Lectures in metric geometry

Anton Petrunin

Disclaimer

Considerable part of the text is a compilation from [1–4].

Contents

Chapter 0

Homework assignments

You have to think about problems, but you do not have to solve them. If the problem is solved, you do not have to write the solutions, but try to sketch it (so you can read it yourself in one month or so).

0.1 Due Tue Jan 21

Exercises 1.3.1, 1.4.3, 1.7.2, 1.7.3, 1.7.7, 2.1.7.

Chapter 1

Definitions

1.1 Metric spaces

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

The function

$$\operatorname{dist}_x\colon y \mapsto |x-y|$$

is called the *distance function* from x.

Given $R \in [0,\infty]$ and $x \in \mathcal{X}$, the sets

$$\begin{aligned}
\mathrm{B}(x,R) &= \{\, y\in\mathcal{X} \mid |x-y| < R \,\},\\
\overline{\mathrm{B}}[x,R] &= \{\, y\in\mathcal{X} \mid |x-y| \leqslant R \,\}
\end{aligned}$$

are called, respectively, the *open* and the *closed balls* of radius R with center x. Again, if we need to emphasize that these balls are taken in the metric space $\mathcal{X}$, we write

$$\mathrm{B}(x,R)_{\mathcal{X}} \quad\text{and}\quad \overline{\mathrm{B}}[x,R]_{\mathcal{X}}.$$

1.2 Variations of definition

Recall that a metric is a real-valued function $(x,y)\mapsto|x-y|_{\mathcal{X}}$ that satisfies the following conditions for any three points $x,y,z\in\mathcal{X}$:

(i) $|x-y|_{\mathcal{X}} \geqslant 0$,

(ii) $|x-y|_{\mathcal{X}} = 0 \iff x = y$,

(iii) $|x-y|_{\mathcal{X}} = |y-x|_{\mathcal{X}}$,

(iv) $|x-y|_{\mathcal{X}} + |y-z|_{\mathcal{X}} \geqslant |x-z|_{\mathcal{X}}$,

Pseudometrics. A generalization of a metric in which the distance between two distinct points can be zero is called *pseudometric*. In other words, to define pseudometric, we need to remove condition (ii) from the list.

The following two observations show that nearly any question about pseudometric spaces can be reduced to a question about genuine metric spaces.

Assume $\mathcal{X}$ is a pseudometric space. Set $x \sim y$ if $|x-y| = 0$. Note that if $x \sim x'$, then $|y-x| = |y-x'|$ for any $y \in \mathcal{X}$. Thus, $|*-$ $-*|$ defines a metric on the quotient set $\mathcal{X}/\sim$. In this way we obtain a metric space $\mathcal{X}'$. The space $\mathcal{X}'$ is called the *corresponding metric space* for the pseudometric space $\mathcal{X}$. Often we do not distinguish between $\mathcal{X}'$ and $\mathcal{X}$.

∞-metrics. One may also consider metrics with values in $\mathbb{R}\cup\{\infty\}$; we might call then ∞-metrics or simply metrics.

Again nearly any question about ∞-metric spaces can be reduced to a question about genuine metric spaces.

Indeed, set $x \approx y$ if and only if $|x-y| < \infty$; this is an other equivalence relation on $\mathcal{X}$. The equivalence class of a point $x \in \mathcal{X}$ will be called the *metric component* of x; it will be denoted as $\mathcal{X}_x$. One could think of $\mathcal{X}_x$ as $\mathrm{B}(x,\infty)_{\mathcal{X}}$ — the open ball centered at x and radius ∞ in $\mathcal{X}$; see definition below.

It follows that any ∞-metric space is a *disjoint union* of genuine metric spaces — the metric components of the original ∞-metric space.

1.3 Completeness

Recall that a metric space $\mathcal{X}$ is called *complete* if every Cauchy sequence of points in $\mathcal{X}$ converges in $\mathcal{X}$.

1.3.1. Exercise. *Suppose that ρ is a positive continuous function on a complete metric space $\mathcal{X}$. Show that for any $\varepsilon > 0$ there is a point $x \in \mathcal{X}$ such that*

$$\rho(x) > (1-\varepsilon)\cdot\rho(y)$$

for any point $y \in \mathrm{B}(x,\rho(x))$.

Most of the time we will assume that a metric space is complete. The following construction produces a complete metric space $\bar{\mathcal{X}}$ for any given metric space $\mathcal{X}$. The space $\bar{\mathcal{X}}$ is called *completion* of $\mathcal{X}$; the original space $\mathcal{X}$ forms a dense subset in $\bar{\mathcal{X}}$.

Completion. Given metric space $\mathcal{X}$, consider the set of all Cauchy sequences in $\mathcal{X}$. Note that for any two Cauchy sequences (x_n) and (y_n)

the right hand side below is defined; moreover it defines a pseudometric on the set $\mathcal{C}$ of all Cauchy sequences

$$|(x_n)-(y_n)|_{\mathcal{C}} \stackrel{def}{=\!=} \lim_{n\to\infty} |x_n - y_n|_{\mathcal{X}}.$$

The corresponding metric space is called a completion of $\mathcal{X}$.

It is left as an exercise that completion of $\mathcal{X}$ is complete.

Note that for each point $x \in \mathcal{X}$ one can consider a constant sequence $x_n = x$ which is Cauchy. It defines a natural map $\mathcal{X} \to \bar{\mathcal{X}}$. It is easy to check that this map is distance preserving. In partucular we can (and will) consider $\mathcal{X}$ as a subset of $\bar{\mathcal{X}}$.

1.4 Compactness

Let us recall few equivalent definitions of compact metric spaces.

1.4.1. Definition. *A metric space $\mathcal{K}$ is compact if and only if one of the following equivalent condition holds:*

a) Every open cover of $\mathcal{K}$ has a finite subcover.

b) For any open cover of $\mathcal{K}$ there is $\varepsilon > 0$ such that any ε-ball in $\mathcal{K}$ lie in one element of the cover. (The value ε is called Lebesgue number of the covering.)

c) Every sequence in $\mathcal{K}$ has a convergent subsequence.

d) The space $\mathcal{K}$ is complete and totally bounded*; that is, for any $\varepsilon > 0$, the space $\mathcal{K}$ admits a finite cover by open ε-balls.*[1]

1.4.2. Exercise. *Let $\operatorname{pack}_\varepsilon \mathcal{X}$ be exact upper bound on the number of points $x_1, \dots, x_n \in \mathcal{X}$ such that $|x_i - x_j| > \varepsilon$ for any $i \neq j$.*

Show that a complete space $\mathcal{X}$ is compact if and only of $\operatorname{pack}_\varepsilon \mathcal{X} < \infty$ for any $\varepsilon > 0$.

1.4.3. Exercise. *Let K be a compact metric space and*

$$f\colon K \to K$$

be a distance non-decreasing map. Prove that f is an isometry.

A metric space $\mathcal{X}$ is called *proper* if all closed bounded sets in $\mathcal{X}$ are compact. This condition is equivalent to each of the following statements:

[1] Equivalently, for any $\varepsilon > 0$ there is a finite ε-*net*; that is a finite set of points $x_1, \dots, x_n \in \mathcal{K}$ such that any other point x lies on the distance less than ε from one of x_i.

1. For some (and therefore any) point $p \in \mathcal{X}$ and any $R < \infty$, the closed ball $\overline{\mathrm{B}}[p, R] \subset \mathcal{X}$ is compact.
2. The function $|p - {}|\colon \mathcal{X} \to \mathbb{R}$ is proper for some (and therefore any) point $p \in \mathcal{X}$; that is, for any compact set $K \subset \mathbb{R}$, its inverse image $\{\, x \in \mathcal{X} \mid |p - x|_{\mathcal{X}} \in K \,\}$ is compact.

A metric space $\mathcal{X}$ is called *locally compact* if any point in $\mathcal{X}$ admits a compact neighborhood; in other words, for any point $x \in \mathcal{X}$ a closed ball $\overline{\mathrm{B}}[x, r]$ is compact for some $r > 0$.

1.4.4. Exercise. *Give an example of space which is locally compact but not proper.*

1.5 Geodesics

Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ be a real interval. A globally isometric map $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a *geodesic*[2]; in other words, $\gamma\colon \mathbb{I} \to \mathcal{X}$ is a geodesic if

$$|\gamma(s) - \gamma(t)|_{\mathcal{X}} = |s - t|$$

for any pair $s, t \in \mathbb{I}$.

We say that $\gamma\colon \mathbb{I} \to \mathcal{X}$ is a geodesic from point p to point q if $\mathbb{I} = [a, b]$ and $p = \gamma(a)$, $q = \gamma(b)$. In this case the image of γ is denoted by $[pq]$ and with an abuse of notations we also call it a *geodesic*. Given a geodesic $[pq]$, we can parametrize it by distance to p; this parametrization will be denoted by $\mathrm{geod}_{[pq]}(t)$.

We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$. We also use the following shortcut notation:

$$]pq[= [pq]\backslash\{p, q\}, \qquad]pq] = [pq]\backslash\{p\}, \qquad [pq[= [pq]\backslash\{q\}.$$

In general, a geodesic between p and q need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we mean that we have made a choice of geodesic.

A metric space is called *geodesic* if any pair of its points can be joined by a geodesic.

A *geodesic path* is a geodesic with constant-speed parametrization by $[0, 1]$. Given a geodesic $[pq]$, we denote by $\mathrm{path}_{[pq]}$ the corresponding geodesic path; that is,

$$\mathrm{path}_{[pq]}(t) \stackrel{def}{=\!=} \mathrm{geod}_{[pq]}(t \cdot |p - q|).$$

A curve $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a *local geodesic* if for any $t \in \mathbb{I}$ there is a neighborhood U of t in $\mathbb{I}$ such that the restriction $\gamma|_U$ is a geodesic. A constant-speed parametrization of a local geodesic by the unit interval $[0, 1]$ is called a *local geodesic path*.

[2]Various authors call it differently: *shortest path*, *minimizing geodesic*.

1.6 Length

A *curve* is defined as a continuous map from a real interval to a space. If the real interval is $[0,1]$, then the curve is called a *path*.

1.6.1. Definition. *Let $\mathcal{X}$ be a metric space and $\alpha\colon \mathbb{I}\to\mathcal{X}$ be a curve. We define the* length *of α as*

$$\operatorname{length}\alpha \stackrel{def}{=\!=} \sup_{t_0\leqslant t_1\leqslant\ldots\leqslant t_n}\sum_i|\alpha(t_i)-\alpha(t_{i-1})|.$$

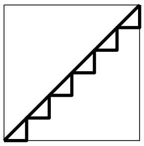
A curve α is called rectifiable *if* $\operatorname{length}\alpha<\infty$.

1.6.2. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\gamma_n\colon [a,b]\to\mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty\colon[a,b]\to\mathcal{X}$; that is, for any fixed $t\in[a,b]$, $\gamma_n(t)\to\gamma_\infty(t)$ as $n\to\infty$. Then

❶ $$\liminf_{n\to\infty}\operatorname{length}\gamma_n\geqslant\operatorname{length}\gamma_\infty.$$

Note that the inequality ❶ might be strict. For example the diagonal γ_∞ of the unit square can be approximated by a stairs-like polygonal curves γ_n with sides parallel to the sides of the square (γ_6 is on the picture). In this case



$$\operatorname{length}\gamma_\infty=\sqrt{2}\quad\text{and}\quad\operatorname{length}\gamma_n=2$$

for any n.

Proof. Fix a partition $a=t_0<t_1<\cdots<t_k=b$. Set

$$\Sigma_n \stackrel{def}{=\!=} |\gamma_n(t_0)-\gamma_n(t_1)|+\cdots+|\gamma_n(t_{k-1})-\gamma_n(t_k)|.$$
$$\Sigma_\infty \stackrel{def}{=\!=} |\gamma_\infty(t_0)-\gamma_\infty(t_1)|+\cdots+|\gamma_\infty(t_{k-1})-\gamma_\infty(t_k)|.$$

Note that for each i we have

$$|\gamma_n(t_{i-1})-\gamma_n(t_i)|\to|\gamma_\infty(t_{i-1})-\gamma_\infty(t_i)|$$

and therefore

$$\Sigma_n\to\Sigma_\infty$$

as $n\to\infty$. Note that

$$\Sigma_n\leqslant\operatorname{length}\gamma_n$$

for each n. Hence

❷
$$\liminf_{n\to\infty} \operatorname{length}\gamma_n \geqslant \Sigma_\infty.$$

If γ_∞ is rectifiable, we can assume that

$$\operatorname{length}\gamma_\infty < \Sigma_\infty + \varepsilon.$$

for any given $\varepsilon > 0$. By ❷ it follows that

$$\liminf_{n\to\infty} \operatorname{length}\gamma_n > \operatorname{length}\gamma_\infty - \varepsilon$$

for any $\varepsilon > 0$; whence ❶ follows.

It remains to consider the case when γ_∞ is not rectifiable; that is, $\operatorname{length}\gamma_\infty = \infty$. In this case we can choose a partition so that $\Sigma_\infty > L$ for any real number L. By ❷ it follows that

$$\liminf_{n\to\infty} \operatorname{length}\gamma_n > L$$

for any given L; whence

$$\liminf_{n\to\infty} \operatorname{length}\gamma_n = \infty$$

and ❶ follows. □

1.7 Length spaces

If for any $\varepsilon > 0$ and any pair of points x and y in a metric space $\mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length}\alpha < |x - y| + \varepsilon,$$

then $\mathcal{X}$ is called a *length space* and the metric on $\mathcal{X}$ is called a *length metric.*

Note that any geodesic space is a length space. As can be seen from the following example, the converse does not hold.

1.7.1. Example. *Let $\mathcal{X}$ be obtained by gluing a countable collection of disjoint intervals $\{\mathbb{I}_n\}$ of length $1 + \frac{1}{n}$, where for each $\mathbb{I}_n$ the left end is glued to p and the right end to q. Then $\mathcal{X}$ carries a natural complete length metric with respect to which $|p - q| = 1$ but there is no geodesic connecting p to q.*

1.7.2. Exercise. *Give an example of a complete length space for which no pair of distinct points can be joined by a geodesic.*

Directly from the definition, it follows that if a path $\alpha\colon [0,1] \to \mathcal{X}$ connects two points x and y (that is, if $\alpha(0) = x$ and $\alpha(1) = y$), then

$$\operatorname{length}\alpha \geqslant |x - y|.$$

Set

$$\|x - y\| = \inf\{\operatorname{length}\alpha\}$$

where the infimum is taken for all paths connecing x and y. It is straightforward to check that $(x, y) \mapsto \|x - y\|$ is an ∞-metric; moreover $(\mathcal{X}, \|{*}-{*}\|)$ is a length space. The metric $\|{*}-{*}\|$ is called *induced length metric.*

1.7.3. Exercise. *Suppose $(\mathcal{X}, |{*} - {*}|)$ is a compact metric space. Show that $(\mathcal{X}, \|{*} - {*}\|)$ is complete.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x - y|_A = \inf_{\alpha}\{\operatorname{length}\alpha\},$$

where the infimum is taken for all paths α from x to y in A.[3]

If $|x - y|_A$ takes finite value for each pair $x, y \in A$, then $|x - y|_A$ defines a metric on A; this metric will be called the *induced length metric* on A.

Let $\mathcal{X}$ be a metric space and $x, y \in \mathcal{X}$.

(i) A point $z \in \mathcal{X}$ is called a *midpoint* between x and y if

$$|x - z| = |y - z| = \tfrac{1}{2}{\cdot}|x - y|.$$

(ii) Assume $\varepsilon \geqslant 0$. A point $z \in \mathcal{X}$ is called an *ε-midpoint* between x and y if

$$|x - z|, \quad |y - z| \leqslant \tfrac{1}{2}{\cdot}|x - y| + \varepsilon.$$

Note that a 0-midpoint is the same as a midpoint.

1.7.4. Lemma. *Let $\mathcal{X}$ be a complete metric space.*

a) Assume that for any pair of points $x, y \in \mathcal{X}$ and any $\varepsilon > 0$ there is an ε-midpoint z. Then $\mathcal{X}$ is a length space.

b) Assume that for any pair of points $x, y \in \mathcal{X}$, there is a midpoint z. Then $\mathcal{X}$ is a geodesic space.

[3] Note that while this notation slightly conflicts with the previously defined notation for distance on a general metric space, we will usually work with ambient length spaces where the meaning will be unambiguous.

Proof. We first prove (a). Let $x, y \in \mathcal{X}$ be a pair of points.

Set $\varepsilon_n = \frac{\varepsilon}{4^n}$, $\alpha(0) = x$ and $\alpha(1) = y$.

Let $\alpha(\frac{1}{2})$ be an ε_1-midpoint between $\alpha(0)$ and $\alpha(1)$. Further, let $\alpha(\frac{1}{4})$ and $\alpha(\frac{3}{4})$ be ε_2-midpoints between the pairs $(\alpha(0), \alpha(\frac{1}{2}))$ and $(\alpha(\frac{1}{2}), \alpha(1))$ respectively. Applying the above procedure recursively, on the n-th step we define $\alpha(\frac{k}{2^n})$, for every odd integer k such that $0 < \frac{k}{2^n} < 1$, as an ε_n-midpoint between the already defined $\alpha(\frac{k-1}{2^n})$ and $\alpha(\frac{k+1}{2^n})$.

In this way we define $\alpha(t)$ for $t \in W$, where W denotes the set of dyadic rationals in $[0, 1]$. Since $\mathcal{X}$ is complete, the map α can be extended continuously to $[0, 1]$. Moreover,

❶
$$\begin{aligned}\operatorname{length}\alpha &\leqslant |x-y| + \sum_{n=1}^{\infty} 2^{n-1}\cdot\varepsilon_n \leqslant \\ &\leqslant |x-y| + \tfrac{\varepsilon}{2}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get (a).

To prove (b), one should repeat the same argument taking midpoints instead of ε_n-midpoints. In this case ❶ holds for $\varepsilon_n = \varepsilon = 0$. □

Since in a compact space a sequence of $\frac{1}{n}$-midpoints z_n contains a convergent subsequence, Lemma 1.7.4 immediately implies

1.7.5. Proposition. *A proper length space is geodesic.*

1.7.6. Hopf–Rinow theorem. *Any complete, locally compact length space is proper.*

Proof. Let $\mathcal{X}$ be a locally compact length space. Given $x \in \mathcal{X}$, denote by $\rho(x)$ the supremum of all $R > 0$ such that the closed ball $\overline{\mathrm{B}}[x, R]$ is compact. Since $\mathcal{X}$ is locally compact,

❷
$$\rho(x) > 0 \quad \text{for any} \quad x \in \mathcal{X}.$$

It is sufficient to show that $\rho(x) = \infty$ for some (and therefore any) point $x \in \mathcal{X}$.

Assume the contrary; that is, $\rho(x) < \infty$. We claim that

❸ $B = \overline{\mathrm{B}}[x, \rho(x)]$ *is compact for any* x.

Indeed, $\mathcal{X}$ is a length space; therefore for any $\varepsilon > 0$, the set $\overline{\mathrm{B}}[x, \rho(x) - \varepsilon]$ is a compact ε-net in B. Since B is closed and hence complete, it must be compact. △

Next we claim that

❹ $|\rho(x) - \rho(y)| \leqslant |x - y|_{\mathcal{X}}$ *for any* $x, y \in \mathcal{X}$*; in particular* $\rho\colon \mathcal{X} \to \mathbb{R}$ *is a continuous function.*

Indeed, assume the contrary; that is, $\rho(x) + |x - y| < \rho(y)$ for some $x, y \in \mathcal{X}$. Then $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$ is a closed subset of $\overline{\mathrm{B}}[y, \rho(y)]$ for some $\varepsilon > 0$. Then compactness of $\overline{\mathrm{B}}[y, \rho(y)]$ implies compactness of $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$, a contradiction. $\triangle$

Set $\varepsilon = \min\{\, \rho(y) \mid y \in B \,\}$; the minimum is defined since B is compact. From ❷, we have $\varepsilon > 0$.

Choose a finite $\frac{\varepsilon}{10}$-net $\{a_1, a_2, \dots, a_n\}$ in B. The union W of the closed balls $\overline{\mathrm{B}}[a_i, \varepsilon]$ is compact. Clearly $\overline{\mathrm{B}}[x, \rho(x) + \frac{\varepsilon}{10}] \subset W$. Therefore $\overline{\mathrm{B}}[x, \rho(x) + \frac{\varepsilon}{10}]$ is compact, a contradiction. $\square$

1.7.7. Exercise. *Construct a geodesic space that is locally compact, but whose completion is neither geodesic nor locally compact.*

Chapter 2

Convergence

2.1 Hausdorff convergence

Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$, consider the distance function to A

$$\operatorname{dist}_A : \mathcal{X} \to [0, \infty)$$

defined as

$$\operatorname{dist}_A(x) \stackrel{def}{=\!=} \inf_{a \in A} \{ |a - x|_{\mathcal{X}} \}.$$

2.1.1. Definition. *Let A and B be two compact subsets of a metric space $\mathcal{X}$. Then the* Hausdorff distance *between A and B is defined as*

$$|A - B|_{\mathcal{H}(\mathcal{X})} \stackrel{def}{=\!=} \sup_{x \in \mathcal{X}} \{ |\operatorname{dist}_A(x) - \operatorname{dist}_B(x)| \}.$$

Suppose A and B be two compact subsets of a metric space $\mathcal{X}$. It is straightforward to check that $|A - B|_{\mathcal{H}(\mathcal{X})} \leqslant R$ if and only if $\operatorname{dist}_A(b) \leqslant R$ for any $b \in B$ and $\operatorname{dist}_B(a) \leqslant R$ for any $a \in A$. In other words, $|A - B|_{\mathcal{H}(\mathcal{X})} < R$ if and only if B lies in a R-neighborhood of A, and A lies in a R-neighborhood of B.

Note that the set of all nonempty compact subsets of a metric space $\mathcal{X}$ equipped with the Hausdorff metric forms a metric space. This new metric space will be denoted as $\mathcal{H}(\mathcal{X})$.

2.1.2. Exercise. *Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$ define its diameter as*

$$\operatorname{diam} A \stackrel{def}{=\!=} \sup_{a, b \in A} |a - b|.$$

Show that

$$\operatorname{diam}\colon \mathcal{H}(\mathcal{X}) \to \mathbb{R}$$

is a continuous function.

2.1.3. Blaschke selection theorem. *Let $\mathcal{X}$ be a metric space. Then the space $\mathcal{H}(\mathcal{X})$ is compact if and only if $\mathcal{X}$ is compact.*

Proof; "only if" part. Note that the map $\iota\colon \mathcal{X} \to \mathcal{H}(\mathcal{X})$, defined as $\iota\colon x \mapsto \{x\}$ (that is, point x mapped to the one-point subset $\{x\}$ of $\mathcal{X}$) is distance preserving. Therefore $\mathcal{X}$ is isometric to the set $\iota(\mathcal{X})$ in $\mathcal{H}(\mathcal{X})$.

Note that for a nonempty subset $A \subset \mathcal{X}$, we have $\operatorname{diam} A = 0$ if and only if A is a one-point set. Therefore, from Exercise 2.1.2, it follows that $\iota(\mathcal{X})$ is closed in $\mathcal{H}(\mathcal{X})$.

Hence $\iota(\mathcal{X})$ is compact, as it is a closed subset of a compact space. Since $\mathcal{X}$ is isometric to $\iota(\mathcal{X})$, "only if" part follows. ☐

To prove "if" part we will need the following two lemmas.

2.1.4. Lemma. *Let $K_1 \supset K_2 \supset \dots$ be a sequence of nonempty compact sets in a metric space $\mathcal{X}$ then $K_\infty = \bigcap_n K_n$ is the Hausdorff limit of K_n; that is, $|K_\infty - K_n|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$.*

Proof. Note that K_∞ is compact; by finite intersection property, K_∞ is nonempty.

If the assertion were false, then there is $\varepsilon > 0$ such that for each n one can choose $x_n \in K_n$ such that $\operatorname{dist}_{K_\infty}(x_n) \geqslant \varepsilon$. Note that $x_n \in K_1$ for each n. Since K_1 is compact, there is a *partial limit*[1] x_∞ of x_n. Clearly $\operatorname{dist}_{K_\infty}(x_\infty) \geqslant \varepsilon$.

On the other hand, since K_n is closed and $x_m \in K_n$ for $m \geqslant n$, we get $x_\infty \in K_n$ for each n. It follows that $x_\infty \in K_\infty$ and therefore $\operatorname{dist}_{K_\infty}(x_\infty) = 0$, a contradiction. ☐

2.1.5. Lemma. *If $\mathcal{X}$ is a compact metric space then $\mathcal{H}(\mathcal{X})$ is complete.*

Proof. Let (Q_n) be a Cauchy sequence in $\mathcal{H}(\mathcal{X})$. Passing to a subsequence of Q_n we may assume that

❶ $$|Q_n - Q_{n+1}|_{\mathcal{H}(\mathcal{X})} \leqslant \tfrac{1}{10^n}$$

for each n.

[1] Partial limit is a limit of a subsequence.

Set

$$K_n = \left\{ x \in \mathcal{X} \,\middle|\, \mathrm{dist}_{Q_n}(x) \leqslant \tfrac{1}{10^n} \right\}$$

Since $\mathcal{X}$ is compact so is each K_n.

Clearly, $|Q_n - K_n|_{\mathcal{H}(\mathcal{X})} \leqslant \frac{1}{10^n}$ and from ❶, we get $K_n \supset K_{n+1}$ for each n. Set

$$K_\infty = \bigcap_{n=1}^{\infty} K_n.$$

Applying Lemma 2.1.4, we get that $|K_n - K_\infty|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$. Since $|Q_n - K_n|_{\mathcal{H}(\mathcal{X})} \leqslant \frac{1}{10^n}$, we get $|Q_n - K_\infty|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$ — hence the lemma. □

2.1.6. Exercise. *Let $\mathcal{X}$ be a complete metric space and K_n be a sequence of compact sets which converges in the sence of Hausdorff. Show that closure of the union $\bigcup_{n=1}^{\infty} K_n$ is compact.*

Use this to show that in Lemma 2.1.5 compactness of $\mathcal{X}$ can be exchanged to completeness.

Proof of "if" part in 2.1.3. According to Lemma 2.1.5, $\mathcal{H}(\mathcal{X})$ is complete. It remains to show that $\mathcal{H}(\mathcal{X})$ is totally bounded (1.4.1d); that is, given $\varepsilon > 0$ there is a finite ε-net in $\mathcal{H}(\mathcal{X})$.

Choose a finite ε-net A in $\mathcal{X}$. Denote by $\mathcal{A}$ the set of all subsets of A. Note that $\mathcal{A}$ is finite set in $\mathcal{H}(\mathcal{X})$. For each compact set $K \subset \mathcal{X}$, consider the subset K' of all points $a \in A$ such that $\mathrm{dist}_K(a) \leqslant \varepsilon$. Then $K' \in \mathcal{A}$ and $|K - K'|_{\mathcal{H}(\mathcal{X})} \leqslant \varepsilon$. In other words $\mathcal{A}$ is a finite ε-net in $\mathcal{H}(\mathcal{X})$. □

Hausdorff metric defines convergence of compact sets which is more important than metric itself.

2.1.7. Exercise. *Let X and Y be two compact subsets in $\mathbb{R}^2$. Assume $|X-Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, is it true that $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, where ∂X denotes the boundary of X.*

Does the converse holds? That is, assume X and Y be two compact subsets in $\mathbb{R}^2$ and $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$; is it true that $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} <$ $< \varepsilon$?

2.2 A variation

It seems that *Hausdorff convergence* was first introduced by Felix Hausdorff in [5], and a couple of years later an equivalent definition was given by Wilhelm Blaschke in [6].

The following refinement of the definition was introduced by Zdeněk Frolík in [7], and later rediscovered by Robert Wijsman in [8]. This refinement takes an intermediate place between the original Hausdorff convergence and *closed convergence*, also introduced by Hausdorff in [5]; so we still call it Hausdorff convergence.

2.2.1. Definition. *Let $\mathcal{X}$ be a proper metric space. We say that a sequence of closed sets A_n converges to a set A_∞ in the sense of Hausdorff if* $\mathrm{dist}_{A_n}(x) \to \mathrm{dist}_{A_\infty}(x)$ *for any* $x \in \mathcal{X}$.

2.2.2. Exercise. *Let $\mathcal{X}$ be a proper metric space and $(A_n)_{n=1}^\infty$ be a sequence of closed sets in $\mathcal{X}$. Assume that for some (and therefore any) point $x \in \mathcal{X}$, the sequence* $\mathrm{dist}_{A_n}(x)$ *is bounded. Show that the sequence $(A_n)_{n=1}^\infty$ has a convergent subsequence in the sense of Definition 2.2.1.*

Bibliography

[1] Stephanie Alexander, Vitali Kapovitch, and Anton Petrunin. *An invitation to Alexandrov geometry*. SpringerBriefs in Mathematics. CAT(0) spaces. Springer, Cham, 2019, pp. xii+88.

[2] Stephanie Alexander, Vitali Kapovitch, and Anton Petrunin. *Alexandrov geometry: preliminary version no. 1.* 2019. arXiv: `1903.08539 [math.DG]`.

[3] A. Petrunin and A. Yashinski. “Piecewise isometric mappings”. *Algebra i Analiz* 27.1 (2015), pp. 218–247.

[4] A. Petrunin and S. Zamora Barrera. *Differential geometry of curves and surfaces: a working approach.* URL: `https://anton-petrunin.github.io/comparison-geometry/curves-and-surfaces.pdf`.

[5] Felix Hausdorff. *Grundzüge der Mengenlehre*. Chelsea Publishing Company, New York, N. Y., 1949, pp. viii+476.

[6] Wilhelm Blaschke. *Kreis und Kugel*. Chelsea Publishing Co., New York, 1949, pp. x+169.

[7] Zdeněk Frolık. “Concerning topological convergence of sets”. *Czechoslovak Math. J* 10(85) (1960), pp. 168–180.

[8] R. A. Wijsman. “Convergence of sequences of convex sets, cones and functions. II”. *Trans. Amer. Math. Soc.* 123 (1966), pp. 32–45.