Lectures in metric geometry

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Disclaimer

Considerable part of the text is a compilation from [1, 2, 15, 17, 18] and its drafts.

Contents

Chapter 0

Homework assignments

It is better to think about all the problems, but you do not have to solve *all* of them. If a problem is solved, you do not have to write its solutions, but try sketch it.

0.1 Due Tue Jan 21

Exercises: ~~1.3.1,~~ 1.4.3, 1.7.2, ~~1.7.3,~~ 1.7.8, 2.1.7.

0.2 Due Tue Jan 28

Exercises: 1.3.1, 1.7.3, 2.1.8, 3.4.4, 3.5.1, 3.5.2a.

0.3 Due Tue Feb 4

Exercises: 1.8.3, 3.5.2b, ~~3.6.3,~~ ~~3.6.4,~~ 4.4.1, 4.4.3.

0.4 Due Tue Feb 11

Finish exercises 1.8.3 , 3.5.2b, 3.6.3, 3.6.4.
Exercises: 4.3.3, 4.5.1, 5.2.2, 5.3.2.

Chapter 1

Definitions

1.1 Metric spaces

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

The function

$$\operatorname{dist}_x \colon y \mapsto |x-y|$$

is called the *distance function* from x.

Given $R \in [0,\infty]$ and $x \in \mathcal{X}$, the sets

$$\mathrm{B}(x,R) = \{y \in \mathcal{X} \mid |x-y| < R\},$$
$$\overline{\mathrm{B}}[x,R] = \{y \in \mathcal{X} \mid |x-y| \leqslant R\}$$

are called, respectively, the *open* and the *closed balls* of radius R with center x. Again, if we need to emphasize that these balls are taken in the metric space $\mathcal{X}$, we write

$$\mathrm{B}(x,R)_{\mathcal{X}} \quad \text{and} \quad \overline{\mathrm{B}}[x,R]_{\mathcal{X}}.$$

1.2 Variations of definition

Recall that a metric is a real-valued function $(x,y) \mapsto |x-y|_{\mathcal{X}}$ that satisfies the following conditions for any three points $x,y,z \in \mathcal{X}$:

(i) $|x-y|_{\mathcal{X}} \geqslant 0$,

(ii) $|x-y|_{\mathcal{X}} = 0 \iff x = y$,

(iii) $|x-y|_{\mathcal{X}} = |y-x|_{\mathcal{X}}$,

(iv) $|x-y|_{\mathcal{X}} + |y-z|_{\mathcal{X}} \geqslant |x-z|_{\mathcal{X}}$,

Pseudometrics. A generalization of a metric in which the distance between two distinct points can be zero is called *pseudometric*. In other words, to define pseudometric, we need to remove condition (ii) from the list.

The following two observations show that nearly any question about pseudometric spaces can be reduced to a question about genuine metric spaces.

Assume $\mathcal{X}$ is a pseudometric space. Set $x \sim y$ if $|x-y|=0$. Note that if $x \sim x'$, then $|y-x| = |y-x'|$ for any $y \in \mathcal{X}$. Thus, $|*-$ $-*|$ defines a metric on the quotient set $\mathcal{X}/\sim$. In this way we obtain a metric space $\mathcal{X}'$. The space $\mathcal{X}'$ is called the *corresponding metric space* for the pseudometric space $\mathcal{X}$. Often we do not distinguish between $\mathcal{X}'$ and $\mathcal{X}$.

∞-metrics. One may also consider metrics with values in $\mathbb{R}\cup\{\infty\}$; we might call them ∞-metrics or simply metrics.

Again nearly any question about ∞-metric spaces can be reduced to a question about genuine metric spaces.

Indeed, set $x \approx y$ if and only if $|x-y| < \infty$; this is an other equivalence relation on $\mathcal{X}$. The equivalence class of a point $x \in \mathcal{X}$ will be called the *metric component* of x; it will be denoted as $\mathcal{X}_x$. One could think of $\mathcal{X}_x$ as $\mathrm{B}(x,\infty)_{\mathcal{X}}$ — the open ball centered at x and radius ∞ in $\mathcal{X}$.

It follows that any ∞-metric space is a *disjoint union* of genuine metric spaces — the metric components of the original ∞-metric space.

1.2.1. Exercise. *Given two sets A and B on the plane, set*

$$|A-B| = \mu(A\backslash B) + \mu(B\backslash A),$$

where μ denotes the Lebesgue measure.

a) Show that $|*-*|$ is a pseudometric on the set of bounded measurable sets of the plane.

b) Show that $|*-*|$ is an ∞-metric on the set of all open sets of the plane.

1.3 Completeness

Recall that a metric space $\mathcal{X}$ is called *complete* if every Cauchy sequence of points in $\mathcal{X}$ converges in $\mathcal{X}$.

1.3.1. Exercise. *Suppose that ρ is a positive continuous function on a complete metric space $\mathcal{X}$. Show that for any $\varepsilon > 0$ there is a point $x \in \mathcal{X}$ such that*

$$\rho(x) < (1+\varepsilon)\cdot\rho(y)$$

for any point $y \in \mathrm{B}(x, \rho(x))$*.*

Most of the time we will assume that a metric space is complete. The following construction produces a complete metric space $\bar{\mathcal{X}}$ for any given metric space $\mathcal{X}$. The space $\bar{\mathcal{X}}$ is called *completion* of $\mathcal{X}$; the original space $\mathcal{X}$ forms a dense subset in $\bar{\mathcal{X}}$.

Completion. Given metric space $\mathcal{X}$, consider the set of all Cauchy sequences in $\mathcal{X}$. Note that for any two Cauchy sequences (x_n) and (y_n) the right hand side in ❶ is defined; moreover it defines a pseudometric on the set $\mathcal{C}$ of all Cauchy sequences

❶ $$|(x_n) - (y_n)|_{\mathcal{C}} \stackrel{def}{=\!=} \lim_{n \to \infty} |x_n - y_n|_{\mathcal{X}}.$$

The corresponding metric space is called a completion of $\mathcal{X}$.

It is left as an exercise that completion of $\mathcal{X}$ is complete.

Note that for each point $x \in \mathcal{X}$ one can consider a constant sequence $x_n = x$ which is Cauchy. It defines a natural map $\mathcal{X} \to \bar{\mathcal{X}}$. It is easy to check that this map is distance preserving. In partucular we can (and will) consider $\mathcal{X}$ as a subset of $\bar{\mathcal{X}}$.

1.4 Compactness

Let us recall few equivalent definitions of compact metric spaces.

1.4.1. Definition. *A metric space $\mathcal{K}$ is compact if and only if one of the following equivalent condition holds:*

a) Every open cover of $\mathcal{K}$ has a finite subcover.

b) For any open cover of $\mathcal{K}$ there is $\varepsilon > 0$ such that any ε-ball in $\mathcal{K}$ lie in one element of the cover. (The value ε is called Lebesgue number of the covering.)

c) Every sequence in $\mathcal{K}$ has a convergent subsequence.

d) The space $\mathcal{K}$ is complete and totally bounded*; that is, for any $\varepsilon > 0$, the space $\mathcal{K}$ admits a finite cover by open ε-balls.*[1]

Let $\operatorname{pack}_\varepsilon \mathcal{X}$ be exact upper bound on the number of points $x_1, \dots$ $\dots, x_n \in \mathcal{X}$ such that $|x_i - x_j| \geqslant \varepsilon$ for any $i \neq j$.

If $n = \operatorname{pack}_\varepsilon \mathcal{X} < \infty$, then the collection of points $x_1, \dots, x_n$ is called a *maximal ε-packing*. Note that n is the maximal number of open disjoint $\frac{\varepsilon}{2}$-balls in $\mathcal{X}$.

1.4.2. Exercise. *Show that a complete space $\mathcal{X}$ is compact if and only of* $\operatorname{pack}_\varepsilon \mathcal{X} < \infty$ *for any $\varepsilon > 0$.*

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[1] Equivalently, for any $\varepsilon > 0$ there is a finite *ε-net*; that is a finite set of points $x_1, \dots, x_n \in \mathcal{K}$ such that any other point x lies on the distance less than ε from one of x_i.

Show that any maximal ε-packing is an ε-net.

1.4.3. Exercise. *Let $\mathcal{K}$ be a compact metric space and*

$$f\colon \mathcal{K} \to \mathcal{K}$$

be a distance non-decreasing map. Prove that f is an isometry.

A metric space $\mathcal{X}$ is called *proper* if all closed bounded sets in $\mathcal{X}$ are compact. This condition is equivalent to each of the following statements:

- ⋄ For some (and therefore any) point $p \in \mathcal{X}$ and any $R < \infty$, the closed ball $\overline{\mathrm{B}}[p, R]_{\mathcal{X}}$ is compact.
- ⋄ The function $\mathrm{dist}_p\colon \mathcal{X} \to \mathbb{R}$ is proper for some (and therefore any) point $p \in \mathcal{X}$; that is, for any compact set $K \subset \mathbb{R}$, its inverse image

  $$\mathrm{dist}_p^{-1}(K) = \{\, x \in \mathcal{X} \;:\; |p - x|_{\mathcal{X}} \in K \,\}$$

  is compact.

A metric space $\mathcal{X}$ is called *locally compact* if any point in $\mathcal{X}$ admits a compact neighborhood; in other words, for any point $x \in \mathcal{X}$ a closed ball $\overline{\mathrm{B}}[x, r]$ is compact for some $r > 0$.

1.5 Geodesics

Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ a real interval. A globally isometric map $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a *geodesic*[2]; in other words, $\gamma\colon \mathbb{I} \to \mathcal{X}$ is a geodesic if

$$|\gamma(s) - \gamma(t)|_{\mathcal{X}} = |s - t|$$

for any pair $s, t \in \mathbb{I}$.

We say that $\gamma\colon \mathbb{I} \to \mathcal{X}$ is a geodesic from point p to point q if $\mathbb{I} = [a, b]$ and $p = \gamma(a)$, $q = \gamma(b)$. In this case the image of γ is denoted by $[pq]$ and with an abuse of notations we also call it a *geodesic*. Given a geodesic $[pq]$, we can parametrize it by distance to p; this parametrization will be denoted by $\mathrm{geod}_{[pq]}(t)$.

We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$. We also use the following shortcut notation:

$$]pq[= [pq]\backslash\{p, q\}, \qquad]pq] = [pq]\backslash\{p\}, \qquad [pq[= [pq]\backslash\{q\}.$$

In general, a geodesic from p to q need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we assume mean that we have made a choice of geodesic.

[2] Various authors call it differently: *shortest path*, *minimizing geodesic*.

A metric space is called *geodesic* if any pair of its points can be joined by a geodesic.

A *geodesic path* is a geodesic with constant-speed parametrization by $[0,1]$. Given a geodesic $[pq]$, we denote by $\mathrm{path}_{[pq]}$ the corresponding geodesic path; that is,

$$\mathrm{path}_{[pq]}(t) \stackrel{def}{=\!=} \mathrm{geod}_{[pq]}(t{\cdot}|p-q|).$$

A curve $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a *local geodesic* if for any $t \in \mathbb{I}$ there is a neighborhood U of t in $\mathbb{I}$ such that the restriction $\gamma|_U$ is a geodesic. A constant-speed parametrization of a local geodesic by the unit interval $[0,1]$ is called a *local geodesic path*.

1.6 Length

A *curve* is defined as a continuous map from a real interval to a space. If the real interval is $[0,1]$, then the curve is called a *path*.

1.6.1. Definition. *Let $\mathcal{X}$ be a metric space and $\alpha\colon \mathbb{I} \to \mathcal{X}$ be a curve. We define the* length *of α as*

$$\mathrm{length}\,\alpha \stackrel{def}{=\!=} \sup_{t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n} \sum_i |\alpha(t_i) - \alpha(t_{i-1})|.$$

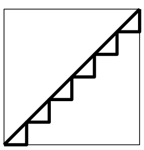
A curve α is called rectifiable *if* $\mathrm{length}\,\alpha < \infty$.

1.6.2. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\gamma_n\colon \mathbb{I} \to \mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty\colon \mathbb{I} \to \mathcal{X}$; that is, for any fixed $t \in \mathbb{I}$, $\gamma_n(t) \to \gamma_\infty(t)$ as $n \to \infty$. Then

❶ $$\liminf_{n\to\infty} \mathrm{length}\,\gamma_n \geqslant \mathrm{length}\,\gamma_\infty.$$

Note that the inequality ❶ might be strict. For example the diagonal γ_∞ of the unit square can be approximated by a stairs-like polygonal curves γ_n with sides parallel to the sides of the square (γ_6 is on the picture). In this case



$$\mathrm{length}\,\gamma_\infty = \sqrt{2} \quad \text{and} \quad \mathrm{length}\,\gamma_n = 2$$

for any n.

Proof. Fix a sequence $t_0 < t_1 < \dots < t_k$ in $\mathbb{I}$. Set

$$\Sigma_n \stackrel{def}{=\!=} |\gamma_n(t_0) - \gamma_n(t_1)| + \dots + |\gamma_n(t_{k-1}) - \gamma_n(t_k)|.$$
$$\Sigma_\infty \stackrel{def}{=\!=} |\gamma_\infty(t_0) - \gamma_\infty(t_1)| + \dots + |\gamma_\infty(t_{k-1}) - \gamma_\infty(t_k)|.$$

Note that for each i we have

$$|\gamma_n(t_{i-1}) - \gamma_n(t_i)| \to |\gamma_\infty(t_{i-1}) - \gamma_\infty(t_i)|$$

and therefore

$$\Sigma_n \to \Sigma_\infty$$

as $n \to \infty$. Note that

$$\Sigma_n \leqslant \operatorname{length} \gamma_n$$

for each n. Hence

$$❷ \qquad \liminf_{n\to\infty} \operatorname{length} \gamma_n \geqslant \Sigma_\infty.$$

If γ_∞ is rectifiable, we can assume that

$$\operatorname{length} \gamma_\infty < \Sigma_\infty + \varepsilon.$$

for any given $\varepsilon > 0$. By ❷ it follows that

$$\liminf_{n\to\infty} \operatorname{length} \gamma_n > \operatorname{length} \gamma_\infty - \varepsilon$$

for any $\varepsilon > 0$; whence ❶ follows.

It remains to consider the case when γ_∞ is not rectifiable; that is, $\operatorname{length} \gamma_\infty = \infty$. In this case we can choose a partition so that $\Sigma_\infty > L$ for any real number L. By ❷ it follows that

$$\liminf_{n\to\infty} \operatorname{length} \gamma_n > L$$

for any given L; whence

$$\liminf_{n\to\infty} \operatorname{length} \gamma_n = \infty$$

and ❶ follows. □

1.7 Length spaces

If for any $\varepsilon > 0$ and any pair of points x and y in a metric space $\mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length} \alpha < |x - y| + \varepsilon,$$

then $\mathcal{X}$ is called a *length space* and the metric on $\mathcal{X}$ is called a *length metric*.

Note that any geodesic space is a length space. As can be seen from the following example, the converse does not hold.

1.7.1. Example. *Let $\mathcal{X}$ be obtained by gluing a countable collection of disjoint intervals $\{\mathbb{I}_n\}$ of length $1+\frac{1}{n}$, where for each $\mathbb{I}_n$ the left end is glued to p and the right end to q.*

Observe that the space $\mathcal{X}$ carries a natural complete length metric with respect to which $|p-q| = 1$ but there is no geodesic connecting p to q.

1.7.2. Exercise. *Give an example of a complete length space for which no pair of distinct points can be joined by a geodesic.*

Directly from the definition, it follows that if a path $\alpha\colon [0,1] \to \mathcal{X}$ connects two points x and y (that is, if $\alpha(0) = x$ and $\alpha(1) = y$), then

$$\operatorname{length}\alpha \geqslant |x-y|.$$

Set

$$\|x-y\| = \inf\{\operatorname{length}\alpha\}$$

where the greatest lower bound is taken for all paths connecing x and y. It is straightforward to check that $(x,y) \mapsto \|x-y\|$ is an ∞-metric; moreover $(\mathcal{X}, \|{*}-{*}\|)$ is a length space. The metric $\|{*}-{*}\|$ is called *induced length metric.*

1.7.3. Exercise. *Suppose $(\mathcal{X}, |{*}-{*}|)$ is a complete metric space. Show that $(\mathcal{X}, \|{*}-{*}\|)$ is complete.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x-y|_A = \inf_\alpha\{\operatorname{length}\alpha\},$$

where the greatest lower bound is taken for all paths α from x to y in A.[3]

Let $\mathcal{X}$ be a metric space and $x, y \in \mathcal{X}$.

(i) A point $z \in \mathcal{X}$ is called a *midpoint* between x and y if

$$|x-z| = |y-z| = \tfrac{1}{2}\cdot|x-y|.$$

[3] This notation slightly conflicts with the previously defined notation for distance $|x-y|_{\mathcal{X}}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.

(ii) Assume $\varepsilon \geqslant 0$. A point $z \in \mathcal{X}$ is called an *ε-midpoint* between x and y if

$$|x - z|, \quad |y - z| \leqslant \tfrac{1}{2}\cdot|x - y| + \varepsilon.$$

Note that a 0-midpoint is the same as a midpoint.

1.7.4. Lemma. *Let $\mathcal{X}$ be a complete metric space.*

a) Assume that for any pair of points $x, y \in \mathcal{X}$ and any $\varepsilon > 0$ there is an ε-midpoint z. Then $\mathcal{X}$ is a length space.

b) Assume that for any pair of points $x, y \in \mathcal{X}$, there is a midpoint z. Then $\mathcal{X}$ is a geodesic space.

Proof. We first prove (a). Let $x, y \in \mathcal{X}$ be a pair of points.

Set $\varepsilon_n = \frac{\varepsilon}{4^n}$, $\alpha(0) = x$ and $\alpha(1) = y$.

Let $\alpha(\frac{1}{2})$ be an ε_1-midpoint between $\alpha(0)$ and $\alpha(1)$. Further, let $\alpha(\frac{1}{4})$ and $\alpha(\frac{3}{4})$ be ε_2-midpoints between the pairs $(\alpha(0), \alpha(\frac{1}{2}))$ and $(\alpha(\frac{1}{2}), \alpha(1))$ respectively. Applying the above procedure recursively, on the n-th step we define $\alpha(\frac{k}{2^n})$, for every odd integer k such that $0 < \frac{k}{2^n} < 1$, as an ε_n-midpoint between the already defined $\alpha(\frac{k-1}{2^n})$ and $\alpha(\frac{k+1}{2^n})$.

In this way we define $\alpha(t)$ for $t \in W$, where W denotes the set of dyadic rationals in $[0, 1]$. Since $\mathcal{X}$ is complete, the map α can be extended continuously to $[0, 1]$. Moreover,

$$\begin{aligned} \text{length}\,\alpha &\leqslant |x - y| + \sum_{n=1}^{\infty} 2^{n-1}\cdot\varepsilon_n \leqslant \\ &\leqslant |x - y| + \tfrac{\varepsilon}{2}. \end{aligned} \tag{❶}$$

Since $\varepsilon > 0$ is arbitrary, we get (a).

To prove (b), one should repeat the same argument taking midpoints instead of ε_n-midpoints. In this case ❶ holds for $\varepsilon_n = \varepsilon = 0$. $\square$

Since in a compact space a sequence of $\frac{1}{n}$-midpoints z_n contains a convergent subsequence, Lemma 1.7.4 immediately implies

1.7.5. Proposition. *A proper length space is geodesic.*

1.7.6. Hopf–Rinow theorem. *Any complete, locally compact length space is proper.*

It is instructive to solve the following exercise before reading the proof.

1.7.7. Exercise. *Give an example of space which is locally compact but not proper.*

Proof. Let $\mathcal{X}$ be a locally compact length space. Given $x \in \mathcal{X}$, denote by $\rho(x)$ the supremum of all $R > 0$ such that the closed ball $\overline{\mathrm{B}}[x, R]$ is compact. Since $\mathcal{X}$ is locally compact,

❷ $$\rho(x) > 0 \quad \text{for any} \quad x \in \mathcal{X}.$$

It is sufficient to show that $\rho(x) = \infty$ for some (and therefore any) point $x \in \mathcal{X}$.

Assume the contrary; that is, $\rho(x) < \infty$. We claim that

❸ $B = \overline{\mathrm{B}}[x, \rho(x)]$ *is compact for any* x.

Indeed, $\mathcal{X}$ is a length space; therefore for any $\varepsilon > 0$, the set $\overline{\mathrm{B}}[x, \rho(x) - \varepsilon]$ is a compact ε-net in B. Since B is closed and hence complete, it must be compact. △

Next we claim that

❹ $|\rho(x) - \rho(y)| \leqslant |x - y|_{\mathcal{X}}$ *for any* $x, y \in \mathcal{X}$*; in particular* $\rho \colon \mathcal{X} \to \mathbb{R}$ *is a continuous function.*

Indeed, assume the contrary; that is, $\rho(x) + |x - y| < \rho(y)$ for some $x, y \in \mathcal{X}$. Then $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$ is a closed subset of $\overline{\mathrm{B}}[y, \rho(y)]$ for some $\varepsilon > 0$. Then compactness of $\overline{\mathrm{B}}[y, \rho(y)]$ implies compactness of $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$, a contradiction. △

Set $\varepsilon = \min\{\, \rho(y) \,:\, y \in B \,\}$; the minimum is defined since B is compact. From ❷, we have $\varepsilon > 0$.

Choose a finite $\frac{\varepsilon}{10}$-net $\{a_1, a_2, \dots, a_n\}$ in B. The union W of the closed balls $\overline{\mathrm{B}}[a_i, \varepsilon]$ is compact. Clearly $\overline{\mathrm{B}}[x, \rho(x) + \frac{\varepsilon}{10}] \subset W$. Therefore $\overline{\mathrm{B}}[x, \rho(x) + \frac{\varepsilon}{10}]$ is compact, a contradiction. □

1.7.8. Exercise. *Construct a geodesic space that is locally compact, but whose completion is neither geodesic nor locally compact.*

1.8 Subsets in normed spaces

Recall that a function $v \mapsto |v|$ on a vector space $\mathcal{V}$ is called *norm* if it satisfies the following condition for any two vectors $v, w \in \mathcal{V}$ and a scalar α:

- ◇ $|v| \geqslant 0$;
- ◇ $|\alpha{\cdot}v| = |\alpha|{\cdot}|v|$;
- ◇ $|v| + |w| \geqslant |v + w|$.

It is straightforward to check that for any normed space the function $(v, w) \mapsto |v - w|$ defines a metric on it. Therefore any normed space is an example of metric space (which is in fact geodesic). The following lemma says in particular that any metric space is isometric to a subset of a normed space.

1.8.1. Lemma. *Suppose $\mathcal{X}$ is a bounded separable space; that is,* $\operatorname{diam}\mathcal{X}$ *is finite and $\mathcal{X}$ contains a countable, dense set $\{w_n\}$. Given $x \in \mathcal{X}$, set $a_n(x) = |w_n - x|_{\mathcal{X}}$. Then*

$$\iota\colon x \mapsto (a_1(x), a_2(x), \dots)$$

defines a distance preserving embedding $\iota\colon \mathcal{X} \hookrightarrow \ell^\infty$.

Proof. By the triangle inequality

$$|a_n(x) - a_n(y)| \leqslant |x - y|_{\mathcal{X}}.$$

Therefore ι is short.

Again by triangle inequality we have

$$|a_n(x) - a_n(y)| \geqslant |x - y|_{\mathcal{X}} - 2{\cdot}|w_n - x|_{\mathcal{X}}.$$

Since the set $\{w_n\}$ is dense, we can choose w_n arbitrary close to x. Whence the value $|a_n(x) - a_n(y)|$ can be chosen arbitrary close to $|x - y|_{\mathcal{X}}$. In other words

$$\sup_n\{\, ||w_n - x|_{\mathcal{X}} - |w_n - y|_{\mathcal{X}}|\,\} \geqslant |x - y|_{\mathcal{X}};$$

hence ι is distance non-decreasing. $\square$

The following exercise generalizes the lemma to arbitrary separable spaces.

1.8.2. Exercise. *Suppose $\{w_n\}$ is a countable, dense set in a metric space $\mathcal{X}$. Choose $x_0 \in \mathcal{X}$; given $x \in \mathcal{X}$, set*

$$a_n(x) = |w_n - x|_{\mathcal{X}} - |w_n - x_0|_{\mathcal{X}}.$$

Show that $\iota\colon x \mapsto (a_1(x), a_2(x), \dots)$ defines a distance preserving embedding $\iota\colon \mathcal{X} \hookrightarrow \ell^\infty$.

1.8.3. Exercise. *Show that any compact metric space is isometric $\mathcal{K}$ to a subspace of a compact geodesic space.*

The lemma above was proved by Maurice René Fréchet in the paper where he defined metric space [8]. Nearly identical construction was rediscovered later by Kazimierz Kuratowski [13]. Namely he made the following claim:

1.8.4. Lemma. *Let $\mathcal{X}$ be arbitrary metric space. Denote by $\ell^\infty(\mathcal{X})$ the space of all bounded functions of $\mathcal{X}$ equipped with sup-norm.*

Then for any point $x_0 \in \mathcal{X}$, the map $\iota\colon \mathcal{X} \to \ell^\infty(\mathcal{X})$ defied by

$$\iota\colon x \mapsto (\mathrm{dist}_x - \mathrm{dist}_{x_0})$$

is distance preserving.

Note that this claim implies that *any metric space is isometric to a subset of a normed vector space.*

Chapter 2

Space of sets

2.1 Hausdorff convergence

Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$, consider the distance function to A

$$\operatorname{dist}_A : \mathcal{X} \to [0, \infty)$$

defined as

$$\operatorname{dist}_A(x) \stackrel{def}{=\!=} \inf_{a \in A} \{ |a - x|_{\mathcal{X}} \}.$$

2.1.1. Definition. *Let A and B be two compact subsets of a metric space $\mathcal{X}$. Then the* Hausdorff distance *between A and B is defined as*

$$|A - B|_{\mathcal{H}(\mathcal{X})} \stackrel{def}{=\!=} \sup_{x \in \mathcal{X}} \{ |\operatorname{dist}_A(x) - \operatorname{dist}_B(x)| \}.$$

Suppose A and B be two compact subsets of a metric space $\mathcal{X}$. It is straightforward to check that $|A - B|_{\mathcal{H}(\mathcal{X})} \leqslant R$ if and only if $\operatorname{dist}_A(b) \leqslant R$ for any $b \in B$ and $\operatorname{dist}_B(a) \leqslant R$ for any $a \in A$. In other words, $|A - B|_{\mathcal{H}(\mathcal{X})} < R$ if and only if B lies in a R-neighborhood of A, and A lies in a R-neighborhood of B.

Note that the set of all nonempty compact subsets of a metric space $\mathcal{X}$ equipped with the Hausdorff metric forms a metric space. This new metric space will be denoted as $\mathcal{H}(\mathcal{X})$.

2.1.2. Exercise. *Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$ define its diameter as*

$$\operatorname{diam} A \stackrel{def}{=\!=} \sup_{a,b \in A} |a - b|.$$

Show that

$$\operatorname{diam}\colon \mathcal{H}(\mathcal{X}) \to \mathbb{R}$$

is a 2*-Lipschitz function; that is,* $|\operatorname{diam} A - \operatorname{diam} B| \leqslant 2\cdot|A-B|_{\mathcal{H}(\mathcal{X})}$.

2.1.3. Blaschke selection theorem. *Let $\mathcal{X}$ be a metric space. Then the space $\mathcal{H}(\mathcal{X})$ is compact if and only if $\mathcal{X}$ is compact.*

Note that the theorem implies that from any sequence of compact sets in $\mathcal{X}$ one can select a subsequence converging in the sense of Hausdorff; by that reason it is called a selection theorem.

Proof; "only if" part. Note that the map $\iota\colon \mathcal{X} \to \mathcal{H}(\mathcal{X})$, defined as $\iota\colon x \mapsto \{x\}$ (that is, point x mapped to the one-point subset $\{x\}$ of $\mathcal{X}$) is distance preserving. Therefore $\mathcal{X}$ is isometric to the set $\iota(\mathcal{X})$ in $\mathcal{H}(\mathcal{X})$.

Note that for a nonempty subset $A \subset \mathcal{X}$, we have $\operatorname{diam} A = 0$ if and only if A is a one-point set. Therefore, from Exercise 2.1.2, it follows that $\iota(\mathcal{X})$ is closed in $\mathcal{H}(\mathcal{X})$.

Hence $\iota(\mathcal{X})$ is compact, as it is a closed subset of a compact space. Since $\mathcal{X}$ is isometric to $\iota(\mathcal{X})$, "only if" part follows. □

To prove "if" part we will need the following two lemmas.

2.1.4. Lemma. *Let $K_1 \supset K_2 \supset \dots$ be a sequence of nonempty compact sets in a metric space $\mathcal{X}$ then $K_\infty = \bigcap_n K_n$ is the Hausdorff limit of K_n; that is, $|K_\infty - K_n|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$.*

Proof. Note that K_∞ is compact; by finite intersection property, K_∞ is nonempty.

If the assertion were false, then there is $\varepsilon > 0$ such that for each n one can choose $x_n \in K_n$ such that $\operatorname{dist}_{K_\infty}(x_n) \geqslant \varepsilon$. Note that $x_n \in K_1$ for each n. Since K_1 is compact, there is a *partial limit*[1] x_∞ of x_n. Clearly $\operatorname{dist}_{K_\infty}(x_\infty) \geqslant \varepsilon$.

On the other hand, since K_n is closed and $x_m \in K_n$ for $m \geqslant n$, we get $x_\infty \in K_n$ for each n. It follows that $x_\infty \in K_\infty$ and therefore $\operatorname{dist}_{K_\infty}(x_\infty) = 0$, a contradiction. □

2.1.5. Lemma. *If $\mathcal{X}$ is a compact metric space, then $\mathcal{H}(\mathcal{X})$ is complete.*

Proof. Let (Q_n) be a Cauchy sequence in $\mathcal{H}(\mathcal{X})$. Passing to a subsequence of Q_n we may assume that

❶ $$|Q_n - Q_{n+1}|_{\mathcal{H}(\mathcal{X})} \leqslant \tfrac{1}{10^n}$$

[1] Partial limit is a limit of a subsequence.

for each n.

Set

$$K_n = \left\{ x \in \mathcal{X} \ : \ \mathrm{dist}_{Q_n}(x) \leqslant \tfrac{1}{10^n} \right\}$$

Since $\mathcal{X}$ is compact so is each K_n.

Clearly, $|Q_n - K_n|_{\mathcal{H}(\mathcal{X})} \leqslant \frac{1}{10^n}$ and from ❶, we get $K_n \supset K_{n+1}$ for each n. Set

$$K_\infty = \bigcap_{n=1}^{\infty} K_n.$$

Applying Lemma 2.1.4, we get that $|K_n - K_\infty|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$. Since $|Q_n - K_n|_{\mathcal{H}(\mathcal{X})} \leqslant \frac{1}{10^n}$, we get $|Q_n - K_\infty|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$ — hence the lemma. ☐

2.1.6. Exercise. *Let $\mathcal{X}$ be a complete metric space and K_n be a sequence of compact sets which converges in the sence of Hausdorff. Show that closure of the union $\bigcup_{n=1}^{\infty} K_n$ is compact.*

Use this to show that in Lemma 2.1.5 compactness of $\mathcal{X}$ can be exchanged to completeness.

Proof of "if" part in 2.1.3. According to Lemma 2.1.5, $\mathcal{H}(\mathcal{X})$ is complete. It remains to show that $\mathcal{H}(\mathcal{X})$ is totally bounded (1.4.1d); that is, given $\varepsilon > 0$ there is a finite ε-net in $\mathcal{H}(\mathcal{X})$.

Choose a finite ε-net A in $\mathcal{X}$. Denote by $\mathcal{A}$ the set of all subsets of A. Note that $\mathcal{A}$ is finite set in $\mathcal{H}(\mathcal{X})$. For each compact set $K \subset \mathcal{X}$, consider the subset K' of all points $a \in A$ such that $\mathrm{dist}_K(a) \leqslant \varepsilon$. Then $K' \in \mathcal{A}$ and $|K - K'|_{\mathcal{H}(\mathcal{X})} \leqslant \varepsilon$. In other words $\mathcal{A}$ is a finite ε-net in $\mathcal{H}(\mathcal{X})$. ☐

Hausdorff metric defines convergence of compact sets which is more important than metric itself.

2.1.7. Exercise. *Let X and Y be two compact subsets in $\mathbb{R}^2$. Assume $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, is it true that $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, where ∂X denotes the boundary of X.*

Does the converse holds? That is, assume X and Y be two compact subsets in $\mathbb{R}^2$ and $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$; is it true that $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} <$ $< \varepsilon$?

2.1.8. Exercise. *Let $\mathcal{C}$ be a subspace of $\mathcal{H}(\mathbb{R}^2)$ formed by all compact convex subsets in $\mathbb{R}^2$. Show that perimeter*[2] *and area are continuous*

[2]If the set degenerates to a line segment of length ℓ, then its perimeter is defined as $2 \cdot \ell$.

on $\mathcal{C}$. That is, if a sequence of convex compact plane sets X_n converges to X_∞ in the sense of Hausdorff, then

$$\operatorname{perim} X_n \to \operatorname{perim} X_\infty \quad \textit{and} \quad \operatorname{area} X_n \to \operatorname{area} X_\infty$$

as $n \to \infty$.

The above exercise can be used in a proof of isoperimetrical inequality in the plane; it states that *among the plane figures bounded by closed curves of length at most ℓ the round disc has maximal area.*

Indeed it is sufficient to consider only convex figures of given perimeter; if a figure is not convex pass to its convex hull and observe that it has larger area and smaller perimeter. Further the exercise guarantees existence of a figure D_ℓ with perimeter ℓ and maximal area. It remains to show that D_ℓ is a round disc. The latter is easy to show, see for example Steiner's 4-joint method [3].

2.2 A variation

It seems that *Hausdorff convergence* was first introduced by Felix Hausdorff [11], and a couple of years later an equivalent definition was given by Wilhelm Blaschke [3].

The following refinement of the definition was introduced by Zdeněk Frolík in [9], and later rediscovered by Robert Wijsman in [21]. This refinement takes an intermediate place between the original Hausdorff convergence and *closed convergence*, also introduced by Hausdorff in [11]; so we still call it Hausdorff convergence.

2.2.1. Definition. *Let (A_n) be a sequence of closed sets in a metric space $\mathcal{X}$. We say that (A_n) converges to a closed set A_∞ in the sense of Hausdorff if $\operatorname{dist}_{A_n}(x) \to \operatorname{dist}_{A_\infty}(x)$ for any $x \in \mathcal{X}$.*

For example, suppose $\mathcal{X}$ is the Euclidean plane and A_n is the circle with radius n and center at $(n, 0)$. If we use the standard definition (2.1.1), then the sequence (A_n) diverges, but it converges to the y-axis in the sense of Definition 2.2.1.

The following exercise is analogous to the Blaschke selection theorem (2.1.3).

2.2.2. Exercise. *Let $\mathcal{X}$ be a proper metric space and $(A_n)_{n=1}^\infty$ be a sequence of closed sets in $\mathcal{X}$. Assume that for some (and therefore any) point $x \in \mathcal{X}$, the sequence $a_n = \operatorname{dist}_{A_n}(x)$ is bounded. Show that the sequence $(A_n)_{n=1}^\infty$ has a convergent subsequence in the sense of Definition 2.2.1.*

Chapter 3

Space of spaces

3.1 Gromov–Hausdorff metric

The goal of this section is to cook up a metric space out of metric spaces. More precisely, we want to define the so called Gromov–Hausdorff metric on the set of *isometry classes* of compact metric spaces. (Being isometric is an equivalence relation, and an isometry class is an equivalence class with respect to this equivalence relation.)

The obtained metric space will be denoted as $\mathcal{M}$. Given two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, denote by $[\mathcal{X}]$ and $[\mathcal{Y}]$ their isometry classes; that is, $\mathcal{X}' \in [\mathcal{X}]$ if and only if $\mathcal{X}' \stackrel{iso}{=\!=} \mathcal{X}$. Pedantically, the Gromov–Hausdorff distance from $[\mathcal{X}]$ to $[\mathcal{Y}]$ should be denoted as $|[\mathcal{X}]-[\mathcal{Y}]|_{\mathcal{M}}$; but we will often write it as $|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}}$ and say (not quite correctly) "$|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}}$ is the Gromov–Hausdorff distance from $\mathcal{X}$ to $\mathcal{Y}$". In other words, from now on the term *metric space* might stands for *isometry class of this metric space.*

The metric on $\mathcal{M}$ is maximal metric such that *the distance between subspaces in a metric space is not greater than the Hausdorff distance between them.* Here is a formal definition:

3.1.1. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact metric spaces. The Gromov–Hausdorff distance $|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}}$ between them is defined by the following relation.*

Given $r > 0$, we have that $|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}} < r$ if and only if there exist a metric space $\mathcal{Z}$ and subspaces $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{Z}$ that are isometric to $\mathcal{X}$ and $\mathcal{Y}$ respectively and such that $|\mathcal{X}'-\mathcal{Y}'|_{\mathcal{H}(\mathcal{Z})} < r$. (Here $|\mathcal{X}'-\mathcal{Y}'|_{\mathcal{H}(\mathcal{Z})}$ denotes the Hausdorff distance between sets $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{Z}$.)

Bit later (see 3.4.1) we will show that *Hausdorff metric* is indeed a metric.

We say that a sequence of (isometry classes of) compact metric spaces $\mathcal{X}_n$ *converges in the sense of Gromov–Hausdorff* to the (isometry classes of) compact metric space $\mathcal{X}_\infty$ if $|\mathcal{X}_n-\mathcal{X}_\infty|_{\mathcal{M}} \to 0$ as $n \to \infty$; in this case we write $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$.

3.2 Reformulations

Let us discuss few alternative ways to define the Gromov–Hausdorff metric.

Metrics on disjoined union. Definition 3.1.1 deals with a huge class of metric spaces, namely, all metric spaces $\mathcal{Z}$ that contain subspaces isometric to $\mathcal{X}$ and $\mathcal{Y}$. It is possible to reduce this class to metrics on the disjoint unions of $\mathcal{X}$ and $\mathcal{Y}$. More precisely,

3.2.1. Proposition. *The Gromov–Hausdorff distance between two compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is the infimum of $r > 0$ such that there exists a metric $|{*}-{*}|_{\mathcal{W}}$ on the disjoint union $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Y}$ such that the restrictions of $|{*}-{*}|_{\mathcal{W}}$ to $\mathcal{X}$ and $\mathcal{Y}$ coincide with $|{*}-{*}|_{\mathcal{X}}$ and $|{*}-{*}|_{\mathcal{Y}}$ and $|\mathcal{X}-\mathcal{Y}|_{\mathcal{H}(\mathcal{W})} < r$.*

Proof. Identify $\mathcal{X} \sqcup \mathcal{Y}$ with $\mathcal{X}' \cup \mathcal{Y}' \subset \mathcal{Z}$ (the notation is from Definition 3.1.1).

More formally, fix isometries $f\colon \mathcal{X} \to \mathcal{X}'$ and $g\colon \mathcal{Y} \to \mathcal{Y}'$, then define the distance between $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ by $|x-y|_{\mathcal{W}} = |f(x)-g(y)|_{\mathcal{Z}} + \varepsilon$ for small enuf $\varepsilon > 0$.[1] This yields a metric on $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Y}$ for which $|\mathcal{X}-\mathcal{Y}|_{\mathcal{H}(\mathcal{W})} < r$. □

Fixed ambient space. The following proposition says that the space $\mathcal{Z}$ in Definition 3.1.1 can be exchanged to a fixed space, namely ℓ^∞ — the space of bounded infinite sequences with the metric defined by sup-norm.

3.2.2. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be comact metric spaces. Then*

$$|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}} = \inf\{|\mathcal{X}'-\mathcal{Y}'|_{\mathcal{H}(\ell^\infty)}\}$$

where the infimum is taken over all pairs of sets $\mathcal{X}'$ and $\mathcal{Y}'$ in ℓ^∞ which isometric to $\mathcal{X}$ and $\mathcal{Y}$ correspondingly.

Proof of 3.2.2. By the definition, we have that

$$|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}} \le \inf\{|\mathcal{X}'-\mathcal{Y}'|_{\mathcal{H}(\ell^\infty)}\}.$$

[1] We add ε to ensure that $d(x,y) > 0$ for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$; so $|x-y|_{\mathcal{W}}$ is indeed a metric.

Let $\mathcal{W}$ be an arbitrary metric space with the underlying set $\mathcal{X}\sqcup\mathcal{Y}$. Note $\mathcal{W}$ is compact since it is union of two compact subsets $\mathcal{X},\mathcal{Y}\subset\mathcal{W}$. In particular, $\mathcal{W}$ is separable.

By Lemma 1.8.1, there is an distance preserving embedding $\iota\colon\mathcal{W}\to\to\ell^\infty$. It remains to apply Proposition 3.2.1. □

3.3 Almost isometries

3.3.1. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces and $\varepsilon>0$. A map[2] $f\colon\mathcal{X}\to\mathcal{Y}$ is called an ε-isometry if*

$$|f(x)-f(x')|_{\mathcal{Y}}\lessgtr|x-x'|_{\mathcal{X}}\pm\varepsilon$$

for any $x,x'\in\mathcal{X}$ and if $f(\mathcal{X})$ is an ε-net in $\mathcal{Y}$.

3.3.2. Exercise.

a) *Let $f\colon\mathcal{X}\to\mathcal{Y}$ and $g\colon\mathcal{Y}\to\mathcal{Z}$ be two ε-isometries. Show that $g\circ f\colon\mathcal{X}\to\mathcal{Z}$ is a $(3{\cdot}\varepsilon)$-isometry.*

b) *Assume $f\colon\mathcal{X}\to\mathcal{Y}$ is an ε-isometry. Show that there is a $(3{\cdot}\varepsilon)$-isometry $g\colon\mathcal{Y}\to\mathcal{X}$.*

c) *Assume $|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}}<\varepsilon$, show that there is a $(2{\cdot}\varepsilon)$-isometry $f\colon\mathcal{X}\to\mathcal{Y}$.*

3.3.3. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces and let $f\colon\mathcal{X}\to\mathcal{Y}$ be an ε-isometry. Then*

$$|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}}\leqslant 2{\cdot}\varepsilon.$$

Proof. Consider the set $\mathcal{W}=\mathcal{X}\sqcup\mathcal{Y}$. Note that the following defines a metric on $\mathcal{W}$:

◇ For any $x,x'\in\mathcal{X}$

$$|x-x'|_{\mathcal{W}}=|x-x'|_{\mathcal{X}};$$

◇ For any $y,y'\in\mathcal{Y}$,

$$|y-y'|_{\mathcal{W}}=|y-y'|_{\mathcal{Y}}$$

◇ For any $x\in\mathcal{X}$ and $y\in\mathcal{Y}$,

$$|x-y|_{\mathcal{W}}=\varepsilon+\inf_{x'\in\mathcal{X}}\{|x-x'|_{\mathcal{X}}+|f(x')-y|_{\mathcal{Y}}\}.$$

[2]possibly noncontinuous

Since $f(\mathcal{X})$ is an ε-net in $\mathcal{Y}$, for any $y \in \mathcal{Y}$ there is $x \in \mathcal{X}$ such that $|f(x) - y|_{\mathcal{Y}} \leqslant \varepsilon$; therefore $|x - y|_{\mathcal{W}} \leqslant 2{\cdot}\varepsilon$. On the other hand for any $x \in \mathcal{X}$, we have $|x - y|_{\mathcal{W}} \leqslant \varepsilon$ for $y = f(x) \in \mathcal{Y}$.

It follows that $|\mathcal{X} - \mathcal{Y}|_{\mathcal{H}(\mathcal{W})} \leqslant 2{\cdot}\varepsilon$. □

The Gromov–Hausdorff metric defines Gromov–Hausdorff convegence and this is the only thing it is good for. In other words in all applications, we use only topology on $\mathcal{M}$ and we do not care about particular value of Gromov–Hausdorff distance between spaces.

In order to determine that a given sequence of metric spaces $(\mathcal{X}_n)$ converges in the Gromov–Hausdorff sense to $\mathcal{X}_\infty$, it is sufficient to estimate distances $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{M}}$ and check if $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{M}} \to 0$. This problem turns to be simpler than finding Gromov–Hausdorff distance between a particular pair of spaces. The following proposition gives one way to do this.

3.3.4. Proposition. *A sequence of compact metric spaces $(\mathcal{X}_n)$ converges to $\mathcal{X}_\infty$ in the sense of Gromov–Hausdorff if and only if there is a sequence $\varepsilon_n \to 0+$ and an ε_n-isometry $f_n\colon \mathcal{X}_n \to \mathcal{X}_\infty$ for each n.*

Proof. Follows from Propsition 3.3.3 and Exercise 3.3.2c □

3.4 It is a metric

3.4.1. Theorem. *The set of isometry classes of compact metric spaces equipped with Gromov–Hausdorff metric forms a metric space (which is denoted by $\mathcal{M}$).*

Proof. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be arbitrary compact metric spaces. We need to check the following:

(i) $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} \geqslant 0$;
(ii) $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} = 0$ if and only if $\mathcal{X}$ is isometric to $\mathcal{Y}$;
(iii) $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} = |\mathcal{Y} - \mathcal{X}|_{\mathcal{M}}$;
(iv) $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} + |\mathcal{Y} - \mathcal{Z}|_{\mathcal{M}} \geqslant |\mathcal{X} - \mathcal{Z}|_{\mathcal{M}}$.

Note that *(i)*, *(iii)* and "if"-part of *(ii)* follow directly from Definition 3.1.1.

(iv). Choose arbitrary $a, b \in \mathbb{R}$ such that

$$a > |\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} \quad \text{and} \quad b > |\mathcal{Y} - \mathcal{Z}|_{\mathcal{M}}.$$

Choose two metrics on $\mathcal{U} = \mathcal{X} \sqcup \mathcal{Y}$ and $\mathcal{V} = \mathcal{Y} \sqcup \mathcal{Z}$ so that $|\mathcal{X} - \mathcal{Y}|_{\mathcal{H}(\mathcal{U})} < a$ and $|\mathcal{Y} - \mathcal{Z}|_{\mathcal{H}(\mathcal{V})} < b$ and the inclusions $\mathcal{X} \hookrightarrow \mathcal{U}$, $\mathcal{Y} \hookrightarrow \mathcal{U}$, $\mathcal{Y} \hookrightarrow \mathcal{V}$ and $\mathcal{Z} \hookrightarrow \mathcal{V}$ are distance preserving.

Consider the metric on $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Z}$ so that inclusions $\mathcal{X} \hookrightarrow \mathcal{W}$ and $\mathcal{Z} \hookrightarrow \mathcal{W}$ are distance preserving and

$$|x - z|_{\mathcal{W}} = \inf_{y \in \mathcal{Y}} \{|x - y|_{\mathcal{U}} + |y - z|_{\mathcal{V}}\}.$$

Note that $|* - *|_{\mathcal{W}}$ is indeed a metric and

$$|\mathcal{X} - \mathcal{Z}|_{\mathcal{H}(\mathcal{W})} < a + b.$$

Property *(iv)* follows since the last inequality holds for any $a > |\mathcal{X} - \mathcal{Y}|_{\mathcal{M}}$ and $b > |\mathcal{Y} - \mathcal{Z}|_{\mathcal{M}}$.

"Only if"-part of (ii). According to Exercise 3.3.2c, for any sequence $\varepsilon_n \to 0+$ there is a sequence of ε_n-isometries $f_n \colon \mathcal{X} \to \mathcal{Y}$.

Since $\mathcal{X}$ is compact, we can choose a countable dense set S in $\mathcal{X}$. Use a diagonal procedure if necessary, to pass to a subsequence of (f_n) such that for every $x \in S$ the sequence $(f_n(x))$ converges in $\mathcal{Y}$. Consider the pointwise limit map $f_\infty \colon S \to \mathcal{Y}$ defined by

$$f_\infty(x) = \lim_{n \to \infty} f_n(x)$$

for every $x \in S$. Since

$$|f_n(x) - f_n(x')|_{\mathcal{Y}} \lessgtr |x - x'|_{\mathcal{X}} \pm \varepsilon_n,$$

we have

$$|f_\infty(x) - f_\infty(x')|_{\mathcal{Y}} = \lim_{n \to \infty} |f_n(x) - f_n(x')|_{\mathcal{Y}} = |x - x'|_{\mathcal{X}}$$

for all $x, x' \in S$; that is, $f_\infty \colon S \to \mathcal{Y}$ is a distance-preserving map. Therefore f_∞ can be extended to a distance-preserving map from all of $\mathcal{X}$ to $\mathcal{Y}$. The later is done by setting

$$f_\infty(x) = \lim_{n \to \infty} f_\infty(x_n)$$

for some (and therefore any) sequence of points (x_n) in S which converges to x in $\mathcal{X}$. (Note that if $x_n \to x$, then (x_n) is Cauchy. Since f_∞ is distance preserving, $y_n = f_\infty(x_n)$ is also a Cauchy sequence in $\mathcal{Y}$; therefore it converges.)

This way we obtain a distance preserving map $f_\infty \colon \mathcal{X} \to \mathcal{Y}$. It remains to show that f_∞ is surjective; that is, $f_\infty(\mathcal{X}) = \mathcal{Y}$.

Note that in the same way we can obtain a distance preserving map $g_\infty \colon \mathcal{Y} \to \mathcal{X}$. If f_∞ is not surjective, then neither is $f_\infty \circ g_\infty \colon \mathcal{Y} \to \mathcal{Y}$. So $f_\infty \circ g_\infty$ is a distance preserving map from a compact space to itself which is not an isometry. The later contradicts Exercise 1.4.3. $\square$

3.4.2. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two compact metric spaces. Prove that*

$$|\operatorname{diam}\mathcal{X} - \operatorname{diam}\mathcal{Y}| \leqslant 2\cdot|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}}.$$

In other words, $\operatorname{diam}\colon \mathcal{M} \to \mathbb{R}$ *is a 2-Lipschitz function.*

3.4.3. Exercise. *Show that $\mathcal{M}$ is a length space.*

Given two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, we will write $\mathcal{X} \leqslant \mathcal{Y}$ if there is a noncontracting map $f\colon \mathcal{X} \to \mathcal{Y}$; that is, if

$$|x - x'|_{\mathcal{X}} \leqslant |f(x) - f(x')|_{\mathcal{Y}}$$

for any $x, x' \in \mathcal{X}$.

Further, given $\varepsilon > 0$, we will write $\mathcal{X} \leqslant \mathcal{Y} + \varepsilon$ if there is a map $f\colon \mathcal{X} \to \mathcal{Y}$ such that

$$|x - x'|_{\mathcal{X}} \leqslant |f(x) - f(x')|_{\mathcal{Y}} + \varepsilon$$

for any $x, x' \in \mathcal{X}$.

3.4.4. Exercise. *Show that*

$$|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}'} = \inf\{\,\varepsilon > 0 \;:\; \mathcal{X} \leqslant \mathcal{Y} + \varepsilon \quad \textit{and} \quad \mathcal{Y} \leqslant \mathcal{X} + \varepsilon\,\}$$

defines a metric on the space of (isometry classes) of compact metric spaces.

Moreover $|{*} - {*}|_{\mathcal{M}'}$ is equivalent to the Gromov–Haudorff metric; that is,

$$|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{M}} \to 0 \qquad \Longleftrightarrow \qquad |\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{M}'} \to 0$$

as $n \to \infty$.

3.5 Uniformly totally bonded families

Let $\mathcal{Q}$ be a set of (isometry classes) of compact metric spaces. Suppose that there is a sequence $\varepsilon_n \to 0$ such that for any positive integer n each space $\mathcal{X}$ in $\mathcal{Q}$ admits an ε_n-net with at most n points. Then we say that $\mathcal{Q}$ is *uniformly totally bonded.*

Observe that in this case $\operatorname{diam}\mathcal{X} < \varepsilon_1$ for any $\mathcal{X}$ in $\mathcal{Q}$; that is diameters of spaces in $\mathcal{Q}$ are bounded above.

Fix a real constant C. A measure μ on a metric space $\mathcal{X}$ is called *C-doubling* if

$$\mu[\mathrm{B}(p, 2\cdot r) < C\cdot\mu[\mathrm{B}(p, r)]$$

for any point $p \in \mathcal{X}$ and any positive real r. A measure is called *doubling* if it is C*-doubling* for a some real constant C.

3.5.1. Exercise. *Let $\mathcal{Q}(C, D)$ be the set of all the compact metric spaces with diameter at most D that admit a C-doubling measure. Show that $\mathcal{Q}(C, D)$ is totally bounded.*

Recall that we write $\mathcal{X} \leqslant \mathcal{Y}$ if there is a distance non-decreasing map $\mathcal{X} \to \mathcal{Y}$.

3.5.2. Exercise.

a) Let $\mathcal{Y}$ be a compact metric space. Show that the set of all spaces $\mathcal{X}$ such that $\mathcal{X} \leqslant \mathcal{Y}$ is uniformly totally bounded.

b) Show that for any uniformly totally bounded set $\mathcal{Q} \subset \mathcal{M}$ there is a compact space $\mathcal{Y}$ such that $\mathcal{X} \leqslant \mathcal{Y}$ for any $\mathcal{X}$ in $\mathcal{Q}$.

3.6 Gromov's selection theorem

The following theorem is analogous to Blaschke selection theorems (2.1.3).

3.6.1. Gromov selection theorem. *Let $\mathcal{Q}$ be a closed and totally bounded subset of $\mathcal{M}$. Then $\mathcal{Q}$ is compact.*

3.6.2. Lemma. *$\mathcal{M}$ is complete.*

Proof. Let $(\mathcal{X}_n)$ be a Cauchy sequence in $\mathcal{M}$. Passing to a subsequence if necessary, we can assume that $|\mathcal{X}_n - \mathcal{X}_{n+1}|_{\mathcal{M}} < \frac{1}{2^n}$ for each n. In particular, for each n one can equip $\mathcal{W}_n = \mathcal{X}_n \sqcup \mathcal{X}_{n+1}$ with a metric such that inclusions $\mathcal{X}_n \hookrightarrow \mathcal{W}_n$ and $\mathcal{X}_{n+1} \hookrightarrow \mathcal{W}_n$ are distance preserving, and

$$|\mathcal{X}_n - \mathcal{X}_{n+1}|_{\mathcal{H}(\mathcal{W}_n)} < \tfrac{1}{2^n}$$

for each n.

Set $\mathcal{W}$ to be the disjoint union of all $\mathcal{X}_n$. Let us equip $\mathcal{W}$ with a metric defined the following way:

- ◇ for any fixed n and any two points $x_n, x_n' \in \mathcal{X}_n$ set
$$|x_n - x_n'|_{\mathcal{W}} = |x_n - x_n'|_{\mathcal{X}_n}$$
- ◇ for any positive integers $m > n$ and any two points $x_n \in \mathcal{X}_n$ and $x_m \in \mathcal{X}_m$ set
$$|x_n - x_m|_{\mathcal{W}} = \inf \left\{ \sum_{i=n}^{m-1} |x_i - x_{i+1}|_{\mathcal{W}_i} \right\},$$
where the infimum is taken for all sequences $x_i \in \mathcal{X}_i$.

Observe that $| * - * |_{\mathcal{W}}$ is indeed a metric.

Let $\bar{\mathcal{W}}$ be the completion of $\mathcal{W}$. Note that $|\mathcal{X}_m - \mathcal{X}_n| < \frac{1}{2^{n-1}}$ if $m > n$. Therefore the union of $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \dots \cup \mathcal{X}_n$ forms a $\frac{1}{2^{n-1}}$-net in $\bar{\mathcal{W}}$. Since each $\mathcal{X}_i$ is compact, we get that $\bar{\mathcal{W}}$ admits a compact ε-net for any $\varepsilon > 0$. Whence $\bar{\mathcal{W}}$ is compact.

According to Blaschke selection theorem (2.1.3), we can pass to a subsequence of $(\mathcal{X}_n)$ that converges in $\mathcal{H}(\bar{\mathcal{W}})$ and therefore in $\mathcal{M}$. □

Proof of 3.6.1; "only if" part. If there is no sequence $\varepsilon_n \to 0$ as described in the problem, then for a fixed fixed $\delta > 0$ there is a sequence of spaces $\mathcal{X}_n \in \mathcal{Q}$ such that

$$\operatorname{pack}_\delta \mathcal{X}_n \to \infty \quad \text{as} \quad n \to \infty.$$

Since $\mathcal{Q}$ is compact, this sequence has a partial limit say $\mathcal{X}_\infty \in \mathcal{Q}$. Observe that $\operatorname{pack}_\delta \mathcal{X}_\infty = \infty$. Therefore $\mathcal{X}_\infty$ — a contradiction.

"If" part. Without loss of generality, we may assume that there is a sequence $\varepsilon_n \to 0$ such that $\mathcal{Q}$ is the set of all compact metric spaces $\mathcal{X}$ such that $\operatorname{pack}_{\varepsilon_n} \mathcal{X} \leqslant n$.

Note that $\operatorname{diam} \mathcal{X} \leqslant \varepsilon_1$ for any $\mathcal{X} \in \mathcal{Q}$. Given positive integer n consider set of all metric spaces $\mathcal{W}_n$ with number of points at most n and diameter $\leqslant \varepsilon_1$. Note that $\mathcal{W}_n$ is compact for each n.

Further a maximal ε_n-packing of any $\mathcal{X} \in \mathcal{Q}$ forms a subspace from $\mathcal{W}_n$. Therefore $\mathcal{W}_n \cap \mathcal{Q}$ is a comapct ε_n-net in $\mathcal{Q}$. That is, $\mathcal{Q}$ has compact ε-net for any $\varepsilon > 0$. The ince $\mathcal{Q}$ is a closed set □

In the following exercises *converge* means *converge in the sense of Gromov–Hausdorff.*

3.6.3. Exercise.

a) Show that a sequence of compact simply connected length spaces can not converge to a circle.

b) Construct a sequence of compact simply connected length spaces that converges to a compact nonsimply connected space.

3.6.4. Exercise.

a) Show that a sequence of lenght metrics on the 2-sphere can not converge to a the unit disc.

b) Construct a sequence of lenght metrics on the 3-sphere that converges to a unit 3-ball.

3.7 Remarks

Suppose $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$, then there is a metric on the disjoint union

$$\boldsymbol{X} = \bigsqcup_{n\in\mathbb{N}\cup\{\infty\}} \mathcal{X}_n$$

such that the restriction of metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric and and $\mathcal{X}_n \xrightarrow{\text{H}} \mathcal{X}_\infty$ as subsets in $\boldsymbol{X}$.

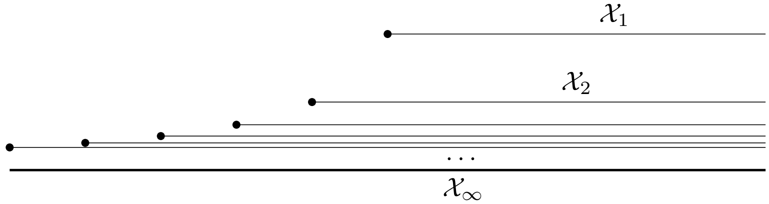
Indeed, since $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$, there is a metric on $\mathcal{V}_n = \mathcal{X}_n \sqcup \mathcal{X}_\infty$ such that the restriction of metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric and $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{H}(\mathcal{V}_n)} < \varepsilon_n$ for some sequence $\varepsilon_n \to 0$. Arguing as in the proof of (iv) in Theorem 3.4.1 we define metric on $\boldsymbol{X}$ by setting

$$\begin{aligned}
|x_m - x_n|_{\boldsymbol{X}} &= \inf_{x_\infty} \left\{\, |x_m - x_\infty|_{\mathcal{V}_m} + |x_n - x_\infty|_{\mathcal{V}_n} \;:\; \right\},\\
|x_n - x_\infty|_{\boldsymbol{X}} &= |x_n - x_\infty|_{\mathcal{V}_n}\\
|x_n - x_n'|_{\boldsymbol{X}} &= |x_n - x_n'|_{\mathcal{X}_n}
\end{aligned}$$

where $x_n, x_n' \in \mathcal{X}_n$ for every $n \in \mathbb{N}\cup\{\infty\}$.

In other words, the metric on $\boldsymbol{X}$ defines convergence $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$. This metric makes possible to talk about limits of sequences $x_n \in \mathcal{X}_n$ as $n \to \infty$, as well as weak limit of a sequence of measures μ_n on $\mathcal{X}_n$ and so on. By that reason it might be useful to fix such metric on $\boldsymbol{X}$. This approach can be also used to define Gromov–Hausdorff convergence of noncompact spaces which will be discussed latter.

We may consider a metric on $\boldsymbol{X}$ such that $\mathcal{X}_n \xrightarrow{\text{H}} \mathcal{X}_\infty$ without assuming that all the spaces $\mathcal{X}_n$ and $\mathcal{X}_\infty$ are compact; in this case we need to use the variation of Hausdorff convergence described in Section 2.2. The limit spaces for this generalized convergence is not uniquely defined. For example if each space $\mathcal{X}_n$ in the sequence is isometric to the half-line, then its limit might be isometric to the half-line or to whole line. The first convergence is evident and the second could be guessed from the diagram.



Often the isometry class of the limit can be fixed by marking a point p_n in each space $\mathcal{X}_n$, it is called *pointed Gromov–Haudorff convergence*

— we say that $(\mathcal{X}_n, p_n)$ converges to $(\mathcal{X}_\infty, p_\infty)$ if there is a metric on $\boldsymbol{X}$ such that $\mathcal{X}_n \xrightarrow{\text{H}} \mathcal{X}_\infty$ and $p_n \to p_\infty$. For example the sequence $(\mathcal{X}_n, p_n) = (\mathbb{R}_+, 0)$ converges to $(\mathbb{R}_+, 0)$, while $(\mathcal{X}_n, p_n) = (\mathbb{R}_+, n)$ converges to $(\mathbb{R}, 0)$.

This convergence works nicely for proper metric spaces. The following theorem is an analog of Gromov's selection theorem for pointed Gromov–Haudorff convergence.

3.7.1. Theorem. *Let $\mathcal{Q}$ be a set of isometry classes of pointed proper metric spaces $(\mathcal{X}, p)$. Assume that for any $R > 0$, the R-balls in the spaces centered at the marked points form a uniformly totally bounded family of spaces. Then $\mathcal{Q}$ is precompact with respect to pointed Gromov–Haudorff convergence.*

Chapter 4

Ultralimits

Here we introduce ultralimits of sequences of points, metric spaces and functions. The ultralimits of metric spaces can be considered as a variation of Gromov–Hausdorff convergence. Our presentation is based on [12].

Our use of ultralimits is very limited; we use them only as a canonical way to pass to a convergent subsequence. (In principle, we could avoid selling our souls to the set-theoretical devil, but in this case we must say "pass to convergent subsequence" too many times.)

4.1 Ultrafilters

We will need the existence of a nonprinciple ultrafilter ω, which we fix once and for all.

Recall that $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N} = \{1, 2, \dots\}$

4.1.1. Definition. *A finitely additive measure ω on $\mathbb{N}$ is called an* ultrafilter *if it satisfies*

a) $\omega(S) = 0$ or 1 for any subset $S \subset \mathbb{N}$.

An ultrafilter ω is called nonprinciple *if in addition*

b) $\omega(F) = 0$ for any finite subset $F \subset \mathbb{N}$.

If $\omega(S) = 0$ for some subset $S \subset \mathbb{N}$, we say that S is *ω-small.* If $\omega(S) = 1$, we say that S contains *ω-almost all* elements of $\mathbb{N}$.

Classical definition. More commonly, a nonprinciple ultrafilter is defined as a collection, say $\mathfrak{F}$, of sets in $\mathbb{N}$ such that

1. if $P \in \mathfrak{F}$ and $Q \supset P$, then $Q \in \mathfrak{F}$,
2. if $P, Q \in \mathfrak{F}$, then $P \cap Q \in \mathfrak{F}$,
3. for any subset $P \subset \mathbb{N}$, either P or its complement is an element of $\mathfrak{F}$.

4. if $F \subset \mathbb{N}$ is finite, then $F \notin \mathfrak{F}$.

Setting $P \in \mathfrak{F} \Leftrightarrow \omega(P) = 1$ makes these two definitions equivalent.

A nonempty collection of sets $\mathfrak{F}$ that does not include the empty set and satisfies only conditions 1 and 2 is called a *filter*; if in addition $\mathfrak{F}$ satisfies Condition 3 it is called an *ultrafilter*. From Zorn's lemma, it follows that every filter contains an ultrafilter. Thus there is an ultrafilter $\mathfrak{F}$ contained in the filter of all complements of finite sets; clearly this $\mathfrak{F}$ is nonprinciple.

Stone–Čech compactification. Given a set $S \subset \mathbb{N}$, consider subset Ω_S of all ultrafilters ω such that $\omega(S) = 1$. It is straightforward to check that the sets Ω_S for all $S \subset \mathbb{N}$ form a topology on the set of ultrafilters on $\mathbb{N}$. The obtained space is called *Stone–Čech compactification* of $\mathbb{N}$; it is usually denoted as $\beta\mathbb{N}$.

There is a natural embedding $\mathbb{N} \hookrightarrow \beta\mathbb{N}$ defined as $n \mapsto \omega_n$, where ω_n is the principle ultrafilter such that $\omega_n(S) = 1$ if and only if $n \in S$. Using the described embedding, we can (and will) consider $\mathbb{N}$ as a subset of $\beta\mathbb{N}$.

The space $\beta\mathbb{N}$ is the maximal compact Hausdorff space that contains $\mathbb{N}$ as an everywhere dense subset. More precisely, for any compact Hausdorff space $\mathcal{X}$ and a map $f\colon \mathbb{N} \to \mathcal{X}$ there is unique continuous map $\bar{f}\colon \beta\mathbb{N} \to X$ such that the restriction $\bar{f}|_{\mathbb{N}}$ coincides with f.

4.2 Ultralimits of points

Fix an ultrafilter ω. Assume (x_n) is a sequence of points in a metric space $\mathcal{X}$. Let us define the *ω-limit* of (x_n) as the point x_ω such that for any $\varepsilon > 0$, ω-almost all elements of (x_n) lie in $\mathrm{B}(x_\omega, \varepsilon)$; that is,

$$\omega\{\, n \in \mathbb{N} : |x_\omega - x_n| < \varepsilon \,\} = 1.$$

In this case, we will write

$$x_\omega = \lim_{n \to \omega} x_n \quad \text{or} \quad x_n \to x_\omega \text{ as } n \to \omega.$$

For example if ω is the principle ultrafilter such that $\omega(\{n\}) = 1$ for some $n \in \mathbb{N}$, then $x_\omega = x_n$.

Note that ω-limits of a sequence and its subsequence may differ. For example, in general

$$\lim_{n \to \omega} x_n \neq \lim_{n \to \omega} x_{2 \cdot n}.$$

4.2.1. Proposition. *Let ω be a nonprinciple ultrafilter. Assume (x_n) is a sequence of points in a metric space $\mathcal{X}$ and $x_n \to x_\omega$ as $n \to \omega$.*

Then x_ω is a partial limit of the sequence (x_n); that is, there is a subsequence $(x_n)_{n\in S}$ that converges to x_ω in the usual sense.

Remark. A nonprinciple ultrafilter ω is called *selective* if for any partition of $\mathbb{N}$ into sets $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ such that $\omega(C_\alpha) = 0$ for each α, there is a set $S \subset \mathbb{N}$ such that $\omega(S) = 1$ and $S \cap C_\alpha$ is a one-point set for each $\alpha \in \mathcal{A}$.

The existence of a selective ultrafilter follows from the continuum hypothesis; it was proved by Walter Rudin in [19].

For a selective ultrafilter ω, there is a stronger version of Proposition 4.2.1; namely we can assume that the subsequence $(x_n)_{n\in S}$ can be chosen so that $\omega(S) = 1$. (So, if needed, you may assume that the ultrafilter ω is selective and use this stronger version of the proposition.)

Proof. Given $\varepsilon > 0$, set $S_\varepsilon = \{\, n \in \mathbb{N} : |x_n - x_\omega| < \varepsilon \,\}$.

Note that $\omega(S_\varepsilon) = 1$ for any $\varepsilon > 0$. Since ω is nonprinciple, the set S_ε is infinite. Therefore we can choose an increasing sequence (n_k) such that $n_k \in S_{\frac{1}{k}}$ for each $k \in \mathbb{N}$. Clearly $x_{n_k} \to x_\omega$ as $k \to \infty$. $\square$

The following proposition is analogous to the statement that any sequence in a compact metric space has a convergent subsequence; it can be proved the same way.

4.2.2. Proposition. *Let $\mathcal{X}$ be a compact metric space. Then any sequence of points (x_n) in $\mathcal{X}$ has unique ω-limit x_ω.*

In particular, a bounded sequence of real numbers has a unique ω-limit.

Alternatively, the sequence (x_n) can be regarded as a map $\mathbb{N} \to \mathcal{X}$. In this case the map $\mathbb{N} \to \mathcal{X}$ can be extended to a continuous map from the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Then the ω-limit x_ω can be regarded as the image of ω.

The following lemma is an ultralimit analog of Cauchy convergence test.

4.2.3. Lemma. *Let (x_n) be a sequence of points in a complete space $\mathcal{X}$. Assume for each subsequence (y_n) of (x_n), the ω-limit*

$$y_\omega = \lim_{n\to\omega} y_n \in \mathcal{X}$$

is defined and does not depend on the choice of subsequence, then the sequence (x_n) converges in the usual sense.

Proof. Assume that (x_n) is a Cauchy sequence. Then for some $\varepsilon > 0$, there is a subsequence (y_n) of (x_n) such that $|x_n - y_n| \geqslant \varepsilon$ for all n.

It follows that $|x_\omega - y_\omega| \geqslant \varepsilon$, a contradiction. $\square$

4.3 Ultralimits of spaces

From now on, ω denotes a nonprinciple ultrafilter on the set of natural numbers.

Let $\mathcal{X}_n$ be a sequence of metric spaces. Consider all sequences of points $x_n \in \mathcal{X}_n$. On the set of all such sequences, define a pseudometric by

$$❶ \qquad |(x_n) - (y_n)| = \lim_{n\to\omega} |x_n - y_n|.$$

Note that the ω-limit on the right hand side is always defined and takes a value in $[0, \infty]$.

Set $\mathcal{X}_\omega$ to be the corresponding metric space; that is, the underlying set of $\mathcal{X}_\omega$ is formed by classes of equivalence of sequences of points $x_n \in \mathcal{X}_n$ defined by

$$(x_n) \sim (y_n) \;\Leftrightarrow\; \lim_{n\to\omega} |x_n - y_n| = 0$$

and the distance is defined by ❶.

The space $\mathcal{X}_\omega$ is called *ω-limit* of $\mathcal{X}_n$. Typically $\mathcal{X}_\omega$ will denote the ω-limit of sequence $\mathcal{X}_n$; we may also write

$$\mathcal{X}_n \to \mathcal{X}_\omega \;\text{ as }\; n \to \omega \;\text{ or }\; \mathcal{X}_\omega = \lim_{n\to\omega} \mathcal{X}_n.$$

Given a sequence $x_n \in \mathcal{X}_n$, we will denote by x_ω its equivalence class which is a point in $\mathcal{X}_\omega$; equivalently we will write

$$x_n \to x_\omega \;\text{ as }\; n \to \omega \;\text{ or }\; x_\omega = \lim_{n\to\omega} x_n.$$

4.3.1. Observation. *The ω-limit of any sequence of metric spaces is complete.*

Proof. Let $\mathcal{X}_n$ be a sequence of metric spaces and $\mathcal{X}_n \to \mathcal{X}_\omega$ as $n \to \omega$.

Fix a Cauchy sequence $x_m \in \mathcal{X}_\omega$. Passing to a subsequence we can assume that $|x_m - x_{m-1}|_{\mathcal{X}_\omega} < \frac{1}{2^m}$ for any m.

Let us choose double sequence $x_{n,m} \in \mathcal{X}_n$ such that for any fixed m we have $x_{n,m} \to x_m$ as $n \to \omega$. Note that $|x_{n,m} - x_{n,m-1}| < \frac{1}{2^m}$ for ω-almost all n. It follows that we can choose a nested sequence of sets

$$\mathbb{N} = S_1 \supset S_2 \supset \dots$$

such that

- $\diamond$ $\omega(S_m) = 1$ for each m,
- $\diamond$ $k \geqslant m$ for any $k \in S_m$, and

◇ if $n \in S_m$, then
$$|x_{n,m} - x_{n,m-1}| < \tfrac{1}{2^m}$$

Consider the sequence $y_n = x_{n,m(n)}$, where $m(n)$ is the largest value such that $m(n) \in S_m$. Denote by $y \in \mathcal{X}_\omega$ its ω-limit.

Observe that by construction $x_n \to y$ as $n \to \infty$. Hence the statement follows. □

4.3.2. Observation. *The ω-limit of any sequence of length spaces is geodesic.*

Proof. If $\mathcal{X}_n$ is a sequence length spaces, then for any sequence of pairs $x_n, y_n \in X_n$ there is a sequence of $\tfrac{1}{n}$-midpoints z_n.

Let $x_n \to x_\omega$, $y_n \to y_\omega$ and $z_n \to z_\omega$ as $n \to \omega$. Note that z_ω is a midpoint of x_ω and y_ω in $\mathcal{X}^\omega$.

By Observation 4.3.1, $\mathcal{X}^\omega$ is complete. Applying Lemma 1.7.4 we get the statement. □

A geodesic space $\mathcal{T}$ is called a *metric tree* if any pair of points in $\mathcal{T}$ are connected by a unique geodesic, and the union of any two geodesics $[xy]$, and $[yz]$ contain the geodesic $[xz]_{\mathcal{T}}$. In other words any triangle in $\mathcal{T}$ is a tripod; that is for any three geodesics $[xy]$, $[yz]$, and $[zx]$ have a common point.

4.3.3. Exercise. *Show that an ultralimit of metric trees is a metric tree.*

4.4 Ultrapower

If all the metric spaces in the sequence are identical $\mathcal{X}_n = \mathcal{X}$, its ω-limit $\lim_{n\to\omega} \mathcal{X}_n$ is denoted by $\mathcal{X}^\omega$ and called ω-power of $\mathcal{X}$.

4.4.1. Exercise. *For any point $x \in \mathcal{X}$, consider the constant sequence $x_n = x$ and set $\iota(x) = \lim_{n\to\omega} x_n \in \mathcal{X}^\omega$.*

a) Show that $\iota\colon \mathcal{X} \to \mathcal{X}^\omega$ is distance preserving embedding. (So we can and will consider $\mathcal{X}$ as a subset of $\mathcal{X}^\omega$.)

b) Show that ι is onto if and only if $\mathcal{X}$ compact.

c) Show that if $\mathcal{X}$ is proper, then $\iota(\mathcal{X})$ forms a metric component of $\mathcal{X}^\omega$; that is, a subset of $\mathcal{X}^\omega$ that lie on finite distance from a given point.

4.4.2. Observation. *Let $\mathcal{X}$ be a complete metric space. Then $\mathcal{X}^\omega$ is geodesic space if and only if $\mathcal{X}$ is a length space.*

Proof. Assume $\mathcal{X}^\omega$ is geodesic space. Then any pair of points $x, y \in \mathcal{X}$ has a midpoint $z_\omega \in \mathcal{X}^\omega$. Fix a sequence of points $z_n \in \mathcal{X}$ such that $z_n \to z_\omega$ as $n \to \omega$.

Note that $|x - z_n|_{\mathcal{X}} \to \frac{1}{2}\cdot|x - y|_{\mathcal{X}}$ and $|y - z_n|_{\mathcal{X}} \to \frac{1}{2}\cdot|x - y|_{\mathcal{X}}$ as $n \to \omega$. In particular, for any $\varepsilon > 0$, the point z_n is an ε-midpoint of x and y for ω-almost all n. It remains to apply Lemma 1.7.4.

The "if"-part follows from Observation 4.3.2. $\square$

4.4.3. Exercise. *Assume $\mathcal{X}$ is a complete length space and $p, q \in \mathcal{X}$ cannot be joined by a geodesic in $\mathcal{X}$. Then there are at least two distinct geodesics between p and q in the ultrapower $\mathcal{X}^\omega$.*

4.4.4. Exercise. *Construct a proper metric space $\mathcal{X}$ such that $\mathcal{X}^\omega$ is not proper; that is, there is a point $p \in \mathcal{X}^\omega$ and $R < \infty$ such that the closed ball $\overline{\mathrm{B}}[p, R]_{\mathcal{X}^\omega}$ is not compact.*

4.5 Tangent and asymptotic spaces

Choose a space $\mathcal{X}$ and a sequence of $\lambda_n > 0$. Consider the sequence of scalings $\mathcal{X}_n = \lambda_n\cdot\mathcal{X} = (\mathcal{X}, \lambda_n\cdot|{*} - {*}|_{\mathcal{X}})$.

Choose a point $p \in \mathcal{X}$ and denote by p_n the corresponding point in $\mathcal{X}_n$. Consider the ω-limit $\mathcal{X}_\omega$ of $\mathcal{X}_n$ (one may denote it by $\lambda_\omega\cdot\mathcal{X}$); set p_ω to be the ω-limit of p_n.

If $\lambda_n \to 0$ as $n \to \omega$, then the metric component of p_ω in $\mathcal{X}_\omega$ is called *ω-tangent space* at p and denoted by $\mathrm{T}_p\mathcal{X}$.

If $\lambda_n \to \infty$ as $n \to \omega$, then the metric component of p_ω in called *ω-asymptotic space*[1] and denoted by $\operatorname{Asym}\mathcal{X}$. Note that the space $\operatorname{Asym}\mathcal{X}$ and its point p_ω does not depend on the choice of $p \in \mathcal{X}$.

4.5.1. Exercise. *Let $\mathcal{L}$ be the Lobachevsky plane; $\mathcal{T} = \operatorname{Asym}\mathcal{L}$.*

a) Show that $\mathcal{T}$ is a complete metric tree.

b) Show that $\mathcal{T}$ has continuum degree *at any point; that is, for any point $t \in \mathcal{T}$ the set of connected components of the complement $\mathcal{T}\backslash\{t\}$ has cardinality continuum.*

c) Show that $\mathcal{T}$ is homogeneous; that is given two points $s, t \in \mathcal{T}$ there is an isometry of $\mathcal{T}$ that maps s to t.

d) Prove (a–c) if $\mathcal{L}$ is Lobachevsky space and/or for the infinite 3-regular[2] tree with unit edge.

[1] Often it is called *asymptotic cone* despite that it is not a cone in general; this name is used since in good cases it has a cone structure.

[2] that is, degree of any vertex is 3.

As it shown in [7], the properties (a) and (b) describe the tree $\mathcal{T}$ up to isometry. In particular, the asymptotic space of Lobachevsky plane does not depend on the choice of ultrafilter and the sequence $\lambda_n \to \infty$. In general, the tangent and asymptotic spaces depend on number of choices — we need to fix a sequence λ_n and an nonprinciple ultrafiler ω.

Chapter 5

Urysohn space

We discuss a construction introduced by Pavel Urysohn [20]. Our presentation is very close to the one in [10].

This subject is closely related to the so called *Rado graph*, also known as *Erdős–Rényi graph* or *random graph*; a good survey this subject is written by Peter Cameron [6].

5.1 Existance

Suppose a metric space $\mathcal{X}$ is a subspace of a pseudometric space $\mathcal{X}'$. In this case we may say that $\mathcal{X}'$ is an *extension* of $\mathcal{X}$. If $\operatorname{diam}\mathcal{X}' \leqslant d$, then we say that $\mathcal{X}'$ is a *d-extension*.

If the complement $\mathcal{X}'\backslash\mathcal{X}$ contains a single point, say p, we say that $\mathcal{X}'$ is a *one-point extension* of $\mathcal{X}$. In this case, to define metric on $\mathcal{X}'$, it is sufficient to specify the distance function from p; that is, a function $f\colon\mathcal{X}\to\mathbb{R}$ defined by

$$f(x) = |p-x|_{\mathcal{X}'}.$$

The function f can not be taken arbitrary — the triangle inequality implies that

$$f(x)+f(y) \geqslant |x-y|_{\mathcal{X}} \geqslant |f(x)-f(y)|$$

for any $x,y\in\mathcal{X}$. In particular f is a non-negative 1-Lipschitz function on $\mathcal{X}$. For a d-extension we need to assume in addition that $\operatorname{diam}\mathcal{X} \leqslant d$ and $f(x)\leqslant d$ for any $x\in\mathcal{X}$.

Any function f of that type will be called *extension function* or *d-extension function* correspondingly.

5.1.1. Definition. *A metric space $\mathcal{U}$ is called* universal *if for any finite subspace $\mathcal{F} \subset \mathcal{U}$ and any extension function $f\colon \mathcal{F} \to \mathbb{R}$ there is a point $p \in \mathcal{U}$ such that $|p - x| = f(x)$ for any $x \in \mathcal{F}$.*

If instead of extension functions we consider only d-extension functions and assume in addition that $\operatorname{diam} \mathcal{U} \leqslant d$, *then we arrive to a definition of* d-universal space.

If in addition $\mathcal{U}$ is separable and complete, then it is called Urysohn space *or* d-Urysohn space.

5.1.2. Proposition. *Given a positive d, there is a separable d-universal metric space.*

Moreover, a separable universal space metric exists.

Proof. Let $\mathcal{X}$ be a compact metric space such that $\operatorname{diam} \mathcal{X} \leqslant d$. Denote by $\mathcal{X}^d$ the space of all d-extension functions on $\mathcal{X}$ equipped with the metric defined by sup-norm. Note that the map $\mathcal{X} \to \mathcal{X}^d$ defined by $x \mapsto \operatorname{dist}_x$ is a distance preserving embedding, so we can (and will) treat $\mathcal{X}$ as a subspace of $\mathcal{X}^d$, or, equivalently, $\mathcal{X}^d$ is an extension of $\mathcal{X}$.

Let us iterate this construction. Start with a one-point space $\mathcal{X}_0$ and consider a sequence of spaces $(\mathcal{X}_n)$ defined by $\mathcal{X}_{n+1} = \mathcal{X}_n^d$. Note that the sequence is nested, that is $\mathcal{X}_0 \subset \mathcal{X}_1 \subset \dots$ and the union

$$\mathcal{X}_\infty = \bigcup_n \mathcal{X}_n;$$

comes with metric such that $|x - y|_{\mathcal{X}_\infty} = |x - y|_{\mathcal{X}_n}$ if $x, y \in \mathcal{X}_n$.

Note that if $\mathcal{X}$ is compact, then so is $\mathcal{X}^d$. It follows that each space $\mathcal{X}_n$ is compact. Since $\mathcal{X}_\infty$ is a countable union of compact spaces, it is separable.

Any finite subspace $\mathcal{F}$ of $\mathcal{X}_\infty$ lies in some $\mathcal{X}_n$ for $n < \infty$. By construction, there is a point $p \in \mathcal{X}_{n+1}$ that meets the condition in Definiton 5.1.1. That is, $\mathcal{X}_\infty$ is d-universal.

A construction of a universal separable metric space is done along the same lines, but one has the sequence should be defined by $\mathcal{X}_{n+1} = \mathcal{X}_n^{d_n}$ for some sequence $d_n \to \infty$. $\square$

5.1.3. Proposition. *A completion of d-universlal space is d-universal.*

A completion of universal space universal.

Note that 5.1.2 and 5.1.3 imply the following:

5.1.4. Theorem. *Urysohn space, and d-Urysohn space for any $d > 0$, exist.*

Proof. Suppose $\mathcal{V}$ be a d-universal space; denote by $\mathcal{U}$ its completion; so $\mathcal{V}$ is a dense subset in a complete space $\mathcal{U}$.

Observe that $\mathcal{U}$ is *approximately d-universal*; that is, if $\mathcal{F} \subset \mathcal{U}$ is a finite set, and $f\colon \mathcal{F} \to \mathbb{R}$ is a d-extension function, then there exists $p \in \mathcal{U}$ such that

$$|p - x| \lesssim f(x) \pm \varepsilon.$$

for any $x \in \mathcal{F}$.

Therefore there is a sequence of points $p_n \in \mathcal{U}$ such that for any $x \in \mathcal{F}$,

$$|p_n - x| \lesssim f(x) \pm \tfrac{1}{2^n}.$$

Moreover, we can assume that

❶ $$|p_n - p_{n+1}| < \tfrac{1}{2^n}$$

for all large n. Indeed, consider the sets $\mathcal{F}_n = \mathcal{F} \cup \{p_n\}$ and the functions f_n defined by $f_n(x) = f(x)$ for any $x \in \mathcal{F}$, and

$$f_n(p_n) = \max\left\{\, \big||p_n - x| - f(x)\big| \;:\; x \in \mathcal{F} \,\right\}.$$

Observe that f_n is a an d-extension function for large n and $f_n(p_n) < < \frac{1}{2^n}$. By applying approximate universal property recursively we get ❶.

By ❶, (p_n) is a Cauchy sequence and its limit meets the condition in the definition of universal space (5.1.1). □

5.2 Universality

5.2.1. Proposition. *Let $\mathcal{U}$ a Urysohn space. Then any separable metric space $\mathcal{S}$ admits a distance preserving embedding $\mathcal{S} \hookrightarrow \mathcal{U}$.*

Moreover, for any finite subspace $\mathcal{F} \subset \mathcal{S}$, any distance preserving embedding $\mathcal{F} \hookrightarrow \mathcal{U}$ can be extended to an distance preserving embedding $\mathcal{S} \hookrightarrow \mathcal{U}$.

If $\mathcal{U}$ is d-Urysohn, then the statements hold provided $\operatorname{diam} \mathcal{S} \leqslant d$.

Proof. The required isometry will be denoted by $x \mapsto x'$.

Choose a dense sequence of points $s_1, s_2, \ldots \in \mathcal{S}$. We may assume that $\mathcal{F} = \{s_1, \ldots, s_n\}$, so $s_i' \in \mathcal{U}$ are defined for $i \leqslant n$.

The sequence s_i' for $i > n$ can be defined recursively using universality of $\mathcal{U}$. Namely that $s_1', \ldots, s_{i-1}'$ are already defined. Since $\mathcal{U}$ is universal, there is a point $s_i' \in \mathcal{U}$ such that

$$|s_i' - s_j'|_{\mathcal{U}} = |s_i - s_j|_{\mathcal{S}}$$

for any $j < i$.

We constructed a distance preserving map $s_i \mapsto s'_i$, it remains to extend it to a continuous map on whole $\mathcal{S}$.

The first statement follows if $\mathcal{F} = \varnothing$. $\square$

5.2.2. Exercise. *Show that any two distinct points in an Urysohn space can be jointed by infinite number of geodesics.*

5.2.3. Exercise. *Show that Urysohn space is simply connected.*

5.3 Uniqueness

5.3.1. Theorem. *Suppose $\mathcal{F} \subset \mathcal{U}$ and $\mathcal{F}' \subset \mathcal{U}'$ be finite isometric subspaces in a pair of (d-)Urysohn spaces $\mathcal{U}$ and $\mathcal{U}'$. Then any isometry $\mathcal{F} \to \mathcal{F}'$ can be extended to an isometry $\mathcal{U} \to \mathcal{U}'$.*

Note that 5.2.1 implies that there are distance-preserving maps $\mathcal{U} \to \mathcal{U}'$ and $\mathcal{U}' \to \mathcal{U}$, but it does not immideately implies existence of an isometry. The following construction use the same idea as in the proof of 5.2.1, but we need to apply it *back and forth* to ensure that the constructed distance-preserving map is onto.

Proof. The required isometry $\mathcal{U} \leftrightarrow \mathcal{U}'$ will be denoted by $u \leftrightarrow u'$.

Choose a dense sequences $a_1, a_2, \dots \in \mathcal{U}$ and $b'_1, b'_2, \dots \in \mathcal{U}$. Let us define recursively $a'_1, b_1, a'_2, b_2, \dots$ — on the odd step we define the images of $a_1, a_2, \dots$ and on the even steps we define invese images of $b'_1, b'_2, \dots$. The same argument as in the proof of 5.2.1 shows that we can construct two sequences $a'_1, a'_2, \dots \in \mathcal{U}'$ and $b_1, b_2, \dots \in \mathcal{U}$ such that

$$\begin{aligned}
|a_i - a_j|_{\mathcal{U}} &= |a'_i - a'_j|_{\mathcal{U}'} \\
|a_i - b_j|_{\mathcal{U}} &= |a'_i - b'_j|_{\mathcal{U}'} \\
|b_i - b_j|_{\mathcal{U}} &= |b'_i - b'_j|_{\mathcal{U}'}
\end{aligned}$$

for all i and j.

Let us extend the constructed distance preserving bijection defined by $a_i \leftrightarrow a'_i$ and $b_i \leftrightarrow b'_i$ continuousely to whole $\mathcal{U}$. Observe that the image of this bijection is dense in $\mathcal{U}'$ therefore the constructed map $\mathcal{U} \to \mathcal{U}'$ is a bijection. $\square$

Further the Urysohn space will be denoted by $\mathcal{U}$, and the d-Urysohn space will be denoted by $\mathcal{U}_d$. Observe that 5.3.1 implies that the spaces $\mathcal{U}$ and $\mathcal{U}_d$ are n-point homogeneous; that is,

$\diamond$ Any distance preserving map from a finite set F

5.3.2. Exercise. *Let S be a sphere of radius $\frac{d}{2}$ in $\mathcal{U}_d$; that is,*

$$S = \left\{\, x \in \mathcal{U}_d \;:\; |p - x|_{\mathcal{U}_d} = \tfrac{d}{2} \,\right\}$$

for some point $p \in \mathcal{U}_d$. Show that S is isometric to $\mathcal{U}_d$.

5.3.3. Exercise. *Modify the proofs of 5.1.3 and 5.3.1 to prove the following theorem.*

5.3.4. Theorem. *Let $K \subset \mathcal{U}$ be a compact set. Show that any distance-preserving map $f\colon K \to \mathcal{U}$ can be extended to an isometry of $\mathcal{U}$.*

Appendix A

Semisolutions

Exercise 1.3.1. Assume the statement is wrong. Then for any point $x \in \mathcal{X}$, there is a point $x' \in \mathcal{X}$ such that

$$|x - x'| < \rho(x) \quad \text{and} \quad \rho(x') \leqslant \frac{\rho(x)}{1+\varepsilon}.$$

Consider a sequence of points (x_n) such that $x_{n+1} = x'_n$. Clearly

$$|x_{n+1} - x_n| \leqslant \frac{\rho(x_0)}{\varepsilon\cdot(1+\varepsilon)^n} \quad \text{and} \quad \rho(x_n) \leqslant \frac{\rho(x_0)}{(1+\varepsilon)^n}.$$

Therefore (x_n) is Cauchy. Since $\mathcal{X}$, the sequence (x_n); denote its limit by x_∞. Since ρ is a continuous function we get

$$\begin{aligned}\rho(x_\infty) &= \lim_{n\to\infty} \rho(x_n) = \\ &= 0.\end{aligned}$$

The latter contradicts that $\rho > 0$. $\square$

Exercise 1.4.3. Given any pair of point $x_0, y_0 \in \mathcal{K}$, consider two sequences $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ such that $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ for each n.

Since $\mathcal{K}$ is compact, we can choose an increasing sequence of integers n_k such that both sequences $(x_{n_i})_{i=1}^\infty$ and $(y_{n_i})_{i=1}^\infty$ converge. In particular, both are Cauchy sequences; that is,

$$|x_{n_i} - x_{n_j}|_{\mathcal{K}}, |y_{n_i} - y_{n_j}|_{\mathcal{K}} \to 0 \quad \text{as} \quad \min\{i,j\} \to \infty.$$

Since f is non-contracting, we get

$$|x_0 - x_{|n_i - n_j|}| \leqslant |x_{n_i} - x_{n_j}|.$$

It follows that there is a sequence $m_i \to \infty$ such that

$$x_{m_i} \to x_0 \quad \text{and} \quad y_{m_i} \to y_0 \quad \text{as} \quad i \to \infty. \tag{$*$}$$

Set

$$\ell_n = |x_n - y_n|_{\mathcal{K}}.$$

Since f is non-contracting, the sequence (ℓ_n) is non-decreasing.

By $(*)$, $\ell_{m_i} \to \ell_0$ as $m_i \to \infty$. It follows that (ℓ_n) is a constant sequence.

In particular

$$|x_0 - y_0|_{\mathcal{K}} = \ell_0 = \ell_1 = |f(x_0) - f(y_0)|_{\mathcal{K}}$$

for any pair of points (x_0, y_0) in $\mathcal{K}$. That is, f is distance preserving, in particular injective.

From $(*)$, we also get that $f(\mathcal{K})$ is everywhere dense. Since $\mathcal{K}$ is compact $f\colon \mathcal{K} \to \mathcal{K}$ is surjective. Hence the result follows. □

This is a basic lemma in the introduction to Gromov–Hausdorff distance [see 7.3.30 in 5]. I learned this proof from Travis Morrison, a student in my MASS class at Penn State, Fall 2011.

Note that as an easy corollary one can see that any surjective non-expanding map from a compact metric space to itself is an isometry.

Exercise 1.7.2. We assume that the space is not trivial, otherwise a one-point space is an example.

Consider the unit ball (B, ρ_0) in the space c_0 of all sequences converging to zero equipped with the sup-norm.

Consider another metric ρ_1 which is different from ρ_0 by the conformal factor

$$\varphi(\boldsymbol{x}) = 2 + \tfrac{1}{2}\cdot x_1 + \tfrac{1}{4}\cdot x_2 + \tfrac{1}{8}\cdot x_3 + \dots,$$

where $\boldsymbol{x} = (x_1, x_2 \dots) \in B$. That is, if $\boldsymbol{x}(t)$, $t \in [0, \ell]$, is a curve parametrized by ρ_0-length then its ρ_1-length is

$$\operatorname{length}_{\rho_1} \boldsymbol{x} = \int\limits_0^\ell \varphi \circ \boldsymbol{x}.$$

Note that the metric ρ_1 is bi-Lipschitz to ρ_0.

Assume $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t)$ are two curves parametrized by ρ_0-length that differ only in the m-th coordinate, denoted by $x_m(t)$ and $x'_m(t)$ correspondingly. Note that if $x'_m(t) \leqslant x_m(t)$ for any t and the function $x'_m(t)$ is locally 1-Lipschitz at all t such that $x'_m(t) < x_m(t)$, then

$$\operatorname{length}_{\rho_1} \boldsymbol{x}' \leqslant \operatorname{length}_{\rho_1} \boldsymbol{x}.$$

Moreover this inequality is strict if $x'_m(t) < x_m(t)$ for some t.

Fix a curve $\boldsymbol{x}(t)$, $t \in [0,\ell]$, parametrized by ρ_0-length. We can choose m large, so that $x_m(t)$ is sufficiently close to 0 for any t. In particular, for some values t, we have $y_m(t) < x_m(t)$, where

$$y_m(t) = (1 - \tfrac{t}{\ell})\cdot x_m(0) + \tfrac{t}{\ell}\cdot x_m(\ell) - \tfrac{1}{100}\cdot \min\{t, \ell - t\}.$$

Consider the curve $\boldsymbol{x}'(t)$ as above with

$$x'_m(t) = \min\{x_m(t), y_m(t)\}.$$

Note that $\boldsymbol{x}'(t)$ and $\boldsymbol{x}(t)$ have the same end points, and by the above

$$\operatorname{length}_{\rho_1} \boldsymbol{x}' < \operatorname{length}_{\rho_1} \boldsymbol{x}.$$

That is, for any curve $\boldsymbol{x}(t)$ in (B, ρ_1), we can find a shorter curve $\boldsymbol{x}'(t)$ with the same end points. In particular, (B, ρ_1) has no geodesics. $\square$

This example was suggested by Fedor Nazarov [14].

Exercise 1.7.3. Choose a Cauchy sequence (x_n) in $(\mathcal{X}, \| * - * \|)$; it sufficient to show that a subsequence of (x_n) converges.

Note that the sequence (x_n) is Cauchy in $(\mathcal{X}, | * - * |)$; denote its limit by x_∞.

After passing to a subsequence, we can assume that $\|x_n - x_{n+1}\| < \frac{1}{2^n}$. It follows that there is a 1-Lipschitz path γ in $(\mathcal{X}, \| * - * \|)$ such that $x_n = \gamma(\frac{1}{2^n})$ for each n and $x_\infty = \gamma(0)$.

It follows that

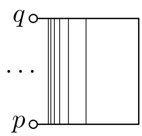
$$\begin{aligned}\|x_\infty - x_n\| &\leqslant \operatorname{length} \gamma|_{[0,\frac{1}{2^n}]} \leqslant \\ &\leqslant \tfrac{1}{2^n}.\end{aligned}$$

In particular x_n converges. $\square$

Source: [16, Lemma 2.3].

Exercise 1.7.8. Consider the following subset of $\mathbb{R}^2$ equipped with the induced length metric

$$\mathcal{X} = \big((0,1] \times \{0,1\}\big) \cup \big(\{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\} \times [0,1]\big)$$



Note that $\mathcal{X}$ is locally compact and geodesic.

Its completion $\bar{\mathcal{X}}$ is isometric to the closure of $\mathcal{X}$ equipped with the induced length metric; $\bar{\mathcal{X}}$ is obtained from $\mathcal{X}$ by adding two points $p = (0,0)$ and $q = (0,1)$.

The point p admits no compact neighborhood in $\bar{\mathcal{X}}$ and there is no geodesic connecting p to q in $\bar{\mathcal{X}}$. $\square$

This exercise and its solution is taken from [4].

Exercise 1.8.3. By Frechet lemma (1.8.1) we can identify $\mathcal{K}$ with a compact subset of ℓ^∞.

Denote by $\mathcal{L} = \operatorname{Conv}\mathcal{K}$ the minimal convex closed set in ℓ^∞ that contains $\mathcal{K}$. Observe that $\mathcal{L}$ is a length space.

It remains to show that since $\mathcal{K}$, so is $\mathcal{L}$ — do it.

Exercise 2.1.7. The answer is "no" in both parts.

For the first part let X be a unit disc and Y a finite ε-net in X. Evidently $|X-Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, but $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} \approx 1$.

For the second part take X to be a unit disc and $Y = \partial X$ to be its boundary circle. Note that $\partial X = \partial Y$ in particular $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} = 0$ while $|X-Y|_{\mathcal{H}(\mathbb{R}^2)} = 1$. □

A more interesting example for the second part can be build on *lakes of Wada* — and example of three open bounded topological disks in the plane that have identical boundary.

Exercise 2.1.8. Let A be a compact convex set in the plane. Denote by A^r the closed r-neighborhood of A. Recall that by Steiner's formula we have

$$\operatorname{area} A^r = \operatorname{area} A + r\cdot \operatorname{perim} A + \pi\cdot r^2.$$

Taking derivative and applying coarea formula, we get

$$\operatorname{perim} A^r = \operatorname{perim} A + 2\cdot\pi\cdot r.$$

Observe that if A lies in a compact set B bounded by a colsed curve, then

$$\operatorname{perim} A \leqslant \operatorname{perim} B.$$

Indeed the closest-point projection $\mathbb{R}^2 \to A$ is short and it maps ∂B onto ∂A.

It remains to observe that if $A_n \to A_\infty$, then for any $r > 0$ we have that

$$A^r_\infty \supset A_n \quad \text{and} \quad A_\infty \subset A^r_n$$

for all large n.

Exercise 3.4.4. In order to check that $|{*}-{*}|_{\mathcal{M}'}$ is a metric, it is sufficient to show that

$$|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}'} = 0 \quad \Longrightarrow \quad \mathcal{X} \overset{iso}{=\!=} \mathcal{Y};$$

the remaining conditions are trivial.

If $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}'} = 0$, then there is a sequence of maps $f_n\colon \mathcal{X} \to \mathcal{Y}$ such that

$$|f_n(x) - f_n(x')|_{\mathcal{Y}} \geqslant |x - x'|_{\mathcal{X}} - \tfrac{1}{n}.$$

Choose a countable dense set S in $\mathcal{X}$. Passing to a subsequence of f_n we can assume that $f_n(x)$ converges for any $x \in S$ as $n \to \infty$; denote its limit by $f_\infty(x)$.

For each point $x \in \mathcal{X}$ choose a sequence $x_m \in S$ converging to x. Since $\mathcal{Y}$ is compact, we can assume in addition that $y_m = f_\infty(x_m)$ converges in $\mathcal{Y}$. Set $f_\infty(x) = y$. Note that the map $f_\infty \colon \mathcal{X} \to \mathcal{Y}$ is distance non-decreasing.

The same way we can construct a distance non-decreasing map $g_\infty \colon \mathcal{Y} \to \mathcal{X}$.

By Exercise 1.4.3, the compositions $f_\infty \circ g_\infty \colon \mathcal{Y} \to \mathcal{Y}$ and $g_\infty \circ$ $\circ f_\infty \colon \mathcal{X} \to \mathcal{X}$ are isometrises. Therefore f_∞ and g_∞ are isometries as well.

(The proof of the second part is coming.)

Exercise 3.5.1. Choose a space $\mathcal{X}$ in $\mathcal{Q}(C, D)$, denote a C-doubling measure by μ. Without loss of generality we may assume that $\mu(\mathcal{X}) =$ $= 1$.

The doubling condition implies that

$$\mu[\mathrm{B}(p, \tfrac{D}{2^n})] \geqslant \tfrac{1}{C^n}$$

for any point $x \in \mathcal{X}$. It follows that

$$\operatorname{pack}_{\frac{D}{2^n}} \mathcal{X} \leqslant C^n.$$

By Exercise 1.4.2, for any $\varepsilon \geqslant \frac{D}{2^{n-1}}$, the space $\mathcal{X}$ admits an ε-net with at most C^n points. Whence $\mathcal{Q}(C, D)$ is uniformly totally bounded.

Exercise 3.5.2. Since $\mathcal{Y}$ is compact, it has a finite ε-net for any $\varepsilon > 0$. For each $\varepsilon > 0$ choose a finite ε-net $\{y_1, \dots, y_{n_\varepsilon}\}$ in $\mathcal{Y}$.

Suppose $f \colon \mathcal{X} \to \mathcal{Y}$ be a distance non-decreasing map. Choose one point x_i in each nonempty subset $B_i = f^{-1}[\mathrm{B}(y_i, \varepsilon)]$. Note that the subset B_i has diameter at most $2{\cdot}\varepsilon$ and

$$\mathcal{X} = \bigcup_i B_i.$$

Therefore the set of points $\{x_i\}$ forms a $2{\cdot}\varepsilon$ net in $\mathcal{X}$. Whence (a) follows.

(b). Let $\mathcal{Q}$ be a uniformly totally bounded family of spaces. Suppose that each space in $\mathcal{Q}$ has an $\frac{1}{2^n}$-net with at most M_n points; we may assume that $M_0 = 1$.

Consider the space $\mathcal{Y}$ of all infinite integer sequences $m_0, m_1, \dots$ such that $1 \leqslant m_n \leqslant M_n$ for any n. Given two sequences (ℓ_n), and

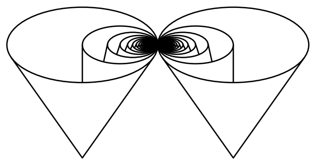
(m_n) of points in $\mathcal{Y}$, set

$$|(\ell_n) - (m_n)|_{\mathcal{Y}} = \tfrac{1}{2^{n-1}},$$

where n is minimal index such that $\ell_n \neq m_n$.

It remains to observe that $\mathcal{Y}$ is compact and any space $\mathcal{X}$ in $\mathcal{Q}$ admits a distance non-decreasing map $\mathcal{X} \to \mathcal{Y}$.

Exercise 3.6.3*b*. Let $\mathcal{V}$ be a cone over Hawaiian earring. Consider the *doubled cone* $\mathcal{W}$ — two copies of $\mathcal{V}$ with glued base points earrings (see the diagrm).



The space $\mathcal{W}$ can be equipped with length metric for example the induced length metric from the shown embedding.

Note that $\mathcal{V}$ is simply connected, but $\mathcal{W}$ is not — it is a good exercise in topology.

If we delete from the earrings all small circles, then the obtained double cone becomes simply connected and it remains to be close to $\mathcal{W}$ in the sense of Gromov–Hausdorff.

Exercise 3.6.4*b*. Make fine burrows in the standard 3-ball without changing its topology, but at the same time come sufficiently close to any point in the ball.

Consider the *doubling* of the obtained ball along its boundary; that is, two copies of the ball with identified corresponding points on their boundaries. The obtained space is homeomorphic to $\mathbb{S}^3$. Note that the burrows can be made so that the obtained space is sufficiently close to the original ball in the Gromov–Hausdorff metric. □

Sourse: [5, Ex. 7.5.17].

Exercise 4.4.1. Part (a) follows directly from the definitions. Further we consider $\mathcal{X}$ as a subset of $\mathcal{X}^\omega$.

(b). Suppose $\mathcal{X}$ compact. Given a sequence (x_n) in $\mathcal{X}$, denote its ω-limit in $\mathcal{X}^\omega$ by x^ω and its ω-limit in $\mathcal{X}$ by x_ω.

Observe that $x^\omega = \iota(x_\omega)$. Therefore ι is onto.

If $\mathcal{X}$ is not compact, we can choose a sequence (x_n) such that $|x_m - x_n| > \varepsilon$ for fixed $\varepsilon > 0$ and $m \neq n$. Observe that

$$\lim_{n\to\omega} |x_n - y|_{\mathcal{X}} \geqslant \tfrac{\varepsilon}{2}$$

for any $y \in \mathcal{X}$. It follows that x_ω lies on the distance at least $\frac{\varepsilon}{2}$ from $\mathcal{X}$.

(c). A sequence of points (x_n) in $\mathcal{X}$ will be called ω-bounded if there is a real constant C such that

$$|p - x_n|_{\mathcal{X}} \leqslant C$$

for ω-almost all n.

The same argument as in (b) shows that any ω-bounded sequence has its ω-limit in $\mathcal{X}$. Further if (x_n) is not ω-bounded, then

$$\lim_{n \to \omega} |p - x_n|_{\mathcal{X}} = \infty;$$

that is x_ω does not lie in the metric component of p in $\mathcal{X}^\omega$.

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