Lectures on metric geometry

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Part I

Pure metric geometry

Lecture 1

Definitions

In this lecture we give some conventions used further and remind some the definitions related to metric spaces.

1.1 Metric spaces

The distance between two points x and y in a metric space $\mathcal{X}$ will be denoted by $|x-y|$ or $|x-y|_{\mathcal{X}}$. The latter notation is used if we need to emphasize that the distance is taken in the space $\mathcal{X}$. Let us recall the definition of metric.

1.1.1. Definition. *A* metric *on a set $\mathcal{X}$ is a real-valued function $(x,y) \mapsto |x-y|_{\mathcal{X}}$ that satisfies the following conditions for any three points $x, y, z \in \mathcal{X}$:*
(i) $|x-y|_{\mathcal{X}} \geqslant 0$,
(ii) $|x-y|_{\mathcal{X}} = 0 \iff x = y$,
(iii) $|x-y|_{\mathcal{X}} = |y-x|_{\mathcal{X}}$,
(iv) $|x-y|_{\mathcal{X}} + |y-z|_{\mathcal{X}} \geqslant |x-z|_{\mathcal{X}}$,
A set $\mathcal{X}$ with a metric on it is called metric space*; most of the time we keep the same notation for the metric space and its underlying set.*

1.1.2. Exercise. *Let p, q, x, and y be points in a metric space $\mathcal{X}$. Show that*

$$|p-q|_{\mathcal{X}} + |x-y|_{\mathcal{X}} \leqslant |p-x|\mathcal{X} + |p-y|_{\mathcal{X}} + |q-x|_{\mathcal{X}} + |q-y|_{\mathcal{X}}.$$

The function

$$\operatorname{dist}_x\colon y \mapsto |x-y|$$

is called the *distance function* from x.

Given $R \in [0, \infty]$ and $x \in \mathcal{X}$, the sets

$$\mathrm{B}(x, R) = \{y \in \mathcal{X} \mid |x - y| < R\},$$
$$\overline{\mathrm{B}}[x, R] = \{y \in \mathcal{X} \mid |x - y| \leqslant R\}$$

are called, respectively, the *open* and the *closed balls* of radius R with center x. Again, if we need to emphasize that these balls are taken in the metric space $\mathcal{X}$, we write

$$\mathrm{B}(x, R)_{\mathcal{X}} \quad \text{and} \quad \overline{\mathrm{B}}[x, R]_{\mathcal{X}}.$$

1.2 Variations of definition

Pseudometrics. A metric for which the distance between two distinct points can be zero is called a *pseudometric*. In other words, to define pseudometric, we need to remove condition (ii) from 1.1.1.

The following observation show that nearly any question about pseudometric spaces can be reduced to a question about genuine metric spaces.

Assume $\mathcal{X}$ is a pseudometric space. Consider an equivalence relation $\sim$ on $\mathcal{X}$ defined by $x \sim y$ if and only if $|x - y| = 0$. Note that if $x \sim x'$, then $|y - x| = |y - x'|$ for any $y \in \mathcal{X}$. Thus, $|* - *|$ defines a metric on the quotient set $\mathcal{X}/\!\sim$. This way we obtain a metric space $\mathcal{X}'$. The space $\mathcal{X}'$ is called the *corresponding metric space* for the pseudometric space $\mathcal{X}$. Often we do not distinguish between $\mathcal{X}'$ and $\mathcal{X}$.

∞-metrics. One may also consider metrics with values in $\mathbb{R} \cup \{\infty\}$; we might call them ∞-*metrics*, but most of the time we use the term *metric*.

Again nearly any question about ∞-metric spaces can be reduced to a question about genuine metric spaces.

Indeed, set $x \approx y$ if and only if $|x - y| < \infty$; this is an other equivalence relation on $\mathcal{X}$. The equivalence class of a point $x \in \mathcal{X}$ will be called the *metric component* of x; it will be denoted by $\mathcal{X}_x$. One could think of $\mathcal{X}_x$ as $\mathrm{B}(x, \infty)_{\mathcal{X}}$ — the open ball centered at x and radius ∞ in $\mathcal{X}$.

It follows that any ∞-metric space is a *disjoint union* of genuine metric spaces — the metric components of the original ∞-metric space.

1.2.1. Exercise. *Given two sets A and B on the plane, set*

$$|A - B| = \mu(A \backslash B) + \mu(B \backslash A),$$

where μ denotes the Lebesgue measure.

(a) *Show that* $|{*}-{*}|$ *is a pseudometric on the set of bounded measurable sets of the plane.*
(b) *Show that* $|{*}-{*}|$ *is an* ∞*-metric on the set of all open sets of the plane.*

1.3 Completeness

A metric space $\mathcal{X}$ is called *complete* if every Cauchy sequence of points in $\mathcal{X}$ converges in $\mathcal{X}$.

1.3.1. Exercise. *Suppose that* ρ *is a positive continuous function on a complete metric space* $\mathcal{X}$. *Show that for any* $\varepsilon > 0$ *there is a point* $x \in \mathcal{X}$ *such that*

$$\rho(x) < (1+\varepsilon)\cdot\rho(y)$$

for any point $y \in \mathrm{B}(x, \rho(x))$.

Most of the time we will assume that a metric space is complete. The following construction produces a complete metric space $\bar{\mathcal{X}}$ for any given metric space $\mathcal{X}$.

Completion. Given metric space $\mathcal{X}$, consider the set $\mathcal{C}$ of all Cauchy sequences in $\mathcal{X}$. Note that for any two Cauchy sequences (x_n) and (y_n) the right hand side in ❶ is defined; moreover it defines a pseudometric on $\mathcal{C}$

❶ $$|(x_n)-(y_n)|_{\mathcal{C}} := \lim_{n\to\infty} |x_n - y_n|_{\mathcal{X}}.$$

The corresponding metric space $\bar{\mathcal{X}}$ is called a *completion* of $\mathcal{X}$.

Note that the original space $\mathcal{X}$ forms a dense subset in its completion $\bar{\mathcal{X}}$. More precisely, for each point $x \in \mathcal{X}$ one can consider a constant sequence $x_n = x$ which is Cauchy. It defines a natural map $\mathcal{X} \to \bar{\mathcal{X}}$. It is easy to check that this map is distance-preserving. In partucular we can (and will) consider $\mathcal{X}$ as a subset of $\bar{\mathcal{X}}$.

1.3.2. Exercise. *Show that completion of a metric space is complete.*

1.4 Compact spaces

Let us recall few equivalent definitions of compact metric spaces.

1.4.1. Definition. *A metric space* $\mathcal{K}$ *is compact if and only if one of the following equivalent condition holds:*
(a) *Every open cover of* $\mathcal{K}$ *has a finite subcover.*

(b) *For any open cover of $\mathcal{K}$ there is $\varepsilon > 0$ such that any ε-ball in $\mathcal{K}$ lie in one element of the cover. (The value ε is called a* Lebesgue number *of the covering.)*

(c) *Every sequence of points in $\mathcal{K}$ has a subsequence that converges in $\mathcal{K}$.*

(d) *The space $\mathcal{K}$ is complete and* totally bounded*; that is, for any $\varepsilon > 0$, the space $\mathcal{K}$ admits a finite cover by open ε-balls.*

A subset N of a metric space $\mathcal{K}$ is called *ε-net* if any other point x lies on the distance less than ε from a point in N. Note that totally bounded spaces can be defined as spaces that admit a finite ε-net for any $\varepsilon > 0$.

1.4.2. Exercise. *Show that a space $\mathcal{K}$ is totally bounded if and only if it contains a compact ε-net for any $\varepsilon > 0$.*

Let $\operatorname{pack}_\varepsilon \mathcal{X}$ be exact upper bound on the number of points $x_1, \dots$ $\dots, x_n \in \mathcal{X}$ such that $|x_i - x_j| \geqslant \varepsilon$ if $i \neq j$.

If $n = \operatorname{pack}_\varepsilon \mathcal{X} < \infty$, then the collection of points $x_1, \dots, x_n$ is called a *maximal ε-packing.* Note that n is the maximal number of open disjoint $\frac{\varepsilon}{2}$-balls in $\mathcal{X}$.

1.4.3. Exercise. *Show that a complete space $\mathcal{X}$ is compact if and only of* $\operatorname{pack}_\varepsilon \mathcal{X} < \infty$ *for any $\varepsilon > 0$.*

Show that any maximal ε-packing is an ε-net.

1.4.4. Exercise. *Let $\mathcal{K}$ be a compact metric space and*

$$f\colon \mathcal{K} \to \mathcal{K}$$

be a distance-nondecreasing map. Prove that f is an isometry*; that is, f is a distance-preserving bijection.*

1.4.5. Advanced exercise. *Show that for any compact length-metric space X there is unique number $\ell = \ell(X)$ such that for any finite collection of points there is a point z that lies of average distance ℓ from the collection; that is, for any $x_1, \dots, x_n \in X$ there is $z \in X$ such that*

$$\tfrac{1}{n} \cdot \sum_i |x_i - z|_X = \ell.$$

1.5 Proper spaces

A metric space $\mathcal{X}$ is called *proper* if all closed bounded sets in $\mathcal{X}$ are compact. This condition is equivalent to each of the following statements:

- ◇ For some (and therefore any) point $p \in \mathcal{X}$ and any $R < \infty$, the closed ball $\overline{\mathrm{B}}[p, R]_{\mathcal{X}}$ is compact.
- ◇ The function $\mathrm{dist}_p \colon \mathcal{X} \to \mathbb{R}$ is proper for some (and therefore any) point $p \in \mathcal{X}$; that is, for any compact set $K \subset \mathbb{R}$, its inverse image

$$\mathrm{dist}_p^{-1}(K) = \{\, x \in \mathcal{X} \;:\; |p - x|_{\mathcal{X}} \in K \,\}$$

  is compact.

A metric space $\mathcal{X}$ is called *locally compact* if any point in $\mathcal{X}$ admits a compact neighborhood; in other words, for any point $x \in \mathcal{X}$ a closed ball $\overline{\mathrm{B}}[x, r]$ is compact for some $r > 0$.

1.6 Geodesics

Let $\mathcal{X}$ be a metric space and $\mathbb{I}$ a real interval. A globally isometric map $\gamma \colon \mathbb{I} \to \mathcal{X}$ is called a *geodesic*[1]; in other words, $\gamma \colon \mathbb{I} \to \mathcal{X}$ is a geodesic if

$$|\gamma(s) - \gamma(t)|_{\mathcal{X}} = |s - t|$$

for any pair $s, t \in \mathbb{I}$.

We say that $\gamma \colon \mathbb{I} \to \mathcal{X}$ is a geodesic from point p to point q if $\mathbb{I} = [a, b]$ and $p = \gamma(a)$, $q = \gamma(b)$. In this case the image of γ is denoted by $[pq]$ and with an abuse of notations we also call it a *geodesic*.

We may write $[pq]_{\mathcal{X}}$ to emphasize that the geodesic $[pq]$ is in the space $\mathcal{X}$. We also use the following shortcut notation:

$$]pq[\,=\, [pq]\backslash\{p, q\}, \qquad]pq] \,=\, [pq]\backslash\{p\}, \qquad [pq[\,=\, [pq]\backslash\{q\}.$$

In general, a geodesic from p to q need not exist and if it exists, it need not be unique. However, once we write $[pq]$ we assume mean that we have made a choice of geodesic.

A metric space is called *geodesic* if any pair of its points can be joined by a geodesic.

A *geodesic path* is a geodesic with constant-speed parametrization by $[0, 1]$.

A curve $\gamma \colon \mathbb{I} \to \mathcal{X}$ is called a *local geodesic* if for any $t \in \mathbb{I}$ there is a neighborhood U of t in $\mathbb{I}$ such that the restriction $\gamma|_U$ is a geodesic. A constant-speed parametrization of a local geodesic by the unit interval $[0, 1]$ is called a *local geodesic path*.

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[1] Various authors call it differently: *shortest path*, *minimizing geodesic*.

1.7 Metric trees

A geodesic space $\mathcal{T}$ is called a *metric tree* if any pair of points in $\mathcal{T}$ are connected by a unique geodesic, and the union of any two geodesics $[xy]$, and $[yz]$ contain the geodesic $[xz]_{\mathcal{T}}$. In other words any triangle in $\mathcal{T}$ is a tripod; that is for any three geodesics $[xy]$, $[yz]$, and $[zx]$ have a common point.

1.7.1. Exercise. *Show that spheres in metric trees are ultrametric spaces; that is, if Σ is a sphere in a metric tree $\mathcal{T}$, then*

$$|x-z|_{\mathcal{T}} \leqslant \max\{\,|x-y|_{\mathcal{T}}, |y-z|_{\mathcal{T}}\,\}$$

for any $x, y, z \in \Sigma$.

1.8 Length

A *curve* is defined as a continuous map from a real interval to a space. If the real interval is $[0,1]$, then the curve is called a *path*.

1.8.1. Definition. *Let $\mathcal{X}$ be a metric space and $\alpha\colon \mathbb{I} \to \mathcal{X}$ be a curve. We define the* length *of α as*

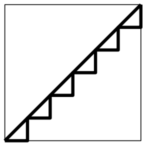
$$\operatorname{length}\alpha := \sup_{t_0\leqslant t_1\leqslant\ldots\leqslant t_n} \sum_i |\alpha(t_i)-\alpha(t_{i-1})|.$$

A curve α is called rectifiable *if* $\operatorname{length}\alpha < \infty$.

1.8.2. Theorem. *Length is a lower semi-continuous with respect to pointwise convergence of curves.*

More precisely, assume that a sequence of curves $\gamma_n\colon \mathbb{I}\to\mathcal{X}$ in a metric space $\mathcal{X}$ converges pointwise to a curve $\gamma_\infty\colon\mathbb{I}\to\mathcal{X}$; that is, for any fixed $t\in\mathbb{I}$, $\gamma_n(t)\to\gamma_\infty(t)$ as $n\to\infty$. Then

❶ $$\varliminf_{n\to\infty} \operatorname{length}\gamma_n \geqslant \operatorname{length}\gamma_\infty.$$



Note that the inequality ❶ might be strict. For example the diagonal γ_∞ of the unit square can be approximated by a stairs-like polygonal curves γ_n with sides parallel to the sides of the square (γ_6 is on the picture). In this case

$$\operatorname{length}\gamma_\infty = \sqrt{2} \quad\text{and}\quad \operatorname{length}\gamma_n = 2$$

for any n.

Proof. Fix a sequence $t_0 < t_1 < \dots < t_k$ in $\mathbb{I}$. Set

$$\begin{aligned}\Sigma_n &:= |\gamma_n(t_0) - \gamma_n(t_1)| + \dots + |\gamma_n(t_{k-1}) - \gamma_n(t_k)|.\\ \Sigma_\infty &:= |\gamma_\infty(t_0) - \gamma_\infty(t_1)| + \dots + |\gamma_\infty(t_{k-1}) - \gamma_\infty(t_k)|.\end{aligned}$$

Note that for each i we have

$$|\gamma_n(t_{i-1}) - \gamma_n(t_i)| \to |\gamma_\infty(t_{i-1}) - \gamma_\infty(t_i)|$$

and therefore

$$\Sigma_n \to \Sigma_\infty$$

as $n \to \infty$. Note that

$$\Sigma_n \leqslant \operatorname{length}\gamma_n$$

for each n. Hence

❷ $$\varliminf_{n\to\infty} \operatorname{length}\gamma_n \geqslant \Sigma_\infty.$$

If γ_∞ is rectifiable, we can assume that

$$\operatorname{length}\gamma_\infty < \Sigma_\infty + \varepsilon.$$

for any given $\varepsilon > 0$. By ❷ it follows that

$$\varliminf_{n\to\infty} \operatorname{length}\gamma_n > \operatorname{length}\gamma_\infty - \varepsilon$$

for any $\varepsilon > 0$; whence ❶ follows.

It remains to consider the case when γ_∞ is not rectifiable; that is, $\operatorname{length}\gamma_\infty = \infty$. In this case we can choose a partition so that $\Sigma_\infty > L$ for any real number L. By ❷ it follows that

$$\varliminf_{n\to\infty} \operatorname{length}\gamma_n > L$$

for any given L; whence

$$\varliminf_{n\to\infty} \operatorname{length}\gamma_n = \infty$$

and ❶ follows. □

1.9 Length spaces

If for any $\varepsilon > 0$ and any pair of points x and y in a metric space $\mathcal{X}$, there is a path α connecting x to y such that

$$\operatorname{length}\alpha < |x - y| + \varepsilon,$$

then $\mathcal{X}$ is called a *length space* and the metric on $\mathcal{X}$ is called a *length metric.*

If $\mathcal{X}$ is an ∞-metric space, then we assume that x and y lie in one metric component; that is, $|x-y|_{\mathcal{X}} < \infty$. In other words an ∞-metric space $\mathcal{X}$ is a length space if each metric component of $\mathcal{X}$ is a length space.

Note that any geodesic space is a length space. As can be seen from the following example, the converse does not hold.

1.9.1. Example. *Let $\mathcal{X}$ be obtained by gluing a countable collection of disjoint intervals $\{\mathbb{I}_n\}$ of length $1+\frac{1}{n}$, where for each $\mathbb{I}_n$ the left end is glued to p and the right end to q.*

Observe that the space $\mathcal{X}$ carries a natural complete length metric with respect to which $|p-q| = 1$ but there is no geodesic connecting p to q.

1.9.2. Exercise. *Give an example of a complete length space $\mathcal{X}$ such that no pair of distinct points in $\mathcal{X}$ can be joined by a geodesic.*

Directly from the definition, it follows that if a path $\alpha\colon [0,1] \to \mathcal{X}$ connects two points x and y (that is, if $\alpha(0) = x$ and $\alpha(1) = y$), then

$$\operatorname{length}\alpha \geqslant |x-y|.$$

Set

$$\|x-y\| = \inf\{\operatorname{length}\alpha\}$$

where the greatest lower bound is taken for all paths connecing x and y. It is straightforward to check that $(x,y) \mapsto \|x-y\|$ is an ∞-metric; moreover $(\mathcal{X}, \|{*}-{*}\|)$ is a length space. The metric $\|{*}-{*}\|$ is called *induced length metric.*

1.9.3. Exercise. *Let $\mathcal{X}$ be a length space. Show that for any compact subset K in $\mathcal{X}$ there is a compact path connected subset $K' \supset K$.*

1.9.4. Exercise. *Suppose $(\mathcal{X}, |{*}-{*}|)$ is a complete metric space. Show that $(\mathcal{X}, \|{*}-{*}\|)$ is complete.*

Let A be a subset of a metric space $\mathcal{X}$. Given two points $x, y \in A$, consider the value

$$|x-y|_A = \inf_{\alpha}\{\operatorname{length}\alpha\},$$

where the greatest lower bound is taken for all paths α from x to y in A. In other words $|{*}-{*}|_A$ denotes the induced length metric on the subspace A.[2]

[2] This notation slightly conflicts with the previously defined notation for distance $|x-y|_{\mathcal{X}}$ in a metric space $\mathcal{X}$. However, most of the time we will work with ambient length spaces where the meaning will be unambiguous.

Let $\mathcal{X}$ be a metric space and $x, y \in \mathcal{X}$.

(i) A point $z \in \mathcal{X}$ is called a *midpoint* between x and y if

$$|x - z| = |y - z| = \tfrac{1}{2}\cdot|x - y|.$$

(ii) Assume $\varepsilon \geqslant 0$. A point $z \in \mathcal{X}$ is called an ε-*midpoint* between x and y if

$$|x - z|, \quad |y - z| \leqslant \tfrac{1}{2}\cdot|x - y| + \varepsilon.$$

Note that a 0-midpoint is the same as a midpoint.

1.9.5. Lemma. *Let $\mathcal{X}$ be a complete metric space.*

(a) Assume that for any pair of points $x, y \in \mathcal{X}$ and any $\varepsilon > 0$ there is an ε-midpoint z. Then $\mathcal{X}$ is a length space.

(b) Assume that for any pair of points $x, y \in \mathcal{X}$, there is a midpoint z. Then $\mathcal{X}$ is a geodesic space.

Proof. We first prove *(a)*. Let $x, y \in \mathcal{X}$ be a pair of points.

Set $\varepsilon_n = \frac{\varepsilon}{4^n}$, $\alpha(0) = x$ and $\alpha(1) = y$.

Let $\alpha(\frac{1}{2})$ be an ε_1-midpoint between $\alpha(0)$ and $\alpha(1)$. Further, let $\alpha(\frac{1}{4})$ and $\alpha(\frac{3}{4})$ be ε_2-midpoints between the pairs $(\alpha(0), \alpha(\frac{1}{2}))$ and $(\alpha(\frac{1}{2}), \alpha(1))$ respectively. Applying the above procedure recursively, on the n-th step we define $\alpha(\frac{k}{2^n})$, for every odd integer k such that $0 < \frac{k}{2^n} < 1$, as an ε_n-midpoint between the already defined $\alpha(\frac{k-1}{2^n})$ and $\alpha(\frac{k+1}{2^n})$.

In this way we define $\alpha(t)$ for $t \in W$, where W denotes the set of dyadic rationals in $[0, 1]$. Since $\mathcal{X}$ is complete, the map α can be extended continuously to $[0, 1]$. Moreover,

❶
$$\begin{aligned} \operatorname{length}\alpha &\leqslant |x - y| + \sum_{n=1}^{\infty} 2^{n-1}\cdot\varepsilon_n \leqslant \\ &\leqslant |x - y| + \tfrac{\varepsilon}{2}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get *(a)*.

To prove *(b)*, one should repeat the same argument taking midpoints instead of ε_n-midpoints. In this case ❶ holds for $\varepsilon_n = \varepsilon = 0$. $\square$

Since in a compact space a sequence of $\frac{1}{n}$-midpoints z_n contains a convergent subsequence, Lemma 1.9.5 immediately implies

1.9.6. Proposition. *A proper length space is geodesic.*

1.9.7. Hopf–Rinow theorem. *Any complete, locally compact length space is proper.*

It is instructive to solve the following exercise before reading the proof.

1.9.8. Exercise. *Give an example of space which is locally compact but not proper.*

Proof. Let $\mathcal{X}$ be a locally compact length space. Given $x \in \mathcal{X}$, denote by $\rho(x)$ the supremum of all $R > 0$ such that the closed ball $\overline{\mathrm{B}}[x, R]$ is compact. Since $\mathcal{X}$ is locally compact,

❷ $$\rho(x) > 0 \quad \text{for any} \quad x \in \mathcal{X}.$$

It is sufficient to show that $\rho(x) = \infty$ for some (and therefore any) point $x \in \mathcal{X}$.

Assume the contrary; that is, $\rho(x) < \infty$. We claim that

❸ $B = \overline{\mathrm{B}}[x, \rho(x)]$ *is compact for any* x.

Indeed, $\mathcal{X}$ is a length space; therefore for any $\varepsilon > 0$, the set $\overline{\mathrm{B}}[x, \rho(x) - \varepsilon]$ is a compact ε-net in B. Since B is closed and hence complete, it must be compact. △

Next we claim that

❹ $|\rho(x) - \rho(y)| \leqslant |x - y|_{\mathcal{X}}$ *for any* $x, y \in \mathcal{X}$*; in particular* $\rho\colon \mathcal{X} \to \mathbb{R}$ *is a continuous function.*

Indeed, assume the contrary; that is, $\rho(x) + |x - y| < \rho(y)$ for some $x, y \in \mathcal{X}$. Then $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$ is a closed subset of $\overline{\mathrm{B}}[y, \rho(y)]$ for some $\varepsilon > 0$. Then compactness of $\overline{\mathrm{B}}[y, \rho(y)]$ implies compactness of $\overline{\mathrm{B}}[x, \rho(x) + \varepsilon]$, a contradiction. △

Set $\varepsilon = \min\{\, \rho(y) : y \in B \,\}$; the minimum is defined since B is compact. From ❷, we have $\varepsilon > 0$.

Choose a finite $\frac{\varepsilon}{10}$-net $\{a_1, a_2, \dots, a_n\}$ in B. The union W of the closed balls $\overline{\mathrm{B}}[a_i, \varepsilon]$ is compact. Clearly $\overline{\mathrm{B}}[x, \rho(x) + \frac{\varepsilon}{10}] \subset W$. Therefore $\overline{\mathrm{B}}[x, \rho(x) + \frac{\varepsilon}{10}]$ is compact, a contradiction. □

1.9.9. Exercise. *Construct a geodesic space $\mathcal{X}$ that is locally compact, but whose completion $\bar{\mathcal{X}}$ is neither geodesic nor locally compact.*

Lecture 2

Universal space

In this lecture we introduce a construction of Maurice René Fréchet [**frechet**] and its variation given by Kazimierz Kuratowski [**kuratowski**]. It produces a distance-preserving map from any metric space to a normed space.

Further we discuss a construction of Pavel Urysohn [**urysohn**] which answers a question of Maurice Fréchet. It produces a separable metric space that includes a subspace isometric to any separable metric space; in addition it is homogenius in a very strong sense, see 2.4.4.

The idea of this construction was reused in graph theory; it produces the so called *Rado graph*, also known as *Erdős–Rényi graph* or *random graph*; a good survey on the subject is given by Peter Cameron [**cameron**].

We follow presentation given by Mikhael Gromov [**gromov-2007**].

2.1 Embedding in a normed space

Recall that a function $v \mapsto |v|$ on a vector space $\mathcal{V}$ is called *norm* if it satisfies the following condition for any two vectors $v, w \in \mathcal{V}$ and a scalar α:

- $|v| \geqslant 0$;
- $|\alpha{\cdot}v| = |\alpha|{\cdot}|v|$;
- $|v| + |w| \geqslant |v + w|$.

It is straightforward to check that for any normed space the function $(v, w) \mapsto |v - w|$ defines a metric on it. Therefore any normed space is an example of metric space (in fact it is a geodesic space). The following lemma says in particular that any separable metric space is isometric to a subset of a normed space.

Recall that *diameter* of a metric space $\mathcal{X}$ (briefly $\operatorname{diam}\mathcal{X}$) is defined as least upper bound on the distances between pairs of its points; that is,

$$\operatorname{diam}\mathcal{X} = \sup\{\, |x-y|_{\mathcal{X}} \;:\; x,y\in\mathcal{X}\,\}.$$

2.1.1. Lemma. *Suppose $\mathcal{X}$ is a* bounded separable *metric space; that is,* $\operatorname{diam}\mathcal{X}$ *is finite and $\mathcal{X}$ contains a countable, dense set $\{w_n\}$. Given $x\in\mathcal{X}$, set $a_n(x)=|w_n-x|_{\mathcal{X}}$. Then*

$$\iota\colon x\mapsto (a_1(x),a_2(x),\dots)$$

defines a distance-preserving embedding $\iota\colon\mathcal{X}\hookrightarrow\ell^\infty$.

Proof. By the triangle inequality

$$|a_n(x)-a_n(y)|\leqslant |x-y|_{\mathcal{X}}.$$

Therefore ι is *short*; that is, ι is distance-nonexpanding.

Again by triangle inequality we have

$$|a_n(x)-a_n(y)|\geqslant |x-y|_{\mathcal{X}}-2{\cdot}|w_n-x|_{\mathcal{X}}.$$

Since the set $\{w_n\}$ is dense, we can choose w_n arbitrary close to x. Whence the value $|a_n(x)-a_n(y)|$ can be chosen arbitrary close to $|x-y|_{\mathcal{X}}$. In other words

$$\sup_n\{\, ||w_n-x|_{\mathcal{X}}-|w_n-y|_{\mathcal{X}}|\,\}\geqslant |x-y|_{\mathcal{X}};$$

hence ι is distance-nondecreasing. □

The following exercise generalizes the lemma to arbitrary separable spaces.

2.1.2. Exercise. *Suppose $\{w_n\}$ is a countable, dense set in a metric space $\mathcal{X}$. Choose $x_0\in\mathcal{X}$; given $x\in\mathcal{X}$, set*

$$a_n(x)=|w_n-x|_{\mathcal{X}}-|w_n-x_0|_{\mathcal{X}}.$$

Show that $\iota\colon x\mapsto (a_1(x),a_2(x),\dots)$ defines a distance-preserving embedding $\iota\colon\mathcal{X}\hookrightarrow\ell^\infty$.

2.1.3. Exercise. *Show that any compact metric space $\mathcal{K}$ is isometric to a subspace of a compact geodesic space.*

2.1.4. Lemma. *Let $\mathcal{X}$ be arbitrary metric space. Denote by $\ell^\infty(\mathcal{X})$ the space of all bounded functions on $\mathcal{X}$ equipped with sup-norm.*

Then for any point $x_0 \in \mathcal{X}$, the map $\iota\colon \mathcal{X} \to \ell^\infty(\mathcal{X})$ defied by

$$\iota\colon x \mapsto (\mathrm{dist}_x - \mathrm{dist}_{x_0})$$

is distance-preserving.

Note that this claim implies that *any metric space is isometric to a subset of a normed vector space.*

Remarks. The statement in 2.1.2 was proved by Maurice René Fréchet in the paper where he defined metric space [**frechet**]. The statement in 2.1.4 can be proved by nearly identical construction; this variation was given by Kazimierz Kuratowski [**kuratowski**].

Note that this exercise says that ℓ^∞ is universal in the following sense: it includes an isometric copy of any separable metric space. Note that ℓ^∞ is not separable. Maurice René Fréchet asked if there is a separable universal space. Pavel Urysohn answed to this question by constructing a separable univeral space with number of interesting properties. His construction will be discussed in the remaining sections of this lecture.

2.2 Urysohn space

Suppose a metric space $\mathcal{X}$ is a subspace of a pseudometric space $\mathcal{X}'$. In this case we may say that $\mathcal{X}'$ is an *extension* of $\mathcal{X}$. If $\operatorname{diam} \mathcal{X}' \leqslant d$, then we say that $\mathcal{X}'$ is a *d-extension.*

If the complement $\mathcal{X}'\backslash\mathcal{X}$ contains a single point, say p, we say that $\mathcal{X}'$ is a *one-point extension* of $\mathcal{X}$. In this case, to define metric on $\mathcal{X}'$, it is sufficient to specify the distance function from p; that is, a function $f\colon \mathcal{X} \to \mathbb{R}$ defined by

$$f(x) = |p - x|_{\mathcal{X}'}.$$

The function f cannot be taken arbitrary — the triangle inequality implies that

$$f(x) + f(y) \geqslant |x - y|_{\mathcal{X}} \geqslant |f(x) - f(y)|$$

for any $x, y \in \mathcal{X}$. In particular f is a non-negative 1-Lipschitz function on $\mathcal{X}$. For a d-extension we need to assume in addition that $\operatorname{diam} \mathcal{X} \leqslant$ $\leqslant d$ and $f(x) \leqslant d$ for any $x \in \mathcal{X}$.

Any function f of that type will be called *extension function* or *d-extension function* respectively.

2.2.1. Definition. *A metric space $\mathcal{U}$ is called* universal *if for any finite subspace $\mathcal{F} \subset \mathcal{U}$ and any extension function $f\colon \mathcal{F} \to \mathbb{R}$ there is a point $p \in \mathcal{U}$ such that $|p - x| = f(x)$ for any $x \in \mathcal{F}$.*

If instead of extension functions we consider only d-extension functions and assume in addition that $\operatorname{diam}\mathcal{U} \leqslant d$*, then we arrive to a definition of* d-universal space.

If in addition $\mathcal{U}$ is separable and complete, then it is called Urysohn space *or* d-Urysohn space.

2.2.2. Proposition. *Given $d > 0$, there is a separable d-universal metric space. Moreover, a separable universal space metric exists.*

Proof. Let $\mathcal{X}$ be a compact metric space such that $\operatorname{diam}\mathcal{X} \leqslant d$. Denote by $\mathcal{X}^d$ the space of all d-extension functions on $\mathcal{X}$ equipped with the metric defined by the sup-norm. Note that the map $\mathcal{X} \to \mathcal{X}^d$ defined by $x \mapsto \operatorname{dist}_x$ is a distance-preserving embedding, so we can (and will) treat $\mathcal{X}$ as a subspace of $\mathcal{X}^d$, or, equivalently, $\mathcal{X}^d$ is an extension of $\mathcal{X}$.

Let us iterate this construction. Start with a one-point space $\mathcal{X}_0$ and consider a sequence of spaces $(\mathcal{X}_n)$ defined by $\mathcal{X}_{n+1} = \mathcal{X}_n^d$. Note that the sequence is nested, that is $\mathcal{X}_0 \subset \mathcal{X}_1 \subset \dots$ and the union

$$\mathcal{X}_\infty = \bigcup_n \mathcal{X}_n;$$

comes with metric such that $|x-y|_{\mathcal{X}_\infty} = |x-y|_{\mathcal{X}_n}$ if $x, y \in \mathcal{X}_n$.

Note that if $\mathcal{X}$ is compact, then so is $\mathcal{X}^d$. It follows that each space $\mathcal{X}_n$ is compact. Since $\mathcal{X}_\infty$ is a countable union of compact spaces, it is separable.

Any finite subspace $\mathcal{F}$ of $\mathcal{X}_\infty$ lies in some $\mathcal{X}_n$ for $n < \infty$. By construction, there is a point $p \in \mathcal{X}_{n+1}$ that meets the condition in Definiton 2.2.1. That is, $\mathcal{X}_\infty$ is d-universal.

A construction of a universal separable metric space is done along the same lines, but the sequence should be defined by $\mathcal{X}_{n+1} = \mathcal{X}_n^{d_n}$ for some sequence $d_n \to \infty$; also the point p should be taken from $\mathcal{X}_{n+k}$ for sufficiently large k. ☐

2.2.3. Proposition. *A completion of d-universlal space is d-universal.*

A completion of universal space universal.

Proof. Suppose $\mathcal{V}$ be a d-universal space; denote by $\mathcal{U}$ its completion; so $\mathcal{V}$ is a dense subset in a complete space $\mathcal{U}$.

Observe that $\mathcal{U}$ is *approximately d-universal*; that is, if $\mathcal{F} \subset \mathcal{U}$ is a finite set, $\varepsilon > 0$, and $f\colon \mathcal{F} \to \mathbb{R}$ is a d-extension function, then there exists $p \in \mathcal{U}$ such that

$$|p - x| \lesssim f(x) \pm \varepsilon.$$

for any $x \in \mathcal{F}$.

Therefore there is a sequence of points $p_n \in \mathcal{U}$ such that for any $x \in \mathcal{F}$,

$$|p_n - x| \lesssim f(x) \pm \tfrac{1}{2^n}.$$

Moreover, we can assume that

$$❶ \qquad |p_n - p_{n+1}| < \tfrac{1}{2^n}$$

for all large n. Indeed, consider the sets $\mathcal{F}_n = \mathcal{F} \cup \{p_n\}$ and the functions $f_n \colon \mathcal{F}_n \to \mathbb{R}$ defined by $f_n(x) = f(x)$ for any $x \in \mathcal{F}$, and

$$f_n(p_n) = \max\left\{\,\big||p_n - x| - f(x)\big| \;:\; x \in \mathcal{F}\,\right\}.$$

Observe that f_n is a an d-extension function for large n and $f_n(p_n) <$ $< \frac{1}{2^n}$. Therefore applying approximate universal property recursively we get ❶.

By ❶, (p_n) is a Cauchy sequence and its limit meets the condition in the definition of universal space (2.2.1). $\square$

Note that 2.2.2 and 2.2.3 imply the following:

2.2.4. Theorem. *Urysohn space, and d-Urysohn space for any $d > 0$, exist.*

2.3 Universality

2.3.1. Proposition. *Let $\mathcal{U}$ be an Urysohn space. Then any separable metric space $\mathcal{S}$ admits a distance-preserving embedding $\mathcal{S} \hookrightarrow \mathcal{U}$.*

Moreover, for any finite subspace $\mathcal{F} \subset \mathcal{S}$, any distance-preserving embedding $\mathcal{F} \hookrightarrow \mathcal{U}$ can be extended to a distance preserving embedding $\mathcal{S} \hookrightarrow \mathcal{U}$.

If $\mathcal{U}$ is d-Urysohn, then the statements hold provided $\operatorname{diam} \mathcal{S} \leqslant d$.

Proof. We will prove the second statement, the first statement is its partial case for $\mathcal{F} = \varnothing$.

The required isometry will be denoted by $x \mapsto x'$.

Choose a dense sequence of points $s_1, s_2, \ldots \in \mathcal{S}$. We may assume that $\mathcal{F} = \{s_1, \ldots, s_n\}$, so $s_i' \in \mathcal{U}$ are defined for $i \leqslant n$.

The sequence s_i' for $i > n$ can be defined recursively using universality of $\mathcal{U}$. Namely suppose that $s_1', \ldots, s_{i-1}'$ are already defined. Since $\mathcal{U}$ is universal, there is a point $s_i' \in \mathcal{U}$ such that

$$|s_i' - s_j'|_{\mathcal{U}} = |s_i - s_j|_{\mathcal{S}}$$

for any $j < i$.

We constructed a distance-preserving map $s_i \mapsto s'_i$, it remains to extend it to a continuous map on whole $\mathcal{S}$. $\square$

2.3.2. Exercise. *Show that any two distinct points in an Urysohn space can be jointed by infinite number of geodesics.*

2.3.3. Exercise. *Modify the proofs of 2.2.3 and 2.3.1 to prove the following theorem.*

2.3.4. Theorem. *Let K be a compact set in a separable space $\mathcal{S}$. Then any distance-preserving map from K to an Urysohn space can be extended to a distance-preserving map on whole $\mathcal{S}$.*

2.3.5. Exercise. *Show that (d-) Urysohn space is simply connected.*

2.4 Uniqueness and homogeneity

2.4.1. Theorem. *Suppose $\mathcal{F} \subset \mathcal{U}$ and $\mathcal{F}' \subset \mathcal{U}'$ be finite isometric subspaces in a pair of (d-)Urysohn spaces $\mathcal{U}$ and $\mathcal{U}'$. Then any isometry $\mathcal{F} \to \mathcal{F}'$ can be extended to an isometry $\mathcal{U} \to \mathcal{U}'$.*

In particular (d-)Urysohn space is unique up to isometry.

Note that 2.3.1 implies that there are distance-preserving maps $\mathcal{U} \to \mathcal{U}'$ and $\mathcal{U}' \to \mathcal{U}$, but it does not solely imply existence of an isometry. The following construction use the same idea as in the proof of 2.3.1, but we need to apply it *back-and-forth* to ensure that the constructed distance-preserving map is onto.

Proof. The required isometry $\mathcal{U} \leftrightarrow \mathcal{U}'$ will be denoted by $u \leftrightarrow u'$.

Choose dense sequences $a_1, a_2, \dots \in \mathcal{U}$ and $b'_1, b'_2, \dots \in \mathcal{U}'$. Let us define recursively $a'_1, b_1, a'_2, b_2, \dots$ — on the odd step we define the images of $a_1, a_2, \dots$ and on the even steps we define invese images of $b'_1, b'_2, \dots$. The same argument as in the proof of 2.3.1 shows that we can construct two sequences $a'_1, a'_2, \dots \in \mathcal{U}'$ and $b_1, b_2, \dots \in \mathcal{U}$ such that

$$\begin{aligned}
|a_i - a_j|_{\mathcal{U}} &= |a'_i - a'_j|_{\mathcal{U}'} \\
|a_i - b_j|_{\mathcal{U}} &= |a'_i - b'_j|_{\mathcal{U}'} \\
|b_i - b_j|_{\mathcal{U}} &= |b'_i - b'_j|_{\mathcal{U}'}
\end{aligned}$$

for all i and j.

It remains to observe that the constructed distance-preserving bijection defined by $a_i \leftrightarrow a'_i$ and $b_i \leftrightarrow b'_i$ extends continuousely to an isometry $\mathcal{U} \leftrightarrow \mathcal{U}'$. $\square$

Observe that 2.4.1 implies that the Urysohn space (as well as the d-Urysohn space) is finite-set homogeneous; that is,

- $\diamond$ any distance-preserving map from a finite subset to the whole space can be extended to an isometry.

2.4.2. Open question. *Is there a* noncomplete *universal space that is finite-set homogeneous?*

This is question of Pavel Urysohn; it appeared already in [**urysohn**] and reappeared in [**gromov-2007**] with a missing key word. In fact I do not see an example of a 1-point homogeneous universal space.

2.4.3. Exercise. *Let S_r be a sphere of radius r in the d-Urysohn space $\mathcal{U}_d$; that is,*

$$S_r = \left\{\, x \in \mathcal{U}_d \;:\; |p - x|_{\mathcal{U}_d} = r \,\right\}$$

for some point $p \in \mathcal{U}_d$. Show that S_r is isometric to $\mathcal{U}_d$ if $d \geqslant r \geqslant \frac{d}{2}$.

Use it to show that $\mathcal{U}_d$ is not countable-set homogeneous; that is, there is an distance-preserving map from a countable subset of $\mathcal{U}_d$ to $\mathcal{U}_d$ that cannot be extended to an isometry of $\mathcal{U}_d$.

In fact the Urysohn space is compact-set homogeneous; more precisely the following theorem holds.

2.4.4. Theorem. *Let K be a compact set in an (d-)Uryson space $\mathcal{U}$. Then any distance-preserving map $K \to \mathcal{U}$ can be exteded to an isometry of $\mathcal{U}$.*

A proof can be obtained by modifying the proofs of 2.2.3 and 2.4.1 the same way as it is done in 2.3.3.

2.4.5. Exercise. *Which of the following metric spaces*

(a) Euclidean palne,
(b) Hilbert space,
(c) ℓ^∞,
(d) ℓ^1

are 1-point set homogenius, finite set homogenius, compact set homogenious, countable homogeneous?

Lecture 3

Injective spaces

In this lecture we discuss *injective spaces* also known as *hyperconvex spaces*. They are metric analog of convex sets in Euclidean space. The so called *injective envelop* is a minimal injective space that contains a given metric space as a subspace; it is a direct analog of convex hull of a set in a Euclidean space.

This type of spaces were introduced by Nachman Aronszajn and Prom Panitchpakdi [**aronszajn-panitchpakdi**] and injective envelop was introduced by John Isbell [**isbell**]; it was rediscovered number of times since then.

We follow closely the paper of John Isbell [**isbell**].

3.1 Admissible and extremal functions

Let $\mathcal{X}$ be a metric space. A function $r\colon \mathcal{X}\to\mathbb{R}$ is called *admissible* if the following inequality

❶ $$r(x)+r(y)\geqslant|x-y|$$

holds for any $x,y\in\mathcal{X}$.

3.1.1. Observation.

(a) Any admissible is nonnegative.

(b) If $\mathcal{X}$ is a geodesic space, then a function $r\colon\mathcal{X}\to\mathbb{R}$ is admissible if and only if

$$\overline{\mathrm{B}}[x,r(x)]\cap\overline{\mathrm{B}}[y,r(y)]\neq\varnothing$$

for any $x,y\in\mathcal{X}$.

Proof. For *(a)*, take $x=y$ in ❶. Part *(b)* follows from the triangle inequality and the definition of geodesic. □

A minimal admissible function will be called *extremal*. More precisely, an admissible function $r\colon \mathcal{X} \to \mathbb{R}$ is extremal if

$$s \leqslant r \quad \Longrightarrow \quad s = r$$

for any admissible function $s\colon \mathcal{X} \to \mathbb{R}$.

3.1.2. Exercise. *Let r and s be two extremal functions of a metric space $\mathcal{X}$. Suppose that $r \geqslant s - c$ for some constant c. Show that $c \geqslant 0$ and $r \leqslant s + c$.*

3.1.3. Observations. *Let $\mathcal{X}$ be a metric space.*

(a) *For any point $p \in \mathcal{X}$ the distance function $r = \operatorname{dist}_p$ is extremal.*

(b) *Any extremal function r on $\mathcal{X}$ is 1-Lipschitz; that is,*

$$|r(p) - r(q)| \leqslant |p - q|$$

for any $p, q \in \mathcal{X}$.[1]

(c) *Let r be an extremal function on $\mathcal{X}$. Then for any point $p \in \mathcal{X}$ and $\delta > 0$, there is a point $q \in \mathcal{X}$ such that*

$$r(p) + r(q) < |p - q|_{\mathcal{X}} + \delta.$$

Moreover if $\mathcal{X}$, then there is q such that

$$r(p) + r(q) = |p - q|_{\mathcal{X}}.$$

(d) *For any admissible function s there is an extremal function r such that $r \leqslant s$.*

Proof; (a). By the triangle inequality, ❶ holds. Further if $s \leqslant r$ is another admissible function then $s(p) = 0$ and ❶ implies that $s(x) \geqslant$ $\geqslant |p - x|$. Whence $s = r$.

3.1.3b. By *(a)*, dist_p is an extremal function. Since r is extremal,

$$r \geqslant \operatorname{dist}_p - r(p).$$

By 3.1.2, we get that

$$r \leqslant \operatorname{dist}_p + r(p),$$

or, equivalently,

$$r(q) - r(p) \leqslant |p - q|$$

[1] In other words, any extremal function is an extension function; see the definition on page 19.

for any $p, q \in \mathcal{X}$. The same way we can show that

$$r(p) - r(q) \leqslant |p - q|$$

Whence the statement follows.

3.1.3c. Again, by *(a)*, dist_p is an extremal function. Arguing by contradiction, assume

$$r(q) \geqslant \mathrm{dist}_p(q) - r(p) + \delta$$

for any q. By 3.1.2, we get that

$$r(q) \leqslant \mathrm{dist}_p(q) + r(p) - \delta$$

for any q. Taking $q = p$, we get $r(p) \leqslant r(p) - \delta$, a contradiction.

If $\mathcal{X}$ is compact, then passing to a partial limit of the obtained points q as $\delta \to 0$; we get that

$$r(p) + r(q) \leqslant |p - q|_{\mathcal{X}}.$$

Since r is admissible, the opposite inequality holds; whence the second statement follows.

(d). Follows by Zorn's lemma. □

3.2 Injective spaces

3.2.1. Definition. *A metric space $\mathcal{Y}$ is called* injective *if for any metric space $\mathcal{X}$, any its subspace $\mathcal{A}$ any short map $f \colon \mathcal{A} \to \mathcal{Y}$ can be extended to a short map $F \colon \mathcal{X} \to \mathcal{Y}$; that is, $f = F|_{\mathcal{A}}$.*

3.2.2. Exercise. *Show that any injective space is*

(a) complete,

(b) geodesic, and

(c) contractible.

3.2.3. Exercise. *Show that the following spaces are injective:*

(a) the real line;

(b) complete metric tree;

(c) plane with the metric induced by ℓ^∞-norm.

3.2.4. Exercise. *Suppose that a metric space $\mathcal{X}$ satisfies the following property: For any subspace $\mathcal{A}$ in $\mathcal{X}$ and any other metric space $\mathcal{Y}$, any short map $f\colon \mathcal{A}\to\mathcal{Y}$ can be extended to a short map $F\colon\mathcal{X}\to\mathcal{Y}$.*

Show that $\mathcal{X}$ is an ultrametric space*; that is, the following strong version of the triangle inequality*

$$|x-z|_{\mathcal{X}}\leqslant \max\{\,|x-y|_{\mathcal{X}},|y-z|_{\mathcal{X}}\,\}$$

holds for any three points $x,y,z\in\mathcal{X}$.

3.2.5. Theorem. *For any metric space $\mathcal{Y}$ the following condition are equivalent:*

(a) $\mathcal{Y}$ is injective

(b) If $r\colon\mathcal{Y}\to\mathbb{R}$ is an extremal function then there is a point $p\in\mathcal{Y}$ such that

$$|p-x|\leqslant r(x)$$

for any $x\in\mathcal{Y}$.

(c) $\mathcal{Y}$ is hyperconvex*; that is, if $\{\overline{\mathrm{B}}[x_\alpha,r_\alpha]\}_{\alpha\in\mathcal{A}}$ is a family of closed balls in $\mathcal{Y}$ such that*

$$r_\alpha+r_\beta\geqslant|x_\alpha-x_\beta|$$

for any $\alpha,\beta\in\mathcal{A}$, then all the balls in the family $\{\overline{\mathrm{B}}[x_\alpha,r_\alpha]\}_{\alpha\in\mathcal{A}}$ have a common point.

Proof. We will prove implications *(a)*⇒*(b)*⇒*(c)*⇒*(a)*.

(a)⇒*(b)*. Since $\mathcal{Y}$ is injective for any extension function $r\colon\mathcal{Y}\to\mathbb{R}$ there is a point $p\in\mathcal{Y}$ such that

$$|p-x|\leqslant r(x)$$

for any $x\in\mathcal{Y}$. By 3.1.3b, any extremal function is an extension function, whence the implication follow.

(b)⇒*(c)*. By 3.1.1b, part *(c)* is equivalent to the following statement:

◇ If $r\colon\mathcal{Y}\to\mathbb{R}$ is an admissible function, then there is a point $p\in\mathcal{Y}$ such that

❶ $$|p-x|\leqslant r(x)$$

for any $x\in\mathcal{Y}$.

Indeed, set $r(x) = \inf\{\, r_\alpha \,:\, x_\alpha = x \,\}$. The condition in *(c)* imply that r is admissible. It remains to observe that $p \in \overline{\mathrm{B}}[x_\alpha, r_\alpha]$ for every α if and only if ❶ holds.

By 3.1.3d, for any admissible function r there is an extramal function $\bar r \leqslant r$; whence *(b)⇒(c)*.

(c)⇒(a). Arguing by contradiction, suppose $\mathcal{Y}$ is not injective; that is, there is a metric space $\mathcal{X}$ with a subset $\mathcal{A}$ such that a short map $f\colon \mathcal{A} \to \mathcal{Y}$ cannot be extended to a short map $F\colon \mathcal{X} \to \mathcal{Y}$. By Zorn's lemma we may assume that $\mathcal{A}$ is a maximal subset; that is, the domain of f cannot be enlarged by a single point.[2]

Fix a point p in the complement $\mathcal{X}\backslash\mathcal{A}$. To extend f to p, we need to choose $f(p)$ in the intersection of the balls $\overline{\mathrm{B}}[f(x), r(x)]$, where $r(x) = |p - x|$. Therefore this intersection for all $x \in \mathcal{A}$ have to be empty.

Since f is short, we have that

$$\begin{aligned} r(x) + r(y) \geqslant |x-y|_{\mathcal{X}} &\geqslant \\ &\geqslant |f(x) - f(y)|_{\mathcal{Y}}. \end{aligned}$$

Therefore by *(c)* the balls $\overline{\mathrm{B}}[f(x), r(x)]$ have a common point — a contradiction. □

3.2.6. Exercise. *Suppose a length space $\mathcal{W}$ have two subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X}\cup\mathcal{Y} = \mathcal{W}$ and $\mathcal{X}\cup\mathcal{Y}$ is one-point set. Assume $\mathcal{X}$ and $\mathcal{Y}$ are injective. Show that $\mathcal{W}$ is injective*

3.2.7. Exercise. *Show that the Urysohn space is* finitely hypercovex *but not* countably hyperconvex*; that is, the condition in 3.2.5c holds for any finite family of balls, but may not hold for a countable family. Conclude that the Urysohn space is not injective.*

3.3 Space of extremal functions

Let $\mathcal{X}$ be a metric space. Consider the space $\operatorname{Inj}\mathcal{X}$ of extremal functions on $\mathcal{X}$ equipped with sup-norm; that is,

$$|f - g|_{\operatorname{Inj}\mathcal{X}} := \sup\{\, |f(x) - g(x)| \,:\, x \in \mathcal{X} \,\}.$$

Recall that by 3.1.3a, any distance function is extremal. It follwos that the map $x \mapsto \operatorname{dist}_x$ produces a distance-preserving embedding $\mathcal{X} \hookrightarrow \operatorname{Inj}\mathcal{X}$. So we can (and will) treat $\mathcal{X}$ as a subspace of $\operatorname{Inj}\mathcal{X}$, or, equivalently, $\operatorname{Inj}\mathcal{X}$ as an extension of $\mathcal{X}$.

[2]In this case $\mathcal{A}$ must be closed, but we will not use it.

Since any extremal fucntion is 1-Lipschitz, for any $f \in \operatorname{Inj}\mathcal{X}$ and $p \in \mathcal{X}$, we have that $f(x) \leqslant f(p) + \operatorname{dist}_p(x)$. By 3.1.2, we also get $f(x) \geqslant -f(p) + \operatorname{dist}_p(x)$. Therefore

❶
$$\begin{aligned}|f-p|_{\operatorname{Inj}\mathcal{X}} &= \sup\{\,|f(x)-\operatorname{dist}_p(x)| \,:\, x\in\mathcal{X}\,\} = \\ &= f(p).\end{aligned}$$

In particular, the statement in 3.1.3c can be written as

$$|f-p|_{\operatorname{Inj}\mathcal{X}} + |f-q|_{\operatorname{Inj}\mathcal{X}} < |p-q|_{\operatorname{Inj}\mathcal{X}} + \delta.$$

3.3.1. Exercise. *Let $\mathcal{X}$ be a metric space. Show that* $\operatorname{Inj}\mathcal{X}$ *is compact if and only if so is $\mathcal{X}$.*

3.3.2. Exercise. *Suppose that $\mathcal{X}$ is*

(a) a metric space with exactly tree points a, b, c such that

$$|a-b|_{\mathcal{X}} = |b-c|_{\mathcal{X}} = |c-a|_{\mathcal{X}} = 1.$$

(b) a metric space with exactly four points p, q, x, y such that

$$|p-x|_{\mathcal{X}} = |p-y|_{\mathcal{X}} = |q-x|_{\mathcal{X}} = |q-x|_{\mathcal{X}} = 1$$

and

$$|p-q|_{\mathcal{X}} = |x-y|_{\mathcal{X}} = 2.$$

Describe the set of all extremal functions on $\mathcal{X}$ and the metric space $\operatorname{Inj}\mathcal{X}$ *in each case.*

3.3.3. Proposition. *For any metric space $\mathcal{X}$, its extension* $\operatorname{Inj}\mathcal{X}$ *is injective.*

3.3.4. Lemma. *Let $\mathcal{X}$ be a metric space. Suppose that r is an extremal function on* $\operatorname{Inj}\mathcal{X}$*. Then the restriction $r|_{\mathcal{X}}$ is an extremal function on $\mathcal{X}$. In other words, $r|_{\mathcal{X}} \in \operatorname{Inj}\mathcal{X}$*

Proof. Arguing by contradiction, suppose that there is an admissible function $s\colon \mathcal{X} \to \mathbb{R}$ such that $s(x) \leqslant r(x)$ for any $x \in \mathcal{X}$ and $s(p) < r(p)$ for some point $p \in \mathcal{X}$. Consider another function $\bar r\colon \operatorname{Inj}\mathcal{X} \to \mathbb{R}$ such that $\bar r(f) = r(f)$ if $f \neq p$ and $\bar r(p) := s(p)$.

Let us show that $\bar r$ is admissible; that is,

❷
$$\bar r(f) + \bar r(g) \geqslant |f-g|_{\operatorname{Inj}\mathcal{X}}$$

for any $f, g \in \operatorname{Inj}\mathcal{X}$.

Since r is admissible and $\bar{r} = r$ on $(\operatorname{Inj}\mathcal{X})\backslash\{p\}$, it is sufficient to prove ❷ if $f \neq g = p$. By ❶, we have $|f - p|_{\operatorname{Inj}\mathcal{X}} = f(p)$. Therefore ❷ boils down to the following inequality

❸ $$r(f) + s(p) \geqslant f(p).$$

for any $f \in \operatorname{Inj}\mathcal{X}$.

Fix small $\delta > 0$. Let $q \in \mathcal{X}$ be the point provided by 3.1.3c. Then

$$r(f) + s(p) \geqslant [r(f) - r(q)] + [r(q) + s(p)] \geqslant$$

since r is 1-Lipschitz, and $r(q) \geqslant s(q)$, we can continue

$$\geqslant -|q - f|_{\operatorname{Inj}\mathcal{X}} + [s(q) + s(p)] \geqslant$$

by ❶ and since s is admissible

$$\geqslant -f(q) + |p - q| >$$

by 3.1.3c

$$> f(p) - \delta.$$

Since $\delta > 0$ is arbitrary, ❸ and ❷ follow.

Summarizing: the function $\bar{r}$ is admissible, $\bar{r} \leqslant r$ and $\bar{r}(p) < r(p)$; that is, r is not extremal — a contradiction. □

Proof of 3.3.3. By 3.2.5b, it is sufficient to show that for any extremal function $r\colon \operatorname{Inj}\mathcal{X} \to \mathbb{R}$, there is $s \in \operatorname{Inj}\mathcal{X}$ such that

❹ $$r(f) \geqslant |s - f|_{\operatorname{Inj}\mathcal{X}}$$

for any $f \in \operatorname{Inj}\mathcal{X}$.

Let us show that one can take $s = r|_{\mathcal{X}}$. By 3.3.4, s is extremal; that is, $s \in \operatorname{Inj}\mathcal{X}$.

Since r is 1-Lipschitz (3.1.3b) we have that

$$s(x) - f(x) = r(x) - |f - x|_{\operatorname{Inj}\mathcal{X}} \leqslant r(f).$$

for any x. Since r is admissible we have that

$$s(x) - f(x) = r(x) - |f - x|_{\operatorname{Inj}\mathcal{X}} \geqslant -r(f).$$

for any x. That is, $|s(x) - f(x)| \leqslant r(f)$ for any $x \in \mathcal{X}$. Recall that

$$|s - f|_{\operatorname{Inj}\mathcal{X}} := \sup\{\, |s(x) - f(x)| \;:\; x \in \mathcal{X}\,\};$$

hence ❹ follows. □

3.3.5. Exercise. *Let $\mathcal{X}$ be a compact metric space. Show that for any two points $f, g \in \operatorname{Inj}\mathcal{X}$ lie on a geodesic $[pq]$ with the ends on $\mathcal{X}$.*

3.4 Injective envelop

An extension $\mathcal{E}$ of a metric space $\mathcal{X}$ will be called its *injective envelop* if $\mathcal{E}$ is a an injective space and there is no injective proper subspace of $\mathcal{E}$ that contains $\mathcal{X}$.

Two injective envelopes $e\colon \mathcal{X} \hookrightarrow \mathcal{E}$ and $f\colon \mathcal{X} \hookrightarrow \mathcal{F}$ are called equivalent if there is an isometry $\iota\colon \mathcal{E} \to \mathcal{F}$ such that $f = \iota \circ e$.

3.4.1. Theorem. *For any metric space $\mathcal{X}$, its extension $\operatorname{Inj}\mathcal{X}$ is an injective envelop.*

Moreover, any other injective envelop of $\mathcal{X}$ is equivalent to $\operatorname{Inj}\mathcal{X}$.

Proof. Suppose $S \subset \operatorname{Inj}\mathcal{X}$ is an injective subspace containing $\mathcal{X}$. Since S is injective, there is a short map $w\colon \operatorname{Inj}\mathcal{X} \to S$ that fixes all points in $\mathcal{X}$.

Suppose that $w\colon f \mapsto f'$; observe that $f(x) \geqslant f'(x)$ for any $x \in \mathcal{X}$. Since f is extremal, $f = f'$; that is, w is the identity map and therefore $S = \operatorname{Inj}\mathcal{X}$.

Assume we have another injective envelop $e\colon \mathcal{X} \hookrightarrow \mathcal{E}$. Then there are short maps $v\colon \mathcal{E} \to \operatorname{Inj}\mathcal{X}$ and $w\colon \operatorname{Inj}\mathcal{X} \to \mathcal{E}$ such that $x = v \circ e(x)$ and $e(x) = w(x)$ for any $x \in \mathcal{X}$. From above, the $v \circ w$ is the identity on $\operatorname{Inj}\mathcal{X}$. In particular w is distance-preserving.

The composition $w \circ v\colon \mathcal{E} \to \mathcal{E}$ is a short map that fixes points in $e(\mathcal{X})$. Since $e\colon \mathcal{X} \hookrightarrow \mathcal{E}$ is an injective envelop, the composition $w \circ v$ and therefore w are onto. Whence w is an isometry. $\square$

Lecture 4

Space of sets

4.1 Hausdorff convergence

Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$, consider the distance function to A

$$\operatorname{dist}_A : \mathcal{X} \to [0, \infty)$$

defined as

$$\operatorname{dist}_A(x) := \inf_{a \in A} \{ |a - x|_{\mathcal{X}} \}.$$

4.1.1. Definition. *Let A and B be two compact subsets of a metric space $\mathcal{X}$. Then the* Hausdorff distance *between A and B is defined as*

$$|A - B|_{\mathcal{H}(\mathcal{X})} := \sup_{x \in \mathcal{X}} \{ |\operatorname{dist}_A(x) - \operatorname{dist}_B(x)| \}.$$

Suppose A and B be two compact subsets of a metric space $\mathcal{X}$. It is straightforward to check that $|A - B|_{\mathcal{H}(\mathcal{X})} \leqslant R$ if and only if $\operatorname{dist}_A(b) \leqslant R$ for any $b \in B$ and $\operatorname{dist}_B(a) \leqslant R$ for any $a \in A$. In other words, $|A - B|_{\mathcal{H}(\mathcal{X})} < R$ if and only if B lies in a R-neighborhood of A, and A lies in a R-neighborhood of B.

Note that the set of all nonempty compact subsets of a metric space $\mathcal{X}$ equipped with the Hausdorff metric forms a metric space. This new metric space will be denoted as $\mathcal{H}(\mathcal{X})$.

4.1.2. Exercise. *Let $\mathcal{X}$ be a metric space. Given a subset $A \subset \mathcal{X}$ define its diameter as*

$$\operatorname{diam} A := \sup_{a, b \in A} |a - b|.$$

Show that

$$\operatorname{diam}\colon \mathcal{H}(\mathcal{X}) \to \mathbb{R}$$

is a 2*-Lipschitz function; that is,* $|\operatorname{diam} A - \operatorname{diam} B| \leqslant 2\cdot|A-B|_{\mathcal{H}(\mathcal{X})}$.

4.1.3. Blaschke selection theorem. *Let $\mathcal{X}$ be a metric space. Then the space $\mathcal{H}(\mathcal{X})$ is compact if and only if $\mathcal{X}$ is compact.*

Note that the theorem implies that from any sequence of compact sets in $\mathcal{X}$ one can select a subsequence converging in the sense of Hausdorff; by that reason it is called a selection theorem.

Proof; "only if" part. Note that the map $\iota\colon \mathcal{X} \to \mathcal{H}(\mathcal{X})$, defined as $\iota\colon x \mapsto \{x\}$ (that is, point x mapped to the one-point subset $\{x\}$ of $\mathcal{X}$) is distance-preserving. Therefore $\mathcal{X}$ is isometric to the set $\iota(\mathcal{X})$ in $\mathcal{H}(\mathcal{X})$.

Note that for a nonempty subset $A \subset \mathcal{X}$, we have $\operatorname{diam} A = 0$ if and only if A is a one-point set. Therefore, from Exercise 4.1.2, it follows that $\iota(\mathcal{X})$ is closed in $\mathcal{H}(\mathcal{X})$.

Hence $\iota(\mathcal{X})$ is compact, as it is a closed subset of a compact space. Since $\mathcal{X}$ is isometric to $\iota(\mathcal{X})$, "only if" part follows. □

To prove "if" part we will need the following two lemmas.

4.1.4. Lemma. *Let $K_1 \supset K_2 \supset \dots$ be a sequence of nonempty compact sets in a metric space $\mathcal{X}$ then $K_\infty = \bigcap_n K_n$ is the Hausdorff limit of K_n; that is, $|K_\infty - K_n|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$.*

Proof. Note that K_∞ is compact; by finite intersection property, K_∞ is nonempty.

If the assertion were false, then there is $\varepsilon > 0$ such that for each n one can choose $x_n \in K_n$ such that $\operatorname{dist}_{K_\infty}(x_n) \geqslant \varepsilon$. Note that $x_n \in K_1$ for each n. Since K_1 is compact, there is a *partial limit*[1] x_∞ of x_n. Clearly $\operatorname{dist}_{K_\infty}(x_\infty) \geqslant \varepsilon$.

On the other hand, since K_n is closed and $x_m \in K_n$ for $m \geqslant n$, we get $x_\infty \in K_n$ for each n. It follows that $x_\infty \in K_\infty$ and therefore $\operatorname{dist}_{K_\infty}(x_\infty) = 0$, a contradiction. □

4.1.5. Lemma. *If $\mathcal{X}$ is a compact metric space, then $\mathcal{H}(\mathcal{X})$ is complete.*

Proof. Let (Q_n) be a Cauchy sequence in $\mathcal{H}(\mathcal{X})$. Passing to a subsequence of Q_n we may assume that

➊ $$|Q_n - Q_{n+1}|_{\mathcal{H}(\mathcal{X})} \leqslant \tfrac{1}{10^n}$$

[1]Partial limit is a limit of a subsequence.

for each n.

Set

$$K_n = \left\{ x \in \mathcal{X} \, : \, \operatorname{dist}_{Q_n}(x) \leqslant \tfrac{1}{10^n} \right\}$$

Since $\mathcal{X}$ is compact so is each K_n.

Clearly, $|Q_n - K_n|_{\mathcal{H}(\mathcal{X})} \leqslant \frac{1}{10^n}$ and from ❶, we get $K_n \supset K_{n+1}$ for each n. Set

$$K_\infty = \bigcap_{n=1}^{\infty} K_n.$$

Applying Lemma 4.1.4, we get that $|K_n - K_\infty|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$. Since $|Q_n - K_n|_{\mathcal{H}(\mathcal{X})} \leqslant \frac{1}{10^n}$, we get $|Q_n - K_\infty|_{\mathcal{H}(\mathcal{X})} \to 0$ as $n \to \infty$ — hence the lemma. □

4.1.6. Exercise. *Let $\mathcal{X}$ be a complete metric space and K_n be a sequence of compact sets which converges in the sence of Hausdorff. Show that closure of the union $\bigcup_{n=1}^{\infty} K_n$ is compact.*

Use this to show that in Lemma 4.1.5 compactness of $\mathcal{X}$ can be exchanged to completeness.

Proof of "if" part in 4.1.3. According to Lemma 4.1.5, $\mathcal{H}(\mathcal{X})$ is complete. It remains to show that $\mathcal{H}(\mathcal{X})$ is totally bounded (1.4.1d); that is, given $\varepsilon > 0$ there is a finite ε-net in $\mathcal{H}(\mathcal{X})$.

Choose a finite ε-net A in $\mathcal{X}$. Denote by $\mathcal{A}$ the set of all subsets of A. Note that $\mathcal{A}$ is finite set in $\mathcal{H}(\mathcal{X})$. For each compact set $K \subset \mathcal{X}$, consider the subset K' of all points $a \in A$ such that $\operatorname{dist}_K(a) \leqslant \varepsilon$. Then $K' \in \mathcal{A}$ and $|K - K'|_{\mathcal{H}(\mathcal{X})} \leqslant \varepsilon$. In other words $\mathcal{A}$ is a finite ε-net in $\mathcal{H}(\mathcal{X})$. □

Hausdorff metric defines convergence of compact sets which is more important than metric itself.

4.1.7. Exercise. *Let X and Y be two compact subsets in $\mathbb{R}^2$. Assume $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$.*

(a) *Show that $|\operatorname{Conv} X - \operatorname{Conv} Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, where $\operatorname{Conv} X$ denoted a convex hull of X.*

(b) *Is it true that $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, where ∂X denotes the boundary of X.*
Does the converse holds? That is, assume X and Y be two compact subsets in $\mathbb{R}^2$ and $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$; is it true that $|X - Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$?

4.1.8. Exercise. *Let $\mathcal{C}$ be a subspace of $\mathcal{H}(\mathbb{R}^2)$ formed by all compact convex subsets in $\mathbb{R}^2$. Show that perimeter*[2] *and area are continuous on $\mathcal{C}$. That is, if a sequence of convex compact plane sets X_n converges to X_∞ in the sense of Hausdorff, then*

$$\operatorname{perim} X_n \to \operatorname{perim} X_\infty \quad \textit{and} \quad \operatorname{area} X_n \to \operatorname{area} X_\infty$$

as $n \to \infty$.

The above exercise can be used in a proof of isoperimetrical inequality in the plane; it states that *among the plane figures bounded by closed curves of length at most ℓ the round disc has maximal area.*

Indeed it is sufficient to consider only convex figures of given perimeter; if a figure is not convex pass to its convex hull and observe that it has larger area and smaller perimeter. Further the exercise guarantees existence of a figure D_ℓ with perimeter ℓ and maximal area. It remains to show that D_ℓ is a round disc. The latter is easy to show, see for example Steiner's 4-joint method [**blaschke**].

4.2 A variation

It seems that Hausdorff convergence was first introduced by Felix Hausdorff [**hausdorff**], and a couple of years later an equivalent definition was given by Wilhelm Blaschke [**blaschke**].

The following refinement of the definition was introduced by Zdeněk Frolík in [**frolik**], and later rediscovered by Robert Wijsman in [**wijsman**]. This refinement takes an intermediate place between the original Hausdorff convergence and *closed convergence*, also introduced by Hausdorff in [**hausdorff**]; so we still call it Hausdorff convergence.

4.2.1. Definition. *Let (A_n) be a sequence of closed sets in a metric space $\mathcal{X}$. We say that (A_n) converges to a closed set A_∞ in the sense of Hausdorff if $\operatorname{dist}_{A_n}(x) \to \operatorname{dist}_{A_\infty}(x)$ for any $x \in \mathcal{X}$.*

For example, suppose $\mathcal{X}$ is the Euclidean plane and A_n is the circle with radius n and center at $(n, 0)$. If we use the standard definition (4.1.1), then the sequence (A_n) diverges, but it converges to the y-axis in the sense of Definition 4.2.1.

The following exercise is analogous to the Blaschke selection theorem (4.1.3).

4.2.2. Exercise. *Let $\mathcal{X}$ be a proper metric space and $(A_n)_{n=1}^\infty$ be a sequence of closed sets in $\mathcal{X}$. Assume that for some (and therefore*

[2] If the set degenerates to a line segment of length ℓ, then its perimeter is defined as $2{\cdot}\ell$.

any) point $x \in \mathcal{X}$*, the sequence* $a_n = \operatorname{dist}_{A_n}(x)$ *is bounded. Show that the sequence* $(A_n)_{n=1}^{\infty}$ *has a convergent subsequence in the sense of Definition 4.2.1.*

Lecture 5

Space of spaces

5.1 Gromov–Hausdorff metric

The goal of this section is to cook up a metric space out of metric spaces. More precisely, we want to define the so called Gromov–Hausdorff metric on the set of *isometry classes* of compact metric spaces. (Being isometric is an equivalence relation, and an isometry class is an equivalence class with respect to this equivalence relation.)

The obtained metric space will be denoted as $\mathcal{M}$. Given two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, denote by $[\mathcal{X}]$ and $[\mathcal{Y}]$ their isometry classes; that is, $\mathcal{X}' \in [\mathcal{X}]$ if and only if $\mathcal{X}' \stackrel{iso}{=\!=} \mathcal{X}$. Pedantically, the Gromov–Hausdorff distance from $[\mathcal{X}]$ to $[\mathcal{Y}]$ should be denoted as $|[\mathcal{X}]-[\mathcal{Y}]|_{\mathcal{M}}$; but we will often write it as $|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}}$ and say (not quite correctly) "$|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}}$ is the Gromov–Hausdorff distance from $\mathcal{X}$ to $\mathcal{Y}$". In other words, from now on the term *metric space* might stands for *isometry class of this metric space.*

The metric on $\mathcal{M}$ is maximal metric such that *the distance between subspaces in a metric space is not greater than the Hausdorff distance between them.* Here is a formal definition:

5.1.1. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact metric spaces. The Gromov–Hausdorff distance $|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}}$ between them is defined by the following relation.*

Given $r > 0$, we have that $|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}} < r$ if and only if there exist a metric space $\mathcal{Z}$ and subspaces $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{Z}$ that are isometric to $\mathcal{X}$ and $\mathcal{Y}$ respectively and such that $|\mathcal{X}'-\mathcal{Y}'|_{\mathcal{H}(\mathcal{Z})} < r$. (Here $|\mathcal{X}'-\mathcal{Y}'|_{\mathcal{H}(\mathcal{Z})}$ denotes the Hausdorff distance between sets $\mathcal{X}'$ and $\mathcal{Y}'$ in $\mathcal{Z}$.)

Bit later (see 5.4.1) we will show that Hausdorff metric is indeed a metric.

We say that a sequence of (isometry classes of) compact metric spaces $\mathcal{X}_n$ *Gromov–Hausdorff convergence* to the (isometry classes of) compact metric space $\mathcal{X}_\infty$ if $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{M}} \to 0$ as $n \to \infty$; in this case we write $\mathcal{X}_n \xrightarrow{\mathrm{GH}} \mathcal{X}_\infty$.

5.1.2. Exercise. *Let $\mathcal{X}_\infty, \mathcal{X}_1, \mathcal{X}_2$ be compact metric spaces. Suppose $\mathcal{X}_n \xrightarrow{\mathrm{GH}} \mathcal{X}_\infty$. Show that $\operatorname{Inj}\mathcal{X}_n \xrightarrow{\mathrm{GH}} \operatorname{Inj}\mathcal{X}_\infty$. (Recall that $\operatorname{Inj}\mathcal{X}$ denotes the injective envelop of $\mathcal{X}$ that is defined on page 29.)*

5.2 Reformulations

Let us discuss few alternative ways to define the Gromov–Hausdorff metric.

Metrics on disjoined union. Definition 5.1.1 deals with a huge class of metric spaces, namely, all metric spaces $\mathcal{Z}$ that contain subspaces isometric to $\mathcal{X}$ and $\mathcal{Y}$. It is possible to reduce this class to metrics on the disjoint unions of $\mathcal{X}$ and $\mathcal{Y}$. More precisely,

5.2.1. Proposition. *The Gromov–Hausdorff distance between two compact metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is the infimum of $r > 0$ such that there exists a metric $|{*}-{*}|_{\mathcal{W}}$ on the disjoint union $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Y}$ such that the restrictions of $|{*}-{*}|_{\mathcal{W}}$ to $\mathcal{X}$ and $\mathcal{Y}$ coincide with $|{*}-{*}|_{\mathcal{X}}$ and $|{*}-{*}|_{\mathcal{Y}}$ and $|\mathcal{X}-\mathcal{Y}|_{\mathcal{H}(\mathcal{W})} < r$.*

Proof. Identify $\mathcal{X} \sqcup \mathcal{Y}$ with $\mathcal{X}' \cup \mathcal{Y}' \subset \mathcal{Z}$ (the notation is from Definition 5.1.1).

More formally, fix isometries $f\colon \mathcal{X} \to \mathcal{X}'$ and $g\colon \mathcal{Y} \to \mathcal{Y}'$, then define the distance between $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ by $|x-y|_{\mathcal{W}} = |f(x) - g(y)|_{\mathcal{Z}} + \varepsilon$ for small enuf $\varepsilon > 0$.[1] This yields a metric on $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Y}$ for which $|\mathcal{X}-\mathcal{Y}|_{\mathcal{H}(\mathcal{W})} < r$. □

Fixed ambient space. The following proposition says that the space $\mathcal{Z}$ in Definition 5.1.1 can be exchanged to a fixed space, namely ℓ^∞ — the space of bounded infinite sequences with the metric defined by sup-norm.

5.2.2. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be comact metric spaces. Then*

$$|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}} = \inf\{|\mathcal{X}'-\mathcal{Y}'|_{\mathcal{H}(\ell^\infty)}\}$$

where the infimum is taken over all pairs of sets $\mathcal{X}'$ and $\mathcal{Y}'$ in ℓ^∞ which isometric to $\mathcal{X}$ and $\mathcal{Y}$ respectively.

[1] We add ε to ensure that $d(x,y) > 0$ for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$; so $|x-y|_{\mathcal{W}}$ is indeed a metric.

Proof of 5.2.2. By the definition, we have that

$$|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}} \le \inf\{|\mathcal{X}'-\mathcal{Y}'|_{\mathcal{H}(\ell^\infty)}\}.$$

Let $\mathcal{W}$ be an arbitrary metric space with the underlying set $\mathcal{X}\sqcup\mathcal{Y}$. Note $\mathcal{W}$ is compact since it is union of two compact subsets $\mathcal{X},\mathcal{Y}\subset\mathcal{W}$. In particular, $\mathcal{W}$ is separable.

By Lemma 2.1.1, there is a distance-preserving embedding $\iota\colon\mathcal{W}\to \to\ell^\infty$. It remains to apply Proposition 5.2.1. □

5.3 Almost isometries

5.3.1. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces and $\varepsilon>0$. A map*[2] *$f\colon\mathcal{X}\to\mathcal{Y}$ is called an ε-isometry if*

$$|f(x)-f(x')|_{\mathcal{Y}} \lesseqgtr |x-x'|_{\mathcal{X}}\pm\varepsilon$$

for any $x,x'\in\mathcal{X}$ and if $f(\mathcal{X})$ is an ε-net in $\mathcal{Y}$.

5.3.2. Exercise.

(a) Let $f\colon\mathcal{X}\to\mathcal{Y}$ and $g\colon\mathcal{Y}\to\mathcal{Z}$ be two ε-isometries. Show that $g\circ f\colon\mathcal{X}\to\mathcal{Z}$ is a $(3{\cdot}\varepsilon)$-isometry.

(b) Assume $f\colon\mathcal{X}\to\mathcal{Y}$ is an ε-isometry. Show that there is a $(3{\cdot}\varepsilon)$-isometry $g\colon\mathcal{Y}\to\mathcal{X}$.

(c) Assume $|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}}<\varepsilon$, show that there is a $(2{\cdot}\varepsilon)$-isometry $f\colon\mathcal{X}\to\mathcal{Y}$.

5.3.3. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces and let $f\colon\mathcal{X}\to\mathcal{Y}$ be an ε-isometry. Then*

$$|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}}\leqslant 2{\cdot}\varepsilon.$$

Proof. Consider the set $\mathcal{W}=\mathcal{X}\sqcup\mathcal{Y}$. Note that the following defines a metric on $\mathcal{W}$:

◇ For any $x,x'\in\mathcal{X}$

$$|x-x'|_{\mathcal{W}}=|x-x'|_{\mathcal{X}};$$

◇ For any $y,y'\in\mathcal{Y}$,

$$|y-y'|_{\mathcal{W}}=|y-y'|_{\mathcal{Y}}$$

[2] possibly noncontinuous

◇ For any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,

$$|x - y|_{\mathcal{W}} = \varepsilon + \inf_{x' \in \mathcal{X}} \{|x - x'|_{\mathcal{X}} + |f(x') - y|_{\mathcal{Y}}\}.$$

Since $f(\mathcal{X})$ is an ε-net in $\mathcal{Y}$, for any $y \in \mathcal{Y}$ there is $x \in \mathcal{X}$ such that $|f(x) - y|_{\mathcal{Y}} \leqslant \varepsilon$; therefore $|x - y|_{\mathcal{W}} \leqslant 2{\cdot}\varepsilon$. On the other hand for any $x \in \mathcal{X}$, we have $|x - y|_{\mathcal{W}} \leqslant \varepsilon$ for $y = f(x) \in \mathcal{Y}$.

It follows that $|\mathcal{X} - \mathcal{Y}|_{\mathcal{H}(\mathcal{W})} \leqslant 2{\cdot}\varepsilon$. □

The Gromov–Hausdorff metric defines Gromov–Hausdorff convegence and this is the only thing it is good for. In other words in all applications, we use only topology on $\mathcal{M}$ and we do not care about particular value of Gromov–Hausdorff distance between spaces.

In order to determine that a given sequence of metric spaces $(\mathcal{X}_n)$ converges in the Gromov–Hausdorff sense to $\mathcal{X}_\infty$, it is sufficient to estimate distances $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{M}}$ and check if $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{M}} \to 0$. This problem turns to be simpler than finding Gromov–Hausdorff distance between a particular pair of spaces. The following proposition gives one way to do this.

5.3.4. Proposition. *A sequence of compact metric spaces $(\mathcal{X}_n)$ converges to $\mathcal{X}_\infty$ in the sense of Gromov–Hausdorff if and only if there is a sequence $\varepsilon_n \to 0+$ and an ε_n-isometry $f_n\colon \mathcal{X}_n \to \mathcal{X}_\infty$ for each n.*

Proof. Follows from Propsition 5.3.3 and Exercise 5.3.2*c* □

5.4 It is a metric

5.4.1. Theorem. *The set of isometry classes of compact metric spaces equipped with Gromov–Hausdorff metric forms a metric space (which is denoted by $\mathcal{M}$).*

Proof. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be arbitrary compact metric spaces. We need to check the following:

(i) $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} \geqslant 0$;
(ii) $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} = 0$ if and only if $\mathcal{X}$ is isometric to $\mathcal{Y}$;
(iii) $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} = |\mathcal{Y} - \mathcal{X}|_{\mathcal{M}}$;
(iv) $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} + |\mathcal{Y} - \mathcal{Z}|_{\mathcal{M}} \geqslant |\mathcal{X} - \mathcal{Z}|_{\mathcal{M}}$.

Note that *(i)*, *(iii)* and "if"-part of *(ii)* follow directly from Definition 5.1.1.

(iv). Choose arbitrary $a, b \in \mathbb{R}$ such that

$$a > |\mathcal{X} - \mathcal{Y}|_{\mathcal{M}} \quad \text{and} \quad b > |\mathcal{Y} - \mathcal{Z}|_{\mathcal{M}}.$$

Choose two metrics on $\mathcal{U} = \mathcal{X} \sqcup \mathcal{Y}$ and $\mathcal{V} = \mathcal{Y} \sqcup \mathcal{Z}$ so that $|\mathcal{X} - \mathcal{Y}|_{\mathcal{H}(\mathcal{U})} < a$ and $|\mathcal{Y} - \mathcal{Z}|_{\mathcal{H}(\mathcal{V})} < b$ and the inclusions $\mathcal{X} \hookrightarrow \mathcal{U}$, $\mathcal{Y} \hookrightarrow \mathcal{U}$, $\mathcal{Y} \hookrightarrow \mathcal{V}$ and $\mathcal{Z} \hookrightarrow \mathcal{V}$ are distance-preserving.

Consider the metric on $\mathcal{W} = \mathcal{X} \sqcup \mathcal{Z}$ so that inclusions $\mathcal{X} \hookrightarrow \mathcal{W}$ and $\mathcal{Z} \hookrightarrow \mathcal{W}$ are distance-preserving and

$$|x - z|_{\mathcal{W}} = \inf_{y \in \mathcal{Y}} \{|x - y|_{\mathcal{U}} + |y - z|_{\mathcal{V}}\}.$$

Note that $|* - *|_{\mathcal{W}}$ is indeed a metric and

$$|\mathcal{X} - \mathcal{Z}|_{\mathcal{H}(\mathcal{W})} < a + b.$$

Property *(iv)* follows since the last inequality holds for any $a > |\mathcal{X} - \mathcal{Y}|_{\mathcal{M}}$ and $b > |\mathcal{Y} - \mathcal{Z}|_{\mathcal{M}}$.

"Only if"-part of (ii). According to Exercise 5.3.2*c*, for any sequence $\varepsilon_n \to 0+$ there is a sequence of ε_n-isometries $f_n \colon \mathcal{X} \to \mathcal{Y}$.

Since $\mathcal{X}$ is compact, we can choose a countable dense set S in $\mathcal{X}$. Use a diagonal procedure if necessary, to pass to a subsequence of (f_n) such that for every $x \in S$ the sequence $(f_n(x))$ converges in $\mathcal{Y}$. Consider the pointwise limit map $f_\infty \colon S \to \mathcal{Y}$ defined by

$$f_\infty(x) = \lim_{n \to \infty} f_n(x)$$

for every $x \in S$. Since

$$|f_n(x) - f_n(x')|_{\mathcal{Y}} \lessgtr |x - x'|_{\mathcal{X}} \pm \varepsilon_n,$$

we have

$$|f_\infty(x) - f_\infty(x')|_{\mathcal{Y}} = \lim_{n \to \infty} |f_n(x) - f_n(x')|_{\mathcal{Y}} = |x - x'|_{\mathcal{X}}$$

for all $x, x' \in S$; that is, $f_\infty \colon S \to \mathcal{Y}$ is a distance-preserving map. Therefore f_∞ can be extended to a distance-preserving map from all of $\mathcal{X}$ to $\mathcal{Y}$. The later is done by setting

$$f_\infty(x) = \lim_{n \to \infty} f_\infty(x_n)$$

for some (and therefore any) sequence of points (x_n) in S which converges to x in $\mathcal{X}$. (Note that if $x_n \to x$, then (x_n) is Cauchy. Since f_∞ is distance-preserving, $y_n = f_\infty(x_n)$ is also a Cauchy sequence in $\mathcal{Y}$; therefore it converges.)

This way we obtain a distance-preserving map $f_\infty \colon \mathcal{X} \to \mathcal{Y}$. It remains to show that f_∞ is surjective; that is, $f_\infty(\mathcal{X}) = \mathcal{Y}$.

Note that in the same way we can obtain a distance-preserving map $g_\infty \colon \mathcal{Y} \to \mathcal{X}$. If f_∞ is not surjective, then neither is $f_\infty \circ g_\infty \colon \mathcal{Y} \to \mathcal{Y}$.

So $f_\infty\circ g_\infty$ is a distance-preserving map from a compact space to itself which is not an isometry. The later contradicts Exercise 1.4.4. $\square$

5.4.2. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two compact metric spaces. Prove that*

$$|\operatorname{diam}\mathcal{X}-\operatorname{diam}\mathcal{Y}|\leqslant 2\cdot|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}}.$$

In other words, $\operatorname{diam}\colon\mathcal{M}\to\mathbb{R}$ *is a* 2*-Lipschitz function.*

5.4.3. Exercise. *Show that $\mathcal{M}$ is a length space.*

Given two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, we will write $\mathcal{X}\leqslant\mathcal{Y}$ if there is a noncontracting map $f\colon\mathcal{X}\to\mathcal{Y}$; that is, if

$$|x-x'|_{\mathcal{X}}\leqslant|f(x)-f(x')|_{\mathcal{Y}}$$

for any $x,x'\in\mathcal{X}$.

Further, given $\varepsilon>0$, we will write $\mathcal{X}\leqslant\mathcal{Y}+\varepsilon$ if there is a map $f\colon\mathcal{X}\to\mathcal{Y}$ such that

$$|x-x'|_{\mathcal{X}}\leqslant|f(x)-f(x')|_{\mathcal{Y}}+\varepsilon$$

for any $x,x'\in\mathcal{X}$.

5.4.4. Exercise.

(a) *Show that*

$$|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}'}=\inf\{\,\varepsilon>0\;:\;\mathcal{X}\leqslant\mathcal{Y}+\varepsilon\quad\textit{and}\quad\mathcal{Y}\leqslant\mathcal{X}+\varepsilon\,\}$$

defines a metric on the space of (isometry classes) of compact metric spaces.

(b) *Moreover $|{*}-{*}|_{\mathcal{M}'}$ is equivalent to the Gromov–Haudorff metric; that is,*

$$|\mathcal{X}_n-\mathcal{X}_\infty|_{\mathcal{M}}\to 0\qquad\Longleftrightarrow\qquad|\mathcal{X}_n-\mathcal{X}_\infty|_{\mathcal{M}'}\to 0$$

as $n\to\infty$.

5.5 Uniformly totally bonded families

Let $\mathcal{Q}$ be a set of (isometry classes) of compact metric spaces. Suppose that there is a sequence $\varepsilon_n\to 0$ such that for any positive integer n each space $\mathcal{X}$ in $\mathcal{Q}$ admits an ε_n-net with at most n points. Then we say that $\mathcal{Q}$ is *uniformly totally bonded.*

Observe that in this case $\operatorname{diam}\mathcal{X}<\varepsilon_1$ for any $\mathcal{X}$ in $\mathcal{Q}$; that is diameters of spaces in $\mathcal{Q}$ are bounded above.

Fix a real constant C. A measure μ on a metric space $\mathcal{X}$ is called *C-doubling* if

$$\mu[\mathrm{B}(p,2{\cdot}r) < C{\cdot}\mu[\mathrm{B}(p,r)]$$

for any point $p\in\mathcal{X}$ and any positive real r. A measure is called *doubling* if it is *C-doubling* for a some real constant C.

5.5.1. Exercise. *Let $\mathcal{Q}(C,D)$ be the set of all the compact metric spaces with diameter at most D that admit a C-doubling measure. Show that $\mathcal{Q}(C,D)$ is totally bounded.*

Recall that we write $\mathcal{X}\leqslant\mathcal{Y}$ if there is a distance-nondecreasing map $\mathcal{X}\to\mathcal{Y}$.

5.5.2. Exercise.

(a) Let $\mathcal{Y}$ be a compact metric space. Show that the set of all spaces $\mathcal{X}$ such that $\mathcal{X}\leqslant\mathcal{Y}$ is uniformly totally bounded.

(b) Show that for any uniformly totally bounded set $\mathcal{Q}\subset\mathcal{M}$ there is a compact space $\mathcal{Y}$ such that $\mathcal{X}\leqslant\mathcal{Y}$ for any $\mathcal{X}$ in $\mathcal{Q}$.

5.6 Gromov's selection theorem

The following theorem is analogous to Blaschke selection theorems (4.1.3).

5.6.1. Gromov selection theorem. *Let $\mathcal{Q}$ be a closed and totally bounded subset of $\mathcal{M}$. Then $\mathcal{Q}$ is compact.*

5.6.2. Lemma. *$\mathcal{M}$ is complete.*

Proof. Let $(\mathcal{X}_n)$ be a Cauchy sequence in $\mathcal{M}$. Passing to a subsequence if necessary, we can assume that $|\mathcal{X}_n-\mathcal{X}_{n+1}|_{\mathcal{M}}<\tfrac{1}{2^n}$ for each n. In particular, for each n one can equip $\mathcal{W}_n=\mathcal{X}_n\sqcup\mathcal{X}_{n+1}$ with a metric such that inclusions $\mathcal{X}_n\hookrightarrow\mathcal{W}_n$ and $\mathcal{X}_{n+1}\hookrightarrow\mathcal{W}_n$ are distance-preserving, and

$$|\mathcal{X}_n-\mathcal{X}_{n+1}|_{\mathcal{H}(\mathcal{W}_n)}<\tfrac{1}{2^n}$$

for each n.

Set $\mathcal{W}$ to be the disjoint union of all $\mathcal{X}_n$. Let us equip $\mathcal{W}$ with a metric defined the following way:

◇ for any fixed n and any two points $x_n,x_n'\in\mathcal{X}_n$ set

$$|x_n-x_n'|_{\mathcal{W}}=|x_n-x_n'|_{\mathcal{X}_n}$$

◇ for any positive integers $m > n$ and any two points $x_n \in \mathcal{X}_n$ and $x_m \in \mathcal{X}_m$ set

$$|x_n - x_m|_{\mathcal{W}} = \inf\left\{\sum_{i=n}^{m-1} |x_i - x_{i+1}|_{\mathcal{W}_i}\right\},$$

where the infimum is taken for all sequences $x_i \in \mathcal{X}_i$.

Observe that $|*-*|_{\mathcal{W}}$ is indeed a metric.

Let $\bar{\mathcal{W}}$ be the completion of $\mathcal{W}$. Note that $|\mathcal{X}_m - \mathcal{X}_n| < \frac{1}{2^{n-1}}$ if $m > n$. Therefore the union of $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \dots \cup \mathcal{X}_n$ forms a $\frac{1}{2^{n-1}}$-net in $\bar{\mathcal{W}}$. Since each $\mathcal{X}_i$ is compact, we get that $\bar{\mathcal{W}}$ admits a compact ε-net for any $\varepsilon > 0$. Whence $\bar{\mathcal{W}}$ is compact.

According to Blaschke selection theorem (4.1.3), we can pass to a subsequence of $(\mathcal{X}_n)$ that converges in $\mathcal{H}(\bar{\mathcal{W}})$ and therefore in $\mathcal{M}$. □

Proof of 5.6.1; "only if" part. If there is no sequence $\varepsilon_n \to 0$ as described in the problem, then for a fixed $\delta > 0$ there is a sequence of spaces $\mathcal{X}_n \in \mathcal{Q}$ such that

$$\operatorname{pack}_\delta \mathcal{X}_n \to \infty \quad \text{as} \quad n \to \infty.$$

Since $\mathcal{Q}$ is compact, this sequence has a partial limit say $\mathcal{X}_\infty \in \mathcal{Q}$. Observe that $\operatorname{pack}_\delta \mathcal{X}_\infty = \infty$. Therefore $\mathcal{X}_\infty$ — a contradiction.

"If" part. Without loss of generality, we may assume that there is a sequence $\varepsilon_n \to 0$ such that $\mathcal{Q}$ is the set of all compact metric spaces $\mathcal{X}$ such that $\operatorname{pack}_{\varepsilon_n} \mathcal{X} \leqslant n$.

Note that $\operatorname{diam} \mathcal{X} \leqslant \varepsilon_1$ for any $\mathcal{X} \in \mathcal{Q}$. Given positive integer n consider set of all metric spaces $\mathcal{W}_n$ with number of points at most n and diameter $\leqslant \varepsilon_1$. Note that $\mathcal{W}_n$ is compact for each n.

Further a maximal ε_n-packing of any $\mathcal{X} \in \mathcal{Q}$ forms a subspace from $\mathcal{W}_n$. Therefore $\mathcal{W}_n \cap \mathcal{Q}$ is a comapct ε_n-net in $\mathcal{Q}$. That is, $\mathcal{Q}$ has compact ε-net for any $\varepsilon > 0$. The ince $\mathcal{Q}$ is a closed set □

In the following exercises *converge* means *converge in the sense of Gromov–Hausdorff.*

5.6.3. Exercise.
(a) *Show that a sequence of compact simply connected length spaces cannot converge to a circle.*
(b) *Construct a sequence of compact simply connected length spaces that converges to a compact nonsimply connected space.*

5.6.4. Exercise.
(a) *Show that a sequence of lenght metrics on the 2-sphere cannot converge to a the unit disc.*
(b) *Construct a sequence of lenght metrics on the 3-sphere that converges to a unit 3-ball.*

5.7 Remarks

Suppose $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$, then there is a metric on the disjoint union

$$\boldsymbol{X} = \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} \mathcal{X}_n$$

such that the restriction of metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric and $\mathcal{X}_n \xrightarrow{\text{H}} \mathcal{X}_\infty$ as subsets in $\boldsymbol{X}$.

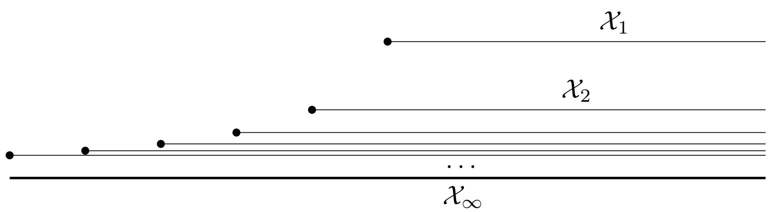
Indeed, since $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$, there is a metric on $\mathcal{V}_n = \mathcal{X}_n \sqcup \mathcal{X}_\infty$ such that the restriction of metric on each $\mathcal{X}_n$ and $\mathcal{X}_\infty$ coincides with its original metric and $|\mathcal{X}_n - \mathcal{X}_\infty|_{\mathcal{H}(\mathcal{V}_n)} < \varepsilon_n$ for some sequence $\varepsilon_n \to 0$. Arguing as in the proof of (iv) in Theorem 5.4.1 we define metric on $\boldsymbol{X}$ by setting

$$\begin{aligned} |x_m - x_n|_{\boldsymbol{X}} &= \inf_{x_\infty} \left\{ |x_m - x_\infty|_{\mathcal{V}_m} + |x_n - x_\infty|_{\mathcal{V}_n} \ : \ \right\}, \\ |x_n - x_\infty|_{\boldsymbol{X}} &= |x_n - x_\infty|_{\mathcal{V}_n} \\ |x_n - x_n'|_{\boldsymbol{X}} &= |x_n - x_n'|_{\mathcal{X}_n} \end{aligned}$$

where $x_n, x_n' \in \mathcal{X}_n$ for every $n \in \mathbb{N} \cup \{\infty\}$.

In other words, the metric on $\boldsymbol{X}$ defines convergence $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}_\infty$. This metric makes possible to talk about limits of sequences $x_n \in \mathcal{X}_n$ as $n \to \infty$, as well as weak limit of a sequence of measures μ_n on $\mathcal{X}_n$ and so on. By that reason it might be useful to fix such metric on $\boldsymbol{X}$. This approach can be also used to define Gromov–Hausdorff convergence of noncompact spaces which will be discussed latter.

We may consider a metric on $\boldsymbol{X}$ such that $\mathcal{X}_n \xrightarrow{\text{H}} \mathcal{X}_\infty$ without assuming that all the spaces $\mathcal{X}_n$ and $\mathcal{X}_\infty$ are compact; in this case we need to use the variation of Hausdorff convergence described in Section 4.2. The limit spaces for this generalized convergence is not uniquely defined. For example if each space $\mathcal{X}_n$ in the sequence is isometric to the half-line, then its limit might be isometric to the half-line or to whole line. The first convergence is evident and the second could be guessed from the diagram.



Often the isometry class of the limit can be fixed by marking a point p_n in each space $\mathcal{X}_n$, it is called *pointed Gromov–Haudorff convergence*

— we say that $(\mathcal{X}_n,p_n)$ converges to $(\mathcal{X}_\infty,p_\infty)$ if there is a metric on $\boldsymbol{X}$ such that $\mathcal{X}_n \xrightarrow{\text{H}} \mathcal{X}_\infty$ and $p_n\to p_\infty$. For example the sequence $(\mathcal{X}_n,p_n)=(\mathbb{R}_+,0)$ converges to $(\mathbb{R}_+,0)$, while $(\mathcal{X}_n,p_n)=(\mathbb{R}_+,n)$ converges to $(\mathbb{R},0)$.

This convergence works nicely for proper metric spaces. The following theorem is an analog of Gromov's selection theorem for pointed Gromov–Haudorff convergence.

5.7.1. Theorem. *Let $\mathcal{Q}$ be a set of isometry classes of pointed proper metric spaces $(\mathcal{X},p)$. Assume that for any $R>0$, the R-balls in the spaces centered at the marked points form a uniformly totally bounded family of spaces. Then $\mathcal{Q}$ is precompact with respect to pointed Gromov–Haudorff convergence.*

Lecture 6

Ultralimits

Here we introduce ultralimits of sequences of points, metric spaces and functions. The ultralimits of metric spaces can be considered as a variation of Gromov–Hausdorff convergence. Our presentation is based on [**kleiner-leeb**].

Our use of ultralimits is very limited; we use them only as a canonical way to pass to a convergent subsequence. (In principle, we could avoid selling our souls to the set-theoretical devil, but in this case we must say "pass to convergent subsequence" too many times.)

6.1 Ultrafilters

We will need the existence of a nonprinciple ultrafilter ω, which we fix once and for all.

Recall that $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N} = \{1, 2, \dots\}$

6.1.1. Definition. *A finitely additive measure ω on $\mathbb{N}$ is called an* ultrafilter *if it satisfies*

(a) $\omega(S) = 0$ or 1 for any subset $S \subset \mathbb{N}$.

An ultrafilter ω is called nonprinciple *if in addition*

(b) $\omega(F) = 0$ for any finite subset $F \subset \mathbb{N}$.

If $\omega(S) = 0$ for some subset $S \subset \mathbb{N}$, we say that S is *ω-small.* If $\omega(S) = 1$, we say that S contains *ω-almost all* elements of $\mathbb{N}$.

Classical definition. More commonly, a nonprinciple ultrafilter is defined as a collection, say $\mathfrak{F}$, of sets in $\mathbb{N}$ such that

1. if $P \in \mathfrak{F}$ and $Q \supset P$, then $Q \in \mathfrak{F}$,
2. if $P, Q \in \mathfrak{F}$, then $P \cap Q \in \mathfrak{F}$,
3. for any subset $P \subset \mathbb{N}$, either P or its complement is an element of $\mathfrak{F}$.

4. if $F \subset \mathbb{N}$ is finite, then $F \notin \mathfrak{F}$.

Setting $P \in \mathfrak{F} \Leftrightarrow \omega(P) = 1$ makes these two definitions equivalent.

A nonempty collection of sets $\mathfrak{F}$ that does not include the empty set and satisfies only conditions 1 and 2 is called a *filter*; if in addition $\mathfrak{F}$ satisfies Condition 3 it is called an *ultrafilter*. From Zorn's lemma, it follows that every filter contains an ultrafilter. Thus there is an ultrafilter $\mathfrak{F}$ contained in the filter of all complements of finite sets; clearly this $\mathfrak{F}$ is nonprinciple.

Stone–Čech compactification. Given a set $S \subset \mathbb{N}$, consider subset Ω_S of all ultrafilters ω such that $\omega(S) = 1$. It is straightforward to check that the sets Ω_S for all $S \subset \mathbb{N}$ form a topology on the set of ultrafilters on $\mathbb{N}$. The obtained space is called *Stone–Čech compactification* of $\mathbb{N}$; it is usually denoted as $\beta\mathbb{N}$.

There is a natural embedding $\mathbb{N} \hookrightarrow \beta\mathbb{N}$ defined as $n \mapsto \omega_n$, where ω_n is the principle ultrafilter such that $\omega_n(S) = 1$ if and only if $n \in S$. Using the described embedding, we can (and will) consider $\mathbb{N}$ as a subset of $\beta\mathbb{N}$.

The space $\beta\mathbb{N}$ is the maximal compact Hausdorff space that contains $\mathbb{N}$ as an everywhere dense subset. More precisely, for any compact Hausdorff space $\mathcal{X}$ and a map $f\colon \mathbb{N} \to \mathcal{X}$ there is unique continuous map $\bar{f}\colon \beta\mathbb{N} \to X$ such that the restriction $\bar{f}|_{\mathbb{N}}$ coincides with f.

6.2 Ultralimits of points

Fix an ultrafilter ω. Assume (x_n) is a sequence of points in a metric space $\mathcal{X}$. Let us define the ω-*limit* of (x_n) as the point x_ω such that for any $\varepsilon > 0$, ω-almost all elements of (x_n) lie in $\mathrm{B}(x_\omega, \varepsilon)$; that is,

$$\omega\{\, n \in \mathbb{N} : |x_\omega - x_n| < \varepsilon \,\} = 1.$$

In this case, we will write

$$x_\omega = \lim_{n\to\omega} x_n \quad \text{or} \quad x_n \to x_\omega \text{ as } n \to \omega.$$

For example if ω is the principle ultrafilter such that $\omega(\{n\}) = 1$ for some $n \in \mathbb{N}$, then $x_\omega = x_n$.

Note that ω-limits of a sequence and its subsequence may differ. For example, in general

$$\lim_{n\to\omega} x_n \neq \lim_{n\to\omega} x_{2\cdot n}.$$

6.2.1. Proposition. *Let ω be a nonprinciple ultrafilter. Assume (x_n) is a sequence of points in a metric space $\mathcal{X}$ and $x_n \to x_\omega$ as $n \to \omega$.*

Then x_ω is a partial limit of the sequence (x_n); that is, there is a subsequence $(x_n)_{n\in S}$ that converges to x_ω in the usual sense.

Remark. A nonprinciple ultrafilter ω is called *selective* if for any partition of $\mathbb{N}$ into sets $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ such that $\omega(C_\alpha) = 0$ for each α, there is a set $S \subset \mathbb{N}$ such that $\omega(S) = 1$ and $S \cap C_\alpha$ is a one-point set for each $\alpha \in \mathcal{A}$.

The existence of a selective ultrafilter follows from the continuum hypothesis; it was proved by Walter Rudin in [**rudin**].

For a selective ultrafilter ω, there is a stronger version of Proposition 6.2.1; namely we can assume that the subsequence $(x_n)_{n\in S}$ can be chosen so that $\omega(S) = 1$. (So, if needed, you may assume that the ultrafilter ω is selective and use this stronger version of the proposition.)

Proof. Given $\varepsilon > 0$, set $S_\varepsilon = \{\, n \in \mathbb{N} : |x_n - x_\omega| < \varepsilon \,\}$.

Note that $\omega(S_\varepsilon) = 1$ for any $\varepsilon > 0$. Since ω is nonprinciple, the set S_ε is infinite. Therefore we can choose an increasing sequence (n_k) such that $n_k \in S_{\frac{1}{k}}$ for each $k \in \mathbb{N}$. Clearly $x_{n_k} \to x_\omega$ as $k \to \infty$. □

The following proposition is analogous to the statement that any sequence in a compact metric space has a convergent subsequence; it can be proved the same way.

6.2.2. Proposition. *Let $\mathcal{X}$ be a compact metric space. Then any sequence of points (x_n) in $\mathcal{X}$ has unique ω-limit x_ω.*

In particular, a bounded sequence of real numbers has a unique ω-limit.

Alternatively, the sequence (x_n) can be regarded as a map $\mathbb{N} \to \mathcal{X}$. In this case the map $\mathbb{N} \to \mathcal{X}$ can be extended to a continuous map from the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Then the ω-limit x_ω can be regarded as the image of ω.

The following lemma is an ultralimit analog of Cauchy convergence test.

6.2.3. Lemma. *Let (x_n) be a sequence of points in a complete space $\mathcal{X}$. Assume for each subsequence (y_n) of (x_n), the ω-limit*

$$y_\omega = \lim_{n\to\omega} y_n \in \mathcal{X}$$

is defined and does not depend on the choice of subsequence, then the sequence (x_n) converges in the usual sense.

Proof. Assume that (x_n) is a Cauchy sequence. Then for some $\varepsilon > 0$, there is a subsequence (y_n) of (x_n) such that $|x_n - y_n| \geqslant \varepsilon$ for all n.

It follows that $|x_\omega - y_\omega| \geqslant \varepsilon$, a contradiction. □

6.3 Ultralimits of spaces

From now on, ω denotes a nonprinciple ultrafilter on the set of natural numbers.

Let $\mathcal{X}_n$ be a sequence of metric spaces. Consider all sequences of points $x_n \in \mathcal{X}_n$. On the set of all such sequences, define a pseudometric by

$$❶ \qquad |(x_n) - (y_n)| = \lim_{n\to\omega} |x_n - y_n|.$$

Note that the ω-limit on the right hand side is always defined and takes a value in $[0, \infty]$.

Set $\mathcal{X}_\omega$ to be the corresponding metric space; that is, the underlying set of $\mathcal{X}_\omega$ is formed by classes of equivalence of sequences of points $x_n \in \mathcal{X}_n$ defined by

$$(x_n) \sim (y_n) \ \Leftrightarrow \ \lim_{n\to\omega} |x_n - y_n| = 0$$

and the distance is defined by ❶.

The space $\mathcal{X}_\omega$ is called *ω-limit* of $\mathcal{X}_n$. Typically $\mathcal{X}_\omega$ will denote the ω-limit of sequence $\mathcal{X}_n$; we may also write

$$\mathcal{X}_n \to \mathcal{X}_\omega \ \text{ as } \ n \to \omega \ \text{ or } \ \mathcal{X}_\omega = \lim_{n\to\omega} \mathcal{X}_n.$$

Given a sequence $x_n \in \mathcal{X}_n$, we will denote by x_ω its equivalence class which is a point in $\mathcal{X}_\omega$; equivalently we will write

$$x_n \to x_\omega \ \text{ as } \ n \to \omega \ \text{ or } \ x_\omega = \lim_{n\to\omega} x_n.$$

6.3.1. Observation. *The ω-limit of any sequence of metric spaces is complete.*

Proof. Let $\mathcal{X}_n$ be a sequence of metric spaces and $\mathcal{X}_n \to \mathcal{X}_\omega$ as $n \to \omega$.

Fix a Cauchy sequence $x_m \in \mathcal{X}_\omega$. Passing to a subsequence we can assume that $|x_m - x_{m-1}|_{\mathcal{X}_\omega} < \frac{1}{2^m}$ for any m.

Let us choose double sequence $x_{n,m} \in \mathcal{X}_n$ such that for any fixed m we have $x_{n,m} \to x_m$ as $n \to \omega$. Note that $|x_{n,m} - x_{n,m-1}| < \frac{1}{2^m}$ for ω-almost all n. It follows that we can choose a nested sequence of sets

$$\mathbb{N} = S_1 \supset S_2 \supset \dots$$

such that

- ⋄ $\omega(S_m) = 1$ for each m,
- ⋄ $k \geqslant m$ for any $k \in S_m$, and

◇ if $n \in S_m$, then

$$|x_{n,m} - x_{n,m-1}| < \tfrac{1}{2^m}$$

Consider the sequence $y_n = x_{n,m(n)}$, where $m(n)$ is the largest value such that $m(n) \in S_m$. Denote by $y \in \mathcal{X}_\omega$ its ω-limit.

Observe that by construction $x_n \to y$ as $n \to \infty$. Hence the statement follows. □

6.3.2. Observation. *The ω-limit of any sequence of length spaces is geodesic.*

Proof. If $\mathcal{X}_n$ is a sequence length spaces, then for any sequence of pairs $x_n, y_n \in X_n$ there is a sequence of $\frac{1}{n}$-midpoints z_n.

Let $x_n \to x_\omega$, $y_n \to y_\omega$ and $z_n \to z_\omega$ as $n \to \omega$. Note that z_ω is a midpoint of x_ω and y_ω in $\mathcal{X}^\omega$.

By Observation 6.3.1, $\mathcal{X}^\omega$ is complete. Applying Lemma 1.9.5 we get the statement. □

6.3.3. Exercise. *Show that an ultralimit of metric trees is a metric tree.*

6.4 Ultrapower

If all the metric spaces in the sequence are identical $\mathcal{X}_n = \mathcal{X}$, its ω-limit $\lim_{n\to\omega} \mathcal{X}_n$ is denoted by $\mathcal{X}^\omega$ and called ω-power of $\mathcal{X}$.

6.4.1. Exercise. *For any point $x \in \mathcal{X}$, consider the constant sequence $x_n = x$ and set $\iota(x) = \lim_{n\to\omega} x_n \in \mathcal{X}^\omega$.*

(a) Show that $\iota\colon \mathcal{X} \to \mathcal{X}^\omega$ is distance-preserving embedding. (So we can and will consider $\mathcal{X}$ as a subset of $\mathcal{X}^\omega$.)

(b) Show that ι is onto if and only if $\mathcal{X}$ compact.

(c) Show that if $\mathcal{X}$ is proper, then $\iota(\mathcal{X})$ forms a metric component of $\mathcal{X}^\omega$; that is, a subset of $\mathcal{X}^\omega$ that lie on finite distance from a given point.

6.4.2. Observation. *Let $\mathcal{X}$ be a complete metric space. Then $\mathcal{X}^\omega$ is geodesic space if and only if $\mathcal{X}$ is a length space.*

Proof. Assume $\mathcal{X}^\omega$ is geodesic space. Then any pair of points $x, y \in \mathcal{X}$ has a midpoint $z_\omega \in \mathcal{X}^\omega$. Fix a sequence of points $z_n \in \mathcal{X}$ such that $z_n \to z_\omega$ as $n \to \omega$.

Note that $|x - z_n|_{\mathcal{X}} \to \frac{1}{2}\cdot|x - y|_{\mathcal{X}}$ and $|y - z_n|_{\mathcal{X}} \to \frac{1}{2}\cdot|x - y|_{\mathcal{X}}$ as $n \to \omega$. In particular, for any $\varepsilon > 0$, the point z_n is an ε-midpoint of x and y for ω-almost all n. It remains to apply Lemma 1.9.5.

The "if"-part follows from Observation 6.3.2. □

6.4.3. Exercise. *Assume $\mathcal{X}$ is a complete length space and $p,q\in\mathcal{X}$ cannot be joined by a geodesic in $\mathcal{X}$. Then there are at least two distinct geodesics between p and q in the ultrapower $\mathcal{X}^\omega$.*

6.4.4. Exercise. *Construct a proper metric space $\mathcal{X}$ such that $\mathcal{X}^\omega$ is not proper; that is, there is a point $p\in\mathcal{X}^\omega$ and $R<\infty$ such that the closed ball $\overline{\mathrm{B}}[p,R]_{\mathcal{X}^\omega}$ is not compact.*

6.5 Tangent and asymptotic spaces

Choose a space $\mathcal{X}$ and a sequence of $\lambda_n>0$. Consider the sequence of scalings $\mathcal{X}_n=\lambda_n{\cdot}\mathcal{X}=(\mathcal{X},\lambda_n{\cdot}|{*}-{*}|_{\mathcal{X}})$.

Choose a point $p\in\mathcal{X}$ and denote by p_n the corresponding point in $\mathcal{X}_n$. Consider the ω-limit $\mathcal{X}_\omega$ of $\mathcal{X}_n$ (one may denote it by $\lambda_\omega{\cdot}\mathcal{X}$); set p_ω to be the ω-limit of p_n.

If $\lambda_n\to 0$ as $n\to\omega$, then the metric component of p_ω in $\mathcal{X}_\omega$ is called *ω-tangent space* at p and denoted by $\mathrm{T}_p^{\lambda_\omega}\mathcal{X}$ (or $\mathrm{T}_p^{\omega}\mathcal{X}$ if $\lambda_n=n$).

If $\lambda_n\to\infty$ as $n\to\omega$, then the metric component of p_ω in called *ω-asymptotic space*[1] and denoted by $\operatorname{Asym}\mathcal{X}$. Note that the space $\operatorname{Asym}\mathcal{X}$ and its point p_ω does not depend on the choice of $p\in\mathcal{X}$.

6.5.1. Exercise. *Let $\mathcal{L}$ be the Lobachevsky plane; $\mathcal{T}=\operatorname{Asym}\mathcal{L}$.*

(a) Show that $\mathcal{T}$ is a complete metric tree.

(b) Show that $\mathcal{T}$ has continuum degree *at any point; that is, for any point $t\in\mathcal{T}$ the set of connected components of the complement $\mathcal{T}\backslash\{t\}$ has cardinality continuum.*

(c) Show that $\mathcal{T}$ is homogeneous; that is given two points $s,t\in\mathcal{T}$ there is an isometry of $\mathcal{T}$ that maps s to t.

(d) Prove (a)–(c) if $\mathcal{L}$ is Lobachevsky space and/or for the infinite 3-regular[2] tree with unit edge.

As it shown in [**dyubina-polterovich**], the properties *(a)* and *(b)* describe the tree $\mathcal{T}$ up to isometry. In particular, the asymptotic space of Lobachevsky plane does not depend on the choice of ultrafilter and the sequence $\lambda_n\to\infty$. In general, the tangent and asymptotic spaces depend on number of choices — we need to fix a sequence λ_n and an nonprinciple ultrafiler ω.

[1]Often it is called *asymptotic cone* despite that it is not a cone in general; this name is used since in good cases it has a cone structure.

[2]that is, degree of any vertex is 3.

Part II

Alexandrov geometry

Lecture 7

Introduction

7.1 Manifesto

Alexandrov geometry can use "back to Euclid" as a slogan. Alexandrov spaces are defined via axioms similar to those given by Euclid, but certain equalities are changed to inequalities. Depending on the sign of the inequalities, we get Alexandrov spaces with *curvature bounded above* or *curvature bounded below*. The definitions of the two classes of spaces are similar, but their properties and known applications are quite different.

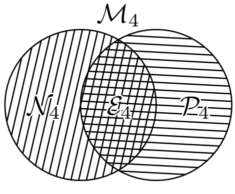
Consider the space $\mathcal{M}_4$ of all isometry classes of 4-point metric spaces. Each element in $\mathcal{M}_4$ can be described by 6 numbers — the distances between all 6 pairs of its points, say $\ell_{i,j}$ for $1 \leqslant i < j \leqslant 4$ modulo permutations of the index set $(1, 2, 3, 4)$. These 6 numbers are subject to 12 triangle inequalities; that is,

$$\ell_{i,j} + \ell_{j,k} \geqslant \ell_{i,k}$$

holds for all i, j and k, where we assume that $\ell_{j,i} = \ell_{i,j}$ and $\ell_{i,i} = 0$.

The space $\mathcal{M}_4$ comes with topology. It can defined as a quotient of the cone in $\mathbb{R}^6$ by permutations of the 4-points of the space. And, the same topology is induced on $\mathcal{M}_4$ by the Gromov–Hausdorff metric.

Consider the subset $\mathcal{E}_4 \subset \mathcal{M}_4$ of all isometry classes of 4-point metric spaces that admit isometric embeddings into Euclidean space.

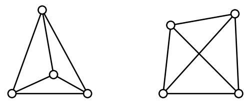


7.1.1. Claim. *The complement $\mathcal{M}_4 \backslash \mathcal{E}_4$ has two connected components.*

A proof of the claim can extracted from 7.3.3.

The definition of Alexandrov spaces is based on this claim. Let us denote one of the components by $\mathcal{P}_4$ and the other by $\mathcal{N}_4$. Here $\mathcal{P}$ and $\mathcal{N}$ stand for *positive* and *negative curvature* because spheres have no quadruples of type $\mathcal{N}_4$ and hyperbolic space has no quadruples of type $\mathcal{P}_4$.

A metric space, with length metric, that has no quadruples of points of type $\mathcal{P}_4$ or $\mathcal{N}_4$ respectively is called an Alexandrov space with non-positive (CAT(0)) or non-negative curvature (CBB(0)).



Let us describe the subdivision into $\mathcal{P}_4$, $\mathcal{E}_4$ and $\mathcal{N}_4$ intuitively. Imagine that you move out of $\mathcal{E}_4$ — your path is a one parameter family of 4-point metric spaces. The last thing you see in $\mathcal{E}_4$ is one of the two plane configurations shown on the diagram. If you see the left configuration then you move into $\mathcal{N}_4$; if it is the one on the right, then you move into $\mathcal{P}_4$. More degenerate pictures can be avoided, for example a triangle with a point on a side. From such a configuration one may move in $\mathcal{N}_4$ and in $\mathcal{P}_4$ (as well as come back to $\mathcal{E}_4$).

Here is an exercise, solving which would force you to rebuild a considerable part of Alexandrov geometry. It might be helpful to spend some time thinking about this exercise before proceeding.

7.1.2. Advanced exercise. *Assume $\mathcal{X}$ is a complete metric space with length metric, containing only quadruples of type $\mathcal{E}_4$. Show that $\mathcal{X}$ is isometric to a convex set in a Hilbert space.*

In the definition above, instead of Euclidean space one can take hyperbolic space of curvature -1. In this case, one obtains the definition of spaces with curvature bounded above or below by -1 (CAT(-1) or CBB(-1)).

To define spaces with curvature bounded above or below by 1 (CAT(1) or CBB(1)), one has to take the unit 3-sphere and specify that only the quadruples of points such that each of the four triangles has perimeter less than $2\cdot\pi$ are checked. The latter condition could be considered as a part of the *spherical triangle inequality.*

7.2 Triangles, hinges and angles

Triangles. For a triple of points $p, q, r \in \mathcal{X}$, a choice of a triple of geodesics $([qr], [rp], [pq])$ will be called a *triangle*; we will use the short notation $[pqr] = ([qr], [rp], [pq])$.

Given a triple $p, q, r \in \mathcal{X}$ there may be no triangle $[pqr]$ simply because one of the pairs of these points cannot be joined by a geodesic.

Also, many different triangles with these vertices may exist, any of which can be denoted by $[pqr]$. However, if we write $[pqr]$, it means that we have made a choice of such a triangle; that is, we have fixed a choice of the geodesics $[qr]$, $[rp]$, and $[pq]$.

The value
$$|p-q|+|q-r|+|r-p|$$
will be called the *perimeter of the triangle* $[pqr]$.

Model triangles. Let $\mathcal{X}$ be a metric space and $p,q,r\in\mathcal{X}$. Let us define the *model triangle* $[\tilde p\tilde q\tilde r]$ (briefly, $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$) to be a triangle in the Euclidean plane $\mathbb{E}^2$ such that
$$\begin{aligned}|\tilde p-\tilde q|_{\mathbb{E}^2}&=|p-q|_{\mathcal{X}},\\|\tilde q-\tilde r|_{\mathbb{E}^2}&=|q-r|_{\mathcal{X}},\\|\tilde r-\tilde p|_{\mathbb{E}^2}&=|r-p|_{\mathcal{X}}.\end{aligned}$$

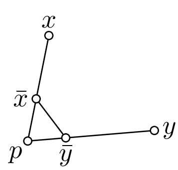
In the same way we can define the *hyperbolic* and the *spherical model triangles* $\tilde\triangle(pqr)_{\mathbb{H}^2}$, $\tilde\triangle(pqr)_{\mathbb{S}^2}$ in the hyperbolic plane $\mathbb{H}^2$ and the unit sphere $\mathbb{S}^2$. In the latter case the model triangle is said to be defined if in addition
$$|p-q|+|q-r|+|r-p|<2{\cdot}\pi.$$
In this case the model triangle again exists and is unique up to an isometry of $\mathbb{S}^2$.

Model angles. If $[\tilde p\tilde q\tilde r]=\tilde\triangle(pqr)_{\mathbb{E}^2}$ and $|p-q|,|p-r|>0$, the angle measure of $[\tilde p\tilde q\tilde r]$ at $\tilde p$ will be called the *model angle* of the triple p, q, r and will be denoted by $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{E}^2}$. In the same way we define $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{H}^2}$ and $\tilde\measuredangle(p\,{}^q_r)_{\mathbb{S}^2}$; in the latter case we assume in addition that the model triangle $\tilde\triangle(pqr)_{\mathbb{S}^2}$ is defined.

We may use the notation $\tilde\measuredangle(p\,{}^q_r)$ if it is evident which of the model spaces $\mathbb{H}^2$, $\mathbb{E}^2$ or $\mathbb{S}^2$ is meant.

Hinges. Let $p,x,y\in\mathcal{X}$ be a triple of points such that p is distinct from x and y. A pair of geodesics $([px],[py])$ will be called a *hinge* and will be denoted by $[p\,{}^x_y]=([px],[py])$.

Angles. Given a hinge $[p\,{}^x_y]$, we define its *angle* as the limit



❶
$$\measuredangle[p\,{}^x_y]:=\lim_{\bar x,\bar y\to p}\tilde\measuredangle(p\,{}^{\bar x}_{\bar y}),$$
where $\bar x\in\,]px]$ and $\bar y\in\,]py]$. The angle $\measuredangle[p\,{}^x_y]$ is defined if the limit exists.

It is straightforward to check that in ❶, one can use $\tilde\measuredangle(p\,{}^{\bar x}_{\bar y})_{\mathbb{S}^2}$ or $\tilde\measuredangle(p\,{}^{\bar x}_{\bar y})_{\mathbb{H}^2}$ or $\tilde\measuredangle(p\,{}^{\bar x}_{\bar y})_{\mathbb{E}^2}$, the result will be the same.

7.2.1. Exercise. *Give an example of a hinge $[p\,{}^{x}_{y}]$ in a metric space with undefined angle $\measuredangle[p\,{}^{x}_{y}]$.*

7.2.2. Exercise. *Suppose that for three geodesics $[px]$, $[py]$, and $[pz]$ in a metric space, the angles $\alpha = \measuredangle[p\,{}^{x}_{y}]$, $\beta = \measuredangle[p\,{}^{y}_{z}]$, and $\gamma = \measuredangle[p\,{}^{z}_{x}]$ are defined. Show that α, β and γ satisfy all triangle inequalities:*

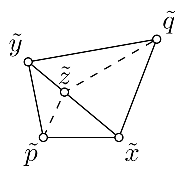
$$\alpha \leqslant \beta + \gamma, \qquad \beta \leqslant \gamma + \alpha, \qquad \gamma \leqslant \alpha + \beta,$$

7.3 Definitions

Curvature bounded above. Given a quadruple of points p, q, x, y in a metric space $\mathcal{X}$, consider two model triangles $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{E}^2}$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)_{\mathbb{E}^2}$ with common side $[\tilde x\tilde y]$.

If the inequality

$$|p - q|_{\mathcal{X}} \leqslant |\tilde p - \tilde z|_{\mathbb{E}^2} + |\tilde z - \tilde q|_{\mathbb{E}^2}$$



holds for any point $\tilde z \in [\tilde x\tilde y]$, then we say that the quadruple p, q, x, y *satisfies* CAT(0) *comparison.*

If we do the same for spherical model triangles $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{S}^2}$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)_{\mathbb{S}^2}$, then we arrive at the definition of CAT(1) comparison. If one of the spherical model triangles is undefined,[1] then it is assumed that CAT(1) comparison automatically holds for this quadruple.

We can do the same for the *model plane* of curvature κ; that is, a sphere if $\kappa > 0$, Euclidean plane if $\kappa = 0$ and Lobachevsky plane if $\kappa < 0$. In this case we arrive at the definition of CAT(κ) comparison. However we will mostly consider CAT(0) comparison and occasionally CAT(1) comparison; so, if you see CAT(κ), you can assume that κ is 0 or 1.

If all quadruples in a metric space $\mathcal{X}$ satisfy CAT(κ) comparison, then we say that the space $\mathcal{X}$ is CAT(κ) (we use CAT(κ) as an adjective).

Here CAT is an acronym for Cartan, Alexandrov, and Toponogov. It was coined by Mikhael Gromov in 1987, but it should be pronounced

[1] That is, if

$$|p - x| + |p - y| + |x - y| \geqslant 2\cdot\pi \quad\text{or}\quad |q - x| + |q - y| + |x - y| \geqslant 2\cdot\pi.$$

as "cat" in the sense of "miauw". Originally, Alexandrov called these spaces "$\mathfrak{R}_\kappa$ domain"; this term is still in use.)

7.3.1. Exercise. *Show that a metric space $\mathcal{X}$ is* CAT(0) *if and only if for any quadruple of points p, q, x, y in $\mathcal{X}$ there is a quadruple $\tilde p, \tilde q, \tilde x, \tilde y$ in $\mathbb{E}^2$ such that*

$$\begin{aligned} |\tilde p - \tilde q| &= |p - q|, & |\tilde x - \tilde y| &= |x - y|, \\ |\tilde p - \tilde x| &\leqslant |p - x|, & |\tilde p - \tilde y| &\leqslant |p - y|, \\ |\tilde q - \tilde x| &\leqslant |q - x|, & |\tilde q - \tilde y| &\leqslant |q - y|. \end{aligned}$$

Curvature bounded below. If the inequality

$$\tilde\measuredangle(p\,{}^x_y)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^y_z)_{\mathbb{E}^2} + \tilde\measuredangle(p\,{}^z_x)_{\mathbb{E}^2} \leqslant 2\cdot\pi$$

holds for points p, x, y, z in a metric space $\mathcal{X}$, then we say that the quadruple p, x, y, z *satisfies* CBB(0) *comparison.*

If we do the same for spherical or hyperbolic model angles, then we arrive at the definition of CBB(1) or CBB(−1) comparison. Here CBB(κ) is an abbreviation of *curvature bounded below by κ.* If one of one of the model angles is undefined, then we assume that CBB(1) comparison automatically holds for this quadruple.

We can do the same for the model plane of curvature κ. In this case we arrive at the definition of CAT(κ) comparison. But we will mostly consider CBB(0) comparison and occasionally CBB(1) comparison; so, if you see CBB(κ), you can assume that κ is 0 or 1.

If all quadruples in a metric space $\mathcal{X}$ satisfy CBB(κ) comparison, then we say that the space $\mathcal{X}$ is CBB(κ). (Again — CBB(κ) is an adjective.)

7.3.2. Exercise. *Show that a metric space $\mathcal{X}$ is* CBB(0) *if and only if for any quadruple of points $p, x, y, z \in \mathcal{X}$, there is a quadruple of points $\tilde p, \tilde x, \tilde y, \tilde z \in \mathbb{E}^2$ such that*

$$\begin{aligned} |p - x|_{\mathcal{X}} &= |\tilde p - \tilde x|_{\mathbb{E}^2}, & |p - y|_{\mathcal{X}} &= |\tilde p - \tilde y|_{\mathbb{E}^2}, & |p - z|_{\mathcal{X}} &= |\tilde p - \tilde z|_{\mathbb{E}^2}, \\ |x - y|_{\mathcal{X}} &\leqslant |\tilde x - \tilde y|_{\mathbb{E}^2}, & |y - z|_{\mathcal{X}} &\leqslant |\tilde y - \tilde z|_{\mathbb{E}^2}, & |z - x|_{\mathcal{X}} &\leqslant |\tilde z - \tilde x|_{\mathbb{E}^2} \end{aligned}$$

for all i and j.

7.3.3. Exercise. *Suppose that a quadruple of points satisfies* CAT(0) *and* CBB(0) *for all labeling. Show that the quadruple is isometric to a subset of Euclidean space.*

Observe that in order to check CAT(κ) or CBB(κ) comparison, it is sufficient to know the 6 distances between all pairs of points in the quadruple. This observation implies the following.

7.3.4. Proposition. *Any Gromov–Hausdorff limit (as well as ultra limit) of a sequence of* $\mathrm{CAT}(\kappa)$ *or* $\mathrm{CBB}(\kappa)$ *spaces is* $\mathrm{CAT}(\kappa)$ *or* $\mathrm{CBB}(\kappa)$ *respectively.*

In the proposition above, it does not matter which definition of convergence for metric spaces you use, as long as any quadruple of points in the limit space can be arbitrarily well approximated by quadruples in the sequence of metric spaces.

7.4 Products and cones

Given two metric spaces $\mathcal{U}$ and $\mathcal{V}$, the *product space* $\mathcal{U}\times\mathcal{V}$ is defined as the set of all pairs (u,v) where $u\in\mathcal{U}$ and $v\in\mathcal{V}$ with the metric defined by formula

$$|(u_1,v_1)-(u_2,v_2)|_{\mathcal{U}\times\mathcal{V}}=\sqrt{|u_1-u_2|^2_{\mathcal{U}}+|v_1-v_2|^2_{\mathcal{V}}}.$$

7.4.1. Proposition. *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be* $\mathrm{CAT}(0)$ *spaces. Then the product space* $\mathcal{U}\times\mathcal{V}$ *is* $\mathrm{CAT}(0)$.

Proof. Fix a quadruple in $\mathcal{U}\times\mathcal{V}$:

$$p=(p_1,p_2),\qquad q=(q_1,q_2),\qquad x=(x_1,x_2),\qquad y=(y_1,y_2).$$

For the quadruple p_1,q_1,x_1,y_1 in $\mathcal{U}$, construct two model triangles $[\tilde p_1\tilde x_1\tilde y_1]=\tilde\triangle(p_1x_1y_1)_{\mathbb{E}^2}$ and $[\tilde q_1\tilde x_1\tilde y_1]=\tilde\triangle(q_1x_1y_1)_{\mathbb{E}^2}$. Similarly, for the quadruple p_2,q_2,x_2,y_2 in $\mathcal{V}$ construct two model triangles $[\tilde p_2\tilde x_2\tilde y_2]$ and $[\tilde q_2\tilde x_2\tilde y_2]$.

Consider four points in $\mathbb{E}^4=\mathbb{E}^2\times\mathbb{E}^2$

$$\tilde p=(\tilde p_1,\tilde p_2),\qquad \tilde q=(\tilde q_1,\tilde q_2),\qquad \tilde x=(\tilde x_1,\tilde x_2),\qquad \tilde y=(\tilde y_1,\tilde y_2).$$

Note that the triangles $[\tilde p\tilde x\tilde y]$ and $[\tilde q\tilde x\tilde y]$ in $\mathbb{E}^4$ are isometric to the model triangles $\tilde\triangle(pxy)_{\mathbb{E}^2}$ and $\tilde\triangle(qxy)_{\mathbb{E}^2}$.

If $\tilde z=(\tilde z_1,\tilde z_2)\in[\tilde x\tilde y]$, then $\tilde z_1\in[\tilde x_1\tilde y_1]$ and $\tilde z_2\in[\tilde x_2\tilde y_2]$ and

$$\begin{aligned}|\tilde z-\tilde p|^2_{\mathbb{E}^4}&=|\tilde z_1-\tilde p_1|^2_{\mathbb{E}^2}+|\tilde z_2-\tilde p_2|^2_{\mathbb{E}^2},\\ |\tilde z-\tilde q|^2_{\mathbb{E}^4}&=|\tilde z_1-\tilde q_1|^2_{\mathbb{E}^2}+|\tilde z_2-\tilde q_2|^2_{\mathbb{E}^2},\\ |p-q|^2_{\mathcal{U}\times\mathcal{V}}&=|p_1-q_1|^2_{\mathcal{U}}+|p_2-q_2|^2_{\mathcal{V}}.\end{aligned}$$

Therefore $\mathrm{CAT}(0)$ comparison for the quadruples p_1,q_1,x_1,y_1 in $\mathcal{U}$ and p_2,q_2,x_2,y_2 in $\mathcal{V}$ implies $\mathrm{CAT}(0)$ comparison for the quadruples p,q,x,y in $\mathcal{U}\times\mathcal{V}$. □

7.4.2. Exercise. *Assume $\mathcal{U}$ and $\mathcal{V}$ are* CBB(0) *spaces. Show that the product space $\mathcal{U}\times\mathcal{V}$ is* CBB(0)*.*

The *cone* $\mathcal{V}=\operatorname{Cone}\mathcal{U}$ over a metric space $\mathcal{U}$ is defined as the metric space whose underlying set consists of equivalence classes in $[0,\infty)\times\mathcal{U}$ with the equivalence relation "$\sim$" given by $(0,p)\sim(0,q)$ for any points $p,q\in\mathcal{U}$, and whose metric is given by the cosine rule

$$|(p,s)-(q,t)|_{\mathcal{V}}=\sqrt{s^2+t^2-2{\cdot}s{\cdot}t{\cdot}\cos\alpha},$$

where $\alpha=\min\{\pi,|p-q|_{\mathcal{U}}\}$.

The point in the cone $\mathcal{V}$ formed by the equivalence class of $0\times\mathcal{U}$ is called the *tip of the cone* and is denoted by 0 or $0_{\mathcal{V}}$. The distance $|0-v|_{\mathcal{V}}$ is called the norm of v and is denoted by $|v|$ or $|v|_{\mathcal{V}}$. The space $\mathcal{U}$ can be identified with the subset $x\in\mathcal{V}$ such that $|x|=1$.

The points in the cone $\mathcal{V}$ can be multiplied by a real number $\lambda\geqslant 0$; namely, if $x=(x',r)$, then $\lambda{\cdot}x:=(x',\lambda{\cdot}r)$.

7.4.3. Proposition. *Let $\mathcal{U}$ be a metric space. Then* $\operatorname{Cone}\mathcal{U}$ *is* CAT(0) *if and only if $\mathcal{U}$ is* CAT(1)*.*

Proof; "if" part. Given a point $x\in\operatorname{Cone}\mathcal{U}$, denote by x' its projection to $\mathcal{U}$ and by $|x|$ the distance from x to the tip of the cone; if x is the tip, then $|x|=0$ and we can take any point of $\mathcal{U}$ as x'.

Let p, q, x, y be a quadruple in $\operatorname{Cone}\mathcal{U}$. Assume that the spherical model triangles $[\tilde p'\tilde x'\tilde y']=\tilde\triangle(p'x'y')_{\mathbb{S}^2}$ and $[\tilde q'\tilde x'\tilde y']=\tilde\triangle(q'x'y')_{\mathbb{S}^2}$ are defined. Consider the following points in $\mathbb{E}^3=\operatorname{Cone}\mathbb{S}^2$:

$$\tilde p=|p|{\cdot}\tilde p',\qquad \tilde q=|q|{\cdot}\tilde q',\qquad \tilde x=|x|{\cdot}\tilde x',\qquad \tilde y=|y|{\cdot}\tilde y'.$$

Note that $[\tilde p\tilde x\tilde y]\stackrel{iso}{=\!=}\tilde\triangle(pxy)_{\mathbb{E}^2}$ and $[\tilde q\tilde x\tilde y]\stackrel{iso}{=\!=}\tilde\triangle(qxy)_{\mathbb{E}^2}$. Further note that if $\tilde z\in[\tilde x\tilde y]_{\mathbb{E}^3}$, then $\tilde z'=\tilde z/|\tilde z|$ lies on the geodesic $[\tilde x'\tilde y']_{\mathbb{S}^2}$. Therefore the CAT(1) comparison for $|p'-q'|$ with $\tilde z'\in[\tilde x'\tilde y']_{\mathbb{S}^2}$ implies the CAT(0) comparison for $|p-q|$ with $\tilde z\in[\tilde x\tilde y]_{\mathbb{E}^3}$.

"Only-if" part. Suppose that $\tilde p'$, $\tilde q'$, $\tilde x'$, $\tilde y'$ are defined as above. Assume all these points lie in a half-space of $\mathbb{E}^3=\operatorname{Cone}\mathbb{S}^2$ with origin at its boundary. Then we can choose positive values a, b, c, and d such that the points $a{\cdot}\tilde p'$, $b{\cdot}\tilde q'$, $c{\cdot}\tilde x'$, $d{\cdot}\tilde y'$ lie in one plane. Consider the corresponding points $a{\cdot}p'$, $b{\cdot}q'$, $c{\cdot}x'$, $d{\cdot}y'$ in $\operatorname{Cone}\mathcal{U}$. Applying the CAT(0) comparison for the these points leads to CAT(1) comparison for the quadruple p', q', x', y' in $\mathcal{U}$.

It remains to consider the case when $\tilde p'$, $\tilde q'$, $\tilde x'$, $\tilde y'$ do not in a half-space. Fix $\tilde z'\in[\tilde x'\tilde y']_{\mathbb{S}^2}$. Observe that

$$|\tilde p'-\tilde x'|_{\mathbb{S}^2}+|\tilde q'-\tilde x'|_{\mathbb{S}^2}\leqslant|\tilde p'-\tilde z'|_{\mathbb{S}^2}+|\tilde q'-\tilde z'|_{\mathbb{S}^2}$$

or

$$|\tilde{p}' - \tilde{y}'|_{\mathbb{S}^2} + |\tilde{q}' - \tilde{y}'|_{\mathbb{S}^2} \leqslant |\tilde{p}' - \tilde{z}'|_{\mathbb{S}^2} + |\tilde{q}' - \tilde{z}'|_{\mathbb{S}^2}.$$

That is, in this case, the CAT(1) comparison follow from the triangle inequality. □

7.5 Geodesics

7.5.1. Proposition. *Let $\mathcal{X}$ be a complete length* CAT(0) *space. Then any two points in $\mathcal{X}$ are joint by a unique geodesic.*

Proof. Fix two points $x, y \in \mathcal{X}$. Choose a sequence of approximate midpoints p_n for x and y; that is,

❶ $$|x - p_n| \to \tfrac{1}{2}\cdot|x - y| \quad \text{and} \quad |y - p_n| \to \tfrac{1}{2}\cdot|x - y|$$

as $n \to \infty$.

Consider model triangles $[\tilde{p}_n\tilde{x}\tilde{y}] = \tilde{\triangle}(p_n x y)$. Let $\tilde{z}$ be the midpoint of $\tilde{x}$ and $\tilde{y}$. By ❶, we have that

$$|\tilde{p}_n - \tilde{z}| \to 0$$

as $n \to \infty$.

By CAT(0) comparison,

$$|p_n - p_m|_{\mathcal{X}} \leqslant |\tilde{p}_n - \tilde{z}| + |\tilde{p}_m - \tilde{z}|.$$

Therefore $|p_n - p_m| \to 0$ as $m, n \to \infty$; that is, (p_n) is Cauchy. Clearly the limit of the sequence (p_n) is a midpoint of x and y. Applying 1.9.5b, we get that $\mathcal{X}$ is geodesic.

It remains to prove uniqueness. Suppose there are two geodescics between x and y. Then we can choose two points $p \neq q$ on these geodesics such that $|x - p| = |x - q|$ and therefore $|y - p| = |y - q|$.

Observe that the model triangles $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}(pxy)$ and $[\tilde{q}\tilde{x}\tilde{y}] = \tilde{\triangle}(qxy)$ are degenerate and moreover $\tilde{p} = \tilde{q}$. Applying CAT(0) comparison with $\tilde{z} = \tilde{p} = \tilde{q}$, we get that $|p - q| = 0$, a contradiction. □

The following exercise is an analogous statement for CBB spaces. In general complete length CBB(0) space might fail to be geodesic and uniqueness of geodesic usually does not hold.

7.5.2. Exercise. *Let $\mathcal{X}$ be a complete length* CBB(0) *space. Show that if two geodesics from x to y share yet another point z, then they coincide.*

7.6 Alexandrov's lemma

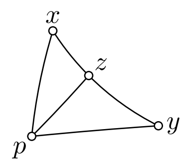
7.6.1. Lemma. *Let p, x, y, z be distinct points in a metric space such that $z \in \,]xy[$. Then the following expressions for the Euclidean model angles have the same sign:*

(a) $\tilde\measuredangle(x\,{}^{p}_{y}) - \tilde\measuredangle(x\,{}^{p}_{z})$,

(b) $\tilde\measuredangle(z\,{}^{p}_{x}) + \tilde\measuredangle(z\,{}^{p}_{y}) - \pi$.

Moreover,

$$\tilde\measuredangle(p\,{}^{x}_{y}) \geqslant \tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{z}_{y}),$$



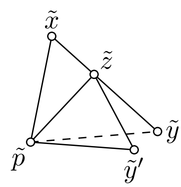
with equality if and only if the expressions in (a) and (b) vanish.

The same holds for the hyperbolic and spherical model angles, but in the latter case one has to assume in addition that

$$|p - z| + |p - y| + |x - y| < 2{\cdot}\pi.$$

Proof. Consider the model triangle $[\tilde x \tilde p \tilde z] = \tilde\triangle(xpz)$. Take a point $\tilde y$ on the extension of $[\tilde x \tilde z]$ beyond $\tilde z$ so that $|\tilde x - \tilde y| = |x - y|$ (and therefore $|\tilde x - \tilde z| = |x - z|$).

Since increasing the opposite side in a plane triangle increases the corresponding angle, the following expressions have the same sign:



(i) $\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(x\,{}^{p}_{y})$,

(ii) $|\tilde p - \tilde y| - |p - y|$,

(iii) $\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] - \tilde\measuredangle(z\,{}^{p}_{y})$.

Since

$$\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}] = \measuredangle[\tilde x\,{}^{\tilde p}_{\tilde z}] = \tilde\measuredangle(x\,{}^{p}_{z})$$

and

$$\measuredangle[\tilde z\,{}^{\tilde p}_{\tilde y}] = \pi - \measuredangle[\tilde z\,{}^{\tilde x}_{\tilde p}] = \pi - \tilde\measuredangle(z\,{}^{x}_{p}),$$

the first statement follows.

For the second statement, construct a model triangle $[\tilde p \tilde z \tilde y'] = \tilde\triangle(pzy)_{\mathbb{E}^2}$ on the opposite side of $[\tilde p \tilde z]$ from $[\tilde x \tilde p \tilde z]$. Note that

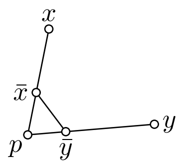
$$\begin{aligned} |\tilde x - \tilde y'| &\leqslant |\tilde x - \tilde z| + |\tilde z - \tilde y'| = \\ &= |x - z| + |z - y| = \\ &= |x - y|. \end{aligned}$$

Therefore

$$\begin{aligned}\tilde\measuredangle(p\,{}^{x}_{z}) + \tilde\measuredangle(p\,{}^{z}_{y}) &= \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde z}] + \measuredangle[\tilde p\,{}^{\tilde z}_{\tilde y'}] = \\ &= \measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y'}] \leqslant \\ &\leqslant \tilde\measuredangle(p\,{}^{x}_{y}).\end{aligned}$$

Equality holds if and only if $|\tilde x - \tilde y'| = |x - y|$, as required. ☐

7.6.2. Exercise. *Given $[p\,{}^{x}_{y}]$ in a metric space $\mathcal{X}$, consider the function*



$$f\colon (|p-\bar x|, |p-\bar y|) \mapsto \tilde\measuredangle(p\,{}^{\bar x}_{\bar y})$$

where $\bar x \in\,]px]$ and $\bar y \in\,]py]$.

(a) Suppose $\mathcal{X}$ is CAT(0). *Show that f is nondecreasing in each argument.*

(b) Suppose $\mathcal{X}$ is CBB(0). *Show that f is nonincreasing in each argument.*

Conclude that for any hinge in a CAT(0) *or* CBB(0) *space has defined angle.*

7.6.3. Exercise. *Fix a point p in a be a complete length* CAT(0) *space $\mathcal{X}$. Given a point $x \in \mathcal{X}$, denote by γ_x a (necessary unique) geodesic path from p to x.*

Show that the family of maps $h_t\colon \mathcal{X} \to \mathcal{X}$ defined by

$$h_t(x) = \gamma_x(t)$$

is a homotopy; it is called geodesic homotopy. *Conclude that $\mathcal{X}$ is contractible.*

The geodesic homotopy introduced in the previous exercise should help to solve the next one.

7.6.4. Exercise. *Let $\mathcal{X}$ be a complete length* CAT(0) *space. Assume $\mathcal{X}$ is a topological manifold. Show that any geodesic in $\mathcal{X}$ can be extended as a two-side infinite geodesic.*

7.7 Thin and fat triangles

Recall that a *triangle* $[xyz]$ in a space $\mathcal{X}$ is a triple of minimizing geodesics $[xy]$, $[yz]$ and $[zx]$. Consider the model triangle $[\tilde x\tilde y\tilde z] = \tilde\triangle(xyz)_{\mathbb{E}^2}$ in the Euclidean plane. The *natural map* $[\tilde x\tilde y\tilde z] \to [xyz]$ sends a point $\tilde p \in [\tilde x\tilde y] \cup [\tilde y\tilde z] \cup [\tilde z\tilde x]$ to the corresponding point $p \in$

$[xy] \cup [yz] \cup [zx]$; that is, if $\tilde{p}$ lies on $[\tilde{y}\tilde{z}]$, then $p \in [yz]$ and $|\tilde{y} - \tilde{p}| = |y - p|$ (and therefore $|\tilde{z} - \tilde{p}| = |z - p|$.

In the same way, the natural map can be defined for the spherical model triangle $\tilde{\triangle}(xyz)_{\mathbb{S}^2}$.

7.7.1. Definition. *A triangle $[xyz]$ in the metric space $\mathcal{X}$ is called* thin *(or* fat*) if the natural map $\tilde{\triangle}(xyz)_{\mathbb{E}^2} \to [xyz]$ is distance nonincreasing (or respectively distance nondecreasing).*

Analogously, a triangle $[xyz]$ is called spherically thin *or* spherically fat *if the natural map from the spherical model triangle $\tilde{\triangle}(xyz)_{\mathbb{S}^2}$ to $[xyz]$ is distance nonincreasing or nondecreasing.*

7.7.2. Proposition. *A geodesic space is* CAT(0) *(*CAT(1)*) if and only if all its triangles are thin (respectively, all its triangles of perimeter $< 2{\cdot}\pi$ are spherically thin).*

Proof; "if" part. Apply the triangle inequality and thinness of triangles $[pxy]$ and $[qxy]$, where p, q, x and y are as in the definition of CAT(κ) comparison (page 60).

"Only if" part. Applying CAT(0) comparison to a quadruple p, q, x, y with $q \in [xy]$ shows that any triangle satisfies *point-side comparison*, that is, the distance from a vertex to a point on the opposite side is no greater than the corresponding distance in the Euclidean model triangle.

Now consider a triangle $[xyz]$ and let $p \in [xy]$ and $q \in [xz]$. Let $\tilde{p}$, $\tilde{q}$ be the corresponding points on the sides of the model triangle $\tilde{\triangle}(xyz)_{\mathbb{E}^2}$. Applying 7.6.2$a$, we get that

$$\tilde{\measuredangle}(x\,{}^{y}_{z})_{\mathbb{E}^2} \geqslant \tilde{\measuredangle}(x\,{}^{p}_{q})_{\mathbb{E}^2}.$$

Therefore $|\tilde{p} - \tilde{q}|_{\mathbb{E}^2} \geqslant |p - q|$.

The CAT(1) argument is the same. □

7.7.3. Exercise. *Show that any triangle is a* CBB(0) *space is fat.*

7.7.4. Exercise. *Suppose $\gamma_1, \gamma_2 \colon [0,1] \to \mathcal{U}$ be two geodesic paths in a complete length* CAT(0) *space $\mathcal{U}$. Show that*

$$t \mapsto |\gamma_1(t) - \gamma_2(t)|_{\mathcal{U}}$$

is a convex function.

7.7.5. Exercise. *Let A be a convex closed set in a proper length* CAT(0) *space $\mathcal{U}$; that is, if $x, y \in A$, then $[xy] \subset A$. Show that for any $r > 0$ the closed r-neighborhood of A is convex; that is, the set*

$$A_r = \{\, x \in \mathcal{U} \,:\, \operatorname{dist}_A x \leqslant r \,\}$$

is convex.

7.7.6. Exercise. *Let $\mathcal{U}$ be a proper length* CAT(0) *space and $K \subset \mathcal{U}$ be a closed convex set. Show that:*

(a) *For each point $p \in \mathcal{U}$ there is unique point $p^* \in K$ that minimizes the distance $|p - p^*|$.*
(b) *The closest-point projection $p \mapsto p^*$ defined by (a) is short.*

Recall that a set A in a metric space $\mathcal{U}$ is called *locally convex* if for any point $p \in A$ there is an open neighborhood $\mathcal{U} \ni p$ such that any geodesic in $\mathcal{U}$ with ends in A lies in A.

7.7.7. Exercise. *Let $\mathcal{U}$ be a proper length* CAT(0) *space. Show that any closed, connected, locally convex set in $\mathcal{U}$ is convex.*

7.8 Other descriptions

In this section we will list few ways to describe CAT(0) and CBB(0) spaces. We do not give proofs of these statements, altho they are not hard; see [**alexander-kapovitch-petrunin-2025**] and the references therein.

These conditions will not be used in the sequel, but they might help to build right intuition.

Convexity of function. The following condition might help to adapt intuition from real analysis.

Let $\mathcal{X}$ be a metric space and $\lambda \in \mathbb{R}$. A function $f\colon \mathcal{X} \to \mathbb{R}$ is called *λ-convex* (*λ-concave*) if the real-to-real function

$$t \mapsto f \circ \gamma(\gamma) - \tfrac{\lambda}{2}\cdot t^2$$

is convex (respectively concave) for any geodesic $\gamma\colon \mathbb{I} \to \mathbb{R}$.

The λ-convex and λ-concave functions can be thought as functions satisfying inequalities $f'' \geqslant \lambda$ and respectively $f'' \leqslant \lambda$ in a generalized sense. Note that a smooth real-to-real function f is λ-convex (λ-concave) if it satisfies inequality $f'' \geqslant \lambda$ (respectively $f'' \leqslant \lambda$).

7.8.1. Proposition. *Let $\mathcal{X}$ be a geodesic space. Then $\mathcal{X}$ is* CAT(0) *(respectively* CBB(0)*) if and only if for any point $p \in \mathcal{X}$ the function*

$$f(x) = \tfrac{1}{2}\cdot|p - x|_{\mathcal{X}}$$

is 1-convex (respectively 1-concave).

Angle comparison. The following condition might help to adapt intuition from Euclidean geometry.

Recall that in CAT(0) and CBB(0) spaces any hinge has defined angle; see 7.6.2.

7.8.2. Proposition. *Let $\mathcal{X}$ be a geodesic space such that any hinge in $\mathcal{X}$ has defined angle. Then*

(a) $\mathcal{X}$ is CAT(0) *if and only if*

$$\measuredangle[p\,{}^x_y] \leqslant \tilde\measuredangle(p\,{}^x_y).$$

(b) $\mathcal{X}$ is CBB(0) *if and only if*

$$\measuredangle[p\,{}^x_y] \geqslant \tilde\measuredangle(p\,{}^x_y)$$

and

$$\measuredangle[p\,{}^x_y] + \measuredangle[p\,{}^x_z] = \pi$$

for any adjacent hinges $[p\,{}^x_y]$ *and* $[p\,{}^x_z]$*; that is, the union of the sides* $[px]$ *and* $[pz]$ *of the hinges form a geodesic* $[xy]$*.*

It is unknown if the condition on adjacent hinges in *(b)* can be removed (even in the two-dimesional case).

Kirszbraun property. We include the following condition only because it is beautiful.

The following theorem was proved by Mojżesz Kirszbraun [**kirszbraun**] and rediscovered later by Frederick Valentine [**valentine**].

7.8.3. Theorem. *Let $A \subset \mathbb{E}^m$. Then any short map $f\colon A \to \mathbb{E}^n$ admits a short extension $f\colon \mathbb{E}^m \to \mathbb{E}^n$.*

The conclusion of the theorem holds for some other metric spaces instead of $\mathbb{E}^m$ and $\mathbb{E}^n$. For example instead of $\mathbb{E}^n$ one might take any injective space (3.2.1) and instead of $\mathbb{E}^m$ one may take any compact ultrametric space (3.2.4). On the other hand existance of extension to/from a Euclidean space is much weaker condition than in 3.2.1 and 3.2.4. As the following theorems state, these conditions are closely related to the CBB(0) and CAT(0) conditions.

7.8.4. Theorem. *Let $\mathcal{X}$ be a complete length space and $n \geqslant 2$. Then $\mathcal{X}$ is* CBB(0) *if and only if for any set $A \subset \mathcal{X}$, any short map $f\colon A \to \mathbb{E}^n$ admits a short extension $F\colon \mathcal{X} \to \mathbb{E}^n$.*

7.8.5. Theorem. *Let $\mathcal{Y}$ be a metric space with and $m \geqslant 2$. Assume any two points in $\mathcal{Y}$ are joint by unique geodesic. Then $\mathcal{Y}$ is* CAT(0) *if and only if for any set $A \subset \mathbb{E}^m$, any short map $f\colon \mathbb{E}^m \to \mathcal{Y}$ admits a short extension $F\colon \mathbb{E}^m \to \mathcal{Y}$.*

7.9 History

The idea that the essence of curvature lies in a condition on quadruples of points apparently originated with Abraham Wald. It is found in his publication on "coordinate-free differential geometry" [**wald**] written under the supervision of Karl Menger; the story of this discovery can be found in [**menger**]. In 1941, similar definitions were rediscovered independently by Alexandr Danilovich Alexandrov [**alexandrov:def**]. In Alexandrov's work the first fruitful applications of this approach were given. Mainly:

- ◇ Alexandrov's embedding theorem — *metrics of non-negative curvature on the sphere, and only they, are isometric to closed convex surfaces in Euclidean 3-space.*
- ◇ Gluing theorem, which tells when the sphere obtained by gluing of two discs along their boundaries has non-negative curvature in the sense of Alexandrov.

These two results together gave a very intuitive geometric tool for studying embeddings and bending of surfaces in Euclidean space, and changed this subject dramatically. They formed the foundation of the branch of geometry now called *Alexandrov geometry.*

The study of spaces with curvature bounded above started later. The first paper on the subject was written by Alexandrov [**alexandrov:strong-angle**]. It was based on work of Herbert Busemann, who studied spaces satisfying a weaker condition [**busemann-CBA**].

Lecture 8

Gluing and billiards

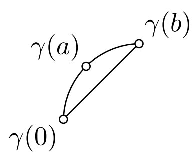
This chapter is nearly a copy of [**alexander-kapovitch-petrunin-2019**]; here we prove Reshetnyak's gluing theorem for $\mathrm{CAT}(0)$ spaces and apply it to a problem in billiards.

8.1 Inheritance lemma

The inheritance lemma 8.1.2 proved below plays a central role in the theory of $\mathrm{CAT}(\kappa)$ spaces.

8.1.1. Proposition. *Suppose $\mathcal{U}$ is a proper length* $\mathrm{CAT}(0)$ *space. Then any local geodesic in $\mathcal{U}$ is a geodesic.*

Analogously, if $\mathcal{U}$ is a proper length $\mathrm{CAT}(1)$ *space, then any local geodesic in $\mathcal{U}$ which is shorter than π is a geodesic.*



Proof. Suppose $\gamma\colon [0,\ell]\to\mathcal{U}$ is a local geodesic that is not a geodesic. Choose a to be the maximal value such that γ is a geodesic on $[0,a]$. Further choose $b>a$ so that γ is a geodesic on $[a,b]$.

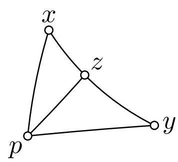
Since the triangle $[\gamma(0)\gamma(a)\gamma(b)]$ is thin and $|\gamma(0)-\gamma(b)|<b$ we have

$$|\gamma(a-\varepsilon)-\gamma(a+\varepsilon)|<2{\cdot}\varepsilon$$

for all small $\varepsilon>0$. That is, γ is not length-minimizing on the interval $[a-\varepsilon,a+\varepsilon]$ for any $\varepsilon>0$, a contradiction.

The spherical case is done in the same way. $\square$

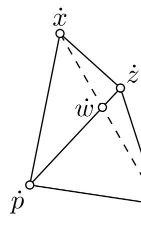
Now let us formulate the main result of this section.



8.1.2. Inheritance lemma. *Assume that a triangle $[pxy]$ in a metric space is* decomposed *into two triangles $[pxz]$ and $[pyz]$; that is, $[pxz]$ and $[pyz]$ have a common side $[pz]$, and the sides $[xz]$ and $[zy]$ together form the side $[xy]$ of $[pxy]$.*

If both triangles $[pxz]$ and $[pyz]$ are thin, then the triangle $[pxy]$ is also thin.

Analogously, if $[pxy]$ has perimeter $< 2\cdot\pi$ and both triangles $[pxz]$ and $[pyz]$ are spherically thin, then triangle $[pxy]$ is spherically thin.



Proof. Construct the model triangles $[\dot p\dot x\dot z] = \tilde\triangle(pxz)_{\mathbb{E}^2}$ and $[\dot p\dot y\dot z] = \tilde\triangle(pyz)_{\mathbb{E}^2}$ so that $\dot x$ and $\dot y$ lie on opposite sides of $[\dot p\dot z]$.

Let us show that

$$❶ \qquad \tilde\measuredangle(z\,{}^p_x) + \tilde\measuredangle(z\,{}^p_y) \geqslant \pi.$$

If not, hen for some point $\dot w \in [\dot p\dot z]$, we have

$$|\dot x - \dot w| + |\dot w - \dot y| < |\dot x - \dot z| + |\dot z - \dot y| = |x - y|.$$

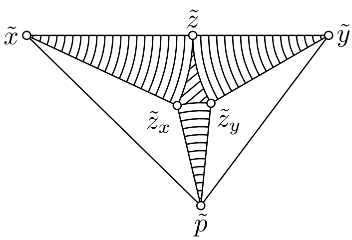
Let $w \in [pz]$ correspond to $\dot w$; that is, $|z - w| = |\dot z - \dot w|$. Since $[pxz]$ and $[pyz]$ are thin, we have

$$|x - w| + |w - y| < |x - y|,$$

contradicting the triangle inequality.

Denote by $\dot D$ the union of two solid triangles $[\dot p\dot x\dot z]$ and $[\dot p\dot y\dot z]$. Further, denote by $\tilde D$ the solid triangle $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)_{\mathbb{E}^2}$. By ❶, there is a short map $F\colon \tilde D \to \dot D$ that sends

$$\tilde p \mapsto \dot p, \qquad \tilde x \mapsto \dot x, \qquad \tilde z \mapsto \dot z, \qquad \tilde y \mapsto \dot y.$$



Indeed, by Alexandrov's lemma (7.6.1), there are nonoverlapping triangles

$$[\tilde p\tilde x\tilde z_x] \overset{iso}{=\!=} [\dot p\dot x\dot z]$$

and

$$[\tilde p\tilde y\tilde z_y] \overset{iso}{=\!=} [\dot p\dot y\dot z]$$

inside the triangle $[\tilde p\tilde x\tilde y]$.

Connect the points in each pair $(\tilde z, \tilde z_x)$, $(\tilde z_x, \tilde z_y)$ and $(\tilde z_y, \tilde z)$ with arcs of circles centered at $\tilde y$, $\tilde p$, and $\tilde x$ respectively. Define F as follows:

- ⋄ Map Conv$[\tilde p\tilde x\tilde z_x]$ isometrically onto Conv$[\dot p\dot x\dot z]$; similarly map Conv$[\tilde p\tilde y\tilde z_y]$ onto Conv$[\dot p\dot y\dot z]$.
- ⋄ If x is in one of the three circular sectors, say at distance r from its center, set $F(x)$ to be the point on the corresponding segment $[pz]$, $[xz]$ or $[yz]$ whose distance from the left-hand endpoint of the segment is r.
- ⋄ Finally, if x lies in the remaining curvilinear triangle $\tilde z\tilde z_x\tilde z_y$, set $F(x) = z$.

By construction, F satisfies the conditions.

By assumption, the natural maps $[\dot p\dot x\dot z] \to [pxz]$ and $[\dot p\dot y\dot z] \to [pyz]$ are short. By composition, the natural map from $[\tilde p\tilde x\tilde y]$ to $[pyz]$ is short, as claimed.

The spherical case is done along the same lines. □

8.2 Reshetnyak's gluing

Suppose $\mathcal{U}^1$ and $\mathcal{U}^2$ are proper length spaces with isometric closed convex sets $A^i \subset \mathcal{U}^i$ and let $\iota\colon A^1 \to A^2$ be an isometry. Consider the space $\mathcal{W}$ of all equivalence classes in $\mathcal{U}^1 \sqcup \mathcal{U}^2$ with the equivalence relation given by $a \sim \iota(a)$ for any $a \in A^1$.

It is straightforward to see that $\mathcal{W}$ is a proper length space when equipped with the following metric

$$\begin{aligned}
|x - y|_{\mathcal{W}} &:= |x - y|_{\mathcal{U}^i} \\
&\qquad \text{if} \quad x, y \in \mathcal{U}^i, \quad \text{and} \\
|x - y|_{\mathcal{W}} &:= \min\left\{\, |x - a|_{\mathcal{U}^1} + |y - \iota(a)|_{\mathcal{U}^2} \;:\; a \in A^1 \,\right\} \\
&\qquad \text{if} \quad x \in \mathcal{U}^1 \quad \text{and} \quad y \in \mathcal{U}^2.
\end{aligned}$$

Abusing notation, we denote by x and y the points in $\mathcal{U}^1 \sqcup \mathcal{U}^2$ and their equivalence classes in $\mathcal{U}^1 \sqcup \mathcal{U}^2/\!\sim$.

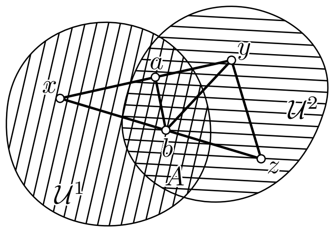
The space $\mathcal{W}$ is called the *gluing* of $\mathcal{U}^1$ and $\mathcal{U}^2$ along ι. If one applies this construction to two copies of one space $\mathcal{U}$ with a set $A \subset \mathcal{U}$ and the identity map $\iota\colon A \to A$, then the obtained space is called the *double* of $\mathcal{U}$ along A.

We can (and will) identify $\mathcal{U}^i$ with its image in $\mathcal{W}$; this way both subsets $A^i \subset \mathcal{U}^i$ will be identified and denoted further by A. Note that $A = \mathcal{U}^1 \cap \mathcal{U}^2 \subset \mathcal{W}$, therefore A is also a convex set in $\mathcal{W}$.

8.2.1. Reshetnyak gluing. *Suppose $\mathcal{U}^1$ and $\mathcal{U}^2$ are proper length* CAT(0) *spaces with isometric closed convex sets $A^i \subset \mathcal{U}^i$, and $\iota\colon A^1 \to A^2$ is an isometry. Then the gluing of $\mathcal{U}^1$ and $\mathcal{U}^2$ along ι is a* CAT(0) *proper length space.*

Proof. By construction of the gluing space, the statement can be reformulated in the following way:

8.2.2. Reformulation of 8.2.1. *Let $\mathcal{W}$ be a proper length space which has two closed convex sets $\mathcal{U}^1, \mathcal{U}^2 \subset \mathcal{W}$ such that $\mathcal{U}^1 \cup \mathcal{U}^2 = \mathcal{W}$ and $\mathcal{U}^1$, $\mathcal{U}^2$ are* CAT(0)*. Then $\mathcal{W}$ is* CAT(0)*.*



It suffices to show that any triangle $[xyz]$ in $\mathcal{W}$ is thin. This is obviously true if all three points x, y, z lie in one of $\mathcal{U}^i$. Thus, without loss of generality, we may assume that $x \in \mathcal{U}^1$ and $y, z \in \in \mathcal{U}^2$.

Choose points $a, b \in A = \mathcal{U}^1 \cap \cap \mathcal{U}^2$ that lie respectively on the sides $[xy], [xz]$. Note that

- ◇ the triangle $[xab]$ lies in $\mathcal{U}^1$,
- ◇ both triangles $[yab]$ and $[ybz]$ lie in $\mathcal{U}^2$.

In particular each triangle $[xab]$, $[yab]$ and $[ybz]$ is thin.

Applying the inheritance lemma (8.1.2) twice, we get that $[xyb]$ and consequently $[xyz]$ is thin. □

8.2.3. Exercise. *Suppose $\mathcal{U}$ is a geodesic space and $A \subset \mathcal{U}$ is a closed subset. Assume that the doubling of $\mathcal{U}$ in A is* CAT(0)*. Show that A is a convex set of $\mathcal{U}$.*

8.3 Puff pastry

In this section we introduce the notion of Reshetnyak puff pastry. This construction will be used in the next section to prove the collision theorem (8.5.1).

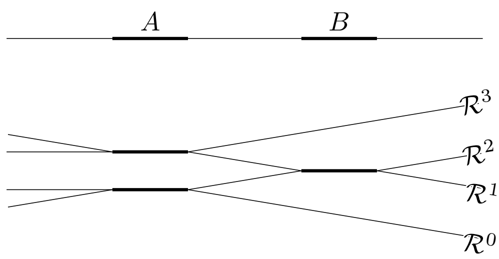
Let $\boldsymbol{A} = (A^1, \dots, A^N)$ be an array of convex closed sets in the Euclidean space $\mathbb{E}^m$. Consider an array of $N+1$ copies of $\mathbb{E}^m$. Assume that the space $\mathcal{R}$ is obtained by gluing successive pairs of spaces along $A^1, \dots, A^N$ respectively.

The resulting space $\mathcal{R}$ will be called the *Reshetnyak puff pastry* for the array $\boldsymbol{A}$. The copies of $\mathbb{E}^m$ in the puff pastry $\mathcal{R}$ will be called *levels*; they will be denoted by $\mathcal{R}^0, \dots, \mathcal{R}^N$. The point in the k-th level $\mathcal{R}^k$ that corresponds to $x \in \mathbb{E}^m$ will be denoted by x^k.

Given $x \in \mathbb{E}^m$, any point $x^k \in \mathcal{R}$ is called a *lifting* of x. The map $x \mapsto x^k$ defines an isometry $\mathbb{E}^m \to \mathcal{R}^k$; in particular we can talk about liftings of subsets in $\mathbb{E}^m$.

Note that:

- ◇ The intersection $A^1 \cap \dots \cap A^N$ admits a unique lifting in $\mathcal{R}$.



Puff pastry for (A, B, A).

- ⋄ Moreover, $x^i = x^j$ for some $i < j$ if and only if

$$x \in A^{i+1} \cap \dots \cap A^j.$$

- ⋄ The restriction $\mathcal{R}^k \to \mathbb{E}^m$ of the natural projection $x^k \mapsto x$ is an isometry.

8.3.1. Observation. *Any Reshetnyak puff pastry is a proper length* CAT(0) *space.*

Proof. Apply Reshetnyak gluing theorem (8.2.1) recursively for the convex sets in the array. □

8.3.2. Proposition. *Assume $(A^1, \dots, A^N)$ and $(\check{A}^1, \dots, \check{A}^N)$ are two arrays of convex closed sets in $\mathbb{E}^m$ such that $A^k \subset \check{A}^k$ for each k. Let $\mathcal{R}$ and $\check{\mathcal{R}}$ be the corresponding Reshetnyak puff pastries. Then the map $\mathcal{R} \to \check{\mathcal{R}}$ defined by $x^k \mapsto \check{x}^k$ is short.*

Moreover, if

$$|x^i - y^j|_{\mathcal{R}} = |\check{x}^i - \check{y}^j|_{\check{\mathcal{R}}} \tag{❶}$$

for some $x, y \in \mathbb{E}^m$ and $i, j \in \{0, \dots, n\}$, then the unique geodesic $[\check{x}^i \check{y}^j]_{\check{\mathcal{R}}}$ is the image of the unique geodesic $[x^i y^j]_{\mathcal{R}}$ under the map $x^i \mapsto \check{x}^i$.

Proof. The first statement in the proposition follows from the construction of Reshetnyak puff pastries.

By Observation 8.3.1, $\mathcal{R}$ and $\check{\mathcal{R}}$ are proper length CAT(0) spaces; hence $[x^i y^j]_{\mathcal{R}}$ and $[\check{x}^i \check{y}^j]_{\check{\mathcal{R}}}$ are unique. By ❶, since the map $\mathcal{R} \to \check{\mathcal{R}}$ is short, the image of $[x^i y^j]_{\mathcal{R}}$ is a geodesic of $\check{\mathcal{R}}$ joining $\check{x}^i$ to $\check{y}^j$. Hence the second statement follows. □

8.3.3. Definition. *Consider a Reshetnyak puff pastry $\mathcal{R}$ with the levels $\mathcal{R}^0, \dots, \mathcal{R}^N$. We say that $\mathcal{R}$ is* end-to-end convex *if $\mathcal{R}^0 \cup \mathcal{R}^N$,* *the union of its lower and upper levels, forms a convex set in $\mathcal{R}$; that is, if $x, y \in \mathcal{R}^0 \cup \mathcal{R}^N$, then $[xy]_{\mathcal{R}} \subset \mathcal{R}^0 \cup \mathcal{R}^N$.*

Note that if $\mathcal{R}$ is the Reshetnyak puff pastry for an array of convex sets $\boldsymbol{A} = (A^1, \dots, A^N)$, then $\mathcal{R}$ is end-to-end convex if and only if the union of the lower and the upper levels $\mathcal{R}^0 \cup \mathcal{R}^N$ is isometric to the double of $\mathbb{E}^m$ along the nonempty intersection $A^1 \cap \dots \cap A^N$.

8.3.4. Observation. *Let $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ be arrays of convex bodies in $\mathbb{E}^m$. Assume that the array $\boldsymbol{A}$ is obtained by inserting in $\check{\boldsymbol{A}}$ several copies of the bodies which were already listed in $\check{\boldsymbol{A}}$.*

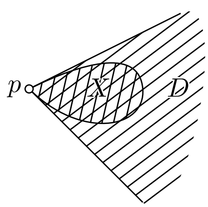
For example, if $\check{\boldsymbol{A}} = (A, C, B, C, A)$, by placing B in the second place and A in the fourth place, we obtain $\boldsymbol{A} = (A, B, C, A, B, C, A)$.

Denote by $\check{\mathcal{R}}$ and $\mathcal{R}$ the Reshetnyak puff pastries for $\check{\boldsymbol{A}}$ and $\boldsymbol{A}$ respectively.

If $\check{\mathcal{R}}$ is end-to-end convex, then so is $\mathcal{R}$.

Proof. Without loss of generality we may assume that $\boldsymbol{A}$ is obtained by inserting one element in $\check{\boldsymbol{A}}$, say at the place number k.

Note that $\check{\mathcal{R}}$ is isometric to the puff pastry for $\boldsymbol{A}$ with A^k replaced by $\mathbb{E}^m$. It remains to apply Proposition 8.3.2. □



Let X be a convex set in a Euclidean space. By a *dihedral angle* we understand an intersection of two half-spaces; the intersection of corresponding hyperplanes is called the *edge* of the angle. We say that a dihedral angle D supports X at a point $p \in X$ if D contains X and the edge of D contains p.

8.3.5. Lemma. *Let A and B be two convex sets in $\mathbb{E}^m$. Assume that any dihedral angle supporting $A \cap B$ has angle measure at least α. Then the Reshetnyak puff pastry for the array*

$$(\underbrace{A, B, A, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \ \textit{times}}).$$

is end-to-end convex.

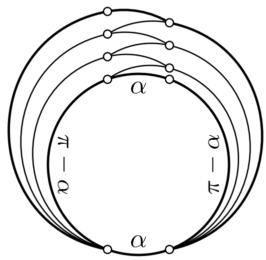
The proof of the lemma is based on a partial case, which we formulate as a sublemma.

8.3.6. Sublemma. *Let $\ddot{A}$ and $\ddot{B}$ be two half-planes in $\mathbb{E}^2$, where $\ddot{A} \cap \ddot{B}$ is an angle with measure α. Then the Reshetnyak puff pastry for the array*

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \ \textit{times}})$$

is end-to-end convex.

Proof. Note that the puff pastry $\ddot{\mathcal{R}}$ is isometric to the cone over the space glued from the unit circles as shown on the diagram.

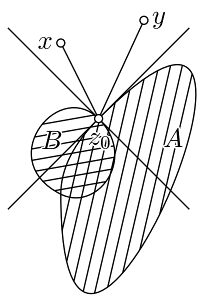


All the short arcs on the diagram have length α; the long arcs have length $\pi-\alpha$, so making a circuit along any path will take $2\cdot\pi$.

Observe that end-to-end convexity of $\ddot{\mathcal{R}}$ is equivalent to the fact that any geodesic shorter than π with the ends on the inner and the outer circles lies completely in the union of these two circles.

The latter holds if the zigzag line in the picture has length at least π. This line is formed by $\lceil\frac{\pi}{\alpha}\rceil$ arcs with length α each. Hence the sublemma. ☐

In the proof of 8.3.5, we will use the following exercise in convex geometry:



8.3.7. Exercise. *Let A and B be two closed convex sets in $\mathbb{E}^m$ and $A\cap B\ne\varnothing$. Given two points $x,y\in\mathbb{E}^m$ let $f(z)=|x-z|+|y-z|$.*

Let $z_0\in A\cap B$ be a point of minimum of $f|_{A\cap B}$.

Show that there are half-spaces $\dot A$ and $\dot B$ such that $\dot A\supset A$ and $\dot B\supset B$ and z_0 is also a point of minimum of the restriction $f|_{\dot A\cap\dot B}$.

Proof of 8.3.5. Fix arbitrary $x,y\in\mathbb{E}^m$. Choose a point $z\in A\cap B$ for which the sum

$$|x-z|+|y-z|$$

is minimal. To show the end-to-end convexity of $\mathcal{R}$, it is sufficient to prove the following:

❷ *The geodesic $[x^0y^N]_{\mathcal{R}}$ contains $z^0=z^N\in\mathcal{R}$.*

Without loss of generality we may assume that $z\in\partial A\cap\partial B$. Indeed, since the puff pastry for the 1-array (B) is end-to-end convex, Proposition 8.3.2 together with Observation 8.3.4 imply ❷ in case z lies in the interior of A. In the same way we can treat the case when z lies in the interior of B.

Note that $\mathbb{E}^m$ admits an isometric splitting $\mathbb{E}^{m-2}\times\mathbb{E}^2$ such that

$$\dot A=\mathbb{E}^{m-2}\times\ddot A$$
$$\dot B=\mathbb{E}^{m-2}\times\ddot B$$

where $\ddot{A}$ and $\ddot{B}$ are half-planes in $\mathbb{E}^2$.

Using Exercise 8.3.7, let us replace each A by $\dot{A}$ and each B by $\dot{B}$ in the array, to get the array

$$(\underbrace{\dot{A}, \dot{B}, \dot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \text{ times}}).$$

The corresponding puff pastry $\dot{\mathcal{R}}$ splits as a product of $\mathbb{E}^{m-2}$ and a puff pastry, call it $\ddot{\mathcal{R}}$, glued from the copies of the plane $\mathbb{E}^2$ for the array

$$(\underbrace{\ddot{A}, \ddot{B}, \ddot{A}, \dots}_{\lceil\frac{\pi}{\alpha}\rceil + 1 \text{ times}}).$$

Note that the dihedral angle $\dot{A} \cap \dot{B}$ is at least α. Therefore the angle measure of $\ddot{A} \cap \ddot{B}$ is also at least α. According to Sublemma 8.3.6 and Observation 8.3.4, $\ddot{\mathcal{R}}$ is end-to-end convex.

Since $\dot{\mathcal{R}} \overset{iso}{=\!=} \mathbb{E}^{m-2} \times \ddot{\mathcal{R}}$, the puff pastry $\dot{\mathcal{R}}$ is also end-to-end convex.

It follows that the geodesic $[\dot{x}^0 \dot{y}^N]_{\dot{\mathcal{R}}}$ contains $\dot{z}^0 = \dot{z}^N \in \dot{\mathcal{R}}$. By Proposition 8.3.2, the image of $[\dot{x}^0 \dot{y}^N]_{\dot{\mathcal{R}}}$ under the map $\dot{x}^k \mapsto x^k$ is the geodesic $[x^0 y^N]_{\mathcal{R}}$. Hence Claim ❷, and the lemma follow. □

8.4 Wide corners

We say that a closed convex set $A \subset \mathbb{E}^m$ has *ε-wide corners* for given $\epsilon > 0$ if together with each point p, the set A contains a small right circular cone with tip at p and aperture ε; that is, ε is the maximum angle between two generating lines of the cone.

For example, a plane polygon has ε-wide corners if all its interior angles are at least ε.

We will consider finite collections of closed convex sets $A^1, \dots, A^n \subset$ $\subset \mathbb{E}^m$ such that for any subset $F \subset \{1, \dots, n\}$, the intersection $\bigcap_{i \in F} A^i$ has ε-wide corners. In this case we may say briefly *all intersections of A^i have ε-wide corners.*

8.4.1. Exercise. *Assume $A^1, \dots, A^n \subset \mathbb{E}^m$ are compact, convex sets with a common interior point. Show that all intersections of A^i have ε-wide corners for some positive ε.*

8.4.2. Exercise. *Assume $A^1, \dots, A^n \subset \mathbb{E}^m$ are convex sets with nonempty interior that have a common center of symmetry. Show that all intersections of A^i have ε-wide corners for some positive ε.*

The proof of the following proposition is based on Lemma 8.3.5; this lemma is essentially the case $n = 2$ in the proposition.

8.4.3. Proposition. *Given $\varepsilon > 0$ and a positive integer n, there is an array of integers $\boldsymbol{j}_\varepsilon(n) = (j_1, \dots, j_N)$ such that:*

(a) For each k we have $1 \leqslant j_k \leqslant n$, and each number $1, \dots, n$ appears in $\boldsymbol{j}_\varepsilon$ at least once.

(b) If $A^1, \dots, A^n$ is a collection of closed convex sets in $\mathbb{E}^m$ with a common point and all their intersections have ε-wide corners, then the puff pastry for the array $(A^{j_1}, \dots, A^{j_N})$ is end-to-end convex.

Moreover we can assume that $N \leqslant (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^n$.

Proof. The array $\boldsymbol{j}_\varepsilon(n) = (j_1, \dots, j_N)$ is constructed recursively. For $n = 1$, we can take $\boldsymbol{j}_\varepsilon(1) = (1)$.

Assume that $\boldsymbol{j}_\varepsilon(n)$ is constructed. Let us replace each occurrence of n in $\boldsymbol{j}_\varepsilon(n)$ by the alternating string

$$\underbrace{n, n+1, n, \dots}_{\lceil \frac{\pi}{\varepsilon} \rceil + 1 \text{ times}}.$$

Denote the obtained array by $\boldsymbol{j}_\varepsilon(n+1)$.

By Lemma 8.3.5, end-to-end convexity of the puff pastry for $\boldsymbol{j}_\varepsilon(n+1)$ follows from end-to-end convexity of the puff pastry for the array where each string

$$\underbrace{A^n, A^{n+1}, A^n, \dots}_{\lceil \frac{\pi}{\varepsilon} \rceil + 1 \text{ times}}$$

is replaced by $Q = A^n \cap A^{n+1}$. End-to-end convexity of the latter follows by the assumption on $\boldsymbol{j}_\varepsilon(n)$, since all the intersections of $A^1, \dots, A^{n-1}, Q$ have ε-wide corners.

The upper bound on N follows directly from the construction. $\square$

8.5 Billiards

Let $A^1, A^2, \dots A^n$ be a finite collection of closed convex sets in $\mathbb{E}^m$. Assume that for each i the boundary ∂A^i is a smooth hypersurface.

Consider the billiard table formed by the closure of the complement

$$T = \overline{\mathbb{E}^m \backslash \bigcup_i A^i}.$$

The sets A^i will be called *walls* of the table T and the billiards described above will be called *billiards with convex walls.*

A *billiard trajectory* on the table T is a unit-speed broken line γ that follows the standard law of billiards at the break points on ∂A^i —

in particular, the angle of reflection is equal to the angle of incidence. The break points of the trajectory will be called *collisions*. We assume the trajectory meets only one wall at a time.

Recall that the definition of sets with ε-wide corners is given on page 78.

8.5.1. Collision theorem. *Assume $T \subset \mathbb{E}^m$ is a billiard table with n convex walls. Assume that the walls of T have a common interior point and all their intersections have ε-wide corners. Then the number of collisions of any trajectory in T is bounded by a number N which depends only on n and ε.*

As we will see from the proof, the value N can be found explicitly; $N = (\lceil \frac{\pi}{\varepsilon} \rceil + 1)^{n^2}$ will do.

The collision theorem was proved by Dmitri Burago, Serge Ferleger and Alexey Kononenko in [**burago-ferleger-kononenko-1997**]; we present their proof with minor improvements.

Let us formulate and prove a corollary of the collision theorem; it answers a question formulated by Yakov Sinai [**galperin**].

8.5.2. Corollary. *Consider n homogeneous hard balls moving freely and colliding elastically in $\mathbb{R}^3$. Every ball moves along a straight line with constant speed until two balls collide, and then the new velocities of the two balls are determined by the laws of classical mechanics. We assume that only two balls can collide at the same time.*

Then the total number of collisions cannot exceed some number N that depends on the radii and masses of the balls. If the balls are identical, then N depends only on n.

8.5.3. Exercise. *Show that in the case of identical balls in the one-dimensional space (in $\mathbb{R}$) the total number of collisions cannot exceed $N = \frac{n\cdot(n-1)}{2}$.*

The proof below admits a straightforward generalization to all dimensions.

Proof. Denote by $a_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ the center of the i-th ball. Consider the corresponding point in $\mathbb{R}^{3\cdot N}$

$$\begin{aligned} \boldsymbol{a} &= (a_1, a_2, \dots, a_n) = \\ &= (x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_n, y_n, z_n). \end{aligned}$$

The i-th and j-th ball intersect if

$$|a_i - a_j| \leqslant R_i + R_j,$$

where R_i denotes the radius of the i-th ball. These inequalities define $\frac{n\cdot(n-1)}{2}$ cylinders

$$C_{i,j} = \left\{ (a_1, a_2, \dots, a_n) \in \mathbb{R}^{3\cdot n} \ : \ |a_i - a_j| \leqslant R_i + R_j \right\}.$$

The closure of the complement

$$T = \overline{\mathbb{R}^{3\cdot n} \backslash \bigcup_{i<j} C_{i,j}}$$

is the configuration space of our system. Its points correspond to valid positions of the system of balls.

The evolution of the system of balls is described by the motion of the point $\boldsymbol{a} \in \mathbb{R}^{3\cdot n}$. It moves along a straight line at a constant speed until it hits one of the cylinders $C_{i,j}$; this event corresponds to a collision in the system of balls.

Consider the norm of $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{R}^{3\cdot n}$ defined by

$$\|\boldsymbol{a}\| = \sqrt{M_1 \cdot |a_1|^2 + \dots + M_n \cdot |a_n|^2},$$

where $|a_i| = \sqrt{x_i^2 + y_i^2 + z_i^2}$ and M_i denotes the mass of the i-th ball. In the metric defined by $\|*\|$, the collisions follow the standard law of billiards.

By construction, the number of collisions of hard balls that we need to estimate is the same as the number of collisions of the corresponding billiard trajectory on the table T with $C_{i,j}$ as the walls.

Note that each cylinder $C_{i,j}$ is a convex set; it has smooth boundary, and it is centrally symmetric around the origin. By Exercise 8.4.2, all the intersections of the walls have ε-wide corners for some $\varepsilon > 0$ that depend on the radiuses R_i and the masses M_i. It remains to apply the collision theorem (8.5.1). □

Now we present the proof of the collision theorem (8.5.1) based on the results developed in the previous section.

Proof of 8.5.1. Let us apply induction on n.

Base: $n = 1$. The number of collisions cannot exceed 1. Indeed, by the convexity of A^1, if the trajectory is reflected once in ∂A^1, then it cannot return to A^1.

Step. Assume γ is a trajectory that meets the walls in the order $A^{i_1}, \dots, A^{i_N}$ for a large integer N.

Consider the array

$$\boldsymbol{A}_\gamma = (A^{i_1}, \dots, A^{i_N}).$$

The induction hypothesis implies:

❶ *There is a positive integer M such that any M consecutive elements of $\boldsymbol{A}_\gamma$ contain each A^i at least once.*

Let $\mathcal{R}_\gamma$ be the Reshetnyak puff pastry for $\boldsymbol{A}_\gamma$.

Consider the lift of γ to $\mathcal{R}_\gamma$, defined by $\bar\gamma(t) = \gamma^k(t) \in \mathcal{R}_\gamma$ for any moment of time t between the k-th and $(k+1)$-th collisions. Since γ follows the standard law of billiards at break points, the lift $\bar\gamma$ is locally a geodesic in $\mathcal{R}_\gamma$. By Observation 8.3.1, the puff pastry $\mathcal{R}_\gamma$ is a proper length CAT(0) space. Therefore $\bar\gamma$ is a geodesic.

Since γ does not meet $A^1 \cap \dots \cap A^n$, the lift $\bar\gamma$ does not lie in $\mathcal{R}^0_\gamma \cup \mathcal{R}^N_\gamma$. In particular, $\mathcal{R}_\gamma$ is not end-to-end convex.

Let
$$\boldsymbol{B} = (A^{j_1}, \dots, A^{j_K})$$
be the array provided by Proposition 8.4.3; so $\boldsymbol{B}$ contains each A^i at least once and the puff pastry $\mathcal{R}_{\boldsymbol{B}}$ for $\boldsymbol{B}$ is end-to-end convex. If N is sufficiently large, namely $N \geqslant K \cdot M$, then ❶ implies that $\boldsymbol{A}_\gamma$ can be obtained by inserting a finite number of A^i's in $\boldsymbol{B}$.

By Observation 8.3.4, $\mathcal{R}_\gamma$ is end-to-end convex, a contradiction. □

8.6 Comments

The gluing theorem (8.2.1) was proved by Yuri Reshetnyak in [**reshetnyak:glue**]. It can be extended to the class of geodesic CAT(0) spaces, which by 7.5.1 includes all complete length CAT(0) spaces. It also admits a natural generalization to length CAT(κ) spaces; see the book of Martin Bridson and André Haefliger [**bridson-haefliger**] and our book [**alexander-kapovitch-petrunin-2025**] for details.

Puff pastry is used to bound topological entropy of the billiard flow and to approximate the shortest billiard path that touches given lines in a given order; see the papers of Dmitri Burago with Serge Ferleger and Alexey Kononenko [**burago-ferleger-kononenko-1998**], and with Dimitri Grigoriev and Anatol Slissenko [**burago-grigoriev-slissenko**]. The lecture of Dmitri Burago [**burago-1998**] gives a short survey on the subject.

Note that the interior points of the walls play a key role in the proof despite the fact that trajectories never go inside the walls. In a similar fashion, puff pastry was used by the Stephanie Alexander and Richard Bishop in [**alexander-bishop**] to find the upper curvature bound for warped products.

In [**hass**], Joel Hass constructed an example of a Riemannian metric on the 3-ball with negative curvature and concave boundary. This example might decrease your appetite for generalizing the collision theorem — while locally such a 3-ball looks as good as the billiards table in the theorem, the number of collisions is obviously infinite.

It was show by Dmitri Burago and Sergei Ivanov [**burago-ivanov**] that the number of collisions that may occur between n identical balls in $\mathbb{R}^3$ grows at least exponentially in n; the two-dimensional case is open so far.

Lecture 9

Globalization

This lecture is nearly a copy of [**alexander-kapovitch-petrunin-2019**]; here we introduce locally CAT(0) spaces and prove the globalization theorem that provides a sufficient condition for locally CAT(0) spaces to be globally CAT(0).

9.1 Locally CAT spaces

We say that a space $\mathcal{U}$ is *locally* CAT(0) (or *locally* CAT(1)) if a small closed ball centered at any point p in $\mathcal{U}$ is CAT(0) (or CAT(1), respectively).

For example, the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is locally isometric to $\mathbb{R}$, and so $\mathbb{S}^1$ is locally CAT(0). On the other hand, $\mathbb{S}^1$ is not CAT(0), since closed local geodesics in $\mathbb{S}^1$ are not geodesics, so $\mathbb{S}^1$ does not satisfy Proposition 8.1.1.

If $\mathcal{U}$ is a proper length space, then it is locally CAT(0) (or locally CAT(1)) if and only if each point $p \in \mathcal{U}$ admits an open neighborhood Ω that is geodesic and such that any triangle in Ω is thin (or spherically thin, respectively).

9.2 Space of local geodesic paths

In this section we will study behavior of local geodesics in locally CAT(κ) spaces. The results will be used in the proof of the globalization theorem (9.3.1).

Recall that a *path* is a curve parametrized by $[0,1]$. The space of paths in a metric space $\mathcal{U}$ comes with the natural metric

❶ $$|\alpha - \beta| = \sup\left\{\, |\alpha(t) - \beta(t)|_{\mathcal{U}} \,:\, t \in [0,1] \,\right\}.$$

9.2.1. Proposition. *Let $\mathcal{U}$ be a proper length, locally* $\mathrm{CAT}(\kappa)$ *space.*

Assume $\gamma_n\colon [0,1] \to \mathcal{U}$ is a sequence of local geodesic paths converging to a path $\gamma_\infty\colon [0,1] \to \mathcal{U}$. Then γ_∞ is a local geodesic path. Moreover

$$\operatorname{length}\gamma_n \to \operatorname{length}\gamma_\infty$$

as $n \to \infty$.

Proof; $\mathrm{CAT}(0)$ *case.* Fix $t \in [0,1]$. Let $R > 0$ be sufficiently small, so that $\overline{\mathrm{B}}[\gamma_\infty(t), R]$ forms a proper length $\mathrm{CAT}(0)$ space.

Assume that a local geodesic σ is shorter than $R/2$ and intersects the ball $\mathrm{B}(\gamma_\infty(t), R/2)$. Then σ cannot leave the ball $\overline{\mathrm{B}}[\gamma_\infty(t), R]$. Hence, by Proposition 8.1.1, σ is a geodesic. In particular, for all sufficiently large n, any arc of γ_n of length $R/2$ or less containing $\gamma_n(t)$ is a geodesic.

Since $\mathcal{B} = \overline{\mathrm{B}}[\gamma_\infty(t), R]$ is a proper length $\mathrm{CAT}(0)$ space, by 7.5.1, geodesic segments in $\mathcal{B}$ depend uniquely on their endpoint pairs. Thus there is a subinterval $\mathbb{I}$ of $[0,1]$, that contains a neighborhood of t in $[0,1]$ and such that the arc $\gamma_n|_{\mathbb{I}}$ is minimizing for all large n. It follows that $\gamma_\infty|_{\mathbb{I}}$ is a geodesic, and therefore γ_∞ is a local geodesic.

The $\mathrm{CAT}(1)$ case is done in the same way, but one has to assume in addition that $R < \pi$. $\square$

The following lemma and its proof were suggested to us by Alexander Lytchak. This lemma allows a local geodesic path to be moved continuously so that its endpoints follow given trajectories. This statement was originally proved by Stephanie Alexander and Richard Bishop using a different method [**alexander-bishop-1990**].

9.2.2. Patchwork along a curve. *Let $\mathcal{U}$ be a proper length, locally* $\mathrm{CAT}(0)$ *space, and $\gamma\colon [0,1] \to \mathcal{U}$ be a path.*

Then there is a proper length $\mathrm{CAT}(0)$ *space $\mathcal{N}$, an open set $\hat{\Omega} \subset \mathcal{N}$, and a path $\hat{\gamma}\colon [0,1] \to \hat{\Omega}$, such that there is an open locally isometric immersion $\Phi\colon \hat{\Omega} \looparrowright \mathcal{U}$ satisfying $\Phi \circ \hat{\gamma} = \gamma$.*

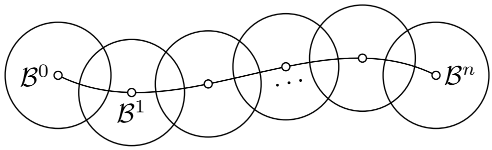
If $\operatorname{length}\gamma < \pi$*, then the same holds in the* $\mathrm{CAT}(1)$ *case. Namely we assume that $\mathcal{U}$ is a proper length, locally* $\mathrm{CAT}(1)$ *space and construct a proper length* $\mathrm{CAT}(1)$ *space $\mathcal{N}$ with the same property as above.*

Proof. Fix $r > 0$ so that for each $t \in [0,1]$, the closed ball $\overline{\mathrm{B}}[\gamma(t), r]$ forms a proper length $\mathrm{CAT}(\kappa)$ space.

Choose a partition $0 = t^0 < t^1 < \dots < t^n = 1$ such that

$$\mathrm{B}(\gamma(t^i), r) \supset \gamma([t^{i-1}, t^i])$$

for all $n > i > 0$. Set $\mathcal{B}^i = \overline{\mathrm{B}}[\gamma(t^i), r]$.



Consider the disjoint union $\bigsqcup_i \mathcal{B}^i = \{(i,x) : x \in \mathcal{B}^i\}$ with the minimal equivalence relation $\sim$ such that $(i,x) \sim (i-1,x)$ for all i. Let $\mathcal{N}$ be the space obtained by gluing the $\mathcal{B}^i$ along $\sim$.

Note that $A^i = \mathcal{B}^i \cap \mathcal{B}^{i-1}$ is convex in $\mathcal{B}^i$ and in $\mathcal{B}^{i-1}$. Applying the Reshetnyak gluing theorem (8.2.1) n times, we conclude that $\mathcal{N}$ is a proper length $\mathrm{CAT}(0)$ space.

For $t \in [t^{i-1}, t^i]$, define $\hat\gamma(t)$ as the equivalence class of $(i, \gamma(t))$ in $\mathcal{N}$. Let $\hat\Omega$ be the ε-neighborhood of $\hat\gamma$ in $\mathcal{N}$, where $\varepsilon > 0$ is chosen so that $\mathrm{B}(\gamma(t), \varepsilon) \subset \mathcal{B}^i$ for all $t \in [t^{i-1}, t^i]$.

Define $\Phi\colon \hat\Omega \to \mathcal{U}$ by sending the equivalence class of (i,x) to x. It is straightforward to check that Φ, $\hat\gamma$ and $\hat\Omega \subset \mathcal{N}$ satisfy the conclusion of the lemma.

The $\mathrm{CAT}(1)$ case is proved in the same way. □

The following two corollaries follow from: (1) patchwork (9.2.2), (2) Proposition 8.1.1, which states that local geodesics are geodesics in any $\mathrm{CAT}(0)$ space, and (3) Proposition 7.5.1 on uniqueness of geodesics.

9.2.3. Corollary. *If $\mathcal{U}$ is a proper length, locally $\mathrm{CAT}(0)$ space, then for any pair of points $p, q \in \mathcal{U}$, the space of all local geodesic paths from p to q is discrete; that is, for any local geodesic path γ connecting p to q, there is $\varepsilon > 0$ such that for any other local geodesic path δ from p to q we have $|\gamma(t) - \delta(t)|_{\mathcal{U}} > \varepsilon$ for some $t \in [0,1]$.*

Analogously, if $\mathcal{U}$ is a proper length, locally $\mathrm{CAT}(1)$ space, then for any pair of points $p, q \in \mathcal{U}$, the space of all local geodesic paths shorter than π from p to q is discrete.

9.2.4. Corollary. *If $\mathcal{U}$ is a proper length, locally $\mathrm{CAT}(0)$ space, then for any path α there is a choice of local geodesic path γ_α connecting the ends of α such that the map $\alpha \mapsto \gamma_\alpha$ is continuous, and if α is a local geodesic path then $\gamma_\alpha = \alpha$.*

Analogously, if $\mathcal{U}$ is a proper length, locally $\mathrm{CAT}(1)$ space, then for any path α shorter than π, there is a choice of local geodesic path γ_α shorter than π connecting the ends of α such that the map $\alpha \mapsto \gamma_\alpha$ is continuous, and if α is a local geodesic path then $\gamma_\alpha = \alpha$.

Proof of 9.2.4. We do the $\mathrm{CAT}(0)$ case; the $\mathrm{CAT}(1)$ case is analogous.

Consider the maximal interval $\mathbb{I} \subset [0,1]$ containing 0 such that there is a continuous one-parameter family of local geodesic paths γ_t for $t \in \mathbb{I}$ connecting $\alpha(0)$ to $\alpha(t)$, with $\gamma_t(0) = \gamma_0(t) = \alpha(0)$ for any t.

By Proposition 9.2.1, $\mathbb{I}$ is closed, so we may assume $\mathbb{I} = [0,s]$ for some $s \in [0,1]$.

Applying patchwork (9.2.2) to γ_s, we find that $\mathbb{I}$ is also open in $[0,1]$. Hence $\mathbb{I} = [0,1]$. Set $\gamma_\alpha = \gamma_1$.

By construction, if α is a local geodesic path, then $\gamma_\alpha = \alpha$.

Moreover, from Corollary 9.2.3, the construction $\alpha \mapsto \gamma_\alpha$ produces close results for sufficiently close paths in the metric defined by ❶; that is, the map $\alpha \mapsto \gamma_\alpha$ is continuous. $\square$

Given a path $\alpha\colon [0,1] \to \mathcal{U}$, we denote by $\bar\alpha$ the same path traveled in the opposite direction; that is,

$$\bar\alpha(t) = \alpha(1-t).$$

The *product* of two paths will be denoted with "$*$"; if two paths α and β connect the same pair of points, then the product $\bar\alpha * \beta$ is a closed curve.

9.2.5. Exercise. *Assume $\mathcal{U}$ is a proper length, locally* CAT(1) *space. Consider the construction $\alpha \mapsto \gamma_\alpha$ provided by Corollary 9.2.4.*

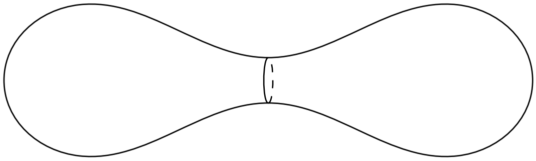
Assume that α and β are two paths connecting the same pair of points in $\mathcal{U}$, where each is shorter than π and the product $\bar\alpha * \beta$ is null-homotopic in the class of closed curves shorter than $2\cdot\pi$. Show that $\gamma_\alpha = \gamma_\beta$.

9.3 Globalization

9.3.1. Globalization theorem. *If a proper length, locally* CAT(0) *space is simply connected, then it is* CAT(0).

Analogously, suppose $\mathcal{U}$ is a proper length, locally CAT(1) *space such that any closed curve $\gamma\colon \mathbb{S}^1 \to \mathcal{U}$ shorter than $2\cdot\pi$ is null-homotopic in the class of closed curves shorter than $2\cdot\pi$. Then $\mathcal{U}$ is* CAT(1).

The surface on the diagram is an example of a simply connected space that is locally CAT(1) but not CAT(1). To contract the marked



curve one has to increase its length to $2\cdot\pi$ or more; in particular the surface does not satisfy the assumption of the globalization theorem.

The proof of the globalization theorem relies on the following theorem, which is essentially [**alexandrov-1957**].

9.3.2. Patchwork globalization theorem. *A proper length, locally* $\mathrm{CAT}(0)$ *space* $\mathcal{U}$ *is* $\mathrm{CAT}(0)$ *if and only if all pairs of points in* $\mathcal{U}$ *are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

Analogously, a proper length, locally $\mathrm{CAT}(1)$ *space* $\mathcal{U}$ *is* $\mathrm{CAT}(1)$ *if and only if all pairs of points in* $\mathcal{U}$ *at distance less than* π *are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

The proof uses a thin-triangle decomposition with the inheritance lemma (8.1.2) and the following construction:

9.3.3. Line-of-sight map. *Let* p *be a point and* α *be a curve of finite length in a length space* $\mathcal{X}$*. Let* $\mathring{\alpha} : [0,1] \to \mathcal{U}$ *be the constant-speed parametrization of* α*. If* $\gamma_t\colon [0,1] \to \mathcal{U}$ *is a geodesic path from* p *to* $\mathring{\alpha}(t)$*, we say*

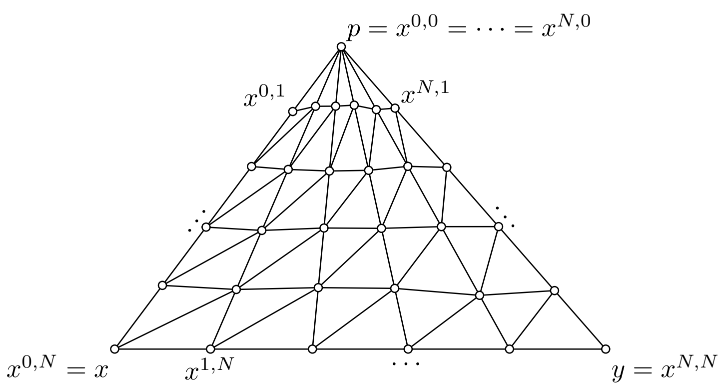
$$[0,1]\times[0,1]\to\mathcal{U}\colon (t,s)\mapsto\gamma_t(s)$$

is a line-of-sight map from p to α*.*

Proof of the patchwork globalization theorem (9.3.2). Note that the implication "only if" follows from 7.5.1 and 7.7.4; it remains to prove the "if" part.

Fix a triangle $[pxy]$ in $\mathcal{U}$. We need to show that $[pxy]$ is thin.

By the assumptions, the line-of-sight map $(t,s)\mapsto\gamma_t(s)$ from p to $[xy]$ is uniquely defined and continuous.



Fix a partition

$$0 = t^0 < t^1 < \ldots < t^N = 1,$$

and set $x^{i,j} = \gamma_{t^i}(t^j)$. Since the line-of-sight map is continuous and $\mathcal{U}$ is locally CAT(0), we may assume that the triangles

$$[x^{i,j} x^{i,j+1} x^{i+1,j+1}] \quad \text{and} \quad [x^{i,j} x^{i+1,j} x^{i+1,j+1}]$$

are thin for each pair i, j.

Now we show that the thin property propagates to $[pxy]$ by repeated application of the inheritance lemma (8.1.2):

- ◇ For fixed i, sequentially applying the lemma shows that the triangles $[px^{i,1}x^{i+1,2}]$, $[px^{i,2}x^{i+1,2}]$, $[px^{i,2}x^{i+1,3}]$, and so on are thin.

In particular, for each i, the long triangle $[px^{i,N}x^{i+1,N}]$ is thin.

- ◇ By the same lemma the triangles $[px^{0,N}x^{2,N}]$, $[px^{0,N}x^{3,N}]$, and so on, are thin.

In particular, $[pxy] = [px^{0,N}x^{N,N}]$ is thin. □

Proof of the globalization theorem; CAT(0) *case.* Let $\mathcal{U}$ be a proper length, locally CAT(0) space that is simply connected. Given a path α in $\mathcal{U}$, denote by γ_α the local geodesic path provided by Corollary 9.2.4. Since the map $\alpha \mapsto \gamma_\alpha$ is continuous, by Corollary 9.2.3 we have $\gamma_\alpha = \gamma_\beta$ for any pair of paths α and β homotopic relative to the ends.

Since $\mathcal{U}$ is simply connected, any pair of paths with common ends are homotopic. In particular, if α and β are local geodesics from p to q, then $\alpha = \gamma_\alpha = \gamma_\beta = \beta$ by Corollary 9.2.4. It follows that any two points $p, q \in \mathcal{U}$ are joined by a unique local geodesic that depends continuously on (p, q).

Since $\mathcal{U}$ is geodesic, it remains to apply the patchwork globalization theorem (9.3.2).

CAT(1) *case.* The proof goes along the same lines, but one needs to use Exercise 9.2.5. □

9.3.4. Corollary. *Any compact length, locally* CAT(0) *space that contains no closed local geodesics is* CAT(0).

Analogously, any compact length, locally CAT(1) *space that contains no closed local geodesics shorter than* $2\cdot\pi$ *is* CAT(1).

Proof. By the globalization theorem (9.3.1), we need to show that the space is simply connected. Assume the contrary. Fix a nontrivial homotopy class of closed curves.

Denote by ℓ the exact lower bound for the lengths of curves in the class. Note that $\ell > 0$; otherwise there would be a closed noncontractible curve in a CAT(0) neighborhood of some point, contradicting 7.6.3.

Since the space is compact, the class contains a length-minimizing curve, which must be a closed local geodesic.

The CAT(1) case is analogous, one only has to consider a homotopy class of closed curves shorter than $2\cdot\pi$. □

9.3.5. Exercise. *Prove that any compact length, locally* CAT(0) *space* $\mathcal{X}$ *that is not* CAT(0) *contains a* geodesic circle*; that is, a simple closed curve γ such that for any two points $p, q \in \gamma$, one of the arcs of γ with endpoints p and q is a geodesic.*

Formulate and prove the analogous statement for CAT(1) *spaces.*

9.3.6. Advanced exercise. *Let $\mathcal{U}$ be a proper length* CAT(0) *space. Assume $\tilde{\mathcal{U}} \to \mathcal{U}$ is a metric double cover branching along a geodesic. (For example 3-dimensional Euclidean space admits a double cover branching along a line.)*

Show that $\tilde{\mathcal{U}}$ is CAT(0)*.*

Hint: Apply the globalization theorem (9.3.1) and that an r-neighborhood of convex set is convex (7.7.5).

9.4 Remarks

Riemannian manifolds with nonpositive sectional curvature are locally CAT(0). The original formulation of the *globalization theorem*, or *Hadamard–Cartan theorem*, states that if M is a complete Riemannian manifold with sectional curvature at most 0, then the exponential map at any point $p \in M$ is a covering; in particular it implies that the universal cover of M is diffeomorphic to the Euclidean space of the same dimension.

In this generality, this theorem appeared in the lectures of Elie Cartan [**cartan**]. This theorem was proved for surfaces in Euclidean 3-space by Hans von Mangoldt [**mangoldt**] and a few years later independently for two-dimensional Riemannian manifolds by Jacques Hadamard [**hadamard**].

Formulations for metric spaces of different generality were proved by Herbert Busemann in [**busemann-CBA**], Willi Rinow in [**rinow**], Mikhael Gromov in [**gromov-1987**]. A detailed proof of Gromov's statement was given by Werner Ballmann in [**ballmann-1995**] when

$\mathcal{U}$ is proper, and by the Stephanie Alexander and Richard Bishop in [**alexander-bishop-1990**] in more generality.

For proper CAT(1) spaces, the globalization theorem was proved by Brian Bowditch [**bowditch**].

The globalization theorem holds for complete length spaces (not necessary proper spaces) [**alexander-kapovitch-petrunin-2025**].

The patchwork globalization (9.3.2) is proved by Alexandrov [**alexandrov-1957**]. For proper spaces one can remove the continuous dependence from the formulation; it follows from uniqueness. For complete spaces the later is not true [**bridson-haefliger**].

For spaces with curvature bounded below globalization requires no additional condition. Namely the following theorem holds; see [**alexander-kapovitch-petrunin-2025**] and the references therein.

9.4.1. Globalization theorem. *Any complete length locally* $\mathrm{CBB}(\kappa)$ *space is* $\mathrm{CBB}(\kappa)$*.*

Lecture 10

Polyhedral spaces

This lecture is nearly a copy of [**alexander-kapovitch-petrunin-2019**]; here we give a condition for polyhedral spaces that grantees that it is CAT(0).

10.1 Space of directions and tangent space

In this section we introduce a metric analog of (unit) tangent bundle that makes sense in Alexandrov geometry.

Let $\mathcal{X}$ be a metric space with defined angles for all hinges; by 7.6.2 it holds for any CBB(κ) or CAT(κ) space. Fix a point $p \in \mathcal{X}$.

Consider the set $\mathfrak{S}_p$ of all nontrivial geodesics that start at p. By 7.2.2, the triangle inequality holds for $\measuredangle$ on $\mathfrak{S}_p$, so $(\mathfrak{S}_p, \measuredangle)$ forms a *pseudometric space*; that is, $\measuredangle$ satisfies all the conditions of a metric on $\mathfrak{S}_p$, except that the angle between distinct geodesics might vanish.

The metric space corresponding to $(\mathfrak{S}_p, \measuredangle)$ is called the *space of geodesic directions* at p, denoted by Σ'_p or $\Sigma'_p\mathcal{X}$. Elements of Σ'_p are called *geodesic directions* at p. Each geodesic direction is formed by an equivalence class of geodesics in $\mathfrak{S}_p$ for the equivalence relation

$$[px] \sim [py] \quad \iff \quad \measuredangle[p\,{}^x_y] = 0.$$

The completion of Σ'_p is called the *space of directions* at p and is denoted by Σ_p or $\Sigma_p\mathcal{X}$. Elements of Σ_p are called *directions* at p.

The Euclidean cone $\operatorname{Cone}\Sigma_p$ over the space of directions Σ_p is called the *tangent space* at p and is denoted by T_p or $\mathrm{T}_p\mathcal{X}$.

10.1.1. Exercise. *Assume $\mathcal{U}$ is a proper length* CAT(0) *space with extendable geodesics; that is, any geodesic is an arc in a local geodesic $\mathbb{R} \to \mathcal{U}$.*

Show that the space of geodesic directions at any point in $\mathcal{U}$ is complete.

Does the statement remain true if $\mathcal{U}$ is complete, but not required to be proper?

The tangent space T_p could also be defined directly, without introducing the space of directions. To do so, consider the set $\mathfrak{T}_p$ of all geodesics with constant-speed parametrizations starting at p. Given $\alpha, \beta \in \mathfrak{T}_p$, set

$$❶ \qquad |\alpha - \beta|_{\mathfrak{T}_p} = \lim_{\varepsilon \to 0} \frac{|\alpha(\varepsilon) - \beta(\varepsilon)|_{\mathcal{X}}}{\varepsilon}$$

Since the angles in $\mathcal{X}$ are defined, ❶ defines a pseudometric on $\mathfrak{T}_p$.

The corresponding metric space admits a natural isometric identification with the cone $\mathrm{T}'_p = \mathrm{Cone}\,\Sigma'_p$. The elements of T'_p are equivalence classes for the relation

$$\alpha \sim \beta \quad \Longleftrightarrow \quad |\alpha(t) - \beta(t)|_{\mathcal{X}} = o(t).$$

The completion of T'_p is therefore naturally isometric to T_p.

Elements of T_p will be called *tangent vectors* at p, regardless of the fact that T_p is only a metric cone and need not be a vector space. Elements of T'_p will be called *geodesic tangent vectors* at p.

10.1.2. Exercise. *Let $\mathcal{X}$ be a complete length* CAT(0) *space. Show that for any point $p \in \mathcal{X}$ the tangent space $\mathrm{T}_p\mathcal{X}$ is isometric to a subset of the ultra-tangent space $\mathrm{T}_p^\omega\mathcal{X}$ (defined on page 54).*

Use 7.3.4 to conclude that $\mathrm{T}_p\mathcal{X}$ is CAT(0)*.*

10.1.3. Exercise. *Let $\mathcal{X}$ be a complete length* CAT(0) *space. Show that for any point $p \in \mathcal{X}$ the tangent space $\mathrm{T}_p\mathcal{X}$ is a length space.*

10.2 Suspension

Suspension a spherical analog of cone construction defined on page 63.

The *suspension* $\mathcal{V} = \mathrm{Susp}\,\mathcal{U}$ over a metric space $\mathcal{U}$ is defined as the metric space whose underlying set consists of equivalence classes in $[0, \pi] \times \mathcal{U}$ with the equivalence relation "$\sim$" given by $(0, p) \sim (0, q)$ and $(\pi, p) \sim (\pi, q)$ for any points $p, q \in \mathcal{U}$, and whose metric is given by the spherical cosine rule

$$\cos|(p, s) - (q, t)|_{\mathrm{Susp}\,\mathcal{U}} = \cos s \cdot \cos t - \sin s \cdot \sin t \cdot \cos \alpha,$$

where $\alpha = \min\{\pi, |p - q|_{\mathcal{U}}\}$.

The points in $\mathcal{V}$ formed by the equivalence classes of $0\times\mathcal{U}$ and $\pi\times\mathcal{U}$ are called the *north* and the *south poles* of the suspension.

10.2.1. Exercise. *Let $\mathcal{U}$ be a metric space. Show that the spaces*

$$\mathbb{R}\times\mathrm{Cone}\,\mathcal{U}\quad\textit{and}\quad\mathrm{Cone}[\mathrm{Susp}\,\mathcal{U}]$$

are isometric.

The following statement is a direct analog of 7.4.3 and it can be proved along the same lines.

10.2.2. Proposition. *Let $\mathcal{U}$ be a metric space. Then* $\mathrm{Susp}\,\mathcal{U}$ *is* CAT(1) *if and only if $\mathcal{U}$ is* CAT(1)*.*

10.3 Definitions

10.3.1. Definition. *A length space $\mathcal{P}$ is called a* (spherical) polyhedral space *if it admits a finite triangulation τ such that every simplex in τ is isometric to a simplex in a Euclidean space (or respectively a unit sphere) of appropriate dimension.*

By a triangulation of a polyhedral space *we will always understand a triangulation as above.*

Note that according to the above definition, all polyhedral spaces are compact. However, most of the statements below admit straightforward generalizations to *locally polyhedral spaces*; that is, complete length spaces, any point of which admits a closed neighborhood isometric to a polyhedral space. The latter class of spaces includes in particular infinite covers of polyhedral spaces.

The *dimension* of a polyhedral space $\mathcal{P}$ is defined as the maximal dimension of the simplices in one (and therefore any) triangulation of $\mathcal{P}$.

Links. Let $\mathcal{P}$ be a polyhedral space and σ be a simplex in a triangulation τ of $\mathcal{P}$.

The simplices that contain σ form an abstract simplicial complex called the *link* of σ, denoted by Link_σ. If m is the dimension of σ, then the set of vertices of Link_σ is formed by the $(m+1)$-simplices that contain σ; the set of its edges are formed by the $(m+2)$-simplices that contain σ; and so on.

The link Link_σ can be identified with the subcomplex of τ formed by all the simplices σ' such that $\sigma\cap\sigma'=\varnothing$ but both σ and σ' are faces of a simplex of τ.

The points in Link_σ can be identified with the normal directions to σ at a point in its interior. The angle metric between directions makes

Link_σ into a spherical polyhedral space. We will always consider the link with this metric.

Tangent space and space of directions. Let $\mathcal{P}$ be a polyhedral space (Euclidean or spherical) and τ be its triangulation. If a point $p \in \mathcal{P}$ lies in the interior of a k-simplex σ of τ then the tangent space $\mathrm{T}_p = \mathrm{T}_p\mathcal{P}$ is naturally isometric to

$$\mathbb{E}^k \times (\mathrm{Cone}\,\mathrm{Link}_\sigma).$$

Equivalently, the space of directions $\Sigma_p = \Sigma_p\mathcal{P}$ can be isometrically identified with the k-times iterated suspension over Link_σ; that is,

$$\Sigma_p \stackrel{iso}{=\!=} \mathrm{Susp}^k(\mathrm{Link}_\sigma).$$

If $\mathcal{P}$ is an m-dimensional polyhedral space, then for any $p \in \mathcal{P}$ the space of directions Σ_p is a spherical polyhedral space of dimension at most $m-1$.

In particular, for any point p in σ, the isometry class of Link_σ together with $k = \dim\sigma$ determines the isometry class of Σ_p, and the other way around — Σ_p and k determines the isometry class of Link_σ.

A small neighborhood of p is isometric to a neighborhood of the tip of $\mathrm{Cone}\,\Sigma_p$. (If $\mathcal{P}$ is a spherical polyhedral space, then a small neighborhood of p is isometric to a neighborhood of the north pole in $\mathrm{Susp}\,\Sigma_p$.) In fact, if this property holds at any point of a compact length space $\mathcal{P}$, then $\mathcal{P}$ is a polyhedral space [**lebedeva-petrunin**].

10.4 CAT test

The following theorem provides a combinatorial description of polyhedral spaces with curvature bounded above.

10.4.1. Theorem. *Let $\mathcal{P}$ be a polyhedral space and τ be its triangulation. Then $\mathcal{P}$ is locally* $\mathrm{CAT}(0)$ *if and only if the link of each simplex in τ has no closed local geodesic shorter than* $2{\cdot}\pi$.

Analogously, let $\mathcal{P}$ be a spherical polyhedral space and τ be its triangulation. Then $\mathcal{P}$ is $\mathrm{CAT}(1)$ *if and only if neither $\mathcal{P}$ nor the link of any simplex in τ has a closed local geodesic shorter than* $2{\cdot}\pi$.

Proof. The "only if" part follows from 8.1.1, 10.2.2, and 7.4.3.

To prove the "if" part, we apply induction on $\dim\mathcal{P}$. The base case $\dim\mathcal{P} = 0$ is evident. Let us start with the CAT(1) case.

Step. Assume that the theorem is proved in the case $\dim\mathcal{P} < m$. Suppose $\dim\mathcal{P} = m$.

Fix a point $p \in \mathcal{P}$. A neighborhood of p is isometric to a neighborhood of the north pole in the suspension over the space of directions Σ_p.

Note that Σ_p is a spherical polyhedral space, and its links are isometric to links of $\mathcal{P}$. By the induction hypothesis, Σ_p is $\mathrm{CAT}(1)$. Thus, by the second part of Exercise 7.4.3, $\mathcal{P}$ is locally $\mathrm{CAT}(1)$.

Applying the second part of Corollary 9.3.4, we get the statement.

The $\mathrm{CAT}(0)$ case is done in exactly the same way except we need to use the first part of Exercise 7.4.3 and the first part of Corollary 9.3.4 on the last step. □

10.4.2. Exercise. *Let $\mathcal{P}$ be a polyhedral space such that any two points can be connected by a unique geodesic. Show that $\mathcal{P}$ is* $\mathrm{CAT}(0)$.

10.4.3. Advanced exercise. *Construct a Euclidean polyhedral metric on $\mathbb{S}^3$ such that the total angle around each edge in its triangulation is at least $2\cdot\pi$.*

10.5 Flag complexes

10.5.1. Definition. *A simplicial complex $\mathcal{S}$ is called* flag *if whenever $\{v^0, \dots, v^k\}$ is a set of distinct vertices of $\mathcal{S}$ that are pairwise joined by edges, then the vertices $v^0, \dots, v^k$ span a k-simplex in $\mathcal{S}$.*

If the above condition is satisfied for $k = 2$, then we say that $\mathcal{S}$ satisfies the no-triangle condition.

Note that every flag complex is determined by its one-skeleton. Moreover, for any graph, its *cliques* (that is, complete subgraphs) define a flag complex. For that reason flag complexes are also called *clique complexes.*

10.5.2. Exercise. *Show that the barycentric subdivision of any simplicial complex is a flag complex.*

Use the flag condition (see 10.5.5 below) to conclude that any finite simplicial complex is homeomorphic to a proper length $\mathrm{CAT}(1)$ *space.*

10.5.3. Proposition. *A simplicial complex $\mathcal{S}$ is flag if and only if $\mathcal{S}$ as well as all the links of all its simplices satisfy the no-triangle condition.*

From the definition of flag complex we get the following.

10.5.4. Observation. *Any link of any simplex in a flag complex is flag.*

Proof of 10.5.3. By Observation 10.5.4, the no-triangle condition holds for any flag complex and the links of all its simplices.

Now assume that a complex $\mathcal{S}$ and all its links satisfy the no-triangle condition. It follows that $\mathcal{S}$ includes a 2-simplex for each triangle. Applying the same observation for each edge we get that $\mathcal{S}$ includes a 3-simplex for any complete graph with 4 vertices. Repeating this observation for triangles, 4-simplices, 5-simplices, and so on, we get that $\mathcal{S}$ is flag. $\square$

All-right triangulation. A triangulation of a spherical polyhedral space is called an *all-right triangulation* if each simplex of the triangulation is isometric to a spherical simplex all of whose angles are right. Similarly, we say that a simplicial complex is equipped with an *all-right spherical metric* if it is a length metric and each simplex is isometric to a spherical simplex all of whose angles are right.

Spherical polyhedral CAT(1) spaces glued from right-angled simplices admit the following characterization discovered by Mikhael Gromov [**gromov-1987**].

10.5.5. Flag condition. *Assume that a spherical polyhedral space $\mathcal{P}$ admits an all-right triangulation τ. Then $\mathcal{P}$ is* CAT(1) *if and only if τ is flag.*

Proof; "only if" part. Assume there are three vertices v^1, v^2 and v^3 of τ that are pairwise joined by edges but do not span a triangle. Note that in this case

$$\measuredangle[v^1\,{}^{v^2}_{v^3}] = \measuredangle[v^2\,{}^{v^3}_{v^1}] = \measuredangle[v^3\,{}^{v^1}_{v^2}] = \pi.$$

Equivalently,

❶ *The product of the geodesics $[v^1v^2]$, $[v^2v^3]$ and $[v^3v^1]$ forms a locally geodesic loop in $\mathcal{P}$ of length $\tfrac{3}{2}\cdot\pi$.*

Now assume that $\mathcal{P}$ is CAT(1). Then by Theorem 10.4.1, $\mathrm{Link}_\sigma \mathcal{P}$ is CAT(1) for every simplex σ in τ.

Each of these links is an all-right spherical complex and by Theorem 10.4.1, none of these links can contain a geodesic circle shorter than $2\cdot\pi$.

Therefore Proposition 10.5.3 and ❶ imply the "only if" part.

"If" part. By Observation 10.5.4 and Theorem 10.4.1, it is sufficient to show that any closed local geodesic γ in a flag complex $\mathcal{S}$ with all-right metric has length at least $2\cdot\pi$.

Recall that the *closed star* of a vertex v (briefly $\overline{\mathrm{Star}}_v$) is formed by all the simplices containing v. Similarly, Star_v, the open star of v, is the union of all simplices containing v with faces opposite v removed.

Choose a vertex v such that Star_v contains a point $\gamma(t_0)$ of γ. Consider the maximal arc γ_v of γ that contains the point $\gamma(t_0)$ and runs in Star_v. Note that the distance $|v-\gamma_v(t)|_{\mathcal{P}}$ behaves in exactly the same way as the distance from the north pole in $\mathbb{S}^2$ to a geodesic in the north hemisphere; that is, there is a geodesic $\tilde\gamma_v$ in the north hemisphere of $\mathbb{S}^2$ such that for any t we have

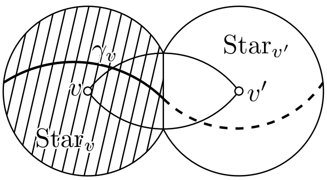
$$|v-\gamma_v(t)|_{\mathcal{P}} = |n-\tilde\gamma_v(t)|_{\mathbb{S}^2},$$

where n denotes the north pole of $\mathbb{S}^2$. In particular,

$$\operatorname{length}\gamma_v = \pi;$$

that is, γ spends time π on every visit to Star_v.

After leaving Star_v, the local geodesic γ has to enter another simplex, say σ'. Since τ is flag, the simplex σ' has a vertex v' not joined to v by an edge; that is,



$$\mathrm{Star}_v \cap \mathrm{Star}_{v'} = \varnothing$$

The same argument as above shows that γ spends time π on every visit to $\mathrm{Star}_{v'}$. Therefore the total length of γ is at least $2\cdot\pi$. $\square$

10.5.6. Exercise. *Assume that a spherical polyhedral space $\mathcal{P}$ admits a triangulation τ such that all edge lengths of all simplices are at least $\frac{\pi}{2}$. Show that $\mathcal{P}$ is* CAT(1) *if τ is flag.*

10.5.7. Exercise. *Let P be a convex polyhedron in $\mathbb{E}^3$ with n faces $F_1,\dots,F_n$. Suppose that each face of P has only obtuse or right angles. Let us take 2^n copies of P indexed by n-bit array. Glue two copies of P along F_i if their arrays differ only in i-th bit. Show that the obtained space is a locally* CAT(0) *topological manifold.*

The space of trees. The following construction is given by Louis Billera, Susan Holmes, and Karen Vogtmann in [**billera-holmes-vogtmann**].

Let $\mathcal{T}_n$ be the set of all metric trees with n end vertices labeled by $a^1,\dots,a^n$. To describe one tree in $\mathcal{T}_n$ we may fix a topological tree t with end vertices $a^1,\dots,a^n$ and all other vertices of degree 3, and prescribe the lengths of $2\cdot n-3$ edges. If the length of an edge vanishes, we assume that this edge degenerates; such a tree can be also described using a different topological tree t'. The subset of $\mathcal{T}_n$ corresponding to the given topological tree t can be identified with the octant

$$\{\,(x_1,\dots,x_{2\cdot n-3})\in\mathbb{R}^{2\cdot n-3}\;:\;x_i\geqslant 0\,\}.$$

Equip each such subset with the metric induced from $\mathbb{R}^{2\cdot n-3}$ and consider the length metric on $\mathcal{T}_n$ induced by these metrics.

10.5.8. Exercise. *Show that $\mathcal{T}_n$ with the described metric is* $\mathrm{CAT}(0)$*.*

10.6 Remarks

Let us formulate a test for spaces with lower curvature bound.

10.6.1. Theorem. *Let $\mathcal{P}$ be a polyhedral space and τ be a triangulation of $\mathcal{P}$. Then $\mathcal{P}$ is* $\mathrm{CBB}(0)$ *if and only if the following conditions hold.*

(a) *τ is* pure*; that is, any simplex in τ is a face of some simplex of dimension exactly m.*
(b) *The link of any simplex of dimension $m-1$ is formed by single point or two points.*
(c) *The link of any simplex of dimension $\leqslant m-2$ is connected.*
(d) *Any link of any simplex of dimension $m-2$ has diameter at most π.*

The proof relies on 9.4.1. The condition *(c)* can be reformulated in the following way:

(c)′ Any path $\gamma\colon [0,1]\to\mathcal{P}$ can be approximated by paths $\gamma_n\colon [0,1]\to\mathcal{P}$ that cross only simplexes of dimension m and $m-1$.

Further, modulo the other conditions, the condition *(d)* is equivalent to the following:

(d)′ The link of any simplex of dimension $m-2$ is isometric to a circle of length $\leqslant 2\cdot\pi$ or a closed real interval of length $\leqslant\pi$.

Lecture 11

Exotic aspherical manifolds

This lecture is nearly a copy of [**alexander-kapovitch-petrunin-2019**]; here we describe a set of rules for gluing Euclidean cubes that produce a locally CAT(0) space and use these rules to construct exotic examples of aspherical manifolds.

11.1 Cubical complexes

The definition of a cubical complex mostly repeats the definition of a simplicial complex, with simplices replaced by cubes.

Formally, a *cubical complex* is defined as a subcomplex of the unit cube in the Euclidean space $\mathbb{R}^N$ of large dimension; that is, a collection of faces of the cube such that together with each face it contains all its sub-faces. Each cube face in this collection will be called a *cube* of the cubical complex.

Note that according to this definition, any cubical complex is finite.

The union of all the cubes in a cubical complex $\mathcal{Q}$ will be called its *underlying space*. A homeomorphism from the underlying space of $\mathcal{Q}$ to a topological space $\mathcal{X}$ is called a *cubulation of* $\mathcal{X}$.

The underlying space of a cubical complex $\mathcal{Q}$ will be always considered with the length metric induced from $\mathbb{R}^N$. In particular, with this metric, each cube of $\mathcal{Q}$ is isometric to the unit cube of the corresponding dimension.

It is straightforward to construct a triangulation of the underlying space of $\mathcal{Q}$ such that each simplex is isometric to a Euclidean simplex. In particular the underlying space of $\mathcal{Q}$ is a Euclidean polyhedral space.

The link of a cube in a cubical complex is defined similarly to the link of a simplex in a simplicial complex. It is a simplicial complex that admits a natural all-right triangulation — each simplex corresponds to an adjusted cube.

Cubical analog of a simplicial complex. Let $\mathcal{S}$ be a finite simplicial complex and $\{v_1, \dots, v_N\}$ be the set of its vertices.

Consider $\mathbb{R}^N$ with the standard basis $\{e_1, \dots, e_N\}$. Denote by $\square^N$ the standard unit cube in $\mathbb{R}^N$; that is,

$$\square^N = \left\{\,(x_1, \dots, x_N) \in \mathbb{R}^N \;:\; 0 \leqslant x_i \leqslant 1 \text{ for each } i\,\right\}.$$

Given a k-dimensional simplex $\langle v_{i_0}, \dots, v_{i_k}\rangle$ in $\mathcal{S}$, mark the $(k+1)$-dimensional faces in $\square^N$ (there are 2^{N-k} of them) which are parallel to the coordinate $(k+1)$-plane spanned by $e_{i_0}, \dots, e_{i_k}$.

Note that the set of all marked faces of $\square^N$ forms a cubical complex; it will be called the *cubical analog* of $\mathcal{S}$ and will be denoted as $\square_{\mathcal{S}}$.

11.1.1. Proposition. *Let $\mathcal{S}$ be a finite connected simplicial complex and $\mathcal{Q} = \square_{\mathcal{S}}$ be its cubical analog. Then the underlying space of $\mathcal{Q}$ is connected and the link of any vertex of $\mathcal{Q}$ is isometric to $\mathcal{S}$ equipped with the spherical right-angled metric.*

In particular, if $\mathcal{S}$ is a flag complex, then $\mathcal{Q}$ is a locally CAT(0) *and therefore its universal cover $\tilde{\mathcal{Q}}$ is* CAT(0).

Proof. The first part of the proposition follows from the construction of $\square_{\mathcal{S}}$.

If $\mathcal{S}$ is flag, then by the flag condition (10.5.5) the link of any cube in $\mathcal{Q}$ is CAT(1). Therefore, by the cone construction (7.4.3) $\mathcal{Q}$ is locally CAT(0). It remains to apply the globalization theorem (9.3.1). □

From Proposition 11.1.1, it follows that the cubical analog of any flag complex is aspherical. The following exercise states that the converse also holds; see [**davis-2001**].

11.1.2. Exercise. *Show that a finite simplicial complex is flag if and only if its cubical analog is aspherical.*

11.2 Construction

By the globalization theorem (9.3.1), any proper length CAT(0) space is contractible. Therefore all proper length, locally CAT(0) spaces are *aspherical*; that is, they have contractible universal covers. This observation can be used to construct examples of aspherical spaces.

Let $\mathcal{X}$ be a proper topological space. Recall that $\mathcal{X}$ is called *simply connected at infinity* if for any compact set $K \subset \mathcal{X}$ there is a bigger compact set $K' \supset K$ such that $\mathcal{X}\backslash K'$ is path connected and any loop which lies in $\mathcal{X}\backslash K'$ is null-homotopic in $\mathcal{X}\backslash K$.

Recall that path connected spaces are not empty by definition. Therefore compact spaces are not simply connected at infinity.

The following example was constructed by Michael Davis in [**davis-1983**].

11.2.1. Proposition. *For any $m \geqslant 4$ there is a closed aspherical m-dimensional manifold whose universal cover is not simply connected at infinity.*

In particular, the universal cover of this manifold is not homeomorphic to the m-dimensional Euclidean space.

The proof requires the following lemma.

11.2.2. Lemma. *Let $\mathcal{S}$ be a finite flag complex, $\mathcal{Q} = \square_{\mathcal{S}}$ be its cubical analog and $\tilde{\mathcal{Q}}$ be the universal cover of $\mathcal{Q}$.*

Assume $\tilde{\mathcal{Q}}$ is simply connected at infinity. Then $\mathcal{S}$ is simply connected.

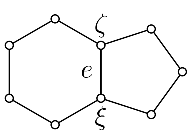
Proof. Assume $\mathcal{S}$ is not simply connected. Equip $\mathcal{S}$ with an all-right spherical metric. Choose a shortest noncontractible circle $\gamma\colon \mathbb{S}^1 \to \mathcal{S}$ formed by the edges of $\mathcal{S}$.

Note that γ forms a one-dimensional subcomplex of $\mathcal{S}$ which is a closed local geodesic. Denote by G the subcomplex of $\mathcal{Q}$ which corresponds to γ.

Fix a vertex $v \in G$; let G_v be the connected component of v in G. Let $\tilde{G}$ be a connected component of the inverse image of G_v in $\tilde{\mathcal{Q}}$ for the universal cover $\tilde{\mathcal{Q}} \to \mathcal{Q}$. Fix a point $\tilde{v} \in \tilde{G}$ in the inverse image of v.

Note that

❶ *$\tilde{G}$ is a convex set in $\tilde{\mathcal{Q}}$.*



Indeed, according to Proposition 11.1.1, $\tilde{\mathcal{Q}}$ is CAT(0). By Exercise 7.7.7, it is sufficient to show that $\tilde{G}$ is locally convex in $\tilde{\mathcal{Q}}$, or equivalently, G is locally convex in $\mathcal{Q}$.

Note that the latter can only fail if γ contains two vertices, say ξ and ζ in $\mathcal{S}$, which are joined by an edge not in γ; denote this edge by e.

Each edge of $\mathcal{S}$ has length $\frac{\pi}{2}$. Therefore each of two circles formed by e and an arc of γ from ξ to ζ is shorter that γ. Moreover, at least one of them is noncontractible since γ is noncontractible. That is, γ is not a shortest noncontractible circle, a contradiction. △

Further, note that $\tilde{G}$ is homeomorphic to the plane, since $\tilde{G}$ is a two-dimensional manifold without boundary which by the above is CAT(0) and hence is contractible.

Denote by C_R the circle of radius R in $\tilde{G}$ centered at $\tilde{v}$. All C_R are homotopic to each other in $\tilde{G}\backslash\{\tilde{v}\}$ and therefore in $\tilde{\mathcal{Q}}\backslash\{\tilde{v}\}$.

Note that the map $\tilde{\mathcal{Q}}\backslash\{\tilde{v}\} \to \mathcal{S}$ which returns the direction of $[\tilde{v}x]$ for any $x \neq \tilde{v}$, maps C_R to a circle homotopic to γ. Therefore C_R is not contractible in $\tilde{\mathcal{Q}}\backslash\{\tilde{v}\}$.

If R is large, the circle C_R lies outside of any fixed compact set K' in $\tilde{\mathcal{Q}}$. From above C_R is not contractible in $\tilde{\mathcal{Q}}\backslash K$ if $K \supset \tilde{v}$. It follows that $\tilde{\mathcal{Q}}$ is not simply connected at infinity, a contradiction. □

The proof of the following exercise is analogous. It will be used later in the proof of Proposition 11.2.4 — a more geometric version of Proposition 11.2.1.

11.2.3. Exercise. *Under the assumptions of Lemma 11.2.2, for any vertex v in $\mathcal{S}$ the complement $\mathcal{S}\backslash\{v\}$ is simply connected.*

Proof of 11.2.1. Let Σ^{m-1} be an $(m-1)$-dimensional smooth homology sphere that is not simply connected, and bounds a contractible smooth compact m-dimensional manifold $\mathcal{W}$.

For $m \geqslant 5$ the existence of such $(\mathcal{W}, \Sigma)$ follows from [**kervaire**]. For $m = 4$ it follows from the construction in [**mazur**].

Pick any triangulation τ of W and let $\mathcal{S}$ be the resulting subcomplex that triangulates Σ.

We can assume that $\mathcal{S}$ is flag; otherwise pass to the barycentric subdivision of τ and apply Exercise 10.5.2.

Let $\mathcal{Q} = \square_{\mathcal{S}}$ be the cubical analog of $\mathcal{S}$.

By Proposition 11.1.1, $\mathcal{Q}$ is a homology manifold. It follows that $\mathcal{Q}$ is a piecewise linear manifold with a finite number of singularities at its vertices.

Removing a small contractible neighborhood V_v of each vertex v in $\mathcal{Q}$, we can obtain a piecewise linear manifold $\mathcal{N}$ whose boundary is formed by several copies of Σ.

Let us glue a copy of $\mathcal{W}$ along its boundary to each copy of Σ in the boundary of $\mathcal{N}$. This results in a closed piecewise linear manifold $\mathcal{M}$ which is homotopically equivalent to $\mathcal{Q}$.

Indeed, since both V_v and $\mathcal{W}$ are contractible, the identity map of their common boundary Σ can be extended to a homotopy equivalence $V_v \to \mathcal{W}$ relative to the boundary. Therefore the identity map on $\mathcal{N}$ extends to homotopy equivalences $f\colon \mathcal{Q} \to \mathcal{M}$ and $g\colon \mathcal{M} \to \mathcal{Q}$.

Finally, by Lemma 11.2.2, the universal cover $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ is not simply connected at infinity.

The same holds for the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$. The latter follows since the constructed homotopy equivalences $f\colon \mathcal{Q}\to\mathcal{M}$ and $g\colon \mathcal{M}\to\mathcal{Q}$ lift to *proper maps* $\tilde f\colon \tilde{\mathcal{Q}}\to\tilde{\mathcal{M}}$ and $\tilde g\colon \tilde{\mathcal{M}}\to\tilde{\mathcal{Q}}$; that is, for any compact sets $A\subset\tilde{\mathcal{Q}}$ and $B\subset\tilde{\mathcal{M}}$, the inverse images $\tilde g^{-1}(A)$ and $\tilde f^{-1}(B)$ are compact. □

The following proposition was proved by Fredric Ancel, Michael Davis, and Craig Guilbault [**ancel-davis-guilbault**]; it could be considered as a more geometric version of Proposition 11.2.1.

11.2.4. Proposition. *Given $m\geqslant 5$, there is a Euclidean polyhedral space $\mathcal{P}$ such that:*

(a) $\mathcal{P}$ is homeomorphic to a closed m-dimensional manifold.
(b) $\mathcal{P}$ is locally CAT(0)*.*
(c) The universal cover of $\mathcal{P}$ is not simply connected at infinity.

There are no three-dimensional examples of that type; see [**rolfsen**] by Dale Rolfsen. In [**thurston**], Paul Thurston conjectured that the same holds in the four-dimensional case.

Proof. Apply Exercise 11.2.3 to the barycentric subdivision of the simplicial complex $\mathcal{S}$ provided by Exercise 11.2.5. □

11.2.5. Exercise. *Given an integer $m\geqslant 5$, construct a finite $(m-1)$-dimensional simplicial complex $\mathcal{S}$ such that* $\operatorname{Cone}\mathcal{S}$ *is homeomorphic to $\mathbb{E}^m$ and $\pi_1(\mathcal{S}\backslash\{v\})\neq 0$ for some vertex v in $\mathcal{S}$.*

11.3 Remarks

As was mentioned earlier, the motivation for the notion of CAT(κ) spaces comes from the fact that a Riemannian manifold is locally CAT(κ) if and only if it has $\sec\leqslant\kappa$. This easily follows from Rauch comparison for Jacobi fields and Proposition 7.7.2.

In the globalization theorem (9.3.1), properness can be weakened to completeness; see our book [**alexander-kapovitch-petrunin-2025**] and the references therein.

The condition on polyhedral CAT(κ) spaces given in Theorem 10.4.1 might look easy to use, but in fact, it is hard to check even in very simple cases. For example the description of those coverings of $\mathbb{S}^3$ branching at three great circles which are CAT(1) requires quite a bit of work; see [**charney-davis-1993**] — try to guess the answer before reading.

Another example is the space $\mathcal{B}_4$ that is the universal cover of $\mathbb{C}^4$ infinitely branching in six complex planes $z_i=z_j$ with the induced length metric. So far it is not known if $\mathcal{B}_4$ is CAT(0) [**panov-petrunin-2016**].

Understanding this space could be helpful for studying the braid group on 4 strings. This circle of questions is closely related to the generalization of the flag condition (10.5.5) to spherical simplices with few acute dihedral angles.

The construction used in the proof of Proposition 11.2.1 admits a number of interesting modifications, several of which are discussed in the survey [**davis-2001**] by Michael Davis.

A similar argument was used by Michael Davis, Tadeusz Januszkiewicz, and Jean-François Lafont in [**davis-januszkiewicz-lafont**]. They constructed a closed smooth four-dimensional manifold M with universal cover $\tilde{M}$ diffeomorphic to $\mathbb{R}^4$, such that M admits a polyhedral metric which is locally CAT(0), but does not admit a Riemannian metric with nonpositive sectional curvature. Another example of that type was constructed by Stephan Stadler; see [**stadler**]. There are no lower dimensional examples of this type — the two-dimensional case follows from the classification of surfaces, and the three-dimensional case follows from the geometrization conjecture.

It is noteworthy that any complete, simply connected Riemannian manifold with nonpositive curvature is homeomorphic to the Euclidean space of the same dimension. In fact, by the globalization theorem (9.3.1), the exponential map at a point of such a manifold is a homeomorphism. In particular, there is no Riemannian analog of Proposition 11.2.4.

Recall that a triangulation of an m-dimensional manifold defines a piecewise linear structure if the link of every simplex Δ is homeomorphic to the sphere of dimension $m-1-\dim\Delta$. According to Stone's theorem, see [**stone**, **davis-januszkiewicz**], the triangulation of $\mathcal{P}$ in Proposition 11.2.4 cannot be made piecewise linear — despite the fact that $\mathcal{P}$ is a manifold, its triangulation does not induce a piecewise linear structure.

The flag condition also leads to the so-called *hyperbolization* procedure, a flexible tool for constructing aspherical spaces; a good survey on the subject is given by Ruth Charney and Michael Davis in [**charney-davis-1995**].

The CAT(0) property of a cube complex admits interesting (and useful) geometric descriptions if one exchanged the ℓ^2-metric to a natural ℓ^1 or ℓ^∞ on each cube.

11.3.1. Theorem. *The following three conditions are equivalent.*

(a) A cube complex Q equiped with ℓ^2-metric is CAT(0)*.*

(b) A cube complex Q equiped with ℓ^∞-metric is injective.

(c) A cube complex Q equiped with ℓ^1-metric is median*. The later means that for any three points x, y, z there is a* unique *point m*

(it is called median *of x, y, and z) that lies on* some *geodesics* $[xy]$, $[xz]$ *and* $[yz]$.

A very readable paper on the subject was written by Brian Bowditch [**bowditch-2020**]; two easy parts of the theorem are included in the following exercise.

11.3.2. Exercise. *Prove the implication (b)⇒(a) and/or (c)⇒(a) in the theorem.*

All the topics discussed in this lecture link Alexandrov geometry with the fundamental group. The theory of *hyperbolic groups*, a branch of *geometric group theory*, introduced by Mikhael Gromov [**gromov-1987**], could be considered as a further step in this direction.

Lecture 12

Subsets

This lecture is nearly a copy of [**alexander-kapovitch-petrunin-2019**]; here we give a partial answer to the following question:

Which subsets of Euclidean space, equipped with their induced length-metrics, are CAT(0)*?*

12.1 Motivating examples

Consider three subgraphs of different quadric surfaces:

$$\begin{aligned}
A &= \left\{ (x,y,z)\in\mathbb{E}^3 \;:\; z\leqslant x^2+y^2 \right\},\\
B &= \left\{ (x,y,z)\in\mathbb{E}^3 \;:\; z\leqslant -x^2-y^2 \right\},\\
C &= \left\{ (x,y,z)\in\mathbb{E}^3 \;:\; z\leqslant x^2-y^2 \right\}.
\end{aligned}$$

12.1.1. Question. *Which of the sets A, B and C, if equipped with the induced length metric, are* CAT(0) *and why?*

The answers are given below, but it is instructive to think about these questions before reading further.

A. No, A is not CAT(0).

The boundary ∂A is the paraboloid described by $z=x^2+y^2$; in particular it bounds an open convex set in $\mathbb{E}^3$ whose complement is A. The closest point projection of $A\to\partial A$ is short (Exercise 7.7.6). It follows that ∂A is a convex set in A equipped with its induced length metric.

Therefore if A is CAT(0), then so is ∂A. The latter is not true: ∂A is a smooth convex surface, and has strictly positive curvature by the Gauss formula.

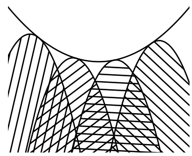
B. Yes, B is CAT(0).

Evidently B is a convex closed set in $\mathbb{E}^3$. Therefore the length metric on B coincides with the Euclidean metric and CAT(0) comparison holds.

C. Yes, C is CAT(0), but the proof is not as easy as before. We give a sketch here; a complete proof of a more general statement is given in Section 12.3.

Set $f_t(x,y) = x^2 - y^2 - 2{\cdot}(x-t)^2$. Consider the one-parameter family of sets

$$V_t = \left\{ (x,y,z) \in \mathbb{E}^3 \;:\; z \leqslant f_t(x,y) \right\}.$$



Each set V_t is a solid paraboloid tangent to ∂C along the parabola $y \mapsto (t, y, t^2 - y^2)$. The set V_t is closed and convex for any t, and

$$C = \bigcup_t V_t.$$

Further note that the function $t \mapsto f_t(x,y)$ is concave for any fixed x, y. Therefore

❶ *if $a < b < c$, then $V_b \supset V_a \cap V_c$.*

Consider the finite union

$$C' = V_{t_1} \cup \dots \cup V_{t_n}.$$

The inclusion ❶ makes it possible to apply Reshetnyak gluing theorem 8.2.1 recursively and show that C' is CAT(0). By approximation, the CAT(0) comparison holds for any 4 points in C; hence C is CAT(0).

Remark. The set C is not convex, but it is *two-convex* as defined in the next section. As you will see, two-convexity is closely related to the inheritance of an upper curvature bound by a subset.

12.2 Two-convexity

12.2.1. Definition. *We say that a subset $K \subset \mathbb{E}^m$ is* two-convex *if the following condition holds for any plane $W \subset \mathbb{E}^m$: If γ is a simple closed curve in $W \cap K$ that is null-homotopic in K, then it is null-homotopic in $W \cap K$, and in particular the disc in W bounded by γ lies in K.*

Note that two-convex sets do not have to be connected or simply connected. The following two propositions follow immediately from the definition.

12.2.2. Proposition. *Any subset in $\mathbb{E}^2$ is two-convex.*

12.2.3. Proposition. *The intersection of an arbitrary collection of two-convex sets in $\mathbb{E}^m$ is two-convex.*

12.2.4. Proposition. *Show that the interior of any two-convex set in $\mathbb{E}^m$ is a two-convex set.*

Proof. Fix a two-convex set $K \subset \mathbb{E}^m$ and a 2-plane W; denote by $\operatorname{Int} K$ the interior of K. Let γ be a closed simple curve in $W \cap \operatorname{Int} K$ that is contractible in the interior of K.

Since K is two-convex, the plane disc D bounded by γ lies in K. The same holds for the translations of D by small vectors. Therefore D lies in $\operatorname{Int} K$; that is, $\operatorname{Int} K$ is two-convex. □

12.2.5. Definition. *Given a subset $K \subset \mathbb{E}^m$, define its two-convex hull (briefly, $\operatorname{Conv}_2 K$) as the intersection of all two-convex subsets containing K.*

Note that by Proposition 12.2.3, the two-convex hull of any set is two-convex. Further, by 12.2.4, the two-convex hull of an open set is open.

The next proposition describes closed two-convex sets with smooth boundary.

12.2.6. Proposition. *Let $K \subset \mathbb{E}^m$ be a closed subset.*

Assume that the boundary of K is a smooth hypersurface S. Consider the unit normal vector field ν on S that points outside of K. Denote by $k_1 \leqslant \ldots \leqslant k_{m-1}$ the principal curvature functions of S with respect to ν (note that if K is convex, then $k_1 \geqslant 0$).

Then K is two-convex if and only if $k_2(p) \geqslant 0$ for any point $p \in S$. Moreover, if $k_2(p) < 0$ at some point p, then Definition 12.2.1 fails for some curve γ forming a triangle in an arbitrary small neighborhood of p.

The proof is taken from [**gromov-1991**], but we added some details.

Proof; "only if" part. If $k_2(p) < 0$ for some $p \in S$, consider the plane W containing p and spanned by the first two principal directions at p. Choose a small triangle in W which surrounds p and move it slightly in the direction of $\nu(p)$. We get a triangle $[xyz]$ which is null-homotopic in K, but the solid triangle $\Delta = \operatorname{Conv}\{x, y, x\}$ bounded by $[xyz]$ does not lie in K completely. Therefore K is not two-convex. (See figure in the "only if" part of the smooth two-convexity theorem (12.3.1).)

"If" part. Recall that a smooth function $f\colon \mathbb{E}^m \to \mathbb{R}$ is called *strongly convex* if its Hessian is positive definite at each point.

Suppose $f\colon \mathbb{E}^m \to \mathbb{R}$ is a smooth strongly convex function such that the restriction $f|_S$ is a Morse function. Note that a generic smooth strongly convex function $f\colon \mathbb{E}^m \to \mathbb{R}$ has this property.

For a critical point p of $f|_S$, the outer normal vector $\nu(p)$ is parallel to the gradient $\nabla_p f$; we say that p is a *positive critical point* if $\nu(p)$ and $\nabla_p f$ point in the same direction, and *negative* otherwise. If f is generic, then we can assume that the sign is defined for all critical points; that is, $\nabla_p f \neq 0$ for any critical point p of $f|_S$.

Since $k_2 \geqslant 0$ and the function f is strongly convex, the negative critical points of $f|_S$ have index at most 1.

Given a real value s, set

$$K_s = \{\, x \in K \;:\; f(x) < s \,\}.$$

Assume $\varphi_0\colon \mathbb{D} \to K$ is a continuous map of the disc $\mathbb{D}$ such that $\varphi_0(\partial\mathbb{D}) \subset K_s$.

Note that by the Morse lemma, there is a homotopy $\varphi_t\colon \mathbb{D} \to K$ rel $\partial\mathbb{D}$ such that $\varphi_1(\mathbb{D}) \subset K_s$.

Indeed, we can construct a homotopy $\varphi_t\colon \mathbb{D} \to K$ that decreases the maximum of $f \circ \varphi$ on $\mathbb{D}$ until the maximum occurs at a critical point p of $f|_S$. This point cannot be negative, otherwise its index would be at least 2. If this critical point is positive, then it is easy to decrease the maximum a little by pushing the disc from S into K in the direction of $-\nabla f_p$.

Consider a closed curve $\gamma\colon \mathbb{S}^1 \to K$ that is null-homotopic in K. Note that the distance function

$$f_0(x) = |\operatorname{Conv}\gamma - x|_{\mathbb{E}^m}$$

is convex. Therefore f_0 can be approximated by smooth strongly convex functions f in general position. From above, there is a disc in K with boundary γ that lies arbitrarily close to $\operatorname{Conv}\gamma$. Since K is closed, the statement follows. $\square$

Note that the "if" part proves a somewhat stronger statement. Namely, any plane curve γ (not necessary simple) which is contractible in K is also contractible in the intersection of K with the plane of γ. The latter condition does not hold for the complement of two planes in $\mathbb{E}^4$, which is two-convex by Proposition 12.2.3; see also Exercise 12.5.3

below. The following proposition shows that there are no such examples in $\mathbb{E}^3$.

12.2.7. Proposition. *Let $\Omega \subset \mathbb{E}^3$ be an open two-convex subset. Then for any plane $W \subset \mathbb{E}^3$, any closed curve in $W \cap \Omega$ that is null-homotopic in Ω is also null-homotopic in $W \cap \Omega$.*

This statement is intuitively obvious, but the proof is not trivial; it use the following classical result. An alternative definition of two-convexity using homology instead of homotopy is mentioned in the last section. For this definition the proof is simpler.

12.2.8. Loop theorem. *Let M be a three-dimensional manifold with nonempty boundary ∂M. Assume $f\colon (\mathbb{D}, \partial\mathbb{D}) \to (M, \partial M)$ is a continuous map from the disc $\mathbb{D}$ such that the boundary curve $f|_{\partial\mathbb{D}}$ is not null-homotopic in ∂M. Then there is an embedding $h\colon (\mathbb{D}, \partial\mathbb{D}) \to (M, \partial M)$ with the same property.*

The theorem is due to Christos Papakyriakopoulos; a proof can be found in [**hatcher**].

Proof of 12.2.7. Fix a closed plane curve γ in $W \cap \Omega$ that is null-homotopic in Ω. Suppose γ is not contractible in $W \cap \Omega$.

Let $\varphi\colon \mathbb{D} \to \Omega$ be a map of the disc with the boundary curve γ.

Since Ω is open we can first change φ slightly so that $\varphi(x) \notin W$ for $1 - \varepsilon < |x| < 1$ for some small $\varepsilon > 0$. By further changing φ slightly we can assume that it is transversal to W on $\operatorname{Int}\mathbb{D}$ and agrees with the previous map near $\partial\mathbb{D}$.

This means that $\varphi^{-1}(W) \cap \operatorname{Int}\mathbb{D}$ consists of finitely many simple closed curves which cut $\mathbb{D}$ into several components. Consider one of the "innermost" components c'; that is, c' is a boundary curve of a disc $\mathbb{D}' \subset \mathbb{D}$, $\varphi(c')$ is a closed curve in W and $\varphi(\mathbb{D}')$ completely lies in one of the two half-spaces with boundary W. Denote this half-space by H.

If $\varphi(c')$ is not contractible in $W \cap \Omega$, then applying the loop theorem to $M^3 = H \cap \Omega$ we conclude that there exists a *simple* closed curve $\gamma' \subset \Omega \cap W$ which is not contractible in $\Omega \cap W$ but is contractible in $\Omega \cap H$. This contradicts two-convexity of Ω.

Hence $\varphi(c')$ is contractible in $W \cap \Omega$. Therefore φ can be changed in a small neighborhood U of $\mathbb{D}'$ so that the new map $\hat\varphi$ maps U to one side of W. In particular, the set $\hat\varphi^{-1}(W)$ consists of the same curves as $\varphi^{-1}(W)$ with the exception of c'.

Repeating this process several times we reduce the problem to the case where $\varphi^{-1}(W) \cap \operatorname{Int}\mathbb{D} = \varnothing$. This means that $\varphi(\mathbb{D})$ lies entirely in one of the half-spaces bounded by W.

Again applying the loop theorem, we obtain a simple closed curve in $W \cap \Omega$ which is not contractible in $W \cap \Omega$ but is contractible in Ω. This again contradicts two-convexity of Ω. Hence γ is contractible in $W \cap \Omega$ as claimed. $\square$

12.3 Sets with smooth boundary

In this section we characterize the subsets with smooth boundary in $\mathbb{E}^m$ that form CAT(0) spaces.

12.3.1. Smooth two-convexity theorem. *Let K be a closed, simply connected subset in $\mathbb{E}^m$ equipped with the induced length metric. Assume K is bounded by a smooth hypersurface. Then K is* CAT(0) *if and only if K is two-convex.*

This theorem is a baby case of a result of Stephanie Alexander, David Berg, and Richard Bishop [**alexander-berg-bishop**], which is briefly discussed at the end of the lecture. The proof below is based on the argument in Section 12.1.

Proof. Denote by S and by Ω the boundary and the interior of K respectively. Since K is connected and S is smooth, Ω is also connected.

Denote by $k_1(p) \leqslant \ldots \leqslant k_{m-1}(p)$ the principal curvatures of S at $p \in S$ with respect to the normal vector $\nu(p)$ pointing out of K. By Proposition 12.2.6, K is two-convex if and only if $k_2(p) \geqslant 0$ for any $p \in S$.

"Only if" part. Assume K is not two-convex. Then by Proposition 12.2.6, there is a triangle $[xyz]$ in K which is null-homotopic in K, but the solid triangle $\Delta = \operatorname{Conv}\{x, y, z\}$ does not lie in K completely. Evidently the triangle $[xyz]$ is not thin in K. Hence K is not CAT(0).

"If" part. Since K is simply connected, by the globalization theorem (9.3.1) it suffices to show that any point $p \in K$ admits a CAT(0) neighborhood.

If $p \in \operatorname{Int} K$, then it admits a neighborhood isometric to a CAT(0) subset of $\mathbb{E}^m$. Fix $p \in S$. Assume that $k_2(p) > 0$. Fix a sufficiently small $\varepsilon > 0$ and set $K' = K \cap \overline{\mathrm{B}}[p, \varepsilon]$. Let us show that

❶ *K' is* CAT(0).

Consider the coordinate system with the origin at p and the principal directions and $\nu(p)$ as the coordinate directions. For small $\varepsilon > 0$, the set K' can be described as a subgraph

$$K' = \left\{ (x_1, \ldots, x_m) \in \overline{\mathrm{B}}[p, \varepsilon] \;:\; x_m \leqslant f(x_1, \ldots, x_{m-1} \right\}.$$

Fix $s \in [-\varepsilon, \varepsilon]$. Since ε is small and $k_2(p) > 0$, the restriction $f|_{x_1=s}$ is concave in the $(m-2)$-dimensional cube defined by the inequalities $|x_i| < 2{\cdot}\varepsilon$ for $2 \leqslant i \leqslant m-1$.

Fix a negative real value $\lambda < k_1(p)$. Given $s \in (-\varepsilon, \varepsilon)$, consider the set

$$V_s = \left\{ (x_1, \dots, x_m) \in K' \;:\; x_m \leqslant f(x_1, \dots, x_{m-1}) + \lambda{\cdot}(x_1 - s)^2 \right\}.$$

Note that the function

$$(x_1, \dots, x_{m-1}) \mapsto f(x_1, \dots, x_{m-1}) + \lambda{\cdot}(x_1 - s)^2$$

is concave near the origin. Since ε is small, we can assume that the V_s are convex subsets of $\mathbb{E}^m$.

Further note that

$$K' = \bigcup_{s \in [-\varepsilon, \varepsilon]} V_s.$$

Also, the same argument as in 12.1.1 shows that

❷ *If $a < b < c$, then $V_b \supset V_a \cap V_c$.*

Given an array of values $s^1 < \dots < s^k$ in $[-\varepsilon, \varepsilon]$, set $V^i = V_{s^i}$ and consider the unions

$$W^i = V^1 \cup \dots \cup V^i$$

equipped with the induced length metric.

Note that the array (s^n) can be chosen in such a way that W^k is arbitrarily close to K' in the sense of Hausdorff.

By Proposition 7.3.4, in order to prove ❶, it is sufficient to show the following:

❸ *All W^i are* CAT(0).

This claim is proved by induction. Base: $W^1 = V^1$ is CAT(0) as a convex subset in $\mathbb{E}^m$.

Step: Assume that W^i is CAT(0). According to ❷,

$$V^{i+1} \cap W^i = V^{i+1} \cap V^i.$$

Moreover, this is a convex set in $\mathbb{E}^m$ and therefore it is a convex set in W^i and in V^{i+1}. By the Reshetnyak gluing theorem, W^{i+1} is CAT(0). Hence the claim follows. △

Note that we have proved the following:

❹ *K' is* CAT(0) *if K is* strongly two-convex*, that is, $k_2(p) > 0$ at any point $p \in S$.*

It remains to show that p admits a CAT(0) neighborhood in the case $k_2(p) = 0$.

Choose a coordinate system $(x_1, \dots, x_m)$ as above, so that the $(x_1, \dots, x_{m-1})$-coordinate hyperplane is the tangent subspace to S at p.

Fix $\varepsilon > 0$ so that a neighborhood of p in S is the graph

$$x_m = f(x_1, \dots, x_{m-1})$$

of a function f defined on the open ball B of radius ε centered at the origin in the $(x_1, \dots, x_{m-1})$-hyperplane. Fix a smooth positive strongly convex function $\varphi\colon B \to \mathbb{R}_+$ such that $\varphi(x) \to \infty$ as x approaches the boundary of B. Note that for $\delta > 0$, the subgraph K_δ defined by the inequality

$$x_m \leqslant f(x_1, \dots, x_{m-1}) - \delta{\cdot}\varphi(x_1, \dots, x_{m-1})$$

is strongly two-convex. By ❹, K_δ is CAT(0).

Finally as $\delta \to 0$, the closed ε-neighborhoods of p in K_δ converge to the closed ε-neighborhood of p in K. By Proposition 7.3.4, the ε-neighborhood of p is CAT(0). □

12.4 Open plane sets

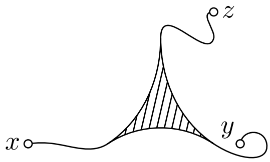
In this section we consider inheritance of upper curvature bounds by subsets of the Euclidean plane.

12.4.1. Theorem. *Let Ω be an open simply connected subset of $\mathbb{E}^2$. Equip Ω with its induced length metric and denote its completion by K. Then K is* CAT(0).

The assumption that the set Ω is open is not critical; instead one can assume that the induced length metric takes finite values at all points of Ω. We sketch the proof given by Richard Bishop in [**bishop**] and leave the details to be finished as an exercise. A generalization of this result is proved by Alexander Lytchak and Stefan Wenger [**lytchak-wenger**]; this paper also contains a far-reaching application.

Sketch of proof. It is sufficient to show that any triangle in K is thin, as defined in 7.7.1.

Note that K admits a length-preserving map to $\mathbb{E}^2$ that extends the embedding $\Omega \hookrightarrow \mathbb{E}^2$. Therefore each triangle $[xyz]$ in K can be mapped to the plane in a length-preserving way. Since Ω is simply connected, any open region, say Δ, that is surrounded by the image of $[xyz]$ lies completely in Ω.



Note that in each triangle $[xyz]$ in K, the sides $[xy]$, $[yz]$ and $[zx]$ intersect each other along a geodesic starting at a common vertex, possibly a one-point geodesic. In other words, every triangle in K looks like the one in the diagram.

Indeed, assuming the contrary, there will be a *lune* in K bounded by two minimizing geodesics with common ends but no other common points. The image of this lune in the plane must have concave sides, since otherwise one could shorten the sides by pushing them into the interior. Evidently, there is no plane lune with concave sides, a contradiction.

Note that it is sufficient to consider only simple triangles $[xyz]$, that is, triangles whose sides $[xy]$, $[yz]$ and $[zx]$ intersect each other only at the common vertices. If this is not the case, chopping the overlapping part of sides reduces to the injective case (this is formally stated in Exercise 12.4.2).

Again, the open region, say Δ, bounded by the image of $[xyz]$ has concave sides in the plane, since otherwise one could shorten the sides by pushing them into Ω. It remains to solve Exercise 12.4.3. $\square$

12.4.2. Exercise. *Assume that $[pq]$ is a common part of the two sides $[px]$ and $[py]$ of the triangle $[pxy]$. Consider the triangle $[qxy]$ whose sides are formed by arcs of the sides of $[pxy]$. Show that if $[qxy]$ is thin, then so is $[pxy]$.*

12.4.3. Exercise. *Assume S is a closed plane region whose boundary is a plane triangle T with concave sides. Equip S with the induced length metric. Show that the triangle T is thin in S.*

Here is a spherical analog of Theorem 12.4.1, which can be proved along the same lines. It will be used in the next section.

12.4.4. Proposition. *Let Θ be an open connected subset of the unit sphere $\mathbb{S}^2$ that does not contain a closed hemisphere. Equip Θ with the induced length metric. Let $\tilde\Theta$ be a metric cover of Θ such that any closed curve in $\tilde\Theta$ shorter than $2\cdot\pi$ is contractible.*

Show that the completion of $\tilde\Theta$ is CAT(1).

12.4.5. Exercise. *Prove the following partial case of the proposition:*

Let K be closed subset of the unit sphere $\mathbb{S}^2$ that does not contain a closed hemisphere. Suppose K is simply connected and bounded by a simple Lipschitz curve. Show that K with induced length metric is CAT(1).

12.5 Shefel's theorem

In this section we will formulate our version of a theorem of Samuel Shefel (12.5.2) and prove a couple of its corollaries.

It seems that Shefel was very intrigued by the survival of metric properties under affine transformation. To describe an instance of such phenomena, note that two-convexity survives under affine transformations of a Euclidean space. Therefore, as a consequence of the smooth two-convexity theorem (12.3.1), the following holds.

12.5.1. Corollary. *Let K be closed connected subset of Euclidean space equipped with the induced length metric. Assume K is* $\mathrm{CAT}(0)$ *and bounded by a smooth hypersurface. Then any affine transformation of K is also* $\mathrm{CAT}(0)$.

By Corollary 12.5.4, an analogous statement holds for sets bounded by Lipschitz surfaces in the three-dimensional Euclidean space. In higher dimensions this is no longer true.

12.5.2. Two-convexity theorem. *Let Ω be a connected open set in $\mathbb{E}^3$. Equip Ω with the induced length metric and denote by $\tilde{K}$ the completion of the universal metric cover of Ω. Then $\tilde{K}$ is* $\mathrm{CAT}(0)$ *if and only if Ω is two-convex.*

The proof of this statement will be given in the following three sections. First we prove its polyhedral analog, then we prove some properties of two-convex hulls in three-dimensional Euclidean space and only then do we prove the general statement.

The following exercise shows that the analogous statement does not hold in higher dimensions.

12.5.3. Exercise. *Let Π_1, Π_2 be two planes in $\mathbb{E}^4$ intersecting at a single point. Let $\tilde{K}$ be the completion of the universal metric cover of $\mathbb{E}^4 \backslash (\Pi_1 \cup \Pi_2)$.*

Show that $\tilde{K}$ is $\mathrm{CAT}(0)$ *if and only if $\Pi_1 \perp \Pi_2$.*

Before coming to the proof of the two-convexity theorem, let us formulate a few corollaries. The following corollary is a generalization of the smooth two-convexity theorem (12.3.1) for three-dimensional Euclidean space.

12.5.4. Corollary. *Let K be a closed subset in $\mathbb{E}^3$ bounded by a Lipschitz hypersurface. Then K with the induced length metric is* $\mathrm{CAT}(0)$ *if and only if the interior of K is two-convex and simply connected.*

Proof. Set $\Omega = \operatorname{Int} K$. Since K is simply connected and bounded by a surface, Ω is also simply connected.

Apply the two-convexity theorem to Ω. Note that the completion of Ω equipped with the induced length metric is isometric to K with the induced length metric. Hence the result. □

Note that the Lipschitz condition is used just once to show that the completion of Ω is isometric to K with the induced length metric. This property holds for a wider class of hypersurfaces; for instance Alexander horned ball might have CAT(0) induced length metric.

Let U be an open set in $\mathbb{R}^2$. A continuous function $f\colon U \to \mathbb{R}$ is called *saddle* if for any linear function $\ell\colon \mathbb{R}^2 \to \mathbb{R}$, the difference $f - \ell$ does not have local maxima or local minima in U. Equivalently, the open subgraph and epigraph of f

$$\left\{\,(x,y,z)\in\mathbb{E}^3\;:\;z<f(x,y),\;(x,y)\in U\,\right\},$$
$$\left\{\,(x,y,z)\in\mathbb{E}^3\;:\;z>f(x,y),\;(x,y)\in U\,\right\}$$

are two-convex.

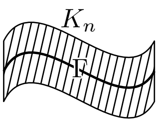
12.5.5. Theorem. *Let $f\colon \mathbb{D}\to\mathbb{R}$ be a Lipschitz function which is saddle in the interior of the closed unit disc $\mathbb{D}$. Then the graph*

$$\Gamma=\left\{\,(x,y,z)\in\mathbb{E}^3\;:\;z=f(x,y)\,\right\},$$

equipped with induced length metric is CAT(0).

Proof. Since the function f is Lipschitz, its graph Γ with the induced length metric is bi-Lipschitz equivalent to $\mathbb{D}$ with the Euclidean metric.

Consider the sequence of sets



$$K_n=\left\{\,(x,y,z)\in\mathbb{E}^3\;:\;z\lesseqgtr f(x,y)\pm\tfrac1n,\;(x,y)\in\mathbb{D}\,\right\}.$$

Note that each K_n is closed and simply connected. By definition K is also two-convex. Moreover the boundary of K_n is a Lipschitz surface.

Equip K_n with the induced length metric. By Corollary 12.5.4, K_n is CAT(0). It remains to note that $K_n \to \Gamma$ in the sense of Gromov–Hausdorff, and apply Proposition 7.3.4. □

12.6 Polyhedral case

Now we are back to the proof of the two-convexity theorem (12.5.2).

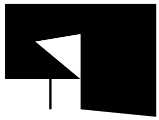
Recall that a subset P of $\mathbb{E}^m$ is called a *polytope* if it can be presented as a union of a finite number of simplices. Similarly, a *spherical polytope* is a union of a finite number of simplices in $\mathbb{S}^m$.

Note that any polytope admits a finite triangulation. Therefore any polytope equipped with the induced length metric forms a Euclidean polyhedral space as defined in 10.3.1.

12.6.1. Lemma. *The two-convexity theorem (12.5.2) holds if the set Ω is the interior of a polytope.*

The statement might look obvious, but there is a hidden obstacle in the proof that is related to the following. Let P be a polytope and Ω its interior, both considered with the induced length metrics. Typically, the completion K of Ω is isometric to P — in this case the lemma follows easily from 10.4.1.

However in general we only have a locally distance-preserving map $K \to P$; it does not have to be onto and it may not be injective. An example can be guessed from the picture. Nevertheless, is easy to see that K is always a polyhedral space.



The proof uses the following two exercises.

12.6.2. Exercise. *Show that any closed path of length $< 2{\cdot}\pi$ in the units sphere $\mathbb{S}^2$ lies in an open hemisphere.*

12.6.3. Exercise. *Assume Ω is an open subset in $\mathbb{E}^3$ that is not two-convex. Show that there is a plane W such that the complement $W\backslash\Omega$ contains an isolated point and a small circle around this point in W is contractible in Ω.*

Proof of 12.6.1. The "only if" part can be proved in the same way as in the smooth two-convexity theorem (12.3.1) with additional use of Exercise 12.6.3.

"If" part. Assume that Ω is two-convex. Denote by $\tilde\Omega$ the universal metric cover of Ω. Let $\tilde K$ and K be the corresponding completions of $\tilde\Omega$ and Ω.

The main step is to show that $\tilde K$ is CAT(0).

Note that K is a polyhedral space and the covering $\tilde\Omega \to \Omega$ extends to a covering map $\tilde K \to K$ which might be branching at some vertices.[1]

Fix a point $\tilde p \in \tilde K\backslash\tilde\Omega$; denote by p the image of $\tilde p$ in K. Note that $\tilde K$ is a ramified cover of K and hence is locally contractible. Thus, any loop in $\tilde K$ is homotopic to a loop in $\tilde\Omega$ which is simply connected. Therefore $\tilde K$ is simply connected too.

[1]For example, if $K = \{\,(x,y,z)\in\mathbb{E}^3 : |z| \leqslant |x|+|y| \leqslant 1\,\}$ and p is the origin, then Σ_p, the space of directions at p, is not simply connected and $\tilde K \to K$ branches at p.

Thus, by the globalization theorem (9.3.1), it is sufficient to show that

❶ *a small neighborhood of $\tilde p$ in $\tilde K$ is* CAT(0).

Recall that $\Sigma_{\tilde p} = \Sigma_{\tilde p}\tilde K$ denotes the space of directions at $\tilde p$. Note that a small neighborhood of $\tilde p$ in $\tilde K$ is isometric to an open set in the cone over $\Sigma_{\tilde p}\tilde K$. By Exercise 7.4.3, ❶ follows once we can show that

❷ $\Sigma_{\tilde p}$ *is* CAT(1).

By rescaling, we can assume that every face of K which does not contain p lies at distance at least 2 from p. Denote by $\mathbb{S}^2$ the unit sphere centered at p, and set $\Theta = \mathbb{S}^2 \cap \Omega$. Note that $\Sigma_p K$ is isometric to the completion of Θ and $\Sigma_{\tilde p}\tilde K$ is the completion of the regular metric covering $\tilde\Theta$ of Θ induced by the universal metric cover $\tilde\Omega \to \Omega$.

By 12.4.4, it remains to show the following:

❸ *Any closed curve in $\tilde\Theta$ shorter than $2{\cdot}\pi$ is contractible.*

Fix a closed curve $\tilde\gamma$ of length $< 2{\cdot}\pi$ in $\tilde\Theta$. Its projection γ in $\Theta \subset \mathbb{S}^2$ has the same length. Therefore, by Exercise 12.6.2, γ lies in an open hemisphere. Then for a plane Π passing close to p, the central projection γ' of γ to Π is defined and lies in Ω. By construction of $\tilde\Theta$, the curve γ and therefore γ' are contractible in Ω. From two-convexity of Ω and Proposition 12.2.7, the curve γ' is contractible in $\Pi \cap \Omega$.

It follows that γ is contractible in Θ and therefore $\tilde\gamma$ is contractible in $\tilde\Theta$. □

12.7 Two-convex hulls

The following proposition describes a construction which produces the two-convex hull $\operatorname{Conv}_2 \Omega$ of an open set $\Omega \subset \mathbb{E}^3$. This construction is very close to the one given by Samuel Shefel [**shefel-1964**].

12.7.1. Proposition. *Let $\Pi_1, \Pi_2 \dots$ be an everywhere dense sequence of planes in $\mathbb{E}^3$. Given an open set Ω, consider the recursively defined sequence of open sets $\Omega = \Omega_0 \subset \Omega_1 \subset \dots$ such that Ω_n is the union of Ω_{n-1} and all the bounded components of $\mathbb{E}^3 \backslash (\Pi_n \cup \Omega_{n-1})$. Then*

$$\operatorname{Conv}_2 \Omega = \bigcup_n \Omega_n.$$

Proof. Set

$$❶ \qquad \Omega' = \bigcup_n \Omega_n.$$

Note that Ω' is a union of open sets, in particular Ω' is open.

Let us show that

❷ $$\operatorname{Conv}_2 \Omega \supset \Omega'.$$

Suppose we already know that $\operatorname{Conv}_2 \Omega \supset \Omega_{n-1}$. Fix a bounded component $\mathfrak{C}$ of $\mathbb{E}^3 \backslash (\Pi_n \cup \Omega_{n-1})$. It is sufficient to show that $\mathfrak{C} \subset \operatorname{Conv}_2 \Omega$.

By 12.2.4, $\operatorname{Conv}_2 \Omega$ is open. Therefore, if $\mathfrak{C} \not\subset \operatorname{Conv}_2 \Omega$, then there is a point $p \in \mathfrak{C} \backslash \operatorname{Conv}_2 \Omega$ lying at maximal distance from Π_n. Denote by W_p the plane containing p which is parallel to Π_n.

Note that p lies in a bounded component of $W_p \backslash \operatorname{Conv}_2 \Omega$. In particular p can be surrounded by a simple closed curve γ in $W_p \cap \operatorname{Conv}_2 \Omega$. Since p lies at maximal distance from Π_n, the curve γ is null-homotopic in $\operatorname{Conv}_2 \Omega$. Therefore $p \in \operatorname{Conv}_2 \Omega$, a contradiction.

By induction, $\operatorname{Conv}_2 \Omega \supset \Omega_n$ for each n. Therefore ❶ implies ❷.

It remains to show that Ω' is two-convex. Assume the contrary; that is, there is a plane Π and a simple closed curve $\gamma \colon \mathbb{S}^1 \to \Pi \cap \Omega'$ which is null-homotopic in Ω', but not null-homotopic in $\Pi \cap \Omega'$.

By approximation we can assume that $\Pi = \Pi_n$ for a large n, and that γ lies in Ω_{n-1}. By the same argument as in the proof of Proposition 12.2.7 using the loop theorem, we can assume that there is an embedding $\varphi \colon \mathbb{D} \to \Omega'$ such that $\varphi|_{\partial\mathbb{D}} = \gamma$ and $\varphi(D)$ lies entirely in one of the half-spaces bounded by Π. By the n-step of the construction, the entire bounded domain U bounded by Π_n and $\varphi(D)$ is contained in Ω' and hence γ is contractible in $\Pi \cap \Omega'$, a contradiction. $\square$

12.7.2. Key lemma. *The two-convex hull of the interior of a polytope in $\mathbb{E}^3$ is also the interior of a polytope.*

Proof. Fix a polytope P in $\mathbb{E}^3$. Set $\Omega = \operatorname{Int} P$. We may assume that Ω is dense in P (if not, redefine P as the closure of Ω). Denote by $F_1, \dots, F_m$ the facets of P. By subdividing F_i if necessary, we may assume that all F_i are convex polygons.

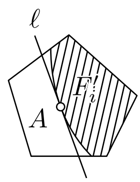
Set $\Omega' = \operatorname{Conv}_2 \Omega$ and let P' be the closure of Ω'. Further, for each i, set $F_i' = F_i \backslash \Omega'$. In other words, F_i' is the subset of the facet F_i which remains on the boundary of P'.

From the construction of the two-convex hull (12.7.1) we have:

❸ *F_i' is a convex subset of F_i.*

Further, since Ω' is two-convex we obtain the following:

❹ *Each connected component of the complement $F_i \backslash F_i'$ is convex.*



Indeed, assume a connected component A of $F_i \backslash F_i'$ fails to be convex. Then there is a supporting line ℓ to F_i' touching F_i' at a single point in the interior of F_i. Then one could rotate the plane of F_i slightly around ℓ and move it parallelly to cut a "cap" from the complement of Ω. The latter means that Ω is not two-convex, a contradiction. △

From ❸ and ❹, we conclude

❺ *F_i' is a convex polygon for each i.*

Consider the complement $\mathbb{E}^3 \backslash \Omega$ equipped with the length metric. By construction of the two-convex hull (12.7.1), the complement $L = \mathbb{E}^3 \backslash (\Omega' \cup P)$ is locally convex; that is, any point of L admits a convex neighborhood.

Summarizing: (1) Ω' is a two-convex open set, (2) the boundary $\partial\Omega'$ contains a finite number of polygons F_i' and the remaining part S of the boundary is locally concave. It remains to show that (1) and (2) imply that S and therefore $\partial\Omega'$ are piecewise linear.

12.7.3. Exercise. *Prove the last statement.* □

12.8 Proof of Shefel's theorem

Proof of 12.5.2. The "only if" part can be proved in the same way as in the smooth two-convexity theorem (12.3.1) with the additional use of Exercise 12.6.3.

"If"-part. Suppose Ω is two-convex. We need to show that $\tilde{K}$ is CAT(0).

Fix a quadruple of points $x^1, x^2, x^3, x^4 \in \tilde{\Omega}$. Let us show that CAT(0) comparison holds for this quadruple.

Fix $\varepsilon > 0$. Choose six broken lines in $\tilde{\Omega}$ connecting all pairs of points x^1, x^2, x^3, x^4, where the length of each broken line is at most ε bigger than the distance between its ends in the length metric on $\tilde{\Omega}$. Denote by X the union of these broken lines. Choose a polytope P in Ω such that its interior $\operatorname{Int} P$ contains the projections of all six broken lines and discs which contract all the loops created by them (it is sufficient to take 3 discs).

Denote by Ω' the two-convex hull of the interior of P. According to the key lemma (12.7.2), Ω' is the interior of a polytope.

Equip Ω' with the induced length metric. Consider the universal metric cover $\tilde{\Omega}'$ of Ω'. (The covering $\tilde{\Omega}' \to \Omega'$ might be nontrivial —

even if $\operatorname{Int} P$ is simply connected, its two-convex hull Ω' might not be simply connected.) Denote by $\tilde{K}'$ the completion of $\tilde{\Omega}'$.

By Lemma 12.6.1, $\tilde{K}'$ is CAT(0).

By construction of $\operatorname{Int} P$, the embedding $\operatorname{Int} P \hookrightarrow \Omega'$ admits a lift $\iota\colon X \hookrightarrow \tilde{K}'$. By construction, ι almost preserves the distances between the points x^1, x^2, x^3, x^4; namely

$$|\iota(x^i) - \iota(x^j)|_L \lesseqgtr |x^i - x^j|_{\operatorname{Int} P} \pm \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and CAT(0) comparison holds in $\tilde{K}'$, we get that CAT(0) comparison holds in Ω for x^1, x^2, x^3, x^4.

The statement follows since the quadruple $x^1, x^2, x^3, x^4 \in \tilde{\Omega}$ is arbitrary. □

12.8.1. Exercise. *Assume $K \subset \mathbb{E}^m$ is a closed set bounded by a Lipschitz hypersurface. Equip K with the induced length metric. Show that if K is* CAT(0)*, then K is two-convex.*

12.9 Remarks

Under the name $(n-2)$-*convex sets*, two-convex sets in $\mathbb{E}^n$ were introduced by Mikhael Gromov in [**gromov-1991**]. In addition to the inheritance of upper curvature bounds by two-convex sets discussed in this lecture, these sets appear as the maximal open sets with vanishing curvature in Riemannian manifolds with non-negative or non-positive sectional curvature; see [**buyalo**], and [**panov-petrunin**].

Two-convex sets could be defined using homology instead of homotopy, as in the formulation of the Leftschetz theorem in [**gromov-1991**]. Namely, we can say that K is two-convex if the following condition holds: if a one-dimensional cycle z has support in the intersection of K with a plane W and bounds in K, then it bounds in $K \cap W$.

The resulting definition is equivalent to the one used above. But unlike our definition it can be generalized to define k-convex sets in $\mathbb{E}^m$ for $k > 2$. With this homological definition one can also avoid the use of the loop theorem, whose proof is quite involved. Nevertheless, we chose the definition using homotopies since it is easier to visualize.

Both definitions work well for open sets; for general sets one should be able to give a similar definition using an appropriate homotopy/homology theory.

In [**alexander-berg-bishop**] the Stephanie Alexander, David Berg and Richard Bishop gave the exact upper bound on Alexandrov's curvature for the Riemannian manifolds with boundary. This theorem includes the smooth two-convexity theorem (12.3.1) as a partial case. Namely they show the following.

12.9.1. Theorem. *Let M be a Riemannian manifold with boundary ∂M. A direction tangent to the boundary will be called concave if there is a short geodesic in this direction which leaves the boundary and goes into the interior of M. A sectional direction (that is, a 2-plane) tangent to the boundary will be called* concave *if all the directions in it are concave.*

Denote by κ an upper bound of sectional curvatures of M and sectional curvatures of ∂M in the concave sectional directions. Then M is locally $\mathrm{CAT}(\kappa)$.

12.9.2. Corollary. *Let M be a Riemannian manifold with boundary ∂M. Assume that all the sectional curvatures of M and ∂M are bounded above by κ. Then M is locally* $\mathrm{CAT}(\kappa)$.

Theorem 12.5.5 is the main statement in Shefel's original paper [**shefel-1965**]. It is related to Alexandrov's theorem about ruled surfaces [**alexandrov-1957-ruled-s**

Let D be an embedded closed disc in $\mathbb{E}^3$. We say that D is *saddle* if each connected component which any plane cuts from D contains a point on the boundary ∂D. If D is locally described by a Lipschitz embedding, then this condition is equivalent to saying that D is two-convex.

12.9.3. Shefel's conjecture. *Any saddle surface in $\mathbb{E}^3$ equipped with the length-metric is locally* $\mathrm{CAT}(0)$.

The conjecture is open even for the surfaces described by a bi-Lipschitz embedding of a disc. From another result of Samuel Shefel [**shefel-1965**], it follows that a saddle surface satisfies the isoperimetric inequality $a \leqslant C\cdot\ell^2$ where a is the area of a disc bounded by a curve of length ℓ and $C = \frac{1}{3\cdot\pi}$. By a result of Alexander Lytchak and Stefan Wenger [**lytchak-wenger**], Shefel's conjecture is equivalent to the isoperimetric inequality with the optimal constant $C = \frac{1}{4\cdot\pi}$.

For more on the subject, see [**petrunin-stadler**] and the references therein.

Part III

Metrics on manifolds

Lecture 13

Besikovitch inequality

13.1 Riemannian spaces

Riemannian spaces are smooth manifolds with metric defined by a metric tensor. These are specially nice length metrics on manifolds. However most of the statements we are going to discuss have counterpart for general length metrics on manifolds.

Let M be a smooth manifold. A *metric tensor* on M is a choice of positive definite quadratic forms g_p on each tangent space T_pM that depends smoothly on the point p. That is, if we fix a local coordinates on M and write g in this coordinates, then each component of g is a smooth function.

A Riemannian manifold (M,g) is a smooth manifold M with a choice metric tensor g on it. The metric tensor g can be used to define length of curves, distances, and volume of regions in M. In particular any Riemannian manifold is a metric space which might be called *Riemannian space.*

Lengths and distances. If $\gamma\colon [a,b] \to M$ is a piecewise smooth curve, then

$$\operatorname{length}_g \gamma = \int\limits_a^b \sqrt{g(\gamma'(t),\gamma'(t))}\cdot dt.$$

Further we can define a metric on M as least lower bound to lengths of piecewise smooth curves connecting two given points; the described distance between points x and y will be denoted by $|x-y|_g$ or $\operatorname{dist}_x(y)_g$. The distance function from a point x will be denoted by $(\operatorname{dist}_x)_g$ or dist_x if the choice of g is evident.

The following claim follows from Myers–Steenrod theorem [**myers-steenrod**];

it says that the metric on the Reimannian manifold *remembers* everything about the Reimannian manifold.

13.1.1. Claim. *Let (M, g) be a Riemannian manifold. Then the metric $(x, y) \mapsto |x - y|_g$ defines a length metric. Moreover this metric completely determines the manifold M with its smooth structure and the metric tensor g.*

Volume. If a region R is covered by one chart $\iota\colon U \to M$, then its volume can be defined as an integral

$$\operatorname{vol} R := \int\limits_{\iota^{-1}(R)} \sqrt{\det g}.$$

In the general case we subdivide R into (a countable collection of) regions $R_1, R_2 \dots$ and define

$$\operatorname{vol} R := \operatorname{vol} R_1 + \operatorname{vol} R_2 + \dots$$

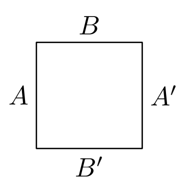
13.2 Besikovitch inequality

13.2.1. Theorem. *Let g be a metric tensor on a unit n-dimensional cube $\square^n$. Suppose that the g-distances between the opposite faces of $\square^n$ are at leat 1; that is, any piecewise smooth curve that connects opposite faces has g-length at least 1. Then* $\operatorname{vol}(\square^n, g) \geqslant 1$.

This theorem was proved by Abram Besicovitch [**besicovitch**].

Proof. We will consider the case $n = 2$; the other cases are proved the same way.

Denote by A, A', and B, B' the opposite faces of the square $\square$. Consider two function



$$f_A(x) := \min\{\operatorname{dist}_A(x)_g, 1\},$$
$$f_B(x) := \min\{\operatorname{dist}_B(x)_g, 1\}.$$

Define $f\colon \square \to \square$ as a map with coordinate funcions f_A and f_B; that is, $f(x) := (f_A(x), f_B(x))$.

Observe that f maps each face to itself. Indeed,

$$x \in A \quad\Longrightarrow\quad \operatorname{dist}_A(x)_g = 0 \quad\Longrightarrow\quad f_A(x) = 0 \quad\Longrightarrow\quad f(x) \in A.$$

Similarly if $x \in B$, then $f(x) \in B$. Further,

$$x \in A' \quad\Longrightarrow\quad \operatorname{dist}_A(x)_g \geqslant 1 \quad\Longrightarrow\quad f_A(x) = 1 \quad\Longrightarrow\quad f(x) \in A'.$$

Similarly if $x \in B'$, then $f(x) \in B'$.

Therefore

$$f_t(x) = t{\cdot}x + (1-t){\cdot}f(x)$$

defines a homotopy of maps of pair of spaces $(\square, \partial\square)$ from f to the identity map. It follows that degree of f is 1; that is, f sends the fundamental class of $(\square, \partial\square)$ to itself. In particular f is onto.

Suppose that Jacobian matrix $\mathrm{Jac}_p f$ of f is defined at $p \in \square$. Choose an orthonormal basis in T_p with respect to g and the standard basis in the target $\square$. Observe that the differentials $d_p f_A$ and $d_p f_B$ written in these basises are the rows of $\mathrm{Jac}_p f$. Evidently $|d_p f_A| \leqslant 1$ and $|d_p f_B| \leqslant 1$. Since the determinant of a matrix is the volume of the parallelepiped spanned on its rows, we get

$$|\det(\mathrm{Jac}_p f)| \leqslant |d_p f_A|{\cdot}|d_p f_B| \leqslant 1.$$

Since $f\colon \square \to \square$ is a Lipschitz onto map, the *area formula* implies that

$$\mathrm{vol}(\square, g) \geqslant \mathrm{vol}\,\square = 1.$$

$\square$

13.2.2. Theorem. *Let (M, g) be Riemannian manifold and its boundary admits a degree 1 map $\partial M \to \partial\square^n$. Suppose $d_1, \dots, d_n$ the distances between the inverse images of pairs of opposite faces of $\square^n$ in ∂M. Then*

$$\mathrm{vol}(M, g) \geqslant d_1 \cdots d_n.$$

Moreover, in the case of equality, $(\square^n, g)$ is isometric to the product $[0, d_1] \times \dots \times [0, d_n]$.

The first part of the stated generalization can be proved along the same lines as 1.2.1.

13.2.3. Exercise. *Prove the second part of 1.2.2.*

13.2.4. Exercise. *Suppose g is a metric tensor on a regular hexagon $\hexagon$ such that g-distances between the opposite sides are at least 1. Is there a positive lower bound on* $\mathrm{area}(\hexagon, g)$*?*

13.2.5. Exercise. *Let V be a compact set in $\mathbb{E}^d$ bounded by a hypersurface Σ. Suppose g is a Riemannian metric on V such that*

$$|p-q|_g \geqslant |p-q|_{\mathbb{E}^d}$$

for any two points $p, q \in \Sigma$. Show that

$$\mathrm{vol}(V, g) \geqslant \mathrm{vol}(V)_{\mathbb{E}^d}.$$

13.2.6. Exercise. *Suppose that sphere with Riemannian matric* $(\mathbb{S}^2, g)$ *admits an involution* ι *such that* $|x - \iota(x)|_g \geqslant 1$.

Show that $\mathrm{area}(\mathbb{S}^2, g) \geqslant \frac{1}{1000}$*; try to show that* $\mathrm{area}(\mathbb{S}^2, g) \geqslant \frac{1}{2}$ *or* $\mathrm{area}(\mathbb{S}^2, g) \geqslant 1$.

13.2.7. Advanced exercise. *Construct a metric tensor* g *on* $\mathbb{S}^3$ *such that (1)* $\mathrm{vol}(\mathbb{S}^3, g)$ *arbitrary small and (2) there is an involution* $\iota\colon \mathbb{S}^3 \to \mathbb{S}^3$ *such that* $|x - \iota(x)|_g \geqslant 1$ *for any* $x \in \mathbb{S}^3$.

13.2.8. Exercise. *Suppose a sequence of Riemannian spaces* $\mathcal{M}_n$ stably converges *in the sense of Gromov–Hausdorff to a Riemannian spaces* $\mathcal{M}_\infty$ *as* $n \to \infty$*; that is, the corresponding Hausdorff approximations can be chousen to be homeomorphisms. Show that*

$$\varliminf_{n\to\infty} \mathrm{vol}\,\mathcal{M}_n \geqslant \mathrm{vol}\,\mathcal{M}_\infty.$$

Show that the statement does not hold if we do not assume that the convergence is stable.

13.3 Systolic inequality

Let $\mathcal{M}$ be a compact Riemannian space. The *systole* of $\mathcal{M}$ (briefly $\mathrm{sys}\,\mathcal{M}$) is defined to be the least length of a noncontractible closed curve in $\mathcal{M}$.

Let Λ be a set of closed n-dimensional manifolds. We say that a systolic inequality holds for Λ if there is a constant c such that for any $M \in \Lambda$ and any metric tenor g on M we have

$$\mathrm{sys}(M, g) \leqslant c \cdot \sqrt[n]{\mathrm{vol}(M, g)}.$$

13.3.1. Exercise. *Use 1.2.1 to show that systolic inequality holds for the 2-torus* $\mathbb{T}^2$.

13.3.2. Exercise. *Use 1.2.1 to show that systolic inequality holds for the real projective palane* $\mathbb{R}\mathrm{P}^2$.

13.3.3. Exercise. *Use 1.2.2 to show that systolic inequality holds for the set of all closed surfaces of positive genus.*

13.3.4. Exercise. *Show that no systolic inequality holds for* $\mathbb{S}^2 \times \mathbb{S}^1$.

In the following lecture we will show that systolic inequality holds for many manifolds, in particular for torus of arbitrary dimension.

13.4 Remarks

The optimal constants in the systolic inequality are known only in the following three cases:

- ⋄ For real projective plane $\mathbb{RP}^2$ the constant is $\frac{\pi}{2}$ — the equality holds for a quotient of a round sphere by isometric involution. The statement was proved by Pao Ming Pu [**pu**].
- ⋄ For torus $\mathbb{T}^2$ the constant is $\frac{2}{\sqrt{3}}$ — the equality holds for a flat torus obtained from a regular hexagon by identifying opposite sides; this is the so called *Loewner's torus inequality.*
- ⋄ For the Klein bottle $\mathbb{RP}^2\#\mathbb{RP}^2$ the constant is $\frac{\pi}{2\cdot\sqrt{2}}$ — the equality holds for a certain nonsmooth metric The statement was proved by Christophe Bavard [**bavard**].

The proofs of these results use the so called *uniformization theorem* available in the 2-dimensional case only. These proofs are beautiful, but they too far from metric geometry. A good survey on the subject is written by Christopher Croke and Mikhail Katz [**croke-katz**].

The results discussed in this lecture admit number generalizations to general length metrics on manifolds (not necessary induced by a metric tensor). Instead of volume one may use the so called *Hausdorff measure* which we are about to discuss briefly. Hausdorff measure is only one of many analogs of volume that can be defined in a general to metric space.

Let $\mathcal{X}$ be a metric space and $W\subset\mathcal{X}$. The α-dimensional Hausdorff measure of W is defined as

$$\mu_\alpha W := \lim_{\varepsilon\to 0}\inf\left\{\sum_{n\in\mathbb{N}}(\operatorname{diam}A_n)^\alpha \ :\ \begin{matrix}\operatorname{diam}A_n<\varepsilon \text{ for}\\ \text{for each } n,\text{all } A_n\\ \text{are closed, and}\\ \bigcup\limits_{n\in\mathbb{N}}A_n\supset W.\end{matrix}\right\}.$$

For properties of Hausdorff measure we refer to a classical book of Herbert Federer [**federer**]; in particular it shown that μ_α is indeed a measure and μ_α-measurable sets include all Borel sets.

The following proposition trivially follows from the definitions

13.4.1. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces, $A\subset\mathcal{X}$ and $f\colon\mathcal{X}\to\mathcal{Y}$ be a L-Lipschitz map. Then*

$$\mu_\alpha[f(A)]\leqslant L^\alpha\cdot\mu_\alpha A.$$

The following exercise proves an weak analog of the Besikovitch inequality that works for arbitrary metric spaces.

13.4.2. Exercise. *Let M be manifold with boundary and ρ is a pseudometric on M. Suppose ∂M admits a degree 1 map $\partial M \to \partial\square^n$. Suppose $d_1, \dots, d_n$ the ρ-distances between the inverse images of pairs of opposite faces of $\square^n$ in M. Then*

$$\mu_n(M, \rho) \geqslant d_1 \cdots d_n.$$

13.4.3. Exercise. *Let $\mathcal{M}$ an n-dimensional Riemannian manifold. Show that*

$$\omega_n{\cdot}\mu_n\, A = \mathrm{vol}_n\, A$$

for any Borel set $A \subset \mathcal{M}$, where ω_n denotes the volume of a ball in the n-dimensional Euclidean space with diameter 1.

Note that $\omega_n < 1$ for $n \geqslant 2$. Therefore by 1.4.3, the conclusions in 1.4.2 (as well as its assumptions) are weaker than in 1.2.2.

One may define systolic inequality on n-dimensional manifolds taking μ_n instead of volume. The exercises 1.3.1–1.3.4 admit straightforward generalization for this setting.

13.4.4. Exercise. *Let X be a contractible metric space with zero $(n+1)$-dimensional Hausdorff measure. Assume that $\Delta_1, \Delta_2 \subset X$ are two embedded n-disks having the same boundary. Show that $\Delta_1 = \Delta_2$.*

Lecture 14

Width and systole

This lecture is based on a paper of Alexander Nabutovsky [**nabutovsky**].

14.1 Nerves and partition of unity

Let $\{V_1, \dots, V_k\}$ be a finite open cover of a compact metric space $\mathcal{X}$. Consider an abstract simplicial complex $\mathcal{N}$, with one vertex v_i for each set V_i such that a simplex with vertexes $v_{i_1}, \dots, v_{i_m}$ is included in $\mathcal{N}$ if the intersection $V_{i_1} \cap \dots \cap V_{i_m}$ is nonempty. The obtained simplicial complex $\mathcal{N}$ called the *nerve of the covering* $\{V_i\}$.

Note that $\mathcal{N}$ is a finite simplicial complex; it is a subcomplex of a simplex with the vertixes $\{v_1, \dots, v_k\}$. The nerve $\mathcal{N}$ has dimension at most n if and only if the covering $\{V_1, \dots, V_k\}$ has multiplicity is at most $n+1$; that is, any point $x \in \mathcal{X}$ belongs to at most $n+1$ sets of the covering.

14.1.1. Proposition. *Let $\{V_1, \dots, V_k\}$ be a finite open covering of a compact metric space $\mathcal{X}$. Then there are Lipschitz functions $\psi_i \colon \mathcal{X} \to \to [0,1]$ such that if $\psi_i(x) > 0$ then $x \in V_i$ and*

$$\sum_i \psi_i(x) = 1$$

for any $x \in \mathcal{X}$.

Proof. Consider functions $\varphi_i \colon \mathcal{X} \to \mathbb{R}$ defined as

$$\varphi_i(x) = \operatorname{dist}_{(\mathcal{X} \setminus V_i)} x.$$

Note φ_i is 1-Lipschitz for any i and $\varphi_i(x) > 0$ if and only if $x \in V_i$. Since $\{V_i\}$ is a covering, we have that

$$\sum_i \varphi_i(x) > 0 \quad \text{for any} \quad x \in \mathcal{X}.$$

Set

$$\psi_k(x) = \frac{\varphi_k(x)}{\sum_i \varphi_i(x)}.$$

Observe that by construction the functions ψ_i meet the conditions in the proposition. $\square$

A collection of functions $\{\psi_i\}$ that meets the conditions in 2.1.1 is called a *partition of unity subordinate to the open covering* $\{V_1, \dots, V_k\}$.

Suppose $\{\psi_i\}$ is a partition of unity subordinate to the open covering $\{V_1, \dots, V_k\}$. Note that for any point $x \in \mathcal{X}$, the set

$$\{\, v_i \,:\, \psi_i(x) > 0 \,\}$$

is formed by vertexes of a simplex in the nerve. Therefore

$$\psi\colon x \mapsto \psi_1(x)\cdot v_1 + \psi_2(x)\cdot v_2 + \dots + \psi_k(x)\cdot v_n.$$

describes a Lipschitz map from $\mathcal{X}$ to the nerve $\mathcal{N}$ of $\{V_i\}$; here the point x is mapped to the point in $\mathcal{N}$ with barycentric coordinates $\psi_i(x)$. In other words we proved the following:

14.1.2. Proposition. *Let $\mathcal{N}$ be a nerve of an open covering $\{V_1, \dots$ $\dots, V_k\}$ of a compact metric space $\mathcal{X}$. Denote by v_i the vertex of $\mathcal{N}$ that corresponds to V_i.*

Then there is a Lipschitz map $\psi\colon \mathcal{X} \to \mathcal{N}$ such that $\psi(V_i) \subset \mathrm{Star}_{v_i}$ for every i; that is, for any $x \in V_i$ the point $\psi(x)$ lies the interior of some simplex with vertex v_i.

14.2 Width

Suppose A is a subset of a metric space $\mathcal{X}$. The radius of A (briefly $\operatorname{rad} A$) is defined as the least upper bound on the values $R > 0$ such that $\mathrm{B}(x, R) \supset A$ for some $x \in \mathcal{X}$.

14.2.1. Definition. *Let $\mathcal{X}$ be a metric space. The n-th width of $\mathcal{X}$ (briefly $\mathrm{width}_n \mathcal{X}$) is defined as the least upper bound on values $R > 0$ such that $\mathcal{X}$ admits a finite open covering $\{V_i\}$ with multiplicity at most $n + 1$ and $\operatorname{rad} V_i < R$ for each i.*

Remarks.

- ◇ Observe that
$$\operatorname{width}_0 \mathcal{X} \geqslant \operatorname{width}_1 \mathcal{X} \geqslant \dots$$
for any compact metric space $\mathcal{X}$. Moreover, if $\mathcal{X}$ is connected, then
$$\operatorname{width}_0 \mathcal{X} = \operatorname{rad} \mathcal{X}.$$
- ◇ Usually width is defined using diameter instead of radius, but the result differ at most twice. Namely if r is the radius-width and d — diameter-width for the same n, then $r \leqslant d \leqslant 2\cdot r$.
- ◇ Note that *Lebesgue covering dimension* of $\mathcal{X}$ can be defined as the least number n such that $\operatorname{width}_n \mathcal{X} = 0$. Another closely related notion is the so called *macroscopic dimesion on scale* R; it is defined as the least number n such that $\operatorname{width}_n \mathcal{X} < R$.

14.2.2. Exercise. *Suppose $\mathcal{X}$ is a compact metric space such that any closed curve γ in $\mathcal{X}$ can be contracted in its R-neighborhood. Show that $\mathcal{X}$ has macroscopic dimension at most 1 on scale* $100\cdot R$.

What about quasiconverse? That is, suppose a simply connected compact metric space $\mathcal{X}$ has macroscopic dimension at most 1 on scale R, is it true that any closed curve γ in $\mathcal{X}$ can be contracted in its $100\cdot R$*-neighborhood?*

The following exercise provides an equivalent definition; it also provides a good reason for the name *width*.

14.2.3. Exercise. *Suppose $\mathcal{X}$ is a compact metric space. Show that* $\operatorname{width}_n \mathcal{X} < R$ *if and only if there is a finite n-dimensional simplicial complex $\mathcal{N}$ and a continuous map $\psi\colon \mathcal{X} \to \mathcal{N}$ such that*
$$\operatorname{rad}[\psi^{-1}(s)] < R$$
for any $s \in \mathcal{N}$.

14.3 Riemannian polyhedrons

A *Riemannian polyhedron* is defined as a finite simplicial complex with a metric tensor on each simplex such that the restriction of the metric tensor to a subsymplex coinsides with the metric on the subsmplex. The dimension of Riemannian polyhedron is defined as the largest dimension it its triangulation. For Riemannian polhedron one can define length of curves and volume the same way as for Riemannian manifolds.

Further we will apply the notion of width to compact Riemannian polyhedrons. If $\mathcal{P}$ is an n-dimensional Riemannian polyhedron, then we suppose that
$$\operatorname{width} \mathcal{P} := \operatorname{width}_{n-1} \mathcal{P}.$$

Let $\mathcal{P}$ be an n-dimensional Riemnnian polyhedron. Let us define *volume profile* of $\mathcal{P}$ as a function returning largest volume of r-ball in $\mathcal{P}$; that is, $\mathrm{VolPro}_{\mathcal{P}}\colon \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\mathrm{VolPro}_{\mathcal{P}}(r) := \sup \{ \mathrm{vol}_n \mathrm{B}(p,r) \ : \ p \in \mathcal{P} \} .$$

Note that $\mathrm{VolPro}_{\mathcal{P}}$ is a nondecreasing function,

$$\mathrm{VolPro}_{\mathcal{P}}(r) \leqslant \mathrm{vol}_n \mathcal{P}.$$

moreover, if $\mathcal{P}$ is connected, then equality holds for all larger r.

Note that if $\mathcal{P}$ is a 1-dimensional connected Riemannian polyhedron, then

$$\mathrm{width}\, \mathcal{P} = \mathrm{width}_0\, \mathcal{P} = \mathrm{rad}\, \mathcal{P}.$$

14.3.1. Exercise. *Suppose $\mathcal{P}$ be a 1-dimensional Riemannian polyhedron. Suppose* $\mathrm{VolPro}_{\mathcal{P}}(R) < R$ *for some* $R > 0$*. Show that*

$$\mathrm{width}\, \mathcal{P} < R.$$

Try to show that $c = \frac{1}{2}$ is the optimal constant such that

$$\mathrm{width}\, \mathcal{P} < c{\cdot}R.$$

14.4 Volume profile bounds width

14.4.1. Theorem. *Let $\mathcal{P}$ be an n-dimensional Reimannian polyhedron. If the inequality*

$$R > n{\cdot} \sqrt[n]{\mathrm{VolPro}_{\mathcal{P}}(R)}$$

holds for some $R > 0$*, then*

$$\mathrm{width}\, \mathcal{P} \leqslant R.$$

Since $\mathrm{VolPro}_{\mathcal{P}}(r) \leqslant \mathrm{vol}\, \mathcal{P}$ for any r, we get the following:

14.4.2. Corollary. *For any n-dimensional Reimannian polyhedron $\mathcal{P}$, we have*

$$\mathrm{width}\, \mathcal{P} \leqslant n{\cdot} \sqrt[n]{\mathrm{vol}\, \mathcal{P}}.$$

In the proof of 2.4.1, we will use the following three technical statements, the proofs are omitted, but they are not hard.

14.4.3. Smoothing procedure. *Let $\mathcal{P}$ be a Reimannian polyhedron and $f\colon \mathcal{P} \to \mathbb{R}$ be a 1-Lipschitz function. Then for any $\delta > 0$ there is a 1-Lipschitz function $\tilde{f}\colon \mathcal{P} \to \mathbb{R}$ that is smooth on each simplex of the triangulation and δ-close to f.*

14.4.4. Sard's theorem. *Let $\mathcal{P}$ be an n-dimensional Reimannian polyhedron and $f\colon \mathcal{P} \to \mathbb{R}$ be a function that is smooth on each simplex. Then for almost all values a, each component of the inverse image $f^{-1}\{a\}$ equipped with the induced metric is a Reimannian polyhedron of dimension at most $n-1$.*

14.4.5. Coarea inequality. *Let $\mathcal{P}$ be an n-dimensional Reimannian polyhedron and $f\colon \mathcal{P} \to \mathbb{R}$ be a 1-Lipschitz function that is smooth on each simplex. Set $V = \operatorname{vol}_n(f^{-1}[a,b])$. Then*

$$\int_a^b \operatorname{vol}_{n-1}(f^{-1}\{x\})\cdot dx \geqslant V.$$

In particular there is a subset of positive measure $A \subset [a,b]$ such that the inequality

$$\operatorname{vol}_{n-1}(f^{-1}\{x\}) \geqslant \frac{V}{b-a}$$

holds for any $x \in A$.

14.4.6. Definition. *Let $\mathcal{P}$ be an n-dimensional Riemannian polyhedron. An $(n-1)$-dimensional subpolyhedron $\mathcal{Q} \subset \mathcal{P}$ is called R-*separating *if $\operatorname{rad} U < R$ for each connected component U of the complement $\mathcal{P}\backslash\mathcal{Q}$.*

14.4.7. Lemma. *Let $\mathcal{P}$ be an n-dimensional Riemannian polyhedron. Then given $R > 0$ and $\varepsilon > 0$ there is a R-separating subpolyhedron $\mathcal{Q} \subset \mathcal{P}$ such that for any $r_0 < r_1 \leqslant R$ we have*

$$\operatorname{VolPro}_{\mathcal{Q}}(r_0) < \tfrac{1}{r_1 - r_0}\cdot \operatorname{VolPro}_{\mathcal{P}}(r_1) + \varepsilon.$$

The idea of the proof is borrowed from the theory minimal surfaces. Namely if a point p lies on a area minimizing surface Σ, then $\operatorname{area}(\Sigma \cap \mathrm{B}(p,r)) \leqslant \frac{1}{2}\cdot \operatorname{area}\mathbb{S}^2\cdot r^2$ for all small $r > 0$.

Proof. Choose a small $\delta > 0$. Applying the smoothing procedure, we can exchange each distance function dist_p on $\mathcal{P}$ by δ-close smooth 1-Lipschitz function, which will be denoted by $\widetilde{\operatorname{dist}}_p$.

By Sard's theorem, almost all level sets

$$\tilde{S}_c(p) = \left\{\, x \in \mathcal{P} \,:\, \widetilde{\operatorname{dist}}_p(x) = c \,\right\}$$

are smooth Riemannian polyhedrons of dimension at most $n-1$. Since δ is small, the coarea inequality implies that for we can choose $c \in \in (r_0 + \delta, r_1 - \delta)$ such that $\tilde{S}_c(p)$ is a subpolyhedron and

$$\begin{aligned}\operatorname{vol}_{n-1} \tilde{S}_c(p) &\leqslant \tfrac{1}{r_1 - r_0 - 2\cdot\delta} \cdot \operatorname{vol}_n[\mathrm{B}(p, r_1)] < \\ &< \tfrac{1}{r_1 - r_0} \cdot \operatorname{VolPro}_{\mathcal{P}}(r_1) + \tfrac{\varepsilon}{2}.\end{aligned}$$

Suppose $\mathcal{Q}$ is an R-separating subpolyhedron in $\mathcal{P}$ with almost minimal volume, say its volume is at most $\frac{\varepsilon}{2}$-far from the greatest lower bound. Note that cutting from $\mathcal{Q}$ everything inside $\tilde{S}_c(p)$ and adding $\tilde{S}_c(p)$ keeps it to be R-separating subpolyhedron. Since $\mathcal{Q}$ has almost minimal volume, we have

$$\operatorname{vol}_{n-1}[\mathcal{Q} \cap \mathrm{B}(p, r_0)_{\mathcal{P}}] - \tfrac{\varepsilon}{2} \leqslant \operatorname{vol}_{n-1} S_c(p).$$

Therefore

$$\text{❶} \qquad \operatorname{vol}_{n-1}[\mathcal{Q} \cap \mathrm{B}(p, r_0)_{\mathcal{P}}] \leqslant \tfrac{1}{r_1 - r_0} \cdot \operatorname{VolPro}_{\mathcal{P}}(r_1) + \varepsilon$$

Recall that $\mathcal{Q}$ is equipped with the induced length metric; therefore $|p-q|_{\mathcal{Q}} \geqslant |p-q|_{\mathcal{P}}$ for any $p, q \in \mathcal{Q}$; in particular,

$$\mathrm{B}(p, r_0)_{\mathcal{Q}} \subset \mathcal{Q} \cap \mathrm{B}(p, r_0)_{\mathcal{P}}$$

for any $p \in \mathcal{Q}$ and $r \geqslant 0$. Hence ❶ implies the lemma. ☐

14.4.8. Lemma. *Let $\mathcal{Q}$ be an R-separating subpolyhedron in an n-dimensional Riemannian polyhedron $\mathcal{P}$. Suppose* $\operatorname{width} \mathcal{Q} \leqslant R$. *Then* $\operatorname{width} \mathcal{P} \leqslant R$

Proof. Start with an open covering $\{V_1, \dots, V_k\}$ of $\mathcal{Q}$ of multiplicity $\leqslant n$ with radiuses of the sets in the intrinsic metric $\leqslant R$.

Note that $\{V_1, \dots, V_k\}$ can be converted into an open covering of a small neighbourhood of $\mathcal{Q}$ in $\mathcal{P}$ without increasing the multiplicity. This is can be done by setting

$$V_i' = \bigcup_{x \in V_i} \mathrm{B}(x, r_x),$$

where $r_x = \frac{1}{10} \cdot \inf\{\, |x-y| \,:\, y \in \mathcal{Q} \backslash V_i \,\}$.

Adding to $\{V_i'\}$ all the components of $\mathcal{P}\backslash\mathcal{Q}$, we increase the multiplicity by at most 1 and obtain a covering of $\mathcal{P}$. The statement follows since $\dim\mathcal{P} = \dim\mathcal{Q} + 1$. □

Proof of 2.4.1. We apply induction on the dimension $n = \dim\mathcal{P}$. The base case $n = 1$ is given in 2.3.1.

Suppose that the $(n-1)$-dimensional case is proved. Consider an n-dimensional Riemannian polyhedron $\mathcal{P}$ and suppose

$$n\cdot \sqrt[n]{\operatorname{VolPro}\mathcal{P}(R)} < R$$

for some $R > 0$. Let $\mathcal{Q}$ be an R-separating subpolyhedron in $\mathcal{P}$ provided by 2.4.7 for a small $\varepsilon > 0$. Applying 2.4.7 for $r = \frac{n-1}{n}\cdot R$ and R, we have that

$$\begin{aligned}\operatorname{VolPro}_{\mathcal{Q}}(r) &< \frac{1}{R-r}\cdot \operatorname{VolPro}_{\mathcal{P}}(R) + \varepsilon < \\ &< \frac{n}{R}\cdot\left(\frac{R}{n}\right)^n = \\ &= \left(\frac{r}{n-1}\right)^{n-1};\end{aligned}$$

that is, $(n-1)\cdot \sqrt[n-1]{\operatorname{VolPro}\mathcal{Q}(r)} < r$. Since $\dim\mathcal{Q} \leqslant n-1$, by the induction hypothesis, we get that

$$\operatorname{width}\mathcal{Q} \leqslant r < R.$$

It remains to apply 2.4.8. □

14.5 Width bounds systole

14.5.1. Theorem. *Suppose $\mathcal{M}$ is a aspherical n-dimensional Riemnnian manifold. Then*

$$\operatorname{sys}\mathcal{M} \leqslant 6\cdot \operatorname{width}\mathcal{M}.$$

14.5.2. Lemma. *Let M be an aspherical space and L be a connected CW-complex. Denote by L^k the k-skeleton of L. Then any continuous map $f\colon L^2 \to M$ can be extended to a continuous map $\bar f\colon L \to M$*

Moreover, if $p \in L$ is a 0-cell and $q \in M$. Then a continuous maps of pairs $\varphi_0, \varphi_1\colon (L,p) \to (M,q)$ are homotopic if and only if φ_0 and φ_1 induce the same homomorphism on fundamental groups $\pi_1(L,p) \to \pi_1(M,q)$.

Proof. Since M is aspherical, any continuous map $\partial\mathbb{D}^n\colon \to M$ for $n \geqslant 3$ is hull-homotopic; that is, it can be extended to a map $\mathbb{D}^n\colon \to M$.

It makes possible to extend f to L^3, L^4, and so on. Therefore f can be extended to whole L.

The only-if part on the second part of lemma is trivial; let us show the if part.

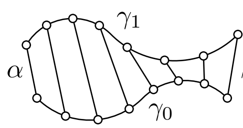
Sine L is connected, we can assume that p forms the only 0-cell in L; otherwise we can collapse a maximal sub-tree of the 1-skeleton in L to p. Therefore L^1 is formed by loops that generates $\pi_1(L,p)$.

By assumption, the restrictions of φ_0 and φ_1 to L^1 are homotopic. In other words the homotopy $\Phi\colon [0,1]\times L$ is defined on the 2-skeleton of $[0,1]\times L$. It remains to apply the first part of the lemma. $\square$

14.5.3. Lemma. *Suppose γ_0,γ_1 are two paths between points in a Riemannian space $\mathcal{M}$ such that $|\gamma_0(t)-\gamma_1(t)|_{\mathcal{M}} < r$ for any $t\in[0,1]$. Let α be a geodesic path from $\gamma_0(0)$ to $\gamma_1(0)$ and β be a geodesic path from $\gamma_0(1)$ to $\gamma_1(1)$. If $2\cdot r < \operatorname{sys}\mathcal{M}$, then there is a homotopy γ_t from γ_0 to γ_1 such that $\alpha(t) = \gamma_t(0)$ and $\beta(t)\mapsto\gamma_t(1)$.*

Proof. Set $s = \operatorname{sys}\mathcal{M}$; since $2\cdot r < s$, we have that $\varepsilon = \frac{1}{10}(s-2\cdot r) > 0$.

Note that we can assume that γ_0 and γ_1 are rectifiable; if not we can homotopy each into a broken geodesic line kipping the assumptions true.



Choose a fine partition $0 = t_0 < t_1 < \dots$ $\dots < t_n = 1$. Consider a sequence of geodesic paths α_i from $\gamma_0(t_i)$ to $\gamma_1(t_i)$; we can assume that $\alpha_0 = \alpha$ and $\alpha_n = \beta$. We can assume that each arc $\gamma_j|_{[t_{i-1},t_i]}$ has length smaller than ε. Therefore every quadrilateral formed by concatenation of α_{i-1}, $\gamma_1|_{[t_{i-1},t_i]}$, reversed α_i, and reversed arc $\gamma_0|_{[t_{i-1},t_i]}$ has length smaller than s. It follows that this curve is contractible. Applying this observation for each quadrilateral, we get the statement. $\square$

Proof of 2.5.1. Let $\mathcal{N}$ be the nerve of a covering $\{V_i\}$ of $\mathcal{M}$ and $\psi\colon\mathcal{M}\to\mathcal{N}$ be the map provided by 2.1.2. As usual, we denote by v_i the vertex of $\mathcal{N}$ that corresponds to V_i.

Set $R = \operatorname{width}\mathcal{M}$ and $s = \operatorname{sys}\mathcal{M}$. Assume we chose $\{V_i\}$ as in the definition of width (2.2.1). For each i choose a point $p_i\in\mathcal{M}$ such that $V_i\subset \mathrm{B}(p_i,R)$. Observe that in this case $\dim\mathcal{N} < n$; therefore ψ kills the fundamental class of $\mathcal{M}$.

Let us construct a continuous map $f\colon\mathcal{N}\to\mathcal{M}$ such that $f\circ\psi$ is homotopic to the identity map on $\mathcal{M}$.

Note that once f is constructed, the theorem is proved, . Indeed, since ψ kills the fundamental class of $\mathcal{M}$, so does $f\circ\psi$. Therefore $[\mathcal{M}]=0$ — a contradiction.

First set $f(v_i)=p$. It defines the map f on the 0-skeleton $\mathcal{N}^0$ of the nerve $\mathcal{N}$. Further we will be define f step by step on the skeletons of higher dimensions $\mathcal{N}^1,\mathcal{N}^2,\dots$

Let us map each edge $[v_iv_j]$ in $\mathcal{N}$ to a geodesic $[p_ip_j]$. It defines the map on the 1-skeleton $\mathcal{N}^1$ of the nerve $\mathcal{N}$. Note that image of each edge is shorter that $2\cdot R$.

Suppose $[v_iv_jv_k]$ is a triangle in $\mathcal{N}$. Note that perimeter of the triangle $[p_ip_jp_k]$ can not exceed $6\cdot R$. Since $6\cdot R<s$, the contour of $[p_ip_jp_k]$ is contractible. Therefore we can extend f to each triangle of $\mathcal{N}$. It defines the map f on $\mathcal{N}^2$.

Finally, since $\mathcal{M}$ is aspherical, by 2.5.2, the map f can be extended to $\mathcal{N}^3$, $\mathcal{N}^4$ and so on.

It remains to show that $f\circ\psi$ is homotopic to the identity map. Choose a CW structure on $\mathcal{M}$ with sufficiently small cell, so that each cell lies in one of V_i. Note that ψ is homotopic to a map ψ_1 that sends $\mathcal{M}^k$ to $\mathcal{N}^k$ for any k. Moreover we may assume that (1) if a 0-cell x of $\mathcal{M}$ maps to a v_i, then $x\in V_i$ and (2) each 1-cell of $\mathcal{M}$ maps to an edge of $\mathcal{N}$. Choose a 1-cell e in $\mathcal{M}$; by the construction, $f\circ\psi_1$ maps e to a geodesic $[p_ip_j]$ and e lies $\mathrm{B}(p_i,R)$. Observe that $[p_ip_j]$ is shorter than $2\cdot R$. It follows that the distance between points on $[p_ip_j]$ and e can not exceed $3\cdot R$. Choose a geodesic path α_i from every 0 cell x_i of $\mathcal{M}$ to $p_j=f\circ\psi_1(x_i)$. It defines a homotopy on $\mathcal{M}^0$. Since $6\cdot R<s$, 2.5.3 implies that this homotopy can be extended to $\mathcal{M}^1$. By 2.5.2, it can be extended to whole $\mathcal{M}$. □

14.5.4. Exercise. *Analyze the proof of 2.5.1 and improve its inequality to*

$$\operatorname{sys}\mathcal{M}\leqslant 4\cdot\operatorname{width}\mathcal{M}.$$

14.5.5. Exercise. *Modify the proof of 2.5.1 to prove the following:*

Suppose that $\mathcal{M}$ is a closed n-dimensional Reimannian manifold with injectivity radius *at least r; that is, if $|p-q|_{\mathcal{M}}<r$, then there is geodesic $[pq]_{\mathcal{M}}$ is uniquely defined. Show that*

$$\operatorname{width}\mathcal{M}\geqslant\tfrac{r}{2\cdot(n+1)}.$$

Use 2.4.2 to conclude that

$$\operatorname{vol}\mathcal{M}\geqslant\varepsilon_n\cdot r^n$$

for some $\varepsilon_n>0$ that depends only on n.

The second statement in the exercise is a theorem of Marcel Berger [**berger-n**].

14.6 Essential manifolds

To generalize 2.5.1 bit further, we need the following definition.

14.6.1. Definition. *A closed manifold $\mathcal{M}$ is called* essential *if it admits a continuous map $\iota\colon \mathcal{M} \to \mathcal{K}$ to an aspherical topological space $\mathcal{K}$ such that ι sends the fundamental class of $\mathcal{M}$ to a nonzero homology class in $\mathcal{K}$.*[1]

Assume that the manifold $\mathcal{M}$ is essential and $\iota\colon \mathcal{M} \to \mathcal{K}$ as in the definition. Following the proof of 2.5.1, we can homotope the map $f\circ\psi\colon \mathcal{M} \to \mathcal{M}$ to the identity on the 2-skeleton of $\mathcal{M}$; further since $\mathcal{K}$ is aspherical we can homotopy the composition $\iota\circ f\circ\psi$ to ι. Existence of this extension implies that ι kills the fundamental class of $\mathcal{M}$ — a contradiction. So, taking 2.5.4 into account, we proved the following generalization of 2.5.1:

14.6.2. Theorem. *Suppose $\mathcal{M}$ is an essential Riemnnian space. Then*

$$\operatorname{sys}\mathcal{M} \leqslant 4\cdot \operatorname{width}\mathcal{M}.$$

As a corollary form 2.6.2 and 2.4.2 we get the so called *Gromov's systolic inequality*:

14.6.3. Theorem. *Suppose $\mathcal{M}$ is an essential n-dimensional Riemannian space. Then*

$$\operatorname{sys}\mathcal{M} \leqslant 4\cdot n\cdot \sqrt[n]{\operatorname{vol}\mathcal{M}}.$$

Note that any closed aspherical manifold is essential — in this case one can take ι to be the identity map on $\mathcal{M}$. The real projective space $\mathbb{RP}^n$ provides an interesting example of an essential manifold which is not aspherical. Indeed, the infinite dimensional projective space $\mathbb{RP}^\infty$ is aspherical and for the natural embedding $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^\infty$ the image $\mathbb{RP}^n$ does not bound in $\mathbb{RP}^\infty$. The following exercise provides more examples of that type:

14.6.4. Exercise. *Show that connected sum of an essential manifold with any closed manifold is essential.*

14.6.5. Exercise. *Show that product of two essential manifolds is essential.*

[1] We assume that the coefficients are $\mathbb{Z}_2$, but one can play with them if necessary.

Show that product of nonessential closed manifold of dimension at least 1 with any closed manifold is not essential.

14.7 Remarks

Theorem 2.6.3 was proved originally by Mikhael Gromov [**gromov-1983**] with much worse constant. The given proof is a result of a sequence of simplifications given by Larry Guth [**guth**], Panos Papasoglu [**papasoglu**], Alexander Nabutovsky and Roman Karasev [**nabutovsky**].

In [**nabutovsky**] the calculations were optimized better which gave the constants $c_n = \sqrt[n]{n!} = \frac{n}{e} + o(n)$ in 2.4.2 instead of n. As a result, we have a stronger statement in 2.6.3:

$$\operatorname{sys}\mathcal{M} \leqslant 4\cdot c_n\cdot \sqrt[n]{\operatorname{vol}\mathcal{M}}.$$

A wide open conjecture says that for any n-dimensional essential manifold we have

$$❶ \qquad \frac{\operatorname{sys}\mathcal{M}}{\sqrt[n]{\operatorname{vol}\mathcal{M}}} \leqslant \frac{\operatorname{sys}\mathbb{R}\mathrm{P}^n}{\sqrt[n]{\operatorname{vol}\mathbb{R}\mathrm{P}^n}},$$

where we assume that the n-dimensional real projective space $\mathbb{R}\mathrm{P}^n$ is equipped with canonical metric. In other words, the ratio in the right hand side of ❶ is the optimal constant in the Gromov's systolic inequality; this ratio grows as $O(\sqrt{n})$. (The ratio for n-dimensional flat torus also grows as $O(\sqrt{n})$ as well.)

For a given group G there is a path connected aspherical space $\mathcal{K}$ with fundamental group G; moreover

- ◇ $\mathcal{K}$ can be chosen to be a CW-complex;
- ◇ $\mathcal{K}$ is uniquely defined up to homotopy equivalence;
- ◇ if $\mathcal{L}$ is a connected finite CW-complex. Then any homomorphism $\pi_1\mathcal{L} \to \pi_1\mathcal{K}$ is induced by a continuous map $\varphi\colon \mathcal{L} \to \mathcal{K}$. Moreover, φ is uniquely defined up to homotopy equivalence.

The space $\mathcal{K}$ is called an *Eilenberg–MacLane space of type* $K(G,1)$, or simply a $K(G,1)$ space.

The following proposition provides an alternative definition of essential manifold, it follows from the observations above.

14.7.1. Proposition. *Suppose $\mathcal{M}$ is a closed manifold, $\mathcal{K}$ is a $K(\pi_1(\mathcal{M}),1)$ space and a map $\iota\colon \mathcal{M} \to \mathcal{K}$ induces an isomorphism of fundamental groups. Then $\mathcal{M}$ is essential if and only if ι sends the fundamental class of $\mathcal{M}$ to a nonzero homology class in $\mathcal{K}$.*

Appendix A

Semisolutions

1.3.1. Assume the statement is wrong. Then for any point $x \in \mathcal{X}$, there is a point $x' \in \mathcal{X}$ such that

$$|x - x'| < \rho(x) \quad \text{and} \quad \rho(x') \leqslant \frac{\rho(x)}{1+\varepsilon}.$$

Consider a sequence of points (x_n) such that $x_{n+1} = x'_n$. Clearly

$$|x_{n+1} - x_n| \leqslant \frac{\rho(x_0)}{\varepsilon\cdot(1+\varepsilon)^n} \quad \text{and} \quad \rho(x_n) \leqslant \frac{\rho(x_0)}{(1+\varepsilon)^n}.$$

Therefore (x_n) is a Cauchy sequence. Since $\mathcal{X}$ is complete, the sequence (x_n) converges; denote its limit by x_∞. Since ρ is a continuous function we get

$$\begin{aligned}\rho(x_\infty) &= \lim_{n\to\infty} \rho(x_n) = \\ &= 0.\end{aligned}$$

The latter contradicts that $\rho > 0$.

1.4.4. Given any pair of point $x_0, y_0 \in \mathcal{K}$, consider two sequences $x_0, x_1, \dots$ and $y_0, y_1, \dots$ such that $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ for each n.

Since $\mathcal{K}$ is compact, we can choose an increasing sequence of integers n_k such that both sequences $(x_{n_i})_{i=1}^\infty$ and $(y_{n_i})_{i=1}^\infty$ converge. In particular, both are Cauchy; that is,

$$|x_{n_i} - x_{n_j}|_{\mathcal{K}}, |y_{n_i} - y_{n_j}|_{\mathcal{K}} \to 0 \quad \text{as} \quad \min\{i,j\} \to \infty.$$

Since f is non-contracting, we get

$$|x_0 - x_{|n_i - n_j|}| \leqslant |x_{n_i} - x_{n_j}|.$$

It follows that there is a sequence $m_i \to \infty$ such that

$$(*) \qquad x_{m_i} \to x_0 \quad \text{and} \quad y_{m_i} \to y_0 \quad \text{as} \quad i \to \infty.$$

Set

$$\ell_n = |x_n - y_n|_{\mathcal{K}}.$$

Since f is non-contracting, the sequence (ℓ_n) is nondecreasing.

By $(*)$, $\ell_{m_i} \to \ell_0$ as $m_i \to \infty$. It follows that (ℓ_n) is a constant sequence.

In particular

$$|x_0 - y_0|_{\mathcal{K}} = \ell_0 = \ell_1 = |f(x_0) - f(y_0)|_{\mathcal{K}}$$

for any pair of points (x_0, y_0) in $\mathcal{K}$. That is, f is distance-preserving, in particular injective.

From $(*)$, we also get that $f(\mathcal{K})$ is everywhere dense. Since $\mathcal{K}$ is compact $f\colon \mathcal{K} \to \mathcal{K}$ is surjective. Hence the result follows.

Remarks. This is a basic lemma in the introduction to Gromov–Hausdorff distance [**burago-burago-ivanov**]. This proof is not quite standard, I learned this proof from Travis Morrison, a student in my MASS class at Penn State, Fall 2011.

Note that as an easy corollary one can see that any surjective non-expanding map from a compact metric space to itself is an isometry.

1.4.5. If such number does not exist then the ranges of average distance functions have empty intersection. Since X is a compact length-metric space, the range of any continuous function on X is a closed interval. By 1-dimesional Helly's theorem, there is a pair of such range intervals that do not intersect. That is, for two point-arrays $(x_1, \dots, x_n)$ and $(y_1, \dots, y_m)$ and their average distance functions

$$f(z) = \tfrac{1}{n} \cdot \sum_i |x_i - z|_X \quad \text{and} \quad h(z) = \tfrac{1}{m} \cdot \sum_j |y_j - z|_X,$$

we have

$$(*) \qquad \min \{ f(z) \;:\; z \in X \} > \max \{ h(z) \;:\; z \in X \}.$$

Note that

$$\tfrac{1}{m} \cdot \sum_j f(y_j) = \tfrac{1}{m \cdot n} \cdot \sum_{i,j} |x_i - y_j|_X = \tfrac{1}{n} \cdot \sum_i h(x_i);$$

that is, the average value of $f(y_j)$ coincides with the average value of $h(x_i)$, which contradicts $(*)$.

Remarks. This is a result of Oliver Gross [**gross**]. The value ℓ is called the *rendezvous value* of X; in fact it is uniquely defined.

1.9.2. We assume that the space is nontrivial, otherwise a one-point space is an example.

Consider the unit ball (B,ρ_0) in the space c_0 of all sequences converging to zero equipped with the sup-norm.

Consider another metric ρ_1 which is different from ρ_0 by the conformal factor

$$\varphi(\boldsymbol{x}) = 2 + \tfrac{1}{2}{\cdot}x_1 + \tfrac{1}{4}{\cdot}x_2 + \tfrac{1}{8}{\cdot}x_3 + \dots,$$

where $\boldsymbol{x} = (x_1, x_2 \dots) \in B$. That is, if $\boldsymbol{x}(t)$, $t \in [0,\ell]$, is a curve parametrized by ρ_0-length then its ρ_1-length is defined by

$$\operatorname{length}_{\rho_1}\boldsymbol{x} := \int_0^\ell \varphi\circ\boldsymbol{x}(t){\cdot}dt.$$

Note that the metric ρ_1 is bi-Lipschitz to ρ_0.

Assume $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t)$ are two curves parametrized by ρ_0-length that differ only in the m-th coordinate, denoted by $x_m(t)$ and $x'_m(t)$ respectively. Note that if $x'_m(t) \leqslant x_m(t)$ for any t and the function $x'_m(t)$ is locally 1-Lipschitz at all t such that $x'_m(t) < x_m(t)$, then

$$\operatorname{length}_{\rho_1}\boldsymbol{x}' \leqslant \operatorname{length}_{\rho_1}\boldsymbol{x}.$$

Moreover this inequality is strict if $x'_m(t) < x_m(t)$ for some t.

Fix a curve $\boldsymbol{x}(t)$, $t \in [0,\ell]$, parametrized by ρ_0-length. We can choose m large, so that $x_m(t)$ is sufficiently close to 0 for any t. In particular, for some values t, we have $y_m(t) < x_m(t)$, where

$$y_m(t) = (1 - \tfrac{t}{\ell}){\cdot}x_m(0) + \tfrac{t}{\ell}{\cdot}x_m(\ell) - \tfrac{1}{100}\cdot\min\{t, \ell - t\}.$$

Consider the curve $\boldsymbol{x}'(t)$ as above with

$$x'_m(t) = \min\{x_m(t), y_m(t)\}.$$

Note that $\boldsymbol{x}'(t)$ and $\boldsymbol{x}(t)$ have the same end points, and by the above

$$\operatorname{length}_{\rho_1}\boldsymbol{x}' < \operatorname{length}_{\rho_1}\boldsymbol{x}.$$

That is, for any curve $\boldsymbol{x}(t)$ in (B,ρ_1), we can find a shorter curve $\boldsymbol{x}'(t)$ with the same end points. In particular, (B,ρ_1) has no geodesics.

Remarks. This solution was suggested by Fedor Nazarov [**nazarov**].

1.9.3. Choose a sequence $\varepsilon_n \to 0$ and a ε_n-net N_n of K for each n. Assume N_0 is a one-point set, so $\varepsilon_0 > \operatorname{diam} K$. Connect each point $x \in N_{k+1}$ to a point $y \in N_k$ by a curve of length at most ε_k.

Consider the union K' of all these curves with K; observe that K' is compact and path connected.

Source: This problem was stated by Eugene Bilokopytov [**bilokopytov**].

1.9.4. Choose a Cauchy sequence (x_n) in $(\mathcal{X}, \| * - * \|)$; it sufficient to show that a subsequence of (x_n) converges.

Note that the sequence (x_n) is Cauchy in $(\mathcal{X}, | * - * |)$; denote its limit by x_∞.

After passing to a subsequence, we can assume that $\|x_n - x_{n+1}\| < \frac{1}{2^n}$. It follows that there is a 1-Lipschitz path γ in $(\mathcal{X}, \| * - * \|)$ such that $x_n = \gamma(\frac{1}{2^n})$ for each n and $x_\infty = \gamma(0)$.

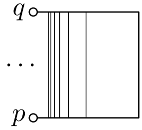
It follows that

$$\begin{aligned} \|x_\infty - x_n\| &\leqslant \operatorname{length} \gamma|_{[0,\frac{1}{2^n}]} \leqslant \\ &\leqslant \tfrac{1}{2^n}. \end{aligned}$$

In particular x_n converges.

Source: [**petrunin-stadler**].

1.9.9. Consider the following subset of $\mathbb{R}^2$ equipped with the induced length metric



$$\mathcal{X} = \big((0,1] \times \{0,1\}\big) \cup \big(\{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\} \times [0,1]\big)$$

Note that $\mathcal{X}$ is locally compact and geodesic.

Its completion $\bar{\mathcal{X}}$ is isometric to the closure of $\mathcal{X}$ equipped with the induced length metric. Note that $\bar{\mathcal{X}}$ is obtained from $\mathcal{X}$ by adding two points $p = (0,0)$ and $q = (0,1)$.

Observe that the point p admits no compact neighborhood in $\bar{\mathcal{X}}$ and there is no geodesic connecting p to q in $\bar{\mathcal{X}}$.

Source: [**bridson-haefliger**].

2.1.3. By Frechet lemma (2.1.1) we can identify $\mathcal{K}$ with a compact subset of ℓ^∞.

Denote by $\mathcal{L} = \operatorname{Conv} \mathcal{K}$ — it is defined as the minimal convex closed set in ℓ^∞ that contains $\mathcal{K}$. (In other words, $\mathcal{L}$ is the intersection of all convex closed sets that contain $\mathcal{K}$.) Observe that $\mathcal{L}$ is a length space.

It remains to show that $\mathcal{L}$ is compact.

By construction $\mathcal{L}$ is closed subset of ℓ^∞, in particular it is a complete space. By 1.4.1d, it remains to show that $\mathcal{L}$ is totally bounded.

Recall that Minkowski sum $A+B$ of two sets A and B in a vector space is defined by

$$A+B=\{\,a+b\ :\ a\in A,\ b\in B\,\}.$$

Observe that Minkowski sum of two convex sets is convex.

Denote by $\bar{B}_\varepsilon$ the closed ε-ball in ℓ^∞ centered at the origin. Choose a finite ε-net N in $\mathcal{K}$ for some $\varepsilon>0$. Note that $P=\operatorname{Conv}N$ is a convex polyhedron, in particular $\operatorname{Conv}N$ is compact.

Observe that $N+\bar{B}_\varepsilon$ is closed ε-neighborhood of N. It follows that $N+\bar{B}_\varepsilon\supset K$ and therefore $P+\bar{B}_\varepsilon\supset\mathcal{L}$. In particular P is a $2{\cdot}\varepsilon$-net in $\mathcal{L}$; since P is compact and $\varepsilon>0$ is arbitrary, $\mathcal{L}$ is totally bounded (see 1.4.2).

Remark. Another solution follows since injective envelop of compact space is compact; see 3.2.2*b*, 3.3.1, and 3.3.3.

2.3.2. Choose a separable space $\mathcal{X}$ that has infinite number of geodesics between a pair of points, say a square will ℓ^∞-metric in $\mathbb{R}^2$. Apply to $\mathcal{X}$ universality of Urysohn space (2.3.1).

2.3.3. First let us prove the following claim:

◇ Suppose $f\colon K\to\mathbb{R}$ is an extension function defined on a compact subset K of the Urysohn space $\mathcal{U}$. Then there is a point $p\in\mathcal{U}$ such that $|p-x|=f(x)$ for any $x\in K$.

Without loss of generality we may assume that $f(x)>0$ for any $x\in K$. Since K is compact, we may fix $\varepsilon>0$ such that $f(x)>\varepsilon$.

Consider the sequenc $\varepsilon_n=\frac{\varepsilon}{100\cdot 2^n}$. Choose a sequence of ε_n-nets $N_n\subset K$. Applying universality of $\mathcal{U}$ recursively, we may choose a point p_n such that $|p_n-x|=f(x)$ for any $x\in N_n$ and $|p_n-p_{n-1}|=$ $=10{\cdot}\varepsilon_{n-1}$. Observe that the sequence (p_n) is Cauchy and its limit p meets $|p-x|=f(x)$ for any $x\in K$.

Now, choose a sequence of points (x_n) in $\mathcal{S}$. Applying the claim, we may extend the map from K to $K\cup\{x_1\}$, and further to $K\cup\{x_1,x_2\}$, and so on. As a result we extend the distance-preserving map f to whole sequence (x_n). It remains to extend it continuisly to whole space $\mathcal{S}$.

2.3.5. It is sufficient to show that any compact subspace $\mathcal{K}$ of Urysohn space can be contracted to a point.

Note that any compact space $\mathcal{K}$ can be extended to a contractible compact space $\mathcal{K}'$; for example we may embed $\mathcal{K}$ into ℓ^∞ and pass to its convex hull, as it was done in 2.1.3.

By 2.4.4, there is an isometric embedding of $\mathcal{K}'$ that agrees with inclusion of $\mathcal{K}$. Since $\mathcal{K}$ is contractible in $\mathcal{K}'$, it is contractible in $\mathcal{U}$.

A better way. One can contract whole Urysohn space using the following construction.

Note that points in the space $\mathcal{X}_\infty$ constructed in the proof of 2.2.2 can be multiplied number $t \in [0,1]$ — simply multiply each function by t. That defines a map

$$\lambda_t \colon \mathcal{X}_\infty \to \mathcal{X}_\infty$$

that scales all distances by factor t. The map λ_t can be extended to the completion of $\mathcal{X}_\infty$, which is isometic to $\mathcal{U}_d$ (or $\mathcal{U}$).

Observe that the map λ_1 is the identity and λ_0 maps whole space to a single point, say x_0 — this is the only point of $\mathcal{X}_0$. Further note that the map $(t,p) \mapsto \lambda_t(p)$ is continuous — in particular $\mathcal{U}_d$ and $\mathcal{U}$ are contractible.

As a bonus, observe that for any point $p \in \mathcal{U}_d$ the curve $t \mapsto \lambda_t(p)$ is a geodesic path from p to x_0.

Source: [**gromov-2007**].

2.4.3. Observe that S_r satisfies the definition of d-Urysohn space and apply the uniqueness (2.4.1).

3.1.2. Note that if $c < 0$, then $r > s$. The latter is impossible since r is extremal and s is admissible.

Observe that the function $\bar r = \min\{\, r, s + c\}$ is admissible. Indeed if $\bar r(x) = r(x)$ and $\bar r(y) = r(y)$ then

$$\bar r(x) + \bar r(y) = r(x) + r(y) \geqslant |x - y|.$$

Further if $\bar r(x) = s(x) + c$ then

$$\begin{aligned}\bar r(x) + \bar r(y) &\geqslant [s(x) + c] + [s(y) - c] = \\ &= s(x) + s(y) \geqslant \\ &\geqslant |x - y|.\end{aligned}$$

Since r is extremal, we have $r = \bar r$; that is, $r \leqslant s + c$.

3.2.2. Choose an injective space $\mathcal{Y}$.

(a). Fix a Cauchy sequence (x_n) in $\mathcal{Y}$; we need to show that it has a limit $x_\infty \in \mathcal{Y}$. Consider metric on $\mathcal{X} = \mathbb{N} \cup \{\infty\}$ defined by

$$\begin{aligned}|m - n|_{\mathcal{X}} &= |x_m - x_n|_{\mathcal{Y}}, \\ |m - \infty|_{\mathcal{X}} &= \lim_{n\to\infty} |x_m - x_n|_{\mathcal{Y}}.\end{aligned}$$

Since the sequence is Cauchy, so is the sequence $\ell_n = |p - x_n|_{\mathcal{Y}}$. Therefore the last limit is defined.

By construction the map $n \mapsto x_n$ is distance-preserving on $\mathbb{N} \subset \mathcal{X}$. Since $\mathcal{Y}$ is injective, this map can be exteneded to ∞ as a short map; set $\infty \mapsto x_\infty$. Since $|x_n - x_\infty|_{\mathcal{Y}} \leqslant |n - \infty|_{\mathcal{X}}$ and $|n - \infty|_{\mathcal{X}} \to 0$, we get that $x_n \to x_\infty$ as $n \to \infty$.

(b). Applying the definition of injective space, we get a midpoint for any pair of points in $\mathcal{Y}$. By *(a)*, $\mathcal{Y}$ is a complete space. It remains to apply 1.9.5b.

(c). Let $k\colon \mathcal{Y} \hookrightarrow \ell^\infty(\mathcal{Y})$ be the Kuratowski embedding (2.1.4). Observe that $\ell^\infty(\mathcal{Y})$ is contractible; in particular, there is a homotopy $k_t\colon \mathcal{Y} \hookrightarrow \ell^\infty(\mathcal{Y})$ such that $k_0 = k$ and k_1 is a constant map. (In fact one can take $k_t = (1-t)\cdot k$.)

Since k is distance-preserving and $\mathcal{Y}$ is injective, there is a short map $f\colon \ell^\infty(\mathcal{Y}) \to \mathcal{Y}$ such that the composition $f \circ k$ is the identity map on $\mathcal{Y}$. The composition $f \circ k_t\colon \mathcal{Y} \hookrightarrow \mathcal{Y}$ is a needed homotopy.

3.2.3. Suppose that a short map $f\colon A \to \mathcal{Y}$ is defined on a subset A of a metric space $\mathcal{X}$. We need to construct a short extension F of f.

(a). Suppose $\mathcal{Y} = \mathbb{R}$. Without loss of generality, we may assume that $A \neq \varnothing$, otherwise map whole $\mathcal{X}$ to a single point. Set

$$F(x) = \inf\{\, f(a) - |a - x| \;:\; a \in A \,\}.$$

Observe that F is short and $F(a) = f(a)$ for any $a \in A$.

(b). Suppose $\mathcal{Y}$ is a complete metric tree. Fix points $p \in \mathcal{X}$ and $q \in \mathcal{Y}$. Given a point $a \in A$, let $x_a \in \overline{\mathrm{B}}[f(a), |a-p|]$ be the point closest to $f(x)$. Note that $x_a \in [q\, f(a)]$ and either $x_a = q$ or x_a lies on distance $|a - p|$ from $f(a)$.

Note that the geodesics $[q\, x_a]$ are nested; that is, for any $a, b \in A$ we have either $[q\, x_a] \subset [q\, x_b]$ or $[q\, x_b] \subset [q\, x_a]$. Moreover, in the first case we have $|x_b - f(a)| \leqslant |p - a|$ and in the second $|x_a - f(b)| \leqslant |p - b|$.

It follows that the closure of the union of all geodesics $[q\, x_a]$ for $a \in \mathcal{A}$ is a geodesic. Denote by x its endpoint; it exists since $\mathcal{Y}$ is complete. It remains to observe that $|x - f(a)| \leqslant |p - a|$ for any $a \in \mathcal{A}$; that is, one can take $f(p) = x$.

3.2.4. Choose three points $x, y, z \in \mathcal{X}$ and set $\mathcal{A} = \{x, z\}$. Let $f\colon \mathcal{A} \to \mathcal{Y}$ be an isometry. Then $F(y) = f(x)$ or $F(y) = f(z)$. If $f(y) = f(x)$, then

$$\begin{aligned}|y - z|_{\mathcal{X}} \geqslant |F(y) - f(z)|_{\mathcal{Y}} &= \\ &= |x - z|_{\mathcal{X}}.\end{aligned}$$

Analogously if $f(y) = f(z)$, then $|x - y|_{\mathcal{X}} \geqslant |x - z|_{\mathcal{X}}$.

It remains to observe that the strong triangle inequality holds in both cases.

(c). In this case $\mathcal{Y} = (\mathbb{R}^2, \ell^\infty)$. Note that the map $\mathcal{X} \to (\mathbb{R}^2, \ell^\infty)$ is short if and only if both of its coordinate projections are short. It remains to apply *(a)*.

3.3.2; *(a).* Let f be an extremal function. Observe that at least two of the numbers $f(a) + f(b)$, $f(b) + f(c)$, and $f(c) + f(a)$ are 1. It follows that for some $x \in [0, \frac{1}{2}]$, we have

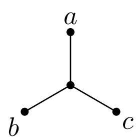
$$f(a) = 1 \pm x, \qquad f(b) = 1 \pm x, \qquad f(c) = 1 \pm x,$$

where we have one "minus" and two "pluses" in these three formulas.

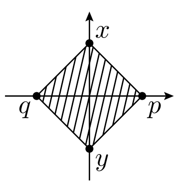
Suppose that

$$g(a) = 1 \pm y, \qquad g(b) = 1 \pm y, \qquad g(c) = 1 \pm y$$

is another extramal function. Then $|f - g| = |x - y|$ if g has "minus" at the same place as f and $|f - g| = |x + y|$ otherwise.



It follows that $\operatorname{Inj}\mathcal{X}$ is isometric to a tripod; that is, $\operatorname{Inj}\mathcal{X}$ is formed by three segments of length $\frac{1}{2}$ glued at one end.



(b). Assume f is an extramal function. Observe that $f(x) + f(y) = f(p) + f(q) = 2$; in particular, two values $a = f(x) - 1$ and $b = f(p) - 1$ completely describe the function f. Since f is extremal, we also have that

$$(1 \pm a) + (1 \pm b) \geqslant 1$$

for all 4 choices of signs; that is, $|a| + |b| \leqslant 1$.

It follows that $\operatorname{Inj}\mathcal{X}$ is isometric to the rhombus $|a| + |b| \leqslant 1$ in the (a, b)-plane with the metric induced by the ℓ^∞-norm.

3.3.5. Recall that

$$|f - g|_{\operatorname{Inj}\mathcal{X}} = \sup\{\, |f(x) - g(x)| \,:\, x \in \mathcal{X} \,\}$$

and

$$|f - p|_{\operatorname{Inj}\mathcal{X}} = f(p)$$

for any $f, g \in \operatorname{Inj}\mathcal{X}$ and $p \in \mathcal{X}$.

Since $\mathcal{X}$ is compact we can find a point $p \in \mathcal{X}$ such that

$$|f - g|_{\operatorname{Inj}\mathcal{X}} = |f(p) - g(p)| = \big||f - p|_{\operatorname{Inj}\mathcal{X}} - |g - p|_{\operatorname{Inj}\mathcal{X}}\big|.$$

Without loss of generality we may assume that

$$|f - p|_{\operatorname{Inj}\mathcal{X}} = |g - p|_{\operatorname{Inj}\mathcal{X}} + |f - g|_{\operatorname{Inj}\mathcal{X}}.$$

Applying 3.1.3c, we can find a point $q \in \mathcal{X}$ such that

$$|q-p|_{\mathrm{Inj}\,\mathcal{X}} = |f-p|_{\mathrm{Inj}\,\mathcal{X}} + |f-q|_{\mathrm{Inj}\,\mathcal{X}},$$

whence the result.

Since $\mathrm{Inj}\,\mathcal{X}$ is injective (3.3.3), by 3.2.2b it has to be geodesic. It remains to note that the concatenation of geodesics $[pq]$, $[gf]$, and $[fq]$ forms a required geodesic $[pq]$.

4.1.7; *(a).* Denote by X_r the r neighborhood of a set $X \subset \mathbb{R}^2$. Observe that

$$(\mathrm{Conv}\,X)_r = \mathrm{Conv}(X_r),$$

and try to use it.

(b). The answer is "no" in both parts.

For the first part let X be a unit disc and Y a finite ε-net in X. Evidently $|X-Y|_{\mathcal{H}(\mathbb{R}^2)} < \varepsilon$, but $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} \approx 1$.

For the second part take X to be a unit disc and $Y = \partial X$ to be its boundary circle. Note that $\partial X = \partial Y$ in particular $|\partial X - \partial Y|_{\mathcal{H}(\mathbb{R}^2)} = 0$ while $|X-Y|_{\mathcal{H}(\mathbb{R}^2)} = 1$.

Remark. A more interesting example for the second part can be build on *lakes of Wada* — and example of three open bounded topological disks in the plane that have identical boundary.

4.1.8. Let A be a compact convex set in the plane. Denote by A^r the closed r-neighborhood of A. Recall that by Steiner's formula we have

$$\mathrm{area}\,A^r = \mathrm{area}\,A + r\cdot \mathrm{perim}\,A + \pi\cdot r^2.$$

Taking derivative and applying coarea formula, we get

$$\mathrm{perim}\,A^r = \mathrm{perim}\,A + 2\cdot\pi\cdot r.$$

Observe that if A lies in a compact set B bounded by a colsed curve, then

$$\mathrm{perim}\,A \leqslant \mathrm{perim}\,B.$$

Indeed the closest-point projection $\mathbb{R}^2 \to A$ is short and it maps ∂B onto ∂A.

It remains to observe that if $A_n \to A_\infty$, then for any $r > 0$ we have that

$$A^r_\infty \supset A_n \quad \text{and} \quad A_\infty \subset A^r_n$$

for all large n.

5.4.4a. In order to check that $|{*}-{*}|_{\mathcal{M}'}$ is a metric, it is sufficient to show that

$$|\mathcal{X}-\mathcal{Y}|_{\mathcal{M}'} = 0 \quad \Longrightarrow \quad \mathcal{X} \stackrel{iso}{=\!=} \mathcal{Y};$$

the remaining conditions are trivial.

If $|\mathcal{X} - \mathcal{Y}|_{\mathcal{M}'} = 0$, then there is a sequence of maps $f_n\colon \mathcal{X} \to \mathcal{Y}$ such that

$$|f_n(x) - f_n(x')|_{\mathcal{Y}} \geqslant |x - x'|_{\mathcal{X}} - \tfrac{1}{n}.$$

Choose a countable dense set S in $\mathcal{X}$. Passing to a subsequence of f_n we can assume that $f_n(x)$ converges for any $x \in S$ as $n \to \infty$; denote its limit by $f_\infty(x)$.

For each point $x \in \mathcal{X}$ choose a sequence $x_m \in S$ converging to x. Since $\mathcal{Y}$ is compact, we can assume in addition that $y_m = f_\infty(x_m)$ converges in $\mathcal{Y}$. Set $f_\infty(x) = y$. Note that the map $f_\infty\colon \mathcal{X} \to \mathcal{Y}$ is distance-nondecreasing.

The same way we can construct a distance-nondecreasing map $g_\infty\colon \mathcal{Y} \to \mathcal{X}$.

By 1.4.4, the compositions $f_\infty \circ g_\infty\colon \mathcal{Y} \to \mathcal{Y}$ and $g_\infty \circ f_\infty\colon \mathcal{X} \to \mathcal{X}$ are isometrises. Therefore f_∞ and g_∞ are isometries as well.

5.5.1. Choose a space $\mathcal{X}$ in $\mathcal{Q}(C, D)$, denote a C-doubling measure by μ. Without loss of generality we may assume that $\mu(\mathcal{X}) = 1$.

The doubling condition implies that

$$\mu[\mathrm{B}(p, \tfrac{D}{2^n})] \geqslant \tfrac{1}{C^n}$$

for any point $x \in \mathcal{X}$. It follows that

$$\operatorname{pack}_{\frac{D}{2^n}} \mathcal{X} \leqslant C^n.$$

By 1.4.3, for any $\varepsilon \geqslant \frac{D}{2^{n-1}}$, the space $\mathcal{X}$ admits an ε-net with at most C^n points. Whence $\mathcal{Q}(C, D)$ is uniformly totally bounded.

5.5.2. Since $\mathcal{Y}$ is compact, it has a finite ε-net for any $\varepsilon > 0$. For each $\varepsilon > 0$ choose a finite ε-net $\{y_1, \dots, y_{n_\varepsilon}\}$ in $\mathcal{Y}$.

Suppose $f\colon \mathcal{X} \to \mathcal{Y}$ be a distance-nondecreasing map. Choose one point x_i in each nonempty subset $B_i = f^{-1}[\mathrm{B}(y_i, \varepsilon)]$. Note that the subset B_i has diameter at most $2\cdot\varepsilon$ and

$$\mathcal{X} = \bigcup_i B_i.$$

Therefore the set of points $\{x_i\}$ forms a $2\cdot\varepsilon$ net in $\mathcal{X}$. Whence *(a)* follows.

(b). Let $\mathcal{Q}$ be a uniformly totally bounded family of spaces. Suppose that each space in $\mathcal{Q}$ has an $\frac{1}{2^n}$-net with at most M_n points; we may assume that $M_0 = 1$.

Consider the space $\mathcal{Y}$ of all infinite integer sequences $m_0, m_1, \dots$ such that $1 \leqslant m_n \leqslant M_n$ for any n. Given two sequences (ℓ_n), and (m_n) of points in $\mathcal{Y}$, set

$$|(\ell_n) - (m_n)|_{\mathcal{Y}} = \tfrac{C}{2^n},$$

where n is minimal index such that $\ell_n \neq m_n$ and C is a positive constant.

Observe that $\mathcal{Y}$ is compact. Indeed it is complete and the sequences constant starting from index n form a finite $\frac{C}{2^n}$-net in $\mathcal{Y}$.

Given a space $\mathcal{X}$ in $\mathcal{Q}$, choose a sequnce of $\frac{1}{2^n}$ nets $N_n \subset \mathcal{X}$ for each natural n. We can assume that $|N_n| \leqslant M_n$; let us enumerate the points in N_n by $\{1, \dots, M_n\}$. Consider the map $f : \mathcal{X} \to \mathcal{Y}$ defined by $f : x \to (m_1(x), m_2(x), \dots)$ where $m_n(x)$ is a number of the point in N_n that lies on the distance $< \frac{1}{2^n}$ from x.

If $\frac{1}{2^{n-2}} \geqslant |x - x'|_{\mathcal{X}} > \frac{1}{2^{n-1}}$, then $m_n(x) \neq m_n(x')$. It follows that $|f(x) - f(x')|_{\mathcal{Y}} \geqslant \frac{C}{2^n}$. In particular, if $C > 10$, then

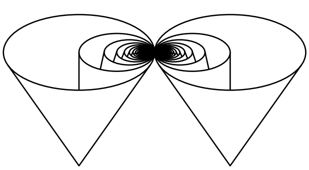
$$|f(x) - f(x')|_{\mathcal{Y}} \geqslant |x - x'|_{\mathcal{X}}$$

for any $x, x' \in \mathcal{X}$. That is, f is a distance-nondecreasing map $\mathcal{X} \to \mathcal{Y}$.

5.6.3, *(a).* Suppose $\mathcal{X}_n \xrightarrow{\text{GH}} \mathcal{X}$ and $\mathcal{X}_n$ are simply connected length metric space. It is sufficient to show that any nontrivial covering map $f : \tilde{\mathcal{X}} \to \mathcal{X}$ corresponds to a nontrivial covering map $f_n : \tilde{\mathcal{X}}_n \to \mathcal{X}_n$ for large n.

The latter can be constructed by covering $\mathcal{X}_n$ by small balls that lie close to sets in $\mathcal{X}$ evenly covered by f, prepare few copies of these sets and glue them the same way as the inverse images of the evenly covered sets in $\mathcal{X}$ glued to obtain $\tilde{\mathcal{X}}$.

(b). Let $\mathcal{V}$ be a cone over Hawaiian earring. Consider the *doubled cone* $\mathcal{W}$ — two copies of $\mathcal{V}$ with glued base points earrings (see the diagram).



The space $\mathcal{W}$ can be equipped with length metric for example the induced length metric from the shown embedding.

Note that $\mathcal{V}$ is simply connected, but $\mathcal{W}$ is not — it is a good exercise in topology.

If we delete from the earrings all small circles, then the obtained double cone becomes simply connected and it remains to be close to $\mathcal{W}$ in the sense of Gromov–Hausdorff.

Remark. Note that from part *(b)*, the limit does not admit a nontrivial covering. So if we define fundamental group right — as the inverse image of groups of deck transformations for all its coverings, then one may say that Gromov–Haudorff limit of simply connected length spaces is simply connected.

5.6.4, *(a).* Suppose that a metric on $\mathbb{S}^2$ is close to the disc $\mathbb{D}^2$. Note that $\mathbb{S}^2$ contains a circle γ that is close to the boundary curve of $\mathbb{D}^2$. By Jordan curve theorem, γ divides $\mathbb{S}^2$ into two discs, say D_1 and D_2.

By 5.6.3b, the Gromov–Hausdorff limit of D_1 and D_2 have to contain whole $\mathbb{D}^2$, otherwise the limit would admit a nontrivial covering. Consider points $p_1 \in D_1$ and $p_2 \in D_2$ that a close to the center of $\mathbb{D}^2$. On one hand the distance $|p_1 - p_2|_n$ have to be very small. On the other hand, any curve from p_1 to p_2 must cross γ, so it has legnth about 2 at least — a contradiction.

(b). Make fine burrows in the standard 3-ball without changing its topology, but at the same time come sufficiently close to any point in the ball.

Consider the *doubling* of the obtained ball along its boundary; that is, two copies of the ball with identified corresponding points on their boundaries. The obtained space is homeomorphic to $\mathbb{S}^3$. Note that the burrows can be made so that the obtained space is sufficiently close to the original ball in the Gromov–Hausdorff metric.

Source: [**burago-burago-ivanov**].

6.4.1. Part *(a)* follows directly from the definitions. Further we consider $\mathcal{X}$ as a subset of $\mathcal{X}^\omega$.

(b). Suppose $\mathcal{X}$ compact. Given a sequence (x_n) in $\mathcal{X}$, denote its ω-limit in $\mathcal{X}^\omega$ by x^ω and its ω-limit in $\mathcal{X}$ by x_ω.

Observe that $x^\omega = \iota(x_\omega)$. Therefore ι is onto.

If $\mathcal{X}$ is not compact, we can choose a sequence (x_n) such that $|x_m - x_n| > \varepsilon$ for fixed $\varepsilon > 0$ and $m \neq n$. Observe that

$$\lim_{n\to\omega} |x_n - y|_{\mathcal{X}} \geqslant \tfrac{\varepsilon}{2}$$

for any $y \in \mathcal{X}$. It follows that x_ω lies on the distance at least $\frac{\varepsilon}{2}$ from $\mathcal{X}$.

(c). A sequence of points (x_n) in $\mathcal{X}$ will be called ω-bounded if there is a real constant C such that

$$|p - x_n|_{\mathcal{X}} \leqslant C$$

for ω-almost all n.

The same argument as in *(b)* shows that any ω-bounded sequence has its ω-limit in $\mathcal{X}$. Further if (x_n) is not ω-bounded, then

$$\lim_{n\to\omega} |p - x_n|_{\mathcal{X}} = \infty;$$

that is x_ω does not lie in the metric component of p in $\mathcal{X}^\omega$.

6.3.3. Observe that if a path γ in a metric tree from p to q pass thru a point x on distance ℓ from $[pq]$, then

❶ $$\operatorname{length}\gamma \geqslant |p-q| + 2{\cdot}\ell.$$

Suppose that $\mathcal{T}_n$ is a sequence of metric trees that ω-converges to $\mathcal{T}_\omega$. By 6.3.2, the space $\mathcal{T}_\omega$.

The uniqueness will follow from ❶. Indeed, if for a geodesic $[p_\omega q_\omega]$ there is another geodesic γ_ω connecting its ends, then it have to pass thru a point $x_\omega \notin [p_\omega q_\omega]$. Choose a sequences $p_n, q_n, x_n \in \mathcal{T}_n$ such that $p_n \to p_\omega$, $q_n \to q_\omega$, $x_n \to x_\omega$ and $n \to \omega$. Then

$$\begin{aligned} |p_\omega - q_\omega| &= \operatorname{length}\gamma \geqslant \lim_{n\to\omega}(|p_n - x_n| + |q_n - x_n|) \geqslant \\ &\geqslant \lim_{n\to\omega}(|p_n - q_n| + 2{\cdot}\ell_n) = \\ &= |p_\omega - q_\omega| + 2{\cdot}\ell_\omega. \end{aligned}$$

Since $x_\omega \notin [p_\omega q_\omega]$, we have that $\ell_\omega > 0$ — a contradiction.

To prove the last property consider sequence of centers of tripods m_n for points $x_n, y_n, z_n \in \mathcal{T}_n$ and observe that its ultralimit m_ω is a the ceter of tripod with ends at $x_\omega, y_\omega, z_\omega \in \mathcal{T}_\omega$.

7.2.2. Let us show that $\gamma \leqslant \alpha+\beta$; the rest of inequalities can be done the same way. Since $\gamma \leqslant \pi$, we may assume that $\alpha + \beta < \pi$.

Denote by γ_x, γ_y, and γ_z the geodesics $[px]$, $[py]$, and $[pz]$ parameterized from p by arc-length. By triangle inequality, for any $\varepsilon > 0$ and all sufficiently small $t, \tau, s \in \mathbb{R}_+$ we have

$$\begin{aligned} |\gamma_x(t) - \gamma_z(\tau)| &\leqslant |\gamma_x(t) - \gamma_y(s)| + |\gamma_y(s) - \gamma_z(\tau)| < \\ &< \sqrt{t^2 + s^2 - 2{\cdot}t{\cdot}s{\cdot}\cos(\alpha+\varepsilon)} + \\ &\quad + \sqrt{s^2 + \tau^2 - 2{\cdot}s{\cdot}\tau{\cdot}\cos(\beta+\varepsilon)} \leqslant \end{aligned}$$

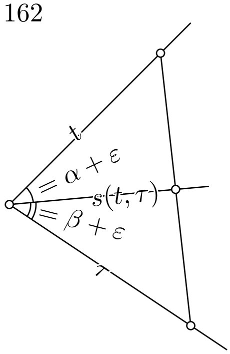
Below we define $s(t,\tau)$ so that for $s = s(t,\tau)$, this chain of inequalities can be continued as follows:

$$\leqslant \sqrt{t^2 + \tau^2 - 2{\cdot}t{\cdot}\tau{\cdot}\cos(\alpha+\beta+2{\cdot}\varepsilon)}.$$

Thus for any $\varepsilon > 0$,

$$\gamma \leqslant \alpha + \beta + 2{\cdot}\varepsilon.$$

Hence the result.



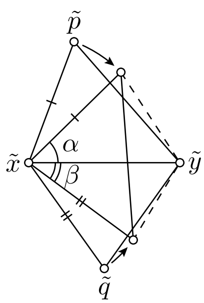
To define $s(t,\tau)$, consider three rays $\tilde\gamma_x$, $\tilde\gamma_y$, $\tilde\gamma_z$ on a Euclidean plane starting at one point, such that $\measuredangle(\tilde\gamma_x,\tilde\gamma_y)=\alpha+\varepsilon$, $\measuredangle(\tilde\gamma_y,\tilde\gamma_z)=\beta+\varepsilon$ and $\measuredangle(\tilde\gamma_x,\tilde\gamma_z)=\alpha+\beta+2{\cdot}\varepsilon$. We parametrize each ray by the distance from the starting point. Given two positive numbers $t,\tau\in\mathbb{R}_+$, let $s=s(t,\tau)$ be the number such that $\tilde\gamma_y(s)\in[\tilde\gamma_x(t)\ \tilde\gamma_z(\tau)]$. Clearly $s\leqslant\max\{t,\tau\}$, so t,τ,s may be taken sufficiently small.

Remark. Note that for the Euclidean space the statement implies that central angle defines a metric on unit sphere. This statement is not quite trivial; moreover, it is straightforward to modify Euclidean proof so it will work in Alexandrov settings.

7.3.1; *"only if" part.* Let us start with two model triangles $[\tilde x\tilde y\tilde p]=\tilde\triangle(xyp)$ and $[\tilde x\tilde y\tilde q]=\tilde\triangle(xyq)$ such that $\tilde p$ and $\tilde q$ lie on the opposite sides of the line $\tilde x\tilde y$.

Suppose $[\tilde x\tilde y]$ intersects $[\tilde p\tilde q]$ at a point $\tilde z$. In this case by CAT(0) comparison we have that

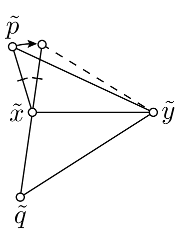
$$|\tilde p-\tilde q|_{\mathbb{E}^2}=|\tilde p-\tilde z|_{\mathbb{E}^2}-|\tilde z-\tilde q|_{\mathbb{E}^2}\leqslant|p-q|_{\mathcal{X}}.$$



Let us fix points $\tilde x$ and $\tilde y$, and the distances from $\tilde x$ to the remaining three points and reduce the angles $\alpha=\measuredangle[\tilde x\,{}^{\tilde p}_{\tilde y}]$ and $\beta=\measuredangle[\tilde x\,{}^{\tilde q}_{\tilde y}]$. It results in decreasing distances $|\tilde p-\tilde q|$, $|\tilde p-\tilde y|$, and $|\tilde q-\tilde y|$. If $\alpha=\beta=0$, then

$$\begin{aligned}|\tilde p-\tilde q|_{\mathbb{E}^2}&=\Big||\tilde x-\tilde p|_{\mathbb{E}^2}-|\tilde x-\tilde q|_{\mathbb{E}^2}\Big|=\\&=\Big||x-p|_{\mathcal{X}}-|x-q|_{\mathcal{X}}\Big|\geqslant\\&\geqslant|p-q|_{\mathcal{X}}.\end{aligned}$$

By the intermediate value theorem, there are intermediate values of α and β so that $|\tilde p-\tilde q|_{\mathbb{E}^2}=|p-q|_{\mathcal{X}}$. By construction, $|\tilde x-\tilde p|_{\mathbb{E}^2}=|x-p|_{\mathcal{X}}$, $|\tilde x-\tilde q|_{\mathbb{E}^2}=|x-q|_{\mathcal{X}}$, $|\tilde y-\tilde p|_{\mathbb{E}^2}\leqslant|y-p|_{\mathcal{X}}$, $|\tilde y-\tilde q|_{\mathbb{E}^2}\leqslant|y-q|_{\mathcal{X}}$.



Now suppose $[\tilde p\tilde q]$ does not intersect $[\tilde x\tilde y]$. Without loss of generality, we may assume that $[\tilde p\tilde q]$ crosses the line $\tilde x\tilde y$ behind $\tilde x$.

Let us rotate $\tilde p$ around $\tilde x$ so that $\tilde x$ will lie between $\tilde p$ and $\tilde q$. It will result in decreasing the distance $|\tilde p-\tilde y|$, by triangle inequality we have that

$$\begin{aligned}|\tilde p-\tilde q|_{\mathbb{E}^2}&=|\tilde p-\tilde x|_{\mathbb{E}^2}+|\tilde x-\tilde q|_{\mathbb{E}^2}=\\&=|p-x|_{\mathcal{X}}+|x-q|_{\mathcal{X}}\geqslant\\&\geqslant|p-q|_{\mathcal{X}}.\end{aligned}$$

Repeating the argument above produces the needed configuration.

"If" part. Suppose $\tilde p,\tilde q,\tilde x,\tilde y\in\mathbb{E}^2$ satisfies the conditions

$$\begin{aligned}|\tilde p-\tilde q|&=|p-q|, & |\tilde x-\tilde y|&=|x-y|,\\|\tilde p-\tilde x|&\leqslant|p-x|, & |\tilde p-\tilde y|&\leqslant|p-y|,\\|\tilde q-\tilde x|&\leqslant|q-x|, & |\tilde q-\tilde y|&\leqslant|q-y|.\end{aligned}$$

Fix $\tilde z\in[\tilde x\tilde y]$. By triangle inequality

$$|\tilde p-\tilde z|+|\tilde z-\tilde q|\geqslant|\tilde p-\tilde q|=|p-q|.$$

Note that if $|\tilde p'-\tilde x|\geqslant|\tilde p-\tilde x|$ and $|\tilde p'-\tilde y|\geqslant|\tilde p-\tilde y|$, then $|\tilde p'-\tilde z|\geqslant|\tilde p-\tilde z|$. In particular if $[\tilde x\tilde y\tilde p']=\tilde\triangle(xyp)$ and $[\tilde x\tilde y\tilde q']=\tilde\triangle(xyq)$, then

$$|\tilde p'-\tilde z|+|\tilde q'-\tilde z|\geqslant|\tilde p-\tilde z|+|\tilde z-\tilde q|.$$

Whence the "if" part follows.

7.3.2. Set $\tilde\alpha=\tilde\measuredangle(p\,{}^x_y)$, $\tilde\beta=\tilde\measuredangle(p\,{}^y_z)$ and $\tilde\gamma=\tilde\measuredangle(p\,{}^z_x)$.

If $\mathcal{X}$ is CBB(0), then

$$\tilde\alpha+\tilde\beta+\tilde\gamma\leqslant2\cdot\pi.$$

Note that we can find α,β,γ such that

$$\tilde\alpha\leqslant\alpha\leqslant\pi,\quad\tilde\beta\leqslant\beta\leqslant\pi,\quad\tilde\gamma\leqslant\gamma\leqslant\pi,$$

and

$$\alpha+\beta+\gamma=2\cdot\pi.$$

Consider a model configuration $\tilde p$, $\tilde x$, $\tilde y$, $\tilde z\in\mathbb{E}^2$ such that

$$\begin{aligned}|\tilde p-\tilde x|_{\mathbb{E}^2}&=|p-x|_{\mathcal{X}}, & |\tilde p-\tilde y|_{\mathbb{E}^2}&=|p-y|_{\mathcal{X}}, & |\tilde p-\tilde z|_{\mathbb{E}^2}&=|p-z|_{\mathcal{X}},\\\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}]&=\alpha, & \measuredangle[\tilde p\,{}^{\tilde y}_{\tilde z}]&=\beta, & \measuredangle[\tilde p\,{}^{\tilde z}_{\tilde x}]&=\gamma.\end{aligned}$$

Since increasing angle in a triangle increase the opposite side, we have

$$|x-y|_{\mathcal{X}}\leqslant|\tilde x-\tilde y|_{\mathbb{E}^2},\quad|y-z|_{\mathcal{X}}\leqslant|\tilde y-\tilde z|_{\mathbb{E}^2},\quad|z-x|_{\mathcal{X}}\leqslant|\tilde z-\tilde x|_{\mathbb{E}^2}.$$

Whence the "only-if" part follows.

Now suppse that we have a model configuration $\tilde p,\tilde x,\tilde y,\tilde z\in\mathbb{E}^2$ such that

$$\begin{aligned}&|p-x|_{\mathcal{X}}=|\tilde p-\tilde x|_{\mathbb{E}^2}, &&|p-y|_{\mathcal{X}}=|\tilde p-\tilde y|_{\mathbb{E}^2}, &&|p-z|_{\mathcal{X}}=|\tilde p-\tilde z|_{\mathbb{E}^2},\\ &|x-y|_{\mathcal{X}}\leqslant|\tilde x-\tilde y|_{\mathbb{E}^2}, &&|y-z|_{\mathcal{X}}\leqslant|\tilde y-\tilde z|_{\mathbb{E}^2}, &&|z-x|_{\mathcal{X}}\leqslant|\tilde z-\tilde x|_{\mathbb{E}^2}.\end{aligned}$$

Set

$$\alpha=\measuredangle[\tilde p\,{}^{\tilde x}_{\tilde y}],\qquad\beta=\measuredangle[\tilde p\,{}^{\tilde y}_{\tilde z}],\qquad\gamma=\measuredangle[\tilde p\,{}^{\tilde z}_{\tilde x}].$$

Observe that

$$\alpha+\beta+\gamma\leqslant 2\cdot\pi.$$

Since increasing a side in a triangle increase the opposite angle, we have that

$$\tilde\alpha\leqslant\alpha,\quad\tilde\beta\leqslant\beta,\quad\tilde\gamma\leqslant\gamma.$$

Whence the "if" part follows.

7.3.3. Set $\tilde\alpha=\tilde\measuredangle(p\,{}^{x}_{q})$, $\tilde\beta=\tilde\measuredangle(p\,{}^{y}_{q})$ and $\tilde\gamma=\tilde\measuredangle(p\,{}^{x}_{y})$.

Note that the quadruple p,x,y,z is euclidean if

$$\tilde\alpha+\tilde\beta+\tilde\gamma\leqslant 2\cdot\pi \qquad ❷$$

and the triple of numbers $\tilde\alpha,\tilde\beta,\tilde\gamma$ satisfies all triangle inequalities. Without loss of generality we may assume that $\tilde\alpha\leqslant\tilde\beta\leqslant\tilde\gamma$; in this case the triangle inequities hold if

$$\tilde\gamma\leqslant\tilde\alpha+\tilde\beta. \qquad ❸$$

Note that the inequality ❷ follow from CBB(0) comparison.

Consider two model triangles $[\tilde x\tilde y\tilde p]=\tilde\triangle(xyp)$ and $[\tilde x\tilde y\tilde q]=\tilde\triangle(xyq)$ such that $\tilde p$ and $\tilde q$ lie on the opposite sides of the line $\tilde x\tilde y$.

Suppose $[\tilde x\tilde y]$ intersects $[\tilde p\tilde q]$ at a point $\tilde z$. In this case by CAT(0) comparison we have that

$$|\tilde x-\tilde y|_{\mathbb{E}^2}=|\tilde x-\tilde z|_{\mathbb{E}^2}-|\tilde z-\tilde y|_{\mathbb{E}^2}\leqslant|x-y|_{\mathcal{X}}.$$

Which is equivalent to ❸.

If $[\tilde x\tilde y]$ crosses the line $[\tilde p\tilde q]$ behind $\tilde p$, then $\tilde\alpha+\tilde\beta>\pi$ and therefore ❸ follows from ❷.

Finally if $[\tilde x\tilde y]$ crosses the line $[\tilde p\tilde q]$ behind $\tilde q$, then by CBB(0) comparison with center at q, we have that

$$\tilde\measuredangle(q\,{}^{x}_{p})+\tilde\measuredangle(q\,{}^{y}_{p})+\tilde\measuredangle(q\,{}^{x}_{y})\leqslant 2\cdot\pi$$

It follows that
$$|\tilde{x}-\tilde{y}|_{\mathbb{E}^2}\geqslant|x-y|_{\mathcal{X}}$$
and therefore
$$\tilde{\gamma}\leqslant\measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{y}}].$$
Since $\measuredangle[\tilde{p}\,{}^{\tilde{x}}_{\tilde{y}}]=\tilde{\alpha}+\tilde{\beta}$ we get ❸.

7.4.2. We will use the charcterization of CBB(0) space provided by 7.3.2; the rest is nearly identical to the proof of 7.4.1.

Fix a quadruple in $\mathcal{U}\times\mathcal{V}$:
$$p=(p_1,p_2),\qquad x=(x_1,x_2),\qquad y=(y_1,y_2),\qquad z=(z_1,z_2).$$
For the quadruple p_1,x_1,y_1,z_1 in $\mathcal{U}$, construct model configurations $\tilde{p}_1,\tilde{x}_1,\tilde{y}_1,\tilde{z}_1$ in $\mathbb{E}^2$ provided by 7.3.2. Similarly, for the quadruple p_2,q_2,x_2,y_2 in $\mathcal{V}$ construct model configurations $\tilde{p}_2,\tilde{x}_2,\tilde{y}_2,\tilde{z}_2$ in $\mathbb{E}^2$

Consider four points in $\mathbb{E}^4=\mathbb{E}^2\times\mathbb{E}^2$
$$\tilde{p}=(\tilde{p}_1,\tilde{p}_2),\qquad \tilde{x}=(\tilde{x}_1,\tilde{x}_2),\qquad \tilde{y}=(\tilde{y}_1,\tilde{y}_2),\qquad \tilde{z}=(\tilde{z}_1,\tilde{z}_2).$$
The inequalities in 7.3.2 imply that
$$\begin{gathered}|p-x|_{\mathcal{X}}=|\tilde{p}-\tilde{x}|_{\mathbb{E}^4},\quad |p-y|_{\mathcal{X}}=|\tilde{p}-\tilde{y}|_{\mathbb{E}^4},\quad |p-z|_{\mathcal{X}}=|\tilde{p}-\tilde{z}|_{\mathbb{E}^4},\\ |x-y|_{\mathcal{X}}\leqslant|\tilde{x}-\tilde{y}|_{\mathbb{E}^4},\quad |y-z|_{\mathcal{X}}\leqslant|\tilde{y}-\tilde{z}|_{\mathbb{E}^4},\quad |z-x|_{\mathcal{X}}\leqslant|\tilde{z}-\tilde{x}|_{\mathbb{E}^4}\end{gathered}$$
It remains to observe that one can move $\tilde{z}$ into the plane of $\tilde{p}$, $\tilde{x}$, and $\tilde{y}$ keeping the distance $|\tilde{p}-\tilde{z}|_{\mathbb{E}^4}$ and nondecreasing the rest of distances.

7.5.2. Suppose that there are distinct geodesics. Then there are two points p and q on different geodesics such that $|p-x|=|q-x|$. Without loss of generality we may assume that $|z-x|<|p-x|$; in other words z lies between p and x on the first geodesic and z lies between q and x on the second geodesic. Observe that
$$\tilde{\measuredangle}(z\,{}^{x}_{p})=\tilde{\measuredangle}(z\,{}^{x}_{q})=\pi.$$
By comparison, we have
$$\tilde{\measuredangle}(z\,{}^{x}_{p})+\tilde{\measuredangle}(z\,{}^{x}_{q})+\tilde{\measuredangle}(z\,{}^{p}_{q})\leqslant2{\cdot}\pi.$$
It follows that $\tilde{\measuredangle}(z\,{}^{p}_{q})=0$. Since $|z-p|=|z-q|$, it implies that $p=q$ — a contradiction.

7.6.3. Use 7.6.2*a*, to show that the map $(t,x)\mapsto\gamma_x(t)$ is continuous; that is $h_t(x)=\gamma_x(t)$ defines a homotopy.

It remains to observe that $h_1(x)=x$ and $h_0(x)=p$ for any x.

7.6.4. Suppose that a geodesic $[pq]$ is not expendable behind q. Denote by h_t the geodesic homotopy with the center at p; see 7.6.3.

Since $[pq]$ is not extendable, $q \notin \operatorname{Im} h_t$ for any $t < 1$. In particular the local homology groups vanish at p; the latter does not hold for a manifold — a contradiction.

7.7.3. Apply 7.6.2*b* twice.

More precisely, consider a triangle $[xyz]$ in the space; let $[\tilde{x}\tilde{y}\tilde{z}] = \tilde{\triangle}(xyz)$. Choose points $p \in [xy]$ and $q \in [xz]$; consider the corresponding points $\tilde{p} \in [\tilde{x}\tilde{y}]$ and $\tilde{q} \in [\tilde{x}\tilde{z}]$. We need to show that

❹ $$|\tilde{p} - \tilde{q}|_{\mathbb{E}^2} \leqslant |p - q|_{\mathcal{X}}.$$

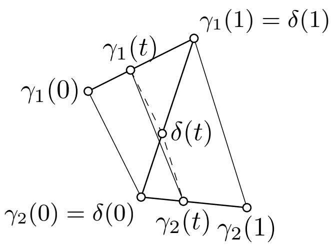
By 7.6.2*b*, we have

$$\tilde{\measuredangle}(x\,{}^{p}_{q}) \geqslant \tilde{\measuredangle}(x\,{}^{y}_{z}).$$

Whence ❹ follows.

7.7.4. It is sufficient to prove the Jensen inequality; that is,

$$|\gamma_1(t) - \gamma_2(t)| \leqslant (1-t)\cdot|\gamma_1(0) - \gamma_2(0)| + t\cdot|\gamma_1(1) - \gamma_2(1)|.$$



Let δ be the geodesic path from $\gamma_2(0)$ to $\gamma_1(1)$. From 7.6.2*a*, we have

$$|\gamma_1(t) - \delta(t)| \leqslant (1-t)\cdot|\gamma_1(0) - \delta(0)|$$
$$|\delta(t) - \gamma_2(t)| \leqslant t\cdot|\delta(1) - \gamma_2(1)|$$

It remains to sum it up and apply the triangle inequality.

Remark. Note that in the Euclidean space the proof is just as hard.

7.7.5. Let $p, q \in A_r$; that is, there are points $p^*, q^* \in A$ such that $|p - p^*|, |q - q^*| \leqslant r$. Consider a geodesic path γ from p to q and a geodesic path γ^* from p^* to q^*. Set $f(t) = |\gamma(t) - \gamma^*(t)|$.

Observe that $f(0), f(t) \leqslant r$. By 7.7.4, f is convex. Therefore $f(t) \leqslant r$ for any $t \in [0,1]$.

Since A is convex γ^* runs in A. Therefore $f(t) \geqslant \operatorname{dist}_A \circ \gamma(t)$; that is, γ runs in A_r.

7.7.6; *(a).* Assume there are two point $x, y \in K$ that minimize the distance to p; suppose $\ell = |p - x| = |p - y|$. Since K is convex, the geodesic $[xy]$ lies in K. Let m be a midpoint of $[xy]$.

Use thinness of $[pxy]$ to show that $|p - m| < \ell$. It follows that x does not minimize the distance to p — a contradiction.

(b). Let p^* and q^* be the closest point projections of p and q to K. Assume all four points p, q, p^*, q^* are distinct. Consider two model triangles $[\tilde{p}\tilde{p}^*\tilde{q}^*] = \tilde{\triangle}(pp^*q^*)$ and $[\tilde{p}\tilde{q}\tilde{q}^*] = \tilde{\triangle}(pqq^*)$ such that the points $\tilde{p}^*$ and $\tilde{q}$ lie on the opposite sides from the line $\tilde{p}\tilde{q}^*$.

Use thinness of $[pp^*q^*]$ and $[pqq^*]$ to show that $\measuredangle[\tilde{p} * {}^{\tilde{p}}_{\tilde{q}^*}] \geqslant \tfrac{\pi}{2}$ and $\measuredangle[\tilde{q} * {}^{\tilde{p}^*}_{\tilde{q}}] \geqslant \tfrac{\pi}{2}$. Finally observe that

$$|p - q|_{\mathcal{U}} = |\tilde{p} - \tilde{q}|_{\mathbb{E}^2} \geqslant |\tilde{p}^* - \tilde{q}^*|_{\mathbb{E}^2} = |p^* - q^*|_{\mathcal{U}}.$$

If some of the points p, q, p^*, q^* coincide, then the proof is easier.

7.7.7. Fix a closed, connected, locally convex set K.

Let us show that $f = \operatorname{dist}_K$ is convex in a neighborhood $\Omega \supset K$; that is, dist_K is convex along any geodesic completely contained in Ω. It is sufficient to show that for any a point $p \in K$ the function f is convex in a ball $B_p = \mathrm{B}(p, r_p)$ if $K \cap \overline{\mathrm{B}}[p, 2{\cdot}r_p]$ is convex.

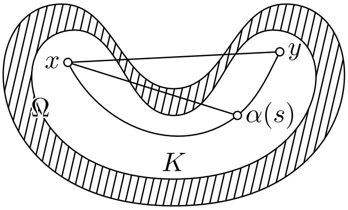
By 7.7.4 for any geodesic path γ_0 in B and any geodesic path γ_1 in K we have that the function $t \mapsto |\gamma_0(t) - \gamma_1(t)|$ is convex. We may choose γ_1 in such a way that its ends realize the distances from the ends of γ_0 to K; that is,

$$\begin{aligned} |\gamma_0(0) - \gamma_1(0)| &= f \circ \gamma_0(0), \\ |\gamma_0(1) - \gamma_1(1)| &= f \circ \gamma_0(1). \end{aligned}$$

Observe that

$$|\gamma_0(t) - \gamma_1(t)| \geqslant f \circ \gamma_0(t)$$

for any t. Whence Jensen's inequality holds for $f \circ \gamma$ if γ is any geodesic in B_p.



Since K is locally convex, it is locally path connected. Since K is connected, the latter implies that K is path connected.

Fix two points $x, y \in K$. Let us connect x to y by a path $\alpha\colon [0,1] \to K$. By 7.5.1 and 7.7.4, the geodesic $[x\,\alpha(s)]$ is uniquely defined and depends continuously on s.

If $[xy] = [x\,\alpha(1)]$ does not completely lie in K, then there is a value $s \in [0,1]$ such that $[x\,\alpha(s)]$ lies in Ω, but does not completely lie in K. Therefore f is convex along $[x\alpha(s)]$. Note that $f(x) = f(\alpha(s)) = 0$ and $f \geqslant 0$, therefore $f(z) = 0$ for any $z \in [x\,\alpha(s)]$. In other words, $[x\,\alpha(s)] \subset K$ — a contradiction.

Remark. The statement generalizes a theorem of Heinrich Tietze [**tietze**]; our proof is nearly identical to the original.

8.2.3. If A is not convex, then there is a geodesic $[xy]$ with the ends in A and the remaining points outside of A. Observe that in the doubling, say $\mathcal{W}$, two copies of this geodesics connect the same pair of points x and y. By 7.5.1, $\mathcal{W}$ is not CAT(0).

8.3.7. By approximation, it is sufficient to consider the case when A and B have smooth boundary.

If $[xy] \cap A \cap B \neq \varnothing$, then $z_0 \in [xy]$ and $\dot{A}, \dot{B}$ can be chosen to be arbitrary half-spaces containing A and B respectively.

In the remaining case $[xy] \cap A \cap B = \varnothing$, we have $z_0 \in \partial(A \cap B)$. Consider the solid ellipsoid

$$C = \{\, z \in \mathbb{E}^m \;:\; f(z) \leqslant f(z_0) \,\}.$$

Note that C is compact, convex and has smooth boundary.

Suppose $z_0 \in \partial A \cap \operatorname{Int} B$. Then A and C touch at z_0 and we can set $\dot{A}$ to be the uniquely defined supporting half-space to A at z_0 and $\dot{B}$ to be any half-space containing B. The case $z_0 \in \partial B \cap \operatorname{Int} A$ is treated similarly.

Finally, suppose $z_0 \in \partial A \cap \partial B$. Then the set $\dot{A}$ (respectively, $\dot{B}$) is defined as the unique supporting half-space to A (respectively, B) at z_0 containing A (respectively, B).

Suppose $f(z) < f(z_0)$ for some $z \in \dot{A} \cap \dot{B}$. Since f is concave, $f(\bar{z}) < f(z_0)$ for any $\bar{z} \in [zz_0[$. Since $[zz_0[\cap A \cap B \neq \varnothing$, the latter contradicts the fact that z_0 is minimum point of f on $A \cap B$.

8.4.1. Fix two open balls $B_1 = \mathrm{B}(0, r_1)$ and $B_2 = \mathrm{B}(0, r_2)$ such that

$$B_1 \subset A^i \subset B_2$$

for each wall A^i.

Suppose X is an intersections of the walls. Observe that

$$B_1 \subset X \subset B_2.$$

Therefore if $x \in X$, then X contains the convex hull $\operatorname{Conv}(B_1 \cup \{x\}$; therefore all intersections of the walls have ε-wide corners for $\varepsilon = 2 \cdot \arcsin \frac{r_1}{r_2}$.

8.4.2. Note that any centrally symmetric convex closed set in Euclidean space is a product of a compact centrally symmetric convex set and a subspace.

It follows that there is $R < \infty$ such that if X is an intersection of an arbitrary number of walls, then for any point $p \in X$ there is an isometry of X that moves p to a point in the ball $\mathrm{B}(0, R)$.

It remains to apply the argument in 8.4.1.

8.5.3. Note that we can assume that the balls have zero radiuses.

Observe that at each collision the balls exchange their velocities. Let us also change their labels at each collision. Note that after the relabeling, the coordinates functions $t \mapsto x_i(t)$ of the balls are linear functions in time.[1]

It remains to show n lines on the plane have at most $\frac{n\cdot(n-1)}{2}$ intersections. It follows since any pair of lines have at most one intersection.

Remarks. For nonidentical balls, the problem is a bit more interesting; Grant Sanderson has couple of funny movies on a partial case of this problem [**sanderson**].

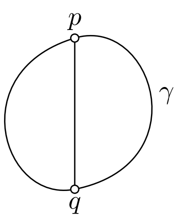
Recall that in the 3-dimensional case the number of collisions grows exponentially in n; the two-dimensional case is open [**burago-ivanov**].

9.2.5. Note that the existence of a null-homotopy is equivalent to the following. There are two one-parameter families of paths α_τ and β_τ, $\tau \in [0,1]$ such that:

◇ $\operatorname{length}\alpha_\tau$, $\operatorname{length}\beta_\tau < \pi$ for any τ.
◇ $\alpha_\tau(0) = \beta_\tau(0)$ and $\alpha_\tau(1) = \beta_\tau(1)$ for any τ.
◇ $\alpha_0(t) = \beta_0(t)$ for any t.
◇ $\alpha_1(t) = \alpha(t)$ and $\beta_1(t) = \beta(t)$ for any t.

By Corollary 9.2.3, the construction in Corollary 9.2.4 produces the same result for α_τ and β_τ. Hence the result.

9.3.5. The following proof works for compact locally simply connected metric spaces; it includes compact length, locally CAT(0) spaces.



By the globalization theorem there is a nontrivial homotopy class of closed curves.

Consider a shortest noncontractible closed curve γ in $\mathcal{X}$; note that such a curve exists.

Indeed, let L be the infimum of lengths of all noncontractible closed curves in $\mathcal{X}$. Compactness and local contractibility of $\mathcal{X}$ imply that any two sufficiently close closed curves in $\mathcal{X}$ are homotopic. Then choosing a sequence of unit speed noncontractible curves whose lengths converge to L, an Arzelá–Ascoli type of argument shows that these curves subconverge to a noncontractible curve of length L.

Assume that γ is not a geodesic circle, that is, there are two points p and q on γ such that the distance $|p-q|$ is shorter then the lengths of the arcs, say α_1 and α_2, of γ from p to q. Consider the products, say γ_1 and γ_2, of $[qp]$ with α_1 and α_2. Then

[1] We use here that radiuses vanish, otherwise $\tilde{x}_i = x_i - 2\cdot k_i\cdot r$ are linear, where k_i is the number of i-th ball counted from left.

◇ γ_1 or γ_2 is noncontractible,

◇ $\operatorname{length}\gamma_1, \operatorname{length}\gamma_2 < \operatorname{length}\gamma$,

a contradiction.

In the CAT(1) case we also have a geodesic circle. The proof is done nearly the same way, but we need to consider the homotopy classes of closed curves shorter than $2\cdot\pi$. One also need to apply 9.2.5, to show that curves γ_1 and γ_2 are not contractible in the class of curves shorter than $2\cdot\pi$.

Remarks. The statement of the exercise fails if the requirement that $\mathcal{X}$ be compact is replaced by the assumption that it is proper. For example, the surface of revolution of the graph of $y = e^x$ around the x-axis is locally CAT(0) but has no closed geodesics.

9.3.6. Consider a closed ε-neighborhood A of the geodesic. Note that A_ε is convex. By the Reshetnyak gluing theorem, the double $\mathcal{W}_\varepsilon$ of $\mathcal{U}$ along A_ε is CAT(0).

Consider the space $\mathcal{W}'_\varepsilon$ obtained by doubly covering $\mathcal{U}\backslash A_\varepsilon$ and gluing back A_ε.

Observe that $\mathcal{W}'_\varepsilon$ is locally isometric to $\mathcal{W}_\varepsilon$. That is, for any point $p' \in \mathcal{W}'_\varepsilon$ there is a point $p \in \mathcal{W}_\varepsilon$ such that the δ-neighborhood of p' is isometric to the δ-neighborhood of p for all small $\delta > 0$.

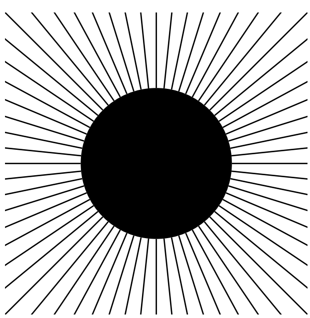
Further observe that $\mathcal{W}'_\varepsilon$ is simply connected since it admits a deformation retraction onto A_ε, which is contractible. By the globalization theorem, $\mathcal{W}'_\varepsilon$ is CAT(0).

It remains to note that $\tilde{\mathcal{U}}$ can be obtained as a limit of $\mathcal{W}'_\varepsilon$ as $\varepsilon \to 0$, and apply Proposition 7.3.4.

10.1.1. Recall that by Proposition 8.1.1, any local geodesic shorter in $\mathcal{U}$ is a geodesic.

Consider a sequence of directions ξ_n at p of geodesics $[pq_n]$. Since the geodesics are extendable, we can assume that the distances $|p - - q_n|_{\mathcal{U}} = 1$ for any n.

Since $\mathcal{U}$ is proper, we can pass to a converging subsequence of (q_n); denote its limit by q. Since $q_n \to q$, the comparison implies that $\measuredangle[p\,{}^{q_n}_{q}] \to 0$ as $n \to \infty$. Therefore the direction ξ of $[pq]$ is the limit of directions ξ_n.



Note that the unit disc in the plane with attached half-line to each point is a complete CAT(0) length space with extendable geodesics. However, the space of geodesic directions on the boundary of the disc is not complete — there is no geodesic tangent to the boundary of the disc. This provides a counterexample to the statement of the exercise if $\mathcal{U}$ is not assumed to be proper.

10.1.2. Given a constant speed geodesic α starting at p, consider sequence of points $x_n = \alpha(\tfrac{1}{n})$. Note that $n{\cdot}|p - x_n|$ is constant. Therefore if we consider x_n as a point in $n{\cdot}\mathcal{X}$, then this sequence has an ω-limit $\iota(\alpha)$ in T_p^ω.

Observe that ι defines a distance-preserving map $\mathrm{T}'_p \to \mathrm{T}_p^\omega$. Since T_p^ω is complete, this map can be extended to T_p. Whence the statement follows.

Since $\mathcal{X}$ is CAT(0), so is $n{\cdot}\mathcal{X}$, and by 7.3.4 so is $\mathrm{T}_p^\omega\mathcal{X}$. Since $\mathrm{T}_p\mathcal{X}$ is naturally isometric to a subspace of T_p^ω, we get that $\mathrm{T}_p\mathcal{X}$ is CAT(0) as well.

Remark. The ultratangent space might be larger than tangent space. For example, let Ш be a *comb* with a spine formed by a real line and a half-line (a tooth) attached to each point of the spine. Then for $p = 0$ on the spine, T_pШ is formed by three half-lines meeting at one point, while T_p^ωШ is isometric to Ш.

10.1.3. Observe that it is sufficient to show that the space of directions Σ_p is a π-*length space*; the latter means that the defining condition of length space holds for pairs of points on distance less than π.

Since Σ_p is complete, the same argument as in 1.9.5*a*, shows that it sufficient to prove existence of almost midpoints for pairs of point on distance less than π; that is, if $\measuredangle(\xi,\zeta) < \pi$, then, given $\varepsilon > 0$, there is $\mu \in \Sigma_p$ such that

❺ $$\measuredangle(\xi,\mu), \quad \measuredangle(\mu,\zeta) \leqslant \tfrac{1}{2}{\cdot}\measuredangle(\xi,\zeta) + \varepsilon.$$

Without loss of generality we may assume that ζ and ξ are geodesic directions; so there are geodesics $[px]$ and $[pz]$ that start from p in these directions; in particular, $\measuredangle[p\,{}^x_z] = \measuredangle(\xi,\zeta)$. Fix small $r > 0$ and choose points $\bar{x} = [px]$ and $\bar{z} = [pz]$ on the distance r from p. Since r is small, we can assume that
$$\measuredangle[p\,{}^x_z] + \varepsilon > \tilde\measuredangle(p\,{}^{\bar{x}}_{\bar{z}}).$$

Take a midpoint m of $[\bar{x}\bar{y}]$. By Alexandrov's lemma (7.6.1)
$$\tilde\measuredangle(p\,{}^{\bar{x}}_m), \quad \tilde\measuredangle(p\,{}^m_{\bar{z}}) \leqslant \tfrac{1}{2}{\cdot}\tilde\measuredangle(p\,{}^{\bar{x}}_{\bar{z}}).$$

By comparison
$$\tilde\measuredangle(p\,{}^{\bar{x}}_m) \geqslant \measuredangle[p\,{}^{\bar{x}}_m] \quad\text{and}\quad \tilde\measuredangle(p\,{}^{\bar{z}}_m) \geqslant \measuredangle[p\,{}^{\bar{z}}_m].$$

Whence ❺ holds for the direction μ of $[pm]$.

10.4.2. Assume $\mathcal{P}$ is not CAT(0). Then by 10.4.1, a link Σ of some simplex contains a closed geodesic α with length $4\cdot\ell < 2\cdot\pi$. We can assume that Σ has minimal possible dimension; so, by 10.4.1, Σ is locally CAT(1).

Divide α into two equal arcs α_1 and α_2.

Assume α_1 and α_2 are length minimizing; parameterize them by $[-\ell,\ell]$. Fix a small $\delta > 0$ and consider two curves in $\operatorname{Cone}\Sigma$ written in polar coordinates as

$$\gamma_i(t) = (\alpha_i(\arctan \tfrac{t}{\delta}), \sqrt{\delta^2 + t^2}).$$

Observe that both curves γ_1 and γ_2 are geodesics in $\operatorname{Cone}\Sigma$ and have common ends.

Observe that a small neighborhood of the tip of $\operatorname{Cone}\Sigma$ admits an isometric embedding into $\mathcal{P}$. Hence we can construct two geodesics γ_1 and γ_2 in $\mathcal{P}$ with common endpoints.

It remains to consider the case when α_1 (and therefore α_2) is not length minimizing.

Pass to its maximal length minimizing arc $\bar\alpha_1$ of α_1. Since Σ is locally CAT(1), 9.2.3 implies that there is another geodesic $\bar\alpha_2$ in Σ_p that shares endpoints with $\bar\alpha_1$. It remains to repeat the above construction for the pair $\bar\alpha_1$, $\bar\alpha_2$.

Remark. By 7.5.1, the given condition is a necessary and sufficient.

10.5.6. Use induction on the dimension to prove that if in a spherical simplex $\triangle$ every edge is at least $\frac{\pi}{2}$, then all dihedral angles of $\triangle$ are at least $\frac{\pi}{2}$.

The rest of the proof goes along the same lines as the proof of the flag condition (10.5.5). The only difference is that a geodesic may spend time *at least* π on each visit to Star_v.

Remark. Note that it is not sufficient to assume only that the all dihedral angles of the simplices are at least $\frac{\pi}{2}$. Indeed, the two-dimensional sphere with removed interior of a small rhombus is a spherical polyhedral space glued from four triangles with all the angles at least $\frac{\pi}{2}$. On the other hand the boundary of the rhombus is closed local geodesic in this space. Therefore the space cannot be CAT(1).

10.5.8. The space $\mathcal{T}_n$ has a natural cone structure with the vertex formed by the completely degenerate tree — all its edges have zero length.

Note that the space Σ over which the cone is taken comes naturally with a triangulation with all-right spherical simplicies. Each simplex corresponds to a combinatorics of a possibly degenerate tree.

Note that the link of any simplex of this triangulation satisfies the no-triangle condition (10.5.1). Indeed, fix a simplex $\triangle$ of the complex; suppose it is described by a possibly degenerate topological tree t. A triangle in the link of $\triangle$ can be described by three ways to resolve a degeneracy of t by adding one edge, such that (1) any pair of these resolutions can be done simultaneously, but (2) all three cannot be done simultaneously. Direct inspection shows that this is impossible.

Therefore, by Proposition 10.5.3 our complex is flag. It remains to apply the flag condition (10.5.5), and then 7.4.3.

11.1.2. If the complex $\mathcal{S}$ is flag, then its cubical analog $\square_{\mathcal{S}}$ is locally CAT(0) and therefore aspherical.

Assume now that the complex $\mathcal{S}$ is not flag. Extend it to a flag complex $\mathcal{T}$ by gluing a simplex in every clique (that is, a complete subgraph) of its one-skeleton.

Note that the cubical analog $\square_{\mathcal{S}}$ is a proper subcomplex in $\square_{\mathcal{T}}$. Since $\mathcal{T}$ is flag, $\tilde{\square}_{\mathcal{T}}$, the universal cover of $\square_{\mathcal{T}}$, is CAT(0). Let $\tilde{\square}_{\mathcal{S}}$ be the inverse image of $\square_{\mathcal{S}}$ in $\tilde{\square}_{\mathcal{T}}$.

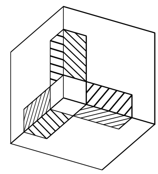
Choose a cube Q with minimal dimension in $\tilde{\square}_{\mathcal{T}}$ which is not present in $\tilde{\square}_{\mathcal{S}}$. By 7.7.7, Q is a convex set in $\tilde{\square}_{\mathcal{T}}$. The closest point projection $\tilde{\square}_{\mathcal{T}} \to Q$ is a retraction. It follows that the boundary ∂Q is not contractible in $\tilde{\square}_{\mathcal{T}} \backslash \operatorname{Int} Q$. Therefore the spheroid ∂Q is not contractible in $\tilde{\square}_{\mathcal{S}}$. That is, a covering of $\square_{\mathcal{S}}$ is not aspherical and therefore $\square_{\mathcal{S}}$ is not as well.

11.2.3. The solution goes along the same lines as the proof of Lemma 11.2.2, but few changes are needed.

The cycle γ is taken in the complement $\mathcal{S}\backslash\{v\}$ (or, alternatively, in the link of v in $\mathcal{S}$). Instead of a vertex, one has to take edge e in $\tilde{Q}$ that corresponds to v; so we show existence of large cycle in $\tilde{Q}$ that is not contractible in $\tilde{Q}\backslash e$. The last change is not principle: it is more visual to think that G is made from the squares parallel to the squares of the cubical complex which meet the edges of the complex orthogonally at their midpoints (in this case formally speaking G is not a subcomplex of the cubical analog).

11.3.2; *(b)⇒(a).* By 3.2.2*c*, Q is contractible. Therefore the globalization theorem and flag condition (9.3.1 and 10.5.5) imply that it is sufficient to show that each link in Q is flag. Further, by 10.5.3 it is sufficient to show that link of each cube in Q satisfies no-triangle condition.

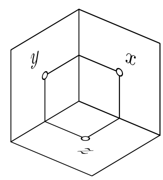
Arguing by contradiction, we can assume that no-triangle condition does not hold at a vertex v; that is, a zero-dimensional cube. In this case v is a vertex of there edges e_x, e_y, and e_z; each pair of edges belong to one of the squares s_x, s_y, and s_z with complementary index, but the squares s_x, s_y, s_z do not belong to one cube. For higher dimensional cubes we have a product of this configuration with a cube.



Let m_x, m_y and m_z be the midpoints of e_x, e_y, and e_z respectively. Consider 3 balls with centers m_x, m_y and m_z and radius $\frac{1}{4}$. Observe that each pair of balls have a common point; but all three together have no points of intersection. By 3.2.5c, the latter implies that (Q, ℓ^∞) is not an injective space — a contradiction.

(c)⇒(a). Observe that median point $m(x,y,z)$ of depends continuously on triple of points (x,y,z) and $m(x,x,y) = x$.

Given a loop $\gamma\colon [0,1] \to Q$ with base at $p = \gamma(0) = \gamma(1)$, consider the map $(a,b) \mapsto m(p, \gamma(a), \gamma(b))$ of the triangle $\triangle$ defined by $0 \leqslant a \leqslant$ $\leqslant b \leqslant 1$. Note that boundary of triangle runs along γ. It follows that γ is null homotopic and therefore Q is simply connected.



It remains to check that all links of Q satisfy no-triangle condition.

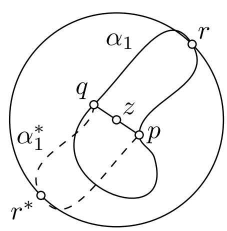
Assume that a link of Q does not satisfy the no-triangle condition. The same way as in the previous problem, we can assume that it is a link of a vertex; so we have a configuration of three squares s_x, s_y, and s_z, three edges e_x, e_y, and e_z, and one common vertex v as above. Observe that the centers x, y, and z of the squares s_x, s_y, and s_z. Observe that the geodesics $[xy]_{\ell^1}$, $[xz]_{\ell^1}$, and $[yz]_{\ell^1}$ are uniquely defined and they have no common point. It follows that the triple (x,y,z) does not have a median; that is, (Q, ℓ^1) is not a median space — a contradiction.

12.6.2. Let α be a closed curve in $\mathbb{S}^2$ of length $2{\cdot}\ell$.

Assume $\ell < \pi$. Let α_1 be a subarc of α of length ℓ, with endpoints p and q. Since $|p-q| \leqslant \ell < \pi$, there is a unique geodesic $[pq]$ in $\mathbb{S}^2$. Let z be the midpoint of $[pq]$.

We claim that α lies in the open hemisphere H centered at z.

Assume the contrary; that is, α meets the equator ∂H at a point r. Without loss of generality we may assume that $r \in \alpha_1$.



The arc α_1 together with its reflection α_1^* in z form a closed curve of length $2{\cdot}\ell$ which meets r and its antipodal point r'. Thus

$$\ell = \operatorname{length} \alpha_1 \geqslant |r - r'| = \pi$$

— a contradiction.

Solution with the Crofton formula. Let α be a closed curve in $\mathbb{S}^2$ of length $\leqslant 2{\cdot}\pi$. We wish to prove that α is contained in a hemisphere in $\mathbb{S}^2$. By approximation it suffices to prove this for smooth curves α of length $< 2{\cdot}\pi$ with transverse self-intersections.

Given $v \in \mathbb{S}^2$, denote by $v^\perp$ the equator in $\mathbb{S}^2$ with the pole at v. Further, $\#X$ will denote the number of points in the set X.

Obviously, if $\#(\alpha \cap v^\perp) = 0$, then α is contained in one of the hemispheres determined by $v^\perp$. Note that $\#(\alpha \cap v^\perp)$ is even for almost all v.

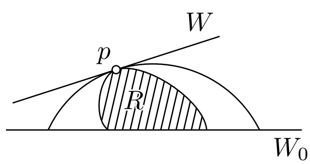
Therefore, if α does not lie in a hemisphere, then $\#(\alpha \cap v^\perp) \geqslant 2$ for almost all $v \in \mathbb{S}^2$.

By the Crofton formula we have that

$$\begin{aligned}\operatorname{length}(\alpha) &= \frac{1}{4}\cdot \int\limits_{\mathbb{S}^2} \#(\alpha \cap v^\perp)\cdot d_v \operatorname{area} \geqslant \\ &\geqslant 2{\cdot}\pi.\end{aligned}$$

12.6.3. Since Ω is not two-convex, we can choose a simple closed curve γ that lies in the intersection of a plane W_0 and Ω, and is contractible in Ω but not contractible in $\Omega \cap W_0$.

Let $\varphi\colon \mathbb{D} \to \Omega$ be a disc that shrinks γ. Applying the loop theorem (arguing as in the proof of Proposition 12.2.7), we can assume that φ is an embedding and $\varphi(\mathbb{D})$ lies on one side of W_0.



Let Q be the bounded closed domain cut from $\mathbb{E}^3$ by $\varphi(\mathbb{D})$ and W_0. By assumption it contains a point that is not in Ω. Changing W_0, γ and φ slightly, we can assume that such a point lies in the interior of Q.

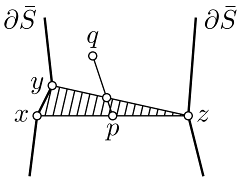
Fix a circle Γ in W_0 that surrounds $Q \cap W_0$. Since Q lies in a half-space with boundary W_0, there is a smallest spherical dome with boundary Γ that includes the set $R = Q\backslash\Omega$.

The dome has to touch R at some point p. The plane W tangent to the dome at p has the required property — the point p is an isolated point of the complement $W\backslash\Omega$. Further, by construction a small circle around p in W is contractible in Ω.

12.7.3. The proof is simple and visual, but it is hard to write it formally in a non-tedious way.

Consider the surface $\bar{S}$ formed by the closure of the remaining part S of the boundary. Note that the boundary ∂S of $\bar{S}$ is a collection of closed polygonal lines.

Assume $\bar{S}$ is not piecewise linear. Show that there is a line segment $[pq]$ in $\mathbb{E}^3$ that is tangent to $\bar{S}$ at some point p and has no common points with $\bar{S}$ except p.



Since $\bar{S}$ is locally concave, there is a local inner supporting plane Π at p that contains the segment $[pq]$.

Show that $\Pi\cap\bar{S}$ contains a segment $[xy] \ni \ni p$ with the ends in $\partial\bar{S}$. Denote by Π^+ the half-plane in Π that contains $[pq]$ and has $[xy]$ in its boundary.

Use the fact that $[pq]$ is tangent to S to show that there is a point $z \in \partial\bar{S}$ such that the line segment $[xz]$ or $[yz]$ lies in $\partial\bar{S} \cap \Pi^+$.

From the latter statement and local convexity of $\bar{S}$, it follows that the solid triangle $[xyz]$ lies in $\bar{S}$. In particular, all points on $[pq]$ sufficiently close to p lie in $\bar{S}$ — a contradiction.

1.2.3. Let us use the same notation as in the proof of 1.2.1.

Consider the map $s\colon x \mapsto (\mathrm{dist}_A(x), \mathrm{dist}_B(x))$. From the proof of 1.2.1 we get that $\operatorname{Im} s \supset \square$. Observe that in the case of equality we have that $\operatorname{Im} s = \square$. Indeed, the same argument shows that

$$\mathrm{vol}(s^{-1}(\square), g) \geqslant \mathrm{vol}\,\square = 1.$$

The set $s^{-1}(\mathbb{R}^1\backslash\square)$ is an open subset of $\square$. If it is nonempty, then it has positive volume. In this case

$$\mathrm{vol}(\square, g) > \mathrm{vol}(s^{-1}(\square), g) \geqslant 1$$

— a contradiction.

Summarizing above discussion, there is a geodesic path of g-length 1 connecting a point on one face of cube to the opposite face.

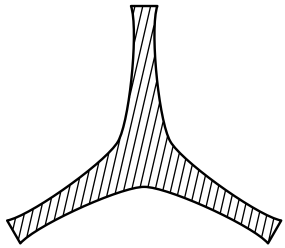
Moreover, for any pair of opposite faces and a point $p \in \square$, there is a geodesic path of g-length 1 from one face to the other that pass thru p. The latter can be shown by cutting $\square$ into two rectangles by a level surface of dist_A thru p, applying the above statement to both rectangles and taking the concatination of the obtained geodesic paths with end at p. (The level surface might cut a rectangle with some topology, so have to apply 1.2.2 instead of 1.2.1).

Let γ be such geodesic path from A to A'. Observe that $\gamma'(t) = = \nabla_{\gamma(t)}\mathrm{dist}_A$. Therefore dist_A is differentiable at every point $p \in \square$. It follows that the map s is differentiable.

Further checking the equality case in each inequality in the proof of 1.2.1, we get that s is a bijection and the equalities

$$|d_p\mathrm{dist}_A| = 1, \quad |d_p\mathrm{dist}_B| = 1, \quad \text{and} \quad \langle d_p\mathrm{dist}_A, d_p\mathrm{dist}_B\rangle = 0$$

hold for almost all $p \in \square$. Since $d_p\mathrm{dist}_A$ and $d_p\mathrm{dist}_B$ are well defined, we get that the equalities hold everywhere. That is s is an isometry.



1.2.4. Consider the hexagon with flat matric and curved sides shown on the diagram. Observe that its area can be made arbitrary small while keeping the distances from the opposite sides at least 1.

1.2.5. Without loss of generality, we may assume that V lies in a unit cube $\square$. Consider a noncontinuous metric tensor $\bar g$ on $\square$ that coincides with g inside V and with the canonical flat metric tensor outside of V.

Observe that the $\bar g$-distances between opposite faces of $\square$ are at least 1. Indeed this is true for the Euclidean metric and the assumption $|p-q|_g \geqslant |p-q|_{\mathbb{E}^d}$ guarantees that one cannot make a shortcut in V. Therefore the $\bar g$-distances between every pair of opposite faces is at least as large as 1 which is the Euclidean distance.

This metric tensor $\bar g$ is not continuous at Σ, but the same argument as in 1.2.1 can be applied to show that $\mathrm{vol}(\square, \bar g) \geqslant \mathrm{vol}\,\square$. Whence the statement follows.

1.2.6. Let $x \in \mathbb{S}^2$ be a point that minimize the distance $|x - x'|_g$. Consider a minimizing geodesic γ from x to x'. We can assume that

$$|x - x'|_g = \mathrm{length}\,\gamma = 1.$$

Let γ' be the antipodal arc to γ. Note that γ' intersects γ only at the common endpoints x and x'. Indeed, if $p' = q$ for some $p, q \in \gamma$, then $|p - q| \geqslant 1$. Since $\mathrm{length}\,\gamma = 1$, the points p and q must be the ends of γ.

It follows that γ together with γ' forms a closed simple curve in $\mathbb{S}^2$ that divides the sphere into two disks D and D'.

Let us divide γ into two equal arcs γ_1 and γ_2; each of length $\frac{1}{2}$. Suppose that $p, q \in \gamma_1$, then

$$\begin{aligned}|p - q'|_g &\geqslant |q - q'|_g - |p - q|_g \geqslant \\ &\geqslant 1 - \tfrac{1}{2} = \tfrac{1}{2}.\end{aligned}$$

That is, the minimal distance from γ_1 to γ_1' is at least $\frac{1}{2}$. The same way we get that the minimal distance from γ_2 to γ_2' is at least $\frac{1}{2}$. By Besicovitch inequality, we get that

$$\mathrm{area}(D, g) \geqslant \tfrac{1}{4} \quad \text{and} \quad \mathrm{area}(D', g) \geqslant \tfrac{1}{4}.$$

Therefore

$$\operatorname{area}(\mathbb{S}^2, g) \geqslant \tfrac{1}{2}.$$

A better estimate. Let us indicate how to improve the obtained bound to

$$\operatorname{area}(\mathbb{S}^2, g) \geqslant 1.$$

Suppose x, x', γ and γ' are as above. Consider the function

$$f(z) = \min_t \{\, |\gamma'(t) - z|_g + t \,\}.$$

Observe that f is 1-Lipschitz.

Show that two points $\gamma'(c)$ and $\gamma(1-c)$ lie on one connected component of the level set $L_c = \left\{\, z \in \mathbb{S}^2 \;:\; f(z) = c \,\right\}$; in particular

$$\operatorname{length} L_c \geqslant 2\cdot|\gamma'(c) - \gamma(1-c)|_g.$$

By the triangle inequality, we have that

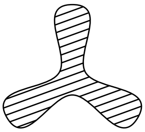
$$\begin{aligned}|\gamma'(c) - \gamma(1-c)|_g &\geqslant 1 - |\gamma(c) - \gamma(1-c)|_g = \\ &= 1 - |1 - 2\cdot c|.\end{aligned}$$

It remains to apply the coarea formula

$$\operatorname{area}(\mathbb{S}^2, g) \geqslant \int\limits_0^1 \operatorname{length} L_c \cdot dc.$$

Remarks. The bound $\frac{1}{2}$ was proved by Marcel Berger [**berger**]. Christopher Croke conjectured that the optimal bound is $\frac{4}{\pi}$ and the round sphere is the only space that achieves this [**croke**].

1.2.7. Given $\varepsilon > 0$, construct a disk Δ in the plane with



$$\operatorname{length} \partial\Delta < 10 \quad \text{and} \quad \operatorname{area} \Delta < \varepsilon$$

that admits an continuous involution ι such that

$$|\iota(x) - x| \geqslant 1$$

for any $x \in \partial\Delta$.

An example of Δ can be guessed from the picture; the invoultion ι makes a length preserving half turn of its boundary $\partial\Delta$.

Take the product $\Delta \times \Delta \subset \mathbb{R}^4$; it is homeomorphic to the 4-ball. Note that

$$\operatorname{vol}_3[\partial(\Delta \times \Delta)] = 2\cdot \operatorname{area} \Delta \cdot \operatorname{length} \partial\Delta < 20\cdot\varepsilon.$$

The boundary $\partial(\Delta\times\Delta)$ is homeomorphic to $\mathbb{S}^3$ and the restriction of the involution $(x,y)\mapsto(\iota(x),\iota(y))$ has the needed property.

All we have to do now is to smooth $\partial(\Delta\times\Delta)$ a little bit.

Remark. This example is given by Christopher Croke [**croke**]. Note that according to 2.6.3, the involution ι cannot be made isometric.

1.2.8. Note that if $\mathcal{M}_\infty$ is $e^{\pm\varepsilon}$-bilipschitz to a cube, then applying Besicovitch inequality, we get that

$$\varliminf_{n\to\infty}\operatorname{vol}\mathcal{M}_n\geqslant e^{-n\cdot\varepsilon}\cdot\operatorname{vol}\mathcal{M}_\infty.$$

Applying Vitali covering theorem, given $\varepsilon>0$, we can cover whole volume of $\mathcal{M}_\infty$ by $e^{\pm\varepsilon}$-bilipschitz cubes. Applying the above observation and summing up the results, we get that

$$\varliminf_{n\to\infty}\operatorname{vol}\mathcal{M}_n\geqslant e^{-n\cdot\varepsilon}\cdot\operatorname{vol}\mathcal{M}_\infty.$$

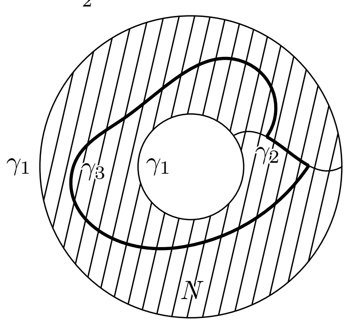
The statement follows since ε is arbitrary positive number.

Remark. A more general result was obtaind by Sergei Ivanov [**ivanov-1997**]. Note that the statement does not hold without stability of the convergence. In fact any compact metric space can be approximated by Riemannian surface with arbitrary small area.

1.3.1. Set $s=\operatorname{sys}(\mathbb{T}^2,g)$.

Cut $\mathbb{T}^2$ along a shortest closed noncontractible curve γ_1. We get an anulus with a Riemnnian metric on it (N,g). Denote by A and A' the two components of its boundary.

Assume that γ_2 is a shortest path that runs from A to A' in (N,g). The image of γ_2 in $\mathbb{T}^2$ connects two points in γ_1; further we will use the same notation for γ_2 and its image in $\mathbb{T}^2$. Connect $\gamma_2(0)$ to $\gamma_2(1)$ by a shorter arc in γ_1. Note that the obtained closed curve is noncontractible in $\mathbb{T}^2$. Therefore its length is at least s. The arc of γ_1 has length at most half of length γ_1. Whence length $\gamma_2\geqslant\frac{s}{2}$. In particular the distance from A to A' in (N,g) is at least $\frac{s}{2}$.



Let us cut (N,g) by γ_2, we obtain a square $(\square,g)$ with Riemnnian metric on it. Let us keep the notation A and A' for the pair of opposite sides in $(\square,g)$ that correspond to A and A' in (N,g). From above we have that distance from A to A' is at least $\frac{s}{2}$.

Denote by B and B' the remaining pair of opposite sides $(\square,g)$. Suppose that γ_3 is a path connecting these

sides. Map it the curves γ_i back to the
torus and let us keep for them the same notation. The path γ_3 connects two points on γ_2. Since γ_2 is shortest, the arc of γ_2 between this pair of points cannot be longer than γ_3. This arc together with γ_3 forms a closed noncontractible curve, so its length has to be at least s. It follows that $\operatorname{length}\gamma_3 \geqslant \frac{s}{2}$. That is distance from B to B' in $(\square, g)$ is at least $\frac{s}{2}$.

Applying Besikovitch inequality, we get that

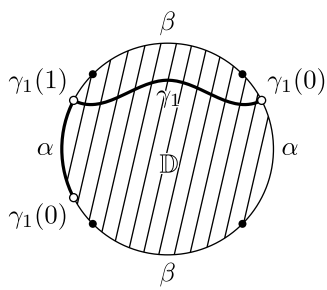
$$\operatorname{area}(\mathbb{T}^2, g) = \operatorname{area}(\square, g) \geqslant \tfrac{1}{4}{\cdot}s^2.$$

Remark. Alternatively one may notice that any curve in (N, g) that is bordant to A has length at least $\frac{s}{2}$. Therefore the level sets defined by $\operatorname{dist}_A(x)_{(N,g)} = t$ have length at least $\frac{s}{2}$ if $0 \leqslant t \leqslant \frac{s}{2}$. Applying coarea fromula we get that

$$\operatorname{area}(\mathbb{T}^2, g) = \operatorname{area}(N, g) \geqslant \tfrac{1}{2}{\cdot}s^2.$$

This estimate is twice better then the one above, but it is still far from the optimal bound $\frac{2}{\sqrt{3}}{\cdot}s^2$ in proved by Loewner inequality

1.3.2. Set $s = \operatorname{sys}(\mathbb{R}\mathrm{P}^2, g)$. Cut $(\mathbb{R}\mathrm{P}^2, g)$ along a shortest noncontractible curve γ. We obtain $(\mathbb{D}^2, g)$ — a disc with metric tensor which we still denote by g. Divide γ into two equal arcs α and β. Denote by A and A' the two connected components of the inverse image of α. Similarly denote by B and B' the two connected components of the inverse image of β.



Let γ_1 be a path from A to A'; map it to $\mathbb{R}\mathrm{P}^2$ and keep the same notation for it. Note that γ_1 together with a subarc of α forms a closed noncontractible curve in $\mathbb{R}\mathrm{P}^2$. Since $\operatorname{length}\alpha = \frac{s}{2}$, we have that $\operatorname{length}\gamma_1 \geqslant \frac{s}{2}$. It follows that the distance between A and A' in $(\mathbb{D}^2, g)$ is at least $\frac{s}{2}$. The same way we show that the distance between B and B' in $(\mathbb{D}^2, g)$ is at least $\frac{s}{2}$.

Note that $(\mathbb{D}^2, g)$ can be paraneterized by a square with sides A, B, A' and B' and apply 1.2.1 to show that

$$\operatorname{area}(\mathbb{R}\mathrm{P}^2, g) = \operatorname{area}(\mathbb{D}^2, g) \geqslant \tfrac{1}{4}{\cdot}s^2.$$

Remark. For the optimal constant was found by Pao Ming Pu see the discussion on page 9. His proof shows that any Riemannian metric on the disc with the boundary globally isometric to a unit circle with angle metric has area at least as large as the unit hemisphere. It is expected that the same inequality holds for any compact surface bounded by a single curve (not necessary a disc); this is the so called the *filling area conjecture* mentioned in [**gromov-1983**].

1.3.3. Cut the surface along a shortest noncontractible curve γ. We might get a surface with one or two components of the boundary. In these two cases repeat the arguments in 1.3.2 or 1.3.1 using 1.2.2 instead of 1.2.1.

1.3.4. Consider the product of small 2-sphere with a unit circle.

2.2.2. The following claim resembles Besikovitch inequality; it is key to the proof:

(∗) Let a be a positive real number. Assume that a closed curve γ in a metric space $\mathcal{X}$ can be sudivided into 4 arcs α, β, α', and β' in such a way that

- $|x-x'|>a$ for any $x\in\alpha$ and $x'\in\alpha'$ and
- $|y-y'|>a$ for any $y\in\beta$ and $y'\in\beta'$.

Then γ is not contractable in its $\frac{a}{2}$-neighborhood.

To prove (∗), consider two functions defined on $\mathcal{X}$ as follows:

$$\begin{aligned}w_1(x)&=\min\{\,a,\operatorname{dist}_\alpha(x)\,\}\\ w_2(x)&=\min\{\,a,\operatorname{dist}_\beta(x)\,\}\end{aligned}$$

and the map $\boldsymbol{w}\colon\mathcal{X}\to[0,a]\times[0,a]$, defined by

$$\boldsymbol{w}\colon x\mapsto(w_1(x),w_2(x)).$$

Note that

$$\begin{aligned}\boldsymbol{w}(\alpha)&=0\times[0,a], &\qquad \boldsymbol{w}(\beta)&=[0,a]\times 0,\\ \boldsymbol{w}(\alpha')&=a\times[0,a], &\qquad \boldsymbol{w}(\beta')&=[0,a]\times a,\end{aligned}$$

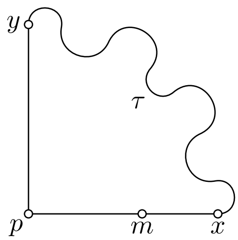
Therefore, the composition $\boldsymbol{w}\circ\gamma$ is a degree 1 map

$$\mathbb{S}^1\to\partial([0,a]\times[0,a]).$$

It follows that if $h\colon\mathbb{D}\to\mathcal{X}$ shrinks γ, then there is a point $z\in\mathbb{D}$ such that $\boldsymbol{w}\circ h(z)=(\frac{a}{2},\frac{a}{2})$. Therefore $h(z)$ lies at distance at least $\frac{a}{2}$ from α, β, α', β' and therefore from γ. Hence the claim (∗) follows.

Coming back to the problem, let $\{W_i\}$ be an open covering of the real line with multiplicity 2 and $\operatorname{rad}W_i<R$ for each i; for example one may take $W_i=((i-\frac{2}{3})\cdot R,(i+\frac{2}{3})\cdot R)$.

Choose a point $p \in \mathcal{X}$. Denote by $\{V_j\}$ the connected components of $\operatorname{dist}_p^{-1}(W_i)$ for all i. Note that $\{V_j\}$ is an open finite cover of $\mathcal{X}$ with multiplicity at most 2. It remains to show that $\operatorname{rad} V_j < 100{\cdot}R$ for each j.

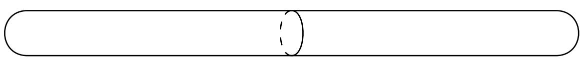


Aarguing by contradiction assume there is a pair of points $x, y \in V_i$ such that $|x - y|_{\mathcal{X}} \geqslant 100{\cdot}R$. Connect x to y with a curve τ in V_j. Consider the closed curve σ formed by τ and two geodesics $[px]$, $[py]$.

Note that $|p - x| > 40$. Therefore there is a point m on $[px]$ such that $|m - x| = 20$.

By the triangle inequality, the subsdivision of σ into the arcs $[pm]$, $[mx]$, τ and $[yp]$ satisfy the conditions of the claim $(*)$ for $a = 10{\cdot}R$. Hence the statement follows.

The quasiconverse does not hold. As an example take a surface that looks like a long cylinder with two hats, it is a smooth surface diffeomorphic to a sphere. Assuming the cylinder is thin, it has macroscopic



dimension 1 at a given scale. However a circle formed by a section of cylinder around its midpoint by a plane parallel to the base is a circle that cannot be contracted in its small neighborhood.

Sourse: [**gromov-1983**].

2.2.3, *"only if" part.* Suppose $\operatorname{width}_n \mathcal{X} < R$. Consider a covering $\{V_1, \dots, V_k\}$ of $\mathcal{X}$ guaranteed by the definition of width. Let $\mathcal{N}$ be its nerve and $\psi\colon \mathcal{X} \to \mathcal{N}$ be the map provided by 2.1.2.

Since the multiplicity of the covering is at most $n + 1$, we ahve $\dim \mathcal{N} \leqslant n$.

Note that if $x \in \mathcal{N}$ lies in a star of a vertex v_i, then $\psi^{-1}\{x\} \subset V_i$; in particular $\operatorname{rad}[\psi^{-1}\{x\}] < R$.

"If" part. Choose $x \in \mathcal{N}$. Since the inverse image $\psi^{-1}\{x\}$ is compact, ψ is continuous, and $\operatorname{rad}[\psi^{-1}\{x\}] < R$, there is a neighborhood $U \ni x$ such that the $\operatorname{rad}[\psi^{-1}(U)] < R$.

Since $\mathcal{X}$ is compact, there is a finite cover $\{U_i\}$ of $\mathcal{N}$ such that $\psi^{-1}(U_i) \subset \mathcal{X}$ has radius smaller than R for each i. Since $\mathcal{N}$ has dimension n, we can inscribe[2] in $\{U_i\}$ a finite open cover $\{W_i\}$ with multiplicity at most $n + 1$. It remains to observe that $V_i = \psi^{-1}(W_i)$

[2] Recall that a covering $\{W_i\}$ is inscribed in the covering $\{U_i\}$ if for every W_i is a subset of some U_j.

defines a finite open cover of $\mathcal{X}$ with radius less than R and multiplicity at most $n+1$.

2.3.1. Assume that $\mathcal{P}$ is connected.

Let us show that $\operatorname{diam}\mathcal{P} < R$. If this is not the case, then there are points $p,q\in\mathcal{P}$ on distance R from each other. Let γ be a geodesic from p to q. Clearly $\operatorname{length}\gamma \geqslant R$ and γ lies in $\mathrm{B}(p,R)$ except for the endpoint q. Therefore $\operatorname{length}[\mathrm{B}(p,R)_{\mathcal{P}}] \geqslant R$. Since $\operatorname{VolPro}_{\mathcal{P}}(R) \geqslant$ $\geqslant \operatorname{length}[\mathrm{B}(p,R)_{\mathcal{P}}]$, the latter contradicts $\operatorname{VolPro}_{\mathcal{P}}(R) < R$.

In general case, we get that each connected component of $\mathcal{P}$ has radius smaller that R. Whence the width of $\mathcal{P}$ is smaller that R.

Second part. Again, we can assume that $\mathcal{P}$ is connected.

The examples of line segment or a circle show that the constant $c=\frac{1}{2}$ cannot be improved. It remains to show that the inequality holds with $c=\frac{1}{2}$.

Choose $p\in\mathcal{P}$ such that the value

$$\rho(p)=\max\{\,|p-q|_{\mathcal{P}}\ :\ q\in\mathcal{P}\,\}$$

is minimal. Suppose $\rho(p)\geqslant\frac{1}{2}\cdot R$. Observe that there is a point $x\in\mathcal{P}\backslash\{p\}$ that lies on any shortest path starting from p and length $\geqslant\frac{1}{2}\cdot R$. Otherwise for any $r\in(0,\frac{1}{2}\cdot R)$ there would be at least two points on distance r from p; by coarea inequality we get that the total length of $\mathcal{P}\cap\mathrm{B}(p,\frac{1}{2}\cdot R)$ is at least R — a contradiction.

Moving p toward to x reduce $\rho(p)$ which contradicts the choice of p.

2.6.4. Suppose M is an essential manifold and N is arbitrary closed manifold. Observe that shrinking N to a point produces a map $f\colon N\# M\to M$ of degree 1; that is, the fundamental class of $N\# M$ maps to the fundamental class of M.

Since M is essential, there is an aspherical space K and a map $\iota\colon M\to K$ that sends fundamental class of M to nonzero homology class in K. From above, the composition $\iota\circ f\colon N\# M\to K$ sends fundamental class of $N\# M$ to the same homology class in K.

Remark. Note that we only used that there is a map $N\# M\to K$ of degree 1. If essential manifold is defined using homologies with integer coefficients, then existence of map of nonzero degree is sufficient.

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