# Pure metric geometry: introductory lectures

Anton Petrunin

We discuss only domestic affairs of metric spaces; applications are given only as illustrations.

These notes are based on the introductory part of my course in metric geometry given at Penn State, Spring 2020. The complete lectures can be found on the author's website; it includes an introduction to Alexandrov geometry based on [1] and metric geometry on manifolds [34] based on a simplified proof of Gromov's systolic inequality given by Alexander Nabutovsky [28].

A part of the text is a compilation from [1, 2, 30, 32, 33] and its drafts.

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# Contents

# Lecture 1

# Definitions

In this lecture we make some conventions used further and remind some definitions related to metric spaces.

We assume some prior knowledge of metric spaces. For a more detailed introduction, we recommend the first couple of chapters in the book by Dmitri Burago, Yuri Burago, and Sergei Ivanov [8].

## A Metric spaces

The distance between two points $x$ and $y$ in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

Let us recall the definition of metric.

**1.1. Definition.** *A metric on a set $\mathcal{X}$ is a real-valued function $(x,y)\mapsto|x-y|_{\mathcal{X}}$ that satisfies the following conditions for any three points $x,y,z\in\mathcal{X}$:*

*(a)* $|x-y|_{\mathcal{X}}\geqslant 0$,
*(b)* $|x-y|_{\mathcal{X}}=0 \iff x=y$,
*(c)* $|x-y|_{\mathcal{X}}=|y-x|_{\mathcal{X}}$,
*(d)* $|x-y|_{\mathcal{X}}+|y-z|_{\mathcal{X}}\geqslant|x-z|_{\mathcal{X}}$.

Recall that a metric space is a set with a metric on it. The elements of the set are called points. Most of the time we keep the same notation for the metric space and its underlying set, but if needed we may use $\underline{\mathcal{X}}$ for the underlying set of metric space $\mathcal{X}$.

The function

$$\operatorname{dist}_x\colon y\mapsto|x-y|$$

is called the distance function from $x$.

Given $R \in [0, \infty]$ and $x \in \mathcal{X}$, the sets

$$\mathrm{B}(x, R) = \{y \in \mathcal{X} \mid |x - y| < R\},$$
$$\overline{\mathrm{B}}[x, R] = \{y \in \mathcal{X} \mid |x - y| \leqslant R\}$$

are called, respectively, the open and the closed balls of radius $R$ with center $x$. If we need to emphasize that these balls are taken in the metric space $\mathcal{X}$, we write

$$\mathrm{B}(x, R)_{\mathcal{X}} \quad \text{and} \quad \overline{\mathrm{B}}[x, R]_{\mathcal{X}}.$$

**1.2. Exercise.** *Show that the following inequality*

$$|p - q|_{\mathcal{X}} + |x - y|_{\mathcal{X}} \leqslant |p - x|_{\mathcal{X}} + |p - y|_{\mathcal{X}} + |q - x|_{\mathcal{X}} + |q - y|_{\mathcal{X}}$$

*holds for any four points $p$, $q$, $x$, and $y$ in a metric space $\mathcal{X}$.*

**1.3. Exercise.** *Let $A$ and $B$ be two disjoint closed sets in a metric space $\mathcal{X}$. Construct a continuous function $f \colon \mathcal{X} \to [0, 1]$ such that $A = f^{-1}\{0\}$ and $B = f^{-1}\{1\}$.*

**1.4. Advanced exercise.** *Let $f \colon A \to \mathbb{R}$ be a continuous function defined on a closed set $A$ in a metric space $\mathcal{X}$. Show that it admits a continuous extension to the whole space; that is, there is a continuous function $F \colon \mathcal{X} \to \mathbb{R}$ such that $F(a) = f(a)$ for any $a \in A$.*

## B Variations

**Pseudometris.** A metric for which the distance between two distinct points can be zero is called a pseudometric. In other words, to define pseudometric, we need to remove condition *(b)* from 1.1.

Assume $\mathcal{X}$ is a pseudometric space. Consider an equivalence relation $\sim$ on $\mathcal{X}$ defined by

$$x \sim y \quad \Longleftrightarrow \quad |x - y| = 0.$$

Note that if $x \sim x'$, then $|y - x| = |y - x'|$ for any $y \in \mathcal{X}$. Thus, $|* - *|$ defines a metric on the quotient set $\mathcal{X}/{\sim}$. This way we obtain a metric space $\mathcal{X}'$. The space $\mathcal{X}'$ is called the corresponding metric space for the pseudometric space $\mathcal{X}$. Often we do not distinguish between $\mathcal{X}'$ and $\mathcal{X}$.

This construction shows that nearly any question about pseudometric spaces can be reduced to a question about genuine metric spaces.

**$\infty$-metrics.** One may also consider metrics with values in $[0,\infty]$; that is, we allow infinite distance between points. We might call them $\infty$-metrics, but most of the time we use the term metric.

Again nearly any question about $\infty$-metric spaces can be reduced to a question about genuine metric spaces.

Set

$$x \approx y \qquad \Longleftrightarrow \qquad |x-y| < \infty;$$

it defines another equivalence relation on $\mathcal{X}$. The equivalence class of a point $x \in \mathcal{X}$ will be called the metric component of $x$; it will be denoted by $\mathcal{X}_x$. One could think of $\mathcal{X}_x$ as $\mathrm{B}(x,\infty)_{\mathcal{X}}$ — the open ball centered at $x$ and radius $\infty$ in $\mathcal{X}$.

It follows that any $\infty$-metric space is a disjoint union of genuine metric spaces — the metric components of the original $\infty$-metric space.

**1.5. Exercise.** *Given two sets $A$ and $B$ on the plane, set*

$$|A-B| = \mu(A \vartriangle B),$$

*where $\mu$ denotes the Lebesgue measure and $\vartriangle$ denotes symmetric difference*

$$A \vartriangle B := (A \cup B) \setminus (B \cap A) = (A \setminus B) \cup (B \setminus A).$$

(a) *Show that $|{*}-{*}|$ is a pseudometric on the set of bounded closed subsets.*
(b) *Show that $|{*}-{*}|$ is an $\infty$-metric on the set of all open subsets.*

## C Completeness

A metric space $\mathcal{X}$ is called complete if every Cauchy sequence of points in $\mathcal{X}$ converges in $\mathcal{X}$.

**1.6. Exercise.** *Suppose that $\rho$ is a positive continuous function on a complete metric space $\mathcal{X}$ and $\varepsilon > 0$. Show that there is a point $x \in \mathcal{X}$ such that*

$$\rho(x) < (1+\varepsilon)\cdot\rho(y)$$

*for any point $y \in \mathrm{B}(x, \rho(x))$.*

Most of the time we will assume that a metric space is complete. The following construction produces a complete metric space $\bar{\mathcal{X}}$ for any given metric space $\mathcal{X}$.

**Completion.** Given a metric space $\mathcal{X}$, consider the set $\mathcal{C}$ of all Cauchy sequences in $\mathcal{X}$. Note that for any two Cauchy sequences $(x_n)$ and $(y_n)$

the right-hand side in ❶ is defined; moreover, it defines a pseudometric on $\mathcal{C}$

❶ $$|(x_n)-(y_n)|_{\mathcal{C}} := \lim_{n\to\infty} |x_n-y_n|_{\mathcal{X}}.$$

The corresponding metric space is called the completion of $\mathcal{X}$; it will be denoted by $\bar{\mathcal{X}}$.

For each point $x\in\mathcal{X}$, one can consider a constant sequence $x_n=x$ which is Cauchy. It defines a natural inclusion map $\mathcal{X}\hookrightarrow\bar{\mathcal{X}}$. It is easy to check that this map is distance-preserving. In particular, we can (and will) consider $\mathcal{X}$ as a subset of $\bar{\mathcal{X}}$; note that $\mathcal{X}$ is a dense subset in its completion $\bar{\mathcal{X}}$

**1.7. Exercise.** *Show that the completion of a metric space is complete.*

# D Compact spaces

Let us recall a few statements about compact metric spaces.

**1.8. Definition.** *A metric space $\mathcal{K}$ is compact if and only if one of the following equivalent conditions holds:*

*(a) Every open cover of $\mathcal{K}$ has a finite subcover.*

*(b) Every sequence of points in $\mathcal{K}$ has a subsequence that converges in $\mathcal{K}$.*

*(c) The space $\mathcal{K}$ is complete and totally bounded; that is, for any $\varepsilon>0$, the space $\mathcal{K}$ admits a finite cover by open $\varepsilon$-balls.*

**1.9. Lebesgue lemma.** *Let $\mathcal{K}$ be a compact metric space. Then for any open cover of $\mathcal{K}$ there is $\varepsilon>0$ such that any $\varepsilon$-ball in $\mathcal{K}$ lies in one element of the cover.*

*The value $\varepsilon$ is called a Lebesgue number of the covering.*

A subset $N$ of a metric space $\mathcal{K}$ is called $\varepsilon$-net if any point $x\in\mathcal{K}$ lies at the distance less than $\varepsilon$ from a point in $N$. More generally, a subset $N$ is called $\varepsilon$-net of a subset $S\subset\mathcal{K}$ if any point $x\in S$ lies at the distance less than $\varepsilon$ from a point in $N$.

Note that totally bounded spaces can be defined as spaces that admit a finite $\varepsilon$-net for any $\varepsilon>0$.

**1.10. Exercise.** *Show that a space $\mathcal{K}$ is totally bounded if and only if it contains a compact $\varepsilon$-net for any $\varepsilon>0$.*

Let $\operatorname{pack}_\varepsilon\mathcal{X}$ be the exact upper bound on the number of points $x_1,\dots,x_n\in\mathcal{X}$ such that $|x_i-x_j|\geqslant\varepsilon$ if $i\neq j$.

If $n = \operatorname{pack}_\varepsilon \mathcal{X} < \infty$, then the collection of points $x_1, \dots, x_n$ is called a maximal $\varepsilon$-packing. If $\mathcal{X}$ is a length space (see Section 1I) then $n$ is the maximal number of disjoint open $\frac{\varepsilon}{2}$-balls in $\mathcal{X}$.

**1.11. Exercise.** *Show that any maximal $\varepsilon$-packing is an $\varepsilon$-net. Conclude that a complete space $\mathcal{X}$ is compact if and only if* $\operatorname{pack}_\varepsilon \mathcal{X} < \infty$ *for any $\varepsilon > 0$.*

**1.12. Exercise.** *Let $\mathcal{K}$ be a compact metric space and*

$$f\colon \mathcal{K} \to \mathcal{K}$$

*be a distance-noncontracting map. Prove that $f$ is an isometry; that is, $f$ is a distance-preserving bijection.*

A metric space $\mathcal{X}$ is called locally compact if any point in $\mathcal{X}$ admits a compact neighborhood; equivalently, for any point $x \in \mathcal{X}$ a closed ball $\overline{\mathrm{B}}[x, r]$ is compact for some $r > 0$.

# E Proper spaces

A metric space $\mathcal{X}$ is called proper if all closed bounded sets in $\mathcal{X}$ are compact. It is straightforward to check that this condition is equivalent to each of the following statements:

- ◇ For some (and therefore any) point $p \in \mathcal{X}$ and any $R < \infty$, the closed ball $\overline{\mathrm{B}}[p, R]_{\mathcal{X}}$ is compact.
- ◇ The function $\operatorname{dist}_p\colon \mathcal{X} \to \mathbb{R}$ is proper for some (and therefore any) point $p \in \mathcal{X}$. (Recall that a function $f\colon \mathcal{X} \to \mathbb{R}$ is proper if for any compact set $K \subset \mathbb{R}$, its inverse image $f^{-1}(K)$ is compact.)

**1.13. Exercise.** *Give an example of a metric space that is locally compact but not proper.*

# F Geodesics

Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ a real interval. A distance-preserving map $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a geodesic[1]; in other words, $\gamma\colon \mathbb{I} \to \mathcal{X}$ is a geodesic if

$$|\gamma(s) - \gamma(t)|_{\mathcal{X}} = |s - t|$$

for any pair $s, t \in \mathbb{I}$.

[1]Others call it differently: shortest path, minimizing geodesic. Also, note that the meaning of the term geodesic is different from what is used in Riemannian geometry, altho they are closely related.

If $\gamma\colon [a,b] \to \mathcal{X}$ is a geodesic and $p = \gamma(a)$, $q = \gamma(b)$, then we say that $\gamma$ is a geodesic from $p$ to $q$. In this case, the image of $\gamma$ is denoted by $[pq]$, and, with abuse of notations, we also call it a geodesic. We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$.

In general, a geodesic from $p$ to $q$ need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we assume that we have chosen such geodesic.

A geodesic path is a geodesic with constant-speed parameterization by the unit interval $[0,1]$.

A metric space is called geodesic if any pair of its points can be joined by a geodesic.

An $\infty$-metric space $\mathcal{X}$ is called geodesic if each metric component of $\mathcal{X}$ is geodesic.

**1.14. Exercise.** *Let $f$ be a centrally symmetric positive continuous function on $\mathbb{S}^2$. Given two points $x, y \in \mathbb{S}^2$, set*

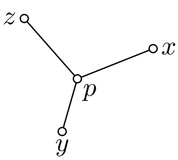
$$\|x - y\| = \int\limits_{B(x,\frac{\pi}{2})\backslash B(y,\frac{\pi}{2})} f.$$

*Show that $(\mathbb{S}^2, \|{*} - {*}\|)$ is a geodesic space, and the geodesics in $(\mathbb{S}^2, \|{*} - {*}\|)$ run along great circles of $\mathbb{S}^2$.*

## G Metric trees

A geodesic space $\mathcal{T}$ is called a metric tree if any pair of points in $\mathcal{T}$ are connected by a unique geodesic, and the union of any two geodesics $[xy]_{\mathcal{T}}$, and $[yz]_{\mathcal{T}}$ contain the geodesic $[xz]_{\mathcal{T}}$.

The latter means that any triangle in $\mathcal{T}$ is a tripod; that is, for any three points $x$, $y$, and $z$ there is a point $p$ such that




$$[xy] \cup [yz] \cup [zx] = [px] \cup [py] \cup [pz].$$

**1.15. Exercise.** *Let $p$, $x$, $y$, and $z$ be points in a metric tree.*

*(a) Consider three numbers*

$$a = |p - x| + |y - z|, \quad b = |p - y| + |z - x|, \quad c = |p - z| + |x - y|.$$

*Suppose that $a \leqslant b \leqslant c$. Show that $b = c$.*

*(b) Consider three numbers*

$$\begin{aligned}\alpha &= \tfrac{1}{2}\cdot(|p - y| + |p - z| - |y - z|),\\ \beta &= \tfrac{1}{2}\cdot(|p - x| + |p - z| - |x - z|),\\ \gamma &= \tfrac{1}{2}\cdot(|p - x| + |p - y| - |x - y|).\end{aligned}$$

*Suppose that $\alpha \leqslant \beta \leqslant \gamma$. Show that $\alpha = \beta$.*

The set

$$S(p,r)_{\mathcal{X}} = \{\, x \in \mathcal{X} \;:\; |p-x|_{\mathcal{X}} = r \,\}$$

will be called a s p h e r e with center $p$ and radius $r$ in a metric space $\mathcal{X}$.

**1.16. Exercise.** *Show that spheres in metric trees are ultrametric spaces. That is,*

$$|x-z| \leqslant \max\{\, |x-y|, |y-z| \,\}$$

*for any $x, y, z \in S(p,r)_{\mathcal{T}}$.*

# H Length

A c u r v e is defined as a continuous map from a real interval $\mathbb{I}$ to a metric space. If $\mathbb{I} = [0,1]$, then the curve is called a p a t h.

**1.17. Definition.** *Let $\mathcal{X}$ be a metric space and $\alpha \colon \mathbb{I} \to \mathcal{X}$ be a curve. We define the l e n g t h of $\alpha$ as*

$$\operatorname{length}\alpha := \sup_{t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n} \sum_i |\alpha(t_i) - \alpha(t_{i-1})|.$$

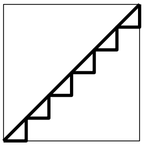
*A curve $\alpha$ is called r e c t i f i a b l e if* $\operatorname{length}\alpha < \infty$.

**1.18. Theorem.** *Length is a lower semi-continuous with respect to the pointwise convergence of curves.*

*More precisely, assume that a sequence of curves $\gamma_n \colon \mathbb{I} \to \mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty \colon \mathbb{I} \to \mathcal{X}$; that is, for any fixed $t \in \mathbb{I}$ we have $\gamma_n(t) \to \gamma_\infty(t)$ as $n \to \infty$. Then*

❶
$$\varliminf_{n\to\infty} \operatorname{length}\gamma_n \geqslant \operatorname{length}\gamma_\infty.$$

Note that the inequality ❶ might be strict. For example, the diagonal $\gamma_\infty$ of the unit square can be approximated by stairs-like polygonal curves $\gamma_n$ with sides parallel to the sides of the square ($\gamma_6$ is on the picture). In this case



$$\operatorname{length}\gamma_\infty = \sqrt{2} \quad \text{and} \quad \operatorname{length}\gamma_n = 2$$

for any $n$.

*Proof.* Fix a sequence $t_0 \leqslant t_1 \leqslant \ldots \leqslant t_k$ in $\mathbb{I}$. Set

$$\Sigma_n := |\gamma_n(t_0) - \gamma_n(t_1)| + \cdots + |\gamma_n(t_{k-1}) - \gamma_n(t_k)|.$$
$$\Sigma_\infty := |\gamma_\infty(t_0) - \gamma_\infty(t_1)| + \cdots + |\gamma_\infty(t_{k-1}) - \gamma_\infty(t_k)|.$$

Note that for each $i$ we have

$$|\gamma_n(t_{i-1}) - \gamma_n(t_i)| \to |\gamma_\infty(t_{i-1}) - \gamma_\infty(t_i)|$$

and therefore

$$\Sigma_n \to \Sigma_\infty$$

as $n \to \infty$. Note that

$$\Sigma_n \leqslant \operatorname{length} \gamma_n$$

for each $n$. Hence,

$$\varliminf_{n\to\infty} \operatorname{length} \gamma_n \geqslant \Sigma_\infty.$$

Since the partition was arbitrary, by the definition of length, the inequality ❶ is obtained. □

# I Length spaces

Let $\mathcal{X}$ be a metric space. If for any $\varepsilon > 0$ and any pair of points $x, y \in \mathcal{X}$, there is a path $\alpha$ connecting $x$ to $y$ such that

$$\operatorname{length} \alpha < |x - y| + \varepsilon,$$

then $\mathcal{X}$ is called a length space and the metric on $\mathcal{X}$ is called a length metric.

An $\infty$-metric space is a length space if each of its metric components is a length space. In other words, if $\mathcal{X}$ is an $\infty$-metric space, then in the above definition we assume in addition that $|x-y|_{\mathcal{X}} < \infty$.

Note that any geodesic space is a length space. The following example shows that the converse does not hold.

**1.19. Example.** *Suppose a space $\mathcal{X}$ is obtained by gluing a countable collection of disjoint intervals $\{\mathbb{I}_n\}$ of length $1 + \frac{1}{n}$, where for each $\mathbb{I}_n$ the left end is glued to $p$ and the right end to $q$.*

*Observe that the space $\mathcal{X}$ carries a natural complete length metric with respect to which $|p - q|_{\mathcal{X}} = 1$ but there is no geodesic connecting $p$ to $q$.*

**1.20. Exercise.** *Give an example of a complete length space $\mathcal{X}$ such that no pair of distinct points in $\mathcal{X}$ can be joined by a geodesic.*

Directly from the definition, it follows that if $\alpha\colon [0,1] \to \mathcal{X}$ is a path from $x$ to $y$ (that is, $\alpha(0) = x$ and $\alpha(1) = y$), then

$$\operatorname{length}\alpha \geqslant |x - y|.$$

Set

$$\|x - y\| = \inf\{\operatorname{length}\alpha\}$$

where the greatest lower bound is taken for all paths from $x$ to $y$. It is straightforward to check that $(x, y) \mapsto \|x - y\|$ is an $\infty$-metric; moreover, $(\mathcal{X}, \|{*} - {*}\|)$ is a length space. The metric $\|{*} - {*}\|$ is called the induced length metric.

**1.21. Exercise.** *Let $\mathcal{X}$ be a complete length space. Show that for any compact subset $K$ in $\mathcal{X}$ there is a compact path-connected subset $K'$ that contains $K$.*

**1.22. Exercise.** *Suppose $(\mathcal{X}, |{*}-{*}|)$ is a complete metric space. Show that $(\mathcal{X}, \|{*} - {*}\|)$ is complete.*

Let $A$ be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x - y|_A = \inf_\alpha\{\operatorname{length}\alpha\},$$

where the greatest lower bound is taken for all paths $\alpha$ from $x$ to $y$ in $A$. In other words $|{*} - {*}|_A$ denotes the induced length metric on the subspace $A$.[2]

Let $x$ and $y$ be points in a metric space $\mathcal{X}$.

(i) A point $z \in \mathcal{X}$ is called a midpoint between $x$ and $y$ if

$$|x - z| = |y - z| = \tfrac{1}{2}{\cdot}|x - y|.$$

(ii) Assume $\varepsilon \geqslant 0$. A point $z \in \mathcal{X}$ is called an $\varepsilon$-midpoint between $x$ and $y$ if

$$|x - z|, \quad |y - z| \leqslant \tfrac{1}{2}{\cdot}|x - y| + \varepsilon.$$

Note that a 0-midpoint is the same as a midpoint.

**1.23. Lemma.** *Let $\mathcal{X}$ be a complete metric space.*

*(a) Assume that for any pair of points $x, y \in \mathcal{X}$, and any $\varepsilon > 0$, there is an $\varepsilon$-midpoint $z$. Then $\mathcal{X}$ is a length space.*

*(b) Assume that for any pair of points $x, y \in \mathcal{X}$, there is a midpoint $z$. Then $\mathcal{X}$ is a geodesic space.*

[2] The notation $|{*}-{*}|_A$ conflicts with the previously defined notation for distance $|x-y|_{\mathcal{X}}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.

*Proof.* We first prove *(a)*. Let $x, y \in \mathcal{X}$ be a pair of points.

Set $\varepsilon_n = \frac{\varepsilon}{4^n}$, $\alpha(0) = x$ and $\alpha(1) = y$.

Let $\alpha(\frac{1}{2})$ be an $\varepsilon_1$-midpoint between $\alpha(0)$ and $\alpha(1)$. Further, let $\alpha(\frac{1}{4})$ and $\alpha(\frac{3}{4})$ be $\varepsilon_2$-midpoints between the pairs $(\alpha(0), \alpha(\frac{1}{2}))$ and $(\alpha(\frac{1}{2}), \alpha(1))$ respectively. Applying the above procedure recursively, on the $n$-th step we define $\alpha(\frac{k}{2^n})$, for every odd integer $k$ such that $0 < \frac{k}{2^n} < 1$, as an $\varepsilon_n$-midpoint of the already defined $\alpha(\frac{k-1}{2^n})$ and $\alpha(\frac{k+1}{2^n})$.

This way we define $\alpha(t)$ for $t \in W$, where $W$ denotes the set of dyadic rationals in $[0,1]$. Moreover, $\alpha\colon W \to \mathcal{X}$ is Lipschitz (it has Lipschitz constant $1 + \frac{\varepsilon}{|x-y|}$). Since $\mathcal{X}$ is complete, the map $\alpha$ can be extended continuously to $[0,1]$. Moreover,

❶
$$\begin{aligned}\operatorname{length}\alpha &\leqslant |x-y| + \sum_{n=1}^{\infty} 2^{n-1}\cdot\varepsilon_n \leqslant \\ &\leqslant |x-y| + \tfrac{\varepsilon}{2}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get *(a)*.

To prove *(b)*, one should repeat the same argument taking midpoints instead of $\varepsilon_n$-midpoints. In this case, ❶ holds for $\varepsilon_n = \varepsilon = 0$. □

Since in a compact space a sequence of $\frac{1}{n}$-midpoints $z_n$ contains a convergent subsequence, 1.23 immediately implies the following.

**1.24. Proposition.** *Any proper length space is geodesic.*

**1.25. Hopf–Rinow theorem.** *Any complete, locally compact length space is proper.*

Before reading the proof, it is instructive to solve 1.13.

*Proof.* Let $\mathcal{X}$ be a locally compact length space. Given $x \in \mathcal{X}$, denote by $\rho(x)$ the least upper bound of all $R > 0$ such that the closed ball $\overline{\mathrm{B}}[x, R]$ is compact. Since $\mathcal{X}$ is locally compact,

❷
$$\rho(x) > 0 \quad \text{for any} \quad x \in \mathcal{X}.$$

It is sufficient to show that $\rho(x) = \infty$ for some (and therefore any) point $x \in \mathcal{X}$.

❸ *If $\rho(x) < \infty$, then $B = \overline{\mathrm{B}}[x, \rho(x)]$ is compact.*

Indeed, $\mathcal{X}$ is a length space; therefore for any $\varepsilon > 0$, the set $\overline{\mathrm{B}}[x, \rho(x) - \varepsilon]$ is a compact $\varepsilon$-net in $B$. Since $B$ is closed and hence complete, it must be compact; see 1.8*c* and 1.10. △

❹ *$|\rho(x)-\rho(y)| \leqslant |x-y|_{\mathcal{X}}$ for any $x, y \in \mathcal{X}$; in particular, $\rho\colon \mathcal{X} \to \mathbb{R}$ is a continuous function.*

Indeed, assume the contrary; that is, $\rho(x) + |x-y| < \rho(y)$ for some $x, y \in \mathcal{X}$. Then $\overline{\mathrm{B}}[x, \rho(x)+\varepsilon]$ is a closed subset of $\overline{\mathrm{B}}[y, \rho(y)]$ for some $\varepsilon > 0$. Then compactness of $\overline{\mathrm{B}}[y, \rho(y)]$ implies compactness of $\overline{\mathrm{B}}[x, \rho(x)+\varepsilon]$ — a contradiction. △

Set $\varepsilon = \min\{\,\rho(y) : y \in B\,\}$; the minimum is defined since $B$ is compact and $\rho$ is continuous. From ❷, we have $\varepsilon > 0$.

Choose a finite $\frac{\varepsilon}{10}$-net $\{a_1, a_2, \dots, a_n\}$ in $B = \overline{\mathrm{B}}[x, \rho(x)]$. The union $W$ of the closed balls $\overline{\mathrm{B}}[a_i, \varepsilon]$ is compact. Clearly $\overline{\mathrm{B}}[x, \rho(x)+\frac{\varepsilon}{10}] \subset W$. Therefore, $\overline{\mathrm{B}}[x, \rho(x)+\frac{\varepsilon}{10}]$ is compact, a contradiction. □

**1.26. Exercise.** *Construct a geodesic space $\mathcal{X}$ that is locally compact, but whose completion $\bar{\mathcal{X}}$ is neither geodesic nor locally compact.*

**1.27. Advanced exercise.** *Show that for any compact connected space $\mathcal{X}$ there is a number $\ell$ such that for any finite collection of points there is a point $z$ that lies on average distance $\ell$ from the collection; that is, for any $x_1, \dots, x_n \in \mathcal{X}$ there is $z \in \mathcal{X}$ such that*

$$\tfrac{1}{n}\cdot\sum_i |x_i - z|_{\mathcal{X}} = \ell.$$

# Lecture 2

# Universal spaces

## A Embedding in a normed space

Recall that a function $v \mapsto |v|$ on a vector space $\mathcal{V}$ is called norm if it satisfies the following condition for any two vectors $v, w \in \mathcal{V}$ and a scalar $\alpha$:

- ◇ $|v| \geqslant 0$;
- ◇ $|\alpha \cdot v| = |\alpha| \cdot |v|$;
- ◇ $|v| + |w| \geqslant |v + w|$.

As an example, consider $\ell^\infty$ — the space of real sequences equipped with sup-norm; that is, the norm of $\boldsymbol{a} = a_1, a_2, \dots$ is defined by

$$|\boldsymbol{a}|_{\ell^\infty} = \sup_n \{ |a_n| \}.$$

It is straightforward to check that for any normed space the function $(v, w) \mapsto |v - w|$ defines a metric on it. Therefore, any normed space is an example of metric space (in fact, it is a geodesic space). Often we do not distinguish between normed space and the corresponding metric space. (In fact by Mazur–Ulam theorem, the metric remembers the affine structure of the space; so to recover the original normed space we only need to specify the origin. A slick proof of Mazur–Ulam theorem was given by Jussi Väisälä [41].)

Now let us show that reasonable metric spaces are isometric to subsets of $\ell^\infty$.

Recall that diameter of a metric space $\mathcal{X}$ (briefly $\operatorname{diam} \mathcal{X}$) is defined as the least upper bound on the distances between pairs of its points; that is,

$$\operatorname{diam} \mathcal{X} = \sup \{ |x - y|_{\mathcal{X}} \, : \, x, y \in \mathcal{X} \}.$$

**2.1. Lemma.** *Suppose $\mathcal{X}$ is a* b o u n d e d s e p a r a b l e *metric space; that is,* $\operatorname{diam}\mathcal{X}$ *is finite and $\mathcal{X}$ contains a countable, dense set $\{w_n\}$. Given $x \in \mathcal{X}$, set $a_n(x) = |w_n - x|_{\mathcal{X}}$. Then*

$$\iota\colon x \mapsto (a_1(x), a_2(x), \dots)$$

*defines a distance-preserving embedding $\iota\colon \mathcal{X} \hookrightarrow \ell^\infty$.*

*Proof.* By the triangle inequality

❶ $$|a_n(x) - a_n(y)| \leqslant |x - y|_{\mathcal{X}}.$$

Therefore, $\iota$ is s h o r t (in other words, $\iota$ is distance-noncontracting).

Again by triangle inequality we have

$$|a_n(x) - a_n(y)| \geqslant |x - y|_{\mathcal{X}} - 2{\cdot}|w_n - x|_{\mathcal{X}}.$$

Since the set $\{w_n\}$ is dense, we can choose $w_n$ arbitrarily close to $x$. Whence the value $|a_n(x) - a_n(y)|$ can be chosen arbitrarily close to $|x - y|_{\mathcal{X}}$. In other words

$$\sup_n \left\{ \big||w_n - x|_{\mathcal{X}} - |w_n - y|_{\mathcal{X}}\big| \right\} \geqslant |x - y|_{\mathcal{X}}.$$

Hence

❷ $$\sup_n \{ |a_n(x) - a_n(y)| \} \geqslant |x - y|_{\mathcal{X}};$$

that is, $\iota$ is distance-noncontracting.

Finally, observe that ❶ and ❷ imply the lemma. □

**2.2. Exercise.** *Show that any compact metric space $\mathcal{K}$ is isometric to a subspace of a compact geodesic space.*

The following exercise generalizes the lemma to arbitrary separable spaces.

**2.3. Exercise.** *Suppose $\{w_n\}$ is a countable, dense set in a metric space $\mathcal{X}$. Choose $x_0 \in \mathcal{X}$; given $x \in \mathcal{X}$, set*

$$a_n(x) = |w_n - x|_{\mathcal{X}} - |w_n - x_0|_{\mathcal{X}}.$$

*Show that $\iota\colon x \mapsto (a_1(x), a_2(x), \dots)$ defines a distance-preserving embedding $\iota\colon \mathcal{X} \hookrightarrow \ell^\infty$.*

The following lemma implies that *any metric space is isometric to a subset of a normed vector space*; its proof is nearly identical to the proof of 2.3.

**2.4. Lemma.** *Let $\mathcal{X}$ be arbitrary metric space. Denote by $\ell^\infty(\mathcal{X})$ the space of all bounded functions on $\mathcal{X}$ equipped with sup-norm.*

*Then for any point $x_0 \in \mathcal{X}$, the map $\iota\colon \mathcal{X} \to \ell^\infty(\mathcal{X})$ defined by*

$$\iota\colon x \mapsto (\mathrm{dist}_x - \mathrm{dist}_{x_0})$$

*is distance-preserving.*

# B Extension property

If a metric space $\mathcal{X}$ is a subspace of a pseudometric space $\mathcal{X}'$, then we say that $\mathcal{X}'$ is an extension of $\mathcal{X}$. If in addition, $\operatorname{diam}\mathcal{X}' \leqslant d$, then we say that $\mathcal{X}'$ is a $d$-extension.

If the complement $\mathcal{X}'\backslash\mathcal{X}$ contains a single point, say $p$, we say that $\mathcal{X}'$ is a one-point extension of $\mathcal{X}$. In this case, to define a metric on $\mathcal{X}'$, it is sufficient to specify the distance function from $p$; that is, a function $f\colon \mathcal{X} \to \mathbb{R}$ defined by

$$f(x) = |p - x|_{\mathcal{X}'}.$$

Any function $f$ of that type will be called extension function or $d$-extension function respectively.

The extension function $f$ cannot be taken arbitrary — the triangle inequality implies that

$$f(x) + f(y) \geqslant |x - y|_{\mathcal{X}} \geqslant |f(x) - f(y)|$$

for any $x, y \in \mathcal{X}$. In particular, $f$ is a non-negative 1-Lipschitz function on $\mathcal{X}$. For a $d$-extension, we need to assume in addition that $\operatorname{diam}\mathcal{X} \leqslant d$ and $f(x) \leqslant d$ for any $x \in \mathcal{X}$. It is easy to see that these conditions are necessary and sufficient.

**2.5. Exercise.** *Let $\mathcal{X}$ be a subspace of metric space $\mathcal{Y}$. Assume $f$ is an extension function on $\mathcal{X}$.*

*(a) Show that*

$$\bar f(y) := \inf_{x\in\mathcal{X}} \{\, f(x) + |x - y|_{\mathcal{Y}} \,\}$$

*defines an extension function on $\mathcal{Y}$.*

*(b) Assume that $\operatorname{diam}\mathcal{Y} \leqslant d$ and $f(x) \leqslant d$ for any $x \in \mathcal{X}$. Show that*

$$\bar f_d := \min\{\, \bar f, d \,\}$$

*is a $d$-extension function on $\mathcal{Y}$.*

The functions $\bar{f}$ and $\bar{f}_d$ in the above exercise are called Katětov extensions of $f$.

**2.6. Definition.** *A metric space $\mathcal{U}$ meets the extension property if for any finite subspace $\mathcal{F} \subset \mathcal{U}$ and any extension function $f\colon \mathcal{F} \to \mathbb{R}$ there is a point $p \in \mathcal{U}$ such that $|p - x| = f(x)$ for any $x \in \mathcal{F}$.*

*If we assume in addition that $\operatorname{diam} \mathcal{U} \leqslant d$ and instead of extension functions we consider only $d$-extension functions, then we arrive at a definition of $d$-extension property.*

*If in addition, $\mathcal{U}$ is separable and complete, then it is called Urysohn space or $d$-Urysohn space respectively.*

**2.7. Proposition.** *There is a separable metric space with the (d-) extension property (for any $d \geqslant 0$).*

*Proof.* Choose $d \geqslant 0$. Let us construct a separable metric space with the $d$-extension property.

Let $\mathcal{X}$ be a metric space such that $\operatorname{diam} \mathcal{X} \leqslant d$. Denote by $\mathcal{X}^d$ the space of all $d$-extension functions on $\mathcal{X}$ equipped with the metric defined by the sup-norm. Note that the map $\mathcal{X} \to \mathcal{X}^d$ defined by $x \mapsto \operatorname{dist}_x$ is a distance-preserving embedding, so we can (and will) treat $\mathcal{X}$ as a subspace of $\mathcal{X}^d$, or, equivalently, $\mathcal{X}^d$ is an extension of $\mathcal{X}$.

Let us iterate this construction. Start with a one-point space $\mathcal{X}_0$ and consider a sequence of spaces $(\mathcal{X}_n)$ defined by $\mathcal{X}_{n+1} = \mathcal{X}_n^d$. Note that the sequence is nested; that is, $\mathcal{X}_0 \subset \mathcal{X}_1 \subset \dots$ and the union

$$\mathcal{X}_\infty = \bigcup_n \mathcal{X}_n;$$

comes with metric such that $|x - y|_{\mathcal{X}_\infty} = |x - y|_{\mathcal{X}_n}$ if $x, y \in \mathcal{X}_n$.

Note that if $\mathcal{X}$ is compact, then so is $\mathcal{X}^d$. It follows that each space $\mathcal{X}_n$ is compact. In particular, $\mathcal{X}_\infty$ is a countable union of compact spaces; therefore $\mathcal{X}_\infty$ is separable.

Any finite subspace $\mathcal{F}$ of $\mathcal{X}_\infty$ lies in some $\mathcal{X}_n$ for $n < \infty$. By construction, there is a point $p \in \mathcal{X}_{n+1}$ that meets the condition in 2.6 for any extension function $f\colon \mathcal{F} \to \mathbb{R}$. That is, $\mathcal{X}_\infty$ has the $d$-extension property.

The construction of a separable metric space with the extension property requires only two changes. First, the sequence should be defined by $\mathcal{X}_{n+1} = \mathcal{X}_n^{d_n}$, where $d_n$ is an increasing sequence such that $d_n \to \infty$. Second, the point $p$ should be taken in $\mathcal{X}_{n+k}$ for sufficiently large $k$, so that $d_{n+k} > \max\{f(x)\}$ (here one has to apply 2.5$a$). $\square$

Given a metric space $\mathcal{X}$, denote by $\mathcal{X}^\infty$ the space of all extension functions on $\mathcal{X}$ equipped with the metric defined by the sup-norm.

**2.8. Exercise.** *Construct a proper length space $\mathcal{X}$ such that $\mathcal{X}^\infty$ is not separable.*

**2.9. Proposition.** *If a metric space $\mathcal{V}$ meets the (d-) extension property, then so does its completion.*

*Proof.* Let us assume $\mathcal{V}$ meets the extension property. We will show that its completion $\mathcal{U} = \bar{\mathcal{V}}$ meets the extension property as well. The $d$-extension case can be proved along the same lines.

Note that $\mathcal{V}$ is a dense subset in a complete space $\mathcal{U}$. Observe that $\mathcal{U}$ has the approximate extension property; that is, if $\mathcal{F} \subset \mathcal{U}$ is a finite set, $\varepsilon > 0$, and $f\colon \mathcal{F} \to \mathbb{R}$ is an extension function, then there exists $p \in \mathcal{U}$ such that

$$|p - x| \lesssim f(x) \pm \varepsilon \qquad ❶$$

for any $x \in \mathcal{F}$. Indeed, the Katětov extension $\bar{f}\colon \mathcal{U} \to \mathbb{R}$ of $f$; see 2.5. Since $\mathcal{V}$ is dense in $\mathcal{U}$, we can choose a finite set $\mathcal{F}' \in \mathcal{V}$ such that for any $x \in \mathcal{F}$ there is $x' \in \mathcal{F}'$ with $|x - x'| < \frac{\varepsilon}{2}$. It remains to observe that the point $p$ provided by the extension property for the restriction $\bar{f}|_{\mathcal{F}'}$ meets ❶.

It follows that there is a sequence of points $p_n \in \mathcal{U}$ such that for any $x \in \mathcal{F}$,

$$|p_n - x| \lesssim f(x) \pm \tfrac{1}{2^n}.$$

Moreover, we can assume that

$$|p_n - p_{n+1}| < \tfrac{1}{2^n} \qquad ❷$$

for all large $n$. Indeed, consider the sets $\mathcal{F}_n = \mathcal{F} \cup \{p_n\}$ and the functions $f_n\colon \mathcal{F}_n \to \mathbb{R}$ defined by $f_n(x) = f(x)$ if $x \ne p_n$ and

$$f_n(p_n) = \max\left\{\, \big||p_n - x| - f(x)\big| \;:\; x \in \mathcal{F} \,\right\}.$$

Observe that $f_n$ is an extension function for large $n$ and $f_n(p_n) < \frac{1}{2^n}$. Therefore, applying the approximate extension property recursively we get ❷.

By ❷, the sequence $p_n$ is Cauchy and its limit meets the condition in the definition of extension property (2.6). ☐

Note that 2.7 and 2.9 imply the following:

**2.10. Theorem.** *Urysohn space and d-Urysohn space exist for any $d > 0$.*

Here is a slightly stronger statement:

**2.11. Theorem.** *Any separable metric space $\mathcal{X}$ admits a distance-preserving embedding into an Urysohn space $\mathcal{U}$ such that any isometry of $\mathcal{X}$ can be extended to an isometry of $\mathcal{U}$.*

*Sketch of proof.* Denote by $\mathcal{X}^\infty$ the space of all extension functions on $\mathcal{X}$ equipped with the metric defined by the sup-norm. Note that $x \mapsto \mathrm{dist}_x$ defines a distance preserving inclusion $\mathcal{X} \hookrightarrow \mathcal{X}^\infty$, and any isometry $\mathcal{X} \to \mathcal{X}$ can be extended to a unique isometry $\mathcal{X}^\infty \to \mathcal{X}^\infty$.

Given a separable metric space $\mathcal{X} = \mathcal{X}_0$ consider a nested sequence of spaces

$$\mathcal{X}_0 \subset \mathcal{X}_1 \subset \dots$$

such that $\mathcal{X}_{n+1} = \mathcal{X}_n^\infty$. It is easy to modify the proofs of 2.7 and 2.9 to show that of the completion of the union $\bigcup_n \mathcal{X}_n$ is an Urysohn space, say $\mathcal{U}$, that comes with a distance-preserving inclusion $\mathcal{X} \hookrightarrow \mathcal{U}$.

From above, for any isometry $f\colon \mathcal{X} \to \mathcal{X}$ there is a unique sequence of isometries $f_n\colon \mathcal{X}_n \to \mathcal{X}_n$ such that $f_{n+1}$ is an extension of $f_n$ for any $n$. Passing to a limit we get an isometry of $\mathcal{U}$. $\square$

# C Universality

A metric space will be called universal if it includes as a subspace an isometric copy of any separable metric space. In 2.3, we proved that $\ell^\infty$ is a universal space. The following proposition shows that an Urysohn space is universal as well. Unlike $\ell^\infty$, Urysohn spaces are separable; so it might be considered as a *better* universal space. Theorem 2.19 will give another reason why Urysohn spaces are better.

**2.12. Proposition.** *An Urysohn space is universal. That is, if $\mathcal{U}$ is an Urysohn space, then any separable metric space $\mathcal{S}$ admits a distance-preserving embedding $\mathcal{S} \hookrightarrow \mathcal{U}$.*

*Moreover, for any finite subspace $\mathcal{F} \subset \mathcal{S}$, any distance-preserving embedding $\mathcal{F} \hookrightarrow \mathcal{U}$ can be extended to a distance-preserving embedding $\mathcal{S} \hookrightarrow \mathcal{U}$.*

*A $d$-Urysohn space is $d$-universal; that is, the above statements hold provided that* $\operatorname{diam} \mathcal{S} \leqslant d$.

*Proof.* We will prove the second statement; the first statement is its partial case for $\mathcal{F} = \varnothing$.

The required isometry will be denoted by $x \mapsto x'$.

Choose a dense sequence of points $s_1, s_2, \ldots \in \mathcal{S}$. We may assume that $\mathcal{F} = \{s_1, \ldots, s_n\}$, so $s_i' \in \mathcal{U}$ are defined for $i \leqslant n$.

The sequence $s_i'$ for $i > n$ can be defined recursively using the extension property in $\mathcal{U}$. Namely, suppose that $s_1', \ldots, s_{i-1}'$ are already defined. Since $\mathcal{U}$ meets the extension property, there is a point $s'_i \in \mathcal{U}$ such that

$$|s'_i - s'_j|_{\mathcal{U}} = |s_i - s_j|_{\mathcal{S}}$$

for any $j < i$.

The constructed map $s_i \mapsto s'_i$ is distance-preserving. Therefore it can be continuously extended to the whole $\mathcal{S}$. It remains to observe that the constructed map $\mathcal{S} \hookrightarrow \mathcal{U}$ is distance-preserving. □

**2.13. Exercise.** *Show that any two distinct points in an Urysohn space can be joined by an infinite number of geodesics.*

**2.14. Exercise.** *Modify the proofs of 2.9 and 2.12 to prove the following theorem.*

**2.15. Theorem.** *Let $K$ be a compact set in a separable space $\mathcal{S}$. Then any distance-preserving map from $K$ to an Urysohn space can be extended to a distance-preserving map on whole $\mathcal{S}$.*

**2.16. Exercise.** *Show that (d-) Urysohn space is simply-connected.*

# D Uniqueness and homogeneity

**2.17. Theorem.** *Suppose $\mathcal{F} \subset \mathcal{U}$ and $\mathcal{F}' \subset \mathcal{U}'$ be finite isometric subspaces in a pair of (d-)Urysohn spaces $\mathcal{U}$ and $\mathcal{U}'$. Then any isometry $\iota\colon \mathcal{F} \leftrightarrow \mathcal{F}'$ can be extended to an isometry $\mathcal{U} \leftrightarrow \mathcal{U}'$.*

*In particular, (d-)Urysohn space is unique up to isometry.*

While 2.12 implies that there are distance-preserving maps $\mathcal{U} \to \mathcal{U}'$ and $\mathcal{U}' \to \mathcal{U}$, it does not solely imply the existence of an isometry $\mathcal{U} \leftrightarrow \mathcal{U}'$. Its construction uses the idea of 2.12, but it is applied back-and-forth to ensure that the obtained distance-preserving map is onto.

*Proof.* Choose dense sequences $a_1, a_2, \dots \in \mathcal{U}$ and $b'_1, b'_2, \dots \in \mathcal{U}'$. We can assume that $\mathcal{F} = \{a_1, \dots, a_n\}$, $\mathcal{F}' = \{b'_1, \dots, b'_n\}$ and $\iota(a_i) = b_i$ for $i \leqslant n$.

The required isometry $\mathcal{U} \leftrightarrow \mathcal{U}'$ will be denoted by $u \leftrightarrow u'$. Set $a'_i = b'_i$ if $i \leqslant n$.

Let us define recursively $a'_{n+1}, b_{n+1}, a'_{n+2}, b_{n+2}, \dots$ — on the odd step we define the images of $a_{n+1}, a_{n+2}, \dots$ and on the even steps we define inverse images of $b'_{n+1}, b'_{n+2}, \dots$ The same argument as in the proof of 2.12 shows that we can construct two sequences $a_1', a_2', \dots \in \mathcal{U}'$ and $b_1, b_2, \dots \in \mathcal{U}$ such that

$$\begin{aligned}|a_i - a_j|_{\mathcal{U}} &= |a_i' - a_j'|_{\mathcal{U}'}\\ |a_i - b_j|_{\mathcal{U}} &= |a_i' - b_j'|_{\mathcal{U}'}\\ |b_i - b_j|_{\mathcal{U}} &= |b_i' - b_j'|_{\mathcal{U}'}\end{aligned}$$

for all $i$ and $j$.

It remains to observe that the constructed distance-preserving bijection defined by $a_i \leftrightarrow a_i'$ and $b_i \leftrightarrow b_i'$ extends continuously to an isometry $\mathcal{U} \leftrightarrow \mathcal{U}'$. □

Observe that 2.17 implies that the Urysohn space (as well as the $d$-Urysohn space) is finite-set-homogeneous; that is,

◇ any distance-preserving map from a finite subset to the whole space can be extended to an isometry.

Recall that $S(p,r)_{\mathcal{X}}$ denotes the sphere of radius $r$ centered at $p$ in a metric space $\mathcal{X}$; that is,

$$S(p,r)_{\mathcal{X}} = \{\, x \in \mathcal{X} \,:\, |p - x|_{\mathcal{X}} = r \,\}.$$

**2.18. Exercise.** *Choose $d \in [0,\infty]$. Denote by $\mathcal{U}_d$ the $d$-Urysohn space, so $\mathcal{U}_\infty$ is the Urysohn space.*

*(a) Assume that $L = S(p,r)_{\mathcal{U}_d} \neq \varnothing$. Show that $L$ is isometric to $\mathcal{U}_\ell$; find $\ell$ in terms of $r$ and $d$.*

*(b) Let $\ell = |p - q|_{\mathcal{U}_d}$. Show that the subset $M \subset \mathcal{U}_d$ of midpoints between $p$ and $q$ is isometric to $\mathcal{U}_\ell$.*

*(c) Show that $\mathcal{U}_d$ is not countable-set-homogeneous; that is, there is a distance-preserving map from a countable subset of $\mathcal{U}_d$ to $\mathcal{U}_d$ that cannot be extended to an isometry of $\mathcal{U}_d$.*

In fact the Urysohn space is compact-set-homogeneous; more precisely the following theorem holds.

**2.19. Theorem.** *Let $K$ be a compact set in a ($d$-)Urysohn space $\mathcal{U}$. Then any distance-preserving map $K \to \mathcal{U}$ can be extended to an isometry of $\mathcal{U}$.*

A proof can be obtained by modifying the proofs of 2.9 and 2.17 the same way as it is done in 2.14.

**2.20. Exercise.** *Let $S$ be a unit sphere in the Urysohn space $\mathcal{U}$. Show that for any two distinct points $x, y \in \mathcal{U}$ there is a point $z \in S$ such that $|x - z| \neq |y - z|$.*

*Conclude that two isometries of $\mathcal{U}$ coincide if they coincide on $S$.*

**2.21. Exercise.** *Let $B$ be an open unit ball in the Urysohn space $\mathcal{U}$. Show that $\mathcal{U}\setminus B$ is isometric to $\mathcal{U}$.*

*Use it to construct an isometry of a unit sphere $S$ in $\mathcal{U}$ that cannot be extended to an isometry of $\mathcal{U}$.*

The following exercise answers a question posted by Pavel Urysohn [39, §2(6)]. It was solved by Miroslav Katětov [21], but long after that, it was again mentioned as an open problem [13, p. 83].

**2.22. Exercise.**

(a) *Show that there is a distance-preserving inclusion of the Urysohn space $\iota\colon\mathcal{U}\hookrightarrow\mathcal{U}$ such that $\mathcal{U}'=\iota(\mathcal{U})$ is nowhere dense in $\mathcal{U}$ and any isometry of $\mathcal{U}'$ can be extended to an isometry of the whole $\mathcal{U}$.*

(b) *Consider a nested sequence $\mathcal{U}_0\subset\mathcal{U}_1\subset\dots$ of Urysohn spaces with each inclusion $\mathcal{U}_n\hookrightarrow\mathcal{U}_{n+1}$ as in (a). Show that the union $\bigcup_n\mathcal{U}_n$ is a noncomplete finite-set-homogeneous metric space that meets the extension property.*

**2.23. Exercise.** *Which of the following metric spaces are one-point-homogeneous, finite-set-homogeneous, compact-set-homogeneous, countable-set-homogeneous?*

(a) *Euclidean plane,*

(b) *Hilbert space $\ell^2$,*

(c) *$\ell^\infty$,*

(d) *$\ell^1$.*

**2.24. Exercise.** *Show that any separable one-point-homogeneous metric tree is isometric to the real line $\mathbb{R}$ or the one-point space.*

# E Remarks

The statement in 2.3 was proved by Maurice René Fréchet in the paper where he first defined metric spaces [11]; its extension 2.4 was given by Kazimierz Kuratowski [23]. The question about the existence of a separable universal space was posted by Maurice René Fréchet and answered by Pavel Urysohn [39].

The idea of Urysohn's construction was reused in graph theory; it produces the so-called Rado graph, also known as Erdős–Rényi graph or random graph; a good survey on the subject is given by Peter Cameron [9]. In fact the Urysohn space is the random metric space in *certain sense* [42].

You might wonder what is the topology of the Urysohn space. This question was answered by Vladimir Uspenskij [40]: *the Urysohn space is homeomorphic to the countable product of the real lines.*

The finite-set-homogeneous spaces include Euclidean spaces, hyperbolic spaces, standard spheres, and projective spaces all with standard metrics of constant curvatures. In fact from *Gleason–Yamabe theorem* it follows that these are the only examples of locally compact three-point-homogeneous length spaces; a proof was given by Jacques Tits [38]. The answer might be the same for complete all-set-homogeneous length spaces.

# Lecture 3

# Injective spaces

Injective spaces (also known as hyperconvex spaces) are the metric analog of convex sets.

## A Admissible and extremal functions

Let $\mathcal{X}$ be a metric space. A function $r\colon \mathcal{X}\to\mathbb{R}$ is called admissible if the following inequality

$$r(x)+r(y)\geqslant |x-y|_{\mathcal{X}} \qquad ❶$$

holds for any $x,y\in\mathcal{X}$.

**3.1. Observation.**

(a) *Any admissible function is nonnegative.*

(b) *If $\mathcal{X}$ is a geodesic space, then a function $r\colon \mathcal{X}\to\mathbb{R}$ is admissible if and only if*

$$\overline{\mathrm{B}}[x,r(x)]\cap\overline{\mathrm{B}}[y,r(y)]\neq\varnothing$$

*for any $x,y\in\mathcal{X}$.*

*Proof.* For *(a)*, take $x=y$ in ❶.

Part *(b)* follows from the triangle inequality and the existence of a geodesic $[xy]$. $\square$

A minimal admissible function will be called extremal. More precisely, an admissible function $r\colon \mathcal{X}\to\mathbb{R}$ is extremal if for any admissible function $s\colon \mathcal{X}\to\mathbb{R}$ we have

$$s\leqslant r \quad\Longrightarrow\quad s=r.$$

**3.2. Key exercise.** *Let $r$ be an extremal function and $s$ an admissible function on a metric space $\mathcal{X}$. Suppose that $r \geqslant s - c$ for some constant $c$. Show that $r \leqslant s + c$; in particular, $c \geqslant 0$.*

**3.3. Observations.** *Let $\mathcal{X}$ be a metric space.*

*(a) For any point $p \in \mathcal{X}$ the distance function $r = \operatorname{dist}_p$ is extremal.*

*(b) Any extremal function $r$ on $\mathcal{X}$ is 1-Lipschitz; that is,*

$$|r(p) - r(q)| \leqslant |p - q|$$

*for any $p, q \in \mathcal{X}$. In other words, any extremal function is an extension function; see the definition in 2B.*

*(c) An admissible function $r$ on $\mathcal{X}$ is extremal if and only if for any point $p \in \mathcal{X}$ and any $\delta > 0$, there is a point $q \in \mathcal{X}$ such that*

$$r(p) + r(q) < |p - q|_{\mathcal{X}} + \delta.$$

*(d) If $\mathcal{X}$ is compact, then an admissible function $r$ on $\mathcal{X}$ is extremal if and only if for any point $p \in \mathcal{X}$ there is a point $q \in \mathcal{X}$ such that*

$$r(p) + r(q) = |p - q|_{\mathcal{X}}.$$

*(e) For any admissible function $s$ there is an extremal function $r$ such that $r \leqslant s$.*

*Proof; (a).* By the triangle inequality, ❶ holds; that is, $r = \operatorname{dist}_p$ is an admissible function.

Further, if $s \leqslant r$ is another admissible function, then $s(p) = 0$ and ❶ implies that $s(x) \geqslant |p - x|$. Whence $s = r$.

*(b).* By *(a)*, $\operatorname{dist}_p$ is admissible. Since $r$ is admissible, we have that

$$r \geqslant \operatorname{dist}_p - r(p).$$

Since $r$ is extremal, 3.2 implies that

$$r \leqslant \operatorname{dist}_p + r(p),$$

or, equivalently,

$$r(q) - r(p) \leqslant |p - q|$$

for any $p, q \in \mathcal{X}$. The same way we can show that $r(p) - r(q) \leqslant |p - q|$. Whence the statement follows.

*(c).* Assume $r$ is extremal. Arguing by contradiction, assume there is $\delta > 0$ such that

$$r(q) \geqslant \operatorname{dist}_p(q) - r(p) + \delta$$

for any $q$. By *(a)*, $\operatorname{dist}_p$ is extremal; in particular, admissible. Therefore 3.2 implies that

$$r(q) \leqslant \operatorname{dist}_p(q) + r(p) - \delta$$

for any $q$. Taking $q = p$, we get $r(p) \leqslant r(p) - \delta$, a contradiction.

Now suppose $r$ is not extremal; that is, there is an admissible function $s \leqslant r$ such that $r(p) - s(p) = \delta > 0$ for some $p$. Then, for any $q$, we have

$$r(p) + r(q) \geqslant s(p) + s(q) + \delta \geqslant |p - q|_{\mathcal{X}} + \delta$$

— a contradiction.

*(d).* The if part follows from *(c)*.

Denote by $q_n$ the point provided by *(c)* for $\delta = \frac{1}{n}$. Let $q$ be a partial limit of $q_n$. Then

$$r(p) + r(q) \leqslant |p - q|_{\mathcal{X}}.$$

Since $r$ is admissible, the opposite inequality holds; whence the only-if part follows.

*(e).* Follows by Zorn's lemma. □

**3.4. Exercise.** *Consider the unit circle $\mathbb{S}^1 = \{\, (x, y) \,:\, x^2 + y^2 = 1 \,\}$ in the plane with induced length metric. Show that $r\colon \mathbb{S}^1 \to \mathbb{R}$ is extremal if and only if it is 1-Lipschitz and*

$$r(p) + r(-p) = \pi$$

*for any $p \in \mathbb{S}^1$.*

**3.5. Exercise.** *Given a real-valued function $s$ on a metric space $\mathcal{X}$, consider the function*

$$s^*(x) = \sup\{\, |z - y|_{\mathcal{X}} - s(y) \,:\, y \in \mathcal{X} \,\}$$

*Show that if $s$ is admissible then so is $\frac{1}{2}{\cdot}(s + s^*)$.*

# B Injective spaces

**3.6. Definition.** *A metric space $\mathcal{Y}$ is called injective if for any metric space $\mathcal{X}$ and any of its subspaces $\mathcal{A}$ any short map $f\colon \mathcal{A} \to \mathcal{Y}$ can be extended to a short map $F\colon \mathcal{X} \to \mathcal{Y}$; that is, $f = F|_{\mathcal{A}}$.*

**3.7. Exercise.** *Show that any injective space is* *(a) complete,* *(b) geodesic, and* *(c) contractible.*

**3.8. Exercise.** *Show that the following spaces are injective:*

(a) *the real line;*
(b) *complete metric tree;*
(c) *coordinate plane with the metric induced by the $\ell^\infty$-norm.*

The following two exercise deals with ultrametric spaces which in some sense are dual to the injective spaces. Recall that if the following inequality

$$|x-z|_{\mathcal{X}} \leqslant \max\{\,|x-y|_{\mathcal{X}}, |y-z|_{\mathcal{X}}\,\}$$

holds for any three points $x, y, z$ in a metric space $\mathcal{X}$, then $\mathcal{X}$ is called an ultrametric space.

**3.9. Exercise.** *Suppose that a metric space $\mathcal{X}$ satisfies the following property: For any subspace $\mathcal{A}$ in $\mathcal{X}$ and any other metric space $\mathcal{Y}$, any short map $f\colon \mathcal{A}\to\mathcal{Y}$ can be extended to a short map $F\colon \mathcal{X}\to\mathcal{Y}$.*

*Show that $\mathcal{X}$ is an ultrametric space.*

A subspace $\mathcal{S}$ of a metric space $\mathcal{X}$ is called its short retract if there is a short map $\mathcal{X}\to\mathcal{S}$ that is the identity on $\mathcal{S}$.

**3.10. Exercise.** *Show that any compact subspace $\mathcal{K}$ of an ultrametric space $\mathcal{X}$ is its short retract.*

*Construct an example of a complete ultrametric space $\mathcal{X}$ with a closed subset $Q$ that is not its short retract.*

**3.11. Theorem.** *For any metric space $\mathcal{Y}$ the following condition are equivalent:*

(a) *$\mathcal{Y}$ is injective*
(b) *If $r\colon \mathcal{Y}\to\mathbb{R}$ is an extremal function, then there is a point $p\in\mathcal{Y}$ such that*

$$|p-x|\leqslant r(x)$$

*for any $x\in\mathcal{Y}$.*
(c) *$\mathcal{Y}$ is hyperconvex; that is, if $\left\{\,\overline{\mathrm{B}}[x_\alpha, r_\alpha] \,:\, \alpha\in\mathcal{A}\,\right\}$ is a family of closed balls in $\mathcal{Y}$ such that*

$$r_\alpha + r_\beta \geqslant |x_\alpha - x_\beta|$$

*for any $\alpha,\beta\in\mathcal{A}$, then all the balls in the family $\{\overline{\mathrm{B}}[x_\alpha, r_\alpha]\}_{\alpha\in\mathcal{A}}$ have a common point.*

*Proof.* We will prove implications *(a)⇒(b)⇒(c)⇒(a).*

*(a)⇒(b).* Let us apply the definition of injective space to a one-point extension of $\mathcal{Y}$. It follows that for any extension function $r\colon \mathcal{Y}\to\mathbb{R}$ there is a point $p\in\mathcal{Y}$ such that

$$|p-x|\leqslant r(x)$$

for any $x\in\mathcal{Y}$. By 3.3*b*, any extremal function is an extension function, whence the implication follows.

*(b)⇒(c).* By 3.1*b*, part *(c)* is equivalent to the following statement:

◇ If $r\colon \mathcal{Y}\to\mathbb{R}$ is an admissible function, then there is a point $p\in\mathcal{Y}$ such that

$$❶\qquad |p-x|\leqslant r(x)$$

for any $x\in\mathcal{Y}$.

Indeed, set $r(x) := \inf\{\, r_\alpha : x_\alpha = x\,\}$. (If $x_\alpha\neq x$ for any $\alpha$, then $r(x)=\infty$.) The condition in *(c)* implies that $r$ is admissible. It remains to observe that $p\in\overline{\mathrm{B}}[x_\alpha,r_\alpha]$ for every $\alpha$ if and only if ❶ holds.

By 3.3*e*, for any admissible function $r$ there is an extremal function $\bar r\leqslant r$; hence *(b)⇒(c).*

*(c)⇒(a).* Arguing by contradiction, suppose $\mathcal{Y}$ is not injective; that is, there is a metric space $\mathcal{X}$ with a subset $\mathcal{A}$ such that a short map $f\colon\mathcal{A}\to\mathcal{Y}$ cannot be extended to a short map $F\colon\mathcal{X}\to\mathcal{Y}$. By Zorn's lemma, we may assume that $\mathcal{A}$ is a maximal subset; that is, the domain of $f$ cannot be enlarged by a single point.[1]

Fix a point $p$ in the complement $\mathcal{X}\setminus\mathcal{A}$. To extend $f$ to $p$, we need to choose $f(p)$ in the intersection of the balls $\overline{\mathrm{B}}[f(x),r(x)]$, where $r(x)=|p-x|$. Therefore, this intersection for all $x\in\mathcal{A}$ has to be empty.

Since $f$ is short, we have that

$$\begin{aligned} r(x)+r(y)&\geqslant |x-y|_{\mathcal{X}}\geqslant\\ &\geqslant |f(x)-f(y)|_{\mathcal{Y}}.\end{aligned}$$

Therefore, by *(c)* the balls $\overline{\mathrm{B}}[f(x),r(x)]$ have a common point — a contradiction. □

**3.12. Exercise.** *Suppose a length space $\mathcal{W}$ has two subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X}\cup\mathcal{Y}=\mathcal{W}$ and $\mathcal{X}\cap\mathcal{Y}$ is a one-point set. Assume $\mathcal{X}$ and $\mathcal{Y}$ are injective. Show that $\mathcal{W}$ is injective*

[1] In this case, $\mathcal{A}$ must be closed, but we will not use it.

**3.13. Exercise.** *Show that the d-Urysohn space is finitely hyperconvex but not countably hyperconvex; that is, the condition in 3.11c holds for any finite family of balls, but may not hold for a countable family. Conclude that the d-Urysohn space is not injective.*

*Try to do the same for the Urysohn space.*

# C Space of extremal functions

Let $\mathcal{X}$ be a metric space. Consider the space $\operatorname{Ext}\mathcal{X}$ of extremal functions on $\mathcal{X}$ equipped with sup-norm; that is,

$$|f-g|_{\operatorname{Ext}\mathcal{X}} := \sup\{\,|f(x)-g(x)|\;:\;x\in\mathcal{X}\,\}.$$

Recall that by 3.3$a$, any distance function is extremal. It follows that the map $x\mapsto\operatorname{dist}_x$ produces a distance-preserving embedding $\mathcal{X}\hookrightarrow\operatorname{Ext}\mathcal{X}$. So we can (and will) treat $\mathcal{X}$ as a subspace of $\operatorname{Ext}\mathcal{X}$, or, equivalently, $\operatorname{Ext}\mathcal{X}$ as an extension of $\mathcal{X}$.

Since any extremal function is 1-Lipschitz, for any $f\in\operatorname{Ext}\mathcal{X}$ and $p\in\mathcal{X}$, we have that $f(x)\leqslant f(p)+\operatorname{dist}_p(x)$. By 3.2, we also get $f(x)\geqslant -f(p)+\operatorname{dist}_p(x)$. Therefore

$$\begin{aligned}|f-p|_{\operatorname{Ext}\mathcal{X}} &= \sup\{\,|f(x)-\operatorname{dist}_p(x)|\;:\;x\in\mathcal{X}\,\}=\\ &=f(p).\end{aligned} \tag{❶}$$

In particular, the statement in 3.3$c$ can be written as

$$|f-p|_{\operatorname{Ext}\mathcal{X}}+|f-q|_{\operatorname{Ext}\mathcal{X}}<|p-q|_{\operatorname{Ext}\mathcal{X}}+\delta.$$

**3.14. Exercise.** *Let* $\mathcal{X}$ *be a metric space. Show that* $\operatorname{Ext}\mathcal{X}$ *is compact if and only if so is* $\mathcal{X}$*.*

**3.15. Exercise.** *Describe the set of all extremal functions on a metric space* $\mathcal{X}$ *and the metric space* $\operatorname{Ext}\mathcal{X}$ *in each of the following cases:*

(a) $\mathcal{X}$ *is a metric space with exactly two points* $v,w$ *on distance 1 from each other.*

(b) $\mathcal{X}$ *is a metric space with exactly three points* $a,b,c$ *such that*

$$|a-b|_{\mathcal{X}}=|b-c|_{\mathcal{X}}=|c-a|_{\mathcal{X}}=1.$$

(c) $\mathcal{X}$ *is a metric space with exactly four points* $p,q,x,y$ *such that*

$$|p-x|_{\mathcal{X}}=|p-y|_{\mathcal{X}}=|q-x|_{\mathcal{X}}=|q-y|_{\mathcal{X}}=1$$

*and*

$$|p-q|_{\mathcal{X}}=|x-y|_{\mathcal{X}}=2.$$

**3.16. Exercise.** *Assume $\mathcal{X}$ is a compact metric space. Recall that the map $x \mapsto \operatorname{dist}_x$ gives an isometric embedding $\mathcal{X} \hookrightarrow \ell^\infty(\mathcal{X})$; so we can think that $\mathcal{X}$ is a subset of $\ell^\infty(\mathcal{X})$.*

*Given two points $x, y \in \mathcal{X}$, denote by $G_{x,y}$ the union of all geodesics from $x$ to $y$ in $\ell^\infty(\mathcal{X})$. Show that $\operatorname{Ext}\mathcal{X}$ is isometric to*

$$G = \bigcap_{x\in\mathcal{X}} \left(\bigcup_{y\in\mathcal{X}} G_{x,y}\right).$$

**3.17. Proposition.** *For any metric space $\mathcal{X}$, its extension $\operatorname{Ext}\mathcal{X}$ is injective.*

**3.18. Lemma.** *Let $\mathcal{X}$ be a metric space. Suppose $r \in \operatorname{Ext}(\operatorname{Ext}\mathcal{X})$; that is, $r$ is an extremal function on $\operatorname{Ext}\mathcal{X}$. Then $r|_{\mathcal{X}} \in \operatorname{Ext}\mathcal{X}$; that is, the restriction of $r$ to $\mathcal{X}$ is an extremal function.*

*Proof.* Arguing by contradiction, suppose that there is an admissible function $s\colon \mathcal{X} \to \mathbb{R}$ such that $s(x) \leqslant r(x)$ for any $x \in \mathcal{X}$ and $s(p) < < r(p)$ for some point $p \in \mathcal{X}$. Consider another function $\bar r\colon \operatorname{Ext}\mathcal{X} \to \mathbb{R}$ such that $\bar r(f) := r(f)$ if $f \ne p$ and $\bar r(p) := s(p)$.

Let us show that $\bar r$ is admissible; that is,

$$❷ \qquad |f - g|_{\operatorname{Ext}\mathcal{X}} \leqslant \bar r(f) + \bar r(g)$$

for any $f, g \in \operatorname{Ext}\mathcal{X}$.

Since $r$ is admissible and $\bar r = r$ on $(\operatorname{Ext}\mathcal{X}) \setminus \{p\}$, it is sufficient to prove ❷ if $f \ne g = p$. By ❶, we have $|f - p|_{\operatorname{Ext}\mathcal{X}} = f(p)$. Therefore, ❷ boils down to the following inequality

$$❸ \qquad r(f) + s(p) \geqslant f(p).$$

for any $f \in \operatorname{Ext}\mathcal{X}$.

Fix small $\delta > 0$. Let $q \in \mathcal{X}$ be the point provided by 3.3*c*. Then

$$r(f) + s(p) \geqslant [r(f) - r(q)] + [r(q) + s(p)] \geqslant$$

since $r$ is 1-Lipschitz, and $r(q) \geqslant s(q)$, we can continue

$$\geqslant -|q - f|_{\operatorname{Ext}\mathcal{X}} + [s(q) + s(p)] \geqslant$$

by ❶ and since $s$ is admissible

$$\geqslant -f(q) + |p - q| >$$

and by 3.3$c$

$$> f(p) - \delta.$$

Since $\delta > 0$ is arbitrary, ❸ and ❷ follow.

Summarizing: the function $\bar{r}$ is admissible, $\bar{r} \leqslant r$ and $\bar{r}(p) < r(p)$; that is, $r$ is not extremal — a contradiction. □

*Proof of 3.17.* Choose a function $r \in \operatorname{Ext}(\operatorname{Ext}\mathcal{X})$. By 3.18, $s := r|_{\mathcal{X}} \in \operatorname{Ext}\mathcal{X}$; that is, $s$ is extremal. By 3.11$b$, it is sufficient to show that

❹ $$r(f) \geqslant |s - f|_{\operatorname{Ext}\mathcal{X}}$$

for any $f \in \operatorname{Ext}\mathcal{X}$.

Since $r$ is 1-Lipschitz (3.3$b$) we have that

$$s(x) - f(x) = r(x) - |f - x|_{\operatorname{Ext}\mathcal{X}} \leqslant r(f).$$

for any $x \in \mathcal{X}$. Since $r$ is admissible we have that

$$s(x) - f(x) = r(x) - |f - x|_{\operatorname{Ext}\mathcal{X}} \geqslant -r(f).$$

for any $x \in \mathcal{X}$. That is, $|s(x) - f(x)| \leqslant r(f)$ for any $x \in \mathcal{X}$. Recall that

$$|s - f|_{\operatorname{Ext}\mathcal{X}} := \sup\{\,|s(x) - f(x)| \,:\, x \in \mathcal{X}\,\};$$

whence ❹ follows. □

**3.19. Exercise.** *Let $\mathcal{X}$ be a compact metric space. Show that for any two points $f, g \in \operatorname{Ext}\mathcal{X}$ lie on a geodesic $[pq]$ with $p, q \in \mathcal{X}$.*

A metric space $\mathcal{X}$ is called $\delta$-hyperbolic if

$$|p - q| + |x - y| \leqslant \max\{\,|p - x| + |q - y|,\, |p - y| + |q - x|\,\} + 2{\cdot}\delta$$

for any $p, q, x, y \in \mathcal{X}$.

**3.20. Advanced exercise.** *Show that $\operatorname{Ext}\mathcal{X}$ is $\delta$-hyperbolic if and only if $\mathcal{X}$ is.*

## D Injective envelope

An extension $\mathcal{E}$ of a metric space $\mathcal{X}$ will be called its injective envelope if $\mathcal{E}$ is an injective space, and there is no proper injective subspace of $\mathcal{E}$ that contains $\mathcal{X}$.

Two injective envelopes $e\colon \mathcal{X}\hookrightarrow\mathcal{E}$ and $f\colon \mathcal{X}\hookrightarrow\mathcal{F}$ are called equivalent if there is an isometry $\iota\colon\mathcal{E}\to\mathcal{F}$ such that $f=\iota\circ e$.

**3.21. Theorem.** *For any metric space $\mathcal{X}$, its extension $\operatorname{Ext}\mathcal{X}$ is an injective envelope.*

*Moreover, any other injective envelope of $\mathcal{X}$ is equivalent to $\operatorname{Ext}\mathcal{X}$.*

*Proof.* Suppose $S\subset\operatorname{Ext}\mathcal{X}$ is an injective subspace containing $\mathcal{X}$. Since $S$ is injective, there is a short map $w\colon \operatorname{Ext}\mathcal{X}\to S$ that fixes all points in $\mathcal{X}$.

Suppose that $w\colon f\mapsto f'$; observe that $f(x)\geqslant f'(x)$ for any $x\in\mathcal{X}$. Since $f$ is extremal, $f=f'$; that is, $w$ is the identity map, and therefore $S=\operatorname{Ext}\mathcal{X}$.

Assume we have another injective envelope $e\colon\mathcal{X}\hookrightarrow\mathcal{E}$. Then there are short maps $v\colon\mathcal{E}\to\operatorname{Ext}\mathcal{X}$ and $w\colon\operatorname{Ext}\mathcal{X}\to\mathcal{E}$ such that $x=v\circ e(x)$ and $e(x)=w(x)$ for any $x\in\mathcal{X}$. From above, the composition $v\circ w$ is the identity on $\operatorname{Ext}\mathcal{X}$. In particular, $w$ is distance-preserving.

The composition $w\circ v\colon\mathcal{E}\to\mathcal{E}$ is a short map that fixes points in $e(\mathcal{X})$. Since $e\colon\mathcal{X}\hookrightarrow\mathcal{E}$ is an injective envelope, the composition $w\circ v$ and therefore $w$ are onto. Whence $w$ is an isometry. ☐

**3.22. Exercise.** *Suppose $\mathcal{X}$ is a subspace of a metric space $\mathcal{U}$. Show that the inclusion $\mathcal{X}\hookrightarrow\mathcal{U}$ can be extended to a distance-preserving inclusion $\operatorname{Ext}\mathcal{X}\hookrightarrow\operatorname{Ext}\mathcal{U}$.*

**3.23. Exercise.** *Let $\mathcal{X}$ be a metric space. Show that for any two points $x,y\in\mathcal{X}$ one can choose a path $\gamma_{x,y}\colon[0,1]\to\operatorname{Ext}\mathcal{X}$ such that $\gamma_{x,y}(t)\equiv\gamma_{y,x}(1-t)$ and*

$$|\gamma_{x,y}(t)-\gamma_{p,q}(t)|_{\operatorname{Ext}\mathcal{X}}\leqslant(1-t)\cdot|p-x|_{\mathcal{X}}+t\cdot|q-y|_{\mathcal{X}}$$

*for any $x,y,p,q\in\mathcal{X}$.*

# E Remarks

Injective spaces were introduced by Nachman Aronszajn and Prom Panitchpakdi [3]. The injective envelope was introduced by John Isbell [18]. It was rediscovered a couple of times since then; as a result, the injective envelope has many other names including t i g h t  s p a n and h y p e r c o n v e x  h u l l.

# Lecture 4

# Space of sets

## A Hausdorff distance

Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$, consider the distance function to $A$

$$\operatorname{dist}_A : \mathcal{X} \to [0, \infty)$$

defined as

$$\operatorname{dist}_A(x) := \inf_{a \in A} \{\, |a - x|_{\mathcal{X}} \,\}.$$

**4.1. Definition.** *Let $A$ and $B$ be two compact subsets of a metric space $\mathcal{X}$. Then the Hausdorff distance between $A$ and $B$ is defined as*

$$|A - B|_{\operatorname{Haus}\mathcal{X}} := \sup_{x \in \mathcal{X}} \{\, |\operatorname{dist}_A(x) - \operatorname{dist}_B(x)| \,\}.$$

The following observation gives a useful reformulation of the definition:

**4.2. Observation.** *Suppose $A$ and $B$ be two compact subsets of a metric space $\mathcal{X}$. Then $|A - B|_{\operatorname{Haus}\mathcal{X}} < R$ if and only if and only if $B$ lies in an $R$-neighborhood of $A$, and $A$ lies in an $R$-neighborhood of $B$.*

Note that the set of all nonempty compact subsets of a metric space $\mathcal{X}$ equipped with the Hausdorff metric forms a metric space. This new metric space will be denoted as $\operatorname{Haus}\mathcal{X}$.

**4.3. Exercise.** *Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$ define its diameter as*

$$\operatorname{diam} A := \sup_{a, b \in A} |a - b|.$$

*Show that*

$$\operatorname{diam}\colon \operatorname{Haus}\mathcal{X} \to \mathbb{R}$$

*is a 2-Lipschitz function; that is,*

$$|\operatorname{diam} A - \operatorname{diam} B| \leqslant 2\cdot|A-B|_{\operatorname{Haus}\mathcal{X}}$$

*for any two compact nonempty sets $A, B \subset \mathcal{X}$.*

**4.4. Exercise.** *Let $A$ and $B$ be two compact subsets in the Euclidean plane $\mathbb{R}^2$. Assume $|A-B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$.*

*(a) Show that $|\operatorname{Conv} A - \operatorname{Conv} B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$, where $\operatorname{Conv} A$ denoted the convex hull of $A$.*

*(b) Is it true that $|\partial A - \partial B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$, where $\partial A$ denotes the boundary of $A$.*
*Does the converse hold? That is, assume $A$ and $B$ be two compact subsets in $\mathbb{R}^2$ and $|\partial A - \partial B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$; is it true that $|A-B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$?*

Note that part *(a)* implies that $A \mapsto \operatorname{Conv} A$ defines a short map $\operatorname{Haus}\mathbb{R}^2 \to \operatorname{Haus}\mathbb{R}^2$.

**4.5. Exercise.** *Let $A$ and $B$ be two compact subsets in metric space $\mathcal{X}$. Show that*

$$|A-B|_{\operatorname{Haus}\mathcal{X}} = \sup_f \{ \max_{a\in A}\{f(a)\} - \max_{b\in B}\{f(b)\} \},$$

*where the least upper bound is taken for all 1-Lipschitz functions $f$.*

**4.6. Advanced exercise.**

*(a) Construct a family of compact sets $C_t \subset \mathbb{S}^1$, $t \in [0,1]$ that is continuous in the Hausdorff topology, but does not admit a section. That is, there is no path $c\colon [0,1] \to \mathbb{S}^1$ such that $c(t) \in C_t$ for all $t$.*

*(b) Show that any family of compact sets $C_t \subset \mathbb{R}^1$, $t \in [0,1]$ that is continuous in the Hausdorff topology, admits a section. That is, there is path $c\colon [0,1] \to \mathbb{R}^1$ such that $c(t) \in C_t$ for all $t$.*

## B Hausdorff convergence

**4.7. Blaschke selection theorem.** *A metric space $\mathcal{X}$ is compact if and only if so is $\operatorname{Haus}\mathcal{X}$.*

The Hausdorff metric can be used to define convergence. Namely, suppose $K_1, K_2, \dots,$ and $K_\infty$ are compact sets in a metric space $\mathcal{X}$.

If $|K_\infty - K_n|_{\mathrm{Haus}\,\mathcal{X}} \to 0$ as $n \to \infty$, then we say that the sequence $K_n$ converges to $K_\infty$ in the sense of Hausdorff; or we can say that $K_\infty$ is Hausdorff limit of the sequence $K_n$.

Note that the theorem implies that from any sequence of compact sets in $\mathcal{X}$ one can select a subsequence that converges in the sense of Hausdorff; for that reason, it is called a *selection* theorem.

*Proof; if part.* Consider the map $\iota$ that sends each point $x \in \mathcal{X}$ to the one-point subset $\{x\}$ of $\mathcal{X}$. Note that $\iota\colon \mathcal{X} \to \mathrm{Haus}\,\mathcal{X}$ is distance-preserving.

Suppose that $A \subset \mathcal{X}$. Note that $\operatorname{diam} A = 0$ if and only if $A$ is a one-point set. By 4.3, $\iota(\mathcal{X})$ is a closed subset of the compact space $\mathrm{Haus}\,\mathcal{X}$. It follows that $\iota(\mathcal{X})$, and therefore $\mathcal{X}$, are compact. □

To prove the "if" part we will need the following two lemmas.

**4.8. Monotone convergence.** *Let $K_1 \supset K_2 \supset \dots$ be a nested sequence of nonempty compact sets in a metric space $\mathcal{X}$. Then $K_\infty = \bigcap_n K_n$ is the Hausdorff limit of $K_n$; that is, $|K_\infty - K_n|_{\mathrm{Haus}\,\mathcal{X}} \to 0$ as $n \to \infty$.*

*Proof.* By finite intersection property, $K_\infty$ is a nonempty compact set.

If the assertion were false, then there is $\varepsilon > 0$ such that for each $n$ one can choose $x_n \in K_n$ such that $\mathrm{dist}_{K_\infty}(x_n) \geqslant \varepsilon$. Note that $x_n \in K_1$ for each $n$. Since $K_1$ is compact, there is a partial limit[1] $x_\infty$ of $x_n$. Clearly $\mathrm{dist}_{K_\infty}(x_\infty) \geqslant \varepsilon$.

On the other hand, since $K_n$ is closed and $x_m \in K_n$ for $m \geqslant n$, we get $x_\infty \in K_n$ for each $n$. It follows that $x_\infty \in K_\infty$ and therefore $\mathrm{dist}_{K_\infty}(x_\infty) = 0$ — a contradiction. □

**4.9. Lemma.** *If $\mathcal{X}$ is a compact metric space, then* $\mathrm{Haus}\,\mathcal{X}$ *is complete.*

*Proof.* Let $(Q_n)$ be a Cauchy sequence in $\mathrm{Haus}\,\mathcal{X}$. Passing to a subsequence of $Q_n$ we may assume that

$$❶ \qquad |Q_n - Q_{n+1}|_{\mathrm{Haus}\,\mathcal{X}} \leqslant \tfrac{1}{10^n}$$

for each $n$.

Denote by $K_n$ the closed $\tfrac{1}{10^n}$-neighborhood of $Q_n$; that is,

$$K_n = \left\{\, x \in \mathcal{X} \;:\; \mathrm{dist}_{Q_n}(x) \leqslant \tfrac{1}{10^n} \,\right\}$$

Since $\mathcal{X}$ is compact so is each $K_n$.

[1] Partial limit is a limit of a subsequence.

By 4.2, $|Q_n - K_n|_{\mathrm{Haus}\,\mathcal{X}} \leqslant \frac{1}{10^n}$. From ❶, we get $K_n \supset K_{n+1}$ for each $n$. Set

$$K_\infty = \bigcap_{n=1}^{\infty} K_n.$$

By the monotone convergence (4.8), $|K_n - K_\infty|_{\mathrm{Haus}\,\mathcal{X}} \to 0$ as $n \to \infty$. Since $|Q_n - K_n|_{\mathrm{Haus}\,\mathcal{X}} \leqslant \frac{1}{10^n}$, we get $|Q_n - K_\infty|_{\mathrm{Haus}\,\mathcal{X}} \to 0$ as $n \to \infty$ — hence the lemma. □

**4.10. Exercise.** *Let $\mathcal{X}$ be a complete metric space and $K_1, K_2, \dots$ be a sequence of compact sets that converges in the sense of Hausdorff. Show that the union $K_1 \cup K_2 \cup \dots$ has compact closure.*

*Use this statement to show that in Lemma 4.9 compactness of $\mathcal{X}$ can be exchanged to completeness.*

*Proof of only-if part in 4.7.* According to Lemma 4.9, $\mathrm{Haus}\,\mathcal{X}$ is complete. It remains to show that $\mathrm{Haus}\,\mathcal{X}$ is totally bounded (1.8*c*); that is, given $\varepsilon > 0$ there is a finite $\varepsilon$-net in $\mathrm{Haus}\,\mathcal{X}$.

Choose a finite $\varepsilon$-net $A$ in $\mathcal{X}$. Denote by $B$ the set of all subsets of $A$. Note that $B$ is a finite set in $\mathrm{Haus}\,\mathcal{X}$. For each compact set $K \subset \mathcal{X}$, consider the subset $K'$ of all points $a \in A$ such that $\mathrm{dist}_K(a) \leqslant \varepsilon$. Observe that $K' \in B$ and $|K - K'|_{\mathrm{Haus}\,\mathcal{X}} \leqslant \varepsilon$. In other words, $B$ is a finite $\varepsilon$-net in $\mathrm{Haus}\,\mathcal{X}$. □

**4.11. Exercise.** *Let $\mathcal{X}$ be a complete metric space. Show that $\mathcal{X}$ is a length space if and only if so is* $\mathrm{Haus}\,\mathcal{X}$.

## C An application

The following statement is called isoperimetric inequality in the plane.

**4.12. Theorem.** *Among the plane figures bounded by closed curves of length at most $\ell$, the round disk has the maximal area.*

In this section, we will sketch a proof of the isoperimetric inequality that uses the Hausdorff convergence. It is based on the following exercise.

**4.13. Exercise.** *Let $\mathcal{C}$ be a subspace of* $\mathrm{Haus}\,\mathbb{R}^2$ *formed by all compact convex subsets in $\mathbb{R}^2$. Show that perimeter*[2] *and area are continuous*

---

[2]If the set degenerates to a line segment of length $\ell$, then its perimeter is defined as $2{\cdot}\ell$.

*on $\mathcal{C}$. That is, if a sequence of convex compact plane sets $X_n$ converges to $X_\infty$ in the sense of Hausdorff, then*

$$\operatorname{perim} X_n \to \operatorname{perim} X_\infty \quad \textit{and} \quad \operatorname{area} X_n \to \operatorname{area} X_\infty$$

*as $n \to \infty$.*

*Semiproof of 4.12.* It is sufficient to consider only convex figures of the given perimeter; if a figure is not convex, pass to its convex hull and observe that it has a larger area and smaller perimeter.

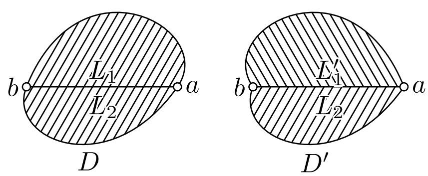
Note that the selection theorem (4.7) together with the exercise imply the existence of figure $D$ with perimeter $\ell$ and maximal area.

It remains to show that $D$ is a round disk. This is a problem in elementary geometry.

Let us cut $D$ along a chord $[ab]$ into two lenses, $L_1$ and $L_2$. Denote by $L_1'$ the reflection of $L_1$ across the perpendicular bisector of $[ab]$. Note that $D$ and $D' = L_1' \cup L_2$ have the same perimeter and area. That is, $D'$ has perimeter $\ell$ and maximal possible area; in particular, $D'$ is convex.

The following exercise will finish the proof. □

**4.14. Exercise.** *Suppose $D$ is a convex figure such that for any chord $[ab]$ of $D$ the above construction produces a convex figure $D'$. Show that $D$ is a round disk.*




Another popular way to prove that $D$ is a round disk is given by the so-called Steiner's 4-joint method [5].

## D Remarks

It seems that Hausdorff convergence was first introduced by Felix Hausdorff [15]. A couple of years later an equivalent definition was given by Wilhelm Blaschke [5].

The following refinement was introduced by Zdeněk Frolík [12], later it was rediscovered by Robert Wijsman [43]. This refinement is also called Hausdorff convergence; in fact, it takes an intermediate place between the original Hausdorff convergence and closed convergence, also introduced by Hausdorff in [15].

**4.15. Definition.** *Let $A_1, A_2, \dots$ be a sequence of closed sets in a metric space $\mathcal{X}$. We say that the sequence $A_n$ converges to a closed set*

*$A_\infty$ in the sense of Hausdorff if for any $x \in \mathcal{X}$, we have $\operatorname{dist}_{A_n}(x) \to$ $\to \operatorname{dist}_{A_\infty}(x)$ as $n \to \infty$.*

For example, suppose $\mathcal{X}$ is the Euclidean plane and $A_n$ is the circle with radius $n$ and center at the point $(n, 0)$. If we use the standard definition (4.1), then the sequence $(A_n)$ diverges, but it converges to the $y$-axis in the sense of Definition 4.15.

Further, consider the sequence of one-point sets $B_n = \{(n, 0)\}$ in the Euclidean plane. It diverges in the sense of the standard definition, but, in the sense of 4.15, it converges to the empty set; indeed, for any point $x$ we have $\operatorname{dist}_{B_n}(x) \to \infty$ as $n \to \infty$ and $\operatorname{dist}_\varnothing(x) = \infty$ for any $x$.

The following exercise is analogous to the Blaschke selection theorem (4.7) for the modified Hausdorff convergence.

**4.16. Exercise.** *Let $\mathcal{X}$ be a proper metric space and $A_1, A_2, \ldots$ be a sequence of closed sets in $\mathcal{X}$. Show that the sequence $A_1, A_2, \ldots$ has a convergent subsequence in the sense of Definition 4.15.*

# Lecture 5

# Space of spaces

## A Gromov–Hausdorff metric

The goal of this section is to cook up a metric space out of metric spaces. More precisely, we want to define the so-called Gromov–Hausdorff metric on the set of *isometry classes* of compact metric spaces. (Being isometric is an equivalence relation, and an isometry class is an equivalence class with respect to this relation.)

The obtained metric space will be denoted by GH. Given two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, denote by $[\mathcal{X}]$ and $[\mathcal{Y}]$ their isometry classes; that is, $\mathcal{X}' \in [\mathcal{X}]$ if and only if $\mathcal{X}' \overset{iso}{=\!=} \mathcal{X}$. Pedantically, the Gromov–Hausdorff distance from $[\mathcal{X}]$ to $[\mathcal{Y}]$ should be denoted as $|[\mathcal{X}]-[\mathcal{Y}]|_{\mathrm{GH}}$; but we will write it as $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}$ and say (not quite correctly) that $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}$ *is the Gromov–Hausdorff distance from* $\mathcal{X}$ *to* $\mathcal{Y}$. In other words, from now on the term *metric space* might also stand for its *isometry class*.

The metric on GH is defined as the maximal metric such that *the distance between subspaces in a metric space is not greater than the Hausdorff distance between them.* Here is a formal definition:

**5.1. Definition.** *The Gromov–Hausdorff distance $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}$ between compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is defined by the following relation.*

*Given $r > 0$, we have that $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}} < r$ if and only if there exists a metric space $\mathcal{W}$ and subspaces $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{W}$ that are isometric to $\mathcal{X}$ and $\mathcal{Y}$ respectively such that $|\mathcal{X}'-\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}} < r$. (Here $|\mathcal{X}'-\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}}$ denotes the Hausdorff distance between sets $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{W}$.)*

**5.2. Theorem.** *The set of isometry classes of compact metric spaces equipped with Gromov–Hausdorff metric forms a metric space (which* *is denoted by* GH*).*

*In other words, for arbitrary compact metric spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ the following conditions hold:*

(a) $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}} \geqslant 0$;

(b) $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}} = 0$ *if and only if* $\mathcal{X}$ *is isometric to* $\mathcal{Y}$;

(c) $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}} = |\mathcal{Y}-\mathcal{X}|_{\mathrm{GH}}$;

(d) $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}} + |\mathcal{Y}-\mathcal{Z}|_{\mathrm{GH}} \geqslant |\mathcal{X}-\mathcal{Z}|_{\mathrm{GH}}$.

Note that *(a)*, *(c)*, and the "if"-part of *(b)* follow directly from Definition 5.1. Part *(d)* will be proved in Section 5B. The "only-if"-part of *(b)* will be proved in Section 5C.

Recall that $a{\cdot}\mathcal{X}$ denotes $\mathcal{X}$ rescaled by factor $a>0$; that is, $a{\cdot}\mathcal{X}$ is a metric space with the underlying set of $\mathcal{X}$ and the metric defined by

$$|x-y|_{a\cdot\mathcal{X}} := a{\cdot}|x-y|_{\mathcal{X}}.$$

**5.3. Exercise.** *Let $\mathcal{X}$ be a compact metric space, $\mathcal{O}$ be the one-point metric space.*

*Prove that*

(a) $|\mathcal{X}-\mathcal{O}|_{\mathrm{GH}} = \frac{1}{2}\cdot\operatorname{diam}\mathcal{X}$.

(b) $|a{\cdot}\mathcal{X}-b{\cdot}\mathcal{X}|_{\mathrm{GH}} = \frac{1}{2}{\cdot}|a-b|\cdot\operatorname{diam}\mathcal{X}$.

(c) $\iota[\mathcal{O}]=[\mathcal{O}]$ *for any isometry* $\iota\colon \mathrm{GH}\to\mathrm{GH}$.

**5.4. Exercise.** *Find subsets $A, B\subset\mathbb{R}^2$ such that*

$$|A-B|_{\mathrm{GH}} > |A-\iota(B)|_{\mathrm{Haus}\,\mathbb{R}^2}$$

*for any isometry $\iota$ of $\mathbb{R}^2$.*

**5.5. Exercise.** *Let $\mathcal{A}_r$ be a rectangle $1$ by $r$ in the Euclidean plane and $\mathcal{B}_r$ be a closed line interval of length $r$. Show that*

$$|\mathcal{A}_r-\mathcal{B}_r|_{\mathrm{GH}} > \tfrac{1}{10}$$

*for all large $r$.*

**5.6. Advanced exercise.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact metric spaces; denote by $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$ their injective envelopes (see 3C). Show that*

$$|\hat{\mathcal{X}}-\hat{\mathcal{Y}}|_{\mathrm{GH}} \leqslant 2{\cdot}|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}.$$

*In other words $\mathcal{X}\mapsto\hat{\mathcal{X}}$ defines a 2-Lipschitz map* GH $\to$ GH.

## B Approximations and almost isometries

**5.7. Definition.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces. A relation $\approx$ between points in $\mathcal{X}$ and $\mathcal{Y}$ is called $\varepsilon$-approximation if the following conditions hold:*

- ⋄ *For any $x \in \mathcal{X}$ there is $y \in \mathcal{Y}$ such that $x \approx y$.*
- ⋄ *For any $y \in \mathcal{Y}$ there is $x \in \mathcal{X}$ such that $x \approx y$.*
- ⋄ *If for some $x, x' \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$ we have $x \approx y$ and $x' \approx y'$, then*
$$\big||x - x'|_{\mathcal{X}} - |y - y'|_{\mathcal{Y}}\big| < 2{\cdot}\varepsilon.$$

**5.8. Exercise.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be two compact metric spaces. Show that*
$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} < \varepsilon$$
*if and only if there is an $\varepsilon$-approximation between $\mathcal{X}$ and $\mathcal{Y}$.*

*In other words $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}}$ is the greatest lower bound of values $\varepsilon > 0$ such that there is an $\varepsilon$-approximation between $\mathcal{X}$ and $\mathcal{Y}$.*

*Proof of 5.2d.* Suppose that

- ⋄ $\approx_1$ is a relation between points in $\mathcal{X}$ and $\mathcal{Y}$,
- ⋄ $\approx_2$ is a relation between points in $\mathcal{Y}$ and $\mathcal{Z}$.

Consider the relation $\approx_3$ between points in $\mathcal{X}$ and $\mathcal{Z}$ such that $x \approx_3 z$ if and only if there is $y \in \mathcal{Y}$ such that $x \approx_1 y$ and $y \approx_2 z$.

It is straightforward to check that if $\approx_1$ is an $\varepsilon_1$-approximation and $\approx_2$ is an $\varepsilon_2$-approximation, then $\approx_3$ is an $(\varepsilon_1 + \varepsilon_2)$-approximation.

Applying 5.8, we get that if
$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} < \varepsilon_1 \quad \text{and} \quad |\mathcal{Y} - \mathcal{Z}|_{\mathrm{GH}} < \varepsilon_2,$$
then
$$|\mathcal{X} - \mathcal{Z}|_{\mathrm{GH}} < \varepsilon_1 + \varepsilon_2.$$
Hence 5.2$d$ follows. □

The following weakened version of isometry is closely related to $\varepsilon$-approximations.

**5.9. Definition.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces and $\varepsilon > 0$. A map[1] $f\colon \mathcal{X} \to \mathcal{Y}$ is called an $\varepsilon$-isometry if $f(\mathcal{X})$ is an $\varepsilon$-net in $\mathcal{Y}$ and*
$$\big||x - x'|_{\mathcal{X}} - |f(x) - f(x')|_{\mathcal{Y}}\big| < \varepsilon.$$
*for any $x, x' \in \mathcal{X}$.*

**5.10. Exercise.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact metric spaces.*

[1] possibly noncontinuous

(a) *If* $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}<\varepsilon$, *then there is a* $2{\cdot}\varepsilon$*-isometry* $f\colon\mathcal{X}\to\mathcal{Y}$.
(b) *If there is an* $\varepsilon$*-isometry* $f\colon\mathcal{X}\to\mathcal{Y}$, *then* $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}<\varepsilon$.

# C Optimal realization

Note that

$$|\mathcal{X}'-\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}}\geqslant|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}},$$

where $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}'$, $\mathcal{Y}'$, and $\mathcal{W}$ are as in 5.1. The following proposition states that equality holds for some choice of $\mathcal{X}'$, $\mathcal{Y}'$, and $\mathcal{W}$.

**5.11. Proposition.** *For any two compact metric spaces* $\mathcal{X}$ *and* $\mathcal{Y}$ *there is a metric space* $\mathcal{W}$ *with subsets* $\mathcal{X}'$ *and* $\mathcal{Y}'$ *such that* $\mathcal{X}'\stackrel{iso}{=\!=}\mathcal{X}$, $\mathcal{Y}'\stackrel{iso}{=\!=}\mathcal{Y}$, *and*

$$|\mathcal{X}'-\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}}=|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}.$$

Let us introduce the so-called *appropriate functions* and use them in a reinterpretation of Gromov–Hausdorff distance.

Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}'$, $\mathcal{Y}'$, and $\mathcal{W}$ are as in 5.1. By passing to the subspace $\mathcal{X}'\cup\mathcal{Y}'$ in $\mathcal{W}$, we can assume that $\mathcal{W}=\mathcal{X}'\cup\mathcal{Y}'$. Note that in this case the metric on $\mathcal{W}$ is completely determined by the function

$$f(x,y)=|x-y|_{\mathcal{W}};$$

a function $f\colon\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ that can appear this way will be called a p p r o p r i a t e.

Note that a function $f\colon\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ is appropriate if and only if $x\mapsto f(x,y)$ and $y\mapsto f(x,y)$ are extension functions; that is, if

❶
$$\begin{aligned}f(x,y)+f(x,y')&\geqslant|y-y'|_{\mathcal{Y}}\geqslant|f(x,y)-f(x,y')|,\\ f(x,y)+f(x',y)&\geqslant|x-x'|_{\mathcal{X}}\geqslant|f(x,y)-f(x',y)|;\end{aligned}$$

for any $x,x',\in\mathcal{X}$ and $y,y'\in\mathcal{X}$; see 2B. In other words, the following defines a pseudometric on $\mathcal{X}\sqcup\mathcal{Y}$

$$|x-y|_{\mathcal{X}\sqcup\mathcal{Y}}=\begin{cases}|x-y|_{\mathcal{X}} & \text{if } x,y\in\mathcal{X},\\ |x-y|_{\mathcal{Y}} & \text{if } x,y\in\mathcal{Y},\\ f(x,y) & \text{if } x\in\mathcal{X} \text{ and } y\in\mathcal{Y},\end{cases}$$

and the corresponding metric space $\mathcal{W}$ contains isometric copies of $\mathcal{X}$ and $\mathcal{Y}$.

Given an appropriate function $f\colon\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$, set

$$\begin{aligned}a_f&=\max_{x\in\mathcal{X}}\{\min_{y\in\mathcal{Y}}\{f(x,y)\}\},\\ b_f&=\max_{y\in\mathcal{Y}}\{\min_{x\in\mathcal{X}}\{f(x,y)\}\}.\end{aligned}$$

**5.12. Observation.** *If $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}'$, $\mathcal{Y}'$, and $\mathcal{W}$ as above then*

$$|\mathcal{X}'-\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}}=\inf_f\{a_f,b_f\}.$$

*Proof of 5.11.* By ❶, any appropriate functions $f\colon\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ is 2-Lipschitz. Observe that the functional $f\mapsto\min\{a_f,b_f\}$ is continuous. Applying the Arzelà–Ascoli theorem, we can get an appropriate function $f\colon\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ with minimal possible value $\min\{a_f,b_f\}$. It remains to apply 5.12. □

**5.13. Exercise.** *Construct three compact metric spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ such that for any metric space $\mathcal{W}$ with subsets $\mathcal{X}'$, $\mathcal{Y}'$, and $\mathcal{Z}'$ such that $\mathcal{X}'\overset{iso}{=\!=}\mathcal{X}$, $\mathcal{Y}'\overset{iso}{=\!=}\mathcal{Y}$, and $\mathcal{Z}'\overset{iso}{=\!=}\mathcal{Z}$ at least one of the following three inequalities is strict*

$$\begin{aligned}|\mathcal{X}'-\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{W}}&\geqslant|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}},\\|\mathcal{Y}'-\mathcal{Z}'|_{\mathrm{Haus}\,\mathcal{W}}&\geqslant|\mathcal{Y}-\mathcal{Z}|_{\mathrm{GH}},\\|\mathcal{Z}'-\mathcal{X}'|_{\mathrm{Haus}\,\mathcal{W}}&\geqslant|\mathcal{Z}-\mathcal{X}|_{\mathrm{GH}}.\end{aligned}$$

# D Convergence

The Gromov–Hausdorff metric is used to define Gromov–Hausdorff convergence. Namely, a sequence of compact metric spaces $\mathcal{X}_n$ converges to compact metric spaces $\mathcal{X}_\infty$ in the sense of Gromov–Hausdorff if

$$|\mathcal{X}_n-\mathcal{X}_\infty|_{\mathrm{GH}}\to0\quad\text{as}\quad n\to\infty.$$

This convergence is more important than the metric — in all applications, we use only the topology on GH and we do not care about the particular value of Gromov–Hausdorff distance between spaces. The following observation follows from 5.10:

**5.14. Observation.** *A sequence of compact metric spaces $(\mathcal{X}_n)$ converges to $\mathcal{X}_\infty$ in the sense of Gromov–Hausdorff if and only if there is a sequence $\varepsilon_n\to0+$ and an $\varepsilon_n$-isometry $f_n\colon\mathcal{X}_n\to\mathcal{X}_\infty$ for each $n$.*

In the following exercises, *convergence* is understood in the sense of Gromov–Hausdorff.

**5.15. Exercise.**

*(a) Show that a sequence of compact simply-connected length spaces cannot converge to a circle.*

(b) *Construct a sequence of compact simply-connected length spaces that converges to a compact non-simply-connected space.*

**5.16. Exercise.**

(a) *Show that a sequence of length metrics on the 2-sphere cannot converge to the unit disk.*

(b) *Construct a sequence of length metrics on the 3-sphere that converges to a unit 3-ball.*

# E Uniformly totally bonded families

**5.17. Definition.** *A family $\mathcal{Q}$ of (isometry classes) of compact metric spaces is called uniformly totally bonded if it meets the following two conditions:*

(a) *spaces in $\mathcal{Q}$ have uniformly bounded diameters; that is, there is $D \in \mathbb{R}$ such that*

$$\operatorname{diam} \mathcal{X} \leqslant D$$

*for any space $\mathcal{X}$ in $\mathcal{Q}$.*

(b) *For any $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that any space $\mathcal{X}$ in $\mathcal{Q}$ admits an $\varepsilon$-net with at most $n$ points.*

**5.18. Exercise.** *Let $\mathcal{Q}$ be a family of compact spaces with uniformly bounded diameters. Show that $\mathcal{Q}$ is uniformly totally bonded if for any $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that*

$$\operatorname{pack}_\varepsilon \mathcal{X} \leqslant n$$

*for any space $\mathcal{X}$ in $\mathcal{Q}$.*

Fix a real constant $C$. A Borel measure $\mu$ on a metric space $\mathcal{X}$ is called $C$-doubling if

$$\mu[\mathrm{B}(p, 2\cdot r)] < C\cdot\mu[\mathrm{B}(p, r)]$$

for any point $p \in \mathcal{X}$ and any $r > 0$. A Borel measure is called doubling if it is $C$-doubling for some real constant $C$.

**5.19. Exercise.** *Let $\mathcal{Q}(C, D)$ be the set of all the compact metric spaces with diameter at most $D$ that admit a $C$-doubling measure. Show that $\mathcal{Q}(C, D)$ is totally bounded.*

Given two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, we will write $\mathcal{X} \leqslant \mathcal{Y}$ if there is a distance-noncontracting map $f\colon \mathcal{X} \to \mathcal{Y}$; that is, if

$$|x - x'|_{\mathcal{X}} \leqslant |f(x) - f(x')|_{\mathcal{Y}}$$

for any $x, x' \in \mathcal{X}$.

**5.20. Exercise.**

(a) *Let $\mathcal{Y}$ be a compact metric space. Show that the set of all spaces $\mathcal{X}$ such that $\mathcal{X} \leqslant \mathcal{Y}$ is uniformly totally bounded.*

(b) *Show that for any uniformly totally bounded set $\mathcal{Q} \subset \mathrm{GH}$ there is a compact space $\mathcal{Y}$ such that $\mathcal{X} \leqslant \mathcal{Y}$ for any $\mathcal{X}$ in $\mathcal{Q}$.*

# F Gromov selection theorem

The following theorem is analogous to Blaschke selection theorems (4.7).

**5.21. Gromov selection theorem.** *Let $\mathcal{Q}$ be a closed subset of* GH. *Then $\mathcal{Q}$ is compact if and only if the spaces in $Q$ are uniformly totally bounded.*

**5.22. Lemma.** *The space* GH *is complete.*

Let us define gluing of metric spaces that will be used in the proof of the lemma.

Suppose $\mathcal{U}$ and $\mathcal{V}$ are metric spaces with isometric closed sets $A \subset \mathcal{U}$ and $A' \subset \mathcal{V}$; let $\iota\colon A \to A'$ be an isometry. Consider the space $\mathcal{W}$ of all equivalence classes in $\mathcal{U} \sqcup \mathcal{V}$ with the equivalence relation given by $a \sim \iota(a)$ for any $a \in A$.

It is straightforward to check that the following defines a metric on $\mathcal{W}$:

$$
\begin{aligned}
|u - u'|_{\mathcal{W}} &:= |u - u'|_{\mathcal{U}} \\
|v - v'|_{\mathcal{W}} &:= |v - v'|_{\mathcal{V}} \\
|u - v|_{\mathcal{W}} &:= \min\{\, |u - a|_{\mathcal{U}} + |v - \iota(a)|_{\mathcal{V}} \;:\; a \in A \,\}
\end{aligned}
$$

where $u, u' \in \mathcal{U}$ and $v, v' \in \mathcal{V}$.

The space $\mathcal{W}$ is called the g l u i n g of $\mathcal{U}$ and $\mathcal{V}$ along $\iota$; briefly, we can write $\mathcal{W} = \mathcal{U} \sqcup_\iota \mathcal{V}$. If one applies this construction to two copies of one space $\mathcal{U}$ with a set $A \subset \mathcal{U}$ and the identity map $\iota\colon A \to A$, then the obtained space is called the d o u b l i n g of $\mathcal{U}$ along $A$; this space can be denoted by $\sqcup^2_A \mathcal{U}$.

Note that the inclusions $\mathcal{U} \hookrightarrow \mathcal{W}$ and $\mathcal{V} \hookrightarrow \mathcal{W}$ are distance preserving. Therefore we can and will consider $\mathcal{U}$ and $\mathcal{V}$ as the subspaces of $\mathcal{W}$; this way the subsets $A$ and $A'$ will be identified and denoted further by $A$. Note that $A = \mathcal{U} \cap \mathcal{V} \subset \mathcal{W}$.

*Proof.* Let $\mathcal{X}_1, \mathcal{X}_2, \dots$ be a Cauchy sequence in GH. Passing to a subsequence if necessary, we can assume that $|\mathcal{X}_n - \mathcal{X}_{n+1}|_{\mathrm{GH}} < \frac{1}{2^n}$

for each $n$. In particular, for each $n$ there is a metric space $\mathcal{V}_n$ with distance preserving inclusions $\mathcal{X}_n \hookrightarrow \mathcal{V}_n$ and $\mathcal{X}_{n+1} \hookrightarrow \mathcal{V}_n$ such that

$$|\mathcal{X}_n - \mathcal{X}_{n+1}|_{\operatorname{Haus}\mathcal{V}_n} < \tfrac{1}{2^n}$$

for each $n$. Moreover, we may assume that $\mathcal{V}_n = \mathcal{X}_n \cup \mathcal{X}_{n+1}$.

Let us glue $\mathcal{V}_1$ to $\mathcal{V}_2$ along $\mathcal{X}_2$; to the obtained space glue $\mathcal{V}_3$ along $\mathcal{X}_3$, and so on. The obtained metric space $\mathcal{W}$ has an underlying set formed by the disjoint union of all $\mathcal{X}_n$ such that each inclusion $\mathcal{X}_n \hookrightarrow \hookrightarrow \mathcal{W}$ is distance preserving and

$$|\mathcal{X}_n - \mathcal{X}_{n+1}|_{\operatorname{Haus}\mathcal{W}} < \tfrac{1}{2^n}$$

for each $n$. In particular,

$$❶ \qquad |\mathcal{X}_m - \mathcal{X}_n|_{\operatorname{Haus}\mathcal{W}} < \tfrac{1}{2^{n-1}}$$

if $m > n$.

Denote by $\bar{\mathcal{W}}$ the completion of $\mathcal{W}$. Observe that the union $\mathcal{X}_1 \cup \cup \mathcal{X}_2 \cup \ldots \cup \mathcal{X}_n$ is compact and ❶ implies that it forms a $\frac{1}{2^{n-1}}$-net in $\bar{\mathcal{W}}$. Whence $\bar{\mathcal{W}}$ is compact; see 1.8*c* and 1.10.

Applying the Blaschke selection theorem (4.7), we can pass to a subsequence of $\mathcal{X}_n$ that converges in $\operatorname{Haus}\bar{\mathcal{W}}$; denote its limit by $\mathcal{X}_\infty$. It remains to observe that $\mathcal{X}_\infty$ is the Gromov–Hausdorff limit of $\mathcal{X}_n$. □

*Proof of 5.21; only-if part.* Suppose that there is no sequence $\varepsilon_n \to 0$ as described in 5.17. Observe that in this case there is a sequence of spaces $\mathcal{X}_n \in \mathcal{Q}$ such that

$$\operatorname{pack}_\delta \mathcal{X}_n \to \infty \quad \text{as} \quad n \to \infty$$

for some fixed $\delta > 0$.

Since $\mathcal{Q}$ is compact, this sequence has a partial limit, say $\mathcal{X}_\infty \in \mathcal{Q}$. Observe that $\operatorname{pack}_\delta \mathcal{X}_\infty = \infty$. Therefore, $\mathcal{X}_\infty$ is not compact — a contradiction.

*If part.* Given a positive integer $n$ consider the set of all metric spaces $\mathcal{W}_n$ with the number of points at most $n$ and diameter $\leqslant D$. Note that $\mathcal{W}_n$ is a compact set in GH for each $n$.

Let $D$ and $n = n(\varepsilon)$ be as in the definition of uniformly totally bonded families (5.17).

Note that an $\varepsilon$-net of any $\mathcal{X} \in \mathcal{Q}$ belongs to $\mathcal{W}_{n(\varepsilon)}$. Therefore, $\mathcal{W}_{n(\varepsilon)}$ is a compact $\varepsilon$-net of $\mathcal{Q}$ for any $\varepsilon > 0$. Since $\mathcal{Q}$ is closed in a complete space GH, it implies that $\mathcal{Q}$ is compact. □

**5.23. Exercise.** *Show that the space* GH *is*

*(a) length,*
*(b) geodesic.*

**5.24. Exercise.** *For two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $\mathcal{X} \leqslant \mathcal{Y}+\varepsilon$ if there is a map $f\colon \mathcal{X} \to \mathcal{Y}$ such that*

$$|x - x'|_{\mathcal{X}} \leqslant |f(x) - f(x')|_{\mathcal{Y}} + \varepsilon$$

*for any $x, x' \in \mathcal{X}$.*

*(a) Show that*

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}'} = \inf\{\,\varepsilon > 0 \,:\, \mathcal{X} \leqslant \mathcal{Y} + \varepsilon \quad \textit{and} \quad \mathcal{Y} \leqslant \mathcal{X} + \varepsilon\,\}$$

*defines a metric on the space of (isometry classes) of compact metric spaces.*

*(b) Moreover $|{*} - {*}|_{\mathrm{GH}'}$ is equivalent to the Gromov–Hausdorff metric; that is,*

$$|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \to 0 \qquad \Longleftrightarrow \qquad |\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}'} \to 0$$

*as $n \to \infty$.*

# G Universal ambient space

Recall that a metric space is called universal if it contains an isometric copy of any separable metric space (in particular, any compact metric space). Examples of universal spaces include $\mathcal{U}_\infty$ — the Urysohn space and $\ell^\infty$ — the space of bounded infinite sequences with the metric defined by sup-norm; see 2.12 and 2.3.

The following proposition says that the space $\mathcal{W}$ in Definition 5.1 can be exchanged to a fixed universal space.

**5.25. Proposition.** *Let $\mathcal{U}$ be a universal space. Then for any compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$ we have*

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} = \inf\{|\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{U}}\}$$

*where the greatest lower bound is taken over all pairs of sets $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{U}$ which isometric to $\mathcal{X}$ and $\mathcal{Y}$ respectively.*

*Proof of 5.25.* By the definition (5.1), we have that

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} \leqslant \inf\{|\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{U}}\};$$

it remains to prove the opposite inequality.

Suppse $|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}<\varepsilon$; let $\mathcal{X}'$, $\mathcal{Y}'$ and $\mathcal{W}$ be as in 5.1. We can assume that $\mathcal{W}=\mathcal{X}'\cup\mathcal{Y}'$; otherwise pass to the subspace $\mathcal{X}'\cup\mathcal{Y}'$ of $\mathcal{W}$. In this case, $\mathcal{W}$ is compact; in particular, it is separable.

Since $\mathcal{U}$ is universal, there is a distance-preserving embedding of $\mathcal{W}$ in $\mathcal{U}$; let us keep the same notation for $\mathcal{X}'$, $\mathcal{Y}'$, and their images. It follows that

$$|\mathcal{X}'-\mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{U}}<\varepsilon,$$

— hence the result. □

**5.26. Exercise.** *Let $\mathcal{U}_\infty$ be the Urysohn space. Given two compact sets $A$ and $B$ in $\mathcal{U}_\infty$ define*

$$\|A-B\|=\inf\{|A-\iota(B)|_{\mathrm{Haus}\,\mathcal{U}_\infty}\},$$

*where the greatest lower bound is taken for all isometrics $\iota$ of $\mathcal{U}_\infty$. Show that $\|{*}-{*}\|$ defines a pseudometric*[2] *on nonempty compact subsets of $\mathcal{U}_\infty$ and its corresponding metric space is isometric to* GH.

# H Remarks

Suppose $\mathcal{X}_n\xrightarrow{\mathrm{GH}}\mathcal{X}_\infty$, then there is a metric on the disjoint union

$$\boldsymbol{X}=\bigsqcup_{n\in\mathbb{N}\cup\{\infty\}}\mathcal{X}_n$$

that satisfies the following property:

**5.27. Property.** *The restriction of metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric, and $\mathcal{X}_n\xrightarrow{\mathrm{H}}\mathcal{X}_\infty$ as subsets in $\boldsymbol{X}$.*

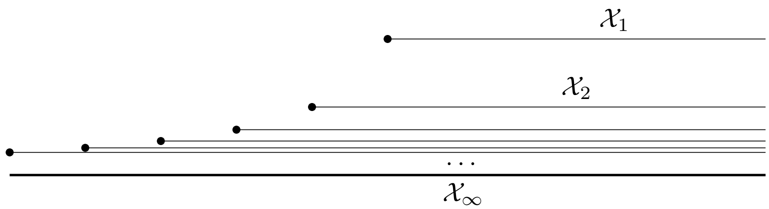
Indeed, since $\mathcal{X}_n\xrightarrow{\mathrm{GH}}\mathcal{X}_\infty$, there is a metric on $\mathcal{V}_n=\mathcal{X}_n\sqcup\mathcal{X}_\infty$ such that the restriction of metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric, and $|\mathcal{X}_n-\mathcal{X}_\infty|_{\mathrm{Haus}\,\mathcal{V}_n}<\varepsilon_n$ for some sequence $\varepsilon_n\to 0$. Gluing all $\mathcal{V}_n$ along $\mathcal{X}_\infty$, we obtain the required space $\boldsymbol{X}$.

In other words, the metric on $\boldsymbol{X}$ *defines* the convergence $\mathcal{X}_n\xrightarrow{\mathrm{GH}}$ $\xrightarrow{\mathrm{GH}}\mathcal{X}_\infty$. This metric makes it possible to talk about limits of sequences $x_n\in\mathcal{X}_n$ as $n\to\infty$, as well as weak limits of a sequence of Borel measures $\mu_n$ on $\mathcal{X}_n$ and so on.

For that reason, it is useful to define convergence by specifying the metric on $\boldsymbol{X}$ that satisfies the property for the variation of Hausdorff convergence described in Section 4D.

[2]The value $\|A-B\|$ is called Hausdorff distance up to isometry from $A$ to $B$ in $\mathcal{U}_\infty$.

This approach is more flexible; in particular, it can be used to define Gromov–Hausdorff convergence of arbitrary metric spaces (not necessarily compact). A limit space for this generalized convergence is not uniquely defined. For example, if each space $\mathcal{X}_n$ in the sequence is isometric to the half-line, then its limit might be isometric to the half-line or the whole line. The first convergence is evident and the second could be guessed from the diagram.




Often the isometry class of the limit can be fixed by marking a point $p_n$ in each space $\mathcal{X}_n$, it is called *pointed Gromov–Hausdorff convergence* — we say that $(\mathcal{X}_n, p_n)$ converges to $(\mathcal{X}_\infty, p_\infty)$ if there is a metric on $\boldsymbol{X}$ as in 5.27 such that $\mathcal{X}_n \xrightarrow{\text{H}} \mathcal{X}_\infty$ and $p_n \to p_\infty$. For example, the sequence $(\mathcal{X}_n, p_n) = (\mathbb{R}_+, 0)$ converges to $(\mathbb{R}_+, 0)$, while $(\mathcal{X}_n, p_n) = (\mathbb{R}_+, n)$ converges to $(\mathbb{R}, 0)$.

The pointed convergence works nicely for proper metric spaces; the following theorem is an analog of Gromov's selection theorem for this convergence.

**5.28. Theorem.** *Let $\mathcal{Q}$ be a set of isometry classes of pointed proper metric spaces $(\mathcal{X}, p)$. Assume that for any $R > 0$, the $R$-balls in the spaces centered at the marked points form a uniformly totally bounded family of spaces. Then $\mathcal{Q}$ is precompact with respect to the pointed Gromov–Hausdorff convergence.*

# Lecture 6

# Ultralimits

Ultralimits provide a very general way to pass to a limit. This procedure works for *any* sequence of metric spaces, its result reminds limit in the sense of Gromov–Hausdorff, but has some strange features; for example, the limit of a constant sequence of spaces $\mathcal{X}_n = \mathcal{X}$ is *not* $\mathcal{X}$ (see 6.13$b$).

In geometry, ultralimits are used only as a canonical way to pass to a convergent subsequence. It is useful in the proofs where one needs to repeat "pass to convergent subsequence" too many times.

This lecture is based on the introductory part of the paper by Bruce Kleiner and Bernhard Leeb [22].

## A Faces of ultrafilters

Recall that $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N} = \{1, 2, \dots\}$

**6.1. Definition.** *A finitely additive measure $\omega$ on $\mathbb{N}$ is called an ultrafilter if it satisfies the following condition:*

*(a) $\omega(\mathbb{N}) = 1$ and $\omega(S) = 0$ or $1$ for any subset $S \subset \mathbb{N}$.*

*An ultrafilter $\omega$ is called nonprincipal if in addition*

*(b) $\omega(F) = 0$ for any finite subset $F \subset \mathbb{N}$.*

If $\omega(S) = 0$ for some subset $S \subset \mathbb{N}$, we say that $S$ is $\omega$-small. If $\omega(S) = 1$, we say that $S$ contains $\omega$-almost all elements of $\mathbb{N}$.

**6.2. Advanced exercise.** *Let $\omega$ be an ultrafilter on $\mathbb{N}$ and $f\colon \mathbb{N} \to \mathbb{N}$. Suppose that $\omega(S) \leqslant \omega(f^{-1}(S))$ for any set $S \subset \mathbb{N}$. Show that $f(n) = n$ for $\omega$-almost all $n \in \mathbb{N}$.*

**Classical definition.** More commonly, a nonprincipal ultrafilter is defined as a collection, say $\mathfrak{F}$, of sets in $\mathbb{N}$ such that

1. if $P \in \mathfrak{F}$ and $Q \supset P$, then $Q \in \mathfrak{F}$,
2. if $P, Q \in \mathfrak{F}$, then $P \cap Q \in \mathfrak{F}$,
3. for any subset $P \subset \mathbb{N}$, either $P$ or its complement is an element of $\mathfrak{F}$.
4. if $F \subset \mathbb{N}$ is finite, then $F \notin \mathfrak{F}$.

Setting $P \in \mathfrak{F} \Leftrightarrow \omega(P) = 1$ makes these two definitions equivalent.

A nonempty collection of sets $\mathfrak{F}$ that does not include the empty set and satisfies only conditions 1 and 2 is called a filter; if in addition $\mathfrak{F}$ satisfies condition 3 it is called an ultrafilter. From Zorn's lemma, it follows that every filter contains an ultrafilter. Thus there is an ultrafilter $\mathfrak{F}$ contained in the filter of all complements of finite sets; clearly, this ultrafilter $\mathfrak{F}$ is nonprincipal.

**Stone–Čech compactification.** Given a set $S \subset \mathbb{N}$, consider subset $\Omega_S$ of all ultrafilters $\omega$ such that $\omega(S) = 1$. It is straightforward to check that the sets $\Omega_S$ for all $S \subset \mathbb{N}$ form a topology on the set of ultrafilters on $\mathbb{N}$. The obtained space is called Stone–Čech compactification of $\mathbb{N}$; it is usually denoted as $\beta\mathbb{N}$.

Let $\omega_n$ denotes the principal ultrafilter such that $\omega_n(\{n\}) = 1$; that is, $\omega_n(S) = 1$ if and only if $n \in S$. Note that $n \mapsto \omega_n$ defines a natural embedding $\mathbb{N} \hookrightarrow \beta\mathbb{N}$. Using the described embedding, we can (and will) consider $\mathbb{N}$ as a subset of $\beta\mathbb{N}$.

The space $\beta\mathbb{N}$ is the maximal compact Hausdorff space that contains $\mathbb{N}$ as an everywhere dense subset. More precisely, for any compact Hausdorff space $\mathcal{X}$ and a map $f \colon \mathbb{N} \to \mathcal{X}$ there is a unique continuous map $\bar{f} \colon \beta\mathbb{N} \to X$ such that the restriction $\bar{f}|_{\mathbb{N}}$ coincides with $f$.

# B Ultralimits of points

Let us fix a nonprincipal ultrafilter $\omega$ once and for all.

Assume $x_n$ is a sequence of points in a metric space $\mathcal{X}$. Let us define the $\omega$-limit of a sequence $x_1, x_2, \dots$ as the point $x_\omega$ such that for any $\varepsilon > 0$, point $x_n$ lie in $\mathrm{B}(x_\omega, \varepsilon)$ for $\omega$-almost all $n$; that is,

$$\omega\,\{\, n \in \mathbb{N} \,:\, |x_\omega - x_n| < \varepsilon \,\} = 1.$$

In this case, we will write

$$x_\omega = \lim_{n \to \omega} x_n \quad \text{or} \quad x_n \to x_\omega \text{ as } n \to \omega.$$

For example, if $\omega_n$ is the *principal* ultrafilter such that $\omega_n\{n\} = 1$ for some $n \in \mathbb{N}$, then $x_{\omega_n} = x_n$.

The sequence $x_n$ can be regarded as a map $\mathbb{N} \to \mathcal{X}$ defined by $n \mapsto x_n$. If $\mathcal{X}$ is compact, then the map $\mathbb{N} \to \mathcal{X}$ can be extended to a continuous map $\beta\mathbb{N} \to \mathcal{X}$ from the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Then the $\omega$-limit $x_\omega$ can be regarded as the image of $\omega$.

Note that the $\omega$-limits of a sequence and its subsequence may differ. For example, sequence $y_n = -(-1)^n$ is a subsequence of $x_n = (-1)^n$, but for any ultrafilter $\omega$, we have

$$\lim_{n\to\omega} x_n \neq \lim_{n\to\omega} y_n.$$

**6.3. Proposition.** *Let $\omega$ be a nonprincipal ultrafilter. Assume $x_n$ is a sequence of points in a metric space $\mathcal{X}$ and $x_n \to x_\omega$ as $n \to \omega$. Then $x_\omega$ is a partial limit of the sequence $x_n$; that is, there is a subsequence $(x_n)_{n\in S}$ that converges to $x_\omega$ in the usual sense.*

*Proof.* Given $\varepsilon > 0$, set $S_\varepsilon = \{\, n \in \mathbb{N} : |x_n - x_\omega| < \varepsilon \,\}$.

Note that $\omega(S_\varepsilon) = 1$ for any $\varepsilon > 0$. Since $\omega$ is nonprincipal, the set $S_\varepsilon$ is infinite. Therefore, we can choose an increasing sequence $n_k$ such that $n_k \in S_{\frac{1}{k}}$ for each $k \in \mathbb{N}$. Clearly $x_{n_k} \to x_\omega$ as $k \to \infty$. $\square$

The following proposition is analogous to the statement that any sequence in a compact metric space has a convergent subsequence; it can be proved the same way.

**6.4. Proposition.** *Let $\mathcal{X}$ be a compact metric space. Then any sequence $x_n$ of points in $\mathcal{X}$ has a unique $\omega$-limit $x_\omega$.*

*In particular, a bounded sequence of real numbers has a unique $\omega$-limit.*

The following lemma is an ultralimit analog of the Cauchy convergence test.

**6.5. Lemma.** *Let $x_n$ be a sequence of points in a complete space $\mathcal{X}$. Assume for each subsequence $y_n$ of $x_n$, the $\omega$-limit*

$$y_\omega = \lim_{n\to\omega} y_n \in \mathcal{X}$$

*is defined and does not depend on the choice of subsequence, then the sequence $x_n$ converges in the usual sense.*

*Proof.* If $x_n$ is not a Cauchy sequence, then for some $\varepsilon > 0$, there is a subsequence $y_n$ of $x_n$ such that $|x_n - y_n| \geqslant \varepsilon$ for all $n$.

It follows that $|x_\omega - y_\omega| \geqslant \varepsilon$ — a contradiction. $\square$

**6.6. Exercise.** *Denote by $S$ the space of bounded sequences of real numbers. Show that there is a linear functional $L\colon S \to \mathbb{R}$ such that for any sequence $\boldsymbol{s} = (s_1, s_2, \dots) \in S$ the image $L(\boldsymbol{s})$ is a partial limit of $s_1, s_2, \dots$*

**6.7. Exercise.** *Suppose that $f\colon \mathbb{N}\to\mathbb{N}$ is a map such that*

$$\lim_{n\to\omega} x_n = \lim_{n\to\omega} x_{f(n)}$$

*for any bounded sequence $x_n$ of real numbers. Show that $f(n)=n$ for $\omega$-almost all $n\in\mathbb{N}$.*

# C An illustration

**6.8. Claim.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact spaces. Suppose that for every $n\in\mathbb{N}$ there is a $\tfrac1n$-isometry $f_n\colon \mathcal{X}\to\mathcal{Y}$. Then there is an isometry $\mathcal{X}\to\mathcal{Y}$.*

We give a proof of this claim only as an illustration for ulralimits.

*Proof.* Consider the $\omega$-limit $f_\omega$ of $f_n$; according to 6.4, $f_\omega$ is defined. Since

$$|f_n(x)-f_n(x')| \lesseqgtr |x-x'| \pm \tfrac1n$$

we get that

$$|f_\omega(x)-f_\omega(x')| = |x-x'|$$

for any $x,x'\in\mathcal{X}$; that is, $f_\omega$ is distance-preserving.

Further, since $f_n$ is a $\tfrac1n$-isometry, for any $y\in\mathcal{Y}$ there is a sequence $x_n\in\mathcal{X}$ such that $|f_n(x_n)-y|\leqslant\tfrac1n$. Therefore,

$$f_\omega(x_\omega)=y,$$

where $x_\omega$ is the $\omega$-limit of $x_n$; that is, $f_\omega$ is onto.

It follows that $f_\omega\colon \mathcal{X}\to\mathcal{Y}$ is an isometry. □

# D Ultralimits of spaces

Recall that $\omega$ denotes a nonprincipal ultrafilter on the set of natural numbers.

Let $\mathcal{X}_n$ be a sequence of metric spaces. Consider all sequences of points $x_n\in\mathcal{X}_n$. On the set of all such sequences, define a pseudometric by

❶ $$|(x_n)-(y_n)| = \lim_{n\to\omega} |x_n-y_n|_{\mathcal{X}_n}.$$

Note that the $\omega$-limit on the right-hand side is always defined and takes a value in $[0,\infty]$. (The $\omega$-convergence to $\infty$ is defined analogously to the usual convergence to $\infty$).

Set $\mathcal{X}_\omega$ to be the corresponding metric space; that is, the underlying set of $\mathcal{X}_\omega$ is formed by classes of equivalence of sequences of points $x_n \in \mathcal{X}_n$ defined by

$$(x_n) \sim (y_n) \Leftrightarrow \lim_{n\to\omega} |x_n - y_n| = 0$$

and the distance is defined by ❶.

The space $\mathcal{X}_\omega$ is called the $\omega$-limit of $\mathcal{X}_n$. Typically $\mathcal{X}_\omega$ will denote the $\omega$-limit of sequence $\mathcal{X}_n$; we may also write

$$\mathcal{X}_n \to \mathcal{X}_\omega \quad \text{as} \quad n \to \omega \quad \text{or} \quad \mathcal{X}_\omega = \lim_{n\to\omega} \mathcal{X}_n.$$

Given a sequence $x_n \in \mathcal{X}_n$, we will denote by $x_\omega$ its equivalence class which is a point in $\mathcal{X}_\omega$; it can be written as

$$x_n \to x_\omega \quad \text{as} \quad n \to \omega, \quad \text{or} \quad x_\omega = \lim_{n\to\omega} x_n.$$

**6.9. Observation.** *The $\omega$-limit of any sequence of metric spaces is complete.*

We will repeat the proof of 1.7 using a slightly different language.

*Proof.* Let $\mathcal{X}_n$ be a sequence of metric spaces and $\mathcal{X}_n \to \mathcal{X}_\omega$ as $n \to \omega$. Choose a Cauchy sequence $x_1, x_2, \ldots \in \mathcal{X}_\omega$. Passing to a subsequence, we can assume that $|x_k - x_m|_{\mathcal{X}_\omega} < \frac{1}{k}$ if $k < m$.

Choose a double sequence $x_{n,m} \in \mathcal{X}_n$ such that for any fixed $m$ we have $x_{n,m} \to x_m$ as $n \to \omega$. Note that for any $k < m$ the inequality $|x_{n,k} - x_{n,m}| < \frac{1}{k}$ holds for $\omega$-almost all $n$. It follows that we can choose a nested sequence of sets

$$\mathbb{N} = S_1 \supset S_2 \supset \ldots$$

such that

- ◇ $\omega(S_m) = 1$ for each $m$,
- ◇ $\bigcap_m S_m = \varnothing$, and
- ◇ $|x_{n,k} - x_{n,l}| < \frac{1}{k}$ for $k < l \leqslant m$ and $n \in S_m$.

Consider the sequence $y_n = x_{n,m(n)}$, where $m(n)$ is the largest value such that $n \in S_{m(n)}$. Denote by $y_\omega \in \mathcal{X}_\omega$ the $\omega$-limit of $y_n$.

Observe that $|y_m - x_{n,m}| < \frac{1}{m}$ for $\omega$-almost all $n$. It follows that $|x_m - y_\omega| \leqslant \frac{1}{m}$ for any $m$. Therefore, $x_n \to y_\omega$ as $n \to \infty$. That is, any Cauchy sequence in $\mathcal{X}_\omega$ converges. $\square$

**6.10. Observation.** *The $\omega$-limit of any sequence of length spaces is geodesic.*

*Proof.* If $\mathcal{X}_n$ is a sequence of length spaces, then for any sequence of pairs $x_n, y_n \in X_n$ there is a sequence of $\frac{1}{n}$-midpoints $z_n$.

Let $x_n \to x_\omega$, $y_n \to y_\omega$ and $z_n \to z_\omega$ as $n \to \omega$. Note that $z_\omega$ is a midpoint of $x_\omega$ and $y_\omega$ in $\mathcal{X}_\omega$.

By Observation 6.9, $\mathcal{X}_\omega$ is complete. Applying Lemma 1.23 we get the statement. □

**6.11. Exercise.** *Show that an ultralimit of metric trees is a metric tree.*

**6.12. Exercise.** *Suppose that $\mathcal{X}_\infty$ and $\mathcal{X}_1, \mathcal{X}_2, \dots$ are compact metric spaces. Assume $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$. Show that $\mathcal{X}_\omega \overset{iso}{=\!=} \mathcal{X}_\infty$.*

## E Ultrapower

If all the metric spaces in the sequence are identical $\mathcal{X}_n = \mathcal{X}$, its $\omega$-limit $\lim_{n\to\omega} \mathcal{X}_n$ is denoted by $\mathcal{X}^\omega$ and called $\omega$-power of $\mathcal{X}$.

**6.13. Exercise.** *For any point $x \in \mathcal{X}$, consider the constant sequence $x_n = x$ and set $\iota(x) = \lim_{n\to\omega} x_n \in \mathcal{X}^\omega$.*

- *(a) Show that $\iota\colon \mathcal{X} \to \mathcal{X}^\omega$ is distance-preserving embedding. (So we can and will consider $\mathcal{X}$ as a subset of $\mathcal{X}^\omega$.)*
- *(b) Show that $\iota$ is onto if and only if $\mathcal{X}$ is compact.*
- *(c) Show that if $\mathcal{X}$ is proper, then $\iota(\mathcal{X})$ forms a metric component of $\mathcal{X}^\omega$; that is, a subset of $\mathcal{X}^\omega$ that lies at a finite distance from a given point.*

Note that *(b)* implies that the inclusion $\mathcal{X} \hookrightarrow \mathcal{X}^\omega$ is not onto if the space $\mathcal{X}$ is not compact. However, the spaces $\mathcal{X}$ and $\mathcal{X}^\omega$ might be isometric; here is an example:

**6.14. Exercise.** *Let $\mathcal{X}$ be a countable set with discrete metric; that is $|x - y|_{\mathcal{X}} = 1$ if $x \neq y$. Show that*

- *(a) $\mathcal{X}^\omega$ is not isometric to $\mathcal{X}$.*
- *(b) $\mathcal{X}^\omega$ is isometric to $(\mathcal{X}^\omega)^\omega$.*

**6.15. Exercise.** *Given a nonprincipal ultrafilter $\omega$, construct an ultrafilter $\omega_1$ such that*

$$\mathcal{X}^{\omega_1} \overset{iso}{=\!=} (\mathcal{X}^\omega)^\omega$$

*for any metric space $\mathcal{X}$.*

**6.16. Observation.** *Let $\mathcal{X}$ be a complete metric space. Then $\mathcal{X}^\omega$ is geodesic space if and only if $\mathcal{X}$ is a length space.*

*Proof.* The "if"-part follows from 6.10; it remains to prove the "only-if"-part

Assume $\mathcal{X}^\omega$ is geodesic space. Then any pair of points $x, y \in \mathcal{X}$ has a midpoint $z_\omega \in \mathcal{X}^\omega$. Fix a sequence of points $z_n \in \mathcal{X}$ such that $z_n \to z_\omega$ as $n \to \omega$.

Note that $|x - z_n|_{\mathcal{X}} \to \frac{1}{2}\cdot|x - y|_{\mathcal{X}}$ and $|y - z_n|_{\mathcal{X}} \to \frac{1}{2}\cdot|x - y|_{\mathcal{X}}$ as $n \to \omega$. In particular, for any $\varepsilon > 0$, the point $z_n$ is an $\varepsilon$-midpoint of $x$ and $y$ for $\omega$-almost all $n$. It remains to apply 1.23. □

**6.17. Exercise.** *Assume $\mathcal{X}$ is a complete length space and $p, q \in \mathcal{X}$ cannot be joined by a geodesic in $\mathcal{X}$. Then there are at least two distinct geodesics between $p$ and $q$ in the ultrapower $\mathcal{X}^\omega$.*

**6.18. Exercise.** *Construct a proper metric space $\mathcal{X}$ such that $\mathcal{X}^\omega$ is not proper; that is, there is a point $p \in \mathcal{X}^\omega$ and $R < \infty$ such that the closed ball $\overline{\mathrm{B}}[p, R]_{\mathcal{X}^\omega}$ is not compact.*

# F Tangent and asymptotic spaces

Choose a space $\mathcal{X}$ and a sequence $\lambda_n$ of positive numbers. Consider the sequence of rescalings $\mathcal{X}_n = \lambda_n\cdot\mathcal{X} = (\mathcal{X}, \lambda_n\cdot|* - *|_{\mathcal{X}})$.

Choose a point $p \in \mathcal{X}$ and denote by $p_n$ the corresponding point in $\mathcal{X}_n$. Consider the $\omega$-limit $\mathcal{X}_\omega$ of $\mathcal{X}_n$ (one may denote it by $\lambda_\omega\cdot\mathcal{X}$); set $p_\omega$ to be the $\omega$-limit of $p_n$.

If $\lambda_n \to \infty$ as $n \to \omega$, then the metric component of $p_\omega$ in $\mathcal{X}_\omega$ is called $\lambda_\omega$-tangent space at $p$ and denoted by $\mathrm{T}_p^{\lambda_\omega}\mathcal{X}$ (or $\mathrm{T}_p^{\omega}\mathcal{X}$ if $\lambda_n = n$).

If $\lambda_n \to 0$ as $n \to \omega$, then the metric component of $p_\omega$ is called $\lambda_\omega$-asymptotic space[1] and denoted by $\operatorname{Asym}\mathcal{X}$ or $\operatorname{Asym}^{\lambda_\omega}\mathcal{X}$. Note that the space $\operatorname{Asym}\mathcal{X}$ and its point $p_\omega$ does not depend on the choice of $p \in \mathcal{X}$.

The following exercise states that the constructions above depend on the sequence $\lambda_n$ and a nonprincipal ultrafilter $\omega$.

**6.19. Exercise.** *Construct a metric space $\mathcal{X}$ with a point $p$ such that the tangent space $\mathrm{T}_p^{\lambda_\omega}\mathcal{X}$ depends on the sequence $\lambda_n$ and/or ultrafilter $\omega$.*

For nice spaces, different choices may give the same space.

**6.20. Exercise.** *Let $\mathcal{L}$ be the Lobachevsky plane; $\mathcal{T} = \operatorname{Asym}\mathcal{L}$.*

*(a) Show that $\mathcal{T}$ is a complete metric tree.*

[1] Often it is called an asymptotic cone despite that it is not a cone in general; this name is used since in good cases it has a cone structure.

(b) *Show that $\mathcal{T}$ is one-point-homogeneous; that is, given two points $s, t \in \mathcal{T}$ there is an isometry of $\mathcal{T}$ that maps $s$ to $t$.*

(c) *Show that $\mathcal{T}$ has continuum degree at any point; that is, for any point $t \in \mathcal{T}$ the set of connected components of the complement $\mathcal{T} \setminus \{t\}$ has cardinality continuum.*

(d) *Prove (a)–(c) if $\mathcal{L}$ is Lobachevsky space and/or for the infinite 3-regular*[2] *metric tree with unit edge length.*

# G Remarks

A nonprincipal ultrafilter $\omega$ is called selective if for any partition of $\mathbb{N}$ into sets $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ such that $\omega(C_\alpha) = 0$ for each $\alpha$, there is a set $S \subset \mathbb{N}$ such that $\omega(S) = 1$ and $S \cap C_\alpha$ is a one-point set for each $\alpha \in \mathcal{A}$.

The existence of a selective ultrafilter follows from the continuum hypothesis [36].

If needed, we may assume that the chosen ultrafilter $\omega$ is selective. In this case *the subsequence $(x_n)_{n\in S}$ in 6.3 can be chosen so that $\omega(S) = 1$.*

[2] that is, the degree of any vertex is 3.

# Appendix A

# Semisolutions

**1.2.** Add four triangle inequalities (1.1$d$).

**1.3.** Consider the function

$$f(x) = \tfrac{\mathrm{dist}_A x}{\mathrm{dist}_A x + \mathrm{dist}_B x},$$

where $\mathrm{dist}_A x := \inf_{a\in A} |a - x|$. Show that $f$ is continuous and satisfies the needed property.

**1.4.** Use 1.3 to construct an approximation of the needed function and pass to a limit. Alternatively, read about the *Tietze extension theorem*.

**1.5**;*(a).* Note that if $\mu(A) = \mu(B) = 0$, then $|A - B| = 0$. Therefore, 1.1$b$ does not hold for bounded closed subsets. It is straightforward to check that for bounded measurable sets the remaining conditions in 1.1 hold true.

*(b).* Note that the distance from the empty set to the whole plane is infinite; so the value $|A - B|$ might be infinite. It is straightforward to check the remaining conditions in 1.1.

**1.6.** Assume the statement is wrong. Then for any point $x \in \mathcal{X}$, there is a point $x' \in \mathcal{X}$ such that

$$|x - x'| < \rho(x) \quad \text{and} \quad \rho(x') \leqslant \frac{\rho(x)}{1+\varepsilon}.$$

Consider a sequence $x_n$ of points such that $x_{n+1} = x_n'$. Clearly

$$|x_{n+1} - x_n| \leqslant \frac{\rho(x_0)}{\varepsilon\cdot(1+\varepsilon)^n} \quad \text{and} \quad \rho(x_n) \leqslant \frac{\rho(x_0)}{(1+\varepsilon)^n}.$$

Therefore, the sequence $x_n$ is Cauchy. Since $\mathcal{X}$ is complete, $x_n$ converges; denote its limit by $x_\infty$. Since $\rho$ is a continuous function we get

$$\begin{aligned}\rho(x_\infty) &= \lim_{n\to\infty} \rho(x_n) = \\ &= 0.\end{aligned}$$

The latter contradicts that $\rho > 0$.

**1.7.** Let $\bar{\mathcal{X}}$ be the completion of $\mathcal{X}$. By the definition, for any $y \in \bar{\mathcal{X}}$ there is a Cauchy sequence $x_n$ in $\mathcal{X}$ that converges to $y$.

Choose a Cauchy sequence $y_m$ in $\bar{\mathcal{X}}$. From above, we can choose points $x_{n,m} \in \mathcal{X}$ such that $x_{n,m} \to y_m$ for any $m$. Choose $z_m = x_{n_m,m}$ such that $|y_m - z_m| < \frac{1}{m}$. Observe that $z_m$ is Cauchy. Therefore, its limit $z_\infty$ lie in $\bar{\mathcal{X}}$. Finally, show that $x_m \to z_\infty$.

**1.10.** A compact $\varepsilon$-net $N$ in $\mathcal{K}$ contains a finite $\varepsilon$ net $F$. Show and use that $F$ is a $2{\cdot}\varepsilon$-net of $\mathcal{K}$.

**1.12.** Given a pair of points $x_0, y_0 \in \mathcal{K}$, consider two sequences $x_0, x_1, \dots$ and $y_0, y_1, \dots$ such that $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ for each $n$.

Since $\mathcal{K}$ is compact, we can choose an increasing sequence of integers $n_k$ such that both sequences $(x_{n_i})_{i=1}^\infty$ and $(y_{n_i})_{i=1}^\infty$ converge. In particular, both are Cauchy; that is,

$$|x_{n_i} - x_{n_j}|_{\mathcal{K}}, |y_{n_i} - y_{n_j}|_{\mathcal{K}} \to 0 \quad \text{as} \quad \min\{i,j\} \to \infty.$$

Since $f$ is distance-noncontracting, we get

$$|x_0 - x_{|n_i - n_j|}| \leqslant |x_{n_i} - x_{n_j}|.$$

It follows that there is a sequence $m_i \to \infty$ such that

$$(*) \qquad x_{m_i} \to x_0 \quad \text{and} \quad y_{m_i} \to y_0 \quad \text{as} \quad i \to \infty.$$

Set

$$\ell_n = |x_n - y_n|_{\mathcal{K}}.$$

Since $f$ is distance-noncontracting, the sequence $(\ell_n)$ is nondecreasing.

By $(*)$, $\ell_{m_i} \to \ell_0$ as $m_i \to \infty$. It follows that $(\ell_n)$ is a constant sequence.

In particular,

$$|x_0 - y_0|_{\mathcal{K}} = \ell_0 = \ell_1 = |f(x_0) - f(y_0)|_{\mathcal{K}}$$

for any pair of points $(x_0, y_0)$ in $\mathcal{K}$. That is, the map $f$ is distance-preserving and, in particular, injective.

From $(*)$, we also get that $f(\mathcal{K})$ is everywhere dense. Since $\mathcal{K}$ is compact $f\colon \mathcal{K} \to \mathcal{K}$ is surjective — hence the result.

*Remarks.* This is a basic lemma in the introduction to Gromov–Hausdorff distance [see 7.3.30 in 8]. The presented proof is not quite standard, I learned it from Travis Morrison, a student in my MASS class at Penn State, Fall 2011.

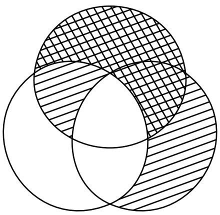
Note that this exercise implies that *any surjective non-expanding map from a compact metric space to itself is an isometry.*

**1.13.** Consider an infinite discrete space.

**1.14.** The conditions *(a)–(c)* in Definition 1.1 are evident.

The triangle inequality *(d)* follows since

$$(*) \qquad [B(x,\tfrac{\pi}{2}) \setminus B(y,\tfrac{\pi}{2})] \cup [B(y,\tfrac{\pi}{2}) \setminus B(z,\tfrac{\pi}{2})] \supseteq B(x,\tfrac{\pi}{2}) \setminus B(z,\tfrac{\pi}{2}).$$



Observe that $B(x,\frac{\pi}{2}) \setminus B(y,\frac{\pi}{2})$ does not overlap with $B(y,\frac{\pi}{2}) \setminus B(z,\frac{\pi}{2})$ and we get equality in $(*)$ if and only if $y$ lies on the great circle arc from $x$ to $z$. Therefore, the second statement follows.

*Remarks.* This construction was given by Aleksei Pogorelov [35]. It is closely related to the construction given by David Hilbert [16] which was the motivating example for his 4-th problem.

**1.15.** Without loss of generality, we may assume that non of the points $p, x, y, z$ lie on a geodesic between other two.

Let $K$ be the set in the tree covered by all six geodesics with the endpoints $p, x, y, z$. Observe that $K$ looks like an H or like an X; make a conclusion.

*Remarks.* In fact a four-point metric space admits an isometric embedding into a metric tree if and only if one of these two equivalent conditions hold. Moreover, a finite metric space admits an isometric embedding into a metric tree if every its 4-point subspace admits such embedding.

The value $\frac{1}{2}\cdot(|p-x|+|p-y|-|x-y|)$ is called Gromov's product of $x$ and $y$ with the origin at $p$; usually it is denoted by $(x|y)_p$.

**1.16.** Apply 1.15.

**1.20.** Formally speaking, a one-point space is a solution, but we will construct a nontrivial example.

Consider the unit ball $(B, \rho_0)$ in the space $c_0$ of all sequences converging to zero equipped with the sup-norm.

Consider another metric $\rho_1$ which is different from $\rho_0$ by the conformal factor

$$\varphi(\boldsymbol{x}) = 2 + \tfrac{1}{2}{\cdot}x_1 + \tfrac{1}{4}{\cdot}x_2 + \tfrac{1}{8}{\cdot}x_3 + \dots,$$

where $\boldsymbol{x} = (x_1, x_2 \dots) \in B$. That is, if $t \mapsto \boldsymbol{x}(t)$ for $t \in [0, \ell]$ is a curve parametrized by $\rho_0$-length, then its $\rho_1$-length is defined by

$$\operatorname{length}_{\rho_1} \boldsymbol{x} := \int\limits_0^\ell \varphi \circ \boldsymbol{x}(t){\cdot}dt.$$

Note that the metric $\rho_1$ is bilipschitz to $\rho_0$.

Assume $t \mapsto \boldsymbol{x}(t)$ and $t \mapsto \boldsymbol{x}'(t)$ are two curves parametrized by $\rho_0$-length that differ only in the $m$-th coordinate; denote them by $x_m(t)$ and $x'_m(t)$ respectively. Note that if $x'_m(t) \leqslant x_m(t)$ for any $t$ and the function $x'_m(t)$ is locally 1-Lipschitz at all $t$ such that $x'_m(t) < x_m(t)$, then

$$\operatorname{length}_{\rho_1} \boldsymbol{x}' \leqslant \operatorname{length}_{\rho_1} \boldsymbol{x}.$$

Moreover, this inequality is strict if $x'_m(t) < x_m(t)$ for some $t$.

Fix a curve $\boldsymbol{x}(t)$, $t \in [0, \ell]$, parametrized by $\rho_0$-length. We can choose large $m$ so that $x_m(t)$ is sufficiently close to 0 for any $t$. In this case, it is easy to construct a function $t \mapsto x'_m$ that meets the above properties. It follows that for any curve $\boldsymbol{x}(t)$ in $(B, \rho_1)$, we can find a shorter curve $\boldsymbol{x}'(t)$ with the same endpoints. In particular, $(B, \rho_1)$ has no geodesics.

*Remark.* This solution was suggested by Fedor Nazarov [29].

**1.21.** Choose a sequence of positive numbers $\varepsilon_n \to 0$ and a finite $\varepsilon_n$-net $N_n$ of $K$ for each $n$. Assume $N_0$ is a one-point set, so $\varepsilon_0 > \operatorname{diam} K$. If $|x - y| < \varepsilon_k$ for some $x \in N_{k+1}$ and $y \in N_k$, then connect them by a curve of length at most $\varepsilon_k$.

Consider the union $K'$ of all these curves with $K$; observe that $K'$ is compact and path-connected.

*Source:* This problem was suggested by Eugene Bilokopytov [4].

**1.22.** Choose a Cauchy sequence $x_n$ in $(\mathcal{X}, \|{*} - {*}\|)$; it is sufficient to show that a subsequence of $x_n$ converges.

Note that the sequence $x_n$ is Cauchy in $(\mathcal{X}, |{*} - {*}|)$; denote its limit by $x_\infty$.

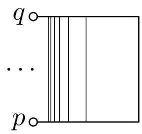
Passing to a subsequence, we can assume that $\|x_n - x_{n+1}\| < \frac{1}{2^n}$. It follows that there is a 1-Lipschitz path $\gamma$ in $(\mathcal{X}, \|{*} - {*}\|)$ such that $x_n = \gamma(\frac{1}{2^n})$ for each $n$ and $x_\infty = \gamma(0)$.

It follows that

$$\begin{aligned}\|x_\infty - x_n\| &\leqslant \operatorname{length}\gamma|_{[0,\frac{1}{2^n}]} \leqslant \\ &\leqslant \tfrac{1}{2^n}.\end{aligned}$$

In particular, $x_n$ converges to $x_\infty$ in $(\mathcal{X}, \| * - * \|)$.

*Source:* [17, Corollary]; see also [31, Lemma 2.3].




**1.26.** Consider the following subset of $\mathbb{R}^2$ equipped with the induced length metric

$$\mathcal{X} = \big((0,1] \times \{0,1\}\big) \cup \big(\{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\} \times [0,1]\big)$$

Note that $\mathcal{X}$ is locally compact and geodesic.

Its completion $\bar{\mathcal{X}}$ is isometric to the closure of $\mathcal{X}$ equipped with the induced length metric. Note that $\bar{\mathcal{X}}$ is obtained from $\mathcal{X}$ by adding two points $p = (0,0)$ and $q = (0,1)$.

Observe that the point $p$ admits no compact neighborhood in $\bar{\mathcal{X}}$ and there is no geodesic connecting $p$ to $q$ in $\bar{\mathcal{X}}$.

*Source:* [7, I.3.6(4)].

**1.27.** If such a number does not exist, then the ranges of average distance functions have an empty intersection. Since $\mathcal{X}$ is a compact length space, the range of any continuous function on $\mathcal{X}$ is a closed interval. By 1-dimensional Helly's theorem, there is a pair of such range intervals that do not intersect. That is, for two point-arrays $(x_1, \dots, x_n)$ and $(y_1, \dots, y_m)$ and their average distance functions

$$f(z) = \tfrac{1}{n} \cdot \sum_i |x_i - z|_{\mathcal{X}} \quad \text{and} \quad h(z) = \tfrac{1}{m} \cdot \sum_j |y_j - z|_{\mathcal{X}},$$

we have

$$(*) \qquad \min\{\, f(z) \;:\; z \in \mathcal{X}\,\} > \max\{\, h(z) \;:\; z \in \mathcal{X}\,\}.$$

Note that

$$\tfrac{1}{m} \cdot \sum_j f(y_j) = \tfrac{1}{m \cdot n} \cdot \sum_{i,j} |x_i - y_j|_{\mathcal{X}} = \tfrac{1}{n} \cdot \sum_i h(x_i);$$

that is, the average value of $f(y_j)$ coincides with the average value of $h(x_i)$, which contradicts $(*)$.

*Remark.* The value $\ell$ is uniquely defined; it is called the rendezvous value of $\mathcal{X}$. This is a result of Oliver Gross [14].

**2.2.** By the Fréchet lemma (2.1) we can identify $\mathcal{K}$ with a compact subset of $\ell^\infty$.

Denote by $\mathcal{L} = \operatorname{Conv} \mathcal{K}$ — it is defined as the minimal convex closed set in $\ell^\infty$ that contains $\mathcal{K}$. (In other words, $\mathcal{L}$ is the minimal closed set containing $\mathcal{K}$ such that if $x, y \in \mathcal{L}$, then $t{\cdot}x + (1-t){\cdot}y \in \mathcal{L}$ for any $t \in [0,1]$.)

Observe that $\mathcal{L}$ is a length space. It remains to show that $\mathcal{L}$ is compact.

By construction, $\mathcal{L}$ is a closed subset of $\ell^\infty$; in particular, it is complete. By 1.8$c$, it remains to show that $\mathcal{L}$ is totally bounded.

Recall that Minkowski sum $A + B$ of two sets $A$ and $B$ in a vector space is defined by

$$A + B = \{\, a + b \;:\; a \in A,\ b \in B \,\}.$$

Observe that the Minkowski sum of two convex sets is convex.

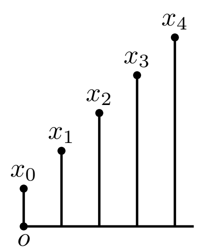
Denote by $\bar{B}_\varepsilon$ the closed $\varepsilon$-ball in $\ell^\infty$ centered at the origin. Choose a finite $\varepsilon$-net $N$ in $\mathcal{K}$ for some $\varepsilon > 0$. Note that $P = \operatorname{Conv} N$ is a convex polyhedron; in particular, $\operatorname{Conv} N$ is compact.

Observe that $N + \bar{B}_\varepsilon$ is a closed $\varepsilon$-neighborhood of $N$. It follows that $N + \bar{B}_\varepsilon \supset K$ and therefore $P + \bar{B}_\varepsilon \supset \mathcal{L}$. In particular, $P$ is a $2{\cdot}\varepsilon$-net in $\mathcal{L}$; since $P$ is compact and $\varepsilon > 0$ is arbitrary, $\mathcal{L}$ is totally bounded (see 1.10).

*Remark.* Alternatively, one may use that *the injective envelope of a compact space is compact*; see 3.7$b$, 3.14, and 3.17.

**2.3.** Modify the proof of 2.1.

**2.8.** Consider the metric tree $\mathcal{T}$ shown on the diagram; it is a half-line $[0,\infty)$ with attached an interval of length $n+1$ to each integer point $n$. Denote by $o$ the origin of the half-line and by $x_n$ the endpoint of $n^{\text{th}}$ interval.




Observe that if $m \neq n$, then

$$|x_m - x_n|_{\mathcal{T}} \geqslant |o - x_n|_{\mathcal{T}} + 1.$$

Show and use that for any binary sequence $\varepsilon_n$ there is an extension function $f$ such that

$$f(x_n) = |o - x_n|_{\mathcal{T}} + \varepsilon_n.$$

*Remark.* An if-and-only-if condition on $\mathcal{X}$ that have separable $\mathcal{X}^\infty$ was found by Julien Melleray [27, 2.8].

**2.13.** Choose a separable space $\mathcal{X}$ that has an infinite number of geodesics between a pair of points with the given distance between them; say a square in $\mathbb{R}^2$ with $\ell^\infty$-metric will do. Apply to $\mathcal{X}$ universality of Urysohn space (2.12).

**2.14.** First let us prove the following claim:

◇ Suppose $f\colon K\to\mathbb{R}$ is an extension function defined on a compact subset $K$ of the Urysohn space $\mathcal{U}$. Then there is a point $p\in\mathcal{U}$ such that $|p-x|=f(x)$ for any $x\in K$.

Without loss of generality, we may assume that $f>0$. Since $K$ is compact, we may fix $\varepsilon>0$ such that $f(x)>\varepsilon$ for any $x\in K$.

Consider the sequence $\varepsilon_n=\frac{\varepsilon}{100\cdot 2^n}$. Choose a sequence of $\varepsilon_n$-nets $N_n\subset K$. Applying universality of $\mathcal{U}$ recursively, we may choose a point $p_n$ such that $|p_n-x|=f(x)$ for any $x\in N_n$ and $|p_n-p_{n-1}|= = 10\cdot\varepsilon_{n-1}$. Observe that the sequence $p_n$ is Cauchy and its limit $p$ meets $|p-x|=f(x)$ for any $x\in K$.

Now, choose a sequence $x_n$ of points that is dense in $\mathcal{S}$. Applying the claim, we may extend the map from $K$ to $K\cup\{x_1\}$, further to $K\cup\{x_1,x_2\}$, and so on. As a result, we extend the distance-preserving map $f$ to the whole sequence $x_n$. It remains to extend it continuously to the whole space $\mathcal{S}$.

**2.16.** It is sufficient to show that any compact subspace $\mathcal{K}$ of the Urysohn space $\mathcal{U}$ can be contracted to a point.

Note that any compact space $\mathcal{K}$ can be extended to a contractible compact space $\mathcal{K}'$; for example, we may embed $\mathcal{K}$ into $\ell^\infty$ and pass to its convex hull, as it was done in 2.2.

By 2.19, there is an isometric embedding of $\mathcal{K}'$ that agrees with the inclusion $\mathcal{K}\hookrightarrow\mathcal{U}$. Since $\mathcal{K}$ is contractible in $\mathcal{K}'$, it is contractible in $\mathcal{U}$.

*A better way.* One can contract the whole Urysohn space using the following construction.

Note that points in $\mathcal{X}_\infty$ constructed in the proof of 2.7 can be multiplied by number $t\in[0,1]$ — simply multiply each function by $t$. That defines a map

$$\lambda_t\colon \mathcal{X}_\infty\to\mathcal{X}_\infty$$

that rescales all distances by factor $t$. The map $\lambda_t$ can be extended to the completion of $\mathcal{X}_\infty$, which is isometric to $\mathcal{U}_d$ (or $\mathcal{U}$).

Observe that the map $\lambda_1$ is the identity and $\lambda_0$ maps the whole space to a single point, say $x_0$ — this is the only point of $\mathcal{X}_0$. Further, note that $(t,p)\mapsto\lambda_t(p)$ is a continuous map; in particular, $\mathcal{U}_d$ and $\mathcal{U}$ are contractible.

As a bonus, observe that for any point $p\in\mathcal{U}_d$ the curve $t\mapsto\lambda_t(p)$ is a geodesic path from $p$ to $x_0$.

*Source:* [13, (d) on page 82].

**2.18**; *(a)* and *(b).* Observe that $L$ and $M$ satisfy the definition of $d$-Urysohn space and apply the uniqueness (2.17). Note that

$$\ell = \operatorname{diam} L = \min\{2{\cdot}r, d\}.$$

*(c).* Use *(a)*, maybe twice.

**2.20.** Let $p$ be the center of the sphere; without loss of generality, we can assume that $|p-x| \leqslant |p-y|$. Set $f(p) = 1$, $f(x) = 1 + |p-x|$ and $f(y) = 1 + |p-y| - \varepsilon$. Suppose $\varepsilon > 0$ is sufficiently small; show that $f$ is an extension function on $\{p, x, y\}$.

By the extension property, there is a point $z \in \mathcal{U}$ such that $|p-z| = f(p)$, $|x-z| = f(x)$, and $|y-z| = f(y)$. Whence the statement follows.

*Source:* This problem is taken from a survey of Julien Melleray [27, Prop. 4.3], where it was attributed to Matatyahu Rubin.

**2.21.** Observe that the complement $\mathcal{V} = \mathcal{U} \setminus B$ is complete. Show that it $\mathcal{V}$ satisfies the extension property. Conclude that $\mathcal{V}$ is an Urysohn space and apply 2.17.

For the second part, observe that there is an isometry $\iota\colon \mathcal{U} \to \mathcal{V}$. Moreover, if $p$ is the center of $B$, then we can assume that $\iota$ has a fixed point $x$ such that $|p-x| > 2$.

Consider the unit sphere $S$ centered at $x$. The restriction of $\iota$ to $S$ is an isometry of $S$. Use 2.20 to show that it cannot be extended to an isometry of $\mathcal{U}$.

*Source:* [27, Sec. 4.4].

**2.22.** Apply 2.17 and the construction in 2.11.

**2.23**; *(a).* The Euclidean plane is homogeneous in every sense.

*(b).* The Hilbert space $\ell^2$ is finite-set-homogeneous, but not compact-set-homogeneous, nor countable-set-homogeneous.

*(c).* $\ell^\infty$ is one-point-homogeneous, but not two-point-homogeneous. Try to show that there is no isometry of $\ell^\infty$ such that

$$\begin{aligned}(0,0,0,\dots) &\mapsto (0,0,0,\dots),\\ (1,1,1,\dots) &\mapsto (1,0,0,\dots).\end{aligned}$$

*(d).* $\ell^1$ is one-point-homogeneous, but not two-point-homogeneous. Try to show that there is no isometry of $\ell^\infty$ such that

$$\begin{aligned}(0,0,0,\dots) &\mapsto (0,0,0,\dots),\\ (2,0,0\dots) &\mapsto (1,1,0,\dots).\end{aligned}$$

**2.24.** Let $\mathcal{T}$ be a one-point-homogeneous metric tree. Note that all point in $\mathcal{T}$ have the same degree $d$; that is, for any point $t \in \mathcal{T}$ the set of connected components of the complement $\mathcal{T} \setminus \{t\}$ has the same cardinality $d$.

Show that if $d = 0$, then $\mathcal{T}$ is a one-point space; there is no tree with $d = 1$, and if $d = 2$, then $\mathcal{T} \overset{iso}{=\!=} \mathbb{R}$.

Suppose $d \geqslant 3$. Choose a geodesic $\gamma$ in $\mathcal{T}$. Show that number of connected components of $\mathcal{T} \backslash \gamma$ has cardinality continuum. Observe and use that one can choose a point $p_\alpha$ in each connected component such that $|p_\alpha - p_\beta|_{\mathcal{T}} > 1$ if $\alpha \neq \beta$.

**3.2.** Note that if $c < 0$, then $r > s$. The latter is impossible since $r$ is extremal and $s$ is admissible.

Observe that the function $\bar{r} = \min\{\, r, s + c\,\}$ is admissible. Indeed, choose $x, y \in \mathcal{X}$. If $\bar{r}(x) = r(x)$ and $\bar{r}(y) = r(y)$, then

$$\bar{r}(x) + \bar{r}(y) = r(x) + r(y) \geqslant |x - y|.$$

Further, if $\bar{r}(x) = s(x) + c$, then

$$\begin{aligned}\bar{r}(x) + \bar{r}(y) &\geqslant [s(x) + c] + [s(y) - c] = \\ &= s(x) + s(y) \geqslant \\ &\geqslant |x - y|.\end{aligned}$$

Since $r$ is extremal, we have $r = \bar{r}$; that is, $r \leqslant s + c$.

**3.4;** *only-if part.* Suppose $r$ is extremal. By $3.3b$, $r$ is 1-Lipschitz. Since $\mathbb{S}^1$ is compact, $3.3d$ implies that for any $p \in \mathbb{S}^1$ there is $q \in \mathbb{S}^1$ such that

$$r(p) + r(q) = |p - q|_{\mathbb{S}^1}.$$

Therefore

$$\begin{aligned}\pi &= |p - (-p)|_{\mathbb{S}^1} \leqslant \\ &\leqslant r(p) + r(-p) = \\ &= r(p) + r(q) + r(-p) - r(q) \leqslant \\ &\leqslant |p - q|_{\mathbb{S}^1} + |q - (-p)|_{\mathbb{S}^1} = \\ &= \pi.\end{aligned}$$

It implies that we have equalities in both places. Hence the only-if part follows.

*If part.* Assume $r$ is 1-Lipschitz function such that $r(p) + r(-p) = \pi$. Then

$$\begin{aligned}|p - q|_{\mathbb{S}^1} &= |p - (-p)|_{\mathbb{S}^1} - |q - (-p)|_{\mathbb{S}^1} \geqslant \\ &\geqslant \pi - (r(-p) - r(q)) = \\ &= r(p) + r(q).\end{aligned}$$

Therefore $r$ is admissible.

Finally, if $r$ is not extremal, then there is an admissible function $s \leqslant r$ such that $s(p) < r(p)$ for some $p$. The latter contradicts the equality $r(p) + r(-p) = \pi$.

*Source:* [44, Proposition 2.7].

**3.5.** Show and use that $s^*(x) + s(y) \geqslant |x - y|$ for any $x, y \in \mathcal{X}$.

*Remarks.* It is easy to check that $q\colon s \mapsto \frac{1}{2}\cdot(s + s^*)$ is a short map on the space of admissible functions (with sup-norm). Moreover iterating $q$ and passing to the limit, we get a short retraction from the space of admissible functions to the space of extremal functions on $\mathcal{X}$ [see 3.1 in 24]. The existence of such a map will also follow from 3.21.

**3.7.** Choose an injective space $\mathcal{Y}$.

*(a).* Fix a Cauchy sequence $x_n$ in $\mathcal{Y}$; we need to show that it has a limit $x_\infty \in \mathcal{Y}$. Consider metric on $\mathcal{X} = \mathbb{N} \cup \{\infty\}$ defined by

$$\begin{aligned} |m - n|_{\mathcal{X}} &:= |x_m - x_n|_{\mathcal{Y}}, \\ |m - \infty|_{\mathcal{X}} &:= \lim_{n\to\infty} |x_m - x_n|_{\mathcal{Y}}. \end{aligned}$$

Since the sequence is Cauchy, so is the sequence $\ell_n = |x_m - x_n|_{\mathcal{Y}}$ for any $m$. Therefore, the last limit is defined.

By construction, the map $n \mapsto x_n$ is distance-preserving on $\mathbb{N} \subset \mathcal{X}$. Since $\mathcal{Y}$ is injective, this map can be extended to $\infty$ as a short map; set $\infty \mapsto x_\infty$. Since $|x_n - x_\infty|_{\mathcal{Y}} \leqslant |n - \infty|_{\mathcal{X}}$ and $|n - \infty|_{\mathcal{X}} \to 0$, we get that $x_n \to x_\infty$ as $n \to \infty$.

*(b).* Applying the definition of injective space, we get a midpoint for any pair of points in $\mathcal{Y}$. By *(a)*, $\mathcal{Y}$ is a complete space. It remains to apply 1.23$b$.

*(c).* Let $k\colon \mathcal{Y} \hookrightarrow \ell^\infty(\mathcal{Y})$ be the Kuratowski embedding (2.4). Observe that $\ell^\infty(\mathcal{Y})$ is contractible; in particular, there is a homotopy $k_t\colon \mathcal{Y} \hookrightarrow \ell^\infty(\mathcal{Y})$ such that $k_0 = k$ and $k_1$ is a constant map. (In fact, one can take $k_t = (1 - t)\cdot k$.)

Since $k$ is distance-preserving and $\mathcal{Y}$ is injective, there is a short map $f\colon \ell^\infty(\mathcal{Y}) \to \mathcal{Y}$ such that the composition $f \circ k$ is the identity map on $\mathcal{Y}$. The composition $f \circ k_t\colon \mathcal{Y} \hookrightarrow \mathcal{Y}$ provides the needed homotopy.

**3.8.** Suppose that a short map $f\colon A \to \mathcal{Y}$ is defined on a subset $A$ of a metric space $\mathcal{X}$. We need to construct a short extension $F$ of $f$. Without loss of generality, we may assume that $A \neq \varnothing$, otherwise map the whole $\mathcal{X}$ to a single point. By Zorn's lemma, it is sufficient to enlarge $A$ by a single point $x \notin A$.

*(a).* Suppose $\mathcal{Y} = \mathbb{R}$. Set

$$F(x) = \inf \{\, f(a) - |a - x| \;:\; a \in A \,\}.$$

Observe that $F$ is short and $F(a) = f(a)$ for any $a \in A$.

*(b).* Suppose $\mathcal{Y}$ is a complete metric tree. Fix points $p \in \mathcal{X}$ and $q \in \mathcal{Y}$. Given a point $a \in A$, let $x_a \in \overline{\mathrm{B}}[f(a), |a-p|]$ be the point closest to $f(x)$. Note that $x_a \in [q\, f(a)]$ and either $x_a = q$ or $x_a$ lies on distance $|a-p|$ from $f(a)$.

Note that the geodesics $[q\, x_a]$ are nested; that is, for any $a, b \in A$ we have either $[q\, x_a] \subset [q\, x_b]$ or $[q\, x_b] \subset [q\, x_a]$. Moreover, in the first case we have $|x_b - f(a)| \leqslant |p-a|$ and in the second $|x_a - f(b)| \leqslant |p-b|$.

It follows that the closure of the union of all geodesics $[q\, x_a]$ for $a \in \mathcal{A}$ is a geodesic. Denote by $x$ its endpoint; it exists since $\mathcal{Y}$ is complete. It remains to observe that $|x - f(a)| \leqslant |p - a|$ for any $a \in \mathcal{A}$; that is, one can take $f(p) = x$.

*(c).* In this case, $\mathcal{Y} = (\mathbb{R}^2, \ell^\infty)$. Note that $\mathcal{X} \to (\mathbb{R}^2, \ell^\infty)$ is a short map if and only if both of its coordinate projections are short. It remains to apply *(a)*.

More generally, *any $\ell^\infty$-product of injective spaces is injective*; in particular, if $\mathcal{Y}$ and $\mathcal{Z}$ are injective then the product $\mathcal{Y} \times \mathcal{Z}$ equipped with the metric

$$|(y,z) - (y',z')|_{\mathcal{Y}\times\mathcal{Z}} = \max\{\, |y-y'|_{\mathcal{Y}}, |z-z'|_{\mathcal{Z}} \,\}$$

is injective as well.

**3.9.** Choose three points $x, y, z \in \mathcal{X}$ and set $\mathcal{A} = \{x, z\}$. Let $f\colon \mathcal{A} \to \to \mathcal{A}$ be the identity map. Then $F(y) = x$ or $F(y) = z$. The strong triangle inequality easily follows in both cases.

**3.10**; *main part.* Choose a maximal subset $A \supset K$ that admits a short retraction $f\colon A \to K$; it exists by Zorn's lemma. If $A$ is the whole space, then the problem is solved. Otherwise, choose $p \notin A$.

Choose a sequence of points $a_n \in A$ such that $|a_n - p|$ converge to the exact lower bound on the distances from points in $A$ to $p$. Since $K$ is compact, we can pass to a subsequence of $a_n$ such that $f(a_n)$ converges. Set $f(p) = \lim f(a_n)$.

It remains to check that

$$❶ \qquad |f(a) - f(p)| \leqslant |a-p|$$

for any $a \in A$. Choose $\varepsilon > 0$; note that

$$|a_n - p| < |a-p| + \varepsilon \quad \text{and} \quad |f(a_n) - f(p)| < |f(a) - f(a_n)| + \varepsilon$$

for all large $n$. Therefore,

$$\begin{aligned}|f(a)-f(p)| &\leqslant \max\{\,|f(a)-f(a_n)|,|f(a_n)-f(p)|\,\}\leqslant\\ &\leqslant |f(a)-f(a_n)|+\varepsilon\leqslant\\ &\leqslant |a-a_n|+\varepsilon\leqslant\\ &\leqslant \max\{\,|a-p|,|a_n-p|\,\}+\varepsilon<\\ &< |a-p|+2{\cdot}\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we get ❶.

*Example.* Consider set of $\{\infty,1,2,\dots\}$ with metric defined by

$$|m-n|=1+\frac{1}{\min\{m,n\}}$$

for $m\ne n$. Observe that the space is complete, the subset $\{1,2,\dots\}$ is closed, but it is not a short retract of the ambient space.

**3.12.** Apply 3.11$c$.

**3.13.** Denote by $\mathcal{U}_d$ the $d$-Urysohn space, so $\mathcal{U}_\infty$ is the Urysohn space.

The extension property implies finite hyperconvexity. It remains to show that $\mathcal{U}_d$ is not countably hyperconvex.

Suppose that $d<\infty$. Then $\operatorname{diam}\mathcal{U}_d=d$ and for any point $x\in\mathcal{U}_d$ there is a point $y\in\mathcal{U}_d$ such that $|x-y|_{\mathcal{U}_d}=d$. It follows that there is no point $z\in\mathcal{U}_d$ such that $|z-x|_{\mathcal{U}_d}\leqslant\frac{d}{2}$ for any $x\in\mathcal{U}_d$. Whence $\mathcal{U}_d$ is not countably hyperconvex.

Use 2.18$b$ to reduce the case $d=\infty$ to the case $d<\infty$.

**3.14.** Observe and use that the functions in $\operatorname{Ext}\mathcal{X}$ are 1-Lipschitz and uniformly bounded.

**3.15**; *(a).* Use 3.3$d$ to show that if $f$ is extremal if and only if $f(v)=x$ and $f(w)=1-x$ for some $x\in[0,1]$. Conclude that $\operatorname{Ext}\mathcal{X}$ is isometric to the unit interval $[0,1]$.

*(b).* Let $f$ be an extremal function. By 3.3$d$, at least two of the numbers $f(a)+f(b)$, $f(b)+f(c)$, and $f(c)+f(a)$ are 1. It follows that for some $x\in[0,\frac12]$, we have

$$f(a)=1\pm x,\qquad f(b)=1\pm x,\qquad f(c)=1\pm x,$$

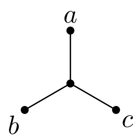
where we have one "minus" and two "pluses" in these three formulas.

Suppose that

$$g(a)=1\pm y,\qquad g(b)=1\pm y,\qquad g(c)=1\pm y$$

is another extremal function. Then $|f-g| = |x-y|$ if $g$ has "minus" at the same place as $f$ and $|f-g| = |x+y|$ otherwise.

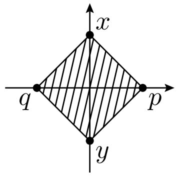
It follows that $\operatorname{Ext}\mathcal{X}$ is isometric to a *tripod* — three segments of length $\frac{1}{2}$ glued at one end.




*(c).* Assume $f$ is an extremal function. Use 3.3$d$ to show that

$$f(x)+f(y)=f(p)+f(q)=2;$$

in particular, two values $a = f(x)-1$ and $b = f(p)-1$ completely describe the function $f$. Since $f$ is extremal, we also have that




$$(1\pm a)+(1\pm b)\geqslant 1$$

for all 4 choices of signs; equivalently,

$$|a|+|b|\leqslant 1.$$

It follows that $\operatorname{Ext}\mathcal{X}$ is isometric to the rhombus $|a|+|b|\leqslant 1$ in the $(a,b)$-plane with the metric induced by the $\ell^\infty$-norm.

*Remarks.* If $\mathcal{X}$ is finite, then $\operatorname{Ext}\mathcal{X}$ has polyhedral structure with $\ell^\infty$ metric on each face; its combinatorics can be encoded by certain graphs with the vertex set $\mathcal{X}$ [see Section 4 in 24].

**3.16.** Recall that $x\mapsto \operatorname{dist}_x$ gives an isometric embedding $\mathcal{X}\hookrightarrow \ell^\infty(\mathcal{X})$; so we can identify $\mathcal{X}$ with a subset of $\ell^\infty(\mathcal{X})$. Further, $\operatorname{Ext}\mathcal{X}$ is a subset of $\ell^\infty(\mathcal{X})$. It is sufficient to show that $\operatorname{Ext}\mathcal{X}=G$.

Use 3.3$d$ to show that $\operatorname{Ext}\mathcal{X}\subset G$.

Given $g\in G$, show that $g(x)=|g-x|_{\ell^\infty(\mathcal{X})}$. Conclude that $g$ is admissible and apply 3.3$d$.

*Source:* The problem has been by Rostislav Matveyev.

**3.19.** Recall that

$$|f-g|_{\operatorname{Ext}\mathcal{X}} = \sup\{\,|f(x)-g(x)| \,:\, x\in\mathcal{X}\,\}$$

and

$$|f-p|_{\operatorname{Ext}\mathcal{X}} = f(p)$$

for any $f,g\in\operatorname{Ext}\mathcal{X}$ and $p\in\mathcal{X}$.

Since $\mathcal{X}$ is compact we can find a point $p\in\mathcal{X}$ such that

$$|f-g|_{\operatorname{Ext}\mathcal{X}} = |f(p)-g(p)| = \big||f-p|_{\operatorname{Ext}\mathcal{X}} - |g-p|_{\operatorname{Ext}\mathcal{X}}\big|.$$

Without loss of generality, we may assume that

$$|f-p|_{\operatorname{Ext}\mathcal{X}} = |g-p|_{\operatorname{Ext}\mathcal{X}} + |f-g|_{\operatorname{Ext}\mathcal{X}}.$$

Applying 3.3$d$, we can find a point $q\in\mathcal{X}$ such that

$$|q-p|_{\operatorname{Ext}\mathcal{X}} = |f-p|_{\operatorname{Ext}\mathcal{X}} + |f-q|_{\operatorname{Ext}\mathcal{X}},$$

whence the result.

Since $\operatorname{Ext}\mathcal{X}$ is injective (3.17), by 3.7$b$ it has to be geodesic. It remains to note that the concatenation of geodesics $[pq]$, $[gf]$, and $[fq]$ is a required geodesic $[pq]$.

**3.20.** The only-if part follows since $\mathcal{X}$ is isometric to a subset of $\operatorname{Ext}\mathcal{X}$.

The if part means that

$$❷\quad |f-g|+|v-w| \leqslant \max\{\,|f-v|+|g-w|,\ |f-w|+|g-v|\,\}+2\cdot\delta$$

for any $f,g,v,w\in\operatorname{Ext}\mathcal{X}$.

Suppose $\mathcal{X}$ is compact. Applying 3.19, we can choose $p,q,x,y\in\mathcal{X}$ such that

$$❸\quad \begin{aligned} |p-f|+|f-g|+|g-q| &= |p-q| \\ |x-v|+|v-w|+|w-y| &= |x-y| \end{aligned}$$

Since $\mathcal{X}$ is $\delta$-hyperbolic, we have

$$|p-q|+|x-y| \leqslant \max\{\,|p-x|+|q-y|,\ |p-y|+|q-x|\,\}+2\cdot\delta.$$

Show that this inequality, together with the triangle inequality and ❸ imply ❷.

For the noncompact case, prove an approimate version of ❸ and apply it the same way.

**3.22.** Show that there is a pair of short maps $\operatorname{Ext}\mathcal{X}\to\operatorname{Ext}\mathcal{U}\to\operatorname{Ext}\mathcal{X}$ such that their composition is the identity on $\mathcal{X}$. Make a conclusion.

**3.23.** By 2.4, the space $\mathcal{X}$ can be considered as a subset in $\ell^\infty(\mathcal{X})$. Given $x,y\in\mathcal{X}$, let $\tilde\gamma_{x,y}(t)=(1-t)\cdot x+t\cdot y\in\ell^\infty(\mathcal{X})$. Observe that $\tilde\gamma_{x,y}$ meets both conditions in the exercise. It remains to apply 3.21.

*Source:* [24, 3.6].

**4.3.** Suppose that $|A-B|_{\operatorname{Haus}\mathcal{X}}<r$. Choose a pair of points $a,a'\in A$ on maximal distance from each other. Observe that there are points $b,b'\in B$ such that $|a-b|_{\mathcal{X}}, |a'-b'|_{\mathcal{X}}<r$. Whence

$$|a-a'|_{\mathcal{X}}-|b-b'|_{\mathcal{X}}\leqslant 2\cdot r$$

and therefore

$$\operatorname{diam} A - \operatorname{diam} B \leqslant 2\cdot|A - B|_{\operatorname{Haus}\mathcal{X}}.$$

It remains to swap $A$ and $B$ and repeat the argument.

**4.4**; *(a)*. Denote by $A^r$ the closed $r$-neighborhood of a set $A \subset \mathbb{R}^2$. Observe that

$$(\operatorname{Conv} A)^r = \operatorname{Conv}(A^r),$$

and try to use it.

*(b)*. The answer is "no" in both parts.

For the first part let $A$ be a unit disk and $B$ a finite $\varepsilon$-net in $A$. Evidently, $|A - B|_{\operatorname{Haus}\mathbb{R}^2} < \varepsilon$, but $|\partial A - \partial B|_{\operatorname{Haus}\mathbb{R}^2} \approx 1$.

For the second part take $A$ to be a unit disk and $B = \partial A$ to be its boundary circle. Note that $\partial A = \partial B$; in particular, $|\partial A - \partial B|_{\operatorname{Haus}\mathbb{R}^2} = 0$ while $|A - B|_{\operatorname{Haus}\mathbb{R}^2} = 1$.

*Remark.* There are the so-called lakes of Wada — an example of three (and more) open bounded topological disks in the plane that have identical boundaries. It can be used to construct more interesting examples for *(b)*.

**4.6**; *(a)*. Given $t \in (0,1]$, consider the real interval $\tilde{C}_t = [\frac{1}{t} + t, \frac{1}{t} + 1]$. Denote by $C_t$ the image of $\tilde{C}_t$ under the covering map $\pi\colon \mathbb{R} \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

Set $C_0 = \mathbb{S}^1$. Note that the Hausdorff distance from $C_0$ to $C_t$ is $\frac{t}{2}$. Therefore $\{C_t\}_{t\in[0,1]}$ is a family of compact subsets in $\mathbb{S}^1$ that is continuous in the sense of Hausdorff.

Assume there is a continuous section $c(t) \in C_t$, for $t \in [0,1]$. Since $\pi$ is a covering map, we can lift the path $c$ to a path $\tilde{c}\colon [0,1] \to \mathbb{R}$ such that $\tilde{c}(t) \in \tilde{C}_t$ for all $t$. In particular $\tilde{c}(t) \to \infty$ as $t \to 0$, a contradiction.

*(b)*. Consider path $c(t) := \min C_t$.

*Source:* The problem had been suggested by Stephan Stadler.

**4.10.** Show that for any $\varepsilon > 0$ there is a positive integer $N$ such that $\bigcup_{n\leqslant N} K_n$ is an $\varepsilon$-net in the union $\bigcup_n K_n$. Observe that $\bigcup_{n\leqslant N} K_n$ is compact and apply 1.10 and 1.8$c$.

**4.11**; *if part.* Choose two compact sets $A, B \subset \mathcal{X}$; suppose that $|A - B|_{\operatorname{Haus}\mathcal{X}} < r$.

Choose finite $\varepsilon$-nets $\{a_1, \dots, a_m\} \subset A$ and $\{b_1, \dots b_n\} \subset B$. For each pair $a_i, b_j$ construct a constant-speed path $\gamma_{i,j}$ from $a_i$ to $b_j$ such that

$$\operatorname{length}\gamma_{i,j} < |a_i - b_j| + \varepsilon.$$

Set
$$C(t) = \{\, \gamma_{i,j}(t) \;:\; |a_i - b_j|_{\mathcal{X}} < r + \varepsilon \,\}.$$
Observe that $C(t)$ is finite; in particular, it is compact.

Show and use that
$$|A - C(t)|_{\mathcal{X}} < t{\cdot}r + 10{\cdot}\varepsilon,$$
$$|C(t) - B|_{\mathcal{X}} < (1-t){\cdot}r + 10{\cdot}\varepsilon.$$

Apply 4.10 and 1.23.

*Only-if part.* Choose points $p, q \in \mathcal{X}$. Show that the existence of $\varepsilon$-midpoints between $\{p\}$ and $\{q\}$ in $\operatorname{Haus}\mathcal{X}$ implies the existence of $\varepsilon$-midpoints between $p$ and $q$ in $\mathcal{X}$. Apply 1.23.

**4.13.** Let $A$ be a compact convex set in the plane. Denote by $A^r$ the closed $r$-neighborhood of $A$. Recall that by Steiner's formula we have
$$\operatorname{area} A^r = \operatorname{area} A + r{\cdot}\operatorname{perim} A + \pi{\cdot}r^2.$$

Taking derivative and applying the coarea formula, we get
$$\operatorname{perim} A^r = \operatorname{perim} A + 2{\cdot}\pi{\cdot}r.$$

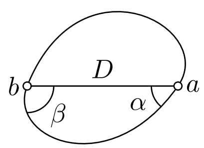
Observe that if $A$ lies in a compact set $B$ bounded by a closed curve, then
$$\operatorname{perim} A \leqslant \operatorname{perim} B.$$

Indeed the closest-point projection $\mathbb{R}^2 \to A$ is short and it maps $\partial B$ onto $\partial A$.

It remains to use the following observation: if $A_n \to A_\infty$, then for any $r > 0$ we have that the inclusions
$$A_\infty^r \supset A_n \quad \text{and} \quad A_\infty \subset A_n^r$$

hold for all large $n$.

**4.14.** Note that almost all points on $\partial D$ have a defined tangent line. In particular, for almost all pairs of points $a, b \in \partial D$ the two angles $\alpha$ and $\beta$ between the chord $[ab]$ and $\partial D$ are defined.




The convexity of $D'$ implies that $\alpha = \beta$; here we measure the angles $\alpha$ and $\beta$ on one side from $[ab]$. Show that if the identity $\alpha = \beta$ holds for almost all chords, then $D$ is a round disk.

**4.16.** Observe that all functions $\operatorname{dist}_{A_n}$ are Lipschitz.

If they are not uniformly bounded on compact sets, then we can pass to a subsequence of $A_n$ so that $\operatorname{dist}_{A_n}(x)\to\infty$ for any $x$; in this case $A_n$ converges to the empty set.

If the functions are uniformly bounded, then, passing to a subsequence, we may assume that the sequence $\operatorname{dist}_{A_n}$ converges to some function $f$.

Set $A_\infty=f^{-1}\{0\}$. It remains to show that $f=\operatorname{dist}_{A_\infty}$.

**5.3**; *(a).* Apply the definition for space $\mathcal{Z}$ obtained from $\mathcal{X}$ by adding a point that lies at distance $\tfrac12\cdot\operatorname{diam}\mathcal{X}$ from each point of $\mathcal{X}$.

*(b).* Given a point $x\in\mathcal{X}$, denote by $a{\cdot}x$ and $b{\cdot}x$ the corresponding points in $a{\cdot}\mathcal{X}$ and $b{\cdot}\mathcal{X}$ respectively. Show that there is a metric on $\mathcal{Z}=a{\cdot}\mathcal{X}\sqcup b{\cdot}\mathcal{X}$ such that

$$|a{\cdot}x-b{\cdot}x|_{\mathcal{Z}}=\tfrac{|b-a|}{2}\cdot\operatorname{diam}\mathcal{X}$$

for any $x$ and the inclusions $a{\cdot}\mathcal{X}\hookrightarrow\mathcal{Z}$, $b{\cdot}\mathcal{X}\hookrightarrow\mathcal{Z}$ are distance preserving.

*(c).* Use *(a)* and *(b)* to show that the isometry class of $\mathcal{O}$ is completely determined by the following property

$$|\mathcal{X}-\mathcal{Y}|_{\mathrm{GH}}\leqslant\max\{\,|\mathcal{O}-\mathcal{X}|_{\mathrm{GH}},|\mathcal{O}-\mathcal{Y}|_{\mathrm{GH}}\,\}.$$

for any $\mathcal{X}$ and $\mathcal{Y}$.

*Remark.* In fact, *the isometry group of space* GH *is trivial.* The latter was proved by George Lowther [20, 26].

**5.4.** Check a one-point set and the vertices of an equilateral triangle. You may use 5.3$a$.

**5.5.** Arguing by contradiction, we can identify $\mathcal{A}_r$ and $\mathcal{B}_r$ with subspaces of a space $\mathcal{Z}$ such that

$$|\mathcal{A}_r-\mathcal{B}_r|_{\operatorname{Haus}\mathcal{Z}}<\tfrac{1}{10}$$

for large $r$; see the definition of Gromov–Hausdorff metric (5.1).

Set $n=\lceil r\rceil$. Note that there are $2{\cdot}n$ integer points in $\mathcal{A}_r$: $a_1={}$ $=(0,0)$, $a_2=(1,0),\dots,a_{2\cdot n}=(n,1)$. Choose a point $b_i\in\mathcal{B}_r$ that lies at the minimal distance from $a_i$. Note that $|b_i-b_j|>\tfrac45$ if $i\ne j$. It follows that $r>\tfrac45{\cdot}(2{\cdot}n-1)$. The latter contradicts $n=\lceil r\rceil$ for large $r$.

*Remark.* Try to show that $|\mathcal{A}_r-\mathcal{B}_r|_{\mathrm{GH}}=\tfrac12$ for all large $r$.

**5.6.** Suppose that $|\mathcal{X}-\mathcal{Y}|_{\mathcal{U}}<\varepsilon$; we need to show that

$$|\hat{\mathcal{X}}-\hat{\mathcal{Y}}|_{\mathrm{GH}}<2{\cdot}\varepsilon.$$

Denote by $\hat{\mathcal{U}}$ the injective envelope of $\mathcal{U}$. Recall that $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$ can be considered as subspaces of $\hat{\mathcal{U}}$, $\hat{\mathcal{X}}$, and $\hat{\mathcal{Y}}$ respectively.

According to 3.22, the inclusions $\mathcal{X} \hookrightarrow \mathcal{U}$ and $\mathcal{Y} \hookrightarrow \mathcal{U}$ can be extended to distance-preserving inclusions $\hat{\mathcal{X}} \hookrightarrow \hat{\mathcal{U}}$ and $\hat{\mathcal{Y}} \hookrightarrow \hat{\mathcal{U}}$. Therefore, we can and will consider $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$ as subspaces of $\hat{\mathcal{U}}$.

Given $f \in \hat{\mathcal{U}}$, let us find $g \in \hat{\mathcal{X}}$ such that

❹ $$|f(u) - g(u)| < 2{\cdot}\varepsilon$$

for any $u \in \mathcal{U}$. Note that the restriction $f|_{\mathcal{X}}$ is admissible on $\mathcal{X}$. By 3.3$e$, there is $g \in \hat{\mathcal{X}}$ such that

❺ $$g(x) \leqslant f(x)$$

for any $x \in \mathcal{X}$.

Recall that any extremal function is 1-Lipschitz; in particular, $f$ and $g$ are 1-Lipschitz on $\mathcal{U}$. Therefore, ❺ and $|\mathcal{X} - \mathcal{Y}|_{\mathcal{U}} < \varepsilon$ imply that

$$g(u) < f(u) + 2{\cdot}\varepsilon$$

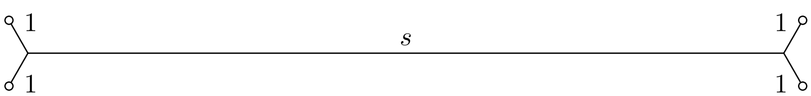
for any $u \in \mathcal{U}$. By 3.2, we also have

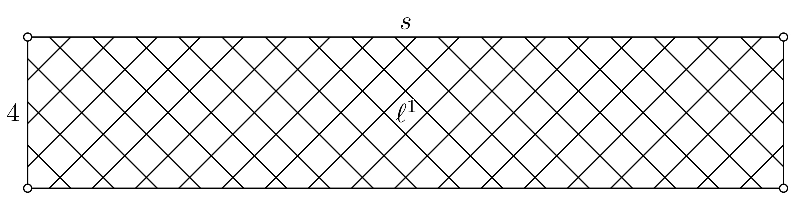
$$g(u) > f(u) - 2{\cdot}\varepsilon$$

for any $u \in \mathcal{U}$. Whence ❹ follows.

It follows that $\hat{\mathcal{Y}}$ lies in a $2{\cdot}\varepsilon$-neighborhood of $\hat{\mathcal{X}}$ in $\hat{\mathcal{U}}$. The same way we show that $\hat{\mathcal{X}}$ lies in a $2{\cdot}\varepsilon$-neighborhood of $\hat{\mathcal{Y}}$ in $\hat{\mathcal{U}}$. The latter means that $|\hat{\mathcal{X}} - \hat{\mathcal{Y}}|_{\mathrm{Haus}\,\hat{\mathcal{U}}} < 2{\cdot}\varepsilon$, and therefore $|\hat{\mathcal{X}} - \hat{\mathcal{Y}}|_{\mathrm{GH}} < 2{\cdot}\varepsilon$.

*Remark.* This problem was discussed by Urs Lang, Maël Pavón, and Roger Züst [25, 3.1]. They also show that the constant 2 is opti-




mal. To see this, look at the injective envelopes of two 4-point metric spaces shown on the diagram and observe that the Gromov–Hausdorff distance between the 4-point metric spaces is 1, while the distance between their injective envelopes approaches 2 as $s \to \infty$.

**5.8**; *only-if part.* Let us identify $\mathcal{X}$ and $\mathcal{Y}$ with subspaces of a metric space $\mathcal{Z}$ such that

$$|\mathcal{X} - \mathcal{Y}|_{\operatorname{Haus} \mathcal{Z}} < \varepsilon.$$

Set $x \approx y$ if and only if $|x - y|_{\mathcal{Z}} < \varepsilon$. It remains to check that $\approx$ is an $\varepsilon$-approximation.

*If part.* Show that we can assume that

$$R = \{\, (x, y) \in \mathcal{X} \times \mathcal{Y} \,:\, x \approx y \,\}$$

is a compact subset of $\mathcal{X} \times \mathcal{Y}$. Conclude that

$$\big||x - x'|_{\mathcal{X}} - |y - y'|_{\mathcal{Y}}\big| < 2{\cdot}\varepsilon'$$

for some $\varepsilon' < \varepsilon$.

Show that there is a metric on $\mathcal{Z} = \mathcal{X} \sqcup \mathcal{Y}$ such that the inclusions $\mathcal{X} \hookrightarrow \mathcal{Z}$ and $\mathcal{Y} \hookrightarrow \mathcal{Z}$ are distance preserving and $|x - y|_{\mathcal{Z}} = \varepsilon'$ if $x \approx y$. Conclude that

$$|\mathcal{X} - \mathcal{Y}|_{\operatorname{Haus} \mathcal{Z}} \leqslant \varepsilon' < \varepsilon.$$

**5.10**; *(a).* Let $\approx$ be an $\varepsilon$-approximation provided by 5.8. For any $x \in \mathcal{X}$ choose a point $f(x) \in \mathcal{Y}$ such that $x \approx f(x)$. Show that $x \mapsto f(x)$ is an $2{\cdot}\varepsilon$-isometry.

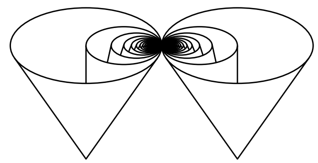
*(b).* Let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Set $x \approx y$ if $|y - f(x)|_{\mathcal{Y}} < \varepsilon$. Show that $\approx$ is an $\varepsilon$-approximation. Apply 5.8.

**5.13.** Consider the product space $[0, \varepsilon] \times \mathbb{Z}_n$ with the natural $\ell^\infty$-product metric and make three variations of it by changing the size of the segments.

**5.15**; *(a).* Suppose $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}$ and $\mathcal{X}_n$ are simply-connected length metric space. It is sufficient to show that any nontrivial covering map $f \colon \tilde{\mathcal{X}} \to \mathcal{X}$ corresponds to a nontrivial covering map $f_n \colon \tilde{\mathcal{X}}_n \to \mathcal{X}_n$ for large $n$.

The latter can be constructed by covering $\mathcal{X}_n$ with small balls that lie close to sets in $\mathcal{X}$ evenly covered by $f$, prepare a few copies of these sets and glue them the same way as the inverse images of the evenly covered sets in $\mathcal{X}$ glued to obtain $\tilde{\mathcal{X}}$.

*(b).* Let $\mathcal{V}$ be a cone over Hawaiian earrings. Consider the doubled cone $\mathcal{W}$ — two copies of $\mathcal{V}$ with glued base points (see the diagram).



The space $\mathcal{W}$ can be equipped with a length metric (for example, the induced length metric from the shown embedding).

Show that $\mathcal{V}$ is simply-connected, but $\mathcal{W}$ is not; follows from the van Kampen theorem.

If we delete from the earrings all small circles, then the obtained double cone becomes simply-connected and it remains to be close to $\mathcal{W}$. That is $\mathcal{W}$ is a Gromov–Hausdorff limit of simply-connected spaces.

*Remark.* Note that the limit space in *(b)*, does not admit a nontrivial covering.

**5.16,** *(a).* Suppose that a metric on $\mathbb{S}^2$ is close to the unit disk $\mathbb{D}^2$. Note that $\mathbb{S}^2$ contains a circle $\gamma$ that is close to the boundary curve of $\mathbb{D}^2$. By the Jordan curve theorem, $\gamma$ divides $\mathbb{S}^2$ into two disks, say $D_1$ and $D_2$.

By 5.15$a$, the Gromov–Hausdorff limits of $D_1$ and $D_2$ have to contain the whole $\mathbb{D}^2$, otherwise the limit would admit a nontrivial covering.

Consider points $p_1 \in D_1$ and $p_2 \in D_2$ that are close to the center of $\mathbb{D}^2$. If $n$ is large, the distance $|p_1 - p_2|_n$ has to be very small. On the other hand, any curve from $p_1$ to $p_2$ must cross $\gamma$, so it has length about 2 — a contradiction.

*(b).* Make fine burrows in the standard 3-ball without changing its topology, but at the same time come sufficiently close to any point in the ball.

Consider the doubling of the obtained ball along its boundary; that is, two copies of the ball with identified corresponding points on their boundaries. The obtained space is homeomorphic to $\mathbb{S}^3$. Note that the burrows can be made so that the obtained space is sufficiently close to the original ball in the Gromov–Hausdorff metric.

*Source:* [8, Exercises 7.5.13 and 7.5.17].

**5.18.** Apply 1.11.

**5.19.** Choose a space $\mathcal{X}$ in $\mathcal{Q}(C,D)$, denote a $C$-doubling measure by $\mu$. Without loss of generality, we may assume that $\mu(\mathcal{X}) = 1$.

The doubling condition implies that

$$\mu[\mathrm{B}(p, \tfrac{D}{2^n})] \geqslant \tfrac{1}{C^n}$$

for any point $x \in \mathcal{X}$. It follows that

$$\operatorname{pack}_{\frac{D}{2^n}} \mathcal{X} \leqslant C^n.$$

By 1.11, for any $\varepsilon \geqslant \frac{D}{2^{n-1}}$, the space $\mathcal{X}$ admits an $\varepsilon$-net with at most $C^n$ points. Whence $\mathcal{Q}(C, D)$ is uniformly totally bounded.

**5.20**; *(a)*. Choose $\varepsilon > 0$. Since $\mathcal{Y}$ is compact, we can choose a finite $\varepsilon$-net $\{y_1, \dots, y_n\}$ in $\mathcal{Y}$.

Suppose $f\colon \mathcal{X} \to \mathcal{Y}$ be a distance-noncontracting map. Choose one point $x_i$ in each nonempty subset $B_i = f^{-1}[\mathrm{B}(y_i, \varepsilon)]$. Note that the subset $B_i$ has diameter at most $2{\cdot}\varepsilon$ and

$$\mathcal{X} = \bigcup_i B_i.$$

Therefore, the set of points $\{x_i\}$ is a $2{\cdot}\varepsilon$-net in $\mathcal{X}$.

*(b)*. Let $\mathcal{Q}$ be a uniformly totally bounded family of spaces. Suppose that each space in $\mathcal{Q}$ has an $\frac{1}{2^n}$-net with at most $M_n$ points; we may assume that $M_0 = 1$.

Consider the space $\mathcal{Y}$ of all infinite integer sequences $m_0, m_1, \dots$ such that $1 \leqslant m_n \leqslant M_n$ for any $n$. Given two sequences $\boldsymbol{\ell} = (\ell_1, \ell_2, \dots)$, and $\boldsymbol{m} = (m_1, m_2, \dots)$ of points in $\mathcal{Y}$, set

$$|\boldsymbol{\ell} - \boldsymbol{m}|_{\mathcal{Y}} = \tfrac{C}{2^n},$$

where $n$ is the minimal index such that $\ell_n \neq m_n$ and $C$ is a positive constant.

Observe that $\mathcal{Y}$ is compact. Indeed it is complete and the sequences with constant tails, starting from index $n$, form a finite $\frac{C}{2^n}$-net in $\mathcal{Y}$.

Given a space $\mathcal{X}$ in $\mathcal{Q}$, choose a sequence of $\frac{1}{2^n}$ nets $N_n \subset \mathcal{X}$ for each $n$. We can assume that $|N_n| \leqslant M_n$; let us label the points in $N_n$ by $\{1, \dots, M_n\}$. Consider the map $f\colon \mathcal{X} \to \mathcal{Y}$ defined by $f : x \mapsto (m_1(x), m_2(x), \dots)$ where $m_n(x)$ is the label of a point in $N_n$ that lies at the distance $< \frac{1}{2^n}$ from $x$.

If $\frac{1}{2^{n-2}} \geqslant |x - x'|_{\mathcal{X}} > \frac{1}{2^{n-1}}$, then $m_n(x) \neq m_n(x')$. It follows that $|f(x) - f(x')|_{\mathcal{Y}} \geqslant \frac{C}{2^n}$. In particular, if $C > 10$, then

$$|f(x) - f(x')|_{\mathcal{Y}} \geqslant |x - x'|_{\mathcal{X}}$$

for any $x, x' \in \mathcal{X}$. That is, $f$ is a distance-noncontracting map $\mathcal{X} \to \mathcal{Y}$.

**5.23**; *(a)* Apply 4.11, 5.25, 5.22, and 1.23.

*(b)*. Choose two compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$. Show that there are subsets $\mathcal{X}'$, and $\mathcal{Y}'$ in the Urysohn space $\mathcal{U}$ that are isometric to $\mathcal{X}$ and $\mathcal{Y}$ respectively, and such that

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}} = |\mathcal{X}' - \mathcal{Y}'|_{\mathrm{Haus}\,\mathcal{U}}.$$

Further, construct a sequence of compact sets $\mathcal{Z}_n \subset \mathcal{U}$ such that $\mathcal{Z}_n$ is an $\frac{1}{2^n}$-midpoint of $\mathcal{X}'$, and $\mathcal{Y}'$ in Haus$\mathcal{U}$ and

$$|\mathcal{Z}_n - \mathcal{Z}_{n+1}|_{\mathrm{Haus}\,\mathcal{U}} < \tfrac{1}{2^n}$$

for any $n$.

Observe that the sequence $\mathcal{Z}_n$ converges in GH, and its limit by $\mathcal{Z}$ is a midpoint of $\mathcal{X}$ and $\mathcal{Y}$. Finally, apply 5.22 and 1.23.

*Source:* [19].

**5.24**; *(a).* To check that $|{*}-{*}|_{\mathrm{GH}'}$ is a metric, it is sufficient to show that

$$|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}'} = 0 \quad \Longrightarrow \quad \mathcal{X} \overset{iso}{=\!=} \mathcal{Y};$$

the remaining conditions are trivial.

If $|\mathcal{X} - \mathcal{Y}|_{\mathrm{GH}'} = 0$, then there is a sequence of maps $f_n\colon \mathcal{X} \to \mathcal{Y}$ such that

$$|f_n(x) - f_n(x')|_{\mathcal{Y}} \geqslant |x - x'|_{\mathcal{X}} - \tfrac{1}{n}.$$

Choose a countable dense subset $S \subset \mathcal{X}$ and pass to a subsequence such that $f_n(x)$ converges for any $x \in S$; denote by $f_\infty\colon S \to \mathcal{Y}$ the limit map. Note that $f_\infty$ is distance-noncontracting, and it can be extended to a distance-noncontracting map $f_\infty\colon \mathcal{X} \to \mathcal{Y}$.

The same way we can construct a distance-noncontracting map $g_\infty\colon \mathcal{Y} \to \mathcal{X}$.

By 1.12, the compositions $f_\infty \circ g_\infty\colon \mathcal{Y} \to \mathcal{Y}$ and $g_\infty \circ f_\infty\colon \mathcal{X} \to \mathcal{X}$ are isometries. Therefore, $f_\infty$ and $g_\infty$ are isometries as well.

*(b).* The implication

$$|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \to 0 \quad \Rightarrow \quad |\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}'} \to 0$$

follows from 5.10$a$.

Now suppose $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}'} \to 0$. Show that $\{\mathcal{X}_n\}$ is a uniformly totally bonded family.

If $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \not\to 0$, then we can pass to a subsequence such that $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathrm{GH}} \geqslant \varepsilon$ for some $\varepsilon > 0$. By Gromov selection theorem, we can assume that $\mathcal{X}_n$ converges in the sense of Gromov–Hausdorff. From the first implication, the limit $\mathcal{X}'_\infty$ has to be isometric to $\mathcal{X}_\infty$; on the other hand, $|\mathcal{X}'_\infty - \mathcal{X}_\infty|_{\mathrm{GH}} \geqslant \varepsilon$ — a contradiction.

**5.26.** Apply 2.19 and 5.25.

**6.2.** Let $F = \{\, n \in \mathbb{N} : f(n) = n \,\}$; we need to show that $\omega(F) = 1$.

Consider an oriented graph $\Gamma$ with vertex set $\mathbb{N}\backslash F$ such that $m$ is connected to $n$ if $f(m) = n$. Show that each connected component of

$\Gamma$ has at most one cycle. Use it to subdivide vertices of $\Gamma$ into three sets $S_1$, $S_2$, and $S_3$ such that $f(S_i) \cap S_i = \varnothing$ for each $i$.

Conclude that $\omega(S_1) = \omega(S_2) = \omega(S_3) = 0$ and hence

$$\omega(F) = \omega(\mathbb{N} \setminus (S_1 \cup S_2 \cup S_3)) = 1.$$

*Source:* The presented proof was given by Robert Solovay [37], but the key statement is due to Miroslav Katětov [21].

**6.6.** Choose a nonprincipal ultrafilter $\omega$ and set $L(\boldsymbol{s}) = s_\omega$. It remains to observe that $L$ is linear.

**6.7.** Use 6.2.

**6.11.** Let $\gamma$ be a path from $p$ to $q$ in a metric tree $\mathcal{T}$. Assume that $\gamma$ passes thru a point $x$ on distance $\ell$ from $[pq]$. Then

$$\text{❻} \qquad \operatorname{length}\gamma \geqslant |p-q| + 2{\cdot}\ell.$$

Suppose that $\mathcal{T}_n$ is a sequence of metric trees that $\omega$-converges to $\mathcal{T}_\omega$. By 6.10, the space $\mathcal{T}_\omega$ is geodesic.

The uniqueness of geodesics follows from ❻. Indeed, if for a geodesic $[p_\omega q_\omega]$ there is another geodesic $\gamma_\omega$ connecting its ends, then it has to pass thru a point $x_\omega \notin [p_\omega q_\omega]$. Choose sequences $p_n, q_n, x_n \in \mathcal{T}_n$ such that $p_n \to p_\omega$, $q_n \to q_\omega$, and $x_n \to x_\omega$ as $n \to \omega$. Then

$$\begin{aligned} |p_\omega - q_\omega| &= \operatorname{length}\gamma \geqslant \lim_{n\to\omega}(|p_n - x_n| + |q_n - x_n|) \geqslant \\ &\geqslant \lim_{n\to\omega}(|p_n - q_n| + 2{\cdot}\ell_n) = \\ &= |p_\omega - q_\omega| + 2{\cdot}\ell_\omega. \end{aligned}$$

Since $x_\omega \notin [p_\omega q_\omega]$, we have that $\ell_\omega > 0$ — a contradiction.

It remains to show that any geodesic triangle $\mathcal{T}_\omega$ is a tripod. Consider the sequence of centers of tripods $m_n$ for given sequences of points $x_n, y_n, z_n \in \mathcal{T}_n$. Observe that its ultralimit $m_\omega$ is the center of the tripod with ends at $x_\omega, y_\omega, z_\omega \in \mathcal{T}_\omega$.

**6.12.** Construct $\boldsymbol{X}$ and distance-preserving embeddings $\mathcal{X}_n \hookrightarrow \boldsymbol{X}$ that satisfy 5.27. Given $x_\infty \in \mathcal{X}_\infty$, choose a sequence $x_n \in \mathcal{X}_n$ such that $x_n \to x_\infty$ in $\boldsymbol{X}$. Let $x_\omega$ be $\omega$-limit of the sequence $x_n$ in $\boldsymbol{X}$. Note that $x_\omega \in \mathcal{X}_\infty$. Show that the map $x_\infty \mapsto x_\omega$ is defined; that is, it does not depend on the choice of the sequence $x_n$. Further, show that the map $x_\infty \mapsto x_\omega$ is an isometry of $\mathcal{X}_\infty$. Make a conclusion.

**6.13.** Further, we consider $\mathcal{X}$ as a subset of $\mathcal{X}^\omega$.

*(a).* Follows directly from the definitions.

*(b).* Suppose $\mathcal{X}$ compact. Given a sequence $x_1, x_2, \ldots \in \mathcal{X}$, denote its $\omega$-limit in $\mathcal{X}^\omega$ by $x^\omega$ and its $\omega$-limit in $\mathcal{X}$ by $x_\omega$.

Observe that $x^\omega = \iota(x_\omega)$. Therefore, $\iota$ is onto.

If $\mathcal{X}$ is not compact, we can choose a sequence $x_n$ such that $|x_m - x_n| > \varepsilon$ for fixed $\varepsilon > 0$ and all $m \neq n$. Observe that

$$\lim_{n\to\omega} |x_n - y|_{\mathcal{X}} \geqslant \tfrac{\varepsilon}{2}$$

for any $y \in \mathcal{X}$. It follows that $x_\omega$ lies at the distance $\geqslant \frac{\varepsilon}{2}$ from $\mathcal{X}$.

*(c).* A sequence of points $x_n$ in $\mathcal{X}$ will be called $\omega$-bounded if there is a real constant $C$ such that

$$|p - x_n|_{\mathcal{X}} \leqslant C$$

for $\omega$-almost all $n$.

The same argument as in *(b)* shows that any $\omega$-bounded sequence has its $\omega$-limit in $\mathcal{X}$. Further, if $(x_n)$ is not $\omega$-bounded, then

$$\lim_{n\to\omega} |p - x_n|_{\mathcal{X}} = \infty;$$

that is, $x_\omega$ does not lie in the metric component of $p$ in $\mathcal{X}^\omega$.

**6.14.** Let us identify points in $\mathcal{X}$ with nonnegative integers. Consider the set $\mathcal{A}$ of all sequences $a_n$ such that $a_0 = 0$ and $a_{n+1} = a_n + \varepsilon_n \cdot 2^n$ where $\varepsilon_n \in \{0,1\}$ for any $n$. Observe that $\mathcal{A}$ has cardinality continuum and distinct sequences in $\mathcal{A}$ have distinct $\omega$-limits. Conclude that the cardinality of $\mathcal{X}^\omega$ is at least continuum.

Show and use that the spaces $\mathcal{X}^\omega$ and $(\mathcal{X}^\omega)^\omega$ have discrete metrics and both have cardinality at most continuum.

**6.15.** Choose a bijection $\iota\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Given a set $S \subset \mathbb{N}$, consider the sequence $S_1, S_2, \ldots$ of subsets in $\mathbb{N}$ defined by $m \in S_n$ if $(m,n) = \iota(k)$ for some $k \in S$. Set $\omega_1(S) = 1$ if and only if $\omega(S_n) = 1$ for $\omega$-almost all $n$. It remains to check that $\omega_1$ meets the conditions of the exercise.

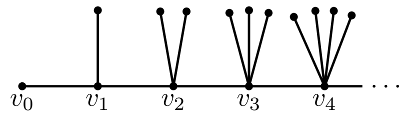
*Comment.* It turns out that $\omega_1 \neq \omega$ for any $\iota$; see the post of Andreas Blass [6].

**6.17.** Apply 6.5 and 6.16.

**6.18.** Consider the infinite metric $\mathcal{T}$ tree with unit edges shown on the diagram. Observe that $\mathcal{T}$ is proper.

Consider the vertex $v_\omega = \lim_{n\to\omega} v_n$ in the ultrapower $\mathcal{T}^\omega$. Observe that $\omega$ has an infinite degree. Conclude that $\mathcal{T}^\omega$ is not locally compact.

**6.19.** Consider a product space $[0,1] \times [0,\frac{1}{2}] \times [0,\frac{1}{4}] \times \ldots$




**6.20**; *(a).* Show that there is $\delta > 0$ such that sides of any geodesic triangle in $\mathcal{L}$ intersect a disk of radius $\delta$. Conclude that any geodesic triangle in $\operatorname{Asym}\mathcal{L}$ is a tripod.

*(b).* Observe that $\mathcal{L}$ is one-point-homogeneous and use it.

*(c).* By *(b)*, it is sufficient to show that $p_\omega$ has a continuum degree.

Choose distinct geodesics $\gamma_1, \gamma_2 \colon [0, \infty) \to L$ that start at a point $p$. Show that the limits of $\gamma_1$ and $\gamma_2$ run in the different connected components of $(\operatorname{Asym}\mathcal{L}) \setminus \{p_\omega\}$. Since there is a continuum of distinct geodesics starting at $p$, we get that the degree of $p_\omega$ is at least continuum.

On the other hand, the set of sequences of points in $\mathcal{L}$ has cardinality continuum. In particular, the set of points in $\operatorname{Asym}\mathcal{L}$ has cardinality at most continuum. It follows that the degree of any vertex is at most continuum.

*(d).* The proof for the Lobachevsky space goes along the same lines.

For the infinite three-regular tree, part *(a)* follows from 6.11. The three-regular tree is not one-point-homogeneous, but it is vertex-homogeneous; the latter is sufficient to prove *(b)*. No changes are needed in *(c)*.

*Remark.* According to the result of Anna Dyubina and Iosif Polterovich [10], the properties *(b)* and *(c)* describe the tree $\mathcal{T}$ up to isometry. In particular, the asymptotic space of the Lobachevsky plane does not depend on the choice of the ultrafilter and the sequence $\lambda_n \to \infty$.

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