Lectures in metric geometry

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Disclaimer

Considerable part of the text is a compilation from [1, 2, 11–13] and its drafts.

Contents

Chapter 0

Homework assignments

It is better to think about all the problems, but you do not have to solve all of them. If a problem is solved, you do not have to write its solutions, but try sketch it.

0.1 Due Tue Jan 21

Exercises: 1.3.1, 1.4.3, 1.7.2, 1.7.3, 1.7.8, 2.1.7.

0.2 Due Tue Jan 28

Exercises: 1.3.1, 1.7.3, TBC.

Chapter 1

Definitions

1.1 Metric spaces

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$.

The function

$$\operatorname{dist}_x\colon y\mapsto |x-y|$$

is called the *distance function* from x.

Given $R\in[0,\infty]$ and $x\in\mathcal{X}$, the sets

$$\begin{aligned}\mathrm{B}(x,R)&=\{y\in\mathcal{X}\mid |x-y|<R\},\\ \overline{\mathrm{B}}[x,R]&=\{y\in\mathcal{X}\mid |x-y|\leqslant R\}\end{aligned}$$

are called, respectively, the *open* and the *closed balls* of radius R with center x. Again, if we need to emphasize that these balls are taken in the metric space $\mathcal{X}$, we write

$$\mathrm{B}(x,R)_{\mathcal{X}}\quad\text{and}\quad\overline{\mathrm{B}}[x,R]_{\mathcal{X}}.$$

1.2 Variations of definition

Recall that a metric is a real-valued function $(x,y)\mapsto|x-y|_{\mathcal{X}}$ that satisfies the following conditions for any three points $x,y,z\in\mathcal{X}$:

(i) $|x-y|_{\mathcal{X}}\geqslant 0$,

(ii) $|x-y|_{\mathcal{X}}=0\iff x=y$,

(iii) $|x-y|_{\mathcal{X}}=|y-x|_{\mathcal{X}}$,

(iv) $|x-y|_{\mathcal{X}}+|y-z|_{\mathcal{X}}\geqslant|x-z|_{\mathcal{X}}$,

Pseudometrics. A generalization of a metric in which the distance between two distinct points can be zero is called *pseudometric*. In other words, to define pseudometric, we need to remove condition (ii) from the list.

The following two observations show that nearly any question about pseudometric spaces can be reduced to a question about genuine metric spaces.

Assume $\mathcal{X}$ is a pseudometric space. Set $x \sim y$ if $|x-y| = 0$. Note that if $x \sim x'$, then $|y-x| = |y-x'|$ for any $y \in \mathcal{X}$. Thus, $|*-$ $-*|$ defines a metric on the quotient set $\mathcal{X}/\sim$. In this way we obtain a metric space $\mathcal{X}'$. The space $\mathcal{X}'$ is called the *corresponding metric space* for the pseudometric space $\mathcal{X}$. Often we do not distinguish between $\mathcal{X}'$ and $\mathcal{X}$.

∞-metrics. One may also consider metrics with values in $\mathbb{R} \cup \{\infty\}$; we might call then ∞-metrics or simply metrics.

Again nearly any question about ∞-metric spaces can be reduced to a question about genuine metric spaces.

Indeed, set $x \approx y$ if and only if $|x-y| < \infty$; this is an other equivalence relation on $\mathcal{X}$. The equivalence class of a point $x \in \mathcal{X}$ will be called the *metric component* of x; it will be denoted as $\mathcal{X}_x$. One could think of $\mathcal{X}_x$ as $\mathrm{B}(x, \infty)_{\mathcal{X}}$ — the open ball centered at x and radius ∞ in $\mathcal{X}$.

It follows that any ∞-metric space is a *disjoint union* of genuine metric spaces — the metric components of the original ∞-metric space.

1.3 Completeness

Recall that a metric space $\mathcal{X}$ is called *complete* if every Cauchy sequence of points in $\mathcal{X}$ converges in $\mathcal{X}$.

1.3.1. Exercise. *Suppose that ρ is a positive continuous function on a complete metric space $\mathcal{X}$. Show that for any $\varepsilon > 0$ there is a point $x \in \mathcal{X}$ such that*

$$\rho(x) < (1+\varepsilon)\cdot\rho(y)$$

for any point $y \in \mathrm{B}(x, \rho(x))$.

Most of the time we will assume that a metric space is complete. The following construction produces a complete metric space $\bar{\mathcal{X}}$ for any given metric space $\mathcal{X}$. The space $\bar{\mathcal{X}}$ is called *completion* of $\mathcal{X}$; the original space $\mathcal{X}$ forms a dense subset in $\bar{\mathcal{X}}$.

Completion. Given metric space $\mathcal{X}$, consider the set of all Cauchy sequences in $\mathcal{X}$. Note that for any two Cauchy sequences (x_n) and (y_n)

the right hand side in ❶ is defined; moreover it defines a pseudometric on the set $\mathcal{C}$ of all Cauchy sequences

$$❶ \qquad |(x_n) - (y_n)|_{\mathcal{C}} \stackrel{def}{=\!=} \lim_{n\to\infty} |x_n - y_n|_{\mathcal{X}}.$$

The corresponding metric space is called a completion of $\mathcal{X}$.

It is left as an exercise that completion of $\mathcal{X}$ is complete.

Note that for each point $x \in \mathcal{X}$ one can consider a constant sequence $x_n = x$ which is Cauchy. It defines a natural map $\mathcal{X} \to \bar{\mathcal{X}}$. It is easy to check that this map is distance preserving. In partucular we can (and will) consider $\mathcal{X}$ as a subset of $\bar{\mathcal{X}}$.

1.4 Compactness

Let us recall few equivalent definitions of compact metric spaces.

1.4.1. Definition. *A metric space $\mathcal{K}$ is compact if and only if one of the following equivalent condition holds:*

a) Every open cover of $\mathcal{K}$ has a finite subcover.

b) For any open cover of $\mathcal{K}$ there is $\varepsilon > 0$ such that any ε-ball in $\mathcal{K}$ lie in one element of the cover. (The value ε is called Lebesgue number of the covering.)

c) Every sequence in $\mathcal{K}$ has a convergent subsequence.

d) The space $\mathcal{K}$ is complete and totally bounded*; that is, for any $\varepsilon > 0$, the space $\mathcal{K}$ admits a finite cover by open ε-balls.*[1]

Let $\operatorname{pack}_\varepsilon \mathcal{X}$ be exact upper bound on the number of points $x_1, \dots$ $\dots, x_n \in \mathcal{X}$ such that $|x_i - x_j| \geqslant \varepsilon$ for any $i \neq j$. If $n = \operatorname{pack}_\varepsilon \mathcal{X} < \infty$, then the collection of points $x_1, \dots, x_n$ is called a *maximal ε-packing.*

1.4.2. Exercise. *Show that a complete space $\mathcal{X}$ is compact if and only of* $\operatorname{pack}_\varepsilon \mathcal{X} < \infty$ *for any $\varepsilon > 0$.*

Show that any maximal ε-packing is an ε-net.

1.4.3. Exercise. *Let $\mathcal{K}$ be a compact metric space and*

$$f\colon \mathcal{K} \to \mathcal{K}$$

be a distance non-decreasing map. Prove that f is an isometry.

A metric space $\mathcal{X}$ is called *proper* if all closed bounded sets in $\mathcal{X}$ are compact. This condition is equivalent to each of the following statements:

[1] Equivalently, for any $\varepsilon > 0$ there is a finite *ε-net*; that is a finite set of points $x_1, \dots, x_n \in \mathcal{K}$ such that any other point x lies on the distance less than ε from one of x_i.

- ◇ For some (and therefore any) point $p \in \mathcal{X}$ and any $R < \infty$, the closed ball $\overline{\mathrm{B}}[p, R]_{\mathcal{X}}$ is compact.
- ◇ The function $\mathrm{dist}_p\colon \mathcal{X} \to \mathbb{R}$ is proper for some (and therefore any) point $p \in \mathcal{X}$; that is, for any compact set $K \subset \mathbb{R}$, its inverse image

$$\mathrm{dist}_p^{-1}(K) = \{\, x \in \mathcal{X} \mid |p - x|_{\mathcal{X}} \in K \,\}$$

is compact.

A metric space $\mathcal{X}$ is called *locally compact* if any point in $\mathcal{X}$ admits a compact neighborhood; in other words, for any point $x \in \mathcal{X}$ a closed ball $\overline{\mathrm{B}}[x, r]$ is compact for some $r > 0$.

1.5 Geodesics

Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ a real interval. A globally isometric map $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a *geodesic*[2]; in other words, $\gamma\colon \mathbb{I} \to \mathcal{X}$ is a geodesic if

$$|\gamma(s) - \gamma(t)|_{\mathcal{X}} = |s - t|$$

for any pair $s, t \in \mathbb{I}$.

We say that $\gamma\colon \mathbb{I} \to \mathcal{X}$ is a geodesic from point p to point q if $\mathbb{I} = [a, b]$ and $p = \gamma(a)$, $q = \gamma(b)$. In this case the image of γ is denoted by $[pq]$ and with an abuse of notations we also call it a *geodesic*. Given a geodesic $[pq]$, we can parametrize it by distance to p; this parametrization will be denoted by $\mathrm{geod}_{[pq]}(t)$.

We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$. We also use the following shortcut notation:

$$]pq[\, = [pq]\backslash\{p, q\}, \qquad]pq] = [pq]\backslash\{p\}, \qquad [pq[\, = [pq]\backslash\{q\}.$$

In general, a geodesic from p to q need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we assume mean that we have made a choice of geodesic.

A metric space is called *geodesic* if any pair of its points can be joined by a geodesic.

A *geodesic path* is a geodesic with constant-speed parametrization by $[0, 1]$. Given a geodesic $[pq]$, we denote by $\mathrm{path}_{[pq]}$ the corresponding geodesic path; that is,

$$\mathrm{path}_{[pq]}(t) \stackrel{def}{=\!=} \mathrm{geod}_{[pq]}(t{\cdot}|p - q|).$$

A curve $\gamma\colon \mathbb{I} \to \mathcal{X}$ is called a *local geodesic* if for any $t \in \mathbb{I}$ there is a neighborhood U of t in $\mathbb{I}$ such that the restriction $\gamma|_U$ is a geodesic. A

[2] Various authors call it differently: *shortest path*, *minimizing geodesic*.

constant-speed parametrization of a local geodesic by the unit interval $[0,1]$ is called a *local geodesic path.*

1.6 Length

A *curve* is defined as a continuous map from a real interval to a space. If the real interval is $[0,1]$, then the curve is called a *path.*

1.6.1. Definition. *Let $\mathcal{X}$ be a metric space and $\alpha\colon \mathbb{I}\to\mathcal{X}$ be a curve. We define the* length *of α as*

$$\operatorname{length}\alpha \stackrel{def}{=\!=} \sup_{t_0\leqslant t_1\leqslant\ldots\leqslant t_n} \sum_i |\alpha(t_i)-\alpha(t_{i-1})|.$$

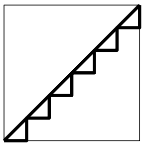
A curve α is called rectifiable *if* $\operatorname{length}\alpha<\infty$.

1.6.2. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\gamma_n\colon\mathbb{I}\to\mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty\colon\mathbb{I}\to\mathcal{X}$; that is, for any fixed $t\in\mathbb{I}$, $\gamma_n(t)\to\gamma_\infty(t)$ as $n\to\infty$. Then

❶ $$\liminf_{n\to\infty}\operatorname{length}\gamma_n\geqslant\operatorname{length}\gamma_\infty.$$

Note that the inequality ❶ might be strict. For example the diagonal γ_∞ of the unit square can be approximated by a stairs-like polygonal curves γ_n with sides parallel to the sides of the square (γ_6 is on the picture). In this case



$$\operatorname{length}\gamma_\infty=\sqrt{2}\quad\text{and}\quad\operatorname{length}\gamma_n=2$$

for any n.

Proof. Fix a sequence $t_0<t_1<\cdots<t_k$ in $\mathbb{I}$. Set

$$\Sigma_n \stackrel{def}{=\!=} |\gamma_n(t_0)-\gamma_n(t_1)|+\cdots+|\gamma_n(t_{k-1})-\gamma_n(t_k)|.$$
$$\Sigma_\infty \stackrel{def}{=\!=} |\gamma_\infty(t_0)-\gamma_\infty(t_1)|+\cdots+|\gamma_\infty(t_{k-1})-\gamma_\infty(t_k)|.$$

Note that for each i we have

$$|\gamma_n(t_{i-1})-\gamma_n(t_i)|\to|\gamma_\infty(t_{i-1})-\gamma_\infty(t_i)|$$

and therefore

$$\Sigma_n\to\Sigma_\infty$$

as $n \to \infty$. Note that

$$\Sigma_n \leqslant \operatorname{length} \gamma_n$$

for each n. Hence

❷ $$\liminf_{n\to\infty} \operatorname{length} \gamma_n \geqslant \Sigma_\infty.$$

If γ_∞ is rectifiable, we can assume that

$$\operatorname{length} \gamma_\infty < \Sigma_\infty + \varepsilon.$$

for any given $\varepsilon > 0$. By ❷ it follows that

$$\liminf_{n\to\infty} \operatorname{length} \gamma_n > \operatorname{length} \gamma_\infty - \varepsilon$$

for any $\varepsilon > 0$; whence ❶ follows.

It remains to consider the case when γ_∞ is not rectifiable; that is, $\operatorname{length} \gamma_\infty = \infty$. In this case we can choose a partition so that $\Sigma_\infty > L$ for any real number L. By ❷ it follows that

$$\liminf_{n\to\infty} \operatorname{length} \gamma_n > L$$

for any given L; whence

$$\liminf_{n\to\infty} \operatorname{length} \gamma_n = \infty$$

and ❶ follows. □

1.7 Length spaces

If for any $\varepsilon > 0$ and any pair of points x and y in a metric space $\mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length} \alpha < |x - y| + \varepsilon,$$

then $\mathcal{X}$ is called a *length space* and the metric on $\mathcal{X}$ is called a *length metric.*

Note that any geodesic space is a length space. As can be seen from the following example, the converse does not hold.

1.7.1. Example. *Let $\mathcal{X}$ be obtained by gluing a countable collection of disjoint intervals $\{\mathbb{I}_n\}$ of length $1 + \frac{1}{n}$, where for each $\mathbb{I}_n$ the left end is glued to p and the right end to q.*

Observe that the space $\mathcal{X}$ carries a natural complete length metric with respect to which $|p - q| = 1$ but there is no geodesic connecting p to q.

1.7.2. Exercise. *Give an example of a complete length space for which no pair of distinct points can be joined by a geodesic.*

Directly from the definition, it follows that if a path $\alpha\colon [0,1]\to\mathcal{X}$ connects two points x and y (that is, if $\alpha(0)=x$ and $\alpha(1)=y$), then

$$\operatorname{length}\alpha \geqslant |x-y|.$$

Set

$$\|x-y\| = \inf\{\operatorname{length}\alpha\}$$

where the greatest lower bound is taken for all paths connecing x and y. It is straightforward to check that $(x,y)\mapsto\|x-y\|$ is an ∞-metric; moreover $(\mathcal{X},\|{*}-{*}\|)$ is a length space. The metric $\|{*}-{*}\|$ is called *induced length metric.*

1.7.3. Exercise. *Suppose $(\mathcal{X},|{*}-{*}|)$ is a compact metric space. Show that $(\mathcal{X},\|{*}-{*}\|)$ is complete.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x,y\in A$, consider the value

$$|x-y|_A = \inf_{\alpha}\{\operatorname{length}\alpha\},$$

where the greatest lower bound is taken for all paths α from x to y in A.[3]

Let $\mathcal{X}$ be a metric space and $x,y\in\mathcal{X}$.

(i) A point $z\in\mathcal{X}$ is called a *midpoint* between x and y if

$$|x-z| = |y-z| = \tfrac12\cdot|x-y|.$$

(ii) Assume $\varepsilon\geqslant 0$. A point $z\in\mathcal{X}$ is called an *ε-midpoint* between x and y if

$$|x-z|,\quad |y-z| \leqslant \tfrac12\cdot|x-y|+\varepsilon.$$

Note that a 0-midpoint is the same as a midpoint.

1.7.4. Lemma. *Let $\mathcal{X}$ be a complete metric space.*

a) Assume that for any pair of points $x,y\in\mathcal{X}$ and any $\varepsilon>0$ there is an ε-midpoint z. Then $\mathcal{X}$ is a length space.

b) Assume that for any pair of points $x,y\in\mathcal{X}$, there is a midpoint z. Then $\mathcal{X}$ is a geodesic space.

[3] This notation slightly conflicts with the previously defined notation for distance $|x-y|_{\mathcal{X}}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.

Proof. We first prove (a). Let $x, y \in \mathcal{X}$ be a pair of points.

Set $\varepsilon_n = \frac{\varepsilon}{4^n}$, $\alpha(0) = x$ and $\alpha(1) = y$.

Let $\alpha(\frac{1}{2})$ be an ε_1-midpoint between $\alpha(0)$ and $\alpha(1)$. Further, let $\alpha(\frac{1}{4})$ and $\alpha(\frac{3}{4})$ be ε_2-midpoints between the pairs $(\alpha(0), \alpha(\frac{1}{2}))$ and $(\alpha(\frac{1}{2}), \alpha(1))$ respectively. Applying the above procedure recursively, on the n-th step we define $\alpha(\frac{k}{2^n})$, for every odd integer k such that $0 < \frac{k}{2^n} < 1$, as an ε_n-midpoint between the already defined $\alpha(\frac{k-1}{2^n})$ and $\alpha(\frac{k+1}{2^n})$.

In this way we define $\alpha(t)$ for $t \in W$, where W denotes the set of dyadic rationals in $[0, 1]$. Since $\mathcal{X}$ is complete, the map α can be extended continuously to $[0, 1]$. Moreover,

$$\text{❶} \qquad \begin{aligned} \operatorname{length} \alpha &\leqslant |x - y| + \sum_{n=1}^{\infty} 2^{n-1} \cdot \varepsilon_n \leqslant \\ &\leqslant |x - y| + \tfrac{\varepsilon}{2}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get (a).

To prove (b), one should repeat the same argument taking midpoints instead of ε_n-midpoints. In this case ❶ holds for $\varepsilon_n = \varepsilon = 0$. □

Since in a compact space a sequence of $\frac{1}{n}$-midpoints z_n contains a convergent subsequence, Lemma 1.7.4 immediately implies

1.7.5. Proposition. *A proper length space is geodesic.*

1.7.6. Hopf–Rinow theorem. *Any complete, locally compact length space is proper.*

It is instructive to solve the following exercise before reading the proof.

1.7.7. Exercise. *Give an example of space which is locally compact but not proper.*

Proof. Let $\mathcal{X}$ be a locally compact length space. Given $x \in \mathcal{X}$, denote by $\rho(x)$ the supremum of all $R > 0$ such that the closed ball $\overline{\mathrm{B}}[x, R]$ is compact. Since $\mathcal{X}$ is locally compact,

$$\text{❷} \qquad \rho(x) > 0 \ \text{ for any } \ x \in \mathcal{X}.$$

It is sufficient to show that $\rho(x) = \infty$ for some (and therefore any) point $x \in \mathcal{X}$.

Assume the contrary; that is, $\rho(x) < \infty$. We claim that

❸ $B = \overline{\mathrm{B}}[x, \rho(x)]$ *is compact for any* x.

Indeed, $\mathcal{X}$ is a length space; therefore for any $\varepsilon > 0$, the set $\overline{\mathrm{B}}[x, \rho(x) - \varepsilon]$ is a compact ε-net in B. Since B is closed and hence complete, it must be compact. △

Next we claim that

❹ $|\rho(x) - \rho(y)| \leqslant |x - y|_{\mathcal{X}}$ *for any* $x, y \in \mathcal{X}$*; in particular* $\rho\colon \mathcal{X} \to \mathbb{R}$ *is a continuous function.*

Indeed, assume the contrary; that is, $\rho(x) + |x - y| < \rho(y)$ for some $x, y \in \mathcal{X}$. Then $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$ is a closed subset of $\overline{\mathrm{B}}[y, \rho(y)]$ for some $\varepsilon > 0$. Then compactness of $\overline{\mathrm{B}}[y, \rho(y)]$ implies compactness of $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$, a contradiction. △

Set $\varepsilon = \min\{\, \rho(y) \mid y \in B \,\}$; the minimum is defined since B is compact. From ❷, we have $\varepsilon > 0$.

Choose a finite $\tfrac{\varepsilon}{10}$-net $\{a_1, a_2, \dots, a_n\}$ in B. The union W of the closed balls $\overline{\mathrm{B}}[a_i, \varepsilon]$ is compact. Clearly $\overline{\mathrm{B}}[x, \rho(x) + \tfrac{\varepsilon}{10}] \subset W$. Therefore $\overline{\mathrm{B}}[x, \rho(x) + \tfrac{\varepsilon}{10}]$ is compact, a contradiction. □

1.7.8. Exercise. *Construct a geodesic space that is locally compact, but whose completion is neither geodesic nor locally compact.*

1.8 Subsets in normed spaces

Recall that a function $v \mapsto |v|$ on a vector space $\mathcal{V}$ is called *norm* if it satisfies the following condition for any two vectors $v, w \in \mathcal{V}$ and a scalar α:

- ⋄ $|v| \geqslant 0$;
- ⋄ $|\alpha \cdot v| = |\alpha| \cdot |v|$;
- ⋄ $|v| + |w| \geqslant |v + w|$.

It is straightforward to check that for any normed space the function $(v, w) \mapsto |v - w|$ defines a metric on it. Therefore any normed space is an example of metric space (which is in fact geodesic). The following lemma says in particular that any metric space is isometric to a subset of a normed space.

1.8.1. Lemma. *Suppose* $\mathcal{X}$ *is a bounded separable space; that is,* $\operatorname{diam} \mathcal{X}$ *is finite and* $\mathcal{X}$ *contains a countable, dense set* $\{w_n\}$*. Given* $x \in \mathcal{X}$*, set* $a_n(x) = |w_n - x|_{\mathcal{X}}$*. Then*

$$\iota\colon x \mapsto (a_1(x), a_2(x), \dots)$$

defines a distance preserving embedding $\iota\colon \mathcal{X} \hookrightarrow \ell^\infty$*.*

Proof. By the triangle inequality

$$|a_n(x) - a_n(y)| \leqslant |x - y|_{\mathcal{X}}.$$

Therefore ι is short.

Again by triangle inequality we have

$$|a_n(x) - a_n(y)| \geqslant |x - y|_{\mathcal{X}} - 2{\cdot}|w_n - x|_{\mathcal{X}}.$$

Since the set $\{w_n\}$ is dense, we can choose w_n arbitrary close to x. Whence the value $|a_n(x) - a_n(y)|$ can be chosen arbitrary close to $|x - y|_{\mathcal{X}}$. In other words

$$\sup_n \{\, ||w_n - x|_{\mathcal{X}} - |w_n - y|_{\mathcal{X}}|\,\} \geqslant |x - y|_{\mathcal{X}};$$

hence ι is distance non-decreasing. □

The following exercise generalizes the lemma to arbitrary separable spaces.

1.8.2. Exercise. *Suppose $\{w_n\}$ is a countable, dense set in a metric space $\mathcal{X}$. Choose $x_0 \in \mathcal{X}$; given $x \in \mathcal{X}$, set*

$$a_n(x) = |w_n - x|_{\mathcal{X}} - |w_n - x_0|_{\mathcal{X}}.$$

Show that $\iota\colon x \mapsto (a_1(x), a_2(x), \dots)$ defines a distance preserving embedding $\iota\colon \mathcal{X} \hookrightarrow \ell^\infty$.

1.8.3. Exercise. *Show that any compact metric space is isometric to a subspace of a compact geodesic space.*

The lemma above was proved by Maurice René Fréchet in the paper where he defined metric space [6]. Nearly identical construction was rediscovered later by Kazimierz Kuratowski [9]. Namely he made the following claim:

1.8.4. Lemma. *Let $\mathcal{X}$ be arbitrary metric space. Denote by $\ell^\infty(\mathcal{X})$ the space of all bounded functions of $\mathcal{X}$ equipped with sup-norm.*

Then for any point $x_0 \in \mathcal{X}$, the map $\iota\colon \mathcal{X} \to \ell^\infty(\mathcal{X})$ defied by

$$\iota\colon x \mapsto (\mathrm{dist}_x - \mathrm{dist}_{x_0})$$

is distance preserving.

Note that this claim implies that *any metric space is isometric to a subset of a normed vector space.*

Chapter 2

Convergence

2.1 Hausdorff convergence

Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$, consider the distance function to A

$$\mathrm{dist}_A : \mathcal{X} \to [0, \infty)$$

defined as

$$\mathrm{dist}_A(x) \stackrel{def}{=\!=} \inf_{a \in A} \{\, |a - x|_{\mathcal{X}} \,\}.$$

2.1.1. Definition. *Let A and B be two compact subsets of a metric space $\mathcal{X}$. Then the* Hausdorff distance *between A and B is defined as*

$$|A - B|_{\mathcal{H}(\mathcal{X})} \stackrel{def}{=\!=} \sup_{x \in \mathcal{X}} \{\, |\mathrm{dist}_A(x) - \mathrm{dist}_B(x)| \,\}.$$

Suppose A and B be two compact subsets of a metric space $\mathcal{X}$. It is straightforward to check that $|A - B|_{\mathcal{H}(\mathcal{X})} \leqslant R$ if and only if $\mathrm{dist}_A(b) \leqslant R$ for any $b \in B$ and $\mathrm{dist}_B(a) \leqslant R$ for any $a \in A$. In other words, $|A - B|_{\mathcal{H}(\mathcal{X})} < R$ if and only if B lies in a R-neighborhood of A, and A lies in a R-neighborhood of B.

Note that the set of all nonempty compact subsets of a metric space $\mathcal{X}$ equipped with the Hausdorff metric forms a metric space. This new metric space will be denoted as $\mathcal{H}(\mathcal{X})$.

2.1.2. Exercise. *Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$ define its diameter as*

$$\mathrm{diam}\, A \stackrel{def}{=\!=} \sup_{a, b \in A} |a - b|.$$

Show that

$$\operatorname{diam}\colon \mathcal{H}(\mathcal{X}) \to \mathbb{R}$$

is a 2-Lipschitz function; that is, $|\operatorname{diam} A - \operatorname{diam} B| \leqslant 2\cdot|A-B|_{\mathcal{H}(\mathcal{X})}$.

2.1.3. Blaschke selection theorem. *Let $\mathcal{X}$ be a metric space. Then the space $\mathcal{H}(\mathcal{X})$ is compact if and only if $\mathcal{X}$ is compact.*

Proof; "only if" part. Note that the map $\iota\colon \mathcal{X} \to \mathcal{H}(\mathcal{X})$, defined as $\iota\colon x \mapsto \{x\}$ (that is, point x mapped to the one-point subset $\{x\}$ of $\mathcal{X}$) is distance preserving. Therefore $\mathcal{X}$ is isometric to the set $\iota(\mathcal{X})$ in $\mathcal{H}(\mathcal{X})$.

Note that for a nonempty subset $A \subset \mathcal{X}$, we have $\operatorname{diam} A = 0$ if and only if A is a one-point set. Therefore, from Exercise 2.1.2, it follows that $\iota(\mathcal{X})$ is closed in $\mathcal{H}(\mathcal{X})$.

Hence $\iota(\mathcal{X})$ is compact, as it is a closed subset of a compact space. Since $\mathcal{X}$ is isometric to $\iota(\mathcal{X})$, "only if" part follows. □

To prove "if" part we will need the following two lemmas.

2.1.4. Lemma. *Let $K_1 \supset K_2 \supset \dots$ be a sequence of nonempty compact sets in a metric space $\mathcal{X}$ then $K_\infty = \bigcap_n K_n$ is the Hausdorff limit of K_n; that is, $|K_\infty - K_n|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$.*

Proof. Note that K_∞ is compact; by finite intersection property, K_∞ is nonempty.

If the assertion were false, then there is $\varepsilon > 0$ such that for each n one can choose $x_n \in K_n$ such that $\operatorname{dist}_{K_\infty}(x_n) \geqslant \varepsilon$. Note that $x_n \in K_1$ for each n. Since K_1 is compact, there is a *partial limit*[1] x_∞ of x_n. Clearly $\operatorname{dist}_{K_\infty}(x_\infty) \geqslant \varepsilon$.

On the other hand, since K_n is closed and $x_m \in K_n$ for $m \geqslant n$, we get $x_\infty \in K_n$ for each n. It follows that $x_\infty \in K_\infty$ and therefore $\operatorname{dist}_{K_\infty}(x_\infty) = 0$, a contradiction. □

2.1.5. Lemma. *If $\mathcal{X}$ is a compact metric space then $\mathcal{H}(\mathcal{X})$ is complete.*

Proof. Let (Q_n) be a Cauchy sequence in $\mathcal{H}(\mathcal{X})$. Passing to a subsequence of Q_n we may assume that

❶ $$|Q_n - Q_{n+1}|_{\mathcal{H}(\mathcal{X})} \leqslant \tfrac{1}{10^n}$$

for each n.

[1] Partial limit is a limit of a subsequence.

Set

$$K_n = \left\{ x \in \mathcal{X} \,\middle|\, \operatorname{dist}_{Q_n}(x) \leqslant \tfrac{1}{10^n} \right\}$$

Since $\mathcal{X}$ is compact so is each K_n.

Clearly, $|Q_n - K_n|_{\mathcal{H}(\mathcal{X})} \leqslant \frac{1}{10^n}$ and from ❶, we get $K_n \supset K_{n+1}$ for each n. Set

$$K_\infty = \bigcap_{n=1}^{\infty} K_n.$$

Applying Lemma 2.1.4, we get that $|K_n - K_\infty|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$. Since $|Q_n - K_n|_{\mathcal{H}(\mathcal{X})} \leqslant \frac{1}{10^n}$, we get $|Q_n - K_\infty|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$ — hence the lemma. □

2.1.6. Exercise. *Let $\mathcal{X}$ be a complete metric space and K_n be a sequence of compact sets which converges in the sence of Hausdorff. Show that closure of the union $\bigcup_{n=1}^{\infty} K_n$ is compact.*

Use this to show that in Lemma 2.1.5 compactness of $\mathcal{X}$ can be exchanged to completeness.

Proof of "if" part in 2.1.3. According to Lemma 2.1.5, $\mathcal{H}(\mathcal{X})$ is complete. It remains to show that $\mathcal{H}(\mathcal{X})$ is totally bounded (1.4.1d); that is, given $\varepsilon > 0$ there is a finite ε-net in $\mathcal{H}(\mathcal{X})$.

Choose a finite ε-net A in $\mathcal{X}$. Denote by $\mathcal{A}$ the set of all subsets of A. Note that $\mathcal{A}$ is finite set in $\mathcal{H}(\mathcal{X})$. For each compact set $K \subset \mathcal{X}$, consider the subset K' of all points $a \in A$ such that $\operatorname{dist}_K(a) \leqslant \varepsilon$. Then $K' \in \mathcal{A}$ and $|K - K'|_{\mathcal{H}(\mathcal{X})} \leqslant \varepsilon$. In other words $\mathcal{A}$ is a finite ε-net in $\mathcal{H}(\mathcal{X})$. □

Hausdorff metric defines convergence of compact sets which is more important than metric itself.

2.1.7. Exercise. *Let X and Y be two compact subsets in $\mathbb{R}^2$. Assume $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, is it true that $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, where ∂X denotes the boundary of X.*

Does the converse holds? That is, assume X and Y be two compact subsets in $\mathbb{R}^2$ and $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$; is it true that $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} <$ $< \varepsilon$?

2.2 A variation

It seems that *Hausdorff convergence* was first introduced by Felix Hausdorff [8], and a couple of years later an equivalent definition was given by Wilhelm Blaschke [3].

The following refinement of the definition was introduced by Zdeněk Frolík in [7], and later rediscovered by Robert Wijsman in [14]. This refinement takes an intermediate place between the original Hausdorff convergence and *closed convergence*, also introduced by Hausdorff in [8]; so we still call it Hausdorff convergence.

2.2.1. Definition. *Let (A_n) be a sequence of closed sets in a metric space $\mathcal{X}$. We say that (A_n) converges to a closed set A_∞ in the sense of Hausdorff if* $\operatorname{dist}_{A_n}(x) \to \operatorname{dist}_{A_\infty}(x)$ *for any $x \in \mathcal{X}$.*

For example, suppose $\mathcal{X}$ is the Euclidean plane and A_n is the circle with radius n and center at $(n, 0)$. If we use the standard definition (2.1.1), then the sequence (A_n) diverges, but it converges to the y-axis in the sense of Definition 2.2.1.

The following exercise is analogous to the Blaschke selection theorem (2.1.3).

2.2.2. Exercise. *Let $\mathcal{X}$ be a proper metric space and $(A_n)_{n=1}^{\infty}$ be a sequence of closed sets in $\mathcal{X}$. Assume that for some (and therefore any) point $x \in \mathcal{X}$, the sequence $a_n = \operatorname{dist}_{A_n}(x)$ is bounded. Show that the sequence $(A_n)_{n=1}^{\infty}$ has a convergent subsequence in the sense of Definition 2.2.1.*

2.3 Gromov–Hausdorff metric

The goal of this section is to cook up a metric space out of metric spaces. More precisely, we want to define the so called Gromov–Hausdorff metric on the set of *isometry classes* of compact metric spaces. (Being isometric is an equivalence relation, and an isometry class is an equivalence class with respect to this equivalence relation.)

The obtained metric space will be denoted as $\mathcal{M}$. Given two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, denote by $[\mathcal{X}]$ and $[\mathcal{Y}]$ their isometry classes; that is, $\mathcal{X}' \in [\mathcal{X}]$ if and only if $\mathcal{X}' \stackrel{iso}{=\!=} \mathcal{X}$. Pedantically, the Gromov–Hausdorff distance from $[\mathcal{X}]$ to $[\mathcal{Y}]$ should be denoted as $|[\mathcal{X}] - [\mathcal{Y}]|_{\mathcal{M}}$; but we will often write it as $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}}$ and say (not quite correctly) "$|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}}$ is the Gromov–Hausdorff distance from $\mathcal{X}$ to $\mathcal{Y}$". In other words, from now on the term *metric space* might stands for *isometry class of this metric space.*

The metric on $\mathcal{M}$ is maximal metric such that *the distance between subspaces in a metric space is not greater than the Hausdorff distance between them.* Here is a formal definition:

2.3.1. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact metric spaces. The Gromov–Hausdorff distance $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}}$ between them is defined by the following relation.*

Given $r > 0$, we have that $|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}} < r$ if and only if there exist a metric space $\mathcal{Z}$ and subspaces $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{Z}$ that are isometric to $\mathcal{X}$ and $\mathcal{Y}$ respectively and such that $|\mathcal{X}'-\mathcal{Y}'|_{\mathcal{H}(\mathcal{Z})} < r$. (Here $|\mathcal{X}'-\mathcal{Y}'|_{\mathcal{H}(\mathcal{Z})}$ denotes the Hausdorff distance between sets $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{Z}$.)

Bit later (see **??**) we will show that *Hausdorff metric* is indeed a metric.

We say that a sequence of (isometry classes of) compact metric spaces $\mathcal{X}_n$ *converges in the sense of Gromov–Hausdorff* to the (isometry classes of) compact metric space $\mathcal{X}_\infty$ if $|\mathcal{X}_n-\mathcal{X}_\infty|_{\mathcal{M}} \to 0$ as $n \to \infty$; in this case we write $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$.

Let us discuss few reformulations of the definition.

Metrics on disjoined union. Definition 2.3.1 deals with a huge class of metric spaces, namely, all metric spaces $\mathcal{Z}$ that contain subspaces isometric to $\mathcal{X}$ and $\mathcal{Y}$. It is possible to reduce this class to metrics on the disjoint unions of $\mathcal{X}$ and $\mathcal{Y}$. More precisely,

2.3.2. Proposition. *The Gromov–Hausdorff distance between two compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is the infimum of $r > 0$ such that there exists a metric $|{*}-{*}|_{\mathcal{W}}$ on the disjoint union $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Y}$ such that the restrictions of $|{*}-{*}|_{\mathcal{W}}$ to $\mathcal{X}$ and $\mathcal{Y}$ coincide with $|{*}-{*}|_{\mathcal{X}}$ and $|{*}-{*}|_{\mathcal{Y}}$ and $|\mathcal{X}-\mathcal{Y}|_{\mathcal{H}(\mathcal{W})} < r$.*

Proof. Identify $\mathcal{X} \sqcup \mathcal{Y}$ with $\mathcal{X}' \cup \mathcal{Y}' \subset \mathcal{Z}$ (the notation is from Definition 2.3.1).

More formally, fix isometries $f\colon \mathcal{X} \to \mathcal{X}'$ and $g\colon \mathcal{Y} \to \mathcal{Y}'$, then define the distance between $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ by $|x-y|_{\mathcal{W}} = |f(x)-g(y)|_{\mathcal{Z}} + \varepsilon$ for small enuf $\varepsilon > 0$.[2] This yields a metric on $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Y}$ for which $|\mathcal{X}-\mathcal{Y}|_{\mathcal{H}(\mathcal{W})} < r$. □

Fixed ambient space. The following proposition says that the space $\mathcal{Z}$ in Definition 2.3.1 can be exchanged to a fixed space, namely ℓ^∞ — the space of bounded infinite sequences with the metric defined by sup-norm.

2.3.3. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be comact metric spaces. Then*

$$|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}} = \inf\{|\mathcal{X}'-\mathcal{Y}'|_{\mathcal{H}(\ell^\infty)}\}$$

where the infimum is taken over all pairs of sets $\mathcal{X}'$ and $\mathcal{Y}'$ in ℓ^∞ which isometric to $\mathcal{X}$ and $\mathcal{Y}$ correspondingly.

[2] We add ε to ensure that $d(x,y) > 0$ for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$; so $|x-y|_{\mathcal{W}}$ is indeed a metric.

Proof of 2.3.3. By the definition, we have that

$$|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} \le \inf\{|\mathcal{X}' - \mathcal{Y}'|_{\mathcal{H}(\ell^\infty)}\}.$$

Let $\mathcal{W}$ be an arbitrary metric space with the underlying set $\mathcal{X} \sqcup \mathcal{Y}$. Note $\mathcal{W}$ is compact since it is union of two compact subsets $\mathcal{X}, \mathcal{Y} \subset \mathcal{W}$. In particular, $\mathcal{W}$ is separable.

By Lemma 1.8.1, there is an distance preserving embedding $\iota\colon \mathcal{W} \to \to \ell^\infty$. It remains to apply Proposition 2.3.2. □

2.4 Almost isometries

2.4.1. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces and $\varepsilon > 0$. A map*[3] *$f\colon \mathcal{X} \to \mathcal{Y}$ is called an ε-isometry if*

$$|f(x) - f(x')|_{\mathcal{Y}} \lessgtr |x - x'|_{\mathcal{X}} \pm \varepsilon$$

for any $x, x' \in \mathcal{X}$ and if $f(\mathcal{X})$ is an ε-net in $\mathcal{Y}$.

2.4.2. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ be two ε-isometries. Show that $g \circ f\colon \mathcal{X} \to \mathcal{Z}$ is a $(3{\cdot}\varepsilon)$-isometry.*

2.4.3. Exercise. *Assume $f\colon \mathcal{X} \to \mathcal{Y}$ is an ε-isometry. Show that there is a $(3{\cdot}\varepsilon)$-isometry $g\colon \mathcal{Y} \to \mathcal{X}$.*

2.4.4. Exercise. *Assume $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} < \varepsilon$, show that there is a $(2{\cdot}\varepsilon)$-isometry $f\colon \mathcal{X} \to \mathcal{Y}$.*

2.4.5. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces and let $f\colon \mathcal{X} \to \mathcal{Y}$ be an ε-isometry. Then*

$$|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} \leqslant 2{\cdot}\varepsilon.$$

Proof. Consider the set $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Y}$. Note that the following defines a metric on $\mathcal{W}$:

◇ For any $x, x' \in \mathcal{X}$

$$|x - x'|_{\mathcal{W}} = |x - x'|_{\mathcal{X}};$$

◇ For any $y, y' \in \mathcal{Y}$,

$$|y - y'|_{\mathcal{W}} = |y - y'|_{\mathcal{Y}}$$

[3] possibly noncontinuous

◇ For any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,

$$|x - y|_{\mathcal{W}} = \varepsilon + \inf_{x' \in \mathcal{X}} \{|x - x'|_{\mathcal{X}} + |f(x') - y|_{\mathcal{Y}}\}.$$

Since $f(\mathcal{X})$ is an ε-net in $\mathcal{Y}$, for any $y \in \mathcal{Y}$ there is $x \in \mathcal{X}$ such that $|f(x) - y|_{\mathcal{Y}} \leqslant \varepsilon$; therefore $|x - y|_{\mathcal{W}} \leqslant 2{\cdot}\varepsilon$. On the other hand for any $x \in \mathcal{X}$, we have $|x - y|_{\mathcal{W}} \leqslant \varepsilon$ for $y = f(x) \in \mathcal{Y}$.

It follows that $|\mathcal{X} - \mathcal{Y}|_{\mathcal{H}(\mathcal{W})} \leqslant 2{\cdot}\varepsilon$. □

Appendix A

Semisolutions

Exercise 1.4.3. Given any pair of point $x_0, y_0 \in \mathcal{K}$, consider two sequences $x_0, x_1, \dots$ and $y_0, y_1, \dots$ such that $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ for each n.

Since $\mathcal{K}$ is compact, we can choose an increasing sequence of integers n_k such that both sequences $(x_{n_i})_{i=1}^{\infty}$ and $(y_{n_i})_{i=1}^{\infty}$ converge. In particular, both are Cauchy sequences; that is,

$$|x_{n_i} - x_{n_j}|_{\mathcal{K}}, |y_{n_i} - y_{n_j}|_{\mathcal{K}} \to 0 \quad \text{as} \quad \min\{i, j\} \to \infty.$$

Since f is non-contracting, we get

$$|x_0 - x_{|n_i - n_j|}| \leqslant |x_{n_i} - x_{n_j}|.$$

It follows that there is a sequence $m_i \to \infty$ such that

$$(*) \qquad x_{m_i} \to x_0 \quad \text{and} \quad y_{m_i} \to y_0 \quad \text{as} \quad i \to \infty.$$

Set

$$\ell_n = |x_n - y_n|_{\mathcal{K}}.$$

Since f is non-contracting, the sequence (ℓ_n) is non-decreasing.

By $(*)$, $\ell_{m_i} \to \ell_0$ as $m_i \to \infty$. It follows that (ℓ_n) is a constant sequence.

In particular

$$|x_0 - y_0|_{\mathcal{K}} = \ell_0 = \ell_1 = |f(x_0) - f(y_0)|_{\mathcal{K}}$$

for any pair of points (x_0, y_0) in $\mathcal{K}$. That is, f is distance preserving, in particular injective.

From $(*)$, we also get that $f(\mathcal{K})$ is everywhere dense. Since $\mathcal{K}$ is compact $f\colon \mathcal{K} \to \mathcal{K}$ is surjective. Hence the result follows. $\square$

This is a basic lemma in the introduction to Gromov–Hausdorff distance [see 7.3.30 in 5]. I learned this proof from Travis Morrison, a student in my MASS class at Penn State, Fall 2011.

Note that as an easy corollary one can see that any surjective non-expanding map from a compact metric space to itself is an isometry.

Exercise 1.7.2. We assume that the space is not trivial, otherwise a one-point space is an example.

Consider the unit ball (B,ρ_0) in the space c_0 of all sequences converging to zero equipped with the sup-norm.

Consider another metric ρ_1 which is different from ρ_0 by the conformal factor

$$\varphi(\boldsymbol{x}) = 2 + \tfrac{1}{2}\cdot x_1 + \tfrac{1}{4}\cdot x_2 + \tfrac{1}{8}\cdot x_3 + \dots,$$

where $\boldsymbol{x} = (x_1, x_2 \dots) \in B$. That is, if $\boldsymbol{x}(t)$, $t \in [0,\ell]$, is a curve parametrized by ρ_0-length then its ρ_1-length is

$$\operatorname{length}_{\rho_1} \boldsymbol{x} = \int_0^\ell \varphi\circ\boldsymbol{x}.$$

Note that the metric ρ_1 is bi-Lipschitz to ρ_0.

Assume $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t)$ are two curves parametrized by ρ_0-length that differ only in the m-th coordinate, denoted by $x_m(t)$ and $x'_m(t)$ correspondingly. Note that if $x'_m(t) \leqslant x_m(t)$ for any t and the function $x'_m(t)$ is locally 1-Lipschitz at all t such that $x'_m(t) < x_m(t)$, then

$$\operatorname{length}_{\rho_1} \boldsymbol{x}' \leqslant \operatorname{length}_{\rho_1} \boldsymbol{x}.$$

Moreover this inequality is strict if $x'_m(t) < x_m(t)$ for some t.

Fix a curve $\boldsymbol{x}(t)$, $t \in [0,\ell]$, parametrized by ρ_0-length. We can choose m large, so that $x_m(t)$ is sufficiently close to 0 for any t. In particular, for some values t, we have $y_m(t) < x_m(t)$, where

$$y_m(t) = (1-\tfrac{t}{\ell})\cdot x_m(0) + \tfrac{t}{\ell}\cdot x_m(\ell) - \tfrac{1}{100}\cdot\min\{t, \ell - t\}.$$

Consider the curve $\boldsymbol{x}'(t)$ as above with

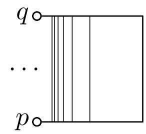
$$x'_m(t) = \min\{x_m(t), y_m(t)\}.$$

Note that $\boldsymbol{x}'(t)$ and $\boldsymbol{x}(t)$ have the same end points, and by the above

$$\operatorname{length}_{\rho_1} \boldsymbol{x}' < \operatorname{length}_{\rho_1} \boldsymbol{x}.$$

That is, for any curve $\boldsymbol{x}(t)$ in (B,ρ_1), we can find a shorter curve $\boldsymbol{x}'(t)$ with the same end points. In particular, (B,ρ_1) has no geodesics. $\square$

This example was suggested by Fedor Nazarov [10].



Exercise 1.7.8. Consider the following subset of $\mathbb{R}^2$ equipped with the induced length metric

$$\mathcal{X} = \big((0,1] \times \{0,1\}\big) \cup \big(\{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\} \times [0,1]\big)$$

Note that $\mathcal{X}$ is locally compact and geodesic.

Its completion $\bar{\mathcal{X}}$ is isometric to the closure of $\mathcal{X}$ equipped with the induced length metric; $\bar{\mathcal{X}}$ is obtained from $\mathcal{X}$ by adding two points $p = (0,0)$ and $q = (0,1)$.

The point p admits no compact neighborhood in $\bar{\mathcal{X}}$ and there is no geodesic connecting p to q in $\bar{\mathcal{X}}$. □

This exercise and solution is taken from [4].

Exercise 2.1.7. The answer is "no" in both parts.

For the first part let X be a unit disc and Y a finite ε-net in X. Evidently $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, but $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} \approx 1$.

For the second part take X to be a unit disc and $Y = \partial X$ to be its boundary circle. Note that $\partial X = \partial Y$ in particular $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} = 0$ while $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} = 1$. □

A more interesting example for the second part can be build on *lakes of Wada* — and example of three open bounded topological disks in the plane that have identical boundary.

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