An invitation to Alexandrov geometry: spaces with curvature bounded below

Vitali Kapovitch and Anton Petrunin
with with an appendix by
Nina Lebedeva and Anton Petrunin

Contents

Preface

As in our previous invitation [2], we try to demonstrate the beauty and power of Alexandrov geometry by reaching interesting applications and theorems with a minimum of preparation. This time we do spaces with curvature bounded below in the sense of Alexandrov.

This subject is more technical, this time we jumped over proofs of couple of technical results, namely existence part in generalized Picard's theorem (5.3) and Perelman's theorem about conic neighborhoods (10.3). The rest of our presentation is nearly rigorous.

In Lecture 1, we discuss necessary preliminaries and fix notations.

Lecture 2 introduces the main object of our study — spaces with curvature bounded below in the sense of Alexandrov.

In Lecture 3 we formulate and prove the globalization theorem — local Alexandrov condition implies global. To simplify the presentation we consider only compact case, but this case is leading.

In Lecture 4 we do beginning of calculus — tangent space and space of directions, differential, and gradient.

Lecture 5 introduces gradient flow — this is the main technical tool in the theory.

Lecture 6 proves the line splitting theorem. It provides the first application of gradient flow.

In Lecture 7 we introduce and discuss dimension of Alexandrov spaces.

Lecture 8 shows that lower curvature bound survives in the Gromov–Hausdorff limit. Further, we introduce volume, prove the Bishop–Gromov inequality and use it to prove Gromov's selection theorem.

Lecture 9 starts with Perelman's construction of concave functions. Further, we apply it with Gromov's selection theorem to prove the homotopy finiteness theorem. This proof illustrates the main source of applications of Alexandrov geometry.

In Lecture 10 we introduce boundary of finite-dimensioanal Alexandrov space and prove the doubling theorem.

Lecture 11 we show that quotient Alexandrov space by isometric group action is an Alexandrov space and give several applications of this statement. These proofs illustrate another source of applications of Alexandrov geometry.

In Lecture 12 we show that surface of a convex body in Euclidean space is an Alexandrov space. This is historically the first serious application of Alexandrov geometry.

Finally, Appendix A sketches Alexandrov embedding theorem of convex polyhedra. Historically, this theorem is the first remarkable result in Alexandrov geometry that dates back to 1941. The proof is very well written by Alexandrov, but we decided to include its sketch here due to its beauty and importance. This appendix was written by Nina Lebedeva and the second author for a book about .

Let us give a list of available texts on Alexandrov spaces with curvature bounded below:

- ◇ The first introduction to Alexandrov geometry is given in the original paper of Yuriy Burago, Michael Gromov, and Grigory Perelman [11] and its extension [46] written by Perelman.
- ◇ A brief and reader-friendly introduction was written by Katsuhiro Shiohama [60, Sections 1–8].
- ◇ Another reader-friendly introduction, written by Dmiti Burago, Yuriy Burago, and Sergei Ivanov [10, Chapter 10].
- ◇ Survey by Conrad Plaut [58].
- ◇ Survey by the second author [51].

Acknowledgments. Our notes were shaped in a number of lectures given by the authors at different occasions in Penn State, including the MASS program, at the Summer School “Algebra and Geometry” in Yaroslavl, at SPbSU, and University of Toronto. We want to thank these institution for hospitality and support.

We were partially supported by the following grants: Vitali Kapovitch — NSERC Discovery grants; Anton Petrunin — NSF grant DMS-2005279.

Lecture 1

Preliminaries

A Prerequisite

We assume that the reader is familiar with the following topics in metric geometry:

- ◇ Compactness and proper metric spaces; recall that a metric space is proper if all its closed balls (with finite radius) are compact.
- ◇ Complete metric spaces and completion.
- ◇ Curves, semicontinuity of length and rectifiability.
- ◇ Hausdorff and Gromov–Hausdorff convergence. These are discussed briefly in 1I–1L, but it is better to have prior acquaintance with these convergences.

These topics are treated in [10] and [55]. Occasionally, we use some topology; any introductory text in algebraic topology should be sufficient.

B Notations

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

Given radius $r \in [0,\infty]$ and center $x \in \mathcal{X}$, the sets

$$\mathrm{B}(x,r) = \{\, y \in \mathcal{X} \,:\, |x-y| < r \,\},$$
$$\overline{\mathrm{B}}[x,r] = \{\, y \in \mathcal{X} \,:\, |x-y| \leqslant r \,\}$$

are called, respectively, the open and the closed balls. The notations $\mathrm{B}(x,r)_{\mathcal{X}}$ and $\overline{\mathrm{B}}[x,r]_{\mathcal{X}}$ might be used if we need to emphasize that these balls are taken in the metric space $\mathcal{X}$.

We will denote by $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$ the n-dimensional sphere (with angle metric), Euclidean space, and Lobachevsky space respectively. More generally, $\mathbb{M}^n(\kappa)$ will denote the model n-space of curvature κ; that is,

- ◇ if $\kappa > 0$, then $\mathbb{M}^n(\kappa)$ is the n-sphere of radius $\frac{1}{\sqrt{\kappa}}$, so $\mathbb{S}^n = \mathbb{M}^n(1)$
- ◇ $\mathbb{M}^n(0) = \mathbb{E}^n$,
- ◇ if $\kappa < 0$, then $\mathbb{M}^n(\kappa)$ is the Lobachevsky n-space $\mathbb{H}^n$ rescaled by factor $\frac{1}{\sqrt{-\kappa}}$; in particular $\mathbb{M}^n(-1) = \mathbb{H}^n$.

C Length spaces

Let $\mathcal{X}$ be a metric space. If for any $\varepsilon > 0$ and any pair of points $x, y \in \mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length}\alpha < |x - y| + \varepsilon,$$

then $\mathcal{X}$ is called a length space and the metric on $\mathcal{X}$ is called a length metric.

1.1. Exercise. *Let $\mathcal{X}$ be a complete length space. Show that for any compact subset $K \subset \mathcal{X}$ there is a compact path-connected subset $K' \subset \mathcal{X}$ that contains K.*

Induced length metric. Directly from the definition, it follows that if $\alpha\colon [0,1] \to \mathcal{X}$ is a path from x to y (that is, $\alpha(0) = x$ and $\alpha(1) = y$), then

$$\operatorname{length}\alpha \geqslant |x - y|.$$

Set

$$\|x - y\| = \inf\{\operatorname{length}\alpha\}$$

where the greatest lower bound is taken for all paths from x to y. It is straightforward to check that $(x, y) \mapsto \|x - y\|$ is an ∞-metric; that is, $(x, y) \mapsto \|x - y\|$ is a metric in the extended positive reals $[0, \infty]$. The metric $\|{*} - {*}\|$ is called the induced length metric.

1.2. Exercise. *Suppose $(\mathcal{X}, |{*} - {*}|)$ is a complete metric space. Show that $(\mathcal{X}, \|{*} - {*}\|)$ is complete; that is, any Cauchy sequence of points in $(\mathcal{X}, \|{*} - {*}\|)$ converges in $(\mathcal{X}, \|{*} - {*}\|)$.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x - y|_A = \inf_\alpha\{\operatorname{length}\alpha\},$$

where the greatest lower bound is taken for all paths α from x to y in A. In other words, $|{*} - {*}|_A$ denotes the induced length metric on the subspace A. (The notation $|{*}-{*}|_A$ conflicts with the previously defined notation for distance $|x-y|_{\mathcal{X}}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.)

D Geodesics

Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ a real interval. A distance-preserving map $\gamma\colon \mathbb{I}\to\mathcal{X}$ is called a geodesic[1]; in other words, $\gamma\colon \mathbb{I}\to\mathcal{X}$ is a geodesic if

$$|\gamma(s)-\gamma(t)| = |s-t|$$

for any pair $s,t\in\mathbb{I}$.

If $\gamma\colon [a,b]\to\mathcal{X}$ is a geodesic such that $p=\gamma(a)$, $q=\gamma(b)$, then we say that γ is a geodesic from p to q. In this case, the image of γ is denoted by $[pq]$, and, with abuse of notations, we also call it a geodesic. We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$.

In general, a geodesic from p to q need not exist and if it exists, it need not be unique; for example, any meridian is a geodesic between poles on the sphere. However, once we write $[pq]$ we assume that we have chosen such a geodesic.

A geodesic path is a geodesic with constant-speed parameterization by the unit interval $[0,1]$.

A metric space is called geodesic if any pair of its points can be joined by a geodesic.

Evidently, any geodesic space is a length space.

1.3. Exercise. *Show that any proper length space is geodesic.*

E Menger's lemma

1.4. Lemma. *Let $\mathcal{X}$ be a complete metric space. Assume that for any pair of points $x,y\in\mathcal{X}$, there is a midpoint z. Then $\mathcal{X}$ is a geodesic space.*

This lemma is due to Karl Menger [41, Section 6].

Proof. Choose $x,y\in\mathcal{X}$; set $\gamma(0)=x$, and $\gamma(1)=y$.

Let $\gamma(\tfrac12)$ be a midpoint between $\gamma(0)$ and $\gamma(1)$. Further, let $\gamma(\tfrac14)$ and $\gamma(\tfrac34)$ be midpoints between the pairs $(\gamma(0),\gamma(\tfrac12))$ and $(\gamma(\tfrac12),\gamma(1))$

[1] Others call it differently: *shortest path*, *minimizing geodesic*. Also, note that the meaning of the term *geodesic* is different from what is used in Riemannian geometry, altho they are closely related.

$$x=\gamma(0)\qquad \gamma(\tfrac14)\qquad \gamma(\tfrac12)\qquad \gamma(\tfrac34)\qquad \gamma(1)=y$$

respectively. Applying the above procedure recursively, on the n-th step we define $\gamma(\frac{k}{2^n})$, for every odd integer k such that $0<\frac{k}{2^n}<1$, as a midpoint of the already defined $\gamma(\frac{k-1}{2^n})$ and $\gamma(\frac{k+1}{2^n})$.

This way we define $\gamma(t)$ for all dyadic rationals t in $[0,1]$. Moreover, γ has Lipschitz constant $|x-y|$. Since $\mathcal{X}$ is complete, the map γ can be extended continuously to $[0,1]$. Moreover,

$$\operatorname{length}\gamma\leqslant|x-y|.$$

Therefore γ is a geodesic path from x to y. $\square$

1.5. Exercise. *Let $\mathcal{X}$ be a complete metric space. Assume that for any pair of points $x,y\in\mathcal{X}$, there is an almost midpoint; that is, given $\varepsilon>0$, there is a point z such that*

$$|x-z|<\tfrac12\cdot|x-y|+\varepsilon\quad\textit{and}\quad|y-z|<\tfrac12\cdot|x-y|+\varepsilon.$$

Show that $\mathcal{X}$ is a length space.

F Triangles and model tangles

Triangles. Given a triple of distinct points p,q,r in a metric space $\mathcal{X}$, a choice of geodesics $([qr],[rp],[pq])$ will be called a triangle; we will use the short notation $[pqr]=[pqr]_{\mathcal{X}}=([qr],[rp],[pq])$.

Given a triple $p,q,r\in\mathcal{X}$ there may be no triangle $[pqr]$ simply because one of the pairs of these points cannot be joined by a geodesic. Also, many different triangles with these vertices may exist, any of which can be denoted by $[pqr]$. If we write $[pqr]$, it means that we have chosen such a triangle.

Model triangles. Given three points p,q,r in a metric space $\mathcal{X}$, let us define its model triangle $[\tilde p\tilde q\tilde r]$ (briefly, $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$) to be a triangle in the Euclidean plane $\mathbb{E}^2$ such that

$$|\tilde p-\tilde q|_{\mathbb{E}^2}=|p-q|_{\mathcal{X}},\qquad|\tilde q-\tilde r|_{\mathbb{E}^2}=|q-r|_{\mathcal{X}},\qquad|\tilde r-\tilde p|_{\mathbb{E}^2}=|r-p|_{\mathcal{X}}.$$

In the same way, we can define the hyperbolic and the spherical model triangles $\tilde\triangle(pqr)_{\mathbb{H}^2}$, $\tilde\triangle(pqr)_{\mathbb{S}^2}$ in the Lobachevsky plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2$. In the latter case, the model triangle is said to be defined if in addition

$$|p-q|+|q-r|+|r-p|<2\cdot\pi.$$

In this case, the model triangle again exists and is unique up to an isometry of $\mathbb{S}^2$.

Model angles. If $[\tilde p\tilde q\tilde r] = \tilde\triangle(pqr)_{\mathbb{E}^2}$ and $|p-q|, |p-r| > 0$, the angle measure of $[\tilde p\tilde q\tilde r]$ at $\tilde p$ will be called the model angle of the triple p, q, r and will be denoted by $\tilde\measuredangle(p\,^q_r)_{\mathbb{E}^2}$.

For example, if $|p-q| = |q-r| = |r-p|$, then $\tilde\measuredangle(p\,^q_r)_{\mathbb{E}^2} = \frac{\pi}{3}$ regardless of existence and relative position of geodesics $[pq]$ and $[pr]$.

The same way we define $\tilde\measuredangle(p\,^q_r)_{\mathbb{M}^2(\kappa)}$; in particular, $\tilde\measuredangle(p\,^q_r)_{\mathbb{H}^2}$ and $\tilde\measuredangle(p\,^q_r)_{\mathbb{S}^2}$. We may use the notation $\tilde\measuredangle(p\,^q_r)$ if it is evident which of the model spaces is meant.

1.6. Exercise. *Show that for any triple of point p, q, and r, the function*

$$\kappa \mapsto \tilde\measuredangle(p\,^q_r)_{\mathbb{M}^2(\kappa)}$$

is nondecreasing in its domain of definition.

G Hinges and their angle measure

Hinges. Let $p, x, y \in \mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair of geodesics $([px], [py])$ will be called a hinge and will be denoted by $[p\,^x_y] = ([px], [py])$.

Angles. The angle measure of a hinge $[p\,^x_y]$ is defined as the following limit

$$\measuredangle[p\,^x_y] = \lim_{\bar x, \bar y \to p} \tilde\measuredangle(p\,^{\bar x}_{\bar y}),$$

where $\bar x \in \,]px]$ and $\bar y \in \,]py]$.

Note that if $\measuredangle[p\,^x_y]$ is defined, then

$$0 \leqslant \measuredangle[p\,^x_y] \leqslant \pi.$$

1.7. Exercise. *Suppose that in the above definition, one uses spherical or hyperbolic model angles instead of Euclidean. Show that it does not change the value $\measuredangle[p\,^x_y]$.*

1.8. Exercise. *Give an example of a hinge $[p\,^x_y]$ in a metric space with an undefined angle measure $\measuredangle[p\,^x_y]$.*

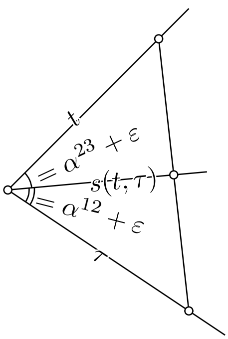
H Triangle inequality for angles

1.9. Proposition. *Let $[px_1]$, $[px_2]$, and $[px_3]$ be three geodesics in a metric space. Suppose all the angle measures $\alpha_{ij} = \measuredangle[p\,{}^{x_i}_{x_j}]$ are defined. Then*

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23}.$$

Proof. Since $\alpha_{13} \leqslant \pi$, we can assume that $\alpha_{12} + \alpha_{23} < \pi$. Denote by γ_i the unit-speed parametrization of $[px_i]$ from p to x_i. Given any $\varepsilon > 0$, for all sufficiently small $t, \tau, s \in \mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned}|\gamma_1(t) - \gamma_3(\tau)| &\leqslant |\gamma_1(t) - \gamma_2(s)| + |\gamma_2(s) - \gamma_3(\tau)| <\\ &< \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha_{12} + \varepsilon)} +\\ &\quad + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\alpha_{23} + \varepsilon)} \leqslant\end{aligned}$$



Below we define $s(t, \tau)$ so that for $s = s(t, \tau)$, this chain of inequalities can be continued as follows:

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\alpha_{13} \leqslant \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon.$$

Hence the result follows.

To define $s(t, \tau)$, consider three half-lines $\tilde\gamma_1$, $\tilde\gamma_2$, $\tilde\gamma_3$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde\gamma_1, \tilde\gamma_2) = \alpha_{12} + \varepsilon$, $\measuredangle(\tilde\gamma_2, \tilde\gamma_3) = \alpha_{23} + \varepsilon$, and $\measuredangle(\tilde\gamma_1, \tilde\gamma_3) = \alpha_{12} + \alpha_{23} + 2{\cdot}\varepsilon$. We parametrize each half-line by the distance from the starting point. Given two positive numbers $t, \tau \in \mathbb{R}_{\geqslant 0}$, let $s = s(t, \tau)$ be the number such that $\tilde\gamma_2(s) \in [\tilde\gamma_1(t)\ \tilde\gamma_3(\tau)]$. Clearly, $s \leqslant \max\{t, \tau\}$, so t, τ, s may be taken sufficiently small. $\square$

1.10. Exercise. *Prove that the sum of adjacent angles is at least π.*

More precisely: suppose two hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ are adjacent; that is, they share side $[pz]$, and the union of two sides $[px]$ and $[py]$ form a geodesic $[xy]$. Show that

$$\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] \geqslant \pi$$

whenever each angle on the left-hand side is defined.

Give an example showing that the inequality can be strict.

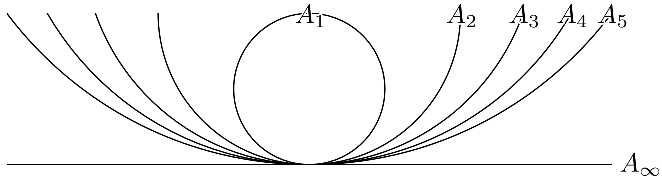
1.11. Exercise. *Assume that the angle measure of $[q\,{}^p_x]$ is defined. Let γ be the unit speed parametrization of $[qx]$ from q to x. Show that*

$$|p - \gamma(t)| \leqslant |q - p| - t{\cdot}\cos(\measuredangle[q\,{}^p_x]) + o(t).$$

I Hausdorff convergence

1.12. Definition. *Let $A_1, A_2, \ldots$ be a sequence of closed sets in a metric space $\mathcal{X}$. We say that the sequence A_n converges to a closed set A_∞ in the sense of Hausdorff if, for any $x \in \mathcal{X}$, we have $\operatorname{dist}_{A_n}(x) \to \operatorname{dist}_{A_\infty}(x)$ as $n \to \infty$.*

For example, suppose $\mathcal{X}$ is the Euclidean plane and A_n is the circle with radius n and center at the point $(0, n)$; it converges to the x-axis.



Further, consider the sequence of one-point sets $B_n = \{(n, 0)\}$ in the Euclidean plane. It converges to the empty set; indeed, for any point x we have $\operatorname{dist}_{B_n}(x) \to \infty$ as $n \to \infty$ and $\operatorname{dist}_\varnothing(x) = \infty$ for any x.

The following exercise is an extension of the so-called Blaschke selection theorem to our version of Hausdorff convergence.

1.13. Exercise. *Show that any sequence of closed sets in a proper metric space has a convergent subsequence in the sense of Hausdorff.*

J Hausdorff metric

1.14. Definition. *Let A and B be two nonempty compact subsets of a metric space $\mathcal{X}$. Then the Hausdorff distance between A and B is defined as*

$$|A - B|_{\operatorname{Haus}\mathcal{X}} := \sup_{x \in \mathcal{X}} \{\, |\operatorname{dist}_A(x) - \operatorname{dist}_B(x)| \,\}.$$

The following observation gives a useful reformulation of the definition:

1.15. Observation. *Suppose A and B be two compact subsets of a metric space $\mathcal{X}$. Then $|A - B|_{\operatorname{Haus}\mathcal{X}} < R$ if and only if and only if B lies in an R-neighborhood of A, and A lies in an R-neighborhood of B.*

The following exersice implies that for compact subsets the Hausdorff convergence is the convergence in Hausdorff metric.

1.16. Exercise. *Let $A_1, A_2, \dots$ and A_∞ be compact nonempty sets in a metric space $\mathcal{X}$. Show that $|A_n - A_\infty|_{\mathrm{Haus}\,\mathcal{X}} \to 0$ as $n \to \infty$ if and only if $A_n \to A_\infty$ in the sense of Hausdorff.*

K Gromov–Hausdorff convergence

Let $\mathcal{X}_1, \mathcal{X}_2, \dots$ and $\mathcal{X}_\infty$ be a sequence of complete metric spaces. Suppose that there is a metric on the disjoint union

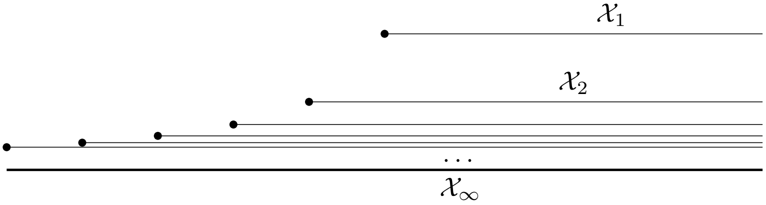
$$\boldsymbol{X} = \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} \mathcal{X}_n$$

that satisfies the following property:

1.17. Property. *The restriction of metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric, and $\mathcal{X}_n \to \mathcal{X}_\infty$ as subsets in $\boldsymbol{X}$ in the sense of Hausdorff.*

In this case we say that the metric on $\boldsymbol{X}$ *defines* a convergence $\mathcal{X}_n \to \mathcal{X}_\infty$ in the sense of Gromov–Hausdorff. The metric on $\bigsqcup \mathcal{X}_n$ makes it possible to talk about limits of sequences $x_n \in \mathcal{X}_n$ as $n \to \infty$, as well as weak limits of a sequence of Borel measures μ_n on $\mathcal{X}_n$ and so on.

The limit space is not uniquely defined by the sequence. For example, if each space $\mathcal{X}_n$ in the sequence is isometric to the half-line, then its limit might be isometric to the half-line or the whole line. The first convergence is evident and the second could be guessed from the diagram.



The following exercise states that if limit compact, then it is unique up to isometry.

1.18. Exercise. *Let $\mathcal{X}_1, \mathcal{X}_2, \dots$ be a sequence of geodesic metric spaces. Suppose $\mathcal{X}_n \to \mathcal{X}_\infty$ is a convergence in the sense of Gromov–Hausdorff. Assume $\mathcal{X}_\infty$ is proper, show that it is geodesic.*

1.19. Exercise. *Let $\mathcal{X}_1, \mathcal{X}_2, \dots$ be a sequence of metric spaces. Suppose $\mathcal{X}_\infty$ and $\mathcal{X}'_\infty$ are limit spaces for some Gromov–Hausdorff con-*

vergences of $\mathcal{X}_n$. Assume $\mathcal{X}_\infty$ is compact, show that it is isometric to $\mathcal{X}'_\infty$.

Pointed convergence. Often the isometry class of the limit can be fixed by marking a point p_n in each space $\mathcal{X}_n$, it is called pointed Gromov–Hausdorff convergence — we say that $(\mathcal{X}_n, p_n)$ converges to $(\mathcal{X}_\infty, p_\infty)$ if there is a metric on $\boldsymbol{X}$ as in 1.17 such that $p_n \to p_\infty$. For example, the sequence $(\mathcal{X}_n, p_n) = (\mathbb{R}_+, 0)$ converges to $(\mathbb{R}_+, 0)$, while $(\mathcal{X}_n, p_n) = (\mathbb{R}_+, n)$ converges to $(\mathbb{R}, 0)$.

L Gromov–Hausdorff metric

In this section we cook up a metric space out of all compact nonempty metric spaces that defines the Gromov–Hausdorff convergence. We want to define the metric on the set of *isometry classes* of compact metric spaces. Further, term *metric space* might also stand for its *isometry class.*

The obtained metric is called Gromov–Hausdorff metric; the corresponding metric space will be denoted by GH. This distance is defined as the maximal metric such that *the distance between subspaces in a metric space is not greater than the Hausdorff distance between them.* Here is a formal definition.

1.20. Definition. *The Gromov–Hausdorff distance $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}}$ between compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is defined by the following relation.*

Given $r > 0$, we have that $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} < r$ if and only if there exists a metric space $\mathcal{W}$ and subspaces $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{W}$ that are isometric to $\mathcal{X}$ and $\mathcal{Y}$, respectively, such that $|\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}} < r$. (Here $|\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}}$ denotes the Hausdorff distance between sets $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{W}$.)

For the proof of the following statement we refer to [10] and [55].

1.21. Proposition. GH *is complete metric space.*

The Gromov–Hausdorff convergence of compact spaces has specially nice properties. From the technical point of view they follow from the next statement that we formulate as an exercise.

1.22. Exercise. *Let f be a distance noncontracting map from a compact metric space $\mathcal{K}$ to itself. Show that f is an isometry; that is, it is a distance-preserving bijection.*

For two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $\mathcal{X} \leqslant \mathcal{Y} + \varepsilon$ if there is a map $f\colon \mathcal{X} \to \mathcal{Y}$ such that

$$|x - x'|_{\mathcal{X}} \leqslant |f(x) - f(x')|_{\mathcal{Y}} + \varepsilon$$

for any $x, x' \in \mathcal{X}$.

1.23. Exercise. *Let $\mathcal{X}_1, \mathcal{X}_2, \dots$ and $\mathcal{X}_\infty$ are compact metric spaces. Show that there is a Gromov–Hausdorff convergence $\mathcal{X}_n \to \mathcal{X}_\infty$ if and only if for some sequence $\varepsilon_n \to 0$, we have*

$$\mathcal{X}_\infty \leqslant \mathcal{X}_n + \varepsilon_n \quad \textit{and} \quad \mathcal{X}_n \leqslant \mathcal{X}_\infty + \varepsilon_n.$$

Lecture 2

Definitions

A Four-point comparison

Recall that model angle $\tilde\measuredangle(p\,{}^x_y)$ is defined on pager 5.

Let p, x, y, z be a quadruple of points in a metric space. If the inequality

$$\text{❶}\qquad \tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^y_z)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^z_x)_{\mathbb{E}^2} \leqslant 2\cdot\pi$$

holds, then we say that the quadruple meets $\mathbb{E}^2$-*comparison*.

If instead of $\mathbb{E}^2$, we use $\mathbb{S}^2$ or $\mathbb{H}^2$, then we get the definition of $\mathbb{S}^2$- or $\mathbb{H}^2$-comparisons. Note that $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2}$ and $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2}$ are defined if $p \ne x$, $p \ne y$, but for $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2}$ we require in addition that

$$|p - x| + |p - y| + |x - y| < 2\cdot\pi;$$

if this inequality does not hold, than we assume that $\mathbb{S}^2$-comparison holds for this quadruple.

More generally, one may apply this definition to $\mathbb{M}^2(\kappa)$. This way we define $\mathbb{M}^2(\kappa)$-comparison for any real κ. However, if you see $\mathbb{M}^2(\kappa)$-comparison, it is usually safe to assume that $\kappa = -1$, 0, or 1 (applying rescaling, the $\mathbb{M}^2(\kappa)$-comparison can be reduced to these three cases).

2.1. Definition. *A metric space $\mathcal{X}$ has curvature $\geqslant \kappa$ in the sense of Alexandrov if $\mathbb{M}^2(\kappa)$-comparison holds for any quadruple in $\mathcal{X}$ such that each model angle in ❶ is defined.*

While this definition can be applied to any metric space, we will use it mostly for geodesic space that are complete (and often compact or proper). If a complete geodesic space has curvature $\geqslant \kappa$ in the sense of Alexandrov, then it will be called $\mathrm{A\scriptstyle LEX}(\kappa)$ space; here $\mathrm{A\scriptstyle LEX}(\kappa)$ is an adjective. An $\mathcal{X}$ is $\mathrm{ALEX}(\kappa)$ space for some κ, then it will be called Alexandrov space.

2.2. Exercise. *Show that* $\mathbb{E}^n$ *is* $\mathrm{ALEX}(0)$.

2.3. Exercise. *Show that a metric space $\mathcal{X}$ has nonnegative curvature in the sense of Alexandrov if and only if for any quadruple of points $p, x_1, x_2, x_3 \in \mathcal{X}$ there is a quadruple of points $q, y_1, y_2, y_3 \in \mathbb{E}^3$ such that*

$$|p - x_i|_{\mathcal{X}} \geqslant |q - y_i|_{\mathbb{E}^2} \quad \textit{and} \quad |x_i - x_j|_{\mathcal{X}} \leqslant |y_i - y_j|_{\mathbb{E}^2}$$

for all i and j.

B Alexandrov's lemma

Recall that $[xy]$ denotes a geodesic from x to y; set

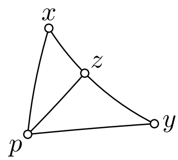
$$]xy] = [xy] \setminus \{x\}, \quad [xy[= [xy] \setminus \{y\}, \quad]xy[= [xy] \setminus \{x, y\}.$$

2.4. Lemma. *Let p, x, y, z be distinct points in a metric space such that $z \in\ ]xy[$. Then the following expressions have the same sign:*

(a) $\tilde\measuredangle(x\,{}^p_y) - \tilde\measuredangle(x\,{}^p_z)$,

(b) $\tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) - \pi$.

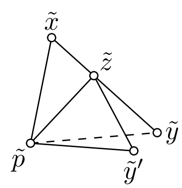
The same holds for the hyperbolic and spherical model angles, but in the latter case, one has to assume in addition that



$$|p - z| + |p - y| + |x - y| < 2{\cdot}\pi.$$

Proof. Consider the model triangle $[\tilde x\tilde p\tilde z] = \tilde\triangle(xpz)$. Take a point $\tilde y$ on the extension of $[\tilde x\tilde z]$ beyond $\tilde z$ so that $|\tilde x - \tilde y| = |x - y|$ (and therefore $|\tilde x - \tilde z| = |x - z|$).

Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:



(i) $\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(x\,{}^p_y)$,

(ii) $|\tilde p - \tilde y| - |p - y|$,

(iii) $\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(z\,{}^p_y)$.

Since

$$\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] = \measuredangle[\tilde x\,{}^{\tilde p}_{\tilde z}] = \tilde\measuredangle(x\,{}^{p}_{z})$$

and

$$\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] = \pi - \measuredangle[\tilde z\,{}^{\tilde x}_{\tilde p}] = \pi - \tilde\measuredangle(z\,{}^{x}_{p}),$$

the statement follows.

The spherical and hyperbolic cases can be proved in the same way.

$\square$

2.5. Exercise. *Assume p, x, y, z are as in Alexandrov's lemma. Show that*

$$\tilde\measuredangle(p\,{}^{x}_{y}) \geqslant \tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{z}_{y}),$$

with equality if and only if the expressions in (a) and (b) vanish.

Note that

$$p \in \,]xy[\quad\Longrightarrow\quad \tilde\measuredangle(p\,{}^{x}_{y}) = \pi.$$

Applying it with Alexandrov's lemma and $\mathbb{E}^2$-comparison, we get the following.

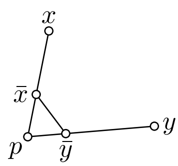
2.6. Claim. *If p, x, y, z are points in an* Alex(0) *space. Suppose $p \in \,]xy[$, then*

$$\tilde\measuredangle(x\,{}^{y}_{z}) \leqslant \tilde\measuredangle(x\,{}^{p}_{z}).$$

2.7. Exercise. *Let $[p\,{}^{x}_{y}]$ be a hinge in an* Alex(0) *space. Consider the function*

$$f\colon (|p-\bar x|, |p-\bar y|) \mapsto \tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$

where $\bar x \in \,]px]$ and $\bar y \in \,]py]$. Show that f is nonincreasing in each argument.



Note that 2.7 implies the following.

2.8. Claim. *For any hinge $[p\,{}^{x}_{y}]$ in an* Alex(0) *space, the angle measure $\measuredangle[p\,{}^{x}_{y}]$ is defined, and*

$$\measuredangle[p\,{}^{x}_{y}] \geqslant \tilde\measuredangle(p\,{}^{x}_{y}).$$

2.9. Exercise. *Let $[p\,{}^{x}_{y}]$ be a hinge in an* Alex(0) *space. Suppose $\measuredangle[p\,{}^{x}_{y}] = 0$; show that $[px] \subset [py]$ or $[py] \subset [px]$.*

2.10. Exercise. *Let $[xy]$ be a geodesic in an* Alex(0) *space. Suppose $z \in \,]xy[$ show that there is a unique geodesic $[xz]$ and $[xz] \subset [xy]$.*

Recall that adjacent hinges are defined in 1.10.

2.11. Exercise. *Let $[p\,{}^x_z]$ and $[p\,{}^y_z]$ be adjacent hinges in an* Alex(0) *space. Show that*

$$\measuredangle[p\,{}^x_z]+\measuredangle[p\,{}^y_z]=\pi.$$

2.12. Exercise. *Let $\mathcal{L}$ be an* Alex(0) *space. Show that*

$$\tilde\measuredangle(x\,{}^y_p)=\tilde\measuredangle(x\,{}^v_p)\quad\iff\quad\tilde\measuredangle(x\,{}^y_p)=\tilde\measuredangle(x\,{}^w_p)$$

for any points p,x,y,v,w in $\mathcal{L}$ such that $v,w\in\,]xy[$.

2.13. Exercise. *Let $\mathcal{L}$ be an* Alex(0) *space. Suppose hinges $[x_n\,{}^{y_n}_{z_n}]$ in $\mathcal{L}$ converge to the hinge $[x_\infty\,{}^{y_\infty}_{z_\infty}]$; that is geodesics $[x_ny_n]$ and $[x_nz_n]$ converge to the geodesics $[x_\infty y_\infty]$ and $[x_\infty z_\infty]$ in the Hausdorff sense. Show that*

$$\varliminf_{n\to\infty}\measuredangle[x_n\,{}^{y_n}_{z_n}]\geqslant\measuredangle[x_\infty\,{}^{y_\infty}_{z_\infty}].$$

C Hinge comparison

Let $[p\,{}^x_y]$ be a hinge in an Alex(0) space $\mathcal{L}$. By 2.9, the angle measure $\measuredangle[p\,{}^x_y]$ is defined and

$$\measuredangle[p\,{}^x_y]\geqslant\tilde\measuredangle(p\,{}^x_y).$$

Further, according to 2.11, we have

$$\measuredangle[p\,{}^x_z]+\measuredangle[p\,{}^y_z]=\pi$$

for adjacent hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ in $\mathcal{L}$.

The following theorem implies that a geodesic space has nonnegative curvature in the sense of Alexandrov if the above conditions hold for all its hinges.

2.14. Theorem. *A complete geodesic space $\mathcal{L}$ is* Alex(0) *if the following conditions hold.*

(a) For any hinge $[x\,{}^p_y]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and

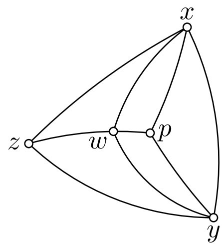
$$\measuredangle[x\,{}^p_y]\geqslant\tilde\measuredangle(x\,{}^p_y).$$

(b) For any two adjacent hinges $[p\,{}^x_z]$ and $[p\,{}^y_z]$ in $\mathcal{L}$, we have

$$\measuredangle[p\,{}^x_z]+\measuredangle[p\,{}^y_z]\leqslant\pi.$$

Proof. Consider a point $w\in\,]pz[$ close to p. From *(b)*, it follows that

$$\measuredangle[w\,{}^x_z]+\measuredangle[w\,{}^x_p]\leqslant\pi\quad\text{and}\quad\measuredangle[w\,{}^y_z]+\measuredangle[w\,{}^y_p]\leqslant\pi.$$



Since $\measuredangle[w\,{}^{x}_{y}] \leqslant \measuredangle[w\,{}^{x}_{p}] + \measuredangle[w\,{}^{y}_{p}]$ (see 1.9), we get

$$\measuredangle[w\,{}^{x}_{z}] + \measuredangle[w\,{}^{y}_{z}] + \measuredangle[w\,{}^{x}_{y}] \leqslant 2{\cdot}\pi.$$

Applying *(a)*,

$$\tilde\measuredangle(w\,{}^{x}_{z}) + \tilde\measuredangle(w\,{}^{y}_{z}) + \tilde\measuredangle(w\,{}^{x}_{y}) \leqslant 2{\cdot}\pi.$$

Passing to the limits $w \to p$, we have

$$\tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{y}_{z}) + \tilde\measuredangle(p\,{}^{x}_{y}) \leqslant 2{\cdot}\pi.$$

□

D Equivalent conditions

The following theorem summarizes 2.6, 2.8, 2.11, 2.14.

2.15. Theorem. *Let $\mathcal{L}$ be a complete geodesic space. Then the following conditions are equivalent.*

(a) *$\mathcal{L}$ is* Alex(0).

(b) *(adjacent angle comparison) for any geodesic $[xy]$ and point $z \in {]xy[}$, $z \neq p$ in $\mathcal{L}$, we have*

$$\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) \leqslant \pi.$$

(c) *(point-on-side comparison) for any geodesic $[xy]$ and $z \in {]xy[}$ in $\mathcal{L}$, we have*

$$\tilde\measuredangle(x\,{}^{p}_{y}) \leqslant \tilde\measuredangle(x\,{}^{p}_{z}).$$

(d) *(hinge comparison) for any hinge $[x\,{}^{p}_{y}]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^{p}_{y}]$ is defined and*

$$\measuredangle[x\,{}^{p}_{y}] \geqslant \tilde\measuredangle(x\,{}^{p}_{y}).$$

Moreover,

$$\measuredangle[z\,{}^{p}_{y}] + \measuredangle[z\,{}^{p}_{x}] \leqslant \pi$$

for any adjacent hinges $[z\,{}^{p}_{y}]$ and $[z\,{}^{p}_{x}]$.

Moreover, the implications (a)⇒(b)⇒(c)⇒(d) hold in any space, not necessarily geodesic.

2.16. Advanced Exercise. *Construct a complete geodesic space $\mathcal{X}$ that is not* Alex(0)*, but satisfies the following condition: for any three points $p, x, y \in \mathcal{X}$ there is a geodesic $[xy]$ such that for any $z \in {]xy[}$*

$$\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) \leqslant \pi.$$

2.17. Exercise. *Let $\mathcal{W}$ be $\mathbb{R}^2$ with the metric induced by a norm. Suppose that $\mathcal{W}$ is* Alex(0)*. Show that $\mathcal{W}$ is isometric to the Euclidean plane $\mathbb{E}^2$.*

E Function comparison

Real-to-real functions. Choose $\lambda\in\mathbb{R}$. Let $s\colon\mathbb{I}\to\mathbb{R}$ be a locally Lipschitz function defined on an interval $\mathbb{I}$. The following statement are equivalent; if one (and therefore any) of them hold for s, then we say that s is λ-concave.

- ◇ We have inequality $s''\leqslant\lambda$, where the second derivative s'' is understood in the sense of distributions.
- ◇ The function $t\mapsto s(t)-\lambda\cdot\frac{t^2}{2}$ is concave.
- ◇ The Jensen inequality

  $$s(a\cdot t_0+(1-a)\cdot t_1)\geqslant a\cdot s(t_0)+(1-a)\cdot s(t_1)+\tfrac{\lambda}{2}\cdot a\cdot(1-a)\cdot(t_1-t_0)^2$$

  holds for any $t_0,t_1\in\mathbb{I}$ and $a\in[0,1]$.
- ◇ for any $t_0\in\mathbb{I}$ there is a quadratic polynomial $\ell=\frac{\lambda}{2}\cdot t^2+a\cdot t+b$ that (locally) supports s at t_0 from above; that is, $\ell(t_0)=s(t_0)$ and $\ell(t)\geqslant s(t)$ for any t (in a neighborhood of t_0)

The equivalence of these definitions is assumed to be known. [1]

We will also use that *λ-concave functions are one-side differentiable.*

Functions on metric space. A function on a metric space $\mathcal{L}$ will usually mean a *locally Lipschitz real-valued function defined on an open subset of* $\mathcal{L}$. The domain of definition of a function f will be denoted by $\operatorname{Dom} f$.

Let f be a function on a metric space $\mathcal{L}$. We say that f is λ-concave (briefly $f''\leqslant\lambda$) if for any unit-speed geodesic $\gamma\colon\mathbb{I}\to\operatorname{Dom} f$ the real-to-real function $t\mapsto f\circ\gamma(t)$ is λ-concave.

The following proposition is simple, but conceptual — it reformulates a global geometric condition into an infinitesimal condition on distance functions.

2.18. Proposition. *A complete geodesic space $\mathcal{L}$ in* $\mathrm{A\scriptstyle LEX}(0)$ *if and only if $f''\leqslant 1$ for any function f of the following type*

$$f\colon x\mapsto\tfrac12\cdot|p-x|^2.$$

Proof. Choose a unit-speed geodesic γ in $\mathcal{L}$ and two points $x=\gamma(t_0)$, $y=\gamma(t_1)$ for some $t_0<t_1$. Consider the model triangle $[\tilde p\tilde x\tilde y]=$ $=\tilde\triangle(pxy)$. Let $\tilde\gamma\colon[t_0,t_1]\to\mathbb{E}^2$ be the unit-speed parametrization of $[\tilde x\tilde y]$ from $\tilde x$ to $\tilde y$.

[1] V: I was asked by several people for references to this.

Set

$$\tilde r(t) := |\tilde p - \tilde\gamma(t)|, \qquad\qquad r(t) := |p - \gamma(t)|.$$

Clearly, $\tilde r(t_0) = r(t_0)$ and $\tilde r(t_1) = r(t_1)$. Note that the point-on-side comparison (2.15c) is equivalent to

❶ $$t_0 \leqslant t \leqslant t_1 \qquad\Longrightarrow\qquad \tilde r(t) \leqslant r(t)$$

for any γ and $t_0 < t_1$.

Set

$$\tilde h(t) = \tfrac12\cdot\tilde r^2(t) - \tfrac12\cdot t^2, \qquad\qquad h = \tfrac12\cdot r^2(t) - \tfrac12\cdot t^2.$$

Note that $\tilde h$ is linear, $\tilde h(t_0) = h(t_0)$ and $\tilde h(t_1) = h(t_1)$. Observe that the Jensen inequality for the function h is equivalent to ❶. Hence the proposition follows. □

F Comments

The 4-point comparison is closely related to the so-called CAT comparison, which defines *upper* curvature bound in the sense of Alexandrov; this is the subject of out previous invitation [2].

In both comparisons we check certain condition on 6 distances between every pair of a 4-point subset. Michael Gromov [20, Section 1.19_+] suggested to consider other conditions of that type for n-point subsets; see [18, 21, 34–39, 54, 63] for development of this idea.

We have chosen complete geodesic spaces with curvature at least κ as the main object of study — Alex(κ) spaces. Instead, we could choose complete *length* spaces with curvature at least κ. This option is slightly more general, but many statements can be reduced to geodesic case. in particular, *any complete length spaces $\mathcal{L}$ with curvature $\geqslant \kappa$ can be embedded into an* Alex(κ) *space* — the ultrapower of $\mathcal{L}$; see [3, 4.11+8.4]. Also *any point p in $\mathcal{L}$ can be connected by geodesics to most other points* [3, 8.11].

All the discussed statements admit natural generalizations to spaces with curvature $\geqslant \kappa$ in the sense of Alexandrov. The proof are nearly the same, but the formulas are getting more complicated. It is common practce in Alexandrov geometry to write proofs for nonnegative curvature and leave the general curvature bound as an exercise. Sometime theses exercises are nontrivial.

For example, the function comparison for Alex(-1) spaces states that $f'' \leqslant f$ for any function of the type $f = \cosh\circ\mathrm{dist}_p$. (The inequality used here will be defined in Section 4C.)

Similarly, the function comparison for $\mathrm{A{\scriptstyle LEX}}(1)$ states that for any point p, we have $f'' \leqslant -f$ for the function $f = -\cos \circ \mathrm{dist}_p$ defined in $\mathrm{B}(p, \pi)$. The geometric meaning of these inequalities remains the same: *distance functions are more concave than distance functions in* $\mathbb{M}^2(\kappa)$.

Lecture 3

Globalization

A Globalization

A complete geodesic metric space $\mathcal{L}$ is l o c a l l y ALEX(0) if any point $p \in \mathcal{L}$ admits a neighborhood $U \ni p$ such that the $\mathbb{E}^2$-comparison holds for any quadruple of points in U.

3.1. Globalization theorem. *Any compact locally* ALEX(0) *space is* ALEX(0).

Proof modulo the key lemma. Note that condition 2.14*b* holds in $\mathcal{L}$ (the proof is the same). It remains to prove that 2.14*a* holds in $\mathcal{L}$; that is,

$$❶ \qquad \measuredangle[x\,{}^p_y] \geqslant \tilde{\measuredangle}(x\,{}^p_y).$$

First note that ❶ holds for hinges in a small neighborhood of any point; this can be proved the same way as 2.8 and 2.11, applying the local version of the $\mathbb{E}^2$-comparison. Since $\mathcal{L}$ is compact, there is $\varepsilon > 0$ such that ❶ holds if $|x-p| + |p-y| < \varepsilon$. Applying the key lemma several times we get that ❶ holds for any given hinge. □

3.2. Key lemma. *Let $\mathcal{L}$ be locally* ALEX(0)*. Assume that the comparison*

$$\measuredangle[x\,{}^p_q] \geqslant \tilde{\measuredangle}(x\,{}^p_q)$$

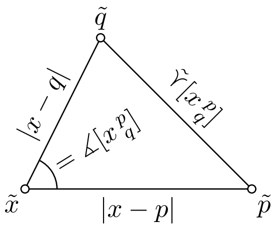
holds for any hinge $[x\,{}^p_q]$ with $|x-y| + |x-q| < \tfrac{2}{3}\cdot\ell$. Then the comparison

$$\measuredangle[x\,{}^p_q] \geqslant \tilde{\measuredangle}(x\,{}^p_q)$$

holds for any hinge $[x\,{}^p_q]$ with $|x-p| + |x-q| < \ell$.

Let $[x\,{}^{p}_{q}]$ be a hinge in a metric space $\mathcal{L}$ with defined angle measure. Denote by $\tilde{\Upsilon}[x\,{}^{p}_{q}]$ its model side; this is the opposite side in a flat triangle with the same angle and two adjacent sides as in $[x\,{}^{p}_{q}]$.

More precisely, consider the model hinge $[\tilde{x}\,{}^{\tilde{p}}_{\tilde{q}}]$ in $\mathbb{E}^2$ that is defined by



$$\begin{aligned}\measuredangle[\tilde{x}\,{}^{\tilde{p}}_{\tilde{q}}]_{\mathbb{E}^2} &= \measuredangle[x\,{}^{p}_{q}]_{\mathcal{L}},\\ |\tilde{x}-\tilde{p}|_{\mathbb{E}^2} &= |x-p|_{\mathcal{L}},\\ |\tilde{x}-\tilde{q}|_{\mathbb{E}^2} &= |x-q|_{\mathcal{L}};\end{aligned}$$

then

$$\tilde{\Upsilon}[x\,{}^{p}_{q}]_{\mathcal{L}} := |\tilde{p}-\tilde{q}|_{\mathbb{E}^2}.$$

Note that

$$\tilde{\Upsilon}[x\,{}^{p}_{q}] \geqslant |p-q| \quad\iff\quad \measuredangle[x\,{}^{p}_{q}] \geqslant \tilde{\measuredangle}(x\,{}^{p}_{q}).$$

We will use it in the following proof.

Proof. It is sufficient to prove the inequality

❷ $$\tilde{\Upsilon}[x\,{}^{p}_{q}] \geqslant |p-q|$$

for any hinge $[x\,{}^{p}_{q}]$ with $|x-p|+|x-q|<\ell$.

Consider a hinge $[x\,{}^{p}_{q}]$ such that

$$\tfrac{2}{3}\cdot\ell \leqslant |p-x|+|x-q| < \ell.$$

First, let us construct a new hinge $[x'\,{}^{p}_{q}]$ with

❸ $$|p-x|+|x-q| \geqslant |p-x'|+|x'-q|,$$

such that

❹ $$\tilde{\Upsilon}[x\,{}^{p}_{q}] \geqslant \tilde{\Upsilon}[x'\,{}^{p}_{q}].$$

Construction. Assume $|x-q| \geqslant |x-p|$; otherwise, switch the roles of p and q. Take $x' \in [xq]$ such that

❺ $$|p-x|+3\cdot|x-x'| = \tfrac{2}{3}\cdot\ell.$$

Choose a geodesic $[x'p]$ and consider the hinge $[x'\,{}^{p}_{q}]$ formed by $[x'p]$ and $[x'q] \subset [xq]$. The triangle inequality implies ❸. Further, note that

$$|p-x|+|x-x'| < \tfrac{2}{3}\cdot\ell, \qquad |p-x'|+|x'-x| < \tfrac{2}{3}\cdot\ell.$$

In particular,

❻ $$\measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde{\measuredangle}(x\,{}^{p}_{x'}) \quad\text{and}\quad \measuredangle[x'\,{}^{p}_{x}] \geqslant \tilde{\measuredangle}(x'\,{}^{p}_{x}).$$

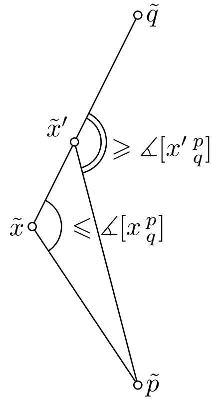
Now, let $[\tilde x\tilde x'\tilde p]=\tilde\triangle(xx'p)$. Take $\tilde q$ on the extension of $[\tilde x\tilde x']$ beyond x' such that $|\tilde x-\tilde q|=|x-q|$ (and therefore $|\tilde x'-\tilde q|=|x'-q|$). By ❻,

$$\measuredangle[x\,{}^{p}_{q}]=\measuredangle[x\,{}^{p}_{x'}]\geqslant\tilde\measuredangle(x\,{}^{p}_{x'})\quad\Rightarrow\quad\tilde\Upsilon[x\,{}^{q}_{p}]\geqslant|\tilde p-\tilde q|.$$

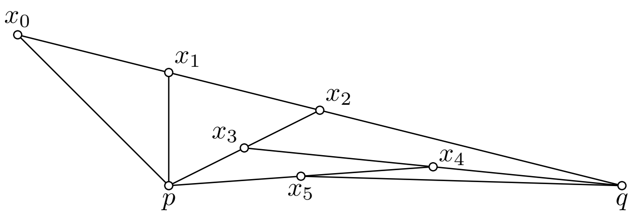
Hence

$$\begin{aligned}\measuredangle[\tilde x'\,{}^{\tilde p}_{\tilde q}]&=\pi-\tilde\measuredangle(x'\,{}^{p}_{x})\geqslant\\&\geqslant\pi-\measuredangle[x'\,{}^{p}_{x}]=\\&=\measuredangle[x'\,{}^{p}_{q}],\end{aligned}$$

and ❹ follows.



Let us continue the proof. Set $x_0=x$. Let us apply inductively the above construction to get a sequence of hinges $[x_n\,{}^{p}_{q}]$ with $x_{n+1}=x_n'$. From ❹, we have that the sequence $s_n=\tilde\Upsilon[x_n\,{}^{p}_{q}]$ is nonincreasing.



The sequence might terminate at some n only if $|p-x_n|+|x_n-q|<\tfrac23\cdot\ell$. In this case, by the assumptions of the lemma, $\tilde\Upsilon[x_n\,{}^{p}_{q}]\geqslant|p-q|$. Since the sequence s_n is nonincreasing, inequality ❷ follows.

Otherwise, the sequence $r_n=|p-x_n|+|x_n-q|$ is nonincreasing, and $r_n\geqslant\tfrac23\cdot\ell$ for all n. Note that by construction, the distances $|x_n-x_{n+1}|$, $|x_n-p|$, and $|x_n-q|$ are bounded away from zero for all large n. Indeed, since on each step, we move x_n toward to the point p or q that is further away, the distances $|x_n-p|$ and $|x_n-q|$ become about the same. Namely, by ❺, we have that $|p-x_n|-|x_n-q|\leqslant\tfrac29\cdot\ell$ for all large n. Since $|p-x_n|+|x_n-q|\geqslant\tfrac23\cdot\ell$, we have $|x_n-p|\geqslant\tfrac{\ell}{100}$ and $|x_n-q|\geqslant\tfrac{\ell}{100}$. Further, since $r_n\geqslant\tfrac23\cdot\ell$, ❺ implies that $|x_n-x_{n+1}|>\tfrac{\ell}{100}$.

Since the sequence r_n is nonincreasing, it converges. In particular, $r_n-r_{n+1}\to0$ as $n\to\infty$. It follows that $\tilde\measuredangle(x_n\,{}^{p_n}_{x_{n+1}})\to\pi$, where $p_n=p$ if $x_{n+1}\in[x_nq]$, and otherwise $p_n=q$. Since $\measuredangle[x_n\,{}^{p_n}_{x_{n+1}}]\geqslant\tilde\measuredangle(x_n\,{}^{p_n}_{x_{n+1}})$, we have $\measuredangle[x_n\,{}^{p_n}_{x_{n+1}}]\to\pi$ as $n\to\infty$.

It follows that

$$r_n-s_n=|p-x_n|+|x_n-q|-\tilde\Upsilon[x_n\,{}^{p}_{q}]\to0.$$

Together with the triangle inequality

$$|p - x_n| + |x_n - q| \geqslant |p - q|$$

this yields

$$\lim_{n\to\infty} \tilde{\Upsilon}[x_n\,{}^p_q] \geqslant |p - q|.$$

Finally, the monotonicity of the sequence $s_n = \tilde{\Upsilon}[x_n\,{}^p_q]$ implies ❷. $\square$

B On general curvature bound

The globalization theorem can be generalized to any curvature bound κ. The case $\kappa \leqslant 0$ is proved in the same way, but the case $\kappa > 0$ requires modifications; most general statements are given in 3.9 and ??.

By 1.6, we have

$$\tilde{\measuredangle}(x\,{}^y_z)_{\mathbb{M}^2(\kappa)} \leqslant \tilde{\measuredangle}(x\,{}^y_z)_{\mathbb{M}^2(\mathrm{K})}$$

if $\kappa \leqslant \mathrm{K}$ and the right-hand side is defined. It follows that a $\mathrm{A\small{LEX}}(\mathrm{K})$ space is *locally* $\mathrm{A\small{LEX}}(\kappa)$. Therefore, the globalization theorem imply the following.

3.3. Claim. *Suppose* $\mathrm{K} > \kappa$, *then any* $\mathrm{A\small{LEX}}(\mathrm{K})$ *space is* $\mathrm{A\small{LEX}}(\kappa)$.

In other words expressin *curvature bounded below by* κ makes sense for geodesic spaces. However, as you can see from the following exercise, it does not make much sense in general.

3.4. Exercise. *Let $\mathcal{X}$ be the set $\{p, x_1, x_2, x_3\}$ with the metric defined by*

$$|p - x_i| = \pi, \quad |x_i - x_j| = 2{\cdot}\pi$$

for all $i \neq j$. Show that $\mathcal{X}$ has curvature $\geqslant 1$, but does not have curvature $\geqslant 0$.

3.5. Exercise. *Let p and q be points in an* $\mathrm{A\small{LEX}}(1)$ *space $\mathcal{L}$. Suppose $|p-q| > \pi$. Denote by m the midpoint of $[pq]$. Show that for any hinge $[m\,{}^x_p]$ we have either $\measuredangle[m\,{}^x_p] = 0$ or $\measuredangle[m\,{}^x_p] = \pi$. Conclude that $\mathcal{L}$ is isometric to a line interval or a circle.*

3.6. Exercise. *Suppose $\mathcal{L}$ is an* $\mathrm{A\small{LEX}}(1)$ *and* $\operatorname{diam}\mathcal{L} \leqslant \pi$. *Show that*

$$|x - y| + |y - z| + |z - x| \leqslant 2{\cdot}\pi$$

for any triple of points $x, y, z \in \mathcal{L}$.

C Remarks

The following question about 2.14a is a long-standing open problem (possibly dating back to Alexandrov); it was stated in [10, footnote in 4.1.5] but is much much older.

3.7. Open question. *Let $\mathcal{L}$ be a complete geodesic space (you can also assume that $\mathcal{L}$ is homeomorphic to $\mathbb{S}^2$ or $\mathbb{R}^2$) such that for any hinge $[x\,{}^p_y]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and*

$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle(x\,{}^p_y).$$

Is it true that $\mathcal{L}$ is an Alexandrov space?

The globalization theorem is also known as the *generalized Toponogov theorem.* Its two-dimensional case was proved by Paolo Pizzetti [56]; later it was reproved independently by Alexandr Alexandrov [7]. Victor Toponogov [62] proved it for Riemannian manifolds of all dimensions.

We took the proof from our book [3] and reduced generality. This proof is based on simplifications obtained by Conrad Plaut [57] and Dmitry Burago, Yuriy Burago, and Sergei Ivanov [10]. The same proof was rediscovered independently by Urs Lang and Viktor Schroeder [32]. Another proof was obtained by Katsuhiro Shiohama [60].

The compactness condition in our version of the theorem can be traded for completeness. The proof uses the following statement where $r(x)$ measures the size of a neighborhood of x where the comparison holds.

3.8. Exercise. *Let $\mathcal{X}$ be a complete metric space. Suppose $r\colon \mathcal{X} \to \mathbb{R}$ is a positive continuous function. Show that for any $\varepsilon > 0$ there is a point $p \in \mathcal{X}$ such that*

$$r(x) > (1-\varepsilon)\cdot r(p)$$

for any $x \in \overline{\mathrm{B}}[p, \tfrac{1}{\varepsilon}\cdot r(p)]$.

It implies the following general version of the globalization theorem.

3.9. Theorem. *Any locally* $\mathrm{A\scriptstyle LEX}(\kappa)$ *length space is* $\mathrm{A\scriptstyle LEX}(\kappa)$.

Slightly more general versions were proved in the papers by Michael Gromov, Yuriy Burago, and Grigory Perelman [11] and by the second author [50]. Their assumptions overlap, but neither implies the other.

Lecture 4

Calculus

A Space of directions

By 2.8, angle measure of a hinge in an Alecandrov space $\mathcal{L}$ is defined. Given $p \in \mathcal{L}$, consider the set $\mathfrak{S}_p$ of all nontrivial unit-speed geodesics starting at p. By 1.9, the triangle inequality holds for $\measuredangle$ on $\mathfrak{S}_p$, that is, $(\mathfrak{S}_p, \measuredangle)$ forms a semimetric space.

The metric space corresponding to $(\mathfrak{S}_p, \measuredangle)$ is called the space of geodesic directions at p, denoted by Σ'_p or $\Sigma'_p\mathcal{L}$. The elements of Σ'_p are called geodesic directions at p. Each geodesic direction is formed by an equivalence class of geodesics starting from p for the equivalence relation

$$[px] \sim [py] \qquad \Longleftrightarrow \qquad \measuredangle[p\,{}^x_y] = 0;$$

the direction of $[px]$ is denoted by $\uparrow_{[px]}$. By 2.9,

$$[px] \sim [py] \qquad \Longleftrightarrow \qquad [px] \subset [py] \quad \text{or} \quad [px] \supset [py].$$

The completion of Σ'_p is called the space of directions at p and is denoted by Σ_p or $\Sigma_p\mathcal{L}$. The elements of Σ_p are called directions at p.

4.1. Exercise. *Let $\mathcal{L}$ be an Alexandrov space. Assume that a sequence of geodesics $[px_n]$ converge to a geodesic $[px_\infty]$ in the sense of Hausdorff, and $x_\infty \neq p$.*

(a) Suppose Σ_p is compact. Show that $\uparrow_{[px_n]} \to \uparrow_{[px_\infty]}$ as $n \to \infty$.

(b) Give an example showing that without compactness of Σ_p, part (a) does not hold.

B Tangent space

The Euclidean cone $\mathcal{V} = \operatorname{Cone}\mathcal{X}$ over a metric space $\mathcal{X}$ is defined as the metric space whose underlying set consists of equivalence classes in $[0,\infty)\times\mathcal{X}$ with the equivalence relation "$\sim$" given by $(0,p)\sim(0,q)$ for any points $p,q\in\mathcal{X}$, and whose metric is given by the cosine rule

$$|(s,p)-(t,q)|_{\mathcal{V}} = \sqrt{s^2+t^2-2{\cdot}s{\cdot}t{\cdot}\cos\theta},$$

where $\theta=\min\{\pi,|p-q|_{\mathcal{X}}\}$.

Note that $\operatorname{Cone}\mathbb{S}^n$ *is isometric to* $\mathbb{E}^{n+1}$ — this is a leading example. Next, we need to generalize several notions from Euclidean space to Euclidean cones.

The point in $\mathcal{V}$ that corresponds $(t,x)\in[0,\infty)\times\mathcal{X}$ will be denoted by $t{\cdot}x$. The point in $\mathcal{V}$ formed by the equivalence class of $\{0\}\times\mathcal{X}$ is called the origin of the cone and is denoted by 0 or $0_{\mathcal{V}}$. For $v\in\mathcal{V}$ the distance $|0-v|_{\mathcal{V}}$ is called the norm of v and is denoted by $|v|$ or $|v|_{\mathcal{V}}$. The scalar product $\langle v,w\rangle$ of $v=s{\cdot}p$ and $w=t{\cdot}q$ is defined by

$$\langle v,w\rangle := |v|{\cdot}|w|{\cdot}\cos\theta$$

where $\theta=\min\{\pi,|p-q|_{\mathcal{X}}\}$. The value θ is undefined if $v=0$ or $w=0$; in this cases we assume that $\langle v,w\rangle:=0$.

4.2. Exercise. *Show that* $\operatorname{Cone}\mathcal{X}$ *is geodesic if and only if* $\mathcal{X}$ *is* π*-geodesic; that is any two points* $x,y\in\mathcal{X}$ *such that* $|x-y|_{\mathcal{X}}<\pi$ *can be joined by a geodesic in* $\mathcal{X}$*.*

Tangent space. The Euclidean cone $\operatorname{Cone}\Sigma_p$ over the space of directions Σ_p is called the tangent space at p and denoted by T_p or $\mathrm{T}_p\mathcal{L}$ if we need to emphasise that p is a point of space $\mathcal{L}$. The elements of $\mathrm{T}_p\mathcal{X}$ will be called tangent vectors at p (despite that T_p is only a cone — not a vector space). The space of directions Σ_p can be (and will be) identified with the unit sphere in T_p.

4.3. Proposition. *Tangent spaces to an Alexandrov space have nonnegative curvature in the sense of Alexandrov.*

Proof. Consider the tangent space $\mathrm{T}_p=\operatorname{Cone}\Sigma_p$ of an Alexandrov space $\mathcal{L}$ at a point p. We need to show that the $\mathbb{E}^2$-comparison holds for a given quadruple $v_0,v_1,v_2,v_3\in\mathrm{T}_p$.

Recall that the space of geodesic directions Σ'_p is dense in Σ_p. It follows that the subcone $\mathrm{T}'_p=\operatorname{Cone}\Sigma'_p$ is dense in T_p. Therefore, it is sufficient to consider the case $v_0,v_1,v_2,v_3\in\mathrm{T}'_p$.

For each i, choose a geodesic γ_i from p in the direction of v_i; assume γ_i has speed $|v_i|$ for each i. Since the angles are defined, we have

❶ $$|\gamma_i(\varepsilon)-\gamma_j(\varepsilon)|_{\mathcal{L}}=\varepsilon\cdot|v_i-v_j|_{\mathrm{T}_p}+o(\varepsilon)$$

for $\varepsilon>0$. The quadruple $\gamma_0(\varepsilon)$, $\gamma_1(\varepsilon)$, $\gamma_2(\varepsilon)$, $\gamma_3(\varepsilon)$ meets the $\mathbb{M}^2(\kappa)$-comparison. After rescaling all the distances by $\frac{1}{\varepsilon}$, it becomes the $\mathbb{M}^2(\varepsilon^2\cdot\kappa)$-comparison. Passing to the limit as $\varepsilon\to 0$ and applying ❶, we get the $\mathbb{E}^2$-comparison for v_0, v_1, v_2, v_3. □

4.4. Exercise. *Let p be a point in a finite-dimensional Alexandrov space $\mathcal{L}$, and let $\lambda_n\to\infty$. Show that there is a pointed Gromov–Hausdorff convergence $(\lambda\cdot\mathcal{L},p)\to(\mathrm{T}_p,0)$ in the sense of Gromov–Hausdorff. Moreover, the convergence can be chosen so that*

$$\lambda_n\cdot\gamma(t/\lambda_n)\to t\cdot\gamma^+(0)$$

for any geodesic γ that starts at p.

C Semiconcave functions

Recall that λ-concave functions were defined in Section 2E; also recall that we assume that these functions are locally Lipschitz and defined on an open domain.

Let f be locally Lipschitz real-valued function defined in an open subset $\operatorname{Dom} f$ of an Alexandrov space $\mathcal{L}$. We will write $f''\leqslant\varphi$ if for any point $x\in\operatorname{Dom} f$ and any $\varepsilon>0$ there is a neighborhood $U\ni x$ such that the restriction $f|_U$ is $(\varphi(x)+\varepsilon)$-concave. Here we assume that φ is continuous and defined in $\operatorname{Dom} f$.

If $f''\leqslant\varphi$ for some continuous function φ, then f is called s e m i - c o n c a v e.

4.5. Exercise. *Let f be a d i s t a n c e f u n c t i o n on an Alex0 space $\mathcal{L}$; that is, $f(x)\equiv|p-x|$ for some $p\in\mathcal{L}$. Show that $f''\leqslant\frac{1}{f}$. In particular, f is semiconcave in $\mathcal{L}\setminus\{p\}$.*

D Differential

Let $\mathcal{X}$ be a space with defined angles. Let f be a semiconcave function on $\mathcal{X}$ and $p\in\operatorname{Dom} f$. Choose a unit-speed geodesic γ that starts at p; let $\xi\in\Sigma_p$ be its direction. Define

$$(\boldsymbol{d}_pf)(\xi):=(f\circ\gamma)^+(0),$$

here $(f\circ\gamma)^+$ denotes the right derivative of $(f\circ\gamma)$; it is defined since f is semiconcave.

By the following exercise, the value $(\boldsymbol{d}_pf)(\xi)$ is defined; that is, it does not depend on the choice of γ. Moreover, $\boldsymbol{d}_pf$ is a Lipschitz function on Σ'_p. It follows that the function $\boldsymbol{d}_pf\colon \Sigma'_p\to\mathbb{R}$ can be extended to a Lipschitz function $\boldsymbol{d}_pf\colon \Sigma_p\to\mathbb{R}$. Further, we can extend it to the tangent space by setting

$$(\boldsymbol{d}_pf)(r\cdot\xi) := r\cdot(\boldsymbol{d}_pf)(\xi)$$

for any $r\geqslant 0$ and $\xi\in\Sigma_p$. The obtained function $\boldsymbol{d}_pf\colon \mathrm{T}_p\to\mathbb{R}$ is Lipschitz; it is called the differential of f at p.

4.6. Exercise. *Let f be a semiconcave function on an Alexandrov space. Suppose γ_1 and γ_2 are unit-speed geodesics that start at $p\in$ $\in\mathrm{Dom}\,f$; denote by θ the angle between γ_1 and γ_2 at p. Show that*

$$|(f\circ\gamma_1)^+(0)-(f\circ\gamma_2)^+(0)|\leqslant L\cdot\theta,$$

where L is the Lipschitz constant of f in a neighborhood of p.

4.7. Exercise. *Let p and q be distinct points in an Alexandrov space $\mathcal{L}$. Show that*

(a) *If ξ is the direction of a geodesic $[pq]$ at p, then*

$$\boldsymbol{d}_p\mathrm{dist}_q(v)\leqslant -\langle\xi,v\rangle$$

for any $v\in\mathrm{T}_p$.

(b) *Suppose $\mathcal{L}$ is proper. Let Ξ be the set of all direction of geodesics from q to p. Then*

$$\boldsymbol{d}_p\mathrm{dist}_q(v) = -\max_{\xi\in\Xi}\langle\xi,v\rangle$$

for any $v\in\mathrm{T}_p$.

E Gradient

4.8. Definition. *Let f be a semiconcave function on an Alexandrov space. A tangent vector $g\in\mathrm{T}_p$ is called a gradient of f at p (briefly, $g=\nabla_pf$) if*

(a) *$(\boldsymbol{d}_pf)(w)\leqslant\langle g,w\rangle$ for any $w\in\mathrm{T}_p$, and*

(b) *$(\boldsymbol{d}_pf)(g)=\langle g,g\rangle$.*

The following exercise provides a key property of gradients that will be important latter; see the first distance estimate (5.6).

4.9. Exercise. *Let f be a λ-concave function on an Alexandrov space. Suppose that gradients $\nabla_x f$ and $\nabla_y f$ are defined. Show that*

$$\langle \uparrow_{[xy]}, \nabla_x f\rangle + \langle \uparrow_{[yx]}, \nabla_y f\rangle + \lambda{\cdot}|x-y| \geqslant 0.$$

4.10. Proposition. *Suppose that a semiconcave function f is defined in a neighborhood of a point p in an Alexandrov space. Then the gradient $\nabla_p f$ is uniquely defined.*

Moreover, if $\boldsymbol{d}_p f \leqslant 0$, then we have $\nabla_p f = 0$; otherwise, $\nabla_p f = s{\cdot}\bar\xi$, where $s = \boldsymbol{d}_p f(\bar\xi)$ and $\bar\xi \in \Sigma_p$ is the direction that maximize the value $\boldsymbol{d}_p f(\xi)$ for $\xi \in \Sigma_p$.

4.11. Key lemma. *Let f be a λ-concave function that is defined in a neighborhood of a point p in an Alexandrov space $\mathcal{L}$. Then for any $u, v \in \mathrm{T}_p$, we have*

$$s{\cdot}\sqrt{|u|^2 + 2{\cdot}\langle u, v\rangle + |v|^2} \geqslant (\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v),$$

where

$$s = \sup\{\,(\boldsymbol{d}_p f)(\xi) \,:\, \xi \in \Sigma_p\,\}.$$

Note that if $\mathrm{T}_p \stackrel{iso}{=\!=} \mathbb{E}^m$ and $\boldsymbol{d}_p f$ is a concave function, then $2{\cdot}(\boldsymbol{d}_p f)(\frac{u+v}{2}) \geqslant (\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v)$. The latter implies the statement since $|u+v| = \sqrt{|u|^2 + 2{\cdot}\langle u, v\rangle + |v|^2}$. In general, T_p is not geodesic (and not even a length space), so concavity of $\boldsymbol{d}_p f$ does not make sense. The key lemma however says that in a certain sense $\boldsymbol{d}_p f$ behaves as a concave function.

It is instructive to solve the following exercise before reading the proof of the key lemma.

4.12. Exercise. *Let $[q\,{}^p_x]$ be a hinge in an Alexandrov space and $y \in\,]qp[$. Suppose that γ is the unit speed parametrization of $[qx]$ from q to x. Show that*

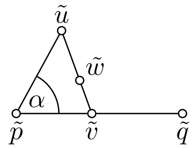
$$|y - \gamma(t)| = |y - q| - t{\cdot}\cos(\measuredangle[q\,{}^p_x]) + o(t).$$

Conclude that

$$(\boldsymbol{d}_q \mathrm{dist}_y)(w) = -\langle \uparrow_{[qp]}, w\rangle$$

for any $w \in \mathrm{T}_x$

Proof of 4.11. We will assume $\kappa = 0$; the general case requires only minor modifications. We can assume that $v \neq 0$, $w \neq 0$, and $\alpha = \measuredangle(u, v) > 0$; otherwise, the statement is trivial.



Consider a model configuration of five points: $\tilde p$, $\tilde u$, $\tilde v$, $\tilde q$, $\tilde w \in \mathbb{E}^2$ such that

- ◇ $\measuredangle[\tilde p\,{}^{\tilde u}_{\tilde v}] = \alpha$,
- ◇ $|\tilde p - \tilde u| = |u|$,
- ◇ $|\tilde p - \tilde v| = |v|$,
- ◇ $\tilde q$ lies on an extension of $[\tilde p\tilde v]$ so that $\tilde v$ is the midpoint of $[\tilde p\tilde q]$,
- ◇ $\tilde w$ is the midpoint between $\tilde u$ and $\tilde v$.

Note that

❶ $$|\tilde p - \tilde w| = \tfrac12\cdot\sqrt{|u|^2 + 2\cdot\langle u, v\rangle + |v|^2}.$$

Since the geodesic space of directions Σ'_p is dense in Σ_p, we can assume that there are geodesics in the directions of u and v. Choose such geodesics γ_u and γ_v and assume that they are parametrized with speed $|u|$ and $|v|$ respectively. For all small $t > 0$, consider points $u_t, v_t, q_t, w_t \in \mathcal{L}$ such that

- ◇ $v_t = \gamma_v(t)$, $\quad q_t = \gamma_v(2\cdot t)$
- ◇ $u_t = \gamma_u(t)$.
- ◇ w_t is the midpoint of $[u_t v_t]$.

Clearly

$$|p - u_t| = t\cdot|u|, \qquad |p - v_t| = t\cdot|v|, \qquad |p - q_t| = 2\cdot t\cdot|v|.$$

Since $\measuredangle(u, v)$ is defined, we have

$$|u_t - v_t| = t\cdot|\tilde u - \tilde v| + o(t), \qquad |u_t - q_t| = t\cdot|\tilde u - \tilde q| + o(t).$$

From the point-on-side and hinge comparisons (2.15c+2.15d), we have

$$\tilde\measuredangle(v_t\,{}^{p}_{w_t}) \geqslant \tilde\measuredangle(v_t\,{}^{p}_{u_t}) \geqslant \measuredangle[\tilde v\,{}^{\tilde p}_{\tilde u}] + \tfrac{o(t)}{t}$$

and

$$\tilde\measuredangle(v_t\,{}^{q_t}_{w_t}) \geqslant \tilde\measuredangle(v_t\,{}^{q_t}_{u_t}) \geqslant \measuredangle[\tilde v\,{}^{\tilde q}_{\tilde u}] + \tfrac{o(t)}{t}.$$

Clearly, $\measuredangle[\tilde v\,{}^{\tilde p}_{\tilde u}] + \measuredangle[\tilde v\,{}^{\tilde q}_{\tilde u}] = \pi$. From the adjacent angle comparison (2.15b), $\tilde\measuredangle(v_t\,{}^{p}_{u_t}) + \tilde\measuredangle(v_t\,{}^{u_t}_{q_t}) \leqslant \pi$. Hence $\tilde\measuredangle(v_t\,{}^{p}_{w_t}) \to \measuredangle[\tilde v\,{}^{\tilde p}_{\tilde w}]$ as $t \to 0+$ and thus

$$|p - w_t| = t\cdot|\tilde p - \tilde w| + o(t).$$

Without loss of generality, we can assume that $f(p) = 0$. Since f is λ-concave, we have

$$\begin{aligned} 2\cdot f(w_t) &\geqslant f(u_t) + f(v_t) + \tfrac{\lambda}{4}\cdot|u_t - v_t|^2 = \\ &= t\cdot[(\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v)] + o(t). \end{aligned}$$

Applying λ-concavity of f, we have

$$\begin{aligned}(\boldsymbol{d}_p f)(\uparrow_{[pw_t]}) &\geqslant \frac{f(w_t) - \frac{\lambda}{2}\cdot|p-w_t|^2}{|p-w_t|} \geqslant \\ &\geqslant \frac{t\cdot[(\boldsymbol{d}_p f)(u) + (\boldsymbol{d}_p f)(v)] + o(t)}{2\cdot t\cdot|\tilde{p}-\tilde{w}| + o(t)}.\end{aligned}$$

By ❶, the key lemma follows. □

Proof of 4.10; uniqueness. If $g, g' \in \mathrm{T}_p$ are two gradients of f, then

$$\langle g, g\rangle = (\boldsymbol{d}_p f)(g) \leqslant \langle g, g'\rangle, \qquad \langle g', g'\rangle = (\boldsymbol{d}_p f)(g') \leqslant \langle g, g'\rangle.$$

Therefore,

$$|g - g'|^2 = \langle g, g\rangle - 2\cdot\langle g, g'\rangle + \langle g', g'\rangle \leqslant 0.$$

It follows that $g = g'$.

Existence. Note first that if $\boldsymbol{d}_p f \leqslant 0$, then one can take $\nabla_p f = 0$.

Otherwise, if $s = \sup\{\,(\boldsymbol{d}_p f)(\xi) \,:\, \xi \in \Sigma_p\,\} > 0$, it is sufficient to show that there is $\overline{\xi} \in \Sigma_p$ such that

❷ $$(\boldsymbol{d}_p f)\left(\overline{\xi}\right) = s.$$

Indeed, suppose $\overline{\xi}$ exists. Applying 4.11 for $u = \overline{\xi}$, $v = \varepsilon\cdot w$ with $\varepsilon \to 0+$, we get

$$(\boldsymbol{d}_p f)(w) \leqslant \langle w, s\cdot\overline{\xi}\rangle$$

for any $w \in \mathrm{T}_p$; that is, $s\cdot\overline{\xi}$ is the gradient at p.

Take a sequence of directions $\xi_n \in \Sigma_p$, such that $(\boldsymbol{d}_p f)(\xi_n) \to s$. Applying 4.11 for $u = \xi_n$ and $v = \xi_m$, we get

$$s \geqslant \frac{(\boldsymbol{d}_p f)(\xi_n) + (\boldsymbol{d}_p f)(\xi_m)}{\sqrt{2 + 2\cdot\cos\measuredangle(\xi_n, \xi_m)}}.$$

Therefore $\measuredangle(\xi_n, \xi_m) \to 0$ as $n, m \to \infty$; that is, the sequence ξ_n is Cauchy. Clearly, $\overline{\xi} = \lim_n \xi_n$ meets ❷. □

4.13. Exercise. *Let f and g be locally Lipschitz semiconcave functions defined in a neighborhood of a point p in an Alexandrov space. Show that*

$$|\nabla_p f - \nabla_p g|^2_{\mathrm{T}_p} \leqslant s\cdot(|\nabla_p f| + |\nabla_p g|),$$

where

$$s = \sup\{\,|(\boldsymbol{d}_p f)(\xi) - (\boldsymbol{d}_p g)(\xi)| \,:\, \xi \in \Sigma_p\,\}.$$

Conclude that if the sequence of restrictions $\boldsymbol{d}_p f_n|_{\Sigma_p}$ *converges uniformly, then* $\nabla_p f_n$ *converges as* $n \to \infty$. *Here we assume that all functions* f_1, $f_2, \dots$ *are semiconcave and locally Lipschitz.*

4.14. Exercise. *Let* f *be a locally Lipschitz* λ*-concave function on an Alexandrov space* $\mathcal{L}$.

(a) *Suppose* $s \geqslant 0$. *Show that* $|\nabla_x f| > s$ *if and only if for some point* y *we have*

$$f(y) - f(x) > s\cdot\ell + \lambda\cdot\tfrac{\ell^2}{2},$$

were $\ell = |x - y|$.

(b) *Show that* $x \mapsto |\nabla_x f|$ *is lower semicontinuous; that is, if* $x_n \to \to x_\infty$, *then*

$$|\nabla_{x_\infty} f| \leqslant \varliminf_{n\to\infty} |\nabla_{x_n} f|.$$

Lecture 5

Gradient flow

A Velocity of curve

Let α be a curve in an Alexandrov space $\mathcal{L}$. If for any choice of geodesics $[p\,\alpha(t_0+\varepsilon)]$ the vectors

$$\tfrac{1}{\varepsilon}\cdot|p-\alpha(t_0+\varepsilon)|\cdot\uparrow_{[p\alpha(t_0+\varepsilon)]}$$

converge as $\varepsilon\to 0+$, then their limit in T_p is called the right derivative of α at t_0; it will be denoted by $\alpha^+(t_0)$. In addition, $\alpha^+(t_0):=0$ if $\frac{1}{\varepsilon}\cdot|p-\alpha(t_0+\varepsilon)|\to 0$ as $\varepsilon\to 0+$.

The tangent vector $v=|p-x|\cdot\uparrow_{[px]}$ can be called logarithm of x at p (briefly, $\log_p x$); it is a tangent vector at p of a geodesic path from p to x.

5.1. Claim. *Let α be a curve in an Alexandrov space $\mathcal{L}$. Suppose f a semiconcave Lipschitz function defined in a neighborhood of $p=\alpha(0)$, and $\alpha^+(0)$ is defined. Then*

$$(f\circ\alpha)^+(0)=(\boldsymbol{d}_pf)(\alpha^+(0)).$$

Proof. Without loss of generality, we can assume that $f(p)=0$. Suppose f and therefore $\boldsymbol{d}_pf$ are L-Lipschitz.

Choose a constant-speed geodesic γ that starts from p, such that the distance $s=|\alpha^+(0)-\gamma^+(0)|_{\mathrm{T}_p}$ is small. By the definition of differential,

$$(f\circ\gamma)^+(0)=\boldsymbol{d}_pf(\gamma^+(0)).$$

By comparison and the definition of α^+,

$$|\alpha(\varepsilon)-\gamma(\varepsilon)|_{\mathcal{L}}\leqslant s\cdot\varepsilon+o(\varepsilon)$$

for $\varepsilon > 0$. Therefore,

$$|f \circ \alpha(\varepsilon) - f \circ \gamma(\varepsilon)| \leqslant L{\cdot}s{\cdot}\varepsilon + o(\varepsilon).$$

Suppose $(f \circ \alpha)^+(0)$ is defined. Then

$$|(f \circ \alpha)^+(0) - (f \circ \gamma)^+(0)| \leqslant L{\cdot}s.$$

Since $\boldsymbol{d}_p f$ is L-Lipschitz, we also get

$$|\boldsymbol{d}_p f(\alpha^+(0)) - \boldsymbol{d}_p f(\gamma^+(0))| \leqslant L{\cdot}s.$$

It follows that the needed identity holds up to error $2{\cdot}L{\cdot}s$. The statement follows since $s > 0$ can be chosen arbitrarily.

The same argument is applicable if in the place of $(f \circ \alpha)^+(0)$ we use any limit of $\frac{1}{\varepsilon_n}{\cdot}[f \circ \alpha(\varepsilon_n) - f(p)]$ for a sequence $\varepsilon_n \to 0+$. It proves that all such limits coincide; in particular, $(f \circ \alpha)^+(0)$ is defined and equals to $(\boldsymbol{d}_p f)(\alpha^+(0))$. □

B Gradient curves

5.2. Definition. *Let $f\colon \mathcal{L} \to \mathbb{R}$ be a locally Lipschitz and semiconcave function on an Alexandrov space $\mathcal{L}$.*

A locally Lipschitz curve $\alpha\colon [t_{\min}, t_{\max}) \to \operatorname{Dom} f$ will be called an f-gradient curve if

$$\alpha^+ = \nabla_\alpha f;$$

that is, for any $t \in [t_{\min}, t_{\max})$, $\alpha^+(t)$ is defined and $\alpha^+(t) = \nabla_{\alpha(t)} f$.

A complete proof of the following theorem is given in [3]; it mimics the proof of the standard Picard theorem on the existence and uniqueness of solutions of ordinary differential equations. We omit the proof of existence as it is rather lengthy; the uniqueness will be proved in the next section.

5.3. Picard theorem. *Let $f\colon \mathcal{L} \to \mathbb{R}$ be a locally Lipschitz and λ-concave function on an Alexandrov space $\mathcal{L}$. Then for any $p \in \operatorname{Dom} f$, there are unique $t_{\max} \in (0, \infty]$ and f-gradient curve $\alpha\colon [0, t_{\max}) \to \mathcal{L}$ with $\alpha(0) = p$ such that any sequence $t_n \to t_{\max}-$, the sequence $\alpha(t_n)$ does not have a limit point in $\operatorname{Dom} f$.*

Note that the theorem says that the future of a gradient curve is determined by its present, but it says nothing about its past.

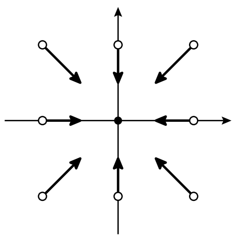
Here is an example showing that the past is not determined by the present. Consider the function $f\colon x \mapsto -|x|$ on the real line $\mathbb{R}$. The tangent space $\mathrm{T}_x\mathbb{R}$ can be identified with $\mathbb{R}$. Note that

$$\nabla_x f = \begin{cases} 1 & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x > 0. \end{cases}$$

So, the f-gradient curves go to the origin with unit speed and then stand there forever. In particular, if α is an f-gradient curve that starts at x, then $\alpha(t) = 0$ for any $t \geqslant |x|$.

Here is a slightly more interesting example; it shows that gradient curves can merge even in the region where $|\nabla f| \neq 0$.

5.4. Example. *Consider the function $f\colon (x, y) \mapsto -|x| - |y|$ on the (x, y)-plane. Note that f is concave; its gradient field is sketched on the figure.*



Let α be an f-gradient curve that starts at (x, y) for $x > y > 0$. Then

$$\alpha(t) = \begin{cases} (x - t, y - t) & \textit{for} \quad 0 \leqslant t \leqslant x - y, \\ (x - t, 0) & \textit{for} \quad x - y \leqslant t \leqslant x, \\ (0, 0) & \textit{for} \quad x \leqslant t. \end{cases}$$

C Distance estimates

5.5. Observation. *Let α be a gradient curve of a λ-concave function f defined on an Alexandrov space. Choose a point p; let $\ell(t) := \operatorname{dist}_p \circ \alpha(t)$ and $q = \alpha(t_0)$. Then*

$$\ell^+(t_0) \leqslant -\left(f(p) - f(q) - \tfrac{\lambda}{2}\cdot\ell^2(t_0)\right)/\ell(t_0)$$

Proof. Let γ be the unit-speed parametrization of $[qp]$ from q to p, so $q = \gamma(0)$. Then

$$\begin{aligned} \ell^+(t_0) &= (\boldsymbol{d}_q \operatorname{dist}_p)(\nabla_q f) \leqslant \\ &\leqslant -\langle \uparrow_{[qp]}, \nabla_q f\rangle \leqslant \\ &\leqslant -\boldsymbol{d}_q f(\uparrow_{[qp]}) = \\ &= -(f \circ \gamma)^+(0) \leqslant \\ &\leqslant -\left(f(p) - f(q) - \tfrac{\lambda}{2}\cdot\ell^2(t_0)\right)/\ell(t_0) \end{aligned}$$

In the above calculations we consequently applied 5.1, 4.7, the definition of gradient, the definition of differential, and concavity of $t \mapsto \mapsto f \circ \gamma(t) - \frac{\lambda}{2} \cdot t^2$. □

Note that the following estimate implies uniqueness in the Picard theorem (5.3).

5.6. First distance estimate. *Let f be a λ-concave locally Lipschitz function on an Alexandrov space $\mathcal{L}$. Then*

$$|\alpha(t) - \beta(t)| \leqslant e^{\lambda \cdot t} \cdot |\alpha(0) - \beta(0)|$$

for any $t \geqslant 0$ and any two f-gradient curves α and β.

Moreover, the statement holds for a locally Lipschitz λ-concave function defined in an open domain if there is a geodesic $[\alpha(t)\, \beta(t)]$ in $\operatorname{Dom} f$ *for any t.*

Proof. Fix a choice of geodesic $[\alpha(t)\, \beta(t)]$ for each t. Let $\ell(t) = |\alpha(t) - \beta(t)|$. Note that

$$\ell^+(t) \leqslant -\langle \uparrow_{[\alpha(t)\beta(t)]}, \nabla_{\alpha(t)} f \rangle - \langle \uparrow_{[\beta(t)\alpha(t)]}, \nabla_{\beta(t)} f \rangle \leqslant \lambda \cdot \ell(t).$$

Here one has to apply 5.5 for distance to the midpoint m of $[\alpha(t)\, \beta(t)]$, and then apply the triangle inequality. Hence the result. □

The following exercise describes a global geometric property of a gradient curve without direct reference to its function. It uses the notion of *self-contracting curves* introduced by Aris Daniilidis, Olivier Ley, and Stéphane Sabourau [14].

5.7. Exercise. *Let $f \colon \mathcal{L} \to \mathbb{R}$ be a locally Lipschitz and concave function on an Alexandrov space $\mathcal{L}$. Then*

$$|\alpha(t_1) - \alpha(t_3)|_{\mathcal{L}} \geqslant |\alpha(t_2) - \alpha(t_3)|_{\mathcal{L}}.$$

for any f-gradient curve α and $t_1 \leqslant t_2 \leqslant t_3$.

5.8. Exercise. *Let f be a locally Lipschitz concave function defined on an Alexandrov space $\mathcal{L}$. Suppose $\hat\alpha \colon [0, \ell] \to \mathcal{L}$ is an arc-length reparametrization of an f-gradient curve. Show that $(f \circ \hat\alpha)$ is concave.*

The following exercise implies that gradient curves for a uniformly converging sequence of λ-concave functions converge to the gradient curves of the limit function.

5.9. Exercise. *Let f and g be λ-concave locally Lipschitz functions on an Alexandrov space $\mathcal{L}$. Suppose $\alpha, \beta \colon [0, t_{\max}) \to \mathcal{L}$ are respectively*

f- and g-gradient curves. Assume $|f-g|<\varepsilon$; let $\ell\colon t\mapsto|\alpha(t)-\beta(t)|$. Show that

$$\ell^+\leqslant\lambda{\cdot}\ell+\tfrac{2\cdot\varepsilon}{\ell}.$$

Conclude that if $\alpha(0)=\beta(0)$ and $t_{\max}<\infty$, then

$$|\alpha(t)-\beta(t)|\leqslant c{\cdot}\sqrt{\varepsilon{\cdot}t}$$

for some constant $c=c(t_{\max},\lambda)$.

D Gradient flow

Let $\mathcal{L}$ be an Alexandrov space and f be a locally Lipschitz semiconcave function defined on an open subset of $\mathcal{L}$. If there is an f-gradient curve α such that $\alpha(0)=x$ and $\alpha(t)=y$, then we will write

$$\mathrm{Flow}_f^t(x)=y.$$

The partially defined map Flow_f^t from $\mathcal{L}$ to itself is called the f-gradient flow for time t. Note that

$$\mathrm{Flow}_f^{t_1+t_2}=\mathrm{Flow}_f^{t_1}\circ\mathrm{Flow}_f^{t_2}.$$

In other words, one may think that gradient flow is an action of the *semigroup* $(\mathbb{R}_{\geqslant0},+)$ on the space.

From the first distance estimate 5.6, it follows that for any $t\geqslant0$, the domain of definition of Flow_f^t is an open subset of $\mathcal{L}$. In some cases, it is globally defined. For example, if f is a λ-concave function defined on the whole space $\mathcal{L}$, then $\mathrm{Flow}_f^t(x)$ is defined for all $x\in\mathcal{L}$ and $t\geqslant0$; see [3, 16.19].

Now let us reformulate the statements about gradient curves obtained earlier using this new terminology. From the first distance estimate, we have the following.

5.10. Proposition. *Let $\mathcal{L}$ be an Alexandrov space and $f\colon\mathcal{L}\to\mathbb{R}$ be a semiconcave function. Then the map $x\mapsto\mathrm{Flow}_f^t(x)$ is locally Lipschitz.*

Moreover, if f is λ-concave, then Flow_f^t is $e^{\lambda\cdot t}$-Lipschitz.

The next proposition follows from 5.9.

5.11. Proposition. *Let $\mathcal{L}$ be an Alexandrov space. Suppose $f_n\colon\mathcal{L}\to\mathbb{R}$ is a sequence of λ-concave functions that converges to $f_\infty\colon\mathcal{L}\to\mathbb{R}$. Then for any $x\in\mathcal{L}$ and $t\geqslant0$, we have*

$$\mathrm{Flow}_{f_n}^t(x)\to\mathrm{Flow}_{f_\infty}^t(x)$$

as $n\to\infty$.

E Gradient exponent

One of the technical difficulties in Alexandrov's geometry comes from nonextendability of geodesics. In particular, the exponential map, $\exp_p\colon \mathrm{T}_p \to \mathcal{L}$, if defined in the usual way, can be undefined in an arbitrary small neighborhood of the origin.

We construct its analog, the gradient exponential map

$$\mathrm{gexp}_p\colon \mathrm{T}_p \to \mathcal{L},$$

which essentially solves this problem. It shares many properties with the ordinary exponential map, and better in certain respects, even in the Riemannian universe.

Let $\mathcal{L}$ be Alexandrov's space and $p \in \mathcal{L}$, consider the function $f = \mathrm{dist}_p^2/2$. Suppose $i_\lambda\colon \lambda{\cdot}\mathcal{L} \to \mathcal{L}$ sends a point in the rescaled copy $\lambda{\cdot}\mathcal{L}$ to the corresponding point in $\mathcal{L}$. Consider the one parameter family of maps

$$\Phi_f^t \circ i_{e^t}\colon e^t{\cdot}\mathcal{L} \to \mathcal{L}$$

where Φ_f^t denotes gradient flow. Note that $(e^t{\cdot}\mathcal{L}, p) \xrightarrow{\mathrm{GH}} (\mathrm{T}_p, o_p)$ as $t \to \infty$. Let us define the *gradient exponential map* as the limit

$$\mathrm{gexp}_p = \lim_{t\to\infty} \Phi_f^t \circ i_{e^t}.$$

5.12. Proposition. *Let $\mathcal{L}$ be a proper* $\mathrm{A\small LEX}(0)$ *space. Then for any $p \in \mathcal{L}$ the gradient exponent* $\mathrm{gexp}_p\colon \mathrm{T}_p \to \mathcal{L}$ *is defined. Moreover,* gexp_p *is a short map and*

$$\mathrm{gexp}_p(\gamma^+(0)) = \gamma(1)$$

for any geodesic path γ that starts at p.

The last statement in the proposition says that it is appropriate to use term *exponent* for gexp.

Proof. Note that $f'' \leqslant 1$. By the first distance estimate, we have that Φ_f^t is an e^t-Lipschitz. Therefore, the compositions $\Phi_f^t \circ i_{e^t}\colon e^t{\cdot}\mathcal{L} \to \mathcal{L}$ are short. Hence a partial limit $\mathrm{gexp}_p\colon \mathrm{T}_p\mathcal{L} \to \mathcal{L}$ exists, and it is a short map.

Clearly for any partial limit we have

$$\Phi_f^t \circ \mathrm{gexp}_p(v) = \mathrm{gexp}_p(e^t{\cdot}v).$$

Since Φ^t is e^t-Lipschitz, it follows that gexp_p is uniquely defined. $\square$

F Remarks

??? gradient exponent for $\kappa \neq 0$ and for nonproper.

The gradient exponential map gexp_p for a point p a Riemannian manifold (M, g) coincides with the Riemannian exponential map inside the cut locus of p but is different from the Riemannian exponential outside it.

quasigeodesics

Lecture 6

Line splitting

A Busemann function

A half-line is a distance-preserving map from $\mathbb{R}_{\geqslant 0} = [0, \infty)$ to a metric space. In other words, a half-line is a geodesic defined on the real half-line $\mathbb{R}_{\geqslant 0}$. If $\gamma\colon [0, \infty) \to \mathcal{X}$ is a half-line, then the limit

❶ $$\operatorname{bus}_\gamma(x) = \lim_{t\to\infty} |\gamma(t) - x| - t$$

is called the Busemann function of γ.

The Busemann function $\operatorname{bus}_\gamma$ mimics behavior of the distance function from the ideal point of γ.

6.1. Proposition. *For any half-line γ in a metric space $\mathcal{X}$, its Busemann function $\operatorname{bus}_\gamma\colon \mathcal{X} \to \mathbb{R}$ is defined. Moreover, $\operatorname{bus}_\gamma$ is 1-Lipschitz and $\operatorname{bus}_\gamma \circ\gamma(t) + t = 0$ for any t.*

Proof. By the triangle inequality, the function

$$t \mapsto |\gamma(t) - x| - t$$

is nonincreasing for any fixed x.

Since $t = |\gamma(0) - \gamma(t)|$, the triangle inequality implies that

$$|\gamma(t) - x| - t \geqslant -|\gamma(0) - x|.$$

Thus the limit in ❶ is defined, and it is 1-Lipschitz as a limit of 1-Lipschitz functions. The last statement follows since $|\gamma(t) - \gamma(t_0)| = t - t_0$ for all large t. □

6.2. Exercise. *Any Busemann function on an* $\text{A}\textsc{lex}(0)$ *space is concave.*

B Splitting theorem

A line is a distance-preserving map from $\mathbb{R}$ to a metric space. In other words, a line is a geodesic defined on the real line $\mathbb{R}$.

6.3. Exercise. *Let γ be a line in a metric space $\mathcal{X}$. Show that for any point x we have*

$$\mathrm{bus}_+(x) + \mathrm{bus}_-(x) \geqslant 0$$

where, bus_+ and bus_-, are the Busemann functions asociated with half-lines $\gamma : [0, \infty) \to \mathcal{L}$ and $\gamma : (-\infty, 0] \to \mathcal{L}$ respectively.

Let $\mathcal{X}$ be a metric space and $A, B \subset \mathcal{X}$. We will write

$$\mathcal{X} = A \oplus B$$

if there are projections $\mathrm{proj}_A \colon \mathcal{X} \to A$ and $\mathrm{proj}_B \colon \mathcal{X} \to B$ such that

$$|x - y|^2 = |\mathrm{proj}_A(x) - \mathrm{proj}_A(y)|^2 + |\mathrm{proj}_B(x) - \mathrm{proj}_B(y)|^2$$

for any two points $x, y \in \mathcal{X}$.

Note that if

$$\mathcal{X} = A \oplus B$$

then

- ◇ A intersects B at a single point,
- ◇ both sets A and B are convex sets in $\mathcal{X}$; the latter means that any geodesic with the ends in A (or B) lies in A (or B).

6.4. Line splitting theorem. *Let γ be a line in a* $\mathrm{ALEX}(0)$ *space $\mathcal{L}$. Then*

$$\mathcal{L} = \mathcal{L}' \oplus \gamma(\mathbb{R})$$

for some subset $\mathcal{L}' \subset \mathcal{L}$.

Before going into the proof, let us state a corollary of the theorem.

6.5. Corollary. *Any* $\mathrm{ALEX}(0)$ *space $\mathcal{L}$ splits isometrically as*

$$\mathcal{L} = \mathcal{L}' \oplus H$$

where $H \subset \mathcal{L}$ is a subset isometric to a Hilbert space, and $\mathcal{L}' \subset \mathcal{L}$ is a convex subset that contains no line.

The following lemma is closely relevant to the first distance estimate (5.6); its proof goes along the same lines.

6.6. Lemma. *Suppose $f\colon\mathcal{L}\to\mathbb{R}$ be a concave 1-Lipschitz function on an* Alex(0) *space $\mathcal{L}$. Consider two f-gradient curves α and β. Then for any $t,s\geqslant 0$ we have*

$$|\alpha(s)-\beta(t)|^2\leqslant|p-q|^2+2{\cdot}(f(p)-f(q)){\cdot}(s-t)+(s-t)^2,$$

where $p=\alpha(0)$ and $q=\beta(0)$.

Proof. Since f is 1-Lipschitz, $|\nabla f|\leqslant 1$. Therefore

$$f\circ\beta(t)\leqslant f(q)+t$$

for any $t\geqslant 0$.

Set $\ell(t)=|p-\beta(t)|$. Applying 5.5, we get

$$\begin{aligned}(\ell^2)^+(t)&\leqslant 2{\cdot}\left(f\circ\beta(t)-f(p)\right)\leqslant\\ &\leqslant 2{\cdot}\left(f(q)+t-f(p)\right).\end{aligned}$$

Therefore

$$\ell^2(t)-\ell^2(0)\leqslant 2{\cdot}\left(f(q)-f(p)\right){\cdot}t+t^2.$$

It proves the needed inequality in case $s=0$. Combining it with the first distance estimate (5.6), we get the result in case $s\leqslant t$. The case $s\geqslant t$ follows by switching the roles of s and t. $\square$

Proof of 6.4. Consider two Busemann functions, bus_+ and bus_-, asociated with half-lines $\gamma:[0,\infty)\to\mathcal{L}$ and $\gamma:(-\infty,0]\to\mathcal{L}$ respectively; that is,

$$\mathrm{bus}_\pm(x):=\lim_{t\to\infty}|\gamma(\pm t)-x|-t.$$

According to 6.2, both functions $\mathrm{bus}_\pm$ are concave.

By 6.3, $\mathrm{bus}_+(x)+\mathrm{bus}_-(x)\geqslant 0$ for any $x\in\mathcal{L}$. On the other hand, by 2.18, $f(t)=\mathrm{dist}_x^2\circ\gamma(t)$ is 2-concave. In particular, $f(t)\leqslant t^2+at+b$ for some constants $a,b\in\mathbb{R}$. Passing to the limit as $t\to\pm\infty$, we have $\mathrm{bus}_+(x)+\mathrm{bus}_-(x)\leqslant 0$. Hence

$$\mathrm{bus}_+(x)+\mathrm{bus}_-(x)=0$$

for any $x\in\mathcal{L}$. In particular, the functions $\mathrm{bus}_\pm$ are affine; that is, they are convex and concave at the same time.

Note that for any x,

$$\begin{aligned}|\nabla_x\mathrm{bus}_\pm|&=\sup\{\,\boldsymbol{d}_x\mathrm{bus}_\pm(\xi)\;:\;\xi\in\Sigma_x\,\}=\\ &=\sup\{\,-\boldsymbol{d}_x\mathrm{bus}_\mp(\xi)\;:\;\xi\in\Sigma_x\,\}\equiv\\ &\equiv 1.\end{aligned}$$

Observe that α is a $\operatorname{bus}_\pm$-gradient curve if and only if α is a geodesic such that $(\operatorname{bus}_\pm \circ\alpha)^+ = 1$. Indeed, if α is a geodesic, then $(\operatorname{bus}_\pm \circ\alpha)^+ \leqslant 1$ and the equality holds only if $\nabla_\alpha \operatorname{bus}_\pm = \alpha^+$. Now suppose $\nabla_\alpha \operatorname{bus}_\pm = \alpha^+$. Then $|\alpha^+| \leqslant 1$ and $(\operatorname{bus}_\pm \circ\alpha)^+ = 1$; therefore

$$\begin{aligned}|t_0 - t_1| \geqslant |\alpha(t_0) - \alpha(t_1)| \geqslant \\ &\geqslant |\operatorname{bus}_\pm \circ\alpha(t_0) - \operatorname{bus}_\pm \circ\alpha(t_1) = \\ &= |t_0 - t_1|.\end{aligned}$$

It follows that for any $t > 0$, the $\operatorname{bus}_\pm$-gradient flows commute; that is,

$$\mathrm{Flow}^t_{\operatorname{bus}_+} \circ \mathrm{Flow}^t_{\operatorname{bus}_-} = \mathrm{id}_{\mathcal{L}}\,.$$

Setting

$$\mathrm{Flow}^t = \begin{cases}\mathrm{Flow}^t_{\operatorname{bus}_+} & \text{if } t \geqslant 0\\ \mathrm{Flow}^t_{\operatorname{bus}_-} & \text{if } t \leqslant 0\end{cases}$$

defines an $\mathbb{R}$-action on $\mathcal{L}$.

Consider the level set $\mathcal{L}' = \operatorname{bus}_+^{-1}(0) = \operatorname{bus}_-^{-1}(0)$; it is a closed convex subset of $\mathcal{L}$, and therefore forms an Alexandrov space. Consider the map $h\colon \mathcal{L}' \times \mathbb{R} \to \mathcal{L}$ defined by $h\colon (x,t) \mapsto \mathrm{Flow}^t(x)$. Note that h is onto. Applying Lemma 6.6 for $\mathrm{Flow}^t_{\operatorname{bus}_+}$ and $\mathrm{Flow}^t_{\operatorname{bus}_-}$ shows that h is short and non-contracting at the same time; that is, h is an isometry. □

Recall that according our definition the real line $\mathbb{R}$ is $\mathrm{ALEX}(1)$. However most of $\mathrm{ALEX}(1)$ spaces have diameter at most π; see 3.5.

6.7. Exercise. *Suppose $\mathcal{X}$ is a complete geodesic space. Show that* $\operatorname{Cone}\mathcal{X}$ *is* $\mathrm{ALEX}(0)$ *if and only if* $\mathcal{X}$ *is* $\mathrm{ALEX}(1)$ *and* $\operatorname{diam}\mathcal{X} \leqslant 1$.

C Comments

The splitting theorem has an interesting history that starts with Stefan Cohn-Vossen [13]. Our proof is based on the idea of Jeff Cheeger and Detlef Gromoll [12].

Lecture 7

Dimension

A Polar vectors

Here we give a corollary of 4.13. It will be used to prove basic properties of the tangent space.

7.1. Anti-sum lemma. *Let $\mathcal{L}$ be an Alexandrov space and $p \in \mathcal{L}$.*

Given two vectors $u, v \in \mathrm{T}_p$, there is a unique vector $w \in \mathrm{T}_p$ such that

$$\langle u, x\rangle + \langle v, x\rangle + \langle w, x\rangle \geqslant 0$$

for any $x \in \mathrm{T}_p$, and

$$\langle u, w\rangle + \langle v, w\rangle + \langle w, w\rangle = 0.$$

7.2. Exercise. *Suppose $u, v, w \in \mathrm{T}_p$ are as in 7.1. Show that*

$$|w|^2 \leqslant |u|^2 + |v|^2 + 2{\cdot}\langle u, v\rangle.$$

If T_p were geodesic, then the lemma would follow from the existence of the gradient, applied to the function $\mathrm{T}_p \to \mathbb{R}$ defined by $x \mapsto -(\langle u, x\rangle + \langle v, x\rangle)$ which is concave. However, the tangent space T_p might fail to be geodesic; see Halbeisen's example [3].

Applying the above lemma for $u = v$, we have the following statement.

7.3. Existence of polar vector. *Let $\mathcal{L}$ be an Alexandrov space and $p \in \mathcal{L}$. Given a vector $u \in \mathrm{T}_p$, there is a unique vector $u^* \in \mathrm{T}_p$ such that $\langle u^*, u^*\rangle + \langle u, u^*\rangle = 0$ and u^* is polar to u; that is, $\langle u^*, x\rangle + \langle u, x\rangle \geqslant 0$ for any $x \in \mathrm{T}_p$.*

In particular, for any vector $u \in \mathrm{T}_p$ there is a polar vector $u^* \in \mathrm{T}_p$ such that $|u^*| \leqslant |u|$.

7.4. Example. *Let $\mathcal{L}$ be the upper half plane in $\mathbb{E}^2$; that is, $\mathcal{L} = \{(x, y) \in \mathbb{E}^2 \mid y \geqslant 0\}$. It is an Alexandrov space. For $p = 0$, the tangent space T_p can be canonically identified with $\mathcal{L}$. If $y > 0$, then $u = (x, y) \in \mathrm{T}_p$ has many polar vectors; it includes $u^* = (-x, 0)$ which is provided by 7.3, but the vector $w = (-x, y)$ is polar as well.*

Proof of 7.1. By 4.12, we can choose two sequences of points a_n, b_n such that

$$\boldsymbol{d}_p\mathrm{dist}_{a_n}(w) = -\langle\uparrow_{[pa_n]}, w\rangle$$
$$\boldsymbol{d}_p\mathrm{dist}_{b_n}(w) = -\langle\uparrow_{[pb_n]}, w\rangle$$

for any $w \in \mathrm{T}_p$ and $\uparrow_{[pa_n]} \to u/|u|$, $\uparrow_{[pb_n]} \to v/|v|$ as $n \to \infty$

Consider a sequence of functions

$$f_n = |u|\cdot\mathrm{dist}_{a_n} + |v|\cdot\mathrm{dist}_{b_n}.$$

Note that

$$(\boldsymbol{d}_p f_n)(x) = -|u|\cdot\langle\uparrow_{[pa_n]}, x\rangle - |v|\cdot\langle\uparrow_{[pb_n]}, x\rangle.$$

Thus we have the following uniform convergence for $x \in \Sigma_p$:

$$(\boldsymbol{d}_p f_n)(x) \to -\langle u, x\rangle - \langle v, x\rangle$$

as $n \to \infty$, According to 4.13, the sequence $\nabla_p f_n$ converges. Let

$$w = \lim_{n\to\infty} \nabla_p f_n.$$

By the definition of gradient,

$$\begin{aligned} \langle w, w\rangle &= \lim_{n\to\infty}\langle\nabla_p f_n, \nabla_p f_n\rangle = \\ &= \lim_{n\to\infty}(\boldsymbol{d}_p f_n)(\nabla_p f_n) = \\ &= -\langle u, w\rangle - \langle v, w\rangle, \end{aligned} \qquad \begin{aligned} \langle w, x\rangle &= \lim_{n\to\infty}\langle\nabla_p f_n, x\rangle \geqslant \\ &\geqslant \lim_{n\to\infty}(\boldsymbol{d}_p f_n)(x) = \\ &= -\langle u, x\rangle - \langle v, x\rangle. \end{aligned}$$

□

B Linear subspace of tangent space

7.5. Definition. *Let $\mathcal{L}$ be an Alexandrov space, $p \in \mathcal{L}$ and $u, v \in \mathrm{T}_p$. We say that vectors u and v are* o p p o s i t e *to each other, (briefly, $u + v = 0$) if $|u| = |v| = 0$ or $\measuredangle(u, v) = \pi$ and $|u| = |v|$.*

The subcone

$$\mathrm{Lin}_p = \{\, v \in \mathrm{T}_p \,:\, \exists\, w \in \mathrm{T}_p \quad \textit{such that} \quad w + v = 0 \,\}$$

will be called the l i n e a r s u b s p a c e *of* T_p.

7.6. Proposition. *Let $\mathcal{L}$ be an Alexandrov space and $p \in \mathcal{L}$. Given two vectors $u, v \in \mathrm{T}_p$, the following statements are equivalent:*
(a) $u + v = 0$;
(b) $\langle u, x\rangle + \langle v, x\rangle = 0$ *for any* $x \in \mathrm{T}_p$;
(c) $\langle u, \xi\rangle + \langle v, \xi\rangle = 0$ *for any* $\xi \in \Sigma_p$.

Proof. The equivalence *(b)*⇔*(c)* is trivial.

The condition $u + v = 0$ is equivalent to

$$\langle u, u\rangle = -\langle u, v\rangle = \langle v, v\rangle;$$

thus *(b)*⇒*(a)*.

Recall that T_p has nonnegative curvature. Note that the hinges $[0\,{}^{u}_{x}]$ and $[0\,{}^{v}_{x}]$ are adjacent. By 2.11, $\measuredangle[0\,{}^{u}_{x}]+\measuredangle[0\,{}^{v}_{x}] = 0$; hence *(a)*⇒*(b)*. □

7.7. Exercise. *Let $\mathcal{L}$ be an Alexandrov space and $p \in \mathcal{L}$. Then for any three vectors $u, v, w \in \mathrm{T}_p$, if $u+v = 0$ and $u+w = 0$ then $v = w$.*

Let $u \in \mathrm{Lin}_p$; that is, $u + v = 0$ for some $v \in \mathrm{T}_p$. Given $s < 0$, let

$$s{\cdot}u := (-s){\cdot}v.$$

So we can multiply any vector in Lin_p by any real number (positive and negative). By 7.7, this multiplication is uniquely defined; by 7.6; we have identity

$$\langle -v, x\rangle = -\langle v, x\rangle;$$

later we will see that it extends to a linear structure on Lin_p.

7.8. Exercise. *Suppose $u, v, w \in \mathrm{T}_p$ are as in 7.1. Show that*

$$\langle u, x\rangle + \langle v, x\rangle + \langle w, x\rangle = 0$$

for any $x \in \mathrm{Lin}_p$.

7.9. Exercise. *Let $\mathcal{L}$ be an Alexandrov space, $p \in \mathcal{L}$ and $u \in \mathrm{T}_p$. Suppose $u^* \in \mathrm{T}_p$ is provided by 7.3; that is,*

$$\langle u^*, u^*\rangle + \langle u, u^*\rangle = 0 \quad \textit{and} \quad \langle u^*, x\rangle + \langle u, x\rangle \geqslant 0$$

for any $x \in \mathrm{T}_p$. Show that

$$u = -u^* \quad \iff \quad |u| = |u^*|.$$

7.10. Theorem. *Let p be a point in an Alexandrov space. Then* Lin_p *is isometric to a Hilbert space.*

Proof. Note that Lin_p is a closed subset of T_p; in particular, it is complete.

If any two vectors in Lin_p can be connected by a geodesic in Lin_p, then the statement follows from the splitting theorem (6.4). By Menger's lemma (1.4), it is sufficient to show that for any two vectors $x, y \in \mathrm{Lin}_p$ there is a midpoint $w \in \mathrm{Lin}_p$.

Choose $w \in \mathrm{T}_p$ to be the anti-sum of $u = -\frac{1}{2}\cdot x$ and $v = -\frac{1}{2}\cdot y$; see 7.1. By 7.2 and 7.8,

$$\begin{aligned} |w|^2 &\leqslant \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle, \\ \langle w, x\rangle &= \tfrac{1}{2}\cdot|x|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle, \\ \langle w, y\rangle &= \tfrac{1}{2}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle, \end{aligned}$$

It follows that

$$\begin{aligned} |x - w|^2 &= |x|^2 + |w|^2 - 2\cdot\langle w, x\rangle \leqslant \\ &\leqslant \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 - \tfrac{1}{2}\cdot\langle x, y\rangle = \\ &= \tfrac{1}{4}\cdot|x - y|^2. \end{aligned}$$

That is, $|x - w| \leqslant \frac{1}{2}\cdot|x - y|$, and similarly $|y - w| \leqslant \frac{1}{2}\cdot|x - y|$. Therefore w is a midpoint of x and y. In addition we get equality

$$|w|^2 = \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle.$$

It remains to show that $w \in \mathrm{Lin}_p$. Let w^* be the polar vector provided by 7.3. Note that

$$|w^*| \leqslant |w|, \quad \langle w^*, x\rangle + \langle w, x\rangle = 0, \quad \langle w^*, y\rangle + \langle w, y\rangle = 0.$$

The same calculation as above shows that w^* is a midpoint of $-x$ and $-y$ and

$$|w^*|^2 = \tfrac{1}{4}\cdot|x|^2 + \tfrac{1}{4}\cdot|y|^2 + \tfrac{1}{2}\cdot\langle x, y\rangle = |w|^2.$$

By 7.9, $w = -w^*$; hence $w \in \mathrm{Lin}_p$. □

7.11. Exercise. *Let p be a point in an Alexandrov space $\mathcal{L}$ and $f = \mathrm{dist}_p$. Denote by S the subset of points $x \in \mathcal{L}$ such that $|\nabla_x f| = 1$.*

(a) *Show that S is a dense G-delta set.*
(b) *Show that*
$$\nabla_x f + \uparrow_{[xp]} = 0$$
for any $x \in S$; in particular, $\uparrow_{[xp]} \in \mathrm{Lin}_x$.
(c) *Show that if $|\nabla_x f| = 1$, then $\boldsymbol{d}_x f(w) = \langle \nabla_x f, w\rangle$ for any $w \in \mathrm{T}_x$.*

Note that 7.11*b* implies the following.

7.12. Corollary. *Given a countable set of points X in an Alexandrov space $\mathcal{L}$ there is a G-delta dense set $S \subset \mathcal{L}$ such that $\uparrow_{[sx]} \in \mathrm{Lin}_s$ for any $s \in S$ and $x \in X$.*

C Linear dimension

Suppose $\mathcal{L}$ is an Alexandrov space. Let us define its linear dimension $\mathrm{LinDim}\,\mathcal{L}$ as the least upper bound on integers m such that the Euclidean space $\mathbb{E}^m$ is isometric to a subspace of the tangent space $\mathrm{T}_p\mathcal{L}$ at some point $p \in \mathcal{L}$. If not stated otherwise, dimension of an Alexandrov space is its linear dimension.

7.13. $(\boldsymbol{n}{+}\mathbf{1})$-comparison. *Let $\mathcal{L}$ be an* $\mathrm{A}\textsc{lex}(0)$ *space. Then for any finite set of points $p, x_1, \dots, x_n \in \mathcal{L}$, there is a model configuration $\tilde p, \tilde x_1, \dots, \tilde x_n \in \mathbb{E}^m$ such that*
$$|\tilde p - \tilde x_i|_{\mathbb{E}^m} = |p - x_i|_{\mathcal{L}} \quad \textit{and} \quad |\tilde x_i - \tilde x_j|_{\mathbb{E}^m} \geqslant |x_i - x_j|_{\mathcal{L}}$$
for any i and j. Moreover, we can assume that $m \leqslant \mathrm{LinDim}\,\mathcal{L}$.

Proof. By 7.12, we can choose a point p' arbitrarily close to p so that $\mathrm{Lin}_{p'} \ni \uparrow_{[p'x_i]}$ for any i. Let us identify $\mathbb{E}^m$ with a subspace of $\mathrm{Lin}_{p'}$ spanned by $\uparrow_{[p'x_1]}, \dots, \uparrow_{[p'x_n]}$. Note that $m \leqslant \mathrm{LinDim}\,\mathcal{L}$.

Set $\tilde p' = 0 \in \mathbb{E}^m$ and $\tilde x_i = |p' - x_n| \cdot \uparrow_{[p'x_n]} \in \mathbb{E}^m$ for every i. Note that
$$|\tilde p' - \tilde x_i|_{\mathbb{E}^m} = |p' - x_i|_{\mathcal{L}}$$
for every i. Applying the comparison $\measuredangle[p'\,{}^{x_i}_{x_j}] \geqslant \tilde\measuredangle(p'\,{}^{x_i}_{x_j})$, we get
$$|\tilde x_i - \tilde x_j|_{\mathbb{E}^m} \geqslant |x_i - x_j|_{\mathcal{L}}$$
for any i and j. Passing to a limit configuration as $p' \to p$ we get the result. □

7.14. Exercise. *Let $\mathcal{L}$ be an* $\mathrm{A}\textsc{lex}(0)$ *space. Suppose* $\mathrm{LinDim}\,\mathcal{L} =$ $= m < \infty$. *Show that $\mathrm{T}_p\mathcal{L} \overset{iso}{=\!=} \mathbb{E}^m$ for a G-delta dense set of points $p \in \mathcal{L}$.*

7.15. Exercise. *Show that a 1-dimensional Alexandrov space is homeomorphic to a 1-dimensional manifold, possibly with nonempty boundary.*

7.16. Exercise. *Let $\mathcal{L}$ be an* ALEX(0) *space.*

Show that $\operatorname{LinDim}\mathcal{L} \geqslant m$ *if and only if for some $m+2$ points p, $x_0, \dots, x_m \in \mathcal{L}$ we have*

$$\tilde{\measuredangle}(p\,{}^{x_i}_{x_j}) > \tfrac{\pi}{2}$$

for any pair $i \neq j$.[1]

D Space of directions

A metric space $\mathcal{X}$ will be called ℓ-geodesic if any two points $x, y \in \mathcal{X}$ such that $|x-y| < \ell$ can be connected by a geodesic. Note that geodesic spaces are ∞-geodesic.

7.17. Theorem. *Let $\mathcal{L}$ be a finite-dimensional Alexandrov space. Then for any point $p \in \mathcal{L}$, its space of directions Σ_p is a compact π-geodesic space with curvature $\geqslant 1$.*

7.18. Exercise. *Prove the following in the assumptions of 7.17.*

(a) The tangent space T_p is a proper geodesic space.

(b) $\dim \Sigma_p = \dim \mathcal{L} - 1$.

(c) If $\dim \mathcal{L} > 1$, *then Σ_p is geodesic.*

Using 7.18*b*, one can prove results for all finite-dimensiona Alexandrov spaces via induction on its dimension. Such proofs will be indicated below.

Proof. Note that 4.3 and **??** imply that Σ_p has curvature $\geqslant 1$.

Compactness. Choose $\varepsilon > 0$; suppose $\mathcal{L}$ is m-dimensional. Assume can choose n directions $\xi_1, \dots, \xi_n \in \Sigma_p$ such that $\measuredangle(\xi_i, \xi_j) > \varepsilon$ for any $i \neq j$. Without loss of generality, we may assume that each direction is geodesic; that is, there is a point $x_i \in \mathcal{L}$ such that $\xi_i = \uparrow_{[px_i]}$.

Choose $y_i \in [px_i]$ such that $|p - y_i| = r$ for each i and small fixed $r > 0$. Since r is small, we can assume that $\tilde{\measuredangle}(p\,{}^{y_i}_{y_j}) > \varepsilon$ for any $i \neq j$. By 7.12, we can choose p' arbitrarily close to p such that $\uparrow_{[p'y_i]} \in \operatorname{Lin} p'$ for any i. Since $|p'-p|$ is small, $\tilde{\measuredangle}(p'\,{}^{y_i}_{y_j}) > \varepsilon$ for any $i \neq j$. By comparison,

$$\measuredangle[p'\,{}^{y_i}_{y_j}] > \varepsilon.$$

[1] *If $m = \operatorname{LinDim}\mathcal{L}$ then the map $q \mapsto (|x_1 - q|, \dots, |x_m - q|)$ induces a bi-Lipschitz embedding of a neighborhood of p to $\mathbb{E}^m$.* (We mention it without proof, altho it is not hard to prove.)

Therefore $n \leqslant \operatorname{pack}_\varepsilon \mathbb{S}^{m-1}$.

Since $\mathbb{S}^{m-1}$ is compact, $\operatorname{pack}_\varepsilon \mathbb{S}^{m-1} < \infty$. By the definition, the space of directions Σ_p is complete. Applying 8.8, we get that Σ_p is compact.

Geodesicness. Now we will show that *if Σ_p is compact, then it is π-geodesic*; we will not use the finiteness of dimension directly.

Choose two geodesic directions $\xi = \uparrow_{[px]}$ and $\zeta = \uparrow_{[py]}$; let

$$\alpha = \tfrac{1}{2}\cdot\measuredangle[p\,{}^x_y] = \tfrac{1}{2}\cdot|\xi - \zeta|_{\Sigma_p}.$$

Suppose $\alpha < \pi$. Let us show that it is sufficient to construct an almost midpoint $\mu = \uparrow_{[pz]}$ of ξ and ζ in Σ_p; that is, we need to show that for any $\varepsilon > 0$ there is a geodesic $[pz]$ such that

$$\measuredangle[p\,{}^x_z] \leqslant \alpha + \varepsilon \quad \text{and} \quad \measuredangle[p\,{}^y_z] \leqslant \alpha + \varepsilon.$$

Indeed, once it is done, the compactness of Σ_p can be used to get an actual midpoint for any two directions in Σ_p. After that Menger's lemma (1.4) will finish the proof.

Choose a sequence of small positive numbers $r_n \to 0$ Consider sequnces $x_n \in [px]$ and $y_n \in [py]$ such that $|p - x_n| = |p - y_n| = r_n$. Let m_n be a midpoint of $[x_n\, y_n]$.

Since Σ_p is compact, we can pass to a sequence of r_n such that $\uparrow_{[pm_n]}$ converges; denote its limit by μ. Choose a geodesic $[pz]$ that runs at small angle from μ. Let us show that $\uparrow_{[pz]}$ is the needed almost midpoint.

Evidently, $\tilde\measuredangle(p\,{}^{x_n}_{m_n}) = \tilde\measuredangle(p\,{}^{y_n}_{m_n})$. By 2.5, we have

$$\tilde\measuredangle(p\,{}^{x_n}_{m_n}) + \tilde\measuredangle(p\,{}^{y_n}_{m_n}) \leqslant \tilde\measuredangle(p\,{}^{x_n}_{y_n}).$$

Let $z_n \in [pz]$ be the point such that $|p - z_n| = |p - m_n|$. By construction, for all large n, we have $\measuredangle[p\,{}^{z}_{m_n}] \approx 0$ with arbitrary small given error. By comparison, the value $\frac{|z_n - m_n|}{|p - z_n|}$ can be assumed to be arbitrary small for all large n. Applying this observation and the definition of angle measure, we also have the following approximations

$$\begin{aligned} \tilde\measuredangle(p\,{}^{x_n}_{y_n}) &\approx \measuredangle[p\,{}^{x_n}_{y_n}], \\ \tilde\measuredangle(p\,{}^{x_n}_{m_n}) \approx \tilde\measuredangle(p\,{}^{x_n}_{z_n}) &\approx \measuredangle[p\,{}^{x_n}_{z_n}], \\ \tilde\measuredangle(p\,{}^{m_n}_{y_n}) \approx \tilde\measuredangle(p\,{}^{z_n}_{y_n}) &\approx \measuredangle[p\,{}^{z_n}_{y_n}], \end{aligned}$$

again, with arbitrary given error and all large n. It follows that $\uparrow_{[pz]}$ is an almost midpoint of $\uparrow_{[px]}$ and $\uparrow_{[py]}$, as required. ☐

Notice that the comparison gives lower bounds on $\measuredangle[p\,{}^x_z]$ and $\measuredangle[p\,{}^y_z]$, but in the proof we needed upper bounds which were obtained by from the definition of angle measure and compactness of space of directions.

E Comments

Corollary 7.12 is the key to the proof that *all reasonable definitions of dimension give the same result for Alexandrov spaces.* More precisely, we have the following theorem [3, 15.16].

7.19. Theorem. *For any Alexandrov space $\mathcal{L}$, we have*

$$\operatorname{LinDim}\mathcal{L} = \operatorname{TopDim}\mathcal{L} = \operatorname{HausDim}\mathcal{L},$$

where TopDim *and* HausDim *stands for Lebesgue covering dimension and Hausdorff dimension, respectively.*

7.20. Open question. *Let p be a point in an Alexandrov space $\mathcal{L}$. Suppose that $0 \neq v \in \operatorname{Lin}_p$. Is it true that the tangent space T_p splits in the direction of v?*

Halbeisen's example [3, 23] shows that compactness of space of directions is essential in the proof that space of directions is π-geodesic (see 7.17).

7.21. Open question. *Let $\mathcal{L}$ be a proper Alexandrov space. Is it true that for any $p \in \mathcal{L}$, the tangent space T_p is a length space?*

Lecture 8

Limit spaces

A Survival of curvature bounds

8.1. Theorem. *Let $\mathcal{X}_n \to \mathcal{X}_\infty$ be a convergence in the sense of Gromov–Hausdorff. Suppose that each for each n, the space $\mathcal{X}_n$ has curvature $\geqslant \kappa$ in the sense of Alxanrov. Then the same holds for $\mathcal{X}_\infty$.*

Proof. Choose a quadruple of points $p_\infty, x_\infty, y_\infty, z_\infty \in \mathcal{X}_\infty$.

By the definition of Gromov–Hausdorff convergence, we can choose a quadruple p_n, x_n, y_n, $z_n \in \mathcal{X}_n$ such that each of 6 distances between pairs of p_n, x_n, y_n, z_n converge to the distance between the corresponding pairs of $p_\infty, x_\infty, y_\infty, z_\infty$.

By the assumption, $\mathbb{M}^2(\kappa)$-comparison holds for each p_n, x_n, y_n, z_n. Passing to the limit, we get the $\mathbb{M}^2(\kappa)$-comparison for p_∞, x_∞, y_∞, z_∞. $\square$

8.2. Exercise. *Suppose that a sequence $\mathcal{L}_1, \mathcal{L}_2, \dots$ of* $\mathrm{A\scriptstyle LEX}(\kappa)$ *spaces that converges to $\mathcal{L}_\infty$ in the sense of Gromov–Hausdorff. Show that $\mathcal{L}_\infty$ is* $\mathrm{A\scriptstyle LEX}(\kappa)$ *and*

$$\operatorname{LinDim} \mathcal{L}_\infty \leqslant \varliminf_{n\to\infty} \operatorname{LinDim} \mathcal{L}_n.$$

B Volume

Fix a positive integer m. The m-dimensional Hausdorff measure of a Borel set B in a metric space will be called its m-volume; it will be denoted by $\operatorname{vol}_m B$. We assume that the Hausdorff measure is calibrated so that the unit cube in $\mathbb{E}^m$ has unit volume.

This definition will be used mostly in m-dimensional Alexandrov spaces. In this case, we may write $\operatorname{vol} B$ instead of $\operatorname{vol}_m B$.

8.3. Bishop–Gromov inequality. *Let $\mathcal{L}$ be an* ALEX(0) *space. Suppose $\mathcal{L} = m < \infty$. Then*

$$\operatorname{vol} \mathrm{B}(p, R) \leqslant \omega_m \cdot R^m,$$

where ω_m denotes the volume of the unit ball in $\mathbb{E}^m$. Moreover, the function

$$R \mapsto \frac{\operatorname{vol} B(p, R)}{R^m}$$

is nonincreasing.

Proof. Given $x \in \mathcal{L}$ choose a geodesic path γ_x from p to x. Let $\log_p \colon \mathcal{L} \to \mathrm{T}_p$ be defined by $\log_p \colon x \mapsto \gamma_x^+(0)$. By comparison, $\log_p$ is distance-noncontracting. Note that $\log_p$ maps $\mathrm{B}(p, R)_{\mathcal{L}}$ to $\mathrm{B}(0, R)_{\mathrm{T}_p}$.

If $\mathrm{T}_p \stackrel{iso}{=\!=} \mathbb{E}^m$, then $\operatorname{vol} \mathrm{B}(0, R)_{\mathrm{T}_p} = \omega_m \cdot R^m$, and the first statement follows. Otherwise, by 7.14, we can find a point p' arbitrarily close to p such that $\mathrm{T}_{p'} \stackrel{iso}{=\!=} \mathbb{E}^m$. If $\varepsilon > |p - p'|$, then $\mathrm{B}(p, R) \subset \mathrm{B}(p', R + \varepsilon)$. Therefore,

$$\operatorname{vol} \mathrm{B}(p, R) \leqslant \omega_m \cdot (R + \varepsilon)^m$$

for any $\varepsilon > 0$. Hence the first statement follows in the general case.

For the second statement, choose $0 < R_1 < R_2$. Consider the map $w \colon \mathcal{L} \to \mathcal{L}$ defined by $w \colon x \mapsto \gamma_x(\tfrac{R_1}{R_2})$. (The map w mimics the dilation with center at p and coefficient $\tfrac{R_1}{R_2}$.) By comparison,

$$|w(x) - w(y)| \geqslant \tfrac{R_1}{R_2} \cdot |x - y|.$$

Since $\mathrm{B}(p, R_1) \supset w[\mathrm{B}(p, R_2)]$, we get

$$\operatorname{vol} \mathrm{B}(p, R_1) \geqslant (\tfrac{R_1}{R_2})^m \cdot \operatorname{vol} \mathrm{B}(p, R_2)$$

□

8.4. Exercise. *Show that any finite-dimensional Alexandrov space is proper.*

The following exercise generalizes the theorem to ALEX(−1) case. This statement is sufficient for most applications, but a more exact statement is given in 8.10.

8.5. Exersice. *Let $\mathcal{L}$ be an* ALEX(−1) *space. Suppose $\mathcal{L} = m < \infty$. Show that*

$$\operatorname{vol} \mathrm{B}(p, R) \leqslant (\sinh R)^m,$$

for a fixed constant c. Moreover, the function

$$R \mapsto \frac{\operatorname{vol} B(p,R)}{(\sinh R)^m}$$

for a fixed positive function

C Gromov's selection theorem

8.6. Theorem. *Let $D, \kappa \in \mathbb{R}$ and m be a positive integer. Then any sequence of m-dimensional $\mathrm{ALEX}(\kappa)$ spaces with diameters at most D has a converging subsequence in the sense of Gromov–Hausdorff.*

Let X be a subset of a metric space $\mathcal{W}$. Recall that a set $Z \subset \mathcal{W}$ is called ε-net of X if for any point $x \in X$, there is a point $z \in Z$ such that $|x - z| < \varepsilon$.

We will use the following characterization of compact sets: *a closed subset X of a complete metric space is compact if and only if X admits a finite ε-net for any $\varepsilon > 0$.* The following statement is slightly more general.

8.7. Claim. *A closed subset X of a complete metric space is compact if and only if it admits a compact ε-net for any $\varepsilon > 0$.*

Proof. Let Z be a compact ε-net of X. Since Z is compact, it admits a finite ε-net F. Note that F is a $2{\cdot}\varepsilon$-net of X. Since $\varepsilon > 0$ is arbitrary, we get the result. □

Let $\operatorname{pack}_\varepsilon \mathcal{X}$ be the exact upper bound on the number of points $x_1, \dots, x_n \in \mathcal{X}$ such that $|x_i - x_j| \geqslant \varepsilon$ if $i \neq j$.

If $n = \operatorname{pack}_\varepsilon \mathcal{X} < \infty$, then the collection of points $x_1, \dots, x_n$ is called a maximal ε-packing.

8.8. Exercise. *Show that any maximal ε-packing $x_1, \dots, x_n$ is an ε-net. Conclude that a complete metric space $\mathcal{X}$ is compact if and only if $\operatorname{pack}_\varepsilon \mathcal{X} < \infty$ for any $\varepsilon > 0$.*

Proof of 8.6. Denote by $\boldsymbol{K}$ the set of isometry classes of $\mathrm{ALEX}(0)$ spaces with dimension $\leqslant m$ and diameter $\leqslant D$. By 8.2, $\boldsymbol{K}$ is a closed subset of GH.

Choose a space $\mathcal{L} \in \boldsymbol{K}$; suppose $x_1, \dots, x_n \in \mathcal{L}$ is a collection of points such that $|x_i - x_j| \geqslant \varepsilon$ for all $i \neq j$. Note that the balls $B_i = \mathrm{B}(x_i, \frac{\varepsilon}{2})$ do not overlap.

By Bishop–Gromov inequality, we get

$$\operatorname{vol} B_i \geqslant \left(\tfrac{\varepsilon}{2\cdot D}\right)^m \cdot \operatorname{vol} \mathcal{L}$$

for any i and any small $\varepsilon > 0$. It follows that $n \leqslant (\frac{2\cdot D}{\varepsilon})^m$; that is,

$$\operatorname{pack}_\varepsilon \mathcal{L} \leqslant N(\varepsilon) := (\tfrac{2\cdot D}{\varepsilon})^m$$

for all small $\varepsilon > 0$.

Choose a maximal ε-packing $x_1, \dots, x_n \in \mathcal{L}$. By 8.8, $\mathcal{F}_\varepsilon := \{x_1, \dots \dots, x_n\}$ is an ε-net of $\mathcal{L}$. Observe that $|\mathcal{F}_\varepsilon - \mathcal{L}|_{\mathrm{GH}} \leqslant \varepsilon$. Further, note that the set $\boldsymbol{F}_\varepsilon$ of finite metric spaces with diameter $\leqslant D$ and at most $N(\varepsilon)$ points forms a compact subset in GH.

Summarizing, for any $\varepsilon > 0$ we can find a compact ε-net $\boldsymbol{F}_\varepsilon \subset \mathrm{GH}$ of $\boldsymbol{K}$. It remains to apply 1.21 and 8.7.

The $\mathrm{A{\scriptsize LEX}}(\kappa)$ case follows from the $\mathrm{A{\scriptsize LEX}}(-1)$ case. It can be proved applying 8.5 instead of 8.3. $\square$

8.9. Exercise. *Let $\mathcal{L}$ be an* ALEX(0) *space with dimension m and diameter $\leqslant D$. Suppose* $\operatorname{vol}\mathcal{L} \geqslant v_0 > 0$. *Show that*

$$\operatorname{pack}_\varepsilon \mathcal{L} \geqslant \frac{c}{\varepsilon^m}$$

for some constant $c = c(m, D, v_0) > 0$.

Conclude that if $\mathcal{L}_n$ is a sequence of m-dimensional ALEX(0) *spaces with diameter $\leqslant D$, and volume $\geqslant v_0$, then its Gromov–Hausdorff limit $\mathcal{L}_\infty$ (if it is defined) has dimension m.*

D Comments

Let us state a version of Bishop–Gromov inequality for ALEX(κ) spaces. The proof requires in addition the so-called *coarea formula* for Alexandrov spaces. A weaker inequality 8.5 is sufficient for the sequel.

8.10. Bishop–Gromov inequality. *Let p be a point in an m-dimensional* ALEX(κ) *space. Consider the function $v(R) = \operatorname{vol}_m \mathrm{B}(p, R)$; denote by $\tilde{v}(R)$ the volume of R ball in $\mathbb{M}(\kappa)$. Then*

$$v(R) \leqslant \tilde{v}(R)$$

for any $R > 0$ and the function

$$R \mapsto \frac{v(R)}{\tilde{v}(R)}$$

is nonincreasing; if $\kappa > 0$, then one has to assume that $R < \frac{\pi}{\sqrt{\kappa}}$.

The same inequality holds for complete m-dimensional Riemannian manifolds with Ricci curvature at least $(m-1)\cdot\kappa$.

Gromov's selection theorem is the main source of applications of Alexandrov spaces. The homotopy-type finiteness theorem (9.5) in the next lecture illustrates this technique.

A version Gromov's selection theorem (as well as Bishop–Gromov inequality) holds for m-dimensional Riemannian manifolds with a lower bound on Ricci curvature. It motivates the study of the so-called $\mathrm{CD}(K,m)$ spaces; CD stands for curvature-dimension condition. This theory has serious applications in Alexandrov geometry; in particular, it provides a version of Liouville theorem about phase-space volume of geodesic flow in Alexandrov space [9].

Lecture 9

Homotopy finiteness theorem

A Controlled concavity

While Alexandrov spaces have plenty of semiconcave functions (for instance, square of distance function), it is not at all easy to construct a strictly concave one.

9.1. Theorem. *Let $\mathcal{L}$ be a complete finite-dimensional Alexandrov space. Then for any point $p \in \mathcal{L}$, there is a strictly concave function f defined in an open neighborhood of p.*

Moreover, given $0 \neq v \in T_p$, the differential, $\boldsymbol{d}_p f$, can be chosen arbitrarily close to $x \mapsto -\langle v, x\rangle$.

Proof. Fix small $r > 0$ and large c; consider the real-to-real function

$$\varphi_{r,c}(x) = (x - r) - c(x - r)^2/r,$$

so we have $\varphi_{r,c}(r) = 0$, $\varphi'_{r,c}(r) = 1$, and $\varphi''_{r,c}(r) = -2c/r$.

Let γ be a unit-speed geodesic, fix a point q and let

$$\alpha(t) = \measuredangle(\gamma^+(t), \uparrow_{[\gamma(t)q]}).$$

If $|q - \gamma(t)|$ is sufficiently close to r, then direct calculations show that

$$(\varphi_{r,c} \circ \operatorname{dist}_q \circ \gamma)''(t) \leqslant \frac{3 - c\cdot \cos^2[\alpha(t)]}{r}.$$

(Since c is large, this inequality implies that $\varphi_{r,c} \circ \operatorname{dist}_q \circ \gamma$ is strictly concave at t unless γ runs nearly perpendicular to the direction to q.)

Now, assume $\{q_1, \dots, q_N\}$ is a finite set of points such that $|p - q_i| = r$ for any i. For a geodesic γ, set $\alpha_i(t) = \measuredangle(\gamma^+(t), \uparrow_{[\gamma(t)q_i]})$. Assume we have a collection $\{q_i\}$ such that for any geodesic γ in $\mathrm{B}(p,\varepsilon)$ we have $\max_i\{|\alpha_i(t) - \frac{\pi}{2}|\} \geqslant \varepsilon > 0$. We can assume that $c > 3N/\cos^2\varepsilon$; then the inequality above implies that the function

$$f = \sum_i \varphi_{r,c} \circ \mathrm{dist}_{q_i}$$

is strictly concave in $\mathrm{B}(p,\varepsilon')$ for some positive $\varepsilon' < \varepsilon$.

To construct the needed collection $\{q_i\}$, note that for small $r > 0$, one can choose $N \geqslant c/\delta^{(m-1)}$ points $\{q_i\}$ such that $|p - q_i| = r$ and $\tilde\measuredangle(p\,{}^{q_j}_{q_i}) > \delta$ (here $c = c(\Sigma_p) > 0$). On the other hand, suppose γ runs from x to y. If $|\alpha_i(t) - \frac{\pi}{2}| < \varepsilon \ll \delta$, then applying the $(n{+}1)$-point comparison to $\gamma(t)$, x, y and all $\{q_i\}$ we get that $N \leqslant c(m)/\delta^{(m-2)}$. Therefore, for small $\delta > 0$ and yet smaller $\varepsilon > 0$, the set $\{q_i\}$ forms the needed collection.

If r is small, then points q_i can be chosen so that all directions $\uparrow_{[pq_i]}$ will be ε-close to a given direction ξ and therefore the second property follows. □

Note that in 9.1 the function f can be chosen to have maximum value 0 at p, $f(p) = 0$ and with $\boldsymbol{d}_p f(x) \approx -|x|$. It can be constructed by taking the minimum of the functions in the theorem. Then the set $\Omega = \{\, x \in \mathcal{L} : f(x) \geqslant -\varepsilon \,\}$ forms an open convex neighborhood of p for any small $\varepsilon > 0$, so we get the following.

9.2. Corollary. *Any point p of a finite-dimensional Alexandrov space admits an arbitrary small convex neighborhood Ω and a strictly concave function f defined in a neighborhood of the closure $\bar\Omega$ such that p is the maximum point of f and $f|_{\partial\Omega} = 0$.*

B Liftings

Suppose that the Gromov–Haudorff distance $|\mathcal{L} - \mathcal{L}'|_{\mathrm{GH}}$ is sufficienlty small, so we may think that both spaces $\mathcal{L}$ and $\mathcal{L}'$ lie on small Haudorff distance in an ambient metric space $\mathcal{W}$. In particular, we may choose small $\varepsilon > 0$, so that for any point $p \in \mathcal{L}$, we can find a point $p' \in \mathcal{L}'$ such that $|p - p'|_{\mathcal{W}} < \varepsilon$; the point p' will be called an lifting (or ε-lifting) of p in $\mathcal{L}'$. We may choose a lifting $p' \in \mathcal{L}'$ for every point $p \in \mathcal{L}$, in this case the map $p \mapsto p'$ is called a (ε-)lifting map.

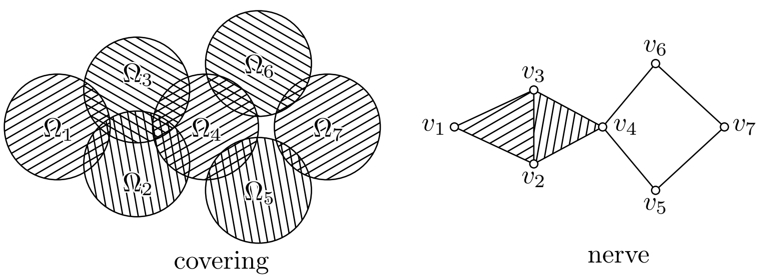
Note that the lifting is not uniquely defined. The lifting maps is not assumed to be continuous. When we talk about liftings, we assume that $\varepsilon > 0$, the inclusions $\mathcal{L}, \mathcal{L}' \hookrightarrow \mathcal{W}$, as well as $\mathcal{W}$ are chosen.

Choose a compact m-dimensional Alexandrov space $\mathcal{L}$. Suppose $\mathcal{L}'$ is another compact m-dimensional Alexandrov space such that $|\mathcal{L} - \mathcal{L}'|_{\mathrm{GH}}$ is sufficiently small — smaller than some $\varepsilon = \varepsilon(\mathcal{L}) > 0$. Then the construction in $\mathcal{L}$ from the previous section can be repeated in $\mathcal{L}'$ for the liftings of all points and the same function φ. It produces a strictly concave function defined in a controlled neighborhood of the lifting p' of p.

The result of this and related constructions will be called liftings, say we can talk about a lifting from $\mathcal{L}$ to $\mathcal{L}'$ of a function provided by 9.1 (if the Gromov–Hausdorff distance $|\mathcal{L} - \mathcal{L}'|_{\mathrm{GH}}$ is small, then these liftings are stricly concave) and a lifting of a convex neighborhood from 9.2. Here one cannot use 9.1 and 9.2 as black boxes — one has to understand the construction, but it is straightforward.

C Nerves

Let $\{\Omega_1, \dots, \Omega_k\}$ be a finite open cover of a compact metric space $\mathcal{X}$. Consider an abstract simplicial complex $\mathcal{N}$, with one vertex v_i for each set Ω_i such that a simplex with vertices $v_{i_1}, \dots, v_{i_m}$ is included in $\mathcal{N}$ if the intersection $\Omega_{i_1} \cap \dots \cap \Omega_{i_m}$ is nonempty. The obtained simplicial



complex $\mathcal{N}$ is called the nerve of the covering $\{\Omega_i\}$. Evidently $\mathcal{N}$ is a finite simplicial complex — it is a subcomplex of a simplex with the vertices $\{v_1, \dots, v_k\}$. Recall that Star_{v_i} denotes the union of all simplexes in $\mathcal{N}$ that shares vertex v_i.

The next statement follows from [25, 4G.3].

9.3. Nerve theorem. *Let $\{\Omega_1, \dots, \Omega_k\}$ be an open cover of a compact metric space $\mathcal{X}$ and let $\mathcal{N}$ be the corresponging nerve with vertices $\{v_1, \dots, v_k\}$. Suppose that every nonempty finite intersection $\Omega_{\alpha_1} \cap \dots \cap \Omega_{\alpha_k}$ is contractible. Then $\mathcal{X}$ is homotopy equivalent to the nerve $\mathcal{N}$ of the cover.*

Moreover homotopy equivalences $a\colon \mathcal{X} \to \mathcal{N}$ and $b\colon \mathcal{N} \to \mathcal{X}$ can be chosen so that if $x \in \Omega_i$, then $a(x) \in \mathrm{Star}_{v_i}$, and if $y \in \mathcal{N}$ lies in the *simplex with vertices $v_{i_1},\dots,v_{i_m}$, then $b(y)\in\Omega_{i_1}\cup\dots\cup\Omega_{i_m}$.*

D Homotopy stability

9.4. Theorem. *Let $\mathcal{L}_1,\mathcal{L}_2,\dots,\mathcal{L}_\infty$ be a sequence of m-dimensional* Alex(κ) *spaces, and $m<\infty$. Suppose $\mathcal{L}_n\xrightarrow{\text{GH}}\mathcal{L}_\infty$ as $n\to\infty$. Then there is a $\mathcal{L}_\infty$ is homotopically equivalent to $\mathcal{L}_n$ for all large n.*

Moreover, given $\varepsilon>0$ there are maps $h_n\colon\mathcal{L}_\infty\to\mathcal{L}_n$ that are homotopy equivalences and ε-liftings for all large n.

Applying this theorem with the Gromov's selection theorem (8.6) and Exercise 8.9, we get the following.

9.5. Theorem. *There are only finitely many homotopy types of m-dimensional* Alex(κ) *spaces with diameter $\leqslant D$, and volume $\geqslant v_0$; here we assume that an integer m, and $v_0,D\in\mathbb{R}$ such that $v_0>0$ are given.*

Proof of 9.5 modulo 9.4. Assume the contrary, then we can choose a sequence of spaces $\mathcal{L}_1,\mathcal{L}_2,\dots$ that have different homotopy types and satisfy the assumptions of the theorem. By Gromov's compactness theorem, we can assume that $\mathcal{L}_n$ converges to say $\mathcal{L}_\infty$ in the sense of Gromov–Hausdorff.

By 8.9, $\operatorname{LinDim}\mathcal{L}_\infty=m$. It remains to apply 9.4. □

Proof of 9.4. Since $\mathcal{L}_\infty$ is compact, applying 9.2, we can find a finite open cover of $\mathcal{L}_\infty$ by convex open sets $\Omega_1,\dots,\Omega_k$ such that for each Ω_i there is a strictly concave function f_i that is defined in a neighborhood of $\bar\Omega_i$ and such that $f_i|_{\partial\Omega_i}=0$.

Subtracting from functions f_i some small value $\varepsilon>0$, we can ensure that $\bigcap_{i\in S}\Omega_i\ne\varnothing$ if and only if $\bigcap_{i\in S}\bar\Omega_i\ne\varnothing$.

Suppose that $W=\bigcap_{i\in S}\Omega_i\ne\varnothing$. Then W is contractible. Indeed the function

$$f_S:=\min_{i\in S}f_i$$

is strictly concave and it vanished on the boundary of W. The f_S-gradient flow $(t,x)\mapsto\mathrm{Flow}^t_{f_S}(x)$ defines a homotopy $[0,\infty)\times W\to W$. Note that $\mathrm{Flow}^t_{f_S}(x)$ converges to the (necessarily unique) maximum point of f_S as $t\to\infty$. Therefore, in the obtained homotoly we can parametrize $[0,\infty)$ by $[0,1)$ and extend the homotopy by continiously to $[0,1]$; this way we get that W is contractible. In other words, the cover $\{\Omega_1,\dots,\Omega_k\}$ meets the assumptions of the nerve theorem. Therefore $\mathcal{L}_\infty$ is homotopy equivalent to the nerve $\mathcal{N}$ of the cover.

The functions f_i and sets Ω_i can be lifted to $\mathcal{L}_n$ keeping its properties for all large n. More precisely, there are liftings $f_{i,n}$ of all f_i to $\mathcal{L}_n$ which are strictly concave for all large n and such that $\bar{\Omega}_{i,n} = \{\, x \in \mathcal{L}_n \,:\, f_{i,n}(x) \geqslant 0 \,\}$ is a compact convex set and $\Omega_{i,n} = \{\, x \in \mathcal{L}_n \,:\, f_{i,n}(x) > 0 \,\}$ is an open convex set for each i.

Notice that $\{\Omega_{1,n}, \dots, \Omega_{k,n}\}$ is an open cover of $\mathcal{L}_n$ for all large n. Indeed suppose we have $p_n \in \mathcal{L}_n \setminus (\Omega_{1,n} \cup \dots \cup \Omega_{k,n})$ for arbitrary large n. Since $\mathcal{L}_\infty$ is compact, there is a limit point $p_\infty \in \mathcal{L}_\infty$ for a subsequnce of p_n. But $p_\infty \in \Omega_i$ for some i and therefore $p_n \in \Omega_{i,n}$ for arbitrary large n — a contradiction.

In a similar fashion, we can show that if n is large, then any collection $\{\Omega_{i,n}\}_{i \in S}$ has a common point in $\mathcal{L}_n$ if and only if $\{\Omega_i\}_{i \in S}$ has a common point in $\mathcal{L}_\infty$. Here we have to use that $\bigcap_{i \in S} \Omega_i \neq \varnothing$ if and only if $\bigcap_{i \in S} \bar{\Omega}_i \neq \varnothing$.

It follows that for any large n the following two covers the same nerve

- $\{\Omega_1, \dots, \Omega_k\}$ of $\mathcal{L}_\infty$ and
- $\{\Omega_{1,n}, \dots, \Omega_{k,n}\}$ of $\mathcal{L}_n$.

Therefore, $\mathcal{L}_n$ is homotopy equivalent to $\mathcal{N}$ for all large n — a contradiction. □

E Comments

The construction of strictly concave function is due to Grigori Perelman [44, 47].

Let us list some results that can be proved by applying Gromov's selection theorem in the same fashion as in the proof of homotopy-type finiteness theorem (9.5). The following theorem can be proved using this technique, altho Gromov's original proof [19] did not use Alexandrov geometry directly.

9.6. Betti-number theorem. *There is a constant $c = c(m, D, \kappa)$ such that*

$$\beta_0(M) + \beta_1(M) + \dots + \beta_m(M) \leqslant c$$

for any closed m-dimensional Riemannian manifold M with sectional curvature $\geqslant \kappa$ and diameter $\leqslant D$. Here $\beta_i(M)$ denotes i^{th} Betti number of M.

The following result of the second author [53], and it uses the same technique.

9.7. Scalar curvature bound. *There is a constant $c = c(m, D, \kappa)$ such that*

$$\int_M \mathrm{Sc} \leqslant c$$

for any closed m-dimensional Riemannian manifold M with sectional curvature $\geqslant \kappa$ and diameter $\leqslant D$. Here Sc *denotes the scalar curvature.*

The following theorem is a more exact version of 9.4. It will play an important role in the following lecture.

9.8. Stability theorem. *Let $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\infty$ be a sequence of m-dimensional* ALEX(κ)*, and $m < \infty$. Suppose $\mathcal{L}_n \xrightarrow{\text{GH}} \mathcal{L}_\infty$ as $n \to \infty$. Then there is a $\mathcal{L}_\infty$ is homotopically equivalent to $\mathcal{L}_n$ for all large n.*

Moreover, given $\varepsilon > 0$ there are maps $h_n \colon \mathcal{L}_\infty \to \mathcal{L}_n$ that are homeomorphisms and ε-liftings for all large n.

This theorem was proved by Grigori Perelman [46]; the proof was rewritten with more details by the first author [28]. Around the same time, he made informal annoncemnt that the homeomorphisms in the theorem can be made bi-Lipschitz with constants that depend on $\mathcal{L}_\infty$; but the proof was not written and it save to consider it as a conjecture.

The last statement in the theorem implies the following finiteness result.

9.9. Homeorphism-type finiteness. *There are only finitely many homeomorphism types of closed m-dimensional manifolds that admit a Riemannian metric with sectional curvature $\geqslant \kappa$, and diameter $\leqslant D$.*

In fact, this theorem implies diffeomorphism-type finiteness in all dimensions except 4.

Lecture 10

Boundary

A Definition

Let us give an inductive definition the boundary for finite-dimensional Alexandrov spaces.

Suppose $\mathcal{L}$ is a 1-dimensional Alexandrov space. By Exercise 7.15, $\mathcal{L}$ is homeomorphic to a 1-dimensional manifold (possibly with non-empty boundary). It allows us to define the boundary $\partial\mathcal{L} \subset \mathcal{L}$ as the boundary of a manifold.

Now assume that the notion of boundary is already defined in dimensions $1, \dots, m-1$. Suppose $\mathcal{L}$ is m-dimensional Alexandrov space. We say that $p \in \mathcal{L}$ belongs to the boundary (briefly $p \in \partial\mathcal{L}$) if $\partial\Sigma_p \neq \varnothing$. By 7.17 and 7.18*b*, Σ_p is $(m-1)$-dimensional Alexandrov space; therefore its boundary is already defined.

10.1. Exercise. *Show that for closed convex set $K \subset \mathbb{E}^m$ with non-empty interior, the topological boundary of K as a subset of $\mathbb{E}^m$ coincides with the boundary K described above.*

10.2. Exercise. *Let $\mathcal{L}$ be a finite-dimensional Alexandrov space. Suppose $\mathcal{L} \stackrel{iso}{=\!=} \mathcal{L}_1 \times \mathcal{L}_2$. Show that*

$$\partial\mathcal{L} = (\partial\mathcal{L}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \partial\mathcal{L}_2).$$

B Conic neighborhoods

The following statement is close relative of Perelman's stability theorem 9.8, but its proof is slightly simpler [45]. We are going to use this result without proof.

Recall that logarithm $\log_p x\colon \mathcal{L}\to \mathrm{T}_p$ is defined on page 33.

10.3. Theorem. *Any point p in a finite-dimensional Alexandrov space $\mathcal{L}$ and all sufficiently small $\varepsilon>0$ there is a homeomorphism $h_\varepsilon\colon \mathrm{B}(p,\varepsilon)_{\mathcal{L}}\to \mathrm{B}(0,\varepsilon)_{\mathrm{T}_p}$ such that $0=h_\varepsilon(p)$.*

Moreover, we may assume that

$$\sup_{x\in \mathrm{B}(p,\varepsilon)}\left\{\tfrac{1}{\varepsilon}\cdot|\log_p x-h_\varepsilon(x)|_{\mathrm{T}_p}\right\}\to 0\quad \textit{as}\quad \varepsilon\to 0.$$

This statement is often used together with *uniqueness of conic neighborhood* stated below.

Suppose that a neighborhood U of a point x in a metric space $\mathcal{X}$ admits a homeomorphism to $\operatorname{Cone}\Sigma$ such that x maps to the origin of the cone. In this case, we say that U has a conic neighborhood of x.

10.4. Uniqueness of conic neighborhood. *Any two conic neighborhoods of a given point in a metric space are pointed homeomorphic; that is, there is a homeomorphism between neighborhoods that maps the origin of the cone to the origin.*

10.5. Advanced exercise. *Prove 10.4 or read the original proof by Kyung Whan Kwun [31].*

10.6. Exercise. *Suppose $x\mapsto x'$ is a homeomorphism between finite-dimensional Alexandrov spaces $\mathcal{L}$ and $\mathcal{L}'$. Show that*

(a) $\mathrm{T}_x\cong \mathrm{T}_{x'}$,

(b) $\operatorname{Susp}\Sigma_x\cong \operatorname{Susp}\Sigma_{x'}$.

(c) but in general $\Sigma_x\ncong \Sigma_{x'}$.

C Topology

The following theorem states that boundary is a topological property despite our definition used geometry.

10.7. Theorem. *Let $\mathcal{L}$ and $\mathcal{L}'$ be homeomorphic finite-dimensional Alexandrov spaces. Then $\dim\mathcal{L}=\dim\mathcal{L}'$ and*

$$\partial\mathcal{L}\ne\varnothing\quad\iff\quad\partial\mathcal{L}'\ne\varnothing$$

Before diving into the proof, it is instructive to solve the following exercise.

10.8. Exercise. *Construct two plane subsets K_1 and K_2 such that $K_1\ncong K_2$, but $\mathbb{R}\times K_1\cong\mathbb{R}\times K_2$.*

Let $\mathcal{L}$ be an m-dimensional Alexandrov space and $m < \infty$. Define rank of $\mathcal{L}$ (briefly, $\operatorname{rank}\mathcal{L}$) as the minimal value k such that $\mathcal{L} \overset{iso}{=\!=} \mathbb{R}^{m-k} \times \mathcal{K}$, where $\mathcal{K}$ is a k-dimensional Alexandrov space.

In the following proof we will apply induction on the rank of space.

Proof. The first statement follows from 7.19.

Suppose we have a counterexample, say $\partial\mathcal{L} \neq \varnothing$, but $\partial\mathcal{L}' = \varnothing$. Let $k := \operatorname{rank}\mathcal{L}$ and $k' := \operatorname{rank}\mathcal{L}'$. We can assume that the pair (k, k') is minimal in the lexicographic order; in particular, k is minimal. Let $x \mapsto x'$ be a homeomorphism from $\mathcal{L}$ to $\mathcal{L}'$.

Choose $x \in \partial\mathcal{L}$. Since $\partial\mathcal{L}' = \varnothing$, we have $x' \notin \partial\mathcal{L}'$. Note that

$$\operatorname{rank} \mathrm{T}_x \leqslant k \quad \text{and} \quad \operatorname{rank} \mathrm{T}_{x'} \leqslant k',$$

By 10.6a, $\mathrm{T}_x \cong \mathrm{T}_{x'}$. Note that $\partial \mathrm{T}_x \neq \varnothing$ and $\partial \mathrm{T}_{x'} = \varnothing$. Therefore, we may assume that $\mathcal{L}$ and $\mathcal{L}'$ are Euclidean cones and the homeomorphism sends the origin to the origin. The remaining part of the proof is divided in three cases.

Case 1. Suppose $k > 1$. Let $\mathcal{L} \overset{iso}{=\!=} \mathbb{R}^{m-k} \times \mathcal{C}$, where $\mathcal{C}$ a k-dimensional $\mathrm{ALEX}(0)$ cone. Note that $\operatorname{rank} \mathrm{T}_y \leqslant \operatorname{rank}\mathcal{L}$ for any $y \in \mathcal{L}$ and the equality holds only if y projects to the origin of $\mathcal{C}$.

Since $k > 1$ we can find $z \in \partial\mathcal{C}$ such that $z \neq 0$. Choose y that projects to z, so $\operatorname{rank} \mathrm{T}_y < \operatorname{rank}\mathcal{L}$. By 10.6$a$, $\mathrm{T}_y \cong \mathrm{T}_{y'}$, $\partial\mathrm{T}_y \neq \varnothing$ and $\partial\mathrm{T}_{y'} = \varnothing$. The latter contradicts contradicts minimality of k.

Case 2. Suppose $k \leqslant 1$ and $k' > 1$. Since $\partial\mathcal{L} \neq \varnothing$, we get that $k = 1$ and $\mathcal{L} = \mathbb{R}^{m-1} \times \mathbb{R}_{\geqslant 0}$.

Let $\mathcal{L}' \overset{iso}{=\!=} \mathbb{R}^{m-k'} \times \mathcal{C}'$, where $\mathcal{C}'$ a k'-dimensional $\mathrm{ALEX}(0)$ cone. Since $\partial\mathcal{L} \cong \mathbb{R}^{m-1}$, the image of $\partial\mathcal{L}$ in $\mathcal{L}'$ does not lie in $\mathbb{R}^{m-k'} \times \{0\}$. In other words, we can choose $y \in \partial\mathcal{L}$ such that its image $y' \in \mathcal{L}'$ has nonzero projection in $\mathcal{C}'$. Observe that $\mathrm{T}_y \cong \mathrm{T}_{y'}$,

$$\operatorname{rank} \mathrm{T}_y \leqslant k = 1, \quad \operatorname{rank} \mathrm{T}_{y'} < k', \quad \partial\mathrm{T}_y = \varnothing, \quad \text{and} \quad \partial\mathrm{T}_{y'} \neq \varnothing$$

— a contradiction.

Case 3. Suppose $k \leqslant 1$ and $k' \leqslant 1$. Since $\partial\mathcal{L} \neq \varnothing$, $k = 1$. By 7.15, $\mathcal{L} \cong \mathbb{R}_{\geqslant 0}$. Therefore, $\mathcal{L}' \cong \mathbb{R}$, and $\mathcal{L} \ncong \mathcal{L}'$ — a contradiction. $\square$

10.9. Exercise. *Let $x \mapsto x'$ be a homeomorphism $\Omega \to \Omega'$ between open subsets in finte-dimensional Alexandrov spaces $\mathcal{L}$ and $\mathcal{L}'$. Show that $x \in \partial\mathcal{L}$ if and only if $x' \in \partial\mathcal{L}'$.*

10.10. Exercise. *Show that boundary of a finite-dimensional Alexandrov space is a closed subset.*

D Tangent space

Let X be a subset in a finite-dimensional Alexandrov space $\mathcal{L}$. Choose $p \in \mathcal{L}$ and $\xi \in \Sigma_p$. Suppose ξ is a limit of directions $\uparrow_{[px_n]}$ for a sequence $x_1, x_2, \ldots \in X$ that converges to p. Then we say that ξ is in the space of directions from p to X; briefly $\xi \in \Sigma_p X$.

Further, the cone $\mathrm{Cone}(\Sigma_p X)$ will be called tangent space to X at p; it will be denoted by $\mathrm{T}_p X$.

10.11. Theorem. *For any finite-dimensional Alexandrov space $\mathcal{L}$, we have*

$$\partial(\Sigma_p\mathcal{L}) = \Sigma_p(\partial\mathcal{L}) \quad \textit{and} \quad \partial(\mathrm{T}_p\mathcal{L}) = \mathrm{T}_p(\partial\mathcal{L}).$$

Proof. Choose a sequence $x_n \in \partial\mathcal{L}$ such that $x_n \to p$ and $\uparrow_{[px_n]} \to \xi$.

Let $\varepsilon_n = 2{\cdot}|p - x_n|$, and let $h_{\varepsilon_n} \colon \mathrm{B}(p, \varepsilon_n)_{\mathcal{L}} \to \mathrm{B}(0, \varepsilon_n)_{\mathrm{T}_p}$ be the homeomorphisms provided by 10.3; in particular, $\frac{2}{\varepsilon_n}{\cdot}h_{\varepsilon_n}(x_n) \to \xi$ as $n \to \infty$. By 10.9, $h_{\varepsilon_n}(x_n) \in \partial\mathrm{T}_p$. By 10.10, $\xi \in \partial\mathrm{T}_p$. Therefore,

$$\partial(\Sigma_p\mathcal{L}) \supset \Sigma_p(\partial\mathcal{L}) \quad \text{and} \quad \partial(\mathrm{T}_p\mathcal{L}) \supset \mathrm{T}_p(\partial\mathcal{L}).$$

Similarly, choose $\xi \in \partial\Sigma_p$. Let $h_{\varepsilon_n} \colon \mathrm{B}(p, \varepsilon_n)_{\mathcal{L}} \to \mathrm{B}(0, \varepsilon_n)_{\mathrm{T}_p}$ be the homeomorphisms provided by 10.3 for a sequence $\varepsilon_n \to 0$ as $n \to \infty$. By 10.9, $x_n = h_{\varepsilon_n}^{-1}(\frac{\varepsilon_n}{2}{\cdot}\xi) \in \partial\mathrm{T}_p$. By 10.3, $\uparrow_{[px_n]} \to \xi$. Hence

$$\partial(\Sigma_p\mathcal{L}) \subset \Sigma_p(\partial\mathcal{L}) \quad \text{and} \quad \partial(\mathrm{T}_p\mathcal{L}) \subset \mathrm{T}_p(\partial\mathcal{L}).$$

□

E Doubling

Let A be a closed subset in a metric space $\mathcal{X}$. The doubling $\mathcal{W}$ of $\mathcal{X}$ across A is two copies of $\mathcal{X}$ glued along A; more precisely, the underlying set of $\mathcal{W}$ is the quotient $\mathcal{X} \times \{0, 1\}/\sim$, where $(a, 0) \sim (a, 1)$ for any $a \in A$ and it has minimal metric such that both maps $\mathcal{X} \to \mathcal{W}$ defined by $x \mapsto (x, 0)$ and $x \mapsto (x, 1)$ are distance-preserving.

The metric on $\mathcal{W}$ can be also defined explicitly by

$$|(x, i) - (y, j)|_{\mathcal{W}} = \begin{cases} |x - y|_{\mathcal{X}} & \text{if} \quad i = j. \\ \inf\{\, |x - a|_{\mathcal{X}} + |y - a|_{\mathcal{X}} \;:\; a \in A \,\} & \text{if} \quad i \neq j. \end{cases}$$

The last part of the following statement is the so-called doubling theorem.

10.12. Theorem. *Let $\mathcal{L}$ be a finite-dimensional Alexandrov space with nonempty boundary. Suppose $f = \frac{1}{2}{\cdot}\mathrm{dist}_p^2$ for some $p \in \mathcal{L}$. Then*

(a) *If* $\dim\mathcal{L}\geqslant 2$*, then* $\operatorname{dist}_{\partial\Sigma_x}(\xi)\leqslant\frac{\pi}{2}$ *for any* $x\in\partial\mathcal{L}$ *and* $\xi\in\Sigma_x$*. Moreover, if* $\operatorname{dist}_{\partial\Sigma_x}(\xi)=\frac{\pi}{2}$*, then* $\measuredangle(\xi,\zeta)\leqslant\frac{\pi}{2}$ *for any* $\zeta\in\Sigma_x$*.*

(b) $\nabla_x f\in\partial\mathrm{T}_x$ *for any* $x\in\partial\mathcal{L}$*.*

(c) *If* α *is an* f*-gradient curve that starts at* $x\in\partial\mathcal{L}$*, then* $\alpha(t)\in\partial\mathcal{L}$ *for any* t*. Moreover, if* $p\in\partial\mathcal{L}$*, then* $\operatorname{gexp}_p(v)\in\partial\mathcal{L}$ *for any* $v\in\partial\mathrm{T}_p$*.*

(d) *The doubling* $\mathcal{W}$ *of* $\mathcal{L}$ *across* $\partial\mathcal{L}$ *is a Alexandrov space with the same curvature bound.*

Proof. We will denote by *(a)*$_m,\dots,$*(d)*$_m$ the corresponding statement assuming $m=\dim\mathcal{L}$.

The proof goes by induction on m. Note that *(d)*$_1$ follows from 7.15 — this is the base of induction. The step of induction is a combination of several implications listed below.

(d)$_{m-1}\Rightarrow$*(a)*$_m$. If $m=2$, then $\dim\Sigma_x=1$; see 7.18*b*. By 7.15, Σ_x isometric to a line segment $[0,\ell]$; we need to show that $\ell\leqslant\pi$.

Suppose $\ell>\pi$, then the tangent space $\mathrm{T}_x=\operatorname{Cone}\Sigma_x$ has several different lines thru the origin. Recall that T_x is a Alexandrov space; see 7.18. By 6.5, T_x is isometric to the Euclidean plane; the latter contradicts that Σ_x is a line segment.

Let $m=\dim\mathcal{L}>2$, so $\dim\Sigma_x>1$. Suppose $\operatorname{dist}_{\partial\Sigma_x}(\xi)>\frac{\pi}{2}$ for some ξ. By *(d)*$_{m-1}$, the doubling Ξ of Σ_x is $\mathrm{ALEX}(1)$. Denote by ξ_0 and ξ_1 the points in Ξ that correspond to ξ. Observe that $|\xi_0-\xi_1|_\Xi>\pi$. The latter contradicts 3.5.

Finally, if $\operatorname{dist}_{\partial\Sigma_x}(\xi)=\frac{\pi}{2}$, then $|\xi_0-\xi_1|_\Xi=\pi$. Therefore $\operatorname{Cone}\Xi$ contains a line in directions of ξ_0 and ξ_1; in other words, Ξ is a spherical suspension with poles ξ_0 and ξ_1. In particular, every point of Ξ lies on distance at most $\frac{\pi}{2}$ from ξ_0 or ξ_1. The natural projection $\Xi\to\Sigma_x$ does not increase distances and sends both ξ_0 and ξ_1 to ξ. Therefore, the second statement follows.

(d)$_{m-1}$+*(a)*$_{m-1}$+*(a)*$_m\Rightarrow$*(b)*$_m$. We can assume that $s=\nabla_x f\neq 0$. By 4.10, $\nabla_x f=s\cdot\bar\xi$, where $s=\boldsymbol{d}_x f(\bar\xi)$ and $\bar\xi\in\Sigma_p$ is the direction that maximize $\boldsymbol{d}_x f(\bar\xi)$.

Let $\zeta\in\partial\Sigma_x$ be a direction that minimize the angle $\measuredangle(\bar\xi,\zeta)$. It is sufficient to show that $\zeta=\bar\xi$.

Assume $\zeta\neq\bar\xi$; let $\eta=\uparrow_{[\zeta\bar\xi]}$. By *(a)*$_m$, $\measuredangle(\bar\xi,\zeta)\leqslant\frac{\pi}{2}$ and *(a)*$_{m-1}$ implies that

$$\text{❶}\qquad\measuredangle(\eta,\nu)\leqslant\tfrac{\pi}{2}$$

for any $\nu\in\Sigma_\zeta\Sigma_x$; if $m=2$, then the last statement is evident.

Consider function $\varphi\colon\Sigma_x\to\mathbb{R}$ defined by $\varphi(\xi):=\boldsymbol{d}_x f(\xi)$. Applying 4.7*a* and ❶, we get that $\boldsymbol{d}_\xi\varphi(\eta)\leqslant 0$. Since $\boldsymbol{d}_x f$ is convex, we have that $\varphi'' + \varphi \leqslant 0$. If $\varphi(\zeta) \leqslant 0$, then it implies that $\varphi(\bar\xi) \leqslant 0$ — a contradiction. If $\varphi(\zeta) > 0$, then $\varphi(\bar\xi) < \varphi(\zeta)$ — a contradiction again.

(b)$_m$ ⇒(c)$_m$. Let α be an f-gradient curve and $\ell(t) = \mathrm{dist}_{\partial L}\alpha(t)$. Note that ℓ is a Lipschitz function.

Choose t; let $x = \alpha(t)$ and $y \in \partial\mathcal{L}$ be a closest point to x. By *(b)$_m$*, we have that $\nabla_y f \in \partial\mathrm{T}_y$. Since the distance $|x-y|$ is minimal, we get $\langle \uparrow_{[yx]}, v\rangle \leqslant 0$ for any $v \in \partial\mathrm{T}_y$. In particular,

$$\langle \uparrow_{[yx]}, \nabla_y f\rangle \leqslant 0$$

Applying Exercise 4.9 to x and y, we get

$$\ell'(t) \leqslant \ell(t)$$

if the left-hand side is defined. Since ℓ is Lipschitz, ℓ' is defined almost everywhere. Integrating the inequality, we get

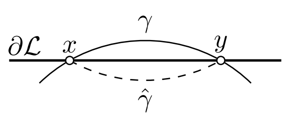
$$\ell(t) \leqslant e^t \cdot \ell(0)$$

for any $t \geqslant 0$. In particular, if $\ell(0) = 0$, then $\ell(t) = 0$ for any $t \geqslant 0$. Since $\partial\mathcal{L}$ is closed (10.10), the statement follows.

(c)$_m$+(d)$_{m-1}$ ⇒(d)$_m$. We will consider the case $\kappa = 0$; other cases can be done the same way, but formulas getting more complicated.

Denote by $\mathcal{L}_0$ and $\mathcal{L}_1$ the two copies of $\mathcal{L}$ in $\mathcal{W}$; let us keep the notation $\partial\mathcal{L}$ for the common boundary of $\mathcal{L}_0$ and $\mathcal{L}_1$. Next claim says that any geodesic in $\mathcal{W}$ either lies in $\partial\mathcal{L}$ or it crosses $\partial\mathcal{L}$ at most once.

❷ *Let γ be a geodesic in $\mathcal{W}$. Then either γ has at most one interior point in $\partial\mathcal{L}$ or $\gamma \subset \partial\mathcal{L}$.*



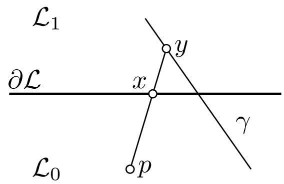
Assume γ shares at least two points with $\partial\mathcal{L}$, say $x = \gamma(t_1)$ and $y = \gamma(t_2)$ and these are not endpoints of γ. Remove from γ the set $\gamma \cap \mathcal{L}_1$ and exchange it to its reflection across $\mathcal{L}_0$; denote the obtained curve by $\hat\gamma$.

Note that any arc of $\hat\gamma$ with endpoint at x or y is a geodesic in $\mathcal{L}_0$. Moreover, the arc of $\hat\gamma$ behind y lies in the image of map $t \mapsto \mathrm{Flow}^t_{f_x}(y)$, where $f_x = \frac{1}{2}\cdot\mathrm{dist}^2_x$. By *(c)*, this arc lies in $\partial\mathcal{L}$.

Now choose a point z on this arc, so $z \in \partial\mathcal{L}$. Applying the same argument, we get that the arc of $\hat\gamma$ before y lies in $\partial\mathcal{L}$. Hence the claim follows. △

Choose a point p in $\mathcal{W}$; let $f := \frac{1}{2}\cdot\mathrm{dist}^2_p$. It is sufficient to show that $(f\circ\gamma)'' \leqslant 1$ for any t. If $p \in \partial\mathcal{L}$, then the statement follows from function comparison in $\mathcal{L}_0$ and $\mathcal{L}_1$. So, we can assume that $p \in \mathcal{L}_0\backslash\partial\mathcal{L}$.

If γ lies in $\partial\mathcal{L}$, then this inequality follows from the comparison in $\mathcal{L}_0$.



Choose $y = \gamma(t_0)$; without loss of generality we can assume that $t_0 = 0$.

If $y \in \mathcal{L}_0 \setminus \partial\mathcal{L}$, then $(f \circ \gamma)''(0) \leqslant 1$ in the barrier sense; it follows from the comparison in $\mathcal{L}_0$.

Assume $y \in \mathcal{L}_1 \setminus \partial\mathcal{L}$. Suppose $[py]$ crosses ∂L at x. Let Σ_x be the space of directions of $\mathcal{L}$ at x and let Ξ be its doubling. By *(d)*$_{m-1}$, Ξ is Alex(1).

Note that $\uparrow_{[xy]}$ and $\uparrow_{[xp]}$ lie in opposite sides of Ξ and

$$|\uparrow_{[xy]} - \uparrow_{[xp]}|_\Xi \geqslant \pi.$$

Otherwise, choose a direction $\xi \in \partial\Sigma$ such that

$$|\uparrow_{[xy]} - \xi|_\Xi + |\xi - \uparrow_{[xp]}|_\Xi < \pi,$$

Consider the radial curve $\alpha(t) = \operatorname{gexp}_x(t{\cdot}\xi)$. By *(c)*$_m$, α lies in ∂L. By 5.12

$$|p - \alpha(s)|_{\mathcal{L}_0} + |y - \alpha(s)|_{\mathcal{L}_1} < |p - y|_{\mathcal{W}}$$

for small values $s > 0$ — a contradition.

Note that $\operatorname{Cone}\Xi$ contains a line with directions $\uparrow_{[xy]}$ and $\uparrow_{[xp]}$. By splitting theorem $\operatorname{Cone}\Xi$ split in these directions; in particular,

$$|\uparrow_{[xy]} - \xi| + |\xi - \uparrow_{[xp]}| = \pi.$$

for any $\xi \in \Xi$. It follows that for any $\xi \in \Xi$ there is $\xi' \in \partial\Sigma_x$ such that ξ and ξ' lie on some geodesic $[\uparrow_{[xy]}\uparrow_{[xp]}]_\Xi$.

Choose such $\xi'(t)$ for $\xi(t) = \uparrow_{[x\gamma(t)]}$, where $t \approx 0$. Consider the radial curve $\alpha_t(s) := \operatorname{gexp}_x(s{\cdot}\xi'(t))$. Applying the comparison and 5.12, we get that

$$\begin{aligned}|p - \gamma(t)|_{\mathcal{W}} &\leqslant |p - \alpha(s)|_{\mathcal{L}_0} + |\alpha(s) - \gamma(t)|_{\mathcal{L}_1} \leqslant \\ &\leqslant \tilde{\gamma}[y\,{}^{p}_{\gamma(t)}].\end{aligned}$$

for suitably chosen parameter s. Hence we get $(f \circ \gamma)''(0) \leqslant 1$ in the barrier sense.

Finally if $\gamma(0) \in \partial\mathcal{L}$, then splitting argument shows that

$$(f \circ \gamma)^+(0) + (f \circ \gamma)^-(0) \leqslant 0.$$

Summarizing, we get that $(f \circ \gamma)'' \leqslant 1$ if γ lies entirely in $\mathcal{L}_0$ or $\mathcal{L}_1$. If γ crosses $\partial\mathcal{L}$, then we know that it happens only once and at the crossing moment t_0 we have $f \circ \gamma^+(t_0) + f \circ \gamma^-(t_0) \leqslant 0$. All this implies that $(f \circ \gamma)'' \leqslant 1$. $\square$

10.13. Exercise. *Let $\mathcal{L}$ be a finite-dimensional* $\mathrm{ALEX}(1)$ *space with nonempty boundary $\partial\mathcal{L}$. Show that $\partial\mathcal{L}$ is connected.*

10.14. Exercise. *Let $\mathcal{L}$ be a finite-dimensional* $\mathrm{ALEX}(0)$ *space with nonempty boundary $\partial\mathcal{L}$. Show that the distance function to the boundary*

$$\mathrm{dist}_{\partial\mathcal{L}}\colon \mathcal{L}\to\mathbb{R}$$

is concave.

10.15. Exercise. *Let $\mathcal{L}$ be a finite-dimensional* $\mathrm{ALEX}(0)$ *space with nonempty boundary $\partial\mathcal{L}$. Suppose γ is a geodesic in $\partial\mathcal{L}$ with the induced length metric. Show that the function $t\mapsto\frac{1}{2}\cdot\mathrm{dist}_p^2\circ\gamma(t)$ is 1-concave for any point p.*

10.16. Exercise. *Let $\mathcal{W}$ be doubling of a finite-dimensional Alexandrov space $\mathcal{L}$ across its boundary, and let $proj\colon\mathcal{W}\to\mathcal{L}$ be the natural projection. Suppose $f\colon\mathcal{L}\to\mathbb{R}$ is a λ-concave function. Show that $f\circ\mathrm{proj}\colon\mathcal{W}\to\mathbb{R}$ is λ-concave if and only if $\nabla_x f\in\partial\mathrm{T}_x$ for any $x\in\partial\mathcal{L}$.*

F Remarks

Theorem 10.3 can be used to prove the following.

10.17. Topological stratification. *Any m-dimensional Alexandrov space with $m<\infty$ can be subdivided into subsets topological manifolds $S_0,\dots,S_m$ such that for every i we have $\dim S_i=i$ or $S_i=\varnothing$. Moreover,*

(a) closure of S_{m-1} is the boundary of the space, and

(b) $S_{m-2}=\varnothing$.

Let us mention that this statement implies that compact finite-dimensional Alexandrov space has homotopy type of a finite CW complex, but it seems to be unknown if it has to be homeomorphic to a CW complex.

The stratification theorem 10.17 can be sharpened as follows.

10.18. Boundary characterization. *Let $\mathcal{L}$ be an m-dimensional Alexandrov space with $m<\infty$. Then the following statements are equivalent.*

(a) $p\in\partial\mathcal{L}$;

(b) $\Sigma_p(\mathcal{L})$ is contractible;

(c) $\tilde H_{m-1}(\Sigma_p\mathcal{L},\mathbb{Z}/2)=0$;

(d) $H_m(\mathcal{L},\mathcal{L}\setminus\{p\},\mathbb{Z}/2)=0$;

Note that doubling of a finite-dimensional Alexandrov space across its boundary is an Alexandrov space without boundary. This observation can be used to deduce a statemet about general finite-dimensional Alexandrov space to an Alexandrov space without boundary; so the following tools become available.

10.19. Fundamental-class lemma. *Any finite-dimensional Alexandrov space has a fundamental class cohomology with $\mathbb{Z}/2$ coefficients; that is, if $\mathcal{L}$ is a compact m-dimensional Alexandrov space, then $\mathcal{L}$*

$$\bar{H}^m(\mathcal{L}, \mathbb{Z}/2) = \mathbb{Z}/2.$$

This lemma was proved by Kartsen Grove and Peter Perersen [22]. Originally it was proved for Alexander–Spanier, but for Alexandrov spaces it is the same as singular cohomology. It implies, for example, that gradient is an onto map for functions of the following type $f = \frac{1}{2}\mathrm{dist}_p$ is surjective on finite-dimensional Alexandrov spaces. It is also used in the proof of the following generalized domain invariance theorem [29, Theorem 3.2].

10.20. Domain invariance. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two m-dimensional Alexandrov spaces with empty boundary; m is finite. Suppose Ω_1 is an open subset in $\mathcal{L}_1$ and $f\colon \Omega_1 \to \mathcal{L}_2$ is an injective continuous map. Then $f(\Omega_1)$ is open in $\mathcal{L}_2$.*

Let f be a semiconcave function. A point $p \in \operatorname{Dom} f$ is called critical point of f if $\boldsymbol{d}_p f \leqslant 0$; otherwise it is called regular.

The following statement plays technical role in the proof of stability theorem, but this is also a useful technical tool.

10.21. Morse lemma. *Let f be a semiconcave function on a finite-dimensional Alexandrov space without boundary. Suppose K is a compact set of regular points of f in its level set $f = a$. Then an open neighborhood Ω of K admits homeomorphism $x \mapsto (h(x), f(x))$ to a product space $\Lambda \times (a - \varepsilon, a + \varepsilon)$.*

Subsets that satisfy condition in 10.12c are called extremal. More precisely, a subset E in an Alexandrov space is called extremal if for any $x \in E$ and f-gradient curve that starts in E remains in E; here f is arbitrary function of the form $\frac{1}{2}\cdot\mathrm{dist}_p^2$.

Extremal subsets were introduced by Grigori Perelman and the second author [44]. They play an important role in Alexandrov geometry and its applications; in particular they will pop up in the next lecture.

The following conjecture is one of the oldest questions in Alexandrov geometry that remains open.

10.22. Conjecture. *Let S be a component of the boundary of a finite-dimensional Alexandrov space. Then S equipped with the induced length metric is an Alexandrov space with the same curvature bound.*

Lecture 11

Quotients

A Quotient space

Suppose that a group G acts isometrically on a metric space $\mathcal{X}$. Note that

$$|G{\cdot}x - G{\cdot}y|_{\mathcal{X}/G} := \inf\{\, |x - g{\cdot}y|_{\mathcal{X}} \;:\; g \in G \,\}$$

defines a semimetric on the orbit space $\mathcal{X}/G$. Moreover, it is a genuine metric if the orbits of the action are closed.

11.1. Theorem. *Assume that group G acts isometrically on a proper* $\mathrm{ALEX}(0)$ *space $\mathcal{L}$, and G has closed orbits. Then the quotient space $\mathcal{L}/G$ is* $\mathrm{ALEX}(0)$.

Proof. Denote by $\sigma\colon \mathcal{L} \to \mathcal{L}/G$ the quotient map.

Fix a quadruple of points $p, x_1, x_2, x_3 \in \mathcal{L}/G$. Choose an arbitrary $\hat{p} \in \mathcal{L}$ such that $\sigma(\hat{p}) = p$. Since $\mathcal{L}$ is proper, we can choose the points $\hat{x}_i \in \mathcal{L}$ such that $\sigma(\hat{x}_i) = x_i$ and

$$|p - x_i|_{\mathcal{L}/G} = |\hat{p} - \hat{x}_i|_{\mathcal{L}}$$

for all i.

Note that

$$|x_i - x_j|_{\mathcal{L}/G} \leqslant |\hat{x}_i - \hat{x}_j|_{\mathcal{L}}$$

for all i and j. Therefore

$$\tilde{\measuredangle}(p\,{}^{x_i}_{x_j}) \leqslant \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_i}_{\hat{x}_j}) \tag{❶}$$

holds for all i and j.

By $\mathbb{E}^2$-comparison in $\mathcal{L}$, we have

$$\tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_1}_{\hat{x}_2}) + \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_2}_{\hat{x}_3}) + \tilde{\measuredangle}(\hat{p}\,{}^{\hat{x}_3}_{\hat{x}_1}) \leqslant 2{\cdot}\pi.$$

Applying ❶, we get

$$\tilde\measuredangle(p\,^{x_1}_{x_2}) + \tilde\measuredangle(p\,^{x_2}_{x_3}) + \tilde\measuredangle(p\,^{x_3}_{x_1}) \leqslant 2\cdot\pi;$$

that is, the $\mathbb{E}^2$-comparison holds for any quadruple in $\mathcal{L}/G$. □

11.2. Advanced exercise. *Let G be a compact Lie group with a bi-invariant Riemannian metric. Show that G is isometric to a quotient of the Hilbert space by an isometric group action.*

Conclude that G is Alex(0).

B Generalizations

A map $\sigma\colon \mathcal{X} \to \mathcal{Y}$ between the metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is called a submetry if

$$\sigma(\mathrm{B}(p,r)_{\mathcal{X}}) = \mathrm{B}(\sigma(p),r)_{\mathcal{Y}}$$

for any $p \in \mathcal{X}$ and $r \geqslant 0$.

Suppose G and $\mathcal{L}$ are as in 11.1. Observe that the quotient map $\sigma\colon \mathcal{L} \to \mathcal{L}/G$ is a submetry. The following two exercises show that this is not the only source of submetries.

11.3. Exercise. *Construct submetries*

(a) $\sigma_1\colon \mathbb{S}^2 \to [0,\pi]$

(b) $\sigma_2\colon \mathbb{S}^2 \to [0,\frac{\pi}{2}]$

(c) $\sigma_n\colon \mathbb{S}^2 \to [0,\frac{\pi}{n}]$ *(for integer $n \geqslant 1$)*

such that the fibers $\sigma_n^{-1}\{x\}$ are connected for any x.

11.4. Exercise. *Let $\sigma\colon \mathbb{E}^2 \to [0,\infty)$ be a submetry. Show that $K = \sigma^{-1}\{0\}$ is a closed convex set in $\mathbb{E}^2$ and $\sigma(x) = \mathrm{dist}_K x$.*

The proof of 11.1 works for submetries. Therefore we get the following.

11.5. Generalization. *Let $\sigma\colon \mathcal{L} \to \mathcal{M}$ be a submetry. Suppose $\mathcal{L}$ is* Alex(0)*, then so is $\mathcal{M}$.*

Theorems 11.1 and 11.5 admit straightforward generalizations to Alex(−1) case. In the Alex(1) case, the proof produces a slightly weaker statement — *any open $\frac{\pi}{2}$-ball in the quotient of* Alex(1) *is* Alex(1); in particular, the quotient space is *locally* Alex(1). If the space is geodesic, then the globalization theorem implies that it is globally Alex(1). The same holds for the targets of submetry from a Alex(1) space. In other words, we have the following two statements.

11.6. Theorem. *Let $\sigma\colon \mathcal{L} \to \mathcal{M}$ be a submetry. Suppose $\mathcal{L}$ is* Alex(κ) *space, then so is $\mathcal{M}$.*

C Hopf's conjecture

Recall that Hopf's conjecture asks *does $\mathbb{S}^2 \times \mathbb{S}^2$ admit a Riemannian metric with positive sectional curvature?* The following partial result was obtained by Wu-Yi Hsiang and Bruce Kleiner [26].

11.7. Theorem. *There is no Riemannian metric on $\mathbb{S}^2 \times \mathbb{S}^2$ with sectional curvature $\geqslant 1$ and a nontrivial isometric $\mathbb{S}^1$-action.*

11.8. Key lemma. *Suppose $\mathbb{S}^1 \curvearrowright \mathbb{S}^3$ is an isometric action without fixed points and $\Sigma = \mathbb{S}^3/\mathbb{S}^1$ is its quotient space. Then there is a distance noncontracting map $\Sigma \to \frac{1}{2}\cdot\mathbb{S}^2$, where $\frac{1}{2}\cdot\mathbb{S}^2$ is the standard 2-sphere rescaled with a factor $\frac{1}{2}$.*

The proof of the lemma is done mostly by calculations; we present it in the form of guided exercise.

11.9. Exercise. *Suppose $\mathbb{S}^1 \curvearrowright \mathbb{S}^3$ is an isometric action without fixed points. Let us think that $\mathbb{S}^3$ is the unit sphere in $\mathbb{R}^4$.*

(a) *Show that one can identify $\mathbb{R}^4$ with $\mathbb{C}^2$ so that the action is given by matrix multiplication*

$$\begin{pmatrix} u^p & 0 \\ 0 & u^q \end{pmatrix},$$

where (p, q) is a pair of relatively prime positive integers and $u \in \mathbb{S}^1 = \{\, z \in \mathbb{C} : |z| = 1 \,\}$. In particular, our $\mathbb{S}^1$ is a subgroup in the torus that acts by matrix multiplication

$$\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix},$$

where $v, w \in \mathbb{S}^1 = \{\, z \in \mathbb{C} : |z| = 1 \,\}$.

Fix p and q as above. Let $\Sigma_{p,q} = \mathbb{S}^3/\mathbb{S}^1$ be the quotient space.

(b) *Show that the $\Sigma_{p,q} = \mathbb{S}^3/\mathbb{S}^1$ is a topological sphere with $\mathbb{S}^1$-symmetry. This symmetry has two fixed points, north pole and south pole, that correspond to the orbits of $(1, 0)$ and $(0, 1)$ in $\mathbb{S}^3$.*

Denote by $S(r)$ the circle of radius r with the center at the north pole of $\Sigma_{p,q}$.

(c) *Show that the inverse image $T(r)$ of $S(r)$ in $\mathbb{S}^3$ is also an orbit of the torus action. Conclude that $a(r) = \pi^2 \cdot \sin r \cdot \cos r$, where $a(r)$ is the area $T(r)$.*

(d) *Let $b_{p,q}(r)$ be the length of the $\mathbb{S}^1$-orbit in $\mathbb{S}^3$ that corresponds to a point on $S(r)$. Show that $b_{p,q} = \pi \cdot \sqrt{(p\cdot \sin r)^2 + (q\cdot \cos r)^2}$.*

(e) *Let $c_{p,q}(r)$ be the length of $S(r)$. Show that $a(r) = c_{p,q}(r)\cdot b_{p,q}(r)$.*

(f) *Show that $c_{p,q}(r)\leqslant c_{1,1}(r)$ for any pair (p,q) of relatively prime positive integers. Use it to show construct a distance noncontracting map $\Sigma_{p,q}\to\tfrac12\cdot\mathbb{S}^2\stackrel{iso}{=\!=}\Sigma_{1,1}$.*

Sketch of 11.7. Assume $\mathcal{M}=(\mathbb{S}^2\times\mathbb{S}^2,g)$ is a counterexample. By the Toponogov theorem, $\mathcal{M}$ is $\mathrm{ALEX}(1)$. By 11.1, the quotient space $\mathcal{L}=\mathcal{M}/\mathbb{S}^1$ is $\mathrm{ALEX}(1)$; evidently, $\mathcal{L}$ is 3-dimensional.

Denote by $F\subset\mathcal{M}$ the fixed point set of the $\mathbb{S}^1$-action. Each connected component of F is either an isolated point or a 2-dimensional geodesic submanifold in $\mathcal{M}$; the latter has to have positive curvature, and therefore it is either $\mathbb{S}^2$ or $\mathbb{R}\mathrm{P}^2$. Notice that

- ◇ each isolated point contributes 1 to the Euler characteristic of $\mathcal{M}$,
- ◇ each sphere contributes 2 to the Euler characteristic of $\mathcal{M}$, and
- ◇ each projective plane contributes 1 to the Euler characteristic of $\mathcal{M}$.

Since $\chi(\mathcal{M})=4$, we are in one of the following three cases:

- ◇ F has exactly 4 isolated points,
- ◇ F has one 2-dimensional submanifold and at least 2 isolated points,
- ◇ F has at least two 2-dimensional submanifolds.

Each case is covered separately.

Case 1. Suppose F has exactly 4 isolated points x_1, x_2, x_3, and x_4. Denote by y_1, y_2, y_3, and y_4 the corresponding points in $\mathcal{L}$. Note that $\Sigma_{y_i}\mathcal{L}$ is isometric to a quotient of $\mathbb{S}^3$ by an isometric $\mathbb{S}^1$-action without fixed points.

By 11.9, each angle $\measuredangle[y_i\,{}^{y_j}_{y_k}]\leqslant\tfrac{\pi}{2}$ for any three distinct points y_i, y_j, y_k. In particular, all four triangles $[y_1y_2y_3]$, $[y_1y_2y_4]$, $[y_1y_3y_4]$, and $[y_2y_3y_4]$ are nondegenerate. By the comparison, the sum of angles in each triangle is strictly greater than π.

Denote by σ the sum of all 12 angles in 4 triangles $[y_1y_2y_3]$, $[y_1y_2y_4]$, $[y_1y_3y_4]$, and $[y_2y_3y_4]$. From above,

$$\sigma>4\cdot\pi.$$

On the other hand, by 11.9 any triangle in $\Sigma_{y_1}\mathcal{L}$ has perimeter at most π. In particular,

$$\measuredangle[y_1\,{}^{y_2}_{y_3}]+\measuredangle[y_1\,{}^{y_3}_{y_4}]+\measuredangle[y_1\,{}^{y_4}_{y_2}]\leqslant\pi.$$

Apply the same argument in $\Sigma_{y_2}\mathcal{L}$, $\Sigma_{y_3}\mathcal{L}$, and $\Sigma_{y_4}\mathcal{L}$. Adding the results we get

$$\sigma\leqslant 4\cdot\pi$$

— a contradiction.

Case 2. Let F contain one surface S. Note that the projection of S to $\mathcal{L}$ forms its boundary $\partial\mathcal{L}$. Note that doubling $\hat{\mathcal{L}}$ of $\mathcal{L}$ across its boundary has 4 singular points — each singular point of $\mathcal{L}$ corresponds to two singular points of $\hat{\mathcal{L}}$.

By the doubling theorem, $\hat{\mathcal{L}}$ is a $\mathrm{ALEX}(1)$ space. Therefore we arrive at a contradiction the same way as in the first case.

Case 3. Suppose F contains at least two surfaces. Then $\partial\mathcal{L}$ has at least two connected components; choose two of them A and B. Denote by γ a geodesic that minimizes the distance from A to B.

Let

$$\dots, \mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{L}_1, \dots$$

be a two-sided infinite sequence of copies on $\partial\mathcal{L}$. Let us glue $\mathcal{L}_i$ to $\mathcal{L}_{i+1}$ along A if i is even and along B if i is odd.

By the doubling theorem, every point in the obtained space $\mathcal{N}$ has a neighborhood that is isometric to a neighborhood of the corresponding point in $\mathcal{L}$ or its doubling. By the globalization theorem, $\mathcal{N}$ is $\mathrm{ALEX}(1)$.

Note that the copies of γ in $\mathcal{L}_i$ form a line in $\mathcal{N}$. By the splitting theorem, $\mathcal{N}$ is isometric to a product $\mathcal{N}' \oplus \mathbb{R}$. Since $\dim\mathcal{N} > 1$, Exercise 7.15 implies that $\operatorname{diam}\mathcal{N} \leqslant \pi$ — a contradiction. □

D Erdős' problem rediscovered

A point p in an Alexandrov space is called extremal if $\measuredangle[p\,{}^x_y] \leqslant \frac{\pi}{2}$ for any hinge $[p\,{}^x_y]$ with the vertex at p.

11.10. Theorem. *Let $\mathcal{L}$ be a compact m-dimensional* $\mathrm{ALEX}(0)$ *space. Then it has at most 2^m extremal points.*

The proof is a translation of the proof of the following classical problem in discrete geometry to Alexandrov's language.

11.11. Erdős' problem. *Let F be a set of points in $\mathbb{E}^m$ such that any triangle formed by three distinct points in F has no obtuse angles. Then $|F| \leqslant 2^m$. Moreover, if $|F| = 2^m$ then F consists of the vertexes of an m-dimensional rectangle.*

This problem was posed by Paul Erdős [17] and solved by Ludwig Danzer and Branko Grünbaum [15]. Grigori Perelman noticed that after proper definitions, the same proof works in Alexandrov spaces [43]; so it proves 11.10.

Proof of 11.10. Let $\{p_1, \dots, p_N\}$ be extremal points in $\mathcal{L}$. For each p_i consider its open Voronoi domain V_i; that is,

$$V_i = \{\, x \in \mathcal{L} : |p_i - x| < |p_j - x| \text{ for any } j \neq i \,\}.$$

Clearly $V_i \cap V_j = \varnothing$ if $i \neq j$.

Suppose $0 < \alpha \leqslant 1$. Given a point $x \in \mathcal{L}$, choose a geodesic $[p_i x]$ and denote by x_i the point on $[p_i x]$ such that $|p_i - x_i| = \alpha{\cdot}|p_i - x|$; let $\Phi_i \colon x \to x_i$ be the corresponding map. By the comparison,

$$|x_i - y_i| \geqslant \alpha{\cdot}|x - y|$$

for any x, y, and i. Therefore

$$\operatorname{vol}(\Phi_i \mathcal{L}) \geqslant \alpha^m \cdot \operatorname{vol} \mathcal{L}.$$

Suppose $\alpha < \frac{1}{2}$. Then $x_i \in V_i$ for any $x \in \mathcal{L}$. Indeed, assume $x_i \notin V_i$, then threre is p_j such that $|p_i - x_i| \geqslant |p_j - x_i|$. Then from the comparison, we have $\tilde\measuredangle(p_j\,{}^{p_i}_{x})_{\mathbb{E}^2} > \frac{\pi}{2}$; that is, p_j does not form a one-point extremal set. It follows that $\operatorname{vol} V_i \geqslant \alpha^m \cdot \operatorname{vol} \mathcal{L}$ for any $0 < \alpha < \frac{1}{2}$; hence

$$\operatorname{vol} V_i \geqslant \tfrac{1}{2^m} \cdot \operatorname{vol} \mathcal{L} \quad \text{and} \quad N \leqslant 2^m.$$

□

E Crystallographic actions

An isometric action $\Gamma \curvearrowright \mathbb{E}^m$ is called crystallographic if it is properly discontinuous (that is, for any compact set $K \subset \mathbb{E}^m$ and $x \in \mathbb{E}^m$ there are only finitely many $g \in \Gamma$ such that $g{\cdot}x \in K$) and cocompact (that is, the quotient space $\mathcal{L} = \mathbb{E}^m/\Gamma$ is compact).

Let F be a maximal finite subgroup of Γ; that is, if H is a finite group H such that $F < H < \Gamma$, then $F = H$. Denote by $\#(\Gamma)$ the number of maximal finite subgroups of Γ up to conjugation.

11.12. Open question. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Is it true that $\#(\Gamma) \leqslant 2^m$?*

Note that any finite subgroup F of Γ fixes an affine subspace A_F in $\mathbb{E}^m$. If F is maximal, then A_F completely describes F. Indeed, since the action is properly discontinuous, the subgroup of Γ that fix A_F has to be finite. This subgroup must contain F, but since F is maximal, it must coinside with F.

Denote by $\#_k(\Gamma)$ the number of maximal finite subgroups $F < \Gamma$ (up to conjugation) such that $\dim A_F = k$.

Choose a finite subgroup $F < \Gamma$; consider a conjugate subgroup $F' = g{\cdot}F{\cdot}g^{-1}$. Note that $A_{F'} = g{\cdot}A_F$. In particular, the subspaces A_F and $A_{F'}$ have the same image in the quotient space $\mathcal{L} = \mathbb{E}^m/\Gamma$. Therefore, to count subgroups up to conjugation, we need to count the images of their fixed set. Therefore, by the lemma below, $\#_0(\Gamma)$ cannot exceed the number of extremal points in $\mathcal{L}=\mathbb{E}^m/\Gamma$. Combining this observation with 11.10, we get the following.

11.13. Proposition. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Then $\#_0(\Gamma)\leqslant 2^m$.*

11.14. Lemma. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action and F be a maximal finite subgroup of Γ that fixes an isolated point p. Then the image of p in the quotient space $\mathcal{L}=\mathbb{E}^m/\Gamma$ is an extremal point.*

Proof. Let q be the image of p. Suppose q is not extremal; that is, $\measuredangle[q\,{}^{y_1}_{y_2}]>\frac{\pi}{2}$ for some hinge $[q\,{}^{y_1}_{y_2}]$ in $\mathcal{L}$.

Choose the inverse images $x_1,x_2\in\mathbb{E}^m$ of $y_1,y_2\in\mathcal{L}$ such that $|p-x_i|_{\mathbb{E}^m}=|q-y_i|_{\mathcal{L}}$. Note that $\measuredangle[p\,{}^{x_1}_{x_2}]\geqslant\measuredangle[q\,{}^{y_1}_{y_2}]>\frac{\pi}{2}$. Moreover, since p is fixed by F, we have

$$❶\qquad \measuredangle[p\,{}^{x_1}_{g\cdot x_2}]>\tfrac{\pi}{2}$$

for any $g\in F$.

Denote by z the barycenter of the orbit $G\cdot x_2$. Note that z is a fixed point of F. By ❶, $z\neq p$; so F must fix the line pz. But p is an isolated fixed point of F — a contradiction. □

11.15. Exercise. *Let $\Gamma \curvearrowright \mathbb{E}^m$ be a crystallographic action. Show that*

(a) $\#_{m-1}(\Gamma)\leqslant 2$, and

(b) if $\#_{m-1}(\Gamma)=1$, then $\#_0(\Gamma)\leqslant 2^{m-1}$.

Construct crystallographic actions with equalities in (a) and (b).

F Remarks

It is expected that *no* Alex(1) *space with a nontrivial isometric $\mathbb{S}^1$-action can be homeomorphic to $\mathbb{S}^2\times\mathbb{S}^2$*; so 11.7 holds for general Alex(1) space. The proof of 11.7 would work if we had the following generalization of 11.8; see [24].

11.16. Open question. *Let Σ be an* Alex(1) *space homeomorphic to $\mathbb{S}^3$. Suppose $\mathbb{S}^1$ acts on Σ isometrically and without fixed points. Is it true that any triangle in $\Sigma/\mathbb{S}^1$ has perimeter at most π? and, what about the existence of distance-noncontracting map $\Sigma/\mathbb{S}^1\to\frac{1}{2}\cdot\mathbb{S}^2$?*

11.17. Advanced exercise. *Suppose that closed local geodesic γ is the fixed point of an isometric $\mathbb{S}^1$ action on an* Alex(1) *space that is homeomorphic to $\mathbb{S}^3$. Show that* $\operatorname{length}\gamma\leqslant 2\cdot\pi$.

It seems that the same question for a $\mathbb{Z}_2$-action is open.

Compact m-dimensional ALEX(0) spaces with the maximal number of extremal points include m-dimensional rectangles and the quotients of flat tori by reflections across a point. (This action has 2^m isolated fixed points; each corresponds to an extremal point in the quotient space $\mathcal{L} = \mathbb{T}^m/\mathbb{Z}_2$.) Nina Lebedeva has proved [33] that *every m-dimensional* ALEX(0) *space with 2^m extremal points is a quotient of Euclidean space by a crystallographic action.*

Counting maximal finite subgroups in a crystallographic group Γ is equivalent to counting the so-called primitive extremal subsets in the quotient space $\mathcal{L} = \mathbb{E}^m/\Gamma$. So, 11.13 would follow from the next conjecture.

11.18. Conjecture. *Any m-dimensional compact* ALEX(0) *space has at most 2^m primitive extremal subset.*

A closed subset E in a finite-dimensional Alexandrov space is called extremal if $\measuredangle[p\,{}^x_y] \leqslant \frac{\pi}{2}$ for any $x \notin E$ and $p \in E$ such that $|x-p|$ takes minimal value. An extremal subset is called primitive if it contains no proper extremal subsets. For example, the whole space and empty set are aslo extremal in any space. Also every vertex, edge, or face (as well as their union) of the cube is an extremal subset of the cube. Vertices of the cube are the only its primitive extremal subsets.

Lecture 12

Surface of convex body

Let us define a convex body as a compact convex subset in $\mathbb{E}^3$ with nonempty interior.

Suppose B is a convex body. Then the surface of B is defined as its boundary ∂B equipped with the induced length metric.

12.1. Exercise. *Show that surface of a convex body is homeomorphic to* $\mathbb{S}^2$.

In this lecture, we will prove that *surface of a convex body is* Alex(0).

A Surface of convex polyhedra

Convex polyhedron is a convex body with finite number of extremal points that called its vertices.

Observe that a surface, say Σ, of a convex polyhedron P admits a triangulation such that each triangle is isometric to a plane triangle. In other words, Σ is a polyhedral surfaces; that is, it is a 2-dimensional manifold with length metric that admits a triangulation such that each triangle is isometric to a solid plane triangle. A triangulation of polyhedral surface will be assumed to satisfy this condition.

The total angle around a vertex p in Σ is defined as the sum of angles at p of all triangles in the triangulation that contain p.

Note that if a point p is not a vertex of P, then

- ◇ p lies in the interior of a face of P, and its neighborhood in Σ is a piece of plane, or
- ◇ p lies on an edge, and its neighborhood is two half-planes glued along the boundary.

In both cases, a neighborhood of p in Σ (with the induced length metric) is isometric to an open domain of the plane.

12.2. Claim. *Let Σ be the surface of a convex polyhedron P. Then, the total angle around a vertex in Σ cannot exceed $2\cdot\pi$.*

In the proof, we will need the triangle inequality for angles (or the spherical triangle inequality). A proof of this statement is given in the classical geometry textbook by Andrei Kiselyov [30, § 47], it also follows from 1.9. (In fact our proof of 1.9 is a straightforward generalization of the argument in [30, § 47].)

12.3. Spherical triangle inequality. *Let w_1, w_2, w_3 be unit vectors in $\mathbb{E}^3$. Denote by $\theta_{i,j}$ the angle between the vectors v_i and v_j. Show that*

$$\theta_{1,3} \leqslant \theta_{1,2} + \theta_{2,3}$$

and in case of equality, the vectors w_1, w_2, w_3 lie in a plane.

Proof of 12.2. Consider the intersection of P with a small sphere centered at p; it is a convex spherical polygon, say F. Applying rescaling we may assume that the sphere has unit radius. Then we need to show that the perimeter of F does not exceed $2\cdot\pi$.

Note that F lies in a hemisphere, say H. Moreover, there is a decreasing sequence

$$H = H_0 \supset H_1 \supset \dots \supset H_n = F,$$

such that H_{i+1} is obtained from H_i by cutting along a chord.

By 12.3, we have

$$2\cdot\pi = \operatorname{perim} H = \operatorname{perim} H_0 \geqslant \operatorname{perim} H_1 \geqslant \dots \geqslant \operatorname{perim} H_n = \operatorname{perim} F$$

— hence the result. □

A vertex of a triangulation of a polyhedral surface is called essential if the total angle around it is not $2\cdot\pi$.

12.4. Exercise. *Let v be a point on the surface Σ of a convex polyhedron P. Show that v is a vertex of P if and only if v is an essential vertex of Σ.*

12.5. Exercise. *Show that geodesics on a surface of convex polyhedron do not pass thru its essential vertices.*

B Curvature

Let p be a vertex of a polyhedron. If θ_p is the total angle around p, then the value $2\cdot\pi - \theta_p$ is called the curvature of the polyhedral surface at p; if p is not a vertex, then its curvature is defined to be zero.

12.6. Exercise. *Assume that the surface of a nondegenerate tetrahedron T has curvature π at each of its vertices. Show that*

(a) all faces of T are congruent;

(b) the line passing thru midpoints of opposite edges of T intersects these edges at right angles.

Claim 12.2 says that *any vertex of a convex polyhedron has nonvanishing curvature.* However this definition works only for polyhedral surfaces. Now we show that it agrees with the 4-point comparison.

12.7. Proposition. *A polyhedral surface with nonnegative curvature at each vertex is* $\mathrm{A}\textsc{lex}(0)$.

Proof. Denote the surface by Σ. By 2.18, it is sufficient to check that $\mathrm{dist}_p^2\circ\gamma$ is 1-concave for any geodesic γ and a point p in Σ.

We can assume that p is not a vertex; the vertex case can be done by approximation. Further, by 12.5, we may assume that γ does not contain vertices.

Given a point $x = \gamma(t_0)$, choose a geodesic $[px]$. Again, by 12.5, $[px]$ does not contain vertices. Therefore a small neighborhood of $U \supset [px]$ can be unfolded on a plane; that is, there is an injective length-preserving map $z \mapsto \tilde z$ of U into the Euclidean plane. Note that this way we map part of γ in U to a line segment. Let

$$\tilde f(t) := \tfrac12\cdot\mathrm{dist}_{\tilde p}^2\circ\tilde\gamma(t).$$

Note that $\tilde f(t_0) \geqslant f(t_0)$. Further, since the unfolding $z \mapsto \tilde z$ preserves lengths of curves, we get $\tilde f(t) \geqslant f(t)$ if t is sufficiently close to t_0. That is, $\tilde f$ is a local upper support of f at t_0. Evidently, $\tilde f'' \equiv 1$; therefore $f'' \leqslant 1$. It remains to apply 2.18. □

12.8. Exercise. *Prove the converse to the proposition; that is, show that if a poyhedral surface is* $\mathrm{A}\textsc{lex}(0)$, *then it has nonnegative curvature at each vertex.*

C Surface of convex body

12.9. Advanced exercise. *Let $K_\infty, K_1, K_2, \dots$ be convex bodies in $\mathbb{E}^3$. Denote by S_n the surface of K_n with induced length metric. Suppose $K_n \to K_\infty$ in the sense of Hausdorff. Show that $S_n \to S_\infty$ in the sense of Gromov–Hausdorff.*

12.10. Proposition. *The surface of a convex body is* $\mathrm{Alex}(0)$.

Note that any convex body is a Hausdorff limit of a sequence of convex polyhedra. Therefore, the proposition follows from 12.7, 12.9, and 8.1.

D Comments

Note that 12.1 and 12.7 imply that surface of convex body is a sphere with nonnegative curvature in the sense of Alexandrov. The selbrated theorem of Alexandrov states that the converse also holds if we allow degeneration of convex bodies to plane figures; the surface of a plane figure is defined as its doubling across the boundary. In other words, any $\mathrm{Alex}(0)$ metric on the sphere is isometric to a surface of convex body. Moreover this convex body is unique up to congruence. The last result is due to Alexei Pogorelov [59].

Originally, Alexandrov proved the statement for polyhedral metrics on the sphere; this proof is sketced in the appendix. Then he used 12.9 to extend the result to arbitrary $\mathrm{Alex}(0)$ metric on the sphere.

12.11. Advanced exercise. *Let Σ be the surface of a nondegenerate convex body $K \subset \mathbb{E}^3$; we assume that Σ is equipped with its induced length metric.*

- *(a) Show that any geodesic γ in Σ is one-side differentiable as a curve in $\mathbb{E}^3$*
- *(b) Let γ_1 and γ_2 be geodesic paths in Σ that start at one point $p = \gamma_1(0) = \gamma_2(0)$. Suppose $x_1 = \gamma_1(1)$, $x_2 = \gamma_2(1)$, $y_1 = p + \gamma_1^+(0)$, and $y_2 = p + \gamma_2^+(0)$. Show that*

$$|x_1 - x_2|_\Sigma \leqslant |y_1 - y_2|_W,$$

where W is the complement to the interior of K.

Appendix A

Alexandrov's embedding theorem

BY NINA LEBEDEVA AND ANTON PETRUNIN

A Introduction

Intrinsic distance between two points on the surface of a convex polyhedron is defined as the length of a shortest curve on the surface between these points.

Recall that the sum of angles at the tip of a convex polyhedral angle is less than $2\cdot\pi$; this statement can be found in a school textbook [30, § 48].

It is easy to see that the surface of a convex polyhedron is homeomorphic to the sphere. Therefore the statements above imply that the surface of a convex polyhedron equipped with its intrinsic metric is an example of a *polyhedral metric on the sphere with the sum of angles around each vertex at most* $2\cdot\pi$; a metric is called polyhedral if the sphere admits a triangulation such that every triangle is congruent to a plane triangle.

Alexandrov's theorem states that the converse holds if one includes in the consideration *twice covered polygons*. In other words, we assume that a polyhedron can degenerate to a plane polygon; in this case, its surface is defined as two copies of the polygon glued along their boundary.

Further, we assume that a polyhedron can degenerate to a plane polygon.

A.1. Alexandrov's theorem.

I. A polyhedral metric on the sphere is isometric to the surface of a convex polyhedron if and only if the sum of angles around each of its vertex is not greater than $2{\cdot}\pi$.

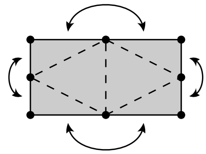
II. Moreover, a convex polyhedron is defined up to congruence by the intrinsic metric on its surface.

A. D. Alexandrov has many remarkable theorems, but in our opinion, this theorem is the most remarkable. At the same time, its proof is elementary; it could be explained to anyone familiar with basic topology.

This theorem has many applications. In particular, it is used in the proof of its generalization [6] that gives a complete description of intrinsic metrics on the sphere that are isometric to convex surfaces in the Euclidean space. The latter statement is fundamental in a branch of modern mathematics — the so-called *Alexandrov geometry*.

The first part is central; it is called the *existence theorem*. The second part is called the *uniqueness theorem*; it is a slight variation of Cauchy's theorem about polyhedrons. (There is another uniqueness theorem of Alexandrov that generalizes Minkowski's theorem about polyhedrons.)

According to the theorem, a convex polyhedron is completely defined by the intrinsic metric of its surface. In particular, knowing the metric we could find the position of the edges. However, in practice, it is not easy to do. For example, the surface glued from a rectangle as shown on the diagram defines a tetrahedron. Some of the glued lines appear inside facets of the tetrahedron and some edges (dashed lines) do not follow the sides of the rectangle.



The theorem was proved by A. D. Alexandrov in 1941 [5]; we will present a sketch of his proof. A complete proof is nicely written by A. D. Alexandrov in his book [4]. Yet another proof was found by Yu. A. Volkov in his thesis [64]; it uses a deformation of three-dimensional polyhedral space.

B Space of polyhedrons and metrics

Space of polyhedrons. Let us denote by Φ the space of all convex polyhedrons in the Euclidean space, including polyhedrons that degenerate to a plane polygon. Polyhedra in Φ will be considered up to a motion of the space, and the whole space Φ will be considered with the natural topology (an intuitive meaning of closeness of two polyhedrons should be sufficient).

Further, denote by Φ_n the polyhedrons in Φ with exactly n vertices. Since any polyhedron has at least 3 vertices, the space Φ admits a subdivision into a countable number of subsets $\Phi_3, \Phi_4, \ldots$

Space of polyhedral metrics. The space of polyhedral metrics on the sphere with the sum of angles around each point at most $2{\cdot}\pi$ will be denoted by Ψ. The metrics in Ψ will be considered up to an isometry, and the whole space Ψ will be equipped with the natural topology (again, an intuitive meaning of closeness of two metrics is sufficient).

A point on the sphere with the sum of angles strictly less than $2{\cdot}\pi$ will be called an essential vertex. The subset of Ψ of all metrics with exactly n essential vertices will be denoted by Ψ_n. It is easy to see that any metric in Ψ has at least 3 essential vertices. Therefore Ψ is subdivided into countably many subsets $\Psi_3, \Psi_4, \ldots$

From a polyhedron to its surface. Recall that the surface of a convex polyhedron is a sphere with a polyhedral metric such that the sum of angles around each point is at most $2{\cdot}\pi$. Therefore passing from a polyhedron to its surface defines a map

$$\iota\colon \Phi \to \Psi.$$

Note that the number of vertices of a polyhedron is equal to the number of essential vertices of its surface. In other words, $\iota(\Phi_n) \subset \Psi_n$ for any $n \geqslant 3$.

C About the proof

Using the notation introduced in the previous section, we can give the following more exact formulation of Alexandrov's theorem:

A.2. Reformulation. *For any integer $n \geqslant 3$, the map ι is a bijection from Φ_n to Ψ_n.*

We sketch the original proof of A. D. Alexandrov. It is based on the construction of a one-parameter family of polyhedrons that starts at arbitrary polyhedron and ends at a polyhedron with its surface isometric to the given one. This type of argument is called the continuity method; it is often used in the theory of differential equations.

The two parts of the first formulation will be proved separately.

Part II. Let us show that the map $\iota\colon \Phi_n \to \Psi_n$ is injective; in other words, a convex polyhedron is defined by the intrinsic metric on its surface up to a motion of the space.

The last statement is analogous to the Cauchy theorem about polyhedrons, and the proof goes along the same lines.

The Cauchy theorem states that facets of a polyhedron together with the gluing rule completely describe a convex polyhedron; its proof is given in many classical popular texts [1, 16, 61].

Part I. Let us prove that $\iota\colon \Phi_n \to \Psi_n$ is surjective. This part of the proof is subdivided into the following lemmas:

A.3. Lemma. *For any integer $n \geqslant 3$, the space Ψ_n is connected.*

The proof of this lemma is not complicated, but it requires ingenuity; it can be done by the direct construction of a one-parameter family of metrics in Ψ_n that connects two given metrics. Such a family can be obtained by a sequential application of the following construction and its inverse.

Let M be a sphere with metric from Ψ_n. Suppose v and w are essential vertices in M. Let us cut M along a shortest line from v to w. Note that the shortest line cannot pass thru an essential vertex of M. Further, note that there is a three-parameter family of patches that can be used to patch the cut so that the obtained metric remains in Ψ_n; in particular, the obtained metric has exactly n essential vertices (after the patching, the vertices v and w may become inessential).

A.4. Lemma. *The map $\iota\colon \Phi_n \to \Psi_n$ is open, that is, it maps any open set in Φ_n to an open set in Ψ_n.*

In particular, for any $n \geqslant 3$, the image $\iota(\Phi_n)$ is open in Ψ_n.

This statement is very close to the so-called invariance of domain theorem; the latter states that a continuous injective map between manifolds of the same dimension is open.

According to part II, ι is injective. The proof of the invariance of domain theorem can be adapted to our case since both spaces Φ_n and Ψ_n are $(3{\cdot}n-6)$-dimensional and both look like manifolds, altho, formally speaking, they are *not* manifolds. In a more technical language, Φ_n and Ψ_n have the natural structure of $(3{\cdot}n-6)$-dimensional orbifolds, and the map ι respects the orbifold structure.

We will only show that both spaces Φ_n and Ψ_n are $(3{\cdot}n-6)$-dimensional.

Choose a polyhedron P in Φ_n. Note that P is uniquely determined by the $3{\cdot}n$ coordinates of its n vertices. We can assume that the first vertex is the origin, the second has two vanishing coordinates and the third has one vanishing coordinate; therefore, all polyhedrons in Φ_n that lie sufficiently close to P can be described by $3{\cdot}n-6$ parameters. If P has no symmetries then this description can be made one-to-one; in this case, a neighborhood of P in Φ_n is a $(3{\cdot}n-6)$-dimensional manifold. If P has a nontrivial symmetry group, then this description is not one-to-one but it does not have an impact on the dimension of Φ_n.

The case of polyhedral metrics is analogous. We need to construct a subdivision of the sphere into plane triangles using only essential vertices. By Euler's formula, there are exactly $3{\cdot}n-6$ edges in this subdivision. Note that the lengths of edges completely describe the metric, and slight changes of these lengths produce a metric with the same property.

A.5. Lemma. *The map $\iota\colon\Phi_n\to\Psi_n$ is closed; that is, the image of a closed set in Φ_n is closed in Ψ_n.*

In particular, for any $n\geqslant 3$, the set $\iota(\Phi_n)$ is closed in Ψ_n.

Choose a closed set Z in Φ_n. Denote by $\bar Z$ the closure of Z in Φ; note that $Z=\Phi_n\cap\bar Z$. Assume $P_1,P_2,\dots\in Z$ is a sequence of polyhedrons that converges to a polyhedron $P_\infty\in\bar Z$. Note that $\iota(P_n)$ converges to $\iota(P_\infty)$ in Ψ. In particular, $\iota(\bar Z)$ is closed in Ψ.

Since $\iota(\Phi_n)\subset\Psi_n$ for any $n\geqslant 3$, we have $\iota(Z)=\iota(\bar Z)\cap\Psi_n$; that is, $\iota(Z)$ is closed in Ψ_n.

Summarizing, $\iota(\Phi_n)$ is a nonempty closed and open set in Ψ_n, and Ψ_n is connected for any $n\geqslant 3$. Therefore, $\iota(\Phi_n)=\Psi_n$; that is, $\iota\colon\Phi_n\to\Psi_n$ is surjective. □

Acknowledgments. We want to thank Stephanie Alexander, Yuri Burago, and Jules Tsukahara for help. The authors were partially supported by RFBR grant 20-01-00070 and NSF grant DMS-2005279.

Semisolutions

1.3. Given a pair of points p and q, choose a sequence of paths γ_n from p to q such that

$$\operatorname{length}\gamma_n \to |p-q| \quad \text{as} \quad n \to \infty;$$

it exists since we are in a length space. Note that we can assume that each γ_n is parametrized proportional to the arc length; in particular, γ_n are equicontinuous. Show that paths γ_n lie in a closed ball, say $\overline{\mathrm{B}}[p,r]$ for some $r<\infty$. Since the space is proper, $\overline{\mathrm{B}}[p,r]$ is compact. By Arzelà–Ascoli theorem, we can pass to a converging subsequence of γ_n. Show that its limit is a geodesic path from p to q.

1.2. Choose a Cauchy sequence x_n in $(\mathcal{X}, \|*-*\|)$; it is sufficient to show that a subsequence of x_n converges.

Observe that the sequence x_n is Cauchy in $(\mathcal{X}, |*-*|)$; denote its limit by x_∞.

Passing to a subsequence, we can assume that $\|x_n - x_{n+1}\| < \frac{1}{2^n}$. It follows that there is a 1-Lipschitz path γ in $(\mathcal{X}, \|*-*\|)$ such that $x_n = \gamma(\frac{1}{2^n})$ for each n and $x_\infty = \gamma(0)$. Therefore,

$$\|x_\infty - x_n\| \leqslant \operatorname{length}\gamma|_{[0,\frac{1}{2^n}]} \leqslant \tfrac{1}{2^n}.$$

In particular, x_n converges to x_∞ in $(\mathcal{X}, \|*-*\|)$.

Source: [27, Corollary]; see also [48, Lemma 2.3].

1.1. Choose a sequence of positive numbers $\varepsilon_n \to 0$ and a finite ε_n-net N_n of K for each n. We can assume that $\varepsilon_0 > \operatorname{diam} K$, and N_0 is a one-point set. If $|x-y| < \varepsilon_k$ for some $x \in N_{k+1}$ and $y \in N_k$, then connect them by a curve of length at most ε_k.

Let K' be the union of all these curves and K. Show that K' is compact and path-connected.

Source: This problem is due to Eugene Bilokopytov [8].

1.5. Choose a sequence $\varepsilon_n > 0$ that converges to zero very fast, say such that $\sum_n 10^n \cdot \varepsilon_n$ is small. Follow the argument in the proof of Menger's lemma, taking ε_n-midpoints at the n^{th} stage.

1.6. Let us write the Riemannian metric on $\mathbb{M}^2(\kappa)$ in the polar coordinates (θ, r); it has form $\left(\begin{smallmatrix} h^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where $h = h(\kappa, r)$. Calculate $h(\kappa, r)$. Show that for fixed r, the function $r \mapsto h(\kappa, r)$ is nonincreasing in the domain of definition. Suppose $\kappa < \mathrm{K}$, consider the map $\mathbb{M}^2(\kappa) \to \mathbb{M}^2(\mathrm{K})$ that sends a point to the point with the same polar coordinates. Show that this map is short in the domain of definition. Use it to prove the statement in the exercise.

1.7. Show and use that $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{S}^2} - \tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} = O(|p - x|^2 + |p - y|^2)$ and $\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} - \tilde\measuredangle(p\,{}^x_y)_{\mathbb{H}^2} = O(|p - x|^2 + |p - y|^2)$.

1.8. Consider a hinge in the plane $\mathbb{R}^2$ with a metric defined by norm, say by ℓ^∞-norm.

1.10. Assume $\measuredangle[p\,{}^x_z] + \measuredangle[p\,{}^y_z] < \pi$. By 1.9, $\measuredangle[p\,{}^x_y] < \pi$. Therefore, $\tilde\measuredangle(p\,{}^{\bar x}_{\bar y}) < \pi$ for some $\bar x \in\,]px]$ and $\bar y \in\,]py]$. Hence

$$|p - \bar x| + |\bar y - p| < |\bar x - \bar y|$$

— a contradiction.

1.11. Denote by α the arc-length parametrization of $[qp]$ from q to p. Choose $\varepsilon > 0$. Observe that

$$|\gamma(t) - \alpha(\tfrac{1}{\varepsilon}\cdot t)|^2 \leqslant t^2\cdot(1 - \tfrac{2}{\varepsilon}\cdot \cos\varphi + \tfrac{1}{\varepsilon^2}) + o(t^2),$$

where $\varphi = \measuredangle[q\,{}^p_x]$. By the triangle inequality

$$|p - \gamma(t)| \leqslant |\gamma(t) - \alpha(\tfrac{1}{\varepsilon}\cdot t)| + |q - p| - \tfrac{1}{\varepsilon}\cdot t.$$

Conclude that

$$|p - \gamma(t)| \leqslant |q - p| - t\cdot \cos\varphi + \delta(\varepsilon)\cdot t + o(t),$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The statement follows since $\varepsilon > 0$ is arbitrary.

2.2. The 4-point comparison (2.1) reduces our question to the following. *Any spherical triangle has perimeter at most* $2\cdot\pi$. Choose a spherical triangle $[xyz]$. Let x' be the antipode of x; that is $x' = -x$. The spherical triangle inequality implies that

$$|x - z|_{\mathbb{S}^2} \leqslant |y - x'|_{\mathbb{S}^2} + |x' - z|_{\mathbb{S}^2}.$$

Observe that

$$|x-y|_{\mathbb{S}^2}+|y-x'|_{\mathbb{S}^2}=\pi,\quad\text{and}\quad |x-z|_{\mathbb{S}^2}+|z-x'|_{\mathbb{S}^2}=\pi.$$

Hence

$$|x-y|_{\mathbb{S}^2}+|x-z|_{\mathbb{S}^2}+|y-z|_{\mathbb{S}^2}\leqslant 2\cdot\pi.$$

2.3. For the only-if part consider the following two cases.

If $\tilde\measuredangle(p\,{}^{x_1}_{x_2})+\tilde\measuredangle(p\,{}^{x_2}_{x_3})\geqslant\pi$, then choose two model triangles $[qy_1y_2]=$ $=\tilde\triangle(px_1x_2)$ and $[qy_2y_3]=\tilde\triangle(px_2x_y)$ that lie on the opposite sides of $[qy_2]$. By the comparison, $|y_1-y_3|\geqslant|x_1-x_3|$. Therefore the obtained configuration meets all the conditions.

If $\tilde\measuredangle(p\,{}^{x_1}_{x_2})+\tilde\measuredangle(p\,{}^{x_2}_{x_3})\geqslant\pi$, then choose two model triangles $[qy_1y_2]=$ $\tilde\triangle(px_1x_2)$ and take y_3 on the extension of $[y_1q]$ behind q such that $|q-y_3|=|p-x_3|$. Then $\measuredangle[q\,{}^{y_2}_{y_3}]\geqslant\tilde\measuredangle(p\,{}^{x_2}_{x_3})$, therefore $|y_2-y_3|\geqslant|x_2-$ $-x_3|$. Further, $|y_2-y_3|=|x_2-p|+|p-x_3|\geqslant|x_2-x_3|$, and again, the obtained configuration meets all the conditions.

To prove the if part, choose a configuration q,y_1,y_2,y_3 that meets all the conditions and maximize the sum

$$|y_1-y_2|+|y_2-y_3|+|y_3-y_1|.$$

Show that that q lies in the solid triangle $y_1y_2y_3$; in particular

$$\measuredangle[q\,{}^{y_1}_{y_2}]+\measuredangle[q\,{}^{y_2}_{y_3}]+\measuredangle[q\,{}^{y_3}_{y_1}]=2\cdot\pi.$$

Moreover, $|q-y_i|=|p-x_i|$ for each i. Applying that increasing the opposite side in a plane triangle increases the corresponding angle, we get

$$\tilde\measuredangle(p\,{}^{x_1}_{x_2})+\tilde\measuredangle(p\,{}^{x_2}_{x_3})+\tilde\measuredangle(p\,{}^{x_3}_{x_1})\leqslant 2\cdot\pi.$$

2.5. Consider model triangles $[\tilde p\tilde x\tilde z]=\tilde\triangle(pxz)$ and $[\tilde p\tilde y\tilde z]=\tilde\triangle(pyz)$ that share side $[\tilde p\tilde z]$ and lie on its opposite sides. Note that

$$\begin{aligned}|\tilde x-\tilde y|_{\mathbb{E}^2}&\geqslant|\tilde x-\tilde y|_{\mathbb{E}^2}+|\tilde x-\tilde y|_{\mathbb{E}^2}=\\&=|x-z|_{\mathcal{X}}+|z-y|_{\mathcal{X}}=\\&=|x-y|_{\mathcal{X}},\end{aligned}$$

where $\mathcal{X}$ is our metric space. It remains to apply the monotonicity of angle in a triangle with respect to its opposite side.

2.7. Apply 2.6.

2.9. Without loss of generality, we can assume that $|p-x|\leqslant|p-y|$. Choose $\bar x\in[px]$; let $\bar y\in[px]$ be such that $|p-\bar x|=|p-\bar y|$. Apply 2.6 to show that $\bar x=\bar y$. Conclude that $[px]\subset[py]$.

2.10. Assume that there are two distinct geodesics from z to x. Then we can choose distinct points p and q one these geodesics such that $|z-p|=|z-q|$. Observe that $\tilde{\measuredangle}(z\,{}^{p}_{q})>0$. By triangle inequality, we get

$$|x-p|+|p-y|\leqslant|x-p|+|p-z|+|z-y|=|x-z|+|z-y|$$

Observe that $\tilde{\measuredangle}(z\,{}^{x}_{y})=\pi$. Therefore $\measuredangle[z\,{}^{x}_{y}]=\pi$ for any geodesic $[zx]$.

2.11. By 1.10, we have

$$\measuredangle[p\,{}^{x}_{z}]+\measuredangle[p\,{}^{y}_{z}]\geqslant\pi.$$

Since $z\in\,]xy[$ we have

$$\tilde{\measuredangle}(z\,{}^{\bar{x}}_{\bar{y}})=\pi$$

for any $\bar{x}\in[xz[$ and $\bar{y}\in\,]zy]$. By comparison, we have that

$$\tilde{\measuredangle}(z\,{}^{\bar{x}}_{\bar{p}})+\tilde{\measuredangle}(z\,{}^{\bar{p}}_{\bar{y}})\leqslant\pi$$

for any $\bar{p}\in\,]zp]$. Passing to the limit as $|z-\bar{x}|\to0$, $|z-\bar{y}|\to0$, and $|z-\bar{p}|\to0$, we get the statement.

2.12. Without loss of generality, we can assume that x, v, w, and y appear on $[xy]$ in this order. By 2.6,

$$\tilde{\measuredangle}(x\,{}^{y}_{p})\geqslant\tilde{\measuredangle}(x\,{}^{w}_{p})\geqslant\tilde{\measuredangle}(x\,{}^{v}_{p}).$$

Hence, $\Rightarrow$ follows.

By Alexandrov's lemma,

$$\begin{aligned}\tilde{\measuredangle}(x\,{}^{y}_{p})=\tilde{\measuredangle}(x\,{}^{v}_{p})\quad&\Longleftrightarrow\quad\tilde{\measuredangle}(y\,{}^{x}_{p})=\tilde{\measuredangle}(y\,{}^{v}_{p}),\\ \tilde{\measuredangle}(x\,{}^{y}_{p})=\tilde{\measuredangle}(x\,{}^{w}_{p})\quad&\Longleftrightarrow\quad\tilde{\measuredangle}(y\,{}^{x}_{p})=\tilde{\measuredangle}(y\,{}^{w}_{p}).\end{aligned}$$

Whence, $\Leftarrow$ follows.

2.13. Suppose $\measuredangle[x_\infty\,{}^{y_\infty}_{z_\infty}]>\alpha$. Then we can choose $\bar{y}_\infty\in\,]x_\infty y_\infty]$ and $\bar{z}_\infty\in\,]x_\infty z_\infty]$ such that $\tilde{\measuredangle}(x_\infty\,{}^{\bar{y}_\infty}_{\bar{z}_\infty})>\alpha$. Now choose $\bar{y}_n\in\,]x_ny_n]$ and $\bar{y}_n\in\,]x_nz_n]$ such that $\bar{y}_n\to\bar{y}_\infty$ and $\bar{z}_n\to\bar{z}_\infty$. Observe that

$$\varliminf_{n\to\infty}\measuredangle[x_n\,{}^{y_n}_{z_n}]\geqslant\varliminf_{n\to\infty}\tilde{\measuredangle}(x_n\,{}^{\bar{y}_n}_{\bar{z}_n})\geqslant\alpha,$$

hence the result.

2.16. The Urysohn space provides an example; see for example [55, Lecture 2].

2.17. Choose a triangle $[0vw]$. Note that $m=\frac{1}{2}(v+w)$ is the midpoint of $[vw]$.

Use comparison, to show that

$$2\cdot|\tfrac12(v+w)|^2+2\cdot|\tfrac12(v-w)|^2\geqslant|v|^2+|w|^2.$$

Note this inequality implies the opposite one; it follows if we rewrite it via $x=\frac12(v+w)$ and $y=\frac12(v-w)$. Hence we have

$$2\cdot|\tfrac12(v+w)|^2+2\cdot|\tfrac12(v-w)|^2=|v|^2+|w|^2$$

for any v,w. That is the norm is quadratic and the statement follows.

3.4. Note that $\mathcal{X}$ has no defined sphericlal model angles; therefore it has curvature $\geqslant 1$.

However, $\mathcal{X}$ does not have curvature $\geqslant 0$ since

$$\tilde\measuredangle(p\,{}^{x_1}_{x_2})_{\mathbb{E}^2}=\tilde\measuredangle(p\,{}^{x_2}_{x_3})_{\mathbb{E}^2}=\tilde\measuredangle(p\,{}^{x_1}_{x_3})_{\mathbb{E}^2}=\pi.$$

3.5. Suppose $\measuredangle[m\,{}^x_p]\neq0$ and $\measuredangle[m\,{}^x_p]\neq\pi$, or equivalently $\measuredangle[m\,{}^x_q]\neq0$.

We can assume that $|p-q|$ only slightly exceeds π, so $|p-m|<\pi$ and $|q-m|<\pi$. We can also assume that $|x-m|<\pi$. Use the comparison to show that

$$|p-x|+|q-x|<|p-q|,$$

and arrive at a contradiction with the triangle inequality.

Extend $[pq]$ to a maximal local geodesic γ. It might be a closed or a line segment. Argue as above to show that any point lies on γ and make a conclusion.

3.6. Arguing by contradiction, suppose

$$\text{❶}\qquad |p-q|+|q-r|+|r-p|>2\cdot\pi$$

for $p,q,r\in\mathcal{L}$. Rescaling the space slightly, we can assume that $\operatorname{diam}\mathcal{L}<\pi$, but the inequality ❶ still holds. By 3.3, after rescaling $\mathcal{L}$ is still $\mathrm{A\scriptstyle LEX}(1)$.

Take $z_0\in[qr]$ on maximal distance from p. Consider the following model configuration: two geodesics $[\tilde p\tilde z_0]$, $[\tilde q\tilde r]$ in $\mathbb{S}^2$ such that

$$\begin{aligned}|\tilde p-\tilde z_0|&=|p-z_0|, & |\tilde q-\tilde r|&=|q-r|,\\ |\tilde z_0-\tilde q|&=|z_0-q|, & |\tilde z_0-\tilde r|&=|z_0-q|,\end{aligned}$$

and

$$\measuredangle[\tilde z_0\,{}^{\tilde q}_{\tilde p}]=\measuredangle[\tilde z_0\,{}^{\tilde r}_{\tilde p}]=\tfrac{\pi}{2}.$$

Let $\tilde{z} \in [\tilde{q}\tilde{r}]$, and let $z \in [qr]$ be the corresponding point. By comparison, $|p - z| \leqslant |\tilde{p} - \tilde{z}|$ for points z near z_0. Moreover, this inequality holds as far as

$$|\tilde{p} - \tilde{z}_0| + |\tilde{z}_0 - \tilde{z}| + |\tilde{p} - \tilde{z}| < 2{\cdot}\pi.$$

But this inequality holds for all $\tilde{z}$ since $|\tilde{p} - \tilde{z}_0| < \pi$, $|\tilde{z}_0 - \tilde{q}| < \pi$, and $|\tilde{z}_0 - \tilde{r}| < \pi$. Hence we get $|p - q| \leqslant |\tilde{p} - \tilde{q}|$ and $|p - r| \leqslant |\tilde{p} - \tilde{r}|$. The latter contradicts ❶.

3.8. Suppose such point does not exists; that is, for any $p \in \mathcal{X}$ there is a point p' such that $r(p') \leqslant (1 - \varepsilon){\cdot}r(p)$ and $|p - p'| < \frac{1}{\varepsilon}{\cdot}r(p)$. Construct a sequence of points $p_0, p_1, \dots$ such that $p_n = p'_{n-1}$ for any n. Show that this sequence is Cauchy; denote its limit by p_∞. Arrive at a contradiction by showing that $r(p_\infty) \leqslant 0$.

4.1; *(a).* Suppose $\uparrow_{[px_n]} \not\to \uparrow_{[px_\infty]}$. Since Σ_p is compact we may pass to a converging subsequence of $\uparrow_{[px_n]}$; denote by ξ its limit. We may assume that $\measuredangle(\uparrow_{[px_\infty]}, \xi) > 0$.

Denote by γ_n and γ_∞ the arc-length parametrization of $[px_n]$ and $[px_\infty]$ from p. Choose a geodesic α that starts from p and goes in a direction sufficiently close to ξ. By comparison we can choose α so that

$$|\alpha(t) - \gamma_n(t)| < \varepsilon{\cdot}t$$

for all large n and all sufficiently small t. Moreover, we can assume that

$$|\alpha(t) - \gamma_\infty(t)| > a{\cdot}t$$

for some fixed $a > 0$ and all small t. These two inequalities imply that

$$|\gamma_n(t) - \gamma_\infty(t)| > \tfrac{a}{2}{\cdot}t$$

for all small t and all large n. On the other hand, by assumption, $|\gamma_n(t) - \gamma_\infty(t)| \to 0$ as $n \to \infty$ — a contradiction.

(b) ???

4.2. Note that any point of $\operatorname{Cone}\mathcal{X}$ can be connected to the origin by a geodesic. Given a nonzero element $v \in \operatorname{Cone}\mathcal{X}$, denote by v' its projection in $\mathcal{X}$.

Suppose $\mathcal{X}$ is π-geodesic. Choose two nonzero elements $v, w \in \operatorname{Cone}\mathcal{X}$; let $\alpha = \measuredangle(v, w) = |v' - w'|_{\mathcal{X}}$. If $\alpha \geqslant \pi$, then the product of geodesics $[v0] \cup [0w]$ forms a geodesic $[vw]$. If $\alpha < \pi$, there is a geodesic $\gamma\colon [0, \alpha] \to \mathcal{X}$ from v' to w'. Consider hinge $[\tilde{o}\,{}^{\tilde{v}}_{\tilde{w}}]$ in the plane such that $\measuredangle[\tilde{o}\,{}^{\tilde{v}}_{\tilde{w}}] = \alpha$, $|\tilde{o} - \tilde{v}| = |v|$, and $|\tilde{o} - \tilde{w}| = |w|$. Let $t \mapsto (\varphi(t), r(t))$ be geodesic $[\tilde{v}\tilde{w}]$ written in the polar coordinates with origin $\tilde{o}$, so that

$\varphi(0)=0$. Show that $t\mapsto r(t)\cdot\gamma\circ\varphi(t)$ is a geodesic from v to w; here we identify $\mathcal{X}$ with the unit sphere in $\operatorname{Cone}\mathcal{X}$.

To prove the converse, try to revert the steps in the argument above.

4.5. From 2.18, this inequality follows in the sense of distributions, and hence in any other sense.

4.6. Since angles are defined, it follows that

$$|\gamma_1(t)-\gamma_2(t)|\leqslant\theta\cdot t$$

for all small $t>0$. Since f is L-Lipschitz, we get

$$|f(\gamma_1(t))-f(\gamma_2(t))|\leqslant L\cdot\theta\cdot t,$$

hence the statement.

4.7; *(a)* Note that we can assume that there is a geodesic in the direction of v, and apply 1.11.

(b). By *(a)* we have an $\leqslant$ inequality. Suppose this inequalities is strict for some v. We can assume that $|v|=1$ and there is a geodesic, say γ in the direction of v. Suppose ... Let $\alpha=$???.

The function $f=\operatorname{dist}_q\circ\gamma$ is Lipschitz; By Rademacher's theorem it is differentaible almost everywhere; moreover,

$$f(t)-f(0)=\int\limits_0^t f'(t)\cdot dt.$$

Suppose $f'(t)$ is defined. Use *(a)* to show that $f'(t)=-\cos\alpha(t)$, where $\alpha(t)$ is the angle between γ and a geodesic from $\gamma(t)$ to q. Note that we can choose a sequence $t_n\to 0$ such that

$$\lim_{n\to\infty}\alpha(t_n)\leqslant\alpha.$$

Consider a sequence of geodsics $[p\gamma(t_n)]$. Since the space is proper, we can pass to its convergent subsequence. Its limit is a geodesic from p to q, denote it by $[pq]$. Observe that $[pq]$ makes angle at most α with γ — a contradiction.

4.9. Let $\gamma\colon[0,\ell]\to\mathcal{L}$ be the geodesic $[xy]$ paramentrized from x to y, and let $\varphi=f\circ\gamma$. Observe that

$$\varphi'(0)=\boldsymbol{d}_xf(\uparrow_{[xy]})\leqslant\langle\uparrow_{[xy]},\nabla_xf\rangle.$$

The same way we get $-\varphi'(\ell) \leqslant \langle \uparrow_{[yx]}, \nabla_y f\rangle$. Since f is λ-concave, we have

$$\begin{aligned} f(y) &\leqslant f(x) + \varphi'(0)\cdot\ell + \tfrac{\lambda}{2}\cdot\ell^2,\\ f(x) &\leqslant f(y) - \varphi'(\ell)\cdot\ell + \tfrac{\lambda}{2}\cdot\ell^2. \end{aligned}$$

Hence the statement follows.

4.12.

4.13. Note that $|(\boldsymbol{d}_p f)(v) - (\boldsymbol{d}_p g)(v)| \leqslant s\cdot|v|$ for any $v \in \mathrm{T}_p$. From the definition of gradient (4.8) we have:

$$\begin{aligned} (\boldsymbol{d}_p f)(\nabla_p g) &\leqslant \langle \nabla_p f, \nabla_p g\rangle, & (\boldsymbol{d}_p g)(\nabla_p f) &\leqslant \langle \nabla_p f, \nabla_p g\rangle,\\ (\boldsymbol{d}_p f)(\nabla_p f) &= \langle \nabla_p f, \nabla_p f\rangle, & (\boldsymbol{d}_p g)(\nabla_p g) &= \langle \nabla_p g, \nabla_p g\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} |\nabla_p f - \nabla_p g|^2 &= \langle \nabla_p f, \nabla_p f\rangle + \langle \nabla_p g, \nabla_p g\rangle - 2\cdot\langle \nabla_p f, \nabla_p g\rangle \leqslant\\ &\leqslant (\boldsymbol{d}_p f)(\nabla_p f) + (\boldsymbol{d}_p g)(\nabla_p g) - (\boldsymbol{d}_p f)(\nabla_p g) - (\boldsymbol{d}_p g)(\nabla_p f) \leqslant\\ &\leqslant s\cdot(|\nabla_p f| + |\nabla_p g|). \end{aligned}$$

4.14. Suppose $|\nabla_x f| > s$. Then we can choose a geodesic γ that starts at x such that $(f\circ\gamma)^+(0) > s$. In particular, there is $\varepsilon > 0$ such that

$$f\circ\gamma(t) > (s+\varepsilon)\cdot t + o(t),$$

hence the only-if part follows.

Now suppose $f(y) - f(x) > s\cdot\ell + \lambda\cdot\frac{\ell^2}{2}$, were $\ell = |x-y|$. Let $\gamma\colon [0,\ell] \to \mathcal{L}$ be a geodesic from x to y. Since $f\circ\gamma$ is λ-concave, we have

$$f\circ\gamma(\ell) \leqslant f\circ\gamma(0) + (f\circ\gamma)^+(0)\cdot\ell + \lambda\cdot\tfrac{\ell^2}{2}.$$

It follows that

$$\boldsymbol{d}_x(\uparrow_{[xy]}) = (f\circ\gamma)^+(0) > s,$$

and by 4.10, $|\nabla_x f| > s$.

5.7. Note that $f\circ\alpha$ is a nondecreasing function. Apply 4.7a and the definition of gradient to show that

$$-\boldsymbol{d}_{\alpha(t)}\mathrm{dist}_{\alpha(t_3)}(\nabla_{\alpha(t)} f) \geqslant \langle \nabla_{\alpha(t)}, \uparrow_{[\alpha(t)\alpha(t_3)]}\rangle \geqslant \boldsymbol{d}_{\alpha(t)}(\uparrow_{[\alpha(t)\alpha(t_3)]}) \geqslant 0$$

for any $t < t_3$. Conclude that the function $t \mapsto \mathrm{dist}_{\alpha(t_3)}\circ\alpha(t)$ is noncreasing for $t \leqslant t_3$.

5.8.

5.9.

6.2. Apply 4.5.

6.3. By the triangle inequality,

$$|\gamma(-t) - x| + |\gamma(t) - x| - 2{\cdot}t \geqslant 0$$

for any $t \geqslant 0$. Passing to the limit as $t \to \infty$, we get the result.

6.7.

7.2. Observe that

$$\begin{aligned}\langle u, u\rangle + \langle v, u\rangle + \langle w, u\rangle &\geqslant 0,\\ \langle u, v\rangle + \langle v, v\rangle + \langle w, v\rangle &\geqslant 0,\\ \langle u, w\rangle + \langle v, w\rangle + \langle w, w\rangle &= 0.\end{aligned}$$

Add the first two inequalities and subtract the last identity.

7.7.

7.8. Show and use that

$$\langle u, x\rangle + \langle v, x\rangle + \langle w, x\rangle \geqslant 0$$

and

$$\langle u, -x\rangle + \langle v, -x\rangle + \langle w, -x\rangle \geqslant 0.$$

7.9. Part $\Rightarrow$ is evident. To prove part $\Leftarrow$, observe that

$$\langle u^*, u^*\rangle = -\langle u, u^*\rangle \leqslant \langle u, u\rangle$$

and since $|u| = |u^*|$, we have equality.

7.11; *(a)*. Let $S_n \subset \mathcal{L}$ be defined by inequality $|\nabla_x f| > 1 - \frac{1}{n}$. Apply 4.14$a$ to show that S_n is open. Choose a point $q \neq p$, observe that $|\nabla_x f| = 1$ for any point $x \in \,]pq[$. Conclude that S_n is dense in $\mathcal{L}$. Observe and use that $S = \bigcap_n S_n$.

(b)+(c). Apply 7.9.

7.14. Apply 7.11b to show that for any finite set of points $p_1, \dots, p_n$ there is a G-delta dense set of points x such that $\mathrm{Lin}_x \ni \uparrow_{[xp_i]}$ for every i. ???

7.15.

7.16.

7.18.

1.19.

1.22. Given a pair of points $x_0, y_0 \in \mathcal{K}$, consider two sequences $x_0, x_1, \dots$ and $y_0, y_1, \dots$ such that $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ for each n.

Since $\mathcal{K}$ is compact, we can choose an increasing sequence of integers n_k such that both sequences $(x_{n_i})_{i=1}^{\infty}$ and $(y_{n_i})_{i=1}^{\infty}$ converge. In particular, both are Cauchy; that is,

$$|x_{n_i} - x_{n_j}|_{\mathcal{K}} \to 0 \quad \text{and} \quad |y_{n_i} - y_{n_j}|_{\mathcal{K}} \to 0$$

as $\min\{i, j\} \to \infty$.

Since f is distance-noncontracting,

$$|x_0 - x_{|n_i - n_j|}| \leqslant |x_{n_i} - x_{n_j}|$$

for any i and j. Therefore, there is a sequence $m_i \to \infty$ such that

$$(*) \qquad x_{m_i} \to x_0 \quad \text{and} \quad y_{m_i} \to y_0$$

as $i \to \infty$.

Since f is distance-noncontracting, the sequence $\ell_n = |x_n - y_n|_{\mathcal{K}}$ is nondecreasing. By $(*)$, $\ell_{m_i} \to \ell_0$ as $m_i \to \infty$. It follows that

$$\ell_0 = \ell_1 = \dots$$

In particular,

$$|x_0 - y_0|_{\mathcal{K}} = \ell_0 = \ell_1 = |f(x_0) - f(y_0)|_{\mathcal{K}}$$

for any pair of points (x_0, y_0) in $\mathcal{K}$. That is, the map f is distance-preserving; hence f is injective. From $(*)$, we also get that $f(\mathcal{K})$ is everywhere dense. Since $\mathcal{K}$ is compact $f\colon \mathcal{K} \to \mathcal{K}$ is surjective — hence the result.

Remarks. This is a basic lemma in the introduction to Gromov–Hausdorff distance [see 7.3.30 in 10]. The presented proof is not quite standard, I learned it from Travis Morrison, a student in my MASS class at Penn State, Fall 2011.

Note that this exercise implies that *any surjective non-expanding map from a compact metric space to itself is an isometry.*

4.4.

8.2.

8.4.

8.8.

8.9.

Comment. A stronger statement holds

$$\operatorname{vol}_m \mathcal{L}_\infty = \lim_{n\to\infty} \operatorname{vol}_m \mathcal{L}_n;$$

in other words, if $\boldsymbol{K} \subset \mathrm{GH}$ denotes the set of isometry classes of all compact $\mathrm{A{\scriptstyle LEX}}(\kappa)$ spaces with dimension $\leqslant m$, then the function $\operatorname{vol}_m \colon \boldsymbol{K} \to \mathbb{R}$ is continuous.

10.1.

10.2.

10.5.

10.6. *(a) (b) (c)*

10.8.

10.9.

10.10.

10.13.

10.14.

10.15.

10.16. Choose a geodesic γ in $\mathcal{W}$. Arguing as in the proof of 10.12d, we get that γ can cross the common boundary of two halves $\mathcal{L}_0$ and $\mathcal{L}_1$ of $\mathcal{W}$ at most once, or it lies in the common boundary.

In the later case λ-concavity of $f \circ \operatorname{proj} \circ \gamma$ follows from λ-concavity of f. In the former case the convexity has to be checked only at the point of crossing; we may assume that it happens at $x = \gamma(0)$. Since $\nabla_x f \in \partial \mathrm{T}_x$ for any $x \in \partial \mathcal{L}$ the f-gradient flows agree on $\mathcal{L}_0$ and $\mathcal{L}_1$.

Assume $f \circ \operatorname{proj} \circ \gamma$ is not λ-concavity at 0. Apply f-gradinent flow to shorten γ keeping its ends as in the proof of 10.15, and arrive at a contradiction.

11.2.

11.3.

11.4.

11.9; *(a).* Our $\mathbb{S}^1$ is a commutative subgroup of $\mathrm{SO}(3)$. Therefore it is a subgroup of a maximal torus in $\mathrm{SO}(3)$. Show that the described torus action is induced by a maximal torus in $\mathrm{SO}(3)$. Use that maximal tori in $\mathrm{SO}(3)$ are conjugate.

(b). Cut $\mathbb{S}^3$ into two solid tori the Clifford torus $\frac{1}{\sqrt{2}}\cdot\mathbb{S}^1\times\mathbb{S}^1$. Observe that the quotient of each solid torus is a disc; conclude that $\Sigma_{p,q}$ is a sphere. The torus action on $\mathbb{S}^3$ induce the needed $\mathbb{S}^1$-cation on $\Sigma_{p,q}$.

(c)+(d)+(e). Straightforward calculations.

(f). Consider the map $\Sigma_{p,q}\to\Sigma_{1,1}$ that sends poles to poles, preserve the distance to the poles and respects the $\mathbb{S}^1$ action.

11.15; *(a).* Suppose $\#_{m-1}(\Gamma)\geqslant 3$; that is $\mathcal{L}=\mathbb{E}^m/\Gamma$ has at least 3 boundary components. Follow Case 3 in the proof 11.7 to glue a train-space from copies of $\mathcal{L}$ using two of these components. Show that the obtained space splits and arrive at a contradiction.

(Alternatively, apply a similar construction to all components of the boundary. Show that obtained space has exponential volume growth; that is, there is $a>1$ such that $\operatorname{vol}\mathrm{B}(p,r)>a^r$ for all large r. Arrive at a contradiction with Bishop–Gromov inequality.)

(b). Apply the doubling theorem as in Case 2 in the proof 11.7.

12.1. We can assume that the origin lies in the interior of the convex body. Consider the central projection from the surface, say Σ to the sphere $\mathbb{S}^2$ centered at the origin. Show that this projection $\Sigma\to\mathbb{S}^2$ is a homeomorphism.

12.4. Follow the argument in 12.2. Show that the inequality is strict if and only if F has opposite points.

12.5. Suppose a geodesic γ pass thru a vertex v. Denote by α and β the angles that γ cuts at v. Since v is essential, $\alpha+\beta<2\cdot\pi$. Therefore $\alpha<\pi$ or $\beta<\pi$. Arrive at a contradiction by showing that γ is not length-minimizing.

12.6; *(a).* Cut the surface of T along three edges coming from one vertex v_1 and unfold the obtained surface on the plane. Show that this way we get a triangle, the three vertices correspond to v_1 and the midpoints of sides correspond to the remaining three vertices. Make a conclusion.

(b). Suppose that $0,v_1,v_2,v_3\in\mathbb{R}^3$ are the vertices of T. From *(a)*, we have that

$$|v_1|=|v_2-v_3|,\quad |v_2|=|v_3-v_1|,\quad |v_3|=|v_1-v_1|.$$

Use it to show that $\langle v_1,v_2+v_3-v_1\rangle=0$. Make a conclusion.

12.8. We need to show that if a polyhedral surface is $\textsc{Alex}(0)$, then the total angle θ of any of its vertex p it at most $2\cdot\pi$.

Assume that $\theta > 2\cdot\pi$, let $\varphi = \max\{\,\pi, \frac{1}{3}\cdot\theta\,\}$. Note that we can choose three points x_1, x_2, and x_3 close to p such that $\measuredangle[p\,{}^{x_i}_{x_j}] = \varphi$ for $i \ne j$. Since the points x_i are close to p, we have $\measuredangle[p\,{}^{x_i}_{x_j}] = \tilde\measuredangle(p\,{}^{x_i}_{x_j})$. The latter contradicts $\mathbb{E}^2$-comparison.

12.9. We will use that the closest-point projection form the Euclidean space to a convex body is short; that is, distance-nonexpanding [49, 13.3].

Assume K_∞ is nondegenerate. Without loss of generality, we may assume that

$$\overline{\mathrm{B}}(0,r) \subset K_\infty \subset \overline{\mathrm{B}}(0,1)$$

for some $r > 0$. Note that there is a sequence $\varepsilon_n \to 0$ such that

$$K_n \subset (1+\varepsilon_n)\cdot K_\infty \quad \text{and} \quad K_\infty \subset (1+\varepsilon_n)\cdot K_n$$

for each large n.

Given $x \in K_n$, denote by $g_n(x)$ the closest-point projection of $(1+\varepsilon_n)\cdot x$ to K_∞. Similarly, given $x \in K_\infty$, denote by $h_n(x)$ the closest point projection of $(1+\varepsilon_n)\cdot x$ to K_n. Note that

$$|g_n(x) - g_n(y)| \leqslant (1+\varepsilon_n)\cdot|x-y|$$

and

$$|h_n(x) - h_n(y)| \leqslant (1+\varepsilon_n)\cdot|x-y|.$$

Denote by Σ_∞ and Σ_n the surface of K_∞ and K_n respectively. The above inequlities imply

$$|g_n(x) - g_n(y)|_{\Sigma_\infty} \leqslant (1+\varepsilon_n)\cdot|x-y|_{\Sigma_n}$$

for any $x, y \in \Sigma_n$, and

$$|h_n(x) - h_n(y)|_{\Sigma_n} \leqslant (1+\varepsilon_n)\cdot|x-y|_{\Sigma_\infty}.$$

for any $x, y \in \Sigma_\infty$.

Note that the maps g_n and h_n are onto. Apply 1.23 to finish the proof.

Alternatively, since the closest-point projection cannot increase the length of curve, we also get

$$\begin{aligned} |x - h_n\circ g_n(x)|_{\Sigma_\infty} &\leqslant 10\cdot\varepsilon_n \\ |y - g_n\circ h_n(y)|_{\Sigma_n} &\leqslant 10\cdot\varepsilon_n. \end{aligned}$$

for all large n. Therefore, g_n is a δ_n-isometry $\Sigma_n \to \Sigma_\infty$ for a sequence $\delta_n \to 0$.

Comments. More generally, if a sequence of m-dimensional $\mathrm{A\scriptstyle LEX}(\kappa)$ spaces $\mathcal{L}_1, \mathcal{L}_2, \dots$ converges to $\mathcal{L}_\infty$ and $\dim \mathcal{L}_\infty = m < \infty$, then $\partial\mathcal{L}_n$ equipped with induced length metric converge to $\partial\mathcal{L}_\infty$. This statement is a partial case of the theorem about extremal subsets proved by the second author [52, 1.2].

12.11; *(a).* By 10.15, the function $f_p\colon t \mapsto \mathrm{dist}_p \circ \gamma(t)$ is semiconcave for any $p \in K$. In particular, one-sided derivatives $f_p^+(t)$ are defined for every t.

Given $x = \gamma(t)$, choose three points $p_1, p_2, p_3 \in K$ in general position; that is, the four points x, p_1, p_2, p_3 do not lie in one plane. Observe that distance functions dist_{p_i} give smooth coordinates in a neighborhood of x. From above the functions f_{p_i} have one-sided derivatives at t. Since the coordinates are smooth we get that $\gamma^+(t)$ is defined as well.

(b). If the plane py_1y_2 supports K, then $\measuredangle[p\,{}^{y_1}_{y_2}]_{\mathbb{E}^3} = \measuredangle[p\,{}^{x_1}_{x_2}]_\Sigma$. In this case, the statement follows from 12.10.

Now suppose that the line segment $[y_1y_2]_{\mathbb{E}^3}$ intersects K. Choose a geodesic $[y_1y_2]_W$; note that it contains a point of K, say z. Now consider one parameter family of points $y_i(t) := \gamma(t) + \gamma^+(t)\cdot(1-t)\cdot|p-$ $- x_i|_\Sigma$. Note that this family is not continuous.

Show that for any point $p \in K$ the function $t \mapsto |p - \gamma_i(t)|_{\mathbb{E}^3}$ is nonincreasing. Conclude that the function $t \mapsto |p - \gamma_i(t)|_W$ is nonincreasing for any $p \in \Sigma$. Therefore,

$$\begin{aligned}
|y_1 - y_2|_W &= |y_1(0) - y_2(0)|_W = \\
&= |y_1(0) - z|_W + |y_2(0) - z|_W \geqslant \\
&\geqslant |y_1(1) - z|_W + |y_2(1) - z|_W \geqslant \\
&\geqslant |x_1 - x_2|_\Sigma.
\end{aligned}$$

The last inequality follows since the closest point projection $W \to \Sigma$ is short.

It remains to consider the case when the plane py_1y_2 does not support K, and $[y_1y_2]_{\mathbb{E}^3}$ does not intersect K. In this case the plane py_1y_2 intersects K along a convex figure F that lies in the solid triangle py_1y_2 and contains its vertex p.

Choose points $y_1' \in [py_1]_{\mathbb{E}^3}$ and $y_2' \in [py_2]_{\mathbb{E}^3}$ such that $[y_1'y_2']$ touches F. Denote by $x_1' \in [px_1]_\Sigma$ and $x_2' \in [px_2]_\Sigma$ the corresponding points; that is, $|p - y_1'|_{\mathbb{E}^3} = |p - x_1'|_\Sigma$ and $|p - y_2'|_{\mathbb{E}^3} = |p - x_2'|_\Sigma$. From above, we have that $|y_1' - y_2'|_{\mathbb{E}^3} \geqslant |x_1' - x_2'|_\Sigma$; in other words,

$$\tilde\measuredangle(p\,{}^{y_1'}_{y_2'}) \geqslant \tilde\measuredangle(p\,{}^{x_1'}_{x_2'});$$

here we think that $[py_1'y_2']$ is a triangle in $\mathbb{E}^3$, but $[px_1'x_2']$ is a triangle in Σ. Note that

$$\tilde\measuredangle(p\,{}^{y_1'}_{y_2'}) = \tilde\measuredangle(p\,{}^{y_1}_{y_2}) \quad\text{and}\quad \tilde\measuredangle(p\,{}^{x_1}_{x_2}) \leqslant \tilde\measuredangle(p\,{}^{x_1'}_{x_2'});$$

the second inequality follows from 2.7. Hence the remaining case follows.

Comments. Part *(a)* was originally proved by Joseph Liberman [40]; the proof of 10.15 is a generalized version of the so-called Liberman lemma — the main tools in studying geodesics on convex surfaces.

Part *(b)* is a result of Anatolii Milka [42, Theorem 2].

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