An invitation to Alexandrov geometry: spaces with curvature bounded below

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with an appendix by Nina Lebedeva and Anton Petrunin

Contents

Preface

As in our previous invitation [3] written jointly with Stephanie Alexander, we try to demonstrate the beauty and power of Alexandrov geometry by reaching interesting applications and theorems with minimal preparation. This time we do spaces with curvature bounded below in the sense of Alexandrov. We have extensively used another book of us with Stephanie Alexander [4].

This subject is more technical; it takes more preparation, and we had to jump over some proofs. Namely, we skip the proof of existence part in generalized Picard's theorem (5.3) and Perelman's theorem about conic neighborhoods (9.1); the rest is nearly rigorous. Some important statements stated as exercises, but they are nearly solved in hints at the end of the book.

In Lecture 1, we discuss necessary preliminaries and fix notations.

Lecture 2 introduces the main object of our study — spaces with curvature bounded below in the sense of Alexandrov.

In Lecture 3, we formulate and prove the globalization theorem — local Alexandrov condition implies global. To simplify the presentation, we consider only the compact case, but this case is leading.

In Lecture 4, we do beginning of calculus — tangent space and space of directions, differential, and gradient.

Lecture 5 introduces gradient flow, which will be further used as the main technical tool.

Lecture 6 proves the line splitting theorem, providing the first application of gradient flow. Furthermore, we introduce and study the linear subspace of tangent space.

In Lecture 7, we introduce linear dimension and volume. Further, we prove the Bishop–Gromov inequality and the right-inverse theorem, introduce the distance chart, and show that all reasonable types of dimension are the same for Alexandrov spaces.

Lecture 8 shows that a lower curvature bound survives in the Gromov–Hausdorff limit and proves Gromov's selection theorem. Further, we present Perelman's construction of strictly concave functions and apply it with Gromov's selection theorem to prove the homotopy finiteness theorem. This proof illustrates the main source of applications of Alexandrov geometry.

In Lecture 9, we introduce the boundary of finite-dimensional Alexandrov spaces and prove the doubling theorem.

In Lecture 10, we show that quotients of Alexandrov spaces by isometric group action are Alexandrov spaces and give several applications of this statement. This is another source of applications of Alexandrov geometry.

Lecture 11 brings us back to the original object of study of Alexandrov. We show that the surface of a convex body in Euclidean space is an Alexandrov space. This is historically the first source of applications of Alexandrov geometry.

Finally, Appendix A sketches Alexandrov's embedding theorem of convex polyhedra. Historically, this theorem is the first remarkable result in Alexandrov geometry, dating back to 1941. The proof is very well written by Alexandrov, but we decided to include its sketch here due to its beauty and importance. This appendix was written by Nina Lebedeva and the second author for for a book about St. Petersburg mathematicians and their discoveries [50].

Let us give a list of available texts on Alexandrov spaces with curvature bounded below:

- ◇ The 2-dimensional theory is treated in the classical book of Alexandr Alexandrov [7].
- ◇ The first introduction to Alexandrov geometry of all dimensions is given in the original paper by Yuriy Burago, Michael Gromov, and Grigory Perelman [15] and its extension [68] written by Perelman.
- ◇ A brief and reader-friendly introduction was written by Katsuhiro Shiohama [87, Sections 1–8].
- ◇ Another reader-friendly introduction, written by Dmitri Burago, Yuriy Burago, and Sergei Ivanov [14, Chapter 10].
- ◇ Survey by Conrad Plaut [83].
- ◇ Survey by the second author [75].

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Lecture 1

Preliminaries

A Prerequisites

We assume that the reader is familiar with the following topics in metric geometry:

- ◇ Compactness and proper metric spaces; recall that a metric space is *proper* if all its closed balls (with finite radius) are compact.
- ◇ Complete metric spaces and completion.
- ◇ Curves, semicontinuity of length and rectifiability.
- ◇ Hausdorff and Gromov–Hausdorff convergence. These are discussed briefly in 1I–1M. The definitions are there, but it would be hard to follow without prior experience.

All these topics are treated in [14] and [80]. Occasionally, we use the Baire category theorem and Rademacher's theorem, but these could be used as black boxes.

We use some topology. Most of the time, any introductory text in algebraic topology should be sufficient. For some examples, we use more advanced results, but these could also be used as black boxes.

Since most of the applications come from Riemannian geometry, it is better to be familiar with the Toponogov comparison theorem and related topics. The classical book by Jeff Cheeger and David Ebin [17] contains more than one needs.

B Notations

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

Given radius $r \in [0,\infty]$ and center $x \in \mathcal{X}$, the sets

$$\mathrm{B}(x,r) = \{\, y \in \mathcal{X} \,:\, |x-y| < r \,\},$$
$$\overline{\mathrm{B}}[x,r] = \{\, y \in \mathcal{X} \,:\, |x-y| \leqslant r \,\}$$

are called, respectively, the open and the closed balls. The notations $\mathrm{B}(x,r)_{\mathcal{X}}$ and $\overline{\mathrm{B}}[x,r]_{\mathcal{X}}$ might be used if we need to emphasize that these balls are taken in the metric space $\mathcal{X}$.

We will denote by $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$ the n-dimensional sphere (with angle metric), Euclidean space, and Lobachevsky space respectively. More generally, $\mathbb{M}^n(\kappa)$ will denote the model n-space of curvature κ; that is,

- ⋄ if $\kappa > 0$, then $\mathbb{M}^n(\kappa)$ is the n-sphere of radius $\frac{1}{\sqrt{\kappa}}$, so $\mathbb{S}^n = \mathbb{M}^n(1)$
- ⋄ $\mathbb{M}^n(0) = \mathbb{E}^n$,
- ⋄ if $\kappa < 0$, then $\mathbb{M}^n(\kappa)$ is the Lobachevsky n-space $\mathbb{H}^n$ rescaled by factor $\frac{1}{\sqrt{-\kappa}}$; in particular $\mathbb{M}^n(-1) = \mathbb{H}^n$.

C Length spaces

Let $\mathcal{X}$ be a metric space. If for any $\varepsilon > 0$ and any pair of points $x, y \in \mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length}\alpha < |x-y| + \varepsilon,$$

then $\mathcal{X}$ is called a length space and the metric on $\mathcal{X}$ is called a length metric.

1.1. Exercise. *Let $\mathcal{X}$ be a complete length space. Show that for any compact subset $K \subset \mathcal{X}$ there is a compact path-connected subset $K' \subset \mathcal{X}$ that contains K.*

Induced length metric. Directly from the definition, it follows that if $\alpha\colon [0,1] \to \mathcal{X}$ is a path from x to y (that is, $\alpha(0) = x$ and $\alpha(1) = y$), then

$$\operatorname{length}\alpha \geqslant |x-y|.$$

Set

$$\|x-y\| = \inf\{\operatorname{length}\alpha\}$$

where the greatest lower bound is taken for all paths from x to y. It is straightforward to check that $(x,y) \mapsto \|x-y\|$ is an ∞-metric; that is, $(x,y) \mapsto \|x-y\|$ is a metric in the extended positive reals $[0,\infty]$. The metric $\|{*}-{*}\|$ is called the induced length metric.

1.2. Exercise. *Suppose $(\mathcal{X}, |{*}-{*}|)$ is a complete metric space. Show that $(\mathcal{X}, \|{*}-{*}\|)$ is complete; that is, any Cauchy sequence of points in $(\mathcal{X}, \|{*}-{*}\|)$ converges in $(\mathcal{X}, \|{*}-{*}\|)$.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x-y|_A = \inf_\alpha \{\, \mathrm{length}\, \alpha \,\},$$

where the greatest lower bound is taken for all paths α from x to y in A. In other words, $|{*}-{*}|_A$ denotes the induced length metric on the subspace A. (The notation $|{*}-{*}|_A$ conflicts with the previously defined notation for distance $|x-y|_{\mathcal{X}}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.)

D Geodesics

Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ a real interval. A distance-preserving map $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a geodesic[1]; in other words, $\gamma\colon \mathbb{I} \to \mathcal{X}$ is a geodesic if

$$|\gamma(s)-\gamma(t)| = |s-t|$$

for any pair $s, t \in \mathbb{I}$.

If $\gamma\colon [a,b] \to \mathcal{X}$ is a geodesic such that $p = \gamma(a)$, $q = \gamma(b)$, then we say that γ is a geodesic from p to q. In this case, the image of γ is denoted by $[pq]$, and, with abuse of notations, we also call it a geodesic. We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$.

In general, a geodesic from p to q need not exist and if it exists, it need not be unique; for example, any meridian is a geodesic between poles on the sphere. However, once we write $[pq]$ we assume that we have chosen such a geodesic.

A geodesic path is a geodesic with constant-speed parameterization by the unit interval $[0,1]$.

A metric space is called geodesic if any pair of its points can be joined by a geodesic.

Evidently, any geodesic space is a length space.

1.3. Exercise. *Show that any proper length space is geodesic.*

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[1]Others call it differently: *shortest path*, *minimizing geodesic*. Also, note that the meaning of the term *geodesic* is different from what is used in Riemannian geometry, altho they are closely related.

E Menger's lemma

1.4. Lemma. *Let $\mathcal{X}$ be a complete metric space. Assume that for any pair of points $x, y \in \mathcal{X}$, there is a midpoint z. Then $\mathcal{X}$ is a geodesic space.*

This lemma is due to Karl Menger [60, Section 6].

Proof. Choose $x, y \in \mathcal{X}$; set $\gamma(0) = x$, and $\gamma(1) = y$.

$$x = \gamma(0) \qquad \gamma(\tfrac{1}{4}) \qquad \gamma(\tfrac{1}{2}) \qquad \gamma(\tfrac{3}{4}) \qquad \gamma(1) = y$$

Let $\gamma(\frac{1}{2})$ be a midpoint between $\gamma(0)$ and $\gamma(1)$. Further, let $\gamma(\frac{1}{4})$ and $\gamma(\frac{3}{4})$ be midpoints between the pairs $(\gamma(0), \gamma(\frac{1}{2}))$ and $(\gamma(\frac{1}{2}), \gamma(1))$ respectively. Applying the above procedure recursively, on the n-th step we define $\gamma(\frac{k}{2^n})$, for every odd integer k such that $0 < \frac{k}{2^n} < 1$, as a midpoint of the already defined $\gamma(\frac{k-1}{2^n})$ and $\gamma(\frac{k+1}{2^n})$.

This way we define $\gamma(t)$ for all dyadic rationals t in $[0, 1]$. Moreover, γ has Lipschitz constant $|x - y|$. Since $\mathcal{X}$ is complete, the map γ can be extended continuously to $[0, 1]$. Moreover,

$$\operatorname{length} \gamma \leqslant |x - y|.$$

Therefore γ is a geodesic path from x to y. $\square$

1.5. Exercise. *Let $\mathcal{X}$ be a complete metric space. Assume that for any pair of points $x, y \in \mathcal{X}$, there is an* a l m o s t m i d p o i n t*; that is, given $\varepsilon > 0$, there is a point z such that*

$$|x - z| < \tfrac{1}{2} \cdot |x - y| + \varepsilon \quad \textit{and} \quad |y - z| < \tfrac{1}{2} \cdot |x - y| + \varepsilon.$$

Show that $\mathcal{X}$ is a length space.

F Triangles and model triangles

Triangles. Given a triple of distinct points p, q, r in a metric space $\mathcal{X}$, a choice of geodesics $([qr], [rp], [pq])$ will be called a t r i a n g l e; we will use the short notation $[pqr] = [pqr]_{\mathcal{X}} = ([qr], [rp], [pq])$.

Given a triple $p, q, r \in \mathcal{X}$ there may be no triangle $[pqr]$ simply because one of the pairs of these points cannot be joined by a geodesic. Also, many different triangles with these vertices may exist, any of which can be denoted by $[pqr]$. If we write $[pqr]$, it means that we have chosen such a triangle.

Model triangles. Given three points p, q, r in a metric space $\mathcal{X}$, let us define its model triangle $[\tilde{p}\tilde{q}\tilde{r}]$ (briefly, $[\tilde{p}\tilde{q}\tilde{r}] = \tilde{\triangle}(pqr)_{\mathbb{E}^2}$) to be a triangle in the Euclidean plane $\mathbb{E}^2$ such that

$$|\tilde{p} - \tilde{q}|_{\mathbb{E}^2} = |p - q|_{\mathcal{X}}, \qquad |\tilde{q} - \tilde{r}|_{\mathbb{E}^2} = |q - r|_{\mathcal{X}}, \qquad |\tilde{r} - \tilde{p}|_{\mathbb{E}^2} = |r - p|_{\mathcal{X}}.$$

In the same way, we can define the hyperbolic and the spherical model triangles $\tilde{\triangle}(pqr)_{\mathbb{H}^2}$, $\tilde{\triangle}(pqr)_{\mathbb{S}^2}$ in the Lobachevsky plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2$. In the latter case, the model triangle is said to be defined if in addition

$$|p - q| + |q - r| + |r - p| < 2\cdot\pi.$$

In this case, the model triangle again exists and is unique up to an isometry of $\mathbb{S}^2$.

Model angles. If $[\tilde{p}\tilde{q}\tilde{r}] = \tilde{\triangle}(pqr)_{\mathbb{E}^2}$ and $|p - q|, |p - r| > 0$, the angle measure of $[\tilde{p}\tilde{q}\tilde{r}]$ at $\tilde{p}$ will be called the model angle of the triple p, q, r and will be denoted by $\tilde{\measuredangle}(p\,{}^q_r)_{\mathbb{E}^2}$.

For example, if $|p - q| = |q - r| = |r - p|$, then $\tilde{\measuredangle}(p\,{}^q_r)_{\mathbb{E}^2} = \frac{\pi}{3}$ regardless of existence and relative position of geodesics $[pq]$ and $[pr]$.

The same way we define $\tilde{\measuredangle}(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$; in particular, $\tilde{\measuredangle}(p\,{}^q_r)_{\mathbb{H}^2}$ and $\tilde{\measuredangle}(p\,{}^q_r)_{\mathbb{S}^2}$. We may use the notation $\tilde{\measuredangle}(p\,{}^q_r)$ if it is evident which of the model spaces is meant.

1.6. Exercise. *Show that for any triple of point p, q, and r, the function*

$$\kappa \mapsto \tilde{\measuredangle}(p\,{}^q_r)_{\mathbb{M}^2(\kappa)}$$

is nondecreasing in its domain of definition.

G Hinges and their angle measure

Hinges. Let $p, x, y \in \mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair of geodesics $([px], [py])$ will be called a hinge and will be denoted by $[p\,{}^x_y] = ([px], [py])$.

Angles. The angle measure of a hinge $[p\,{}^x_y]$ is defined as the following limit

$$\measuredangle[p\,{}^x_y] = \lim_{\bar{x},\bar{y}\to p} \tilde{\measuredangle}(p\,{}^{\bar{x}}_{\bar{y}}),$$

where $\bar{x} \in\,]px]$ and $\bar{y} \in\,]py]$.

Note that if $\measuredangle[p\,{}^x_y]$ is defined, then

$$0 \leqslant \measuredangle[p\,{}^x_y] \leqslant \pi.$$

1.7. Exercise. *Suppose that in the above definition, one uses spherical or hyperbolic model angles instead of Euclidean. Show that it does not change the value* $\measuredangle[p\,{}^x_y]$.

1.8. Exercise. *Give an example of a hinge* $[p\,{}^x_y]$ *in a metric space with an undefined angle measure* $\measuredangle[p\,{}^x_y]$.

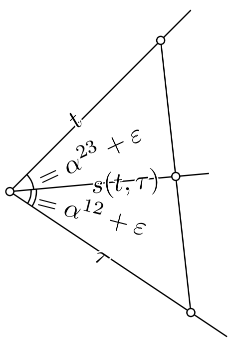
H Triangle inequality for angles

1.9. Proposition. *Let* $[px_1]$, $[px_2]$, *and* $[px_3]$ *be three geodesics in a metric space. Suppose all the angle measures* $\alpha_{ij} = \measuredangle[p\,{}^{x_i}_{x_j}]$ *are defined. Then*

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23}.$$

Proof. Since $\alpha_{13} \leqslant \pi$, we can assume that $\alpha_{12} + \alpha_{23} < \pi$. Denote by γ_i the unit-speed parametrization of $[px_i]$ from p to x_i. Given any $\varepsilon > 0$, for all sufficiently small $t, \tau, s \in \mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned}|\gamma_1(t) - \gamma_3(\tau)| &\leqslant |\gamma_1(t) - \gamma_2(s)| + |\gamma_2(s) - \gamma_3(\tau)| < \\ &< \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha_{12} + \varepsilon)} + \\ &\quad + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\alpha_{23} + \varepsilon)} \leqslant\end{aligned}$$



Below we define $s(t,\tau)$ so that for $s = s(t,\tau)$, this chain of inequalities can be continued as follows:

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon.$$

Hence the result follows.

To define $s(t,\tau)$, consider three half-lines $\tilde\gamma_1$, $\tilde\gamma_2$, $\tilde\gamma_3$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde\gamma_1, \tilde\gamma_2) = \alpha_{12} + \varepsilon$, $\measuredangle(\tilde\gamma_2, \tilde\gamma_3) = \alpha_{23} + \varepsilon$, and $\measuredangle(\tilde\gamma_1, \tilde\gamma_3) = \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon$. We parametrize each half-line by the distance from the starting point. Given two positive numbers $t, \tau \in \mathbb{R}_{\geqslant 0}$, let $s = s(t,\tau)$ be the number such that $\tilde\gamma_2(s) \in [\tilde\gamma_1(t)\ \tilde\gamma_3(\tau)]$. Clearly, $s \leqslant \max\{t,\tau\}$, so t, τ, s may be taken sufficiently small. □

1.10. Exercise. *Prove that the sum of adjacent angles is at least* π.

More precisely: suppose two hinges $[p\,{}^x_z]$ *and* $[p\,{}^y_z]$ *are* ***adjacent****; that is, they share side* $[pz]$*, and the union of two sides* $[px]$ *and* $[py]$ *form a geodesic* $[xy]$*. Show that*

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \geqslant \pi$$

whenever each angle on the left-hand side is defined.

Give an example showing that the inequality can be strict.

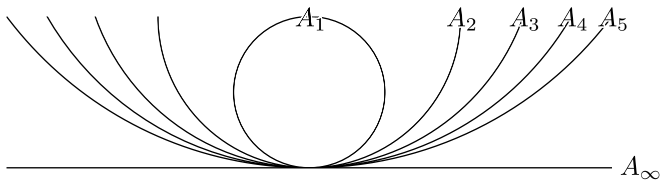
1.11. Exercise. *Assume that the angle measure of* $[q\,{}^p_x]$ *is defined. Let* γ *be the unit speed parametrization of* $[qx]$ *from* q *to* x*. Show that*

$$|p - \gamma(t)| \leqslant |q - p| - t\cdot \cos(\measuredangle[q\,{}^p_x]) + o(t).$$

I Hausdorff convergence

1.12. Definition. *Let* $A_1, A_2, \dots$ *be a sequence of closed sets in a metric space* $\mathcal{X}$*. We say that the sequence* A_n ***converges*** *to a closed set* A_∞ ***in the sense of Hausdorff*** *if, for any* $x \in \mathcal{X}$*, we have* $\operatorname{dist}_{A_n}(x) \to \operatorname{dist}_{A_\infty}(x)$ *as* $n \to \infty$*.*

For example, suppose $\mathcal{X}$ is the Euclidean plane and A_n is the circle with radius n and center at the point $(0, n)$; it converges to the x-axis.



Further, consider the sequence of one-point sets $B_n = \{(n, 0)\}$ in the Euclidean plane. It converges to the empty set; indeed, for any point x we have $\operatorname{dist}_{B_n}(x) \to \infty$ as $n \to \infty$ and $\operatorname{dist}_{\varnothing}(x) = \infty$ for any x.

The following exercise is an extension of the so-called Blaschke selection theorem to our version of Hausdorff convergence.

1.13. Exercise. *Show that any sequence of closed sets in a proper metric space has a convergent subsequence in the sense of Hausdorff.*

J Hausdorff metric

1.14. Definition. *Let A and B be two non-empty compact subsets of a metric space $\mathcal{X}$. Then the Hausdorff distance between A and B is defined as*

$$|A - B|_{\mathrm{Haus}\,\mathcal{X}} := \sup_{x\in\mathcal{X}}\{\,|\mathrm{dist}_A(x) - \mathrm{dist}_B(x)|\,\}.$$

The following observation gives a useful reformulation of the definition:

1.15. Observation. *Suppose A and B be two compact subsets of a metric space $\mathcal{X}$. Then $|A - B|_{\mathrm{Haus}\,\mathcal{X}} < R$ if and only if and only if B lies in an R-neighborhood of A, and A lies in an R-neighborhood of B.*

The following exercise implies that Hausdorff convergence of compact subsets is the convergence in Hausdorff metric.

1.16. Exercise. *Let $A_1, A_2, \dots,$ and A_∞ be compact non-empty sets in a metric space $\mathcal{X}$. Show that $|A_n - A_\infty|_{\mathrm{Haus}\,\mathcal{X}} \to 0$ as $n\to\infty$ if and only if $A_n \to A_\infty$ in the sense of Hausdorff.*

K Gromov–Hausdorff convergence

Let $\mathcal{X}_1, \mathcal{X}_2, \dots,$ and $\mathcal{X}_\infty$ be a sequence of complete metric spaces. Suppose that there is a metric on the disjoint union

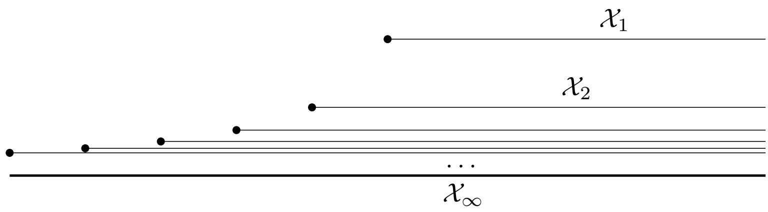
$$\boldsymbol{X} = \bigsqcup_{n\in\mathbb{N}\cup\{\infty\}} \mathcal{X}_n$$

that satisfies the following property:

1.17. Property. *The restriction of the metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric, and $\mathcal{X}_n \to \mathcal{X}_\infty$ as subsets in $\boldsymbol{X}$ in the sense of Hausdorff.*

In this case we say that the metric on $\boldsymbol{X}$ *defines* a convergence $\mathcal{X}_n \to \mathcal{X}_\infty$ in the sense of Gromov–Hausdorff. The metric on $\bigsqcup \mathcal{X}_n$ makes it possible to talk about limits of sequences $x_n \in \mathcal{X}_n$ as $n \to \infty$, as well as weak limits of a sequence of Borel measures μ_n on $\mathcal{X}_n$ and so on.

The limit space is not uniquely defined by the sequence. For example, if each space $\mathcal{X}_n$ in the sequence is isometric to the half-line, then its limit might be isometric to the half-line or the whole line. The



first convergence is evident and the second could be guessed from the diagram.

Note that any sequence of spaces has an empty space as its limit in some Gromov–Hausdorff convergence. Exercise 1.23 states that if the limit is non-empty and compact, then it is unique up to isometry.

1.18. Exercise. *Let $\mathcal{X}_1, \mathcal{X}_2, \dots$ be a sequence of geodesic metric spaces. Suppose $\mathcal{X}_n \to \mathcal{X}_\infty$ is a convergence in the sense of Gromov–Hausdorff. Assume $\mathcal{X}_\infty$ is proper, show that it is geodesic.*

Pointed convergence. Often the isometry class of the limit can be fixed by marking a point p_n in each space $\mathcal{X}_n$. We say that $(\mathcal{X}_n, p_n)$ converges to $(\mathcal{X}_\infty, p_\infty)$ if there is a metric on $\boldsymbol{X}$ as in 1.17 such that $p_n \to p_\infty$. This is called pointed Gromov–Hausdorff convergence. For example, the sequence $(\mathcal{X}_n, p_n) = (\mathbb{R}_{\geqslant 0}, 0)$ converges to $(\mathbb{R}_{\geqslant 0}, 0)$, while $(\mathcal{X}_n, p_n) = (\mathbb{R}_{\geqslant 0}, n)$ converges to $(\mathbb{R}, 0)$ as $n \to \infty$.

L Gromov–Hausdorff metric

In this section we cook up a metric space out of all compact non-empty metric spaces that defines Gromov–Hausdorff convergence. We want to define the metric on the set of *isometry classes* of compact metric spaces. Further, the term *metric space* might also stand for its *isometry class.*

The obtained metric is called the Gromov–Hausdorff metric; the corresponding metric space will be denoted by GH. This distance is defined as the maximal metric such that *the distance between subspaces in a metric space is not greater than the Hausdorff distance between them.* Here is a formal definition.

1.19. Definition. *The Gromov–Hausdorff distance $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}}$ between compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is defined by the following relation.*

Given $r > 0$, we have $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} < r$ if and only if there exists a metric space $\mathcal{W}$ and subspaces $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{W}$ that are isometric to $\mathcal{X}$ and $\mathcal{Y}$, respectively, such that $|\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}} < r$. (Here $|\mathcal{X}' -$

$\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}}$ *denotes the Hausdorff distance between sets* $\mathcal{X}'$ *and* $\mathcal{Y}'$ *in* $\mathcal{W}$.*)*

For the proof of the following statement we refer to [14] and [80].

1.20. Proposition. GH *is a complete metric space.*

Note that this means in particular that if X, Y are compact and $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} = 0$ then X and Y are isometric.

Gromov–Hausdorff convergence of compact spaces has particularly nice properties. From the technical point of view, they follow from the next statement, which we formulate as an exercise.

1.21. Exercise. *Let f be a distance noncontracting map from a compact metric space $\mathcal{K}$ to itself. Show that f is an isometry; that is, it is a distance-preserving bijection.*

For two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $\mathcal{X} \leqslant \mathcal{Y} + \varepsilon$ if there is a map $f\colon \mathcal{X} \to \mathcal{Y}$ such that

$$|x - x'|_{\mathcal{X}} \leqslant |f(x) - f(x')|_{\mathcal{Y}} + \varepsilon$$

for any $x, x' \in \mathcal{X}$.

1.22. Exercise. *Let $\mathcal{X}_1, \mathcal{X}_2, \dots,$ and $\mathcal{X}_\infty$ are compact metric spaces. Show that there is a Gromov–Hausdorff convergence $\mathcal{X}_n \to \mathcal{X}_\infty$ if and only if for some sequence $\varepsilon_n \to 0$, we have*

$$\mathcal{X}_\infty \leqslant \mathcal{X}_n + \varepsilon_n \quad \textit{and} \quad \mathcal{X}_n \leqslant \mathcal{X}_\infty + \varepsilon_n.$$

1.23. Exercise. *Let $\mathcal{X}_1, \mathcal{X}_2, \dots$ be a sequence of metric spaces. Suppose $\mathcal{X}_\infty$ and $\mathcal{X}'_\infty$ are non-empty limit spaces of $\mathcal{X}_n$ for some Gromov–Hausdorff convergences. Assume $\mathcal{X}_\infty$ is compact, show that it is isometric to $\mathcal{X}'_\infty$.*

M Almost isometries

1.24. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called an ε-isometry if the following two conditions hold:*

(a) $f(\mathcal{X})$ is an ε-net in $\mathcal{Y}$; that is, for any $y \in \mathcal{Y}$ there is $x \in \mathcal{X}$ such that $|f(x) - y|_{\mathcal{Y}} < \varepsilon$.

(b) $\big||f(x) - f(x')|_{\mathcal{Y}} - |x - x'|_{\mathcal{X}}\big| \leqslant \varepsilon$ for any $x, x' \in \mathcal{X}$.

When dealing with Gromov–Hausdorff convergence the following lemma is often useful as it allows to bypass constructing explicit metrics on the disjoint unions of $\mathcal{X}_1, \mathcal{X}_2, \dots$, and $\mathcal{X}_\infty$

1.25. Lemma. *Let $\mathcal{X}_1, \mathcal{X}_2, \dots$, and $\mathcal{X}_\infty$ be complete metric spaces, and let $\varepsilon_n \to 0+$ as $n \to \infty$. Suppose that either*

(a) for each n there is an ε_n-isometry $f_n \colon \mathcal{X}_n \to \mathcal{X}_\infty$, or

(b) for each n there is an ε_n-isometry $h_n \colon \mathcal{X}_\infty \to \mathcal{X}_n$.

Then there is a Gromov–Hausdorff convergence $\mathcal{X}_n \to \mathcal{X}_\infty$.

Furthermore, a partial converse also holds.

(c) Suppose we have a Gromov–Hausdorff convergence $\mathcal{X}_n \to \mathcal{X}_\infty$ and $\mathcal{X}_\infty$ is compact. Then there exist $\varepsilon_n \to 0+$ as $n \to \infty$ and ε_n-isometris $f_n \colon \mathcal{X}_n \to \mathcal{X}_\infty$ (and $h_n \colon \mathcal{X}_\infty \to \mathcal{X}_n$) such that $x_n \in \mathcal{X}_n$ converges to $x_\infty \in \mathcal{X}_\infty$ with respect to the convergence $\mathcal{X}_n \to \mathcal{X}_\infty$ if and only if $f_n(x_n) \to x_\infty$ (respectively, $|h_n(x_\infty) - - x_n|_{\mathcal{X}_n} \to 0$) as $n \to \infty$.

Proof. To prove part *(a)* let us construct a common space $\boldsymbol{X}$ for the spaces $\mathcal{X}_1, \mathcal{X}_2, \dots$, and $\mathcal{X}_\infty$ by taking the metric ρ on the disjoint union $\mathcal{X}_\infty \sqcup \mathcal{X}_1 \sqcup \mathcal{X}_2 \sqcup \dots$ that is defined the following way:

$$\begin{aligned}
|x_n - y_n|_{\boldsymbol{X}} &= |x_n - y_n|_{\mathcal{X}_n},\\
|x_\infty - y_\infty|_{\boldsymbol{X}} &= |x_\infty - y_\infty|_{\mathcal{X}_\infty},\\
|x_n - x_\infty|_{\boldsymbol{X}} &= \inf\left\{\, |x_n - y_n|_{\mathcal{X}_n} + \varepsilon_n + |x_\infty - f(y_n)|_{\mathcal{X}_\infty} \;:\; y_n \in \mathcal{X}_n \,\right\},\\
|x_n - x_m|_{\boldsymbol{X}} &= \inf\left\{\, |x_n - y_\infty|_{\boldsymbol{X}} + |x_m - y_\infty|_{\boldsymbol{X}} \;:\; y_\infty \in \mathcal{X}_\infty \,\right\},
\end{aligned}$$

where we assume that $x_\infty, y_\infty \in \mathcal{X}_\infty$, and $x_n, y_n \in \mathcal{X}_n$ for each n. It remains to observe that this indeed defines a metric on $\boldsymbol{X}$, and $\mathcal{X}_n \to \mathcal{X}_\infty$ in the sense of Hausdorff.

The proof of the second part is analogous; one only needs to change one line in the definition of the metric to the following:

$$|x_n - x_\infty|_{\boldsymbol{X}} = \inf\left\{\, |x_n - h(y_\infty)|_{\mathcal{X}_n} + \varepsilon_n + |x_\infty - y_\infty|_{\mathcal{X}_\infty} \;:\; y_\infty \in \mathcal{X}_\infty \,\right\}.$$

We leave part *(c)* as an exercise. □

Lemma 1.25 has a natural analogue for pointed convergence. For simplicity we only state part *(a)* of the lemma. Parts *(b)* and *(c)* can be rephrased similarly; in *(c)* we have to assume that the space is proper.

1.26. Lemma. *Let $(\mathcal{X}_1, p_1), (\mathcal{X}_2, p_2), \dots$, let $(\mathcal{X}_\infty, p_\infty)$ be pointed metric spaces, and let $\varepsilon(n, R) \to 0+$ as $n \to \infty$ for any fixed $R > 0$. Suppose that for each n there is a map $f_n \colon \mathcal{X}_n \to \mathcal{X}_\infty$ such that* *(a)* $f_n(p_n) \to p_\infty$

(b) $\big||f_n(x) - f_n(x')|_{\mathcal{X}_\infty} - |x - x'|_{\mathcal{X}_n}\big| \leqslant \varepsilon(n, R)$ *for any* $x, x' \in$ $\in \mathrm{B}(p_n, R)$.

(c) *For any* $x \in \mathrm{B}(p_\infty, R)$ *there is* $x_n \in \mathrm{B}(p_n, R)$ *such that* $|x -$ $- f_n(x_n)| \leqslant \varepsilon(n, R)$

Then there is a pointed Gromov–Hausdorff convergence $(\mathcal{X}_n, p_n) \to$ $\to (\mathcal{X}_\infty, p_\infty)$.

The proofs of 1.26 and 1.25 are analogous; we leave the former to the reader.

N Remarks

In principle, our definition of Gromov–Hausdorff distance works for complete metric spaces that are not necessarily compact. However, according to the following exercise, it only defines a semimetric; that is, zero Gromov–Hausdorff distance does not imply that the spaces are isometric. For that reason it is not in use.

1.27. Exercise.

(a) *Construct two nonisometric proper (noncompact) metric spaces with vanishing Gromov–Hausdorff distance.*

(b) *Construct two nonisometric complete geodesic metric spaces of bounded diameter with vanishing Gromov–Hausdorff distance.*

Lecture 2

Definitions

In this lecture we give several equivalent definitions of Alexandrov space. Alexandrov's lemma works a the main tool.

A Four-point comparison

Recall that $\tilde\measuredangle(p\,{}^x_y)$ denotes the model angle at p; see page 9.

Let p, x, y, z be a quadruple of points in a metric space. If the inequality

$$\textbf{❶}\qquad \tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^y_z)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^z_x)_{\mathbb{E}^2} \leqslant 2\cdot\pi$$

holds, then we say that the quadruple meets $\mathbb{E}^2$-comparison.

If instead of $\mathbb{E}^2$, we use $\mathbb{S}^2$ or $\mathbb{H}^2$, then we get the definition of $\mathbb{S}^2$- or $\mathbb{H}^2$-comparisons. Recall that $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2}$ and $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2}$ are defined if $p \neq x$, $p \neq y$, but for $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2}$ we require in addition that

$$|p-x| + |p-y| + |x-y| < 2\cdot\pi;$$

if this does not hold for one of the angles, then we assume that $\mathbb{S}^2$-comparison holds for this quadruple.

More generally, one may apply this definition to $\mathbb{M}^2(\kappa)$. This way we define $\mathbb{M}^2(\kappa)$-comparison for any real κ. However, if you see $\mathbb{M}^2(\kappa)$-comparison, it is safe to assume that $\kappa = -1$, 0, or 1 (applying rescaling, the $\mathbb{M}^2(\kappa)$-comparison can be reduced to these three cases).

2.1. Definition. *A metric space $\mathcal{X}$ has curvature $\geqslant \kappa$ in the sense of Alexandrov if $\mathbb{M}^2(\kappa)$-comparison holds for any quadruple of points in $\mathcal{X}$.*

While this definition can be applied to any metric space, we will use it mostly for geodesic space that are complete (and often compact or proper). If a complete geodesic space has curvature $\geqslant \kappa$ in the sense of Alexandrov, then it will be called an $\mathrm{ALEX}(\kappa)$ space; here $\mathrm{ALEX}(\kappa)$ is an adjective. An $\mathcal{X}$ is $\mathrm{ALEX}(\kappa)$ space for some κ, then it will be called an Alexandrov space.

It is common practice in Alexandrov geometry to write proofs for nonnegative curvature and leave the general curvature bound as an exercise. Sometime theses exercises are nontrivial; in this case we add a note. We often formulate statements of $\kappa = 0$ despite that it admits a straightforward generalization to arbitrary curvature bound.

2.2. Exercise. *Show that $\mathbb{E}^n$ is $\mathrm{ALEX}(0)$.*

2.3. Exercise. *Show that a metric space $\mathcal{X}$ has nonnegative curvature in the sense of Alexandrov if and only if for any quadruple of points $p, x_1, x_2, x_3 \in \mathcal{X}$ there is a quadruple of points $q, y_1, y_2, y_3 \in \mathbb{E}^3$ such that*

$$|p - x_i|_{\mathcal{X}} \geqslant |q - y_i|_{\mathbb{E}^2} \quad \textit{and} \quad |x_i - x_j|_{\mathcal{X}} \leqslant |y_i - y_j|_{\mathbb{E}^2}$$

for all i and j.

B Alexandrov's lemma

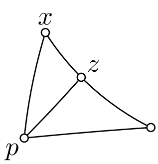
Recall that $[xy]$ denotes a geodesic from x to y; set

$$]xy] = [xy] \setminus \{x\}, \quad [xy[= [xy] \setminus \{y\}, \quad]xy[= [xy] \setminus \{x, y\}.$$

2.4. Lemma. *Let p, x, y, z be distinct points in a metric space such that $z \in]xy[$. Then the following expressions have the same sign:*

(a) $\tilde\measuredangle(x\,{}^p_y) - \tilde\measuredangle(x\,{}^p_z)$,

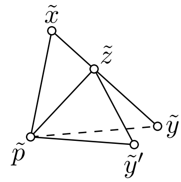
(b) $\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) - \pi$.



The same holds for the hyperbolic and spherical model angles, but in the latter case, one has to assume in addition that

$$|p - z| + |p - y| + |x - y| < 2{\cdot}\pi.$$

Proof. Consider the model triangle $[\tilde x\tilde p\tilde z] = \tilde\triangle(xpz)$. Take a point $\tilde y$ on the extension of $[\tilde x\tilde z]$ beyond $\tilde z$ so that $|\tilde x - \tilde y| = |x - y|$ (and therefore $|\tilde x - \tilde z| = |x - z|$).



Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:
(i) $\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(x\,{}^{p}_{y})$,
(ii) $|\tilde p - \tilde y| - |p - y|$,
(iii) $\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(z\,{}^{p}_{y})$.
Since

$$\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] = \measuredangle[\tilde x\,{}^{\tilde p}_{\tilde z}] = \tilde\measuredangle(x\,{}^{p}_{z})$$

and

$$\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] = \pi - \measuredangle[\tilde z\,{}^{\tilde x}_{\tilde p}] = \pi - \tilde\measuredangle(z\,{}^{x}_{p}),$$

the statement follows.

The spherical and hyperbolic cases can be proved in the same way.

□

2.5. Exercise. *Assume p, x, y, z are as in Alexandrov's lemma. Show that*

$$\tilde\measuredangle(p\,{}^{x}_{y}) \geqslant \tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{z}_{y}),$$

with equality if and only if the expressions in (a) and (b) in 2.4 vanish.

Note that

$$p \in\,]xy[\quad\Longrightarrow\quad \tilde\measuredangle(p\,{}^{x}_{y}) = \pi.$$

Applying it with Alexandrov's lemma and $\mathbb{E}^2$-comparison, we get the following.

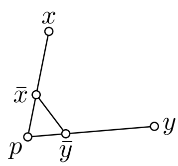
2.6. Claim. *If p, x, y, z are points in an* Alex(0) *space. Suppose $p \in\,]xy[$, then*

$$\tilde\measuredangle(x\,{}^{y}_{z}) \leqslant \tilde\measuredangle(x\,{}^{p}_{z}).$$

2.7. Exercise. *Let $[p\,{}^{x}_{y}]$ be a hinge in an* Alex(0) *space. Consider the function*

$$f\colon (|p - \bar x|, |p - \bar y|) \mapsto \tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x \in\,]px]$ and $\bar y \in\,]py]$. Show that f is nonincreasing in each argument.



Note that 2.7 implies the following.

2.8. Claim. *The angle measure of any hinge in an* Alex(0) *space, is at least as large as the corresponding model angle; that is,*

$$\measuredangle[p\,{}^{x}_{y}] \geqslant \tilde\measuredangle(p\,{}^{x}_{y})$$

for any hinge $[p\,{}^x_y]$ *in an* ALEX(0).

2.9. Exercise. *Let* $[p\,{}^x_y]$ *be a hinge in an* ALEX(0) *space. Suppose* $\measuredangle[p\,{}^x_y]=0$*; show that* $[px]\subset[py]$ *or* $[py]\subset[px]$.

Conclude that geodesic in ALEX(0) *space cannot bifurcate; that is, if two geodesics* $[px]$ *and* $[py]$ *share a nontrivial arc with an end at* p*, then* $[px]\subset[py]$ *or* $[py]\subset[px]$.

2.10. Exercise. *Let* $[xy]$ *be a geodesic in an* ALEX(0) *space. Suppose* $z\in\,]xy[$ *show that there is a unique geodesic* $[xz]$ *and* $[xz]\subset[xy]$.

Recall that adjacent hinges are defined in 1.10.

2.11. Exercise. *Let* $[p\,{}^x_z]$ *and* $[p\,{}^y_z]$ *be adjacent hinges in an* ALEX(0) *space. Show that*

$$\measuredangle[p\,{}^x_z]+\measuredangle[p\,{}^y_z]=\pi.$$

2.12. Exercise. *Let* $\mathcal{A}$ *be an* ALEX(0) *space. Show that*

$$\tilde\measuredangle(x\,{}^y_p)=\tilde\measuredangle(x\,{}^v_p)\quad\Longleftrightarrow\quad\tilde\measuredangle(x\,{}^y_p)=\tilde\measuredangle(x\,{}^w_p)$$

for any points p,x,y,v,w *in* $\mathcal{A}$ *such that* $v,w\in\,]xy[$.

2.13. Exercise. *Let* $\mathcal{A}$ *be an* ALEX(0) *space. Suppose hinges* $[x_n\,{}^{y_n}_{z_n}]$ *in* $\mathcal{A}$ *converge to the hinge* $[x_\infty\,{}^{y_\infty}_{z_\infty}]$*; that is, geodesics* $[x_ny_n]$ *and* $[x_nz_n]$ *converge to the geodesics* $[x_\infty y_\infty]$ *and* $[x_\infty z_\infty]$ *in the Hausdorff sense. Show that*

$$\varliminf_{n\to\infty}\measuredangle[x_n\,{}^{y_n}_{z_n}]\geqslant\measuredangle[x_\infty\,{}^{y_\infty}_{z_\infty}].$$

C Hinge comparison

Let $[p\,{}^x_y]$ be a hinge in an ALEX(0) space $\mathcal{A}$. By 2.9, the angle measure $\measuredangle[p\,{}^x_y]$ is defined and

$$\measuredangle[p\,{}^x_y]\geqslant\tilde\measuredangle(p\,{}^x_y).$$

Further, according to 2.11, we have

$$\measuredangle[p\,{}^x_z]+\measuredangle[p\,{}^y_z]=\pi$$

for adjacent hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ in $\mathcal{A}$.

The following theorem implies that a geodesic space has nonnegative curvature in the sense of Alexandrov if the above conditions hold for all its hinges.

2.14. Theorem. *A complete geodesic space* $\mathcal{A}$ *is* ALEX(0) *if the following conditions hold.*

(a) *For any hinge $[x\,{}^p_y]$ in $\mathcal{A}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and*

$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle(x\,{}^p_y).$$

(b) *For any two adjacent hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ in $\mathcal{A}$, we have*

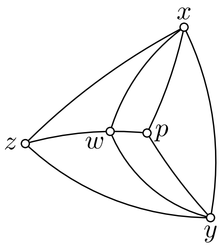
$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \leqslant \pi.$$

Proof. Consider a point $w \in \,]pz[$ close to p. From *(b)*, it follows that

$$\measuredangle[w\,{}^x_z] + \measuredangle[w\,{}^x_p] \leqslant \pi \quad \text{and} \quad \measuredangle[w\,{}^y_z] + \measuredangle[w\,{}^y_p] \leqslant \pi.$$

Since $\measuredangle[w\,{}^x_y] \leqslant \measuredangle[w\,{}^x_p] + \measuredangle[w\,{}^y_p]$ (see 1.9), we get

$$\measuredangle[w\,{}^x_z] + \measuredangle[w\,{}^y_z] + \measuredangle[w\,{}^x_y] \leqslant 2{\cdot}\pi.$$



Applying *(a)*,

$$\tilde\measuredangle(w\,{}^x_z) + \tilde\measuredangle(w\,{}^y_z) + \tilde\measuredangle(w\,{}^x_y) \leqslant 2{\cdot}\pi.$$

Passing to the limits as $w \to p$, we have

$$\tilde\measuredangle(p\,{}^x_z) + \tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^x_y) \leqslant 2{\cdot}\pi.$$

□

D Equivalent conditions

The following theorem summarizes 2.6, 2.8, 2.11, and 2.14.

2.15. Theorem. *Let $\mathcal{A}$ be a complete geodesic space. Then the following conditions are equivalent.*

(a) *$\mathcal{A}$ is* Alex(0).

(b) *(adjacent angle comparison)*

$$\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) \leqslant \pi$$

for any geodesic $[xy]$ and point $z \in \,]xy[$, $z \neq p$ in $\mathcal{A}$.

(c) *(point-on-side comparison)*

$$\tilde\measuredangle(x\,{}^p_y) \leqslant \tilde\measuredangle(x\,{}^p_z)$$

for any geodesic $[xy]$ and $z \in \,]xy[$ in $\mathcal{A}$.

(d) *(hinge comparison) the angle* $\measuredangle[x\,{}^p_y]$ *is defined for any hinge* $[x\,{}^p_y]$ *in* $\mathcal{A}$. *Moreover,*

$$\measuredangle[x\,{}^p_y] \geqslant \tilde{\measuredangle}(x\,{}^p_y)$$

for any hinge $[x\,{}^p_y]$, *and*

$$\measuredangle[z\,{}^p_y] + \measuredangle[z\,{}^p_x] \leqslant \pi$$

for any adjacent hinges $[z\,{}^p_y]$ *and* $[z\,{}^p_x]$.

Moreover, the implications (a)⇒(b)⇒(c)⇒(d) hold in any space, not necessarily geodesic.

2.16. Advanced Exercise. *Construct a complete geodesic space* $\mathcal{X}$ *that is not* Alex(0), *but satisfies the following weaker version of the adjacent angle comparison 2.15b.*

For any three points $p, x, y \in \mathcal{X}$ *there is a geodesic* $[xy]$ *such that for any* $z \in\,]xy[$

$$\tilde{\measuredangle}(z\,{}^p_x) + \tilde{\measuredangle}(z\,{}^p_y) \leqslant \pi.$$

2.17. Exercise. *Let* $\mathcal{W}$ *be* $\mathbb{R}^2$ *with the metric induced by a norm. Show that if* $\mathcal{W}$ *is* Alex(0), *then* $\mathcal{W}$ *is isometric to the Euclidean plane* $\mathbb{E}^2$.

E Function comparison

Real-to-real functions. Choose $\lambda \in \mathbb{R}$. Let $s\colon \mathbb{I} \to \mathbb{R}$ be a locally Lipschitz function defined on an interval $\mathbb{I}$. The following statement are equivalent; if one (and therefore any) of them holds for s, then we say that s is λ-concave.

- ◇ We have inequality $s'' \leqslant \lambda$, where the second derivative s'' is understood in the sense of distributions.
- ◇ The function $t \mapsto s(t) - \lambda\cdot\frac{t^2}{2}$ is concave.
- ◇ The Jensen inequality

$$s(a\cdot t_0 + (1-a)\cdot t_1) \geqslant a\cdot s(t_0) + (1-a)\cdot s(t_1) + \tfrac{\lambda}{2}\cdot a\cdot(1-a)\cdot(t_1 - t_0)^2$$

  holds for any $t_0, t_1 \in \mathbb{I}$ and $a \in [0,1]$.
- ◇ for any $t_0 \in \mathbb{I}$ there is a quadratic polynomial $\ell = \frac{\lambda}{2}\cdot t^2 + a\cdot t + b$ (it is called a barrier) that supports (locally) s at t_0 from above; that is, $\ell(t_0) = s(t_0)$ and $\ell(t) \geqslant s(t)$ for any t (in a neighborhood of t_0)

To prove equivalence, approximate f by smooth functions taking a convolutions $f_n = f * k_n$ for a suitable sequence of kernels k_n. Note that all the conditions are equivalent for f_n; passing to the limit we get the same for f.

We will also use that *λ-concave functions are one-sided differentiable.*

Functions on metric spaces. A function on a metric space $\mathcal{A}$ will usually mean a *locally Lipschitz real-valued function defined on an open subset of $\mathcal{A}$.* The domain of a function f will be denoted by $\operatorname{Dom} f$.

We say that f is λ-concave (briefly $f'' \leqslant \lambda$) if for any unit-speed geodesic $\gamma\colon \mathbb{I} \to \operatorname{Dom} f$ the real-to-real function $t \mapsto f \circ \gamma(t)$ is λ-concave.

The following proposition is simple but conceptual — it reformulates a global geometric condition into an infinitesimal condition on distance functions.

2.18. Proposition. *A complete geodesic space $\mathcal{A}$ in* $\mathrm{A{\scriptstyle LEX}}(0)$ *if and only if $f'' \leqslant 1$ for any function f of the form*

$$f\colon x \mapsto \tfrac{1}{2}\cdot|p - x|^2.$$

Proof. Choose a unit-speed geodesic γ in $\mathcal{A}$ and two points $x = \gamma(t_0)$, $y = \gamma(t_1)$ for some $t_0 < t_1$. Consider the model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$. Let $\tilde{\gamma}\colon [t_0, t_1] \to \mathbb{E}^2$ be the unit-speed parametrization of $[\tilde{x}\tilde{y}]$ from $\tilde{x}$ to $\tilde{y}$.

Set

$$\tilde{r}(t) := |\tilde{p} - \tilde{\gamma}(t)|, \qquad\qquad r(t) := |p - \gamma(t)|.$$

Clearly, $\tilde{r}(t_0) = r(t_0)$ and $\tilde{r}(t_1) = r(t_1)$. Note that the point-on-side comparison (2.15*c*) is equivalent to

❶ $$t_0 \leqslant t \leqslant t_1 \qquad \Longrightarrow \qquad \tilde{r}(t) \leqslant r(t)$$

for any γ and $t_0 < t_1$.

Observe that Jensen's inequality for the function h is equivalent to ❶. Hence the proposition follows. □

F Remarks

Our 4-point comparison in Section 2A is closely related to the so-called CAT comparison, which defines *upper* curvature bound in the sense of Alexandrov; this is the subject of our previous invitation [3].

In both comparisons we check certain conditions on the 6 distances between every pair of points in 4-point sets. Michael Gromov [31, Section 1.19_+] suggested considering other conditions of that type for n-point subsets; see [25, 32, 51–56, 78, 93] for the development of this idea.

We have chosen complete *geodesic* spaces with curvature at least κ as the main object of study (the $\mathrm{ALEX}(\kappa)$ spaces). Instead of the *geodesic* condition, we could assume that they are *length* spaces. This option is more natural and general, but many statements can be reduced to the geodesic case. In particular, suppose $\mathcal{A}$ is a complete length space with curvature $\geqslant \kappa$, then $\mathcal{A}$ *can be isometrically embedded into an* $\mathrm{ALEX}(\kappa)$ *space* — the ultrapower of $\mathcal{A}$; see [4, 4.11+8.4]. Also, by Plaut's theorem, any point p in $\mathcal{A}$ can be connected by geodesics to *most* of points in $\mathcal{A}$ [4, 8.11]; compare to 6.18c.

All the discussed statements admit natural generalizations to spaces with curvature $\geqslant \kappa$ in the sense of Alexandrov. The proof are nearly the same, but the formulas are getting more complicated.

For example, the function comparison for $\mathrm{ALEX}(-1)$ spaces states that $f'' \leqslant f$ for any function of the type $f = \cosh \circ \mathrm{dist}_p$. (The inequality used here will be defined in Section 4C.)

Similarly, the function comparison for $\mathrm{ALEX}(1)$ states that for any point p, we have $f'' \leqslant -f$ for the function $f = -\cos \circ \mathrm{dist}_p$ defined in $\mathrm{B}(p, \pi)$. The geometric meaning of these inequalities remains the same: *distance functions are more concave than distance functions in* $\mathbb{M}^2(\kappa)$.

Lecture 3

Globalization

Globalization theorem states that locally Alexandrov space is globally Alexandrov. We start with the simplest meaningful case of this theorem and indicate a way to extend.

A Globalization

A complete geodesic metric space $\mathcal{A}$ is locally $\mathrm{ALEX}(0)$ if any point $p \in \mathcal{A}$ admits a neighborhood $U \ni p$ such that the $\mathbb{E}^2$-comparison holds for any quadruple of points in U.

3.1. Globalization theorem. *Any compact locally* $\mathrm{ALEX}(0)$ *space is* $\mathrm{ALEX}(0)$.

Proof modulo the key lemma. Note that condition 2.14*b* holds in $\mathcal{A}$ (the proof is the same). It remains to check 2.14*a*; that is,

$$❶ \qquad \measuredangle[x\,{}^{p}_{y}] \geqslant \tilde\measuredangle(x\,{}^{p}_{y})$$

for any hinge $[x\,{}^{p}_{y}]$ in $\mathcal{A}$.

First note that ❶ holds for hinges in a small neighborhood of any point; this can be proved the same way as 2.8 and 2.11, applying the local version of the $\mathbb{E}^2$-comparison. Since $\mathcal{A}$ is compact, there is $\varepsilon > 0$ such that ❶ holds if $|x - p| + |p - y| < \varepsilon$. Applying the key lemma several times we get that ❶ holds for any given hinge. □

3.2. Key lemma. *Let* $\mathcal{A}$ *be locally* $\mathrm{ALEX}(0)$. *Assume that the comparison*

$$\measuredangle[x\,{}^{p}_{q}] \geqslant \tilde\measuredangle(x\,{}^{p}_{q})$$

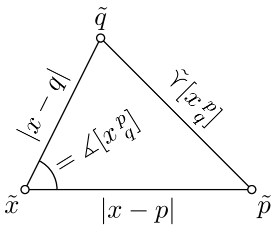
holds for any hinge $[x\,{}^{p}_{q}]$ *with* $|x-y|+|x-q|<\tfrac23\cdot\ell$. *Then the comparison*

$$\measuredangle[x\,{}^{p}_{q}]\geqslant\tilde\measuredangle(x\,{}^{p}_{q})$$

holds for any hinge $[x\,{}^{p}_{q}]$ *with* $|x-p|+|x-q|<\ell$.

Let $[x\,{}^{p}_{q}]$ be a hinge in $\mathcal{A}$. Denote by $\tilde\Upsilon[x\,{}^{p}_{q}]$ its model side; this is the opposite side in a flat triangle with the same angle and two adjacent sides as in $[x\,{}^{p}_{q}]$.

More precisely, consider the model hinge $[\tilde x\,{}^{\tilde p}_{\tilde q}]$ in $\mathbb{E}^2$ that is defined by



$$\begin{aligned}\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde q}]_{\mathbb{E}^2}&=\measuredangle[x\,{}^{p}_{q}]_{\mathcal{A}},\\ |\tilde x-\tilde p|_{\mathbb{E}^2}&=|x-p|_{\mathcal{A}},\\ |\tilde x-\tilde q|_{\mathbb{E}^2}&=|x-q|_{\mathcal{A}};\end{aligned}$$

then

$$\tilde\Upsilon[x\,{}^{p}_{q}]_{\mathcal{A}}:=|\tilde p-\tilde q|_{\mathbb{E}^2}.$$

Note that

$$\tilde\Upsilon[x\,{}^{p}_{q}]\geqslant|p-q|\qquad\Longleftrightarrow\qquad\measuredangle[x\,{}^{p}_{q}]\geqslant\tilde\measuredangle(x\,{}^{p}_{q}).$$

We will use it in the following proof.

Proof. It is sufficient to prove the inequality

❷ $$\tilde\Upsilon[x\,{}^{p}_{q}]\geqslant|p-q|$$

for any hinge $[x\,{}^{p}_{q}]$ with $|x-p|+|x-q|<\ell$.

Consider a hinge $[x\,{}^{p}_{q}]$ such that

$$\tfrac23\cdot\ell\leqslant|p-x|+|x-q|<\ell.$$

First, let us construct a new hinge $[x'\,{}^{p}_{q}]$ with

❸ $$|p-x|+|x-q|\geqslant|p-x'|+|x'-q|,$$

such that

❹ $$\tilde\Upsilon[x\,{}^{p}_{q}]\geqslant\tilde\Upsilon[x'\,{}^{p}_{q}].$$

Construction. Assume $|x-q|\geqslant|x-p|$; otherwise, switch the roles of p and q. Take $x'\in[xq]$ such that

❺ $$|p-x|+3\cdot|x-x'|=\tfrac23\cdot\ell.$$

Choose a geodesic $[x'p]$ and consider the hinge $[x'\,{}^{p}_{q}]$ formed by $[x'p]$ and $[x'q] \subset [xq]$. The triangle inequality implies ❸. Further, note that

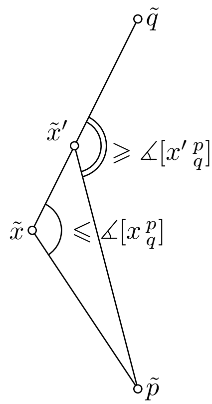
$$|p-x| + |x-x'| < \tfrac{2}{3}\cdot\ell, \qquad |p-x'| + |x'-x| < \tfrac{2}{3}\cdot\ell.$$

In particular,

❻ $$\measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde\measuredangle(x\,{}^{p}_{x'}) \quad\text{and}\quad \measuredangle[x'\,{}^{p}_{x}] \geqslant \tilde\measuredangle(x'\,{}^{p}_{x}).$$

Now, let $[\tilde x\tilde x'\tilde p] = \tilde\triangle(xx'p)$. Take $\tilde q$ on the extension of $[\tilde x\tilde x']$ beyond x' such that $|\tilde x - \tilde q| = |x-q|$ (and therefore $|\tilde x' - \tilde q| = |x'-q|$). By ❻,

$$\measuredangle[x\,{}^{p}_{q}] = \measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde\measuredangle(x\,{}^{p}_{x'}) \quad\Rightarrow\quad \tilde\Upsilon[x\,{}^{q}_{p}] \geqslant |\tilde p - \tilde q|.$$

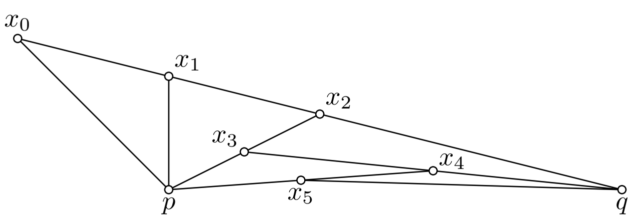


Hence

$$\begin{aligned}\measuredangle[\tilde x'\,{}^{\tilde p}_{\tilde q}] &= \pi - \tilde\measuredangle(x'\,{}^{p}_{x}) \geqslant \\ &\geqslant \pi - \measuredangle[x'\,{}^{p}_{x}] = \\ &= \measuredangle[x'\,{}^{p}_{q}],\end{aligned}$$

and ❹ follows.

Let us continue the proof. Set $x_0 = x$. Let us apply inductively the above construction to get a sequence of hinges $[x_n\,{}^{p}_{q}]$ with $x_{n+1} = x_n'$. From ❹, we have that the sequence $s_n = \tilde\Upsilon[x_n\,{}^{p}_{q}]$ is nonincreasing.



The sequence might terminate at some n only if $|p-x_n|+|x_n-q| < \tfrac{2}{3}\cdot\ell$. In this case, by the assumptions of the lemma, $\tilde\Upsilon[x_n\,{}^{p}_{q}] \geqslant |p-q|$. Since the sequence s_n is nonincreasing, inequality ❷ follows.

Otherwise, the sequence $r_n = |p-x_n| + |x_n - q|$ is nonincreasing, and $r_n \geqslant \tfrac{2}{3}\cdot\ell$ for all n. Note that by construction, the distances $|x_n - x_{n+1}|$, $|x_n - p|$, and $|x_n - q|$ are bounded away from zero for all large n. Indeed, since on each step, we move x_n toward to the point p or q that is further away, the distances $|x_n - p|$ and $|x_n - q|$ become about the same. Namely, by ❺, we have that $|p-x_n| - |x_n - q| \leqslant \tfrac{2}{9}\cdot\ell$

for all large n. Since $|p-x_n|+|x_n-q|\geqslant\frac{2}{3}\cdot\ell$, we have $|x_n-p|\geqslant\frac{\ell}{100}$ and $|x_n-q|\geqslant\frac{\ell}{100}$. Further, since $r_n\geqslant\frac{2}{3}\cdot\ell$, ❺ implies that $|x_n-x_{n+1}|>\frac{\ell}{100}$.

Since the sequence r_n is nonincreasing, it converges. In particular, $r_n-r_{n+1}\to 0$ as $n\to\infty$. It follows that $\tilde\measuredangle(x_n\,{}^{p_n}_{x_{n+1}})\to\pi$, where $p_n=p$ if $x_{n+1}\in[x_nq]$, and otherwise $p_n=q$. Since $\measuredangle[x_n\,{}^{p_n}_{x_{n+1}}]\geqslant\tilde\measuredangle(x_n\,{}^{p_n}_{x_{n+1}})$, we have $\measuredangle[x_n\,{}^{p_n}_{x_{n+1}}]\to\pi$ as $n\to\infty$.

It follows that

$$r_n-s_n=|p-x_n|+|x_n-q|-\tilde\Upsilon[x_n\,{}^p_q]\to 0.$$

Together with the triangle inequality

$$|p-x_n|+|x_n-q|\geqslant|p-q|$$

this yields

$$\lim_{n\to\infty}\tilde\Upsilon[x_n\,{}^p_q]\geqslant|p-q|.$$

Finally, the monotonicity of the sequence $s_n=\tilde\Upsilon[x_n\,{}^p_q]$ implies ❷. □

B General case

The globalization theorem can be generalized to any curvature bound κ. The case $\kappa\leqslant 0$ is proved in the same way, but the case $\kappa>0$ requires modifications.

The compactness condition in our version of the theorem can be traded for completeness. The proof uses the following statement where $r(x)$ measures the size of a neighborhood of x where the comparison holds.

3.3. Exercise. *Let $\mathcal{X}$ be a complete metric space. Suppose $r\colon\mathcal{X}\to\mathbb{R}$ is a positive continuous function. Show that for any $\varepsilon>0$ there is a point $p\in\mathcal{X}$ such that*

$$r(x)>(1-\varepsilon)\cdot r(p)$$

for any $x\in\overline{\mathrm{B}}[p,\frac{1}{\varepsilon}\cdot r(p)]$.

This implies the following general version of the globalization theorem.

3.4. Theorem. *Any locally* $\mathrm{A{\scriptsize LEX}}(\kappa)$ *length space is* $\mathrm{A{\scriptsize LEX}}(\kappa)$*.*

By 1.6, we have

$$\tilde\measuredangle(x\,{}^y_z)_{\mathbb{M}^2(\kappa)}\leqslant\tilde\measuredangle(x\,{}^y_z)_{\mathbb{M}^2(\mathrm{K})}$$

if $\kappa \leqslant \mathrm{K}$ and the right-hand side is defined. It follows that a $\mathrm{A\scriptstyle LEX}(\mathrm{K})$ space is *locally* $\mathrm{A\scriptstyle LEX}(\kappa)$. Therefore, the globalization theorem implies the following.

3.5. Claim. *If* $\mathrm{K} > \kappa$, *then any* $\mathrm{A\scriptstyle LEX}(\mathrm{K})$ *space is* $\mathrm{A\scriptstyle LEX}(\kappa)$.

In other words the expression *curvature bounded below by* κ makes sense for geodesic spaces. However, by the following exercise, it does not make much sense in general.

3.6. Exercise. *Let $\mathcal{X}$ be the set $\{p, x_1, x_2, x_3\}$ with the metric defined by*

$$|p - x_i| = \pi, \quad |x_i - x_j| = 2{\cdot}\pi$$

for all $i \neq j$. Show that $\mathcal{X}$ has curvature $\geqslant 1$, but does not have curvature $\geqslant 0$.

3.7. Exercise. *Let p and q be points in an* $\mathrm{A\scriptstyle LEX}(1)$ *space $\mathcal{A}$. Suppose $|p-q| > \pi$. Denote by m the midpoint of $[pq]$. Show that for any hinge $[m\,{}^x_p]$ we have either $\measuredangle[m\,{}^x_p] = 0$ or $\measuredangle[m\,{}^x_p] = \pi$.*

Conclude that $\mathcal{A}$ is isometric to a line interval or a circle.

3.8. Exercise. *Suppose $\mathcal{A}$ is an* $\mathrm{A\scriptstyle LEX}(1)$ *and* $\operatorname{diam}\mathcal{A} \leqslant \pi$. *Show that*

$$|x - y| + |y - z| + |z - x| \leqslant 2{\cdot}\pi$$

for any triple of points $x, y, z \in \mathcal{A}$.

C Remarks

The following question about 2.14*a* was stated in [14, footnote in 4.1.5] but this is a long-standing open problem (possibly dating back to Alexandrov).

3.9. Open question. *Let $\mathcal{A}$ be a complete geodesic space (you can also assume that $\mathcal{A}$ is homeomorphic to $\mathbb{S}^2$ or $\mathbb{R}^2$) such that for any hinge $[x\,{}^p_y]$ in $\mathcal{A}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and*

$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle(x\,{}^p_y).$$

Is it true that $\mathcal{A}$ is an Alexandrov space?

The globalization theorem is also known as the *generalized Toponogov theorem.* Its two-dimensional case was proved by Paolo Pizzetti [81]; later it was reproved independently by Alexandr Alexandrov [8]. Victor Toponogov [92] proved it for Riemannian manifolds of all dimensions. For Alexandrov spaces of all dimensions, the theorem first appears in the paper of Michael Gromov, Yuriy Burago, and Grigory Perelman [15]. Their statement is slightly more general than 3.4; it is for complete length spaces. Another version for noncomplete, but geodesic spaces was proved by the second author [73].

We took the proof from our book [4], but reduced generality to compact nonnegatively curved spaces. This proof is based on simplifications obtained by Conrad Plaut [82] and Dmitry Burago, Yuriy Burago, and Sergei Ivanov [14]. The same proof was rediscovered independently by Urs Lang and Viktor Schroeder [48]. Another simplified argument was found by Katsuhiro Shiohama [87].

Lecture 4

Calculus

This lecture defines several notions related to the first-order derivatives in Alexandrov spaces; it includes space of directions, tangent space, differential, and gradient.

A Space of directions

Let $\mathcal{A}$ be an Alexandrov space. By 2.8, the angle measure of any hinge in is defined. Given $p \in \mathcal{A}$, consider the set $\mathfrak{S}_p$ of all nontrivial unit-speed geodesics starting at p. By 1.9, the triangle inequality holds for $\measuredangle$ on $\mathfrak{S}_p$, that is, $(\mathfrak{S}_p, \measuredangle)$ forms a semimetric space; that is, $\measuredangle$ behaves like a metric, but might vanish for distinct directions.

The metric space corresponding to $(\mathfrak{S}_p, \measuredangle)$ is called the space of geodesic directions at p, denoted by Σ'_p or $\Sigma'_p\mathcal{A}$. The elements of Σ'_p are called geodesic directions at p. Each geodesic direction is formed by an equivalence class of geodesics starting from p for the equivalence relation

$$[px] \sim [py] \qquad \Longleftrightarrow \qquad \measuredangle[p\,{}^x_y] = 0;$$

the direction of $[px]$ is denoted by $\uparrow_{[px]}$. By 2.9,

$$[px] \sim [py] \qquad \Longleftrightarrow \qquad [px] \subset [py] \quad \text{or} \quad [px] \supset [py].$$

The completion of Σ'_p is called the space of directions at p and is denoted by Σ_p or $\Sigma_p\mathcal{A}$. The elements of Σ_p are called directions at p.

4.1. Exercise. *Let $\mathcal{A}$ be an Alexandrov space. Assume that a sequence of geodesics $[px_n]$ converge to a geodesic $[px_\infty]$ in the sense of Hausdorff, and $x_\infty \neq p$. Suppose Σ_p is compact. Show that $\uparrow_{[px_n]} \to$ $\to \uparrow_{[px_\infty]}$ as $n \to \infty$.*

B Tangent space

The Euclidean cone $\mathcal{V} = \mathrm{Cone}\,\mathcal{X}$ over a metric space $\mathcal{X}$ is defined as the metric space whose underlying set consists of equivalence classes in $[0,\infty)\times\mathcal{X}$ with the equivalence relation "$\sim$" given by $(0,p)\sim(0,q)$ for any points $p,q\in\mathcal{X}$, and whose metric is given by the cosine rule

$$|(s,p)-(t,q)|_{\mathcal{V}} = \sqrt{s^2+t^2-2{\cdot}s{\cdot}t{\cdot}\cos\theta},$$

where $\theta=\min\{\pi,|p-q|_{\mathcal{X}}\}$.

The leading example is

$$\mathrm{Cone}\,\mathbb{S}^n \stackrel{iso}{=\!\!=} \mathbb{E}^{n+1};$$

here "$\stackrel{iso}{=\!\!=}$" stands for "isometric to". Now let us extend several notions from Euclidean space to Euclidean cones.

The point in $\mathcal{V}$ that corresponds $(t,x)\in[0,\infty)\times\mathcal{X}$ will be denoted by $t{\cdot}x$. The point in $\mathcal{V}$ formed by the equivalence class of $\{0\}\times\mathcal{X}$ is called the origin of the cone and is denoted by 0 or $0_{\mathcal{V}}$. For $v\in\mathcal{V}$ the distance $|0-v|_{\mathcal{V}}$ is called the norm of v and is denoted by $|v|$ or $|v|_{\mathcal{V}}$. The scalar product $\langle v,w\rangle$ of $v=s{\cdot}p$ and $w=t{\cdot}q$ is defined by

$$\langle v,w\rangle := |v|{\cdot}|w|{\cdot}\cos\theta$$

where $\theta=\min\{\pi,|p-q|_{\mathcal{X}}\}$. The value θ is undefined if $v=0$ or $w=0$; in these cases we assume that $\langle v,w\rangle:=0$.

4.2. Exercise. *Show that* $\mathrm{Cone}\,\mathcal{X}$ *is geodesic if and only if* $\mathcal{X}$ *is* π-*geodesic; that is, any two points* $x,y\in\mathcal{X}$ *such that* $|x-y|_{\mathcal{X}}<\pi$ *can be joined by a geodesic in* $\mathcal{X}$.

Tangent space. The Euclidean cone $\mathrm{Cone}\,\Sigma_p$ over the space of directions Σ_p is called the tangent space at p and is denoted by T_p or $\mathrm{T}_p\mathcal{A}$. The elements of $\mathrm{T}_p\mathcal{A}$ will be called tangent vectors at p (despite that T_p is only a cone — not a vector space). The space of directions Σ_p can be (and will be) identified with the unit sphere in T_p; that is, with the set $\{\,v\in\mathrm{T}_p : |v|=1\,\}$.

4.3. Proposition. *A tangent space to an Alexandrov space has non-negative curvature in the sense of Alexandrov.*

Halbeisen's example [4] shows that the tangent space T_p at some point of Alexandrov space might fail to be geodesic; in this case T_p is *not* $\mathrm{A\scriptstyle LEX}(0)$.

Proof. Consider the tangent space $\mathrm{T}_p=\mathrm{Cone}\,\Sigma_p$ of an Alexandrov space $\mathcal{A}$ at a point p. We need to show that the $\mathbb{E}^2$-comparison holds for a given quadruple v_0, v_1, v_2, $v_3\in\mathrm{T}_p$.

Recall that the space of geodesic directions Σ'_p is dense in Σ_p. It follows that the subcone $\mathrm{T}'_p = \mathrm{Cone}\,\Sigma'_p$ is dense in T_p. Therefore, it is sufficient to consider the case $v_0, v_1, v_2, v_3 \in \mathrm{T}'_p$.

For each i, choose a geodesic γ_i from p in the direction of v_i; assume γ_i has speed $|v_i|$ for each i. Since the angles are defined, we have

❶ $$|\gamma_i(\varepsilon) - \gamma_j(\varepsilon)|_{\mathcal{A}} = \varepsilon{\cdot}|v_i - v_j|_{\mathrm{T}_p} + o(\varepsilon)$$

for $\varepsilon > 0$. The quadruple $\gamma_0(\varepsilon)$, $\gamma_1(\varepsilon)$, $\gamma_2(\varepsilon)$, $\gamma_3(\varepsilon)$ meets the $\mathbb{M}^2(\kappa)$-comparison. After rescaling all the distances by $\frac{1}{\varepsilon}$, it becomes the $\mathbb{M}^2(\varepsilon^2{\cdot}\kappa)$-comparison. Passing to the limit as $\varepsilon \to 0$ and applying ❶, we get the $\mathbb{E}^2$-comparison for v_0, v_1, v_2, v_3. □

4.4. Exercise. *Let p be a point in an Alexandrov space $\mathcal{A}$, and let $\lambda_n \to \infty$. Suppose Σ_p is compact. Show that there is a pointed Gromov–Hausdorff convergence $(\lambda_n{\cdot}\mathcal{A}, p) \to (\mathrm{T}_p, 0)$. Moreover, for any geodesic γ that starts at p, we have*

$$\iota_n \circ \gamma(t/\lambda_n) \to t{\cdot}\gamma^+(0),$$

where ι_n sends a point in $\mathcal{A}$ to the corresponding point in $\lambda_n{\cdot}\mathcal{A}$.

C Semiconcave functions

Recall that λ-concave functions were defined in Section 2E, and when we say *function* we usually mean a *locally Lipschitz function defined on an open domain.*

Let f be a locally Lipschitz real-valued function defined in an open subset $\mathrm{Dom}\, f$ of an Alexandrov space $\mathcal{A}$. Suppose φ is a continuous function defined in $\mathrm{Dom}\, f$. We will write $f'' \leqslant \varphi$ if for any point $x \in \mathrm{Dom}\, f$ and any $\varepsilon > 0$ there is a neighborhood $U \ni x$ such that the restriction $f|_U$ is $(\varphi(x) + \varepsilon)$-concave.

If $f'' \leqslant \varphi$ for some continuous function φ, then f is called semiconcave.

4.5. Exercise. *Let f be a distance function on an* ALEX(0) *space $\mathcal{A}$; that is, $f(x) \equiv |p - x|$ for some $p \in \mathcal{A}$. Show that $f'' \leqslant \frac{1}{f}$. In particular, f is semiconcave in $\mathcal{A} \setminus \{p\}$.*

D Differential

Let $\mathcal{A}$ be an Alexandrov space. Let f be a semiconcave function on $\mathcal{A}$ and $p \in \mathrm{Dom}\, f$. Choose a unit-speed geodesic γ that starts at p; let

$\xi \in \Sigma_p$ be its direction. Define

$$(\boldsymbol{d}_p f)(\xi) := (f \circ \gamma)^+(0),$$

here $(f \circ \gamma)^+$ denotes the right derivative of $(f \circ \gamma)$; it is defined since f is semiconcave.

By the following exercise, the value $(\boldsymbol{d}_p f)(\xi)$ is defined; that is, it does not depend on the choice of γ. Moreover, $\boldsymbol{d}_p f$ is a Lipschitz function on Σ'_p. It follows that the function $\boldsymbol{d}_p f \colon \Sigma'_p \to \mathbb{R}$ can be extended to a Lipschitz function $\boldsymbol{d}_p f \colon \Sigma_p \to \mathbb{R}$. Further, we can extend it to the tangent space by setting

$$(\boldsymbol{d}_p f)(r{\cdot}\xi) := r{\cdot}(\boldsymbol{d}_p f)(\xi)$$

for any $r \geqslant 0$ and $\xi \in \Sigma_p$. The obtained function $\boldsymbol{d}_p f \colon \mathrm{T}_p \to \mathbb{R}$ is Lipschitz; it is called the differential of f at p.

4.6. Exercise. *Let f be a semiconcave function on an Alexandrov space. Suppose γ_1 and γ_2 are geodesics that start at $p \in \operatorname{Dom} f$; denote by θ the angle between γ_1 and γ_2 at p. Show that*

$$|(f \circ \gamma_1)^+(0) - (f \circ \gamma_2)^+(0)| \leqslant L{\cdot}\theta,$$

where L is the Lipschitz constant of f in a neighborhood of p.

4.7. Exercise. *Let p and q be distinct points in an Alexandrov space $\mathcal{A}$. Show the following.*

(a) $\boldsymbol{d}_p \mathrm{dist}_q(v) \leqslant -\langle \uparrow_{[pq]}, v\rangle$ *for any $v \in \mathrm{T}_p$.*

(b) Suppose $\mathcal{A}$ is proper. Let $\Uparrow_p^q$ be the set of all direction of geodesics from p to q. Then

$$\boldsymbol{d}_p \mathrm{dist}_q(v) = -\max_{\xi \in \Uparrow_p^q} \langle \xi, v\rangle$$

for any $v \in \mathrm{T}_p$.

E Gradient

The following definition generalizes the gradient to semiconcave functions on Alexandrov space. This generalization is not trivial even for concave functions on Euclidean space; we suggest keeping this case in mind.

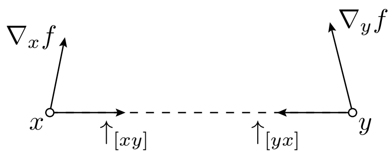
4.8. Definition. *Let f be a semiconcave function on an Alexandrov space. A tangent vector $g \in \mathrm{T}_p$ is called a gradient of f at p (briefly, $g = \nabla_p f$) if*

(a) $(\boldsymbol{d}_p f)(w) \leqslant \langle g, w\rangle$ *for any* $w \in \mathrm{T}_p$, *and*
(b) $(\boldsymbol{d}_p f)(g) = \langle g, g\rangle$.

The following exercise provides a key property of gradients that will be important latter; see the first distance estimate (5.6).

4.9. Exercise. *Let f be a λ-concave function on an Alexandrov space. Suppose that gradients $\nabla_x f$ and $\nabla_y f$ are defined. Show that*

$$\langle \uparrow_{[xy]}, \nabla_x f\rangle + \langle \uparrow_{[yx]}, \nabla_y f\rangle + \lambda{\cdot}|x - y| \geqslant 0.$$



4.10. Proposition. *Suppose that a semiconcave function f is defined in a neighborhood of a point p in an Alexandrov space. Then the gradient $\nabla_p f$ is uniquely defined.*

Moreover, if $\boldsymbol{d}_p f \leqslant 0$, then we have $\nabla_p f = 0$; otherwise, $\nabla_p f = s{\cdot}\overline{\xi}$, where $s = \boldsymbol{d}_p f(\overline{\xi})$ and $\overline{\xi} \in \Sigma_p$ is the direction that maximize the value $\boldsymbol{d}_p f(\xi)$ for $\xi \in \Sigma_p$.

4.11. Key lemma. *Let f be a semiconcave function that is defined in a neighborhood of a point p in an Alexandrov space $\mathcal{A}$. Then for any $u, v \in \mathrm{T}_p$, we have*

$$s{\cdot}\sqrt{|u|^2 + 2{\cdot}\langle u, v\rangle + |v|^2} \geqslant (\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v),$$

where

$$s = \sup\left\{\, (\boldsymbol{d}_p f)(\xi) \;:\; \xi \in \Sigma_p \,\right\}.$$

Note that if $\mathrm{T}_p \overset{iso}{=\!=} \mathbb{E}^m$ and $\boldsymbol{d}_p f$ is a concave function, then $2{\cdot}(\boldsymbol{d}_p f)(\frac{u+v}{2}) \geqslant (\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v)$. The latter implies the statement since $|u + v| = \sqrt{|u|^2 + 2{\cdot}\langle u, v\rangle + |v|^2}$. In general, T_p is not geodesic (and not even a length space), so concavity of $\boldsymbol{d}_p f$ does not make sense. The key lemma however says that in a certain sense $\boldsymbol{d}_p f$ behaves as a concave function.

Solving the following exercise should help to find an approach to the key lemma.

4.12. Exercise. *Let p and q be distinct points in an Alexandrov space $\mathcal{A}$. Suppose the geodesic $[pq]$ can be extended beyond q.*

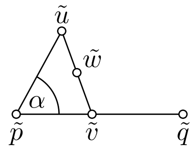
Show that

$$\boldsymbol{d}_p\mathrm{dist}_q(v) = -\langle\uparrow_{[pq]}, v\rangle$$

for any $v \in \mathrm{T}_p$.

Proof of 4.11. We will assume that $\mathcal{A}$ is $\mathrm{A{\scriptsize LEX}}(0)$ and f is concave; the general case requires only minor modifications. We can assume that $v \neq 0$, $w \neq 0$, and $\alpha = \measuredangle(u, v) > 0$; otherwise, the statement is trivial.

Consider a model configuration of five points: $\tilde p$, $\tilde u$, $\tilde v$, $\tilde q$, $\tilde w \in \mathbb{E}^2$ such that



- ⋄ $\measuredangle[\tilde p\,{}^{\tilde u}_{\tilde v}] = \alpha$,
- ⋄ $|\tilde p - \tilde u| = |u|$,
- ⋄ $|\tilde p - \tilde v| = |v|$,
- ⋄ $\tilde q$ lies on an extension of $[\tilde p\tilde v]$ so that $\tilde v$ is the midpoint of $[\tilde p\tilde q]$,
- ⋄ $\tilde w$ is the midpoint between $\tilde u$ and $\tilde v$.

Note that

❶ $$|\tilde p - \tilde w| = \tfrac{1}{2}\cdot\sqrt{|u|^2 + 2\cdot\langle u, v\rangle + |v|^2}.$$

Since the geodesic space of directions Σ'_p is dense in Σ_p, we can assume that there are geodesics in the directions of u and v. Choose such geodesics γ_u and γ_v and assume that they are parametrized with speed $|u|$ and $|v|$ respectively. For all small $t > 0$, consider points $u_t, v_t, q_t, w_t \in \mathcal{A}$ such that

- ⋄ $v_t = \gamma_v(t)$, $\quad q_t = \gamma_v(2\cdot t)$
- ⋄ $u_t = \gamma_u(t)$.
- ⋄ w_t is the midpoint of $[u_t v_t]$.

Clearly

$$|p - u_t| = t\cdot|u|, \qquad |p - v_t| = t\cdot|v|, \qquad |p - q_t| = 2\cdot t\cdot|v|.$$

Since $\measuredangle(u, v)$ is defined, we have

$$|u_t - v_t| = t\cdot|\tilde u - \tilde v| + o(t), \qquad |u_t - q_t| = t\cdot|\tilde u - \tilde q| + o(t).$$

From the point-on-side and hinge comparisons (2.15c+2.15d), we have

$$\tilde\measuredangle(v_t\,{}^{p}_{w_t}) \geqslant \tilde\measuredangle(v_t\,{}^{p}_{u_t}) \geqslant \measuredangle[\tilde v\,{}^{\tilde p}_{\tilde u}] + \tfrac{o(t)}{t}$$

and

$$\tilde\measuredangle(v_t\,{}^{q_t}_{w_t}) \geqslant \tilde\measuredangle(v_t\,{}^{q_t}_{u_t}) \geqslant \measuredangle[\tilde v\,{}^{\tilde q}_{\tilde u}] + \tfrac{o(t)}{t}.$$

Clearly, $\measuredangle[\tilde v\,{}^{\tilde p}_{\tilde u}] + \measuredangle[\tilde v\,{}^{\tilde q}_{\tilde u}] = \pi$. From the adjacent angle comparison (2.15b), $\tilde\measuredangle(v_t\,{}^{p}_{u_t}) + \tilde\measuredangle(v_t\,{}^{u_t}_{q_t}) \leqslant \pi$. Hence $\tilde\measuredangle(v_t\,{}^{p}_{w_t}) \to \measuredangle[\tilde v\,{}^{\tilde p}_{\tilde w}]$ as $t \to 0+$ and thus

$$|p - w_t| = t\cdot|\tilde p - \tilde w| + o(t).$$

Without loss of generality, we can assume that $f(p)=0$. Since f is concave, we have

$$\begin{aligned}2\cdot f(w_t) \geqslant f(u_t)+f(v_t) &= \\ &= t\cdot[(\boldsymbol{d}_p f)(u)+(\boldsymbol{d}_p f)(v)]+o(t).\end{aligned}$$

Applying concavity of f, we have

$$\begin{aligned}(\boldsymbol{d}_p f)(\uparrow_{[pw_t]}) \geqslant \frac{f(w_t)}{|p-w_t|} &\geqslant \\ &\geqslant \frac{t\cdot[(\boldsymbol{d}_p f)(u)+(\boldsymbol{d}_p f)(v)]+o(t)}{2\cdot t\cdot|\tilde p-\tilde w|+o(t)}.\end{aligned}$$

By ❶, the key lemma follows. □

Proof of 4.10; uniqueness. If $g,g'\in \mathrm{T}_p$ are two gradients of f, then

$$\langle g,g\rangle=(\boldsymbol{d}_p f)(g)\leqslant\langle g,g'\rangle, \qquad \langle g',g'\rangle=(\boldsymbol{d}_p f)(g')\leqslant\langle g,g'\rangle.$$

Therefore,

$$|g-g'|^2=\langle g,g\rangle-2\cdot\langle g,g'\rangle+\langle g',g'\rangle\leqslant 0.$$

It follows that $g=g'$.

Existence. If $\boldsymbol{d}_p f\leqslant 0$, then one can take $\nabla_p f=0$.

Suppose $s=\sup\{\,(\boldsymbol{d}_p f)(\xi)\;:\;\xi\in\Sigma_p\,\}>0$, it is sufficient to show that there is $\bar\xi\in\Sigma_p$ such that

❷ $$(\boldsymbol{d}_p f)\left(\bar\xi\right)=s.$$

Indeed, suppose $\bar\xi$ exists. Applying 4.11 for $u=\bar\xi$, $v=\varepsilon\cdot w$ with $\varepsilon\to 0+$, we get

$$(\boldsymbol{d}_p f)(w)\leqslant\langle w,s\cdot\bar\xi\rangle$$

for any $w\in\mathrm{T}_p$; that is, $s\cdot\bar\xi$ is the gradient at p.

Take a sequence of directions $\xi_n\in\Sigma_p$, such that $(\boldsymbol{d}_p f)(\xi_n)\to s$. Applying 4.11 for $u=\xi_n$ and $v=\xi_m$, we get

$$s\geqslant\frac{(\boldsymbol{d}_p f)(\xi_n)+(\boldsymbol{d}_p f)(\xi_m)}{\sqrt{2+2\cdot\cos\measuredangle(\xi_n,\xi_m)}}.$$

Therefore $\measuredangle(\xi_n,\xi_m)\to 0$ as $n,m\to\infty$; that is, $\xi_1,\xi_2,\dots$ is a Cauchy sequence. Clearly, $\bar\xi=\lim_n\xi_n$ meets ❷. □

4.13. Exercise. *Let f and g be locally Lipschitz semiconcave functions defined in a neighborhood of a point p in an Alexandrov space. Show that*

$$|\nabla_p f-\nabla_p g|^2_{\mathrm{T}_p}\leqslant s\cdot(|\nabla_p f|+|\nabla_p g|),$$

where

$$s = \sup\{\, |(\boldsymbol{d}_p f)(\xi) - (\boldsymbol{d}_p g)(\xi)| \;:\; \xi \in \Sigma_p \,\}.$$

Conclude that if the sequence of restrictions $\boldsymbol{d}_p f_n|_{\Sigma_p}$ converges uniformly, then $\nabla_p f_n$ converges as $n \to \infty$. Here we assume that all functions $f_1, f_2, \dots$ are semiconcave and locally Lipschitz.

4.14. Exercise. *Let f be a locally Lipschitz λ-concave function on an Alexandrov space $\mathcal{A}$.*

(a) *Suppose $s \geqslant 0$. Show that $|\nabla_x f| > s$ if and only if for some point y we have*

$$f(y) - f(x) > s \cdot \ell + \lambda \cdot \tfrac{\ell^2}{2},$$

were $\ell = |x - y|$.

(b) *Show that $x \mapsto |\nabla_x f|$ is lower semicontinuous; that is, if $x_n \to \to x_\infty$, then*

$$|\nabla_{x_\infty} f| \leqslant \varliminf_{n\to\infty} |\nabla_{x_n} f|.$$

Lecture 5

Gradient flow

Here we define the gradient flow of a semiconcave function and discuss its properties, most importantly the distance estimates.

A Velocity of curve

Let α be a curve in an Alexandrov space $\mathcal{A}$. If for any choice of geodesics $[p\,\alpha(t_0+\varepsilon)]$ the vectors

$$\tfrac{1}{\varepsilon}\cdot|p-\alpha(t_0+\varepsilon)|\cdot\uparrow_{[p\alpha(t_0+\varepsilon)]}$$

converge as $\varepsilon\to 0+$, then their limit in T_p is called the right derivative of α at t_0; it will be denoted by $\alpha^+(t_0)$. In addition, $\alpha^+(t_0):=0$ if $\frac{1}{\varepsilon}\cdot|p-\alpha(t_0+\varepsilon)|\to 0$ as $\varepsilon\to 0+$.

The tangent vector $v=|p-x|\cdot\uparrow_{[px]}$ can be called the logarithm of x at p (briefly, $\log_p x$); note that $\gamma^+(0)=\log_p x$ for a geodesic path γ from p to x.

5.1. Claim. *Let α be a curve in an Alexandrov space $\mathcal{A}$. Suppose f a semiconcave Lipschitz function defined in a neighborhood of $p=\alpha(0)$, and $\alpha^+(0)$ is defined. Then*

$$(f\circ\alpha)^+(0)=(\boldsymbol{d}_pf)(\alpha^+(0)).$$

Proof. Without loss of generality, we can assume that $f(p)=0$. Suppose f and therefore $\boldsymbol{d}_pf$ are L-Lipschitz.

Choose a constant-speed geodesic γ that starts from p, such that the distance $s=|\alpha^+(0)-\gamma^+(0)|_{\mathrm{T}_p}$ is small. By the definition of differential,

$$(f\circ\gamma)^+(0)=\boldsymbol{d}_pf(\gamma^+(0)).$$

By comparison and the definition of α^+,

$$|\alpha(\varepsilon) - \gamma(\varepsilon)|_{\mathcal{A}} \leqslant s{\cdot}\varepsilon + o(\varepsilon)$$

for $\varepsilon > 0$. Therefore,

$$|f \circ \alpha(\varepsilon) - f \circ \gamma(\varepsilon)| \leqslant L{\cdot}s{\cdot}\varepsilon + o(\varepsilon).$$

Suppose $(f \circ \alpha)^+(0)$ is defined. Then

$$|(f \circ \alpha)^+(0) - (f \circ \gamma)^+(0)| \leqslant L{\cdot}s.$$

Since $\boldsymbol{d}_p f$ is L-Lipschitz, we also get

$$|\boldsymbol{d}_p f(\alpha^+(0)) - \boldsymbol{d}_p f(\gamma^+(0))| \leqslant L{\cdot}s.$$

It follows that the needed identity holds up to error $2{\cdot}L{\cdot}s$. The statement follows since $s > 0$ can be chosen arbitrarily.

The same argument is applicable if in the place of $(f \circ \alpha)^+(0)$ we use any limit of $\frac{1}{\varepsilon_n}{\cdot}[f \circ \alpha(\varepsilon_n) - f(p)]$ for a sequence $\varepsilon_n \to 0+$. It proves that all such limits coincide; in particular, $(f \circ \alpha)^+(0)$ is defined and equals to $(\boldsymbol{d}_p f)(\alpha^+(0))$. $\square$

B Gradient curves

5.2. Definition. *Let $f\colon \mathcal{A} \to \mathbb{R}$ be a locally Lipschitz and semiconcave function on an Alexandrov space $\mathcal{A}$.*

A locally Lipschitz curve $\alpha\colon [t_{\min}, t_{\max}) \to \operatorname{Dom} f$ will be called an f-gradient curve if

$$\alpha^+ = \nabla_\alpha f;$$

that is, for any $t \in [t_{\min}, t_{\max})$, $\alpha^+(t)$ is defined and $\alpha^+(t) = \nabla_{\alpha(t)} f$.

A complete proof of the following theorem is given in [4]; it mimics the proof of the standard Picard theorem on the existence and uniqueness of solutions of ordinary differential equations. The uniqueness will follow from the first distance estimate (5.6) proved in the next section. We omit the proof of existence as it is rather lengthy.

5.3. Picard theorem. *Let $f\colon \mathcal{A} \to \mathbb{R}$ be a locally Lipschitz and λ-concave function on an Alexandrov space $\mathcal{A}$. Then for any $p \in \operatorname{Dom} f$, there are unique $t_{\max} \in (0, \infty]$ and f-gradient curve $\alpha\colon [0, t_{\max}) \to \mathcal{A}$ with $\alpha(0) = p$ such that any sequence $t_n \to t_{\max}-$, the sequence $\alpha(t_n)$ does not have a limit point in $\operatorname{Dom} f$.*

Note that the theorem says that the future of a gradient curve is determined by its present, but it says nothing about its past.

Here is an example showing that the past is not determined by the present. Consider the function $f\colon x\mapsto -|x|$ on the real line $\mathbb{R}$. The tangent space $\mathrm{T}_x\mathbb{R}$ can be identified with $\mathbb{R}$. Note that

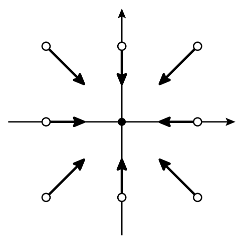
$$\nabla_x f=\begin{cases}1 & \text{if} \quad x<0,\\ 0 & \text{if} \quad x=0,\\ -1 & \text{if} \quad x>0.\end{cases}$$

So, the f-gradient curves go to the origin with unit speed and then stand there forever. In particular, if α is an f-gradient curve that starts at x, then $\alpha(t)=0$ for any $t\geqslant|x|$.

Here is a slightly more interesting example; it shows that gradient curves can merge even in the region where $|\nabla f|\ne 0$.

5.4. Example. *Consider the function $f\colon (x,y)\mapsto -|x|-|y|$ on the (x,y)-plane. Note that f is concave; its gradient field is sketched on the figure.*

Let α be an f-gradient curve that starts at (x,y) for $x>y>0$. Then



$$\alpha(t)=\begin{cases}(x-t,y-t) & \textit{for} \quad 0\leqslant t\leqslant x-y,\\ (x-t,0) & \textit{for} \quad x-y\leqslant t\leqslant x,\\ (0,0) & \textit{for} \quad x\leqslant t.\end{cases}$$

C Distance estimates

5.5. Observation. *Let α be a gradient curve of a λ-concave function f defined on an Alexandrov space. Choose a point p; let $\ell(t):=\operatorname{dist}_p\circ\alpha(t)$ and $q=\alpha(t_0)$. Then*

$$\ell^+(t_0)\leqslant -\left(f(p)-f(q)-\tfrac{\lambda}{2}\cdot\ell^2(t_0)\right)/\ell(t_0)$$

Proof. Let γ be the unit-speed parametrization of $[qp]$ from q to p, so $q=\gamma(0)$. Then

$$\begin{aligned}\ell^+(t_0)&=(\boldsymbol{d}_q\operatorname{dist}_p)(\nabla_q f)\leqslant && \text{(by 5.1)}\\ &\leqslant -\langle\uparrow_{[qp]},\nabla_q f\rangle\leqslant && \text{(by 4.7}a\text{)}\\ &\leqslant -\boldsymbol{d}_q f(\uparrow_{[qp]})= && \text{(by 4.8)}\\ &=-(f\circ\gamma)^+(0)\leqslant\\ &\leqslant -\left(f(p)-f(q)-\tfrac{\lambda}{2}\cdot\ell^2(t_0)\right)/\ell(t_0)\end{aligned}$$

The last two lines follow by the definition of differential, and the concavity of $t \mapsto f\circ\gamma(t) - \frac{\lambda}{2}\cdot t^2$. $\square$

Note that the following estimate implies uniqueness in the Picard theorem (5.3).

5.6. First distance estimate. *Let f be a λ-concave locally Lipschitz function on an Alexandrov space $\mathcal{A}$. Then*

$$|\alpha(t) - \beta(t)| \leqslant e^{\lambda\cdot t}\cdot|\alpha(0) - \beta(0)|$$

for any $t \geqslant 0$ and any two f-gradient curves α and β.

Moreover, the statement holds for a locally Lipschitz λ-concave function defined in an open domain if there is a geodesic $[\alpha(t)\,\beta(t)]$ in Dom f *for any t.*

Proof. Fix a choice of geodesic $[\alpha(t)\,\beta(t)]$ for each t. Let $\ell(t) = |\alpha(t) - \beta(t)|$. Note that

$$\ell^+(t) \leqslant -\langle\uparrow_{[\alpha(t)\beta(t)]}, \nabla_{\alpha(t)} f\rangle - \langle\uparrow_{[\beta(t)\alpha(t)]}, \nabla_{\beta(t)} f\rangle \leqslant \lambda\cdot\ell(t).$$

Here one has to apply 5.5 for distance to the midpoint m of $[\alpha(t)\,\beta(t)]$, then apply the triangle inequality and 4.9. Integrating, we get the result. $\square$

The following exercise describes a global geometric property of a gradient curve without direct reference to its function. It is based on the notion of self-contracting curves introduced by Aris Daniilidis, Olivier Ley, and Stéphane Sabourau [20].

5.7. Exercise. *Let $f\colon \mathcal{A} \to \mathbb{R}$ be a locally Lipschitz and concave function on an Alexandrov space $\mathcal{A}$. Then*

$$|\alpha(t_1) - \alpha(t_3)|_{\mathcal{A}} \geqslant |\alpha(t_2) - \alpha(t_3)|_{\mathcal{A}}.$$

for any f-gradient curve α and $t_1 \leqslant t_2 \leqslant t_3$.

5.8. Exercise. *Let f be a locally Lipschitz concave function defined on an Alexandrov space $\mathcal{A}$. Suppose $\hat\alpha\colon [0,\ell] \to \mathcal{A}$ is an arc-length reparametrization of an f-gradient curve. Show that $f\circ\hat\alpha$ is concave.*

The following exercise implies that gradient curves for a uniformly converging sequence of λ-concave functions converge to the gradient curves of the limit function.

5.9. Exercise. *Let f and g be λ-concave locally Lipschitz functions on an Alexandrov space $\mathcal{A}$. Suppose $\alpha, \beta\colon [0, t_{\max}) \to \mathcal{A}$ are respectively f- and g-gradient curves. Assume $|f - g| < \varepsilon$; let $\ell\colon t \mapsto |\alpha(t) - \beta(t)|$. Show that*

$$\ell^+ \leqslant \lambda\cdot\ell + \tfrac{2\cdot\varepsilon}{\ell}.$$

Conclude that if $\alpha(0)=\beta(0)$ and $t_{\max}<\infty$, then

$$|\alpha(t)-\beta(t)| \leqslant c\cdot\sqrt{\varepsilon\cdot t}$$

for some constant $c=c(t_{\max},\lambda)$.

D Gradient flow

Let $\mathcal{A}$ be an Alexandrov space and f be a locally Lipschitz semiconcave function defined on an open subset of $\mathcal{A}$. If there is an f-gradient curve α such that $\alpha(0)=x$ and $\alpha(t)=y$, then we will write

$$\mathrm{Flow}_f^t(x)=y.$$

The partially defined map Flow_f^t from $\mathcal{A}$ to itself is called the f-gradient flow for time t. Note that

$$\mathrm{Flow}_f^{t_1+t_2}=\mathrm{Flow}_f^{t_1}\circ\mathrm{Flow}_f^{t_2}.$$

In other words, one may of that gradient flow as a partial action of the *semigroup* $(\mathbb{R}_{\geqslant 0},+)$ on the space.

From the first distance estimate 5.6, it follows that for any $t\geqslant 0$, the domain of definition of Flow_f^t is an open subset of $\mathcal{A}$. In some cases, it is globally defined. For example, if f is a λ-concave function defined on the whole space $\mathcal{A}$, then $\mathrm{Flow}_f^t(x)$ is defined for all $x\in\mathcal{A}$ and $t\geqslant 0$; see [4, 16.19].

Now let us reformulate the statements about gradient curves obtained earlier using this new terminology. From the first distance estimate, we have the following.

5.10. Proposition. *Let $\mathcal{A}$ be an Alexandrov space and $f\colon\mathcal{A}\to\mathbb{R}$ be a semiconcave function. Then the map $x\mapsto\mathrm{Flow}_f^t(x)$ is locally Lipschitz.*

Moreover, if f is λ-concave, then Flow_f^t is $e^{\lambda\cdot t}$-Lipschitz.

The next proposition follows from 5.9.

5.11. Proposition. *Let $\mathcal{A}$ be an Alexandrov space. Suppose $f_n\colon\mathcal{A}\to\mathbb{R}$ is a sequence of λ-concave functions that converges to $f_\infty\colon\mathcal{A}\to\mathbb{R}$. Then for any $x\in\mathcal{A}$ and $t\geqslant 0$, we have*

$$\mathrm{Flow}_{f_n}^t(x)\to\mathrm{Flow}_{f_\infty}^t(x)$$

as $n\to\infty$.

There is a more general version of this proposition for a converging sequence $\mathcal{A}_n\to\mathcal{A}_\infty$ of spaces and a converging sequence of functions $f_n\colon\mathcal{A}_n\to\mathbb{R}$; see [4, 16.21].

E Gradient exponent

One of the technical difficulties in Alexandrov geometry comes from nonextendability of geodesics. In particular, the exponential map, $\exp_p\colon \mathrm{T}_p \to \mathcal{A}$, if defined in the usual way, can be undefined in an arbitrarily small neighborhood of the origin.

Next we construct the gradient exponential map

$$\operatorname{gexp}_p\colon \mathrm{T}_p \to \mathcal{A},$$

which essentially solves this problem. It shares many properties with the ordinary exponential map and is even better in certain respects, even in the Riemannian universe.

Let $\mathcal{A}$ be an Alexandrov space and $p \in \mathcal{A}$, consider the function $f = \operatorname{dist}_p^2/2$. Recall that Φ_f^t denotes the gradient flow. Let us define the *gradient exponential map* as the limit

$$\operatorname{gexp}_p(v) = \lim_{n\to\infty} \Phi_f^{t_n}(x_n),$$

where the sequences $x_n \in \mathcal{A}$ and $t_n \geqslant 0$ are chosen so that $t_n \to \infty$ and $e^{t_n} \cdot \log_p x_n \to v$ as $n \to \infty$.

More intuitively, suppose $i_\lambda\colon \lambda{\cdot}\mathcal{A} \to \mathcal{A}$ sends a point in the rescaled copy $\lambda{\cdot}\mathcal{A}$ to the corresponding point in $\mathcal{A}$. By the first distance estimate (5.6), the map

❶ $$\Phi_f^t \circ i_{e^t}\colon e^t{\cdot}\mathcal{A} \to \mathcal{A}$$

is short for any $t \geqslant 0$. If we have a pointed Gromov–Hausdorff convergence $(e^{t_n}{\cdot}\mathcal{A}, p) \to (\mathrm{T}_p, o_p)$, then $\operatorname{gexp}_p\colon \mathrm{T}_p \to \mathcal{A}$ is the limit of $\Phi_f^{tn} \circ i_{e^{tn}}$. This way we get that gexp_p is short as a limit of short maps. This observation is generalized in the following proposition.

5.12. Proposition. *Let $\mathcal{A}$ be a proper* $\mathrm{A\scriptstyle LEX}(0)$ *space. Then for any $p \in \mathcal{A}$ the gradient exponent* $\operatorname{gexp}_p\colon \mathrm{T}_p \to \mathcal{A}$ *is uniquely defined. Moreover,* gexp_p *is a short map and*

$$\operatorname{gexp}_p(\gamma^+(0)) = \gamma(1)$$

for any geodesic path γ that starts at p.

The last statement implies that

$$\operatorname{gexp}_p \circ \log_p = \mathrm{id},$$

so it is appropriate to use term *exponent* for gexp.

Proof. Note that $f'' \leqslant 1$. Since the space is proper we can choose a limit in ❶.

Let γ be a geodesic that stats at p. Observe that $\gamma\circ\ln$ is an f-gradient curve. By the first distance estimate, we have that Φ_f^t is an e^t-Lipschitz. This implies any limit in ❶ has the same value; that is gexp_p is uniquely defined.

Again, since Φ_f^t is an e^t-Lipschitz, we get that gexp_p is a short map □

F Remarks

The idea to use gradient flow in Alexandrov geometry was inspired by the success of Sharafutdinov's retraction in comparison geometry [86]. Originaly, the gradient flow was developed to construct quasigeodesics with given initial data [67, 74, 75], but it turned out that gradient flow and gradient exponent are better tools. Very soon these tools found applications in other types of singular spaces [9, 42, 58, 59, 64, 85].

For a general curvature bound κ, the construction of gradient exponent has to be modified; it is denoted by $\operatorname{gexp}_p^\kappa$ [4, 16.36].

For $\kappa=-1$ we have and $\operatorname{gexp}_p(\gamma^+(0))=\gamma(1)$ for any geodesic path γ that starts at p and

$$|\operatorname{gexp}_p^{-1}v-\operatorname{gexp}_p^{-1}w|_{\mathcal{A}}\leqslant\tilde\Upsilon[0\,{}^v_w]_{\mathbb{H}^2}.$$

Similarly, for $\kappa=1$ we have $\operatorname{gexp}_p(\gamma^+(0))=\gamma(1)$ for any geodesic path γ that starts at p and

$$|\operatorname{gexp}_p^{1}v-\operatorname{gexp}_p^{1}w|_{\mathcal{A}}\leqslant\tilde\Upsilon[0\,{}^v_w]_{\mathbb{S}^2},$$

but this time all this holds only if $|v|,|w|\leqslant\frac{\pi}{2}$ and $\operatorname{length}\gamma\leqslant\frac{\pi}{2}$.

The gradient exponential map in a Riemannian manifold (M,g) coincides with the Riemannian exponential map before the cut locus of but *is different* from the Riemannian exponential after that. The following statement shows that this technique can prove something nontrivial even for Riemannian manifolds.

5.13. Problem. *Let (M,g) be a complete m-dimensional Riemannian with sectional curvature at least 1. Assume M is not homeomorphic to $\mathbb{S}^m$. Show that there is a short onto map $\mathbb{S}^m\to(M,g)$.*

Lecture 6

Line splitting

In this lecture, we prove the line splitting theorem and apply it to study tangent spaces of an Alexandrov space.

A Busemann function

A half-line is a distance-preserving map from $\mathbb{R}_{\geqslant 0} = [0, \infty)$ to a metric space. In other words, a half-line is a geodesic defined on the real half-line $\mathbb{R}_{\geqslant 0}$.

If $\gamma\colon [0, \infty) \to \mathcal{X}$ is a half-line, then the limit

❶ $$\mathrm{bus}_\gamma(x) = \lim_{t\to\infty} |\gamma(t) - x| - t$$

is called the Busemann function of γ.

The Busemann function bus_γ mimics behavior of the distance function from the ideal point of γ.

6.1. Proposition. *For any half-line γ in a metric space $\mathcal{X}$, its Busemann function $\mathrm{bus}_\gamma\colon \mathcal{X} \to \mathbb{R}$ is defined. Moreover, bus_γ is 1-Lipschitz and $\mathrm{bus}_\gamma(\gamma(t)) = -t$ for any t.*

Proof. Since $t = |\gamma(0) - \gamma(t)|$, the triangle inequality implies that, the function

$$t \mapsto |\gamma(t) - x| - t$$

is nonincreasing, and

$$|\gamma(t) - x| - t \geqslant -|\gamma(0) - x|$$

for any $x \in \mathcal{X}$. Therefore, the limit in ❶ is defined, and it is 1-Lipschitz as a limit of 1-Lipschitz functions. The last statement follows since $|\gamma(t) - \gamma(t_0)| = t - t_0$ for all large t. □

6.2. Exercise. *Any Busemann function on an* ALEX(0) *space is concave.*

B Splitting theorem

A line is a distance-preserving map from $\mathbb{R}$ to a metric space. In other words, a line is a geodesic defined on the real line $\mathbb{R}$.

6.3. Exercise. *Let γ be a line in a metric space $\mathcal{X}$. Show that for any point x we have*

$$\mathrm{bus}_+(x) + \mathrm{bus}_-(x) \geqslant 0$$

where, bus_+ *and* bus_-*, are the Busemann functions asociated with half-lines $\gamma : [0, \infty) \to \mathcal{A}$ and $\gamma : (-\infty, 0] \to \mathcal{A}$ respectively.*

Let $\mathcal{X}$ be a metric space and $A, B \subset \mathcal{X}$. A metric space $\mathcal{X}$ is a direct sum of its two A and B, or briefly,

$$\mathcal{X} = A \oplus B$$

if there are projections $\mathrm{proj}_A : \mathcal{X} \to A$ and $\mathrm{proj}_B : \mathcal{X} \to B$ such that

$$|x - y|^2 = |\mathrm{proj}_A(x) - \mathrm{proj}_A(y)|^2 + |\mathrm{proj}_B(x) - \mathrm{proj}_B(y)|^2$$

for any two points $x, y \in \mathcal{X}$.

Note that if $\mathcal{X} = A \oplus B$, then

- ⋄ A intersects B at a single point,
- ⋄ both sets A and B are convex sets in $\mathcal{X}$; the latter means that any geodesic with the ends in A (or B) lies in A (or B).

6.4. Line splitting theorem. *Let γ be a line in a* ALEX(0) *space $\mathcal{A}$. Then*

$$\mathcal{A} = \mathcal{A}' \oplus \gamma(\mathbb{R})$$

for some subset $\mathcal{A}' \subset \mathcal{A}$.

6.5. Corollary. *Any* ALEX(0) *space $\mathcal{A}$ splits isometrically as*

$$\mathcal{A} = \mathcal{A}' \oplus H$$

where $H \subset \mathcal{A}$ is a subset isometric to a Hilbert space, and $\mathcal{A}' \subset \mathcal{A}$ is a convex subset that contains no lines.

The following lemma is closely related to the first distance estimate (5.6); it is also a limit case of 5.12. The proof follows similar lines.

6.6. Lemma. *Suppose $f\colon \mathcal{A}\to\mathbb{R}$ be a concave 1-Lipschitz function on an* ALEX(0) *space $\mathcal{A}$. Consider two f-gradient curves α and β. Then for any $t,s\geqslant 0$ we have*

$$|\alpha(s)-\beta(t)|^2\leqslant|p-q|^2+2{\cdot}(f(p)-f(q)){\cdot}(s-t)+(s-t)^2,$$

where $p=\alpha(0)$ and $q=\beta(0)$.

Proof. Since f is 1-Lipschitz, $|\nabla f|\leqslant 1$. Therefore

$$f\circ\beta(t)\leqslant f(q)+t$$

for any $t\geqslant 0$.

Set $\ell(t)=|p-\beta(t)|$. Applying 5.5, we get

$$\begin{aligned}(\ell^2)^+(t)&\leqslant 2{\cdot}\left(f\circ\beta(t)-f(p)\right)\leqslant\\&\leqslant 2{\cdot}\left(f(q)+t-f(p)\right).\end{aligned}$$

Therefore

$$\ell^2(t)-\ell^2(0)\leqslant 2{\cdot}\left(f(q)-f(p)\right){\cdot}t+t^2.$$

It proves the needed inequality in case $s=0$. Combining it with the first distance estimate (5.6), we get the result in case $s\leqslant t$. The case $s\geqslant t$ follows by switching the roles of s and t. □

Proof of 6.4. Consider two Busemann functions, bus_+ and bus_-, associated with half-lines $\gamma:[0,\infty)\to\mathcal{A}$ and $\gamma:(-\infty,0]\to\mathcal{A}$ respectively; that is,

$$\mathrm{bus}_\pm(x):=\lim_{t\to\infty}|\gamma(\pm t)-x|-t.$$

According to 6.2, both bus_+ and bus_- are concave.

By 6.3, $\mathrm{bus}_+(x)+\mathrm{bus}_-(x)\geqslant 0$ for any $x\in\mathcal{A}$. On the other hand, by 2.18, $f(t)=\mathrm{dist}_x^2\circ\gamma(t)$ is 2-concave. In particular, $f(t)\leqslant t^2+at+b$ for some constants $a,b\in\mathbb{R}$. Therefore, for all large t

$$|\gamma(t)-x|-t+|\gamma(-t)-x|-t\leqslant\sqrt{t^2+at+b}-t+\sqrt{t^2-at+b}-t$$

Passing to the limit as $t\to\infty$, we get that $\mathrm{bus}_+(x)+\mathrm{bus}_-(x)\leqslant 0$. Hence

$$\mathrm{bus}_+(x)+\mathrm{bus}_-(x)=0$$

for any $x\in\mathcal{A}$. In particular, the functions bus_+ and bus_- are affine; that is, they are convex and concave at the same time.

Note that for any x,

$$\begin{aligned}|\nabla_x\mathrm{bus}_\pm|&=\sup\{\,\boldsymbol{d}_x\mathrm{bus}_\pm(\xi)\;:\;\xi\in\Sigma_x\,\}=\\&=\sup\{\,-\boldsymbol{d}_x\mathrm{bus}_\mp(\xi)\;:\;\xi\in\Sigma_x\,\}\equiv\\&\equiv 1.\end{aligned}$$

Observe that α is a $\mathrm{bus}_\pm$-gradient curve if and only if α is a geodesic such that $(\mathrm{bus}_\pm\circ\alpha)^+ = 1$. Indeed, if α is a geodesic, then $(\mathrm{bus}_\pm\circ\alpha)^+ \leqslant 1$ and the equality holds only if $\nabla_\alpha \mathrm{bus}_\pm = \alpha^+$. Now suppose $\nabla_\alpha \mathrm{bus}_\pm = \alpha^+$. Then $|\alpha^+| \leqslant 1$ and $(\mathrm{bus}_\pm\circ\alpha)^+ = 1$; therefore

$$\begin{aligned}|t_0 - t_1| &\geqslant |\alpha(t_0) - \alpha(t_1)| \geqslant \\ &\geqslant |\,\mathrm{bus}_\pm\circ\alpha(t_0) - \mathrm{bus}_\pm\circ\alpha(t_1) = \\ &= |t_0 - t_1|.\end{aligned}$$

It follows that for any $t > 0$, the $\mathrm{bus}_\pm$-gradient flows commute; that is,

$$\mathrm{Flow}^t_{\mathrm{bus}_+}\circ\mathrm{Flow}^t_{\mathrm{bus}_-} = \mathrm{id}_{\mathcal{A}}\,.$$

Setting

$$\mathrm{Flow}^t = \begin{bmatrix}\mathrm{Flow}^t_{\mathrm{bus}_+} & \text{if } t\geqslant 0\\ \mathrm{Flow}^{-t}_{\mathrm{bus}_-} & \text{if } t\leqslant 0\end{bmatrix}$$

defines an $\mathbb{R}$-action on $\mathcal{A}$.

Consider the level set $\mathcal{A}' = \mathrm{bus}_+^{-1}(0) = \mathrm{bus}_-^{-1}(0)$; it is a closed convex subset of $\mathcal{A}$, and therefore forms an Alexandrov space. Consider the map $h\colon \mathcal{A}'\times\mathbb{R}\to\mathcal{A}$ defined by $h\colon (x,t)\mapsto \mathrm{Flow}^t(x)$. Note that h is onto. Applying 6.6 for $\mathrm{Flow}^t_{\mathrm{bus}_+}$ and $\mathrm{Flow}^t_{\mathrm{bus}_-}$ shows that h is distance non-expanding and non-contracting at the same time; that is, h is an isometry. $\square$

Recall that according our definition the real line $\mathbb{R}$ is $\mathrm{A\scriptsize LEX}(1)$. However, most of $\mathrm{A\scriptsize LEX}(1)$ spaces have diameter at most π; see 3.7.

6.7. Exercise. *Suppose $\mathcal{X}$ is a complete geodesic space. Show that* $\mathrm{Cone}\,\mathcal{X}$ *is* $\mathrm{A\scriptsize LEX}(0)$ *if and only if $\mathcal{X}$ is* $\mathrm{A\scriptsize LEX}(1)$ *and* $\mathrm{diam}\,\mathcal{X}\leqslant\pi$.

C Anti-sum

Here we give a corollary of 4.13. It will be used to prove basic properties of the tangent space.

6.8. Anti-sum lemma. *Let $\mathcal{A}$ be an Alexandrov space and $p\in\mathcal{A}$.*

Given two vectors $u,v\in\mathrm{T}_p$, there is a unique vector $w\in\mathrm{T}_p$ such that

$$\langle u,x\rangle + \langle v,x\rangle + \langle w,x\rangle \geqslant 0$$

for any $x\in\mathrm{T}_p$, and

$$\langle u,w\rangle + \langle v,w\rangle + \langle w,w\rangle = 0.$$

6.9. Exercise. *Suppose $u, v, w \in \mathrm{T}_p$ are as in 6.8. Show that*

$$|w|^2 \leqslant |u|^2 + |v|^2 + 2\cdot\langle u, v\rangle.$$

If T_p were geodesic, then the lemma would follow from the existence of the gradient, applied to the function $\mathrm{T}_p \to \mathbb{R}$ defined by $x \mapsto -(\langle u, x\rangle + \langle v, x\rangle)$ which is concave. However, the tangent space T_p might fail to be geodesic; see Halbeisen's example [4].

Applying the above lemma for $u = v$, we have the following statement.

6.10. Existence of polar vector. *Let $\mathcal{A}$ be an Alexandrov space and $p \in \mathcal{A}$. Given a vector $u \in \mathrm{T}_p$, there is a unique vector $u^* \in \mathrm{T}_p$ such that $\langle u^*, u^*\rangle + \langle u, u^*\rangle = 0$ and u^* is polar to u; that is,*

$$\langle u^*, x\rangle + \langle u, x\rangle \geqslant 0$$

for any $x \in \mathrm{T}_p$.

Proof of 6.8. By 4.12, we can choose two sequences of points a_n, b_n such that

$$\begin{aligned}\boldsymbol{d}_p\mathrm{dist}_{a_n}(w) &= -\langle \uparrow_{[pa_n]}, w\rangle\\ \boldsymbol{d}_p\mathrm{dist}_{b_n}(w) &= -\langle \uparrow_{[pb_n]}, w\rangle\end{aligned}$$

for any $w \in \mathrm{T}_p$ and $\uparrow_{[pa_n]} \to u/|u|$, $\uparrow_{[pb_n]} \to v/|v|$ as $n \to \infty$

Consider a sequence of functions

$$f_n = |u|\cdot\mathrm{dist}_{a_n} + |v|\cdot\mathrm{dist}_{b_n}.$$

Note that

$$(\boldsymbol{d}_p f_n)(x) = -|u|\cdot\langle \uparrow_{[pa_n]}, x\rangle - |v|\cdot\langle \uparrow_{[pb_n]}, x\rangle.$$

Thus we have the following uniform convergence for $x \in \Sigma_p$:

$$(\boldsymbol{d}_p f_n)(x) \to -\langle u, x\rangle - \langle v, x\rangle$$

as $n \to \infty$, According to 4.13, the sequence $\nabla_p f_n$ converges. Let

$$w = \lim_{n\to\infty} \nabla_p f_n.$$

By the definition of gradient,

$$\begin{aligned}\langle w, w\rangle &= \lim_{n\to\infty} \langle \nabla_p f_n, \nabla_p f_n\rangle = \\ &= \lim_{n\to\infty} (\boldsymbol{d}_p f_n)(\nabla_p f_n) = \\ &= -\langle u, w\rangle - \langle v, w\rangle,\end{aligned} \qquad \begin{aligned}\langle w, x\rangle &= \lim_{n\to\infty} \langle \nabla_p f_n, x\rangle \geqslant \\ &\geqslant \lim_{n\to\infty} (\boldsymbol{d}_p f_n)(x) = \\ &= -\langle u, x\rangle - \langle v, x\rangle.\end{aligned}$$

□

D Linear subspace

6.11. Definition. *Let $\mathcal{A}$ be an Alexandrov space, $p \in \mathcal{A}$ and $u, v \in$ $\in \mathrm{T}_p$. We say that vectors u and v are opposite to each other, (briefly, $u + v = 0$) if $|u| = |v| = 0$ or $\measuredangle(u, v) = \pi$ and $|u| = |v|$.*

The subcone

$$\mathrm{Lin}_p = \{\, v \in \mathrm{T}_p \,:\, \exists\, w \in \mathrm{T}_p \quad \textit{such that} \quad w + v = 0 \,\}$$

will be called the linear subspace of T_p.

Soon we will introduce a natural linear structure on Lin_p.

6.12. Proposition. *Let $\mathcal{A}$ be an Alexandrov space and $p \in \mathcal{A}$. Given two vectors $u, v \in \mathrm{T}_p$, the following statements are equivalent:*

(a) $u + v = 0$*;*

(b) $\langle u, x\rangle + \langle v, x\rangle = 0$ *for any* $x \in \mathrm{T}_p$*;*

(c) $\langle u, \xi\rangle + \langle v, \xi\rangle = 0$ *for any* $\xi \in \Sigma_p$.

Proof. The equivalence *(b)*$\Leftrightarrow$*(c)* is trivial.

The condition $u + v = 0$ is equivalent to $\langle u, u\rangle = -\langle u, v\rangle = \langle v, v\rangle$; thus, *(b)*$\Rightarrow$*(a)*.

Recall that T_p has nonnegative curvature. Note that the hinges $[0\,{}^{u}_{x}]$ and $[0\,{}^{v}_{x}]$ are adjacent. By 2.11, $\measuredangle[0\,{}^{u}_{x}]+\measuredangle[0\,{}^{v}_{x}] = \pi$; hence *(a)*$\Rightarrow$*(b)*. $\square$

6.13. Exercise. *Let $\mathcal{A}$ be an Alexandrov space and $p \in \mathcal{A}$. Then for any three vectors $u, v, w \in \mathrm{T}_p$, if $u+v = 0$ and $u+w = 0$ then $v = w$.*

Let $u \in \mathrm{Lin}_p$; that is, $u + v = 0$ for some $v \in \mathrm{T}_p$. Given $s < 0$, let

$$s{\cdot}u := (-s){\cdot}v.$$

So we can multiply any vector in Lin_p by any real number (positive and negative). By 6.13, this multiplication is uniquely defined. By 6.12, we have identity

$$\langle -v, x\rangle = -\langle v, x\rangle.$$

6.14. Exercise. *Suppose $u, v, w \in \mathrm{T}_p$ are as in 6.8. Show that*

$$\langle u, x\rangle + \langle v, x\rangle + \langle w, x\rangle = 0$$

for any $x \in \mathrm{Lin}_p$.

6.15. Exercise. *Let $\mathcal{A}$ be an Alexandrov space, $p \in \mathcal{A}$ and $u \in \mathrm{T}_p$. Suppose $u^* \in \mathrm{T}_p$ is provided by 6.10; that is,*

$$\langle u^*, u^*\rangle + \langle u, u^*\rangle = 0 \quad \textit{and} \quad \langle u^*, x\rangle + \langle u, x\rangle \geqslant 0$$

for any $x \in \mathrm{T}_p$. Show that

$$u = -u^* \quad \iff \quad |u| = |u^*|.$$

6.16. Theorem. *Let p be a point in an Alexandrov space. Then* Lin_p *is isometric to a Hilbert space.*

Proof. Note that Lin_p is a closed subset of T_p; in particular, it is complete.

If any two vectors in Lin_p can be connected by a geodesic in Lin_p, then the statement follows from the splitting theorem (6.4). By Menger's lemma (1.4), it is sufficient to show that for any two vectors $x, y \in \mathrm{Lin}_p$ there is a midpoint $w \in \mathrm{Lin}_p$.

Choose $w \in \mathrm{T}_p$ to be the anti-sum of $u = -\tfrac{1}{2}\cdot x$ and $v = -\tfrac{1}{2}\cdot y$; see 6.8. By 6.9 and 6.14,

$$\begin{aligned} |w|^2 &\leqslant \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle, \\ \langle w, x\rangle &= \tfrac{1}{2}\cdot|x|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle, \\ \langle w, y\rangle &= \tfrac{1}{2}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle, \end{aligned}$$

It follows that

$$\begin{aligned} |x - w|^2 &= |x|^2 + |w|^2 - 2\cdot\langle w, x\rangle \leqslant \\ &\leqslant \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 - \tfrac{1}{2}\cdot\langle x, y\rangle = \\ &= \tfrac{1}{4}\cdot|x - y|^2. \end{aligned}$$

That is, $|x - w| \leqslant \tfrac{1}{2}\cdot|x - y|$. Similarly, we get $|y - w| \leqslant \tfrac{1}{2}\cdot|x - y|$. Therefore w is a midpoint of x and y. In addition, we get the equality

$$|w|^2 = \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle.$$

It remains to show that $w \in \mathrm{Lin}_p$. Let w^* be the polar vector provided by 6.10. Note that

$$|w^*| \leqslant |w|, \quad \langle w^*, x\rangle + \langle w, x\rangle = 0, \quad \langle w^*, y\rangle + \langle w, y\rangle = 0.$$

The same calculation as above shows that w^* is a midpoint of $-x$ and $-y$ and

$$|w^*|^2 = \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle = |w|^2.$$

By 6.15, $w = -w^*$; hence $w \in \mathrm{Lin}_p$. □

6.17. Lemma. *Given a point p in an Alexandrov space $\mathcal{A}$, let $f = \mathrm{dist}_p$, and let S be the subset of points $x \in \mathcal{A}$ such that $|\nabla_x f| = 1$. Then S is a dense G-delta set.*

Proof. Let $S_n \subset \mathcal{A}$ be defined by inequality $|\nabla_x f| > 1 - \frac{1}{n}$. By 4.14$a$, S_n is open.

Choose a point $q \neq p$. Observe that $|\nabla_x f| = 1$ for any point $x \in \,]pq[$. It follows that S_n is dense in $\mathcal{A}$.

Since $S = \bigcap_n S_n$, the lemma follows. □

6.18. Exercise. *Let p, f, and S be as in 6.17.*

(a) Show that
$$\nabla_x f + \uparrow_{[xp]} = 0$$
for any $x \in S$; in particular, $\uparrow_{[xp]} \in \mathrm{Lin}_x$.

(b) Show that if $|\nabla_x f| = 1$, then $\boldsymbol{d}_x f(w) = \langle \nabla_x f, w \rangle$ for any $w \in \mathrm{T}_x$.

(c) Show that for any $x \in S$ there is a unique geodesic $[px]$.

Note that 6.18 implies the following.

6.19. Corollary. *Given a countable set of points X in an Alexandrov space $\mathcal{A}$ there is a G-delta dense set $S \subset \mathcal{A}$ such that $\uparrow_{[sx]} \in \mathrm{Lin}_s$ for any $s \in S$ and $x \in X$.*

E Remarks

The splitting theorem has an interesting history that starts with Stefan Cohn-Vossen [19]; who proved its 2-dimensional case. For Riemannian manifolds of higher dimensions it was proved by Victor Toponogov [92]. Then it was generalized by Anatoliy Milka [61] to Alexandrov spaces; historically, it was the first result about Alexandrov spaces of dimension higher than 2. Nearly the same proof is used in [14, 1.5].

Further generalizations of the splitting theorem for Riemannian manifolds with nonnegative Ricci curvature were obtained by Jeff Cheeger and Detlef Gromoll [18]. This was further generalized by Jeff Cheeger and Toby Colding for limits of Riemannian manifolds with almost nonnegative Ricci curvature [16] and to their synthetic generalizations, so-called RCD spaces, by Nicola Gigli [27, 28]. Jost-Hinrich Eschenburg obtained an analogous result for Lorentzian manifolds [24], that is, pseudo-Riemannian manifolds of signature $(1, n)$.

The presented proof is close in spirit to the proof given by Cheeger and Gromoll [18]; it is taken from our book [4].

6.20. Open question. *Let p be a point in an Alexandrov space $\mathcal{A}$. Suppose that $0 \neq v \in \mathrm{Lin}_p$. Is it true that the tangent space T_p splits in the direction of v?*

Halbeisen's example [4, 36] shows that compactness of space of directions is essential in the proof that space of directions is π-geodesic (see 7.5).

6.21. Open question. *Let $\mathcal{A}$ be a proper Alexandrov space. Is it true that for any $p \in \mathcal{A}$, the tangent space T_p is a length space?*

Lecture 7

Dimension and volume

This lecture shows that several different notions of dimension are the same for Alexandrov spaces. Also, we introduce volume and prove the Bishop–Gromov inequality, the right-inverse theorem and introduce the distance chart in finite-dimensional Alexandrov space.

A Linear dimension

Let $\mathcal{A}$ be an Alexandrov space. We define its linear dimension $\operatorname{LinDim}\mathcal{A}$ as the least upper bound on integers m such that the Euclidean space $\mathbb{E}^m$ is isometric to a subspace of the tangent space $\mathrm{T}_p\mathcal{A}$ at some point $p \in \mathcal{A}$. If not stated otherwise, dimension of an Alexandrov space is its linear dimension.

If not stated otherwise, dimension will mean linear dimension. In Section 7F, we will show that linear dimension of Alexandrov space coincides with all reasonable dimensions; after that, we will use $\dim\mathcal{A}$ for $\operatorname{LinDim}\mathcal{A}$.

7.1. $(n{+}1)$-comparison. *Let $\mathcal{A}$ be an* $\mathrm{ALEX}(0)$ *space. Then for any finite set of points $p, x_1, \dots, x_n \in \mathcal{A}$, there exists a model configuration $\tilde{p}, \tilde{x}_1, \dots, \tilde{x}_n \in \mathbb{E}^m$ such that*

$$|\tilde{p} - \tilde{x}_i|_{\mathbb{E}^m} = |p - x_i|_{\mathcal{A}} \quad \textit{and} \quad |\tilde{x}_i - \tilde{x}_j|_{\mathbb{E}^m} \geqslant |x_i - x_j|_{\mathcal{A}}$$

for any i and j. Moreover, we can assume that $m \leqslant \operatorname{LinDim}\mathcal{A}$.

Proof. By 6.19, we can choose a point p' arbitrarily close to p so that $\operatorname{Lin}_{p'} \ni \uparrow_{[p'x_i]}$ for any i. Let us identify $\mathbb{E}^m$ with a subspace of $\operatorname{Lin}_{p'}$ spanned by $\uparrow_{[p'x_1]}, \dots, \uparrow_{[p'x_n]}$. Note that $m \leqslant \operatorname{LinDim}\mathcal{A}$.

Set $\tilde{p}' = 0 \in \mathbb{E}^m$ and $\tilde{x}_i = |p' - x_n| \cdot \uparrow_{[p'x_n]} \in \mathbb{E}^m$ for every i. Note that

$$|\tilde{p}' - \tilde{x}_i|_{\mathbb{E}^m} = |p' - x_i|_{\mathcal{A}}$$

for every i. Applying the comparison $\measuredangle[p'\,{}^{x_i}_{x_j}] \geqslant \tilde{\measuredangle}(p'\,{}^{x_i}_{x_j})$, we get

$$|\tilde{x}_i - \tilde{x}_j|_{\mathbb{E}^m} \geqslant |x_i - x_j|_{\mathcal{A}}$$

for any i and j. Passing to a limit configuration as $p' \to p$ we get the result. $\square$

7.2. Exercise. *Let $\mathcal{A}$ be an* $\mathrm{Alex}(0)$ *space. Suppose* $\operatorname{LinDim}\mathcal{A} = m < \infty$. *Show that* $\mathrm{T}_p\mathcal{A} \overset{iso}{=\!=} \mathbb{E}^m$ *for a G-delta dense set of points* $p \in \mathcal{A}$.

7.3. Exercise. *Show that a 1-dimensional Alexandrov space is homeomorphic to a 1-dimensional manifold, possibly with non-empty boundary.*

7.4. Exercise. *Let $\mathcal{A}$ be an* $\mathrm{Alex}(0)$ *space.*

Show that $\operatorname{LinDim}\mathcal{A} \geqslant m$ *if and only if for some $m+2$ points p, $a_0, \dots, a_m \in \mathcal{A}$ we have*

$$\tilde{\measuredangle}(p\,{}^{a_i}_{a_j}) > \tfrac{\pi}{2}$$

for any pair $i \neq j$.

B Space of directions

A metric space $\mathcal{X}$ will be called ℓ-geodesic if any two points $x, y \in \mathcal{X}$ such that $|x - y| < \ell$ can be connected by a geodesic. For instance, any geodesic space is ∞-geodesic.

7.5. Theorem. *Let $\mathcal{A}$ be a finite-dimensional Alexandrov space. Then for any point $p \in \mathcal{A}$, its space of directions Σ_p is a compact π-geodesic space.*

By 4.4 this immediately gives

7.6. Corollary. *Let p be a point in a finite dimensional Alexandrov space $\mathcal{A}$, and let $\lambda_n \to \infty$. Then there is a pointed Gromov–Hausdorff convergence $(\lambda_n \cdot \mathcal{A}, p) \to (\mathrm{T}_p, 0)$.*

7.7. Exercise. *Let p be a point in a finite-dimensional Alexandrov space $\mathcal{A}$. Prove the following.*

(a) The tangent space T_p *is a proper* $\mathrm{Alex}(0)$ *space.*

(b) $\operatorname{LinDim}\Sigma_p = \operatorname{LinDim}\mathcal{A} - 1$.

(c) If $\operatorname{LinDim}\mathcal{A} > 1$*, then* Σ_p *is geodesic.*

Using 7.7*b*, one can prove results for all finite-dimensional Alexandrov spaces via induction on dimension. Such proofs will be indicated below.

Proof of 7.5. Choose $\varepsilon > 0$; suppose $\mathcal{A}$ is m-dimensional. Assume we can choose n directions $\xi_1, \dots, \xi_n \in \Sigma_p$ such that $\measuredangle(\xi_i, \xi_j) > \varepsilon$ for any $i \neq j$. Without loss of generality, we may assume that each direction is geodesic; that is, there is a point $x_i \in \mathcal{A}$ such that $\xi_i = \uparrow_{[px_i]}$.

Choose $y_i \in [px_i]$ such that $|p - y_i| = r$ for each i and small fixed $r > 0$. Since r is small, we can assume that $\tilde\measuredangle(p\,{}^{y_i}_{y_j}) > \varepsilon$ for any $i \neq j$. By 6.19, we can choose p' arbitrarily close to p such that $\uparrow_{[p'y_i]} \in \operatorname{Lin}_{p'}$ for any i. Since $|p' - p|$ is small, $\tilde\measuredangle(p'\,{}^{y_i}_{y_j}) > \varepsilon$ for any $i \neq j$. By comparison,

$$\measuredangle[p'\,{}^{y_i}_{y_j}] > \varepsilon.$$

Therefore, $n \leqslant \operatorname{pack}_\varepsilon \mathbb{S}^{m-1}$, where $\operatorname{pack}_\varepsilon \mathcal{X}$ is the exact upper bound on the number of points $x_1, \dots, x_k \in \mathcal{X}$ such that $|x_i - x_j| \geqslant \varepsilon$ if $i \neq j$.

Since $\mathbb{S}^{m-1}$ is compact, $\operatorname{pack}_\varepsilon \mathbb{S}^{m-1} < \infty$. By the definition, the space of directions Σ_p is complete. Applying 8.5, we get that Σ_p is compact.

It remains to prove the following claim.

❶ *If* Σ_p *is compact, then it is* π*-geodesic*

Choose two geodesic directions $\xi = \uparrow_{[px]}$ and $\zeta = \uparrow_{[py]}$; let

$$\alpha = \tfrac{1}{2}\cdot\measuredangle[p\,{}^x_y] = \tfrac{1}{2}\cdot|\xi - \zeta|_{\Sigma_p}.$$

Suppose $\alpha < \pi/2$. Let us show that it is sufficient to construct an almost midpoint $\mu = \uparrow_{[pz]}$ of ξ and ζ in Σ_p; that is, we need to show that for any $\varepsilon > 0$ there is a geodesic $[pz]$ such that

$$\measuredangle[p\,{}^x_z] \leqslant \alpha + \varepsilon \quad \text{and} \quad \measuredangle[p\,{}^y_z] \leqslant \alpha + \varepsilon.$$

Indeed, once this is done, the compactness of Σ_p can be used to get an actual midpoint for any two directions in Σ_p. After that, Menger's lemma (1.4) will finish the proof.

Choose a sequence of small positive numbers $r_n \to 0$ Consider sequences $x_n \in [px]$ and $y_n \in [py]$ such that $|p - x_n| = |p - y_n| = r_n$. Let m_n be a midpoint of $[x_n\,y_n]$.

Since Σ_p is compact, we can pass to a sequence of r_n such that $\uparrow_{[pm_n]}$ converges; denote its limit by μ. Choose a geodesic $[pz]$ that runs at a small angle from μ. Let us show that $\uparrow_{[pz]}$ is the needed almost midpoint.

Evidently, $\tilde\measuredangle(p\,{}^{x_n}_{m_n}) = \tilde\measuredangle(p\,{}^{y_n}_{m_n})$. By 2.5, we have

$$\tilde\measuredangle(p\,{}^{x_n}_{m_n}) + \tilde\measuredangle(p\,{}^{y_n}_{m_n}) \leqslant \tilde\measuredangle(p\,{}^{x_n}_{y_n}).$$

Let $z_n \in [pz]$ be the point such that $|p - z_n| = |p - m_n|$. By construction, for all large n, we have $\measuredangle[p\,{}^{z}_{m_n}] \approx 0$ with arbitrary small given error. By comparison, the value $\frac{|z_n - m_n|}{|p - z_n|}$ can be assumed to be arbitrarily small for all large n. Applying this observation and the definition of angle measure, we also have the following approximations

$$\begin{aligned} \tilde\measuredangle(p\,{}^{x_n}_{y_n}) &\approx \measuredangle[p\,{}^{x_n}_{y_n}], \\ \tilde\measuredangle(p\,{}^{x_n}_{m_n}) \approx \tilde\measuredangle(p\,{}^{x_n}_{z_n}) &\approx \measuredangle[p\,{}^{x_n}_{z_n}], \\ \tilde\measuredangle(p\,{}^{m_n}_{y_n}) \approx \tilde\measuredangle(p\,{}^{z_n}_{y_n}) &\approx \measuredangle[p\,{}^{z_n}_{y_n}], \end{aligned}$$

again, with arbitrary given error for all large n. It follows that $\uparrow_{[pz]}$ is an almost midpoint of $\uparrow_{[px]}$ and $\uparrow_{[py]}$, as required. $\square$

In the above proof, the angles $\measuredangle[p\,{}^{x}_{z}]$ and $\measuredangle[p\,{}^{y}_{z}]$ have lower bounds by the comparison, but we needed upper bounds that were extracted from the definition of angle measure and the compactness of space of directions.

C Right-inverse theorem

7.8. Theorem. *Suppose $p, a_0, \dots, a_m$ be points in an Alexandrov space $\mathcal{A}$ such*

$$\tilde\measuredangle(p\,{}^{a_i}_{a_j}) > \tfrac{\pi}{2}$$

for any $i \neq j$. Then the map $f\colon \mathcal{A} \to \mathbb{R}^m$ defined by

$$f\colon x \mapsto (|a_1 - x|, \dots, |a_m - x|)$$

has a right inverse defined in a neighborhood of $f(p)$.

In the proof we construct a local right inverse Φ of f around $f(p)$. The construction uses gradient flow for a suitably chosen family of functions. The structure of the proof can be seen in the following exercise; more details are given in the hints.

7.9. Exercise. *Suppose $p, a_0, \dots, a_m \in \mathcal{A}$ and $f\colon \mathcal{A} \to \mathbb{R}$ are as in 7.8. Assume $\varepsilon > 0$ is sufficiently small. Given $\boldsymbol{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$, consider the function on $\mathcal{A}$ defined by*

$$f_{\boldsymbol{y}}(x) = \min\{\, 0, |a_1 - x| - y_1, \dots, |a_m - x| - y_m \,\} + \varepsilon\cdot|a_0 - x|.$$

(a) *Show that for some fixed $r > 0$ and λ, the function $f_{\boldsymbol{y}}$ is λ-concave in* $\mathrm{B}(p, r)$*,*

(i) $(\boldsymbol{d}_x \operatorname{dist}_{a_i})(\nabla_x f_{\boldsymbol{y}}) < -\varepsilon^2$ *if* $|a_i - x| > y_i$ *and*

(ii) $(\boldsymbol{d}_x \operatorname{dist}_{a_i})(\nabla_x f_{\boldsymbol{y}}) > \varepsilon^2$ *if*

$$|a_i - x| - y_i = \min_j \{|a_j - x| - y_j\} < 0.$$

at any $x \in \mathrm{B}(p, r)$*.*

(b) *Let $\alpha_{\boldsymbol{y}}$ be $f_{\boldsymbol{y}}$-gradient curve that starts at p. Use (a) to show that*

$$\operatorname{dist}_{\boldsymbol{a}}[\alpha_{\boldsymbol{y}}(t_0)] = \boldsymbol{y}$$

if $\frac{1}{\varepsilon^2} \cdot |\operatorname{dist}_{\boldsymbol{a}} p - \boldsymbol{y}| \leqslant t_0 \leqslant \frac{r}{2}$.

(c) *Let* $t_0(\boldsymbol{y}) = \frac{1}{\varepsilon^2} \cdot ||\boldsymbol{a} - p| - \boldsymbol{y}|$*. Use 5.9 to show that the map*

$$\Phi \colon \boldsymbol{y} \mapsto \alpha_{\boldsymbol{y}} \circ t_0(\boldsymbol{y})$$

continuous in $\Omega = \mathrm{B}(|\boldsymbol{a} - p|, \frac{\varepsilon^2}{2} \cdot r) \subset \mathbb{R}^m$ *and* $f \circ \Phi(\boldsymbol{y}) = \boldsymbol{y}$ *for any* $\boldsymbol{y} \in \Omega$*.*

Note that this finishes the proof of 7.8.

D Distance chart

7.10. Theorem. *Suppose $p, a_0, \ldots, a_m$ be points in an m-dimensional Alexandrov space $\mathcal{A}$ such*

$$\tilde{\measuredangle}(p\,{}^{a_i}_{a_j}) > \tfrac{\pi}{2}$$

for any $i \neq j$. Then the map $f \colon \mathcal{A} \to \mathbb{R}^m$ defined by

$$f \colon x \mapsto (|a_1 - x|, \ldots, |a_m - x|)$$

gives a bi-Lipschitz embedding of a neighborhood Ω of p; the restriction $f|_\Omega$ is called distance chart *at p.*

The following exercise guides you to prove the theorem.

7.11. Exercise. *Suppose $p, a_0, \ldots, a_m \in \mathcal{A}$ and $f \colon \mathcal{A} \to \mathbb{R}$ are as in 7.8. Show that there is $\varepsilon > 0$ such that one of the following m inequalities hold*

$$\begin{aligned} &\measuredangle[x\,{}^{y}_{a_1}] < \tfrac{\pi}{2} - \varepsilon, \ \ldots, \ \measuredangle[x\,{}^{y}_{a_m}] < \tfrac{\pi}{2} - \varepsilon, \\ &\measuredangle[y\,{}^{x}_{a_1}] < \tfrac{\pi}{2} - \varepsilon, \ \ldots, \ \measuredangle[y\,{}^{x}_{a_m}] < \tfrac{\pi}{2} - \varepsilon \end{aligned}$$

for any two points x, y in a sufficiently small neighborhood of p.

Use this together with the right-inverse theorem (7.8) to prove 7.10.

E Volume

Fix a positive integer m. The m-dimensional Hausdorff measure of a Borel set B in a metric space will be called its m-volume; it will be denoted by $\mathrm{vol}_m B$. We assume that the Hausdorff measure is calibrated so that the unit cube in $\mathbb{E}^m$ has unit volume.

This definition will be applied mostly to subsets in m-dimensional Alexandrov spaces. In this case, we may write $\mathrm{vol}\, B$ instead of $\mathrm{vol}_m B$.

7.12. Bishop–Gromov inequality. *Let $\mathcal{A}$ be an* $\mathrm{ALEX}(0)$ *space. Suppose* $\dim \mathcal{A} = m < \infty$. *Then*

$$\mathrm{vol}\, \mathrm{B}(p,r) \leqslant \omega_m \cdot r^m,$$

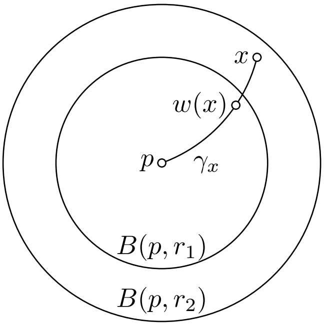
where ω_m denotes the volume of the unit ball in $\mathbb{E}^m$. Moreover, the function

$$r \mapsto \frac{\mathrm{vol}\, B(p,r)}{r^m}$$

is nonincreasing.

Proof. Given $x \in \mathcal{A}$ choose a geodesic path γ_x from p to x. Recall that $\log_p \colon \mathcal{A} \to \mathrm{T}_p$ can be defined by $\log_p \colon x \mapsto \gamma_x^+(0)$. By comparison, $\log_p$ is distance-noncontracting. Note that $\log_p$ maps $\mathrm{B}(p,r)_{\mathcal{A}}$ to $\mathrm{B}(0,r)_{\mathrm{T}_p}$.

If $\mathrm{T}_p \overset{iso}{=\!=} \mathbb{E}^m$, then $\mathrm{vol}\, \mathrm{B}(0,r)_{\mathrm{T}_p} = \omega_m \cdot r^m$, and the first statement follows.



If T_p is not isometric to $\mathbb{E}^m$, then by 7.2, we can find a point p' arbitrarily close to p such that $\mathrm{T}_{p'} \overset{iso}{=\!=} \mathbb{E}^m$. If $\varepsilon > |p - p'|$, then $\mathrm{B}(p,r) \subset \mathrm{B}(p', r+\varepsilon)$. Therefore,

$$\mathrm{vol}\, \mathrm{B}(p,r) \leqslant \omega_m \cdot (r+\varepsilon)^m$$

for any $\varepsilon > 0$. Hence the first statement follows.

Now, suppose $0 < r_1 < r_2$. Consider the map $w \colon \mathcal{A} \to \mathcal{A}$ defined by $w \colon x \mapsto \gamma_x(\tfrac{r_1}{r_2})$. (The map w mimics the dilation with center at p and coefficient $\tfrac{r_1}{r_2}$.) By comparison,

$$|w(x) - w(y)| \geqslant \tfrac{r_1}{r_2} \cdot |x - y|.$$

Observe that $\mathrm{B}(p,r_1) \supset w[\mathrm{B}(p,r_2)]$. Therefore,

$$\mathrm{vol}\, \mathrm{B}(p,r_1) \geqslant (\tfrac{r_1}{r_2})^m \cdot \mathrm{vol}\, \mathrm{B}(p,r_2).$$

□

The following exercise generalizes the Bishop–Gromov inequality to $\mathrm{A\small{LEX}}(-1)$ case. It is sufficient for most applications, but a more exact statement will be given in 7.17 which also includes the case of $\mathrm{A\small{LEX}}(1)$ spaces.

7.13. Exercise. *Show that any finite-dimensional Alexandrov space is proper.*

7.14. Exercise. *Let $\mathcal{A}$ be an* $\mathrm{A\small{LEX}}(-1)$ *space. Suppose $\mathcal{A} = m < \infty$. Show that*

$$\operatorname{vol} \mathrm{B}(p,r) \leqslant \omega_m \cdot (\sinh r)^m,$$

where ω_m denotes the volume of the unit ball in $\mathbb{E}^m$. Moreover, the function

$$r \mapsto \frac{\operatorname{vol} B(p,r)}{(\sinh r)^m}$$

is nonincreasing.

F Other dimensions

Next we want to show that *all reasonable definitions of dimension give the same result for Alexandrov spaces.* More precisely, we have the following theorem; compare to [4, 15.16]. We refer to [41] for definitions of Lebesgue covering dimension TopDim and Hausdorff dimension HausDim.

7.15. Theorem. *For any Alexandrov space $\mathcal{A}$, we have*

$$\operatorname{LinDim} \mathcal{A} = \operatorname{TopDim} \mathcal{A} = \operatorname{HausDim} \mathcal{A}.$$

Proof. Suppose $\operatorname{LinDim} \mathcal{A} = \infty$. By the right-inverse theorem (7.8), $\mathcal{A}$ contains a compact subset K with an arbitrarily large $\operatorname{TopDim} K$. In particular,

$$\operatorname{TopDim} \mathcal{A} = \infty.$$

By Szpilrajn's theorem, $\operatorname{HausDim} K \geqslant \operatorname{TopDim} K$. Thus we also have

$$\operatorname{HausDim} \mathcal{A} = \infty.$$

Now suppose $\operatorname{LinDim} \mathcal{A} = m < \infty$. By the Bishop–Gromov inequality (7.12 and 7.14),

$$\operatorname{HausDim} \mathcal{A} \leqslant m.$$

Since $\mathcal{A}$ is proper (7.13), Szpilrajn's theorem, implies that

$$\operatorname{TopDim}\mathcal{A} \leqslant \operatorname{HausDim}\mathcal{A} \leqslant m.$$

Finally, the right-inverse theorem (7.8) implies that $m \leqslant \operatorname{TopDim}\mathcal{A}$. □

7.16. Exercise. *Let Ω be an open subset of Alexandrov space $\mathcal{A}$. Show that*

$$\operatorname{LinDim}\mathcal{A} = \operatorname{LinDim}\Omega = \operatorname{TopDim}\Omega = \operatorname{HausDim}\Omega.$$

G Remarks

Let us state a version of the Bishop–Gromov inequality for $\mathrm{A{\scriptsize LEX}}(\kappa)$ spaces. Its proof requires additionally the so-called *coarea formula* for Alexandrov spaces. The weaker inequality from 7.14 is sufficient for the sequel.

7.17. Bishop–Gromov inequality. *Given a point p in an m-dimensional $\mathrm{A{\scriptsize LEX}}(\kappa)$ space, consider the function $v(r) = \operatorname{vol}_m \mathrm{B}(p,r)$; denote by $\tilde v(r)$ the volume of r ball in $\mathbb{M}^m(\kappa)$. Then*

$$v(r) \leqslant \tilde v(r)$$

for $r > 0$ and the function

$$r \mapsto \frac{v(r)}{\tilde v(r)}$$

is nonincreasing. If $\kappa > 0$, then one has to assume that $r < \frac{\pi}{\sqrt{\kappa}}$.

This inequality was originally proved for Riemannian manifolds with lower Ricci curvature. The first part is also called Bishop's inequality. It is due to Richard Bishop; see [12] and [11, Corollary 4, p. 256]. The second part is due to Michael Gromov [30].

Theorem 7.15, was essentially proved by Conrad Plaut [82]. At that time, it was not known whether

$$\operatorname{LinDim}\mathcal{A} = \infty \quad \Rightarrow \quad \operatorname{TopDim}\mathcal{A} = \infty$$

for any Alexandrov space $\mathcal{A}$. The latter implication was proved by Grigory Perelman and the second author [67].

Lecture 8

Limit spaces

In this lecture we show that lower curvature bound in the sense of Alexandrov survives under Gromov–Hausdorff limit and prove the Gromov selection theorem. This theorem is the main source of applications of Alexandrov geometry, as an illustration we prove the homotopy stability theorem (8.12) and deduce the homotopy finiteness theorem (8.13) from it.

A Survival of curvature bounds

8.1. Theorem. *Let $\mathcal{X}_n \to \mathcal{X}_\infty$ be a convergence in the sense of Gromov–Hausdorff. Suppose that each for each n, the space $\mathcal{X}_n$ has curvature $\geqslant \kappa$ in the sense of Alexandrov. Then the same holds for $\mathcal{X}_\infty$.*

Proof. Choose a quadruple of points $p_\infty, x_\infty, y_\infty, z_\infty \in \mathcal{X}_\infty$.

By the definition of Gromov–Hausdorff convergence, we can choose points p_n, x_n, y_n, $z_n \in \mathcal{X}_n$ for each n that converge to p_∞, x_∞, y_∞, $z_\infty \in \mathcal{X}_\infty$, respectively. In particular, each of the 6 distances between pairs of p_n, x_n, y_n, z_n converge to the distance between the corresponding pairs of $p_\infty, x_\infty, y_\infty, z_\infty$.

Since $\mathbb{M}^2(\kappa)$-comparison holds for p_n, x_n, y_n, $z_n \in \mathcal{X}_n$, passing to the limit, we get the $\mathbb{M}^2(\kappa)$-comparison for p_∞, x_∞, y_∞, z_∞. □

8.2. Exercise. *Suppose that a sequence $\mathcal{A}_1, \mathcal{A}_2, \dots$ of* Alex(κ) *spaces that converges to $\mathcal{A}_\infty$ in the sense of Gromov–Hausdorff. Show that $\mathcal{A}_\infty$ is* Alex(κ) *and*

$$\dim \mathcal{A}_\infty \leqslant \varliminf_{n\to\infty} \dim \mathcal{A}_n.$$

B Gromov's selection theorem

8.3. Theorem. *Let $D, \kappa \in \mathbb{R}$, and m be a positive integer. Then any sequence of m-dimensional* $\mathrm{A\small LEX}(\kappa)$ *spaces with diameters at most D has a converging subsequence in the sense of Gromov–Hausdorff.*

Let X be a subset of a metric space $\mathcal{W}$. Recall that a set $Z \subset \mathcal{W}$ is called ε-net of X if for any point $x \in X$, there is a point $z \in Z$ such that $|x - z| < \varepsilon$.

We will use the following characterization of compact sets.

8.4. Exercise. *A closed subset X of a complete metric space.*

(a) Show that X is compact if and only if it admits a finite ε-net for any $\varepsilon > 0$.

(b) Show that X is compact if and only if it admits a compact ε-net for any $\varepsilon > 0$.

Recall that $\operatorname{pack}_\varepsilon \mathcal{X}$ is the exact upper bound on the number of points $x_1, \dots, x_n \in \mathcal{X}$ such that $|x_i - x_j| \geqslant \varepsilon$ if $i \neq j$.

If $n = \operatorname{pack}_\varepsilon \mathcal{X} < \infty$, then the collection of points $x_1, \dots, x_n$ is called a maximal ε-packing.

8.5. Exercise. *Show that any maximal ε-packing $x_1, \dots, x_n$ is an ε-net. Conclude that a complete metric space $\mathcal{X}$ is compact if and only if* $\operatorname{pack}_\varepsilon \mathcal{X} < \infty$ *for any $\varepsilon > 0$.*

Proof of 8.3. Denote by $\boldsymbol{K}$ the set of isometry classes of $\mathrm{A\small LEX}(0)$ spaces with dimension $\leqslant m$ and diameter $\leqslant D$. By 8.2, $\boldsymbol{K}$ is a closed subset of GH.

Choose a space $\mathcal{A} \in \boldsymbol{K}$; suppose $x_1, \dots, x_n \in \mathcal{A}$ is a collection of points such that $|x_i - x_j| \geqslant \varepsilon$ for all $i \neq j$. Note that the balls $B_i = \mathrm{B}(x_i, \frac{\varepsilon}{2})$ do not overlap.

By 7.8, $\operatorname{vol} \mathcal{A} > 0$. By Bishop–Gromov inequality, $\operatorname{vol} \mathcal{A} < \infty$, and if $\varepsilon < D$, then

$$\operatorname{vol} B_i \geqslant (\tfrac{\varepsilon}{2 \cdot D})^m \cdot \operatorname{vol} \mathcal{A}$$

for any i. It follows that $n \leqslant (\frac{2 \cdot D}{\varepsilon})^m$; that is,

$$\operatorname{pack}_\varepsilon \mathcal{A} \leqslant N(\varepsilon) := (\tfrac{2 \cdot D}{\varepsilon})^m$$

for all small $\varepsilon > 0$.

Choose a maximal ε-packing $x_1, \dots, x_n \in \mathcal{A}$. By 8.5, $\mathcal{F}_\varepsilon := \{x_1, \dots \dots, x_n\}$ is an ε-net of $\mathcal{A}$. Observe that $|\mathcal{F}_\varepsilon - \mathcal{A}|_{\mathrm{GH}} \leqslant \varepsilon$. Further, note that the set $\boldsymbol{F}_\varepsilon$ of finite metric spaces with diameter $\leqslant D$ and at most $N(\varepsilon)$ points forms a compact subset in GH.

Summarizing, for any $\varepsilon > 0$ we can find a compact ε-net $\boldsymbol{F}_\varepsilon \subset \mathrm{GH}$ of $\boldsymbol{K}$. Since GH is complete (1.20), it remains to apply 8.4*b*.

Note that rescaling reduces $\mathrm{ALEX}(\kappa)$ case to the $\mathrm{ALEX}(-1)$ case. The latter can be proved the same way, using 7.14 instead of 7.12. □

8.6. Exercise.

(a) Let $\mathcal{A}$ be an m-dimensional $\mathrm{ALEX}(0)$ *space with diameter $\leqslant D$. Suppose* $\operatorname{vol}\mathcal{A} \geqslant v_0 > 0$. *Show that*

$$\operatorname{pack}_\varepsilon \mathcal{A} \geqslant \frac{c}{\varepsilon^m}$$

for some constant $c = c(m, D, v_0) > 0$.

(b) Conclude that if $\mathcal{A}_n$ is a sequence of m-dimensional $\mathrm{ALEX}(0)$ *spaces with diameter $\leqslant D$, and volume $\geqslant v_0$, then its Gromov–Hausdorff limit $\mathcal{A}_\infty$ (if it exists) has dimension m.*

8.7. Exercise. *Let $(\mathcal{A}_1, p_1), (\mathcal{A}_2, p_2), \ldots$ be a sequence of m-dimensional* $\mathrm{ALEX}(\kappa)$ *spcaes with marked points. Show that it contains a subsequence pointed-converging in the sense of Gromov–Hausdorff.*

C Controlled concavity

Alexandrov spaces have plenty of semiconcave functions; for instance, squared distance functions. The following theorem provides a source of strictly concave functions defined on small open sets of finite-dimensional Alexandrov spaces.

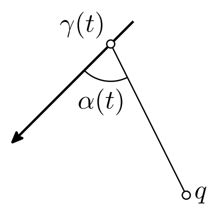
8.8. Theorem. *Let $\mathcal{A}$ be a complete finite-dimensional Alexandrov space. Then for any point $p \in \mathcal{A}$, there is a strictly concave function f defined in an open neighborhood of p.*

Moreover, given $0 \neq v \in \mathrm{T}_p$, the differential, $\mathbf{d}_p f$, can be chosen arbitrarily close to $x \mapsto -\langle v, x\rangle$.

Proof. Fix small $r > 0$ and large c; consider the real-to-real function

$$\varphi_{r,c}(x) = (x - r) - c{\cdot}(x - r)^2/r,$$

so we have $\varphi_{r,c}(r) = 0$, $\varphi'_{r,c}(r) = 1$, and $\varphi''_{r,c}(r) = -2c/r$.



Let γ be a unit-speed geodesic, fix a point q and let

$$\alpha(t) = \measuredangle(\gamma^+(t), \uparrow_{[\gamma(t)q]}).$$

Recall that r is small. If $|q-\gamma(t)|$ is sufficiently close to r, then direct calculations show that

$$(\varphi_{r,c}\circ\operatorname{dist}_q\circ\gamma)''(t)\leqslant\frac{3-c\cdot\cos^2[\alpha(t)]}{r}.$$

(Since c is large, this inequality implies that $\varphi_{r,c}\circ\operatorname{dist}_q\circ\gamma$ is strictly concave at t unless $\alpha(t)\approx\frac{\pi}{2}$.)

Now, assume $\{q_1,\dots,q_N\}$ is a finite set of points such that $|p-$ $-q_i|=r$ for any i. For a geodesic γ, set $\alpha_i(t)=\measuredangle(\gamma^+(t),\uparrow_{[\gamma(t)q_i]})$. Assume we have a collection $\{q_i\}$ such that

$$\max_i\{|\alpha_i(t)-\tfrac{\pi}{2}|\}\geqslant\varepsilon>0$$

for any geodesic γ in $\mathrm{B}(p,\varepsilon)$. We can assume that $c>3N/\cos^2\varepsilon$; then the inequality above implies that the function

$$f=\sum_i\varphi_{r,c}\circ\operatorname{dist}_{q_i}$$

is strictly concave in $\mathrm{B}(p,\varepsilon')$ for some positive $\varepsilon'<\varepsilon$.

The same argument as in 8.6 shows that for small $r>0$, one can choose $N\geqslant c/\delta^{m-1}$ points $\{q_i\}$ such that $|p-q_i|=r$ and $\tilde\measuredangle(p\,{}^{q_j}_{q_i})>\delta$ (here $c=c(\Sigma_p)>0$). On the other hand, suppose γ runs from x to y. If $|\alpha_i(t)-\frac{\pi}{2}|<\varepsilon\ll\delta$, then applying the $(n{+}1)$-point comparison to $\gamma(t)$, x, y and all $\{q_i\}$ we get that $N\leqslant c(m)/\delta^{m-2}$. Therefore, for small $\delta>0$ and yet smaller $\varepsilon>0$, the set $\{q_i\}$ forms the needed collection.

If r is small, then points q_i can be chosen so that all directions $\uparrow_{[pq_i]}$ will be ε-close to a given direction ξ and therefore the second property follows. $\square$

The function f in 8.8 can be chosen to have maximum value 0 at p, $f(p)=0$ and with $\boldsymbol{d}_pf(x)\approx-|x|$. It can be constructed by taking the minimum of the functions in the theorem. Then the set $K=\{\,x\in\mathcal{A}:f(x)\geqslant-\varepsilon\,\}$ forms a closed convex neighborhood of p for any small $\varepsilon>0$, so we get the following.

8.9. Corollary. *Any point p of a finite-dimensional Alexandrov space admits an arbitrarily small convex closed neighborhood K and a strictly concave function f defined in a neighborhood of K such that p is the maximum point of f and $f|_{\partial K}=0$.*

8.10. Exercise. *Construct an Alexandrov space $\mathcal{A}$ such that there is no strictly concave function with an open domain of definition in $\mathcal{A}$.*

D Liftings

Suppose that the Gromov–Haudorff distance $|\mathcal{A}-\mathcal{A}'|_{\mathrm{GH}}$ is sufficiently small, so we may think that both spaces $\mathcal{A}$ and $\mathcal{A}'$ lie at a small Hausdorff distance in an ambient metric space $\mathcal{W}$. In particular, we may choose a small $\varepsilon > 0$, so that for any point $p \in \mathcal{A}$, there is a point $p' \in \mathcal{A}'$ such that $|p-p'|_{\mathcal{W}} < \varepsilon$; the point p' will be called a lifting (or ε-lifting) of p in $\mathcal{A}'$. We may choose a lifting $p' \in \mathcal{A}'$ for every point $p \in \mathcal{A}$, in this case the map $p \mapsto p'$ is called a (ε-)lifting map.

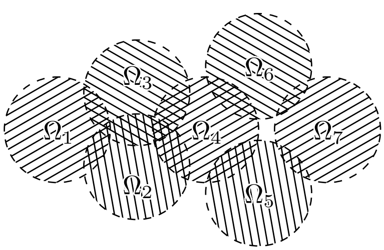
Note that the lifting is not uniquely defined. The lifting map is not assumed to be continuous. When we talk about liftings, we assume that $\varepsilon > 0$, the inclusions $\mathcal{A}, \mathcal{A}' \hookrightarrow \mathcal{W}$, as well as $\mathcal{W}$ are chosen.

Let $\mathcal{A}$ be a compact m-dimensional Alexandrov space. Suppose $\mathcal{A}'$ is another compact m-dimensional Alexandrov space such that $|\mathcal{A}-\mathcal{A}'|_{\mathrm{GH}}$ is sufficiently small — smaller than some $\varepsilon = \varepsilon(\mathcal{A}) > 0$. Then the construction in $\mathcal{A}$ from the previous section can be repeated in $\mathcal{A}'$ for the liftings of all points and the same function φ. It produces a strictly concave function defined in a controlled neighborhood of the lifting p' of p.

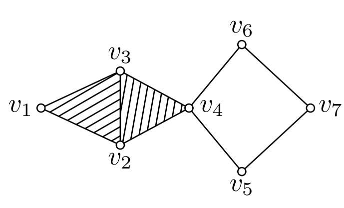
The result of this and related constructions will be called liftings, say we can talk about a lifting from $\mathcal{A}$ to $\mathcal{A}'$ of a function provided by 8.8 (if the Gromov–Hausdorff distance $|\mathcal{A}-\mathcal{A}'|_{\mathrm{GH}}$ is small, then these liftings are stricly concave) and a lifting of a convex neighborhood from 8.9. Here one cannot use 8.8 and 8.9 as black boxes — one has to understand the construction, but it is straightforward.

E Nerves

Let $\{\Omega_1, \dots, \Omega_k\}$ be a finite open cover of a compact metric space $\mathcal{X}$. Consider an abstract simplicial complex $\mathcal{N}$, with one vertex v_i for each set Ω_i such that a simplex with vertices $v_{i_1}, \dots, v_{i_m}$ is included in $\mathcal{N}$ if the intersection $\Omega_{i_1} \cap \dots \cap \Omega_{i_m}$ is non-empty. The obtained simplicial



covering



nerve

complex $\mathcal{N}$ is called the nerve of the covering $\{\Omega_i\}$. Evidently, $\mathcal{N}$

is a finite simplicial complex — it is a subcomplex of a simplex with the vertices $\{v_1,\dots,v_k\}$. Recall that Star_{v_i} denotes the union of all simplices in $\mathcal{N}$ that shares vertex v_i.

The next statement follows from [38, 4G.3].

8.11. Nerve theorem. *Let $\{\Omega_1,\dots,\Omega_k\}$ be an open cover of a compact metric space $\mathcal{X}$ and let $\mathcal{N}$ be the corresponging nerve with vertices $\{v_1,\dots,v_k\}$. Suppose that every non-empty finite intersection $\Omega_{\alpha_1}\cap\dots\cap\Omega_{\alpha_k}$ is contractible. Then $\mathcal{X}$ is homotopy equivalent to the nerve $\mathcal{N}$ of the cover.*

Moreover homotopy equivalences $a\colon\mathcal{X}\to\mathcal{N}$ and $b\colon\mathcal{N}\to\mathcal{X}$ can be chosen so that if $x\in\Omega_i$, then $a(x)\in\mathrm{Star}_{v_i}$, and if $y\in\mathcal{N}$ lies in the simplex with vertices $v_{i_1},\dots,v_{i_m}$, then $b(y)\in\Omega_{i_1}\cup\dots\cup\Omega_{i_m}$.

F Homotopy stability

8.12. Theorem. *Let $\mathcal{A}_1,\mathcal{A}_2,\dots$, and $\mathcal{A}_\infty$ be compact m-dimensional* Alex(κ) *spaces, and $m<\infty$. Suppose $\mathcal{A}_n\to\mathcal{A}_\infty$ as $n\to\infty$ in the sense of Gromov–Hausdorff. Then $\mathcal{A}_\infty$ is homotopically equivalent to $\mathcal{A}_n$ for all large n.*

Moreover, given $\varepsilon>0$ there are maps $h_n\colon\mathcal{A}_\infty\to\mathcal{A}_n$ that are homotopy equivalences and ε-liftings for all large n.

Applying this theorem with Gromov's selection theorem (8.3) and Exercise 8.6, we get the following.

8.13. Theorem. *There are only finitely many homotopy types of m-dimensional* Alex(κ) *spaces with diameter $\leqslant D$, and volume $\geqslant v_0$; here we assume that an integer m, and $v_0>0$ and $D>0$ are given.*

Proof of 8.13 modulo 8.12. Assume the contrary, then we can choose a sequence of spaces $\mathcal{A}_1,\mathcal{A}_2,\dots$ that have different homotopy types and satisfy the assumptions of the theorem. By Gromov's selection theorem, we can assume that $\mathcal{A}_n$ converges to say $\mathcal{A}_\infty$ in the sense of Gromov–Hausdorff.

By 8.6, $\dim\mathcal{A}_\infty=m$. It remains to apply 8.12. □

Proof of 8.12. Since $\mathcal{A}_\infty$ is compact, applying 8.9, we can find a finite open cover of $\mathcal{A}_\infty$ by convex open sets $\Omega_1,\dots,\Omega_k$ such that for each Ω_i there is a strictly concave function f_i that is defined in a neighborhood of $\bar\Omega_i$ and such that $f_i|_{\partial\Omega_i}=0$.

Subtracting from functions f_i some small value $\varepsilon>0$, we can ensure that $\bigcap_{i\in S}\Omega_i\neq\varnothing$ if and only if $\bigcap_{i\in S}\bar\Omega_i\neq\varnothing$.

Suppose that $W = \bigcap_{i\in S}\Omega_i \neq \varnothing$. Then W is contractible. Indeed, the function

$$f_S := \min_{i\in S} f_i$$

is strictly concave and it vanishes on the boundary of W. The f_S-gradient flow $(t,x)\mapsto \mathrm{Flow}^t_{f_S}(x)$ defines a homotopy $[0,\infty)\times W \to W$. By the first distance estimate (5.6), $\mathrm{Flow}^t_{f_S}(x)$ converges to the (necessarily unique) maximum point of f_S as $t\to\infty$. Therefore, in the obtained homotopy we can parametrize $[0,\infty)$ by $[0,1)$ and extend the homotopy continuously to $[0,1]$; thus we get that W is contractible. In other words, the cover $\{\Omega_1,\dots,\Omega_k\}$ meets the assumptions of the nerve theorem (8.11).

The functions f_i and sets Ω_i can be lifted to $\mathcal{A}_n$ keeping their properties for all large n. More precisely, there are liftings $f_{i,n}$ of all f_i to $\mathcal{A}_n$ which are strictly concave for all large n and such that $\bar\Omega_{i,n} = \{\,x\in\mathcal{A}_n : f_{i,n}(x)\geqslant 0\,\}$ is a compact convex set and $\Omega_{i,n} = \{\,x\in\mathcal{A}_n : f_{i,n}(x)>0\,\}$ is an open convex set for each i.

Notice that $\{\Omega_{1,n},\dots,\Omega_{k,n}\}$ is an open cover of $\mathcal{A}_n$ for all large n. Indeed suppose we have $p_n\in\mathcal{A}_n\setminus(\Omega_{1,n}\cup\dots\cup\Omega_{k,n})$ for arbitrary large n. Since $\mathcal{A}_\infty$ is compact, there is a limit point $p_\infty\in\mathcal{A}_\infty$ for a subsequnce of p_n. But $p_\infty\in\Omega_i$ for some i and therefore $p_n\in\Omega_{i,n}$ for arbitrary large n — a contradiction.

In a similar fashion, we can show that if n is large, then any collection $\{\Omega_{i,n}\}_{i\in S}$ has a common point in $\mathcal{A}_n$ if and only if $\{\Omega_i\}_{i\in S}$ has a common point in $\mathcal{A}_\infty$. Here we have to use that $\bigcap_{i\in S}\Omega_i\neq\varnothing$ if and only if $\bigcap_{i\in S}\bar\Omega_i\neq\varnothing$.

It follows that for any large n the covers

- ⋄ $\{\Omega_1,\dots,\Omega_k\}$ of $\mathcal{A}_\infty$ and
- ⋄ $\{\Omega_{1,n},\dots,\Omega_{k,n}\}$ of $\mathcal{A}_n$.

have the same nerve. By the nerve theorem (8.11), $\mathcal{A}_n$ and $\mathcal{A}_\infty$ are homotopically equivalent for all large n — a contradiction. □

Note that the above proof also shows the following.

8.14. Theorem. *Any compact finite-dimensional Alexandrov space is homotopy equivalent to a finite simplicial complex.*

G Remarks

Gromov's selection theorem provides the main source of applications of Alexandrov spaces to Riemannian geometry. The homotopy-type finiteness theorem (8.13) illustrates this technique.

Originally, Gromov's selection theorem was proved for Riemannian manifolds with a lower bound on Ricci curvature [30]. It motivates the study of limits iof manifolds with lower Ricci curvature bounds and their synthetic generalizations, the so-called $\mathrm{CD}(K, m)$ spaces; CD stands for curvature-dimension condition. This theory has significant applications in Alexandrov geometry; in particular, it provides a version of the Liouville theorem about phase-space volume of geodesic flow in Alexandrov space [13].

The construction of a strictly concave function (8.8) is due to Grigory Perelman [66, 69].

Let us list some results that can be proved by applying Gromov's selection theorem in the same fashion as in the proof of homotopy-type finiteness theorem (8.13).

8.15. Betti-number theorem. *There is a constant $c = c(m, D, \kappa)$ such that*

$$\beta_0(M) + \beta_1(M) + \dots + \beta_m(M) \leqslant c$$

for any closed m-dimensional Riemannian manifold M with sectional curvature $\geqslant \kappa$ and diameter $\leqslant D$. Here $\beta_i(M)$ denotes i^{th} Betti number of M.

Gromov's original proof [29] of the Betti-number theorem did not use Alexandrov geometry directly; but it is quite natural to prove it via Gromov's selection theorem. The following result was proved by the second author [77], and it uses the same technique.

8.16. Scalar curvature bound. *There is a constant $c = c(m, D, \kappa)$ such that*

$$\int\limits_M \mathrm{Sc} \leqslant c$$

for any closed m-dimensional Riemannian manifold M with sectional curvature $\geqslant \kappa$ and diameter $\leqslant D$. Here Sc *denotes the scalar curvature.*

The following theorem is a more exact version of 8.12. Its close relative (9.1) will play an important role in the following lecture.

8.17. Stability theorem. *Let $\mathcal{A}_1, \mathcal{A}_2, \dots,$ and $\mathcal{A}_\infty$ be compact m-dimensional $\mathrm{A\small{LEX}}(\kappa)$ spaces, and $m < \infty$. Suppose $\mathcal{A}_n \to \mathcal{A}_\infty$ as $n \to \infty$ in the sense of Gromov–Hausdorff. Then $\mathcal{A}_\infty$ is homeomorphic to $\mathcal{A}_n$ for all large n.*

Moreover, given $\varepsilon > 0$ there are maps $h_n\colon \mathcal{A}_\infty \to \mathcal{A}_n$ that are homeomorphisms and ε-liftings for all large n.

This theorem was proved by Grigory Perelman [68]; the proof was rewritten with more details by the first author [43]. Perelman informally claimed in private conversations that the homeomorphisms in the theorem can be assumed to be bi-Lipschitz with constants that depend on $\mathcal{A}_\infty$; however no written proof has been presented.

Theorem 8.13 was originally proved by Karsten Grove and Peter Petersen [33]. Perelman's stability theorem (8.17) implies the following stronger statement.

8.18. Homeomorphism-type finiteness. *There are only finitely many homeomorphism types of closed m-dimensional manifolds that admit a Riemannian metric with sectional curvature $\geqslant \kappa$, and diameter $\leqslant D$.*

Applying several results in differential topology, this statement can be improved to diffeomorphism-type finiteness in all dimensions m except $m = 4$ because it is known that except for $m = 4$ a closed topological m-manifold admits only finitely many smooth structures; see [45] and [63, 91] for cases $m \geqslant 5$ and $m \leqslant 3$, respectively.

Lecture 9

Boundary

This lecture defines the boundary of a finite-dimensional Alexandrov space. After discussing its properties, we prove the doubling theorem ($9.9d$).

A Definition

Let us give an inductive definition of the boundary of finite-dimensional Alexandrov spaces.

Suppose $\mathcal{A}$ is a 1-dimensional Alexandrov space. By Exercise 7.3, $\mathcal{A}$ is homeomorphic to a 1-dimensional manifold (possibly with non-empty boundary). This allows us to define the boundary $\partial\mathcal{A} \subset \mathcal{A}$ as the boundary of the manifold.

Now assume that the notion of boundary is defined in dimensions $1, \dots, m-1$. Suppose $\mathcal{A}$ is m-dimensional Alexandrov space. We say that $p \in \mathcal{A}$ belongs to the boundary (briefly $p \in \partial\mathcal{A}$) if $\partial\Sigma_p \neq \varnothing$. By 7.5 and $7.7b$, Σ_p is an $(m-1)$-dimensional Alexandrov space; therefore its boundary is already defined and hence this inductive definition makes sense.

It is instructive to check the following statements.

- ◇ For a closed convex set $K \subset \mathbb{E}^m$ with non-empty interior, the topological boundary of K as a subset of $\mathbb{E}^m$ coincides with the boundary K described above.
- ◇ If $\mathcal{A} \stackrel{iso}{=\!=} \mathcal{A}_1 \times \mathcal{A}_2$ is a finite-dimensional Alexandrov space, then

$$\partial\mathcal{A} = (\partial\mathcal{A}_1 \times \mathcal{A}_2) \cup (\mathcal{A}_1 \times \partial\mathcal{A}_2)$$

- ◇ If $\operatorname{Cone}\Sigma$ is an $\mathrm{ALEX}(0)$ space of dimensions $\geqslant 2$ (this necessarily implies that $\operatorname{Cone}\Sigma$ is $\mathrm{ALEX}(1)$ then

$$\partial \operatorname{Cone}\Sigma = \operatorname{Cone}\partial\Sigma,$$

where $\operatorname{Cone}\partial\Sigma = \{\, s\cdot\xi \in \operatorname{Cone}\Sigma \;:\; \xi\in\partial\Sigma\,\}$.

B Conic neighborhoods

The following statement [69] is a close relative of Perelman's stability theorem 8.17. We are going to use this result without proof.

Recall that the logarithm $\log_p x\colon \mathcal{A}\to \mathrm{T}_p$ is defined on page 39.

9.1. Theorem. *For any point p in a finite-dimensional Alexandrov space $\mathcal{A}$ and all sufficiently small $\varepsilon>0$ there is a homeomorphism $h_\varepsilon\colon \mathrm{B}(p,\varepsilon)_{\mathcal{A}}\to \mathrm{B}(0,\varepsilon)_{\mathrm{T}_p}$ such that $0=h_\varepsilon(p)$.*

Moreover, we may assume that

$$\sup_{x\in\mathrm{B}(p,\varepsilon)}\{\tfrac{1}{\varepsilon}\cdot|\log_p x - h_\varepsilon(x)|_{\mathrm{T}_p}\}\to 0 \quad \textit{as} \quad \varepsilon\to 0.$$

This statement is often used together with the *uniqueness of conic neighborhoods* stated below.

Suppose that an open neighborhood U of a point x in a metric space $\mathcal{X}$ admits a homeomorphism to $\operatorname{Cone}\Sigma$ such that x is mapped to the origin of the cone. In this case, we say that U has a conic neighborhood of x.

9.2. Uniqueness of conic neighborhoods. *Any two conic neighborhoods of a given point in a metric space are pointed homeomorphic; that is, there is a homeomorphism between neighborhoods that maps the origin of one cone to the origin of the other.*

9.3. Advanced exercise. *Prove 9.2 or read the proof in [47].*

9.4. Exercise. *Suppose $x\mapsto x'$ is a homeomorphism between finite-dimensional Alexandrov spaces $\mathcal{A}$ and $\mathcal{A}'$. Show that*

(a) $\mathrm{T}_x\cong\mathrm{T}_{x'}$,
(b) $\operatorname{Susp}\Sigma_x\cong\operatorname{Susp}\Sigma_{x'}$.
(c) but in general $\Sigma_x\not\cong\Sigma_{x'}$.

C Topology

The following theorem states that boundary is an object of topology, despite our definition have used geometry.

9.5. Theorem. *Let $\mathcal{A}$ and $\mathcal{A}'$ be homeomorphic finite-dimensional Alexandrov spaces. Then* $\dim\mathcal{A}=\dim\mathcal{A}'$ *and*

$$\partial\mathcal{A}\neq\varnothing \quad\iff\quad \partial\mathcal{A}'\neq\varnothing$$

While working on the proof, keep in mind that there are pairs of spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $\mathcal{K}_1 \not\cong \mathcal{K}_2$, but $\mathbb{R}\times\mathcal{K}_1 \cong \mathbb{R}\times\mathcal{K}_2$. Suspension over the Poincaré homology sphere with $\mathbb{S}^4$ is one of the examples; compare to 9.4*c*.

Let $\mathcal{A}$ be an m-dimensional Alexandrov space and $m<\infty$. Define rank of $\mathcal{A}$ (briefly, $\operatorname{rank}\mathcal{A}$) as the minimal value k such that $\mathcal{A}$ splits isometrically as $\mathbb{R}^{m-k}\times\mathcal{K}$; note that here $\mathcal{K}$ is a k-dimensional Alexandrov space.

In the following proof we will apply induction on the rank of $\mathcal{A}$.

Proof. The first statement follows from 7.15.

Suppose we have a counterexample, say $\partial\mathcal{A}\neq\varnothing$, but $\partial\mathcal{A}'=\varnothing$. Let $k:=\operatorname{rank}\mathcal{A}$ and $k':=\operatorname{rank}\mathcal{A}'$. We can assume that the pair (k,k') is minimal in lexicographic order; in particular, k is minimal. Let $x\mapsto x'$ be a homeomorphism from $\mathcal{A}$ to $\mathcal{A}'$.

Choose $x\in\partial\mathcal{A}$. Since $\partial\mathcal{A}'=\varnothing$, we have $x'\notin\partial\mathcal{A}'$. Note that

$$\operatorname{rank}\mathrm{T}_x\leqslant k\quad\text{and}\quad\operatorname{rank}\mathrm{T}_{x'}\leqslant k',$$

By 9.4*a*, $\mathrm{T}_x\cong\mathrm{T}_{x'}$. Note that $\partial\mathrm{T}_x\neq\varnothing$ and $\partial\mathrm{T}_{x'}=\varnothing$. Therefore, we may assume that $\mathcal{A}$ and $\mathcal{A}'$ are Euclidean cones and the homeomorphism sends the origin to the origin. The remaining part of the proof is divided into three cases.

Case 1. Suppose $k>1$. Let $\mathcal{A}\overset{iso}{=\!=}\mathbb{R}^{m-k}\times\mathcal{C}$, where $\mathcal{C}$ a k-dimensional $\mathrm{Alex}(0)$ cone. Observe that $\operatorname{rank}\mathrm{T}_y\leqslant\operatorname{rank}\mathcal{A}$ for any $y\in\mathcal{A}$ and the equality holds only if y projects to the origin of $\mathcal{C}$.

Since $k>1$ we can find $z\in\partial\mathcal{C}$ such that $z\neq0$. Choose y that projects to z; in particular, $\operatorname{rank}\mathrm{T}_y<\operatorname{rank}\mathcal{A}$. By 9.4*a*, $\mathrm{T}_y\cong\mathrm{T}_{y'}$, $\partial\mathrm{T}_y\neq\varnothing$ and $\partial\mathrm{T}_{y'}=\varnothing$. The latter contradicts the minimality of k.

Case 2. Suppose $k\leqslant1$ and $k'>1$. Since $\partial\mathcal{A}\neq\varnothing$, we get that $k=1$; therefore, $\mathcal{A}=\mathbb{R}^{m-1}\times\mathbb{R}_{\geqslant0}$.

Let $\mathcal{A}'\overset{iso}{=\!=}\mathbb{R}^{m-k'}\times\mathcal{C}'$, where $\mathcal{C}'$ a k'-dimensional $\mathrm{Alex}(0)$ cone. Since $\partial\mathcal{A}\cong\mathbb{R}^{m-1}$, the image of $\partial\mathcal{A}$ in $\mathcal{A}'$ does not lie in $\mathbb{R}^{m-k'}\times\{0\}$. In other words, we can choose $y\in\partial\mathcal{A}$ such that its image $y'\in\mathcal{A}'$ has a nonzero projection in $\mathcal{C}'$. Observe that $\mathrm{T}_y\cong\mathrm{T}_{y'}$,

$$\operatorname{rank}\mathrm{T}_y\leqslant k=1,\quad\operatorname{rank}\mathrm{T}_{y'}<k',\quad\partial\mathrm{T}_y=\varnothing,\quad\text{and}\quad\partial\mathrm{T}_{y'}\neq\varnothing$$

— a contradiction.

Case 3. Suppose $k\leqslant1$ and $k'\leqslant1$. Since $\partial\mathcal{A}\neq\varnothing$, $k=1$. By 7.3, $\mathcal{A}\cong\mathbb{R}^{m-1}\times\mathbb{R}_{\geqslant0}$. Therefore, $\mathcal{A}'\cong\mathbb{R}^m$, and $\mathcal{A}\not\cong\mathcal{A}'$ — a contradiction. $\square$

9.6. Exercise. *Let $x \mapsto x'$ be a homeomorphism $\Omega \to \Omega'$ between open subsets in finite-dimensional Alexandrov spaces $\mathcal{A}$ and $\mathcal{A}'$. Show that $x \in \partial\mathcal{A}$ if and only if $x' \in \partial\mathcal{A}'$.*

9.7. Exercise. *Show that boundary of a finite-dimensional Alexandrov space is a closed subset.*

D Tangent space

Let X be a subset in a finite-dimensional Alexandrov space $\mathcal{A}$. Choose $p \in \mathcal{A}$ and $\xi \in \Sigma_p$. Suppose ξ is a limit of directions $\uparrow_{[px_n]}$ for a sequence $x_1, x_2, \ldots \in X$ that converges to p. Then we say that ξ is in the space of directions from p to X; briefly $\xi \in \Sigma_p X$.

Further, $\mathrm{Cone}(\Sigma_p X)$ will be called tangent space to X at p; it will be denoted by $\mathrm{T}_p X$.

9.8. Theorem. *For any finite-dimensional Alexandrov space $\mathcal{A}$, we have*

$$\partial(\Sigma_p\mathcal{A}) = \Sigma_p(\partial\mathcal{A}) \quad \textit{and} \quad \partial(\mathrm{T}_p\mathcal{A}) = \mathrm{T}_p(\partial\mathcal{A}).$$

Proof. Choose a sequence $x_n \in \partial\mathcal{A}$ such that $x_n \to p$ and $\uparrow_{[px_n]} \to \xi$.

Let $\varepsilon_n = 2\cdot|p - x_n|$, and let $h_{\varepsilon_n} \colon \mathrm{B}(p, \varepsilon_n)_{\mathcal{A}} \to \mathrm{B}(0, \varepsilon_n)_{\mathrm{T}_p}$ be the homeomorphisms provided by 9.1; in particular, $\frac{2}{\varepsilon_n}\cdot h_{\varepsilon_n}(x_n) \to \xi$ as $n \to \infty$. By 9.6, $h_{\varepsilon_n}(x_n) \in \partial\mathrm{T}_p$. By 9.7, $\xi \in \partial\mathrm{T}_p$. Therefore,

$$\partial(\Sigma_p\mathcal{A}) \supset \Sigma_p(\partial\mathcal{A}) \quad \text{and} \quad \partial(\mathrm{T}_p\mathcal{A}) \supset \mathrm{T}_p(\partial\mathcal{A}).$$

Similarly, choose $\xi \in \partial\Sigma_p$. Let $h_{\varepsilon_n} \colon \mathrm{B}(p, \varepsilon_n)_{\mathcal{A}} \to \mathrm{B}(0, \varepsilon_n)_{\mathrm{T}_p}$ be the homeomorphisms provided by 9.1 for a sequence $\varepsilon_n \to 0$ as $n \to \infty$. By 9.6, $x_n = h_{\varepsilon_n}^{-1}(\frac{\varepsilon_n}{2}\cdot\xi) \in \partial\mathcal{A}$. By 9.1, $\uparrow_{[px_n]} \to \xi$. Hence

$$\partial(\Sigma_p\mathcal{A}) \subset \Sigma_p(\partial\mathcal{A}) \quad \text{and} \quad \partial(\mathrm{T}_p\mathcal{A}) \subset \mathrm{T}_p(\partial\mathcal{A}).$$

□

E Doubling

Let A be a closed subset in a metric space $\mathcal{X}$. The doubling $\mathcal{W}$ of $\mathcal{X}$ across A is two copies of $\mathcal{X}$ glued along A; more precisely, the underlying set of $\mathcal{W}$ is the quotient $\mathcal{X} \times \{0, 1\}/\sim$, where $(a, 0) \sim (a, 1)$ for any $a \in A$ and $\mathcal{W}$ is equipped with the minimal metric such that both maps $\mathcal{X} \to \mathcal{W}$ defined by $x \mapsto (x, 0)$ and $x \mapsto (x, 1)$ are distance-preserving.

Alternatively, one may say that $\mathcal{W}$ is equipped with the maximal metric such that the projection $\operatorname{proj}\colon \mathcal{W} \to \mathcal{A}$ defined by $(x,i) \mapsto x$ is a short map. The metric on $\mathcal{W}$ can also be defined explicitly as

$$|(x,i)-(y,j)|_{\mathcal{W}} = \begin{cases} |x-y|_{\mathcal{X}} & \text{if} \quad i=j. \\ \inf\{\,|x-a|_{\mathcal{X}} + |y-a|_{\mathcal{X}} \;:\; a \in A\,\} & \text{if} \quad i \neq j. \end{cases}$$

9.9. Theorem. *Let $\mathcal{A}$ be a finite-dimensional Alexandrov space with non-empty boundary. Suppose $f = \tfrac{1}{2}\cdot\operatorname{dist}_p^2$ for some $p \in \mathcal{A}$. Then*

(a) If $\dim\mathcal{A} \geqslant 2$, then $\operatorname{dist}_{\partial\Sigma_x}(\xi) \leqslant \tfrac{\pi}{2}$ for any $x \in \partial\mathcal{A}$ and $\xi \in \Sigma_x$. Moreover, if $\operatorname{dist}_{\partial\Sigma_x}(\xi) = \tfrac{\pi}{2}$, then $\measuredangle(\xi,\zeta) \leqslant \tfrac{\pi}{2}$ for any $\zeta \in \Sigma_x$.

(b) $\nabla_x f \in \partial\mathrm{T}_x$ for any $x \in \partial\mathcal{A}$.

(c) If α is an f-gradient curve that starts at $x \in \partial\mathcal{A}$, then $\alpha(t) \in \partial\mathcal{A}$ for any t. Moreover, if $p \in \partial\mathcal{A}$, then $\operatorname{gexp}_p(v) \in \partial\mathcal{A}$ for any $v \in \partial\mathrm{T}_p$.

(d) The doubling $\mathcal{W}$ of $\mathcal{A}$ across $\partial\mathcal{A}$ is an Alexandrov space with the same curvature bound.

Part *(d)* is called the d o u b l i n g t h e o r e m.

Proof. We will denote by *(a)*$_m, \dots,$*(d)*$_m$ the corresponding statement assuming $m = \dim\mathcal{A}$.

The proof goes by induction on m. Note that *(d)*$_1$ follows from 7.3 — this is the base. The step is a combination of the implications below.

(d)$_{m-1} \Rightarrow$*(a)*$_m$. Suppose $m=2$, then $\dim\Sigma_x = 1$; see 7.7*b*. By 7.3, Σ_x isometric to a line segment $[0,\ell]$; we need to show that $\ell \leqslant \pi$.

Assume $\ell > \pi$, then the tangent space $\mathrm{T}_x = \operatorname{Cone}\Sigma_x$ has several different lines thru the origin. Recall that T_x is an Alexandrov space; see 7.7. By 6.5, T_x is isometric to the Euclidean plane; the latter contradicts that Σ_x is a line segment.

Now suppose $m > 2$, so $\dim\Sigma_x > 1$. Assume $\operatorname{dist}_{\partial\Sigma_x}(\xi) > \tfrac{\pi}{2}$ for some ξ. By *(d)*$_{m-1}$, the doubling Ξ of Σ_x is $\mathrm{A{\scriptsize LEX}}(1)$. Denote by ξ_0 and ξ_1 the points in Ξ that correspond to ξ. Observe that $|\xi_0 - \xi_1|_{\Xi} > \pi$. The latter contradicts 3.7.

Finally, if $\operatorname{dist}_{\partial\Sigma_x}(\xi) = \tfrac{\pi}{2}$, then $|\xi_0 - \xi_1|_{\Xi} = \pi$. Therefore, $\operatorname{Cone}\Xi$ contains a line in the directions of ξ_0 and ξ_1; in other words, Ξ is a spherical suspension with poles ξ_0 and ξ_1. In particular, every point of Ξ lies on distance at most $\tfrac{\pi}{2}$ from ξ_0 or ξ_1. The natural projection $\Xi \to \Sigma_x$ does not increase distances and sends both ξ_0 and ξ_1 to ξ. Therefore, the second statement follows.

(d)$_{m-1}$+*(a)*$_{m-1}$+*(a)*$_m \Rightarrow$*(b)*$_m$. We can assume that $s = \nabla_x f \neq 0$. By 4.10, $\nabla_x f = s\cdot\overline{\xi}$, where $s = \boldsymbol{d}_x f(\overline{\xi}) > 0$ and $\overline{\xi} \in \Sigma_p$ is the direction that maximize $\boldsymbol{d}_x f(\overline{\xi})$.

Let $\zeta\in\partial\Sigma_x$ be a direction that minimize the angle $\measuredangle(\bar\xi,\zeta)$. It is sufficient to show that $\zeta=\bar\xi$.

Assume $\zeta\ne\bar\xi$; let $\eta={\uparrow}_{[\zeta\bar\xi]_{\Sigma_x}}$. By *(a)*$_m$, $\measuredangle(\bar\xi,\zeta)\leqslant\tfrac\pi2$ and *(a)*$_{m-1}$ implies that

❶ $$\measuredangle(\eta,\nu)\leqslant\tfrac\pi2$$

for any $\nu\in\Sigma_\zeta\Sigma_x$ (if $m=2$, then the last statement is evident).

Let $\varphi\colon\Sigma_x\to\mathbb{R}$ be restriction of $\boldsymbol{d}_xf$ to Σ_x. Applying 4.7*a* and ❶, we get that $\boldsymbol{d}_\xi\varphi(\eta)\leqslant 0$. Since $\boldsymbol{d}_xf$ is convex, we have that $\varphi''+\varphi\leqslant 0$. If $\varphi(\zeta)\leqslant 0$, then it implies that $\varphi(\bar\xi)\leqslant 0$ — a contradiction. If $\varphi(\zeta)>0$, then $\varphi(\bar\xi)<\varphi(\zeta)$ — a contradiction again.

(b)$_m\Rightarrow$*(c)*$_m$. Let α be an f-gradient curve and $\ell(t)=\operatorname{dist}_{\partial\mathcal{A}}\alpha(t)$.

Choose t; let $x=\alpha(t)$ and $y\in\partial\mathcal{A}$ be a closest point to x. By *(b)*$_m$, we have that $\nabla_yf\in\partial\mathrm{T}_y$. Since the distance $|x-y|$ is minimal, we get $\langle{\uparrow}_{[yx]},v\rangle\leqslant 0$ for any $v\in\partial\mathrm{T}_y$. In particular,

$$\langle{\uparrow}_{[yx]},\nabla_yf\rangle\leqslant 0$$

Applying Exercise 4.9 to x and y, we get

$$\ell'(t)\leqslant\ell(t)$$

if the left-hand side is defined. Since ℓ is Lipschitz, ℓ' is defined almost everywhere. Integrating the inequality, we get

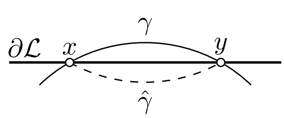
$$\ell(t)\leqslant e^t\cdot\ell(0)$$

for any $t\geqslant 0$. In particular, if $\ell(0)=0$, then $\ell(t)=0$ for any $t\geqslant 0$. Since $\partial\mathcal{A}$ is closed (9.7), the statement follows.

(c)$_m$+*(d)*$_{m-1}\Rightarrow$*(d)*$_m$. We will consider the case $\kappa=0$; other cases can be done in the same way, but formulas get more complicated.

Denote by $\mathcal{A}_0$ and $\mathcal{A}_1$ the two copies of $\mathcal{A}$ in $\mathcal{W}$; let us keep the notation $\partial\mathcal{A}$ for the common boundary of $\mathcal{A}_0$ and $\mathcal{A}_1$.

❷ *Let γ be a geodesic in $\mathcal{W}$. Then either γ has at most one interior point in $\partial\mathcal{A}$ or $\gamma\subset\partial\mathcal{A}$.*



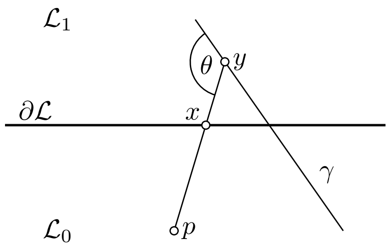
Indeed, assume γ shares at least two points with $\partial\mathcal{A}$, say $x=\gamma(t_1)$ and $y=\gamma(t_2)$ and these are not endpoints of γ. Remove from γ the set $\gamma\cap\mathcal{A}_1$ and exchange it to its reflection across $\partial\mathcal{A}$; denote the obtained curve by $\hat\gamma$.

Note that any arc of $\hat{\gamma}$ with one endpoint in $\partial\mathcal{A}$ is a geodesic in $\mathcal{A}_0$. Since $x, y \in \partial\mathcal{A}$, the arc of $\hat{\gamma}$ behind y lies in the image of map $t \mapsto \mathrm{Flow}^t_{f_x}(y)$, where $f_x = \frac{1}{2}\cdot\mathrm{dist}^2_x$. By *(c)*, this arc lies in $\partial\mathcal{A}$.

Now choose a point z on this arc, so $z \in \partial\mathcal{A}$. Applying the same argument, we get that the arc of $\hat{\gamma}$ before y lies in $\partial\mathcal{A}$. Hence the claim follows. $\triangle$

Choose a point p in $\mathcal{W}$; let $f := \frac{1}{2}\cdot\mathrm{dist}^2_p$. It is sufficient to show that $(f\circ\gamma)'' \leqslant 1$ for any t. If $p \in \partial\mathcal{A}$, then the statement follows from function comparison in $\mathcal{A}_0$ and $\mathcal{A}_1$. So, we can assume that $p \in \mathcal{A}_0 \setminus \partial\mathcal{A}$.

If γ lies in $\partial\mathcal{A}$, then this inequality follows from the comparison in $\mathcal{A}_0$.



Choose $y = \gamma(t_0)$; without loss of generality we can assume that $t_0 = 0$.

If $y \in \mathcal{A}_0 \setminus \partial\mathcal{A}$, then $(f\circ\circ\gamma)''(0) \leqslant 1$ in the barrier sense; it follows from the comparison in $\mathcal{A}_0$.

Assume $y \in \mathcal{A}_1 \setminus \partial\mathcal{A}$. Suppose $[py]$ crosses $\partial\mathcal{A}$ at x. Let Σ_x be the space of directions of $\mathcal{A}$ at x, and let Ξ be its doubling. As before, we denote by Σ_0 and Σ_1 two copies of Σ_x in Ξ and keep notation $\partial\Sigma_x$ for their common boundary. By *(d)*$_{m-1}$, Ξ is $\mathrm{A\scriptstyle LEX}(1)$.

Note that $\uparrow_{[xy]}$ and $\uparrow_{[xp]}$ lie in opposite sides of Ξ and

$$|\uparrow_{[xy]} - \uparrow_{[xp]}|_\Xi \geqslant \pi.$$

Otherwise, we could choose a direction $\xi \in \partial\Sigma$ such that

$$|\uparrow_{[xy]} - \xi|_\Xi + |\xi - \uparrow_{[xp]}|_\Xi < \pi.$$

Furthermore, we could consider the radial curve $\alpha(t) = \mathrm{gexp}_x(t\cdot\xi)$. By *(c)*$_m$, α lies in $\partial\mathcal{A}$. By 5.12

$$|p - \alpha(s)|_{\mathcal{A}_0} + |y - \alpha(s)|_{\mathcal{A}_1} < |p - y|_{\mathcal{W}}$$

for small values $s > 0$ — a contradition.

Note that $\mathrm{Cone}\,\Xi$ contains a line with directions $\uparrow_{[xy]}$ and $\uparrow_{[xp]}$. By splitting theorem $\mathrm{Cone}\,\Xi$ split in these directions; in particular,

$$|\uparrow_{[xy]} - \xi| + |\xi - \uparrow_{[xp]}| = \pi.$$

for any $\xi \in \Xi$. It follows that for any $\xi \in \Xi$ there is $\xi' \in \partial\Sigma_x$ such that ξ and ξ' lie on some geodesic $[\uparrow_{[xy]}\uparrow_{[xp]}]_\Xi$.

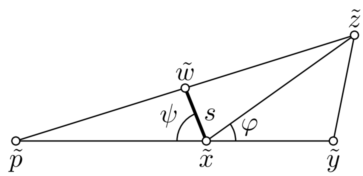
Fix $t \approx 0$ such that $t \neq 0$; let $z = \gamma(t)$. Choose such ξ' for $\xi = \uparrow_{[xz]}$. Consider the radial curve $\alpha(s) := \mathrm{gexp}_x(s\cdot\xi')$. Let us show that

❸ $$\begin{aligned}|p - z|_{\mathcal{W}} &\leqslant |p - \alpha(s)|_{\mathcal{A}_0} + |\alpha(s) - z|_{\mathcal{A}_1} \leqslant \\ &\leqslant \tilde{\Upsilon}[y\,{}^p_z].\end{aligned}$$

for suitable value s.

The first inequality in ❸ is evident. Set $\varphi = \measuredangle[x\,{}^y_z]$ and $\psi = \measuredangle(\uparrow_{[xp]}, \xi')$. The choice of s comes from the model configuration $\tilde{p}$, $\tilde{x}$, $\tilde{y}$, $\tilde{w}$, $\tilde{z} \in \mathbb{E}^2$ such that

$$\begin{aligned}&\tilde{x} \in [\tilde{p}\tilde{y}], \quad |\tilde{p} - \tilde{x}| = |p - x|, \quad |\tilde{p} - \tilde{y}| = |p - y|, \quad |\tilde{x} - \tilde{z}| = |x - z|,\\ &\tilde{w} \in [\tilde{p}\tilde{z}], \quad \measuredangle[\tilde{x}\,{}^{\tilde{y}}_{\tilde{z}}] = \varphi, \quad \measuredangle[\tilde{x}\,{}^{\tilde{p}}_{\tilde{w}}] = \psi, \quad s = |\tilde{x} - \tilde{w}|.\end{aligned}$$



By 5.12, we get

$$\begin{aligned}|p - \alpha(s)|_{\mathcal{A}_0} &\leqslant |\tilde{p} - \tilde{w}|,\\ |\alpha(s) - z|_{\mathcal{A}_1} &\leqslant |\tilde{w} - \tilde{z}|;\end{aligned}$$

by the comparison,

$$|\tilde{p} - \tilde{z}| \leqslant \tilde{\Upsilon}[y\,{}^p_z].$$

9.10. Exercise. *Prove the last inequality.*

Hence we get $(f \circ \gamma)''(0) \leqslant 1$ in the barrier sense.

Finally if $\gamma(0) \in \partial\mathcal{A}$, then splitting argument shows that

$$(f \circ \gamma)^+(0) + (f \circ \gamma)^-(0) \leqslant 0.$$

Summarizing, we get that $(f \circ \gamma)'' \leqslant 1$ on every arc of γ that lies entirely in $\mathcal{A}_0$ or $\mathcal{A}_1$. If γ crosses $\partial\mathcal{A}$, then we know that it happens only once and at the crossing moment t_0 we have $f \circ \gamma^+(t_0) + f \circ \gamma^-(t_0) \leqslant 0$. All this implies that $(f \circ \gamma)'' \leqslant 1$. □

9.11. Exercise. *Let $\mathcal{A}$ be a finite-dimensional* Alex(1) *space of dimension $\geqslant 2$ with non-empty boundary $\partial\mathcal{A}$. Show that $\partial\mathcal{A}$ is connected.*

9.12. Exercise. *Let $\mathcal{A}$ be an m-dimensional* $\mathrm{ALEX}(0)$ *space with non-empty boundary $\partial\mathcal{A}$ for $2 \leqslant m < \infty$. Show that the distance function to the boundary*

$$\mathrm{dist}_{\partial\mathcal{A}} \colon \mathcal{A} \to \mathbb{R}$$

is concave.

9.13. Exercise. *Let $\mathcal{A}$ be a finite-dimensional* $\mathrm{ALEX}(0)$ *space with non-empty boundary $\partial\mathcal{A}$. Suppose γ is a geodesic in $\partial\mathcal{A}$ with the induced length metric. Show that the function $t \mapsto \frac{1}{2}\cdot\mathrm{dist}_p^2 \circ \gamma(t)$ is 1-concave for any point p.*

9.14. Exercise. *Let $\mathcal{W}$ be a doubling of finite-dimensional Alexandrov space $\mathcal{A}$ across its boundary, and let $proj \colon \mathcal{W} \to \mathcal{A}$ be the natural projection. Suppose $f \colon \mathcal{A} \to \mathbb{R}$ is a λ-concave function. Show that $f \circ \mathrm{proj} \colon \mathcal{W} \to \mathbb{R}$ is λ-concave if and only if $\nabla_x f \in \partial \mathrm{T}_x$ for any $x \in \partial\mathcal{A}$.*

F Remarks

Note that the doubling of a finite-dimensional Alexandrov space across its boundary results in an Alexandrov space without boundary. This observation can often be used to reduce a statement about general finite-dimensional Alexandrov spaces to Alexandrov spaces without boundary.

For spaces without boundary the following tools become available.

9.15. Fundamental-class lemma. *Any compact finite-dimensional Alexandrov space $\mathcal{A}$ without boundary has a fundamental class with $\mathbb{Z}/2$ coefficients; that is, if $\mathcal{A}$ is m-dimensional, then*

$$H^m(\mathcal{A}, \mathbb{Z}/2) = \mathbb{Z}/2.$$

This lemma was proved by Karsten Grove and Peter Petersen [34]. Originally it was stated for Alexander–Spanier cohomology. We do not make this distinction because for compact Alexandrov spaces it is the same as singular cohomology. Indeed, both cohomology theories are homotopy invariant [88, Chapter 6], compact Alexandrov spaces are homotopy equivalent to finite simplicial complexes 8.14 and for paracompact CW complexes Alexander–Spanier cohomology is isomorphic to Čech and singular cohomolgy [88, Chapter 6].

This lemma implies, for example, that on finite-dimensional Alexandrov spaces without boundary the gradient flow for a λ-concave function is an onto map; in other words, gradient curves can be extended into the past. It is also used in the proof of the following version of the domain invariance theorem [44, Theorem 3.2].

9.16. Domain invariance. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two m-dimensional Alexandrov spaces with empty boundary; m is finite. Suppose Ω_1 is an open subset in $\mathcal{A}_1$ and $f\colon \Omega_1 \to \mathcal{A}_2$ is an injective continuous map. Then $f(\Omega_1)$ is open in $\mathcal{A}_2$.*

Theorem 9.1 can be used to prove the following.

9.17. Topological stratification. *Any m-dimensional Alexandrov space with $m < \infty$ can be subdivided into topological manifolds $S_0, \dots$ $\dots, S_m$ such that for every i we have $\dim S_i = i$ or $S_i = \varnothing$. Moreover,*

(a) the closure of S_{m-1} is the boundary of the space, and

(b) $S_{m-2} = \varnothing$.

Let us mention that this statement implies that a compact finite-dimensional Alexandrov space has the homotopy type of a finite CW complex, but it seems to be unknown if it has to be homeomorphic to a CW complex.

The stratification theorem 9.17 can be sharpened as follows.

9.18. Boundary characterization. *Let $\mathcal{A}$ be an m-dimensional Alexandrov space with $m < \infty$. Then the following statements are equivalent.*

(a) $p \in \partial\mathcal{A}$;

(b) Σ_p is contractible;

(c) $\tilde{H}_{m-1}(\Sigma_p, \mathbb{Z}/2) = 0$;

(d) $H_m(\mathcal{A}, \mathcal{A} \setminus \{p\}, \mathbb{Z}/2) = 0$;

Let f be a semiconcave function. A point $p \in \operatorname{Dom} f$ is called critical point of f if $\boldsymbol{d}_p f \leqslant 0$; otherwise it is called regular.

The following statement plays a technical role in the proof of stability theorem, but it is also a useful technical tool on its own.

9.19. Morse lemma. *Let f be a semiconcave function on a finite-dimensional Alexandrov space without boundary. Suppose K is a compact set of regular points of f in its level set $f = a$. Then an open neighborhood Ω of K admits a homeomorphism $x \mapsto (h(x), f(x))$ to a product space $\Lambda \times (a - \varepsilon, a + \varepsilon)$.*

Subsets in Alexandrov spaces that satisfy the condition in 9.9c are called extremal. More precisely, a subset E is extremal if for any $x \in E$ and f-gradient curve that starts in E remains in E; here f is arbitrary function of the form $\frac{1}{2}\cdot\operatorname{dist}_p^2$.

Extremal subsets were introduced by Grigory Perelman and the second author [66]. They will pop up in the next lecture.

The following conjecture is one of the oldest questions in Alexandrov geometry that remains open.

9.20. Conjecture. *Let S be a component of the boundary of a finite-dimensional Alexandrov space. Then S equipped with the induced length metric is an Alexandrov space with the same curvature bound.*

The doubling theorem has several generalization [26, 76] that allows to glue nonidentical spaces.

Lecture 10

Quotients

This lecture gives several application of Alexandrov geometry to isometric group actions.

A Quotient space

Suppose that a group G acts isometrically on a metric space $\mathcal{X}$. Note that

$$|G{\cdot}x - G{\cdot}y|_{\mathcal{X}/G} := \inf\{\, |x - g{\cdot}y|_{\mathcal{X}} \;:\; g \in G\,\}$$

defines a semimetric on the orbit space $\mathcal{X}/G$. Moreover, it is a genuine metric if the orbits of the action are closed.

10.1. Theorem. *Suppose that a group G acts isometrically on a proper* $\mathrm{ALEX}(0)$ *space $\mathcal{A}$, and G has closed orbits. Then the quotient space $\mathcal{A}/G$ is* $\mathrm{ALEX}(0)$.

A more general formulation will be in 10.5.

Proof. Denote by $\sigma\colon \mathcal{A} \to \mathcal{A}/G$ the quotient map.

Fix a quadruple of points $p, x_1, x_2, x_3 \in \mathcal{A}/G$. Choose an arbitrary $\hat p \in \mathcal{A}$ such that $\sigma(\hat p) = p$. Since $\mathcal{A}$ is proper, we can choose the points $\hat x_i \in \mathcal{A}$ such that $\sigma(\hat x_i) = x_i$ and

$$|p - x_i|_{\mathcal{A}/G} = |\hat p - \hat x_i|_{\mathcal{A}}$$

for all i.

Note that

$$|x_i - x_j|_{\mathcal{A}/G} \leqslant |\hat x_i - \hat x_j|_{\mathcal{A}}$$

for all i and j. Therefore

❶ $$\tilde\measuredangle(p\,{}^{x_i}_{x_j}) \leqslant \tilde\measuredangle(\hat p\,{}^{\hat x_i}_{\hat x_j})$$

holds for all i and j.

By $\mathbb{E}^2$-comparison in $\mathcal{A}$, we have

$$\tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_1}_{\hat{x}_2}) + \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_2}_{\hat{x}_3}) + \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_3}_{\hat{x}_1}) \leqslant 2{\cdot}\pi.$$

Applying ❶, we get

$$\tilde{\measuredangle}(p\,{}^{x_1}_{x_2}) + \tilde{\measuredangle}(p\,{}^{x_2}_{x_3}) + \tilde{\measuredangle}(p\,{}^{x_3}_{x_1}) \leqslant 2{\cdot}\pi;$$

that is, the $\mathbb{E}^2$-comparison holds for any quadruple in $\mathcal{A}/G$. □

10.2. Advanced exercise. *Let G be a compact Lie group with a bi-invariant Riemannian metric. Show that G is isometric to a quotient of the Hilbert space by an isometric group action.*

Conclude that G is Alex(0).

B Submetries

A map $\sigma\colon \mathcal{X} \to \mathcal{Y}$ between the metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is called a submetry if

$$\sigma(\mathrm{B}(p,r)_{\mathcal{X}}) = \mathrm{B}(\sigma(p),r)_{\mathcal{Y}}$$

for any $p \in \mathcal{X}$ and $r \geqslant 0$.

Suppose G and $\mathcal{A}$ are as in 10.1. Observe that the quotient map $\sigma\colon \mathcal{A} \to \mathcal{A}/G$ is a submetry. The following two exercises show that this is not the only source of submetries.

10.3. Exercise. *Construct submetries*

(a) $\sigma_1\colon \mathbb{S}^2 \to [0,\pi]$,

(b) $\sigma_2\colon \mathbb{S}^2 \to [0,\tfrac{\pi}{2}]$,

(c) $\sigma_n\colon \mathbb{S}^2 \to [0,\tfrac{\pi}{n}]$ *(for integer $n \geqslant 1$)*

such that the fibers $\sigma_n^{-1}\{x\}$ are connected for any x.

10.4. Exercise. *Let $\sigma\colon \mathbb{E}^2 \to [0,\infty)$ be a submetry. Show that $K = \sigma^{-1}\{0\}$ is a closed convex set without interior points and $\sigma(x) = \mathrm{dist}_K x$.*

The proof of 10.1 works for submetries equally well; that is, *if $\sigma\colon \mathcal{A} \to \mathcal{B}$ is a submetry and $\mathcal{A}$ is* Alex(0), *then so is $\mathcal{B}$.* Theorems 10.1 admits straightforward generalizations to Alex(−1) case. In the Alex(1) case, the proof produces a slightly weaker statement — *$\mathbb{S}^2$-comparison holds in any open $\tfrac{\pi}{2}$-ball in the quotient of* Alex(1); in particular, the quotient space is *locally* Alex(1). But since Alex(1) space is geodesic, then so is its quotient. Therefore, the globalization theorem implies that it is globally Alex(1). The same holds for the targets of submetries from a ALEX(1) space. With a bit of extra work, one can extend the statement to nonproper spaces; see [4, 8.34]. Thus, we have the following two statements.

10.5. Theorem. *Let $\sigma\colon \mathcal{A}\to\mathcal{B}$ be a submetry. If $\mathcal{A}$ is* ALEX(κ) *space, then so is $\mathcal{B}$.*

In particular, if G acts isometrically on a proper ALEX(κ) *space $\mathcal{A}$, and G has closed orbits. Then the quotient space $\mathcal{A}/G$ is* ALEX(κ)*.*

C Hopf's conjecture

Does $\mathbb{S}^2\times\mathbb{S}^2$ admit a Riemannian metric with positive sectional curvature? This question is known as Hopf's conjecture. The following partial result was obtained by Wu-Yi Hsiang and Bruce Kleiner [39].

10.6. Theorem. *There is no Riemannian metric on $\mathbb{S}^2\times\mathbb{S}^2$ with sectional curvature $\geqslant 1$ and a nontrivial isometric $\mathbb{S}^1$-action.*

10.7. Key lemma. *Suppose $\mathbb{S}^1\curvearrowright\mathbb{S}^3$ is an isometric action without fixed points and $\Sigma=\mathbb{S}^3/\mathbb{S}^1$ is its quotient space. Then there is a distance noncontracting map $\Sigma\to\tfrac12{\cdot}\mathbb{S}^2$, where $\tfrac12{\cdot}\mathbb{S}^2$ is the standard 2-sphere rescaled with a factor $\tfrac12$.*

The proof of the lemma is done mostly by calculations; it is guided in the following exercise.

10.8. Exercise. *Suppose $\mathbb{S}^1\curvearrowright\mathbb{S}^3$ is an isometric action without fixed points. Let us think that $\mathbb{S}^3$ is the unit sphere in $\mathbb{R}^4$.*

(a) Show that one can identify $\mathbb{R}^4$ with $\mathbb{C}^2$ so that the action is given by matrix multiplication

$$\begin{pmatrix} u^p & 0\\ 0 & u^q\end{pmatrix},$$

where (p,q) is a pair of relatively prime positive integers and $u\in\mathbb{S}^1=\{\,z\in\mathbb{C}\;:\;|z|=1\,\}$. In particular, our $\mathbb{S}^1$ is a subgroup of the torus that acts by matrix multiplication

$$\begin{pmatrix} v & 0\\ 0 & w\end{pmatrix},$$

where $v,w\in\mathbb{S}^1$.

Fix p and q as above. Let $\Sigma_{p,q}=\mathbb{S}^3/\mathbb{S}^1$ be the quotient space.

(b) Show that the $\Sigma_{p,q}=\mathbb{S}^3/\mathbb{S}^1$ is a topological sphere with $\mathbb{S}^1$-symmetry. This symmetry has two fixed points, north pole and south pole, that correspond to the orbits of $(1,0)$ and $(0,1)$ in $\mathbb{S}^3$.

Denote by $S(r)$ the circle of radius r with the center at the north pole of $\Sigma_{p,q}$.

(c) *Show that the inverse image $T(r)$ of $S(r)$ in $\mathbb{S}^3$ is also an orbit of the torus action. Conclude that $a(r) = \pi^2\cdot\sin r\cdot\cos r$ is the area $T(r)$.*

(d) *Let $b_{p,q}(r)$ be the length of the $\mathbb{S}^1$-orbit in $\mathbb{S}^3$ that corresponds to a point on $S(r)$. Show that $b_{p,q} = \pi\cdot\sqrt{(p\cdot\sin r)^2 + (q\cdot\cos r)^2}$.*

(e) *Let $c_{p,q}(r)$ be the length of $S(r)$. Show that $a(r) = c_{p,q}(r)\cdot b_{p,q}(r)$.*

(f) *Show that $c_{p,q}(r) \leqslant c_{1,1}(r)$ for any pair (p,q) of relatively prime positive integers. Use it to construct a distance noncontracting map $\Sigma_{p,q} \to \tfrac{1}{2}\cdot\mathbb{S}^2 \overset{iso}{=\!=} \Sigma_{1,1}$.*

Proof of 10.6. Assume $\mathcal{B} = (\mathbb{S}^2\times\mathbb{S}^2, g)$ is a counterexample. By the Toponogov theorem, $\mathcal{B}$ is $\mathrm{A{\scriptsize LEX}}(1)$. By 10.1, the quotient space $\mathcal{A} = \mathcal{B}/\mathbb{S}^1$ is $\mathrm{A{\scriptsize LEX}}(1)$; evidently, $\mathcal{A}$ is 3-dimensional.

Denote by $F\subset\mathcal{B}$ the fixed point set of the $\mathbb{S}^1$-action. Each connected component of F is either an isolated point or a 2-dimensional geodesic submanifold in $\mathcal{B}$; the latter has to have positive curvature, and therefore it is either $\mathbb{S}^2$ or $\mathbb{RP}^2$. Notice that

- ◇ each isolated point contributes 1 to the Euler characteristic of $\mathcal{B}$,
- ◇ each sphere contributes 2 to the Euler characteristic of $\mathcal{B}$, and
- ◇ each projective plane contributes 1 to the Euler characteristic of $\mathcal{B}$.

Since $\chi(\mathcal{B}) = 4$, we are in one of the following three cases:

- ◇ F has exactly 4 isolated points,
- ◇ F has one 2-dimensional submanifold and at least 2 isolated points,
- ◇ F has at least two 2-dimensional submanifolds.

In each case we will arrive at a contradiction.

Case 1. Suppose F has exactly 4 isolated points x_1, x_2, x_3, and x_4. Denote by y_1, y_2, y_3, and y_4 the corresponding points in $\mathcal{A}$. Note that $\Sigma_{y_i}\mathcal{A}$ is isometric to a quotient of $\mathbb{S}^3$ by an isometric $\mathbb{S}^1$-action without fixed points.

By 10.8, each angle $\measuredangle[y_i\,{}^{y_j}_{y_k}] \leqslant \tfrac{\pi}{2}$ for any three distinct points y_i, y_j, y_k. In particular, all four triangles $[y_1y_2y_3]$, $[y_1y_2y_4]$, $[y_1y_3y_4]$, and $[y_2y_3y_4]$ are nondegenerate. By the comparison, the sum of angles in each triangle is strictly greater than π.

Denote by σ the sum of all 12 angles in the 4 triangles $[y_1y_2y_3]$, $[y_1y_2y_4]$, $[y_1y_3y_4]$, and $[y_2y_3y_4]$. From above,

$$\sigma > 4\cdot\pi.$$

On the other hand, by 10.8 any triangle in $\Sigma_{y_1}\mathcal{A}$ has perimeter at most π. In particular,

$$\measuredangle[y_1\,{}^{y_2}_{y_3}] + \measuredangle[y_1\,{}^{y_3}_{y_4}] + \measuredangle[y_1\,{}^{y_4}_{y_2}] \leqslant \pi.$$

Apply the same argument in $\Sigma_{y_2}\mathcal{A}$, $\Sigma_{y_3}\mathcal{A}$, and $\Sigma_{y_4}\mathcal{A}$; adding the results, we get

$$\sigma \leqslant 4\cdot\pi$$

— a contradiction.

Case 2. Suppose F contains one surface S. Note that the projection of S to $\mathcal{A}$ forms its boundary $\partial\mathcal{A}$. Note that doubling $\mathcal{W}$ of $\mathcal{A}$ across its boundary has 4 singular points — each singular point of $\mathcal{A}$ corresponds to two singular points of $\mathcal{W}$.

By the doubling theorem, $\mathcal{W}$ is a Alex(1) space. Therefore we arrive at a contradiction in the same way as in the first case.

Case 3. Impossible by 9.11. □

D Erdős' problem rediscovered

A point p in an Alexandrov space is called extremal if $\measuredangle[p\,{}^{x}_{y}] \leqslant \frac{\pi}{2}$ for any hinge $[p\,{}^{x}_{y}]$ with the vertex at p.

10.9. Theorem. *Let $\mathcal{A}$ be a compact m-dimensional* Alex(0) *space. Then it has at most 2^m extremal points.*

The proof is a translation of the proof of the following classical problem in discrete geometry to Alexandrov's language.

10.10. Problem. *Let F be a set of points in $\mathbb{E}^m$ such that any triangle formed by three distinct points in F has no obtuse angles. Then $|F| \leqslant 2^m$. Moreover, if $|F| = 2^m$, then F consists of the vertices of an m-dimensional rectangle.*

This problem was posed by Paul Erdős [23] and solved by Ludwig Danzer and Branko Grünbaum [21]. Grigory Perelman noticed that, after proper definitions, the same proof works in Alexandrov spaces [65]; thus, it proves 10.9.

Proof of 10.9. Let $\{p_1, \dots, p_N\}$ be extremal points in $\mathcal{A}$. For each p_i consider its open Voronoi domain V_i; that is,

$$V_i = \{\, x \in \mathcal{A} \,:\, |p_i - x| < |p_j - x| \text{ for any } j \neq i \,\}.$$

Clearly $V_i \cap V_j = \varnothing$ if $i \neq j$.

Suppose $0 < \alpha \leqslant 1$. Given a point $x \in \mathcal{A}$, choose a geodesic $[p_i x]$ and denote by x_i the point on $[p_i x]$ such that $|p_i - x_i| = \alpha{\cdot}|p_i - x|$; let $\Phi_i \colon x \to x_i$ be the corresponding map. By the comparison,

$$|x_i - y_i| \geqslant \alpha{\cdot}|x - y|$$

for any x, y, and i. Therefore

$$\operatorname{vol}(\Phi_i \mathcal{A}) \geqslant \alpha^m \cdot \operatorname{vol} \mathcal{A}.$$

Suppose $\alpha < \frac{1}{2}$. Then $x_i \in V_i$ for any $x \in \mathcal{A}$. Indeed, assume $x_i \notin V_i$, then there is p_j such that $|p_i - x_i| \geqslant |p_j - x_i|$. Then from the comparison, we have $\tilde{\measuredangle}(p_j\,{}^{p_i}_{x})_{\mathbb{E}^2} > \frac{\pi}{2}$; that is, p_j does not form a one-point extremal set. It follows that $\operatorname{vol} V_i \geqslant \alpha^m \cdot \operatorname{vol} \mathcal{A}$ for any $0 < \alpha < \frac{1}{2}$; hence

$$\operatorname{vol} V_i \geqslant \tfrac{1}{2^m} \cdot \operatorname{vol} \mathcal{A} \quad \text{and} \quad N \leqslant 2^m.$$

□

E Crystallographic actions

An isometric action $\Gamma \curvearrowright \mathbb{E}^m$ is called crystallographic if it is properly discontinuous (that is, for any compact set $K \subset \mathbb{E}^m$ and $x \in \in \mathbb{E}^m$ there are only finitely many $g \in \Gamma$ such that $g{\cdot}x \in K$) and cocompact (that is, the quotient space $\mathcal{A} = \mathbb{E}^m/\Gamma$ is compact).

Let F be a maximal finite subgroup of Γ; that is, if $F < H < \Gamma$ for a finite group H, then $F = H$. Denote by $\#(\Gamma)$ the number of maximal finite subgroups of Γ up to conjugation.

10.11. Open question. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Is it true that $\#(\Gamma) \leqslant 2^m$?*

Note that any finite subgroup F of Γ fixes an affine subspace A_F in $\mathbb{E}^m$. If F is maximal, then A_F completely describes F. Indeed, since the action is properly discontinuous, the subgroup of Γ that fix A_F has to be finite. This subgroup must contain F, but since F is maximal, it must coinside with F.

Denote by $\#_k(\Gamma)$ the number of maximal finite subgroups $F < \Gamma$ (up to conjugation) such that $\dim A_F = k$.

Choose a finite subgroup $F < \Gamma$; consider a conjugate subgroup $F' = g{\cdot}F{\cdot}g^{-1}$. Note that $A_{F'} = g{\cdot}A_F$. In particular, the subspaces A_F and $A_{F'}$ have the same image in the quotient space $\mathcal{A} = \mathbb{E}^m/\Gamma$. Therefore, to count subgroups up to conjugation, we need to count the images of their fixed sets. Therefore, by the lemma below (10.13),

$\#_0(\Gamma)$ cannot exceed the number of extremal points in $\mathcal{A} = \mathbb{E}^m/\Gamma$. Combining this observation with 10.9, we get the following.

10.12. Proposition. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Then $\#_0(\Gamma) \leqslant 2^m$.*

10.13. Lemma. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action and F be a maximal finite subgroup of Γ that fixes an isolated point p. Then the image of p in the quotient space $\mathcal{A} = \mathbb{E}^m/\Gamma$ is an extremal point.*

Proof. Let q be the image of p. Suppose q is not extremal; that is, $\measuredangle[q\,{}^{y_1}_{y_2}] > \frac{\pi}{2}$ for some hinge $[q\,{}^{y_1}_{y_2}]$ in $\mathcal{A}$.

Choose the inverse images $x_1, x_2 \in \mathbb{E}^m$ of $y_1, y_2 \in \mathcal{A}$ such that $|p - x_i|_{\mathbb{E}^m} = |q - y_i|_{\mathcal{A}}$. Note that $\measuredangle[p\,{}^{x_1}_{x_2}] \geqslant \measuredangle[q\,{}^{y_1}_{y_2}] > \frac{\pi}{2}$. Moreover, since p is fixed by F, we have

❶ $$\measuredangle[p\,{}^{x_1}_{g\cdot x_2}] > \tfrac{\pi}{2}$$

for any $g \in F$.

Denote by z the barycenter of the orbit $G\cdot x_2$. Note that z is a fixed point of F. By ❶, $z \neq p$; so F must fix the line pz. But p is an isolated fixed point of F — a contradiction. □

10.14. Exercise. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Show that*

(a) $\#_{m-1}(\Gamma) \leqslant 2$, and

(b) if $\#_{m-1}(\Gamma) = 1$, then $\#_0(\Gamma) \leqslant 2^{m-1}$.

Construct crystallographic actions with equalities in (a) and (b).

F Remarks

A more general form of Theorem 10.6 was found by Karsten Grove and Burkhard Wilking [35]; it classifies isometric $\mathbb{S}^1$ actions on 4-dimensional manifolds with nonnegative sectional curvature. This proof is as beautiful as the original work of Wu-Yi Hsiang and Bruce Kleiner.

It is expected that *no* Alex(1) *space with a nontrivial isometric $\mathbb{S}^1$-action can be homeomorphic to* $\mathbb{S}^2 \times \mathbb{S}^2$; so 10.6 holds for general Alex(1) space. The proof of 10.6 would work if we had the following generalization of 10.7; see [37].

10.15. Open question. *Let Σ be an* Alex(1) *space homeomorphic to $\mathbb{S}^3$. Suppose $\mathbb{S}^1$ acts on Σ isometrically and without fixed points. Is it true that any triangle in $\Sigma/\mathbb{S}^1$ has perimeter at most π? and, what about the existence of distance-noncontracting map $\Sigma/\mathbb{S}^1 \to \frac{1}{2}\cdot\mathbb{S}^2$?*

10.16. Advanced exercise. *Suppose $\mathbb{S}^1$ acts isometrically on an* Alex(1) *space $\mathcal{A}$ that is homeomorphic to $\mathbb{S}^3$. Assume its fixed-point set is a closed local geodesic γ. Show that* $\operatorname{length}\gamma \leqslant 2\cdot\pi$.

The same question for a $\mathbb{Z}_2$-action is open [70].

Compact m-dimensional Alex(0) spaces with the maximal number of extremal points include m-dimensional rectangles and the quotients of flat tori by reflections across a point. (This action has 2^m isolated fixed points; each corresponds to an extremal point in the quotient space $\mathcal{A} = \mathbb{T}^m/\mathbb{Z}_2$.) Nina Lebedeva has proved [49] that *every m-dimensional* Alex(0) *space with 2^m extremal points is a quotient of Euclidean space by a crystallographic action.*

Counting maximal finite subgroups in a crystallographic group Γ is equivalent to counting the so-called primitive extremal subsets in the quotient space $\mathcal{A} = \mathbb{E}^m/\Gamma$. So, 10.12 would follow from the next conjecture.

10.17. Conjecture. *Any m-dimensional compact* Alex(0) *space has at most 2^m primitive extremal subset.*

Here is an equivalent definition of extremal sets. A closed subset E in a finite-dimensional Alexandrov space is called e x t r e m a l if $\measuredangle[p\,{}^x_y] \leqslant \frac{\pi}{2}$ for any $x \notin E$ and $p \in E$ such that $|x-p|$ takes a minimal value. An extremal set is called p r i m i t i v e if it contains no proper extremal subsets.

For example, the whole space and the empty set are extremal. Also, every vertex, edge, or face (as well as their union) of the cube is an extremal subset of the cube. Vertices of the cube are its only primitive extremal subsets.

Lecture 11

Surface of convex body

In this lecture, we discuss surfaces of convex bodies; this is historically the first applications of Alexandrov geometry.

A Definitions

Let us define a convex body as a compact convex subset in $\mathbb{E}^3$ with non-empty interior. The surface of a convex body is defined as its boundary equipped with the induced length metric.

11.1. Exercise. *Show that the surface of a convex body is homeomorphic to* $\mathbb{S}^2$.

In this lecture, we will prove that *surface of a convex body is* Alex(0).

B Surface of convex polyhedra

A convex polyhedron is a convex body with a finite number of extremal points, called its vertices.

Observe that the surface, say Σ, of a convex polyhedron P admits a triangulation such that each triangle is isometric to a plane triangle. In other words, Σ is a polyhedral surface; that is, it is a 2-dimensional manifold with a length metric that admits a triangulation such that each triangle is isometric to a solid plane triangle. A triangulation of a polyhedral surface will always be assumed to satisfy this condition.

The total angle around a vertex v in Σ is defined as the sum of angles at v of all triangles in the triangulation that contain v.

Note that if a point $p \in \Sigma$ is not a vertex of P, then

- ◇ p lies in the interior of a face of P, and its neighborhood in Σ is a piece of plane, or
- ◇ p lies on an edge, and its neighborhood is two half-planes glued along the boundary.

In both cases, a neighborhood of p in Σ (with the induced length metric) is isometric to an open domain of the plane. Therefore, the total angle around p should be defined to be $2{\cdot}\pi$.

11.2. Claim. *Let Σ be the surface of a convex polyhedron P. Then, the total angle around any point in Σ cannot exceed $2{\cdot}\pi$.*

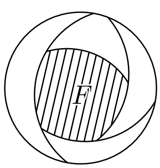
In the proof, we will need the triangle inequality for angles (or the spherical triangle inequality). A proof of this statement is given in the classical geometry textbook by Andrei Kiselyov [46, § 47]; it also follows from 1.9. (In fact our proof of 1.9 is a straightforward generalization of the argument in [46, § 47].)

11.3. Spherical triangle inequality. *Let w_1, w_2, w_3 be unit vectors in $\mathbb{E}^3$. Denote by $\theta_{i,j}$ the angle between the vectors v_i and v_j. Then*

$$\theta_{1,3} \leqslant \theta_{1,2} + \theta_{2,3}$$

and in case of equality, the vectors w_1, w_2, w_3 lie in a plane.

Proof of 11.2. Consider the intersection of P with a small sphere centered at p; it is a convex spherical polygon, say F. Applying rescaling we may assume that the sphere has unit radius. Then we need to show that the perimeter of F does not exceed $2{\cdot}\pi$.



Note that F lies in a hemisphere, say H. Moreover, there is a decreasing sequence of convex spherical polygons

$$H = H_0 \supset \dots \supset H_n = F,$$

such that H_{i+1} is obtained from H_i by cutting along a chord.

By the spherical triangle inequality (11.3), we have

$$2{\cdot}\pi = \operatorname{perim} H = \operatorname{perim} H_0 \geqslant \ldots \geqslant \operatorname{perim} H_n = \operatorname{perim} F$$

— hence the result. □

A vertex of a triangulation of a polyhedral surface is called e s s e n t i a l if the total angle around it is not $2{\cdot}\pi$.

11.4. Exercise. *Let v be a point on the surface Σ of a convex polyhedron P. Show that v is a vertex of P if and only if v is an essential vertex of Σ.*

11.5. Exercise. *Show that geodesics on the surface of a convex polyhedron do not pass thru its essential vertices.*

C Curvature

Let p be a point on the surface of a polyhedron, and θ_p is the total angle around p. The value $2\cdot\pi-\theta_p$ is called the curvature of the polyhedral surface at p. Note that if p is not a vertex, then its curvature is zero.

11.6. Exercise. *Assume that the surface of a nondegenerate tetrahedron T has curvature π at each of its vertices. Show that*

(a) all faces of T are congruent;

(b) the line passing thru midpoints of opposite edges of T intersects these edges at right angles.

Claim 11.2 says that *surfaces of convex polyhedra have nonnegative curvature* in the sense of the above definition. Now we show that this definition agrees with the 4-point comparison.

11.7. Proposition. *A polyhedral surface with nonnegative curvature at each vertex is* Alex(0)*.*

Proof. Denote the surface by Σ. By 2.18, it is sufficient to check that $\mathrm{dist}_p^2\circ\gamma$ is 1-concave for any geodesic γ and a point p in Σ.

We can assume that p is not a vertex; the vertex case can be done by approximation. Further, by 11.5, we may assume that γ does not contain vertices.

Given a point $x=\gamma(t_0)$, choose a geodesic $[px]$. Again, by 11.5, $[px]$ does not contain vertices. Therefore a small neighborhood of $U\supset[px]$ can be unfolded on a plane; that is, there is an injective length-preserving map $z\mapsto\tilde z$ of U into the Euclidean plane. This way we map part of γ in U to a line segment $\tilde\gamma$. Let

$$\tilde f(t):=\tfrac12\cdot\mathrm{dist}_{\tilde p}^2\circ\tilde\gamma(t).$$

Since the geodesic $[px]$ maps to a line segment, we have $\tilde f(t_0)=f(t_0)$. Furthermore, since the unfolding $z\mapsto\tilde z$ preserves lengths of curves, we get $\tilde f(t)\geqslant f(t)$ if t is sufficiently close to t_0. That is, $\tilde f$ is a local upper support of f at t_0. Evidently, $\tilde f''\equiv1$; therefore $f''\leqslant1$. It remains to apply 2.18. $\square$

11.8. Exercise. *Prove the converse to the proposition; that is, show that if a poyhedral surface is* Alex(0)*, then it has nonnegative curvature in the sense defined in this section.*

D Surface of convex body

11.9. Advanced exercise. *Let $K_1, K_2, \dots,$ and K_∞ be convex bodies in $\mathbb{E}^3$. Denote by S_n the surface of K_n with induced length metric. Suppose $K_n \to K_\infty$ in the sense of Hausdorff. Show that $S_n \to S_\infty$ in the sense of Gromov–Hausdorff.*

Note that any convex body is a Hausdorff limit of a sequence of convex polyhedra. Therefore, the next proposition follows from 11.7, 11.9, and 8.1.

11.10. Proposition. *The surface of a convex body is* $\mathrm{ALEX}(0)$.

E Remarks

Note that 11.1 and 11.7 imply that the surface of a convex body is a sphere with nonnegative curvature in the sense of Alexandrov. The celebrated theorem of Alexandrov states that the converse also holds if we allow degeneration of convex bodies to plane figures; the surface of a plane figure is defined as its doubling across the boundary. In other words, any $\mathrm{ALEX}(0)$ metric on the sphere is isometric to a surface of (possibly degenerate) convex body. Moreover this convex body is unique up to congruence. The last result is due to Alexei Pogorelov [84].

Originally, Alexandrov proved the statement for polyhedral metrics on the sphere; this proof is sketched in the appendix. Then he used 11.9 to extend the result to an arbitrary $\mathrm{ALEX}(0)$ metric on the sphere.

11.11. Advanced exercise. *Let S be the surface of a nondegenerate convex body $K \subset \mathbb{E}^3$; we assume that S is equipped with its induced length metric.*

(a) *Show that any geodesic γ in S is one-sided differentiable as a curve in $\mathbb{E}^3$*

(b) *Let γ_1 and γ_2 be geodesic paths in S that start at one point $p = \gamma_1(0) = \gamma_2(0)$. Suppose $x_i = \gamma_i(1)$, and $y_i = p + \gamma_i^+(0)$. Show that*

$$|x_1 - x_2|_S \leqslant |y_1 - y_2|_W,$$

where W is the complement to the interior of K.

Appendix A

Alexandrov's embedding theorem

BY NINA LEBEDEVA AND ANTON PETRUNIN

A Introduction

Intrinsic distance between two points on the surface of a convex polyhedron is defined as the length of a shortest curve on the surface between these points.

Recall that the sum of angles at the tip of a convex polyhedral angle is less than $2\cdot\pi$; this statement can be found in a school textbook [46, § 48].

It is easy to see that the surface of a convex polyhedron is homeomorphic to the sphere. Therefore the statements above imply that the surface of a convex polyhedron equipped with its intrinsic metric is an example of a *polyhedral metric on the sphere with the sum of angles around each vertex at most* $2\cdot\pi$; a metric is called polyhedral if the sphere admits a triangulation such that every triangle is congruent to a plane triangle.

Alexandrov's theorem states that the converse holds if one includes in the consideration *twice covered polygons.* In other words, we assume that a polyhedron can degenerate to a plane polygon; in this case, its surface is defined as two copies of the polygon glued along their boundary.

Further, we assume that a polyhedron can degenerate to a plane polygon.

A.1. Alexandrov's theorem.

I. A polyhedral metric on the sphere is isometric to the surface of a convex polyhedron if and only if the sum of angles around each of its vertex is not greater than $2{\cdot}\pi$.

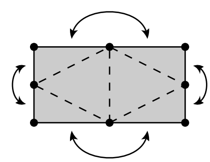
II. Moreover, a convex polyhedron is defined up to congruence by the intrinsic metric on its surface.

A. D. Alexandrov has many remarkable theorems, but in our opinion, this theorem is the most remarkable. At the same time, its proof is elementary; it could be explained to anyone familiar with basic topology.

This theorem has many applications. In particular, it is used in the proof of its generalization [7] that gives a complete description of intrinsic metrics on the sphere that are isometric to convex surfaces in the Euclidean space. The latter statement is fundamental in a branch of modern mathematics — the so-called *Alexandrov geometry.*

The first part is central; it is called the *existence theorem.* The second part is called the *uniqueness theorem*; it is a slight variation of Cauchy's theorem about polyhedrons. (There is another uniqueness theorem of Alexandrov that generalizes Minkowski's theorem about polyhedrons.)

According to the theorem, a convex polyhedron is completely defined by the intrinsic metric of its surface. In particular, knowing the metric we could find the position of the edges. However, in practice, it is not easy to do. For example, the surface glued from a rectangle as shown on the diagram defines a tetrahedron. Some of the glued lines appear inside facets of the tetrahedron and some edges (dashed lines) do not follow the sides of the rectangle.



The theorem was proved by A. D. Alexandrov in 1941 [6]; we will present a sketch of his proof. A complete proof is nicely written by A. D. Alexandrov in his book [5]. Yet another proof was found by Yu. A. Volkov in his thesis [94]; it uses a deformation of three-dimensional polyhedral space.

B Space of polyhedrons and metrics

Space of polyhedrons. Let us denote by Φ the space of all convex polyhedrons in the Euclidean space, including polyhedrons that degenerate to a plane polygon. Polyhedra in Φ will be considered up to a motion of the space, and the whole space Φ will be considered with the natural topology (an intuitive meaning of closeness of two polyhedrons should be sufficient).

Further, denote by Φ_n the polyhedrons in Φ with exactly n vertices. Since any polyhedron has at least 3 vertices, the space Φ admits a subdivision into a countable number of subsets $\Phi_3, \Phi_4, \ldots$

Space of polyhedral metrics. The space of polyhedral metrics on the sphere with the sum of angles around each point at most $2\cdot\pi$ will be denoted by Ψ. The metrics in Ψ will be considered up to an isometry, and the whole space Ψ will be equipped with the natural topology (again, an intuitive meaning of closeness of two metrics is sufficient).

A point on the sphere with the sum of angles strictly less than $2\cdot\pi$ will be called an essential vertex. The subset of Ψ of all metrics with exactly n essential vertices will be denoted by Ψ_n. It is easy to see that any metric in Ψ has at least 3 essential vertices. Therefore Ψ is subdivided into countably many subsets $\Psi_3, \Psi_4, \ldots$

From a polyhedron to its surface. Recall that the surface of a convex polyhedron is a sphere with a polyhedral metric such that the sum of angles around each point is at most $2\cdot\pi$. Therefore passing from a polyhedron to its surface defines a map

$$\iota\colon \Phi \to \Psi.$$

Note that the number of vertices of a polyhedron is equal to the number of essential vertices of its surface. In other words, $\iota(\Phi_n) \subset \Psi_n$ for any $n \geqslant 3$.

C About the proof

Using the notation introduced in the previous section, we can give the following more exact formulation of Alexandrov's theorem:

A.2. Reformulation. *For any integer $n \geqslant 3$, the map ι is a bijection from Φ_n to Ψ_n.*

We sketch the original proof of A. D. Alexandrov. It is based on the construction of a one-parameter family of polyhedrons that starts at arbitrary polyhedron and ends at a polyhedron with its surface isometric to the given one. This type of argument is called the continuity method; it is often used in the theory of differential equations.

The two parts of the first formulation will be proved separately.

Part II. Let us show that the map $\iota\colon \Phi_n \to \Psi_n$ is injective; in other words, a convex polyhedron is defined by the intrinsic metric on its surface up to a motion of the space.

The last statement is analogous to the Cauchy theorem about polyhedrons, and the proof goes along the same lines.

The Cauchy theorem states that facets of a polyhedron together with the gluing rule completely describe a convex polyhedron; its proof is given in many classical popular texts [1, 22, 89].

Part I. Let us prove that $\iota\colon \Phi_n \to \Psi_n$ is surjective. This part of the proof is subdivided into the following lemmas:

A.3. Lemma. *For any integer $n \geqslant 3$, the space Ψ_n is connected.*

The proof of this lemma is not complicated, but it requires ingenuity; it can be done by the direct construction of a one-parameter family of metrics in Ψ_n that connects two given metrics. Such a family can be obtained by a sequential application of the following construction and its inverse.

Let M be a sphere with metric from Ψ_n. Suppose v and w are essential vertices in M. Let us cut M along a shortest line from v to w. Note that the shortest line cannot pass thru an essential vertex of M. Further, note that there is a three-parameter family of patches that can be used to patch the cut so that the obtained metric remains in Ψ_n; in particular, the obtained metric has exactly n essential vertices (after the patching, the vertices v and w may become inessential).

A.4. Lemma. *The map $\iota\colon \Phi_n \to \Psi_n$ is open, that is, it maps any open set in Φ_n to an open set in Ψ_n.*

In particular, for any $n \geqslant 3$, the image $\iota(\Phi_n)$ is open in Ψ_n.

This statement is very close to the so-called invariance of domain theorem; the latter states that a continuous injective map between manifolds of the same dimension is open.

According to part II, ι is injective. The proof of the invariance of domain theorem can be adapted to our case since both spaces Φ_n and Ψ_n are $(3{\cdot}n-6)$-dimensional and both look like manifolds, altho, formally speaking, they are *not* manifolds. In a more technical language, Φ_n and Ψ_n have the natural structure of $(3{\cdot}n-6)$-dimensional orbifolds, and the map ι respects the orbifold structure.

We will only show that both spaces Φ_n and Ψ_n are $(3{\cdot}n-6)$-dimensional.

Choose a polyhedron P in Φ_n. Note that P is uniquely determined by the $3{\cdot}n$ coordinates of its n vertices. We can assume that the first vertex is the origin, the second has two vanishing coordinates and the third has one vanishing coordinate; therefore, all polyhedrons in Φ_n that lie sufficiently close to P can be described by $3{\cdot}n-6$ parameters. If P has no symmetries then this description can be made one-to-one; in this case, a neighborhood of P in Φ_n is a $(3{\cdot}n-6)$-dimensional manifold. If P has a nontrivial symmetry group, then this description is not one-to-one but it does not have an impact on the dimension of Φ_n.

The case of polyhedral metrics is analogous. We need to construct a subdivision of the sphere into plane triangles using only essential vertices. By Euler's formula, there are exactly $3{\cdot}n-6$ edges in this subdivision. Note that the lengths of edges completely describe the metric, and slight changes of these lengths produce a metric with the same property.

A.5. Lemma. *The map $\iota\colon \Phi_n\to\Psi_n$ is closed; that is, the image of a closed set in Φ_n is closed in Ψ_n.*

In particular, for any $n\geqslant 3$, the set $\iota(\Phi_n)$ is closed in Ψ_n.

Choose a closed set Z in Φ_n. Denote by $\bar Z$ the closure of Z in Φ; note that $Z=\Phi_n\cap\bar Z$. Assume $P_1,P_2,\dots\in Z$ is a sequence of polyhedrons that converges to a polyhedron $P_\infty\in\bar Z$. Note that $\iota(P_n)$ converges to $\iota(P_\infty)$ in Ψ. In particular, $\iota(\bar Z)$ is closed in Ψ.

Since $\iota(\Phi_n)\subset\Psi_n$ for any $n\geqslant 3$, we have $\iota(Z)=\iota(\bar Z)\cap\Psi_n$; that is, $\iota(Z)$ is closed in Ψ_n.

Summarizing, $\iota(\Phi_n)$ is a non-empty closed and open set in Ψ_n, and Ψ_n is connected for any $n\geqslant 3$. Therefore, $\iota(\Phi_n)=\Psi_n$; that is, $\iota\colon\Phi_n\to\Psi_n$ is surjective. □

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Semisolutions

1.1. Choose a sequence of positive numbers $\varepsilon_n \to 0$ and a finite ε_n-net N_n of K for each n. We can assume that $\varepsilon_0 > \operatorname{diam} K$, and N_0 is a one-point set. If $|x-y| < \varepsilon_k$ for some $x \in N_{k+1}$ and $y \in N_k$, then connect them by a curve of length at most ε_k.

Let K' be the union of all these curves and K. Show that K' is compact and path-connected.

Source: This problem is due to Eugene Bilokopytov [10].

1.2. Choose a Cauchy sequence x_n in $(\mathcal{X}, \|*-*\|)$; it is sufficient to show that a subsequence of x_n converges.

Observe that the sequence x_n is Cauchy in $(\mathcal{X}, |*-*|)$; denote its limit by x_∞.

Passing to a subsequence, we can assume that $\|x_n - x_{n+1}\| < \frac{1}{2^n}$. It follows that there is a 1-Lipschitz path γ in $(\mathcal{X}, \|*-*\|)$ such that $x_n = \gamma(\frac{1}{2^n})$ for each n and $x_\infty = \gamma(0)$. Therefore,

$$\|x_\infty - x_n\| \leqslant \operatorname{length} \gamma|_{[0,\frac{1}{2^n}]} \leqslant \tfrac{1}{2^n}.$$

In particular, x_n converges to x_∞ in $(\mathcal{X}, \|*-*\|)$.

Source: [40, Corollary]; see also [71, Lemma 2.3].

1.3. Given a pair of points p and q, choose a sequence of paths γ_n from p to q such that

$$\operatorname{length} \gamma_n \to |p-q| \quad \text{as} \quad n \to \infty;$$

these paths exist since we are in a length space. Note that we can assume that each γ_n is parametrized proportionally to the arc length; in particular, γ_n are equicontinuous. Show that paths γ_n lie in a closed ball, say $\overline{\mathrm{B}}[p,r]$ of some radius $r < \infty$. Since the space is proper, $\overline{\mathrm{B}}[p,r]$ is compact. By the Arzelà–Ascoli theorem, we can pass to a converging subsequence of γ_n. Show that its limit is a geodesic path from p to q.

1.5. Choose a sequence $\varepsilon_n > 0$ that converges to zero very fast, say such that $\sum_n 10^n \cdot \varepsilon_n$ is small. Follow the argument in the proof of Menger's lemma, taking ε_n-midpoints at the n^{th} stage.

1.6. Let us write the Riemannian metric on $\mathbb{M}^2(\kappa)$ in polar coordinates (θ, r); it has the form $\left(\begin{smallmatrix} h^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where $h = h(\kappa, r) \geqslant 0$. Calculate $h(\kappa, r)$. Show that for fixed r, the function $r \mapsto h(\kappa, r)$ is nonincreasing in the domain of definition. Suppose $\kappa < \mathrm{K}$, consider the partially defined map $\mathbb{M}^2(\kappa) \to \mathbb{M}^2(\mathrm{K})$ that sends a point to the point with the same polar coordinates. Show that this map is short in the domain of definition. Use it to prove the statement in the exercise.

1.7. Show and use that $\tilde{\measuredangle}(p\,{}^x_y)_{\mathbb{S}^2} - \tilde{\measuredangle}(p\,{}^x_y)_{\mathbb{E}^2} = O(|p-x|^2 + |p-y|^2)$ and $\tilde{\measuredangle}(p\,{}^x_y)_{\mathbb{E}^2} - \tilde{\measuredangle}(p\,{}^x_y)_{\mathbb{H}^2} = O(|p-x|^2 + |p-y|^2)$.

1.8. Consider a hinge in the plane $\mathbb{R}^2$ with a metric defined by norm, say by the ℓ^∞-norm.

1.10. Assume $\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] < \pi$. By 1.9, $\measuredangle[p\,{}^x_y] < \pi$. Therefore, $\tilde{\measuredangle}(p\,{}^{\bar{x}}_{\bar{y}}) < \pi$ for some $\bar{x} \in \,]px]$ and $\bar{y} \in \,]py]$. Hence

$$|p - \bar{x}| + |\bar{y} - p| < |\bar{x} - \bar{y}|$$

— a contradiction.

1.11. Denote by α the arc-length parametrization of $[qp]$ from q to p. Choose $\varepsilon > 0$. Observe that

$$|\gamma(t) - \alpha(\tfrac{1}{\varepsilon}\cdot t)|^2 \leqslant t^2 \cdot (1 - \tfrac{2}{\varepsilon} \cdot \cos\varphi + \tfrac{1}{\varepsilon^2}) + o(t^2),$$

where $\varphi = \measuredangle[q\,{}^p_x]$. By the triangle inequality

$$|p - \gamma(t)| \leqslant |\gamma(t) - \alpha(\tfrac{1}{\varepsilon}\cdot t)| + |q - p| - \tfrac{1}{\varepsilon}\cdot t.$$

Conclude that

$$|p - \gamma(t)| \leqslant |q - p| - t \cdot \cos\varphi + \delta(\varepsilon)\cdot t + o(t),$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The statement follows since $\varepsilon > 0$ is arbitrary.

1.13. Since the space is proper, it is separable; that is, we can choose an countable everywhere dense set $\{x_1, x_2, \dots\}$.

Let $A_1, A_2, \dots$ be a sequence of closed sets. Applying the diagonal procedure, we can pass to a subsequence such that for each i the sequence $\operatorname{dist}_{A_n} x_i$ converges as $n \to \infty$; denote its limit by $f(x_i)$.

Since $\operatorname{dist}_{A_n}$ is 1-Lipschitz for any n, we have

$$|f(x_i) - f(x_j)| \leqslant |x_i - x_j|$$

for all i and j. Suppose $f(x_i) < \infty$ for some i; note that the same holds for any i. Therefore, the function f can be extended to a continuous function defined on the whole ambient space. Show that $A_\infty = f^{-1}\{0\}$ is the limit of A_n in the sense of Hausdorff.

If $f(x_i) = \infty$ for some i, then the same holds for any i. Show that in this case $A_n \to \varnothing$ in the sense of Hausdorff.

1.16. Apply the definition of Hausdorff distance (1.14).

1.18. Given $x_\infty, y_\infty \in \mathcal{X}_\infty$, choose $x_n, y_n \in \mathcal{X}_n$ such that $x_n \to x_\infty$ and $y_n \to y_\infty$. Let z_n be the midpoint of $[x_n y_n]$. Since $\mathcal{X}_\infty$ is proper, we can choose a subsequence of z_m that converges to a point, say $z_\infty \in \mathcal{X}_\infty$. Note that z_∞ is a midpoint of x_∞ and y_∞, then apply Menger's lemma (1.4).

1.21. Given a pair of points $x_0, y_0 \in \mathcal{K}$, consider two sequences $x_0, x_1, \dots$ and $y_0, y_1, \dots$ such that $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ for each n.

Since $\mathcal{K}$ is compact, we can choose an increasing sequence of integers n_k such that both sequences $(x_{n_i})_{i=1}^\infty$ and $(y_{n_i})_{i=1}^\infty$ converge. In particular, both are Cauchy; that is,

$$|x_{n_i} - x_{n_j}|_\mathcal{K} \to 0 \quad \text{and} \quad |y_{n_i} - y_{n_j}|_\mathcal{K} \to 0$$

as $\min\{i, j\} \to \infty$.

Since f is distance-noncontracting,

$$|x_0 - x_{|n_i - n_j|}| \leqslant |x_{n_i} - x_{n_j}|$$

for any i and j. Therefore, there is a sequence $m_i \to \infty$ such that

$$(*) \qquad x_{m_i} \to x_0 \quad \text{and} \quad y_{m_i} \to y_0$$

as $i \to \infty$.

Since f is distance-noncontracting, the sequence $\ell_n = |x_n - y_n|_\mathcal{K}$ is nondecreasing. By $(*)$, $\ell_{m_i} \to \ell_0$ as $m_i \to \infty$. It follows that

$$\ell_0 = \ell_1 = \dots$$

In particular,

$$|x_0 - y_0|_\mathcal{K} = \ell_0 = \ell_1 = |f(x_0) - f(y_0)|_\mathcal{K}$$

for any pair of points (x_0, y_0) in $\mathcal{K}$. That is, the map f is distance-preserving; hence f is injective. From $(*)$, we also get that $f(\mathcal{K})$ is everywhere dense. Since $\mathcal{K}$ is compact $f\colon \mathcal{K} \to \mathcal{K}$ is surjective — hence the result.

Remarks. This is a basic lemma in the introduction to Gromov–Hausdorff distance [see 7.3.30 in 14]. The presented proof is not quite standard; I learned it from Travis Morrison, a student in my MASS class at Penn State, Fall 2011.

Note that this exercise implies that *any surjective non-expanding map from a compact metric space to itself is an isometry.*

1.22. The only-if part is trivial. Let us prove the if part.

If $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \not\to 0$, then we can pass to a subsequence such that $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \geqslant \varepsilon$ for some $\varepsilon > 0$. Show that we can pass to a subsequence again, so that $\mathcal{X}_n$ converges in the sense of Gromov–Hausdorff, say to $\mathcal{Y}$. Observe that $\mathcal{Y} \leqslant \mathcal{X}_\infty$ and $\mathcal{X}_\infty \leqslant \mathcal{Y}$. By 1.21, $\mathcal{Y} \stackrel{iso}{=\!=} \mathcal{X}_\infty$ — a contradiction.

1.23. Show and use that $|\mathcal{X}_\infty - \mathcal{X}'_\infty|_{\mathrm{GH}} < \varepsilon$ for any $\varepsilon > 0$.

1.27.

(a) Consider the graphs of the following functions with the induced metric from $\mathbb{R}^2$.

$$x \mapsto \cos x + \cos \tfrac{x}{\pi} \quad \text{and} \quad x \mapsto \cos x + \sin \tfrac{x}{\pi}.$$

(b) For every rational number $q \in [1, 2]$ consider an interval of length q. Let $\mathcal{X}$ be obtained by identifying all initial points of the intervals to one point and all end points to another.

Let $\mathcal{Y}$ be constructed in the same way but skipping the interval of length 1.5.

2.2. The 4-point comparison (2.1) reduces our question to the following. *Any spherical triangle has perimeter at most* $2{\cdot}\pi$. Choose a spherical triangle $[xyz]$. Let x' be the antipode of x; that is $x' = -x$. The spherical triangle inequality (1.9 or 11.3) implies that

$$|x - z|_{\mathbb{S}^2} \leqslant |y - x'|_{\mathbb{S}^2} + |x' - z|_{\mathbb{S}^2}.$$

Observe that

$$|x - y|_{\mathbb{S}^2} + |y - x'|_{\mathbb{S}^2} = \pi, \quad \text{and} \quad |x - z|_{\mathbb{S}^2} + |z - x'|_{\mathbb{S}^2} = \pi.$$

Hence

$$|x - y|_{\mathbb{S}^2} + |x - z|_{\mathbb{S}^2} + |y - z|_{\mathbb{S}^2} \leqslant 2{\cdot}\pi.$$

2.3. For the only-if part consider the following two cases.

If $\tilde\measuredangle(p\,{}^{x_1}_{x_2}) + \tilde\measuredangle(p\,{}^{x_2}_{x_3}) \geqslant \pi$, then choose two model triangles $[qy_1y_2] = \tilde\triangle(px_1x_2)$ and $[qy_2y_3] = \tilde\triangle(px_2x_y)$ that lie on the opposite sides of

$[qy_2]$. By the comparison, $|y_1-y_3|\geqslant|x_1-x_3|$. Therefore the obtained configuration meets all the conditions.

If $\tilde\measuredangle(p\,^{x_1}_{x_2})+\tilde\measuredangle(p\,^{x_2}_{x_3})\geqslant\pi$, then choose two model triangles $[qy_1y_2]=$ $=\tilde\triangle(px_1x_2)$ and take y_3 on the extension of $[y_1q]$ behind q such that $|q-y_3|=|p-x_3|$. Then $\measuredangle[q\,^{y_2}_{y_3}]\geqslant\tilde\measuredangle(p\,^{x_2}_{x_3})$, therefore $|y_2-y_3|\geqslant|x_2-$ $-x_3|$. Further, $|y_2-y_3|=|x_2-p|+|p-x_3|\geqslant|x_2-x_3|$, and again, the obtained configuration meets all the conditions.

To prove the if part, choose a configuration q,y_1,y_2,y_3 that meets all the conditions and maximize the sum

$$|y_1-y_2|+|y_2-y_3|+|y_3-y_1|.$$

Show that q lies in the solid triangle $y_1y_2y_3$; in particular

$$\measuredangle[q\,^{y_1}_{y_2}]+\measuredangle[q\,^{y_2}_{y_3}]+\measuredangle[q\,^{y_3}_{y_1}]=2\cdot\pi.$$

Moreover, $|q-y_i|=|p-x_i|$ for each i. Applying that increasing the opposite side in a plane triangle increases the corresponding angle, we get

$$\tilde\measuredangle(p\,^{x_1}_{x_2})+\tilde\measuredangle(p\,^{x_2}_{x_3})+\tilde\measuredangle(p\,^{x_3}_{x_1})\leqslant 2\cdot\pi.$$

2.5. Consider model triangles $[\tilde p\tilde x\tilde z]=\tilde\triangle(pxz)$ and $[\tilde p\tilde y\tilde z]=\tilde\triangle(pyz)$ that share side $[\tilde p\tilde z]$ and lie on its opposite sides. Note that

$$\begin{aligned}|\tilde x-\tilde y|_{\mathbb{E}^2}&\geqslant|\tilde x-\tilde y|_{\mathbb{E}^2}+|\tilde x-\tilde y|_{\mathbb{E}^2}=\\&=|x-z|_{\mathcal{X}}+|z-y|_{\mathcal{X}}=\\&=|x-y|_{\mathcal{X}},\end{aligned}$$

where $\mathcal{X}$ is our metric space. It remains to apply the monotonicity of angle in a triangle with respect to its opposite side.

2.7. Apply 2.6.

2.9. Without loss of generality, we can assume that $|p-x|\leqslant|p-y|$. Choose $\bar x\in[px]$; let $\bar y\in[px]$ be such that $|p-\bar x|=|p-\bar y|$. Apply 2.6 to show that $\bar x=\bar y$. Conclude that $[px]\subset[py]$.

2.10. Assume that there are two distinct geodesics from z to x. Then we can choose distinct points p and q on these geodesics such that $|z-p|=|z-q|$. Observe that $\tilde\measuredangle(z\,^p_q)>0$. By the triangle inequality, we get

$$|x-p|+|p-y|\leqslant|x-p|+|p-z|+|z-y|=|x-z|+|z-y|$$

Observe that $\tilde\measuredangle(z\,^x_y)=\pi$. Therefore $\measuredangle[z\,^x_y]=\pi$ for any geodesic $[zx]$.

2.11. By 1.10, we have

$$\measuredangle[p\,{}^{x}_{z}] + \measuredangle[p\,{}^{y}_{z}] \geqslant \pi.$$

Since $z \in]xy[$ we have

$$\tilde\measuredangle(z\,{}^{\bar x}_{\bar y}) = \pi$$

for any $\bar x \in [xz[$ and $\bar y \in]zy]$. By comparison, we have that

$$\tilde\measuredangle(z\,{}^{\bar x}_{\bar p}) + \tilde\measuredangle(z\,{}^{\bar p}_{\bar y}) \leqslant \pi$$

for any $\bar p \in]zp]$. Passing to the limit as $|z - \bar x| \to 0$, $|z - \bar y| \to 0$, and $|z - \bar p| \to 0$, we get the statement.

2.12. Without loss of generality, we can assume that x, v, w, and y appear on $[xy]$ in this order. By 2.6,

$$\tilde\measuredangle(x\,{}^{y}_{p}) \geqslant \tilde\measuredangle(x\,{}^{w}_{p}) \geqslant \tilde\measuredangle(x\,{}^{v}_{p}).$$

Hence, $\Rightarrow$ follows.

By Alexandrov's lemma,

$$\begin{aligned}
\tilde\measuredangle(x\,{}^{y}_{p}) = \tilde\measuredangle(x\,{}^{v}_{p}) \quad&\iff\quad \tilde\measuredangle(y\,{}^{x}_{p}) = \tilde\measuredangle(y\,{}^{v}_{p}),\\
\tilde\measuredangle(x\,{}^{y}_{p}) = \tilde\measuredangle(x\,{}^{w}_{p}) \quad&\iff\quad \tilde\measuredangle(y\,{}^{x}_{p}) = \tilde\measuredangle(y\,{}^{w}_{p}).
\end{aligned}$$

Whence, $\Leftarrow$ follows.

2.13. Suppose $\measuredangle[x_\infty\,{}^{y_\infty}_{z_\infty}] > \alpha$. Then we can choose $\bar y_\infty \in]x_\infty y_\infty]$ and $\bar z_\infty \in]x_\infty z_\infty]$ such that $\tilde\measuredangle(x_\infty\,{}^{\bar y_\infty}_{\bar z_\infty}) > \alpha$. Now choose $\bar y_n \in]x_n y_n]$ and $\bar y_n \in]x_n z_n]$ such that $\bar y_n \to \bar y_\infty$ and $\bar z_n \to \bar z_\infty$. Observe that

$$\varliminf_{n\to\infty} \measuredangle[x_n\,{}^{y_n}_{z_n}] \geqslant \varliminf_{n\to\infty} \tilde\measuredangle(x_n\,{}^{\bar y_n}_{\bar z_n}) \geqslant \alpha,$$

hence the result.

2.16. The Urysohn space provides an example; see for example [80, Lecture 2].

2.17. Choose a triangle $[0vw]$. Note that $m = \tfrac12(v{+}w)$ is the midpoint of $[vw]$.

Use comparison, to show that

$$2{\cdot}|\tfrac12(v+w)|^2 + 2{\cdot}|\tfrac12(v-w)|^2 \geqslant |v|^2 + |w|^2.$$

Note this inequality implies the opposite one; it follows if we rewrite it via $x = \tfrac12(v+w)$ and $y = \tfrac12(v-w)$. Hence we have

$$2{\cdot}|\tfrac12(v+w)|^2 + 2{\cdot}|\tfrac12(v-w)|^2 = |v|^2 + |w|^2$$

for any v, w. That is, the norm is quadratic and the statement follows.

3.3. Suppose such a point does not exist; that is, for any $p \in \mathcal{X}$ there is a point p' such that $r(p') \leqslant (1-\varepsilon)\cdot r(p)$ and $|p - p'| < \frac{1}{\varepsilon}\cdot r(p)$. Construct a sequence of points $p_0, p_1, \dots$ such that $p_n = p'_{n-1}$ for any n. Show that this sequence is Cauchy; denote its limit by p_∞. Arrive at a contradiction by showing that $r(p_\infty) \leqslant 0$.

3.6. Note that $\mathcal{X}$ has no defined sphericlal model angles; therefore it has curvature $\geqslant 1$.

However, $\mathcal{X}$ does not have curvature $\geqslant 0$ since

$$\tilde\measuredangle(p\,{}^{x_1}_{x_2})_{\mathbb{E}^2} = \tilde\measuredangle(p\,{}^{x_2}_{x_3})_{\mathbb{E}^2} = \tilde\measuredangle(p\,{}^{x_1}_{x_3})_{\mathbb{E}^2} = \pi.$$

3.7. Suppose $\measuredangle[m\,{}^{x}_{p}] \neq 0$ and $\measuredangle[m\,{}^{x}_{p}] \neq \pi$, or equivalently $\measuredangle[m\,{}^{x}_{q}] \neq 0$.

We can assume that $|p-q|$ only slightly exceeds π, so $|p-m| < \pi$ and $|q-m| < \pi$. We can also assume that $|x-m| < \pi$. Use the comparison to show that

$$|p-x| + |q-x| < |p-q|,$$

and arrive at a contradiction with the triangle inequality.

Extend $[pq]$ to a maximal local geodesic γ. It might be a closed or a line segment. Argue as above to show that any point lies on γ and make a conclusion.

3.8. Arguing by contradiction, suppose

$$❶ \qquad |p-q| + |q-r| + |r-p| > 2\cdot\pi$$

for $p, q, r \in \mathcal{A}$. Rescaling the space slightly, we can assume that $\operatorname{diam}\mathcal{A} < \pi$, but the inequality ❶ still holds. By 3.5, after rescaling $\mathcal{A}$ is still $\mathrm{A{\scriptsize LEX}}(1)$.

Take $z_0 \in [qr]$ on maximal distance from p. Consider the following model configuration: two geodesics $[\tilde p\tilde z_0]$, $[\tilde q\tilde r]$ in $\mathbb{S}^2$ such that

$$\begin{aligned} |\tilde p - \tilde z_0| &= |p - z_0|, & \qquad |\tilde q - \tilde r| &= |q - r|,\\ |\tilde z_0 - \tilde q| &= |z_0 - q|, & \qquad |\tilde z_0 - \tilde r| &= |z_0 - q|, \end{aligned}$$

and

$$\measuredangle[\tilde z_0\,{}^{\tilde q}_{\tilde p}] = \measuredangle[\tilde z_0\,{}^{\tilde r}_{\tilde p}] = \tfrac{\pi}{2}.$$

Let $\tilde z \in [\tilde q\tilde r]$, and let $z \in [qr]$ be the corresponding point. By comparison, $|p-z| \leqslant |\tilde p - \tilde z|$ for points z near z_0. Moreover, this inequality holds as far as

$$|\tilde p - \tilde z_0| + |\tilde z_0 - \tilde z| + |\tilde p - \tilde z| < 2\cdot\pi.$$

But this inequality holds for all $\tilde{z}$ since $|\tilde{p}-\tilde{z}_0|<\pi$, $|\tilde{z}_0-\tilde{q}|<\pi$, and $|\tilde{z}_0-\tilde{r}|<\pi$. Hence we get $|p-q|\leqslant|\tilde{p}-\tilde{q}|$ and $|p-r|\leqslant|\tilde{p}-\tilde{r}|$. The latter contradicts ❶.

4.1. Suppose $\uparrow_{[px_n]}\not\to\uparrow_{[px_\infty]}$. Since Σ_p is compact, we may pass to a converging subsequence of $\uparrow_{[px_n]}$; denote by ξ its limit. We may assume that $\measuredangle(\uparrow_{[px_\infty]},\xi)>0$.

Denote by γ_n and γ_∞ the arc-length parametrization of $[px_n]$ and $[px_\infty]$ from p. Choose a geodesic α that starts from p and goes in a direction sufficiently close to ξ. By comparison we can choose α so that

$$|\alpha(t)-\gamma_n(t)|<\varepsilon{\cdot}t$$

for all large n and all sufficiently small t. Moreover, we can assume that

$$|\alpha(t)-\gamma_\infty(t)|>a{\cdot}t$$

for some fixed $a>0$ and all small t. These two inequalities imply that

$$|\gamma_n(t)-\gamma_\infty(t)|>\tfrac{a}{2}{\cdot}t$$

for all small t and all large n. On the other hand, by assumption, $|\gamma_n(t)-\gamma_\infty(t)|\to 0$ as $n\to\infty$ — a contradiction.

Comments. The compactness of Σ_p is necessary. An example can be built using iterated warped product of line segments and applying [2, Theorem 1.2]. The space $\mathcal{A}$ can be assumed to be compact.

4.2. Note that any point of $\mathrm{Cone}\,\mathcal{X}$ can be connected to the origin by a geodesic. Given a nonzero element $v\in\mathrm{Cone}\,\mathcal{X}$, denote by v' its projection in $\mathcal{X}$.

Suppose $\mathcal{X}$ is π-geodesic. Choose two nonzero elements $v,w\in$ $\in\mathrm{Cone}\,\mathcal{X}$; let $\alpha=\measuredangle(v,w)=|v'-w'|_{\mathcal{X}}$. If $\alpha\geqslant\pi$, then the product of geodesics $[v0]\cup[0w]$ forms a geodesic $[vw]$. If $\alpha<\pi$, there is a geodesic $\gamma\colon[0,\alpha]\to\mathcal{X}$ from v' to w'. Consider hinge $[\tilde{o}\,{}^{\tilde{v}}_{\tilde{w}}]$ in the plane such that $\measuredangle[\tilde{o}\,{}^{\tilde{v}}_{\tilde{w}}]=\alpha$, $|\tilde{o}-\tilde{v}|=|v|$, and $|\tilde{o}-\tilde{w}|=|w|$. Let $t\mapsto(\varphi(t),r(t))$ be geodesic $[\tilde{v}\tilde{w}]$ written in polar coordinates with origin $\tilde{o}$, so that $\varphi(0)=0$. Show that $t\mapsto r(t){\cdot}\gamma\circ\varphi(t)$ is a geodesic from v to w; here we identify $\mathcal{X}$ with the unit sphere in $\mathrm{Cone}\,\mathcal{X}$.

To prove the converse, try to reverse the steps in the argument above.

4.4. Let $\mathcal{A}_n=\lambda_n{\cdot}\mathcal{A}$. Note that for any n the space $\Sigma_p\mathcal{A}$ is identical to $\Sigma_{\iota_n(p)}\mathcal{A}_n$. In particular, we can identify isometrically $\mathrm{T}_p\mathcal{A}$ with $\mathrm{T}_{\iota_n(p)}(\lambda{\cdot}\mathcal{A})$. So for any geodesic γ that starts at p, the vector $\gamma^+(0)$ corresponds to $\frac{1}{\lambda}{\cdot}(\iota_n\circ\gamma)^+(0))$.

Consider the logarithm maps $f_n = \log_{\iota_n(p)} \colon \mathcal{A}_n \to T_p\mathcal{A}$. We claim that this sequence of maps satisfies the assumptions of Lemma 1.26; the condition in *(a)* is evident.

Note that it is sufficient to check the conditions in *(b)* and *(c)* only for $R = 1$.

Choose $\varepsilon > 0$. By compactness of Σ_p we can find a finite ε-net $\xi_1, \dots, \xi_N$ in Σ_p. Moreover, without loss of generality we can assume that these directions are geodesic; that is, there exist geodesics $\gamma_1, \dots, \gamma_N$ starting at p such that $\xi_i = \gamma_i^+(0)$ for each i.

Choose $T > 0$ such that all γ_i are defined on $[0, T]$. Show that for any $\lambda_n > \frac{1}{T}$ the image under f_n of the union $\bigcup_N \gamma_i([0, T])$ is an ε-net in $\mathrm{B}(0,1)_{T_p}$. This proves *(c)*.

By comparison, we have that

$$|\xi_i - \xi_j|_{\Sigma_p} \geqslant \tilde{\measuredangle}(p\,{}^{\gamma_i(t_i)}_{\gamma_j(t_j)})$$

for all $i \neq j$ and any $t_i, t_j \in (0, T]$. By the definition of an angle, we can assume that T has been chosen so that in addition

$$|\xi_i - \xi_j|_{\Sigma_p} \leqslant \tilde{\measuredangle}(p\,{}^{\gamma_i(t)}_{\gamma_j(t)}) + \varepsilon$$

for all $i \neq j$ and any $t \in (0, T]$.

By construction of the map f_n this implies that

$$||x - x'|_{\mathcal{A}_n} - |f_n(x) - f_n(x')|_{T_p}| < \varepsilon$$

for all $\lambda_n > \frac{1}{T}$ and all points x, x' in $\bigcup_N \gamma_i([0, \frac{1}{\lambda_n}]) \subset \mathrm{B}(p, 1)_{\mathcal{A}_n}$.

Now hinge comparison and the triangle inequality imply that the same holds for arbitrary points x, x' in $\mathrm{B}(p, 1)_{\mathcal{A}_n}$ with ε replaced by 3ε. This verifies *(b)*.

4.5. From 2.18, this inequality follows in the sense of distributions, and hence in any other sense.

4.6. Since angles are defined, it follows that

$$|\gamma_1(t) - \gamma_2(t)| \leqslant \theta{\cdot}t$$

for all small $t > 0$. Since f is L-Lipschitz, we get

$$|f(\gamma_1(t)) - f(\gamma_2(t))| \leqslant L{\cdot}\theta{\cdot}t,$$

hence the statement.

4.7; *(a)* Note that we can assume there is a geodesic in the direction of v, and apply 1.11.

(b). By *(a)*, $\boldsymbol{d}_p\mathrm{dist}_q(v) \leqslant -\max_{\xi\in\Uparrow_p^q}\langle\xi, v\rangle$. Suppose this inequality is strict for some v. We can assume that $|v| = 1$ and there is a geodesic, say γ in the direction of v. Let $\boldsymbol{d}_p\mathrm{dist}_q(v) = -\cos\alpha_0$ for some $\alpha \in [0, \pi]$. Note that any geodesic from p to q makes angle bigger than α_0 with γ.

The function $f = \mathrm{dist}_q \circ \gamma$ is Lipschitz. By Rademacher's theorem it is differentiable almost everywhere; moreover,

$$f(t) - f(0) = \int_0^t f'(t)\cdot dt.$$

Suppose $f'(t)$ is defined. Use *(a)* to show that $f'(t) = -\cos\alpha(t)$, where $\alpha(t)$ is the angle between γ and any geodesic from $\gamma(t)$ to q. Note that we can choose a sequence $t_n \to 0$ such that

$$\lim_{n\to\infty} \alpha(t_n) \leqslant \alpha_0.$$

Consider a sequence of geodsics $[p\,\gamma(t_n)]$. Since the space is proper, we can pass to its convergent subsequence. Its limit is a geodesic from p to q, denote it by $[pq]$.

Use 2.13 to show that $[pq]$ makes an angle at most α_0 with γ — a contradiction.

4.9. Let $\gamma\colon [0, \ell] \to \mathcal{A}$ be the geodesic $[xy]$ parametrized from x to y, and let $\varphi = f \circ \gamma$. Observe that

$$\varphi'(0) = \boldsymbol{d}_x f(\uparrow_{[xy]}) \leqslant \langle\uparrow_{[xy]}, \nabla_x f\rangle.$$

The same way we get $-\varphi'(\ell) \leqslant \langle\uparrow_{[yx]}, \nabla_y f\rangle$. Since f is λ-concave, we have

$$\begin{aligned} f(y) &\leqslant f(x) + \varphi'(0)\cdot\ell + \tfrac{\lambda}{2}\cdot\ell^2, \\ f(x) &\leqslant f(y) - \varphi'(\ell)\cdot\ell + \tfrac{\lambda}{2}\cdot\ell^2. \end{aligned}$$

Hence the statement follows.

4.12. If the space is proper, then the statement follows from *(b)* and 2.10.

To do the general case argue by contradiction. Let z be a point on the extension of $[pq]$ behind q; it exists by the assumption. Note that we can assume that $|v| = 1$ and it is a direction of a geodesic, say $[px]$.

Show that for there is a sequence $x_n \in\,]px]$ such that $|p - x_n| \to 0$ ad $\measuredangle[q\,{}^{x_n}_{p}] > \varepsilon$ for each n and some fixed $\varepsilon > 0$. Observe that $\measuredangle[q\,{}^{x_n}_{z}] < < \pi - \varepsilon$; therefore

$$|z - x_n| < |x_n - q| + |q - z| - \delta$$

for each n and some fixed $\delta > 0$. Pass to the limit as $x_n \to p$ and arrive at a contradiction.

4.13. Note that $|(\boldsymbol{d}_p f)(v) - (\boldsymbol{d}_p g)(v)| \leqslant s\cdot|v|$ for any $v \in \mathrm{T}_p$. From the definition of gradient (4.8) we have:

$$\begin{aligned}
(\boldsymbol{d}_p f)(\nabla_p g) &\leqslant \langle \nabla_p f, \nabla_p g\rangle, & (\boldsymbol{d}_p g)(\nabla_p f) &\leqslant \langle \nabla_p f, \nabla_p g\rangle,\\
(\boldsymbol{d}_p f)(\nabla_p f) &= \langle \nabla_p f, \nabla_p f\rangle, & (\boldsymbol{d}_p g)(\nabla_p g) &= \langle \nabla_p g, \nabla_p g\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\nabla_p f - \nabla_p g|^2 = \langle \nabla_p f, \nabla_p f\rangle + \langle \nabla_p g, \nabla_p g\rangle - 2\cdot\langle \nabla_p f, \nabla_p g\rangle \leqslant\\
&\leqslant (\boldsymbol{d}_p f)(\nabla_p f) + (\boldsymbol{d}_p g)(\nabla_p g) - (\boldsymbol{d}_p f)(\nabla_p g) - (\boldsymbol{d}_p g)(\nabla_p f) \leqslant\\
&\leqslant s\cdot(|\nabla_p f| + |\nabla_p g|).
\end{aligned}$$

4.14. Suppose $|\nabla_x f| > s$. Then we can choose a geodesic γ that starts at x such that $(f\circ\gamma)^+(0) > s$. In particular, there is $\varepsilon > 0$ such that

$$f\circ\gamma(t) > (s+\varepsilon)\cdot t + o(t),$$

hence the only-if part follows.

Now suppose $f(y) - f(x) > s\cdot\ell + \lambda\cdot\tfrac{\ell^2}{2}$, were $\ell = |x-y|$. Let $\gamma\colon [0,\ell] \to \mathcal{A}$ be a geodesic from x to y. Since $f\circ\gamma$ is λ-concave, we have

$$f\circ\gamma(\ell) \leqslant f\circ\gamma(0) + (f\circ\gamma)^+(0)\cdot\ell + \lambda\cdot\tfrac{\ell^2}{2}.$$

It follows that

$$\boldsymbol{d}_x(\uparrow_{[xy]}) = (f\circ\gamma)^+(0) > s,$$

and by 4.10, $|\nabla_x f| > s$.

5.7. Note that $f\circ\alpha$ is a nondecreasing function. Apply 4.7*a* and the definition of gradient to show that

$$-\boldsymbol{d}_{\alpha(t)}\mathrm{dist}_{\alpha(t_3)}(\nabla_{\alpha(t)} f) \geqslant \langle \nabla_{\alpha(t)}, \uparrow_{[\alpha(t)\alpha(t_3)]}\rangle \geqslant \boldsymbol{d}_{\alpha(t)}(\uparrow_{[\alpha(t)\alpha(t_3)]}) \geqslant 0$$

for any $t < t_3$. Conclude that the function $t \mapsto \mathrm{dist}_{\alpha(t_3)}\circ\alpha(t)$ is noncreasing for $t \leqslant t_3$.

5.8. For any $s > s_0$,

$$\begin{aligned}
(f\circ\hat\alpha)^+(s_0) &= |\nabla_{\hat\alpha(s_0)} f| \geqslant\\
&\geqslant (d_{\hat\alpha(s_0)} f)(\uparrow_{[\hat\alpha(s_0)\hat\alpha(s)]}) \geqslant\\
&\geqslant \frac{f\circ\hat\alpha(s) - f\circ\hat\alpha(s_0)}{|\hat\alpha(s) - \hat\alpha(s_0)|}.
\end{aligned}$$

Since $s-s_0\geqslant|\hat\alpha(s)-\hat\alpha(s_0)|$, for any $s>s_0$ we have

$$(f\circ\hat\alpha)^+(s_0)\geqslant\frac{f\circ\hat\alpha(s)-f\circ\hat\alpha(s_0)}{s-s_0}.$$

5.9. Fix t, and let $p=\alpha(t)$ and $q=\beta(t)$. Apply 5.5 to get

$$\begin{aligned}\ell^+&\leqslant-\langle\uparrow_{[pq]},\nabla_p f\rangle-\langle\uparrow_{[qp]},\nabla_q g\rangle\leqslant\\&\leqslant-\Big(f(q)-f(p)-\lambda\cdot\tfrac{\ell^2}{2}\Big)/\ell-\Big(g(p)-g(q)-\lambda\cdot\tfrac{\ell^2}{2}\Big)/\ell\leqslant\\&\leqslant\lambda\cdot\ell+\tfrac{2\cdot\varepsilon}{\ell}.\end{aligned}$$

Integrating this inequality, we get the second statement.

6.2. Apply 4.5.

6.3. By the triangle inequality,

$$|\gamma(-t)-x|+|\gamma(t)-x|-2\cdot t\geqslant0$$

for any $t\geqslant0$. Passing to the limit as $t\to\infty$, we get the result.

6.7. Suppose $\operatorname{Cone}\mathcal X$ is $\mathrm{ALEX}(0)$. Observe that two half-lines in $\operatorname{Cone}\mathcal X$ that start from the origin and go into directions x and $y\in\mathcal X$ form a line if and only if $|x-y|_{\mathcal X}\geqslant\pi$. Apply the splitting theorem to show that for any $x\in\mathcal X$ there is at most one point y such that $|x-{}$ ${}-y|_{\mathcal X}\geqslant\pi$ and in this case we have equality. Conclude that $\operatorname{diam}\mathcal X\leqslant$ $\leqslant\pi$.

Now choose a quadruple of points $p,x_1,x_2,x_3\in\mathcal X$; we will identify $\mathcal X$ with the unit sphere in $\operatorname{Cone}\mathcal X$. Suppose $|p-x_i|<\frac{\pi}{2}$ for any i. Consider the following points in the cone: $y_i=\frac{1}{\cos|p-x_i|_{\mathcal X}}\cdot x_i$, and $q=p$. Show that $\mathbb E^2$-comparison for q,y_1,y_2,y_3 in $\operatorname{Cone}\mathcal X$ implies $\mathbb S^2$-comparsion for p,x_1,x_2,x_3 in $\mathcal X$. Conclude that $\mathcal X$ is locally $\mathrm{ALEX}(1)$. Apply the globalization theorem (3.4).

Now assume $\mathcal X$ is $\mathrm{ALEX}(1)$ and $\operatorname{diam}\mathcal X\leqslant\pi$. By 3.8, the perimeter of any triangle in $\mathcal X$ is at most $2\cdot\pi$. We need to check $\mathbb E^2$-comparison for a given quadruple of points q,y_1,y_2,y_3 in $\operatorname{Cone}\mathcal X$. We can assume that none of these points is the origin; otherwise perturb them a bit.

Set $x_i=y_i/|y_i|$ for each i and $p=q/|q|$; we can assume that p,x_1,x_2,x_3 are distinct in $\mathcal X$, which is the unit sphere in $\operatorname{Cone}\mathcal X$.

Assume the model triangles $\tilde\triangle(px_1x_2)$, $\tilde\triangle(px_2x_3)$, and $\tilde\triangle(px_3x_1)$ are defined; that is, perimeters triangles $[px_1x_2]$, $[px_2x_3]$, and $[px_3x_1]$ are strictly less than $2\cdot\pi$. Note that $\mathbb E^3\overset{iso}{=\!=}\operatorname{Cone}\mathbb S^2$. Use this together with the $\mathbb S^2$-comparison for p,x_1,x_2,x_3 in $\mathcal X$ to show that $\mathbb E^2$-comparison holds for q,y_1,y_2,y_3 in $\operatorname{Cone}\mathcal X$.

Finally, if some of the model triangles are not defined, consider rescaling of $\mathcal{X}$ with a coefficient λ slightly smaller than 1. Apply the argument above to show that the comparison holds for the corresponding points in $\mathrm{Cone}(\lambda{\cdot}\mathcal{X})$ and pass to the limit as $\lambda \to 1$.

Comment. The last part of the proof is close to the argument in 8.1.

6.9. Observe that

$$\begin{aligned}\langle u,u\rangle+\langle v,u\rangle+\langle w,u\rangle&\geqslant 0,\\ \langle u,v\rangle+\langle v,v\rangle+\langle w,v\rangle&\geqslant 0,\\ \langle u,w\rangle+\langle v,w\rangle+\langle w,w\rangle&=0.\end{aligned}$$

Add the first two inequalities and subtract the last identity.

6.13. Apply 6.12 to show that $\langle v,v\rangle=\langle v,w\rangle=\langle w,w\rangle$, and use it.

6.14. Show and use that

$$\langle u,x\rangle+\langle v,x\rangle+\langle w,x\rangle\geqslant 0$$

and

$$\langle u,-x\rangle+\langle v,-x\rangle+\langle w,-x\rangle\geqslant 0.$$

6.15. Part $\Rightarrow$ is evident. To prove part $\Leftarrow$, observe that

$$\langle u^*,u^*\rangle=-\langle u,u^*\rangle\leqslant\langle u,u\rangle$$

and since $|u|=|u^*|$, we have equality.

6.18. Apply 6.15.

7.2. By 7.13, $\mathcal{A}$ is separable; that is, it contains a countable dense set of points. Apply 6.19 to this set.

7.3. Argue as in 3.7.

7.4. The only-if part is trivial. Suppose the configuration p, $a_0,\dots$ $\dots,a_m\in\mathcal{A}$ meets the condition. By 6.18 the directions $\uparrow_{[qa_0]},\dots$ $\dots,\uparrow_{[qa_m]}\in\mathrm{Lin}_q$ for G-delta dense set of points $q\in\mathcal{A}$. If q is sufficiently close to p, then $\tilde\measuredangle(q\,{}^{a_i}_{a_j})>\frac{\pi}{2}$, and therefore, $\measuredangle[q\,{}^{a_i}_{a_j}]>\frac{\pi}{2}$ for $i\neq j$. Conclude that $\dim\mathrm{Lin}_q\geqslant m$ in this case.

7.7; *(a).* Apply 4.2, 4.3, and 7.5.

(b). Apply 7.4 to show that $\mathrm{LinDim}\,\mathrm{T}_p=\mathrm{LinDim}\,\mathcal{A}$ (argue as in 4.3).

(c). By 7.5 for any two points $\xi,\zeta\in\Sigma_p$ such that $|\xi-\zeta|_{\Sigma_p}<\pi$ there is a geodesic $[\xi\zeta]_{\Sigma_p}$. Suppose $|\xi-\zeta|_{\Sigma_p}\geqslant\pi$, then T_p contains a line thru

the origin in the directions ξ and ζ. By *(a)* we can apply the splitting theorem (6.4) to T_p. We get that Σ_p is a spherical suspension with poles ξ and ζ. Hence, $|\xi-\zeta|=\pi$ and there is a geodesic $[\xi\zeta]$.

7.9; *(a)*. By 4.5, each function dist_{a_i} is semiconcave in a small neighborhood of p. Therefore we can choose λ and $r>0$ so that $f_{\boldsymbol{y}}$ is λ-concave in $\mathrm{B}(p,r)$; further we will assume that r is sufficiently small. Choose $\alpha>0$ such that $\tilde\measuredangle(x\,{}^{a_i}_{a_j})>\frac{\pi}{2}+\alpha$ for all $i\ne j$; we may assume that $\alpha<\frac{1}{10}$; in particular,

$$❷\qquad (\boldsymbol{d}_x\mathrm{dist}_{a_j})(\uparrow_{[xa_i]})\geqslant-\cos\tilde\measuredangle(x\,{}^{a_i}_{a_j})>\tfrac{\alpha}{2}$$

for $j\ne i$.

By the definition of gradient and 4.7*a*, we have

$$\begin{aligned}-(\boldsymbol{d}_x\mathrm{dist}_{a_i})(\nabla_x f_{\boldsymbol{y}})&\geqslant\langle\uparrow_{[xa_i]},\nabla_x f_{\boldsymbol{y}}\rangle\geqslant\\&\geqslant(\boldsymbol{d}_x f_{\boldsymbol{y}})(\uparrow_{[xa_i]}).\end{aligned}$$

If $|a_i-x|>y_i$, then

$$\boldsymbol{d}_x f_{\boldsymbol{y}}=\sigma+\varepsilon\cdot\boldsymbol{d}_x\mathrm{dist}_{a_0},$$

where σ is a minimum of a subset of the following functions 0, and $\boldsymbol{d}_x\mathrm{dist}_{a_j}$ for $0\ne j\ne i$. By ❷,

$$(\boldsymbol{d}_x\mathrm{dist}_{a_i})(\nabla_x f_{\boldsymbol{y}})<-\tfrac{\alpha}{2}\cdot\varepsilon.$$

Hence (i) holds for all sufficiently small $\varepsilon>0$.

Now assume that $|a_i-x|-y_i=\min_j\{|a_j-x|-y_j\}<0$. Then

$$\begin{aligned}\boldsymbol{d}_x f_{\boldsymbol{y}}&=\min_{i\in S}\{\,\boldsymbol{d}_x\mathrm{dist}_{a_j}\,\}+\varepsilon\cdot\boldsymbol{d}_x\mathrm{dist}_{a_0}\leqslant\\&\leqslant\boldsymbol{d}_x\mathrm{dist}_{a_i}+\varepsilon\cdot(\boldsymbol{d}_p\mathrm{dist}_{a_0}),\end{aligned}$$

where $j\in S$ if and only if $|a_i-x|-y_i=|a_j-x|-y_j$. Applying ❷, we get

$$\begin{aligned}(\boldsymbol{d}_x\mathrm{dist}_{a_i})(\nabla_x f_{\boldsymbol{y}})&\geqslant\boldsymbol{d}_x f_{\boldsymbol{y}}(\nabla_x f_{\boldsymbol{y}})-\varepsilon\cdot(\boldsymbol{d}_x\mathrm{dist}_{a_0})(\nabla_x f_{\boldsymbol{y}})\geqslant\\&\geqslant\left[(\boldsymbol{d}_x f_{\boldsymbol{y}})(\uparrow_{[xa_0]})\right]^2-2\cdot\varepsilon\geqslant\\&\geqslant\left[\tfrac{\alpha}{2}-\varepsilon\right]^2-2\cdot\varepsilon.\end{aligned}$$

Thus, (ii) holds for all sufficiently small $\varepsilon>0$.

(b) Consider the following real-to-real functions:

$$❸\qquad\begin{aligned}\varphi(t)&:=\max_i\{|a_i-\alpha_{\boldsymbol{y}}(t)|-y_i\},\\ \psi(t)&:=\min_i\{|a_i-\alpha_{\boldsymbol{y}}(t)|-y_i\}.\end{aligned}$$

Use *(a)*, to show that for $t\in[0,t_0]$, we have $\varphi^+(t)<-\frac{1}{10}\cdot\varepsilon^2$ if $\varphi(t)>0$ and $\psi^+(t)>\frac{1}{10}\cdot\varepsilon^2$ if $\psi(t)<0$. Conclude that $\varphi(t_0)=\psi(t_0)=0$; hence the result.

(c) A straightforward application of 5.9 and a reformulation of *(b)*.

Remarks. By 5.9, that the constructed map Φ is bi-Hölder with the exponent $\frac{1}{2}$. In particular, if an infinite-dimensional Alexandrov space $\mathcal{A}$ contains a bi-Hölder copy of Euclidean ball of arbitrary dimension. It seems plausible that $\mathcal{A}$ should contain a bi-Lipschitz copy of Euclidean ball of arbitrary dimension, but this question is open.

7.11. Apply the $(n{+}1)$-comparison (7.1) to show that at least one of the inequalities

$$\measuredangle[x\,{}^{y}_{a_0}]<\tfrac{\pi}{2}-\varepsilon,\ \dots,\ \measuredangle[x\,{}^{y}_{a_m}]<\tfrac{\pi}{2}-\varepsilon,$$

holds. Similarty, we get that at least one of the inequalities

$$\measuredangle[y\,{}^{x}_{a_0}]<\tfrac{\pi}{2}-\varepsilon,\ \dots,\ \measuredangle[y\,{}^{x}_{a_m}]<\tfrac{\pi}{2}-\varepsilon,$$

holds.

Suppose our statement does not hold for x and y in a sufficiently small neighborhood of p. It follows that

❹ $$\measuredangle[y\,{}^{x}_{a_0}]<\tfrac{\pi}{2}-\varepsilon\quad\text{and}\quad\measuredangle[y\,{}^{x}_{a_0}]<\tfrac{\pi}{2}-\varepsilon.$$

Note that $|x-y|$ is small compared to $|a_0-x|$ and $|a_0-y|$. Therefore, the comparison contradicts ❹.

By the construction, f is Lipschitz. From above, we can choose $i>0$ so that $\measuredangle[x\,{}^{y}_{a_i}]<\frac{\pi}{2}-\varepsilon$ (if $\measuredangle[y\,{}^{x}_{a_i}]<\frac{\pi}{2}-\varepsilon$, then swap x and y). By comparison, there is $c>0$ such that $|a_i-y|\leqslant|a_i-x|+c\cdot|x-y|$. Hence f is bi-Lipschitz, and now 7.8 implies 7.10.

7.13. Reuse the argument from the first part of the proof of Bishop–Gromov inequality.

7.14. You should follow the proof Bishop–Gromov inequality, plus prove the following two inequalities

$$\begin{aligned}\sinh r_2\cdot|\log_p x-\log_p y|_{\mathrm{T}_p}&\geqslant|x-y|_{\mathcal{A}}\\ \sinh r_2\cdot|w(x)-w(y)|_{\mathcal{A}}&\geqslant\sinh r_1\cdot|x-y|_{\mathcal{A}}\end{aligned}$$

for any $x,y\in\mathrm{B}(p,r)$.

7.16. Suppose K is a compact set in $\mathcal{A}$ such that $\operatorname{HausDim}K\geqslant m$. Use the map w from the proof of the Bishop–Gromov inequality (7.12

and 7.14) to show that any open ball in $\mathcal{A}$ contains a compact set K' such that $\operatorname{HausDim} K' \geqslant m$.

Use this in addition to the arguments in 7.15.

8.2. Apply 7.4.

8.4; *(a).* Suppose X is compact. Then for any $\varepsilon > 0$ any cover of X by open ε-balls have a finite subcover. Note that the centers of these balls is an ε-net of X.

Suppose X has a finite ε-net. Show that any sequence x_n of points in X has a subsequence such that all of its points lie in one ε-ball. Apply this statement for $\varepsilon = \frac{1}{n}$ together with the diagonal procedure.

(b). Let Z be a compact ε-net of X. By *(a)*, Z admits a finite ε-net F. Note that F is a $2{\cdot}\varepsilon$-net of X. Since $\varepsilon > 0$ is arbitrary, we get the result.

8.5. If $x_1, \dots, x_n$ is not an ε-net, then there is a point y such that $|x_i - y| \geqslant \varepsilon$ for any i. Therefore $x_1, \dots, x_n$ is not a maximal packing — a contradiction.

8.6; *(a)* Apply the Bishop–Gromov inequality (7.12).

(b) By 8.2, $\dim \mathcal{A}_\infty \leqslant m$. To show that $\dim \mathcal{A}_\infty \geqslant m$, apply 6.19 to a maximal packing and use the estimate in *(a)*.

Comment. A stronger statement holds

$$\operatorname{vol}_m \mathcal{A}_\infty = \lim_{n\to\infty} \operatorname{vol}_m \mathcal{A}_n;$$

in other words, if $\boldsymbol{K} \subset \mathrm{GH}$ denotes the set of isometry classes of all compact $\mathrm{A\scriptstyle LEX}(\kappa)$ spaces with dimension $\leqslant m$, then the function $\operatorname{vol}_m\colon \boldsymbol{K} \to \mathbb{R}$ is continuous.

8.7. Argue as in 8.3 to construct a Gromov–Hausdorff convergence of $\overline{\mathrm{B}}(p_n, R)_{\mathcal{A}_n}$ for given $R > 0$, then apply the diagonal procedure to construct the needed convergence.

8.10. Consider the infinite product $\mathbb{S}^1 \times (\tfrac{1}{2}{\cdot}\mathbb{S}^1) \times (\tfrac{1}{4}{\cdot}\mathbb{S}^1) \times \dots$

9.3. Let V and W be two conic neighborhoods of a point p. Without loss of generality, we may assume that $V \Subset W$; that is, the closure of V lies in W.

Construct a sequence of embeddings $f_n\colon V \to W$ such that

- ⋄ For any compact set $K \subset V$ there is a positive integer $n = n_K$ such that $f_n(k) = f_m(k)$ for any $k \in K$ and $m, n \geqslant n_K$.
- ⋄ For any point $w \in W$ there is a point $v \in V$ such that $f_n(v) = w$ for all large n.

Note that once such a sequence is constructed, $f\colon V\to W$ defined by $f(v)=f_n(v)$ for all large values of n gives the needed homeomorphism.

The sequence f_n can be constructed recursively

$$f_{n+1}=\Psi_n\circ f_n\circ\Phi_n,$$

where $\Phi_n\colon V\to V$ and $\Psi_n\colon W\to W$ are homeomorphisms of the form

$$\Phi_n(x)=\varphi_n(x)*x \quad\text{and}\quad \Phi_n(x)=\psi_n(x)\star x,$$

where $\varphi_n\colon V\to\mathbb{R}_{\geqslant 0}$, $\psi_n\colon W\to\mathbb{R}_{\geqslant 0}$ are suitable continuous functions; "$*$" and "$\star$" denote the multiplications in the cone structures of V and W respectively.

Comment. If it is hard to follow, read the original proof by Kyung Whan Kwun [47].

9.4; *(a).* Apply 9.1 and 9.2.

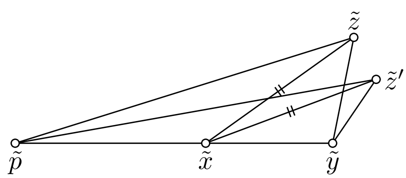
(b). Apply *(a)*.

(c). Recall that the Poincaré homology sphere can be obtained as a quotient space $\Sigma=\mathbb{S}^3/\Gamma$ by an isometric action of a finite group Γ — the so-called binary icosahedral group. By the double suspension theorem, $\operatorname{Susp}^2\Sigma\cong\mathbb{S}^5$. Note that $\operatorname{Susp}^2\Sigma$ is an Alexandrov space and it has a point with space of directions isometric to $\operatorname{Susp}\Sigma$. Observe that $\operatorname{Susp}\Sigma$ is not a manifold; in particular $\operatorname{Susp}\Sigma\ncong\mathbb{S}^4$. Therefore the pair $\operatorname{Susp}^2\Sigma$ and $\mathbb{S}^5$ provides the needed example.

9.6. Apply 9.1, 9.2, and 9.5.

9.7. Let $\mathcal{A}$ be a finite-dimensional Alexandrov space. Choose $x\in\mathcal{A}$. By 9.1, a neighborhood $U\ni x$ is homeomorphic to T_x. Therefore 9.6, implies that $U\cap\partial\mathcal{A}=\varnothing$ if and only if $x\notin\partial\mathcal{A}$; that is, the complement $\mathcal{A}\setminus\partial\mathcal{A}$ is open, and therefore, $\mathcal{A}$ is closed.

9.10. Consider the model triangle $[\tilde x\tilde y\tilde z']=\tilde\triangle(xyz)$.



Show that

$$|\tilde p-\tilde z|\leqslant|\tilde p-\tilde z'|\leqslant\tilde\Upsilon[y\,{}^{p}_{z}].$$

9.11. Assume $\mathcal{A}$ has at least two boundary components, say A and B. Denote by γ a geodesic that minimizes the distance from A to B.

Let

$$\dots, \mathcal{A}_{-1}, \mathcal{A}_0, \mathcal{A}_1, \dots$$

be a two-sided infinite sequence of copies on $\partial\mathcal{A}$. Let us glue $\mathcal{A}_i$ to $\mathcal{A}_{i+1}$ along A if i is even and along B if i is odd.

By the doubling theorem, every point in the obtained space $\mathcal{N}$ has a neighborhood that is isometric to a neighborhood of the corresponding point in $\mathcal{A}$ or its doubling. By the globalization theorem, $\mathcal{N}$ is $\mathrm{ALEX}(1)$.

Note that the copies of γ in $\mathcal{A}_i$ form a line in $\mathcal{N}$. By the splitting theorem, $\mathcal{N}$ is isometric to a product $\mathcal{N}' \oplus \mathbb{R}$. Since $\dim\mathcal{N} > 1$, Exercise 7.3 implies that $\operatorname{diam}\mathcal{N} \leqslant \pi$ — a contradiction.

9.12. Choose x on γ; we can assume that $x = \gamma(0)$. Let $y \in \partial\mathcal{A}$ be a closest point to x. Let $\alpha = \measuredangle(\uparrow_{[xy]}, \gamma^+(0)$.

Suppose $x \notin \partial\mathcal{A}$. Show that $\mathrm{T}_y = \mathbb{R}_{\geqslant 0} \times \mathrm{T}_y\partial\mathcal{A}$ and $\uparrow_{[yx]} \perp \mathrm{T}_y\partial\mathcal{A}$.

Given a vector $v \in \mathrm{T}_y$, denote by $\bar{v}$ its projection to $\mathrm{T}_y\partial\mathcal{A}$. Apply the comparison and 5.12 to show that

$$|\gamma(t) - \operatorname{gexp}_y(\overline{\log_x \gamma(t)})| \leqslant |x - y| + t\cdot \cos\alpha.$$

Conclude that $\gamma''(0) \leqslant 0$ in the barrier sense.

9.13. Suppose γ is defined on the interval $[0,\ell]$. Assume that the function $\rho\colon t \mapsto \tfrac{1}{2}\cdot\operatorname{dist}_p^2 \circ \gamma(t)$ is not 1-concave. Let $\bar\rho\colon [0,\ell] \to \mathbb{R}$ be the minimal 1-concave function such that $\bar\rho \geqslant \rho$. Note that $\bar\rho = \rho$ at the ends of $[0,\ell]$.

Consider the curve $\bar\gamma(t) := \mathrm{Flow}_f^{s(t)}\gamma(t)$; where $f = \tfrac{1}{2}\cdot\operatorname{dist}_p^2$ and $s(t) = \ln\circ\bar\rho(t) - \ln\circ\rho(t)$. Use the first distance estimate to show that $\operatorname{length}\bar\gamma < \operatorname{length}\gamma$ and arrive at a contradiction.

Comment. The statement was proved by Grigory Perelman and the second author [66]; it generalizes a theorem of Joseph Liberman [57] about geodesics on convex surfaces. The original Liberman's version of the following geometric statement. *Suppose that C is the cone over γ with the vertex at p, where γ is a geodesic on a convex surface and p is a point in the convex body bounded by the surface. Then after unfolding C into plane, γ becomes a locally convex curve.* It is instructive to check that this formulation is equivalent to ours for convex bodies.

9.14. Choose a geodesic γ in $\mathcal{W}$. Arguing as in the proof of 9.9*d*, we get that γ can cross the common boundary of two halves $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{W}$ at most once, or it lies in the common boundary.

In the later case λ-concavity of $f\circ\operatorname{proj}\circ\gamma$ follows from λ-concavity of f. In the former case the convexity has to be checked only at the point of crossing; we may assume that it happens at $x = \gamma(0)$. Since $\nabla_x f \in \partial \mathrm{T}_x$ for any $x \in \partial\mathcal{A}$ the f-gradient flows agree on $\mathcal{A}_0$ and $\mathcal{A}_1$.

Assume $f \circ \mathrm{proj} \circ \gamma$ is not λ-concavity at 0. Apply f-gradinent flow to shorten γ keeping its ends as in the proof of 9.13, and arrive at a contradiction.

10.2. Read [90, Section 4] and/or the solution for "Quotient of the Hilbert space" in [79].

10.3; *(a).* Choose an isometric $\mathbb{S}^1$-action on $\mathbb{S}^2$ that fixes the poles of the sphere. Consider the projection to the quotient space $\sigma_1 \colon \mathbb{S}^2 \to$ $\to \mathbb{S}^2/\mathbb{S}^1 = [0, \pi]$.

(b). Take a half-circle γ on $\mathbb{S}^2$ and define $\sigma_2(x) := \mathrm{dist}_\gamma(x)_{\mathbb{S}^2}$.

(c). Consider the subdivision of $\mathbb{S}^2$ into $\mathbb{S}^1$-orbits of the action from *(a)*. Cut $\mathbb{S}^2$ into two hemispheres by meridians rotate one hemisphere by an angle $\alpha = \pi/n$ and glue it back. Observe that there is a submetry σ_n such that the inverse image $\sigma_n^{-1}\{y\}$ is a union of the arcs from the original $\mathbb{S}^1$-orbits.

Note that for $n = 2$ we get the solution in *(b)*.

10.4. Show that for any $x \in \mathbb{E}^2$ there is a half-line $H \ni x$ such that the restriction $\sigma|_H$ is an isometry. Suppose such a half-line H starts at p and passes thru q. Show that $\langle x - p, q - p\rangle \leqslant 0$ for any $x \in \sigma^{-1}\{0\}$. Conclude that $\sigma^{-1}\{0\}$ is a convex closed set. Finally use the definition of submetry to show that $\sigma^{-1}\{0\}$ has no interior points.

10.8; *(a).* Our $\mathbb{S}^1$ is a commutative subgroup of $\mathrm{SO}(3)$. Therefore it is a subgroup of a maximal torus in $\mathrm{SO}(3)$. Show that the described torus action is induced by a maximal torus in $\mathrm{SO}(3)$. Use that maximal tori in $\mathrm{SO}(3)$ are conjugate.

(b). Cut $\mathbb{S}^3$ into two solid tori the Clifford torus $\frac{1}{\sqrt{2}}\cdot\mathbb{S}^1 \times \mathbb{S}^1$. Observe that the quotient of each solid torus is a disc; conclude that $\Sigma_{p,q}$ is a sphere. The torus action on $\mathbb{S}^3$ induce the needed $\mathbb{S}^1$-cation on $\Sigma_{p,q}$.

(c)+(d)+(e). Straightforward calculations.

(f). Consider the map $\Sigma_{p,q} \to \Sigma_{1,1}$ that sends poles to poles, preserve the distance to the poles and respects the $\mathbb{S}^1$ action.

10.14; *(a).* Suppose $\#_{m-1}(\Gamma) \geqslant 3$; that is $\mathcal{A} = \mathbb{E}^m/\Gamma$ has at least 3 boundary components. Follow Case 3 in the proof 10.6 to glue a train-space from copies of $\mathcal{A}$ using two of these components. Show that the obtained space splits and arrive at a contradiction.

(Alternatively, apply a similar construction to all components of the boundary. Show that the obtained space has exponential volume growth; that is, there is $a > 1$ such that $\operatorname{vol} \mathrm{B}(p,r) > a^r$ for all large r. Arrive at a contradiction with the Bishop–Gromov inequality.)

(b). Apply the doubling theorem as in Case 2 in the proof 10.6.

10.16. Show that the quotient space $\Delta = \mathcal{A}/\mathbb{S}^1$ is an $\mathrm{ALEX}(1)$ disc and γ projects isometrically to $\partial\Delta$. It remains to show that the perimeter of Δ cannot exceed $2\cdot\pi$. The latter follows from [75, 3.3.5]; it states that if Δ as an m-dimensional $\mathrm{ALEX}(1)$ space, then $\operatorname{vol}_{m-1} \partial\Delta \leqslant$ $\leqslant \operatorname{vol}_{m-1} \partial\mathbb{S}^{m-1}$.

11.1. We can assume that the origin lies in the interior of the convex body. Consider the central projection from its surface, say Σ, to the sphere $\mathbb{S}^2$ centered at the origin. Show that this projection $\Sigma \to \mathbb{S}^2$ is a homeomorphism.

11.4. Follow the argument in 11.2. Show that the inequality is strict if and only if F has opposite points.

11.5. Suppose a geodesic γ passes thru a vertex v. Denote by α and β the angles that γ cuts at v. Since v is essential, $\alpha+\beta < 2\cdot\pi$. Therefore $\alpha < \pi$ or $\beta < \pi$. Arrive at a contradiction by showing that γ is not length-minimizing.

11.6; *(a).* Cut the surface of T along three edges coming from one vertex v_1 and unfold the obtained surface onto the plane. Show that this way we get a triangle, the three vertices correspond to v_1 and the midpoints of sides correspond to the remaining three vertices. Make a conclusion.

(b). Suppose that $0, v_1, v_2, v_3 \in \mathbb{R}^3$ are the vertices of T. From *(a)*, we have that

$$|v_1| = |v_2 - v_3|, \quad |v_2| = |v_3 - v_1|, \quad |v_3| = |v_1 - v_1|.$$

Use it to show that $\langle v_1, v_2 + v_3 - v_1\rangle = 0$. Make a conclusion.

11.8. We need to show that if a polyhedral surface is $\mathrm{ALEX}(0)$, then the total angle θ at every vertex p it at most $2\cdot\pi$.

Assume that $\theta > 2\cdot\pi$, let $\varphi = \max\{\,\pi, \tfrac{1}{3}\cdot\theta\,\}$. Note that we can choose three points x_1, x_2, and x_3 close to p such that $\measuredangle[p\,{}^{x_i}_{x_j}] = \varphi$ for $i \ne j$. Since the points x_i are close to p, we have $\measuredangle[p\,{}^{x_i}_{x_j}] = \tilde\measuredangle(p\,{}^{x_i}_{x_j})$. The latter contradicts $\mathbb{E}^2$-comparison.

11.9. We will use that the closest-point projection from the Euclidean space to a convex body is short; that is, distance-nonexpanding [72, 13.3].

Assume K_∞ is nondegenerate. Without loss of generality, we may assume that

$$\overline{\mathrm{B}}(0,r) \subset K_\infty \subset \overline{\mathrm{B}}(0,1)$$

for some $r > 0$. Note that there is a sequence $\varepsilon_n \to 0$ such that

$$K_n \subset (1+\varepsilon_n)\cdot K_\infty \quad \text{and} \quad K_\infty \subset (1+\varepsilon_n)\cdot K_n$$

for each large n.

Given $x \in K_n$, denote by $g_n(x)$ the closest-point projection of $(1+\varepsilon_n)\cdot x$ to K_∞. Similarly, given $x \in K_\infty$, denote by $h_n(x)$ the closest point projection of $(1+\varepsilon_n)\cdot x$ to K_n. Note that

$$|g_n(x) - g_n(y)| \leqslant (1+\varepsilon_n)\cdot|x-y|$$

and

$$|h_n(x) - h_n(y)| \leqslant (1+\varepsilon_n)\cdot|x-y|.$$

Denote by Σ_∞ and Σ_n the surface of K_∞ and K_n respectively. The above inequalities imply

$$|g_n(x) - g_n(y)|_{\Sigma_\infty} \leqslant (1+\varepsilon_n)\cdot|x-y|_{\Sigma_n}$$

for any $x, y \in \Sigma_n$, and

$$|h_n(x) - h_n(y)|_{\Sigma_n} \leqslant (1+\varepsilon_n)\cdot|x-y|_{\Sigma_\infty}.$$

for any $x, y \in \Sigma_\infty$.

Note that the maps g_n and h_n are onto. Apply 1.22 to finish the proof.

Alternatively, since the closest-point projection cannot increase the length of curve, we also get

$$\begin{aligned} |x - h_n \circ g_n(x)|_{\Sigma_\infty} &\leqslant 10\cdot\varepsilon_n \\ |y - g_n \circ h_n(y)|_{\Sigma_n} &\leqslant 10\cdot\varepsilon_n. \end{aligned}$$

for all large n. Therefore, g_n is a δ_n-isometry $\Sigma_n \to \Sigma_\infty$ for a sequence $\delta_n \to 0$.

Comments. More generally, if a sequence of m-dimensional $\mathrm{A\small{LEX}}(\kappa)$ spaces $\mathcal{A}_1, \mathcal{A}_2, \dots$ converges to $\mathcal{A}_\infty$ and $\dim \mathcal{A}_\infty = m < \infty$, then $\partial\mathcal{A}_n$ equipped with the induced length metrics converge to $\partial\mathcal{A}_\infty$. This statement is a partial case of the theorem about extremal subsets proved by the second author [76, 1.2].

11.11; *(a).* By 9.13, the function $f_p\colon t\mapsto \mathrm{dist}_p\circ\gamma(t)$ is semiconcave for any $p\in K$. In particular, one-sided derivatives $f_p^+(t)$ are defined for every t.

Given $x=\gamma(t)$, choose three points $p_1,p_2,p_3\in K$ in general position; that is, the four points x,p_1,p_2,p_3 do not lie in one plane. Observe that the distance functions dist_{p_i} give smooth coordinates in a neighborhood of x. From above the functions f_{p_i} have one-sided derivatives at t. Since the coordinates are smooth we get that $\gamma^+(t)$ is defined as well.

(b). If the plane py_1y_2 supports K, then $\measuredangle[p\,{}^{y_1}_{y_2}]_{\mathbb{E}^3}=\measuredangle[p\,{}^{x_1}_{x_2}]_S$. In this case, the statement follows from 11.10.

Now suppose that the line segment $[y_1y_2]_{\mathbb{E}^3}$ intersects K. Choose a geodesic $[y_1y_2]_W$; note that it contains a point of K, say z. Now consider a one-parameter family of points $y_i(t):=\gamma(t)+\gamma^+(t)\cdot(1-t)\cdot$ $\cdot|p-x_i|_S$. Note that this family is not continuous.

Show that for any point $p\in K$, the function $t\mapsto|p-\gamma_i(t)|_{\mathbb{E}^3}$ is nonincreasing. Conclude that the function $t\mapsto|p-\gamma_i(t)|_W$ is nonincreasing for any $p\in S$. Therefore,

$$\begin{aligned}|y_1-y_2|_W&=|y_1(0)-y_2(0)|_W=\\&=|y_1(0)-z|_W+|y_2(0)-z|_W\geqslant\\&\geqslant|y_1(1)-z|_W+|y_2(1)-z|_W\geqslant\\&\geqslant|x_1-x_2|_S.\end{aligned}$$

The last inequality follows since the closest point projection $W\to S$ is short.

It remains to consider the case when the plane py_1y_2 does not support K, and $[y_1y_2]_{\mathbb{E}^3}$ does not intersect K. In this case the plane py_1y_2 intersects K along a convex figure F that lies in the solid triangle py_1y_2 and contains its vertex p.

Choose points $y_1'\in[py_1]_{\mathbb{E}^3}$ and $y_2'\in[py_2]_{\mathbb{E}^3}$ such that $[y_1'y_2']$ touches F. Denote by $x_1'\in[px_1]_S$ and $x_2'\in[px_2]_S$ the corresponding points; that is, $|p-y_1'|_{\mathbb{E}^3}=|p-x_1'|_S$ and $|p-y_2'|_{\mathbb{E}^3}=|p-x_2'|_S$. From the above, we have that $|y_1'-y_2'|_{\mathbb{E}^3}\geqslant|x_1'-x_2'|_S$; in other words,

$$\tilde\measuredangle(p\,{}^{y_1'}_{y_2'})\geqslant\tilde\measuredangle(p\,{}^{x_1'}_{x_2'});$$

here we think that $[py_1'y_2']$ is a triangle in $\mathbb{E}^3$, but $[px_1'x_2']$ is a triangle in S. Note that

$$\tilde\measuredangle(p\,{}^{y_1'}_{y_2'})=\tilde\measuredangle(p\,{}^{y_1}_{y_2})\quad\text{and}\quad\tilde\measuredangle(p\,{}^{x_1}_{x_2})\leqslant\tilde\measuredangle(p\,{}^{x_1'}_{x_2'});$$

the second inequality follows from 2.7. Hence the remaining case follows.

Comments. Part *(a)* is the so-called Liberman lemma — the main tools in studying geodesics on convex surfaces. It was originally proved by Joseph Liberman [57]; the proof of 9.13 is its generalization.

Part *(b)* is the result of Anatolii Milka [62, Theorem 2].

Index

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