

# Lectures in metric geometry

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# Disclaimer

Considerable part of the text is a compilation from [3, 4, 40, 42, 43] and its drafts.

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# Chapter 0

## Homework assignments

It is better to think about all the problems, but you do not have to solve *all* of them. If a problem is solved, you do not have to write its solutions, but try sketch it.

### 0.1 Due Tue Jan 21

Exercises: ~~1.3.1~~, 1.4.4, 1.7.2, ~~1.7.3~~, 1.7.8, 2.1.7.

### 0.2 Due Tue Jan 28

Exercises: 1.3.1, 1.7.3, 2.1.8, 3.4.4, 3.5.1, 3.5.2*a*.

### 0.3 Due Tue Feb 4

Exercises: 1.8.3, 3.5.2*b*, ~~3.6.3~~, ~~3.6.4~~, 4.4.1, 4.4.3.

### 0.4 Due Tue Feb 11

Finish exercises 1.8.3 , 3.5.2*b*, 3.6.3, 3.6.4.

Exercises: 4.3.3, 4.5.1, 5.2.2, 5.3.2.

### 0.5 Due Tue Feb 18

Exercises: 5.2.3, 6.1.2, 6.2.3, 6.2.5, 6.3.1, 6.3.4.

Write down a solution of at least one of the exercises.

## 0.6 Due Tue Feb 25

Finish Exercise 6.3.1*b*. Prepare questions for review on Tuesday.

## 0.7 Due Tue Mar 3

Exercises: 7.3.1, 7.3.2, 7.3.3, 7.4.2, ~~7.5.2~~, 7.7.3.

Write down a solution of at least one of the exercises.

## 0.8 Due Tue Mar 17

Exercises: 7.2.2, 7.5.2, 7.7.4, 8.2.3, 8.3.7, 8.4.2.

Write down a solution of at least one of the exercises.

## 0.9 Due Tue Mar 24

Exercises: 7.6.3, 7.7.5, 7.7.6, 8.5.3, 9.2.5, ~~9.3.6~~.

Write down as many solutions as you can; email it to Zetian Yan (zxy5156) + cc to me (aqp6).

## 0.10 Due Tue Mar 31

Exercises: 9.3.6, 10.1.2, 10.1.3, 10.3.3, 10.4.6, 10.4.7.

Write down as many solutions as you can; email it to Zetian Yan (zxy5156) + cc to me (aqp6).

## Remark

Each working day I will check email before 15:00 and will appear online if you ask (it is easy for me — do not hesitate to ask). We will meet regular hours online (as we did before).



# Part I

## Pure metric geometry



# Chapter 1

## Definitions

### 1.1 Metric spaces

The distance between two points  $x$  and  $y$  in a metric space  $\mathcal{X}$  will be denoted by  $|x - y|$  or  $|x - y|_{\mathcal{X}}$ . The latter notation is used if we need to emphasize that the distance is taken in the space  $\mathcal{X}$ .

The function

$$\text{dist}_x: y \mapsto |x - y|$$

is called the *distance function* from  $x$ .

Given  $R \in [0, \infty]$  and  $x \in \mathcal{X}$ , the sets

$$B(x, R) = \{y \in \mathcal{X} \mid |x - y| < R\},$$

$$\overline{B}[x, R] = \{y \in \mathcal{X} \mid |x - y| \leq R\}$$

are called, respectively, the *open* and the *closed balls* of radius  $R$  with center  $x$ . Again, if we need to emphasize that these balls are taken in the metric space  $\mathcal{X}$ , we write

$$B(x, R)_{\mathcal{X}} \quad \text{and} \quad \overline{B}[x, R]_{\mathcal{X}}.$$

### 1.2 Variations of definition

Recall that a metric is a real-valued function  $(x, y) \mapsto |x - y|_{\mathcal{X}}$  that satisfies the following conditions for any three points  $x, y, z \in \mathcal{X}$ :

- (i)  $|x - y|_{\mathcal{X}} \geq 0$ ,
- (ii)  $|x - y|_{\mathcal{X}} = 0 \iff x = y$ ,
- (iii)  $|x - y|_{\mathcal{X}} = |y - x|_{\mathcal{X}}$ ,
- (iv)  $|x - y|_{\mathcal{X}} + |y - z|_{\mathcal{X}} \geq |x - z|_{\mathcal{X}}$ ,

**Pseudometrics.** A generalization of a metric in which the distance between two distinct points can be zero is called *pseudometric*. In other words, to define pseudometric, we need to remove condition (ii) from the list.

The following two observations show that nearly any question about pseudometric spaces can be reduced to a question about genuine metric spaces.

Assume  $\mathcal{X}$  is a pseudometric space. Set  $x \sim y$  if  $|x - y| = 0$ . Note that if  $x \sim x'$ , then  $|y - x| = |y - x'|$  for any  $y \in \mathcal{X}$ . Thus,  $|\ast - \ast|$  defines a metric on the quotient set  $\mathcal{X}/\sim$ . In this way we obtain a metric space  $\mathcal{X}'$ . The space  $\mathcal{X}'$  is called the *corresponding metric space* for the pseudometric space  $\mathcal{X}$ . Often we do not distinguish between  $\mathcal{X}'$  and  $\mathcal{X}$ .

**$\infty$ -metrics.** One may also consider metrics with values in  $\mathbb{R} \cup \{\infty\}$ ; we might call them  $\infty$ -metrics or simply metrics.

Again nearly any question about  $\infty$ -metric spaces can be reduced to a question about genuine metric spaces.

Indeed, set  $x \approx y$  if and only if  $|x - y| < \infty$ ; this is an other equivalence relation on  $\mathcal{X}$ . The equivalence class of a point  $x \in \mathcal{X}$  will be called the *metric component* of  $x$ ; it will be denoted as  $\mathcal{X}_x$ . One could think of  $\mathcal{X}_x$  as  $B(x, \infty)_{\mathcal{X}}$  — the open ball centered at  $x$  and radius  $\infty$  in  $\mathcal{X}$ .

It follows that any  $\infty$ -metric space is a *disjoint union* of genuine metric spaces — the metric components of the original  $\infty$ -metric space.

**1.2.1. Exercise.** Given two sets  $A$  and  $B$  on the plane, set

$$|A - B| = \mu(A \setminus B) + \mu(B \setminus A),$$

where  $\mu$  denotes the Lebesgue measure.

- (a) Show that  $|\ast - \ast|$  is a pseudometric on the set of bounded measurable sets of the plane.
- (b) Show that  $|\ast - \ast|$  is an  $\infty$ -metric on the set of all open sets of the plane.

## 1.3 Completeness

Recall that a metric space  $\mathcal{X}$  is called *complete* if every Cauchy sequence of points in  $\mathcal{X}$  converges in  $\mathcal{X}$ .

**1.3.1. Exercise.** Suppose that  $\rho$  is a positive continuous function on a complete metric space  $\mathcal{X}$ . Show that for any  $\varepsilon > 0$  there is a point  $x \in \mathcal{X}$  such that

$$\rho(x) < (1 + \varepsilon) \cdot \rho(y)$$

for any point  $y \in B(x, \rho(x))$ .

Most of the time we will assume that a metric space is complete. The following construction produces a complete metric space  $\bar{\mathcal{X}}$  for any given metric space  $\mathcal{X}$ . The space  $\bar{\mathcal{X}}$  is called *completion* of  $\mathcal{X}$ ; the original space  $\mathcal{X}$  forms a dense subset in  $\bar{\mathcal{X}}$ .

**Completion.** Given metric space  $\mathcal{X}$ , consider the set of all Cauchy sequences in  $\mathcal{X}$ . Note that for any two Cauchy sequences  $(x_n)$  and  $(y_n)$  the right hand side in **1** is defined; moreover it defines a pseudometric on the set  $\mathcal{C}$  of all Cauchy sequences

$$\mathbf{1} \quad |(x_n) - (y_n)|_{\mathcal{C}} := \lim_{n \rightarrow \infty} |x_n - y_n|_{\mathcal{X}}.$$

The corresponding metric space is called a completion of  $\mathcal{X}$ .

It is left as an exercise that completion of  $\mathcal{X}$  is complete.

Note that for each point  $x \in \mathcal{X}$  one can consider a constant sequence  $x_n = x$  which is Cauchy. It defines a natural map  $\mathcal{X} \rightarrow \bar{\mathcal{X}}$ . It is easy to check that this map is distance preserving. In particular we can (and will) consider  $\mathcal{X}$  as a subset of  $\bar{\mathcal{X}}$ .

## 1.4 Compactness

Let us recall few equivalent definitions of compact metric spaces.

**1.4.1. Definition.** A metric space  $\mathcal{K}$  is compact if and only if one of the following equivalent condition holds:

- (a) Every open cover of  $\mathcal{K}$  has a finite subcover.
- (b) For any open cover of  $\mathcal{K}$  there is  $\varepsilon > 0$  such that any  $\varepsilon$ -ball in  $\mathcal{K}$  lie in one element of the cover. (The value  $\varepsilon$  is called Lebesgue number of the covering.)
- (c) Every sequence in  $\mathcal{K}$  has a convergent subsequence.
- (d) The space  $\mathcal{K}$  is complete and totally bounded; that is, for any  $\varepsilon > 0$ , the space  $\mathcal{K}$  admits a finite cover by open  $\varepsilon$ -balls.

Alternatively totally bounded spaces can be defined the following way.

A metric space  $\mathcal{K}$  is *totally bounded* if for any  $\varepsilon > 0$  there is a finite  $\varepsilon$ -net; that is, a finite set of points  $x_1, \dots, x_n \in \mathcal{K}$  such that any other point  $x$  lies on the distance less than  $\varepsilon$  from one of  $x_i$ .

**1.4.2. Exercise.** Show that a space  $\mathcal{K}$  is totally bounded if and only if it contains a compact  $\varepsilon$ -net for any  $\varepsilon > 0$ .

Let  $\text{pack}_{\varepsilon} \mathcal{X}$  be exact upper bound on the number of points  $x_1, \dots, x_n \in \mathcal{X}$  such that  $|x_i - x_j| \geq \varepsilon$  for any  $i \neq j$ .

If  $n = \text{pack}_\varepsilon \mathcal{X} < \infty$ , then the collection of points  $x_1, \dots, x_n$  is called a *maximal  $\varepsilon$ -packing*. Note that  $n$  is the maximal number of open disjoint  $\frac{\varepsilon}{2}$ -balls in  $\mathcal{X}$ .

**1.4.3. Exercise.** Show that a complete space  $\mathcal{X}$  is compact if and only of  $\text{pack}_\varepsilon \mathcal{X} < \infty$  for any  $\varepsilon > 0$ .

Show that any maximal  $\varepsilon$ -packing is an  $\varepsilon$ -net.

**1.4.4. Exercise.** Let  $\mathcal{K}$  be a compact metric space and

$$f: \mathcal{K} \rightarrow \mathcal{K}$$

be a distance non-decreasing map. Prove that  $f$  is an isometry.

A metric space  $\mathcal{X}$  is called *proper* if all closed bounded sets in  $\mathcal{X}$  are compact. This condition is equivalent to each of the following statements:

- ◇ For some (and therefore any) point  $p \in \mathcal{X}$  and any  $R < \infty$ , the closed ball  $\overline{B}[p, R]_\mathcal{X}$  is compact.
- ◇ The function  $\text{dist}_p: \mathcal{X} \rightarrow \mathbb{R}$  is proper for some (and therefore any) point  $p \in \mathcal{X}$ ; that is, for any compact set  $K \subset \mathbb{R}$ , its inverse image

$$\text{dist}_p^{-1}(K) = \{x \in \mathcal{X} : |p - x|_\mathcal{X} \in K\}$$

is compact.

A metric space  $\mathcal{X}$  is called *locally compact* if any point in  $\mathcal{X}$  admits a compact neighborhood; in other words, for any point  $x \in \mathcal{X}$  a closed ball  $\overline{B}[x, r]$  is compact for some  $r > 0$ .

## 1.5 Geodesics

Let  $\mathcal{X}$  be a metric space and  $\mathbb{I}$  a real interval. A globally isometric map  $\gamma: \mathbb{I} \rightarrow \mathcal{X}$  is called a *geodesic*<sup>1</sup>; in other words,  $\gamma: \mathbb{I} \rightarrow \mathcal{X}$  is a geodesic if

$$|\gamma(s) - \gamma(t)|_\mathcal{X} = |s - t|$$

for any pair  $s, t \in \mathbb{I}$ .

We say that  $\gamma: \mathbb{I} \rightarrow \mathcal{X}$  is a geodesic from point  $p$  to point  $q$  if  $\mathbb{I} = [a, b]$  and  $p = \gamma(a)$ ,  $q = \gamma(b)$ . In this case the image of  $\gamma$  is denoted by  $[pq]$  and with an abuse of notations we also call it a *geodesic*. Given a geodesic  $[pq]$ , we can parametrize it by distance to  $p$ ; this parametrization will be denoted by  $\text{geod}_{[pq]}(t)$ .

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<sup>1</sup>Various authors call it differently: *shortest path*, *minimizing geodesic*.

We may write  $[pq]_{\mathcal{X}}$  to emphasize that the geodesic  $[pq]$  is in the space  $\mathcal{X}$ . We also use the following shortcut notation:

$$]pq[ = [pq] \setminus \{p, q\}, \quad ]pq] = [pq] \setminus \{p\}, \quad [pq[ = [pq] \setminus \{q\}.$$

In general, a geodesic from  $p$  to  $q$  need not exist and if it exists, it need not be unique. However, once we write  $[pq]$  we assume mean that we have made a choice of geodesic.

A metric space is called *geodesic* if any pair of its points can be joined by a geodesic.

A *geodesic path* is a geodesic with constant-speed parametrization by  $[0, 1]$ . Given a geodesic  $[pq]$ , we denote by  $\text{path}_{[pq]}$  the corresponding geodesic path; that is,

$$\text{path}_{[pq]}(t) := \text{geod}_{[pq]}(t \cdot |p - q|).$$

A curve  $\gamma: \mathbb{I} \rightarrow \mathcal{X}$  is called a *local geodesic* if for any  $t \in \mathbb{I}$  there is a neighborhood  $U$  of  $t$  in  $\mathbb{I}$  such that the restriction  $\gamma|_U$  is a geodesic. A constant-speed parametrization of a local geodesic by the unit interval  $[0, 1]$  is called a *local geodesic path*.

## 1.6 Length

A *curve* is defined as a continuous map from a real interval to a space. If the real interval is  $[0, 1]$ , then the curve is called a *path*.

**1.6.1. Definition.** Let  $\mathcal{X}$  be a metric space and  $\alpha: \mathbb{I} \rightarrow \mathcal{X}$  be a curve. We define the length of  $\alpha$  as

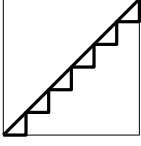
$$\text{length } \alpha := \sup_{t_0 \leq t_1 \leq \dots \leq t_n} \sum_i |\alpha(t_i) - \alpha(t_{i-1})|.$$

A curve  $\alpha$  is called *rectifiable* if  $\text{length } \alpha < \infty$ .

**1.6.2. Theorem.** Length is a lower semi-continuous with respect to pointwise convergence of curves.

More precisely, assume that a sequence of curves  $\gamma_n: \mathbb{I} \rightarrow \mathcal{X}$  in a metric space  $\mathcal{X}$  converges pointwise to a curve  $\gamma_\infty: \mathbb{I} \rightarrow \mathcal{X}$ ; that is, for any fixed  $t \in \mathbb{I}$ ,  $\gamma_n(t) \rightarrow \gamma_\infty(t)$  as  $n \rightarrow \infty$ . Then

$$\textbf{1} \quad \liminf_{n \rightarrow \infty} \text{length } \gamma_n \geq \text{length } \gamma_\infty.$$



Note that the inequality **❶** might be strict. For example the diagonal  $\gamma_\infty$  of the unit square can be approximated by a stairs-like polygonal curves  $\gamma_n$  with sides parallel to the sides of the square ( $\gamma_6$  is on the picture). In this case

$$\text{length } \gamma_\infty = \sqrt{2} \quad \text{and} \quad \text{length } \gamma_n = 2$$

for any  $n$ .

*Proof.* Fix a sequence  $t_0 < t_1 < \dots < t_k$  in  $\mathbb{I}$ . Set

$$\Sigma_n := |\gamma_n(t_0) - \gamma_n(t_1)| + \dots + |\gamma_n(t_{k-1}) - \gamma_n(t_k)|.$$

$$\Sigma_\infty := |\gamma_\infty(t_0) - \gamma_\infty(t_1)| + \dots + |\gamma_\infty(t_{k-1}) - \gamma_\infty(t_k)|.$$

Note that for each  $i$  we have

$$|\gamma_n(t_{i-1}) - \gamma_n(t_i)| \rightarrow |\gamma_\infty(t_{i-1}) - \gamma_\infty(t_i)|$$

and therefore

$$\Sigma_n \rightarrow \Sigma_\infty$$

as  $n \rightarrow \infty$ . Note that

$$\Sigma_n \leq \text{length } \gamma_n$$

for each  $n$ . Hence

$$\text{❷} \quad \liminf_{n \rightarrow \infty} \text{length } \gamma_n \geq \Sigma_\infty.$$

If  $\gamma_\infty$  is rectifiable, we can assume that

$$\text{length } \gamma_\infty < \Sigma_\infty + \varepsilon.$$

for any given  $\varepsilon > 0$ . By **❷** it follows that

$$\liminf_{n \rightarrow \infty} \text{length } \gamma_n > \text{length } \gamma_\infty - \varepsilon$$

for any  $\varepsilon > 0$ ; whence **❶** follows.

It remains to consider the case when  $\gamma_\infty$  is not rectifiable; that is,  $\text{length } \gamma_\infty = \infty$ . In this case we can choose a partition so that  $\Sigma_\infty > L$  for any real number  $L$ . By **❷** it follows that

$$\liminf_{n \rightarrow \infty} \text{length } \gamma_n > L$$

for any given  $L$ ; whence

$$\liminf_{n \rightarrow \infty} \text{length } \gamma_n = \infty$$

and **❶** follows. □



## 1.7 Length spaces

If for any  $\varepsilon > 0$  and any pair of points  $x$  and  $y$  in a metric space  $\mathcal{X}$ , there is a path  $\alpha$  connecting  $x$  to  $y$  such that

$$\text{length } \alpha < |x - y| + \varepsilon,$$

then  $\mathcal{X}$  is called a *length space* and the metric on  $\mathcal{X}$  is called a *length metric*.

Note that any geodesic space is a length space. As can be seen from the following example, the converse does not hold.

**1.7.1. Example.** Let  $\mathcal{X}$  be obtained by gluing a countable collection of disjoint intervals  $\{\mathbb{I}_n\}$  of length  $1 + \frac{1}{n}$ , where for each  $\mathbb{I}_n$  the left end is glued to  $p$  and the right end to  $q$ .

Observe that the space  $\mathcal{X}$  carries a natural complete length metric with respect to which  $|p - q| = 1$  but there is no geodesic connecting  $p$  to  $q$ .

**1.7.2. Exercise.** Give an example of a complete length space for which no pair of distinct points can be joined by a geodesic.

Directly from the definition, it follows that if a path  $\alpha: [0, 1] \rightarrow \mathcal{X}$  connects two points  $x$  and  $y$  (that is, if  $\alpha(0) = x$  and  $\alpha(1) = y$ ), then

$$\text{length } \alpha \geq |x - y|.$$

Set

$$\|x - y\| = \inf\{\text{length } \alpha\}$$

where the greatest lower bound is taken for all paths connecting  $x$  and  $y$ . It is straightforward to check that  $(x, y) \mapsto \|x - y\|$  is an  $\infty$ -metric; moreover  $(\mathcal{X}, \|\ast - \ast\|)$  is a length space. The metric  $\|\ast - \ast\|$  is called *induced length metric*.

**1.7.3. Exercise.** Suppose  $(\mathcal{X}, |\ast - \ast|)$  is a complete metric space. Show that  $(\mathcal{X}, \|\ast - \ast\|)$  is complete.

Let  $A$  be a subset of a metric space  $\mathcal{X}$ . Given two points  $x, y \in A$ , consider the value

$$|x - y|_A = \inf_{\alpha} \{\text{length } \alpha\},$$

where the greatest lower bound is taken for all paths  $\alpha$  from  $x$  to  $y$  in  $A$ .<sup>2</sup>

Let  $\mathcal{X}$  be a metric space and  $x, y \in \mathcal{X}$ .

---

<sup>2</sup>This notation slightly conflicts with the previously defined notation for distance  $|x - y|_{\mathcal{X}}$  in a metric space  $\mathcal{X}$ . However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.

- (i) A point  $z \in \mathcal{X}$  is called a *midpoint* between  $x$  and  $y$  if

$$|x - z| = |y - z| = \frac{1}{2} \cdot |x - y|.$$

- (ii) Assume  $\varepsilon \geq 0$ . A point  $z \in \mathcal{X}$  is called an  $\varepsilon$ -*midpoint* between  $x$  and  $y$  if

$$|x - z|, \quad |y - z| \leq \frac{1}{2} \cdot |x - y| + \varepsilon.$$

Note that a 0-midpoint is the same as a midpoint.

**1.7.4. Lemma.** *Let  $\mathcal{X}$  be a complete metric space.*

- (a) *Assume that for any pair of points  $x, y \in \mathcal{X}$  and any  $\varepsilon > 0$  there is an  $\varepsilon$ -midpoint  $z$ . Then  $\mathcal{X}$  is a length space.*  
 (b) *Assume that for any pair of points  $x, y \in \mathcal{X}$ , there is a midpoint  $z$ . Then  $\mathcal{X}$  is a geodesic space.*

*Proof.* We first prove (a). Let  $x, y \in \mathcal{X}$  be a pair of points.

Set  $\varepsilon_n = \frac{\varepsilon}{4^n}$ ,  $\alpha(0) = x$  and  $\alpha(1) = y$ .

Let  $\alpha(\frac{1}{2})$  be an  $\varepsilon_1$ -midpoint between  $\alpha(0)$  and  $\alpha(1)$ . Further, let  $\alpha(\frac{1}{4})$  and  $\alpha(\frac{3}{4})$  be  $\varepsilon_2$ -midpoints between the pairs  $(\alpha(0), \alpha(\frac{1}{2}))$  and  $(\alpha(\frac{1}{2}), \alpha(1))$  respectively. Applying the above procedure recursively, on the  $n$ -th step we define  $\alpha(\frac{k}{2^n})$ , for every odd integer  $k$  such that  $0 < \frac{k}{2^n} < 1$ , as an  $\varepsilon_n$ -midpoint between the already defined  $\alpha(\frac{k-1}{2^n})$  and  $\alpha(\frac{k+1}{2^n})$ .

In this way we define  $\alpha(t)$  for  $t \in W$ , where  $W$  denotes the set of dyadic rationals in  $[0, 1]$ . Since  $\mathcal{X}$  is complete, the map  $\alpha$  can be extended continuously to  $[0, 1]$ . Moreover,

$$\begin{aligned} \text{length } \alpha &\leq |x - y| + \sum_{n=1}^{\infty} 2^{n-1} \cdot \varepsilon_n \leq \\ \text{①} \quad &\leq |x - y| + \frac{\varepsilon}{2}. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, we get (a).

To prove (b), one should repeat the same argument taking midpoints instead of  $\varepsilon_n$ -midpoints. In this case ① holds for  $\varepsilon_n = \varepsilon = 0$ .  $\square$

Since in a compact space a sequence of  $\frac{1}{n}$ -midpoints  $z_n$  contains a convergent subsequence, Lemma 1.7.4 immediately implies

**1.7.5. Proposition.** *A proper length space is geodesic.*

**1.7.6. Hopf–Rinow theorem.** *Any complete, locally compact length space is proper.*

It is instructive to solve the following exercise before reading the proof.

**1.7.7. Exercise.** Give an example of space which is locally compact but not proper.

*Proof.* Let  $\mathcal{X}$  be a locally compact length space. Given  $x \in \mathcal{X}$ , denote by  $\rho(x)$  the supremum of all  $R > 0$  such that the closed ball  $\overline{B}[x, R]$  is compact. Since  $\mathcal{X}$  is locally compact,

$$\textcircled{2} \quad \rho(x) > 0 \quad \text{for any } x \in \mathcal{X}.$$

It is sufficient to show that  $\rho(x) = \infty$  for some (and therefore any) point  $x \in \mathcal{X}$ .

Assume the contrary; that is,  $\rho(x) < \infty$ . We claim that

$$\textcircled{3} \quad B = \overline{B}[x, \rho(x)] \text{ is compact for any } x.$$

Indeed,  $\mathcal{X}$  is a length space; therefore for any  $\varepsilon > 0$ , the set  $\overline{B}[x, \rho(x) - \varepsilon]$  is a compact  $\varepsilon$ -net in  $B$ . Since  $B$  is closed and hence complete, it must be compact.  $\triangle$

Next we claim that

$$\textcircled{4} \quad |\rho(x) - \rho(y)| \leq |x - y|_{\mathcal{X}} \text{ for any } x, y \in \mathcal{X}; \text{ in particular } \rho: \mathcal{X} \rightarrow \mathbb{R} \text{ is a continuous function.}$$

Indeed, assume the contrary; that is,  $\rho(x) + |x - y| < \rho(y)$  for some  $x, y \in \mathcal{X}$ . Then  $\overline{B}[x, \rho(x) + \varepsilon]$  is a closed subset of  $\overline{B}[y, \rho(y)]$  for some  $\varepsilon > 0$ . Then compactness of  $\overline{B}[y, \rho(y)]$  implies compactness of  $\overline{B}[x, \rho(x) + \varepsilon]$ , a contradiction.  $\triangle$

Set  $\varepsilon = \min \{ \rho(y) : y \in B \}$ ; the minimum is defined since  $B$  is compact. From  $\textcircled{2}$ , we have  $\varepsilon > 0$ .

Choose a finite  $\frac{\varepsilon}{10}$ -net  $\{a_1, a_2, \dots, a_n\}$  in  $B$ . The union  $W$  of the closed balls  $\overline{B}[a_i, \varepsilon]$  is compact. Clearly  $\overline{B}[x, \rho(x) + \frac{\varepsilon}{10}] \subset W$ . Therefore  $\overline{B}[x, \rho(x) + \frac{\varepsilon}{10}]$  is compact, a contradiction.  $\square$

**1.7.8. Exercise.** Construct a geodesic space that is locally compact, but whose completion is neither geodesic nor locally compact.

## 1.8 Subsets in normed spaces

Recall that a function  $v \mapsto |v|$  on a vector space  $\mathcal{V}$  is called *norm* if it satisfies the following condition for any two vectors  $v, w \in \mathcal{V}$  and a scalar  $\alpha$ :

- $\diamond |v| \geq 0;$
- $\diamond |\alpha \cdot v| = |\alpha| \cdot |v|;$
- $\diamond |v| + |w| \geq |v + w|.$

It is straightforward to check that for any normed space the function  $(v, w) \mapsto |v - w|$  defines a metric on it. Therefore any normed space is an example of metric space (which is in fact geodesic). The following lemma says in particular that any metric space is isometric to a subset of a normed space.

**1.8.1. Lemma.** *Suppose  $\mathcal{X}$  is a bounded separable space; that is,  $\text{diam } \mathcal{X}$  is finite and  $\mathcal{X}$  contains a countable, dense set  $\{w_n\}$ . Given  $x \in \mathcal{X}$ , set  $a_n(x) = |w_n - x|_{\mathcal{X}}$ . Then*

$$\iota: x \mapsto (a_1(x), a_2(x), \dots)$$

*defines a distance preserving embedding  $\iota: \mathcal{X} \hookrightarrow \ell^\infty$ .*

*Proof.* By the triangle inequality

$$|a_n(x) - a_n(y)| \leq |x - y|_{\mathcal{X}}.$$

Therefore  $\iota$  is short.

Again by triangle inequality we have

$$|a_n(x) - a_n(y)| \geq |x - y|_{\mathcal{X}} - 2 \cdot |w_n - x|_{\mathcal{X}}.$$

Since the set  $\{w_n\}$  is dense, we can choose  $w_n$  arbitrary close to  $x$ . Whence the value  $|a_n(x) - a_n(y)|$  can be chosen arbitrary close to  $|x - y|_{\mathcal{X}}$ . In other words

$$\sup_n \{ ||w_n - x|_{\mathcal{X}} - |w_n - y|_{\mathcal{X}} | \} \geq |x - y|_{\mathcal{X}};$$

hence  $\iota$  is distance non-decreasing. □

The following exercise generalizes the lemma to arbitrary separable spaces.

**1.8.2. Exercise.** *Suppose  $\{w_n\}$  is a countable, dense set in a metric space  $\mathcal{X}$ . Choose  $x_0 \in \mathcal{X}$ ; given  $x \in \mathcal{X}$ , set*

$$a_n(x) = |w_n - x|_{\mathcal{X}} - |w_n - x_0|_{\mathcal{X}}.$$

*Show that  $\iota: x \mapsto (a_1(x), a_2(x), \dots)$  defines a distance preserving embedding  $\iota: \mathcal{X} \hookrightarrow \ell^\infty$ .*

**1.8.3. Exercise.** *Show that any compact metric space is isometric  $\mathcal{K}$  to a subspace of a compact geodesic space.*

The lemma above was proved by Maurice René Fréchet in the paper where he defined metric space [24]. Nearly identical construction was

rediscovered later by Kazimierz Kuratowski [35]. Namely he made the following claim:

**1.8.4. Lemma.** *Let  $\mathcal{X}$  be arbitrary metric space. Denote by  $\ell^\infty(\mathcal{X})$  the space of all bounded functions of  $\mathcal{X}$  equipped with sup-norm.*

*Then for any point  $x_0 \in \mathcal{X}$ , the map  $\iota: \mathcal{X} \rightarrow \ell^\infty(\mathcal{X})$  defied by*

$$\iota: x \mapsto (\text{dist}_x - \text{dist}_{x_0})$$

*is distance preserving.*

Note that this claim implies that *any metric space is isometric to a subset of a normed vector space.*



# Chapter 2

## Space of sets

### 2.1 Hausdorff convergence

Let  $\mathcal{X}$  be a metric space. Given a subset  $A \subset \mathcal{X}$ , consider the distance function to  $A$

$$\text{dist}_A : \mathcal{X} \rightarrow [0, \infty)$$

defined as

$$\text{dist}_A(x) := \inf_{a \in A} \{ |a - x|_{\mathcal{X}} \}.$$

**2.1.1. Definition.** Let  $A$  and  $B$  be two compact subsets of a metric space  $\mathcal{X}$ . Then the Hausdorff distance between  $A$  and  $B$  is defined as

$$|A - B|_{\mathcal{H}(\mathcal{X})} := \sup_{x \in \mathcal{X}} \{ |\text{dist}_A(x) - \text{dist}_B(x)| \}.$$

Suppose  $A$  and  $B$  be two compact subsets of a metric space  $\mathcal{X}$ . It is straightforward to check that  $|A - B|_{\mathcal{H}(\mathcal{X})} \leq R$  if and only if  $\text{dist}_A(b) \leq R$  for any  $b \in B$  and  $\text{dist}_B(a) \leq R$  for any  $a \in A$ . In other words,  $|A - B|_{\mathcal{H}(\mathcal{X})} < R$  if and only if  $B$  lies in a  $R$ -neighborhood of  $A$ , and  $A$  lies in a  $R$ -neighborhood of  $B$ .

Note that the set of all nonempty compact subsets of a metric space  $\mathcal{X}$  equipped with the Hausdorff metric forms a metric space. This new metric space will be denoted as  $\mathcal{H}(\mathcal{X})$ .

**2.1.2. Exercise.** Let  $\mathcal{X}$  be a metric space. Given a subset  $A \subset \mathcal{X}$  define its diameter as

$$\text{diam } A := \sup_{a, b \in A} |a - b|.$$

Show that

$$\text{diam}: \mathcal{H}(\mathcal{X}) \rightarrow \mathbb{R}$$

is a 2-Lipschitz function; that is,  $|\text{diam } A - \text{diam } B| \leq 2 \cdot |A - B|_{\mathcal{H}(\mathcal{X})}$ .

**2.1.3. Blaschke selection theorem.** Let  $\mathcal{X}$  be a metric space. Then the space  $\mathcal{H}(\mathcal{X})$  is compact if and only if  $\mathcal{X}$  is compact.

Note that the theorem implies that from any sequence of compact sets in  $\mathcal{X}$  one can select a subsequence converging in the sense of Hausdorff; by that reason it is called a selection theorem.

*Proof; “only if” part.* Note that the map  $\iota: \mathcal{X} \rightarrow \mathcal{H}(\mathcal{X})$ , defined as  $\iota: x \mapsto \{x\}$  (that is, point  $x$  mapped to the one-point subset  $\{x\}$  of  $\mathcal{X}$ ) is distance preserving. Therefore  $\mathcal{X}$  is isometric to the set  $\iota(\mathcal{X})$  in  $\mathcal{H}(\mathcal{X})$ .

Note that for a nonempty subset  $A \subset \mathcal{X}$ , we have  $\text{diam } A = 0$  if and only if  $A$  is a one-point set. Therefore, from Exercise 2.1.2, it follows that  $\iota(\mathcal{X})$  is closed in  $\mathcal{H}(\mathcal{X})$ .

Hence  $\iota(\mathcal{X})$  is compact, as it is a closed subset of a compact space. Since  $\mathcal{X}$  is isometric to  $\iota(\mathcal{X})$ , “only if” part follows.  $\square$

To prove “if” part we will need the following two lemmas.

**2.1.4. Lemma.** Let  $K_1 \supset K_2 \supset \dots$  be a sequence of nonempty compact sets in a metric space  $\mathcal{X}$  then  $K_\infty = \bigcap_n K_n$  is the Hausdorff limit of  $K_n$ ; that is,  $|K_\infty - K_n|_{\mathcal{H}(\mathcal{X})} \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* Note that  $K_\infty$  is compact; by finite intersection property,  $K_\infty$  is nonempty.

If the assertion were false, then there is  $\varepsilon > 0$  such that for each  $n$  one can choose  $x_n \in K_n$  such that  $\text{dist}_{K_\infty}(x_n) \geq \varepsilon$ . Note that  $x_n \in K_1$  for each  $n$ . Since  $K_1$  is compact, there is a *partial limit*<sup>1</sup>  $x_\infty$  of  $x_n$ . Clearly  $\text{dist}_{K_\infty}(x_\infty) \geq \varepsilon$ .

On the other hand, since  $K_n$  is closed and  $x_m \in K_n$  for  $m \geq n$ , we get  $x_\infty \in K_n$  for each  $n$ . It follows that  $x_\infty \in K_\infty$  and therefore  $\text{dist}_{K_\infty}(x_\infty) = 0$ , a contradiction.  $\square$

**2.1.5. Lemma.** If  $\mathcal{X}$  is a compact metric space, then  $\mathcal{H}(\mathcal{X})$  is complete.

*Proof.* Let  $(Q_n)$  be a Cauchy sequence in  $\mathcal{H}(\mathcal{X})$ . Passing to a subsequence of  $Q_n$  we may assume that

$$\textcircled{1} \quad |Q_n - Q_{n+1}|_{\mathcal{H}(\mathcal{X})} \leq \frac{1}{10^n}$$

---

<sup>1</sup>Partial limit is a limit of a subsequence.



for each  $n$ .

Set

$$K_n = \left\{ x \in \mathcal{X} : \text{dist}_{Q_n}(x) \leq \frac{1}{10^n} \right\}$$

Since  $\mathcal{X}$  is compact so is each  $K_n$ .

Clearly,  $|Q_n - K_n|_{\mathcal{H}(\mathcal{X})} \leq \frac{1}{10^n}$  and from **1**, we get  $K_n \supset K_{n+1}$  for each  $n$ . Set

$$K_\infty = \bigcap_{n=1}^{\infty} K_n.$$

Applying Lemma 2.1.4, we get that  $|K_n - K_\infty|_{\mathcal{H}(\mathcal{X})} \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $|Q_n - K_n|_{\mathcal{H}(\mathcal{X})} \leq \frac{1}{10^n}$ , we get  $|Q_n - K_\infty|_{\mathcal{H}(\mathcal{X})} \rightarrow 0$  as  $n \rightarrow \infty$  — hence the lemma.  $\square$

**2.1.6. Exercise.** Let  $\mathcal{X}$  be a complete metric space and  $K_n$  be a sequence of compact sets which converges in the sense of Hausdorff. Show that closure of the union  $\bigcup_{n=1}^{\infty} K_n$  is compact.

Use this to show that in Lemma 2.1.5 compactness of  $\mathcal{X}$  can be exchanged to completeness.

*Proof of “if” part in 2.1.3.* According to Lemma 2.1.5,  $\mathcal{H}(\mathcal{X})$  is complete. It remains to show that  $\mathcal{H}(\mathcal{X})$  is totally bounded (1.4.1d); that is, given  $\varepsilon > 0$  there is a finite  $\varepsilon$ -net in  $\mathcal{H}(\mathcal{X})$ .

Choose a finite  $\varepsilon$ -net  $A$  in  $\mathcal{X}$ . Denote by  $\mathcal{A}$  the set of all subsets of  $A$ . Note that  $\mathcal{A}$  is finite set in  $\mathcal{H}(\mathcal{X})$ . For each compact set  $K \subset \mathcal{X}$ , consider the subset  $K'$  of all points  $a \in A$  such that  $\text{dist}_K(a) \leq \varepsilon$ . Then  $K' \in \mathcal{A}$  and  $|K - K'|_{\mathcal{H}(\mathcal{X})} \leq \varepsilon$ . In other words  $\mathcal{A}$  is a finite  $\varepsilon$ -net in  $\mathcal{H}(\mathcal{X})$ .  $\square$

Hausdorff metric defines convergence of compact sets which is more important than metric itself.

**2.1.7. Exercise.** Let  $X$  and  $Y$  be two compact subsets in  $\mathbb{R}^2$ . Assume  $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$ , is it true that  $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$ , where  $\partial X$  denotes the boundary of  $X$ .

Does the converse holds? That is, assume  $X$  and  $Y$  be two compact subsets in  $\mathbb{R}^2$  and  $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$ ; is it true that  $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$ ?

**2.1.8. Exercise.** Let  $\mathcal{C}$  be a subspace of  $\mathcal{H}(\mathbb{R}^2)$  formed by all compact convex subsets in  $\mathbb{R}^2$ . Show that perimeter<sup>2</sup> and area are continuous

<sup>2</sup>If the set degenerates to a line segment of length  $\ell$ , then its perimeter is defined as  $2 \cdot \ell$ .

on  $\mathcal{C}$ . That is, if a sequence of convex compact plane sets  $X_n$  converges to  $X_\infty$  in the sense of Hausdorff, then

$$\text{perim } X_n \rightarrow \text{perim } X_\infty \quad \text{and} \quad \text{area } X_n \rightarrow \text{area } X_\infty$$

as  $n \rightarrow \infty$ .

The above exercise can be used in a proof of isoperimetrical inequality in the plane; it states that *among the plane figures bounded by closed curves of length at most  $\ell$  the round disc has maximal area*.

Indeed it is sufficient to consider only convex figures of given perimeter; if a figure is not convex pass to its convex hull and observe that it has larger area and smaller perimeter. Further the exercise guarantees existence of a figure  $D_\ell$  with perimeter  $\ell$  and maximal area. It remains to show that  $D_\ell$  is a round disc. The latter is easy to show, see for example Steiner's 4-joint method [11].

## 2.2 A variation

It seems that *Hausdorff convergence* was first introduced by Felix Hausdorff [31], and a couple of years later an equivalent definition was given by Wilhelm Blaschke [11].

The following refinement of the definition was introduced by Zdeněk Frolík in [25], and later rediscovered by Robert Wijsman in [50]. This refinement takes an intermediate place between the original Hausdorff convergence and *closed convergence*, also introduced by Hausdorff in [31]; so we still call it Hausdorff convergence.

**2.2.1. Definition.** Let  $(A_n)$  be a sequence of closed sets in a metric space  $\mathcal{X}$ . We say that  $(A_n)$  converges to a closed set  $A_\infty$  in the sense of Hausdorff if  $\text{dist}_{A_n}(x) \rightarrow \text{dist}_{A_\infty}(x)$  for any  $x \in \mathcal{X}$ .

For example, suppose  $\mathcal{X}$  is the Euclidean plane and  $A_n$  is the circle with radius  $n$  and center at  $(n, 0)$ . If we use the standard definition (2.1.1), then the sequence  $(A_n)$  diverges, but it converges to the  $y$ -axis in the sense of Definition 2.2.1.

The following exercise is analogous to the Blaschke selection theorem (2.1.3).

**2.2.2. Exercise.** Let  $\mathcal{X}$  be a proper metric space and  $(A_n)_{n=1}^\infty$  be a sequence of closed sets in  $\mathcal{X}$ . Assume that for some (and therefore any) point  $x \in \mathcal{X}$ , the sequence  $a_n = \text{dist}_{A_n}(x)$  is bounded. Show that the sequence  $(A_n)_{n=1}^\infty$  has a convergent subsequence in the sense of Definition 2.2.1.

# Chapter 3

## Space of spaces

### 3.1 Gromov–Hausdorff metric

The goal of this section is to cook up a metric space out of metric spaces. More precisely, we want to define the so called Gromov–Hausdorff metric on the set of *isometry classes* of compact metric spaces. (Being isometric is an equivalence relation, and an isometry class is an equivalence class with respect to this equivalence relation.)

The obtained metric space will be denoted as  $\mathcal{M}$ . Given two metric spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , denote by  $[\mathcal{X}]$  and  $[\mathcal{Y}]$  their isometry classes; that is,  $\mathcal{X}' \in [\mathcal{X}]$  if and only if  $\mathcal{X}' \stackrel{iso}{=} \mathcal{X}$ . Pedantically, the Gromov–Hausdorff distance from  $[\mathcal{X}]$  to  $[\mathcal{Y}]$  should be denoted as  $||[\mathcal{X}] - [\mathcal{Y}]|_{\mathcal{M}}$ ; but we will often write it as  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}}$  and say (not quite correctly) “ $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}}$  is the Gromov–Hausdorff distance from  $\mathcal{X}$  to  $\mathcal{Y}$ ”. In other words, from now on the term *metric space* might stands for *isometry class of this metric space*.

The metric on  $\mathcal{M}$  is maximal metric such that *the distance between subspaces in a metric space is not greater than the Hausdorff distance between them*. Here is a formal definition:

**3.1.1. Definition.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be compact metric spaces. The Gromov–Hausdorff distance  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}}$  between them is defined by the following relation.*

*Given  $r > 0$ , we have that  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} < r$  if and only if there exist a metric space  $\mathcal{Z}$  and subspaces  $\mathcal{X}'$  and  $\mathcal{Y}'$  in  $\mathcal{Z}$  that are isometric to  $\mathcal{X}$  and  $\mathcal{Y}$  respectively and such that  $|\mathcal{X}' - \mathcal{Y}'|_{\mathcal{H}(\mathcal{Z})} < r$ . (Here  $|\mathcal{X}' - \mathcal{Y}'|_{\mathcal{H}(\mathcal{Z})}$  denotes the Hausdorff distance between sets  $\mathcal{X}'$  and  $\mathcal{Y}'$  in  $\mathcal{Z}$ .)*

Bit later (see 3.4.1) we will show that *Hausdorff metric* is indeed a metric.

We say that a sequence of (isometry classes of) compact metric spaces  $\mathcal{X}_n$  *converges in the sense of Gromov–Hausdorff* to the (isometry classes of) compact metric space  $\mathcal{X}_\infty$  if  $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{M}} \rightarrow 0$  as  $n \rightarrow \infty$ ; in this case we write  $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$ .

## 3.2 Reformulations

Let us discuss few alternative ways to define the Gromov–Hausdorff metric.

**Metrics on disjoint union.** Definition 3.1.1 deals with a huge class of metric spaces, namely, all metric spaces  $\mathcal{Z}$  that contain subspaces isometric to  $\mathcal{X}$  and  $\mathcal{Y}$ . It is possible to reduce this class to metrics on the disjoint unions of  $\mathcal{X}$  and  $\mathcal{Y}$ . More precisely,

**3.2.1. Proposition.** *The Gromov–Hausdorff distance between two compact metric spaces  $\mathcal{X}$  and  $\mathcal{Y}$  is the infimum of  $r > 0$  such that there exists a metric  $|\ast - \ast|_{\mathcal{W}}$  on the disjoint union  $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Y}$  such that the restrictions of  $|\ast - \ast|_{\mathcal{W}}$  to  $\mathcal{X}$  and  $\mathcal{Y}$  coincide with  $|\ast - \ast|_{\mathcal{X}}$  and  $|\ast - \ast|_{\mathcal{Y}}$  and  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{H}(\mathcal{W})} < r$ .*

*Proof.* Identify  $\mathcal{X} \sqcup \mathcal{Y}$  with  $\mathcal{X}' \cup \mathcal{Y}' \subset \mathcal{Z}$  (the notation is from Definition 3.1.1).

More formally, fix isometries  $f: \mathcal{X} \rightarrow \mathcal{X}'$  and  $g: \mathcal{Y} \rightarrow \mathcal{Y}'$ , then define the distance between  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$  by  $|x - y|_{\mathcal{W}} = |f(x) - g(y)|_{\mathcal{Z}} + \varepsilon$  for small enuf  $\varepsilon > 0$ .<sup>1</sup> This yields a metric on  $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Y}$  for which  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{H}(\mathcal{W})} < r$ .  $\square$

**Fixed ambient space.** The following proposition says that the space  $\mathcal{Z}$  in Definition 3.1.1 can be exchanged to a fixed space, namely  $\ell^\infty$  — the space of bounded infinite sequences with the metric defined by sup-norm.

**3.2.2. Proposition.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be compact metric spaces. Then*

$$|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} = \inf\{|\mathcal{X}' - \mathcal{Y}'|_{\mathcal{H}(\ell^\infty)}\}$$

*where the infimum is taken over all pairs of sets  $\mathcal{X}'$  and  $\mathcal{Y}'$  in  $\ell^\infty$  which isometric to  $\mathcal{X}$  and  $\mathcal{Y}$  correspondingly.*

*Proof of 3.2.2.* By the definition, we have that

$$|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} \leq \inf\{|\mathcal{X}' - \mathcal{Y}'|_{\mathcal{H}(\ell^\infty)}\}.$$

---

<sup>1</sup>We add  $\varepsilon$  to ensure that  $d(x, y) > 0$  for any  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ ; so  $|x - y|_{\mathcal{W}}$  is indeed a metric.

Let  $\mathcal{W}$  be an arbitrary metric space with the underlying set  $\mathcal{X} \sqcup \mathcal{Y}$ . Note  $\mathcal{W}$  is compact since it is union of two compact subsets  $\mathcal{X}, \mathcal{Y} \subset \mathcal{W}$ . In particular,  $\mathcal{W}$  is separable.

By Lemma 1.8.1, there is an distance preserving embedding  $\iota: \mathcal{W} \rightarrow \ell^\infty$ . It remains to apply Proposition 3.2.1.  $\square$

### 3.3 Almost isometries

**3.3.1. Definition.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be metric spaces and  $\varepsilon > 0$ . A map<sup>2</sup>  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is called an  $\varepsilon$ -isometry if

$$|f(x) - f(x')|_{\mathcal{Y}} \leq |x - x'|_{\mathcal{X}} \pm \varepsilon$$

for any  $x, x' \in \mathcal{X}$  and if  $f(\mathcal{X})$  is an  $\varepsilon$ -net in  $\mathcal{Y}$ .

**3.3.2. Exercise.**

- (a) Let  $f: \mathcal{X} \rightarrow \mathcal{Y}$  and  $g: \mathcal{Y} \rightarrow \mathcal{Z}$  be two  $\varepsilon$ -isometries. Show that  $g \circ f: \mathcal{X} \rightarrow \mathcal{Z}$  is a  $(3 \cdot \varepsilon)$ -isometry.
- (b) Assume  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is an  $\varepsilon$ -isometry. Show that there is a  $(3 \cdot \varepsilon)$ -isometry  $g: \mathcal{Y} \rightarrow \mathcal{X}$ .
- (c) Assume  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} < \varepsilon$ , show that there is a  $(2 \cdot \varepsilon)$ -isometry  $f: \mathcal{X} \rightarrow \mathcal{Y}$ .

**3.3.3. Proposition.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be metric spaces and let  $f: \mathcal{X} \rightarrow \mathcal{Y}$  be an  $\varepsilon$ -isometry. Then

$$|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} \leq 2 \cdot \varepsilon.$$

*Proof.* Consider the set  $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Y}$ . Note that the following defines a metric on  $\mathcal{W}$ :

◇ For any  $x, x' \in \mathcal{X}$

$$|x - x'|_{\mathcal{W}} = |x - x'|_{\mathcal{X}};$$

◇ For any  $y, y' \in \mathcal{Y}$ ,

$$|y - y'|_{\mathcal{W}} = |y - y'|_{\mathcal{Y}}$$

◇ For any  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ ,

$$|x - y|_{\mathcal{W}} = \varepsilon + \inf_{x' \in \mathcal{X}} \{|x - x'|_{\mathcal{X}} + |f(x') - y|_{\mathcal{Y}}\}.$$

---

<sup>2</sup>possibly noncontinuous

Since  $f(\mathcal{X})$  is an  $\varepsilon$ -net in  $\mathcal{Y}$ , for any  $y \in \mathcal{Y}$  there is  $x \in \mathcal{X}$  such that  $|f(x) - y|_{\mathcal{Y}} \leq \varepsilon$ ; therefore  $|x - y|_{\mathcal{W}} \leq 2 \cdot \varepsilon$ . On the other hand for any  $x \in \mathcal{X}$ , we have  $|x - y|_{\mathcal{W}} \leq \varepsilon$  for  $y = f(x) \in \mathcal{Y}$ .

It follows that  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{H}(\mathcal{W})} \leq 2 \cdot \varepsilon$ .  $\square$

The Gromov–Hausdorff metric defines Gromov–Hausdorff convergence and this is the only thing it is good for. In other words in all applications, we use only topology on  $\mathcal{M}$  and we do not care about particular value of Gromov–Hausdorff distance between spaces.

In order to determine that a given sequence of metric spaces  $(\mathcal{X}_n)$  converges in the Gromov–Hausdorff sense to  $\mathcal{X}_\infty$ , it is sufficient to estimate distances  $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{M}}$  and check if  $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{M}} \rightarrow 0$ . This problem turns to be simpler than finding Gromov–Hausdorff distance between a particular pair of spaces. The following proposition gives one way to do this.

**3.3.4. Proposition.** *A sequence of compact metric spaces  $(\mathcal{X}_n)$  converges to  $\mathcal{X}_\infty$  in the sense of Gromov–Hausdorff if and only if there is a sequence  $\varepsilon_n \rightarrow 0+$  and an  $\varepsilon_n$ -isometry  $f_n: \mathcal{X}_n \rightarrow \mathcal{X}_\infty$  for each  $n$ .*

*Proof.* Follows from Proposition 3.3.3 and Exercise 3.3.2c  $\square$

## 3.4 It is a metric

**3.4.1. Theorem.** *The set of isometry classes of compact metric spaces equipped with Gromov–Hausdorff metric forms a metric space (which is denoted by  $\mathcal{M}$ ).*

*Proof.* Let  $\mathcal{X}, \mathcal{Y}$  and  $\mathcal{Z}$  be arbitrary compact metric spaces. We need to check the following:

- (i)  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} \geq 0$ ;
- (ii)  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} = 0$  if and only if  $\mathcal{X}$  is isometric to  $\mathcal{Y}$ ;
- (iii)  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} = |\mathcal{Y} - \mathcal{X}|_{\mathcal{M}}$ ;
- (iv)  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} + |\mathcal{Y} - \mathcal{Z}|_{\mathcal{M}} \geq |\mathcal{X} - \mathcal{Z}|_{\mathcal{M}}$ .

Note that (i), (iii) and “if”-part of (ii) follow directly from Definition 3.1.1.

(iv). Choose arbitrary  $a, b \in \mathbb{R}$  such that

$$a > |\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} \quad \text{and} \quad b > |\mathcal{Y} - \mathcal{Z}|_{\mathcal{M}}.$$

Choose two metrics on  $\mathcal{U} = \mathcal{X} \sqcup \mathcal{Y}$  and  $\mathcal{V} = \mathcal{Y} \sqcup \mathcal{Z}$  so that  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{H}(\mathcal{U})} < a$  and  $|\mathcal{Y} - \mathcal{Z}|_{\mathcal{H}(\mathcal{V})} < b$  and the inclusions  $\mathcal{X} \hookrightarrow \mathcal{U}$ ,  $\mathcal{Y} \hookrightarrow \mathcal{U}$ ,  $\mathcal{Y} \hookrightarrow \mathcal{V}$  and  $\mathcal{Z} \hookrightarrow \mathcal{V}$  are distance preserving.

Consider the metric on  $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Z}$  so that inclusions  $\mathcal{X} \hookrightarrow \mathcal{W}$  and  $\mathcal{Z} \hookrightarrow \mathcal{W}$  are distance preserving and

$$|x - z|_{\mathcal{W}} = \inf_{y \in \mathcal{Y}} \{|x - y|_{\mathcal{U}} + |y - z|_{\mathcal{V}}\}.$$

Note that  $|\ast - \ast|_{\mathcal{W}}$  is indeed a metric and

$$|\mathcal{X} - \mathcal{Z}|_{\mathcal{H}(\mathcal{W})} < a + b.$$

Property (iv) follows since the last inequality holds for any  $a > |\mathcal{X} - \mathcal{Y}|_{\mathcal{M}}$  and  $b > |\mathcal{Y} - \mathcal{Z}|_{\mathcal{M}}$ .

*“Only if”-part of (ii).* According to Exercise 3.3.2c, for any sequence  $\varepsilon_n \rightarrow 0+$  there is a sequence of  $\varepsilon_n$ -isometries  $f_n: \mathcal{X} \rightarrow \mathcal{Y}$ .

Since  $\mathcal{X}$  is compact, we can choose a countable dense set  $S$  in  $\mathcal{X}$ . Use a diagonal procedure if necessary, to pass to a subsequence of  $(f_n)$  such that for every  $x \in S$  the sequence  $(f_n(x))$  converges in  $\mathcal{Y}$ . Consider the pointwise limit map  $f_\infty: S \rightarrow \mathcal{Y}$  defined by

$$f_\infty(x) = \lim_{n \rightarrow \infty} f_n(x)$$

for every  $x \in S$ . Since

$$|f_n(x) - f_n(x')|_{\mathcal{Y}} \leq |x - x'|_{\mathcal{X}} \pm \varepsilon_n,$$

we have

$$|f_\infty(x) - f_\infty(x')|_{\mathcal{Y}} = \lim_{n \rightarrow \infty} |f_n(x) - f_n(x')|_{\mathcal{Y}} = |x - x'|_{\mathcal{X}}$$

for all  $x, x' \in S$ ; that is,  $f_\infty: S \rightarrow \mathcal{Y}$  is a distance-preserving map. Therefore  $f_\infty$  can be extended to a distance-preserving map from all of  $\mathcal{X}$  to  $\mathcal{Y}$ . The later is done by setting

$$f_\infty(x) = \lim_{n \rightarrow \infty} f_\infty(x_n)$$

for some (and therefore any) sequence of points  $(x_n)$  in  $S$  which converges to  $x$  in  $\mathcal{X}$ . (Note that if  $x_n \rightarrow x$ , then  $(x_n)$  is Cauchy. Since  $f_\infty$  is distance preserving,  $y_n = f_\infty(x_n)$  is also a Cauchy sequence in  $\mathcal{Y}$ ; therefore it converges.)

This way we obtain a distance preserving map  $f_\infty: \mathcal{X} \rightarrow \mathcal{Y}$ . It remains to show that  $f_\infty$  is surjective; that is,  $f_\infty(\mathcal{X}) = \mathcal{Y}$ .

Note that in the same way we can obtain a distance preserving map  $g_\infty: \mathcal{Y} \rightarrow \mathcal{X}$ . If  $f_\infty$  is not surjective, then neither is  $f_\infty \circ g_\infty: \mathcal{Y} \rightarrow \mathcal{Y}$ . So  $f_\infty \circ g_\infty$  is a distance preserving map from a compact space to itself which is not an isometry. The later contradicts Exercise 1.4.4.  $\square$

**3.4.2. Exercise.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be two compact metric spaces. Prove that

$$|\text{diam } \mathcal{X} - \text{diam } \mathcal{Y}| \leq 2 \cdot |\mathcal{X} - \mathcal{Y}|_{\mathcal{M}}.$$

In other words,  $\text{diam}: \mathcal{M} \rightarrow \mathbb{R}$  is a 2-Lipschitz function.

**3.4.3. Exercise.** Show that  $\mathcal{M}$  is a length space.

Given two metric spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , we will write  $\mathcal{X} \leq \mathcal{Y}$  if there is a noncontracting map  $f: \mathcal{X} \rightarrow \mathcal{Y}$ ; that is, if

$$|x - x'|_{\mathcal{X}} \leq |f(x) - f(x')|_{\mathcal{Y}}$$

for any  $x, x' \in \mathcal{X}$ .

Further, given  $\varepsilon > 0$ , we will write  $\mathcal{X} \leq \mathcal{Y} + \varepsilon$  if there is a map  $f: \mathcal{X} \rightarrow \mathcal{Y}$  such that

$$|x - x'|_{\mathcal{X}} \leq |f(x) - f(x')|_{\mathcal{Y}} + \varepsilon$$

for any  $x, x' \in \mathcal{X}$ .

**3.4.4. Exercise.** Show that

$$|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}'} = \inf \{ \varepsilon > 0 : \mathcal{X} \leq \mathcal{Y} + \varepsilon \text{ and } \mathcal{Y} \leq \mathcal{X} + \varepsilon \}$$

defines a metric on the space of (isometry classes) of compact metric spaces.

Moreover  $|\ast - \ast|_{\mathcal{M}'}$  is equivalent to the Gromov–Haudorff metric; that is,

$$|\mathcal{X}_n - \mathcal{X}_{\infty}|_{\mathcal{M}} \rightarrow 0 \iff |\mathcal{X}_n - \mathcal{X}_{\infty}|_{\mathcal{M}'} \rightarrow 0$$

as  $n \rightarrow \infty$ .

## 3.5 Uniformly totally bonded families

Let  $\mathcal{Q}$  be a set of (isometry classes) of compact metric spaces. Suppose that there is a sequence  $\varepsilon_n \rightarrow 0$  such that for any positive integer  $n$  each space  $\mathcal{X}$  in  $\mathcal{Q}$  admits an  $\varepsilon_n$ -net with at most  $n$  points. Then we say that  $\mathcal{Q}$  is *uniformly totally bonded*.

Observe that in this case  $\text{diam } \mathcal{X} < \varepsilon_1$  for any  $\mathcal{X}$  in  $\mathcal{Q}$ ; that is diameters of spaces in  $\mathcal{Q}$  are bounded above.

Fix a real constant  $C$ . A measure  $\mu$  on a metric space  $\mathcal{X}$  is called *C-doubling* if

$$\mu[B(p, 2 \cdot r)] < C \cdot \mu[B(p, r)]$$



for any point  $p \in \mathcal{X}$  and any positive real  $r$ . A measure is called *doubling* if it is  $C$ -doubling for a some real constant  $C$ .

**3.5.1. Exercise.** Let  $\mathcal{Q}(C, D)$  be the set of all the compact metric spaces with diameter at most  $D$  that admit a  $C$ -doubling measure. Show that  $\mathcal{Q}(C, D)$  is totally bounded.

Recall that we write  $\mathcal{X} \leq \mathcal{Y}$  if there is a distance non-decreasing map  $\mathcal{X} \rightarrow \mathcal{Y}$ .

**3.5.2. Exercise.**

- (a) Let  $\mathcal{Y}$  be a compact metric space. Show that the set of all spaces  $\mathcal{X}$  such that  $\mathcal{X} \leq \mathcal{Y}$  is uniformly totally bounded.
- (b) Show that for any uniformly totally bounded set  $\mathcal{Q} \subset \mathcal{M}$  there is a compact space  $\mathcal{Y}$  such that  $\mathcal{X} \leq \mathcal{Y}$  for any  $\mathcal{X}$  in  $\mathcal{Q}$ .

## 3.6 Gromov's selection theorem

The following theorem is analogous to Blaschke selection theorems (2.1.3).

**3.6.1. Gromov selection theorem.** Let  $\mathcal{Q}$  be a closed and totally bounded subset of  $\mathcal{M}$ . Then  $\mathcal{Q}$  is compact.

**3.6.2. Lemma.**  $\mathcal{M}$  is complete.

*Proof.* Let  $(\mathcal{X}_n)$  be a Cauchy sequence in  $\mathcal{M}$ . Passing to a subsequence if necessary, we can assume that  $|\mathcal{X}_n - \mathcal{X}_{n+1}|_{\mathcal{M}} < \frac{1}{2^n}$  for each  $n$ . In particular, for each  $n$  one can equip  $\mathcal{W}_n = \mathcal{X}_n \sqcup \mathcal{X}_{n+1}$  with a metric such that inclusions  $\mathcal{X}_n \hookrightarrow \mathcal{W}_n$  and  $\mathcal{X}_{n+1} \hookrightarrow \mathcal{W}_n$  are distance preserving, and

$$|\mathcal{X}_n - \mathcal{X}_{n+1}|_{\mathcal{H}(\mathcal{W}_n)} < \frac{1}{2^n}$$

for each  $n$ .

Set  $\mathcal{W}$  to be the disjoint union of all  $\mathcal{X}_n$ . Let us equip  $\mathcal{W}$  with a metric defined the following way:

- ◇ for any fixed  $n$  and any two points  $x_n, x'_n \in \mathcal{X}_n$  set

$$|x_n - x'_n|_{\mathcal{W}} = |x_n - x'_n|_{\mathcal{X}_n}$$

- ◇ for any positive integers  $m > n$  and any two points  $x_n \in \mathcal{X}_n$  and  $x_m \in \mathcal{X}_m$  set

$$|x_n - x_m|_{\mathcal{W}} = \inf \left\{ \sum_{i=n}^{m-1} |x_i - x_{i+1}|_{\mathcal{W}_i} \right\},$$

where the infimum is taken for all sequences  $x_i \in \mathcal{X}_i$ .

Observe that  $|\ast - \ast|_{\mathcal{W}}$  is indeed a metric.

Let  $\bar{\mathcal{W}}$  be the completion of  $\mathcal{W}$ . Note that  $|\mathcal{X}_m - \mathcal{X}_n| < \frac{1}{2^{n-1}}$  if  $m > n$ . Therefore the union of  $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \dots \cup \mathcal{X}_n$  forms a  $\frac{1}{2^{n-1}}$ -net in  $\bar{\mathcal{W}}$ . Since each  $\mathcal{X}_i$  is compact, we get that  $\bar{\mathcal{W}}$  admits a compact  $\varepsilon$ -net for any  $\varepsilon > 0$ . Whence  $\bar{\mathcal{W}}$  is compact.

According to Blaschke selection theorem (2.1.3), we can pass to a subsequence of  $(\mathcal{X}_n)$  that converges in  $\mathcal{H}(\bar{\mathcal{W}})$  and therefore in  $\mathcal{M}$ .  $\square$

*Proof of 3.6.1; “only if” part.* If there is no sequence  $\varepsilon_n \rightarrow 0$  as described in the problem, then for a fixed  $\delta > 0$  there is a sequence of spaces  $\mathcal{X}_n \in \mathcal{Q}$  such that

$$\text{pack}_{\delta} \mathcal{X}_n \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Since  $\mathcal{Q}$  is compact, this sequence has a partial limit say  $\mathcal{X}_{\infty} \in \mathcal{Q}$ . Observe that  $\text{pack}_{\delta} \mathcal{X}_{\infty} = \infty$ . Therefore  $\mathcal{X}_{\infty}$  — a contradiction.

*“If” part.* Without loss of generality, we may assume that there is a sequence  $\varepsilon_n \rightarrow 0$  such that  $\mathcal{Q}$  is the set of all compact metric spaces  $\mathcal{X}$  such that  $\text{pack}_{\varepsilon_n} \mathcal{X} \leq n$ .

Note that  $\text{diam } \mathcal{X} \leq \varepsilon_1$  for any  $\mathcal{X} \in \mathcal{Q}$ . Given positive integer  $n$  consider set of all metric spaces  $\mathcal{W}_n$  with number of points at most  $n$  and diameter  $\leq \varepsilon_1$ . Note that  $\mathcal{W}_n$  is compact for each  $n$ .

Further a maximal  $\varepsilon_n$ -packing of any  $\mathcal{X} \in \mathcal{Q}$  forms a subspace from  $\mathcal{W}_n$ . Therefore  $\mathcal{W}_n \cap \mathcal{Q}$  is a compact  $\varepsilon_n$ -net in  $\mathcal{Q}$ . That is,  $\mathcal{Q}$  has compact  $\varepsilon$ -net for any  $\varepsilon > 0$ . The incc  $\mathcal{Q}$  is a closed set  $\square$

In the following exercises *converge* means *converge in the sense of Gromov–Hausdorff*.

### 3.6.3. Exercise.

- (a) Show that a sequence of compact simply connected length spaces can not converge to a circle.
- (b) Construct a sequence of compact simply connected length spaces that converges to a compact nonsimply connected space.

### 3.6.4. Exercise.

- (a) Show that a sequence of length metrics on the 2-sphere can not converge to a the unit disc.
- (b) Construct a sequence of length metrics on the 3-sphere that converges to a unit 3-ball.

### 3.7 Remarks

Suppose  $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$ , then there is a metric on the disjoint union

$$\mathbf{X} = \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} \mathcal{X}_n$$

such that the restriction of metric on each  $\mathcal{X}_n$  and  $\mathcal{X}_\infty$  coincides with its original metric and  $\mathcal{X}_n \xrightarrow{\text{H}} \mathcal{X}_\infty$  as subsets in  $\mathbf{X}$ .

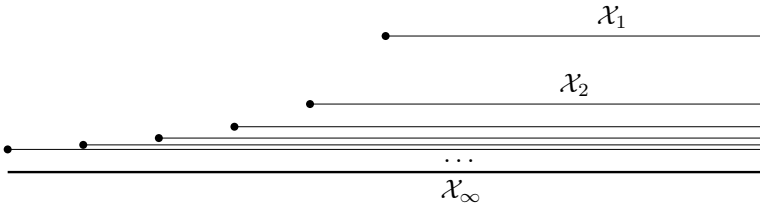
Indeed, since  $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$ , there is a metric on  $\mathcal{V}_n = \mathcal{X}_n \sqcup \mathcal{X}_\infty$  such that the restriction of metric on each  $\mathcal{X}_n$  and  $\mathcal{X}_\infty$  coincides with its original metric and  $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{H}(\mathcal{V}_n)} < \varepsilon_n$  for some sequence  $\varepsilon_n \rightarrow 0$ . Arguing as in the proof of (iv) in Theorem 3.4.1 we define metric on  $\mathbf{X}$  by setting

$$\begin{aligned} |x_m - x_n|_{\mathbf{X}} &= \inf_{x_\infty} \{ |x_m - x_\infty|_{\mathcal{V}_m} + |x_n - x_\infty|_{\mathcal{V}_n} : \}, \\ |x_n - x_\infty|_{\mathbf{X}} &= |x_n - x_\infty|_{\mathcal{V}_n} \\ |x_n - x'_n|_{\mathbf{X}} &= |x_n - x'_n|_{\mathcal{X}_n} \end{aligned}$$

where  $x_n, x'_n \in \mathcal{X}_n$  for every  $n \in \mathbb{N} \cup \{\infty\}$ .

In other words, the metric on  $\mathbf{X}$  defines convergence  $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$ . This metric makes possible to talk about limits of sequences  $x_n \in \mathcal{X}_n$  as  $n \rightarrow \infty$ , as well as weak limit of a sequence of measures  $\mu_n$  on  $\mathcal{X}_n$  and so on. By that reason it might be useful to fix such metric on  $\mathbf{X}$ . This approach can be also used to define Gromov–Hausdorff convergence of noncompact spaces which will be discussed latter.

We may consider a metric on  $\mathbf{X}$  such that  $\mathcal{X}_n \xrightarrow{\text{H}} \mathcal{X}_\infty$  without assuming that all the spaces  $\mathcal{X}_n$  and  $\mathcal{X}_\infty$  are compact; in this case we need to use the variation of Hausdorff convergence described in Section 2.2. The limit spaces for this generalized convergence is not uniquely defined. For example if each space  $\mathcal{X}_n$  in the sequence is isometric to the half-line, then its limit might be isometric to the half-line or to whole line. The first convergence is evident and the second could be guessed from the diagram.



Often the isometry class of the limit can be fixed by marking a point  $p_n$  in each space  $\mathcal{X}_n$ , it is called *pointed Gromov–Hausdorff convergence*

— we say that  $(\mathcal{X}_n, p_n)$  converges to  $(\mathcal{X}_\infty, p_\infty)$  if there is a metric on  $\mathbf{X}$  such that  $\mathcal{X}_n \xrightarrow{\text{H}} \mathcal{X}_\infty$  and  $p_n \rightarrow p_\infty$ . For example the sequence  $(\mathcal{X}_n, p_n) = (\mathbb{R}_+, 0)$  converges to  $(\mathbb{R}_+, 0)$ , while  $(\mathcal{X}_n, p_n) = (\mathbb{R}_+, n)$  converges to  $(\mathbb{R}, 0)$ .

This convergence works nicely for proper metric spaces. The following theorem is an analog of Gromov's selection theorem for pointed Gromov–Haudorff convergence.

**3.7.1. Theorem.** *Let  $\mathcal{Q}$  be a set of isometry classes of pointed proper metric spaces  $(\mathcal{X}, p)$ . Assume that for any  $R > 0$ , the  $R$ -balls in the spaces centered at the marked points form a uniformly totally bounded family of spaces. Then  $\mathcal{Q}$  is precompact with respect to pointed Gromov–Haudorff convergence.*

# Chapter 4

## Ultralimits

Here we introduce ultralimits of sequences of points, metric spaces and functions. The ultralimits of metric spaces can be considered as a variation of Gromov–Hausdorff convergence. Our presentation is based on [34].

Our use of ultralimits is very limited; we use them only as a canonical way to pass to a convergent subsequence. (In principle, we could avoid selling our souls to the set-theoretical devil, but in this case we must say “pass to convergent subsequence” too many times.)

### 4.1 Ultrafilters

We will need the existence of a nonprinciple ultrafilter  $\omega$ , which we fix once and for all.

Recall that  $\mathbb{N}$  denotes the set of natural numbers,  $\mathbb{N} = \{1, 2, \dots\}$

**4.1.1. Definition.** *A finitely additive measure  $\omega$  on  $\mathbb{N}$  is called an ultrafilter if it satisfies*

(a)  $\omega(S) = 0$  or  $1$  for any subset  $S \subset \mathbb{N}$ .

*An ultrafilter  $\omega$  is called nonprinciple if in addition*

(b)  $\omega(F) = 0$  for any finite subset  $F \subset \mathbb{N}$ .

If  $\omega(S) = 0$  for some subset  $S \subset \mathbb{N}$ , we say that  $S$  is  $\omega$ -small. If  $\omega(S) = 1$ , we say that  $S$  contains  $\omega$ -almost all elements of  $\mathbb{N}$ .

**Classical definition.** More commonly, a nonprinciple ultrafilter is defined as a collection, say  $\mathfrak{F}$ , of sets in  $\mathbb{N}$  such that

1. if  $P \in \mathfrak{F}$  and  $Q \supset P$ , then  $Q \in \mathfrak{F}$ ,
2. if  $P, Q \in \mathfrak{F}$ , then  $P \cap Q \in \mathfrak{F}$ ,
3. for any subset  $P \subset \mathbb{N}$ , either  $P$  or its complement is an element of  $\mathfrak{F}$ .

4. if  $F \subset \mathbb{N}$  is finite, then  $F \notin \mathfrak{F}$ .

Setting  $P \in \mathfrak{F} \Leftrightarrow \omega(P) = 1$  makes these two definitions equivalent.

A nonempty collection of sets  $\mathfrak{F}$  that does not include the empty set and satisfies only conditions 1 and 2 is called a *filter*; if in addition  $\mathfrak{F}$  satisfies Condition 3 it is called an *ultrafilter*. From Zorn's lemma, it follows that every filter contains an ultrafilter. Thus there is an ultrafilter  $\mathfrak{F}$  contained in the filter of all complements of finite sets; clearly this  $\mathfrak{F}$  is nonprincipal.

**Stone-Čech compactification.** Given a set  $S \subset \mathbb{N}$ , consider subset  $\Omega_S$  of all ultrafilters  $\omega$  such that  $\omega(S) = 1$ . It is straightforward to check that the sets  $\Omega_S$  for all  $S \subset \mathbb{N}$  form a topology on the set of ultrafilters on  $\mathbb{N}$ . The obtained space is called *Stone-Čech compactification* of  $\mathbb{N}$ ; it is usually denoted as  $\beta\mathbb{N}$ .

There is a natural embedding  $\mathbb{N} \hookrightarrow \beta\mathbb{N}$  defined as  $n \mapsto \omega_n$ , where  $\omega_n$  is the principle ultrafilter such that  $\omega_n(S) = 1$  if and only if  $n \in S$ . Using the described embedding, we can (and will) consider  $\mathbb{N}$  as a subset of  $\beta\mathbb{N}$ .

The space  $\beta\mathbb{N}$  is the maximal compact Hausdorff space that contains  $\mathbb{N}$  as an everywhere dense subset. More precisely, for any compact Hausdorff space  $\mathcal{X}$  and a map  $f: \mathbb{N} \rightarrow \mathcal{X}$  there is unique continuous map  $\bar{f}: \beta\mathbb{N} \rightarrow \mathcal{X}$  such that the restriction  $\bar{f}|_{\mathbb{N}}$  coincides with  $f$ .

## 4.2 Ultralimits of points

Fix an ultrafilter  $\omega$ . Assume  $(x_n)$  is a sequence of points in a metric space  $\mathcal{X}$ . Let us define the  $\omega$ -*limit* of  $(x_n)$  as the point  $x_\omega$  such that for any  $\varepsilon > 0$ ,  $\omega$ -almost all elements of  $(x_n)$  lie in  $B(x_\omega, \varepsilon)$ ; that is,

$$\omega \{ n \in \mathbb{N} : |x_\omega - x_n| < \varepsilon \} = 1.$$

In this case, we will write

$$x_\omega = \lim_{n \rightarrow \omega} x_n \quad \text{or} \quad x_n \rightarrow x_\omega \text{ as } n \rightarrow \omega.$$

For example if  $\omega$  is the principle ultrafilter such that  $\omega(\{n\}) = 1$  for some  $n \in \mathbb{N}$ , then  $x_\omega = x_n$ .

Note that  $\omega$ -limits of a sequence and its subsequence may differ. For example, in general

$$\lim_{n \rightarrow \omega} x_n \neq \lim_{n \rightarrow \omega} x_{2 \cdot n}.$$

**4.2.1. Proposition.** *Let  $\omega$  be a nonprincipal ultrafilter. Assume  $(x_n)$  is a sequence of points in a metric space  $\mathcal{X}$  and  $x_n \rightarrow x_\omega$  as  $n \rightarrow \omega$ .*

Then  $x_\omega$  is a partial limit of the sequence  $(x_n)$ ; that is, there is a subsequence  $(x_n)_{n \in S}$  that converges to  $x_\omega$  in the usual sense.

**Remark.** A nonprinciple ultrafilter  $\omega$  is called *selective* if for any partition of  $\mathbb{N}$  into sets  $\{C_\alpha\}_{\alpha \in \mathcal{A}}$  such that  $\omega(C_\alpha) = 0$  for each  $\alpha$ , there is a set  $S \subset \mathbb{N}$  such that  $\omega(S) = 1$  and  $S \cap C_\alpha$  is a one-point set for each  $\alpha \in \mathcal{A}$ .

The existence of a selective ultrafilter follows from the continuum hypothesis; it was proved by Walter Rudin in [46].

For a selective ultrafilter  $\omega$ , there is a stronger version of Proposition 4.2.1; namely we can assume that the subsequence  $(x_n)_{n \in S}$  can be chosen so that  $\omega(S) = 1$ . (So, if needed, you may assume that the ultrafilter  $\omega$  is selective and use this stronger version of the proposition.)

*Proof.* Given  $\varepsilon > 0$ , set  $S_\varepsilon = \{n \in \mathbb{N} : |x_n - x_\omega| < \varepsilon\}$ .

Note that  $\omega(S_\varepsilon) = 1$  for any  $\varepsilon > 0$ . Since  $\omega$  is nonprinciple, the set  $S_\varepsilon$  is infinite. Therefore we can choose an increasing sequence  $(n_k)$  such that  $n_k \in S_{\frac{1}{k}}$  for each  $k \in \mathbb{N}$ . Clearly  $x_{n_k} \rightarrow x_\omega$  as  $k \rightarrow \infty$ .  $\square$

The following proposition is analogous to the statement that any sequence in a compact metric space has a convergent subsequence; it can be proved the same way.

**4.2.2. Proposition.** *Let  $\mathcal{X}$  be a compact metric space. Then any sequence of points  $(x_n)$  in  $\mathcal{X}$  has unique  $\omega$ -limit  $x_\omega$ .*

*In particular, a bounded sequence of real numbers has a unique  $\omega$ -limit.*

Alternatively, the sequence  $(x_n)$  can be regarded as a map  $\mathbb{N} \rightarrow \mathcal{X}$ . In this case the map  $\mathbb{N} \rightarrow \mathcal{X}$  can be extended to a continuous map from the Stone-Ćech compactification  $\beta\mathbb{N}$  of  $\mathbb{N}$ . Then the  $\omega$ -limit  $x_\omega$  can be regarded as the image of  $\omega$ .

The following lemma is an ultralimit analog of Cauchy convergence test.

**4.2.3. Lemma.** *Let  $(x_n)$  be a sequence of points in a complete space  $\mathcal{X}$ . Assume for each subsequence  $(y_n)$  of  $(x_n)$ , the  $\omega$ -limit*

$$y_\omega = \lim_{n \rightarrow \omega} y_n \in \mathcal{X}$$

*is defined and does not depend on the choice of subsequence, then the sequence  $(x_n)$  converges in the usual sense.*

*Proof.* Assume that  $(x_n)$  is a Cauchy sequence. Then for some  $\varepsilon > 0$ , there is a subsequence  $(y_n)$  of  $(x_n)$  such that  $|x_n - y_n| \geq \varepsilon$  for all  $n$ .

It follows that  $|x_\omega - y_\omega| \geq \varepsilon$ , a contradiction.  $\square$

### 4.3 Ultralimits of spaces

From now on,  $\omega$  denotes a nonprincipal ultrafilter on the set of natural numbers.

Let  $\mathcal{X}_n$  be a sequence of metric spaces. Consider all sequences of points  $x_n \in \mathcal{X}_n$ . On the set of all such sequences, define a pseudometric by

$$\bullet \quad |(x_n) - (y_n)| = \lim_{n \rightarrow \omega} |x_n - y_n|.$$

Note that the  $\omega$ -limit on the right hand side is always defined and takes a value in  $[0, \infty]$ .

Set  $\mathcal{X}_\omega$  to be the corresponding metric space; that is, the underlying set of  $\mathcal{X}_\omega$  is formed by classes of equivalence of sequences of points  $x_n \in \mathcal{X}_n$  defined by

$$(x_n) \sim (y_n) \Leftrightarrow \lim_{n \rightarrow \omega} |x_n - y_n| = 0$$

and the distance is defined by  $\bullet$ .

The space  $\mathcal{X}_\omega$  is called  $\omega$ -limit of  $\mathcal{X}_n$ . Typically  $\mathcal{X}_\omega$  will denote the  $\omega$ -limit of sequence  $\mathcal{X}_n$ ; we may also write

$$\mathcal{X}_n \rightarrow \mathcal{X}_\omega \text{ as } n \rightarrow \omega \text{ or } \mathcal{X}_\omega = \lim_{n \rightarrow \omega} \mathcal{X}_n.$$

Given a sequence  $x_n \in \mathcal{X}_n$ , we will denote by  $x_\omega$  its equivalence class which is a point in  $\mathcal{X}_\omega$ ; equivalently we will write

$$x_n \rightarrow x_\omega \text{ as } n \rightarrow \omega \text{ or } x_\omega = \lim_{n \rightarrow \omega} x_n.$$

**4.3.1. Observation.** *The  $\omega$ -limit of any sequence of metric spaces is complete.*

*Proof.* Let  $\mathcal{X}_n$  be a sequence of metric spaces and  $\mathcal{X}_n \rightarrow \mathcal{X}_\omega$  as  $n \rightarrow \omega$ .

Fix a Cauchy sequence  $x_m \in \mathcal{X}_\omega$ . Passing to a subsequence we can assume that  $|x_m - x_{m-1}|_{\mathcal{X}_\omega} < \frac{1}{2^m}$  for any  $m$ .

Let us choose double sequence  $x_{n,m} \in \mathcal{X}_n$  such that for any fixed  $m$  we have  $x_{n,m} \rightarrow x_m$  as  $n \rightarrow \omega$ . Note that  $|x_{n,m} - x_{n,m-1}| < \frac{1}{2^m}$  for  $\omega$ -almost all  $n$ . It follows that we can choose a nested sequence of sets

$$\mathbb{N} = S_1 \supset S_2 \supset \dots$$

such that

- ◊  $\omega(S_m) = 1$  for each  $m$ ,
- ◊  $k \geq m$  for any  $k \in S_m$ , and



◇ if  $n \in S_m$ , then

$$|x_{n,m} - x_{n,m-1}| < \frac{1}{2^m}$$

Consider the sequence  $y_n = x_{n,m(n)}$ , where  $m(n)$  is the largest value such that  $m(n) \in S_m$ . Denote by  $y \in \mathcal{X}_\omega$  its  $\omega$ -limit.

Observe that by construction  $x_n \rightarrow y$  as  $n \rightarrow \infty$ . Hence the statement follows.  $\square$

**4.3.2. Observation.** *The  $\omega$ -limit of any sequence of length spaces is geodesic.*

*Proof.* If  $\mathcal{X}_n$  is a sequence length spaces, then for any sequence of pairs  $x_n, y_n \in X_n$  there is a sequence of  $\frac{1}{n}$ -midpoints  $z_n$ .

Let  $x_n \rightarrow x_\omega$ ,  $y_n \rightarrow y_\omega$  and  $z_n \rightarrow z_\omega$  as  $n \rightarrow \omega$ . Note that  $z_\omega$  is a midpoint of  $x_\omega$  and  $y_\omega$  in  $\mathcal{X}^\omega$ .

By Observation 4.3.1,  $\mathcal{X}^\omega$  is complete. Applying Lemma 1.7.4 we get the statement.  $\square$

A geodesic space  $\mathcal{T}$  is called a *metric tree* if any pair of points in  $\mathcal{T}$  are connected by a unique geodesic, and the union of any two geodesics  $[xy]$ , and  $[yz]$  contain the geodesic  $[xz]_\mathcal{T}$ . In other words any triangle in  $\mathcal{T}$  is a tripod; that is for any three geodesics  $[xy]$ ,  $[yz]$ , and  $[zx]$  have a common point.

**4.3.3. Exercise.** *Show that an ultralimit of metric trees is a metric tree.*

**4.3.4. Exercise.** *Show that spheres in metric trees are ultrametric spaces; that is, if  $\Sigma$  is a sphere in a metric tree  $\mathcal{T}$ , then*

$$|x - z|_\mathcal{T} \leq \max\{|x - y|_\mathcal{T}, |y - z|_\mathcal{T}\}$$

for any  $x, y, z \in \Sigma$ .

## 4.4 Ultrapower

If all the metric spaces in the sequence are identical  $\mathcal{X}_n = \mathcal{X}$ , its  $\omega$ -limit  $\lim_{n \rightarrow \omega} \mathcal{X}_n$  is denoted by  $\mathcal{X}^\omega$  and called  $\omega$ -power of  $\mathcal{X}$ .

**4.4.1. Exercise.** *For any point  $x \in \mathcal{X}$ , consider the constant sequence  $x_n = x$  and set  $\iota(x) = \lim_{n \rightarrow \omega} x_n \in \mathcal{X}^\omega$ .*

(a) *Show that  $\iota: \mathcal{X} \rightarrow \mathcal{X}^\omega$  is distance preserving embedding. (So we can and will consider  $\mathcal{X}$  as a subset of  $\mathcal{X}^\omega$ .)*

(b) *Show that  $\iota$  is onto if and only if  $\mathcal{X}$  compact.*

- (c) Show that if  $\mathcal{X}$  is proper, then  $\iota(\mathcal{X})$  forms a metric component of  $\mathcal{X}^\omega$ ; that is, a subset of  $\mathcal{X}^\omega$  that lie on finite distance from a given point.

**4.4.2. Observation.** Let  $\mathcal{X}$  be a complete metric space. Then  $\mathcal{X}^\omega$  is geodesic space if and only if  $\mathcal{X}$  is a length space.

*Proof.* Assume  $\mathcal{X}^\omega$  is geodesic space. Then any pair of points  $x, y \in \mathcal{X}$  has a midpoint  $z_\omega \in \mathcal{X}^\omega$ . Fix a sequence of points  $z_n \in \mathcal{X}$  such that  $z_n \rightarrow z_\omega$  as  $n \rightarrow \omega$ .

Note that  $|x - z_n|_{\mathcal{X}} \rightarrow \frac{1}{2} \cdot |x - y|_{\mathcal{X}}$  and  $|y - z_n|_{\mathcal{X}} \rightarrow \frac{1}{2} \cdot |x - y|_{\mathcal{X}}$  as  $n \rightarrow \omega$ . In particular, for any  $\varepsilon > 0$ , the point  $z_n$  is an  $\varepsilon$ -midpoint of  $x$  and  $y$  for  $\omega$ -almost all  $n$ . It remains to apply Lemma 1.7.4.

The “if”-part follows from Observation 4.3.2.  $\square$

**4.4.3. Exercise.** Assume  $\mathcal{X}$  is a complete length space and  $p, q \in \mathcal{X}$  cannot be joined by a geodesic in  $\mathcal{X}$ . Then there are at least two distinct geodesics between  $p$  and  $q$  in the ultrapower  $\mathcal{X}^\omega$ .

**4.4.4. Exercise.** Construct a proper metric space  $\mathcal{X}$  such that  $\mathcal{X}^\omega$  is not proper; that is, there is a point  $p \in \mathcal{X}^\omega$  and  $R < \infty$  such that the closed ball  $\bar{B}[p, R]_{\mathcal{X}^\omega}$  is not compact.

## 4.5 Tangent and asymptotic spaces

Choose a space  $\mathcal{X}$  and a sequence of  $\lambda_n > 0$ . Consider the sequence of scalings  $\mathcal{X}_n = \lambda_n \cdot \mathcal{X} = (\mathcal{X}, \lambda_n \cdot |\cdot - \cdot|_{\mathcal{X}})$ .

Choose a point  $p \in \mathcal{X}$  and denote by  $p_n$  the corresponding point in  $\mathcal{X}_n$ . Consider the  $\omega$ -limit  $\mathcal{X}_\omega$  of  $\mathcal{X}_n$  (one may denote it by  $\lambda_\omega \cdot \mathcal{X}$ ); set  $p_\omega$  to be the  $\omega$ -limit of  $p_n$ .

If  $\lambda_n \rightarrow 0$  as  $n \rightarrow \omega$ , then the metric component of  $p_\omega$  in  $\mathcal{X}_\omega$  is called  $\omega$ -*tangent space* at  $p$  and denoted by  $T_p^\omega \mathcal{X}$ .

If  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \omega$ , then the metric component of  $p_\omega$  in  $\mathcal{X}_\omega$  is called  $\omega$ -*asymptotic space*<sup>1</sup> and denoted by  $\text{Asym } \mathcal{X}$ . Note that the space  $\text{Asym } \mathcal{X}$  and its point  $p_\omega$  does not depend on the choice of  $p \in \mathcal{X}$ .

**4.5.1. Exercise.** Let  $\mathcal{L}$  be the Lobachevsky plane;  $\mathcal{T} = \text{Asym } \mathcal{L}$ .

- (a) Show that  $\mathcal{T}$  is a complete metric tree.  
 (b) Show that  $\mathcal{T}$  has continuum degree at any point; that is, for any point  $t \in \mathcal{T}$  the set of connected components of the complement  $\mathcal{T} \setminus \{t\}$  has cardinality continuum.

<sup>1</sup>Often it is called *asymptotic cone* despite that it is not a cone in general; this name is used since in good cases it has a cone structure.

- (c) Show that  $\mathcal{T}$  is homogeneous; that is given two points  $s, t \in \mathcal{T}$  there is an isometry of  $\mathcal{T}$  that maps  $s$  to  $t$ .
- (d) Prove (a)–(c) if  $\mathcal{L}$  is Lobachevsky space and/or for the infinite 3-regular<sup>2</sup> tree with unit edge.

As it shown in [23], the properties (a) and (b) describe the tree  $\mathcal{T}$  up to isometry. In particular, the asymptotic space of Lobachevsky plane does not depend on the choice of ultrafilter and the sequence  $\lambda_n \rightarrow \infty$ . In general, the tangent and asymptotic spaces depend on number of choices — we need to fix a sequence  $\lambda_n$  and an nonprinciple ultrafilter  $\omega$ .

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<sup>2</sup>that is, degree of any vertex is 3.



# Chapter 5

## Urysohn space

We discuss a construction introduced by Pavel Urysohn [47]. It produces a separable metric space that includes a subspace isometric to any separable metric space; in addition it is homogenous in a very strong sense, see 5.3.3.<sup>1</sup> This construction answers a question of Maurice Fréchet. Note that Fréchet lemma (1.8.1 and 1.8.2) says that  $\ell^\infty$  includes an isometric copy of any separable metric space, but  $\ell^\infty$  is not separable and it is only 1-point homogeneous.

We follow presentation given by Mikhael Gromov [28].

### 5.1 Construction

Suppose a metric space  $\mathcal{X}$  is a subspace of a pseudometric space  $\mathcal{X}'$ . In this case we may say that  $\mathcal{X}'$  is an *extension* of  $\mathcal{X}$ . If  $\text{diam } \mathcal{X}' \leq d$ , then we say that  $\mathcal{X}'$  is a *d-extension*.

If the complement  $\mathcal{X}' \setminus \mathcal{X}$  contains a single point, say  $p$ , we say that  $\mathcal{X}'$  is a *one-point extension* of  $\mathcal{X}$ . In this case, to define metric on  $\mathcal{X}'$ , it is sufficient to specify the distance function from  $p$ ; that is, a function  $f: \mathcal{X} \rightarrow \mathbb{R}$  defined by

$$f(x) = |p - x|_{\mathcal{X}'}$$

The function  $f$  can not be taken arbitrary — the triangle inequality implies that

$$f(x) + f(y) \geq |x - y|_{\mathcal{X}} \geq |f(x) - f(y)|$$

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<sup>1</sup>The idea of this construction was reused in graph theory; it produces the so called *Rado graph*, also known as *Erdős–Rényi graph* or *random graph*; it is discussed by Peter Cameron [21].

for any  $x, y \in \mathcal{X}$ . In particular  $f$  is a non-negative 1-Lipschitz function on  $\mathcal{X}$ . For a  $d$ -extension we need to assume in addition that  $\text{diam } \mathcal{X} \leq d$  and  $f(x) \leq d$  for any  $x \in \mathcal{X}$ .

Any function  $f$  of that type will be called *extension function* or *d-extension function* correspondingly.

**5.1.1. Definition.** A metric space  $\mathcal{U}$  is called *universal* if for any finite subspace  $\mathcal{F} \subset \mathcal{U}$  and any extension function  $f: \mathcal{F} \rightarrow \mathbb{R}$  there is a point  $p \in \mathcal{U}$  such that  $|p - x| = f(x)$  for any  $x \in \mathcal{F}$ .

If instead of extension functions we consider only  $d$ -extension functions and assume in addition that  $\text{diam } \mathcal{U} \leq d$ , then we arrive to a definition of  $d$ -universal space.

If in addition  $\mathcal{U}$  is separable and complete, then it is called *Urysohn space* or *d-Urysohn space*.

**5.1.2. Proposition.** Given a positive  $d$ , there is a separable  $d$ -universal metric space. Moreover, a separable universal space metric exists.

*Proof.* Let  $\mathcal{X}$  be a compact metric space such that  $\text{diam } \mathcal{X} \leq d$ . Denote by  $\mathcal{X}^d$  the space of all  $d$ -extension functions on  $\mathcal{X}$  equipped with the metric defined by the sup-norm. Note that the map  $\mathcal{X} \rightarrow \mathcal{X}^d$  defined by  $x \mapsto \text{dist}_x$  is a distance preserving embedding, so we can (and will) treat  $\mathcal{X}$  as a subspace of  $\mathcal{X}^d$ , or, equivalently,  $\mathcal{X}^d$  is an extension of  $\mathcal{X}$ .

Let us iterate this construction. Start with a one-point space  $\mathcal{X}_0$  and consider a sequence of spaces  $(\mathcal{X}_n)$  defined by  $\mathcal{X}_{n+1} = \mathcal{X}_n^d$ . Note that the sequence is nested, that is  $\mathcal{X}_0 \subset \mathcal{X}_1 \subset \dots$  and the union

$$\mathcal{X}_\infty = \bigcup_n \mathcal{X}_n;$$

comes with metric such that  $|x - y|_{\mathcal{X}_\infty} = |x - y|_{\mathcal{X}_n}$  if  $x, y \in \mathcal{X}_n$ .

Note that if  $\mathcal{X}$  is compact, then so is  $\mathcal{X}^d$ . It follows that each space  $\mathcal{X}_n$  is compact. Since  $\mathcal{X}_\infty$  is a countable union of compact spaces, it is separable.

Any finite subspace  $\mathcal{F}$  of  $\mathcal{X}_\infty$  lies in some  $\mathcal{X}_n$  for  $n < \infty$ . By construction, there is a point  $p \in \mathcal{X}_{n+1}$  that meets the condition in Definition 5.1.1. That is,  $\mathcal{X}_\infty$  is  $d$ -universal.

A construction of a universal separable metric space is done along the same lines, but the sequence should be defined by  $\mathcal{X}_{n+1} = \mathcal{X}_n^{d_n}$  for some sequence  $d_n \rightarrow \infty$ ; also the point  $p$  should be taken from  $\mathcal{X}_{n+k}$  for sufficiently large  $k$ .  $\square$

**5.1.3. Proposition.** A completion of  $d$ -universal space is  $d$ -universal.

A completion of universal space universal.

*Proof.* Suppose  $\mathcal{V}$  be a  $d$ -universal space; denote by  $\mathcal{U}$  its completion; so  $\mathcal{V}$  is a dense subset in a complete space  $\mathcal{U}$ .

Observe that  $\mathcal{U}$  is *approximately  $d$ -universal*; that is, if  $\mathcal{F} \subset \mathcal{U}$  is a finite set,  $\varepsilon > 0$ , and  $f: \mathcal{F} \rightarrow \mathbb{R}$  is a  $d$ -extension function, then there exists  $p \in \mathcal{U}$  such that

$$|p - x| \leq f(x) \pm \varepsilon.$$

for any  $x \in \mathcal{F}$ .

Therefore there is a sequence of points  $p_n \in \mathcal{U}$  such that for any  $x \in \mathcal{F}$ ,

$$|p_n - x| \leq f(x) \pm \frac{1}{2^n}.$$

Moreover, we can assume that

$$\textcircled{1} \quad |p_n - p_{n+1}| < \frac{1}{2^n}$$

for all large  $n$ . Indeed, consider the sets  $\mathcal{F}_n = \mathcal{F} \cup \{p_n\}$  and the functions  $f_n: \mathcal{F}_n \rightarrow \mathbb{R}$  defined by  $f_n(x) = f(x)$  for any  $x \in \mathcal{F}$ , and

$$f_n(p_n) = \max \{ ||p_n - x| - f(x)| : x \in \mathcal{F} \}.$$

Observe that  $f_n$  is a  $d$ -extension function for large  $n$  and  $f_n(p_n) < \frac{1}{2^n}$ . By applying approximate universal property recursively we get  $\textcircled{1}$ .

By  $\textcircled{1}$ ,  $(p_n)$  is a Cauchy sequence and its limit meets the condition in the definition of universal space (5.1.1).  $\square$

Note that 5.1.2 and 5.1.3 imply the following:

**5.1.4. Theorem.** *Urysohn space, and  $d$ -Urysohn space for any  $d > 0$ , exist.*

## 5.2 Separable universality

**5.2.1. Proposition.** *Let  $\mathcal{U}$  be an Urysohn space. Then any separable metric space  $\mathcal{S}$  admits a distance preserving embedding  $\mathcal{S} \hookrightarrow \mathcal{U}$ .*

*Moreover, for any finite subspace  $\mathcal{F} \subset \mathcal{S}$ , any distance preserving embedding  $\mathcal{F} \hookrightarrow \mathcal{U}$  can be extended to an distance preserving embedding  $\mathcal{S} \hookrightarrow \mathcal{U}$ .*

*If  $\mathcal{U}$  is  $d$ -Urysohn, then the statements hold provided  $\text{diam } \mathcal{S} \leq d$ .*

*Proof.* We will prove the second statement, the first statement is its partial case for  $\mathcal{F} = \emptyset$ .

The required isometry will be denoted by  $x \mapsto x'$ .

Choose a dense sequence of points  $s_1, s_2, \dots \in \mathcal{S}$ . We may assume that  $\mathcal{F} = \{s_1, \dots, s_n\}$ , so  $s'_i \in \mathcal{U}$  are defined for  $i \leq n$ .

The sequence  $s'_i$  for  $i > n$  can be defined recursively using universality of  $\mathcal{U}$ . Namely suppose that  $s'_1, \dots, s'_{i-1}$  are already defined. Since  $\mathcal{U}$  is universal, there is a point  $s'_i \in \mathcal{U}$  such that

$$|s'_i - s'_j|_{\mathcal{U}} = |s_i - s_j|_{\mathcal{S}}$$

for any  $j < i$ .

We constructed a distance preserving map  $s_i \mapsto s'_i$ , it remains to extend it to a continuous map on whole  $\mathcal{S}$ .  $\square$

**5.2.2. Exercise.** Show that any two distinct points in an Urysohn space can be jointed by infinite number of geodesics.

**5.2.3. Exercise.** Modify the proofs of 5.1.3 and 5.2.1 to prove the following theorem.

**5.2.4. Theorem.** Let  $K$  be a compact set in a separable space  $\mathcal{S}$ . Then any distance-preserving map from  $K$  to an Urysohn space can be extended to a distance-preserving map on whole  $\mathcal{S}$ .

**5.2.5. Exercise.** Show that Urysohn space is simply connected.

## 5.3 Uniqueness and homogeneity

**5.3.1. Theorem.** Suppose  $\mathcal{F} \subset \mathcal{U}$  and  $\mathcal{F}' \subset \mathcal{U}'$  be finite isometric subspaces in a pair of  $(d-)$ Urysohn spaces  $\mathcal{U}$  and  $\mathcal{U}'$ . Then any isometry  $\mathcal{F} \rightarrow \mathcal{F}'$  can be extended to an isometry  $\mathcal{U} \rightarrow \mathcal{U}'$ .

In particular  $(d-)$ Urysohn space is unique up to isometry.

Note that 5.2.1 implies that there are distance-preserving maps  $\mathcal{U} \rightarrow \mathcal{U}'$  and  $\mathcal{U}' \rightarrow \mathcal{U}$ , but it does not solely imply existence of an isometry. The following construction use the same idea as in the proof of 5.2.1, but we need to apply it *back-and-forth* to ensure that the constructed distance-preserving map is onto.

*Proof.* The required isometry  $\mathcal{U} \leftrightarrow \mathcal{U}'$  will be denoted by  $u \leftrightarrow u'$ .

Choose dense sequences  $a_1, a_2, \dots \in \mathcal{U}$  and  $b'_1, b'_2, \dots \in \mathcal{U}$ . Let us define recursively  $a'_1, b_1, a'_2, b_2, \dots$  — on the odd step we define the images of  $a_1, a_2, \dots$  and on the even steps we define invese images of  $b'_1, b'_2, \dots$ . The same argument as in the proof of 5.2.1 shows that we



can construct two sequences  $a'_1, a'_2, \dots \in \mathcal{U}'$  and  $b_1, b_2, \dots \in \mathcal{U}$  such that

$$\begin{aligned} |a_i - a_j|_{\mathcal{U}} &= |a'_i - a'_j|_{\mathcal{U}'} \\ |a_i - b_j|_{\mathcal{U}} &= |a'_i - b'_j|_{\mathcal{U}'} \\ |b_i - b_j|_{\mathcal{U}} &= |b'_i - b'_j|_{\mathcal{U}'} \end{aligned}$$

for all  $i$  and  $j$ .

Let us extend the constructed distance preserving bijection defined by  $a_i \leftrightarrow a'_i$  and  $b_i \leftrightarrow b'_i$  continuously to whole  $\mathcal{U}$ . Observe that the image of this bijection is dense in  $\mathcal{U}'$  therefore the constructed map  $\mathcal{U} \rightarrow \mathcal{U}'$  is a bijection.  $\square$

Further the Urysohn space will be denoted by  $\mathcal{U}$ , and the  $d$ -Urysohn space will be denoted by  $\mathcal{U}_d$ . Observe that 5.3.1 implies that the spaces  $\mathcal{U}$  and  $\mathcal{U}_d$  are finite-set homogeneous; that is,

- ◇ any distance preserving map from a finite subset to the whole space can be extended to an isometry.

It is unknown if there is a separable universal space that is finite-set homogeneous (this question appeared already in [47] and reappeared in [28, p. 83] with a missing key word). In fact I do not see an example of a 1-point homogeneous universal space.

**5.3.2. Exercise.** Let  $S$  be a sphere of radius  $\frac{d}{2}$  in  $\mathcal{U}_d$ ; that is,

$$S = \{ x \in \mathcal{U}_d : |p - x|_{\mathcal{U}_d} = \frac{d}{2} \}$$

for some point  $p \in \mathcal{U}_d$ . Show that  $S$  is isometric to  $\mathcal{U}_d$ .

Use it to show that  $\mathcal{U}_d$  is not countable-set homogeneous; that is, there is an distance preserving map from a countable subset of  $\mathcal{U}_d$  to  $\mathcal{U}_d$  that can not be extended to an isometry of  $\mathcal{U}_d$ .

In fact the Urysohn space is compact-set homogeneous; more precisely the following theorem holds. A proof can be obtained by modifying the proofs of 5.1.3 and 5.3.1 the same way as it is done in 5.2.3.

**5.3.3. Theorem.** Let  $K$  be a compact set in an  $(d)$ -Uryson space  $\mathcal{U}$ . Then any distance preserving map  $K \rightarrow \mathcal{U}$  can be extended to an isometry of  $\mathcal{U}$ .



# Chapter 6

## Injective spaces

In this chapter we discuss *injective spaces* also known as *hyperconvex spaces*. They are metric analog of convex sets in Euclidean space. The so called *injective envelop* is a minimal injective space that contains a given metric space as a subpace; it is a direct analog of convex hull of a set in a Euclidean space.

This type of spaces were introduced by Nachman Aronszajn and Prom Panitchpakdi [8] and injective envelop was introduced by John Isbell [32]; it was rediscovered number of times since then.

### 6.1 Admissible functions

Let  $\mathcal{X}$  be a metric space. A function  $r: \mathcal{X} \rightarrow \mathbb{R}$  is called *admissible* if the following inequality

$$\bullet \quad r(x) + r(y) \geq |x - y|$$

holds for any  $x, y \in \mathcal{X}$ .

#### 6.1.1. Observation.

- (a) Any admissible is nonnegative.
- (b) If  $\mathcal{X}$  is a geodesic space, then a function  $r: \mathcal{X} \rightarrow \mathbb{R}$  is admissible if and only if

$$\overline{B}[x, r(x)] \cap \overline{B}[y, r(y)] \neq \emptyset$$

for any  $x, y \in \mathcal{X}$ .

*Proof.* For (a), take  $x = y$  in  $\bullet$ . Part (b) follows from the triangle inequality and the definition of geodesic.  $\square$

A minimal admissible function will be called *extremal*. More precisely, an admissible function  $r: \mathcal{X} \rightarrow \mathbb{R}$  is extremal if for any other admissible function  $s: \mathcal{X} \rightarrow \mathbb{R}$  such that  $s \leq r$  we have  $s = r$ .

**6.1.2. Exercise.** Let  $r$  and  $s$  be two extremal functions of a metric space  $\mathcal{X}$ . Suppose that  $r \geq s - c$  for some constant  $c$ . Show that  $c \geq 0$  and  $r \leq s + c$ .

**6.1.3. Observation.**

- (a) For any point  $p$  in a metric space  $\mathcal{X}$  the distance function  $r = \text{dist}_p$  is extremal.
- (b) For any admissible function  $s$  there is an extremal function  $r$  such that  $r \leq s$ .

*Proof;* (a). By the triangle inequality, **1** holds. Further if  $s \leq r$  is another admissible function then  $s(p) = 0$  and **1** implies that  $s(x) \geq |p - x|$ . Whence  $s = r$ .

(b). Follows from Zorn's lemma. □

**6.1.4. Lemma.** Any extremal function  $r$  on  $\mathcal{X}$  is 1-Lipschitz; that is,

$$|r(x) - r(y)| \leq |x - y|$$

for any  $x, y \in \mathcal{X}$ .

In other words, any extremal function is an extension function; see definition on page 46.

*Proof.* Arguing by contradiction, assume that the inequality does not hold; so we can choose two points  $p, q \in \mathcal{X}$  such that

$$r(p) - r(q) > |p - q|.$$

Consider another function  $s$  such that  $s = r$  at all points except  $p$  and  $s(p) := r(q) + |p - q|$ , so

$$s(p) - r(q) = |p - q|$$

Observe that  $0 < s(p) < r(p)$ .

By triangle inequality,  $s$  remains to be admissible. Indeed, since  $r$  is admissible, for any  $x \neq p$  we have

$$\begin{aligned} s(p) + s(x) &= [s(p) - r(q)] + [r(x) + r(q)] \geq \\ &\geq |p - q| + |x - q| \geq \\ &\geq |p - x|. \end{aligned}$$

For  $p = x$  the inequality trivially holds and for the remaining pairs of points the inequality holds since it holds for  $r$ . □

## 6.2 Injective spaces

**6.2.1. Definition.** A metric space  $\mathcal{Y}$  is called *injective* if for any metric space  $\mathcal{X}$ , any its subspace  $\mathcal{A}$  any short map  $f: \mathcal{A} \rightarrow \mathcal{Y}$  can be extended to a short map  $F: \mathcal{X} \rightarrow \mathcal{Y}$ ; that is,  $f = F|_{\mathcal{A}}$ .

**6.2.2. Exercise.** Show that any injective space is

- (a) complete,
- (b) geodesic, and
- (c) contractible.

**6.2.3. Exercise.** Suppose that a metric space  $\mathcal{X}$  satisfies the following property:

For any subspace  $\mathcal{A}$  in  $\mathcal{X}$  and any other metric space  $\mathcal{Y}$ , any short map  $f: \mathcal{A} \rightarrow \mathcal{Y}$  can be extended to a short map  $F: \mathcal{X} \rightarrow \mathcal{Y}$ .

Show that  $\mathcal{X}$  is an ultrametric space; that is, the following strong version of triangle inequality

$$|x - z|_{\mathcal{X}} \leq \max\{|x - y|_{\mathcal{X}}, |y - z|_{\mathcal{X}}\}$$

holds for any three points  $x, y, z \in \mathcal{X}$ .

**6.2.4. Theorem.** For any metric space  $\mathcal{Y}$  the following condition are equivalent:

- (a)  $\mathcal{Y}$  is injective
- (b) If  $r: \mathcal{Y} \rightarrow \mathbb{R}$  is an extremal function then there is a point  $p \in \mathcal{Y}$  such that

$$|p - x| \leq r(x)$$

for any  $x \in \mathcal{Y}$ .

- (c)  $\mathcal{Y}$  is hyperconvex; that is, if  $\{\overline{B}[x_{\alpha}, r_{\alpha}]\}_{\alpha \in \mathcal{A}}$  is a family of closed balls in  $\mathcal{Y}$  such that

$$r_{\alpha} + r_{\beta} \geq |x_{\alpha} - x_{\beta}|$$

for any  $\alpha, \beta \in \mathcal{A}$ , then all the balls in the family have a common point.

*Proof.* We will prove implications  $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$ .

$(a) \Rightarrow (b)$ . Since  $\mathcal{Y}$  is injective for any extension function  $r: \mathcal{Y} \rightarrow \mathbb{R}$  there is a point  $p \in \mathcal{Y}$  such that

$$|p - x| \leq r(x)$$

for any  $x \in \mathcal{Y}$ . By 6.1.4, any extremal function is an extension function, whence the implication follow.

$(b) \Rightarrow (c)$ . By 6.1.1b, part (c) is equivalent to the following statement:

- ◇ If  $r: \mathcal{Y} \rightarrow \mathbb{R}$  is an admissible function, then there is a point  $p \in \mathcal{Y}$  such that

$$\textcircled{1} \quad |p - x| \leq r(x)$$

for any  $x \in \mathcal{Y}$ .

Indeed, set  $r(x) = \inf \{ r_\alpha : x_\alpha = x \}$ . The condition in (c) imply that  $r$  is admissible. It remains to observe that  $p \in \bar{B}[x_\alpha, r_\alpha]$  for every  $\alpha$  if and only if  $\textcircled{1}$  holds.

By 6.1.3b, for any admissible function  $r$  there is an extremal function  $\bar{r} \leq r$ ; whence (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a). Arguing by contradiction, suppose  $\mathcal{Y}$  is not injective; that is, there is a metric space  $\mathcal{X}$  with a subset  $\mathcal{A}$  such that a short map  $f: \mathcal{A} \rightarrow \mathcal{Y}$  can not be extended to a short map  $F: \mathcal{X} \rightarrow \mathcal{Y}$ . By Zorn's lemma we may assume that  $\mathcal{A}$  is a maximal subset; that is, the domain of  $f$  can not be enlarged by a single point.<sup>1</sup>

Fix a point  $p$  in the complement  $\mathcal{X} \setminus \mathcal{A}$ . To extend  $f$  to  $p$ , we need to choose  $f(p)$  in the intersection of the balls  $\bar{B}[f(x), r(x)]$ , where  $r(x) = |p - x|$ . Therefore this intersection for all  $x \in \mathcal{A}$  have to be empty.

Since  $f$  is short, we have that

$$\begin{aligned} r(x) + r(y) &\geq |x - y|_{\mathcal{X}} \geq \\ &\geq |f(x) - f(y)|_{\mathcal{Y}}. \end{aligned}$$

Therefore by (c) the balls  $\bar{B}[f(x), r(x)]$  have a common point — a contradiction.  $\square$

**6.2.5. Exercise.** Show that the following spaces are injective:

- (a) the real line;
- (b) complete metric tree;
- (c) plane with the metric induced by  $\ell^\infty$ -norm.

## 6.3 Injective envelop

Let  $\mathcal{X}$  be a metric space. Consider the space  $\text{Inj } \mathcal{X}$  of extremal functions on  $\mathcal{X}$  equipped with sup-norm; that is,

$$|f - g|_{\text{Inj } \mathcal{X}} := \sup \{ |f(x) - g(x)| : x \in \mathcal{X} \}.$$

Recall that by 6.1.3a, any distance function is extremal. It follows that the map  $x \mapsto \text{dist}_x$  produces a distance-preserving embedding

<sup>1</sup>In this case  $\mathcal{A}$  must be closed, but we will not use it.

$\mathcal{X} \hookrightarrow \text{Inj } \mathcal{X}$ . So we can (and will) treat  $\mathcal{X}$  as a subspace of  $\text{Inj } \mathcal{X}$ , or, equivalently,  $\text{Inj } \mathcal{X}$  as an extension of  $\mathcal{X}$ .

Since any extremal function is 1-Lipschitz, for any  $f \in \text{Inj } \mathcal{X}$  and  $p \in \mathcal{X}$ , we have that  $f(x) \leq f(p) + \text{dist}_p(x)$ . By 6.1.2, we also get  $f(x) \geq -f(p) + \text{dist}_p(x)$ . Therefore

$$\textcircled{1} \quad |f - p|_{\text{Inj } \mathcal{X}} = \sup \{ |f(x) - \text{dist}_p(x)| : x \in \mathcal{X} \} = f(p).$$

**6.3.1. Exercise.** Suppose that  $\mathcal{X}$  is

(a) a metric space with exactly three points  $a, b, c$  such that

$$|a - b|_{\mathcal{X}} = |b - c|_{\mathcal{X}} = |c - a|_{\mathcal{X}} = 1.$$

(b) a metric space with exactly four points  $p, q, x, y$  such that

$$|p - x|_{\mathcal{X}} = |p - y|_{\mathcal{X}} = |q - x|_{\mathcal{X}} = |q - y|_{\mathcal{X}} = 1$$

and

$$|p - q|_{\mathcal{X}} = |x - y|_{\mathcal{X}} = 2.$$

Describe the set of all extremal functions on  $\mathcal{X}$  and the metric space  $\text{Inj } \mathcal{X}$  in each case.

**6.3.2. Proposition.** For any metric space  $\mathcal{X}$ , its extension  $\text{Inj } \mathcal{X}$  is injective.

**6.3.3. Lemma.** Given a point  $p$  in a metric space  $\mathcal{X}$ , a positive  $\delta$  and  $f \in \text{Inj } \mathcal{X}$ , there is a point  $q \in \mathcal{X}$  such that

$$f(p) + f(q) < |p - q|_{\mathcal{X}} + \delta,$$

or equivalently

$$|f - p|_{\text{Inj } \mathcal{X}} + |f - q|_{\text{Inj } \mathcal{X}} < |p - q|_{\text{Inj } \mathcal{X}} + \delta.$$

Moreover if  $\mathcal{X}$  is compact, then for any  $p \in \mathcal{X}$  and  $f \in \text{Inj } \mathcal{X}$ , there is  $q \in \mathcal{X}$  such that

$$f(p) + f(q) = |p - q|_{\mathcal{X}},$$

or equivalently

$$|f - p|_{\text{Inj } \mathcal{X}} + |f - q|_{\text{Inj } \mathcal{X}} = |p - q|_{\text{Inj } \mathcal{X}}.$$

*Proof.* By 6.1.3a,  $\text{dist}_p$  is an extremal function; it remains to apply 6.1.2 to the functions  $\text{dist}_p$  and  $f$ .  $\square$

**6.3.4. Exercise.** Let  $\mathcal{X}$  be a compact space. Show that for any two points  $f, g \in \text{Inj } \mathcal{X}$  there are points  $p, q \in \mathcal{X}$  such that

$$|p - f|_{\text{Inj } \mathcal{X}} + |f - g|_{\text{Inj } \mathcal{X}} + |g - q|_{\text{Inj } \mathcal{X}} = |p - q|_{\text{Inj } \mathcal{X}}.$$

**6.3.5. Lemma.** Let  $\mathcal{X}$  be a metric space. Suppose that  $r$  is an extremal function on  $\text{Inj } \mathcal{X}$ . Then the restriction  $r|_{\mathcal{X}}$  is an extremal function on  $\mathcal{X}$ . In other words,  $r|_{\mathcal{X}} \in \text{Inj } \mathcal{X}$ .

*Proof.* Arguing by contradiction, suppose that there is an admissible function  $s: \mathcal{X} \rightarrow \mathbb{R}$  such that  $s(x) \leq r(x)$  for any  $x \in \mathcal{X}$  and  $s(p) < r(p)$  for some point  $p \in \mathcal{X}$ . Consider another function  $\bar{r}: \text{Inj } \mathcal{X} \rightarrow \mathbb{R}$  such that  $\bar{r} = r$  at all points except  $p$  and  $\bar{r}(p) := s(p)$ .

Let us show that  $\bar{r}$  is admissible, that is

$$\textcircled{2} \quad \bar{r}(f) + \bar{r}(g) \geq |f - g|_{\text{Inj } \mathcal{X}}$$

for any  $f, g \in \text{Inj } \mathcal{X}$ .

Since  $r$  is admissible and  $\bar{r} = r$  on  $(\text{Inj } \mathcal{X}) \setminus \{p\}$ , it is sufficient to prove  $\textcircled{2}$  if  $f \neq g = p$ . By  $\textcircled{1}$ , we have  $|f - p|_{\text{Inj } \mathcal{X}} = f(p)$ . Therefore  $\textcircled{2}$  boils down to the following inequality

$$\textcircled{3} \quad r(f) + s(p) \geq f(p).$$

for any  $f \in \text{Inj } \mathcal{X}$ .

Fix small  $\delta > 0$ . Let  $q \in \mathcal{X}$  be the point provided by 6.3.3. Then

$$r(f) + s(p) \geq [r(f) - r(q)] + [r(q) + s(p)] \geq$$

since  $r$  is 1-Lipschitz, and  $r(q) \geq s(q)$ , we can continue

$$\geq -|q - f|_{\text{Inj } \mathcal{X}} + [s(q) + s(p)] =$$

by  $\textcircled{1}$  and since  $s$  is admissible

$$= f(q) + |p - q| >$$

by 6.3.3

$$> f(p) - \delta.$$

Since  $\delta > 0$  is arbitrary,  $\textcircled{3}$  and  $\textcircled{2}$  follow.

Summarizing, the function  $\bar{r}$  is admissible,  $\bar{r} \leq r$  and  $\bar{r}(p) < r(p)$ ; that is,  $r$  is not extremal — a contradiction.  $\square$



*Proof of 6.3.2.* By 6.2.4b, it is sufficient to show that for any extremal function  $r$  on  $\text{Inj } \mathcal{X}$ , there is a point  $\bar{r} \in \text{Inj } \mathcal{X}$  such that

$$\textcircled{4} \quad r(f) \geq |\bar{r} - f|_{\text{Inj } \mathcal{X}}$$

for any  $f \in \text{Inj } \mathcal{X}$ .

Let us show that one can take  $\bar{r} = r|_{\mathcal{X}}$ . By 6.3.5,  $\bar{r}$  is extremal; that is,  $\bar{r} \in \text{Inj } \mathcal{X}$ .

Since  $r$  is 1-Lipschitz (6.1.4) we have that

$$\bar{r}(x) - f(x) = r(x) - |f - x|_{\text{Inj } \mathcal{X}} \leq r(f).$$

for any  $x$ . Since  $r$  is admissible we have that

$$\bar{r}(x) - f(x) = r(x) - |f - x|_{\text{Inj } \mathcal{X}} \geq -r(f).$$

for any  $x$ . That is,  $|\bar{r}(x) - f(x)| \leq r(f)$  for any  $x \in \mathcal{X}$ . By the definition, we have

$$|\bar{r} - f|_{\text{Inj } \mathcal{X}} = \sup \{ |\bar{r}(x) - f(x)| : x \in \mathcal{X} \};$$

hence  $\textcircled{4}$  follows.  $\square$

An extension  $\mathcal{E}$  of a metric space  $\mathcal{X}$  will be called its *injective envelop* if  $\mathcal{E}$  is an injective space and there is no injective proper subspace of  $\mathcal{E}$  that contains  $\mathcal{X}$ .

Two injective envelopes  $e: \mathcal{X} \hookrightarrow \mathcal{E}$  and  $f: \mathcal{X} \hookrightarrow \mathcal{F}$  are called equivalent if there is an isometry  $\iota: \mathcal{E} \rightarrow \mathcal{F}$  such that  $f = \iota \circ e$ .

**6.3.6. Theorem.** *For any metric space  $\mathcal{X}$ , its extension  $\text{Inj } \mathcal{X}$  is an injective envelop.*

*Moreover, any other injective envelop of  $\mathcal{X}$  is equivalent to  $\text{Inj } \mathcal{X}$ .*

*Proof.* Suppose  $S \subset \text{Inj } \mathcal{X}$  is an injective subspace containing  $\mathcal{X}$ . Since  $S$  is injective, there is a short map  $w: \text{Inj } \mathcal{X} \rightarrow S$  that fixes all points in  $\mathcal{X}$ .

Suppose that  $w: f \mapsto f'$ ; observe that  $f(x) \geq f'(x)$  for any  $x \in \mathcal{X}$ . Since  $f$  is extremal,  $f = f'$ ; that is,  $w$  is the identity map and therefore  $S = \text{Inj } \mathcal{X}$ .

Assume we have another injective envelop  $e: \mathcal{X} \hookrightarrow \mathcal{E}$ . Then there are short maps  $v: \mathcal{E} \rightarrow \text{Inj } \mathcal{X}$  and  $w: \text{Inj } \mathcal{X} \rightarrow \mathcal{E}$  such that  $x = v \circ e(x)$  and  $e(x) = w(x)$  for any  $x \in \mathcal{X}$ . From above, the  $v \circ w$  is the identity on  $\text{Inj } \mathcal{X}$ . In particular  $w$  is distance preserving.

The composition  $w \circ v: \mathcal{E} \rightarrow \mathcal{E}$  is a short map that fixes points in  $e(\mathcal{X})$ . Since  $e: \mathcal{X} \hookrightarrow \mathcal{E}$  is an injective envelop, the composition  $w \circ v$  and therefore  $w$  are onto. Whence  $w$  is an isometry.  $\square$



## Part II

# Alexandrov geometry



# Chapter 7

## Introduction

### 7.1 Manifesto

Alexandrov geometry can use “back to Euclid” as a slogan. Alexandrov spaces are defined via axioms similar to those given by Euclid, but certain equalities are changed to inequalities. Depending on the sign of the inequalities, we get Alexandrov spaces with *curvature bounded above* or *curvature bounded below*. The definitions of the two classes of spaces are similar, but their properties and known applications are quite different.

Consider the space  $\mathcal{M}_4$  of all isometry classes of 4-point metric spaces. Each element in  $\mathcal{M}_4$  can be described by 6 numbers — the distances between all 6 pairs of its points, say  $\ell_{i,j}$  for  $1 \leq i < j \leq 4$  modulo permutations of the index set  $(1, 2, 3, 4)$ . These 6 numbers are subject to 12 triangle inequalities; that is,

$$\ell_{i,j} + \ell_{j,k} \geq \ell_{i,k}$$

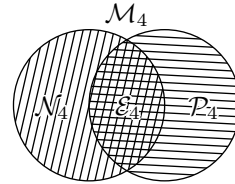
holds for all  $i, j$  and  $k$ , where we assume that  $\ell_{j,i} = \ell_{i,j}$  and  $\ell_{i,i} = 0$ .

The space  $\mathcal{M}_4$  comes with topology. It can be defined as a quotient of the cone in  $\mathbb{R}^6$  by permutations of the 4-points of the space. And, the same topology is induced on  $\mathcal{M}_4$  by the Gromov–Hausdorff metric.

Consider the subset  $\mathcal{E}_4 \subset \mathcal{M}_4$  of all isometry classes of 4-point metric spaces that admit isometric embeddings into Euclidean space.

**7.1.1. Claim.** *The complement  $\mathcal{M}_4 \setminus \mathcal{E}_4$  has two connected components.*

A proof of the claim can be extracted from 7.3.3.



The definition of Alexandrov spaces is based on this claim. Let us denote one of the components by  $\mathcal{P}_4$  and the other by  $\mathcal{N}_4$ . Here  $\mathcal{P}$  and  $\mathcal{N}$  stand for *positive* and *negative curvature* because spheres have no quadruples of type  $\mathcal{N}_4$  and hyperbolic space has no quadruples of type  $\mathcal{P}_4$ .

A metric space, with length metric, that has no quadruples of points of type  $\mathcal{P}_4$  or  $\mathcal{N}_4$  respectively is called an Alexandrov space with non-positive (CAT(0)) or non-negative curvature (CBB(0)).

Let us describe the subdivision into  $\mathcal{P}_4$ ,  $\mathcal{E}_4$  and  $\mathcal{N}_4$  intuitively. Imagine that you move out of  $\mathcal{E}_4$  — your path is a one parameter family of 4-point metric spaces. The last thing you see in  $\mathcal{E}_4$  is one of the two plane configurations



shown on the diagram. If you see the left configuration then you move into  $\mathcal{N}_4$ ; if it is the one on the right, then you move into  $\mathcal{P}_4$ . More degenerate pictures can be avoided, for example a triangle with a point on a side. From such a configuration one may move in  $\mathcal{N}_4$  and in  $\mathcal{P}_4$  (as well as come back to  $\mathcal{E}_4$ ).

Here is an exercise, solving which would force you to rebuild a considerable part of Alexandrov geometry. It might be helpful to spend some time thinking about this exercise before proceeding.

**7.1.2. Advanced exercise.** Assume  $\mathcal{X}$  is a complete metric space with length metric, containing only quadruples of type  $\mathcal{E}_4$ . Show that  $\mathcal{X}$  is isometric to a convex set in a Hilbert space.

In the definition above, instead of Euclidean space one can take hyperbolic space of curvature  $-1$ . In this case, one obtains the definition of spaces with curvature bounded above or below by  $-1$  (CAT( $-1$ ) or CBB( $-1$ )).

To define spaces with curvature bounded above or below by  $1$  (CAT( $1$ ) or CBB( $1$ )), one has to take the unit 3-sphere and specify that only the quadruples of points such that each of the four triangles has perimeter less than  $2 \cdot \pi$  are checked. The latter condition could be considered as a part of the *spherical triangle inequality*.

## 7.2 Triangles, hinges and angles

**Triangles.** For a triple of points  $p, q, r \in \mathcal{X}$ , a choice of a triple of geodesics  $([qr], [rp], [pq])$  will be called a *triangle*; we will use the short notation  $[pqr] = ([qr], [rp], [pq])$ .

Given a triple  $p, q, r \in \mathcal{X}$  there may be no triangle  $[pqr]$  simply because one of the pairs of these points cannot be joined by a geodesic.

Also, many different triangles with these vertices may exist, any of which can be denoted by  $[pqr]$ . However, if we write  $[pqr]$ , it means that we have made a choice of such a triangle; that is, we have fixed a choice of the geodesics  $[qr]$ ,  $[rp]$ , and  $[pq]$ .

The value

$$|p - q| + |q - r| + |r - p|$$

will be called the *perimeter of the triangle*  $[pqr]$ .

**Model triangles.** Let  $\mathcal{X}$  be a metric space and  $p, q, r \in \mathcal{X}$ . Let us define the *model triangle*  $[\tilde{p}\tilde{q}\tilde{r}]$  (briefly,  $[\tilde{p}\tilde{q}\tilde{r}] = \tilde{\Delta}(pqr)_{\mathbb{E}^2}$ ) to be a triangle in the Euclidean plane  $\mathbb{E}^2$  such that

$$\begin{aligned} |\tilde{p} - \tilde{q}|_{\mathbb{E}^2} &= |p - q|_{\mathcal{X}}, \\ |\tilde{q} - \tilde{r}|_{\mathbb{E}^2} &= |q - r|_{\mathcal{X}}, \\ |\tilde{r} - \tilde{p}|_{\mathbb{E}^2} &= |r - p|_{\mathcal{X}}. \end{aligned}$$

In the same way we can define the *hyperbolic* and the *spherical model triangles*  $\tilde{\Delta}(pqr)_{\mathbb{H}^2}$ ,  $\tilde{\Delta}(pqr)_{\mathbb{S}^2}$  in the hyperbolic plane  $\mathbb{H}^2$  and the unit sphere  $\mathbb{S}^2$ . In the latter case the model triangle is said to be defined if in addition

$$|p - q| + |q - r| + |r - p| < 2 \cdot \pi.$$

In this case the model triangle again exists and is unique up to an isometry of  $\mathbb{S}^2$ .

**Model angles.** If  $[\tilde{p}\tilde{q}\tilde{r}] = \tilde{\Delta}(pqr)_{\mathbb{E}^2}$  and  $|p - q|, |p - r| > 0$ , the angle measure of  $[\tilde{p}\tilde{q}\tilde{r}]$  at  $\tilde{p}$  will be called the *model angle* of the triple  $p, q, r$  and will be denoted by  $\tilde{\angle}(p_r^q)_{\mathbb{E}^2}$ . In the same way we define  $\tilde{\angle}(p_r^q)_{\mathbb{H}^2}$  and  $\tilde{\angle}(p_r^q)_{\mathbb{S}^2}$ ; in the latter case we assume in addition that the model triangle  $\tilde{\Delta}(pqr)_{\mathbb{S}^2}$  is defined.

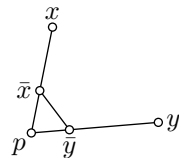
We may use the notation  $\tilde{\angle}(p_r^q)$  if it is evident which of the model spaces  $\mathbb{H}^2$ ,  $\mathbb{E}^2$  or  $\mathbb{S}^2$  is meant.

**Hinges.** Let  $p, x, y \in \mathcal{X}$  be a triple of points such that  $p$  is distinct from  $x$  and  $y$ . A pair of geodesics  $([px], [py])$  will be called a *hinge* and will be denoted by  $[p_y^x] = ([px], [py])$ .

**Angles.** Given a hinge  $[p_y^x]$ , we define its *angle* as the limit

$$\bullet \quad \angle[p_y^x] := \lim_{\bar{x}, \bar{y} \rightarrow p} \tilde{\angle}(p_{\bar{y}}^{\bar{x}}),$$

where  $\bar{x} \in [px]$  and  $\bar{y} \in [py]$ . The angle  $\angle[p_y^x]$  is defined if the limit exists.



It is straightforward to check that in **1**, one can use  $\tilde{\angle}(p_{\frac{x}{y}})_{\mathbb{S}^2}$  or  $\tilde{\angle}(p_{\frac{x}{y}})_{\mathbb{H}^2}$  or  $\tilde{\angle}(p_{\frac{x}{y}})_{\mathbb{E}^2}$ , the result will be the same.

**7.2.1. Exercise.** Give an example of a hinge  $[p_y^x]$  in a metric space with undefined angle  $\angle[p_y^x]$ .

**7.2.2. Exercise.** Suppose that for three geodesics  $[px]$ ,  $[py]$ , and  $[pz]$  in a metric space, the angles  $\alpha = \angle[p_y^x]$ ,  $\beta = \angle[p_z^y]$ , and  $\gamma = \angle[p_x^z]$  are defined. Show that  $\alpha$ ,  $\beta$  and  $\gamma$  satisfy all triangle inequalities:

$$\alpha \leq \beta + \gamma, \quad \beta \leq \gamma + \alpha, \quad \gamma \leq \alpha + \beta,$$

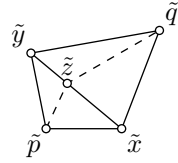
## 7.3 Definitions

**Curvature bounded above.** Given a quadruple of points  $p, q, x, y$  in a metric space  $\mathcal{X}$ , consider two model triangles  $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\Delta}(p_{xy})_{\mathbb{E}^2}$  and  $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\Delta}(q_{xy})_{\mathbb{E}^2}$  with common side  $[\tilde{x}\tilde{y}]$ .

If the inequality

$$|p - q|_{\mathcal{X}} \leq |\tilde{p} - \tilde{z}|_{\mathbb{E}^2} + |\tilde{z} - \tilde{q}|_{\mathbb{E}^2}$$

holds for any point  $\tilde{z} \in [\tilde{x}\tilde{y}]$ , then we say that the quadruple  $p, q, x, y$  satisfies CAT(0) comparison.



If we do the same for spherical model triangles  $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\Delta}(p_{xy})_{\mathbb{S}^2}$  and  $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\Delta}(q_{xy})_{\mathbb{S}^2}$ , then we arrive at the definition of CAT(1) comparison. If one of the spherical model triangles is undefined,<sup>1</sup> then it is assumed that CAT(1) comparison automatically holds for this quadruple.

We can do the same for the *model plane* of curvature  $\kappa$ ; that is, a sphere if  $\kappa > 0$ , Euclidean plane if  $\kappa = 0$  and Lobachevsky plane if  $\kappa < 0$ . In this case we arrive at the definition of CAT( $\kappa$ ) comparison. However we will mostly consider CAT(0) comparison and occasionally CAT(1) comparison; so, if you see CAT( $\kappa$ ), you can assume that  $\kappa$  is 0 or 1.

If all quadruples in a metric space  $\mathcal{X}$  satisfy CAT( $\kappa$ ) comparison, then we say that the space  $\mathcal{X}$  is CAT( $\kappa$ ) (we use CAT( $\kappa$ ) as an adjective).

Here CAT is an acronym for Cartan, Alexandrov, and Toponogov. It was coined by Mikhael Gromov in 1987, but it should be pronounced

<sup>1</sup>That is, if

$$|p - x| + |p - y| + |x - y| \geq 2 \cdot \pi \quad \text{or} \quad |q - x| + |q - y| + |x - y| \geq 2 \cdot \pi.$$



as “cat” in the sense of “miauw”. Originally, Alexandrov called these spaces “ $\mathfrak{R}_\kappa$  domain”; this term is still in use.)

**7.3.1. Exercise.** *Show that a metric space  $\mathcal{X}$  is CAT(0) if and only if for any quadruple of points  $p, q, x, y$  in  $\mathcal{X}$  there is a quadruple  $\tilde{p}, \tilde{q}, \tilde{x}, \tilde{y}$  in  $\mathbb{E}^2$  such that*

$$\begin{aligned} |\tilde{p} - \tilde{q}| &= |p - q|, & |\tilde{x} - \tilde{y}| &= |x - y|, \\ |\tilde{p} - \tilde{x}| &\leq |p - x|, & |\tilde{p} - \tilde{y}| &\leq |p - y|, \\ |\tilde{q} - \tilde{x}| &\leq |q - x|, & |\tilde{q} - \tilde{y}| &\leq |q - y|. \end{aligned}$$

**Curvature bounded below.** If the inequality

$$\tilde{\Delta}(p_y^x)_{\mathbb{E}^2} + \tilde{\Delta}(p_z^y)_{\mathbb{E}^2} + \tilde{\Delta}(p_x^z)_{\mathbb{E}^2} \leq 2 \cdot \pi$$

holds for points  $p, x, y, z$  in a metric space  $\mathcal{X}$ , then we say that the quadruple  $p, x, y, z$  satisfies CBB(0) comparison.

If we do the same for spherical or hyperbolic model angles, then we arrive at the definition of CBB(1) or CBB(−1) comparison. Here CBB( $\kappa$ ) is an abbreviation of *curvature bounded below by  $\kappa$* . If one of one of the model angles is undefined, then we assume that CBB(1) comparison automatically holds for this quadruple.

We can do the same for the model plane of curvature  $\kappa$ . In this case we arrive at the definition of CAT( $\kappa$ ) comparison. But we will mostly consider CBB(0) comparison and occasionally CBB(1) comparison; so, if you see CBB( $\kappa$ ), you can assume that  $\kappa$  is 0 or 1.

If all quadruples in a metric space  $\mathcal{X}$  satisfy CBB( $\kappa$ ) comparison, then we say that the space  $\mathcal{X}$  is CBB( $\kappa$ ). (Again — CBB( $\kappa$ ) is an adjective.)

**7.3.2. Exercise.** *Show that a metric space  $\mathcal{X}$  is CBB(0) if and only if for any quadruple of points  $p, x, y, z \in \mathcal{X}$ , there is a quadruple of points  $\tilde{p}, \tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{E}^2$  such that*

$$\begin{aligned} |p - x|_{\mathcal{X}} &= |\tilde{p} - \tilde{x}|_{\mathbb{E}^2}, & |p - y|_{\mathcal{X}} &= |\tilde{p} - \tilde{y}|_{\mathbb{E}^2}, & |p - z|_{\mathcal{X}} &= |\tilde{p} - \tilde{z}|_{\mathbb{E}^2}, \\ |x - y|_{\mathcal{X}} &\leq |\tilde{x} - \tilde{y}|_{\mathbb{E}^2}, & |y - z|_{\mathcal{X}} &\leq |\tilde{y} - \tilde{z}|_{\mathbb{E}^2}, & |z - x|_{\mathcal{X}} &\leq |\tilde{z} - \tilde{x}|_{\mathbb{E}^2} \end{aligned}$$

for all  $i$  and  $j$ .

**7.3.3. Exercise.** *Suppose that a quadruple of points satisfies CAT(0) and CBB(0) for all labeling. Show that the quadruple is isometric to a subset of Euclidean space.*

Observe that in order to check CAT( $\kappa$ ) or CBB( $\kappa$ ) comparison, it is sufficient to know the 6 distances between all pairs of points in the quadruple. This observation implies the following.

**7.3.4. Proposition.** *Any Gromov–Hausdorff limit (as well as ultra limit) of a sequence of  $\text{CAT}(\kappa)$  or  $\text{CBB}(\kappa)$  spaces is  $\text{CAT}(\kappa)$  or  $\text{CBB}(\kappa)$  correspondingly.*

In the proposition above, it does not matter which definition of convergence for metric spaces you use, as long as any quadruple of points in the limit space can be arbitrarily well approximated by quadruples in the sequence of metric spaces.

## 7.4 Products and cones

Given two metric spaces  $\mathcal{U}$  and  $\mathcal{V}$ , the *product space*  $\mathcal{U} \times \mathcal{V}$  is defined as the set of all pairs  $(u, v)$  where  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$  with the metric defined by formula

$$|(u_1, v_1) - (u_2, v_2)|_{\mathcal{U} \times \mathcal{V}} = \sqrt{|u_1 - u_2|_{\mathcal{U}}^2 + |v_1 - v_2|_{\mathcal{V}}^2}.$$

**7.4.1. Proposition.** *Let  $\mathcal{U}$  and  $\mathcal{V}$  be  $\text{CAT}(0)$  spaces. Then the product space  $\mathcal{U} \times \mathcal{V}$  is  $\text{CAT}(0)$ .*

*Proof.* Fix a quadruple in  $\mathcal{U} \times \mathcal{V}$ :

$$p = (p_1, p_2), \quad q = (q_1, q_2), \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

For the quadruple  $p_1, q_1, x_1, y_1$  in  $\mathcal{U}$ , construct two model triangles  $[\tilde{p}_1 \tilde{x}_1 \tilde{y}_1] = \tilde{\Delta}(p_1 x_1 y_1)_{\mathbb{E}^2}$  and  $[\tilde{q}_1 \tilde{x}_1 \tilde{y}_1] = \tilde{\Delta}(q_1 x_1 y_1)_{\mathbb{E}^2}$ . Similarly, for the quadruple  $p_2, q_2, x_2, y_2$  in  $\mathcal{V}$  construct two model triangles  $[\tilde{p}_2 \tilde{x}_2 \tilde{y}_2]$  and  $[\tilde{q}_2 \tilde{x}_2 \tilde{y}_2]$ .

Consider four points in  $\mathbb{E}^4 = \mathbb{E}^2 \times \mathbb{E}^2$

$$\tilde{p} = (\tilde{p}_1, \tilde{p}_2), \quad \tilde{q} = (\tilde{q}_1, \tilde{q}_2), \quad \tilde{x} = (\tilde{x}_1, \tilde{x}_2), \quad \tilde{y} = (\tilde{y}_1, \tilde{y}_2).$$

Note that the triangles  $[\tilde{p} \tilde{x} \tilde{y}]$  and  $[\tilde{q} \tilde{x} \tilde{y}]$  in  $\mathbb{E}^4$  are isometric to the model triangles  $\tilde{\Delta}(p x y)_{\mathbb{E}^2}$  and  $\tilde{\Delta}(q x y)_{\mathbb{E}^2}$ .

If  $\tilde{z} = (\tilde{z}_1, \tilde{z}_2) \in [\tilde{x} \tilde{y}]$ , then  $\tilde{z}_1 \in [\tilde{x}_1 \tilde{y}_1]$  and  $\tilde{z}_2 \in [\tilde{x}_2 \tilde{y}_2]$  and

$$\begin{aligned} |\tilde{z} - \tilde{p}|_{\mathbb{E}^4}^2 &= |\tilde{z}_1 - \tilde{p}_1|_{\mathbb{E}^2}^2 + |\tilde{z}_2 - \tilde{p}_2|_{\mathbb{E}^2}^2, \\ |\tilde{z} - \tilde{q}|_{\mathbb{E}^4}^2 &= |\tilde{z}_1 - \tilde{q}_1|_{\mathbb{E}^2}^2 + |\tilde{z}_2 - \tilde{q}_2|_{\mathbb{E}^2}^2, \\ |p - q|_{\mathcal{U} \times \mathcal{V}}^2 &= |p_1 - q_1|_{\mathcal{U}}^2 + |p_2 - q_2|_{\mathcal{V}}^2. \end{aligned}$$

Therefore  $\text{CAT}(0)$  comparison for the quadruples  $p_1, q_1, x_1, y_1$  in  $\mathcal{U}$  and  $p_2, q_2, x_2, y_2$  in  $\mathcal{V}$  implies  $\text{CAT}(0)$  comparison for the quadruples  $p, q, x, y$  in  $\mathcal{U} \times \mathcal{V}$ .  $\square$

**7.4.2. Exercise.** Assume  $\mathcal{U}$  and  $\mathcal{V}$  are CBB(0) spaces. Show that the product space  $\mathcal{U} \times \mathcal{V}$  is CBB(0).

The cone  $\mathcal{V} = \text{Cone}\mathcal{U}$  over a metric space  $\mathcal{U}$  is defined as the metric space whose underlying set consists of equivalence classes in  $[0, \infty) \times \mathcal{U}$  with the equivalence relation “ $\sim$ ” given by  $(0, p) \sim (0, q)$  for any points  $p, q \in \mathcal{U}$ , and whose metric is given by the cosine rule

$$|(p, s) - (q, t)|_{\mathcal{V}} = \sqrt{s^2 + t^2 - 2 \cdot s \cdot t \cdot \cos \alpha},$$

where  $\alpha = \min\{\pi, |p - q|_{\mathcal{U}}\}$ .

The point in the cone  $\mathcal{V}$  formed by the equivalence class of  $0 \times \mathcal{U}$  is called the *tip of the cone* and is denoted by  $0$  or  $0_{\mathcal{V}}$ . The distance  $|0 - v|_{\mathcal{V}}$  is called the norm of  $v$  and is denoted by  $|v|$  or  $|v|_{\mathcal{V}}$ . The space  $\mathcal{U}$  can be identified with the subset  $x \in \mathcal{V}$  such that  $|x| = 1$ .

The points in the cone  $\mathcal{V}$  can be multiplied by a real number  $\lambda \geq 0$ ; namely, if  $x = (x', r)$ , then  $\lambda \cdot x := (x', \lambda \cdot r)$ .

**7.4.3. Proposition.** Let  $\mathcal{U}$  be a metric space. Then  $\text{Cone}\mathcal{U}$  is CAT(0) if and only if  $\mathcal{U}$  is CAT(1).

*Proof; “if” part.* Given a point  $x \in \text{Cone}\mathcal{U}$ , denote by  $x'$  its projection to  $\mathcal{U}$  and by  $|x|$  the distance from  $x$  to the tip of the cone; if  $x$  is the tip, then  $|x| = 0$  and we can take any point of  $\mathcal{U}$  as  $x'$ .

Let  $p, q, x, y$  be a quadruple in  $\text{Cone}\mathcal{U}$ . Assume that the spherical model triangles  $[\tilde{p}\tilde{x}'\tilde{y}'] = \tilde{\Delta}(p'x'y')_{\mathbb{S}^2}$  and  $[\tilde{q}'\tilde{x}'\tilde{y}'] = \tilde{\Delta}(q'x'y')_{\mathbb{S}^2}$  are defined. Consider the following points in  $\mathbb{E}^3 = \text{Cone}\mathbb{S}^2$ :

$$\tilde{p} = |p| \cdot \tilde{p}', \quad \tilde{q} = |q| \cdot \tilde{q}', \quad \tilde{x} = |x| \cdot \tilde{x}', \quad \tilde{y} = |y| \cdot \tilde{y}'.$$

Note that  $[\tilde{p}\tilde{x}\tilde{y}] \stackrel{\text{iso}}{=} \tilde{\Delta}(pxy)_{\mathbb{E}^2}$  and  $[\tilde{q}\tilde{x}\tilde{y}] \stackrel{\text{iso}}{=} \tilde{\Delta}(qxy)_{\mathbb{E}^2}$ . Further note that if  $\tilde{z} \in [\tilde{x}\tilde{y}]_{\mathbb{E}^3}$ , then  $\tilde{z}' = \tilde{z}/|\tilde{z}|$  lies on the geodesic  $[\tilde{x}'\tilde{y}']_{\mathbb{S}^2}$ . Therefore the CAT(1) comparison for  $|p' - q'|$  with  $\tilde{z}' \in [\tilde{x}'\tilde{y}']_{\mathbb{S}^2}$  implies the CAT(0) comparison for  $|p - q|$  with  $\tilde{z} \in [\tilde{x}\tilde{y}]_{\mathbb{E}^3}$ .

*“Only-if” part.* Suppose that  $\tilde{p}', \tilde{q}', \tilde{x}', \tilde{y}'$  are defined as above. Assume all these points lie in a half-space of  $\mathbb{E}^3 = \text{Cone}\mathbb{S}^2$  with origin at its boundary. Then we can choose positive values  $a, b, c$ , and  $d$  such that the points  $a \cdot \tilde{p}', b \cdot \tilde{q}', c \cdot \tilde{x}', d \cdot \tilde{y}'$  lie in one plane. Consider the corresponding points  $a \cdot p', b \cdot q', c \cdot x', d \cdot y'$  in  $\text{Cone}\mathcal{U}$ . Applying the CAT(0) comparison for these points leads to CAT(1) comparison for the quadruple  $p', q', x', y'$  in  $\mathcal{U}$ .

It remains to consider the case when  $\tilde{p}', \tilde{q}', \tilde{x}', \tilde{y}'$  do not lie in a half-space. Fix  $\tilde{z}' \in [\tilde{x}'\tilde{y}']_{\mathbb{S}^2}$ . Observe that

$$|\tilde{p}' - \tilde{x}'|_{\mathbb{S}^2} + |\tilde{q}' - \tilde{x}'|_{\mathbb{S}^2} \leq |\tilde{p}' - \tilde{z}'|_{\mathbb{S}^2} + |\tilde{q}' - \tilde{z}'|_{\mathbb{S}^2}$$

or

$$|\tilde{p}' - \tilde{y}'|_{\mathbb{S}^2} + |\tilde{q}' - \tilde{y}'|_{\mathbb{S}^2} \leq |\tilde{p}' - \tilde{z}'|_{\mathbb{S}^2} + |\tilde{q}' - \tilde{z}'|_{\mathbb{S}^2}.$$

That is, in this case, the CAT(1) comparison follow from the triangle inequality.  $\square$

## 7.5 Geodesics

**7.5.1. Proposition.** *Let  $\mathcal{X}$  be a complete length CAT(0) space. Then any two points in  $\mathcal{X}$  are joint by a unique geodesic.*

*Proof.* Fix two points  $x, y \in \mathcal{X}$ . Choose a sequence of approximate midpoints  $p_n$  for  $x$  and  $y$ ; that is,

$$\bullet \quad |x - p_n| \rightarrow \frac{1}{2} \cdot |x - y| \quad \text{and} \quad |y - p_n| \rightarrow \frac{1}{2} \cdot |x - y|$$

as  $n \rightarrow \infty$ .

Consider model triangles  $[\tilde{p}_n \tilde{x} \tilde{y}] = \tilde{\Delta}(p_n xy)$ . Let  $\tilde{z}$  be the midpoint of  $\tilde{x}$  and  $\tilde{y}$ . By  $\bullet$ , we have that

$$|\tilde{p}_n - \tilde{z}| \rightarrow 0$$

as  $n \rightarrow \infty$ .

By CAT(0) comparison,

$$|p_n - p_m|_{\mathcal{X}} \leq |\tilde{p}_n - \tilde{z}| + |\tilde{p}_m - \tilde{z}|.$$

Therefore  $|p_n - p_m| \rightarrow 0$  as  $m, n \rightarrow \infty$ ; that is,  $(p_n)$  is Cauchy. Clearly the limit of the sequence  $(p_n)$  is a midpoint of  $x$  and  $y$ . Applying 1.7.4b, we get that  $\mathcal{X}$  is geodesic.

It remains to prove uniqueness. Suppose there are two geodesics between  $x$  and  $y$ . Then we can choose two points  $p \neq q$  on these geodesics such that  $|x - p| = |x - q|$  and therefore  $|y - p| = |y - q|$ .

Observe that the model triangles  $[\tilde{p} \tilde{x} \tilde{y}] = \tilde{\Delta}(p xy)$  and  $[\tilde{q} \tilde{x} \tilde{y}] = \tilde{\Delta}(q xy)$  are degenerate and moreover  $\tilde{p} = \tilde{q}$ . Applying CAT(0) comparison with  $\tilde{z} = \tilde{p} = \tilde{q}$ , we get that  $|p - q| = 0$ , a contradiction.  $\square$

The following exercise is an analogous statement for CBB spaces. In general complete length CBB(0) space might fail to be geodesic and uniqueness of geodesic usually does not hold.

**7.5.2. Exercise.** *Let  $\mathcal{X}$  be a complete length CBB(0) space. Show that if two geodesics from  $x$  to  $y$  share yet another point  $z$ , then they coincide.*

## 7.6 Alexandrov's lemma

**7.6.1. Lemma.** *Let  $p, x, y, z$  be distinct points in a metric space such that  $z \in ]xy[$ . Then the following expressions for the Euclidean model angles have the same sign:*

- (a)  $\tilde{\angle}(x_p^p) - \tilde{\angle}(x_z^p)$ ,
- (b)  $\tilde{\angle}(z_x^p) + \tilde{\angle}(z_y^p) - \pi$ .

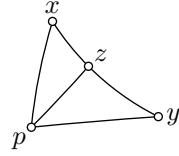
Moreover,

$$\tilde{\angle}(p_x^x) \geq \tilde{\angle}(p_z^x) + \tilde{\angle}(p_y^z),$$

with equality if and only if the expressions in (a) and (b) vanish.

The same holds for the hyperbolic and spherical model angles, but in the latter case one has to assume in addition that

$$|p - z| + |p - y| + |x - y| < 2 \cdot \pi.$$



*Proof.* Consider the model triangle  $[\tilde{x}\tilde{p}\tilde{z}] = \tilde{\Delta}(xpz)$ . Take a point  $\tilde{y}$  on the extension of  $[\tilde{x}\tilde{z}]$  beyond  $\tilde{z}$  so that  $|\tilde{x} - \tilde{y}| = |x - y|$  (and therefore  $|\tilde{x} - \tilde{z}| = |x - z|$ ).

Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:

- (i)  $\angle[\tilde{x}\tilde{p}\tilde{y}] - \tilde{\angle}(x_p^p)$ ,
- (ii)  $|\tilde{p} - \tilde{y}| - |p - y|$ ,
- (iii)  $\angle[\tilde{z}\tilde{p}\tilde{y}] - \tilde{\angle}(z_p^p)$ .

Since

$$\angle[\tilde{x}\tilde{p}\tilde{y}] = \angle[\tilde{x}\tilde{p}\tilde{z}] = \tilde{\angle}(x_p^p)$$

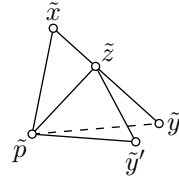
and

$$\angle[\tilde{z}\tilde{p}\tilde{y}] = \pi - \angle[\tilde{z}\tilde{p}\tilde{x}] = \pi - \tilde{\angle}(z_p^p),$$

the first statement follows.

For the second statement, construct a model triangle  $[\tilde{p}\tilde{z}\tilde{y}'] = \tilde{\Delta}(pzy)_{\mathbb{E}^2}$  on the opposite side of  $[\tilde{p}\tilde{z}]$  from  $[\tilde{x}\tilde{p}\tilde{z}]$ . Note that

$$\begin{aligned} |\tilde{x} - \tilde{y}'| &\leq |\tilde{x} - \tilde{z}| + |\tilde{z} - \tilde{y}'| = \\ &= |x - z| + |z - y| = \\ &= |x - y|. \end{aligned}$$



Therefore

$$\begin{aligned}\tilde{Z}(p^x_z) + \tilde{Z}(p^z_y) &= \angle[\tilde{p}^{\tilde{x}}_{\tilde{z}}] + \angle[\tilde{p}^{\tilde{z}}_{\tilde{y}'}] = \\ &= \angle[\tilde{p}^{\tilde{x}}_{\tilde{y}'}] \leq \\ &\leq \tilde{Z}(p^x_y).\end{aligned}$$

Equality holds if and only if  $|\tilde{x} - \tilde{y}'| = |x - y|$ , as required.  $\square$

**7.6.2. Exercise.** Given  $[p^x_y]$  in a metric space  $\mathcal{X}$ , consider the function

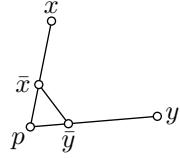
$$f: (|p - \bar{x}|, |p - \bar{y}|) \mapsto \tilde{Z}(p^{\bar{x}}_{\bar{y}})$$

where  $\bar{x} \in ]px]$  and  $\bar{y} \in ]py]$ .

(a) Suppose  $\mathcal{X}$  is CAT(0). Show that  $f$  is nondecreasing in each argument.

(b) Suppose  $\mathcal{X}$  is CBB(0). Show that  $f$  is nonincreasing in each argument.

Conclude that for any hinge in a CAT(0) or CBB(0) space has defined angle.



**7.6.3. Exercise.** Fix a point  $p$  in a complete length CAT(0) space  $\mathcal{X}$ . Given a point  $x \in \mathcal{X}$ , denote by  $\gamma_x$  a (necessary unique) geodesic path from  $p$  to  $x$ .

Show that the family of maps  $h_t: \mathcal{X} \rightarrow \mathcal{X}$  defined by

$$h_t(x) = \gamma_x(t)$$

is a homotopy; it is called geodesic homotopy. Conclude that  $\mathcal{X}$  is contractible.

## 7.7 Thin and fat triangles

Recall that a *triangle*  $[xyz]$  in a space  $\mathcal{X}$  is a triple of minimizing geodesics  $[xy]$ ,  $[yz]$  and  $[zx]$ . Consider the model triangle  $[\tilde{x}\tilde{y}\tilde{z}] = \triangle(xyz)_{\mathbb{E}^2}$  in the Euclidean plane. The *natural map*  $[\tilde{x}\tilde{y}\tilde{z}] \rightarrow [xyz]$  sends a point  $\tilde{p} \in [\tilde{x}\tilde{y}] \cup [\tilde{y}\tilde{z}] \cup [\tilde{z}\tilde{x}]$  to the corresponding point  $p \in [xy] \cup [yz] \cup [zx]$ ; that is, if  $\tilde{p}$  lies on  $[\tilde{y}\tilde{z}]$ , then  $p \in [yz]$  and  $|\tilde{y} - \tilde{p}| = |y - p|$  (and therefore  $|\tilde{z} - \tilde{p}| = |z - p|$ ).

In the same way, the natural map can be defined for the spherical model triangle  $\triangle(xyz)_{\mathbb{S}^2}$ .

**7.7.1. Definition.** A triangle  $[xyz]$  in the metric space  $\mathcal{X}$  is called *thin* (or *fat*) if the natural map  $\triangle(xyz)_{\mathbb{E}^2} \rightarrow [xyz]$  is distance nonincreasing (or correspondingly distance nondecreasing).

Analogously, a triangle  $[xyz]$  is called *spherically thin* or *spherically fat* if the natural map from the spherical model triangle  $\hat{\Delta}(xyz)_{\mathbb{S}^2}$  to  $[xyz]$  is distance nonincreasing or nondecreasing.

**7.7.2. Proposition.** *A geodesic space is CAT(0) (CAT(1)) if and only if all its triangles are thin (respectively, all its triangles of perimeter  $< 2\pi$  are spherically thin).*

*Proof; “if” part.* Apply the triangle inequality and thinness of triangles  $[pxy]$  and  $[qxy]$ , where  $p, q, x$  and  $y$  are as in the definition of CAT( $\kappa$ ) comparison (page 64).

*“Only if” part.* Applying CAT(0) comparison to a quadruple  $p, q, x, y$  with  $q \in [xy]$  shows that any triangle satisfies *point-side comparison*, that is, the distance from a vertex to a point on the opposite side is no greater than the corresponding distance in the Euclidean model triangle.

Now consider a triangle  $[xyz]$  and let  $p \in [xy]$  and  $q \in [xz]$ . Let  $\tilde{p}, \tilde{q}$  be the corresponding points on the sides of the model triangle  $\hat{\Delta}(xyz)_{\mathbb{E}^2}$ . Applying 7.6.2a, we get that

$$\tilde{\Delta}(x^y_z)_{\mathbb{E}^2} \geq \tilde{\Delta}(x^p_q)_{\mathbb{E}^2}.$$

Therefore  $|\tilde{p} - \tilde{q}|_{\mathbb{E}^2} \geq |p - q|$ .

The CAT(1) argument is the same. □

**7.7.3. Exercise.** *Show that any triangle is a CBB(0) space is fat.*

**7.7.4. Exercise.** *Suppose  $\gamma_1, \gamma_2: [0, 1] \rightarrow \mathcal{U}$  be two geodesic paths in a complete length CAT(0) space  $\mathcal{U}$ . Show that*

$$t \mapsto |\gamma_1(t) - \gamma_2(t)|_{\mathcal{U}}$$

*is a convex function.*

**7.7.5. Exercise.** *Let  $A$  be a convex closed set in a proper length CAT(0) space  $\mathcal{U}$ ; that is, if  $x, y \in A$ , then  $[xy] \subset A$ . Show that for any  $r > 0$  the closed  $r$ -neighborhood of  $A$  is convex; that is, the set*

$$A_r = \{x \in \mathcal{U} : \text{dist}_A x \leq r\}$$

*is convex.*

**7.7.6. Exercise.** *Let  $\mathcal{U}$  be a proper length CAT(0) space and  $K \subset \mathcal{U}$  be a closed convex set. Show that:*

- (a) *For each point  $p \in \mathcal{U}$  there is unique point  $p^* \in K$  that minimizes the distance  $|p - p^*|$ .*
- (b) *The closest-point projection  $p \mapsto p^*$  defined by (a) is short.*

## 7.8 Other descriptions

In this section we will list few ways to describe  $\text{CAT}(0)$  and  $\text{CBB}(0)$  spaces. We do not give proofs of these statements, altho they are not hard; see [4] and the reference there in.

These conditions will not be used in the sequel, but they might help to build right intuition.

**Convexity of function.** The following condition might help to adapt intuition from real analysis.

Let  $\mathcal{X}$  be a metric space and  $\lambda \in \mathbb{R}$ . A function  $f: \mathcal{X} \rightarrow \mathbb{R}$  is called  $\lambda$ -convex ( $\lambda$ -concave) if the real-to-real function

$$t \mapsto f \circ \gamma(\gamma) - \frac{\lambda}{2} \cdot t^2$$

is convex (correspondingly concave) for any geodesic  $\gamma: \mathbb{I} \rightarrow \mathbb{R}$ .

The  $\lambda$ -convex and  $\lambda$ -concave functions can be thought as functions satisfying inequalities  $f'' \geq \lambda$  and correspondingly  $f'' \leq \lambda$  in a generalized sense. Note that a smooth real-to-real function  $f$  is  $\lambda$ -convex ( $\lambda$ -concave) if it satisfies inequality  $f'' \geq \lambda$  (correspondingly  $f'' \leq \lambda$ ).

**7.8.1. Proposition.** *Let  $\mathcal{X}$  be a geodesic space. Then  $\mathcal{X}$  is  $\text{CAT}(0)$  (correspondingly  $\text{CBB}(0)$ ) if and only if for any point  $p \in \mathcal{X}$  the function*

$$f(x) = \frac{1}{2} \cdot |p - x|_{\mathcal{X}}$$

*is 1-convex (correspondingly 1-concave).*

**Angle comparison.** The following condition might help to adapt intuition from Euclidean geometry.

Recall that in  $\text{CAT}(0)$  and  $\text{CBB}(0)$  spaces any hinge has defined angle; see 7.6.2.

**7.8.2. Proposition.** *Let  $\mathcal{X}$  be a geodesic space such that any hinge in  $\mathcal{X}$  has defined angle. Then*

*(a)  $\mathcal{X}$  is  $\text{CAT}(0)$  if and only if*

$$\angle[p_y^x] \leq \tilde{\angle}(p_y^x).$$

*(b)  $\mathcal{X}$  is  $\text{CBB}(0)$  if and only if*

$$\angle[p_y^x] \geq \tilde{\angle}(p_y^x)$$

*and*

$$\angle[p_y^x] + \angle[p_z^x] = \pi$$

*for any adjacent hinges  $[p_y^x]$  and  $[p_z^x]$ ; that is, the union of the sides  $[px]$  and  $[pz]$  of the hinges form a geodesic  $[xy]$ .*



It is unknown if the condition on adjacent hinges in (b) can be removed (even in the two-dimensional case).

**Kirszbraun property.** We include the following condition only because it is beautiful.

The following theorem was proved by Mojżesz Kirszbraun [33] and rediscovered later by Frederick Valentine [48].

**7.8.3. Theorem.** *Let  $A \subset \mathbb{E}^m$ . Then any short map  $f: A \rightarrow \mathbb{E}^n$  admits a short extension  $f: \mathbb{E}^m \rightarrow \mathbb{E}^n$ .*

The conclusion of the theorem holds for some other metric spaces instead of  $\mathbb{E}^m$  and  $\mathbb{E}^n$ . For example instead of  $\mathbb{E}^n$  one might take any injective space (6.2.1) and instead of  $\mathbb{E}^m$  one may take any compact ultrametric space (6.2.3). On the other hand existence of extension to/from a Euclidean space is much weaker condition than in 6.2.1 and 6.2.3. As the following theorems state, these conditions are closely related to the CBB(0) and CAT(0) conditions.

**7.8.4. Theorem.** *Let  $\mathcal{X}$  be a complete length space and  $n \geq 2$ . Then  $\mathcal{X}$  is CBB(0) if and only if for any set  $A \subset \mathcal{X}$ , any short map  $f: A \rightarrow \mathbb{E}^n$  admits a short extension  $F: \mathcal{X} \rightarrow \mathbb{E}^n$ .*

**7.8.5. Theorem.** *Let  $\mathcal{Y}$  be a metric space with  $m \geq 2$ . Assume any two points in  $\mathcal{Y}$  are joint by unique geodesic. Then  $\mathcal{Y}$  is CAT(0) if and only if for any set  $A \subset \mathbb{E}^m$ , any short map  $f: \mathbb{E}^m \rightarrow \mathcal{Y}$  admits a short extension  $F: \mathbb{E}^m \rightarrow \mathcal{Y}$ .*

## 7.9 History

The idea that the essence of curvature lies in a condition on quadruples of points apparently originated with Abraham Wald. It is found in his publication on “coordinate-free differential geometry” [49] written under the supervision of Karl Menger; the story of this discovery can be found in [38]. In 1941, similar definitions were rediscovered independently by Alexandr Danilovich Alexandrov [6]. In Alexandrov’s work the first fruitful applications of this approach were given. Mainly:

- ◊ Alexandrov’s embedding theorem — *metrics of non-negative curvature on the sphere, and only they, are isometric to closed convex surfaces in Euclidean 3-space.*
- ◊ Gluing theorem, which tells when the sphere obtained by gluing of two discs along their boundaries has non-negative curvature in the sense of Alexandrov.

These two results together gave a very intuitive geometric tool for studying embeddings and bending of surfaces in Euclidean space, and

changed this subject dramatically. They formed the foundation of the branch of geometry now called *Alexandrov geometry*.

The study of spaces with curvature bounded above started later. The first paper on the subject was written by Alexandrov [7]. It was based on work of Herbert Busemann, who studied spaces satisfying a weaker condition [20].

# Chapter 8

## Gluing and billiards

This chapter is nearly a copy of [3, Chapter 2]; here we prove Reshetnyak's gluing theorem for  $\text{CAT}(0)$  spaces and apply it to a problem in billiards.

### 8.1 Inheritance lemma

The inheritance lemma 8.1.2 proved below plays a central role in the theory of  $\text{CAT}(\kappa)$  spaces.

**8.1.1. Proposition.** *Suppose  $\mathcal{U}$  is a proper length  $\text{CAT}(0)$  space. Then any local geodesic in  $\mathcal{U}$  is a geodesic.*

*Analogously, if  $\mathcal{U}$  is a proper length  $\text{CAT}(1)$  space, then any local geodesic in  $\mathcal{U}$  which is shorter than  $\pi$  is a geodesic.*

*Proof.* Suppose  $\gamma: [0, \ell] \rightarrow \mathcal{U}$  is a local geodesic that is not a geodesic. Choose  $a$  to be the maximal value such that  $\gamma$  is a geodesic on  $[0, a]$ . Further choose  $b > a$  so that  $\gamma$  is a geodesic on  $[a, b]$ .

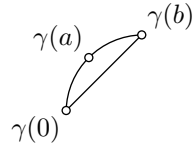
Since the triangle  $[\gamma(0)\gamma(a)\gamma(b)]$  is thin and  $|\gamma(0) - \gamma(b)| < b$  we have

$$|\gamma(a - \varepsilon) - \gamma(a + \varepsilon)| < 2 \cdot \varepsilon$$

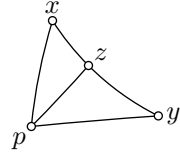
for all small  $\varepsilon > 0$ . That is,  $\gamma$  is not length-minimizing on the interval  $[a - \varepsilon, a + \varepsilon]$  for any  $\varepsilon > 0$ , a contradiction.

The spherical case is done in the same way. □

Now let us formulate the main result of this section.



**8.1.2. Inheritance lemma.** Assume that a triangle  $[pxy]$  in a metric space is decomposed into two triangles  $[pxz]$  and  $[pyz]$ ; that is,  $[pxz]$  and  $[pyz]$  have a common side  $[pz]$ , and the sides  $[xz]$  and  $[zy]$  together form the side  $[xy]$  of  $[pxy]$ .



If both triangles  $[pxz]$  and  $[pyz]$  are thin, then the triangle  $[pxy]$  is also thin.

Analogously, if  $[pxy]$  has perimeter  $< 2 \cdot \pi$  and both triangles  $[pxz]$  and  $[pyz]$  are spherically thin, then triangle  $[pxy]$  is spherically thin.

*Proof.* Construct the model triangles  $[\dot{p}\dot{x}\dot{z}] = \hat{\Delta}(pxz)_{\mathbb{E}^2}$  and  $[\dot{p}\dot{y}\dot{z}] = \hat{\Delta}(pyz)_{\mathbb{E}^2}$  so that  $\dot{x}$  and  $\dot{y}$  lie on opposite sides of  $[\dot{p}\dot{z}]$ .

Let us show that

$$\textcircled{1} \quad \tilde{\angle}(z_x^p) + \tilde{\angle}(z_y^p) \geq \pi.$$



If not, then for some point  $\dot{w} \in [\dot{p}\dot{z}]$ , we have

$$|\dot{x} - \dot{w}| + |\dot{w} - \dot{y}| < |\dot{x} - \dot{z}| + |\dot{z} - \dot{y}| = |x - y|.$$

Let  $w \in [pz]$  correspond to  $\dot{w}$ ; that is,  $|z - w| = |\dot{z} - \dot{w}|$ . Since  $[pxz]$  and  $[pyz]$  are thin, we have

$$|x - w| + |w - y| < |x - y|,$$

contradicting the triangle inequality.

Denote by  $\dot{D}$  the union of two solid triangles  $[\dot{p}\dot{x}\dot{z}]$  and  $[\dot{p}\dot{y}\dot{z}]$ . Further, denote by  $\tilde{D}$  the solid triangle  $[\tilde{p}\tilde{x}\tilde{y}] = \hat{\Delta}(pxy)_{\mathbb{E}^2}$ . By  $\textcircled{1}$ , there is a short map  $F: \tilde{D} \rightarrow \dot{D}$  that sends

$$\tilde{p} \mapsto \dot{p}, \quad \tilde{x} \mapsto \dot{x}, \quad \tilde{z} \mapsto \dot{z}, \quad \tilde{y} \mapsto \dot{y}.$$

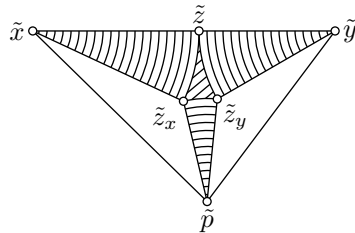
Indeed, by Alexandrov's lemma (7.6.1), there are nonoverlapping triangles

$$[\tilde{p}\tilde{x}\tilde{z}_x] \stackrel{\text{iso}}{=} [\dot{p}\dot{x}\dot{z}]$$

and

$$[\tilde{p}\tilde{y}\tilde{z}_y] \stackrel{\text{iso}}{=} [\dot{p}\dot{y}\dot{z}]$$

inside the triangle  $[\tilde{p}\tilde{x}\tilde{y}]$ .



Connect the points in each pair  $(\tilde{z}, \tilde{z}_x)$ ,  $(\tilde{z}_x, \tilde{z}_y)$  and  $(\tilde{z}_y, \tilde{z})$  with arcs of circles centered at  $\tilde{y}$ ,  $\tilde{p}$ , and  $\tilde{x}$  respectively. Define  $F$  as follows:

- ◇ Map  $\text{Conv}[\tilde{p}\tilde{x}\tilde{z}_x]$  isometrically onto  $\text{Conv}[\dot{p}\dot{x}\dot{z}]$ ; similarly map  $\text{Conv}[\tilde{p}\tilde{y}\tilde{z}_y]$  onto  $\text{Conv}[\dot{p}\dot{y}\dot{z}]$ .
- ◇ If  $x$  is in one of the three circular sectors, say at distance  $r$  from its center, set  $F(x)$  to be the point on the corresponding segment  $[pz]$ ,  $[xz]$  or  $[yz]$  whose distance from the left-hand endpoint of the segment is  $r$ .
- ◇ Finally, if  $x$  lies in the remaining curvilinear triangle  $\tilde{z}\tilde{z}_x\tilde{z}_y$ , set  $F(x) = z$ .

By construction,  $F$  satisfies the conditions.

By assumption, the natural maps  $[\dot{p}\dot{x}\dot{z}] \rightarrow [pxz]$  and  $[\dot{p}\dot{y}\dot{z}] \rightarrow [pyz]$  are short. By composition, the natural map from  $[\tilde{p}\tilde{x}\tilde{y}]$  to  $[pyz]$  is short, as claimed.

The spherical case is done along the same lines.  $\square$

## 8.2 Reshetnyak's gluing

Suppose  $\mathcal{U}^1$  and  $\mathcal{U}^2$  are proper length spaces with isometric closed convex sets  $A^i \subset \mathcal{U}^i$  and let  $\iota: A^1 \rightarrow A^2$  be an isometry. Consider the space  $\mathcal{W}$  of all equivalence classes in  $\mathcal{U}^1 \sqcup \mathcal{U}^2$  with the equivalence relation given by  $a \sim \iota(a)$  for any  $a \in A^1$ .

It is straightforward to see that  $\mathcal{W}$  is a proper length space when equipped with the following metric

$$\begin{aligned}
 |x - y|_{\mathcal{W}} &:= |x - y|_{\mathcal{U}^i} \\
 &\quad \text{if } x, y \in \mathcal{U}^i, \quad \text{and} \\
 |x - y|_{\mathcal{W}} &:= \min \{ |x - a|_{\mathcal{U}^1} + |y - \iota(a)|_{\mathcal{U}^2} : a \in A^1 \} \\
 &\quad \text{if } x \in \mathcal{U}^1 \quad \text{and} \quad y \in \mathcal{U}^2.
 \end{aligned}$$

Abusing notation, we denote by  $x$  and  $y$  the points in  $\mathcal{U}^1 \sqcup \mathcal{U}^2$  and their equivalence classes in  $\mathcal{U}^1 \sqcup \mathcal{U}^2 / \sim$ .

The space  $\mathcal{W}$  is called the *gluing* of  $\mathcal{U}^1$  and  $\mathcal{U}^2$  along  $\iota$ . If one applies this construction to two copies of one space  $\mathcal{U}$  with a set  $A \subset \mathcal{U}$  and the identity map  $\iota: A \rightarrow A$ , then the obtained space is called the *double* of  $\mathcal{U}$  along  $A$ .

We can (and will) identify  $\mathcal{U}^i$  with its image in  $\mathcal{W}$ ; this way both subsets  $A^i \subset \mathcal{U}^i$  will be identified and denoted further by  $A$ . Note that  $A = \mathcal{U}^1 \cap \mathcal{U}^2 \subset \mathcal{W}$ , therefore  $A$  is also a convex set in  $\mathcal{W}$ .

The following theorem was proved by Yuri Reshetnyak in [44].

**8.2.1. Reshetnyak gluing.** *Suppose  $\mathcal{U}^1$  and  $\mathcal{U}^2$  are proper length CAT(0) spaces with isometric closed convex sets  $A^i \subset \mathcal{U}^i$ , and  $\iota: A^1 \rightarrow A^2$  is an isometry. Then the gluing of  $\mathcal{U}^1$  and  $\mathcal{U}^2$  along  $\iota$  is a CAT(0) proper length space.*

*Proof.* By construction of the gluing space, the statement can be reformulated in the following way:

**8.2.2. Reformulation of 8.2.1.** *Let  $\mathcal{W}$  be a proper length space which has two closed convex sets  $\mathcal{U}^1, \mathcal{U}^2 \subset \mathcal{W}$  such that  $\mathcal{U}^1 \cup \mathcal{U}^2 = \mathcal{W}$  and  $\mathcal{U}^1, \mathcal{U}^2$  are CAT(0). Then  $\mathcal{W}$  is CAT(0).*



It suffices to show that any triangle  $[xyz]$  in  $\mathcal{W}$  is thin. This is obviously true if all three points  $x, y, z$  lie in one of  $\mathcal{U}^i$ . Thus, without loss of generality, we may assume that  $x \in \mathcal{U}^1$  and  $y, z \in \mathcal{U}^2$ .

Choose points  $a, b \in A = \mathcal{U}^1 \cap \mathcal{U}^2$  that lie respectively on the sides  $[xy], [xz]$ . Note that

- ◇ the triangle  $[xab]$  lies in  $\mathcal{U}^1$ ,
- ◇ both triangles  $[yab]$  and  $[ybz]$  lie in  $\mathcal{U}^2$ .

In particular each triangle  $[xab]$ ,  $[yab]$  and  $[ybz]$  is thin.

Applying the inheritance lemma (8.1.2) twice, we get that  $[xyb]$  and consequently  $[xyz]$  is thin.  $\square$

**8.2.3. Exercise.** *Suppose  $\mathcal{U}$  is a geodesic space and  $A \subset \mathcal{U}$  is a closed subset. Assume that the doubling of  $\mathcal{U}$  in  $A$  is CAT(0). Show that  $A$  is a convex set of  $\mathcal{U}$ .*

## 8.3 Puff pastry

In this section we introduce the notion of Reshetnyak puff pastry. This construction will be used in the next section to prove the collision theorem (8.5.1).

Let  $\mathbf{A} = (A^1, \dots, A^N)$  be an array of convex closed sets in the Euclidean space  $\mathbb{E}^m$ . Consider an array of  $N+1$  copies of  $\mathbb{E}^m$ . Assume that the space  $\mathcal{R}$  is obtained by gluing successive pairs of spaces along  $A^1, \dots, A^N$  respectively.

The resulting space  $\mathcal{R}$  will be called the *Reshetnyak puff pastry* for the array  $\mathbf{A}$ . The copies of  $\mathbb{E}^m$  in the puff pastry  $\mathcal{R}$  will be called *levels*; they will be denoted by  $\mathcal{R}^0, \dots, \mathcal{R}^N$ . The point in the  $k$ -th level  $\mathcal{R}^k$  that corresponds to  $x \in \mathbb{E}^m$  will be denoted by  $x^k$ .

Given  $x \in \mathbb{E}^m$ , any point  $x^k \in \mathcal{R}$  is called a *lifting* of  $x$ . The map  $x \mapsto x^k$  defines an isometry  $\mathbb{E}^m \rightarrow \mathcal{R}^k$ ; in particular we can talk about liftings of subsets in  $\mathbb{E}^m$ .

Note that:

- ◇ The intersection  $A^1 \cap \dots \cap A^N$  admits a unique lifting in  $\mathcal{R}$ .

Puff pastry for  $(A, B, A)$ .

◇ Moreover,  $x^i = x^j$  for some  $i < j$  if and only if

$$x \in A^{i+1} \cap \dots \cap A^j.$$

◇ The restriction  $\mathcal{R}^k \rightarrow \mathbb{E}^m$  of the natural projection  $x^k \mapsto x$  is an isometry.

**8.3.1. Observation.** Any Reshetnyak puff pastry is a proper length CAT(0) space.

*Proof.* Apply Reshetnyak gluing theorem (8.2.1) recursively for the convex sets in the array.  $\square$

**8.3.2. Proposition.** Assume  $(A^1, \dots, A^N)$  and  $(\check{A}^1, \dots, \check{A}^N)$  are two arrays of convex closed sets in  $\mathbb{E}^m$  such that  $A^k \subset \check{A}^k$  for each  $k$ . Let  $\mathcal{R}$  and  $\check{\mathcal{R}}$  be the corresponding Reshetnyak puff pastries. Then the map  $\mathcal{R} \rightarrow \check{\mathcal{R}}$  defined by  $x^k \mapsto \check{x}^k$  is short.

Moreover, if

$$\textcircled{1} \quad |x^i - y^j|_{\mathcal{R}} = |\check{x}^i - \check{y}^j|_{\check{\mathcal{R}}}$$

for some  $x, y \in \mathbb{E}^m$  and  $i, j \in \{0, \dots, n\}$ , then the unique geodesic  $[\check{x}^i \check{y}^j]_{\check{\mathcal{R}}}$  is the image of the unique geodesic  $[x^i y^j]_{\mathcal{R}}$  under the map  $x^i \mapsto \check{x}^i$ .

*Proof.* The first statement in the proposition follows from the construction of Reshetnyak puff pastries.

By Observation 8.3.1,  $\mathcal{R}$  and  $\check{\mathcal{R}}$  are proper length CAT(0) spaces; hence  $[x^i y^j]_{\mathcal{R}}$  and  $[\check{x}^i \check{y}^j]_{\check{\mathcal{R}}}$  are unique. By  $\textcircled{1}$ , since the map  $\mathcal{R} \rightarrow \check{\mathcal{R}}$  is short, the image of  $[x^i y^j]_{\mathcal{R}}$  is a geodesic of  $\check{\mathcal{R}}$  joining  $\check{x}^i$  to  $\check{y}^j$ . Hence the second statement follows.  $\square$

**8.3.3. Definition.** Consider a Reshetnyak puff pastry  $\mathcal{R}$  with the levels  $\mathcal{R}^0, \dots, \mathcal{R}^N$ . We say that  $\mathcal{R}$  is end-to-end convex if  $\mathcal{R}^0 \cup \mathcal{R}^N$ ,

the union of its lower and upper levels, forms a convex set in  $\mathcal{R}$ ; that is, if  $x, y \in \mathcal{R}^0 \cup \mathcal{R}^N$ , then  $[xy]_{\mathcal{R}} \subset \mathcal{R}^0 \cup \mathcal{R}^N$ .

Note that if  $\mathcal{R}$  is the Reshetnyak puff pastry for an array of convex sets  $\mathbf{A} = (A^1, \dots, A^N)$ , then  $\mathcal{R}$  is end-to-end convex if and only if the union of the lower and the upper levels  $\mathcal{R}^0 \cup \mathcal{R}^N$  is isometric to the double of  $\mathbb{E}^m$  along the nonempty intersection  $A^1 \cap \dots \cap A^N$ .

**8.3.4. Observation.** Let  $\check{\mathbf{A}}$  and  $\mathbf{A}$  be arrays of convex bodies in  $\mathbb{E}^m$ . Assume that the array  $\mathbf{A}$  is obtained by inserting in  $\check{\mathbf{A}}$  several copies of the bodies which were already listed in  $\check{\mathbf{A}}$ .

For example, if  $\check{\mathbf{A}} = (A, C, B, C, A)$ , by placing  $B$  in the second place and  $A$  in the fourth place, we obtain  $\mathbf{A} = (A, B, C, A, B, C, A)$ .

Denote by  $\check{\mathcal{R}}$  and  $\mathcal{R}$  the Reshetnyak puff pastries for  $\check{\mathbf{A}}$  and  $\mathbf{A}$  respectively.

If  $\check{\mathcal{R}}$  is end-to-end convex, then so is  $\mathcal{R}$ .

*Proof.* Without loss of generality we may assume that  $\mathbf{A}$  is obtained by inserting one element in  $\check{\mathbf{A}}$ , say at the place number  $k$ .

Note that  $\check{\mathcal{R}}$  is isometric to the puff pastry for  $\mathbf{A}$  with  $A^k$  replaced by  $\mathbb{E}^m$ . It remains to apply Proposition 8.3.2.  $\square$



Let  $X$  be a convex set in a Euclidean space. By a *dihedral angle* we understand an intersection of two half-spaces; the intersection of corresponding hyperplanes is called the *edge* of the angle. We say that a dihedral angle  $D$  supports  $X$  at a point  $p \in X$  if  $D$  contains  $X$  and the edge of  $D$  contains  $p$ .

**8.3.5. Lemma.** Let  $A$  and  $B$  be two convex sets in  $\mathbb{E}^m$ . Assume that any dihedral angle supporting  $A \cap B$  has angle measure at least  $\alpha$ . Then the Reshetnyak puff pastry for the array

$$\underbrace{(A, B, A, \dots)}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \text{ times}}$$

is end-to-end convex.

The proof of the lemma is based on a partial case, which we formulate as a sublemma.

**8.3.6. Sublemma.** Let  $\check{A}$  and  $\check{B}$  be two half-planes in  $\mathbb{E}^2$ , where  $\check{A} \cap \check{B}$  is an angle with measure  $\alpha$ . Then the Reshetnyak puff pastry for the array

$$\underbrace{(\check{A}, \check{B}, \check{A}, \dots)}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \text{ times}}$$

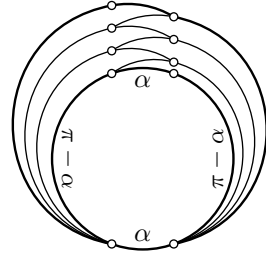


is end-to-end convex.

*Proof.* Note that the puff pastry  $\ddot{\mathcal{R}}$  is isometric to the cone over the space glued from the unit circles as shown on the diagram.

All the short arcs on the diagram have length  $\alpha$ ; the long arcs have length  $\pi - \alpha$ , so making a circuit along any path will take  $2 \cdot \pi$ .

Observe that end-to-end convexity of  $\ddot{\mathcal{R}}$  is equivalent to the fact that any geodesic shorter than  $\pi$  with the ends on the inner and the outer circles lies completely in the union of these two circles.



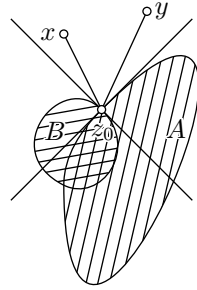
The latter holds if the zigzag line in the picture has length at least  $\pi$ . This line is formed by  $\lceil \frac{\pi}{\alpha} \rceil$  arcs with length  $\alpha$  each. Hence the sublemma.  $\square$

In the proof of 8.3.5, we will use the following exercise in convex geometry:

**8.3.7. Exercise.** Let  $A$  and  $B$  be two closed convex sets in  $\mathbb{E}^m$  and  $A \cap B \neq \emptyset$ . Given two points  $x, y \in \mathbb{E}^m$  let  $f(z) = |x - z| + |y - z|$ .

Let  $z_0 \in A \cap B$  be a point of minimum of  $f|_{A \cap B}$ .

Show that there are half-spaces  $\dot{A}$  and  $\dot{B}$  such that  $\dot{A} \supset A$  and  $\dot{B} \supset B$  and  $z_0$  is also a point of minimum of the restriction  $f|_{\dot{A} \cap \dot{B}}$ .



*Proof of 8.3.5.* Fix arbitrary  $x, y \in \mathbb{E}^m$ . Choose a point  $z \in A \cap B$  for which the sum

$$|x - z| + |y - z|$$

is minimal. To show the end-to-end convexity of  $\mathcal{R}$ , it is sufficient to prove the following:

② The geodesic  $[x^0 y^N]_{\mathcal{R}}$  contains  $z^0 = z^N \in \mathcal{R}$ .

Without loss of generality we may assume that  $z \in \partial A \cap \partial B$ . Indeed, since the puff pastry for the 1-array ( $B$ ) is end-to-end convex, Proposition 8.3.2 together with Observation 8.3.4 imply ② in case  $z$  lies in the interior of  $A$ . In the same way we can treat the case when  $z$  lies in the interior of  $B$ .

Note that  $\mathbb{E}^m$  admits an isometric splitting  $\mathbb{E}^{m-2} \times \mathbb{E}^2$  such that

$$\begin{aligned} \dot{A} &= \mathbb{E}^{m-2} \times \ddot{A} \\ \dot{B} &= \mathbb{E}^{m-2} \times \ddot{B} \end{aligned}$$

where  $\ddot{A}$  and  $\ddot{B}$  are half-planes in  $\mathbb{E}^2$ .

Using Exercise 8.3.7, let us replace each  $A$  by  $\dot{A}$  and each  $B$  by  $\dot{B}$  in the array, to get the array

$$\underbrace{(\dot{A}, \dot{B}, \dot{A}, \dots)}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \text{ times}}.$$

The corresponding puff pastry  $\dot{\mathcal{R}}$  splits as a product of  $\mathbb{E}^{m-2}$  and a puff pastry, call it  $\ddot{\mathcal{R}}$ , glued from the copies of the plane  $\mathbb{E}^2$  for the array

$$\underbrace{(\ddot{A}, \ddot{B}, \ddot{A}, \dots)}_{\lceil \frac{\pi}{\alpha} \rceil + 1 \text{ times}}.$$

Note that the dihedral angle  $\dot{A} \cap \dot{B}$  is at least  $\alpha$ . Therefore the angle measure of  $\ddot{A} \cap \ddot{B}$  is also at least  $\alpha$ . According to Sublemma 8.3.6 and Observation 8.3.4,  $\ddot{\mathcal{R}}$  is end-to-end convex.

Since  $\mathcal{R} \stackrel{\text{iso}}{=} \mathbb{E}^{m-2} \times \ddot{\mathcal{R}}$ , the puff pastry  $\mathcal{R}$  is also end-to-end convex.

It follows that the geodesic  $[x^0 y^N]_{\mathcal{R}}$  contains  $\dot{z}^0 = \dot{z}^N \in \mathcal{R}$ . By Proposition 8.3.2, the image of  $[x^0 y^N]_{\mathcal{R}}$  under the map  $\dot{x}^k \mapsto x^k$  is the geodesic  $[x^0 y^N]_{\mathcal{R}}$ . Hence Claim 2, and the lemma follow.  $\square$

## 8.4 Wide corners

We say that a closed convex set  $A \subset \mathbb{E}^m$  has  $\varepsilon$ -wide corners for given  $\varepsilon > 0$  if together with each point  $p$ , the set  $A$  contains a small right circular cone with tip at  $p$  and aperture  $\varepsilon$ ; that is,  $\varepsilon$  is the maximum angle between two generating lines of the cone.

For example, a plane polygon has  $\varepsilon$ -wide corners if all its interior angles are at least  $\varepsilon$ .

We will consider finite collections of closed convex sets  $A^1, \dots, A^n \subset \mathbb{E}^m$  such that for any subset  $F \subset \{1, \dots, n\}$ , the intersection  $\bigcap_{i \in F} A^i$  has  $\varepsilon$ -wide corners. In this case we may say briefly *all intersections of  $A^i$  have  $\varepsilon$ -wide corners*.

**8.4.1. Exercise.** Assume  $A^1, \dots, A^n \subset \mathbb{E}^m$  are compact, convex sets with a common interior point. Show that all intersections of  $A^i$  have  $\varepsilon$ -wide corners for some positive  $\varepsilon$ .

**8.4.2. Exercise.** Assume  $A^1, \dots, A^n \subset \mathbb{E}^m$  are convex sets with nonempty interior that have a common center of symmetry. Show that all intersections of  $A^i$  have  $\varepsilon$ -wide corners for some positive  $\varepsilon$ .

The proof of the following proposition is based on Lemma 8.3.5; this lemma is essentially the case  $n = 2$  in the proposition.

**8.4.3. Proposition.** *Given  $\varepsilon > 0$  and a positive integer  $n$ , there is an array of integers  $\mathbf{j}_\varepsilon(n) = (j_1, \dots, j_N)$  such that:*

- (a) *For each  $k$  we have  $1 \leq j_k \leq n$ , and each number  $1, \dots, n$  appears in  $\mathbf{j}_\varepsilon$  at least once.*
- (b) *If  $A^1, \dots, A^n$  is a collection of closed convex sets in  $\mathbb{E}^m$  with a common point and all their intersections have  $\varepsilon$ -wide corners, then the puff pastry for the array  $(A^{j_1}, \dots, A^{j_N})$  is end-to-end convex.*

*Moreover we can assume that  $N \leq (\lceil \frac{n}{\varepsilon} \rceil + 1)^n$ .*

*Proof.* The array  $\mathbf{j}_\varepsilon(n) = (j_1, \dots, j_N)$  is constructed recursively. For  $n = 1$ , we can take  $\mathbf{j}_\varepsilon(1) = (1)$ .

Assume that  $\mathbf{j}_\varepsilon(n)$  is constructed. Let us replace each occurrence of  $n$  in  $\mathbf{j}_\varepsilon(n)$  by the alternating string

$$\underbrace{n, n+1, n, \dots}_{\lceil \frac{n}{\varepsilon} \rceil + 1 \text{ times}}$$

Denote the obtained array by  $\mathbf{j}_\varepsilon(n+1)$ .

By Lemma 8.3.5, end-to-end convexity of the puff pastry for  $\mathbf{j}_\varepsilon(n+1)$  follows from end-to-end convexity of the puff pastry for the array where each string

$$\underbrace{A^n, A^{n+1}, A^n, \dots}_{\lceil \frac{n}{\varepsilon} \rceil + 1 \text{ times}}$$

is replaced by  $Q = A^n \cap A^{n+1}$ . End-to-end convexity of the latter follows by the assumption on  $\mathbf{j}_\varepsilon(n)$ , since all the intersections of  $A^1, \dots, A^{n-1}, Q$  have  $\varepsilon$ -wide corners.

The upper bound on  $N$  follows directly from the construction.  $\square$

## 8.5 Billiards

Let  $A^1, A^2, \dots, A^n$  be a finite collection of closed convex sets in  $\mathbb{E}^m$ . Assume that for each  $i$  the boundary  $\partial A^i$  is a smooth hypersurface.

Consider the billiard table formed by the closure of the complement

$$T = \overline{\mathbb{E}^m \setminus \bigcup_i A^i}.$$

The sets  $A^i$  will be called *walls* of the table  $T$  and the billiards described above will be called *billiards with convex walls*.

A *billiard trajectory* on the table  $T$  is a unit-speed broken line  $\gamma$  that follows the standard law of billiards at the break points on  $\partial A^i$  —

in particular, the angle of reflection is equal to the angle of incidence. The break points of the trajectory will be called *collisions*. We assume the trajectory meets only one wall at a time.

Recall that the definition of sets with  $\varepsilon$ -wide corners is given on page 82.

**8.5.1. Collision theorem.** *Assume  $T \subset \mathbb{E}^m$  is a billiard table with  $n$  convex walls. Assume that the walls of  $T$  have a common interior point and all their intersections have  $\varepsilon$ -wide corners. Then the number of collisions of any trajectory in  $T$  is bounded by a number  $N$  which depends only on  $n$  and  $\varepsilon$ .*

As we will see from the proof, the value  $N$  can be found explicitly;  $N = (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^{n^2}$  will do.

The collision theorem was proved by Dmitri Burago, Serge Ferleger and Alexey Kononenko in [16]; we present their proof with minor improvements.

Let us formulate and prove a corollary of the collision theorem; it answers a question formulated by Yakov Sinai [26].

**8.5.2. Corollary.** *Consider  $n$  homogeneous hard balls moving freely and colliding elastically in  $\mathbb{R}^3$ . Every ball moves along a straight line with constant speed until two balls collide, and then the new velocities of the two balls are determined by the laws of classical mechanics. We assume that only two balls can collide at the same time.*

*Then the total number of collisions cannot exceed some number  $N$  that depends on the radii and masses of the balls. If the balls are identical, then  $N$  depends only on  $n$ .*

**8.5.3. Exercise.** *Show that in the case of identical balls in the one-dimensional space (in  $\mathbb{R}$ ) the total number of collisions cannot exceed  $N = \frac{n \cdot (n-1)}{2}$ .*

The proof below admits a straightforward generalization to all dimensions.

*Proof.* Denote by  $a_i = (x_i, y_i, z_i) \in \mathbb{R}^3$  the center of the  $i$ -th ball. Consider the corresponding point in  $\mathbb{R}^{3 \cdot N}$

$$\begin{aligned} \mathbf{a} &= (a_1, a_2, \dots, a_n) = \\ &= (x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_n, y_n, z_n). \end{aligned}$$

The  $i$ -th and  $j$ -th ball intersect if

$$|a_i - a_j| \leq R_i + R_j,$$

where  $R_i$  denotes the radius of the  $i$ -th ball. These inequalities define  $\frac{n \cdot (n-1)}{2}$  cylinders

$$C_{i,j} = \{ (a_1, a_2, \dots, a_n) \in \mathbb{R}^{3 \cdot n} : |a_i - a_j| \leq R_i + R_j \}.$$

The closure of the complement

$$T = \overline{\mathbb{R}^{3 \cdot n} \setminus \bigcup_{i < j} C_{i,j}}$$

is the configuration space of our system. Its points correspond to valid positions of the system of balls.

The evolution of the system of balls is described by the motion of the point  $\mathbf{a} \in \mathbb{R}^{3 \cdot n}$ . It moves along a straight line at a constant speed until it hits one of the cylinders  $C_{i,j}$ ; this event corresponds to a collision in the system of balls.

Consider the norm of  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^{3 \cdot n}$  defined by

$$\|\mathbf{a}\| = \sqrt{M_1 \cdot |a_1|^2 + \dots + M_n \cdot |a_n|^2},$$

where  $|a_i| = \sqrt{x_i^2 + y_i^2 + z_i^2}$  and  $M_i$  denotes the mass of the  $i$ -th ball. In the metric defined by  $\|\cdot\|$ , the collisions follow the standard law of billiards.

By construction, the number of collisions of hard balls that we need to estimate is the same as the number of collisions of the corresponding billiard trajectory on the table  $T$  with  $C_{i,j}$  as the walls.

Note that each cylinder  $C_{i,j}$  is a convex set; it has smooth boundary, and it is centrally symmetric around the origin. By Exercise 8.4.2, all the intersections of the walls have  $\varepsilon$ -wide corners for some  $\varepsilon > 0$  that depend on the radii  $R_i$  and the masses  $M_i$ . It remains to apply the collision theorem (8.5.1).  $\square$

Now we present the proof of the collision theorem (8.5.1) based on the results developed in the previous section.

*Proof of 8.5.1.* Let us apply induction on  $n$ .

*Base:*  $n = 1$ . The number of collisions cannot exceed 1. Indeed, by the convexity of  $A^1$ , if the trajectory is reflected once in  $\partial A^1$ , then it cannot return to  $A^1$ .

*Step.* Assume  $\gamma$  is a trajectory that meets the walls in the order  $A^{i_1}, \dots, A^{i_N}$  for a large integer  $N$ .

Consider the array

$$\mathbf{A}_\gamma = (A^{i_1}, \dots, A^{i_N}).$$

The induction hypothesis implies:

❶ *There is a positive integer  $M$  such that any  $M$  consecutive elements of  $\mathbf{A}_\gamma$  contain each  $A^i$  at least once.*

Let  $\mathcal{R}_\gamma$  be the Reshetnyak puff pastry for  $\mathbf{A}_\gamma$ .

Consider the lift of  $\gamma$  to  $\mathcal{R}_\gamma$ , defined by  $\bar{\gamma}(t) = \gamma^k(t) \in \mathcal{R}_\gamma$  for any moment of time  $t$  between the  $k$ -th and  $(k+1)$ -th collisions. Since  $\gamma$  follows the standard law of billiards at break points, the lift  $\bar{\gamma}$  is locally a geodesic in  $\mathcal{R}_\gamma$ . By Observation 8.3.1, the puff pastry  $\mathcal{R}_\gamma$  is a proper length CAT(0) space. Therefore  $\bar{\gamma}$  is a geodesic.

Since  $\gamma$  does not meet  $A^1 \cap \dots \cap A^n$ , the lift  $\bar{\gamma}$  does not lie in  $\mathcal{R}_\gamma^0 \cup \mathcal{R}_\gamma^N$ . In particular,  $\mathcal{R}_\gamma$  is not end-to-end convex.

Let

$$\mathbf{B} = (A^{i_1}, \dots, A^{i_K})$$

be the array provided by Proposition 8.4.3; so  $\mathbf{B}$  contains each  $A^i$  at least once and the puff pastry  $\mathcal{R}_\mathbf{B}$  for  $\mathbf{B}$  is end-to-end convex. If  $N$  is sufficiently large, namely  $N \geq K \cdot M$ , then ❶ implies that  $\mathbf{A}_\gamma$  can be obtained by inserting a finite number of  $A^i$ 's in  $\mathbf{B}$ .

By Observation 8.3.4,  $\mathcal{R}_\gamma$  is end-to-end convex, a contradiction.  $\square$

## 8.6 Comments

The gluing theorem (8.2.1) extends to the class of geodesic CAT(0) spaces, which by 7.5.1 includes all complete length CAT(0) spaces. It also admits a natural generalization to length CAT( $\kappa$ ) spaces; see the book of Martin Bridson and André Haefliger [13] and our book [4] for details.

Puff pastry is used to bound topological entropy of the billiard flow and to approximate the shortest billiard path that touches given lines in a given order; see the papers of Dmitri Burago with Serge Ferleger and Alexey Kononenko [17], and with Dimitri Grigoriev and Anatol Slissenko [18]. The lecture [14] gives a short survey on the subject.

Note that the interior points of the walls play a key role in the proof despite the fact that trajectories never go inside the walls. In a similar fashion, puff pastry was used by the Stephanie Alexander and Richard Bishop in [1] to find the upper curvature bound for warped products.

In [30], Joel Hass constructed an example of a Riemannian metric on the 3-ball with negative curvature and concave boundary. This example might decrease your appetite for generalizing the collision

theorem — while locally such a 3-ball looks as good as the billiards table in the theorem, the number of collisions is obviously infinite.

It was show by Dmitri Burago and Sergei Ivanov [19] that the number of collisions that may occur between  $n$  identical balls in  $\mathbb{R}^3$  grows at least exponentially in  $n$ ; the two-dimensional case is open so far.





# Chapter 9

## Globalization

This chapter is nearly a copy of [3, Sections 3.1–3.3]; here we introduce locally CAT(0) spaces and prove the globalization theorem that provides a sufficient condition for locally CAT(0) spaces to be globally CAT(0).

### 9.1 Locally CAT spaces

We say that a space  $\mathcal{U}$  is *locally* CAT(0) (or *locally* CAT(1)) if a small closed ball centered at any point  $p$  in  $\mathcal{U}$  is CAT(0) (or CAT(1), respectively).

For example, the circle  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$  is locally isometric to  $\mathbb{R}$ , and so  $\mathbb{S}^1$  is locally CAT(0). On the other hand,  $\mathbb{S}^1$  is not CAT(0), since closed local geodesics in  $\mathbb{S}^1$  are not geodesics, so  $\mathbb{S}^1$  does not satisfy Proposition 8.1.1.

If  $\mathcal{U}$  is a proper length space, then it is locally CAT(0) (or locally CAT(1)) if and only if each point  $p \in \mathcal{U}$  admits an open neighborhood  $\Omega$  that is geodesic and such that any triangle in  $\Omega$  is thin (or spherically thin, respectively).

### 9.2 Space of local geodesic paths

In this section we will study behavior of local geodesics in locally CAT( $\kappa$ ) spaces. The results will be used in the proof of the globalization theorem (9.3.1).

Recall that a *path* is a curve parametrized by  $[0, 1]$ . The space of paths in a metric space  $\mathcal{U}$  comes with the natural metric

$$\bullet \quad |\alpha - \beta| = \sup \{ |\alpha(t) - \beta(t)|_{\mathcal{U}} : t \in [0, 1] \}.$$

**9.2.1. Proposition.** *Let  $\mathcal{U}$  be a proper length, locally  $\text{CAT}(\kappa)$  space.*

*Assume  $\gamma_n: [0, 1] \rightarrow \mathcal{U}$  is a sequence of local geodesic paths converging to a path  $\gamma_\infty: [0, 1] \rightarrow \mathcal{U}$ . Then  $\gamma_\infty$  is a local geodesic path. Moreover*

$$\text{length } \gamma_n \rightarrow \text{length } \gamma_\infty$$

*as  $n \rightarrow \infty$ .*

*Proof;*  $\text{CAT}(0)$  *case.* Fix  $t \in [0, 1]$ . Let  $R > 0$  be sufficiently small, so that  $\overline{B}[\gamma_\infty(t), R]$  forms a proper length  $\text{CAT}(0)$  space.

Assume that a local geodesic  $\sigma$  is shorter than  $R/2$  and intersects the ball  $B(\gamma_\infty(t), R/2)$ . Then  $\sigma$  cannot leave the ball  $\overline{B}[\gamma_\infty(t), R]$ . Hence, by Proposition 8.1.1,  $\sigma$  is a geodesic. In particular, for all sufficiently large  $n$ , any arc of  $\gamma_n$  of length  $R/2$  or less containing  $\gamma_n(t)$  is a geodesic.

Since  $\mathcal{B} = \overline{B}[\gamma_\infty(t), R]$  is a proper length  $\text{CAT}(0)$  space, by 7.5.1, geodesic segments in  $\mathcal{B}$  depend uniquely on their endpoint pairs. Thus there is a subinterval  $\mathbb{I}$  of  $[0, 1]$ , that contains a neighborhood of  $t$  in  $[0, 1]$  and such that the arc  $\gamma_n|_{\mathbb{I}}$  is minimizing for all large  $n$ . It follows that  $\gamma_\infty|_{\mathbb{I}}$  is a geodesic, and therefore  $\gamma_\infty$  is a local geodesic.

The  $\text{CAT}(1)$  case is done in the same way, but one has to assume in addition that  $R < \pi$ .  $\square$

The following lemma and its proof were suggested to us by Alexander Lytchak. This lemma allows a local geodesic path to be moved continuously so that its endpoints follow given trajectories. This statement was originally proved by Stephanie Alexander and Richard Bishop using a different method [2].

**9.2.2. Patchwork along a curve.** *Let  $\mathcal{U}$  be a proper length, locally  $\text{CAT}(0)$  space, and  $\gamma: [0, 1] \rightarrow \mathcal{U}$  be a path.*

*Then there is a proper length  $\text{CAT}(0)$  space  $\mathcal{N}$ , an open set  $\hat{\Omega} \subset \mathcal{N}$ , and a path  $\hat{\gamma}: [0, 1] \rightarrow \hat{\Omega}$ , such that there is an open locally isometric immersion  $\Phi: \hat{\Omega} \rightarrow \mathcal{U}$  satisfying  $\Phi \circ \hat{\gamma} = \gamma$ .*

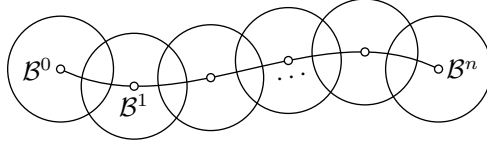
*If  $\text{length } \gamma < \pi$ , then the same holds in the  $\text{CAT}(1)$  case. Namely we assume that  $\mathcal{U}$  is a proper length, locally  $\text{CAT}(1)$  space and construct a proper length  $\text{CAT}(1)$  space  $\mathcal{N}$  with the same property as above.*

*Proof.* Fix  $r > 0$  so that for each  $t \in [0, 1]$ , the closed ball  $\overline{B}[\gamma(t), r]$  forms a proper length  $\text{CAT}(\kappa)$  space.

Choose a partition  $0 = t^0 < t^1 < \dots < t^n = 1$  such that

$$B(\gamma(t^i), r) \supset \gamma([t^{i-1}, t^i])$$

for all  $n > i > 0$ . Set  $\mathcal{B}^i = \overline{B}[\gamma(t^i), r]$ .



Consider the disjoint union  $\bigsqcup_i \mathcal{B}^i = \{(i, x) : x \in \mathcal{B}^i\}$  with the minimal equivalence relation  $\sim$  such that  $(i, x) \sim (i-1, x)$  for all  $i$ . Let  $\mathcal{N}$  be the space obtained by gluing the  $\mathcal{B}^i$  along  $\sim$ .

Note that  $A^i = \mathcal{B}^i \cap \mathcal{B}^{i-1}$  is convex in  $\mathcal{B}^i$  and in  $\mathcal{B}^{i-1}$ . Applying the Reshetnyak gluing theorem (8.2.1)  $n$  times, we conclude that  $\mathcal{N}$  is a proper length CAT(0) space.

For  $t \in [t^{i-1}, t^i]$ , define  $\hat{\gamma}(t)$  as the equivalence class of  $(i, \gamma(t))$  in  $\mathcal{N}$ . Let  $\hat{\Omega}$  be the  $\varepsilon$ -neighborhood of  $\hat{\gamma}$  in  $\mathcal{N}$ , where  $\varepsilon > 0$  is chosen so that  $B(\gamma(t), \varepsilon) \subset \mathcal{B}^i$  for all  $t \in [t^{i-1}, t^i]$ .

Define  $\Phi: \hat{\Omega} \rightarrow \mathcal{U}$  by sending the equivalence class of  $(i, x)$  to  $x$ . It is straightforward to check that  $\Phi, \hat{\gamma}$  and  $\hat{\Omega} \subset \mathcal{N}$  satisfy the conclusion of the lemma.

The CAT(1) case is proved in the same way. □

The following two corollaries follow from: (1) patchwork (9.2.2), (2) Proposition 8.1.1, which states that local geodesics are geodesics in any CAT(0) space, and (3) Proposition 7.5.1 on uniqueness of geodesics.

**9.2.3. Corollary.** *If  $\mathcal{U}$  is a proper length, locally CAT(0) space, then for any pair of points  $p, q \in \mathcal{U}$ , the space of all local geodesic paths from  $p$  to  $q$  is discrete; that is, for any local geodesic path  $\gamma$  connecting  $p$  to  $q$ , there is  $\varepsilon > 0$  such that for any other local geodesic path  $\delta$  from  $p$  to  $q$  we have  $|\gamma(t) - \delta(t)|_{\mathcal{U}} > \varepsilon$  for some  $t \in [0, 1]$ .*

*Analogously, if  $\mathcal{U}$  is a proper length, locally CAT(1) space, then for any pair of points  $p, q \in \mathcal{U}$ , the space of all local geodesic paths shorter than  $\pi$  from  $p$  to  $q$  is discrete.*

**9.2.4. Corollary.** *If  $\mathcal{U}$  is a proper length, locally CAT(0) space, then for any path  $\alpha$  there is a choice of local geodesic path  $\gamma_\alpha$  connecting the ends of  $\alpha$  such that the map  $\alpha \mapsto \gamma_\alpha$  is continuous, and if  $\alpha$  is a local geodesic path then  $\gamma_\alpha = \alpha$ .*

*Analogously, if  $\mathcal{U}$  is a proper length, locally CAT(1) space, then for any path  $\alpha$  shorter than  $\pi$ , there is a choice of local geodesic path  $\gamma_\alpha$  shorter than  $\pi$  connecting the ends of  $\alpha$  such that the map  $\alpha \mapsto \gamma_\alpha$  is continuous, and if  $\alpha$  is a local geodesic path then  $\gamma_\alpha = \alpha$ .*

*Proof of 9.2.4.* We do the CAT(0) case; the CAT(1) case is analogous.

Consider the maximal interval  $\mathbb{I} \subset [0, 1]$  containing 0 such that there is a continuous one-parameter family of local geodesic paths  $\gamma_t$  for  $t \in \mathbb{I}$  connecting  $\alpha(0)$  to  $\alpha(t)$ , with  $\gamma_t(0) = \gamma_0(t) = \alpha(0)$  for any  $t$ .

By Proposition 9.2.1,  $\mathbb{I}$  is closed, so we may assume  $\mathbb{I} = [0, s]$  for some  $s \in [0, 1]$ .

Applying patchwork (9.2.2) to  $\gamma_s$ , we find that  $\mathbb{I}$  is also open in  $[0, 1]$ . Hence  $\mathbb{I} = [0, 1]$ . Set  $\gamma_\alpha = \gamma_1$ .

By construction, if  $\alpha$  is a local geodesic path, then  $\gamma_\alpha = \alpha$ .

Moreover, from Corollary 9.2.3, the construction  $\alpha \mapsto \gamma_\alpha$  produces close results for sufficiently close paths in the metric defined by **1**; that is, the map  $\alpha \mapsto \gamma_\alpha$  is continuous.  $\square$

Given a path  $\alpha: [0, 1] \rightarrow \mathcal{U}$ , we denote by  $\bar{\alpha}$  the same path traveled in the opposite direction; that is,

$$\bar{\alpha}(t) = \alpha(1 - t).$$

The *product* of two paths will be denoted with “ $*$ ”; if two paths  $\alpha$  and  $\beta$  connect the same pair of points, then the product  $\bar{\alpha} * \beta$  is a closed curve.

**9.2.5. Exercise.** Assume  $\mathcal{U}$  is a proper length, locally CAT(1) space. Consider the construction  $\alpha \mapsto \gamma_\alpha$  provided by Corollary 9.2.4.

Assume that  $\alpha$  and  $\beta$  are two paths connecting the same pair of points in  $\mathcal{U}$ , where each is shorter than  $\pi$  and the product  $\bar{\alpha} * \beta$  is null-homotopic in the class of closed curves shorter than  $2 \cdot \pi$ . Show that  $\gamma_\alpha = \gamma_\beta$ .

## 9.3 Globalization

Riemannian manifolds with nonpositive sectional curvature are locally CAT(0). The original formulation of the *globalization theorem*, or *Hadamard–Cartan theorem*, states that if  $M$  is a complete Riemannian manifold with sectional curvature at most 0, then the exponential map at any point  $p \in M$  is a covering; in particular it implies that the universal cover of  $M$  is diffeomorphic to the Euclidean space of the same dimension.

In this generality, this theorem appeared in the lectures of Elie Cartan [22]. This theorem was proved for surfaces in Euclidean 3-space by Hans von Mangoldt [37] and a few years later independently for two-dimensional Riemannian manifolds by Jacques Hadamard [29].

Formulations for metric spaces of different generality were proved by Herbert Busemann in [20], Willi Rinow in [45], Mikhael Gromov in [27, p. 119]. A detailed proof of Gromov’s statement was given

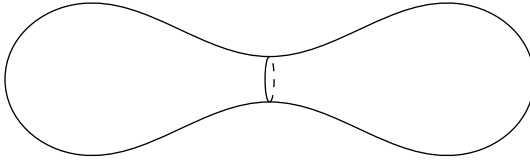
by Werner Ballmann in [9] when  $\mathcal{U}$  is proper, and by the Stephanie Alexander and Richard Bishop in [2] in more generality; also see references in our book [4].

For proper CAT(1) spaces, the globalization theorem was proved by Brian Bowditch [12].

**9.3.1. Globalization theorem.** *If a proper length, locally CAT(0) space is simply connected, then it is CAT(0).*

*Analogously, suppose  $\mathcal{U}$  is a proper length, locally CAT(1) space such that any closed curve  $\gamma: \mathbb{S}^1 \rightarrow \mathcal{U}$  shorter than  $2 \cdot \pi$  is null-homotopic in the class of closed curves shorter than  $2 \cdot \pi$ . Then  $\mathcal{U}$  is CAT(1).*

The surface on the diagram is an example of a simply connected space that is locally CAT(1) but not CAT(1). To contract the marked



curve one has to increase its length to  $2 \cdot \pi$  or more; in particular the surface does not satisfy the assumption of the globalization theorem.

The proof of the globalization theorem relies on the following theorem, which is essentially [5, Satz 9].

**9.3.2. Patchwork globalization theorem.** *A proper length, locally CAT(0) space  $\mathcal{U}$  is CAT(0) if and only if all pairs of points in  $\mathcal{U}$  are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

*Analogously, a proper length, locally CAT(1) space  $\mathcal{U}$  is CAT(1) if and only if all pairs of points in  $\mathcal{U}$  at distance less than  $\pi$  are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

The proof uses a thin-triangle decomposition with the inheritance lemma (8.1.2) and the following construction:

**9.3.3. Line-of-sight map.** *Let  $p$  be a point and  $\alpha$  be a curve of finite length in a length space  $\mathcal{X}$ . Let  $\dot{\alpha}: [0, 1] \rightarrow \mathcal{U}$  be the constant-speed parametrization of  $\alpha$ . If  $\gamma_t: [0, 1] \rightarrow \mathcal{U}$  is a geodesic path from  $p$  to  $\dot{\alpha}(t)$ , we say*

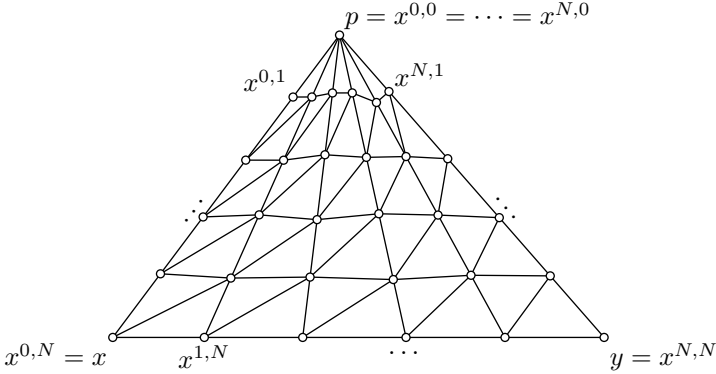
$$[0, 1] \times [0, 1] \rightarrow \mathcal{U}: (t, s) \mapsto \gamma_t(s)$$

*is a line-of-sight map from  $p$  to  $\alpha$ .*

*Proof of the patchwork globalization theorem (9.3.2).* Note that the implication “only if” follows from 7.5.1 and 7.7.4; it remains to prove the “if” part.

Fix a triangle  $[pxy]$  in  $\mathcal{U}$ . We need to show that  $[pxy]$  is thin.

By the assumptions, the line-of-sight map  $(t, s) \mapsto \gamma_t(s)$  from  $p$  to  $[xy]$  is uniquely defined and continuous.



Fix a partition

$$0 = t^0 < t^1 < \dots < t^N = 1,$$

and set  $x^{i,j} = \gamma_{t^i}(t^j)$ . Since the line-of-sight map is continuous and  $\mathcal{U}$  is locally CAT(0), we may assume that the triangles

$$[x^{i,j} x^{i,j+1} x^{i+1,j+1}] \quad \text{and} \quad [x^{i,j} x^{i+1,j} x^{i+1,j+1}]$$

are thin for each pair  $i, j$ .

Now we show that the thin property propagates to  $[pxy]$  by repeated application of the inheritance lemma (8.1.2):

◊ For fixed  $i$ , sequentially applying the lemma shows that the triangles  $[px^{i,1} x^{i+1,2}]$ ,  $[px^{i,2} x^{i+1,3}]$ , and so on are thin. In particular, for each  $i$ , the long triangle  $[px^{i,N} x^{i+1,N}]$  is thin.

◊ By the same lemma the triangles  $[px^{0,N} x^{2,N}]$ ,  $[px^{0,N} x^{3,N}]$ , and so on, are thin.

In particular,  $[pxy] = [px^{0,N} x^{N,N}]$  is thin.  $\square$

*Proof of the globalization theorem; CAT(0) case.* Let  $\mathcal{U}$  be a proper length, locally CAT(0) space that is simply connected. Given a path  $\alpha$  in  $\mathcal{U}$ , denote by  $\gamma_\alpha$  the local geodesic path provided by Corollary 9.2.4. Since the map  $\alpha \mapsto \gamma_\alpha$  is continuous, by Corollary 9.2.3 we have  $\gamma_\alpha = \gamma_\beta$  for any pair of paths  $\alpha$  and  $\beta$  homotopic relative to the ends.

Since  $\mathcal{U}$  is simply connected, any pair of paths with common ends are homotopic. In particular, if  $\alpha$  and  $\beta$  are local geodesics from  $p$  to  $q$ , then  $\alpha = \gamma_\alpha = \gamma_\beta = \beta$  by Corollary 9.2.4. It follows that any two points  $p, q \in \mathcal{U}$  are joined by a unique local geodesic that depends continuously on  $(p, q)$ .

Since  $\mathcal{U}$  is geodesic, it remains to apply the patchwork globalization theorem (9.3.2).

**CAT(1) case.** The proof goes along the same lines, but one needs to use Exercise 9.2.5.  $\square$

**9.3.4. Corollary.** *Any compact length, locally CAT(0) space that contains no closed local geodesics is CAT(0).*

*Analogously, any compact length, locally CAT(1) space that contains no closed local geodesics shorter than  $2\pi$  is CAT(1).*

*Proof.* By the globalization theorem (9.3.1), we need to show that the space is simply connected. Assume the contrary. Fix a nontrivial homotopy class of closed curves.

Denote by  $\ell$  the exact lower bound for the lengths of curves in the class. Note that  $\ell > 0$ ; otherwise there would be a closed noncontractible curve in a CAT(0) neighborhood of some point, contradicting 7.6.3.

Since the space is compact, the class contains a length-minimizing curve, which must be a closed local geodesic.

The CAT(1) case is analogous, one only has to consider a homotopy class of closed curves shorter than  $2\pi$ .  $\square$

**9.3.5. Exercise.** *Prove that any compact length, locally CAT(0) space  $\mathcal{X}$  that is not CAT(0) contains a geodesic circle; that is, a simple closed curve  $\gamma$  such that for any two points  $p, q \in \gamma$ , one of the arcs of  $\gamma$  with endpoints  $p$  and  $q$  is a geodesic.*

*Formulate and prove the analogous statement for CAT(1) spaces.*

**9.3.6. Advanced exercise.** *Let  $\mathcal{U}$  be a proper length CAT(0) space. Assume  $\tilde{\mathcal{U}} \rightarrow \mathcal{U}$  is a metric double cover branching along a geodesic. (For example 3-dimensional Euclidean space admits a double cover branching along a line.)*

*Show that  $\mathcal{U}$  is CAT(0).*

*Hint:* Apply the globalization theorem (9.3.1) and that an  $r$ -neighborhood of convex set is convex (7.7.5).





# Chapter 10

## Polyhedral spaces

### 10.1 Space of directions and tangent space

In this section we introduce a metric analog of (unit) tangent bundle that makes sense in Alexandrov geometry.

Let  $\mathcal{X}$  be a metric space with defined angles for all hinges; by 7.6.2 it holds for any  $\text{CBB}(\kappa)$  or  $\text{CAT}(\kappa)$  space. Fix a point  $p \in \mathcal{X}$ .

Consider the set  $\mathfrak{S}_p$  of all nontrivial geodesics that start at  $p$ . By 7.2.2, the triangle inequality holds for  $\angle$  on  $\mathfrak{S}_p$ , so  $(\mathfrak{S}_p, \angle)$  forms a *pseudometric space*; that is,  $\angle$  satisfies all the conditions of a metric on  $\mathfrak{S}_p$ , except that the angle between distinct geodesics might vanish.

The metric space corresponding to  $(\mathfrak{S}_p, \angle)$  is called the *space of geodesic directions* at  $p$ , denoted by  $\Sigma'_p$  or  $\Sigma'_p\mathcal{X}$ . Elements of  $\Sigma'_p$  are called *geodesic directions* at  $p$ . Each geodesic direction is formed by an equivalence class of geodesics in  $\mathfrak{S}_p$  for the equivalence relation

$$[px] \sim [py] \iff \angle[p^x_y] = 0.$$

The completion of  $\Sigma'_p$  is called the *space of directions* at  $p$  and is denoted by  $\Sigma_p$  or  $\Sigma_p\mathcal{X}$ . Elements of  $\Sigma_p$  are called *directions* at  $p$ .

The Euclidean cone  $\text{Cone } \Sigma_p$  over the space of directions  $\Sigma_p$  is called the *tangent space* at  $p$  and is denoted by  $T_p$  or  $T_p\mathcal{X}$ .

**10.1.1. Exercise.** Assume  $\mathcal{U}$  is a proper length  $\text{CAT}(\kappa)$  space with extendable geodesics; that is, any geodesic is an arc in a local geodesic  $\mathbb{R} \rightarrow \mathcal{U}$ .

Show that the space of geodesic directions at any point in  $\mathcal{U}$  is complete.

Does the statement remain true if  $\mathcal{U}$  is complete, but not required to be proper?

The tangent space  $T_p$  could also be defined directly, without introducing the space of directions. To do so, consider the set  $\mathfrak{T}_p$  of all geodesics with constant-speed parametrizations starting at  $p$ . Given  $\alpha, \beta \in \mathfrak{T}_p$ , set

$$\textcircled{1} \quad |\alpha - \beta|_{\mathfrak{T}_p} = \lim_{\varepsilon \rightarrow 0} \frac{|\alpha(\varepsilon) - \beta(\varepsilon)|_{\mathcal{X}}}{\varepsilon}$$

Since the angles in  $\mathcal{X}$  are defined,  $\textcircled{1}$  defines a pseudometric on  $\mathfrak{T}_p$ .

The corresponding metric space admits a natural isometric identification with the cone  $T'_p = \text{Cone } \Sigma'_p$ . The elements of  $T'_p$  are equivalence classes for the relation

$$\alpha \sim \beta \iff |\alpha(t) - \beta(t)|_{\mathcal{X}} = o(t).$$

The completion of  $T'_p$  is therefore naturally isometric to  $T_p$ .

Elements of  $T_p$  will be called *tangent vectors* at  $p$ , regardless of the fact that  $T_p$  is only a metric cone and need not be a vector space. Elements of  $T'_p$  will be called *geodesic tangent vectors* at  $p$ .

**10.1.2. Exercise.** Let  $\mathcal{X}$  be a complete length CAT(0) space. Show that for any point  $p \in \mathcal{X}$  the tangent space  $T_p \mathcal{X}$  is isometric to a subset of the ultra-tangent space  $T_p^\omega \mathcal{X}$  (defined on page 4.5).

Use 7.3.4 to conclude that  $T_p \mathcal{X}$  is CAT(0).

**10.1.3. Exercise.** Let  $\mathcal{X}$  be a complete length CAT(0) space. Show that for any point  $p \in \mathcal{X}$  the tangent space  $T_p \mathcal{X}$  is a length space.

## 10.2 Suspension

Suspension a spherical analog of cone construction defined on page 67.

The *suspension*  $\mathcal{V} = \text{Susp } \mathcal{U}$  over a metric space  $\mathcal{U}$  is defined as the metric space whose underlying set consists of equivalence classes in  $[0, \pi] \times \mathcal{U}$  with the equivalence relation “ $\sim$ ” given by  $(0, p) \sim (0, q)$  and  $(\pi, p) \sim (\pi, q)$  for any points  $p, q \in \mathcal{U}$ , and whose metric is given by the spherical cosine rule

$$\cos |(p, s) - (q, t)|_{\text{Susp } \mathcal{U}} = \cos s \cdot \cos t - \sin s \cdot \sin t \cdot \cos \alpha,$$

where  $\alpha = \min\{\pi, |p - q|_{\mathcal{U}}\}$ .

The points in  $\mathcal{V}$  formed by the equivalence classes of  $0 \times \mathcal{U}$  and  $\pi \times \mathcal{U}$  are called the *north* and the *south poles* of the suspension.

**10.2.1. Exercise.** Let  $\mathcal{U}$  be a metric space. Show that the spaces

$$\mathbb{R} \times \text{Cone } \mathcal{U} \quad \text{and} \quad \text{Cone}[\text{Susp } \mathcal{U}]$$

are isometric.

The following statement is a direct analog of 7.4.3 and it can be proved along the same lines.

**10.2.2. Proposition.** *Let  $\mathcal{U}$  be a metric space. Then  $\text{Susp}\mathcal{U}$  is CAT(1) if and only if  $\mathcal{U}$  is CAT(1).*

## 10.3 Definitions

**10.3.1. Definition.** *A length space  $\mathcal{P}$  is called a (spherical) polyhedral space if it admits a finite triangulation  $\tau$  such that every simplex in  $\tau$  is isometric to a simplex in a Euclidean space (or correspondingly a unit sphere) of appropriate dimension.*

*By a triangulation of a polyhedral space we will always understand a triangulation as above.*

Note that according to the above definition, all polyhedral spaces are compact. However, most of the statements below admit straightforward generalizations to *locally polyhedral spaces*; that is, complete length spaces, any point of which admits a closed neighborhood isometric to a polyhedral space. The latter class of spaces includes in particular infinite covers of polyhedral spaces.

The *dimension* of a polyhedral space  $\mathcal{P}$  is defined as the maximal dimension of the simplices in one (and therefore any) triangulation of  $\mathcal{P}$ .

**Links.** Let  $\mathcal{P}$  be a polyhedral space and  $\sigma$  be a simplex in a triangulation  $\tau$  of  $\mathcal{P}$ .

The simplices that contain  $\sigma$  form an abstract simplicial complex called the *link* of  $\sigma$ , denoted by  $\text{Link}_\sigma$ . If  $m$  is the dimension of  $\sigma$ , then the set of vertices of  $\text{Link}_\sigma$  is formed by the  $(m+1)$ -simplices that contain  $\sigma$ ; the set of its edges are formed by the  $(m+2)$ -simplices that contain  $\sigma$ ; and so on.

The link  $\text{Link}_\sigma$  can be identified with the subcomplex of  $\tau$  formed by all the simplices  $\sigma'$  such that  $\sigma \cap \sigma' = \emptyset$  but both  $\sigma$  and  $\sigma'$  are faces of a simplex of  $\tau$ .

The points in  $\text{Link}_\sigma$  can be identified with the normal directions to  $\sigma$  at a point in its interior. The angle metric between directions makes  $\text{Link}_\sigma$  into a spherical polyhedral space. We will always consider the link with this metric.

**Tangent space and space of directions.** Let  $\mathcal{P}$  be a polyhedral space (Euclidean or spherical) and  $\tau$  be its triangulation. If a point

$p \in \mathcal{P}$  lies in the interior of a  $k$ -simplex  $\sigma$  of  $\tau$  then the tangent space  $T_p = T_p \mathcal{P}$  is naturally isometric to

$$\mathbb{E}^k \times (\text{Cone Link}_\sigma).$$

Equivalently, the space of directions  $\Sigma_p = \Sigma_p \mathcal{P}$  can be isometrically identified with the  $k$ -times iterated suspension over  $\text{Link}_\sigma$ ; that is,

$$\Sigma_p \stackrel{\text{iso}}{=} \text{Susp}^k(\text{Link}_\sigma).$$

If  $\mathcal{P}$  is an  $m$ -dimensional polyhedral space, then for any  $p \in \mathcal{P}$  the space of directions  $\Sigma_p$  is a spherical polyhedral space of dimension at most  $m - 1$ .

In particular, for any point  $p$  in  $\sigma$ , the isometry class of  $\text{Link}_\sigma$  together with  $k = \dim \sigma$  determines the isometry class of  $\Sigma_p$ , and the other way around —  $\Sigma_p$  and  $k$  determines the isometry class of  $\text{Link}_\sigma$ .

A small neighborhood of  $p$  is isometric to a neighborhood of the tip of  $\text{Cone } \Sigma_p$ . (If  $\mathcal{P}$  is a spherical polyhedral space, then a small neighborhood of  $p$  is isometric to a neighborhood of the north pole in  $\text{Susp } \Sigma_p$ .) In fact, if this property holds at any point of a compact length space  $\mathcal{P}$ , then  $\mathcal{P}$  is a polyhedral space [36].

The following theorem provides a combinatorial description of polyhedral spaces with curvature bounded above.

**10.3.2. Theorem.** *Let  $\mathcal{P}$  be a polyhedral space and  $\tau$  be its triangulation. Then  $\mathcal{P}$  is locally CAT(0) if and only if the link of each simplex in  $\tau$  has no closed local geodesic shorter than  $2\pi$ .*

*Analogously, let  $\mathcal{P}$  be a spherical polyhedral space and  $\tau$  be its triangulation. Then  $\mathcal{P}$  is CAT(1) if and only if neither  $\mathcal{P}$  nor the link of any simplex in  $\tau$  has a closed local geodesic shorter than  $2\pi$ .*

*Proof.* The “only if” part follows from 8.1.1, 10.2.2, and 7.4.3.

To prove the “if” part, we apply induction on  $\dim \mathcal{P}$ . The base case  $\dim \mathcal{P} = 0$  is evident. Let us start with the CAT(1) case.

*Step.* Assume that the theorem is proved in the case  $\dim \mathcal{P} < m$ . Suppose  $\dim \mathcal{P} = m$ .

Fix a point  $p \in \mathcal{P}$ . A neighborhood of  $p$  is isometric to a neighborhood of the north pole in the suspension over the space of directions  $\Sigma_p$ .

Note that  $\Sigma_p$  is a spherical polyhedral space, and its links are isometric to links of  $\mathcal{P}$ . By the induction hypothesis,  $\Sigma_p$  is CAT(1). Thus, by the second part of Exercise 7.4.3,  $\mathcal{P}$  is locally CAT(1).

Applying the second part of Corollary 9.3.4, we get the statement.

The CAT(0) case is done in exactly the same way except we need to use the first part of Exercise 7.4.3 and the first part of Corollary 9.3.4 on the last step.  $\square$

**10.3.3. Exercise.** Let  $\mathcal{P}$  be a polyhedral space such that any two points can be connected by a unique geodesic. Show that  $\mathcal{P}$  is CAT(0).

**10.3.4. Advanced exercise.** Construct a Euclidean polyhedral metric on  $\mathbb{S}^3$  such that the total angle around each edge in its triangulation is at least  $2\cdot\pi$ .

## 10.4 Flag complexes

**10.4.1. Definition.** A simplicial complex  $\mathcal{S}$  is called *flag* if whenever  $\{v^0, \dots, v^k\}$  is a set of distinct vertices of  $\mathcal{S}$  that are pairwise joined by edges, then the vertices  $v^0, \dots, v^k$  span a  $k$ -simplex in  $\mathcal{S}$ .

If the above condition is satisfied for  $k = 2$ , then we say that  $\mathcal{S}$  satisfies the *no-triangle condition*.

Note that every flag complex is determined by its one-skeleton. Moreover, for any graph, its *cliques* (that is, complete subgraphs) define a flag complex. For that reason flag complexes are also called *clique complexes*.

**10.4.2. Exercise.** Show that the barycentric subdivision of any simplicial complex is a flag complex.

Use the flag condition (see 10.4.5 below) to conclude that any finite simplicial complex is homeomorphic to a proper length CAT(1) space.

**10.4.3. Proposition.** A simplicial complex  $\mathcal{S}$  is flag if and only if  $\mathcal{S}$  as well as all the links of all its simplices satisfy the no-triangle condition.

From the definition of flag complex we get the following.

**10.4.4. Observation.** Any link of any simplex in a flag complex is flag.

*Proof of 10.4.3.* By Observation 10.4.4, the no-triangle condition holds for any flag complex and the links of all its simplices.

Now assume that a complex  $\mathcal{S}$  and all its links satisfy the no-triangle condition. It follows that  $\mathcal{S}$  includes a 2-simplex for each triangle. Applying the same observation for each edge we get that  $\mathcal{S}$  includes a 3-simplex for any complete graph with 4 vertices. Repeating this observation for triangles, 4-simplices, 5-simplices, and so on, we get that  $\mathcal{S}$  is flag.  $\square$

**All-right triangulation.** A triangulation of a spherical polyhedral space is called an *all-right triangulation* if each simplex of the triangulation is isometric to a spherical simplex all of whose angles are

right. Similarly, we say that a simplicial complex is equipped with an *all-right spherical metric* if it is a length metric and each simplex is isometric to a spherical simplex all of whose angles are right.

Spherical polyhedral CAT(1) spaces glued from right-angled simplices admit the following characterization discovered by Mikhael Gromov [27, p. 122].

**10.4.5. Flag condition.** *Assume that a spherical polyhedral space  $\mathcal{P}$  admits an all-right triangulation  $\tau$ . Then  $\mathcal{P}$  is CAT(1) if and only if  $\tau$  is flag.*

*Proof; “only if” part.* Assume there are three vertices  $v^1, v^2$  and  $v^3$  of  $\tau$  that are pairwise joined by edges but do not span a triangle. Note that in this case

$$\angle[v^1 v^2_{v^3}] = \angle[v^2 v^3_{v^1}] = \angle[v^3 v^1_{v^2}] = \pi.$$

Equivalently,

❶ *The product of the geodesics  $[v^1 v^2]$ ,  $[v^2 v^3]$  and  $[v^3 v^1]$  forms a locally geodesic loop in  $\mathcal{P}$  of length  $\frac{3}{2} \cdot \pi$ .*

Now assume that  $\mathcal{P}$  is CAT(1). Then by Theorem 10.3.2,  $\text{Link}_\sigma \mathcal{P}$  is CAT(1) for every simplex  $\sigma$  in  $\tau$ .

Each of these links is an all-right spherical complex and by Theorem 10.3.2, none of these links can contain a geodesic circle shorter than  $2 \cdot \pi$ .

Therefore Proposition 10.4.3 and ❶ imply the “only if” part.

*“If” part.* By Observation 10.4.4 and Theorem 10.3.2, it is sufficient to show that any closed local geodesic  $\gamma$  in a flag complex  $\mathcal{S}$  with all-right metric has length at least  $2 \cdot \pi$ .

Recall that the *closed star* of a vertex  $v$  (briefly  $\overline{\text{Star}}_v$ ) is formed by all the simplices containing  $v$ . Similarly,  $\text{Star}_v$ , the open star of  $v$ , is the union of all simplices containing  $v$  with faces opposite  $v$  removed.

Choose a vertex  $v$  such that  $\text{Star}_v$  contains a point  $\gamma(t_0)$  of  $\gamma$ . Consider the maximal arc  $\gamma_v$  of  $\gamma$  that contains the point  $\gamma(t_0)$  and runs in  $\text{Star}_v$ . Note that the distance  $|v - \gamma_v(t)|_{\mathcal{P}}$  behaves in exactly the same way as the distance from the north pole in  $\mathbb{S}^2$  to a geodesic in the north hemisphere; that is, there is a geodesic  $\tilde{\gamma}_v$  in the north hemisphere of  $\mathbb{S}^2$  such that for any  $t$  we have

$$|v - \gamma_v(t)|_{\mathcal{P}} = |n - \tilde{\gamma}_v(t)|_{\mathbb{S}^2},$$

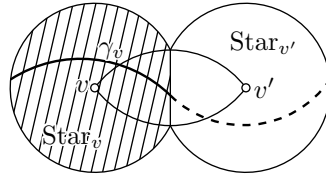
where  $n$  denotes the north pole of  $\mathbb{S}^2$ . In particular,

$$\text{length } \gamma_v = \pi;$$

that is,  $\gamma$  spends time  $\pi$  on every visit to  $\text{Star}_v$ .

After leaving  $\text{Star}_v$ , the local geodesic  $\gamma$  has to enter another simplex, say  $\sigma'$ . Since  $\tau$  is flag, the simplex  $\sigma'$  has a vertex  $v'$  not joined to  $v$  by an edge; that is,

$$\text{Star}_v \cap \text{Star}_{v'} = \emptyset$$



The same argument as above shows that  $\gamma$  spends time  $\pi$  on every visit to  $\text{Star}_{v'}$ . Therefore the total length of  $\gamma$  is at least  $2 \cdot \pi$ .  $\square$

**10.4.6. Exercise.** Assume that a spherical polyhedral space  $\mathcal{P}$  admits a triangulation  $\tau$  such that all edge lengths of all simplices are at least  $\frac{\pi}{2}$ . Show that  $\mathcal{P}$  is CAT(1) if  $\tau$  is flag.

**The space of trees.** The following construction is given by Louis Billera, Susan Holmes, and Karen Vogtmann in [10].

Let  $\mathcal{T}_n$  be the set of all metric trees with  $n$  end vertices labeled by  $a^1, \dots, a^n$ . To describe one tree in  $\mathcal{T}_n$  we may fix a topological trivalent tree  $t$  with end vertices  $a^1, \dots, a^n$  and all other vertices of degree 3, and prescribe the lengths of  $2 \cdot n - 3$  edges. If the length of an edge vanishes, we assume that this edge degenerates; such a tree can be also described using a different topological tree  $t'$ . The subset of  $\mathcal{T}_n$  corresponding to the given topological tree  $t$  can be identified with the octant

$$\{ (x_1, \dots, x_{2 \cdot n - 3}) \in \mathbb{R}^{2 \cdot n - 3} : x_i \geq 0 \}.$$

Equip each such subset with the metric induced from  $\mathbb{R}^{2 \cdot n - 3}$  and consider the length metric on  $\mathcal{T}_n$  induced by these metrics.

**10.4.7. Exercise.** Show that  $\mathcal{T}_n$  with the described metric is CAT(0).





# Appendix A

## Semisolutions

**Exercise 1.3.1.** Assume the statement is wrong. Then for any point  $x \in \mathcal{X}$ , there is a point  $x' \in \mathcal{X}$  such that

$$|x - x'| < \rho(x) \quad \text{and} \quad \rho(x') \leq \frac{\rho(x)}{1 + \varepsilon}.$$

Consider a sequence of points  $(x_n)$  such that  $x_{n+1} = x'_n$ . Clearly

$$|x_{n+1} - x_n| \leq \frac{\rho(x_0)}{\varepsilon \cdot (1 + \varepsilon)^n} \quad \text{and} \quad \rho(x_n) \leq \frac{\rho(x_0)}{(1 + \varepsilon)^n}.$$

Therefore  $(x_n)$  is Cauchy. Since  $\mathcal{X}$ , the sequence  $(x_n)$ ; denote its limit by  $x_\infty$ . Since  $\rho$  is a continuous function we get

$$\begin{aligned} \rho(x_\infty) &= \lim_{n \rightarrow \infty} \rho(x_n) = \\ &= 0. \end{aligned}$$

The latter contradicts that  $\rho > 0$ . □

**Exercise 1.4.4.** Given any pair of point  $x_0, y_0 \in \mathcal{K}$ , consider two sequences  $x_0, x_1, \dots$  and  $y_0, y_1, \dots$  such that  $x_{n+1} = f(x_n)$  and  $y_{n+1} = f(y_n)$  for each  $n$ .

Since  $\mathcal{K}$  is compact, we can choose an increasing sequence of integers  $n_k$  such that both sequences  $(x_{n_i})_{i=1}^\infty$  and  $(y_{n_i})_{i=1}^\infty$  converge. In particular, both are Cauchy sequences; that is,

$$|x_{n_i} - x_{n_j}|_{\mathcal{K}}, |y_{n_i} - y_{n_j}|_{\mathcal{K}} \rightarrow 0 \quad \text{as} \quad \min\{i, j\} \rightarrow \infty.$$

Since  $f$  is non-contracting, we get

$$|x_0 - x_{|n_i - n_j|}| \leq |x_{n_i} - x_{n_j}|.$$

It follows that there is a sequence  $m_i \rightarrow \infty$  such that

$$(*) \quad x_{m_i} \rightarrow x_0 \quad \text{and} \quad y_{m_i} \rightarrow y_0 \quad \text{as} \quad i \rightarrow \infty.$$

Set

$$\ell_n = |x_n - y_n|_{\mathcal{K}}.$$

Since  $f$  is non-contracting, the sequence  $(\ell_n)$  is non-decreasing.

By  $(*)$ ,  $\ell_{m_i} \rightarrow \ell_0$  as  $m_i \rightarrow \infty$ . It follows that  $(\ell_n)$  is a constant sequence.

In particular

$$|x_0 - y_0|_{\mathcal{K}} = \ell_0 = \ell_1 = |f(x_0) - f(y_0)|_{\mathcal{K}}$$

for any pair of points  $(x_0, y_0)$  in  $\mathcal{K}$ . That is,  $f$  is distance preserving, in particular injective.

From  $(*)$ , we also get that  $f(\mathcal{K})$  is everywhere dense. Since  $\mathcal{K}$  is compact  $f: \mathcal{K} \rightarrow \mathcal{K}$  is surjective. Hence the result follows.  $\square$

This is a basic lemma in the introduction to Gromov–Hausdorff distance [see 7.3.30 in 15]. I learned this proof from Travis Morrison, a student in my MASS class at Penn State, Fall 2011.

Note that as an easy corollary one can see that any surjective non-expanding map from a compact metric space to itself is an isometry.

**Exercise 1.7.2.** We assume that the space is not trivial, otherwise a one-point space is an example.

Consider the unit ball  $(B, \rho_0)$  in the space  $c_0$  of all sequences converging to zero equipped with the sup-norm.

Consider another metric  $\rho_1$  which is different from  $\rho_0$  by the conformal factor

$$\varphi(\mathbf{x}) = 2 + \frac{1}{2} \cdot x_1 + \frac{1}{4} \cdot x_2 + \frac{1}{8} \cdot x_3 + \dots,$$

where  $\mathbf{x} = (x_1, x_2, \dots) \in B$ . That is, if  $\mathbf{x}(t)$ ,  $t \in [0, \ell]$ , is a curve parametrized by  $\rho_0$ -length then its  $\rho_1$ -length is

$$\text{length}_{\rho_1} \mathbf{x} = \int_0^\ell \varphi \circ \mathbf{x}.$$

Note that the metric  $\rho_1$  is bi-Lipschitz to  $\rho_0$ .

Assume  $\mathbf{x}(t)$  and  $\mathbf{x}'(t)$  are two curves parametrized by  $\rho_0$ -length that differ only in the  $m$ -th coordinate, denoted by  $x_m(t)$  and  $x'_m(t)$  correspondingly. Note that if  $x'_m(t) \leq x_m(t)$  for any  $t$  and the function  $x'_m(t)$  is locally 1-Lipschitz at all  $t$  such that  $x'_m(t) < x_m(t)$ , then

$$\text{length}_{\rho_1} \mathbf{x}' \leq \text{length}_{\rho_1} \mathbf{x}.$$

Moreover this inequality is strict if  $x'_m(t) < x_m(t)$  for some  $t$ .

Fix a curve  $\mathbf{x}(t)$ ,  $t \in [0, \ell]$ , parametrized by  $\rho_0$ -length. We can choose  $m$  large, so that  $x_m(t)$  is sufficiently close to 0 for any  $t$ . In particular, for some values  $t$ , we have  $y_m(t) < x_m(t)$ , where

$$y_m(t) = (1 - \frac{t}{\ell}) \cdot x_m(0) + \frac{t}{\ell} \cdot x_m(\ell) - \frac{1}{100} \cdot \min\{t, \ell - t\}.$$

Consider the curve  $\mathbf{x}'(t)$  as above with

$$x'_m(t) = \min\{x_m(t), y_m(t)\}.$$

Note that  $\mathbf{x}'(t)$  and  $\mathbf{x}(t)$  have the same end points, and by the above

$$\text{length}_{\rho_1} \mathbf{x}' < \text{length}_{\rho_1} \mathbf{x}.$$

That is, for any curve  $\mathbf{x}(t)$  in  $(B, \rho_1)$ , we can find a shorter curve  $\mathbf{x}'(t)$  with the same end points. In particular,  $(B, \rho_1)$  has no geodesics.  $\square$

This example was suggested by Fedor Nazarov [39].

**Exercise 1.7.3.** Choose a Cauchy sequence  $(x_n)$  in  $(\mathcal{X}, \|* - *\|)$ ; it sufficient to show that a subsequence of  $(x_n)$  converges.

Note that the sequence  $(x_n)$  is Cauchy in  $(\mathcal{X}, \|* - *\|)$ ; denote its limit by  $x_\infty$ .

After passing to a subsequence, we can assume that  $\|x_n - x_{n+1}\| < \frac{1}{2^n}$ . It follows that there is a 1-Lipschitz path  $\gamma$  in  $(\mathcal{X}, \|* - *\|)$  such that  $x_n = \gamma(\frac{1}{2^n})$  for each  $n$  and  $x_\infty = \gamma(0)$ .

It follows that

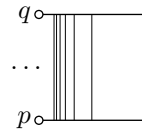
$$\begin{aligned} \|x_\infty - x_n\| &\leq \text{length } \gamma|_{[0, \frac{1}{2^n}]} \leq \\ &\leq \frac{1}{2^n}. \end{aligned}$$

In particular  $x_n$  converges.  $\square$

Source: [41, Lemma 2.3].

**Exercise 1.7.8.** Consider the following subset of  $\mathbb{R}^2$  equipped with the induced length metric

$$\mathcal{X} = ((0, 1] \times \{0, 1\}) \cup (\{1, \frac{1}{2}, \frac{1}{3}, \dots\} \times [0, 1])$$



Note that  $\mathcal{X}$  is locally compact and geodesic.

Its completion  $\bar{\mathcal{X}}$  is isometric to the closure of  $\mathcal{X}$  equipped with the induced length metric;  $\bar{\mathcal{X}}$  is obtained from  $\mathcal{X}$  by adding two points  $p = (0, 0)$  and  $q = (0, 1)$ .

The point  $p$  admits no compact neighborhood in  $\bar{\mathcal{X}}$  and there is no geodesic connecting  $p$  to  $q$  in  $\bar{\mathcal{X}}$ .  $\square$

This exercise and its solution is taken from [13].

**Exercise 1.8.3.** By Frechet lemma (1.8.1) we can identify  $\mathcal{K}$  with a compact subset of  $\ell^\infty$ .

Denote by  $\mathcal{L} = \text{Conv } \mathcal{K}$  — it is defined as the minimal convex closed set in  $\ell^\infty$  that contains  $\mathcal{K}$ . (In other words,  $\mathcal{L}$  is the intersection of all convex closed sets that contain  $\mathcal{K}$ .) Observe that  $\mathcal{L}$  is a length space.

Let us show that since  $\mathcal{K}$  is compact, so is  $\mathcal{L}$ . By construction  $\mathcal{L}$  is closed subset of  $\ell^\infty$ , in particular it is a complete space. By 1.4.1d, it remains to show that  $\mathcal{L}$  is totally bounded.

Recall that Minkowski sum  $A + B$  of two sets  $A$  and  $B$  in a vector space is defined by

$$A + B = \{a + b : a \in A, b \in B\}.$$

Observe that Minkowski sum of two convex sets is convex.

Denote by  $\bar{B}_\varepsilon$  the closed  $\varepsilon$ -ball in  $\ell^\infty$  centered at the origin. Choose a finite  $\varepsilon$ -net  $N$  in  $\mathcal{K}$  for some  $\varepsilon > 0$ . Note that  $P = \text{Conv } N$  is a convex polyhedron, in particular  $\text{Conv } N$  is compact.

Observe that  $N + \bar{B}_\varepsilon$  is closed  $\varepsilon$ -neighborhood of  $N$ ; therefore  $N + \bar{B}_\varepsilon \supset K$ . Therefore  $P + \bar{B}_\varepsilon \supset \mathcal{L}$ ; in particular  $P$  is a  $2 \cdot \varepsilon$ -net in  $\mathcal{L}$ . That is,  $\mathcal{L}$  admits a compact  $\varepsilon$ -net for any  $\varepsilon > 0$ . Therefore  $\mathcal{L}$  is totally bounded (see 1.4.2).

**Exercise 2.1.7.** The answer is “no” in both parts.

For the first part let  $X$  be a unit disc and  $Y$  a finite  $\varepsilon$ -net in  $X$ . Evidently  $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$ , but  $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} \approx 1$ .

For the second part take  $X$  to be a unit disc and  $Y = \partial X$  to be its boundary circle. Note that  $\partial X = \partial Y$  in particular  $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} = 0$  while  $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} = 1$ .  $\square$

A more interesting example for the second part can be build on *lakes of Wada* — an example of three open bounded topological disks in the plane that have identical boundary.

**Exercise 2.1.8.** Let  $A$  be a compact convex set in the plane. Denote by  $A^r$  the closed  $r$ -neighborhood of  $A$ . Recall that by Steiner’s formula we have

$$\text{area } A^r = \text{area } A + r \cdot \text{perim } A + \pi \cdot r^2.$$

Taking derivative and applying coarea formula, we get

$$\text{perim } A^r = \text{perim } A + 2 \cdot \pi \cdot r.$$

Observe that if  $A$  lies in a compact set  $B$  bounded by a closed curve, then

$$\text{perim } A \leq \text{perim } B.$$

Indeed the closest-point projection  $\mathbb{R}^2 \rightarrow A$  is short and it maps  $\partial B$  onto  $\partial A$ .

It remains to observe that if  $A_n \rightarrow A_\infty$ , then for any  $r > 0$  we have that

$$A_\infty^r \supset A_n \quad \text{and} \quad A_\infty \subset A_n^r$$

for all large  $n$ .

**Exercise 3.4.4.** In order to check that  $|\ast - \ast|_{\mathcal{M}'}$  is a metric, it is sufficient to show that

$$|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}'} = 0 \implies \mathcal{X} \stackrel{iso}{=} \mathcal{Y};$$

the remaining conditions are trivial.

If  $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}'} = 0$ , then there is a sequence of maps  $f_n: \mathcal{X} \rightarrow \mathcal{Y}$  such that

$$|f_n(x) - f_n(x')|_{\mathcal{Y}} \geq |x - x'|_{\mathcal{X}} - \frac{1}{n}.$$

Choose a countable dense set  $S$  in  $\mathcal{X}$ . Passing to a subsequence of  $f_n$  we can assume that  $f_n(x)$  converges for any  $x \in S$  as  $n \rightarrow \infty$ ; denote its limit by  $f_\infty(x)$ .

For each point  $x \in \mathcal{X}$  choose a sequence  $x_m \in S$  converging to  $x$ . Since  $\mathcal{Y}$  is compact, we can assume in addition that  $y_m = f_\infty(x_m)$  converges in  $\mathcal{Y}$ . Set  $f_\infty(x) = y$ . Note that the map  $f_\infty: \mathcal{X} \rightarrow \mathcal{Y}$  is distance non-decreasing.

The same way we can construct a distance non-decreasing map  $g_\infty: \mathcal{Y} \rightarrow \mathcal{X}$ .

By Exercise 1.4.4, the compositions  $f_\infty \circ g_\infty: \mathcal{Y} \rightarrow \mathcal{Y}$  and  $g_\infty \circ f_\infty: \mathcal{X} \rightarrow \mathcal{X}$  are isometries. Therefore  $f_\infty$  and  $g_\infty$  are isometries as well.

(The proof of the second part is coming.)

**Exercise 3.5.1.** Choose a space  $\mathcal{X}$  in  $\mathcal{Q}(C, D)$ , denote a  $C$ -doubling measure by  $\mu$ . Without loss of generality we may assume that  $\mu(\mathcal{X}) = 1$ .

The doubling condition implies that

$$\mu[B(p, \frac{D}{2^n})] \geq \frac{1}{C^n}$$

for any point  $x \in \mathcal{X}$ . It follows that

$$\text{pack}_{\frac{D}{2^n}} \mathcal{X} \leq C^n.$$

By Exercise 1.4.3, for any  $\varepsilon \geq \frac{D}{2^{n-1}}$ , the space  $\mathcal{X}$  admits an  $\varepsilon$ -net with at most  $C^n$  points. Whence  $\mathcal{Q}(C, D)$  is uniformly totally bounded.

**Exercise 3.5.2.** Since  $\mathcal{Y}$  is compact, it has a finite  $\varepsilon$ -net for any  $\varepsilon > 0$ . For each  $\varepsilon > 0$  choose a finite  $\varepsilon$ -net  $\{y_1, \dots, y_{n_\varepsilon}\}$  in  $\mathcal{Y}$ .

Suppose  $f: \mathcal{X} \rightarrow \mathcal{Y}$  be a distance non-decreasing map. Choose one point  $x_i$  in each nonempty subset  $B_i = f^{-1}[B(y_i, \varepsilon)]$ . Note that the subset  $B_i$  has diameter at most  $2 \cdot \varepsilon$  and

$$\mathcal{X} = \bigcup_i B_i.$$

Therefore the set of points  $\{x_i\}$  forms a  $2 \cdot \varepsilon$  net in  $\mathcal{X}$ . Whence (a) follows.

(b). Let  $\mathcal{Q}$  be a uniformly totally bounded family of spaces. Suppose that each space in  $\mathcal{Q}$  has an  $\frac{1}{2^n}$ -net with at most  $M_n$  points; we may assume that  $M_0 = 1$ .

Consider the space  $\mathcal{Y}$  of all infinite integer sequences  $m_0, m_1, \dots$  such that  $1 \leq m_n \leq M_n$  for any  $n$ . Given two sequences  $(\ell_n)$ , and  $(m_n)$  of points in  $\mathcal{Y}$ , set

$$|(\ell_n) - (m_n)|_{\mathcal{Y}} = \frac{C}{2^n},$$

where  $n$  is minimal index such that  $\ell_n \neq m_n$  and  $C$  is a positive constant.

Observe that  $\mathcal{Y}$  is compact. Indeed it is complete and the sequences constant starting from index  $n$  form a finite  $\frac{C}{2^n}$ -net in  $\mathcal{Y}$ .

Given a space  $\mathcal{X}$  in  $\mathcal{Q}$ , choose a sequence of  $\frac{1}{2^n}$  nets  $N_n \subset \mathcal{X}$  for each natural  $n$ . We can assume that  $|N_n| \leq M_n$ ; let us enumerate the points in  $N_n$  by  $\{1, \dots, M_n\}$ . Consider the map  $f: \mathcal{X} \rightarrow \mathcal{Y}$  defined by  $f: x \rightarrow (m_1(x), m_2(x), \dots)$  where  $m_n(x)$  is a number of the point in  $N_n$  that lies on the distance  $< \frac{1}{2^n}$  from  $x$ .

If  $\frac{1}{2^{n-2}} \geq |x - x'|_{\mathcal{X}} > \frac{1}{2^{n-1}}$ , then  $m_n(x) \neq m_n(x')$ . It follows that  $|f(x) - f(x')|_{\mathcal{Y}} \geq \frac{C}{2^n}$ . In particular, if  $C > 10$ , then

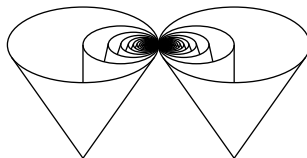
$$|f(x) - f(x')|_{\mathcal{Y}} \geq |x - x'|_{\mathcal{X}}$$

for any  $x, x' \in \mathcal{X}$ . That is,  $f$  is a distance non-decreasing map  $\mathcal{X} \rightarrow \mathcal{Y}$ .

**Exercise 3.6.3, (b).** Suppose  $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}$  and  $\mathcal{X}_n$  are simply connected length metric space. It is sufficient to show that any nontrivial covering map  $f: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  corresponds to a nontrivial covering map  $f_n: \tilde{\mathcal{X}}_n \rightarrow \mathcal{X}_n$  for large  $n$ .

The latter can be constructed by covering  $\mathcal{X}_n$  by small balls that lie close to sets in  $\mathcal{X}$  evenly covered by  $f$ , prepare few copies of these sets and glue them the same way as the inverse images of the evenly covered sets in  $\mathcal{X}$  glued to obtain  $\tilde{\mathcal{X}}$ .

(b). Let  $\mathcal{V}$  be a cone over Hawaiian earring. Consider the *doubled cone*  $\mathcal{W}$  — two copies of  $\mathcal{V}$  with glued base points earrings (see the diagram).



The space  $\mathcal{W}$  can be equipped with length metric for example the induced length metric from the shown embedding.

Note that  $\mathcal{V}$  is simply connected, but  $\mathcal{W}$  is not — it is a good exercise in topology.

If we delete from the earrings all small circles, then the obtained double cone becomes simply connected and it remains to be close to  $\mathcal{W}$  in the sense of Gromov–Hausdorff.

*Comment.* Note that from part (b), the limit does not admit a non-trivial covering. So if we define fundamental group right — as the inverse image of groups of deck transformations for all its coverings, then one may say that Gromov–Hausdorff limit of simply connected length spaces is simply connected.

**Exercise 3.6.4, (a).** Suppose that a metric on  $\mathbb{S}^2$  is close to the disc  $\mathbb{D}^2$ . Note that  $\mathbb{S}^2$  contains a circle  $\gamma$  that is close to the boundary curve of  $\mathbb{D}^2$ . By Jordan curve theorem,  $\gamma$  divides  $\mathbb{S}^2$  into two discs, say  $D_1$  and  $D_2$ .

By 3.6.3b, the Gromov–Hausdorff limit of  $D_1$  and  $D_2$  have to contain whole  $\mathbb{D}^2$ , otherwise the limit would admit a nontrivial covering. Consider points  $p_1 \in D_1$  and  $p_2 \in D_2$  that are close to the center of  $\mathbb{D}^2$ . On one hand the distance  $|p_1 - p_2|_n$  have to be very small. On the other hand, any curve from  $p_1$  to  $p_2$  must cross  $\gamma$ , so it has length about 2 at least — a contradiction.

(b). Make fine burrows in the standard 3-ball without changing its topology, but at the same time come sufficiently close to any point in the ball.

Consider the *doubling* of the obtained ball along its boundary; that is, two copies of the ball with identified corresponding points on their boundaries. The obtained space is homeomorphic to  $\mathbb{S}^3$ . Note that the burrows can be made so that the obtained space is sufficiently close to the original ball in the Gromov–Hausdorff metric.  $\square$

Source: [15, Exercises 7.5.13 and 7.5.17].

**Exercise 4.4.1.** Part (a) follows directly from the definitions. Further we consider  $\mathcal{X}$  as a subset of  $\mathcal{X}^\omega$ .

(b). Suppose  $\mathcal{X}$  compact. Given a sequence  $(x_n)$  in  $\mathcal{X}$ , denote its  $\omega$ -limit in  $\mathcal{X}^\omega$  by  $x^\omega$  and its  $\omega$ -limit in  $\mathcal{X}$  by  $x_\omega$ .

Observe that  $x^\omega = \iota(x_\omega)$ . Therefore  $\iota$  is onto.

If  $\mathcal{X}$  is not compact, we can choose a sequence  $(x_n)$  such that  $|x_m - x_n| > \varepsilon$  for fixed  $\varepsilon > 0$  and  $m \neq n$ . Observe that

$$\lim_{n \rightarrow \omega} |x_n - y|_{\mathcal{X}} \geq \frac{\varepsilon}{2}$$

for any  $y \in \mathcal{X}$ . It follows that  $x_\omega$  lies on the distance at least  $\frac{\varepsilon}{2}$  from  $\mathcal{X}$ .

(c). A sequence of points  $(x_n)$  in  $\mathcal{X}$  will be called  $\omega$ -bounded if there is a real constant  $C$  such that

$$|p - x_n|_{\mathcal{X}} \leq C$$

for  $\omega$ -almost all  $n$ .

The same argument as in (b) shows that any  $\omega$ -bounded sequence has its  $\omega$ -limit in  $\mathcal{X}$ . Further if  $(x_n)$  is not  $\omega$ -bounded, then

$$\lim_{n \rightarrow \omega} |p - x_n|_{\mathcal{X}} = \infty;$$

that is  $x_\omega$  does not lie in the metric component of  $p$  in  $\mathcal{X}^\omega$ .

**Exercise 4.3.3.** Observe that if a path  $\gamma$  in a metric tree from  $p$  to  $q$  pass thru a point  $x$  on distance  $\ell$  from  $[pq]$ , then

$$\textbf{1} \quad \text{length } \gamma \geq |p - q| + 2 \cdot \ell.$$

Suppose that  $\mathcal{T}_n$  is a sequence of metric trees that  $\omega$ -converges to  $\mathcal{T}_\omega$ . By 4.3.2, the space  $\mathcal{T}_\omega$ .

The uniqueness will follow from **1**. Indeed, if for a geodesic  $[p_\omega q_\omega]$  there is another geodesic  $\gamma_\omega$  connecting its ends, then it have to pass thru a point  $x_\omega \notin [p_\omega q_\omega]$ . Choose a sequences  $p_n, q_n, x_n \in \mathcal{T}_n$  such that  $p_n \rightarrow p_\omega$ ,  $q_n \rightarrow q_\omega$ ,  $x_n \rightarrow x_\omega$  and  $n \rightarrow \omega$ . Then

$$\begin{aligned} |p_\omega - q_\omega| &= \text{length } \gamma \geq \lim_{n \rightarrow \omega} (|p_n - x_n| + |q_n - x_n|) \geq \\ &\geq \lim_{n \rightarrow \omega} (|p_n - q_n| + 2\ell_n) = \\ &|p_\omega - q_\omega| + 2 \cdot \ell_\omega. \end{aligned}$$

Since  $x_\omega \notin [p_\omega q_\omega]$ , we have that  $\ell_\omega > 0$  — a contradiction.

To prove the last property consider sequence of centers of tripods  $m_n$  for points  $x_n, y_n, z_n \in \mathcal{T}_n$  and observe that its ultralimit  $m_\omega$  is a the ceter of tripod with ends at  $x_\omega, y_\omega, z_\omega \in \mathcal{T}_\omega$ .

**Exercise 4.5.1.** Coming soon.



**Exercise 5.2.2.** Construct a separable space that has infinite number of geodesics between a pair of points, say a square with  $\ell^\infty$ -metric in  $\mathbb{R}^2$  and apply universality of Urysohn space (5.2.1).

**Exercise 5.2.5.** It is sufficient to show that any compact subspace  $\mathcal{K}$  of Urysohn space can be contracted to a point.

Note that any compact space  $\mathcal{K}$  can be extended to a contractible compact space  $\mathcal{K}'$ ; for example we may embed  $\mathcal{K}$  into  $\ell^\infty$  and pass to its convex hull, as it was done in 1.8.3.

By 5.3.3, there is an isometric embedding of  $\mathcal{K}'$  that agrees with inclusion of  $\mathcal{K}$ . Since  $\mathcal{K}$  is contractible in  $\mathcal{K}'$ , it is contractible in  $\mathcal{U}$ .

In fact one can contract whole Urysohn space using the following construction.

Note that points in the space  $\mathcal{X}_\infty$  constructed in the proof of 5.1.2 can be multiplied by number  $t \in [0, 1]$  — simply multiply each function by factor  $t$ . That defines a map

$$\lambda_t: \mathcal{X}_\infty \rightarrow \mathcal{X}_\infty$$

that scales all distances by factor  $t$ . The map  $\lambda_t$  can be extended to the completion of  $\mathcal{X}_\infty$ , which is isometric to  $\mathcal{U}_d$  (or  $\mathcal{U}$ ).

Observe that the map  $\lambda_1$  is the identity and  $\lambda_0$  maps whole space to a single point, say  $x_0$  — that is the only point of  $\mathcal{X}_0$ . Further note that the map  $(t, p) \mapsto \lambda_t(p)$  is continuous — in particular  $\mathcal{U}_d$  and  $\mathcal{U}$  are contractible.

As a bonus, observe that for any point  $p \in \mathcal{U}_d$  the curve  $t \mapsto \lambda_t(p)$  is a geodesic path from  $p$  to  $x_0$ .

Source: [28, (d) on page 82].

**Exercise 5.3.2.** Observe that  $S$  is an  $d$ -Urysohn space and apply uniqueness (5.3.1).

**Exercise 5.2.3.** The following claim is a key to the proof.

**A.0.1. Claim.** *Suppose  $f: K \rightarrow \mathbb{R}$  is an extension function defined on a compact subset  $K$  of the Urysohn space  $\mathcal{U}$ . Then there is a point  $p \in \mathcal{U}$  such that  $|p - x| = f(x)$  for any  $x \in K$ .*

*Proof.* Without loss of generality we may assume that  $f(x) > 0$  for any  $x \in K$ . Since  $K$  is compact, we may fix  $\varepsilon > 0$  such that  $f(x) > \varepsilon$ .

Consider the sequence  $\varepsilon_n = \frac{\varepsilon}{100 \cdot 2^n}$ . Choose a sequence of  $\varepsilon_n$ -nets  $N_n \subset K$ . Applying universality of  $\mathcal{U}$  recursively, we may choose a point  $p_n$  such that  $|p_n - x| = f(x)$  for any  $x \in N_n$  and  $|p_n - p_{n-1}| = 10 \cdot \varepsilon_{n-1}$ . Observe that the sequence  $(p_n)$  is Cauchy and its limit  $p$  meets  $|p - x| = f(x)$  for any  $x \in K$ .  $\square$

Choose a sequence of points  $(x_n)$  in  $\mathcal{S}$ . Applying the claim, we may extend the map from  $K$  to  $K \cup \{x_1\}$ , and further to  $K \cup \{x_1, x_2\}$ , and so on. As a result we extend the distance-preserving map  $f$  to whole sequence  $(x_n)$ . It remains to extend it continuously to whole space  $\mathcal{S}$ .

**Exercise 6.1.2.** If  $c < 0$  then  $r > s$ . The latter is impossible since  $r$  is extremal and  $s$  is admissible.

Observe that the function  $\bar{r} = \min\{r, s + c\}$  is admissible. Indeed if  $\bar{r}(x) = r(x)$  and  $\bar{r}(y) = r(y)$  then

$$\bar{r}(x) + \bar{r}(y) = r(x) + r(y) \geq |x - y|.$$

Further if  $\bar{r}(x) = s(x) + c$  then

$$\begin{aligned} \bar{r}(x) + \bar{r}(y) &\geq [s(x) + c] + [s(y) - c] = \\ &= s(x) + s(y) \geq \\ &\geq |x - y|. \end{aligned}$$

Since  $r$  is extremal, we have  $r = \bar{r}$ ; that is  $r \leq s + c$ .

**Exercise 6.2.2.** Choose an injective space  $\mathcal{Y}$ .

(a). Fix a Cauchy sequence  $(x_n)$  in  $\mathcal{Y}$ ; we need to show that it has a limit  $x_\infty \in \mathcal{Y}$ . Consider metric on  $\mathcal{X} = \mathbb{N} \cup \{\infty\}$  defined by

$$\begin{aligned} |m - n|_{\mathcal{X}} &= |x_m - x_n|_{\mathcal{Y}}, \\ |m - \infty|_{\mathcal{X}} &= \lim_{n \rightarrow \infty} |x_m - x_n|_{\mathcal{Y}}. \end{aligned}$$

Since the sequence is Cauchy, so is the sequence  $\ell_n = |p - x_n|_{\mathcal{Y}}$ . Therefore the last limit is defined.

By construction the map  $n \mapsto x_n$  is distance preserving on  $\mathbb{N} \subset \mathcal{X}$ . Since  $\mathcal{Y}$  is injective, this map can be extended to  $\infty$  as a short map; set  $\infty \mapsto x_\infty$ . Since  $|x_n - x_\infty|_{\mathcal{Y}} \leq |n - \infty|_{\mathcal{X}}$  and  $|n - \infty|_{\mathcal{X}} \rightarrow 0$ , we get that  $x_n \rightarrow x_\infty$  as  $n \rightarrow \infty$ .

(b). Applying the definition of injective space, we get a midpoint for any pair of points in  $\mathcal{Y}$ . By (a),  $\mathcal{Y}$  is a complete space. It remains to apply 1.7.4b.

(c). Let  $k: \mathcal{Y} \hookrightarrow \ell^\infty(\mathcal{Y})$  be the Kuratowski embedding (1.8.4). Observe that  $\ell^\infty(\mathcal{Y})$  is contractible; in particular, there is a homotopy  $k_t: \mathcal{Y} \hookrightarrow \ell^\infty(\mathcal{Y})$  such that  $k_0 = k$  and  $k_1$  is a constant map. (In fact one can take  $k_t = (1 - t) \cdot k$ .)

Since  $k$  is distance preserving and  $\mathcal{Y}$  is injective, there is a short map  $f: \ell^\infty(\mathcal{Y}) \rightarrow \mathcal{Y}$  such that the composition  $f \circ k$  is the identity map on  $\mathcal{Y}$ . The composition  $f \circ k_t: \mathcal{Y} \hookrightarrow \mathcal{Y}$  is a needed homotopy.

**Exercise 6.2.3.** Choose three points  $x, y, z \in \mathcal{X}$  and set  $\mathcal{A} = \{x, z\}$ . Let  $f: \mathcal{A} \rightarrow \mathcal{Y}$  be an isometry. Then  $F(y) = f(x)$  or  $F(y) = f(z)$ . If  $f(y) = f(x)$ , then

$$\begin{aligned} |y - z|_{\mathcal{X}} &\geq |F(y) - f(z)|_{\mathcal{Y}} = \\ &= |x - z|_{\mathcal{X}}. \end{aligned}$$

Analogously if  $f(y) = f(z)$ , then  $|x - y|_{\mathcal{X}} \geq |x - z|_{\mathcal{X}}$ .

It remains to observe that the strong triangle inequality holds in both cases.

**Exercise 6.2.5.** Suppose a short map  $f: A \rightarrow \mathcal{Y}$  is defined on a subset of a metric space  $\mathcal{X}$ . We need to construct a short extension  $F$  of  $f$ .

(a). In this case  $\mathcal{Y} = \mathbb{R}$ . Without loss of generality, we may assume that  $A \neq \emptyset$ , otherwise map whole  $\mathcal{X}$  to a single point. Set

$$F(x) = \inf \{ f(a) - |a - x| : a \in A \}.$$

Observe that  $F$  is short and  $F(a) = f(a)$  for any  $a \in A$ .

(b). In this case  $\mathcal{Y}$  is a complete metric tree. Fix a point  $p \in \mathcal{X}$  and  $q \in \mathcal{Y}$ . Given a point  $a \in A$ , let  $x_a \in \overline{B}[f(a), |a - p|]$  be the point closest to  $f(x)$ . Note that  $x_a \in [q f(a)]$  and either  $x_a = q$  or  $x_a$  lies on distance  $|a - p|$  from  $f(a)$ .

Note that the geodesics  $[q x_a]$  are nested; that is, for any  $a, b \in A$  we have either  $[q x_a] \subset [q x_b]$  or  $[q x_b] \subset [q x_a]$ . Moreover, in the first case we have  $|x_b - f(a)| \leq |p - a|$  and in the second  $|x_a - f(b)| \leq |p - b|$ .

It follows that the closure of the union of all geodesics  $[q x_a]$  for  $a \in A$  is a geodesic. Denote by  $x$  its end (it exists since  $\mathcal{Y}$  is complete). It remains to observe that  $|x - f(a)| \leq |p - a|$  for any  $a \in A$ ; that is, one can take  $f(p) = x$ .

(c). In this case  $\mathcal{Y} = (\mathbb{R}^2, \ell^\infty)$ . Note that the map  $\mathcal{X} \rightarrow (\mathbb{R}^2, \ell^\infty)$  is short if and only if both of its coordinate projections are short. It remains to apply (a).

**Exercise 6.3.1;** (a). Let  $f$  be an extremal function. Observe that at least two of the numbers  $f(a) + f(b)$ ,  $f(b) + f(c)$ , and  $f(c) + f(a)$  are 1. It follows that for some  $x \in [0, \frac{1}{2}]$ , we have

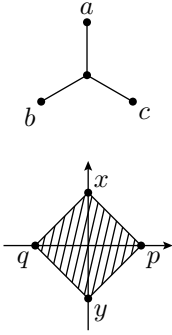
$$f(a) = 1 \pm x, \quad f(b) = 1 \pm x, \quad f(c) = 1 \pm x,$$

where we have one “−” and two “+” in these three formulas.

Suppose that

$$g(a) = 1 \pm y, \quad g(b) = 1 \pm y, \quad g(c) = 1 \pm y$$

is another extremal function. Then  $|f - g| = |x - y|$  if  $g$  has “−” at the same place as  $f$  and  $|f - g| = |x + y|$  otherwise.



It follows that  $\text{Inj } \mathcal{X}$  is isometric to a tripod — that is,  $\text{Inj } \mathcal{X}$  can be made from three segments of length  $\frac{1}{2}$  and by gluing them at one end.

(b). Assume  $f$  is an extremal function. Observe that  $f(x) + f(y) = f(p) + f(q) = 2$ ; in particular, two values  $a = f(x) - 1$  and  $b = f(p) - 1$  completely describe the function  $f$ . Since  $f$  is extremal, we also have that

$$(1 \pm a) + (1 \pm b) \geq 1$$

for all 4 choices of signs; that is,  $|a| + |b| \leq 1$ .

It follows that  $\text{Inj } \mathcal{X}$  is isometric to the rhombus  $|a| + |b| \leq 1$  in the  $(a, b)$ -plane with the metric induced by the  $\ell^\infty$ -norm.

**Exercise 6.3.4.** Recall that

$$|f - g|_{\text{Inj } \mathcal{X}} = \sup \{ |f(x) - g(x)| : x \in \mathcal{X} \}$$

and

$$|f - p|_{\text{Inj } \mathcal{X}} = f(p)$$

for any  $f, g \in \text{Inj } \mathcal{X}$  and  $p \in \mathcal{X}$ .

Since  $\mathcal{X}$  is compact we can find a point  $p \in \mathcal{X}$  such that

$$|f - g|_{\text{Inj } \mathcal{X}} = |f(p) - g(p)| = ||f - p|_{\text{Inj } \mathcal{X}} - |g - p|_{\text{Inj } \mathcal{X}}|.$$

Without loss of generality we may assume that

$$|f - p|_{\text{Inj } \mathcal{X}} = |g - p|_{\text{Inj } \mathcal{X}} + |f - g|_{\text{Inj } \mathcal{X}}.$$

Applying 6.3.3, we can find a point  $q \in \mathcal{X}$  such that

$$|q - p|_{\text{Inj } \mathcal{X}} = |f - p|_{\text{Inj } \mathcal{X}} + |f - q|_{\text{Inj } \mathcal{X}},$$

whence the result.

**Exercise 7.2.2.** Let us show that  $\gamma \leq \alpha + \beta$ ; the rest of inequalities can be done the same way. Since  $\gamma \leq \pi$ , we may assume that  $\alpha + \beta < \pi$ .

Denote by  $\gamma_x$ ,  $\gamma_y$ , and  $\gamma_z$  the geodesics  $[px]$ ,  $[py]$ , and  $[pz]$  parameterized from  $p$  by arc-length. By triangle inequality, for any  $\varepsilon > 0$  and all sufficiently small  $t, \tau, s \in \mathbb{R}_+$  we have

$$\begin{aligned} |\gamma_x(t) - \gamma_z(\tau)| &\leq |\gamma_x(t) - \gamma_y(s)| + |\gamma_y(s) - \gamma_z(\tau)| < \\ &< \sqrt{t^2 + s^2 - 2 \cdot t \cdot s \cdot \cos(\alpha + \varepsilon)} + \\ &\quad + \sqrt{s^2 + \tau^2 - 2 \cdot s \cdot \tau \cdot \cos(\beta + \varepsilon)} \leq \end{aligned}$$

Below we define  $s(t, \tau)$  so that for  $s = s(t, \tau)$ , this chain of inequalities can be continued as follows:

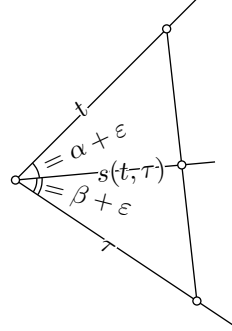
$$\leq \sqrt{t^2 + \tau^2 - 2 \cdot t \cdot \tau \cdot \cos(\alpha + \beta + 2 \cdot \varepsilon)}.$$

Thus for any  $\varepsilon > 0$ ,

$$\gamma \leq \alpha + \beta + 2 \cdot \varepsilon.$$

Hence the result.

To define  $s(t, \tau)$ , consider three rays  $\tilde{\gamma}_x, \tilde{\gamma}_y, \tilde{\gamma}_z$  on a Euclidean plane starting at one point, such that  $\angle(\tilde{\gamma}_x, \tilde{\gamma}_y) = \alpha + \varepsilon$ ,  $\angle(\tilde{\gamma}_y, \tilde{\gamma}_z) = \beta + \varepsilon$  and  $\angle(\tilde{\gamma}_x, \tilde{\gamma}_z) = \alpha + \beta + 2 \cdot \varepsilon$ . We parametrize each ray by the distance from the starting point. Given two positive numbers  $t, \tau \in \mathbb{R}_+$ , let  $s = s(t, \tau)$  be the number such that  $\tilde{\gamma}_y(s) \in [\tilde{\gamma}_x(t), \tilde{\gamma}_z(\tau)]$ . Clearly  $s \leq \max\{t, \tau\}$ , so  $t, \tau, s$  may be taken sufficiently small.



*Remark.* Note that for the Euclidean space the statement implies that central angle defines a metric on unit sphere. This statement is not quite trivial; moreover, it is straightforward to modify Euclidean proof so it will work in Alexandrov settings.

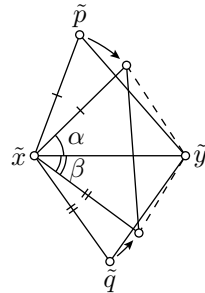
**Exercise 7.3.1;** “only if” part. Let us start with two model triangles  $[\tilde{x}\tilde{y}\tilde{p}] = \tilde{\Delta}(xyp)$  and  $[\tilde{x}\tilde{y}\tilde{q}] = \tilde{\Delta}(xyq)$  such that  $\tilde{p}$  and  $\tilde{q}$  lie on the opposite sides of the line  $\tilde{x}\tilde{y}$ .

Suppose  $[\tilde{x}\tilde{y}]$  intersects  $[\tilde{p}\tilde{q}]$  at a point  $\tilde{z}$ . In this case by CAT(0) comparison we have that

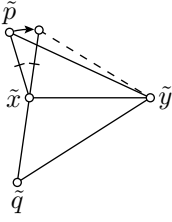
$$|\tilde{p} - \tilde{q}|_{\mathbb{E}^2} = |\tilde{p} - \tilde{z}|_{\mathbb{E}^2} + |\tilde{z} - \tilde{q}|_{\mathbb{E}^2} \leq |p - q|_{\mathcal{X}}.$$

Let us fix points  $\tilde{x}$  and  $\tilde{y}$ , and the distances from  $\tilde{x}$  to the remaining three points and reduce the angles  $\alpha = \angle[\tilde{x}\tilde{y}\tilde{p}]$  and  $\beta = \angle[\tilde{x}\tilde{y}\tilde{q}]$ . It results in decreasing distances  $|\tilde{p} - \tilde{q}|$ ,  $|\tilde{p} - \tilde{y}|$ , and  $|\tilde{q} - \tilde{y}|$ . If  $\alpha = \beta = 0$ , then

$$\begin{aligned} |\tilde{p} - \tilde{q}|_{\mathbb{E}^2} &= \left| |\tilde{x} - \tilde{p}|_{\mathbb{E}^2} - |\tilde{x} - \tilde{q}|_{\mathbb{E}^2} \right| = \\ &= \left| |x - p|_{\mathcal{X}} - |x - q|_{\mathcal{X}} \right| \geq \\ &\geq |p - q|_{\mathcal{X}}. \end{aligned}$$



By the intermediate value theorem, there are intermediate values of  $\alpha$  and  $\beta$  so that  $|\tilde{p} - \tilde{q}|_{\mathbb{E}^2} = |p - q|_{\mathcal{X}}$ . By construction,  $|\tilde{x} - \tilde{p}|_{\mathbb{E}^2} = |x - p|_{\mathcal{X}}$ ,  $|\tilde{x} - \tilde{q}|_{\mathbb{E}^2} = |x - q|_{\mathcal{X}}$ ,  $|\tilde{y} - \tilde{p}|_{\mathbb{E}^2} \leq |y - p|_{\mathcal{X}}$ ,  $|\tilde{y} - \tilde{q}|_{\mathbb{E}^2} \leq |y - q|_{\mathcal{X}}$ .



Now suppose  $[\tilde{p}\tilde{q}]$  does not intersect  $[\tilde{x}\tilde{y}]$ . Without loss of generality, we may assume that  $[\tilde{p}\tilde{q}]$  crosses the line  $\tilde{x}\tilde{y}$  behind  $\tilde{x}$ .

Let us rotate  $\tilde{p}$  around  $\tilde{x}$  so that  $\tilde{x}$  will lie between  $\tilde{p}$  and  $\tilde{q}$ . It will result in decreasing the distance  $|\tilde{p} - \tilde{y}|$ , by triangle inequality we have that

$$\begin{aligned} |\tilde{p} - \tilde{q}|_{\mathbb{E}^2} &= |\tilde{p} - \tilde{x}|_{\mathbb{E}^2} + |\tilde{x} - \tilde{q}|_{\mathbb{E}^2} = \\ &= |p - x|_{\mathcal{X}} + |x - q|_{\mathcal{X}} \geq \\ &\geq |p - q|_{\mathcal{X}}. \end{aligned}$$

Repeating the argument above produces the needed configuration.

*“If” part.* Suppose  $\tilde{p}, \tilde{q}, \tilde{x}, \tilde{y} \in \mathbb{E}^2$  satisfies the conditions

$$\begin{aligned} |\tilde{p} - \tilde{q}| &= |p - q|, & |\tilde{x} - \tilde{y}| &= |x - y|, \\ |\tilde{p} - \tilde{x}| &\leq |p - x|, & |\tilde{p} - \tilde{y}| &\leq |p - y|, \\ |\tilde{q} - \tilde{x}| &\leq |q - x|, & |\tilde{q} - \tilde{y}| &\leq |q - y|. \end{aligned}$$

Fix  $\tilde{z} \in [\tilde{x}\tilde{y}]$ . By triangle inequality

$$|\tilde{p} - \tilde{z}| + |\tilde{z} - \tilde{q}| \geq |\tilde{p} - \tilde{q}| = |p - q|.$$

Note that if  $|\tilde{p}' - \tilde{x}| \geq |\tilde{p} - \tilde{x}|$  and  $|\tilde{p}' - \tilde{y}| \geq |\tilde{p} - \tilde{y}|$ , then  $|\tilde{p}' - \tilde{z}| \geq |\tilde{p} - \tilde{z}|$ . In particular if  $[\tilde{x}\tilde{y}\tilde{p}'] = \triangle(xyp)$  and  $[\tilde{x}\tilde{y}\tilde{q}'] = \triangle(xyq)$ , then

$$|\tilde{p}' - \tilde{z}| + |\tilde{q}' - \tilde{z}| \geq |\tilde{p} - \tilde{z}| + |\tilde{z} - \tilde{q}|.$$

Whence the “if” part follows.

**Exercise 7.3.2.** Set  $\tilde{\alpha} = \tilde{\angle}(p_y^x)$ ,  $\tilde{\beta} = \tilde{\angle}(p_z^y)$  and  $\tilde{\gamma} = \tilde{\angle}(p_x^z)$ .

If  $\mathcal{X}$  is CBB(0), then

$$\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} \leq 2 \cdot \pi.$$

Note that we can find  $\alpha, \beta, \gamma$  such that

$$\tilde{\alpha} \leq \alpha \leq \pi, \quad \tilde{\beta} \leq \beta \leq \pi, \quad \tilde{\gamma} \leq \gamma \leq \pi,$$

and

$$\alpha + \beta + \gamma = 2 \cdot \pi.$$

Consider a model configuration  $\tilde{p}, \tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{E}^2$  such that

$$\begin{aligned} |\tilde{p} - \tilde{x}|_{\mathbb{E}^2} &= |p - x|_{\mathcal{X}}, & |\tilde{p} - \tilde{y}|_{\mathbb{E}^2} &= |p - y|_{\mathcal{X}}, & |\tilde{p} - \tilde{z}|_{\mathbb{E}^2} &= |p - z|_{\mathcal{X}}, \\ \angle[\tilde{p}\tilde{x}\tilde{y}] &= \alpha, & \angle[\tilde{p}\tilde{y}\tilde{z}] &= \beta, & \angle[\tilde{p}\tilde{z}\tilde{x}] &= \gamma. \end{aligned}$$

Since increasing angle in a triangle increase the opposite side, we have

$$|x - y|_{\mathcal{X}} \leq |\tilde{x} - \tilde{y}|_{\mathbb{E}^2}, \quad |y - z|_{\mathcal{X}} \leq |\tilde{y} - \tilde{z}|_{\mathbb{E}^2}, \quad |z - x|_{\mathcal{X}} \leq |\tilde{z} - \tilde{x}|_{\mathbb{E}^2}.$$

Whence the “only-if” part follows.

Now suppose that we have a model configuration  $\tilde{p}, \tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{E}^2$  such that

$$\begin{aligned} |p - x|_{\mathcal{X}} &= |\tilde{p} - \tilde{x}|_{\mathbb{E}^2}, & |p - y|_{\mathcal{X}} &= |\tilde{p} - \tilde{y}|_{\mathbb{E}^2}, & |p - z|_{\mathcal{X}} &= |\tilde{p} - \tilde{z}|_{\mathbb{E}^2}, \\ |x - y|_{\mathcal{X}} &\leq |\tilde{x} - \tilde{y}|_{\mathbb{E}^2}, & |y - z|_{\mathcal{X}} &\leq |\tilde{y} - \tilde{z}|_{\mathbb{E}^2}, & |z - x|_{\mathcal{X}} &\leq |\tilde{z} - \tilde{x}|_{\mathbb{E}^2}. \end{aligned}$$

Set

$$\alpha = \angle[\tilde{p} \tilde{x} \tilde{y}], \quad \beta = \angle[\tilde{p} \tilde{y} \tilde{z}], \quad \gamma = \angle[\tilde{p} \tilde{z} \tilde{x}].$$

Observe that

$$\alpha + \beta + \gamma \leq 2\pi.$$

Since increasing a side in a triangle increase the opposite angle, we have that

$$\tilde{\alpha} \leq \alpha, \quad \tilde{\beta} \leq \beta, \quad \tilde{\gamma} \leq \gamma.$$

Whence the “if” part follows.

**Exercise 7.3.3.** Set  $\tilde{\alpha} = \tilde{\angle}(p_q^x)$ ,  $\tilde{\beta} = \tilde{\angle}(p_q^y)$  and  $\tilde{\gamma} = \tilde{\angle}(p_q^x)$ .

Note that the quadruple  $p, x, y, z$  is euclidean if

$$\textcircled{2} \quad \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} \leq 2\pi$$

and the triple of numbers  $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$  satisfies all triangle inequalities. Without loss of generality we may assume that  $\tilde{\alpha} \leq \tilde{\beta} \leq \tilde{\gamma}$ ; in this case the triangle inequities hold if

$$\textcircled{3} \quad \tilde{\gamma} \leq \tilde{\alpha} + \tilde{\beta}.$$

Note that the inequality  $\textcircled{2}$  follow from CBB(0) comparison.

Consider two model triangles  $[\tilde{x}\tilde{y}\tilde{p}] = \tilde{\Delta}(xyp)$  and  $[\tilde{x}\tilde{y}\tilde{q}] = \tilde{\Delta}(xyq)$  such that  $\tilde{p}$  and  $\tilde{q}$  lie on the opposite sides of the line  $\tilde{x}\tilde{y}$ .

Suppose  $[\tilde{x}\tilde{y}]$  intersects  $[\tilde{p}\tilde{q}]$  at a point  $\tilde{z}$ . In this case by CAT(0) comparison we have that

$$|\tilde{x} - \tilde{y}|_{\mathbb{E}^2} = |\tilde{x} - \tilde{z}|_{\mathbb{E}^2} + |\tilde{z} - \tilde{y}|_{\mathbb{E}^2} \leq |x - y|_{\mathcal{X}}.$$

Which is equivalent to  $\textcircled{3}$ .

If  $[\tilde{x}\tilde{y}]$  crosses the line  $[\tilde{p}\tilde{q}]$  behind  $\tilde{p}$ , then  $\tilde{\alpha} + \tilde{\beta} > \pi$  and therefore  $\textcircled{3}$  follows from  $\textcircled{2}$ .

Finally if  $[\tilde{x}\tilde{y}]$  crosses the line  $[\tilde{p}\tilde{q}]$  behind  $\tilde{q}$ , then by CBB(0) comparison with center at  $q$ , we have that

$$\tilde{\Delta}(q_p^x) + \tilde{\Delta}(q_p^y) + \tilde{\Delta}(q_p^x) \leq 2 \cdot \pi$$

It follows that

$$|\tilde{x} - \tilde{y}|_{\mathbb{E}^2} \geq |x - y|_{\mathcal{X}}$$

and therefore

$$\tilde{\gamma} \leq \angle[\tilde{p}\tilde{x}\tilde{y}].$$

Since  $\angle[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\alpha} + \tilde{\beta}$  we get **9**.

**Exercise 7.4.2.** We will use the characterization of CBB(0) space provided by 7.3.2; the rest is nearly identical to the proof of 7.4.1.

Fix a quadruple in  $\mathcal{U} \times \mathcal{V}$ :

$$p = (p_1, p_2), \quad x = (x_1, x_2), \quad y = (y_1, y_2), \quad z = (z_1, z_2).$$

For the quadruple  $p_1, x_1, y_1, z_1$  in  $\mathcal{U}$ , construct model configurations  $\tilde{p}_1, \tilde{x}_1, \tilde{y}_1, \tilde{z}_1$  in  $\mathbb{E}^2$  provided by 7.3.2. Similarly, for the quadruple  $p_2, x_2, y_2, z_2$  in  $\mathcal{V}$  construct model configurations  $\tilde{p}_2, \tilde{x}_2, \tilde{y}_2, \tilde{z}_2$  in  $\mathbb{E}^2$ .

Consider four points in  $\mathbb{E}^4 = \mathbb{E}^2 \times \mathbb{E}^2$

$$\tilde{p} = (\tilde{p}_1, \tilde{p}_2), \quad \tilde{x} = (\tilde{x}_1, \tilde{x}_2), \quad \tilde{y} = (\tilde{y}_1, \tilde{y}_2), \quad \tilde{z} = (\tilde{z}_1, \tilde{z}_2).$$

The inequalities in 7.3.2 imply that

$$\begin{aligned} |p - x|_{\mathcal{X}} &= |\tilde{p} - \tilde{x}|_{\mathbb{E}^4}, & |p - y|_{\mathcal{X}} &= |\tilde{p} - \tilde{y}|_{\mathbb{E}^4}, & |p - z|_{\mathcal{X}} &= |\tilde{p} - \tilde{z}|_{\mathbb{E}^4}, \\ |x - y|_{\mathcal{X}} &\leq |\tilde{x} - \tilde{y}|_{\mathbb{E}^4}, & |y - z|_{\mathcal{X}} &\leq |\tilde{y} - \tilde{z}|_{\mathbb{E}^4}, & |z - x|_{\mathcal{X}} &\leq |\tilde{z} - \tilde{x}|_{\mathbb{E}^4} \end{aligned}$$

It remains to observe that one can move  $\tilde{z}$  into the plane of  $\tilde{p}, \tilde{x}$ , and  $\tilde{y}$  keeping the distance  $|\tilde{p} - \tilde{z}|_{\mathbb{E}^4}$  and nondecreasing the rest of distances.

**Exercise 7.5.2.** Suppose that there are distinct geodesics. Then there are two points  $p$  and  $q$  on different geodesics such that  $|p - x| = |q - x|$ . Without loss of generality we may assume that  $|z - x| < |p - x|$ ; in other words  $z$  lies between  $p$  and  $x$  on the first geodesic and  $z$  lies between  $q$  and  $x$  on the second geodesic. Observe that

$$\tilde{\Delta}(z_p^x) = \tilde{\Delta}(z_q^x) = \pi.$$

By comparison, we have

$$\tilde{\Delta}(z_p^x) + \tilde{\Delta}(z_q^x) + \tilde{\Delta}(z_q^p) \leq 2 \cdot \pi.$$

It follows that  $\tilde{\Delta}(z_q^p) = 0$ . Since  $|z - p| = |z - q|$ , it implies that  $p = q$  — a contradiction.



**Exercise 7.6.3.** Use 7.6.2a, to show that the map  $(t, x) \mapsto \gamma_x(t)$  is continuous; that is  $h_t(x) = \gamma_x(t)$  defines a homotopy.

It remains to observe that  $h_1(x) = x$  and  $h_0(x) = p$  for any  $x$ .

**Exercise 7.7.3.** Apply 7.6.2b twice.

More precisely, consider a triangle  $[xyz]$  in the space; let  $[\tilde{x}\tilde{y}\tilde{z}] = = \tilde{\Delta}(xyz)$ . Choose points  $p \in [xy]$  and  $q \in [xz]$ ; consider the corresponding points  $\tilde{p} \in [\tilde{x}\tilde{y}]$  and  $\tilde{q} \in [\tilde{x}\tilde{z}]$ . We need to show that

$$\textcircled{4} \quad |\tilde{p} - \tilde{q}|_{\mathbb{E}^2} \leq |p - q|_{\mathcal{X}}.$$

By 7.6.2b, we have

$$\tilde{\Delta}(x \begin{smallmatrix} p \\ q \end{smallmatrix}) \geq \tilde{\Delta}(x \begin{smallmatrix} y \\ z \end{smallmatrix}).$$

Whence  $\textcircled{4}$  follows.

**Exercise 7.7.4.** It is sufficient to prove the Jensen inequality; that is,

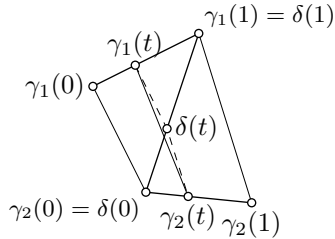
$$|\gamma_1(t) - \gamma_2(t)| \leq (1-t) \cdot |\gamma_1(0) - \gamma_2(0)| + t \cdot |\gamma_1(1) - \gamma_2(1)|.$$

Let  $\delta$  be the geodesic path from  $\gamma_2(0)$  to  $\gamma_1(1)$ . From 7.6.2a, we have

$$|\gamma_1(t) - \delta(t)| \leq (1-t) \cdot |\gamma_1(0) - \delta(0)|$$

$$|\delta(t) - \gamma_2(t)| \leq t \cdot |\delta(1) - \gamma_2(1)|$$

It remains to sum it up and apply the triangle inequality.



*Remark.* Note that in the Euclidean space the proof is just as hard.

**Exercise 7.7.5.** Let  $p, q \in A_r$ ; that is, there are points  $p^*, q^* \in A$  such that  $|p - p^*|, |q - q^*| \leq r$ . Consider a geodesic path  $\gamma$  from  $p$  to  $q$  and a geodesic path  $\gamma^*$  from  $p^*$  to  $q^*$ . Set  $f(t) = |\gamma(t) - \gamma^*(t)|$ .

Observe that  $f(0), f(1) \leq r$ . By 7.7.4,  $f$  is convex. Therefore  $f(t) \leq r$  for any  $t \in [0, 1]$ .

Since  $A$  is convex  $\gamma^*$  runs in  $A$ . Therefore  $f(t) \geq \text{dist}_A(\gamma(t))$ ; that is,  $\gamma$  runs in  $A_r$ .

**Exercise 7.7.6;** (a). Assume there are two point  $x, y \in K$  that minimize the distance to  $p$ ; suppose  $\ell = |p - x| = |p - y|$ . Since  $K$  is convex, the geodesic  $[xy]$  lies in  $K$ . Let  $m$  be a midpoint of  $[xy]$ .

Use thinness of  $[pxy]$  to show that  $|p - m| < \ell$ . It follows that  $x$  does not minimize the distance to  $p$  – a contradiction.

(b). Let  $p^*$  and  $q^*$  be the closest point projections of  $p$  and  $q$  to  $K$ . Assume all four points  $p, q, p^*, q^*$  are distinct. Consider two model

triangles  $[\tilde{p}\tilde{p}^*\tilde{q}^*] = \tilde{\Delta}(pp^*q^*)$  and  $[\tilde{p}\tilde{q}\tilde{q}^*] = \tilde{\Delta}(pqq^*)$  such that the points  $\tilde{p}^*$  and  $\tilde{q}$  lie on the opposite sides from the line  $\tilde{p}\tilde{q}^*$ .

Use thinness of  $[pp^*q^*]$  and  $[pqq^*]$  to show that  $\angle[\tilde{p}^* \frac{\tilde{p}}{\tilde{q}^*}] \geq \frac{\pi}{2}$  and  $\angle[\tilde{q}^* \frac{\tilde{p}}{\tilde{q}}] \geq \frac{\pi}{2}$ . Finally observe that

$$|p - q|_{\mathcal{U}} = |\tilde{p} - \tilde{q}|_{\mathbb{E}^2} \geq |\tilde{p}^* - \tilde{q}^*|_{\mathbb{E}^2} = |p^* - q^*|_{\mathcal{U}}.$$

If some of the points  $p, q, p^*, q^*$  coincide, then the proof is easier.

**Exercise 8.2.3.** If  $A$  is not convex, then there is a geodesic  $[xy]$  with the ends in  $A$  and the remaining points outside of  $A$ . Observe that in the doubling, say  $\mathcal{W}$ , two copies of this geodesics connect the same pair of points  $x$  and  $y$ . By 7.5.1,  $\mathcal{W}$  is not CAT(0).

**Exercise 8.3.7.** By approximation, it is sufficient to consider the case when  $A$  and  $B$  have smooth boundary.

If  $[xy] \cap A \cap B \neq \emptyset$ , then  $z_0 \in [xy]$  and  $\dot{A}, \dot{B}$  can be chosen to be arbitrary half-spaces containing  $A$  and  $B$  respectively.

In the remaining case  $[xy] \cap A \cap B = \emptyset$ , we have  $z_0 \in \partial(A \cap B)$ . Consider the solid ellipsoid

$$C = \{z \in \mathbb{E}^m : f(z) \leq f(z_0)\}.$$

Note that  $C$  is compact, convex and has smooth boundary.

Suppose  $z_0 \in \partial A \cap \text{Int } B$ . Then  $A$  and  $C$  touch at  $z_0$  and we can set  $\dot{A}$  to be the uniquely defined supporting half-space to  $A$  at  $z_0$  and  $\dot{B}$  to be any half-space containing  $B$ . The case  $z_0 \in \partial B \cap \text{Int } A$  is treated similarly.

Finally, suppose  $z_0 \in \partial A \cap \partial B$ . Then the set  $\dot{A}$  (respectively,  $\dot{B}$ ) is defined as the unique supporting half-space to  $A$  (respectively,  $B$ ) at  $z_0$  containing  $A$  (respectively,  $B$ ).

Suppose  $f(z) < f(z_0)$  for some  $z \in \dot{A} \cap \dot{B}$ . Since  $f$  is concave,  $f(\bar{z}) < f(z_0)$  for any  $\bar{z} \in [zz_0[$ . Since  $[zz_0[ \cap A \cap B \neq \emptyset$ , the latter contradicts the fact that  $z_0$  is minimum point of  $f$  on  $A \cap B$ .

**Exercise 8.4.1.** Fix two open balls  $B_1 = B(0, r_1)$  and  $B_2 = B(0, r_2)$  such that

$$B_1 \subset A^i \subset B_2$$

for each wall  $A^i$ .

Suppose  $X$  is an intersections of the walls. Observe that

$$B_1 \subset X \subset B_2.$$

Therefore if  $x \in X$ , then  $X$  contains the convex hull  $\text{Conv}(B_1 \cup \{x\})$ ; therefore all intersections of the walls have  $\varepsilon$ -wide corners for  $\varepsilon = 2 \cdot \arcsin \frac{r_1}{r_2}$ .

**Exercise 8.4.2.** Note that any centrally symmetric convex closed set in Euclidean space is a product of a compact centrally symmetric convex set and a subspace.

It follows that there is  $R < \infty$  such that if  $X$  is an intersection of an arbitrary number of walls, then for any point  $p \in X$  there is an isometry of  $X$  that moves  $p$  to a point in the ball  $B(0, R)$ .

It remains to apply the argument in Exercise 8.4.1.

**Exercise 8.5.3.** Note that we can assume that the balls have zero radiuses.

Observe that at each collision the balls exchange their velocities. Let us also change their labels at each collision. Note that after the relabeling, the coordinates functions  $t \mapsto x_i(t)$  of the balls are linear functions in time.<sup>1</sup>

It remains to show  $n$  lines on the plane have at most  $\frac{n \cdot (n-1)}{2}$  intersections. It follows since any pair of lines have at most one intersection.

*Remark.* Recall that in the 3-dimensional case the number of collisions grows exponentially in  $n$ ; the two-dimensional case is open [19].

**Exercise 9.2.5.** Note that the existence of a null-homotopy is equivalent to the following. There are two one-parameter families of paths  $\alpha_\tau$  and  $\beta_\tau$ ,  $\tau \in [0, 1]$  such that:

- ◊  $\text{length } \alpha_\tau, \text{length } \beta_\tau < \pi$  for any  $\tau$ .
- ◊  $\alpha_\tau(0) = \beta_\tau(0)$  and  $\alpha_\tau(1) = \beta_\tau(1)$  for any  $\tau$ .
- ◊  $\alpha_0(t) = \beta_0(t)$  for any  $t$ .
- ◊  $\alpha_1(t) = \alpha(t)$  and  $\beta_1(t) = \beta(t)$  for any  $t$ .

By Corollary 9.2.3, the construction in Corollary 9.2.4 produces the same result for  $\alpha_\tau$  and  $\beta_\tau$ . Hence the result.

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<sup>1</sup>We use here that radiuses vanish, otherwise  $\tilde{x}_i = x_i - 2 \cdot k_i \cdot r$  are linear, where  $k_i$  is the number of  $i$ -th ball counted from left.



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