbol{d}_p f)(w) \leqslant \langle w, s\cdot\overline{\xi}\rangle$$

for any $w \in \mathrm{T}_p$; that is, $s\cdot\overline{\xi}$ is the gradient at p.

Take a sequence of directions $\xi_n \in \Sigma_p$, such that $(\boldsymbol{d}_p f)(\xi_n) \to s$. Applying 4.10 for $u = \xi_n$ and $v = \xi_m$, we get

$$s \geqslant \frac{(\boldsymbol{d}_p f)(\xi_n) + (\boldsymbol{d}_p f)(\xi_m)}{\sqrt{2 + 2\cdot \cos\measuredangle(\xi_n, \xi_m)}}.$$

Therefore $\measuredangle(\xi_n, \xi_m) \to 0$ as $n, m \to \infty$; that is, $\xi_1, \xi_2, \ldots$ is a Cauchy sequence. Clearly, $\overline{\xi} = \lim_n \xi_n$ meets ❷. ☐

4.12. Exercise! *Let f and g be locally Lipschitz semiconcave functions defined in a neighborhood of a point p in an Alexandrov space. Show that*

$$|\nabla_p f - \nabla_p g|^2_{\mathrm{T}_p} \leqslant s\cdot(|\nabla_p f| + |\nabla_p g|),$$

where

$$s = \sup\{\,|(\boldsymbol{d}_p f)(\xi) - (\boldsymbol{d}_p g)(\xi)| \,:\, \xi \in \Sigma_p\,\}.$$

Conclude that if the sequence of restrictions $\boldsymbol{d}_p f_n|_{\Sigma_p}$ converges uniformly, then $\nabla_p f_n$ converges as $n \to \infty$. Here we assume that all functions f_1, $f_2, \ldots$ are semiconcave and locally Lipschitz.

4.13. Exercise! *Let f be a locally Lipschitz λ-concave function on an Alexandrov space $\mathcal{A}$.*

(a) *Suppose $s \geqslant 0$. Show that $|\nabla_x f| > s$ if and only if for some point y we have*

$$f(y) - f(x) > s\cdot\ell + \lambda\cdot\tfrac{\ell^2}{2},$$

were $\ell = |x - y|$.

(b) Show that $x \mapsto |\nabla_x f|$ is lower semicontinuous; that is,

$$|\nabla_{x_\infty} f| \leqslant \varliminf_{x_n \to x_\infty} |\nabla_{x_n} f|.$$

Lecture 5

Gradient flow

Here we define the gradient flow and prove the distance estimates.

A Velocity of curve

Let α be a curve in an Alexandrov space $\mathcal{A}$. If for any choice of geodesics $[p\,\alpha(t_0+\varepsilon)]$ the vectors

$$\tfrac{1}{\varepsilon}\cdot|p-\alpha(t_0+\varepsilon)|\cdot\uparrow_{[p\alpha(t_0+\varepsilon)]}$$

converge as $\varepsilon\to 0+$, then their limit in T_p is called the right derivative of α at t_0; it will be denoted by $\alpha^+(t_0)$. In addition, we assume that $\alpha^+(t_0):=0$ if $\tfrac{1}{\varepsilon}\cdot|p-\alpha(t_0+\varepsilon)|\to 0$ as $\varepsilon\to 0+$.

The tangent vector $v=|p-x|\cdot\uparrow_{[px]}$ will be called the logarithm of x at p (briefly, $v=\log_p x$). The logarithm is a multivalued function from $\mathcal{A}$ to T_p; so, $v=\log_p x$ and $w=\log_p x$ does *not* mean $v=w$. Note that $\gamma^+(0)=\log_p x$ for any geodesic path γ from p to x.

5.1. Claim. *Let α be a curve in an Alexandrov space $\mathcal{A}$. Suppose f is a semiconcave Lipschitz function defined in a neighborhood of $p=\alpha(0)$, and $\alpha^+(0)$ is defined. Then $(f\circ\alpha)^+(0)$ exists and*

$$(f\circ\alpha)^+(0)=(\boldsymbol{d}_pf)(\alpha^+(0)).$$

Proof. Without loss of generality, we can assume that $f(p)=0$. Suppose f and therefore $\boldsymbol{d}_pf$ are L-Lipschitz.

Let γ be a geodesic with a constant-speed reparametrization that starts from p, and such that the distance $s=|\alpha^+(0)-\gamma^+(0)|_{\mathrm{T}_p}$ is small. By the definition of differential,

$$(f\circ\gamma)^+(0)=\boldsymbol{d}_pf(\gamma^+(0)).$$

By comparison and the definition of α^+,

$$|\alpha(\varepsilon) - \gamma(\varepsilon)|_{\mathcal{A}} \leqslant s\cdot\varepsilon + o(\varepsilon)$$

for $\varepsilon > 0$. Therefore,

$$|f \circ \alpha(\varepsilon) - f \circ \gamma(\varepsilon)| \leqslant L\cdot s\cdot\varepsilon + o(\varepsilon).$$

Suppose $(f \circ \alpha)^+(0)$ is defined. Then

$$|(f \circ \alpha)^+(0) - (f \circ \gamma)^+(0)| \leqslant L\cdot s.$$

Since $\boldsymbol{d}_p f$ is L-Lipschitz, we also get

$$|\boldsymbol{d}_p f(\alpha^+(0)) - \boldsymbol{d}_p f(\gamma^+(0))| \leqslant L\cdot s.$$

It follows that the needed identity holds up to error $2\cdot L\cdot s$. The statement follows since $s > 0$ can be chosen arbitrarily.

The same argument is applicable if in place of $(f \circ \alpha)^+(0)$ we use any limit of $\frac{1}{\varepsilon_n}\cdot[f \circ \alpha(\varepsilon_n) - f(p)]$ for a sequence $\varepsilon_n \to 0+$. It proves that all such limits coincide; in particular, $(f \circ \alpha)^+(0)$ is defined and equals to $(\boldsymbol{d}_p f)(\alpha^+(0))$. $\square$

B Gradient curves

5.2. Definition. *Let f be a semiconcave function on an Alexandrov space $\mathcal{A}$.*

A locally Lipschitz curve $\alpha\colon [t_{\min}, t_{\max}) \to \operatorname{Dom} f$ will be called an f-gradient curve if

$$\alpha^+ = \nabla_\alpha f;$$

that is, for any $t \in [t_{\min}, t_{\max})$, $\alpha^+(t)$ is defined and $\alpha^+(t) = \nabla_{\alpha(t)} f$.

A complete proof of the following theorem is given in our book [6, 16.15]; it mimics the proof of the standard Picard theorem on the existence and uniqueness of solutions of ordinary differential equations. The uniqueness will follow from the first distance estimate (5.6) proved in the next section. We omit the proof of existence as it is rather lengthy.

5.3. Picard theorem. *Let $f\colon \mathcal{A} \to \mathbb{R}$ be a locally Lipschitz and λ-concave function on an Alexandrov space $\mathcal{A}$. Then for any $p \in \operatorname{Dom} f$, there are unique $t_{\max} \in (0, \infty]$ and f-gradient curve $\alpha\colon [0, t_{\max}) \to \mathcal{A}$ with $\alpha(0) = p$ such that for any sequence $t_n \to t_{\max}-$, the sequence $\alpha(t_n)$ does not have a limit point in $\operatorname{Dom} f$.*

According to the theorem, the future of a gradient curve is determined by its present. Let us show that its past is not determined by the present.

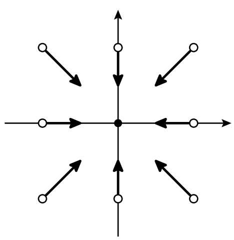
Consider the function $f\colon x\mapsto -|x|$ on the real line $\mathbb{R}$. The tangent space $\mathrm{T}_x\mathbb{R}$ can be identified with $\mathbb{R}$. Note that $\nabla_x f=-\operatorname{sgn} x$; that is,

$$\nabla_x f=\begin{cases}1 & \text{if} \quad x<0,\\ 0 & \text{if} \quad x=0,\\ -1 & \text{if} \quad x>0.\end{cases}$$

So, the f-gradient curves go to the origin with unit speed and then stand there forever. In particular, if α is an f-gradient curve that starts at x, then $\alpha(t)=0$ for any $t\geqslant|x|$.

Here is a slightly more interesting example; it shows that gradient curves can merge even in the region where $|\nabla f|\ne 0$.

5.4. Example. *Consider the function $f\colon(x,y)\mapsto -|x|-|y|$ on the (x,y)-plane. It is concave, and its gradient field is sketched on the figure.*



Let α be an f-gradient curve that starts at (x,y) for $x>y>0$. Then

$$\alpha(t)=\begin{cases}(x-t,y-t) & \textit{for} \quad 0\leqslant t\leqslant x-y,\\ (x-t,0) & \textit{for} \quad x-y\leqslant t\leqslant x,\\ (0,0) & \textit{for} \quad x\leqslant t.\end{cases}$$

C Distance estimates

5.5. Lemma. *Let α be a gradient curve of a λ-concave function f defined on an Alexandrov space. Choose a point p; let $\ell(t):=\operatorname{dist}_p\circ$ $\circ\,\alpha(t)$ and $q=\alpha(t_0)$. Then*

$$\ell^+(t_0)\leqslant-\left(f(p)-f(q)-\tfrac{\lambda}{2}\cdot\ell^2(t_0)\right)/\ell(t_0)$$

Proof. Let γ be the unit-speed parametrization of $[qp]$ from q to p, so $q=\gamma(0)$. Then

$$\begin{aligned}\ell^+(t_0)&=(\boldsymbol{d}_q\operatorname{dist}_p)(\nabla_q f)\leqslant && \text{(by 5.1)}\\ &\leqslant-\langle\uparrow_{[qp]},\nabla_q f\rangle\leqslant && \text{(by 4.6}a\text{)}\\ &\leqslant-\boldsymbol{d}_q f(\uparrow_{[qp]})= && \text{(by 4.7)}\\ &=-(f\circ\gamma)^+(0)\leqslant\\ &\leqslant-\left(f(p)-f(q)-\tfrac{\lambda}{2}\cdot\ell^2(t_0)\right)/\ell(t_0)\end{aligned}$$

The last two lines follow by the definition of differential, and the concavity of $t\mapsto f\circ\gamma(t)-\tfrac{\lambda}{2}{\cdot}t^2$. □

The following estimate implies uniqueness in the Picard theorem (5.3).

5.6. First distance estimate. *Let f be a λ-concave locally Lipschitz function defined on an Alexandrov space $\mathcal{A}$. Then*

$$|\alpha(t)-\beta(t)|\leqslant e^{\lambda\cdot t}{\cdot}|\alpha(0)-\beta(0)|$$

for any $t\geqslant 0$ and any two f-gradient curves α and β.

Moreover, the statement holds for a locally Lipschitz λ-concave function defined in an open domain if there is a geodesic $[\alpha(t)\,\beta(t)]$ in Dom f *for any t.*

Proof. Fix a choice of geodesic $[\alpha(t)\,\beta(t)]$ for each t. Let $\ell(t)=|\alpha(t)-\beta(t)|$. Note that

$$\ell^+(t)\leqslant-\langle\uparrow_{[\alpha(t)\beta(t)]},\nabla_{\alpha(t)}f\rangle-\langle\uparrow_{[\beta(t)\alpha(t)]},\nabla_{\beta(t)}f\rangle\leqslant\lambda{\cdot}\ell(t).$$

Here one has to apply 5.5 for distance to the midpoint m of $[\alpha(t)\,\beta(t)]$, then apply the triangle inequality and 4.8.

Integrating this inequality, we get the result. □

The following exercise describes a global geometric property of a gradient curve without direct reference to its function. It is based on the notion of self-contracting curves introduced by Aris Daniilidis, Olivier Ley, and Stéphane Sabourau [24].

5.7. Exercise. *Let α be a gradient curve of a concave function on an Alexandrov space. Show that*

$$|\alpha(t_1)-\alpha(t_3)|_{\mathcal{A}}\geqslant|\alpha(t_2)-\alpha(t_3)|_{\mathcal{A}}.$$

if $t_1\leqslant t_2\leqslant t_3$.

5.8. Exercise. *Let f be a locally Lipschitz concave function defined on an Alexandrov space $\mathcal{A}$. Suppose $\hat\alpha\colon[0,\ell]\to\mathcal{A}$ is an arc-length reparametrization of an f-gradient curve. Show that $f\circ\hat\alpha$ is concave.*

The following exercise implies that gradient curves for a uniformly converging sequence of λ-concave functions converge to the gradient curves of the limit function.

5.9. Exercise! *Let f and g be λ-concave locally Lipschitz functions on an Alexandrov space $\mathcal{A}$. Suppose $\alpha,\beta\colon[0,t_{\max})\to\mathcal{A}$ are respectively f- and g-gradient curves. Assume $|f-g|<\varepsilon$; let $\ell\colon t\mapsto|\alpha(t)-\beta(t)|$. Show that*

$$\ell^+\leqslant\lambda{\cdot}\ell+\tfrac{2\cdot\varepsilon}{\ell}.$$

Conclude that if $\alpha(0)=\beta(0)$ *and* $t_{\max}<\infty$, *then*

$$|\alpha(t)-\beta(t)|\leqslant c\cdot\sqrt{\varepsilon\cdot t}$$

for some constant $c=c(t_{\max},\lambda)$.

D Gradient flow

Let f be a locally Lipschitz semiconcave function defined on an open subset of an Alexandrov space $\mathcal{A}$. If there is an f-gradient curve α such that $\alpha(0)=x$ and $\alpha(t)=y$, then we will write

$$\mathrm{Flow}_f^t(x)=y.$$

The partially defined map Flow_f^t from $\mathcal{A}$ to itself is called the f-gradient flow for time t. Note that

$$\mathrm{Flow}_f^{t_1+t_2}=\mathrm{Flow}_f^{t_1}\circ\mathrm{Flow}_f^{t_2}.$$

In other words, the gradient flow is a partial action of the *semigroup* $([0,\infty),+)$ on the space.

From the first distance estimate 5.6, it follows that for any $t\geqslant 0$, the domain of definition of Flow_f^t is an open subset of $\mathcal{A}$. For sufficiently nice functions, the gradient flow is globally defined. For example, if f is a λ-concave function and it is defined on the whole space $\mathcal{A}$, then $\mathrm{Flow}_f^t(x)$ is defined for all $x\in\mathcal{A}$ and $t\geqslant 0$; see [6, 16.19].

Using this new terminology, we can reformulate several statements about gradient curves. From the first distance estimate, we have the following.

5.10. Proposition. *Let f be a semiconcave function defined on an Alexandrov space $\mathcal{A}$. Then the map $x\mapsto\mathrm{Flow}_f^t(x)$ is locally Lipschitz.*

Moreover, if f is λ-concave, then Flow_f^t is $e^{\lambda\cdot t}$-Lipschitz.

The next proposition follows from 5.9.

5.11. Proposition. *Let $\mathcal{A}$ be an Alexandrov space. Suppose $f_n\colon\mathcal{A}\to\mathbb{R}$ is a sequence of λ-concave functions that uniformly converges to $f_\infty\colon\mathcal{A}\to\mathbb{R}$. Then for any $x\in\mathcal{A}$ and $t\geqslant 0$, we have*

$$\mathrm{Flow}_{f_n}^t(x)\to\mathrm{Flow}_{f_\infty}^t(x)$$

as $n\to\infty$.

This proposition can be generalized to a converging sequence $\mathcal{A}_n\to\mathcal{A}_\infty$ of spaces and a converging sequence of functions $f_n\colon\mathcal{A}_n\to\mathbb{R}$; see [6, 16.21].

E Gradient exponent

One of the technical difficulties in Alexandrov geometry comes from nonextendability of geodesics. In particular, the exponential map, $\exp_p\colon \mathrm{T}_p \to \mathcal{A}$, if defined in the usual way, can be undefined in an arbitrarily small neighborhood of the origin.

Now we will construct the gradient exponential map

$$\mathrm{gexp}_p\colon \mathrm{T}_p \to \mathcal{A},$$

which essentially solves this problem. It shares many properties with the ordinary exponential map and is even better in certain respects, even in the Riemannian universe.

Let p be a point in an $\mathrm{ALEX}(0)$ space $\mathcal{A}$. Consider the function $f = \mathrm{dist}_p^2/2$. Recall that Flow_f^t denotes the gradient flow. Let us define the *gradient exponential map* as the limit

$$\mathrm{gexp}_p(v) = \lim_{n\to\infty} \mathrm{Flow}_f^{t_n}(x_n),$$

where the sequences $x_n \in \mathcal{A}$ and $t_n \geqslant 0$ are chosen so that $t_n \to \infty$ and $e^{t_n}\cdot \log_p x_n \to v$ as $n \to \infty$.

More intuitively, suppose $i_\lambda\colon \lambda\cdot\mathcal{A} \to \mathcal{A}$ sends a point in the rescaled copy $\lambda\cdot\mathcal{A}$ to the corresponding point in $\mathcal{A}$. By the first distance estimate (5.6), the map

$$\text{➊}\qquad \mathrm{Flow}_f^t \circ i_{e^t}\colon e^t\cdot\mathcal{A} \to \mathcal{A}$$

is short for any $t \geqslant 0$. If we have a pointed Gromov–Hausdorff convergence $(e^{t_n}\cdot\mathcal{A}, p) \to (\mathrm{T}_p, o_p)$, then $\mathrm{gexp}_p\colon \mathrm{T}_p \to \mathcal{A}$ is the limit of $\mathrm{Flow}_f^{t_n} \circ i_{e^{t_n}}$. This way we get that gexp_p is short as a limit of short maps. This observation is generalized in the following proposition.

5.12. Proposition. *Let $\mathcal{A}$ be a proper* $\mathrm{ALEX}(0)$ *space. Then for any $p \in \mathcal{A}$ the gradient exponent* $\mathrm{gexp}_p\colon \mathrm{T}_p \to \mathcal{A}$ *is uniquely defined. Moreover,* gexp_p *is a short map and*

$$\mathrm{gexp}_p(\gamma^+(0)) = \gamma(1)$$

for any geodesic path γ that starts at p.

The last statement implies that

$$\mathrm{gexp}_p \circ \log_p = \mathrm{id},$$

so it is appropriate to use term *exponent* for gexp.

Proof. Note that $f'' \leqslant 1$. Since the space is proper we can choose a limit in ➊.

Let γ be a geodesic that stats at p. Observe that $t \mapsto \gamma \circ \ln(t)$ is an f-gradient curve. By the first distance estimate, we have that Flow_f^t is an e^t-Lipschitz. This implies that any limit in ❶ has the same value; that is, gexp_p is uniquely defined.

Again, since Flow_f^t is an e^t-Lipschitz, we get that gexp_p is short. □

F Remarks

The idea to use gradient flows in Alexandrov geometry was inspired by the success of Sharafutdinov's retraction in comparison geometry [101]. The gradient flow was introduced by the second author to construct quasigeodesics with given initial data [78, 85, 86]. It turned out that gradient flow and gradient exponent are better tools than quiasigeodesics. These tools quickly found applications in other types of singular spaces [12, 48, 69, 70, 75, 100].

For a general lower curvature bound κ, the construction of gradient exponent has to be modified; it is denoted by gexp_p^κ [6, 16.36]. It is done by taking limits of appropriately reparameterized gradient curves of the modified distance function.

For $\kappa = -1$ we have that $\mathrm{gexp}_p(\gamma^+(0)) = \gamma(1)$ for any geodesic path γ that starts at p and

$$|\mathrm{gexp}_p^{-1} v - \mathrm{gexp}_p^{-1} w|_{\mathcal{A}} \leqslant \tilde{\curlyvee}[0\,{}^{v}_{w}]_{\mathbb{H}^2}.$$

In other words gexp_p is short if we equip T_p with the hyperbolic cone metric.

Similarly, for $\kappa = 1$ we have $\mathrm{gexp}_p^1(\gamma^+(0)) = \gamma(1)$ for any geodesic path γ that starts at p and

$$|\mathrm{gexp}_p^1 v - \mathrm{gexp}_p^1 w|_{\mathcal{A}} \leqslant \tilde{\curlyvee}[0\,{}^{v}_{w}]_{\mathbb{S}^2},$$

but this time all this holds only if $|v|, |w| \leqslant \frac{\pi}{2}$ and $\mathrm{length}\,\gamma \leqslant \frac{\pi}{2}$.

The gradient exponential map in a Riemannian manifold (M, g) coincides with the Riemannian exponential map before the cut locus, but *is different* from the Riemannian exponential after that. The following exercise is ment to show that this technique can prove something nontrivial even for Riemannian manifolds.

5.13. Exercise. *Let (M, g) be a complete m-dimensional Riemannian with sectional curvature at least 1. Assume M is not homeomorphic to $\mathbb{S}^m$. Show that there is a short onto map $\mathbb{S}^m \to (M, g)$.*

Lecture 6

Line splitting

In this lecture, we prove the line splitting theorem and apply it to study tangent spaces of Alexandrov spaces.

A Busemann function

A half-line is a geodesic defined on the real half-line $[0, \infty)$; that is, a distance-preserving map from $[0, \infty)$ to a metric space.

If $\gamma\colon [0, \infty) \to \mathcal{X}$ is a half-line, then the limit

❶
$$\mathrm{bus}_\gamma(x) = \lim_{t \to \infty} |\gamma(t) - x| - t$$

is called the Busemann function of γ. It mimics behavior of the distance function from the ideal point of γ.

6.1. Proposition. *For any half-line γ in a metric space $\mathcal{X}$, its Busemann function $\mathrm{bus}_\gamma\colon \mathcal{X} \to \mathbb{R}$ is defined. Moreover, bus_γ is 1-Lipschitz and $\mathrm{bus}_\gamma(\gamma(t)) = -t$ for any t.*

Proof. Since $t = |\gamma(0) - \gamma(t)|$, the triangle inequality implies that

$$t \mapsto |\gamma(t) - x| - t$$

is a nonincreasing function, and

$$|\gamma(t) - x| - t \geqslant -|\gamma(0) - x|$$

for any $x \in \mathcal{X}$. Therefore, the limit in ❶ is defined, and it has to be 1-Lipschitz as a limit of 1-Lipschitz functions. The last statement follows since $|\gamma(t) - \gamma(t_0)| = t - t_0$ for all large t. $\square$

6.2. Exercise! *Show that any Busemann function on an* $\mathrm{Alex}(0)$ *space is concave.*

B Splitting theorem

A line is a distance-preserving map from $\mathbb{R}$ to a metric space. In other words, a line is a geodesic defined on the real line $\mathbb{R}$.

6.3. Exercise! *Let γ be a line in a metric space $\mathcal{X}$. Show that for any point x we have*

$$\mathrm{bus}_+(x) + \mathrm{bus}_-(x) \geqslant 0$$

where, bus_+ *and* bus_-*, are the Busemann functions asociated with half-lines $\gamma : [0, \infty) \to \mathcal{A}$ and $\gamma : (-\infty, 0] \to \mathcal{A}$ respectively.*

Let A and B be two subsets in a metric space $\mathcal{X}$. We say that $\mathcal{X}$ is a direct sum of A and B, or briefly,

$$\mathcal{X} = A \oplus B$$

if there are retractions $\mathrm{proj}_A \colon \mathcal{X} \to A$ and $\mathrm{proj}_B \colon \mathcal{X} \to B$ such that

$$|x - y|^2 = |\mathrm{proj}_A(x) - \mathrm{proj}_A(y)|^2 + |\mathrm{proj}_B(x) - \mathrm{proj}_B(y)|^2$$

for any two points $x, y \in \mathcal{X}$.

6.4. Exercise. *Suppose $\mathcal{X} = A \oplus B$. Show that*

(a) A intersects B at a single point, and

(b) both sets A and B are convex sets in $\mathcal{X}$; the latter means that any geodesic with the endpoints in A (or B) lies in A (respectively B).

6.5. Line splitting theorem. *Let γ be a line in a* $\mathrm{A{\scriptsize LEX}}(0)$ *space $\mathcal{A}$. Then*

$$\mathcal{A} = \mathcal{A}' \oplus \gamma(\mathbb{R})$$

for some subset $\mathcal{A}' \subset \mathcal{A}$.

6.6. Corollary. *Any* $\mathrm{A{\scriptsize LEX}}(0)$ *space $\mathcal{A}$ splits isometrically as*

$$\mathcal{A} = \mathcal{A}' \oplus H$$

where $H \subset \mathcal{A}$ is a subset isometric to a Hilbert space, and $\mathcal{A}' \subset \mathcal{A}$ is a convex subset that contains no lines.

The following lemma is closely related to the first distance estimate (5.6); it is also a limit case of 5.12. The proof follows similar lines.

6.7. Lemma. *Suppose $f\colon\mathcal{A}\to\mathbb{R}$ is a concave 1-Lipschitz function on an* Alex(0) *space $\mathcal{A}$. Consider two f-gradient curves α and β. Then for any $t,s\geqslant 0$ we have*

$$|\alpha(s)-\beta(t)|^2\leqslant|p-q|^2+2{\cdot}(f(p)-f(q)){\cdot}(s-t)+(s-t)^2,$$

where $p=\alpha(0)$ and $q=\beta(0)$.

Proof. Since f is 1-Lipschitz, $|\nabla f|\leqslant 1$. Therefore

$$f\circ\beta(t)\leqslant f(q)+t$$

for any $t\geqslant 0$.

Set $\ell(t)=|p-\beta(t)|$. Applying 5.5, we get

$$\begin{aligned}(\ell^2)^+(t)&\leqslant 2{\cdot}\left(f\circ\beta(t)-f(p)\right)\leqslant\\&\leqslant 2{\cdot}\left(f(q)+t-f(p)\right).\end{aligned}$$

Therefore

$$\ell^2(t)-\ell^2(0)\leqslant 2{\cdot}\left(f(q)-f(p)\right){\cdot}t+t^2.$$

It proves the needed inequality in case $s=0$. Combining it with the first distance estimate (5.6), we get the result in case $s\leqslant t$. The case $s\geqslant t$ follows by switching the roles of s and t. □

Proof of 6.5. Consider two Busemann functions, bus_+ and bus_-, associated with the half-lines $\gamma:[0,\infty)\to\mathcal{A}$ and $\gamma:(-\infty,0]\to\mathcal{A}$ respectively; that is,

$$\mathrm{bus}_\pm(x):=\lim_{t\to\infty}|\gamma(\pm t)-x|-t.$$

According to 6.2, both bus_+ and bus_- are concave.

By 6.3, $\mathrm{bus}_+(x)+\mathrm{bus}_-(x)\geqslant 0$ for any $x\in\mathcal{A}$. On the other hand, by 2.22, $f(t)=\mathrm{dist}_x^2\circ\gamma(t)$ is 2-concave. In particular, $f(t)\leqslant t^2+at+b$ for some constants $a,b\in\mathbb{R}$. Therefore, for all large t

$$|\gamma(t)-x|+|\gamma(-t)-x|-2{\cdot}t\leqslant\sqrt{t^2+at+b}+\sqrt{t^2-at+b}-2{\cdot}t$$

Passing to the limit as $t\to\infty$, we get that $\mathrm{bus}_+(x)+\mathrm{bus}_-(x)\leqslant 0$. Hence

$$\mathrm{bus}_+(x)+\mathrm{bus}_-(x)=0$$

for any $x\in\mathcal{A}$. In particular, the functions bus_+ and bus_- are affine; that is, they are convex and concave at the same time.

For any x,

$$\begin{aligned}|\nabla_x \operatorname{bus}_\pm| &= \sup\{\, \boldsymbol{d}_x \operatorname{bus}_\pm(\xi) \;:\; \xi\in\Sigma_x\,\} =\\ &= \sup\{\, -\boldsymbol{d}_x \operatorname{bus}_\mp(\xi) \;:\; \xi\in\Sigma_x\,\} \equiv\\ &\equiv 1.\end{aligned}$$

A curve α is a $\operatorname{bus}_\pm$-gradient curve if and only if α is a geodesic such that $(\operatorname{bus}_\pm\circ\alpha)^+=1$. Indeed, if α is a geodesic, then $(\operatorname{bus}_\pm\circ\alpha)^+\leqslant 1$ and the equality holds only if $\nabla_\alpha \operatorname{bus}_\pm=\alpha^+$. Now suppose $\nabla_\alpha \operatorname{bus}_\pm = \alpha^+$. Then $|\alpha^+|\leqslant 1$ and $(\operatorname{bus}_\pm\circ\alpha)^+=1$; therefore

$$\begin{aligned}|t_0-t_1| &\geqslant |\alpha(t_0)-\alpha(t_1)| \geqslant\\ &\geqslant |\operatorname{bus}_\pm\circ\alpha(t_0)-\operatorname{bus}_\pm\circ\alpha(t_1)| =\\ &= |t_0-t_1|.\end{aligned}$$

It follows that for any $t>0$, the $\operatorname{bus}_\pm$-gradient flows commute; that is,

$$\operatorname{Flow}^t_{\operatorname{bus}_+}\circ\operatorname{Flow}^t_{\operatorname{bus}_-}=\operatorname{id}_{\mathcal{A}}.$$

Setting

$$\operatorname{Flow}^t=\begin{bmatrix}\operatorname{Flow}^t_{\operatorname{bus}_+} & \text{if } t\geqslant 0\\ \operatorname{Flow}^{-t}_{\operatorname{bus}_-} & \text{if } t\leqslant 0\end{bmatrix}$$

defines an $\mathbb{R}$-action on $\mathcal{A}$.

Consider the level set $\mathcal{A}'=\operatorname{bus}_+^{-1}(0)=\operatorname{bus}_-^{-1}(0)$; it is a closed convex subset of $\mathcal{A}$, and therefore forms an Alexandrov space. Consider the map $h\colon\mathcal{A}'\times\mathbb{R}\to\mathcal{A}$ defined by $h\colon(x,t)\mapsto\operatorname{Flow}^t(x)$. Note that h is onto. Applying 6.7 for $\operatorname{Flow}^t_{\operatorname{bus}_+}$ and $\operatorname{Flow}^t_{\operatorname{bus}_-}$ shows that h is distance non-expanding and non-contracting at the same time; that is, h is an isometry. □

6.8. Exercise. *Suppose $\mathcal{X}$ is a complete geodesic space. Show that* $\operatorname{Cone}\mathcal{X}$ *is* ALEX(0) *if and only if $\mathcal{X}$ is* ALEX(1) *and* $\operatorname{diam}\mathcal{X}\leqslant\pi$.

Recall that according our definition any circle or closed real interval is ALEX(1). Therefore, the condition $\operatorname{diam}\mathcal{X}\leqslant\pi$ is necessary. Nevertheless, according to 3.6, most of ALEX(1) spaces have diameter at most π.

C Anti-sum

The following lemma is a corollary of 4.12. It will be used to prove basic properties of tangent spaces.

6.9. Anti-sum lemma. *Let p be a point in an Alexandrov space $\mathcal{A}$. Given two vectors $u, v \in \mathrm{T}_p$, there is a unique vector $w \in \mathrm{T}_p$ such that*

$$\langle u, x\rangle + \langle v, x\rangle + \langle w, x\rangle \geqslant 0$$

for any $x \in \mathrm{T}_p$, and

$$\langle u, w\rangle + \langle v, w\rangle + \langle w, w\rangle = 0.$$

6.10. Exercise! *Suppose $u, v, w \in \mathrm{T}_p$ are as in 6.9. Show that*

$$|w|^2 \leqslant |u|^2 + |v|^2 + 2{\cdot}\langle u, v\rangle.$$

If T_p were $\mathrm{A{\scriptsize LEX}}(0)$, then the lemma would follow from the existence of the gradient, applied to the function $\mathrm{T}_p \to \mathbb{R}$ defined by $x \mapsto -(\langle u, x\rangle + \langle v, x\rangle)$, which is concave by 6.2. As you will see, a revision of this idea works in the general case, but it cannot work as is since T_p might fail to be geodesic; see Halbeisen's example [6, 13.6].

Applying the above lemma for $u = v$, we have the following statement.

6.11. Existence of polar vector. *Let $\mathcal{A}$ be an Alexandrov space and $p \in \mathcal{A}$. Given a vector $u \in \mathrm{T}_p$, there is a unique vector $u^* \in \mathrm{T}_p$ such that $\langle u^*, u^*\rangle + \langle u, u^*\rangle = 0$ and u^* is p o l a r to u; that is,*

$$\langle u^*, x\rangle + \langle u, x\rangle \geqslant 0$$

for any $x \in \mathrm{T}_p$.

Proof of 6.9. By 4.11, we can choose two sequences of points a_n, b_n such that

$$\boldsymbol{d}_p\mathrm{dist}_{a_n}(w) = -\langle \uparrow_{[pa_n]}, w\rangle \quad \text{and} \quad \boldsymbol{d}_p\mathrm{dist}_{b_n}(w) = -\langle \uparrow_{[pb_n]}, w\rangle$$

for any $w \in \mathrm{T}_p$. Furthemore, we can assume that $\uparrow_{[pa_n]} \to u/|u|$ and $\uparrow_{[pb_n]} \to v/|v|$ as $n \to \infty$

Consider a sequence of functions

$$f_n = |u|{\cdot}\mathrm{dist}_{a_n} + |v|{\cdot}\mathrm{dist}_{b_n}.$$

Note that

$$(\boldsymbol{d}_p f_n)(x) = -|u|{\cdot}\langle \uparrow_{[pa_n]}, x\rangle - |v|{\cdot}\langle \uparrow_{[pb_n]}, x\rangle.$$

Thus we have the following uniform convergence for $x \in \Sigma_p$:

$$(\boldsymbol{d}_p f_n)(x) \rightrightarrows -\langle u, x\rangle - \langle v, x\rangle$$

as $n \to \infty$. According to 4.12, the sequence $\nabla_p f_n$ converges. Let

$$w = \lim_{n\to\infty} \nabla_p f_n.$$

By the definition of gradient,

$$\begin{aligned}\langle w, w\rangle &= \lim_{n\to\infty} \langle \nabla_p f_n, \nabla_p f_n\rangle = \\ &= \lim_{n\to\infty} (\boldsymbol{d}_p f_n)(\nabla_p f_n) = \\ &= -\langle u, w\rangle - \langle v, w\rangle,\end{aligned} \qquad \begin{aligned}\langle w, x\rangle &= \lim_{n\to\infty} \langle \nabla_p f_n, x\rangle \geqslant \\ &\geqslant \lim_{n\to\infty} (\boldsymbol{d}_p f_n)(x) = \\ &= -\langle u, x\rangle - \langle v, x\rangle.\end{aligned}$$

The proof of uniqueness is very similar to the proof of uniqueness of gradients and is left to the reader. ☐

D Linear subspace

6.12. Definition. *Let $\mathcal{A}$ be an Alexandrov space, $p \in \mathcal{A}$ and $u, v \in \in \mathrm{T}_p$. We say that vectors u and v are opposite to each other, (briefly, $u + v = 0$) if $|u| = |v| = 0$ or $\measuredangle(u, v) = \pi$ and $|u| = |v|$.*

The subcone

$$\mathrm{Lin}_p = \{\, v \in \mathrm{T}_p \ :\ \exists\ w \in \mathrm{T}_p \quad \textit{such that} \quad w + v = 0 \,\}$$

will be called the linear subspace of T_p.

Soon we will introduce a natural linear structure on Lin_p.

6.13. Proposition. *Let p be a point in an Alexandrov space. Given two vectors $u, v \in \mathrm{T}_p$, the following statements are equivalent:*

(a) $u + v = 0$;

(b) $\langle u, x\rangle + \langle v, x\rangle = 0$ for any $x \in \mathrm{T}_p$;

(c) $\langle u, \xi\rangle + \langle v, \xi\rangle = 0$ for any $\xi \in \Sigma_p$.

Proof. The equivalence *(b)*⇔*(c)* is trivial.

The condition $u + v = 0$ is equivalent to $\langle u, u\rangle = -\langle u, v\rangle = \langle v, v\rangle$; thus, *(b)*⇒*(a)*.

Suppose *(a)* holds. Recall that T_p has nonnegative curvature. The hinges $[0\,{}^{u}_{x}]$ and $[0\,{}^{v}_{x}]$ are adjacent. By 2.13, $\measuredangle[0\,{}^{u}_{x}] + \measuredangle[0\,{}^{v}_{x}] = \pi$; hence *(a)*⇒*(b)*. ☐

6.14. Exercise! *Let u, v, and w be tangent vectors at a point of an Alexandrov space. Assume $u+v = 0$ and $u+w = 0$. Show that $v = w$.*

Let $u\in\mathrm{Lin}_p$; that is, $u+v=0$ for some $v\in\mathrm{T}_p$. Given $s<0$, let

$$s\cdot u:=(-s)\cdot v.$$

So we can multiply any vector in Lin_p by any real number (positive and negative). By 6.14, this multiplication is uniquely defined. By 6.13, we have identity

$$\langle -v,x\rangle=-\langle v,x\rangle.$$

6.15. Exercise! *Suppose u, v, and w are as in 6.9. Show that*

$$\langle u,x\rangle+\langle v,x\rangle+\langle w,x\rangle=0$$

for any $x\in\mathrm{Lin}_p$.

6.16. Exercise! *Let $\mathcal{A}$ be an Alexandrov space, $p\in\mathcal{A}$ and $u\in\mathrm{T}_p$. Suppose $u^*\in\mathrm{T}_p$ is from 6.11; that is,*

$$\langle u^*,u^*\rangle+\langle u,u^*\rangle=0\quad\textit{and}\quad\langle u^*,x\rangle+\langle u,x\rangle\geqslant 0$$

for any $x\in\mathrm{T}_p$. Show that $u=-u^*$ if and only if $|u|=|u^*|$.

6.17. Theorem. *Let p be a point in an Alexandrov space. Then* Lin_p *is isometric to a Hilbert space.*

Proof. Lin_p is a closed subset of T_p; in particular, it is complete.

If any two vectors in Lin_p can be connected by a geodesic in Lin_p, then the statement follows from the splitting theorem (6.5). By Menger's lemma (1.6), it is sufficient to show that for any pair $x,y\in\mathrm{Lin}_p$ has a midpoint $w\in\mathrm{Lin}_p$.

Choose $w\in\mathrm{T}_p$ to be the anti-sum of $u=-\tfrac12\cdot x$ and $v=-\tfrac12\cdot y$; see 6.9. By 6.10 and 6.15,

$$\begin{aligned}|w|^2&\leqslant\tfrac14\cdot|x|^2+\tfrac14\cdot|y|^2+\tfrac12\cdot\langle x,y\rangle,\\ \langle w,x\rangle&=\tfrac12\cdot|x|^2+\tfrac12\cdot\langle x,y\rangle,\\ \langle w,y\rangle&=\tfrac12\cdot|y|^2+\tfrac12\cdot\langle x,y\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}|x-w|^2&=|x|^2+|w|^2-2\cdot\langle w,x\rangle\leqslant\\ &\leqslant\tfrac14\cdot|x|^2+\tfrac14\cdot|y|^2-\tfrac12\cdot\langle x,y\rangle=\\ &=\tfrac14\cdot|x-y|^2.\end{aligned}$$

That is, $|x-w|\leqslant\frac12\cdot|x-y|$. Similarly, we get $|y-w|\leqslant\frac12\cdot|x-y|$. Therefore w is a midpoint of x and y. In addition, we get the equality

$$|w|^2=\tfrac14\cdot|x|^2+\tfrac14\cdot|y|^2+\tfrac12\cdot\langle x,y\rangle.$$

It remains to show that $w\in\mathrm{Lin}_p$. Let w^* be the polar vector provided by 6.11. Note that

$$|w^*|\leqslant|w|,\quad \langle w^*,x\rangle+\langle w,x\rangle=0,\quad\text{and}\quad \langle w^*,y\rangle+\langle w,y\rangle=0.$$

The same calculation as above shows that w^* is a midpoint of $-x$ and $-y$ and

$$|w^*|^2=\tfrac14\cdot|x|^2+\tfrac14\cdot|y|^2+\tfrac12\cdot\langle x,y\rangle=|w|^2.$$

By 6.16, $w=-w^*$; hence $w\in\mathrm{Lin}_p$. □

6.18. Lemma. *Given a point p in an Alexandrov space $\mathcal{A}$, let $f=$ $=\mathrm{dist}_p$, and let S be the subset of points $x\in\mathcal{A}$ such that $|\nabla_x f|=1$. Then S is a dense G-delta set.*

Proof. Given a positive integer n, consider the set

$$S_n=\left\{\,x\in\mathcal{A}\;:\;|\nabla_x f|>1-\tfrac1n\,\right\}.$$

By 4.13*a*, S_n is open.

Choose a point $q\neq p$. Observe that $|\nabla_x f|=1$ for any point $x\in\,]pq[$. It follows that S_n is dense in $\mathcal{A}$. Since $S=\bigcap_n S_n$, the lemma follows from the Baire category theorem. □

6.19. Exercise! *Let p, f, and S be as in 6.18.*

(a) Show that

$$\nabla_x f+{\uparrow}_{[xp]}=0$$

for any $x\in S$; in particular, ${\uparrow}_{[xp]}\in\mathrm{Lin}_x$.

(b) Show that if $|\nabla_x f|=1$, then $\boldsymbol{d}_x f(w)=\langle\nabla_x f,w\rangle$ for any $w\in\mathrm{T}_x$.

(c) Show that for any $x\in S$ there is a unique geodesic $[px]$.

This exercise implies the following.

6.20. Corollary. *Given a countable set of points X in an Alexandrov space $\mathcal{A}$ there is a G-delta dense set $S\subset\mathcal{A}$ such that ${\uparrow}_{[sx]}\in\mathrm{Lin}_s$ for any $s\in S$ and $x\in X$.*

E Remarks

The history of the splitting theorem starts with Stefan Cohn-Vossen [23], who proved it in the 2-dimensional case. For Riemannian manifolds of higher dimensions, it was proved by Victor Toponogov [106]. Then it was generalized by Anatoliy Milka [72] to Alexandrov spaces; it was the first result about Alexandrov spaces of dimension higher than 2. Nearly the same proof is used in [17, 1.5]. Generalizations to Riemannian manifolds with nonnegative Ricci curvature were obtained by Jeff Cheeger and Detlef Gromoll [22]. This was further generalized by Jeff Cheeger and Toby Colding for the limits of Riemannian manifolds with almost nonnegative Ricci curvature [20]. Nicola Gigli generalized it further to the so-called RCD spaces (spaces with synthetically defined Ricci-curvature bound) [31, 32]. Jost-Hinrich Eschenburg obtained an analogous result for Lorentzian manifolds [28].

The presented proof is close in spirit to the proof given by Cheeger and Gromoll [22]; it is taken from our book [6].

6.21. Open question. *Let p be a point in an Alexandrov space $\mathcal{A}$. Suppose that $0 \neq v \in \mathrm{Lin}_p$. Is it true that the tangent space T_p splits in the direction of v?*

Halbeisen's example [6, 43] shows that compactness of space of directions is essential in the proof that space of directions is π-geodesic (see 7.5).

6.22. Open question. *Let $\mathcal{A}$ be a proper Alexandrov space. Is it true that for any $p \in \mathcal{A}$, the tangent space T_p is a length space?*

Lecture 7

Dimension and volume

In this lecture, we introduce volume, distance charts, and several notions of dimension, and prove the Bishop–Gromov inequality and the right-inverse theorem.

A Linear dimension

Let $\mathcal{A}$ be an Alexandrov space. We define its linear dimension $\operatorname{LinDim}\mathcal{A}$ as the least upper bound on integers m such that the Euclidean space $\mathbb{E}^m$ is isometric to a subspace of the tangent space $\mathrm{T}_p\mathcal{A}$ at *some* point $p \in \mathcal{A}$.

If not stated otherwise, dimension of an Alexandrov space is its linear dimension. In Section 7F, we will show that linear dimension of Alexandrov space coincides with all reasonable dimensions; after that, we will use $\dim \mathcal{A}$ for $\operatorname{LinDim}\mathcal{A}$.

7.1. $(n+1)$-comparison. *Let $\mathcal{A}$ be an* $\mathrm{ALEX}(0)$ *space. Then for any finite set of points $p, x_1, \dots, x_n \in \mathcal{A}$, there exist a model configuration $\tilde{p}, \tilde{x}_1, \dots, \tilde{x}_n \in \mathbb{E}^m$ for some m such that*

$$|\tilde{p} - \tilde{x}_i|_{\mathbb{E}^m} = |p - x_i|_{\mathcal{A}} \quad \textit{and} \quad |\tilde{x}_i - \tilde{x}_j|_{\mathbb{E}^m} \geqslant |x_i - x_j|_{\mathcal{A}}$$

for any i and j. Moreover, we can assume that $m \leqslant \operatorname{LinDim}\mathcal{A}$.

Proof. By 6.20, we can choose a point p' arbitrarily close to p so that $\mathrm{Lin}_{p'} \ni \uparrow_{[p'x_i]}$ for any i. Let us identify $\mathbb{E}^m$ with a subspace of $\mathrm{Lin}_{p'}$ spanned by $\uparrow_{[p'x_1]}, \dots, \uparrow_{[p'x_n]}$. Note that $m \leqslant \operatorname{LinDim}\mathcal{A}$.

Set $\tilde{p}' = 0 \in \mathbb{E}^m$ and $\tilde{x}_i = |p' - x_n| \cdot \uparrow_{[p'x_n]} \in \mathbb{E}^m$ for every i. Note that

$$|\tilde{p}' - \tilde{x}_i|_{\mathbb{E}^m} = |p' - x_i|_{\mathcal{A}}$$

for every i. Applying the comparison $\measuredangle[p'\,{}^{x_i}_{x_j}] \geqslant \tilde\measuredangle(p'\,{}^{x_i}_{x_j})$, we get

$$|\tilde x_i - \tilde x_j|_{\mathbb{E}^m} \geqslant |x_i - x_j|_{\mathcal{A}}$$

for all i and j. Passing to a limit configuration as $p' \to p$ we get the result. □

7.2. Exercise! *Let $\mathcal{A}$ be an* $\mathrm{ALEX}(0)$ *space. Suppose* $\operatorname{LinDim}\mathcal{A} = m < \infty$. *Show that* $\mathrm{T}_p\mathcal{A} \overset{iso}{=\!=} \mathbb{E}^m$ *for a G-delta dense set of points* $p \in \mathcal{A}$.

7.3. Exercise! *Show that a 1-dimensional Alexandrov space is homeomorphic to a 1-dimensional manifold, possibly with nonempty boundary.*

7.4. Exercise. *Let $\mathcal{A}$ be an* $\mathrm{ALEX}(0)$ *space. Show that* $\operatorname{LinDim}\mathcal{A} \geqslant m$ *if and only if there are $m+2$ points $p, a_0, \dots, a_m \in \mathcal{A}$ such that* $\tilde\measuredangle(p\,{}^{a_i}_{a_j}) > \frac{\pi}{2}$ *for any pair $i \neq j$.*

B Space of directions

Recall that a metric space $\mathcal{X}$ is ℓ-geodesic if any two points $x, y \in \mathcal{X}$ such that $|x-y| < \ell$ can be connected by a geodesic.

7.5. Theorem. *Let $\mathcal{A}$ be a finite-dimensional Alexandrov space. Then for any point $p \in \mathcal{A}$, its space of directions Σ_p is a compact π-geodesic space.*

By 4.4, it implies the following.

7.6. Corollary. *Let p be a point in a finite dimensional Alexandrov space $\mathcal{A}$, and let $\lambda_n \to \infty$. Then there is a pointed Gromov–Hausdorff convergence $(\lambda_n{\cdot}\mathcal{A}, p) \to (\mathrm{T}_p, 0)$.*

7.7. Exercise! *Let p be a point in a finite-dimensional Alexandrov space $\mathcal{A}$. Prove the following.*

(a) *The tangent space T_p is a proper* $\mathrm{ALEX}(0)$ *space.*
(b) $\operatorname{LinDim}\Sigma_p = \operatorname{LinDim}\mathcal{A} - 1$.
(c) *If* $\operatorname{LinDim}\mathcal{A} > 1$, *then Σ_p is geodesic.*

Part *(b)* of the exercise, opens a way to use induction on dimension of Alexandrv spece. It will be used extensively in Lecture 9.

Proof of 7.5. Choose $\varepsilon > 0$; suppose $\mathcal{A}$ is m-dimensional. Assume $\operatorname{pack}_\varepsilon \Sigma_p \geqslant n$; that is, we can choose n directions $\xi_1, \dots, \xi_n \in \Sigma_p$ such that $\measuredangle(\xi_i, \xi_j) > \varepsilon$ for any $i \neq j$. Without loss of generality, we may assume that each direction is geodesic; that is, there is a point $x_i \in \mathcal{A}$ such that $\xi_i = \uparrow_{[px_i]}$.

Choose $y_i \in [px_i]$ such that $|p - y_i| = r$ for each i and small fixed $r > 0$. Since r is small, we can assume that $\tilde\measuredangle(p\,{}^{y_i}_{y_j}) > \varepsilon$ if $i \neq j$. By 6.20, we can choose p' arbitrarily close to p such that $\uparrow_{[p'y_i]} \in \mathrm{Lin}_{p'}$ for any i. Since $|p' - p|$ is small, $\tilde\measuredangle(p'\,{}^{y_i}_{y_j}) > \varepsilon$ if $i \neq j$. By comparison,

$$\measuredangle[p'\,{}^{y_i}_{y_j}] > \varepsilon.$$

Therefore, $\mathrm{pack}_\varepsilon \Sigma_p \leqslant \mathrm{pack}_\varepsilon \mathbb{S}^{m-1}$.

Since $\mathbb{S}^{m-1}$ is compact, $\mathrm{pack}_\varepsilon \mathbb{S}^{m-1} < \infty$. By the definition, the space of directions Σ_p is complete. Applying 1.2, we get that Σ_p is compact.

It remains to prove the following claim.

❶ *If Σ_p is compact, then it is π-geodesic*

Choose two geodesic directions $\xi = \uparrow_{[px]}$ and $\zeta = \uparrow_{[py]}$; let

$$\alpha = \tfrac{1}{2}\cdot\measuredangle[p\,{}^x_y] = \tfrac{1}{2}\cdot|\xi - \zeta|_{\Sigma_p}.$$

Suppose $\alpha < \frac{\pi}{2}$.

It is sufficient to construct an almost midpoint $\mu = \uparrow_{[pz]}$ of ξ and ζ in Σ_p; that is, we need to show that for any $\varepsilon > 0$ there is a geodesic $[pz]$ such that

$$\measuredangle[p\,{}^x_z] \leqslant \alpha + \varepsilon \quad \text{and} \quad \measuredangle[p\,{}^y_z] \leqslant \alpha + \varepsilon.$$

Indeed, once this is done, the compactness of Σ_p can be used to get an actual midpoint for any two directions in Σ_p, and Menger's lemma (1.6) finishes the proof.

Choose a sequence of small positive numbers $r_n \to 0$ Consider sequences $x_n \in [px]$ and $y_n \in [py]$ such that $|p - x_n| = |p - y_n| = r_n$. Let m_n be a midpoint of $[x_n\, y_n]$.

Since Σ_p is compact, we can pass to a subsequence of r_n such that $\uparrow_{[pm_n]}$ converges; denote the limit by μ. Choose a geodesic $[pz]$ that runs at a small angle to μ. It remains to show that $\uparrow_{[pz]}$ is the needed almost midpoint.

Evidently, $\tilde\measuredangle(p\,{}^{x_n}_{m_n}) = \tilde\measuredangle(p\,{}^{y_n}_{m_n})$. By 2.7, we have

$$\tilde\measuredangle(p\,{}^{x_n}_{m_n}) + \tilde\measuredangle(p\,{}^{y_n}_{m_n}) \leqslant \tilde\measuredangle(p\,{}^{x_n}_{y_n}).$$

Let $z_n \in [pz]$ be the point such that $|p - z_n| = |p - m_n|$. By construction, for all large n, we have $\measuredangle[p\,{}^{z}_{m_n}] \approx 0$ with arbitrary small given error. By comparison, the value $\frac{|z_n - m_n|}{|p - z_n|}$ can be assumed to be arbitrarily small for all large n. Applying this observation and the definition of angle measure, we also get that the following approximations

$$\begin{aligned}
\tilde\measuredangle(p\,{}^{x_n}_{y_n}) &\approx \measuredangle[p\,{}^{x_n}_{y_n}],\\
\tilde\measuredangle(p\,{}^{x_n}_{m_n}) &\approx \tilde\measuredangle(p\,{}^{x_n}_{z_n}) \approx \measuredangle[p\,{}^{x_n}_{z_n}],\\
\tilde\measuredangle(p\,{}^{m_n}_{y_n}) &\approx \tilde\measuredangle(p\,{}^{z_n}_{y_n}) \approx \measuredangle[p\,{}^{z_n}_{y_n}],
\end{aligned}$$

hold with arbitrary given error and all large n. It follows that $\uparrow_{[pz]}$ is an almost midpoint of $\uparrow_{[px]}$ and $\uparrow_{[py]}$, as required. □

Remark. In the above proof, the angles $\measuredangle[p\,{}^{x}_{z}]$ and $\measuredangle[p\,{}^{y}_{z}]$ have lower bounds by the comparison, but we needed upper bounds. These were extracted from the definition of angle measure and the compactness of the space of directions. Halbeisen's example [6, 13.6] shows that it cannot be done without the compactness condition.

C Right-inverse theorem

7.8. Right-inverse theorem. *Suppose $p, a_0, \dots, a_m$ be points in an Alexandrov space $\mathcal{A}$ such*

$$\tilde\measuredangle(p\,{}^{a_i}_{a_j}) > \tfrac{\pi}{2}$$

for any $i \ne j$. Then the map $f\colon \mathcal{A} \to \mathbb{R}^m$ defined by

$$f\colon x \mapsto (|a_1 - x|, \dots, |a_m - x|)$$

has a continuous right inverse defined in a neighborhood of $f(p)$.

In the proof we construct a local right inverse Φ of f around $f(p)$. The construction uses gradient flow for a suitably chosen family of functions. The frame of the proof can be seen in the following exercise; more details are given in the hints.

7.9. Exercise. *Suppose $p, a_0, \dots, a_m \in \mathcal{A}$ and $f\colon \mathcal{A} \to \mathbb{R}$ are as in 7.8. Assume $\varepsilon > 0$ is sufficiently small. Given $\boldsymbol{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$, consider the function on $\mathcal{A}$ defined by*

$$f_{\boldsymbol{y}}(x) := \min\{\,0, |a_1 - x| - y_1, \dots, |a_m - x| - y_m\,\} + \varepsilon{\cdot}|a_0 - x|.$$

(a) *Show that for some fixed $r > 0$ and λ, the function $f_{\boldsymbol{y}}$ is λ-concave in $\mathrm{B}(p, r)$,*

(i) $(\boldsymbol{d}_x \mathrm{dist}_{a_i})(\nabla_x f_{\boldsymbol{y}}) < -\varepsilon^2$ *if $|a_i - x| > y_i$ and* *(ii)* $(\boldsymbol{d}_x \mathrm{dist}_{a_i})(\nabla_x f_{\boldsymbol{y}}) > \varepsilon^2$ *if*

$$|a_i - x| - y_i = \min_j \{|a_j - x| - y_j\} < 0.$$

at any $x \in \mathrm{B}(p, r)$.

(b) Let $\alpha_{\boldsymbol{y}}$ *be* $f_{\boldsymbol{y}}$*-gradient curve that starts at* p. *Use (a) to show that*

$$\mathrm{dist}_{\boldsymbol{a}}[\alpha_{\boldsymbol{y}}(t_0)] = \boldsymbol{y}$$

if $\frac{1}{\varepsilon^2} \cdot |\mathrm{dist}_{\boldsymbol{a}} p - \boldsymbol{y}| \leqslant t_0 \leqslant \frac{r}{2}$.

(c) Let $t_0(\boldsymbol{y}) = \frac{1}{\varepsilon^2} \cdot \big| |\boldsymbol{a} - p| - \boldsymbol{y} \big|$. *Use 5.9 to show that the map*

$$\Phi \colon \boldsymbol{y} \mapsto \alpha_{\boldsymbol{y}} \circ t_0(\boldsymbol{y})$$

continuous (in fact Hölder) in $\Omega = \mathrm{B}(|\boldsymbol{a} - p|, \frac{\varepsilon^2}{2} \cdot r) \subset \mathbb{R}^m$ *and* $f \circ \Phi(\boldsymbol{y}) = \boldsymbol{y}$ *for any* $\boldsymbol{y} \in \Omega$. *(This finishes the proof of 7.8.)*

D Distance chart

7.10. Theorem. *Suppose* $p, a_0, \dots, a_m$ *be points in an* m*-dimensional Alexandrov space* $\mathcal{A}$ *such*

$$\tilde{\measuredangle}(p\,{}^{a_i}_{a_j}) > \tfrac{\pi}{2}$$

for any $i \neq j$. *Then the map* $f \colon \mathcal{A} \to \mathbb{R}^m$ *defined by*

$$f \colon x \mapsto (|a_1 - x|, \dots, |a_m - x|)$$

gives a bi-Lipschitz embedding of a neighborhood Ω *of* p*; the restriction* $f|_\Omega$ *is called a* d i s t a n c e c h a r t *at* p.

The following exercise guides you to prove the theorem.

7.11. Exercise. *Suppose* $p, a_0, \dots, a_m \in \mathcal{A}$ *and* $f \colon \mathcal{A} \to \mathbb{R}$ *are as in 7.8. Show that there is* $\varepsilon > 0$ *such that one of the inequalities holds in each of the following two strings of* m *inequalities*

$$\begin{aligned}
&\measuredangle[x\,{}^{y}_{a_1}] < \tfrac{\pi}{2} - \varepsilon, \ \dots, \ \measuredangle[x\,{}^{y}_{a_m}] < \tfrac{\pi}{2} - \varepsilon;\\
&\measuredangle[y\,{}^{x}_{a_1}] < \tfrac{\pi}{2} - \varepsilon, \ \dots, \ \measuredangle[y\,{}^{x}_{a_m}] < \tfrac{\pi}{2} - \varepsilon
\end{aligned}$$

for any two points x, y *in a sufficiently small neighborhood of* p.

Use this together with the right-inverse theorem (7.8) to prove 7.10.

E Volume

Fix a positive integer m. The m-dimensional Hausdorff measure of a Borel set B in a metric space will be called its m-volume; it will be denoted by $\operatorname{vol}_m B$. We assume that the Hausdorff measure is calibrated so that the unit cube in $\mathbb{E}^m$ has unit volume.

This definition will be applied mostly to subsets in m-dimensional Alexandrov spaces. In this case, we may write $\operatorname{vol} B$ instead of $\operatorname{vol}_m B$.

7.12. Bishop–Gromov inequality. *Let $\mathcal{A}$ be an* $\mathrm{ALEX}(0)$ *space. Suppose* $\dim \mathcal{A} = m < \infty$. *Then*

$$\operatorname{vol} \mathrm{B}(p,r) \leqslant \omega_m \cdot r^m,$$

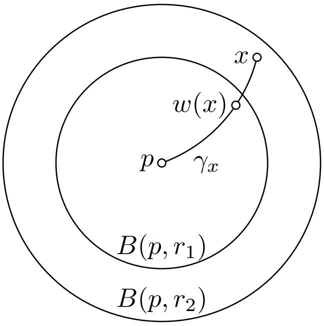
where ω_m denotes the volume of the unit ball in $\mathbb{E}^m$. Moreover, the function

$$r \mapsto \frac{\operatorname{vol} B(p,r)}{r^m}$$

is nonincreasing.

Proof. Given $x \in \mathcal{A}$, choose a geodesic path γ_x from p to x. Recall that $\log_p \colon \mathcal{A} \to \mathrm{T}_p$ can be defined by $\log_p \colon x \mapsto \gamma_x^+(0)$. By comparison, $\log_p$ is distance-noncontracting. Note that $\log_p$ maps $\mathrm{B}(p,r)_{\mathcal{A}}$ to $\mathrm{B}(0,r)_{\mathrm{T}_p}$.

If $\mathrm{T}_p \overset{iso}{=\!=} \mathbb{E}^m$, then $\operatorname{vol} \mathrm{B}(0,r)_{\mathrm{T}_p} = \omega_m \cdot r^m$, and the first statement follows.



If T_p is not isometric to $\mathbb{E}^m$, then by 7.2, we can find a point p' arbitrarily close to p such that $\mathrm{T}_{p'} \overset{iso}{=\!=} \mathbb{E}^m$. If $\varepsilon > |p - p'|$, then $\mathrm{B}(p,r) \subset \mathrm{B}(p', r+\varepsilon)$. Therefore,

$$\operatorname{vol} \mathrm{B}(p,r) \leqslant \omega_m \cdot (r+\varepsilon)^m$$

for any $\varepsilon > 0$. Hence the first statement follows.

Now, suppose $0 < r_1 < r_2$. Consider the map $w \colon \mathcal{A} \to \mathcal{A}$ defined by $w \colon x \mapsto \gamma_x(\tfrac{r_1}{r_2})$. (This map mimics the dilation with center at p and coefficient $\tfrac{r_1}{r_2}$.) By comparison,

$$|w(x) - w(y)| \geqslant \tfrac{r_1}{r_2} \cdot |x - y|.$$

Observe that $\mathrm{B}(p, r_1) \supset w[\mathrm{B}(p, r_2)]$. Therefore,

$$\operatorname{vol} \mathrm{B}(p, r_1) \geqslant (\tfrac{r_1}{r_2})^m \cdot \operatorname{vol} \mathrm{B}(p, r_2).$$

□

The following exercise generalizes the Bishop–Gromov inequality to $\mathrm{Alex}(-1)$ case; it is sufficient for most applications. A more exact statement for $\mathrm{Alex}(\kappa)$ spaces is given in 7.17.

7.13. Exercise! *Let $\mathcal{A}$ be an $\mathrm{Alex}(-1)$ space. Suppose $\mathcal{A} = m < \infty$. Show that*

$$\operatorname{vol} \mathrm{B}(p,r) \leqslant \omega_m \cdot (\sinh r)^m,$$

where ω_m denotes the volume of the unit ball in $\mathbb{E}^m$. Moreover, the function

$$r \mapsto \frac{\operatorname{vol} B(p,r)}{(\sinh r)^m}$$

is nonincreasing.

7.14. Exercise! *Show that any finite-dimensional Alexandrov space is proper.*

F Other dimensions

Now we want to show that *all reasonable definitions of dimension give the same result for Alexandrov spaces.* More precisely, we have the following theorem; compare to [6, 15.16]. We refer to [47] for definitions of Lebesgue covering dimension and Hausdorff dimension, which will be denoted by TopDim and HausDim, respectively.

7.15. Theorem. *For any Alexandrov space $\mathcal{A}$, we have*

$$\operatorname{LinDim} \mathcal{A} = \operatorname{TopDim} \mathcal{A} = \operatorname{HausDim} \mathcal{A}.$$

Proof. Suppose $\operatorname{LinDim} \mathcal{A} = \infty$. By the right-inverse theorem (7.8), $\mathcal{A}$ contains a compact subset K with arbitrarily large $\operatorname{TopDim} K$. In particular,

$$\operatorname{TopDim} \mathcal{A} = \infty.$$

By Szpilrajn's theorem, $\operatorname{HausDim} K \geqslant \operatorname{TopDim} K$. Thus we also have

$$\operatorname{HausDim} \mathcal{A} = \infty.$$

Now suppose $\operatorname{LinDim} \mathcal{A} = m < \infty$. By the Bishop–Gromov inequality (7.12 and 7.13),

$$\operatorname{HausDim} \mathcal{A} \leqslant m.$$

Since $\mathcal{A}$ is proper (7.14), Szpilrajn's theorem, implies that

$$\operatorname{TopDim}\mathcal{A} \leqslant \operatorname{HausDim}\mathcal{A} \leqslant m.$$

Finally, by the right-inverse theorem (7.8), $m \leqslant \operatorname{TopDim}\mathcal{A}$. □

7.16. Exercise. *Let Ω be an open subset of Alexandrov space $\mathcal{A}$. Show that*

$$\operatorname{LinDim}\mathcal{A} = \operatorname{LinDim}\Omega = \operatorname{TopDim}\Omega = \operatorname{HausDim}\Omega.$$

G Remarks

Let us state a version of the Bishop–Gromov inequality for $\mathrm{A{\scriptsize LEX}}(\kappa)$ spaces. Its proof requires additionally the so-called *coarea formula* for Alexandrov spaces. The weaker inequality from 7.13 is sufficient for the sequel.

7.17. Optimal Bishop–Gromov inequality. *Given a point p in an m-dimensional $\mathrm{A{\scriptsize LEX}}(\kappa)$ space, consider the function*

$$v\colon r \to \operatorname{vol}_m \mathrm{B}(p,r);$$

denote by $\tilde v(r)$ the volume of r ball in $\mathbb{M}^m(\kappa)$. Then

$$v(r) \leqslant \tilde v(r)$$

for any $r>0$. Morevoer, the function

$$r \mapsto \frac{v(r)}{\tilde v(r)}$$

is nonincreasing. If $\kappa>0$, then one has to assume that $r<\frac{\pi}{\sqrt{\kappa}}$.

This inequality was originally proved for Riemannian manifolds with lower Ricci curvature. The first part is also known as Bishop's inequality. It is due to Richard Bishop; see [15] and [14, Corollary 4, p. 256]. The second part is due to Michael Gromov [36].

Theorem 7.15, was essentially proved by Conrad Plaut [92]. At that time, it was not known whether

$$\operatorname{LinDim}\mathcal{A} = \infty \quad \Rightarrow \quad \operatorname{TopDim}\mathcal{A} = \infty$$

for any Alexandrov space $\mathcal{A}$. The latter implication was proved by Grigory Perelman and the second author [78].

Lecture 8

Limit spaces

Here we will show that lower curvature bound in the sense of Alexandrov survives under Gromov–Hausdorff limit, present Perelman's construction of strictly concave functions, and prove Gromov's selection theorem.

The suvival of curvature bound provides the main source of applications of Alexandrov geometry; as an illustration we prove the homotopy stability theorem (8.11) and deduce the homotopy finiteness theorem (8.12) from it.

A Survival of curvature bounds

8.1. Theorem. *Let $\mathcal{X}_n \to \mathcal{X}_\infty$ be a convergence in the sense of Gromov–Hausdorff. Suppose that for each n, the space $\mathcal{X}_n$ has curvature $\geqslant \kappa$ in the sense of Alexandrov. Then the same holds for $\mathcal{X}_\infty$.*

Proof. Choose a quadruple of points $p_\infty, x_\infty, y_\infty, z_\infty \in \mathcal{X}_\infty$.

By the definition of Gromov–Hausdorff convergence, we can choose points p_n, x_n, y_n, $z_n \in \mathcal{X}_n$ for each n that converge to p_∞, x_∞, y_∞, $z_\infty \in \mathcal{X}_\infty$, respectively. In particular, each of the 6 distances between pairs of p_n, x_n, y_n, z_n converge to the distance between the corresponding pairs of $p_\infty, x_\infty, y_\infty, z_\infty$.

Since $\mathbb{M}^2(\kappa)$-comparison holds for p_n, x_n, y_n, $z_n \in \mathcal{X}_n$, passing to the limit, we get the $\mathbb{M}^2(\kappa)$-comparison for p_∞, x_∞, y_∞, z_∞. $\square$

8.2. Exercise! *Suppose that a sequence $\mathcal{A}_1, \mathcal{A}_2, \dots$ of* Alex(κ) *spaces converges to $\mathcal{A}_\infty$ in the sense of Gromov–Hausdorff. Show that $\mathcal{A}_\infty$ is* Alex(κ) *and*

$$\dim \mathcal{A}_\infty \leqslant \varliminf_{n\to\infty} \dim \mathcal{A}_n.$$

B Gromov's selection theorem

8.3. Gromov's selection theorem. *Let m be a positive integer, and let $D, \kappa \in \mathbb{R}$. Then any sequence of m-dimensional* $\mathrm{ALEX}(\kappa)$ *spaces with diameters at most D has a converging subsequence in the sense of Gromov–Hausdorff.*

Proof of 8.3. Denote by $\boldsymbol{K}$ the set of all isometry classes of $\mathrm{ALEX}(0)$ spaces with dimension $\leqslant m$ and diameter $\leqslant D$. By 8.2, $\boldsymbol{K}$ is a closed subset of GH.

Choose a space $\mathcal{A} \in \boldsymbol{K}$; suppose $x_1, \dots, x_n \in \mathcal{A}$ is a collection of points such that $|x_i - x_j| > \varepsilon$ for all $i \neq j$. Note that the balls $B_i = \mathrm{B}(x_i, \frac{\varepsilon}{2})$ do not overlap.

By 7.8, $\operatorname{vol} \mathcal{A} > 0$. By Bishop–Gromov inequality, $\operatorname{vol} \mathcal{A} < \infty$, and if $\varepsilon < D$, then

$$\operatorname{vol} B_i \geqslant (\tfrac{\varepsilon}{2 \cdot D})^m \cdot \operatorname{vol} \mathcal{A}$$

for any i. It follows that $n \leqslant (\frac{2 \cdot D}{\varepsilon})^m$; that is,

$$\operatorname{pack}_\varepsilon \mathcal{A} \leqslant N(\varepsilon) := (\tfrac{2 \cdot D}{\varepsilon})^m$$

for all small $\varepsilon > 0$.

Choose a maximal ε-packing $x_1, \dots, x_n \in \mathcal{A}$. By 1.2, $\mathcal{F}_\varepsilon := \{x_1, \dots$ $\dots, x_n\}$ is an ε-net of $\mathcal{A}$. Observe that $|\mathcal{F}_\varepsilon - \mathcal{A}|_{\mathrm{GH}} \leqslant \varepsilon$. Further, note that the set $\boldsymbol{F}_\varepsilon$ of finite metric spaces with diameter $\leqslant D$ and at most $N(\varepsilon)$ points forms a compact subset in GH.

Summarizing, for any $\varepsilon > 0$ we can find a compact ε-net $\boldsymbol{F}_\varepsilon \subset \mathrm{GH}$ of $\boldsymbol{K}$. Since GH is complete (1.22), it remains to apply 1.1*b*.

We finished the proof of the case $\kappa = 0$. In the general case, applying rescaling, we can assume that $\kappa = -1$ and then argue as before, using 7.13 instead of 7.12. □

8.4. Exercise!

(a) *Let $\mathcal{A}$ be an m-dimensional* $\mathrm{ALEX}(0)$ *space with diameter $\leqslant D$. Suppose* $\operatorname{vol} \mathcal{A} \geqslant v_0 > 0$. *Show that*

$$\operatorname{pack}_\varepsilon \mathcal{A} \geqslant \frac{c}{\varepsilon^m}$$

for some constant $c = c(m, D, v_0) > 0$.

(b) *Conclude that if $\mathcal{A}_n$ is a sequence of m-dimensional* $\mathrm{ALEX}(0)$ *spaces with diameter $\leqslant D$, and volume $\geqslant v_0$, then its Gromov–Hausdorff limit $\mathcal{A}_\infty$ (if it exists) has dimension m.*

8.5. Exercise. *Show that any sequence of m-dimensional* $\mathrm{ALEX}(\kappa)$ *spaces with marked points contains a subsequence pointed-converging in the sense of Gromov–Hausdorff (see Section 1L).*

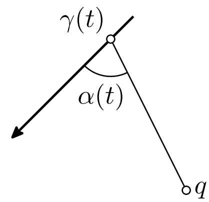
C Controlled concavity

Alexandrov spaces have plenty of semiconcave functions; for instance, squared distance functions. The following theorem provides a source of strictly concave functions defined on small open sets of finite-dimensional Alexandrov spaces.

8.6. Theorem. *Let $\mathcal{A}$ be a complete finite-dimensional Alexandrov space. Then for any point $p \in \mathcal{A}$, there is a strictly concave function f defined in an open neighborhood of p.*

Moreover, given $0 \neq v \in T_p$, the differential, $\boldsymbol{d}_p f$, can be chosen arbitrarily close to $x \mapsto -\langle v, x\rangle$.

Proof. Fix small $r > 0$ and large c; consider the real-to-real function



$$\varphi_{r,c}(x) = (x - r) - c{\cdot}(x - r)^2/r,$$

so we have $\varphi_{r,c}(r) = 0$, $\varphi'_{r,c}(r) = 1$, and $\varphi''_{r,c}(r) = -2c/r$.

Let γ be a unit-speed geodesic, fix a point q and let

$$\alpha(t) = \measuredangle(\gamma^+(t), \uparrow_{[\gamma(t)q]}).$$

Recall that r is small. If $|q - \gamma(t)|$ is sufficiently close to r, then direct calculations show that

$$(\varphi_{r,c} \circ \operatorname{dist}_q \circ \gamma)''(t) \leqslant \frac{3 - c{\cdot} \cos^2[\alpha(t)]}{r}.$$

(Since c is large, this inequality implies that $\varphi_{r,c} \circ \operatorname{dist}_q \circ \gamma$ is strictly concave at t unless $\alpha(t) \approx \frac{\pi}{2}$.)

Now, assume $\{q_1, \dots, q_N\}$ is a finite set of points such that $|p - q_i| = r$ for every i. For a geodesic γ, set $\alpha_i(t) = \measuredangle(\gamma^+(t), \uparrow_{[\gamma(t)q_i]})$. Assume that

❶
$$\max_i\{|\alpha_i(t) - \tfrac{\pi}{2}|\} \geqslant \varepsilon > 0$$

for any geodesic γ in $\mathrm{B}(p, \varepsilon)$. Choose $c > 3N/\cos^2 \varepsilon$; then the inequality above implies that the function

$$f = \sum_i \varphi_{r,c} \circ \operatorname{dist}_{q_i}$$

is strictly concave in $\mathrm{B}(p, \varepsilon')$ for some positive $\varepsilon' < \varepsilon$.

The same argument as in 8.4 shows that for small $r > 0$, one can choose $N \geqslant c/\delta^{m-1}$ points $\{q_i\}$ such that $|p - q_i| = r$ and $\tilde\measuredangle(p\,{}^{q_j}_{q_i}) > \delta$

(here $c = c(\Sigma_p) > 0$). On the other hand, suppose γ runs from x to y. If $|\alpha_i(t) - \frac{\pi}{2}| < \varepsilon \ll \delta$, then applying the $(n{+}1)$-point comparison to $\gamma(t)$, x, y and all $\{q_i\}$ we get that $N \leqslant c(m)/\delta^{m-2}$. Therefore, for small $\delta > 0$ and yet smaller $\varepsilon > 0$, the set $\{q_i\}$ meets ❶.

If r is small, then points q_i can be chosen so that all directions $\uparrow_{[pq_i]}$ will be ε-close to a given direction ξ and therefore the second property follows. □

The function f in 8.6 can be chosen to have maximum value 0 at p, $f(p) = 0$ and with $\boldsymbol{d}_p f(x) \approx -|x|$. It can be constructed by taking the minimum of the functions in the theorem. Then the set $K = \{\, x \in \mathcal{A} : f(x) \geqslant -\varepsilon \,\}$ forms a closed convex neighborhood of p for any small $\varepsilon > 0$, so we get the following.

8.7. Corollary. *Any point p of a finite-dimensional Alexandrov space admits an arbitrarily small convex closed neighborhood K and a strictly concave function f defined in a neighborhood of K such that p is the maximum point of f and $f|_{\partial K} = 0$.*

8.8. Exercise. *Construct an Alexandrov space $\mathcal{A}$ such that there is no open set $\Omega \subset A$ with strictly concave function $f\colon \Omega \to \mathbb{R}$.*

D Liftings

Suppose that the Gromov–Haudorff distance $|\mathcal{A} - \mathcal{A}'|_{\mathrm{GH}}$ is sufficiently small, so we may think that both spaces $\mathcal{A}$ and $\mathcal{A}'$ lie at a small Hausdorff distance in an ambient metric space $\mathcal{W}$. In particular, we may choose a small $\varepsilon > 0$, so that for any point $p \in \mathcal{A}$, there is a point $p' \in \mathcal{A}'$ such that $|p - p'|_{\mathcal{W}} < \varepsilon$; the point p' will be called a lifting (or ε-lifting) of p in $\mathcal{A}'$. We may choose a lifting $p' \in \mathcal{A}'$ for every point $p \in \mathcal{A}$, in this case the map $p \mapsto p'$ is called a (ε-)lifting map.

Let us emphasise that liftings are not uniquely defined, and the lifting map is not assumed to be continuous. Also, to talk about liftings, we have to choose $\varepsilon > 0$, as well as the inclusions $\mathcal{A}, \mathcal{A}' \hookrightarrow \mathcal{W}$.

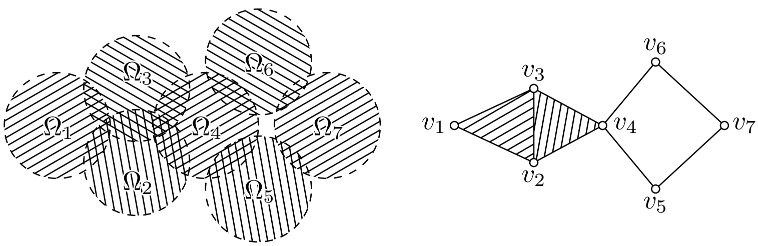
Let $\mathcal{A}$ be a compact m-dimensional Alexandrov space. Suppose $\mathcal{A}'$ is another compact m-dimensional Alexandrov space such that $|\mathcal{A} - \mathcal{A}'|_{\mathrm{GH}}$ is sufficiently small — smaller than some $\varepsilon = \varepsilon(\mathcal{A}) > 0$. If the construction of the previous section is performed in $\mathcal{A}$, then we can be repeated in $\mathcal{A}'$ for the liftings of all points and the same function φ. It produces a strictly concave function defined in a controlled neighborhood of the lifting p' of p.

The results of this and related constructions will be also called liftings, say we can talk about a lifting from $\mathcal{A}$ to $\mathcal{A}'$ of a function provided by 8.6 (if the Gromov–Hausdorff distance $|\mathcal{A}-\mathcal{A}'|_{\mathrm{GH}}$ is small, then these liftings are strictly concave) and a lifting of a convex neighborhood from 8.7. Here one cannot use 8.6 and 8.7 as black boxes — one has to understand the construction, but it is straightforward.

8.9. Exercise. *Give an example of Gromov–Hausdorff convergence of spaces $\mathcal{A}_n \to \mathcal{A}_\infty$ such that all $\mathcal{A}_n$ and $\mathcal{A}_\infty$ are compact finite-dimensional* $\mathrm{Alex}(0)$ *and for any small $\varepsilon > 0$ there is no continuous ε-lifting map $\mathcal{A}_\infty \to \mathcal{A}_n$ for any large n.*

E Nerves

Let $\{\Omega_1, \dots, \Omega_k\}$ be a finite open cover of a compact metric space $\mathcal{X}$. Consider an abstract simplicial complex $\mathcal{N}$, with one vertex v_i for each set Ω_i such that a simplex with vertices $v_{i_1}, \dots, v_{i_m}$ is included in $\mathcal{N}$ if the intersection $\Omega_{i_1} \cap \dots \cap \Omega_{i_m}$ is nonempty. The obtained simplicial



covering nerve

complex $\mathcal{N}$ is called the nerve of the covering $\{\Omega_i\}$. Evidently, $\mathcal{N}$ is a finite simplicial complex — it is a subcomplex of a simplex with the vertices $\{v_1, \dots, v_k\}$. Recall that Star_{v_i} denotes the union of all simplices in $\mathcal{N}$ that share the vertex v_i.

The next statement follows from [45, 4G.3].

8.10. Nerve theorem. *Let $\{\Omega_1, \dots, \Omega_k\}$ be an open cover of a compact metric space $\mathcal{X}$ and let $\mathcal{N}$ be the corresponding nerve with vertices $\{v_1, \dots, v_k\}$. Suppose that every nonempty finite intersection $\Omega_{\alpha_1} \cap \dots$ $\dots \cap \Omega_{\alpha_k}$ is contractible. Then $\mathcal{X}$ is homotopy equivalent to the nerve $\mathcal{N}$ of the cover.*

Moreover homotopy equivalences $a\colon \mathcal{X} \to \mathcal{N}$ and $b\colon \mathcal{N} \to \mathcal{X}$ can be chosen so that if $x \in \Omega_i$, then $a(x) \in \mathrm{Star}_{v_i}$, and if $y \in \mathcal{N}$ lies in the simplex with vertices $v_{i_1}, \dots, v_{i_m}$, then $b(y) \in \Omega_{i_1} \cup \dots \cup \Omega_{i_m}$.

F Homotopy stability

8.11. Theorem. *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots$, and $\mathcal{A}_\infty$ be compact m-dimensional* $\mathrm{Alex}(\kappa)$ *spaces, and $m < \infty$. Suppose $\mathcal{A}_n \to \mathcal{A}_\infty$ as $n \to \infty$ in the sense of Gromov–Hausdorff. Then $\mathcal{A}_\infty$ is homotopy equivalent to $\mathcal{A}_n$ for all large n.*

Moreover, given $\varepsilon > 0$ there are maps $h_n\colon \mathcal{A}_\infty \to \mathcal{A}_n$ that are homotopy equivalences and ε-liftings for all large n.

Applying this theorem with Gromov's selection theorem (8.3) and Exercise 8.4, we get the following.

8.12. Theorem. *Given $\kappa \in \mathbb{R}, D > 0, v_0 > 0$ and a positive integer m, there are only finitely many homotopy types of m-dimensional* $\mathrm{Alex}(\kappa)$ *spaces with diameter $\leqslant D$, and volume $\geqslant v_0$.*

Proof of 8.12 modulo 8.11. Assume the contrary, then we can choose a sequence of m-dimensional $\mathrm{Alex}(\kappa)$ spaces $\mathcal{A}_1, \mathcal{A}_2, \ldots$ that have different homotopy types and satisfy the assumptions of the theorem. By Gromov's selection theorem, we can assume that $\mathcal{A}_n$ converges to some spaces $\mathcal{A}_\infty$ in the sense of Gromov–Hausdorff.

By 8.4, $\dim \mathcal{A}_\infty = m$. It remains to apply 8.11. $\square$

Proof of 8.11. Since $\mathcal{A}_\infty$ is compact, applying 8.7, we can find a finite open cover of $\mathcal{A}_\infty$ by convex open sets $\Omega_1, \ldots, \Omega_k$ such that for each Ω_i there is a strictly concave function f_i that is defined in a neighborhood of the closure $\bar{\Omega}_i$ and such that $f_i|_{\partial\Omega_i} = 0$.

Subtracting from functions f_i some small value $\varepsilon > 0$, we can ensure that $\bigcap_{i\in S} \Omega_i \neq \varnothing$ if and only if $\bigcap_{i\in S} \bar{\Omega}_i \neq \varnothing$.

Suppose that $W = \bigcap_{i\in S} \Omega_i \neq \varnothing$. Then W is contractible. Indeed, the function

$$f_S := \min_{i\in S} f_i$$

is strictly concave and it vanishes on the boundary of W. The f_S-gradient flow $(t, x) \mapsto \mathrm{Flow}_{f_S}^t(x)$ defines a homotopy $[0, \infty) \times W \to \to W$. By the first distance estimate (5.6), $\mathrm{Flow}_{f_S}^t(x)$ converges to the (necessarily unique) maximum point of f_S as $t \to \infty$. Therefore, in the obtained homotopy we can parametrize $[0, \infty)$ by $[0, 1)$ and extend the homotopy continuously to $[0, 1]$; thus we get that W is contractible. In other words, the cover $\{\Omega_1, \ldots, \Omega_k\}$ meets the assumptions of the nerve theorem (8.10).

The functions f_i and sets Ω_i can be lifted to $\mathcal{A}_n$ keeping their properties for all large n. More precisely, there are liftings $f_{i,n}$ of all f_i to $\mathcal{A}_n$ which are strictly concave for all large n and such that

$\bar{\Omega}_{i,n} = \{\, x \in \mathcal{A}_n \,:\, f_{i,n}(x) \geqslant 0 \,\}$ is a compact convex set and $\Omega_{i,n} = \\ = \{\, x \in \mathcal{A}_n \,:\, f_{i,n}(x) > 0 \,\}$ is an open convex set for each i.

Notice that $\{\Omega_{1,n}, \dots, \Omega_{k,n}\}$ is an open cover of $\mathcal{A}_n$ for all large n. Indeed suppose we have $p_n \in \mathcal{A}_n \setminus (\Omega_{1,n} \cup \dots \cup \Omega_{k,n})$ for arbitrary large n. Since $\mathcal{A}_\infty$ is compact, there is a limit point $p_\infty \in \mathcal{A}_\infty$ for a subsequnce of p_n. But $p_\infty \in \Omega_i$ for some i and therefore $p_n \in \Omega_{i,n}$ for arbitrary large n — a contradiction.

In a similar fashion, we can show that if n is large, then any collection $\{\Omega_{i,n}\}_{i\in S}$ has a common point in $\mathcal{A}_n$ if and only if $\{\Omega_i\}_{i\in S}$ has a common point in $\mathcal{A}_\infty$. Here we have to use that $\bigcap_{i\in S} \Omega_i \neq \varnothing$ if and only if $\bigcap_{i\in S} \bar{\Omega}_i \neq \varnothing$.

It follows that for any large n the covers

- ◇ $\{\Omega_1, \dots, \Omega_k\}$ of $\mathcal{A}_\infty$ and
- ◇ $\{\Omega_{1,n}, \dots, \Omega_{k,n}\}$ of $\mathcal{A}_n$.

have the same nerve. By the nerve theorem (8.10), $\mathcal{A}_n$ and $\mathcal{A}_\infty$ are homotopy equivalent for all large n — a contradiction. □

Note that the proof above implies the following.

8.13. Theorem. *Any compact finite-dimensional Alexandrov space is homotopy equivalent to a finite simplicial complex.*

G Remarks

Originally, Gromov's selection theorem was proved for Riemannian manifolds with a lower bound on Ricci curvature [36]. It motivates the study of limits of manifolds with lower Ricci curvature bounds and their synthetic generalizations, the so-called $\mathrm{CD}(K,m)$ spaces; CD stands for curvature-dimension condition. This theory has significant applications in Alexandrov geometry; in particular, it provides a version of the Liouville theorem about phase-space volume of geodesic flow in Alexandrov space [16].

The construction of a strictly concave function (8.6) is due to Grigory Perelman [77, 80].

The homotopy-type finiteness theorem (8.12) illustrates the main source of applications of Alexandrov spaces: to prove a statement about m-dimensional manifolds with lower sectional curvature bound we follow the following steps:

- ◇ Arguing by contradiction, we assume existance of a sequence of manifolds that violates our assumption.
- ◇ Use Gromov's selection theorem to choose a converging subseqence.
- ◇ Use curvature survival, to conclude that the limit space is Alexandrov.

$\diamond$ Try to arrive at a contradiction using Alexandrov geometry.

In principle, the same strategy might work for a sequence of Riemannian manifolds with dimension approaching infinity, but no applications of this kind were found. The following question gives an example where such strategy might be successful in principle.

8.14. Question. *Is it true that no simply connected closed manifold (of arbitrary large dimension) admits a Riemannian metric with sectional curvature and diameter bounded by a fixed positive sufficiently small value?*

If the dimension is bounded, then Gromov's theorem [35] implies that such a manifold can be covered by a compact nil-manifold; in particular, the manifold cannot be simply connected. However, if the dimension is unbounded, then Riemannian manifolds with these conditions may not be covered by compact nil-manifolds; such examples were found by Galina Guzhvina [42].

Let us finish with a list of results that can be proved by applying Gromov's selection theorem in the same fashion as in the proof of homotopy-type finiteness theorem (8.12).

8.15. Betti-number theorem. *For any $\kappa\in\mathbb{R}$, $D>0$ and a positive integer m, there is a constant $c=c(m,D,\kappa)$ such that*

$$\beta_0(M)+\dots+\beta_m(M)\leqslant c$$

for any closed m-dimensional Riemannian manifold M with sectional curvature $\geqslant\kappa$ and diameter $\leqslant D$. Here $\beta_i(M)$ denotes i^{th} Betti number of M.

Gromov's original proof [34] of the Betti-number theorem did not use Alexandrov geometry directly. However, it is easy to produce a proof following the steps above (noone bothered to write it so far). The following result was proved by the second author [88], and it uses the same technique.

8.16. Scalar curvature bound. *Given $\kappa\in\mathbb{R}$, $D>0$ and a positive integer m, there is a constant $c=c(m,D,\kappa)$ such that*

$$\int\limits_M \mathrm{Sc}\leqslant c$$

for any closed m-dimensional Riemannian manifold M with sectional curvature $\geqslant\kappa$ and diameter $\leqslant D$. Here Sc *denotes the scalar curvature.*

The following theorem is a more exact version of 8.11. Its close relative (9.1) will play an important role in the following lecture.

8.17. Stability theorem. *Let $\mathcal{A}_1, \mathcal{A}_2, \dots,$ and $\mathcal{A}_\infty$ be compact m-dimensional* $\mathrm{ALEX}(\kappa)$ *spaces, and $m < \infty$. Suppose $\mathcal{A}_n \to \mathcal{A}_\infty$ as $n \to \infty$ in the sense of Gromov–Hausdorff. Then $\mathcal{A}_\infty$ is homeomorphic to $\mathcal{A}_n$ for all large n.*

Moreover, given $\varepsilon > 0$ there are maps $h_n \colon \mathcal{A}_\infty \to \mathcal{A}_n$ that are homeomorphisms and ε-liftings for all large n.

This theorem was proved by Grigory Perelman [79]. The proof was rewritten with more details by the first author [49]; for more exact statements can see [51] and [52]. In private conversations, Perelman claimed that the homeomorphisms in the theorem can be assumed to be bi-Lipschitz with constants that depend on $\mathcal{A}_\infty$. Since no proof has been written, this statement should be considered as a conjecture; partial results in this direction were obtained by Mohammad Alattar [2].

Theorem 8.12 was originally proved by Karsten Grove and Peter Petersen [39]. Perelman's stability theorem (8.17) implies the following stronger statement.

8.18. Homeomorphism-type finiteness. *For any $\kappa \in \mathbb{R}, D > 0, v > 0$ and a positive integer m, there are only finitely many homeomorphism types of closed m-dimensional manifolds that admit a Riemannian metric with sectional curvature $\geqslant \kappa$, volume $\geqslant v$, and diameter $\leqslant D$.*

This statement can be improved to diffeomorphism-type finiteness in all dimensions $m \neq 4$. Indeed, for $m = 4$ a closed topological m-manifold admits only finitely many smooth structures; see [53] and [74, 105] for cases $m \geqslant 5$ and $m \leqslant 3$, respectively.

Lecture 9

Boundary

This lecture defines the boundary of a finite-dimensional Alexandrov space. After discussing its properties, we prove the doubling theorem (9.9d).

A Definition

The definition of boundary will use induction of dimension, and it will work only for finite-dimensional Alexandrov spaces.

Suppose $\mathcal{A}$ is a 1-dimensional Alexandrov space. By 7.3, $\mathcal{A}$ is homeomorphic to a 1-dimensional manifold (possibly with nonempty boundary). This allows us to define the boundary $\partial\mathcal{A} \subset \mathcal{A}$ as the boundary of the manifold.

Now assume that the notion of boundary is defined in dimensions $1, \dots, m-1$. Suppose $\mathcal{A}$ is m-dimensional Alexandrov space. We say that $p \in \mathcal{A}$ belongs to the boundary (briefly $p \in \partial\mathcal{A}$) if $\partial\Sigma_p \neq \varnothing$. By 7.5 and 7.7$b$, Σ_p is an $(m-1)$-dimensional Alexandrov space; therefore its boundary is already defined and hence this inductive definition makes sense.

It is instructive to check the following statements.

- ◇ For a closed convex set $K \subset \mathbb{E}^m$ with nonempty interior, the topological boundary of K as a subset of $\mathbb{E}^m$ coincides with the boundary K described above.
- ◇ Suppose a finite-dimensional Alexandrov space $\mathcal{A}$ is a direct sum of $\mathcal{A}_1$ and $\mathcal{A}_2$. Then

$$\partial\mathcal{A} = (\partial\mathcal{A}_1 \oplus \mathcal{A}_2) \cup (\mathcal{A}_1 \oplus \partial\mathcal{A}_2)$$

- ◇ If $\operatorname{Cone}\Sigma$ is an $\mathrm{A\scriptsize{LEX}}(0)$ space of dimensions $\geqslant 2$ (this implies that Σ is $\mathrm{A{\scriptsize LEX}}(1)$), then

$$\partial\,\mathrm{Cone}\,\Sigma = \mathrm{Cone}\,\partial\Sigma,$$

where $\mathrm{Cone}\,\partial\Sigma := \{\, s{\cdot}\xi \in \mathrm{Cone}\,\Sigma \;:\; \xi \in \partial\Sigma \,\}$.

B Conic neighborhoods

The following statement is a close relative of Perelman's stability theorem 8.17. We are going to use this result without proof. A proof can be found in [80].

Recall that the logarithm $\log_p\colon \mathcal{A} \to \mathrm{T}_p$ is a mulivalued function defined in 5A.

9.1. Theorem. *For any point p in a finite-dimensional Alexandrov space $\mathcal{A}$ and all sufficiently small $\varepsilon > 0$ there is a homeomorphism $h_\varepsilon\colon \mathrm{B}(p,\varepsilon)_{\mathcal{A}} \to \mathrm{B}(0,\varepsilon)_{\mathrm{T}_p}$ such that $0 = h_\varepsilon(p)$.*

Moreover, we may assume that

$$\sup_{x\in\mathrm{B}(p,\varepsilon)} \{\, \tfrac{1}{\varepsilon}{\cdot}|\log_p x - h_\varepsilon(x)|_{\mathrm{T}_p} \,\} \to 0 \quad \textit{as} \quad \varepsilon \to 0.$$

Note that the last condition automatically implies that h_ε as an $o(\varepsilon)$-sometry.

The above theorem is often used together with the *uniqueness of conic neighborhoods* stated below.

Suppose that an open neighborhood U of a point x in a metric space $\mathcal{X}$ admits a homeomorphism to $\mathrm{Cone}\,\Sigma$ such that x is mapped to the origin of the cone. In this case, we say that U has a conic neighborhood of x.

9.2. Uniqueness of conic neighborhoods. *Any two conic neighborhoods of a given point in a metric space are pointed homeomorphic; that is, there is a homeomorphism between neighborhoods that maps the origin of one cone to the origin of the other.*

9.3. Advanced exercise. *Prove 9.2 or read the proof in [56].*

9.4. Exercise! *Suppose $x \mapsto x'$ is a homeomorphism between finite-dimensional Alexandrov spaces $\mathcal{A}$ and $\mathcal{A}'$. Show that*

(a) $\mathrm{T}_x \cong \mathrm{T}_{x'}$ *(here and below $\cong$ means homeomorphic)*

(b) $\mathrm{Susp}\,\Sigma_x \cong \mathrm{Susp}\,\Sigma_{x'}$.

(c) but in general $\Sigma_x \not\cong \Sigma_{x'}$.

C Topology

The following theorem states that boundary is a topological invariant, despite our definition having used geometry.

9.5. Theorem. *Let $\mathcal{A}$ and $\mathcal{A}'$ be homeomorphic finite-dimensional Alexandrov spaces. Then* $\dim\mathcal{A} = \dim\mathcal{A}'$ *and*

$$\partial\mathcal{A} \neq \varnothing \quad \iff \quad \partial\mathcal{A}' \neq \varnothing.$$

While working on the proof, keep in mind that there are pairs of spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $\mathcal{K}_1 \not\cong \mathcal{K}_2$, but $\mathbb{R}\times\mathcal{K}_1 \cong \mathbb{R}\times\mathcal{K}_2$. For example, one can let $\mathcal{K}_1 = \mathbb{S}^4$ and let $\mathcal{K}_2$ be the suspension over the Poincaré homology sphere; compare to 9.4*c*.

Let $\mathcal{A}$ be an m-dimensional Alexandrov space and $m < \infty$. Define rank of $\mathcal{A}$ (briefly, $\operatorname{rank}\mathcal{A}$) as the minimal value k such that $\mathcal{A}$ splits isometrically as $\mathbb{R}^{m-k}\times\mathcal{K}$; here $\mathcal{K}$ is a k-dimensional Alexandrov space.

In the following proof we will apply induction on the rank of $\mathcal{A}$.

Proof. The first statement follows from 7.15.

Suppose we have a counterexample, say $\partial\mathcal{A} \neq \varnothing$, but $\partial\mathcal{A}' = \varnothing$. Let $k := \operatorname{rank}\mathcal{A}$ and $k' := \operatorname{rank}\mathcal{A}'$. We can assume that the pair (k, k') is minimal in lexicographic order; in particular, k is minimal. Let $x \mapsto x'$ be a homeomorphism from $\mathcal{A}$ to $\mathcal{A}'$.

Choose $x \in \partial\mathcal{A}$. Since $\partial\mathcal{A}' = \varnothing$, we have $x' \notin \partial\mathcal{A}'$. Note that

$$\operatorname{rank}\mathrm{T}_x \leqslant k \quad \text{and} \quad \operatorname{rank}\mathrm{T}_{x'} \leqslant k',$$

By 9.4*a*, $\mathrm{T}_x \cong \mathrm{T}_{x'}$. Note that $\partial\mathrm{T}_x \neq \varnothing$ and $\partial\mathrm{T}_{x'} = \varnothing$. Therefore, we may assume that $\mathcal{A}$ and $\mathcal{A}'$ are Euclidean cones and the homeomorphism sends the origin to the origin. The remaining part of the proof is divided into three cases.

Case 1. Suppose $k > 1$. Let $\mathcal{A} \stackrel{iso}{=\!=} \mathbb{R}^{m-k}\times\mathcal{C}$, where $\mathcal{C}$ a k-dimensional $\mathrm{ALEX}(0)$ cone. Observe that $\operatorname{rank}\mathrm{T}_y \leqslant \operatorname{rank}\mathcal{A}$ for any $y \in \mathcal{A}$ and the equality holds only if y projects to the origin of $\mathcal{C}$.

Since $k > 1$, we can choose $z \neq 0$ that lies in $\partial\mathcal{C}$. Choose y that projects to z; in particular, $\operatorname{rank}\mathrm{T}_y < \operatorname{rank}\mathcal{A}$. By 9.4*a*, $\mathrm{T}_y \cong \mathrm{T}_{y'}$, $\partial\mathrm{T}_y \neq \varnothing$ and $\partial\mathrm{T}_{y'} = \varnothing$. The latter contradicts the minimality of k.

Case 2. Suppose $k \leqslant 1$ and $k' > 1$. Since $\partial\mathcal{A} \neq \varnothing$, we get that $k = 1$; therefore, $\mathcal{A} = \mathbb{R}^{m-1}\times[0,\infty)$.

Let $\mathcal{A}' \stackrel{iso}{=\!=} \mathbb{R}^{m-k'}\times\mathcal{C}'$, where $\mathcal{C}'$ a k'-dimensional $\mathrm{ALEX}(0)$ cone. Since $\partial\mathcal{A} \cong \mathbb{R}^{m-1}$, the image of $\partial\mathcal{A}$ in $\mathcal{A}'$ does not lie in $\mathbb{R}^{m-k'}\times\{0\}$.

In other words, we can choose $y \in \partial\mathcal{A}$ such that its image $y' \in \mathcal{A}'$ has a nonzero projection in $\mathcal{C}'$. Observe that $\mathrm{T}_y \cong \mathrm{T}_{y'}$,

$$\operatorname{rank}\mathrm{T}_y \leqslant k = 1, \quad \operatorname{rank}\mathrm{T}_{y'} < k', \quad \partial\mathrm{T}_y = \varnothing, \quad \text{and} \quad \partial\mathrm{T}_{y'} \neq \varnothing$$

— a contradiction.

Case 3. Suppose $k \leqslant 1$ and $k' \leqslant 1$. Since $\partial\mathcal{A} \neq \varnothing$, we have $k = 1$. By 7.3, $\mathcal{A} \cong \mathbb{R}^{m-1} \times [0, \infty)$. Therefore, $\mathcal{A}' \cong \mathbb{R}^m$, and $\mathcal{A} \not\cong \mathcal{A}'$ — a contradiction. □

9.6. Exercise! *Let $x \mapsto x'$ be a homeomorphism $\Omega \to \Omega'$ between open subsets in finite-dimensional Alexandrov spaces $\mathcal{A}$ and $\mathcal{A}'$. Show that $x \in \partial\mathcal{A}$ if and only if $x' \in \partial\mathcal{A}'$.*

9.7. Exercise! *Show that boundary of a finite-dimensional Alexandrov space is a closed subset.*

D Tangent space

Spaces of directions and tangent spaces of an Alexandrov space have already been defined in Lecture 4. Let us extend these definitions to subsets of an Alexandrov space.

Let X be a subset in a finite-dimensional Alexandrov space $\mathcal{A}$. Choose $p \in \mathcal{A}$ and $\xi \in \Sigma_p$. Suppose ξ is a limit of directions $\uparrow_{[px_n]}$ for a sequence $x_1, x_2, \ldots \in X$ that converges to p. Then we say that ξ is in the space of directions from p to X; briefly $\xi \in \Sigma_p X$.

Further, $\mathrm{Cone}(\Sigma_p X)$ will be called the tangent space to X at p; it will be denoted by $\mathrm{T}_p X$.

Note that $\Sigma_p X$ is a subset of Σ_p and $\mathrm{T}_p X$ is a subcone in T_p.

9.8. Theorem. *For any finite-dimensional Alexandrov space $\mathcal{A}$, we have*

$$\partial(\Sigma_p\mathcal{A}) = \Sigma_p(\partial\mathcal{A}) \quad \textit{and} \quad \partial(\mathrm{T}_p\mathcal{A}) = \mathrm{T}_p(\partial\mathcal{A}).$$

Proof. Choose a sequence $x_n \in \partial\mathcal{A}$ such that $x_n \to p$ and $\uparrow_{[px_n]} \to \xi$.

Let $\varepsilon_n = 2\cdot|p - x_n|$, and let $h_{\varepsilon_n} : \mathrm{B}(p, \varepsilon_n)_{\mathcal{A}} \to \mathrm{B}(0, \varepsilon_n)_{\mathrm{T}_p}$ be the homeomorphisms provided by 9.1; in particular, $\frac{2}{\varepsilon_n}\cdot h_{\varepsilon_n}(x_n) \to \xi$ as $n \to \infty$. By 9.6, $h_{\varepsilon_n}(x_n) \in \partial\mathrm{T}_p$. By 9.7, $\xi \in \partial\mathrm{T}_p$. Therefore,

$$\partial(\Sigma_p\mathcal{A}) \supset \Sigma_p(\partial\mathcal{A}) \quad \text{and} \quad \partial(\mathrm{T}_p\mathcal{A}) \supset \mathrm{T}_p(\partial\mathcal{A}).$$

Similarly, choose $\xi \in \partial\Sigma_p$. Let $h_{\varepsilon_n} : \mathrm{B}(p, \varepsilon_n)_{\mathcal{A}} \to \mathrm{B}(0, \varepsilon_n)_{\mathrm{T}_p}$ be the homeomorphisms provided by 9.1 for a sequence $\varepsilon_n \to 0$ as $n \to \infty$. By 9.6, $x_n = h_{\varepsilon_n}^{-1}(\frac{\varepsilon_n}{2}\cdot\xi) \in \partial\mathcal{A}$. By 9.1, $\uparrow_{[px_n]} \to \xi$. Hence

$$\partial(\Sigma_p\mathcal{A}) \subset \Sigma_p(\partial\mathcal{A}) \quad \text{and} \quad \partial(\mathrm{T}_p\mathcal{A}) \subset \mathrm{T}_p(\partial\mathcal{A}).$$

□

E Doubling

Let A be a closed subset in a metric space $\mathcal{X}$. The doubling $\mathcal{W}$ of $\mathcal{X}$ across A is two copies of $\mathcal{X}$ glued along A; more precisely, the underlying set of $\mathcal{W}$ is the quotient $\mathcal{X}\times\{0,1\}/\sim$, where $(a,0)\sim(a,1)$ for any $a\in A$ and $\mathcal{W}$ is equipped with the minimal metric such that both maps $\mathcal{X}\to\mathcal{W}$ defined by $x\mapsto(x,0)$ and $x\mapsto(x,1)$ are distance-preserving.

Alternatively, one may say that $\mathcal{W}$ is equipped with the maximal metric such that the projection $\mathrm{proj}\colon\mathcal{W}\to\mathcal{A}$ defined by $(x,i)\mapsto x$ is a short map. The metric on $\mathcal{W}$ can also be defined as

$$|(x,i)-(y,j)|_{\mathcal{W}}=\begin{cases}|x-y|_{\mathcal{X}} & \text{if}\quad i=j.\\ \inf\{\,|x-a|_{\mathcal{X}}+|y-a|_{\mathcal{X}}\;:\;a\in A\,\} & \text{if}\quad i\neq j.\end{cases}$$

9.9. Theorem. *Let $\mathcal{A}$ be a finite-dimensional Alexandrov space with nonempty boundary. Suppose $f=\frac12\cdot\mathrm{dist}_p^2$ for some $p\in\mathcal{A}$. Then*

(a) *If $\dim\mathcal{A}\geqslant 2$, then $\mathrm{dist}_{\partial\Sigma_x}(\xi)\leqslant\frac{\pi}{2}$ for any $x\in\partial\mathcal{A}$ and $\xi\in\Sigma_x$. Moreover, if $\mathrm{dist}_{\partial\Sigma_x}(\xi)=\frac{\pi}{2}$, then $\measuredangle(\xi,\zeta)=\frac{\pi}{2}$ for any $\zeta\in\Sigma_x$.*

(b) *$\nabla_x f\in\partial\mathrm{T}_x$ for any $x\in\partial\mathcal{A}$.*

(c) *If α is an f-gradient curve that starts at $x\in\partial\mathcal{A}$, then $\alpha(t)\in\partial\mathcal{A}$ for any t.*
Moreover, if $p\in\partial\mathcal{A}$, then $\mathrm{gexp}_p(v)\in\partial\mathcal{A}$ for any $v\in\partial\mathrm{T}_p$.

(d) *The doubling $\mathcal{W}$ of $\mathcal{A}$ across $\partial\mathcal{A}$ is an Alexandrov space with the same curvature bound.*

Part *(d)* is called the doubling theorem.

Proof. We will denote by *(a)*$_m,\dots,$*(d)*$_m$ the corresponding statement assuming $m=\dim\mathcal{A}$.

The proof goes by induction on m. Statement *(d)*$_1$ follows from 7.3 — this is the base. The induction step is a combination of the implications below.

(d)$_{m-1}\Rightarrow$*(a)*$_m$. Suppose $m=2$, then $\dim\Sigma_x=1$; see 7.7*b*. By 7.3, Σ_x isometric to a line segment $[0,\ell]$; we need to show that $\ell\leqslant\pi$.

Assume $\ell>\pi$, then the tangent space $\mathrm{T}_x=\mathrm{Cone}\,\Sigma_x$ has several different lines thru the origin. Recall that T_x is an Alexandrov space; see 7.7. By 6.6, T_x is isometric to the Euclidean plane; the latter contradicts that Σ_x is a line segment.

Now suppose $m>2$, so $\dim\Sigma_x>1$. Assume $\mathrm{dist}_{\partial\Sigma_x}(\xi)>\frac{\pi}{2}$ for some ξ. By *(d)*$_{m-1}$, the doubling Ξ of Σ_x is $\mathrm{A}\textsc{lex}(1)$. Denote by ξ_0 and ξ_1 the points in Ξ that correspond to ξ. Observe that $|\xi_0-\xi_1|_\Xi>\pi$. The latter contradicts 3.6.

Finally, if $\operatorname{dist}_{\partial\Sigma_x}(\xi)=\frac{\pi}{2}$, then $|\xi_0-\xi_1|_\Xi=\pi$. Therefore, $\operatorname{Cone}\Xi$ contains a line in the directions of ξ_0 and ξ_1. By the splitting theorem (6.5), $\operatorname{Cone}\Xi$ is a direct product of the line with some subcone; in other words, Ξ is a spherical suspension with poles ξ_0 and ξ_1. In particular, every point of Ξ lies on distance at most $\frac{\pi}{2}$ from ξ_0 or ξ_1. The natural projection $\Xi\to\Sigma_x$ does not increase distances and sends both ξ_0 and ξ_1 to ξ. Therefore, the second statement of *(a)*$_m$ follows.

(d)$_{m-1}$+*(a)*$_{m-1}$+*(a)*$_m$ $\Rightarrow$*(b)*$_m$. We can assume that $s=\nabla_x f\neq 0$. By 4.9, $\nabla_x f=s\cdot\bar\xi$, where $s=\boldsymbol{d}_x f(\bar\xi)>0$ and $\bar\xi\in\Sigma_p$ is the direction that maximizes $\boldsymbol{d}_x f(\bar\xi)$.

Let $\zeta\in\partial\Sigma_x$ be a direction that minimizes the angle $\measuredangle(\bar\xi,\zeta)$. It is sufficient to show that $\zeta=\bar\xi$.

Assume $\zeta\neq\bar\xi$; let $\eta=\uparrow_{[\zeta\bar\xi]_{\Sigma_x}}$. By *(a)*$_m$, $\measuredangle(\bar\xi,\zeta)\leqslant\frac{\pi}{2}$ and *(a)*$_{m-1}$ implies that

❶ $$\measuredangle(\eta,\nu)\leqslant\tfrac{\pi}{2}$$

for any $\nu\in\Sigma_\zeta\Sigma_x$ (if $m=2$, then the last statement is evident).

Let $\varphi\colon\Sigma_x\to\mathbb{R}$ be restriction of $\boldsymbol{d}_x f$ to Σ_x. Applying 4.6*a* and ❶, we get that $\boldsymbol{d}_{\bar\xi}\varphi(\eta)\leqslant 0$. Since $\boldsymbol{d}_x f$ is concave, we have that $\varphi''+\varphi\leqslant 0$. If $\varphi(\zeta)\leqslant 0$, then it implies that $\varphi(\bar\xi)\leqslant 0$ — a contradiction to the fact that $s>0$. If $\varphi(\zeta)>0$, then $\varphi(\bar\xi)<\varphi(\zeta)$ — a contradiction again.

(b)$_m$ $\Rightarrow$*(c)*$_m$. Let α be an f-gradient curve and $\ell(t)=\operatorname{dist}_{\partial\mathcal{A}}\alpha(t)$.

Choose t; let $x=\alpha(t)$ and $y\in\partial\mathcal{A}$ be a closest point to x. By *(b)*$_m$, we have that $\nabla_y f\in\partial\mathrm{T}_y$. Since the distance $|x-y|$ is minimal, we get $\langle\uparrow_{[yx]},v\rangle\leqslant 0$ for any $v\in\partial\mathrm{T}_y$. In particular,

$$\langle\uparrow_{[yx]},\nabla_y f\rangle\leqslant 0$$

Applying 4.8 to x and y, we get

$$\ell'(t)\leqslant\ell(t)$$

if the left-hand side is defined. Since ℓ is Lipschitz, ℓ' is defined almost everywhere. Integrating the inequality, we get

$$\ell(t)\leqslant e^t\cdot\ell(0)$$

for any $t\geqslant 0$. In particular, if $\ell(0)=0$, then $\ell(t)=0$ for any $t\geqslant 0$. Since $\partial\mathcal{A}$ is closed (9.7), the first statement follows.

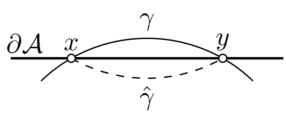
It remains to show $\operatorname{gexp}_p(v)\in\partial\mathcal{A}$ for any $v\in\partial\mathrm{T}_p$. Suppose $v=t\cdot u$ and $|u|=1$. Note that $u\in\partial\mathrm{T}_p$. Therefore, we can choose a sequence of points $x_n\in\partial\mathcal{A}$ such that $\uparrow_{[px_n]}\to v$; let $r_n=|p-x_n|$. From above, the f-gradient curve α_n that start at x_n remains in $\partial\mathcal{A}$.

By the definition of gradient exponent, $\mathrm{gexp}_p(v) = \mathrm{gexp}_p(t{\cdot}u)$ is a limit of $\alpha_n(\ln \frac{t}{r_n})$. Hence the last statement follows.

(c)$_m$+(d)$_{m-1}$ ⇒(d)$_m$. We will consider the case $\kappa = 0$; other cases can be done in the same way, but formulas get more complicated.

Denote by $\mathcal{A}_0$ and $\mathcal{A}_1$ the two copies of $\mathcal{A}$ in $\mathcal{W}$; let us keep the notation $\partial\mathcal{A}$ for the common boundary of $\mathcal{A}_0$ and $\mathcal{A}_1$.

❷ *Let γ be a geodesic in $\mathcal{W}$. Then either γ has at most one interior point in $\partial\mathcal{A}$ or $\gamma \subset \partial\mathcal{A}$.*

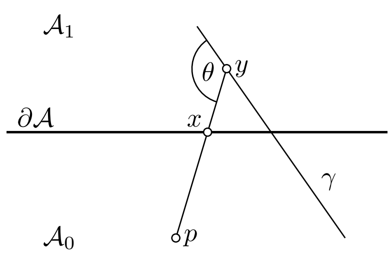


Indeed, assume γ shares at least two points with $\partial\mathcal{A}$, say $x = \gamma(t_1)$ and $y = \gamma(t_2)$ and these are not endpoints of γ. Remove from γ the set $\gamma \cap \mathcal{A}_1$ and exchange it to its reflection across $\partial\mathcal{A}$; denote the obtained curve by $\hat\gamma$.

Any arc of $\hat\gamma$ with one endpoint in $\partial\mathcal{A}$ is a geodesic in $\mathcal{A}_0$. Since $x, y \in \partial\mathcal{A}$, the arc of $\hat\gamma$ behind y lies in the image of map $t \mapsto \mathrm{Flow}^t_{f_x}(y)$, where $f_x = \frac{1}{2}{\cdot}\mathrm{dist}^2_x$. By *(c)*, this arc lies in $\partial\mathcal{A}$.

Now choose a point z on this arc, so $z \in \partial\mathcal{A}$. Applying the same argument, we get that the arc of $\hat\gamma$ before y lies in $\partial\mathcal{A}$. Hence the claim follows. △

Choose a point p in $\mathcal{W}$; let $f := \frac{1}{2}{\cdot}\mathrm{dist}^2_p$. It is sufficient to show that $(f \circ \gamma)'' \leqslant 1$ for any t. If $p \in \partial\mathcal{A}$, then the statement follows from function comparison in $\mathcal{A}_0$ and $\mathcal{A}_1$. So, we can assume that $p \in \mathcal{A}_0 \backslash \partial\mathcal{A}$. Also, we can assume that γ does not lie in $\partial\mathcal{A}$; otherwise, the inequality follows from the comparison in $\mathcal{A}_0$.



Choose $y = \gamma(t_0)$; without loss of generality we can assume that $t_0 = 0$.

If $y \in \mathcal{A}_0 \setminus \partial\mathcal{A}$, then $(f \circ \circ\gamma)''(0) \leqslant 1$ in the barrier sense; it follows from the comparison in $\mathcal{A}_0$.

Assume $y \in \mathcal{A}_1 \setminus \partial\mathcal{A}$. Suppose $[py]$ crosses $\partial\mathcal{A}$ at x. Let Σ_x be the space of directions of $\mathcal{A}$ at x, and let Ξ be its doubling. As before, we denote by Σ_0 and Σ_1 two copies of Σ_x in Ξ and keep notation $\partial\Sigma_x$ for their common boundary. By *(d)$_{m-1}$*, Ξ is Alex(1).

The directions $\uparrow_{[xy]}$ and $\uparrow_{[xp]}$ lie on opposite sides from $\partial\Sigma_x$ and

$$|\uparrow_{[xy]} - \uparrow_{[xp]}|_\Xi \geqslant \pi.$$

Otherwise, we could choose a direction $\xi\in\partial\Sigma$ such that

$$|\uparrow_{[xy]}-\xi|_\Xi+|\xi-\uparrow_{[xp]}|_\Xi<\pi.$$

Furthermore, we could consider the curve $\alpha(t)=\mathrm{gexp}_x(t{\cdot}\xi)$. By *(c)*$_m$, α lies in $\partial\mathcal{A}$. By 5.12

$$|p-\alpha(s)|_{\mathcal{A}_0}+|y-\alpha(s)|_{\mathcal{A}_1}<|p-y|_{\mathcal{W}}$$

for small values $s>0$ — a contradition.

Cone Ξ contains a line with directions $\uparrow_{[xy]}$ and $\uparrow_{[xp]}$. By the splitting theorem, Cone Ξ splits in these directions; in particular,

$$|\uparrow_{[xy]}-\xi|+|\xi-\uparrow_{[xp]}|=\pi.$$

for any $\xi\in\Xi$. It follows that for any $\xi\in\Xi$ there is $\xi'\in\partial\Sigma_x$ such that ξ and ξ' lie on some geodesic $[\uparrow_{[xy]}\uparrow_{[xp]}]_\Xi$.

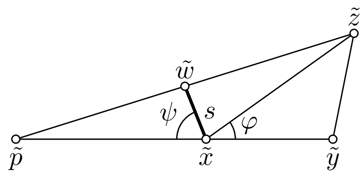
Fix $t\approx 0$ such that $t\ne 0$; let $z=\gamma(t)$. Choose such a direction ξ' for $\xi=\uparrow_{[xz]}$. Consider the curve $\alpha(s):=\mathrm{gexp}_x(s{\cdot}\xi')$. Let us show that

❸
$$\begin{aligned}|p-z|_{\mathcal{W}}&\leqslant|p-\alpha(s)|_{\mathcal{A}_0}+|\alpha(s)-z|_{\mathcal{A}_1}\leqslant\\&\leqslant\tilde\curlyvee[y\,{}^{p}_{z}].\end{aligned}$$

for suitable value s.

The first inequality in ❸ is evident. Set $\varphi=\measuredangle[x\,{}^{y}_{z}]$ and $\psi=$ $=\measuredangle(\uparrow_{[xp]},\xi')$. The choice of s comes from the model configuration $\tilde p$, $\tilde x$, $\tilde y$, $\tilde w$, $\tilde z\in\mathbb{E}^2$ such that

$$\begin{aligned}&\tilde x\in[\tilde p\tilde y], && |\tilde p-\tilde x|=|p-x|, && |\tilde p-\tilde y|=|p-y|, && |\tilde x-\tilde z|=|x-z|,\\ &\tilde w\in[\tilde p\tilde z], && \measuredangle[\tilde x\,{}^{\tilde y}_{\tilde z}]=\varphi, && \measuredangle[\tilde x\,{}^{\tilde p}_{\tilde w}]=\psi, && s=|\tilde x-\tilde w|.\end{aligned}$$



By 5.12, we get

$$\begin{aligned}|p-\alpha(s)|_{\mathcal{A}_0}&\leqslant|\tilde p-\tilde w|,\\|\alpha(s)-z|_{\mathcal{A}_1}&\leqslant|\tilde w-\tilde z|;\end{aligned}$$

by the comparison,
$$|\tilde{p}-\tilde{z}| \leqslant \tilde{\Upsilon}[y\,{}^{p}_{z}].$$

9.10. Exercise. *Prove the last inequality.*

Hence we get $(f\circ\gamma)''(0)\leqslant 1$ in the barrier sense.

Finally if $\gamma(0)\in\partial\mathcal{A}$, then splitting argument shows that
$$(f\circ\gamma)^{+}(0)+(f\circ\gamma)^{-}(0)\leqslant 0.$$

Summarizing, we get that $(f\circ\gamma)''\leqslant 1$ on every arc of γ that lies entirely in $\mathcal{A}_0$ or $\mathcal{A}_1$. If γ crosses $\partial\mathcal{A}$, then we know that it happens only once and at the crossing moment t_0 we have $f\circ\gamma^{+}(t_0)+$ $+f\circ\gamma^{-}(t_0)\leqslant 0$. All this implies that $(f\circ\gamma)''\leqslant 1$. $\square$

9.11. Exercise! *Let $\mathcal{A}$ be a finite-dimensional* $\mathrm{ALEX}(1)$ *space of dimension $\geqslant 2$ with nonempty boundary $\partial\mathcal{A}$. Show that $\partial\mathcal{A}$ is connected.*

9.12. Exercise. *Let $\mathcal{A}$ be an finite-dimensional* $\mathrm{ALEX}(0)$ *space with nonempty boundary $\partial\mathcal{A}$.*

(a) Suppose a geodesic γ in $\mathcal{A}$ has its interior point in $\partial\mathcal{A}$. Show that γ lies in $\partial\mathcal{A}$.

(b) Show that the distance function to the boundary is concave.

9.13. Exercise! *Let $\mathcal{A}$ be a finite-dimensional* $\mathrm{ALEX}(0)$ *space with nonempty boundary $\partial\mathcal{A}$. Suppose γ is a geodesic in $\partial\mathcal{A}$ with the induced length metric. Show that the function $t\mapsto\tfrac{1}{2}\cdot\mathrm{dist}_p^2\circ\gamma(t)$ is 1-concave for any point p.*

9.14. Exercise. *Let $proj\colon\mathcal{W}\to\mathcal{A}$ be the natural projection to a finite-dimensional Alexandrov space $\mathcal{A}$ from its doubling $\mathcal{W}$ across the boundary. Suppose $f\colon\mathcal{A}\to\mathbb{R}$ is a λ-concave function. Show that $f\circ\mathrm{proj}\colon\mathcal{W}\to\mathbb{R}$ is λ-concave if and only if $\nabla_x f\in\partial\mathrm{T}_x$ for any $x\in\partial\mathcal{A}$.*

F Remarks

The doubling theorem has generalizations [30, 87] that allow for gluing multiple nonidentical spaces.

It easily follows by induction on dimension that the doubling of a finite-dimensional Alexandrov space across its boundary results in an Alexandrov space without boundary. This observation can often be used to reduce a statement about general finite-dimensional Alexandrov spaces to Alexandrov spaces without boundary.

For spaces without boundary the following tools become available.

9.15. Fundamental-class lemma. *Any compact finite-dimensional Alexandrov space $\mathcal{A}$ without boundary has a fundamental class with $\mathbb{Z}/2$ coefficients; that is, if $\mathcal{A}$ is m-dimensional, then*

$$H^m(\mathcal{A}, \mathbb{Z}/2) = \mathbb{Z}/2.$$

This lemma was proved by Karsten Grove and Peter Petersen [40]. Originally it was stated for Alexander–Spanier cohomology. We do not make this distinction because for compact Alexandrov spaces it is the same as singular cohomology. Indeed, both cohomology theories are homotopy invariant [103, Chapter 6], compact Alexandrov spaces are homotopy equivalent to finite simplicial complexes (see 8.13) and for paracompact CW complexes Alexander–Spanier cohomology is isomorphic to Čech and singular cohomolgy [103, Chapter 6].

This lemma implies, for example, that on finite-dimensional Alexandrov spaces without boundary the gradient flow for a λ-concave function is an onto map; in other words, gradient curves can be extended into the past. It is also used in the proof of the following version of the domain invariance theorem [50, Theorem 3.2].

9.16. Domain invariance. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two m-dimensional Alexandrov spaces with empty boundary; m is finite. Suppose Ω_1 is an open subset in $\mathcal{A}_1$ and $f\colon \Omega_1 \to \mathcal{A}_2$ is an injective continuous map. Then $f(\Omega_1)$ is open in $\mathcal{A}_2$.*

Theorem 9.1 can be used to prove the following.

9.17. Topological stratification. *Any m-dimensional Alexandrov space with $m < \infty$ can be subdivided into topological manifolds $S_0, \dots$ $\dots, S_m$ such that for every i we have $\dim S_i = i$ or $S_i = \varnothing$. Moreover,*

(a) the closure of S_{m-1} is the boundary of the space, and

(b) $S_{m-2} = \varnothing$.

This statement implies that a compact finite-dimensional Alexandrov space has the homotopy type of a finite CW complex, but it seems to be unknown if it has to be homeomorphic to a CW complex.

The stratification theorem 9.17 can be sharpened as follows.

9.18. Boundary characterization. *Let $\mathcal{A}$ be an m-dimensional Alexandrov space with $m < \infty$. Then the following statements are equivalent.*

(a) $p \in \partial\mathcal{A}$;

(b) Σ_p *is contractible;*
(c) $\tilde{H}_{m-1}(\Sigma_p, \mathbb{Z}/2) = 0$*;*
(d) $H_m(\mathcal{A}, \mathcal{A} \setminus \{p\}, \mathbb{Z}/2) = 0$*;*

Let f be a semiconcave function. A point $p \in \operatorname{Dom} f$ is called critical point of f if $\boldsymbol{d}_p f \leqslant 0$; otherwise it is called regular.

The following statement plays a technical role in the proof of stability theorem, but it is also a useful technical tool on its own.

9.19. Morse lemma. *Let f be a semiconcave function on a finite-dimensional Alexandrov space without boundary. Suppose K is a compact set of regular points of f in its level set $f = a$. Then an open neighborhood Ω of K admits a homeomorphism $x \mapsto (h(x), f(x))$ to a product space $\Lambda \times (a - \varepsilon, a + \varepsilon)$.*

The Morse lemma was originally proved by Grigory Perelman in [80] for special kind of semiconcave functions made out of distance functions. For such functions it holds for all Alexandrov spaces without the requirement that they do not have boundary. For general semiconcave functions on spaces without boundary it follows from [81] where the technical tools from [80] (the notion of inner product between differentials of semiconcave functions and the notion of admissible maps) was generalized to include general semiconcave functions. With those definitions all the main Morse theory results of [80] follow by the same proof by using [81, 2.2] instead of [80, Lemma 1]. On Alexandrov spaces with boundary all the Morse theory results hold if one requires in addition that the semiconcave functions remain semiconcave when canonically extended to the doubling.[1] This is automatic for the special type of semiconcave functions considered in [80] therefore this distinction is not made there.

Subsets in Alexandrov spaces that satisfy the condition in 9.9c are called extremal. More precisely, a subset E is extremal if for any $x \in E$ and f-gradient curve that starts in E remains in E; here f is arbitrary function of the form $\frac{1}{2}\cdot\operatorname{dist}_p^2$.

Extremal subsets were introduced by Grigory Perelman and the second author [77]. They will pop up in the next lecture.

The following conjecture is one of the oldest questions in Alexandrov geometry that remains open.

9.20. Conjecture. *Let S be a component of the boundary of a finite-dimensional Alexandrov space. Then S equipped with the induced length metric is an Alexandrov space with the same curvature bound.*

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[1] Semiconcave functions are often defined to meet this property, but we do not follow this convention.

Lecture 10

Quotients

This lecture gives several applications of Alexandrov geometry to isometric group actions.

A Quotient space

Suppose that a group G acts isometrically on a metric space $\mathcal{X}$. Note that

$$|G{\cdot}x - G{\cdot}y|_{\mathcal{X}/G} := \inf\{\,|x - g{\cdot}y|_{\mathcal{X}} \;:\; g \in G\,\}$$

defines a semimetric on the orbit space $\mathcal{X}/G$. Moreover, if the orbits of the action are closed, then it is a genuine metric.

10.1. Theorem. *Suppose that a group G acts isometrically on a proper* $\mathrm{ALEX}(0)$ *space $\mathcal{A}$, and G has closed orbits. Then the quotient space $\mathcal{A}/G$ is* $\mathrm{ALEX}(0)$.

A more general formulation will be given in 10.5.

Proof. Denote by $\sigma\colon \mathcal{A} \to \mathcal{A}/G$ the quotient map.

Fix a quadruple of points $p, x_1, x_2, x_3 \in \mathcal{A}/G$. Choose $\hat{p} \in \mathcal{A}$ such that $\sigma(\hat{p}) = p$. Since $\mathcal{A}$ is proper, we can choose points $\hat{x}_i \in \mathcal{A}$ such that $\sigma(\hat{x}_i) = x_i$ and

$$|p - x_i|_{\mathcal{A}/G} = |\hat{p} - \hat{x}_i|_{\mathcal{A}}$$

for all i.

Note that

$$|x_i - x_j|_{\mathcal{A}/G} \leqslant |\hat{x}_i - \hat{x}_j|_{\mathcal{A}}$$

for all i and j. Therefore

❶ $$\tilde{\measuredangle}(p\,{}^{x_i}_{x_j}) \leqslant \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_i}_{\hat{x}_j})$$

for all i and j.

By $\mathbb{E}^2$-comparison in $\mathcal{A}$, we have

$$\tilde{\measuredangle}(\hat{p}\,^{\hat{x}_1}_{\hat{x}_2}) + \tilde{\measuredangle}(\hat{p}\,^{\hat{x}_2}_{\hat{x}_3}) + \tilde{\measuredangle}(\hat{p}\,^{\hat{x}_3}_{\hat{x}_1}) \leqslant 2\cdot\pi.$$

Applying ❶, we get

$$\tilde{\measuredangle}(p\,^{x_1}_{x_2}) + \tilde{\measuredangle}(p\,^{x_2}_{x_3}) + \tilde{\measuredangle}(p\,^{x_3}_{x_1}) \leqslant 2\cdot\pi;$$

that is, the $\mathbb{E}^2$-comparison holds for any quadruple in $\mathcal{A}/G$. □

10.2. Very advanced exercise. *Let G be a compact Lie group with a bi-invariant Riemannian metric. Show that G is isometric to a quotient of a Hilbert space by an isometric group action.*

Conclude that G is ALEX(0).

B Submetries

A map $\sigma\colon \mathcal{X} \to \mathcal{Y}$ between metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is called a submetry if

$$\sigma(\mathrm{B}(p,r)_{\mathcal{X}}) = \mathrm{B}(\sigma(p),r)_{\mathcal{Y}}$$

for any $p \in \mathcal{X}$ and $r \geqslant 0$.

Suppose G and $\mathcal{A}$ are as in 10.1. Observe that the quotient map $\sigma\colon \mathcal{A} \to \mathcal{A}/G$ is a submetry. The following two exercises show that this is not the only source of submetries.

10.3. Exercise. *Construct submetries*

(a) $\sigma_1\colon \mathbb{S}^2 \to [0,\pi]$,

(b) $\sigma_2\colon \mathbb{S}^2 \to [0,\frac{\pi}{2}]$,

(c) $\sigma_n\colon \mathbb{S}^2 \to [0,\frac{\pi}{n}]$ *(for integer $n \geqslant 1$)*

such that the fibers $\sigma_n^{-1}\{x\}$ are connected for any x.

10.4. Exercise. *Let $\sigma\colon \mathbb{E}^2 \to [0,\infty)$ be a submetry. Show that $K = \sigma^{-1}\{0\}$ is a closed convex set without interior points and $\sigma(x) = \mathrm{dist}_K x$.*

The proof of 10.1 works for submetries; that is, *if $\sigma\colon \mathcal{A} \to \mathcal{B}$ is a submetry and $\mathcal{A}$ is a proper* ALEX(0) *space, then so is $\mathcal{B}$.* Theorem 10.1 admits a straightforward generalization to ALEX(-1) case.

In the ALEX(1) case, the proof produces a slightly weaker statement — *$\mathbb{S}^2$-comparison holds for a quartuple p, x_1, x_2, x_3 in the quotient of* ALEX(1) *if $|p - x_i| < \frac{\pi}{2}$ for each i.* In particular, the quotient space is *locally* ALEX(1). But since ALEX(1) space is geodesic, then so is its quotient. Therefore, the globalization theorem implies that it is globally $\text{ALEX}(1)$. The same holds for the targets of submetries from an $\text{ALEX}(1)$ space. With a bit of extra work, one can extend the statement to nonproper spaces [6, 8.34]. Thus, we have the following.

10.5. Theorem. *Let $\sigma\colon \mathcal{A} \to \mathcal{B}$ be a submetry. If $\mathcal{A}$ is $\text{ALEX}(\kappa)$ space, then so is $\mathcal{B}$.*

In particular, if G acts isometrically on an $\text{ALEX}(\kappa)$ space $\mathcal{A}$, and G has closed orbits. Then the quotient space $\mathcal{A}/G$ is $\text{ALEX}(\kappa)$.

C Hopf's conjecture

Does $\mathbb{S}^2 \times \mathbb{S}^2$ admit a Riemannian metric with positive sectional curvature? Hopf's conjecture says that the answer should be negative. Let us take a close look at the following partial result obtained by Wu-Yi Hsiang and Bruce Kleiner [46].

10.6. Theorem. *There is no Riemannian metric on $\mathbb{S}^2 \times \mathbb{S}^2$ with sectional curvature $\geqslant 1$ and a nontrivial isometric $\mathbb{S}^1$-action.*

Reacall that a group action $G \curvearrowright \mathcal{X}$ is called effective if for any $g \in G$ there is $x \in \mathcal{X}$ such that $g{\cdot}x \neq x$.

10.7. Key lemma. *Suppose $\mathbb{S}^1 \curvearrowright \mathbb{S}^3$ is an effective isometric action without fixed points and $\Sigma = \mathbb{S}^3/\mathbb{S}^1$ is its quotient space. Then there is a distance noncontracting map $\Sigma \to \frac{1}{2}{\cdot}\mathbb{S}^2$, where $\frac{1}{2}{\cdot}\mathbb{S}^2$ is the standard 2-sphere rescaled with a factor $\frac{1}{2}$.*

The proof of the lemma is guided by the following exercise.

10.8. Exercise! *Suppose $\mathbb{S}^1 \curvearrowright \mathbb{S}^3$ is an effective isometric action without fixed points. Let us think of $\mathbb{S}^3$ as the unit sphere in $\mathbb{R}^4$.*

(a) Show that one can identify $\mathbb{R}^4$ with $\mathbb{C}^2$ so that the action is given by matrix multiplication

$$\begin{pmatrix} u^p & 0 \\ 0 & u^q \end{pmatrix},$$

where (p, q) is a pair of relatively prime positive integers and $u \in \mathbb{S}^1 = \{\, z \in \mathbb{C} \,:\, |z| = 1 \,\}$. In particular, our $\mathbb{S}^1$ is a subgroup of the torus that acts by matrix multiplication

$$\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix},$$

where $v, w \in \mathbb{S}^1$.

Fix p and q as above. Let $\Sigma_{p,q} = \mathbb{S}^3/\mathbb{S}^1$ be the quotient space.

(b) Show that the $\Sigma_{p,q} = \mathbb{S}^3/\mathbb{S}^1$ is a topological sphere with $\mathbb{S}^1$-symmetry. This symmetry has two fixed points, north pole and south pole, that correspond to the orbits of $(1,0)$ and $(0,1)$ in $\mathbb{S}^3$.

Denote by $S(r)$ the circle of radius r with the center at the north pole of $\Sigma_{p,q}$.

(c) Denote by $T(r)$ the inverse image $T(r)$ in $\mathbb{S}^3$, and let $a(r)$ be its area. Show that $T(r)$ is an orbit of the torus action and

$$a(r) = \pi^2 \cdot \sin r \cdot \cos r.$$

(d) Let $b_{p,q}(r)$ be the length of the $\mathbb{S}^1$-orbit in $\mathbb{S}^3$ that corresponds to a point on $S(r)$. Show that

$$b_{p,q} = \pi \cdot \sqrt{(p \cdot \sin r)^2 + (q \cdot \cos r)^2}.$$

(e) Let $c_{p,q}(r)$ be the length of $S(r)$. Show that $a(r) = c_{p,q}(r) \cdot b_{p,q}(r)$.

(f) Show that $c_{p,q}(r) \leqslant c_{1,1}(r)$ for any pair (p,q) of relatively prime positive integers. Use it to construct a distance noncontracting map $\Sigma_{p,q} \to \frac{1}{2} \cdot \mathbb{S}^2 \overset{iso}{=\!=} \Sigma_{1,1}$.

Proof of 10.6. Assume $\mathcal{B} = (\mathbb{S}^2 \times \mathbb{S}^2, g)$ is a counterexample. By the Toponogov theorem, $\mathcal{B}$ is Alex(1). By 10.1, the quotient space $\mathcal{A} = \mathcal{B}/\mathbb{S}^1$ is Alex(1); evidently, $\mathcal{A}$ is 3-dimensional.

Denote by $F \subset \mathcal{B}$ the fixed point set of the $\mathbb{S}^1$-action. Then $\chi(\mathcal{B}) = \chi(F)$. Each connected component of F is either an isolated point or a 2-dimensional geodesic submanifold in $\mathcal{B}$; the latter has to have positive curvature, and therefore it is homeomorphic to $\mathbb{S}^2$ or $\mathbb{R}\mathrm{P}^2$. Notice that

- ⋄ each isolated point contributes 1 to the Euler characteristic of $\mathcal{B}$,
- ⋄ each sphere contributes 2 to the Euler characteristic of $\mathcal{B}$, and
- ⋄ each projective plane contributes 1 to the Euler characteristic of $\mathcal{B}$.

Since $\chi(\mathcal{B}) = 4$, we are in one of the following three cases:

1. F has exactly 4 isolated points,
2. F has one 2-dimensional submanifold and at least 2 isolated points,
3. F has at least two 2-dimensional submanifolds.

In each case we will arrive at a contradiction.

Case 1. Suppose F has exactly 4 isolated points x_1, x_2, x_3, and x_4. Denote by y_1, y_2, y_3, and y_4 the corresponding points in $\mathcal{A}$. Note that $\Sigma_{y_i}\mathcal{A}$ is isometric to a quotient of $\mathbb{S}^3$ by an isometric $\mathbb{S}^1$-action without fixed points.

By 10.8, each angle $\measuredangle[y_i\,{}^{y_j}_{y_k}] \leqslant \frac{\pi}{2}$ for any three distinct points y_i, y_j, y_k. In particular, all four triangles $[y_1y_2y_3]$, $[y_1y_2y_4]$, $[y_1y_3y_4]$, and $[y_2y_3y_4]$ are nondegenerate. By the comparison, the sum of angles in each triangle is strictly greater than π.

Denote by ω the sum of all 12 angles in the 4 triangles $[y_1y_2y_3]$, $[y_1y_2y_4]$, $[y_1y_3y_4]$, and $[y_2y_3y_4]$. From above,

$$\omega > 4\cdot\pi.$$

On the other hand, by 10.8 any triangle in $\Sigma_{y_1}\mathcal{A}$ has perimeter at most π. In particular,

$$\measuredangle[y_1\,{}^{y_2}_{y_3}] + \measuredangle[y_1\,{}^{y_3}_{y_4}] + \measuredangle[y_1\,{}^{y_4}_{y_2}] \leqslant \pi.$$

Apply the same argument in $\Sigma_{y_2}\mathcal{A}$, $\Sigma_{y_3}\mathcal{A}$, and $\Sigma_{y_4}\mathcal{A}$; adding the results, we get

$$\omega \leqslant 4\cdot\pi$$

— a contradiction.

Case 2. Suppose F contains one surface S. Then the projection of S to $\mathcal{A}$ forms its boundary $\partial\mathcal{A}$. The doubling $\mathcal{W}$ of $\mathcal{A}$ across its boundary has at least 4 singular points — each singular point of $\mathcal{A}$ corresponds to two singular points of $\mathcal{W}$.

By the doubling theorem, $\mathcal{W}$ is a Alex(1) space. Therefore we arrive at a contradiction in the same way as in the first case.

Case 3. Impossible by 9.11. □

D Erdős' problem rediscovered

A point p in an Alexandrov space is called extremal if $\measuredangle[p\,{}^{x}_{y}] \leqslant \frac{\pi}{2}$ for any hinge $[p\,{}^{x}_{y}]$ with the vertex at p; equivalently, $\operatorname{diam}\Sigma_p \leqslant \pi/2$.

10.9. Theorem. *Let $\mathcal{A}$ be a compact m-dimensional* Alex(0) *space. Then it has at most 2^m extremal points.*

Proof of 10.9. Let $\{p_1,\dots,p_N\}$ be extremal points in $\mathcal{A}$. For each p_i consider its open Voronoi domain V_i; that is,

$$V_i = \{\, x\in\mathcal{A} \,:\, |p_i - x| < |p_j - x| \text{ for any } j\neq i \,\}.$$

Clearly $V_i \cap V_j = \varnothing$ if $i \neq j$.

Suppose $0 < \alpha \leqslant 1$. Given a point $x\in\mathcal{A}$, choose a geodesic $[p_ix]$ and denote by x_i the point on $[p_ix]$ such that $|p_i - x_i| = \alpha\cdot|p_i - x|$; let $\Phi_i\colon x\to x_i$ be the corresponding map. By the comparison,

$$|x_i - y_i| \geqslant \alpha\cdot|x - y|$$

for any x, y, and i. Therefore

$$\mathrm{vol}(\Phi_i\mathcal{A}) \geqslant \alpha^m \cdot \mathrm{vol}\,\mathcal{A}.$$

Suppose $\alpha < \frac{1}{2}$. Then $x_i \in V_i$ for any $x \in \mathcal{A}$. Indeed, assume $x_i \notin V_i$, then there is p_j such that $|p_i - x_i| \geqslant |p_j - x_i|$. Then by comparison, we have $\tilde\measuredangle(p_j\,{}^{p_i}_{x})_{\mathbb{E}^2} > \frac{\pi}{2}$; that is, p_j is not an extremal point.

It follows that $\mathrm{vol}\,V_i \geqslant \alpha^m \cdot \mathrm{vol}\,\mathcal{A}$ for any $0 < \alpha < \frac{1}{2}$; hence

$$\mathrm{vol}\,V_i \geqslant \tfrac{1}{2^m} \cdot \mathrm{vol}\,\mathcal{A}.$$

Since $V_1, \dots, V_N$ are disjoint subsets of $\mathcal{A}$, we have $N \leqslant 2^m$. □

E Crystallographic actions

An isometric action $\Gamma \curvearrowright \mathbb{E}^m$ is called crystallographic if it is properly discontinuous (that is, for any compact set $K \subset \mathbb{E}^m$ and $x \in \mathbb{E}^m$ there are only finitely many elements $g \in \Gamma$ such that $g \cdot x \in K$) and cocompact (that is, the quotient space $\mathcal{A} = \mathbb{E}^m/\Gamma$ is compact).

Let F be a maximal finite subgroup of Γ; that is, if $F < H < \Gamma$ for a finite group H, then $F = H$. Denote by $\mathfrak{M}(\Gamma)$ the number of maximal finite subgroups of Γ up to conjugation.

10.10. Open question. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Is it true that $\mathfrak{M}(\Gamma) \leqslant 2^m$?*

Note that any finite subgroup F of Γ fixes an affine subspace A_F in $\mathbb{E}^m$. If F is maximal, then A_F completely describes F. Indeed, since the action is properly discontinuous, the subgroup of Γ that fix A_F has to be finite. This subgroup must contain F, but since F is maximal, it must coinside with F.

Denote by $\mathfrak{M}_k(\Gamma)$ the number of maximal finite subgroups $F < \Gamma$ (up to conjugation) such that $\dim A_F = k$.

Choose a finite subgroup $F < \Gamma$; consider a conjugate subgroup $F' = g \cdot F \cdot g^{-1}$. Note that $A_{F'} = g \cdot A_F$. In particular, the subspaces A_F and $A_{F'}$ have the same image in the quotient space $\mathcal{A} = \mathbb{E}^m/\Gamma$. Therefore, to count subgroups up to conjugation, we need to count the images of their fixed sets. By the lemma below (10.12), $\mathfrak{M}_0(\Gamma)$ cannot exceed the number of extremal points in $\mathcal{A} = \mathbb{E}^m/\Gamma$. Combining this observation with 10.9, we get the following.

10.11. Proposition. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Then $\mathfrak{M}_0(\Gamma) \leqslant 2^m$.*

10.12. Lemma. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action and F be a maximal finite subgroup of Γ that fixes an isolated point p. Then the image of p in the quotient space $\mathcal{A} = \mathbb{E}^m/\Gamma$ is an extremal point.*

Proof. Let q be the image of p. Suppose q is not extremal; that is, $\measuredangle[q\,{}^{y_1}_{y_2}] > \frac{\pi}{2}$ for some hinge $[q\,{}^{y_1}_{y_2}]$ in $\mathcal{A}$.

Choose the inverse images $x_1, x_2 \in \mathbb{E}^m$ of $y_1, y_2 \in \mathcal{A}$ such that $|p - x_i|_{\mathbb{E}^m} = |q - y_i|_{\mathcal{A}}$. Note that $\measuredangle[p\,{}^{x_1}_{x_2}] \geqslant \measuredangle[q\,{}^{y_1}_{y_2}] > \frac{\pi}{2}$. Moreover, since p is fixed by F, we have

❶ $$\measuredangle[p\,{}^{x_1}_{g\cdot x_2}] > \tfrac{\pi}{2}$$

for any $g \in F$.

Denote by z the barycenter of the orbit $F\cdot x_2$. Note that z is a fixed point of F. By ❶, $z \neq p$; so F must fix the line pz. But p is an isolated fixed point of F — a contradiction. □

10.13. Exercise. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Show that*

(a) $\mathfrak{M}_{m-1}(\Gamma) \leqslant 2$, and

(b) if $\mathfrak{M}_{m-1}(\Gamma) = 1$, then $\mathfrak{M}_0(\Gamma) \leqslant 2^{m-1}$.

Construct crystallographic actions with equalities in (a) and (b).

F Remarks

Submetries were introduced by Valerii Berestovskii [13] and have attracted attention in various contexts of differential and metric geometry.

A more general form of Theorem 10.6 was found by Karsten Grove and Burkhard Wilking [41]; it classifies isometric $\mathbb{S}^1$ actions on 4-dimensional manifolds with nonnegative sectional curvature. This proof is as beautiful as the original work of Wu-Yi Hsiang and Bruce Kleiner.

It is expected that *no* Alex(1) *space with a nontrivial isometric $\mathbb{S}^1$-action can be homeomorphic to* $\mathbb{S}^2 \times \mathbb{S}^2$; so 10.6 holds for general Alex(1) space. The proof of 10.6 would work if we had the following generalization of 10.7; see [44].

10.14. Open question. *Let Σ be an* Alex(1) *space homeomorphic to $\mathbb{S}^3$. Suppose $\mathbb{S}^1$ acts on Σ isometrically and without fixed points. Is it true that any triangle in $\Sigma/\mathbb{S}^1$ has perimeter at most π?*

And if the answer is, is there a distance-noncontracting map

$$\Sigma/\mathbb{S}^1 \to \tfrac{1}{2}\cdot\mathbb{S}^2?$$

10.15. Advanced exercise. *Suppose $\mathbb{S}^1$ acts isometrically on an* ALEX(1) *space $\mathcal{A}$ that is homeomorphic to $\mathbb{S}^3$. Assume its fixed-point set is a closed local geodesic γ. Show that*

$$\operatorname{length}\gamma \leqslant 2\cdot\pi.$$

An analogous question for a $\mathbb{Z}_2$-action is open [83].

Theorem 10.9 is a translation of the following classical problem in discrete geometry to Alexandrov's language.

10.16. Problem. *Let F be a set of points in $\mathbb{E}^m$ such that any triangle formed by three distinct points in F has no obtuse angles. Then $|F| \leqslant 2^m$. Moreover, if $|F| = 2^m$, then F consists of the vertices of an m-dimensional rectangle.*

This problem was posed by Paul Erdős [27] and solved by Ludwig Danzer and Branko Grünbaum [25]. Grigory Perelman noticed that, after proper definitions, the same proof works in Alexandrov spaces [76]; thus, it proves 10.9. Applying the our argument to the convex hull of F in 10.16 proves that $|F| \leqslant 2^m$; the case of equality requires more work.

Compact m-dimensional ALEX(0) spaces with the maximal number of extremal points include m-dimensional rectangles and the quotients of flat tori by reflections across a point. (This action has 2^m isolated fixed points; each corresponds to an extremal point in the quotient space $\mathcal{A} = \mathbb{T}^m/\mathbb{Z}_2$.) Nina Lebedeva has proved [64] that *every m-dimensional* ALEX(0) *space with 2^m extremal points is a quotient of Euclidean space by a crystallographic action.*

The extremal subsets of Alexandrov space were brifly discussed in 9F. The following definition is more relevant to isometric group actions.

A closed subset E in a finite-dimensional Alexandrov space is called extremal if $\measuredangle[p\,{}^x_y] \leqslant \frac{\pi}{2}$ for any $x \notin E$ and $p \in E$ such that $|x-p|$ takes a minimal value. An extremal set is called minimal if it contains no proper extremal subsets.

For example, the whole space and the empty set are extremal. Also, every vertex, edge, or face (as well as their unions) of the cube is an extremal subset of the cube. Vertices of the cube are its only minimal extremal subsets.

Counting maximal finite subgroups in a crystallographic group Γ (up to conjugation) is equivalent to counting the minimal extremal subsets in the quotient space $\mathcal{A} = \mathbb{E}^m/\Gamma$. So, 10.11 would follow from the next conjecture.

10.17. Conjecture. *Any m-dimensional compact* ALEX(0) *space has at most 2^m minimal extremal subset.*

Let us mention another related conjecture. An extremal set is called primitive if it contains no proper extremal subsets with nonempty relative interior. For example, each face of m-dimensional cube is its primitive extremal subset; therefore the cube has exactly 3^m primitive extremal subset, including the empty set and the whole cube.

10.18. Conjecture. *Any m-dimensional compact* Alex(0) *space has at most 3^m minimal extremal subset.*

Some crude estimates on number of extremal subsets follow from the idea in Gromov's Betti number theorem 8.15.

Lecture 11

Surfaces

This lecture is less rigorous; its aim is to introduce the reader to the geometry of convex surfaces, the main precursor to modern Alexandrov geometry. For a deeper dive into this theory, we recommend the classic and brilliantly written books by Alexandr Alexandrov [7, 9]. Additionally, the book by Alexey Pogorelov [95] is highly recommended, despite being a challenging read.

A Polyhedral surfaces

A *polyhedral surface* is defined as a 2-dimensional manifold (possibly with boundary) equipped with a length metric that admits a finite triangulation, where each triangle is isometric to a Euclidean triangle. A *triangulation* of a polyhedral surface is always assumed to satisfy this condition.

Note that, according to our definition, every polyhedral surface is compact.

Consider a point p on a polyhedral surface $\mathcal{P}$. We can assume that p is a vertex of a triangulation of $\mathcal{P}$, which can be achieved by subdividing the triangulation. Let θ_p denote the *total angle* around p, which is the sum of all angles at p in the triangles that have p as a vertex.

Note that θ_p is independent of the choice of triangulation. If p is an interior point, then the value $2\cdot\pi - \theta_p$ is called the *curvature* at p. If p lies on the boundary of $\mathcal{P}$, then the value $\pi - \theta_p$ is called the *inner turn* at p.

A point with nonzero curvature or inner turn will be called an *essential vertex* of the surface. Note that an essential vertex is a vertex in any triangulation.

11.1. Exercise! *Show that geodesics on a polyhedral surface with nonnegative curvature and nonnegative inner turns can have essential vertices only at their endpoints.*

The following statement is an analog of the Gauss–Bonnet formula.

11.2. Exercise. *Let $K(\mathcal{P})$ and $T(\mathcal{P})$ denote the sum of the curvatures at all interior points and the sum of the inner turns at all boundary points of a polyhedral surface $\mathcal{P}$, respectively. Show that*

$$K(\mathcal{P}) + T(\mathcal{P}) = 2\cdot\pi\cdot\chi(\mathcal{P}),$$

where $\chi(\mathcal{P})$ denotes the Euler characteristic of $\mathcal{P}$.

The following proposition states that this new definition of curvature is consistent with the $\mathrm{A{\scriptsize LEX}}(0)$ comparison.

11.3. Proposition. *A polyhedral surface is* $\mathrm{A{\scriptsize LEX}}(0)$ *if and only if it has nonnegative curvature at every interior point and nonnegative inner turn at each boundary point.*

Proof. By 2.22, it is sufficient to check that $f = \frac{1}{2}\cdot\mathrm{dist}_p^2\circ\gamma$ is 1-concave for any geodesic γ and any point p.

We can assume that p is not a vertex and that the endpoints of γ are not vertices; the case where p or the endpoints of γ are vertices can be handled by approximation. By 11.1, γ does not contain any vertices.

Given a point $x = \gamma(t_0)$, choose a geodesic $[px]$. Again, by 11.1, $[px]$ does not contain any vertices. Therefore, a neighborhood $U \supset [px]$ can be unfolded onto a plane; that is, there exists an injective length-preserving map $z \mapsto \tilde{z}$ of U into the Euclidean plane. In this way, we map the part of γ in U to a line segment $\tilde{\gamma}$. Let

$$\tilde{f}(t) := \tfrac{1}{2}\cdot\mathrm{dist}_{\tilde{p}}^2 \circ \tilde{\gamma}(t).$$

Since the geodesic $[px]$ maps to a line segment, we have $\tilde{f}(t_0) = f(t_0)$. Furthermore, since the unfolding map $z \mapsto \tilde{z}$ preserves the lengths of curves, we obtain $\tilde{f}(t) \geqslant f(t)$ if t is close to t_0. That is, $\tilde{f}$ is a local upper barrier for f at t_0; see 2E. Evidently, $\tilde{f}''(t) \equiv 1$. Therefore, f is 1-concave.

11.4. Exercise. *The converse is left to the reader.* □

B Approximation

The following theorem is a key additional tool in the Alexandrov geometry of surfaces. We will use this result in the proof of 11.21 to reduce questions about ALEX(0) surfaces to polyhedral surfaces with nonnegative curvature.

11.5. Theorem. *Any closed* ALEX(0) *surface is a Gromov–Hausdorff limit of homeomorphic polyhedral surfaces with nonnegative curvature.*

The construction of polyhedral approximations is based on the following exercise.

11.6. Exercise. *Let $\mathcal{P}$ be a closed* ALEX(0) *surface.*

(a) Show that any point p admits an arbitrarily small closed convex polygonal neighborhood N; that is, N is convex and bounded by a broken geodesic.

(b) Given $\delta > 0$, show that $\mathcal{P}$ admits a triangulation τ by convex triangles with positive inner turn at each vertex and diameter smaller than δ.

(c) Suppose that v is a vertex of a triangulation τ of $\mathcal{P}$ by convex triangles. Let θ_v be the sum of angles at v in all the triangles of τ. Show that $\theta_v \leqslant 2\cdot\pi$.

Construction. Let $\mathcal{P}$ be a closed ALEX(0) surface. By part *(b)*, we can triangulate $\mathcal{P}$ by small convex triangles, say diameter of each triangle is less than given $\delta > 0$. Now, replace each triangle in the triangulation with its corresponding model solid triangle, and denote the resulting polyhedral surface by $\tilde{\mathcal{P}}_\delta$.

Note that $\tilde{\mathcal{P}}_\delta$ is homeomorphic to $\mathcal{P}$; moreover, there exists a homeomorphism $\mathcal{P} \to \tilde{\mathcal{P}}_\delta$ that maps each point $x \in \mathcal{P}$ to a point $\tilde{x} \in \tilde{\mathcal{P}}_\delta$ in the corresponding model triangle.

By the hinge comparison (2.17d) and part *(c)* of the exercise, the total angle around each vertex in $\tilde{\mathcal{P}}_\delta$ does not exceed $2\cdot\pi$. Thus, the resulting polyhedral surface $\tilde{\mathcal{P}}_\delta$ has nonnegative curvature. □

Observe that Theorem 11.5 follows from the following statement.

11.7. Claim. *If $\tilde{\mathcal{P}}_\delta$ is provided by the construction, then $\tilde{\mathcal{P}}_\delta \to \mathcal{P}$ as $\delta \to 0$ in the sense of Gromov–Hausdorff.*

Altho this claim looks self-evident, it is not. A remarkably elegant proof was given by Alexandrov [9, VII § 6]. We will outline an alternative proof based on the following exercise and two theorems, which will be stated without proof.

The first theorem is due to Yuri Burago, Mikhael Gromov, and Grigori Perelman [18, 10.8]; it generalizes Alexandrov's theorem for surfaces [9, X § 2]. The second theorem is due to Nan Li [67].

11.8. Theorem. *Let $\mathcal{X}_1, \mathcal{X}_2$ be a sequence of n-dimensional* $\mathrm{Alex}(\kappa)$ *spaces that converges to $\mathcal{X}_\infty$ in the sense of Gromov–Hausdorff. Then the n-volume on $\mathcal{X}_i$ weakly converges to the n-volume on $\mathcal{X}_\infty$.*

11.9. Theorem. *Let $\mathcal{X}$ be an Alexandrov space without boundary, and let $\mathcal{Y}$ be an arbitrary Alexandrov space. Then any short volume-preserving map $\mathcal{X} \to \mathcal{Y}$ is an isometry.*

Suppose a convex solid triangle Δ in an $\mathrm{Alex}(0)$ surface has angles α, β and γ. Let us define its excess by

$$\operatorname{excess}\Delta = \alpha + \beta + \gamma - \pi.$$

Since the angles of a model triangle sum up to π, by the hinge comparison ($2.17d$), the excess is nonnegative.

11.10. Exercise. *Let τ be a triangulation of a closed* $\mathrm{Alex}(0)$ *surface $\mathcal{P}$ by convex triangles $\Delta_1, \dots, \Delta_n$.*

(a) Show that

$$\operatorname{excess}\Delta_1 + \dots + \operatorname{excess}\Delta_n \leqslant 2{\cdot}\pi{\cdot}\chi(\mathcal{P}),$$

where $\chi(\mathcal{P})$ denotes the Euler characteristic of $\mathcal{P}$.

(b) Let x and y be points on the sides of a triangle Δ_i, and let $\tilde{x}$ and $\tilde{y}$ be the corresponding points in the corresponding triangle $\tilde{\Delta}$ in $\tilde{\mathcal{P}}$. Show that

$$|\tilde{x} - \tilde{y}|_{\tilde{\Delta}} \leqslant |x - y|_{\Delta} \leqslant |\tilde{x} - \tilde{y}|_{\tilde{\Delta}} + \operatorname{excess}\Delta{\cdot}\operatorname{diam}\Delta.$$

(c) Let Δ be a solid triangle in the triangulation τ of $\mathcal{P}$, and $\tilde{\Delta}$ — the corresponding triangle in $\tilde{\mathcal{P}}$. Show that

$$\operatorname{area}\tilde{\Delta} \leqslant \operatorname{area}\Delta \leqslant \operatorname{area}\tilde{\Delta} + \tfrac{1}{2}{\cdot}\operatorname{excess}\Delta{\cdot}(\operatorname{diam}\Delta)^2.$$

Note that part *(a)* implies that $\chi(\mathcal{P}) \geqslant 0$. Therefore, $\mathcal{P}$ is homeomorphic to a sphere, projctive plane, torus, or Klein bottle. In the latter two cases, the construction produces a flat surface $\tilde{\mathcal{P}}_\delta$, which has to be isometric to $\mathcal{P}$. Therefore the cases of sphere, projctive plane are more interesting.

Proof of 11.7. Choose a sequence of positive numbers $\delta_n \to 0$; let $\tilde{\mathcal{P}}_{\delta_n}$ be polyhedral spaces provided by the construction and let τ_n be the corresponding triangulation.

According to part *(b)* of the exercise, the spaces $\tilde{\mathcal{P}}_{\delta_n}$ have bounded diameter. Therefore by Gromov's selection theorem, we can pass to a converging sequence of $\tilde{\mathcal{P}}_{\delta_n}$; denote its Gromov–Hausdorff limit by $\tilde{\mathcal{P}}$.

Note that if $\tilde{\mathcal{P}}_\delta$ does not converge to $\mathcal{P}$, then we can assume that $\tilde{\mathcal{P}}$ is not isometric to $\mathcal{P}$.

Choose two points $x, y \in \mathcal{P}$, and connect them by a geodesic. Denote by $s_1, \dots, s_m$ the points of the geodesic on the sides of the triangulation τ_n; we assume that these points appear in the same order on the geodesic. Denote by $\tilde{x}$, $\tilde{y}$, and $\tilde{s}_1, \dots, \tilde{s}_m$ the corresponding points in $\tilde{\mathcal{P}}_{\delta_n}$. By part *(b)* of the exercise,

$$|\tilde{s}_{i-1} - \tilde{s}_i|_{\tilde{\mathcal{P}}_{\delta_n}} \leqslant |s_{i-1} - s_i|_{\mathcal{P}}.$$

Note also that

$$|\tilde{x} - \tilde{s}_1|_{\tilde{\mathcal{P}}_{\delta_n}} \leqslant \delta_n \quad \text{and} \quad |\tilde{s}_m - \tilde{y}|_{\tilde{\mathcal{P}}_{\delta_n}} \leqslant \delta_n$$

Therefore

$$|\tilde{x} - \tilde{y}|_{\tilde{\mathcal{P}}_{\delta_n}} \leqslant |x - y|_{\mathcal{P}} + 2\cdot\delta_n.$$

Passing to the limit, we get a short onto map $\mathcal{P} \to \tilde{\mathcal{P}}$. On the other hand, applying parts *(a)* and *(c)*, we get that

$$\operatorname{area}\mathcal{P} - \pi\cdot\chi(\mathcal{P})\cdot\delta_n^2 \leqslant \operatorname{area}\tilde{\mathcal{P}}_{\delta_n} \leqslant \operatorname{area}\mathcal{P}$$

By Theorem 11.8, $\operatorname{area}\mathcal{P} = \operatorname{area}\tilde{\mathcal{P}}$. Applying Theorem 11.9, we get that the short map $\mathcal{P} \to \tilde{\mathcal{P}}$ is an isometry — a contradiction. □

Remark. The main difficulty in the proof comes from nonconvexity of triangles in the triangulation of $\tilde{\mathcal{P}}_\delta$. If these triangles would be convex, then the first estimate in parts *(b)* and *(a)* would imply that $\tilde{\mathcal{P}}_\delta$ is close to $\mathcal{P}$ in the sense of Gromov–Hausdorff.

C Surface of polyhedrons and bodies

Let us define a convex body as a compact convex subset in $\mathbb{E}^3$ with a nonempty interior. The surface of a convex body is defined as its boundary equipped with the induced length metric.

11.11. Exercise! *Show that the surface of a convex body is homeomorphic to the 2-dimensional sphere.*

A convex polyhedron is a convex body with a finite number of extremal points, called its vertices.

Note that the surface of a convex polyhedron K is a closed polyhedral surface.

11.12. Exercise. *Assume that the surface of a nondegenerate tetrahedron T has curvature π at each of its vertices. Show that*

(a) all faces of T are congruent;
(b) the line containing the midpoints of opposite edges of T intersects these edges at right angles.

11.13. Claim. *The surface $\mathcal{P}$ of any convex polyhedron K has non-negative curvature. Moreover, a point v is a vertex of K if and only if v is an essential vertex of $\mathcal{P}$.*

A proof is given in Kiselyov's school textbook [54, § 48]; one can also deduce it from 1.11.

11.14. Exercise! *Let $K_1, K_2, \dots,$ and K_∞ be convex bodies in $\mathbb{E}^m$. Denote by $\mathcal{P}_n$ the surface of K_n. Suppose $K_n \to K_\infty$ in the sense of Hausdorff. Show that $\mathcal{P}_n \to \mathcal{P}_\infty$ in the sense of Gromov–Hausdorff.*

Since any convex body is a Hausdorff limit of a sequence of convex polyhedrons, the next proposition follows from 11.3, 11.14, and 8.1.

11.15. Proposition. *The surface of a convex body in $\mathbb{E}^3$ is* ALEX(0).

D Uniqueness theorem

11.16. Theorem. *Any two convex polyhedrons in $\mathbb{E}^3$ with isometric surfaces are congruent.*

Moreover, any isometry between the surfaces of convex polyhedrons can be extended to an isometry of the whole $\mathbb{E}^3$.

If one assumes that the isometry between the surfaces is face-to-face, then we get an equivalent reformulation of Cauchy's theorem. Cauchy's argument, with a small addition, proves 11.16.

First, let us remind Cauchy's proof, assuming the reader knows it. If not, then read it in one of the classical texts [1, 26, 104].

Sketch of Cauchy's proof. Suppose K and K' are convex polyhedrons in $\mathbb{E}^3$; denote their surfaces by $\mathcal{P}$ and $\mathcal{P}'$. Suppose there is an isometry $\iota\colon \mathcal{P} \to \mathcal{P}'$ that sends each face of K to a face of K'.

Let us mark an edge of K with "+" (or "−") if the dihedral angle at this edge in K is smaller (respectively, bigger) than the corresponding angle in K'. Further, we consider the graph Γ that is formed by all marked edges. If Γ is empty, then Cauchy's theorem follows; assume the contrary.

The graph Γ is embedded into $\mathcal{P}$, which is homeomorphic to the sphere. In particular, the edges coming from one vertex have a natural cyclic order. Given a vertex v of Γ, we can count the *number of sign* *changes* around v; that is, the number of consequent pairs of edges with different signs.

We need to show two statements:

11.17. Local lemma. *At any vertex of* Γ*, the number of sign changes is at least* 4.

11.18. Global lemma. *No (nonempty) planar graph meets the condition of the local lemma.*

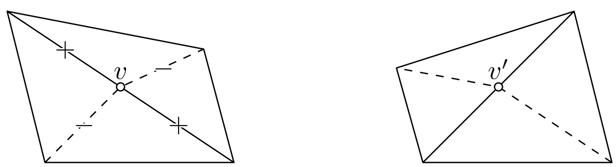
Once the lemmas are proved, Cauchy's theorem follows. □

Once more, the argument above is written only to make sure we are on the same page; it will not work without reading the actual proof.

Alexandrov's addition. We need to remove the assumption that the isometry $\iota\colon \mathcal{P} \to \mathcal{P}'$ is face-to-face. Mark in $\mathcal{P}$ all the edges of K as we did above. In addition, if an edge in K' does not correspond to an edge of K, then mark its inverse image in K with "$-$"; these lines on K will be referred to as fake edges.

The marked lines divide $\mathcal{P}$ into convex polygons, and the restriction of ι to each polygon is a rigid motion. These polygons should be used instead of faces in the Cauchy's argument.

A vertex of the obtained graph can be a vertex of K, or it can be a fake vertex; that is, it might be an intersection of an edge and a fake edge.



For a usual vertex, the local lemma can be proved the same way. For a fake vertex v, it is easy to see that both parts of the edge coming thru v are marked with minus while both of the fake edges at v are marked with plus. Therefore, we still have at least four sign changes at v. The remaining argument works as before. □

Let us also state the following result of Alexey Pogorelov [94, chapter III]; an alternative proof was found by Yurii Volkov [109].

11.19. Theorem. *Any two convex bodies in* $\mathbb{E}^3$ *with isometric surfaces are congruent.*

Moreover, any isometry between surfaces of convex bodies can be extended to an isometry of the whole $\mathbb{E}^3$.

At first glance, this theorem might look like a small improvement of Alexandrov's uniqueness, but this improvement is huge. The proof is quite hard. Let us just mention that it would follow if any two polyhedra K and K' with close surfaces in the sense of Gromov–Hausdorff would be almost congruent; that is, there is a motion μ of $\mathbb{E}^3$ such that the Hausdorff distance from K to $\mu(K')$ is small.

E Existence theorem

By 11.3, 11.13, and 11.11, the surface of a convex polyhedron is an $\mathrm{ALEX}(0)$ and homeomorphic to the sphere. Alexandrov's theorem states that the converse holds if one includes in the consideration *twice covered polygons*. In other words, we have to consider a plane polygon as a degenerate polyhedron; in this case, its surface is defined as the doubling of the polygon across its boundary.

From now on, we assume that a polyhedron can degenerate to a plane polygon.

11.20. Theorem. *A polyhedral metric on the two-sphere is isometric to the surface of a convex polyhedron (possibly degenerate) if and only if it has nonnegative curvature.*

Applying the approximation theorem (11.5) and 11.14, we get the following statement. Here we again assume that a convex body can degenerate to a convex plane figure, and, in this case, its surface is defined as the doubling of the figure across its boundary.

11.21. Corollary. *A metric on the two-sphere is* $\mathrm{ALEX}(0)$ *if and only if it is isometric to the surface of a convex body (possibly degenerate).*

The proof of the existence theorem will be discussed in the following two sections. It is instructive to solve the following exercise before going further.

11.22. Exercise! *Let $\mathcal{P}$ be the 2-sphere equipped with a polyhedral metric with nonnegative curvature.*

(a) *Prove that $\mathcal{P}$ has at least 3 essential vertices.*

(b) *If $\mathcal{P}$ has exactly 3 essential vertices u, v, and w, then it is isometric to the doubling of the solid model triangle $\tilde{\triangle}(uvw)$.*

(c) *If $\mathcal{P}$ has exactly 4 essential vertices, then it is isometric to the surface of a tetrahedron (possibly degenerate to a quadrangle).*

F Reformulation

In this section, we introduce several notions and use them to reformulate the existence theorem (11.23).

Space of polyhedrons. Let us denote by $\boldsymbol{K}$ the space of all convex polyhedrons in the Euclidean space, including polyhedrons that degenerate to a plane polygon. Polyhedrons in $\boldsymbol{K}$ will be considered up to a motion of the space; we will not distinguish between a convex polyhedron and its congruence class.

The space $\boldsymbol{K}$ will be considered with the topology induced by the Hausdorff metric up to a motion; that is, the distance between (equivalence classes of) polyhedrons K and L is defined by

$$|K - L| := \inf_{\mu}\{|K - \mu(L)|_{\mathrm{Haus}}\},$$

where μ runs among all motions of $\mathbb{E}^3$.

We say that a polyhedron K in $\boldsymbol{K}$ has no symmetries if $K \neq \neq \mu(K)$ for any nontrivial motion μ of $\mathbb{E}^3$. The set of all polyhedrons without symmetry in $\boldsymbol{K}$ will be denoted by $\boldsymbol{K}^\circ$. Observe that $\boldsymbol{K}^\circ$ is an open set in $\boldsymbol{K}$.

Further, denote by $\boldsymbol{K}_n$ the polyhedrons in $\boldsymbol{K}$ with exactly n vertices, and let $\boldsymbol{K}_n^\circ = \boldsymbol{K}_n \cap \boldsymbol{K}^\circ$. Since any polyhedron has at least 3 vertices, the space $\boldsymbol{K}$ admits a subdivision into a countable number of subsets $\boldsymbol{K}_3, \boldsymbol{K}_4, \dots$

Space of surfaces. The space of polyhedral surfaces with nonnegative curvature that are homeomorphic to the 2-sphere will be denoted by $\boldsymbol{P}$. The surfaces in $\boldsymbol{P}$ will be considered up to an isometry, and the whole space $\boldsymbol{P}$ will be equipped with the natural topology induced by the Gromov–Hausdorff metric.

We say that a surface $\mathcal{P}$ in $\boldsymbol{P}$ has no symmetries if there is no nontrivial isometry $\mu\colon \mathcal{P} \to \mathcal{P}$. The set of all surfaces without symmetry in $\boldsymbol{P}$ will be denoted by $\boldsymbol{P}^\circ$. Observe that $\boldsymbol{P}^\circ$ is an open set in $\boldsymbol{P}$.

The subset of $\boldsymbol{P}$ of all surfaces with exactly n essential vertices will be denoted by $\boldsymbol{P}_n$; let $\boldsymbol{P}_n^\circ = \boldsymbol{P}_n \cap \boldsymbol{P}^\circ$. By 11.22$a$, any surface in $\boldsymbol{P}$ has at least 3 essential vertices. Therefore $\boldsymbol{P}$ is subdivided into countably many subsets $\boldsymbol{P}_3, \boldsymbol{P}_4, \dots$

From a polyhedron to its surface. Recall that the surface of a convex polyhedron is a sphere with nonnegative curvature. Therefore, passing from a polyhedron to its surface defines a map

$$\iota\colon \boldsymbol{K} \to \boldsymbol{P}.$$

Note that the existence theorem (11.20) follows from the next statement.

11.23. Theorem. *For any integer $n \geqslant 3$, the map ι is a bijection from $\boldsymbol{K}_n$ to $\boldsymbol{P}_n$.*

G About the proof of existence

By 11.14, the map $\iota\colon \boldsymbol{K} \to \boldsymbol{P}$ is continuous. Combining 11.13 with the uniqueness theorem (11.16), we get that $\iota(\boldsymbol{K}_n) \subset \boldsymbol{P}_n$ and the map $\iota\colon \boldsymbol{K}_n \to \boldsymbol{P}_n$ is injective. It remains to prove the following.

11.24. Claim. *For any $n \geqslant 3$, the map $\iota\colon \boldsymbol{K}_n \to \boldsymbol{P}_n$ is surjective.*

The proof is based on the construction of a one-parameter family of polyhedrons that starts at an arbitrary polyhedron and ends at a polyhedron with its surface isometric to the given surface $\mathcal{P}$. This type of argument is called the continuity method; it is often used in the theory of differential equations.

Now let us get into details. First, observe that the second part of the uniqueness theorem (11.16) implies that $\iota(\boldsymbol{K}_n^\circ) \subset \boldsymbol{P}_n^\circ$.

11.25. Lemma. *For any integer $n \geqslant 4$, the space $\boldsymbol{P}_n^\circ$ is connected and dense in $\boldsymbol{P}_n$.*

Note that $\boldsymbol{P}_3^\circ = \varnothing$; indeed the surface of a triangle admits a reflection symmetry. The case $n = 4$ can be deduced from 11.22*c*; thus, we can assume that $n \geqslant 5$.

The second statement is proved by a general-position-type argument.

The proof of the first statement is not complicated, but it requires ingenuity; it can be done by the direct construction of a one-parameter family of surfaces in $\boldsymbol{P}_n^\circ$ that connects two given surfaces. Such a family can be obtained as a sequence of the following deformations (direct or reversed).

Start with a surface $\mathcal{P}$ from $\boldsymbol{P}_n^\circ$. Suppose v and w are essential vertices in $\mathcal{P}$. Let us cut $\mathcal{P}$ along a shortest path from v to w. This way we obtain a sphere with a hole. The hole can be patched by a disc so that the obtained surface remains in $\boldsymbol{P}_n$. In particular, the obtained surface has exactly n essential vertices; note that after the patching, the vertices v and w may become inessential. (There is a three-parameter family of such patches, so we have something to choose from.) Choosing a one-parameter family of such patches, we can get a deformation of $\mathcal{P}$.

Again, applying a general-position-type argument to the above construction, we get a path in $\boldsymbol{P}_n^\circ$, assuming that the starting and ending surfaces are in $\boldsymbol{P}_n^\circ$.

11.26. Lemma. *The map $\iota\colon \boldsymbol{K}_n^\circ \to \boldsymbol{P}_n^\circ$ is open, that is, it maps any open set in $\boldsymbol{K}_n^\circ$ to an open set in $\boldsymbol{P}_n^\circ$.*

In particular, for any $n \geqslant 3$, the image $\iota(\boldsymbol{K}_n^\circ)$ is open in $\boldsymbol{P}_n^\circ$.

This statement follows from the so-called invariance of domain theorem, which states that a *continuous injective map between manifolds of the same dimension is open.*

Recall that ι defines a continuous and injective $\boldsymbol{K}_n^\circ \to \boldsymbol{P}_n^\circ$. It remains to check that both spaces $\boldsymbol{K}_n^\circ$ and $\boldsymbol{P}_n^\circ$ are $(3{\cdot}n{-}6)$-dimensional manifolds.

Choose a polyhedron K in $\boldsymbol{K}_n$. It is uniquely determined by the $3{\cdot}n$ coordinates of its n vertices. We can assume that the first vertex is at the origin, the second has a positive x-coordinate and the remaining two coordinates vanish, and the third has a vanishing z-coordinate and a positive y-coordinate. Therefore, all polyhedrons in $\boldsymbol{K}_n$ that lie sufficiently close to K can be described by $3{\cdot}n - 6$ parameters. If K has no symmetries, then this description is one-to-one; in this case, a neighborhood of K in $\boldsymbol{K}_n$ admits a parametrization by an open set in $\mathbb{R}^{3\cdot n-6}$.

The case of surfaces is analogous. We need to construct a subdivision of the sphere into plane triangles using only essential vertices. By Euler's formula, there are exactly $3{\cdot}n{-}6$ edges in this subdivision. The lengths of the edges completely describe the surface $\mathcal{P}$ and any surface near by. If the surface has no symmetries, then this description is one-to-one, and a neighborhood of $\mathcal{P}$ in $\boldsymbol{P}_n$ admits a parametrization by an open set in $\mathbb{R}^{3\cdot n-6}$.

11.27. Lemma. *The map $\iota\colon \boldsymbol{K}_n \to \boldsymbol{P}_n$ is closed; that is, the image of a closed set in $\boldsymbol{K}_n$ is closed in $\boldsymbol{P}_n$.*

In particular, for any $n \geqslant 3$, the set $\iota(\boldsymbol{K}_n)$ is closed in $\boldsymbol{P}_n$.

Choose a sequence of polyhedrons $K_1, K_2, \dots$ in $\boldsymbol{K}_n$. Assume that the sequence $\mathcal{P}_i = \iota(K_n)$ converges in $\boldsymbol{P}_n$ as $i \to \infty$; denote its limit by $\mathcal{P}_\infty$. We need to construct a polyhedron $K_\infty \in \boldsymbol{K}_n$ such that $\iota(K_\infty) = \mathcal{P}_\infty$; let us do it.

Passing to a subsequence, we can assume that K_i converges in $\boldsymbol{K}$; denote the limit polyhedron by K_∞. Since ι is continuous, $\iota(K_i)$ converges to $\iota(K_\infty)$ in $\boldsymbol{P}$; so, $\iota(K_\infty) = \mathcal{P}_\infty$. Recall that $\iota(\boldsymbol{K}_m) \subset \boldsymbol{P}_m$ for each m; therefore, $K_\infty \in \boldsymbol{K}_n$.

Proof of 11.24. The case $n \leqslant 4$ is already solved in 11.22; so we assume that $n \geqslant 5$. By 11.27 and 11.26, $\iota(\boldsymbol{K}_n^\circ)$ is a nonempty closed and open set in $\boldsymbol{P}_n^\circ$, and $\boldsymbol{P}_n^\circ$ is connected. Therefore, $\iota(\boldsymbol{K}_n^\circ) = \boldsymbol{P}_n^\circ$.

By 11.27, $\iota(\boldsymbol{K}_n)$ is closed in $\boldsymbol{P}_n$. By 11.25, $\boldsymbol{P}_n^\circ$ is dense in $\boldsymbol{P}_n$. Since $\iota(\boldsymbol{K}_n^\circ) = \boldsymbol{P}_n^\circ$, we have $\boldsymbol{P}_n^\circ \subset \iota(\boldsymbol{K}_n)$; therefore, $\iota(\boldsymbol{K}_n) = \boldsymbol{P}_n$; that is, $\iota\colon \boldsymbol{K}_n \to \boldsymbol{P}_n$ is surjective. □

H Ambient space

On one hand the Alexandrov surface theory is simpler since it has extra tools, On the other hand, this tool comes with extra structure, which makes the theory more complicated. The following result of Joseph Liberman [68] gives an example.

11.28. Theorem. *Any geodesic in the surface of a convex body is one-sided differentiable as a curve in* $\mathbb{E}^3$.

Proof. Let γ be a geodesic on the surface of a convex body K. Choose $p \in K$. By 9.13, the function $f_p\colon t \mapsto \mathrm{dist}_p \circ \gamma(t)$ is semiconcave for any $p \in K$. In particular, one-sided derivatives $f_p^+(t)$ are defined for every t.

Given $x = \gamma(t)$, choose three points $p_1, p_2, p_3 \in K$ in general position; that is, the four points x, p_1, p_2, p_3 do not lie in one plane. Observe that the distance functions dist_{p_i} give smooth coordinates in a neighborhood of x. From above the functions f_{p_i} have one-sided derivatives at t. Since the coordinates are smooth, we get that $\gamma^+(t)$ is defined as well. □

11.29. Exercise. *Suppose a plane* Π *cuts from the surface of a convex body* K *a disc* Δ*, and the reflection of* Δ *across* Π *lies in* K*. Show that* Δ *is a convex subset of the surface; that is, if a geodesic has endpoints in* Δ*, then it completely lies in* Δ*.*

The following exercise gives a more exact version of comparison for convex surfaces; it is due to Anatolii Milka [73, Theorem 2].

11.30. Very advanced exercise. *Let* $\mathcal{P}$ *be the surface of a nondegenerate convex body* $K \subset \mathbb{E}^3$*, and let* γ_1 *and* γ_2 *be geodesic paths in* $\mathcal{P}$ *that start at one point* $p = \gamma_1(0) = \gamma_2(0)$*. Suppose* $x_i = \gamma_i(1)$*, and* $y_i = p + \gamma_i^+(0)$*. Show that*

$$|x_1 - x_2|_{\mathcal{P}} \leqslant |y_1 - y_2|_W,$$

where W *is the complement to the interior of* K*.*

I Remarks

The excess of solid triangles can be extended to a sigma-additive measure on Borel subsets of $\mathrm{A{\scriptstyle LEX}}(\kappa)$ surface. This is the so-called curvature measure [9, V]. There have been several attempts to generalize this to higher dimensions, but so far, we only have partial results in this direction [33, 62].

Big part of the theory described in this lecture is generalized to surfaces with bounded integral curvature [10, 97].

The statement of Cauchy's theorem was conjectured by Adrien-Marie Legendre at the end of the 18^{th} century; a formulation was given in the first edition of his geometry textbook [66]. It was motivated by a vague definition in Euclid's Elements, which could be interpreted as *polyhedrons are equal if the same holds for their faces.*

The local lemma was already known to Legendre. Legendre discussed this question with his colleague Joseph-Louis Lagrange, who suggested this problem to Augustin-Louis Cauchy in 1813; soon he solved it [19].

The key observation that the face-to-face condition can be removed was made by Alexandr Alexandrov in 1941; in the same paper he proved the uniqueness theorem [8]. A quite different proof was found by Yurii Volkov in his thesis [108]; it uses a deformation of three-dimensional polyhedral space. (Be aware that the proof of this theorem given in the book by Igor Pak contains an essential mistake [82].)

In Cauchy's proof [19], it was deducted from an analog of the following lemma. Cauchy made a small mistake in its proof that was fixed in a century [98]. Several proofs of the arm lemma can be found in the letters between Isaac Schoenberg and Stanisław Zaremba [99].

11.31. Arm lemma. *Assume that $A=[a_0a_1\dots a_n]$ is a convex polygon in $\mathbb{E}^2$ and $A'=[a_0'a_1'\dots a_n']$ is a polygonal line in $\mathbb{E}^3$ such that $|a_i-a_{i+1}|=|a_i'-a_{i+1}'|$ for any $i\in\{0,\dots,n-1\}$ and $\measuredangle a_i\leqslant\measuredangle a_i'$ for each $i\in\{1,\dots,n-1\}$. Then*

$$|a_0-a_n|\leqslant|a_0'-a_n'|$$

and equality holds if and only if A is congruent to A'.

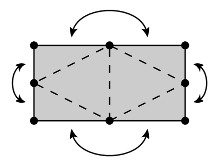
The following variation of the arm lemma makes sense for nonconvex spherical polygons. It is due to Viktor Zalgaller [110]. It can be used instead of the standard arm lemma.

11.32. Another arm lemma. *Let $A=[a_1\dots a_n]$ and $A'=[a_1'\dots$ $\dots a_n']$ be two spherical n-gons (not necessarily convex). Assume that A lies in a half-sphere, the corresponding sides of A and A' are equal,* *and each angle of A is at least the corresponding angle in A'. Then A is congruent to A'.*

Another close relative of the arm lemma is Reshetnyak's majorization theorem [96].

Alexandrov gave two proofs of the global lemma [7, 2.1.2 and 2.1.3]. The first is combinatorial, and the second is more visual. The argument in the second proof was reused by Anton Klyachko [55] in his car-crash lemma.

Proposition 11.15 generalizes to the boundaries of convex bodies in $\mathbb{E}^m$ for any $m \geqslant 2$. It could be considered as a partial case of the conjecture about the boundary of Alexandrov space; see 9.20. Another partial case, for Riemannian manifolds with boundary, is proved by the authors and Stephanie Alexander [4].



According to the uniqueness theorem, a convex polyhedron is completely defined by the intrinsic metric of its surface. In particular, knowing the metric, we could find the position of the edges. However, in practice, it is not easy to do. For example, the surface glued from a rectangle, as shown in the picture, defines a tetrahedron. Some of the glued lines appear inside the facets of the tetrahedron, and some edges (dashed lines) do not follow the sides of the rectangle.

Index

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