Puzzles in geometry that I know and love

Anton Petrunin

November 20, 2018

arXiv:0906.0290 [math.HO]

Contents

Preface

This collection is about ideas, and it is not about theory. An idea might feel more comfortable in a suitable theory, but it has its own live and history and can speak for itself — I hope you will hear it.

I am collecting these problems for fun, but they might be used to improve the problem solving skills in geometry. Every problem has a short elegant solution — this gives a hint which was not available when it was solved for the first time.

How to read it. Open at a random chapter, make sure you like the practice problem — if yes try to solve a random problem in the chapter. A semisolution is given in the end of the chapter, but think before reading, otherwise it might not help.

Acknowledgments. I want to thank everyone who helped me; here is an incomplete list: Stephanie Alexander, Christopher Croke, Bogdan Georgiev, Jouni Luukkainen, Alexander Lytchak, Rostislav Matveyev, Peter Petersen, Idzhad Sabitov, Serge Tabachnikov, Sergio Zamora Barrera.

This collection is partly inspired by connoisseur's collection of puzzles by Peter Winkler [see 1]. Number of problems were suggested on math*overflow* [see 2].

Some problems are marked by $\circ$, $*$, $+$ or $\sharp$.

$\circ$ — easy problem;

$*$ — the solution requires at least two ideas;

$+$ — the solution requires knowledge of a theorem;

$\sharp$ — there are interesting solutions based on different ideas.

Chapter 1

Curves

Recall that a *curve* is a continuous map from a real interval into a space (for example, Euclidean plane) and a *closed curve* is a continuous map defined on a circle. If the map is injective then the curve is called *simple*.

We assume that the reader is familiar with related definitions including length of curve and its curvature. The necessary material is covered in the first couple of lectures of a standard introduction to differential geometry, [see 3, §26–27 or 4, Chapter 1].

We give a practice problem with a solution — after that, you are on your own.

Spiral

The following problem states that if you drive on the plane and turn the steering wheel to the right all the time, then you will not be able to come back to the same place.

Assume γ is a smooth regular plane curve with strictly monotonic curvature. Show that γ has no self-intersections.

Semisolution. The trick is to show that the osculating circles of γ are nested.

Without loss of generality we may assume that the curve is parametrized by its length and its curvature decreases.

Let $z(t)$ be the center of osculating circle at $\gamma(t)$ and $r(t)$ is its radius. Note that

$$z(t) = \gamma(t) + \tfrac{\gamma''(t)}{|\gamma''(t)|^2}, \qquad\qquad r(t) = \tfrac{1}{|\gamma''(t)|}.$$

Straightforward calculations show that

$$|z'(t)| = r'(t).$$

Note that the curve $z(t)$ has no straight arcs; therefore

$$(*) \qquad |z(t_1) - z(t_0)| < r(t_1) - r(t_0).$$

if $t_1 > t_0$.

Denote by D_t the osculating disk of γ at $\gamma(t)$; it has center at $z(t)$ and radius $r(t)$. By $(*)$, D_{t_1} lies in the interior of D_{t_0} for any $t_1 > t_0$. Hence the result follows. □

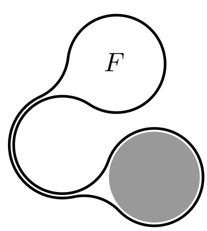
This problem was considered by Peter Tait [see 5] and later rediscovered by Adolf Kneser [see 6]. The osculating circles of the curve give a peculiar decomposition of an annulus into circles; it has the following property: if a smooth function is constant on each osculating circle it must be constant in the annulus [see 7, Lecture 10]. The same idea leads to a solution of the following problem:

Assume that γ is a smooth regular plane curve with strictly monotonic curvature. Show that no line can be tangent to γ at two distinct points.

It is instructive to check that the 3-dimensional analog does not hold; that is, there are self-intersecting smooth regular space curves with strictly monotonic curvature.

Note that if the curve $\gamma(t)$ is defined for $t \in [0, \infty)$ and its curvature converges to ∞ as $t \to \infty$, then the problem implies the convergence of $\gamma(t)$ as $t \to \infty$. The latter could be considered as a continuous analog of the Leibniz's test for alternating series.

Moon in a puddle



A smooth closed simple plane curve with curvature less than 1 bounds a figure F. Prove that F contains a disk of radius 1.

Spring in a tin

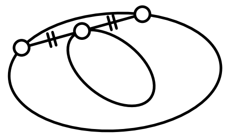
Let α be a closed smooth curve immersed in a unit disk. Prove that the average absolute curvature of α is at least 1, with equality if and only if α is the unit circle possibly traversed more than once.

Curve in a sphere

Show that if a closed curve on the unit sphere intersects every equator then its length is at least $2\cdot\pi$.

Oval in an oval

Consider two closed smooth strictly convex planar curves, one inside the other. Show that there is a chord of the outer curve that is tangent to the inner curve at its midpoint.



Capture a sphere in a knot*

The following formulation uses the notion of smooth isotopy of knots; that is, a one parameter family of embeddings

$$f_t\colon \mathbb{S}^1 \to \mathbb{R}^3, \quad t \in [0,1]$$

such that the map $[0,1] \times \mathbb{S}^1 \to \mathbb{R}^3$ is smooth.

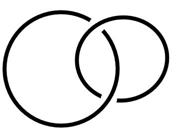
Show that one can not capture a sphere in a knot.

More precisely, let B be the closed unit ball in $\mathbb{R}^3$ and $f\colon \mathbb{S}^1 \to \mathbb{R}^3\backslash B$ be a knot. Show that there is a smooth isotopy

$$f_t\colon \mathbb{S}^1 \to \mathbb{R}^3\backslash B, \quad t \in [0,1],$$

such that $f_0 = f$, the length of f_t is non-increasing with respect to t and $f_1(\mathbb{S}^1)$ can be separated from B by a plane.

Linked circles



Suppose that two linked simple closed curves in $\mathbb{R}^3$ lie at a distance at least 1 from each other. Show that the length of each curve is at least $2\cdot\pi$.

Surrounded area

Consider two simple closed plane curves $\gamma_1, \gamma_2\colon \mathbb{S}^1 \to \mathbb{R}^2$. Assume

$$|\gamma_1(v) - \gamma_1(w)| \leqslant |\gamma_2(v) - \gamma_2(w)|$$

for any $v, w \in \mathbb{S}^1$. Show that the area surrounded by γ_1 does not exceed the area surrounded by γ_2.

Crooked circle

Construct a bounded set in $\mathbb{R}^2$ homeomorphic to an open disk such that its boundary contains no simple curves.

Rectifiable curve

For the following problem we need the notion of *Hausdorff measure.* Fix a compact set $X \subset \mathbb{R}^2$ and $\alpha > 0$. Given $\delta > 0$ consider the value

$$h(\delta) = \inf \left\{ \sum_i (\operatorname{diam} X_i)^\alpha \right\}$$

where the infimum is taken over all finite coverings $\{X_i\}$ of X such that $\operatorname{diam} X_i < \delta$ for each i.

Note that the function $\delta \mapsto h(\delta)$ is not decreasing in δ. In particular, $h(\delta) \to \mathcal{H}_\alpha(X)$ as $\delta \to 0$ for some (possibly infinite) value $\mathcal{H}_\alpha(X)$. This value $\mathcal{H}_\alpha(X)$ is called α-dimensional Hausdorff measure of X.

Let $X \subset \mathbb{R}^2$ be a compact connected set with finite 1-dimensional Hausdorff measure. Show that there is a rectifiable curve passing thru all the points in X.

Typical convex curves

Formally we do not need it in the problem, but it is worth noting that the curvature of a convex curve is defined almost everywhere; it follows from the fact that monotonic functions are differentiable almost everywhere.

Show that most *of the convex closed curves in the plane have vanishing curvature at every point where it is defined.*

We need to explain the meaning of word "most" in the formulation; it use *Hausdorff distance* and *G-delta sets.*

The Hausdorff distance $|A - B|_H$ between two closed bounded sets A and B in the plane is defined as the infimum of the positive numbers r such that the r-neighborhood of A contains B and the r-neighborhood of B contains A.

In particular we can equip the space of all closed plane curves with the Hausdorff metric. The obtained metric space is locally compact. The latter follows from the *selection theorem* [see §18 in 8], which states that the closed subsets of a fixed closed bounded set in the plane form a compact set with respect to the Hausdorff metric.

A G-delta set in a metric space X is defined as a countable intersection of open sets. According to *Baire category theorem*, in locally compact metric spaces X, the intersection of a countable collection of open dense set has to be dense. (The same holds if X is complete, but we will not need it.)

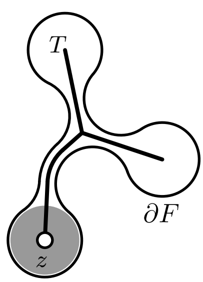
In particular, in X, the intersection of a finite or countable collection of G-delta dense sets is also a G-delta dense set. The later means that G-delta dense sets contain *most* of X. This is the meaning of the word *most* used in the problem.

Semisolutions

Moon in a puddle. In the proof we will use the *cut locus* of F with respect to its boundary[1]; it will be further denoted as T. The cut locus can be defined as the closure of the set of points $x \in F$ for which there exist two or more points in ∂F minimizing the distance to x.

For each point $x \in T$, consider the subset $X \subset \partial F$ where minimal distance to x is attained. If X is not connected then we say that x is a *cut point*; equivalently it means that for any sufficiently small neighborhood $U \ni x$, the complement $U \backslash T$ is disconnected. If X is connected then we say that x is a *focal point*; equivalently it means that the osculating circle to ∂F at any point of X is centered at x.

The trick is to show that T contains a focal point, say z. Since ∂F has curvature of at most 1, the radius of any osculating circle is at lest 1. Hence the distance from ∂F to z is at least 1, and the statement will follow.



After a small perturbation of ∂F we may assume that T is a graph embedded in F with finite number of edges.

Note that T is a deformation retract of F. The retraction $F \to T$ can be obtained the following way: (1) given a point $x \in F \setminus T$, consider the (necessarily unique) point $\hat{x} \in \partial F$ that minimize the distance $|x - \hat{x}|$ and (2) move x along the extension of the line segment $[\hat{x}x]$ behind x until it hits T.

In particular, T is a tree. Therefore T has an end vertex, say z. The point z is focal since there are arbitrary small neighborhoods U of z such that the complements $U \backslash T$ are connected. $\square$

Note that we proved a slightly stronger statement, namely there are at least two points on ∂F which osculating circles lie in F. Note

[1] Also called *medial axis*.

that these points are *vertexes* of ∂F; that is, they are critical points of its curvature.

Note further that inversion respects osculating circles. That is, suppose γ is an osculating circle of curve α at t_0. Assume γ' and α' are inverions of γ and α. Then γ' is an osculating circle of curve α' at t_0. Therefore appling an inversion about a circle with the center in F, we also get a pair of osculating circles of ∂F which surround F. This way we obtain 4 osculating circles that lie on one side of ∂F, which is an interesting generalization of four-vertex theorem.

The case of convex curves of this problem appears in a book of Wilhelm Blaschke [see §24 in 8]. In full generality, the problem was discussed by German Pestov and Vladimir Ionin [see 9]. A solution via curve shortening flow of a weaker statement was given by Konstantin Pankrashkin [see 10]. The statement still holds if the curve fails to be smooth at one point. A spherical version of the later statement was used by Dmitri Panov and me [see 11].

As you can see from the following problem, the 3-dimensional analog of this statement does not hold.

Construct a smooth embedding $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ with all the principle curvatures between -1 and 1 such that it does not surround a ball of radius 1.

Such example can be obtained by fattening a nontrivial contractible 2-complex in $\mathbb{R}^3$ [the Bing's house constructed in 12 will do the job]. This problem is discussed by Vladimir Lagunov in [13] and it was generalized to Riemannian manifolds with boundary by Stephanie Alexander and Richard Bishop [see 14].

A similar argument shows that for any Riemannian metric g on the 2-sphere $\mathbb{S}^2$ and any point $p \in (\mathbb{S}^2, g)$ there is a minimizing geodesic $[pq]$ with conjugate ends. On the other hand, for $(\mathbb{S}^3, g)$ this is not true. Moreover there is a metric g on $\mathbb{S}^3$ with sectional curvature bounded above by arbitrary small $\varepsilon > 0$ and $\operatorname{diam}(\mathbb{S}^3, g) \leqslant 1$. In particular, $(\mathbb{S}^3, g)$ has no minimizing geodesic with conjugate ends. An example was originally constructed by Mikhael Gromov [see 15]; a simplification was given by Peter Buser and Detlef Gromoll [see 16].

Spring in a tin. To solve this problem, you should imagine that you travel on a train along the curve $\alpha(t)$ and watch the position of the center of the disk in the frame of your wagon.

Denote by ℓ the length of α. Equip the plane with complex coordinates so that 0 is the center of the unit disk. We can assume α is equipped with an ℓ-periodic parametrization by arc length.

Consider the curve $\beta(t)=t-\frac{\alpha(t)}{\alpha'(t)}$. Observe that

$$\beta(t+\ell)=\beta(t)+\ell$$

for any t. In particular

$$(*)\qquad \operatorname{length}(\beta|_{[0,\ell]})\geqslant|\beta(\ell)-\beta(0)|=\ell.$$

Also

$$\begin{aligned}|\beta'(t)|&=|\tfrac{\alpha(t)\cdot\alpha''(t)}{\alpha'(t)^2}|\leqslant\\&\leqslant|\alpha''(t)|.\end{aligned}$$

Since $|\alpha''(t)|$ is the absoloute curvature of α at t, the result follows from $(*)$. □

The statement was originally proved by István Fáry in [17]; number of different proofs are discussed by Serge Tabachnikov [see 18 and also 19.5 in 7].

Note that the same argument works for curves in the unit ball.

If instead of the disk, we have a region bounded by a closed convex curve γ, then it is still true that the average curvature of α is at least as big as average curvature of γ. The proof was given by Jeffrey Lagarias and Thomas Richardson [see 19 and also 20].

Curve in a sphere. Let us present two solutions. We assume that α is a closed curve in $\mathbb{S}^2$ of length $2\cdot\ell$ that intersects each equator.

A solution with the Crofton formula. Given a unit vector u denote by e_u the equator with pole at u. Let $k(u)$ the number of intersections between α and e_u.

Note that for almost all $u\in\mathbb{S}^2$, the value $k(u)$ is even or infinite. Since each equator intersects α, we get $k(u)\geqslant 2$ for almost all u.

Then we get

$$\begin{aligned}2\cdot\ell&=\tfrac14\cdot\int\limits_{\mathbb{S}^2}k(u)\cdot d_u\,\mathrm{area}\geqslant\\&\geqslant\tfrac12\cdot\mathrm{area}\,\mathbb{S}^2=\\&=2\cdot\pi.\end{aligned}$$

The first identity above is called the *Crofton formula.* To prove this formula, start with the case when the curve is formed by one geodesic segment, summing up we get it for broken lines and by approximation it holds for all curves. □

A solution by symmetry. Let $\breve\alpha$ be a sub-arc of α of length ℓ, with endpoints p and q. Let z be the midpoint of a minimizing geodesic $[pq]$ in $\mathbb{S}^2$.

Let r be a point of intersection of α with the equator with pole at z. Without loss of generality we may assume that $r \in \breve\alpha$.

The arc $\breve\alpha$ together with its reflection with respect to the point z forms a closed curve of length $2{\cdot}\ell$ passing throgh both r and its antipodal point r^*. Therefore

$$\ell = \operatorname{length} \breve\alpha \geqslant |r - r^*|_{\mathbb{S}^2} = \pi.$$

Here $|r - r^*|_{\mathbb{S}^2}$ denotes the angle metric in the sphere $\mathbb{S}^2$. □

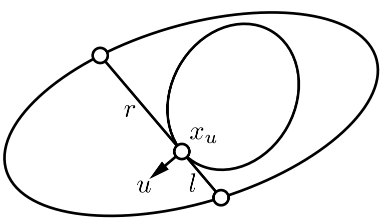
The problem was suggested by Nikolai Nadirashvili. It is nearly equivalent to the following:

🕮 *Show that total curvature of any closed smooth regular space curve is at least* $2{\cdot}\pi$*.*

A way more advanced problem is to show that any embedded circle of total curvature at most $4{\cdot}\pi$ is unknotted. It was solved independently by István Fáry [see 21] and John Milnor [see 22]. Later many interesting generalizations and refinements were found including a generalization to singular spaces by Stephanie Alexander and Richard Bishop [see 23] and the theorem on embedded minimal disk proved by Tobias Ekholm, Brian White and Daniel Wienholtz [see 24].

Oval in an oval. Choose the a chord that minimizes (or maximizes) the ratio in which it divides the bigger oval.

If the chord is not divided into equal parts, then you can rotate it slightly to decrease the ratio. Hence the problem follows. □



Alternative solution. Given a unit vector u, denote by x_u the point on the inner curve with outer normal vector u. Draw a chord of outer curve that is tangent to the inner curve at x_u; denote by $r = r(u)$ and $l = l(u)$ the lengths of the segments of this chord to the right and to the left of x_u, respectively.

Arguing by contradiction, assume $r(u) \neq l(u)$ for all $u \in \mathbb{S}^1$. Since the functions r and l are continuous, we can assume that

$$(*) \qquad r(u) > l(u) \quad \text{for all} \quad u \in \mathbb{S}^1.$$

Prove that each of the following two integrals

$$\tfrac{1}{2}\cdot\int\limits_{\mathbb{S}^1} r^2(u)\cdot du \quad\text{and}\quad \tfrac{1}{2}\cdot\int\limits_{\mathbb{S}^1} l^2(u)\cdot du$$

give the area between the curves. In particular, the integrals are equal. The latter contradicts $(*)$. □

This is a problem by Serge Tabachnikov [see 25]. A closely related *equal tangents problem* is discussed by the same author in [26].

Capture a sphere in a knot. We can assume that the knot lies on the sphere ∂B.

Fix a Möbius transformation $m\colon \mathbb{S}^2 \to \mathbb{S}^2$ close to the identity and not a rotation.

Note that m is a conformal map; that is, there is a function u defined on $\mathbb{S}^2$ as

$$u(x) = \lim_{y,z\to x} \frac{|m(y)-m(z)|}{|y-z|}.$$

(The function u is called the *conformal factor* of m.)

Since the area is preserved, we get

$$\frac{1}{\operatorname{area}\mathbb{S}^2}\cdot\int\limits_{\mathbb{S}^2} u^2 = 1.$$

By Bunyakovsky inequality,

$$\frac{1}{\operatorname{area}\mathbb{S}^2}\cdot\int\limits_{\mathbb{S}^2} u < 1.$$

It follows that after a suitable rotation of $\mathbb{S}^2$, the map m decreases the length of the knot.

Iterate this construction and pass to the limit as $m \to \mathrm{id}$. This way you get a continuous one parameter family of Möbius transformations which shorten the length of the knot. Therefore it drifts the knot to a single hemisphere and allows the ball to escape. □

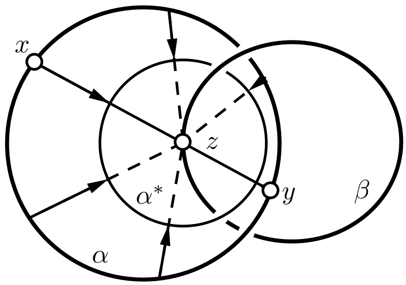
This is a problem by Zarathustra Brady, the given solution is based on the idea of David Eppstein [see 27].

Linked circles. Denote the linked circles by α and β.

Fix a point $x \in \alpha$. Note that there is a point $y \in \alpha$ such that the line segment $[xy]$ intersects β, say at the point z. Indeed, if this is not the case, applying a homothety with center x to α, would shrink it to x without crossing β. The latter contradicts that α and β are linked.

Let α^* be the image of α under the central projection onto the unit sphere around z. Clearly

$$\operatorname{length}\alpha \geqslant \operatorname{length}\alpha^*.$$



Note that α^* passes thru two antipodal points of the sphere, the one corresponding to x and the one corresponding to y. Therefore

$$\operatorname{length}\alpha^* \geqslant 2\cdot\pi.$$

Hence the result follows. □

This problem was proposed by Frederick Gehring [see 7.22 in 28]; solutions and generalizations are surveyed in [29]. The presented solution is attributed to Marvin Ortel in [30] and it is very close to the solution given by Michael Edelstein and Binyamin Schwarz [see 31].

Surrounded area. Let C_1 and C_2 be the compact regions bounded by γ_1 and γ_2 correspondingly.

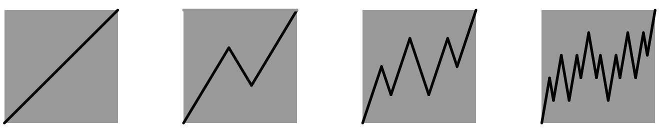
By Kirszbraun theorem, any 1-Lipschitz map $X \to \mathbb{R}^2$ defined on $X \subset \mathbb{R}^2$ can be extended to a 1-Lipschitz map on the whole $\mathbb{R}^2$. In particular, there is a 1-Lipschitz map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\gamma_2(v)) = f(\gamma_1(v))$ for any $v \in \mathbb{S}^1$.

Note that $f(C_2) \supset C_1$. Hence the statement follows. □

The Kirszbraun theorem appears in his thesis [see 32] and was rediscovered later by Frederick Valentine [see 33]. An interesting survey is given by Ludwig Danzer, Branko Grünbaum and Victor Klee [see 34].

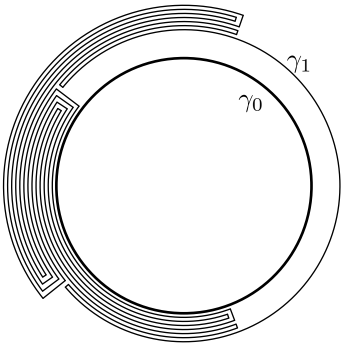
Crooked circle. A continuous function $f\colon [0,1] \to [0,1]$ will be called ε-crooked if $f(0) = 0$, $f(1) = 1$ and for any segment $[a,b] \subset [0,1]$ one can choose $a \leqslant x \leqslant y \leqslant b$ such that

$$|f(y) - f(a)| \leqslant \varepsilon \quad \text{and} \quad |f(x) - f(b)| \leqslant \varepsilon.$$



$\varepsilon = \frac{1}{2}$ $\varepsilon = \frac{1}{3}$ $\varepsilon = \frac{1}{4}$ $\varepsilon = \frac{1}{5}$

A sequence of $\frac{1}{n}$-crooked maps can be constructed recursively. Guess the construction from the diagram.



Now, start with the unit circle, $\gamma_0(t) = (\cos 2\pi t, \sin 2\pi t)$. Fix a sequence of positive numbers ε_n converging to zero very fast. Construct recursively a sequence of simple closed curves $\gamma_n\colon [0,1] \to \mathbb{R}^2$ such that γ_{n+1} runs outside of the disk bounded by γ_n and

$$|\gamma_{n+1}(t) - \gamma_n \circ f_n(t)| < \varepsilon_n,$$

for some ε_n-crooked function f_n. (On the diagram you can see an attempt to draw the first iteration.)

Denote by D the union of all disks bounded by the curves γ_n. Clearly D is homeomorphic to an open disk. For the right choice of the sequence ε_n, the set D is bounded. By construction, the boundary of D contains no simple curves. □

In fact, the only curves in the boundary of the constructed set are constant. Compare to the problem *Simple path* on page 105.

The proof use the so called *pseudo-arc* constructed by Bronisław Knaster [see 35]. The proof resembles construction of the Cantor set. Here are few similar problems:

📖 *Construct three distinct open sets in $\mathbb{R}$ with identical boundaries.*

📖 *Construct three open disks in $\mathbb{R}^2$ having the same boundary.*

These disks are called *lakes of Wada*; it is described by Kunizô Yoneyama [see 36].

📖 *Construct a Cantor set in $\mathbb{R}^3$ with non simply connected complement.*

This example is called *Antoine's necklace* [see 37].

📖 *Construct an open set in $\mathbb{R}^3$ with fundamental group isomorphic to the additive group of rational numbers.*

More advanced examples include *Whitehead manifold*, *Dogbone space*, *Casson handle*; see also the problem "Conic neighborhood" on page 104.

Rectifiable curve. The 1-dimensional Hausdorff measure will be denoted as $\mathcal{H}_1$.

Set $L = \mathcal{H}_1(K)$. Without loss of generality, we may assume that K has diameter 1.

Since K is connected, we get

$$(*) \qquad \mathcal{H}_1(B(x,\varepsilon)\cap K) \geqslant \varepsilon$$

for any $x \in K$ and $0 < \varepsilon < \frac{1}{2}$.

Let $x_1, \dots, x_n$ be a maximal set of points in K with

$$|x_i - x_j| \geqslant \varepsilon$$

for all $i \neq j$. From $(*)$ we have $n \leqslant 2\cdot L/\varepsilon$.

Note that there is a tree T_ε with vertices $x_1, \dots, x_n$ and straight edges with length at most $2\cdot\varepsilon$ each. Therefore the total length of T_ε is below $2\cdot n\cdot\varepsilon \leqslant 4\cdot L$. By construction, T_ε is ε-close to K in the Hausdorff metric.

Clearly, there is a closed curve γ_ε whose image is T_ε and its length is twice the total length of T_ε; that is,

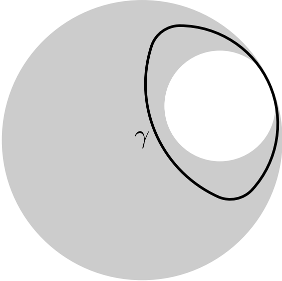
$$\operatorname{length}\gamma_\varepsilon \leqslant 8\cdot L.$$

Passing to a partial limit of γ_ε as $\varepsilon \to 0$, we get the needed curve. □

In terms of measure, the optimal bound is $2\cdot L$; if in addition the diameter D is known then it is $2\cdot L - D$. The problem is due to Samuel Eilenberg and Orville Harrold [see 38]; it also appears in the book of Kenneth Falconer [see Exercise 3.5 in 39].

Typical convex curves. Denote by $\mathfrak{C}$ the space of all closed convex curves in the plane equipped with the Hausdorff metric. Recall that $\mathfrak{C}$ is locally compact. In particular, by the Baire theorem, a countable intersection of everywhere dense open sets is everywhere dense.

Note that if a curve $\gamma \in \mathfrak{C}$ has nonzero second derivative at some point p, then it lies between two circles with one of them tangent to the other from inside at p.



Fix these two circles. It is straightforward to check that there is $\varepsilon > 0$ such that the Hausdorff distance from any convex curve γ squeezed between the circles to any convex n-gon is at least $\frac{\varepsilon}{n^{100}}$.

Fix a countable dense set of convex polygons $\mathfrak{p}_1, \mathfrak{p}_2, \dots$ in $\mathfrak{C}$. Denote by n_i the number of sides in $\mathfrak{p}_i$. For any positive integer k, consider the set $\Omega_k \subset \mathfrak{C}$ defined as

$$\Omega_k = \left\{\ \xi \in \mathfrak{C}\ \middle|\ |\xi - \mathfrak{p}_i|_H < \tfrac{1}{k\cdot n_i^{100}} \quad \text{for some} \quad i\ \right\},$$

where $|* - *|_H$ denotes the Hausdorff distance

From above we get that $\gamma \notin \Omega_k$ for some k.

Note that Ω_k is open and everywhere dense in $\mathfrak{C}$. Therefore

$$\Omega = \bigcap_k \Omega_k$$

is a G-delta dense set. Hence the statement follows. □

This problem states that typical convex curves have an unexpected property. In fact, this is a very common situation — it is hard to see the typical objects and these objects often have surprising properties.

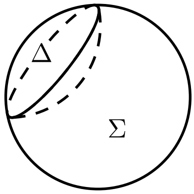
For example, as it was proved by Bernd Kirchheim, Emanuele Spadaro and László Székelyhidi, typical 1-Lipschitz maps from the plane to itself preserve the length of all curves [see 40]. The same way one could show that the boundaries of typical open sets in the plane contain no nontrivial curves, altho the construction of a concrete example is not trivial; see “Crooked circle”, page 8. More problems of that type are surveyed by Tudor Zamfirescu [see 41].

Chapter 2

Surfaces

We assume that the reader is familiar with smooth surfaces and the related definitions including intrinsic metric, geodesics, convex and saddle surfaces as well as different types of curvature. An introductory course in differential geometry should cover all necessary background material; see for example [3, §28–29] or [4].

Convex hat



Let Σ be a smooth closed convex surface in $\mathbb{R}^3$ and Π be a plane that cuts from Σ a disk Δ. Assume that the reflection of Δ with respect to Π lies inside of Σ. Show that Δ is convex *with respect to the intrinsic metric of Σ; that is, if both ends of a minimizing geodesic in Σ lie in Δ, then the whole geodesic lies in Δ.*

Semisolution. Let γ be a minimizing geodesic with ends in Δ.

Assume $\gamma\backslash\Delta \neq \varnothing$. Denote by $\hat\gamma$ the curve formed by $\gamma\cap\Delta$ and the reflection of $\gamma\backslash\Delta$ with respect to Π. Note that

$$\operatorname{length}\hat\gamma = \operatorname{length}\gamma$$

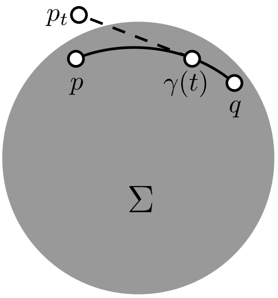
and $\hat\gamma$ runs partly along Σ and partly outside Σ, but does not get inside Σ.

Denote by $\bar\gamma$ the closest point projection of $\hat\gamma$ on Σ. Since Σ is convex, the closest point projection decreases the length. Therefore the curve $\bar\gamma$ lies in Σ, it has the same ends as γ and

$$\operatorname{length}\bar\gamma < \operatorname{length}\gamma.$$

This means that γ is not length minimizing, a contradiction. $\square$

Involute of geodesic



🕮 *Let Σ be a smooth closed strictly convex surface in $\mathbb{R}^3$ and $\gamma\colon [0,\ell] \to \Sigma$ be a unit-speed minimizing geodesic. Set $p = \gamma(0)$, $q = \gamma(\ell)$ and*

$$p_t = \gamma(t) - t{\cdot}\gamma'(t),$$

where $\gamma'(t)$ denotes the velocity vector of γ at t.

Show that for any $t \in (0,\ell)$, one cannot see *q from p_t; that is, the line segment $[p_t q]$ intersects Σ at a point distinct from q.*

Simple geodesic

🕮 *Let Σ be a complete unbounded convex surface in $\mathbb{R}^3$. Show that there is a two-sided infinite geodesic in Σ with no self-intersections.*

Let us review a couple of statements about Gauss curvature which might help to solve the problem [see §28 in 3, for more details].

If Σ is a convex surface in $\mathbb{R}^3$ then its Gauss curvature is nonnegative.

Assume that a simply connected region Ω in the surface Σ is bounded by a closed broken geodesic γ. Denote by $\kappa(\Omega)$ the integral of the Gauss curvaure along Ω.

For any point $p \in \Sigma$ consider the outer unit normal vector $n(p) \in \mathbb{S}^2$. Then

$$\kappa(\Omega) = \operatorname{area}[n(\Omega)]$$

and by the Gauss–Bonnet formula

$$\kappa(\Omega) = 2{\cdot}\pi - \sigma(\gamma),$$

where $\sigma(\gamma)$ denotes the sum of the signed exterior angles of γ. In particular, $|\sigma(\gamma)| \leqslant 2{\cdot}\pi$.

Geodesics for birds

The *total curvature* of a space curve γ is defined as the integral of its curvature. That is, if a curve $\gamma\colon [a,b] \to \mathbb{R}^3$ has unit speed parametrization, then its total curvature equals

$$\int\limits_a^b |\gamma''(t)|{\cdot}dt,$$

the vector $\gamma''(t)$ is called *curvature vector* and its magnitude $|\gamma''(t)|$ is the *curvature* of γ at time t. The above definition has sense for $C^{1,1}$ smooth curves, that is, if $\gamma'(t)$ is locally Lipschitz; in this case the curvature $|\gamma''(t)|$ is defined almost everywhere.

The *geodesics* in the following problem are defined as the curves locally minimizing the length; that is, any sufficiently short arc of the curve containing a given value of the parameter is length minimizing.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth ℓ-Lipschitz function. Let $W \subset \mathbb{R}^3$ be the epigraph of f; that is,

$$W = \left\{\, (x,y,z) \in \mathbb{R}^3 \;\middle|\; z \geqslant f(x,y) \,\right\}.$$

Equip W with the induced intrinsic metric.

Show that any geodesic in W has total curvature at most $2{\cdot}\ell$.

Actually, geodesics in W are $C^{1,1}$-smooth; in particular, the formula for the total curvature mentioned above makes sense. This is an easy exercise in real analysis which can be also taken as granted.

Immersed surface

Let Σ be a smooth connected immersed surface in $\mathbb{R}^3$ with strictly positive Gauss curvature and nonempty boundary $\partial\Sigma$. Assume $\partial\Sigma$ lies on a plane Π and Σ lies entirely in one side of Π. Prove that Σ is an embedded disk.

Periodic asymptote

Let Σ be a closed smooth surface with non-positive curvature and γ be a geodesic in Σ. Assume that γ is not periodic and the curvature of Σ vanish at every point of γ. Show that γ does not have a periodic asymptote; that is, there is no periodic geodesic δ such that the distance from $\gamma(t)$ to δ converges to 0 as $t \to \infty$.

Saddle surface

Recall that a smooth surface Σ in $\mathbb{R}^3$ is called *saddle* at the point p if its principal curvatures at p have opposite signs. We say that Σ is *saddle* if it is saddle at all points.

Let Σ be a saddle surface in $\mathbb{R}^3$ homeomorphic to a disk. Assume that the orthogonal projection to the (x,y)-plane maps the boundary of Σ injectively to a convex closed curve. Show that the orthogonal projection to (x,y)-plane is injective on Σ.

In particular, Σ is the graph $z = f(x, y)$ of a function f defined on a convex figure in the (x, y)-plane.

Asymptotic line

The saddle surfaces are defined in the previous problem.

Recall that an *asymptotic line* on a smooth surface $\Sigma \subset \mathbb{R}^3$ is a curve always tangent to an *asymptotic direction* of Σ; that is, a direction with vanishing normal curvature.

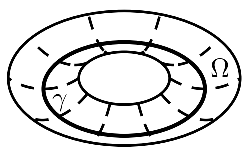
Let $\Sigma \subset \mathbb{R}^3$ be the graph $z = f(x, y)$ of a smooth function f and γ be a closed smooth asymptotic line in Σ. Assume Σ is saddle in a neighborhood of γ. Show that the projection of γ to the (x, y)-plane cannot be star-shaped.

Minimal surface

Recall that a smooth surface in $\mathbb{R}^3$ is called *minimal* if its mean curvature vanishes at all points. The *mean curvature* at each point is defined as the sum of the principal curvatures at that point.

Let Σ be a minimal surface in $\mathbb{R}^3$ having its boundary on a unit sphere. Assume Σ passes thru the center of the sphere. Show that the area of Σ is at least π.

Round gutter*



A round gutter is the surface shown on the picture.

More precisely, consider the torus T; a surface generated by revolving a circle in $\mathbb{R}^3$ around an axis coplanar with the circle. Let $\gamma \subset T$ be one of the circles in T that locally separates positive and negative curvature on T; a plane containing γ is tangent to T at all points of γ. Then a neighborhood of γ in T is called *round gutter* and the circle γ is called its *main latitude.*

Let $\Omega \subset \mathbb{R}^3$ is a round gutter with main latitude γ. Assume $\iota\colon \Omega \to \mathbb{R}^3$ is a smooth length-preserving embedding that is sufficiently close to the identity. Show that γ and $\iota(\gamma)$ are congruent; that is, there is an isometric motion of $\mathbb{R}^3$ sending γ to $\iota(\gamma)$

Non-contractible geodesics

Give an example of a non-flat metric on the 2-torus such that no geodesic is contractible.

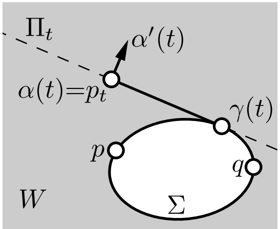
Two disks

Let Σ_1 and Σ_2 be two smoothly embedded open disks in $\mathbb{R}^3$ that have a common closed smooth curve γ. Show that there is a pair of points $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$ with parallel tangent planes.

Semisolutions

Involute of geodesic. Let W be the closed unbounded set formed by Σ and its exterior points. Fix $t \in (0, \ell)$; denote by γ_t the concatenation of the line segment $[p_t\gamma(t)]$ and the arc $\gamma|_{[t,\ell]}$. The key step is to show the following:

($*$) *The curve γ_t is a minimizing geodesic in the intrinsic metric induced on W.*



Try to prove it before reading further.

Let Π_t be the tangent plane to Σ at $\gamma(t)$. Consider the curve $\alpha(t) = p_t$. Note that $\alpha(t) \in \Pi_t$, $\alpha'(t) \perp \Pi_t$ and $\alpha'(t)$ points to the side of Π_t opposite from Σ.

It follows that for any $x \in \Sigma$ the function

$$t \mapsto |x - p_t| \quad \text{and, therefore,} \quad t \mapsto |x - p_t|_W$$

are non-decreasing; here $|x-p_t|_W$ stays for the intrinsic distance from x to p_t in W.

On the other hand, by construction

$$|q - p_t|_W \leqslant |q - p|_\Sigma;$$

therefore, from above

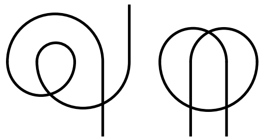
$$|q - p_t|_W = |q - p|_\Sigma$$

for any t. Hence ($*$) follows.

Now assume q is visible from p_t for some t; that is, the line segment $[qp_t]$ intersects Σ only at q. From above, γ_t coincides with the line segment $[qp_t]$. On the other hand γ_t contains $\gamma(t) \in \Sigma$, a contradiction. $\square$

This problem is based on an observation used by Anatoliy Milka in the proof of his (beautiful) generalization of the comparison theorem for convex surfaces [see 42].

Simple geodesic. Look at two combinatoric types of self-intersections shown on the diagram. One of the types can and the other can not appear as self-intersections of a geodesic on an unbounded convex surface. Try to determine which is which before reading further.



Let γ be a two-sided infinite geodesic in Σ. The following is the key statement in the proof.

$(*)$ *The geodesic γ contains at most one simple loop.*

To prove $(*)$, we use the following observation.

$(**)$ *The integral curvature ω of Σ cannot exceed $2{\cdot}\pi$.*

Indeed, since Σ is unbounded and convex, it surrounds a half-line. Consider a coordinate system with this half-line as the positive half of its z-axis. In these coordinates, the surface Σ is described as a graph $z = f(x, y)$ of a convex function f. In particular the outer normal vectors to Σ point to the south hemisphere. Therefore the area of the spherical image of Σ is at most $2{\cdot}\pi$. The area of this image is the integral of the Gauss curvature along Σ. Hence $(**)$ follows.

From the Gauss–Bonnet formula, we get the following. If φ is the angle at the base of a simple geodesic loop then the integral curvature surrounded by the loop equals $\pi+\varphi$. In particular there are no concave loops.

Now assume $(*)$ does not hold, then a geodesic has two simple loops. Note that the disks bounded by the loops have to overlap, otherwise the curvature of Σ would exceed $2{\cdot}\pi$. But if they overlap, then it is easy to show that the curve also contains a concave loop, which contradicts the above observation.[1]

If a geodesic γ has a self-intersection, then it contains a simple loop. From $(*)$, there is only one such loop; it cuts a disk from Σ and goes around it either clockwise or counterclockwise. This way we divide all the self-intersecting geodesics into two sets which we will call *clockwise* and *counterclockwise.*

Note that the geodesic $t \mapsto \gamma(t)$ is clockwise if and only if $t \mapsto \gamma(-t)$ is counterclockwise. The sets of clockwise and counterclockwise are open and the space of geodesics is connected. It follows that there are geodesics which aren't clockwise nor counterclockwise. Those geodesics have no self-intersections. $\square$

[1] This observation implies that the right picture on the above diagram cannot be realized by a geodesic.

The problem is due to Stephan Cohn-Vossen, [see Satz 9 in 43]; generalizations were obtained by Vladimir Streltsov and Alexandr Alexandrov [see 44] and by Victor Bangert [see 45].

Geodesics for birds. Fix a unit-speed geodesic in W, say

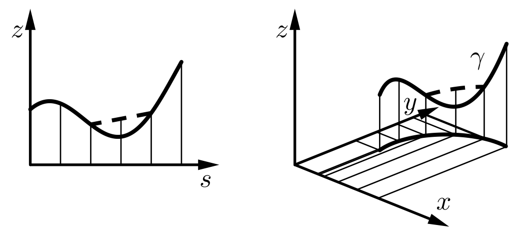
$$\gamma\colon t\mapsto (x(t),y(t),z(t)).$$

We can assume that γ is defined on a closed interval $[a,b]$. The key step is to show the following:

$(*)$ *The function $t\mapsto z$ is concave.*

Parametrize the plane curve $t\mapsto (x(t),y(t))$ by arclength s and reparametrize γ by s.

Note that the function $s\mapsto z$ is concave. If not, one could shorten γ by increasing its z component in a small interval around a point at which the function is not concave, while keeping its endpoints fixed. After the deformation, the curve still lies in W. The latter contradicts that γ is locally length minimizing.



Finally note that concavity of $s\mapsto z$ is equivalent to the concavity of $t\mapsto z$. Hence $(*)$ follows.

Since f is smooth, the curve $\gamma(t)$ is $C^{1,1}$; that is, its first derivative $\gamma'(t)$ is a well defined Lipschitz function. It follows that its second derivative $\gamma''(t)$ is defined almost everywhere.

Since $z(t)$ is concave, we have $z''(t)\leqslant 0$. Since f is ℓ-Lipschitz, $z(t)$ is $\frac{\ell}{\sqrt{1+\ell^2}}$-Lipschitz. It follows that

$$\int_a^b |z''(t)|\leqslant 2\cdot\tfrac{\ell}{\sqrt{1+\ell^2}}.$$

The curvature vector $\gamma''(t)$ is perpendicular to the surface. Since the surface has slope at most ℓ, we get

$$|\gamma''(t)|\leqslant |z''(t)|\cdot\sqrt{1+\ell^2}.$$

Hence

$$\int\limits_a^b |\gamma''(t)| \leqslant 2{\cdot}\ell. \qquad \square$$

The statement holds for general ℓ-Lischitz functions, not necessary smooth. The given bound is optimal, the equality is attained by a both side infinite geodesic on the graph of

$$f(x,y) = -\ell{\cdot}\sqrt{x^2+y^2}.$$

The problem is due to David Berg [see 46], the same bound for convex ℓ-Lipschitz surfaces was proved earlier by Vladimir Usov [see 47]. The observation $(*)$ is called *Liberman's lemma*; it was used yet earlier to bound the total curvature of a geodesic on a convex surface [see 48].[2] This lemma is often useful when working with geodesics on general convex surfaces.

Immersed surface. Let ℓ be a linear function that vanishes on Π and is positive on Σ. We will apply a Morse type argument for the restriction of ℓ to Σ.

Let z_0 be a maximum of ℓ on Σ; set $s_0 = \ell(z_0)$. Given $s < s_0$, denote by Σ_s the connected component of z_0 in $\Sigma \cap \ell^{-1}([s,s_0])$. Note that for all s sufficiently close to s_0 we have

- ◇ Σ_s is an embedded disk;
- ◇ $\partial\Sigma_s$ is a convex plane curve.

Applying an open-closed argument, we get that the same holds for all $s \in [0, s_0)$.

Since Σ is connected, $\Sigma_0 = \Sigma$. Hence the result follows. $\square$

This problem is discussed in the lectures of Mikhael Gromov [see §$\tfrac12$ in 49].

Periodic asymptote. Arguing by contradiction, assume that there is a geodesic γ on the surface Σ with a periodic asymptote δ.

Passing to a finite cover of Σ, we can ensure that the asymptote has no self-intersections. In this case, the restriction $\gamma|_{[a,\infty)}$ has no self-intersections if a is large enuf.

Cut Σ along $\gamma([a,\infty))$ and then cut from the obtained surface an infinite triangle $\triangle$. The triangle $\triangle$ has two sides formed by both sides of cuts along γ; let us denote these sides of $\triangle$ by γ_- and γ_+. Note that

$$(*) \qquad \operatorname{area}\triangle < \operatorname{area}\Sigma < \infty$$

[2] It was a part of the thesis of Joseph Liberman, defended couple of months before his death in the WWII.

and both sides $\gamma_\pm$ are infinite minimizing geodesics in $\triangle$.

Consider the Busemann function f for γ_+ [defined on page 34]; denote by $\ell(t)$ the length of the level curve $f^{-1}(t)$. Let $-\kappa(t)$ be the total curvature of the sup-level set $f^{-1}([t,\infty))$. From the Gauss–Bonnet formula,

$$(**) \qquad \ell'(t) = \kappa(t).$$

The level curve $f^{-1}(t)$ can be parametrized by a unit-speed curve, say $\theta_t\colon [0,\ell(t)] \to \triangle$. By the coarea formula we have

$$\kappa'(t) = -\int\limits_0^{\ell(t)} K_{\theta_t(\tau)}\cdot d\tau,$$

where K_x denotes the Gauss curvature of Σ at the point x. Since $K_{\theta_t(0)} = K_{\theta_t(\ell_t)} = 0$ and the surface is smooth, there is a constant C such that $|K_{\theta_t(\tau)}| \leqslant C\cdot\ell(t)^2$ for all t, τ. Therefore

$$({}^{*}_{**}) \qquad \kappa'(t) \leqslant C\cdot\ell(t)^3$$

Together, $(**)$ and $({}^{*}_{**})$ imply that there is $\varepsilon > 0$ such that

$$\ell(t) \geqslant \frac{\varepsilon}{t-a}$$

for large enuf t. By the coarea formula we get

$$\operatorname{area}\triangle = \int\limits_a^\infty \ell(t) = \infty;$$

the latter contradicts $(*)$. $\square$

I learned the problem from Dmitri Burago and Sergei Ivanov, it originated from a discussion with Keith Burns, Michael Brin and Yakov Pesin.

Here is a motivation: assume Σ is a closed surface with non-positive curvature that is not flat. The space Γ of all unit-speed geodesics $\gamma\colon \mathbb{R} \to \Sigma$ can be identified with the unit tangent bundle $\mathrm{U}\Sigma$. In particular Γ comes with a natural choice of measure. Denote by $\Gamma_0 \subset \Gamma$ the set of geodesics that run in the set of zero curvature all the time. It is expected that Γ_0 has vanishing measure. In all known examples Γ_0 contains only periodic geodesics in only finitely many homotopy classes [see also 50].

Saddle surface. Denote by Σ° the interior of Σ. Fix a plane Π. Note that the intersection $\Pi \cap \Sigma^\circ$ locally either looks like a curve or two curves intersecting transversally; in the latter case Π is tangent to Σ° at the cross-point.

Further note that $\Pi \cap \Sigma^\circ$ has no cycle. Otherwise Σ would fail to be saddle at the point in the disk surrounded by that cycle maximizing the distance to Π.

If Σ is not a graph then there is a point $p \in \Sigma$ with vertical tangent plane; denote this plane by Π. The intersection $\Pi \cap \Sigma$ has cross-point at p.

Since the boundary of Σ projects injectively to a closed convex curve in (x, y)-plane, the intersection of $\Pi \cap \partial\Sigma$ has at most 2 points — these are the only endpoints of $\Pi \cap \Sigma$.

It follows that the connected component of p in $\Pi \cap \Sigma$ is a tree with a vertex of degree 4 at p and at most two end-points, a contradiction. ☐

The described idea can be used to prove the result of Richard Schoen and Shing-Tung Yau [see 51] which gives a sufficient condition for a harmonic map between surfaces to be a diffeomorphism. Unlike the original proof, it requires no calculations.

The proof above is based on the observation that for any saddle surface Σ and plane Π, each connected component of $\Sigma \backslash \Pi$ is either unbounded or intersects the boundary curve. This observation plays a central role in the Sergei Bernstein proof [see 52] of the following problem:

Show that a smooth bounded function $f\colon \mathbb{R}^2 \to \mathbb{R}$ can not have a strictly saddle graph.

One could go further and define *generalized saddle surface* as an arbitrary (non necessarily smooth) surface satisfying the observation above. The geometry of these surfaces is far from being understood, Samuil Shefel has number of beautiful results about them, [see 53, 54, and the references there in]. The problem still holds for these generalized saddle surfaces, but the proof is tricky [see 55]

Asymptotic line. Denote by Π_t the tangent plane to Σ at $\gamma(t)$ and by ℓ_t the tangent line of γ at time t.

Since γ is asymptotic, the plane Π_t rotates around ℓ_t as t changes. Since Σ is saddle, the speed of rotation cannot vanish.[3]

Note that Π_t is a graph of a linear function, say h_t, defined on the (x, y)-plane. Denote by $\bar\ell_t$ the projection of ℓ_t to the (x, y)-plane. The described rotation of Π_t can be expressed algebraicaly: the derivative $\frac{d}{dt}h_t(w)$ vanishes at the point w if and only if $w \in \bar\ell_t$ and the derivative changes sign if w changes the side of $\bar\ell_t$.

[3] By the Beltrami–Enneper theorem, if γ has unit speed, then the speed of rotation is $\pm\sqrt{-K}$, where K is the Gauss curvature which can not vanish on a saddle surface.

Denote by $\bar\gamma$ the projection of γ to the (x,y)-plane. If $\bar\gamma$ is star shaped with respect to a point w, then w can not cross $\bar\gamma_t$. Therefore the function $t\mapsto h_t(w)$ is monotonic on $\mathbb{S}^1$. Observing that this function cannot be constant, we arrive to a contradiction. □

The problem is discussed by Dmitri Panov [see 56].

Minimal surface. Without loss of generality we may assume that the sphere is centered at the origin of $\mathbb{R}^3$.

Let h be the restriction of the function $x\mapsto \tfrac12\cdot|x|^2$ to the surface Σ. Direct calculations show that $\Delta_\Sigma h=2$. Applying the divergence theorem for $\nabla_\Sigma h$ in $\Sigma_r=\Sigma\cap B(0,r)$, we get

$$2\cdot \operatorname{area}\Sigma_r\leqslant r\cdot \operatorname{length}[\partial\Sigma_r].$$

Set $a(r)=\operatorname{area}\Sigma_r$. By the coarea formula, $a'(r)\geqslant \operatorname{length}[\partial\Sigma_r]$ for almost all r. Therefore the function

$$f\colon r\mapsto \frac{\operatorname{area}\Sigma_r}{r^2}$$

is non-decreasing in the interval $(0,1)$.

Since $f(r)\to\pi$ as $r\to 0$, the result follows. □

We described a partial case of the so called *monotonicity formula* for minimal surfaces.

The same argument shows that if 0 is a double point of Σ then $\operatorname{area}\Sigma\geqslant 2\cdot\pi$. This observation was used to prove that the minimal disk bounded by a simple closed curve with total curvature $\leqslant 4\cdot\pi$ is necessarily embedded. It was proved by Tobias Ekholm, Brian White and Daniel Wienholtz [see 24]; an amusing variation of this proof was obtained by Stephan Stadler [see 57]. This result also implies that any embedded circle of total curvature at most $4\cdot\pi$ is unknot. The latter was proved inependently by István Fáry [see 21] and John Milnor [see 22].

Note that if we assume in addition that the surface is a disk, then the statement holds for any saddle surface. Indeed, denote by S_r the sphere of radius r concentrated with the unit sphere. Then according to the problem "A curve in a sphere" [page 7],

$$\operatorname{length}[\partial\Sigma_r]\geqslant 2\cdot\pi\cdot r.$$

Then by the coarea formula we get $\operatorname{area}\Sigma\geqslant\pi$.

On the other hand, there are saddle surfaces homeomorphic to the cylinder having arbitrary small area in the ball.

If Σ does not pass thru the center and we only know the distance, say r, from the center to Σ, then the optimal bound is $\pi\cdot(1-r^2)$. This question was open for about 40 years and proved by Simon Brendle and Pei-Ken Hung [see 58]; their proof is based on a similar idea and quite elementary. Earlier Herbert Alexander, David Hoffman and Robert Osserman proved it for two cases (1) for disks and (2) for arbitrary area minimizing surfaces, any dimension and codimension [see 59, 60].

Round gutter. Without loss of generality, we can assume that the length of γ is $2{\cdot}\pi$ and its intrinsic curvature is 1 at all points.

Let K be the convex hull of $\hat\Omega = \iota(\Omega)$. Part of $\hat\Omega$ touches the boundary of K and the rest lies in the interior of K. Denote by $\hat\gamma$ the curve in $\hat\Omega$ dividing these two parts.

First note that the Gauss curvature of $\hat\Omega$ has to vanish at the points of $\hat\gamma$; in other words, $\hat\gamma = \iota(\gamma)$. Indeed, since $\hat\gamma$ lies on the convex part, the Gauss curvature at the points of $\hat\gamma$ has to be non-negative. On the other hand, $\hat\gamma$ bounds a flat disk in ∂K; therefore its integral intrinsic curvature has to be $2{\cdot}\pi$. If the Gauss curvature is positive at some point of $\hat\gamma$, then by the Gauss–Bonnet formula, the total intrinsic curvature of $\hat\gamma$ has to be smaller than $2{\cdot}\pi$, a contradiction.

On the other hand $\hat\gamma$ is an asymptotic line. Indeed, if the direction of $\hat\gamma$ is not asymptotic at some t_0 then $\hat\gamma(t_0 \pm \varepsilon)$ lies the interior of K for some small $\varepsilon > 0$, a contradiction.

Therefore, as the space curve, $\hat\gamma$ has to be a closed curve with constant curvature 1. Any such curve is congruent to a unit circle. $\square$

It is not known whether $\hat\Omega$ is congruent to Ω or not.

The solution presented above is based on my answer to the question of Joseph O'Rourke [see 61]. Here are some related statements.

- ◇ A gutter is second order rigid; this was proved by Eduard Rembs [see 62 and also page 135 in 63].
- ◇ Any second order rigid surface does not admit analytic deformation [proved by Nikolay Efimov, see page 121 in 63] and for the surfaces of revolution, the assumption of analyticity can be removed [proved by Idzhad Sabitov, see 64].

Non-contractible geodesics. A torus of revolution is an example.

Let T be a torus of revolution; it has a family of*meridians* — a family of circles that form closed geodesics.

Note that a geodesic on T is either a meridian or it intersects meridians transversally. In the latter case all the meridias are crossed by the geodesic in the same direction.

A contractible curve has to cross each meridian an equal number of times in both directions. Hence no geodesic of the torus is contractible. $\square$

I learned this problem from the book of Mikhael Gromov [see 65], where it is attributed to Y. Colin de Verdière. I do not know generic examples of that type.

Two disks. Choose a continuous map $h\colon \Sigma_1 \to \Sigma_2$ being the identity on γ. Let us prove that for some $p_1 \in \Sigma_1$ and $p_2 = h(p_1) \in \Sigma_2$, the tangent planes $\mathrm{T}_{p_1}\Sigma_1$ and $\mathrm{T}_{p_2}\Sigma_2$ are parallel; this fact is stronger than the one required.

Arguing by contradiction, assume that such point does not exist. Then for each $p \in \Sigma_1$ there is unique line $\ell_p \ni p$ parallel to both $\mathrm{T}_p\Sigma_1$ and $\mathrm{T}_{h(p)}\Sigma_2$.

Note that the lines ℓ_p form a tangent line distribution over Σ_1 and ℓ_p is tangent to γ at all $p \in \gamma$.

Let Δ be the disk in Σ_1 bounded by γ. Consider the doubling of Δ along γ; it is diffeomorphic to $\mathbb{S}^2$. The line distribution ℓ lifts to a line distribution on the doubling. The latter contradicts the hairy ball theorem. $\square$

This proof was suggested nearly simultaneously by Steven Sivek and Damiano Testa [see 66].

Note that the same proof works when Σ_i are oriented open surfaces and γ cuts a compact domain in each Σ_i.

There are examples of three disks Σ_1, Σ_2 and Σ_3 with a common closed curve γ such that there is no triple of points $p_i \in \Sigma_i$ with parallel tangent planes. Such examples can be found among ruled surfaces [see 67].

Chapter 3

Comparison geometry

In this chapter we consider Riemannian manifolds with curvature bounds.

This chapter is very demanding; we assume that the reader is familiar with shape operator and second fundamental form, equations of Riccati and Jacobi, comparison theorems and Morse theory. The classical book [68] covers all the necessary material.

Geodesic immersion*

An isometric immesion $\iota\colon N\looparrowright M$ from one Riemannian manifold to another is called *totally geodesic* if it maps any geodesic in N to a geodesic in M.

Let M and N be a simply connected positively curved Riemannian manifolds and $\iota\colon N\looparrowright M$ be a totally geodesic immersion. Assume that

$$\dim N>\tfrac{1}{2}\cdot\dim M.$$

Prove that ι is an embedding.

Semisolution. Set $n=\dim N$, $m=\dim M$.

Fix a smooth increasing strictly concave function φ. Consider the function $f=\varphi\circ\mathrm{dist}_N$.

Note that if f is smooth at some point $x\in M$ then the Hessian of f at x (briefly $\mathrm{hess}_x f$) has at least $n+1$ negative eigenvalues.

Moreover, at any point $x\notin\iota(N)$ the same holds in the barrier sense. That is, there is a smooth function h defined on M such that $h(x)=f(x)$, $h(y)\geqslant f(y)$ for any y and $\mathrm{hess}_x h$ has at least $n+1$ negative eigenvalues.

Use that $m < 2{\cdot}n$ and the described property to prove the following analog of Morse lemma for f.

($*$) *Given $x \notin \iota(N)$ there is a neighborhood $U \ni x$ such that the set*

$$U_- = \{\, z \in U \mid f(z) < f(x) \,\}$$

is simply connected.

Since M is simply connected, any closed curve in $\iota(N)$ can be contracted by a disc, say $s_0\colon \mathbb{D} \to M$.

Applying the claim ($*$), one can construct an f-decreasing homotopy that starts at s_0 and ends in $\iota(N)$. That is, a homotopy $s_t\colon \mathbb{D} \to \to M$, $t \in [0,1]$ such that $s_t(\partial\mathbb{D}) \subset \iota(N)$ for any t and $s_1(\mathbb{D}) \subset \iota(N)$. It follows that $\iota(N)$ is simply connected.

Finally assume that a and b are distinct points in N such that $\iota(a) = \iota(b)$. If γ is a path from a to b in N then the loop $\iota\circ\gamma$ is not contractible in $\iota(N)$. That is, if $\iota\colon N \to M$ has a self-intersection, then the image $\iota(N)$ is not simply connected. Hence the result follows. □

The statement was proved by Fuquan Fang, Sérgio Mendonça and Xiaochun Rong [see 69]. The main idea was discovered by Burkhard Wilking [see 70].

Geodesic hypersurface

The totally geodesic embedding is defined before the previous problem.

Assume a compact connected positively curved manifold M has a totally geodesic embedded hypersurface. Show that M or its double cover is homeomorphic to the sphere.

If convex, then embedded

Let M be a complete simply connected Riemannian manifold with non-positive curvature and dimension at least 3. Prove that any immersed locally convex compact hypersurface Σ in M is embedded.

Let us summarize some statements about complete simply connected Riemannian manifolds with non-positive curvature.

By Cartan–Hadamard theorem, for any point $p \in M$ the exponential map $\exp_p\colon \mathrm{T}_p \to M$ is a diffeomorphism. In particular, M is diffeomorphic to the Euclidean space of the same dimension. Moreover, any geodesic in M is minimizing, and any two points in M are connected by a unique minimizing geodesic, Further, M is a CAT[0] space; that is, it satisfies a global angle comparison which we are about to describe. Assume $[xyz]$ is a triangle in M; that is, three distinct points connected pairwise by geodesics. Consider its model triangle $[\tilde x\tilde y\tilde z]$ in the Euclidean plane; that is, a triangle with the coresponding side lengths as in $[xyz]$. Then each angle in $[xyz]$ can not axceed the corresponding angle in $[\tilde x\tilde y\tilde z]$. This inequality can be written as

$$\tilde\measuredangle(y\,{}^{x}_{z}) \geqslant \measuredangle[y\,{}^{x}_{z}],$$

where $\measuredangle[y\,{}^{x}_{z}]$ denotes the angle of the hinge $[y\,{}^{x}_{z}]$ formed by two geodesics $[yx]$ and $[yz]$ and $\tilde\measuredangle(y\,{}^{x}_{z})$ denotes the corresponding angle in the model triangle $[\tilde x\tilde y\tilde z]$.

From this comparison it follows that any connected closed locally convex sets in M is globally convex. In particular, if Σ is embedded then it bounds a convex set.

Immersed ball*

Prove that any immersed locally convex hypersurface $\iota\colon\Sigma\looparrowright M$ in a compact positively curved manifold M of dimension $m\geqslant 3$, is the boundary of an immersed ball. That is, there is an immersion of a closed ball $f\colon\bar B^m\looparrowright M$ and a diffeomorphism $h\colon\Sigma\to\partial\bar B^m$ such that $\iota=f\circ h$.

Minimal surface in the sphere

A smooth n-dimensional surface Σ in an m-dimensional Riemannian manifold M is called *minimal* if it locally minimizes the n-dimensional area; that is, sufficiently small regions of Σ do not admit area decreasing deformations with fixed boundary.

The minimal surfaces can be also defined via mean curvature vector as follows. Let $\mathrm{T}=\mathrm{T}\,\Sigma$ and $\mathrm{N}=\mathrm{N}\,\Sigma$ correspondingly tangent and normal bundle. Let s denotes the second fundamental form of Σ; it is a quadratic from on T with values in N, see the remark after problem “Hypercurve” below. Given an orthonormal basis (e_i) in T_x, set

$$H_x=\sum_i s(e_i,e_i).$$

The vector H_x lies in the normal space N_x and it does not depend on the choice of orthonormal basis (e_i). This vector H_x is called the mean curvature vector at $x\in\Sigma$.

We say that Σ is *minimal* if $H\equiv 0$.

Let Σ be a closed n-dimensional minimal surface in the unit m-dimensional sphere $\mathbb{S}^m$. Prove that $\operatorname{vol}_n \Sigma \geqslant \operatorname{vol}_n \mathbb{S}^n$.

Hypercurve

The Riemannian curvature tensor R can be viewed as an operator $\mathbf{R}$ on the space of tangent bi-vectors $\bigwedge^2 \mathrm{T}$; it is uniquely defined by identity

$$\langle \mathbf{R}(X \wedge Y), V \wedge W\rangle = \langle R(X,Y)V, W\rangle.$$

The operator $\mathbf{R}\colon \bigwedge^2 \mathrm{T} \to \bigwedge^2 \mathrm{T}$ is called *curvature operator* and it is said to be *positive definite* if $\langle \mathbf{R}(\varphi), \varphi\rangle > 0$ for all non zero bi-vector $\varphi \in \bigwedge^2 \mathrm{T}$.

Let $M^m \hookrightarrow \mathbb{R}^{m+2}$ be a closed smooth m-dimensional submanifold and let g be the induced Riemannian metric on M^m. Assume that sectional curvature of g is positive. Prove that the curvature operator of g is positive definite.

The second fundamental form for manifolds of arbitrary codimension which we are about to describe might help to solve this problem.

Assume M is a smooth submanifold in $\mathbb{R}^m$. Given a point $p \in M$ denote by T_p and $\mathrm{N}_p = \mathrm{T}_p^\perp$ the tangent and normal spaces of M at p. The *second fundamental form* of M at p is defined as

$$s(X,Y) = (\nabla_X Y)^\perp,$$

where $(\nabla_X Y)^\perp$ denotes the orthogonal projection of covariant derivative $\nabla_X Y$ onto the normal bundle.

The curvature tensor of M can be found from the second fundamental form using the following formula

$$\langle R(X,Y)V, W\rangle = \langle s(X,W), s(Y,V)\rangle - \langle s(X,V), s(Y,W)\rangle,$$

which is direct generalization of the formula for Gauss curvature of a surface.

Horo-sphere

We say that a Riemannian manifold has negatively pinched sectional curvature, if its sectional curvatures at all points in all sectional directions lie in an interval $[-a^2, -b^2]$, for fixed constants $a > b > 0$.

Let M be a complete Riemannian manifold and γ a ray in M; that is, $\gamma\colon [0,\infty) \to M$ is a minimizing unit-speed geodesic.

The *Busemann function* $\operatorname{bus}_\gamma\colon M \to \mathbb{R}$ is defined by

$$\operatorname{bus}_\gamma(p) = \lim_{t\to\infty} \left(|p - \gamma(t)|_M - t\right).$$

From the triangle inequality, the expression under the limit is non-increasing in t; therefore the limit above is defined for any p.

A *horo-sphere* in M is defined as a level set of a Busemann function in M.

We say that a complete Riemannian manifold M has *polynomial volume growth* if for some (and therefore any) $p \in M$, we have

$$\operatorname{vol} B(p,r)_M \leqslant C\cdot(r^k + 1),$$

where $B(p,r)_M$ denotes the ball in M and C, k are constants.

Let M be a complete simply connected manifold with negatively pinched sectional curvature and $\Sigma \subset M$ be an horo-sphere in M. Show that Σ with the induced intrinsic metric has polynomial volume growth.

Minimal spheres

Recall that two subsets A and B in a metric space X are called *equidistant* if the distance function $\operatorname{dist}_A\colon X \to \mathbb{R}$ is constant on B and dist_B is constant on A.

The minimal surfaces are defined on page 33.

Show that a 4-dimensional compact positively curved Riemannian manifold cannot contain infinite number of mutually equidistant minimal 2-spheres.

Positive curvature and symmetry^{+}

Assume $\mathbb{S}^1$ acts isometrically on a 4-dimensional positively curved closed Riemannian manifold. Show that the action has at most 3 isolated fixed points.

The following statement might be useful. If (M,g) is a Riemannian manifold with sectional curvature $\geqslant \kappa$ that admits a continuous isometric action of a compact group G, then the quotient space $A = (M,g)/G$ is an Alexandrov space with curvature $\geqslant \kappa$; that is, the conclusion of Toponogov comparison theorem holds in A.

For more on Alexandrov geometry read our book [71].

Energy minimizer

Let F be a smooth map from a closed Riemannian manifold M to a Riemannian manifold N. The energy functional of F is defined as

$$E(F) = \int\limits_M |d_x F|^2 \cdot d_x \operatorname{vol}_M .$$

We assume that

$$|d_xF|^2 = \sum_{i,j} a_{i,j}^2,$$

where $(a_{i,j})$ denote the components of the differential d_xF written in the orthonormal bases of the tangent spaces T_xM and $\mathrm{T}_{F(x)}N$.

Show that the identity map on $\mathbb{R}\mathrm{P}^m$ is energy minimizing in its homotopy class. Here we assume that $\mathbb{R}\mathrm{P}^m$ is equipped with canonical metric.

Curvature against injectivity radius$^+$

Let (M,g) be a closed Riemannian m-dimensional manifold. Assume average of sectional curvatures of (M,g) is 1. Show that the injectivity radius of (M,g) is at most π.

Solutions of this and the previous problems use that geodesic flow on the tangent bundle to a Riemannian manifold preserves the volume form; this is a corollary of Liouville's theorem on phase volume.

Approximation of a quotient

Let (M,g) be a compact Riemannian manifold and G be a compact Lie group acting by isometries on (M,g). Construct a sequence of metrics g_n on a fixed manifold N such that (N,g_n) converges to the quotient space $(M,g)/G$ in the sense of Gromov–Hausdorff.

Polar points$^\sharp$

Let M be a compact Riemannian manifold with sectional curvature at least 1 and the dimension at least 2. Prove that for any point $p\in M$ there is a point $p^*\in M$ such that

$$|p-x|_M + |x-p^*|_M \leqslant \pi$$

for any $x\in M$.

Isometric section*

Let M and W be compact Riemannian manifolds, $\dim W > \dim M$ *and $s\colon W\to M$ be a Riemannian submersion. Assume that W has positive sectional curvature. Show that s does not admit an isometric section; that is, there is no isometric embedding $\iota\colon M\hookrightarrow W$ such that $s\circ\iota(p)=p$ for any $p\in M$.*

Warped product

Let (M,g) and (N,h) be Riemannian manifolds and f be a smooth positive function defined on M. Consider the product manifold $W = M\times N$. Given a tangent vector $X\in \mathrm{T}_{(p,q)}W = \mathrm{T}_pM\times\mathrm{T}_pN$ denote by $X_M\in\mathrm{T}M$ and $X_N\in\mathrm{T}N$ its projections. Let us equip W with the Riemannian metric defined as

$$s(X,Y)=g(X_M,Y_M)+f^2\cdot h(X_N,Y_N).$$

The obtained Riemannian manifold (W,s) is called *warped product* of M and N with respect to $f\colon M\to\mathbb{R}$; it can be written as

$$(W,g)=(N,h)\times_f(M,g).$$

Assume M is an oriented 3-dimensional Riemannian manifold with positive scalar curvature and $\Sigma\subset M$ is an oriented smooth hypersurface that is area minimizing in its homology class.

Show that there is a positive smooth function $f\colon\Sigma\to\mathbb{R}$ such that the warped product $\mathbb{S}^1\times_f\Sigma$ has positive scalar curvature; here Σ is equipped with the Riemannian metric induced from M.

No approximation$^{\sharp}$

Prove that if $p\ne 2$, then $\mathbb{R}^m$ equipped with the metric induced by the ℓ^p-norm cannot be a Gromov–Hausdorff limit of m-dimensional Riemannian manifolds (M_n,g_n) with $\mathrm{Ric}_{g_n}\geqslant C$ for some fixed real constant C.

Area of spheres

Let M be a complete non-compact Riemannian manifold with non-negative Ricci curvature and $p\in M$. Show that there is $\varepsilon>0$ such that

$$\operatorname{area}[\partial B(p,r)]>\varepsilon$$

for all sufficiently large r.

Flat coordinate planes

Let g be a complete Riemannian metric on $\mathbb{R}^3$, such that the coordinate planes $x=0$, $y=0$ and $z=0$ are flat and totally geodesic. Assume the sectional curvature of g is either non-negative or non-positive. Show that in both cases g is flat.

Two-convexity♯

An open subset V with smooth boundary in the Euclidean space is called *two-convex* if at most one principle curvatures in the outward direction to V is negative.

The two-convexity of V is equivalent to the following property. For any plane Π and any closed curve γ in the intersection $V \cap \Pi$, if γ is contactable in V then it is contactable in $\Pi \cap V$.

Let K be a closed set bounded by a smooth surface in $\mathbb{R}^4$. Assume K contains two coordinate planes

$$\{(x, y, 0, 0) \in \mathbb{R}^4\} \quad \textit{and} \quad \{(0, 0, z, t) \in \mathbb{R}^4\}$$

in its interior and also belongs to the closed 1-neighborhood of these two planes.

Show that the complement of K is not two-convex.

Semisolutions

Geodesic hypersurface. Let Σ be the totally geodesic embedded hypersurface in the positively curved manifold M. Without loss of generality, we can assume that Σ is connected.[1]

The complement $M \backslash \Sigma$ has one or two connected components. First let us show that if the number of connected components is two, then M is homeomorphic to a sphere.

By cutting M along Σ we get two manifolds with geodesic boundaries. It is sufficient to show that each of them is homeomorphic to a Euclidean ball.

Fix one of these manifolds; denote it by N. Denote by $f\colon N \to \mathbb{R}$ the distance functions to the boundary ∂N. By Riccati equation $\operatorname{hess} f \leqslant 0$ at any smooth point, and for any point the same holds in the barrier sense [defined on page 31]. It follows that f is concave.

Fix an increasing strictly concave function $\varphi\colon \mathbb{R} \to \mathbb{R}$. Note that $\varphi \circ f$ is strictly concave in the interior of N.

Fix a compact subset K in the interior of N and smooth $\varphi \circ f$ in a neighborhood of K keeping it concave. This can be done by applying the smoothing theorem of Greene and Wu [see Theorem 2 in 72].

After the smoothing, the obtained strictly concave function, say h has single critical point which is its maximum. In particular by Morse lemma, we get that if the set

$$N'_s = \{\, x \in N \mid h(x) \geqslant s \,\}$$

[1] In fact, by Frankel's theorem [see page 44] Σ is connected.

is not empty and lies in K then it is diffeomorphic to a Euclidean ball.

For appropriately chosen set K and the smoothing h, the set N'_s can be made arbitrary close to N; moreover, its boundary $\partial N'_s$ can be made C^∞-close to ∂N. It follows that N are diffeomorphic to a Euclidean ball. This finishes the proof of the first case.

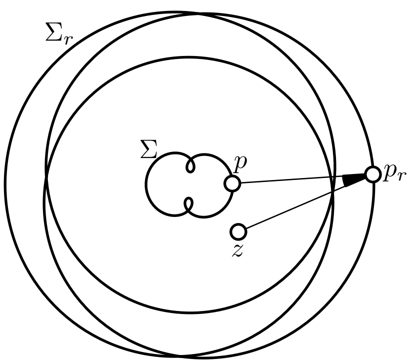
Now assume $M\backslash\Sigma$ is connected. In this case there is a double cover $\tilde M$ of M that induce a double cover $\tilde\Sigma$ of Σ, so $\tilde M$ contains a geodesic hypersurface $\tilde\Sigma$ that divides $\tilde M$ into two connected components. From the case which already has been considered, $\tilde M$ is homeomorphic to a sphere; hence the second case follows. □

The problem was suggested by Peter Petersen.

If convex, then embedded. Set

$$m = \dim\Sigma = \dim M - 1.$$

Given a point p on Σ denote by p_r the point on distance r from p that lies on the geodesic starting from p in the outer normal direction to Σ. Note that for fixed $r \geqslant 0$, the points p_r sweep an immersed locally convex hypersurface which we denote by Σ_r.



Fix $z \in M$. Denote by d the maximal distance from z to the points in Σ. Note that any point on Σ_r lies on a distance at least $r - d$ from z.

By comparison,

$$\measuredangle[p_r\,{}^z_p] \leqslant \arcsin\tfrac{d}{r}.$$

In particular, for large r, each infinite geodesic starting at z intersects Σ_r transversally.

The space of geodesics starting from z is homeomorphic to the sphere $\mathbb{S}^m$. Therefore the map that send a point $x \in \Sigma_r$ to a geodesic from z thru x induce a local diffeomorphism $\varphi_r\colon \Sigma \to \mathbb{S}^m$.

Since $m \geqslant 2$, the sphere $\mathbb{S}^m$ is simply connected. Since Σ is connected, the map φ_r is a diffeomorphism. It follows that Σ_r is star-shaped with center at z. In particular Σ_r is embedded. Since Σ_r is locally convex, it bounds a convex region.

The latter statement holds for all $r \geqslant 0$; this can be shown by applying the open-closed argument. Hence the result follows. □

The problem is due to Stephanie Alexander [see 73].

Immersed ball. Equip Σ with the induced intrinsic metric. Denote by κ the lower bound for principle curvatures of Σ. Note that we can assume that $\kappa > 0$.

Fix sufficiently small $\varepsilon = \varepsilon(M, \kappa) > 0$. Given $p \in \Sigma$ denote by $\Delta(p)$ the ε-ball in Σ centered at p. Consider the lift $\tilde{h}_p\colon \Delta(p) \to \mathrm{T}_{h(p)}$ along the exponential map $\exp_{h(p)}\colon \mathrm{T}_{h(p)} \to M$. More precisely:

1. Connect each point $q \in \Delta(p) \subset \Sigma$ to p by a minimizing geodesic path $\gamma_q\colon [0,1] \to \Sigma$
2. Consider the lifting $\tilde{\gamma}_q$ in $\mathrm{T}_{h(p)}$; that is, the curve such that $\tilde{\gamma}_q(0) = 0$ and $\exp_{h(p)} \circ \tilde{\gamma}_q(t) = \gamma_q(t)$ for any $t \in [0,1]$.
3. Set $\tilde{h}(q) = \tilde{\gamma}_q(1)$.

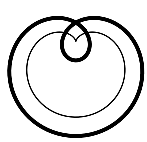
Show that all the hypersurfaces $\tilde{h}_p(\Delta(p)) \subset \mathrm{T}_{h(p)}$ have principle curvatures at least $\frac{\kappa}{2}$.

Use the same idea as in the solution of "Immersed surface" [page 20] to show that one can fix $\varepsilon = \varepsilon(M, \kappa) > 0$ such that the restriction of $\tilde{h}_p|_{\Delta(p)}$ is injective. Conclude that the restriction $h|_{\Delta(p)}$ is injective for any $p \in \Sigma$. (Here we use that $m \geqslant 3$.)

Now consider locally equidistant surfaces Σ_t in the inward direction for small t. The principle curvatures of Σ_t remain at least κ in the barrier sense; that is, at any point p, the surface Σ_t can be supported by a smooth surface with principle curvatures at least κ at p. By the same argument as above, any ε-ball in Σ_t is embedded.

Applying open-closed argument we get a one parameter family of locally convex locally equidistant surfaces Σ_t for t in the maximal interval $[0, a]$, where the surface Σ_a degenerates to a point, say p.

To construct the immersion $\partial \bar{B}^m \looparrowright M$, take the point p as the image of the center $\bar{B}^m$ and take the surfaces Σ_t as the restrictions of the embedding to the spheres; the existence of the immersion follows from the Morse lemma. □



As you see from the picture, the analogous statement does not hold in the two-dimensional case.

The proof presented above was indicated in the lectures of Mikhael Gromov [see 49] and written rigorously by Jost Eschenburg [see 74].

A variation of Gromov's proof was obtained independently by Ben Andrews [see 75]. Instead of equidistant deformation, he uses the so called *inverse mean curvature flow*; this way one has to perform some calculations to show that convexity survives in the flow, but one does not have to worry about non-smoothness of the hypersurfaces Σ_t.

Minimal surface in the sphere. Fix a geodesic n-dimensional sphere $\tilde{\Sigma} = \mathbb{S}^n \subset \mathbb{S}^m$.

Given $r \in (0, \frac{\pi}{2}]$, denote by U_r and $\tilde{U}_r$ the closed tubular r-neighborhood of Σ and $\tilde{\Sigma}$ in $\mathbb{S}^m$ correspondingly.

Note that

$$(*) \qquad U_{\frac{\pi}{2}} = \tilde{U}_{\frac{\pi}{2}} = \mathbb{S}^m.$$

Indeed, clearly $\tilde{U}_{\frac{\pi}{2}} = \mathbb{S}^m$. If $U_{\frac{\pi}{2}} \neq \mathbb{S}^m$, fix $x \in \mathbb{S}^m \backslash U_r$. Choose a closest point $y \in \Sigma$ to x. Since $r = |x - y|_{\mathbb{S}^m} > \frac{\pi}{2}$ the r-sphere $\mathrm{S}_r \subset \mathbb{S}^m$ with center x is concave. Note that S_r supports Σ at y; in particular the mean curvature vector of Σ at y can not vanish, a contradiction.

By Riccati equation,

$$H_r(x) \geqslant \tilde{H}_r$$

for any $x \in \partial U_r$, where $H_r(x)$ denotes the mean curvature of ∂U_r at a point x and $\tilde{H}_r$ is the mean curvature of $\partial \tilde{U}_r$, the latter is the same at all points.

Set

$$a(r) = \mathrm{vol}_{m-1}\, \partial U_r, \qquad \tilde{a}(r) = \mathrm{vol}_{m-1}\, \partial \tilde{U}_r,$$
$$v(r) = \mathrm{vol}_m\, U_r, \qquad \tilde{v}(r) = \mathrm{vol}_m\, \tilde{U}_r.$$

by the coarea formula,

$$\tfrac{d}{dr} v(r) \overset{a.e.}{=\!=} a(r), \qquad \tfrac{d}{dr} \tilde{v}(r) = \tilde{a}(r).$$

Note that

$$\begin{aligned} \tfrac{d}{dr} a(r) &\leqslant \int\limits_{\partial U_r} H_r(x) \cdot d_x \,\mathrm{vol}_{m-1} \leqslant \\ &\leqslant a(r) \cdot \tilde{H}_r \end{aligned}$$

and

$$\tfrac{d}{dr} \tilde{a}(r) = \tilde{a}(r) \cdot \tilde{H}_r.$$

It follows that

$$\frac{v''(r)}{v(r)} \leqslant \frac{\tilde{v}''(r)}{\tilde{v}(r)}$$

for almost all r. Therefore

$$v(r) \leqslant \frac{\mathrm{area}\,\Sigma}{\mathrm{area}\,\tilde{\Sigma}} \cdot \tilde{v}(r)$$

for any $r > 0$.

According to $(*)$,

$$v(\tfrac{\pi}{2}) = \tilde{v}(\tfrac{\pi}{2}) = \operatorname{vol} \mathbb{S}^m.$$

Hence the result follows. □

This problem is the geometric lemma in the proof given by Frederick Almgren of his isoperimetric inequality [see 76]. The argument is similar to the proof of isoperimetric inequality for manifolds with positive Ricci curvature given by Mikhael Gromov [see 77].

Hypercurve. Fix $p \in M$. Denote by s the second fundamental form of M at p. Recall that s is a symmetric bi-linear form on the tangent space T_pM of M with values in the normal space N_pM to M, see page 34.

By Gauss formula

$$\langle R(X,Y)Y,X\rangle = \langle s(X,X), s(Y,Y)\rangle - \langle s(X,Y), s(X,Y)\rangle,$$

Since the sectional curvature of M is positive, we get

$$(*) \qquad \langle s(X,X), s(Y,Y)\rangle > 0$$

for any pair of nonzero vectors $X, Y \in \mathrm{T}_pM$.

The normal space N_pM is two-dimensional. By $(*)$ there is an orthonormal basis e_1, e_2 in N_pM such that the real-valued quadratic forms

$$s_1(X,X) = \langle s(X,X), e_1\rangle, \qquad s_2(X,X) = \langle s(X,X), e_2\rangle$$

are positive definite.

Note that the curvature operators $\mathbf{R}_1$ and $\mathbf{R}_2$, defined by the following identity

$$\mathbf{R}_i(X\wedge Y), V\wedge W\rangle = s_i(X,W)\cdot s_i(Y,V) - s_i(X,V)\cdot s_i(Y,W),$$

are positive. Finally, note that $\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2$ is the curvature operator of M at p. □

The problem is due to Alan Weinstein [see 78]. Note that from [79]/[80] it follows that that the universal cover of M is homeomorphic/diffeomorphic to a standard sphere.

Horo-sphere. Set $m = \dim \Sigma = \dim M - 1$.

Let $\operatorname{bus}\colon M \to \mathbb{R}$ be the Busemann function such that

$$\Sigma = \operatorname{bus}^{-1}\{0\}.$$

Set $\Sigma_r=\mathrm{bus}^{-1}\{r\}$, so $\Sigma_0=\Sigma$.

Let us equip each Σ_r with induced Riemannian metric. Note that all Σ_r have bounded curvature. In particular, the unit balls in Σ_r has volume bounded above by a universal constant, say v_0.

Given $x\in\Sigma$ denote by γ_x the unit-speed geodesic such that $\gamma_x(0)=x$ and $\mathrm{bus}(\gamma_x(t))=t$ for any t. Consider the map $\varphi_r\colon\Sigma\to\Sigma_r$ defined as $\varphi_r\colon x\mapsto\gamma_x(r)$. In other words φ_r is the closest point projection from Σ to Σ_r.

Notice that φ_r is a bi-Lipschitz map with the Lipschitz constants $e^{a\cdot r}$ and $e^{b\cdot r}$. In particular, the ball of radius R in Σ is mapped by φ_r to a ball of radius $e^{a\cdot r}\cdot R$ in Σ_r. Therefore

$$\mathrm{vol}_m B(x,R)_\Sigma\leqslant e^{m\cdot b\cdot r}\cdot\mathrm{vol}_m B(\varphi_r(x),e^{a\cdot r}\cdot R)_{\Sigma_r}$$

for any $R,r>0$. Taking $R=e^{-a\cdot r}$, we get

$$\mathrm{vol}_m B(x,R)_\Sigma\leqslant v_0\cdot R^{m\cdot\frac{b}{a}}$$

for any $R\geqslant 1$. Hence the statement follows. $\square$

The problem was suggested by Vitali Kapovitch.

There are examples of horo-spheres as above with degree of polynomial growth higher than m. For example, consider the horo-sphere Σ in the the complex hyperbolic space of real dimension 4. Clearly $m=\dim\Sigma=3$, but the degree of its volume growth is 4.

In this case Σ is isometric to the Heisenberg group.[2] It is instructive to show that any such metric has volume growth of degree 4.

Minimal spheres. Assuming the contrary, we can choose a pair of sufficiently close minimal spheres Σ and Σ' in the 4-dimesional manifold M; we can assume that the distance a between Σ and Σ' is strictly smaller than the injectivity radius of the manifold. Note that in this case there is a unique bijection $\Sigma\to\Sigma'$, denoted by $p\mapsto p'$ such that the distance $|p-p'|_M=a$ for any $p\in\Sigma$.

Let $\iota_p\colon\mathrm{T}_p\to\mathrm{T}_{p'}$ be the parallel translation along the (necessary unique) minimizing geodesic $[pp']$. Note that there is a pair (p,p') such that $\iota_p(\mathrm{T}_p\Sigma)=\mathrm{T}_{p'}\Sigma'$. Indeed, if this is not the case, then $\iota_p(\mathrm{T}_p\Sigma)\cap\mathrm{T}_{p'}\Sigma'$ forms a continuous line distribution over Σ'. Since Σ' is a two-sphere, the latter contradicts the hairy ball theorem.

[2] *Heisenberg group* is the group of 3×3 upper triangular matrices of the form

$$\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}$$

under the operation of matrix multiplication.

Consider pairs of unit-speed geodesics α and α' in Σ and Σ' that start at p and p' correspondingly and go in the parallel directions, say ν and ν'. Set $\ell_\nu(t) = |\alpha(t) - \alpha'(t)|$.

Use the second variation formula together with the lower bound on Ricci curvature to show that $\ell_\nu''(0)$ has negative average for all tangent directions ν to Σ at p. In particular $\ell_\nu''(0) < 0$ for a vector ν as above; consider the corresponding pair α and α'. It follows that there are points $v = \alpha(\varepsilon) \in \Sigma$ near p and $v' = \alpha'(\varepsilon) \in \Sigma'$ near p' such that

$$|v - v'| < |p - p'|,$$

a contradiction. □

Likely, any compact positively curved 4-dimensional manifold cannot contain a pair of equidistant spheres. The argument above implies that the distance between such a pair has to exceed the injectivity radius of the manifold.

The problem was suggested by Dmitri Burago. Here is a short list of classical problems with use second variation formula in similar fashion:

🕮 *Any compact even-dimensional orientable manifold with positive sectional curvature is simply connected.*

This is called Synge's lemma [see 81].

🕮 *Any two compact minimal hypersurfaces in a Riemannian manifold with positive Ricci curvature must intersect.*

🕮 *Assume Σ_1 and Σ_2 be two compact geodesic submanifolds in a manifold with positive sectional curvature M and*

$$\dim \Sigma_1 + \dim \Sigma_2 \geqslant \dim M.$$

Show that $\Sigma_1 \cap \Sigma_2 \neq \varnothing$.

These two statements proved by Theodore Frankel [see 82].

🕮 *Let (M, g) be a closed Riemannian manifold with negative Ricci curvature. Prove that (M, g) does not admit an isometric $\mathbb{S}^1$-action.*

This is a theorem of Salomon Bochner [see 83].

The problem "Geodesic immersion" [page 31] can be considered as further development of the idea.

Positive curvature and symmetry. Let M be a 4-dimensional Riemannian manifold with isometric $\mathbb{S}^1$-action. Consider the quotient space $X = M/\mathbb{S}^1$. Note that X is a positively curved 3-dimensional

Alexandrov space. In particular the angle $\measuredangle[x\,{}^y_z]$ between any two geodesics $[xy]$ and $[xz]$ is defined. Further, for any non-degenerate triangle $[xyz]$ formed by the minimizing geodesics $[xy]$, $[yz]$ and $[zx]$ in X we have

$$(*)\qquad \measuredangle[x\,{}^y_z]+\measuredangle[y\,{}^z_x]+\measuredangle[z\,{}^x_y]>\pi.$$

Assume $p\in X$ corresponds to a fixed point $\bar p\in M$ of the $\mathbb{S}^1$-action. Each direction of geodesic starting from p in X corresponds to $\mathbb{S}^1$-orbit of the induced isometric action $\mathbb{S}^1\curvearrowright\mathbb{S}^3$ on the sphere of unit vectors at $\bar p$. Any such action is conjugate to the action $\mathbb{S}^1_{p,q}\curvearrowright\mathbb{S}^3\subset\mathbb{C}^2$ induced by complex matrices $\left(\begin{smallmatrix} z^p & 0\\ 0 & z^q\end{smallmatrix}\right)$ with $|z|=1$ and some relatively prime positive integers p,q. The possible quotient spaces $\Sigma_{p,q}=\mathbb{S}^3/\mathbb{S}^1_{p,q}$ have diameter $\frac{\pi}{2}$ and perimeter of any triangle in $\Sigma_{p,q}$ is at most π; this is straightforward to check, but requires work.

It follows that for any three geodesics $[px]$, $[py]$ and $[pz]$ in X we have

$$(**)\qquad \measuredangle[p\,{}^x_y]+\measuredangle[p\,{}^y_z]+\measuredangle[p\,{}^z_x]\leqslant\pi.$$

and

$$({}^{\,*}_{**})\qquad \measuredangle[p\,{}^x_y],\ \measuredangle[p\,{}^y_z],\ \measuredangle[p\,{}^z_x]\leqslant\tfrac{\pi}{2}.$$

Arguing by contradiction, assume that there are 4 fixed points q_1, q_2, q_3 and q_4. Connect each pair by a minimizing geodesic $[q_iq_j]$.

Denote by ω the sum of all 12 angles of the type $\measuredangle[q_i\,{}^{q_j}_{q_k}]$. By $({}^{\,*}_{**})$, each triangle $[q_iq_jq_k]$ is non-degenerate. Therefore by $(*)$, we have

$$\omega>4{\cdot}\pi.$$

Applying $(**)$ at each vertex q_i, we have

$$\omega\leqslant 4{\cdot}\pi,$$

a contradiction. □

The problem is due to Wu-Yi Hsiang and Bruce Kleiner [see 84]. The connection of this proof to Alexandrov geometry was noticed by Karsten Grove [see 85]. An interesting new twist of the idea is given by Karsten Grove and Burkhard Wilking [see 86].

Energy minimizer. Denote by $\mathcal{U}$ the unit tangent bundle over $\mathbb{R}\mathrm{P}^m$ and by $\mathcal{L}$ the space of projective lines in $\ell\colon\mathbb{R}\mathrm{P}^1\to\mathbb{R}\mathrm{P}^m$. The spaces $\mathcal{U}$ and $\mathcal{L}$ have dimensions $2{\cdot}m-1$ and $2{\cdot}(m-1)$ correspondingly.

According to Liouville's theorem on phase volume, the identity

$$\int\limits_{\mathcal{U}} f(v)\cdot d_v\,\mathrm{vol}_{2\cdot m-1} = \int\limits_{\mathcal{L}} d_\ell\,\mathrm{vol}_{2\cdot(m-1)}\cdot \int\limits_{\mathbb{R}\mathrm{P}^1} f(\ell'(t))\cdot dt$$

holds for any integrable function $f\colon \mathcal{U}\to\mathbb{R}$.

Let $F\colon \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^m$ be a smooth map. Note that up to a multiplicative constant, the energy of F can be expressed the following way

$$\int\limits_{\mathcal{U}} |dF(v)|^2\cdot d_v\,\mathrm{vol}_{2m-1} = \int\limits_{\mathcal{L}} d_\ell\,\mathrm{vol}_{2\cdot(m-1)}\cdot \int\limits_{\mathbb{R}\mathrm{P}^1} |[d(F\circ\ell)](t)|^2\cdot dt.$$

Notice that any noncontractable curve in $\mathbb{R}\mathrm{P}^m$ has length at least π. Therefore, by Bunyakovsky inequality, we get

$$\begin{aligned}\int\limits_{\mathbb{R}\mathrm{P}^1} |[d(F\circ\ell)](t)|^2\cdot dt \geqslant \tfrac{1}{\pi}\cdot\Bigg(\int\limits_{\mathbb{R}\mathrm{P}^1} |[d(F\circ\ell)](t)|\cdot dt\Bigg)^2 &=\\ = \tfrac{1}{\pi}\cdot(\mathrm{length}\,F\circ\ell)^2 &\geqslant\\ \geqslant \pi.\end{aligned}$$

for any line $\ell\colon \mathbb{R}\mathrm{P}^1\to\mathbb{R}\mathrm{P}^m$. Hence the result follows. □

The problem is due to Christopher Croke [see 87]. He uses the same idea to show that the identity map on $\mathbb{C}\mathrm{P}^m$ is energy minimizing in its homotopy class. For $\mathbb{S}^m$, an analogous statement does not hold if $m\geqslant 3$. In fact, if a closed Riemannian manifold M has dimension at least 3 and $\pi_1 M = \pi_2 M = 0$, then the identity map on M is homotopic to a map with arbitrary small energy; the latter was shown by Brian White [see 88].

The same idea is used to prove the so called Loewner's inequality [see 89].

Let g be a Riemannian metric on $\mathbb{R}\mathrm{P}^m$ that is conformally equivalent to the canonical metric g_0. Assume that the length of any noncontractable curve in $(\mathbb{R}\mathrm{P}^m, g)$ has length at least π. Show that

$$\mathrm{vol}(\mathbb{R}\mathrm{P}^m, g)\geqslant \mathrm{vol}(\mathbb{R}\mathrm{P}^m, g_0).$$

A more advanced application is the sharp isoperimetric inequality for 4-dimensional Hadamard manifolds proved by Christopher Croke [see 90 and also 91].

Curvature against injectivity radius. We will show that if the injectivity radius of the manifold (M,g) is at least π, then the average of sectional curvatures on (M,g) is at most 1. This is equivalent to the problem.

Fix a point $p\in M$ and two orthonormal vectors $U,V\in \mathrm{T}_pM$. Consider the geodesic γ in M such that $\gamma'(0)=U$.

Set $U_t=\gamma'(t)\in \mathrm{T}_{\gamma(t)}$ and let $V_t\in \mathrm{T}_{\gamma(t)}$ be the parallel translation of $V=V_0$ along γ.

Consider the field $W_t=\sin t\cdot V_t$ on γ. Set

$$\begin{aligned}\gamma_\tau(t)&=\exp_{\gamma(t)}(\tau\cdot W_t),\\ \ell(\tau)&=\mathrm{length}(\gamma_\tau|_{[0,\pi]}),\\ q(U,V)&=\ell''(0).\end{aligned}$$

Note that

$$(*)\qquad q(U,V)=\int_0^\pi[(\cos t)^2-K(U_t,V_t)\cdot(\sin t)^2]\cdot dt,$$

where $K(U,V)$ is the sectional curvature in the direction spanned by U and V.

Since any geodesics of length π is minimizing, we get $q(U,V)\geqslant 0$ for any pair of orthonormal vectors U and V. It follows that average value of the right hand side in $(*)$ is non-negative.

By Liouville's theorem on phase volume, while taking the average of $(*)$, we can switch the order of integrals; therefore

$$0\leqslant \tfrac{\pi}{2}\cdot(1-\bar K),$$

where $\bar K$ denotes the average of sectional curvatures on (M,g). Hence the result follows. □

The problem illustrates the idea of Eberhard Hopf [see 92] which was developed further by Leon Green [see 93]. Hopf used it to show that a metric on torus without conjugate points must be flat and Green showed that average of sectional curvature on closed manifold without conjugate points can not be positive.

More applications of Liouville's theorem on phase volume discussed in the comments the solution of "Energy minimizer", page 46.

Approximation of a quotient. Note that G admits an embedding into a compact connected Lie group H; in fact we can assume that $H=\mathrm{SO}(n)$, for large enuf n.

Fix a $\kappa\leqslant 0$ such that the curvature bound of (M,g) is bounded below by κ.

The bi-invariant metric h on H is non-negatively curved. Therefore for any positive integer n the product $(H,\frac1n\cdot h)\times(M,g)$ is a Riemannian manifold with curvature bounded below by κ.

The diagonal action of G on $(H, \frac{1}{n}\cdot h)\times(M,g)$ is isometric and free. Therefore the quotient $(H, \frac{1}{n}\cdot h)\times(M,g)/G$ is a Riemannian manifold, say (N, g_n). By O'Nail's formula, (N, g_n) has curvature bounded below by κ.

It remains to observe that the spaces (N, g_n) converge to $(M,g)/G$ as $n\to\infty$. $\square$

This construction is called *Cheeger's trick*. The earliest use of this trick I found in [94]; it was used there to show that Berger's spheres have positive curvature. This trick is used in the constructions of most of the known examples of positively and non-negatively curved manifolds [see 95–99].

The quotient space $(M,g)/G$ has finite dimension and curvature bounded below in the sense of Alexandrov. It is expected that not all finite dimensional Alexandrov spaces admit approximation by Riemannian manifolds with curvature bounded below [some partial results are discussed in 100, 101].

Polar points. Fix a unit-speed geodesic γ that starts at p; that is, $\gamma(0)=p$. Set $p^*=\gamma(\pi)$.

Applying Toponogov comparison theorem for the triangle $[pp^*x]$, we get

$$|p^*-x'|_g+|p-x'|_g>\pi.$$

That is, p^* is a solution. $\square$

Alternative proof. Assume the contrary; that is, for any $x\in M$ there is a point x' such that

$$|x-x'|_g+|p-x'|_g>\pi.$$

Given $x\in M$ denote by $f(x)$ a point that maximize the following sum

$$|x-f(x)|_g+|p-f(x)|_g.$$

Show that the f is uniquely defined and continuous.

Fix sufficiently small $\varepsilon>0$. Prove that the set $W_\varepsilon=M\backslash B(p,\varepsilon)$ is homeomorphic to a ball and the map f sends W_ε into itself.

By Brouwer's fixed-point theorem, $x=f(x)$ for some x. In this case

$$\begin{aligned}|x-f(x)|_g+|p-f(x)|_g=|p-x|_g\leqslant\\ \leqslant\pi,\end{aligned}$$

a contradiction. $\square$

The problem is due to Anatoliy Milka [see 102].

Isometric section. Arguing by contradiction, assume there is an isometric section $\iota\colon M\to W$. It makes possible to treat M as a submanifold in W.

Given $p\in M$, denote by N^1_p the unit normal space to M at p. Given $v\in\mathrm{N}^1_p$ and real value k, set

$$p^{k\cdot v}=s\circ\exp_p(k\cdot v).$$

Note that

$$(*)\qquad p^{0\cdot v}=p\ \text{ for any }\ p\in M\ \text{ and }\ v\in\mathrm{N}^1_p.$$

Fix sufficiently small $\delta>0$. By Rauch comparison, if $w\in\mathrm{N}^1_q$ is the parallel translation of $v\in\mathrm{N}^1_q$ along a minimizing geodesic from p to q in M, then

$$(**)\qquad |p^{k\cdot v}-q^{k\cdot w}|_M<|p-q|_M$$

assuming $|k|\leqslant\delta$. The same comparison implies that

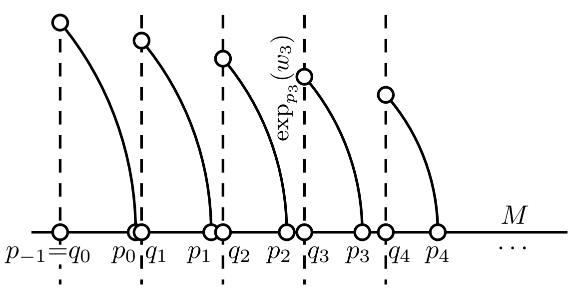
$$({}^{\,*}_{**})\qquad |p^{k\cdot v}-q^{k'\cdot w}|^2_M<|p-q|^2_M+(k-k')^2$$

assuming $|k|,|k'|\leqslant\delta$.

Choose p and $v\in\mathrm{N}^1_p$ so that $r=|p-p^{\delta\cdot v}|$ takes the maximal possible value. From $(**)$ it follows that $r>0$.

Let γ be the extension of the unit-speed minimizing geodesic from p_v to p; denote by v_t the parallel translation of v to $\gamma(t)$ along γ.

We can choose the parameter of γ so that $p=\gamma(0)$, $p^v=\gamma(-r)$. Set $p_n=\gamma(n\cdot r)$, so $p=p_0$ and $p^v=p_{-1}$. Fix large integer N and set $w_n=(1-\frac{n}{N})\cdot v_{n\cdot r}$ and $q_n=p_n^{w_n}$.



From $({}^{\,*}_{**})$, there is a constant C independent of N such that

$$|q_k-q_{k+1}|<r+\tfrac{C}{N^2}\cdot\delta^2.$$

Therefore

$$|q_{k+1}-p_{k+1}|>|q_k-p_k|-\tfrac{C}{N^2}\cdot\delta^2.$$

By induction, we get

$$|q_N - p_N| > r - \tfrac{C}{N}\cdot\delta^2.$$

Since N is large we get

$$|q_N - p_N| > 0.$$

Note that $w_N = 0$; therefore by $(*)$, we get $q_N = p_N^0 = p_N$, a contradiction. □

This is the core of the solution of Soul conjecture by Grigori Perelman [see 103].

Warped product. Given $x \in \Sigma$, denote by ν_x the normal vector to Σ at x that agrees with the orientations of Σ and M. Denote by κ_x the non-negative principle curvature of Σ at x; since Σ is minimal the other principle curvature has to be $-\kappa_x$.

Consider the warped product $W = \mathbb{S}^1 \times_f \Sigma$ for some positive smooth function $f\colon \Sigma \to \mathbb{R}$. Assume that a point $y \in W$ projects to a point $x \in \Sigma$. Straightforward computations show that

$$\begin{aligned}\mathrm{Sc}_W(y) = \mathrm{Sc}_\Sigma(x) - 2\cdot\frac{\Delta f(x)}{f(x)} &= \\ &= \mathrm{Sc}_M(x) - 2\cdot\mathrm{Ric}(\nu_x) - 2\cdot\kappa_x^2 - 2\cdot\frac{\Delta f(x)}{f(x)},\end{aligned}$$

where Sc and Ric denote the scalar and Ricci curvature correspondingly.

Consider linear operator L on the space of smooth functions on Σ defined as

$$(Lf)(x) = -[\mathrm{Ric}(\nu_x) + \kappa_x^2]\cdot f(x) - (\Delta f)(x)$$

It is sufficient to find a smooth function f on Σ such that

$$(*) \qquad f(x) > 0 \quad\text{and}\quad (Lf)(x) \geqslant 0$$

for any $x \in \Sigma$.

Fix a smooth function $f\colon \Sigma \to \mathbb{R}$. Extend the field $f(x)\cdot\nu_x$ on Σ to a smooth field, say v, on whole M. Denote by ι_t the flow along v for time t and set $\Sigma_t = \iota_t(\Sigma)$.

Informal end of proof. Denote by $H_t(x)$ the mean curvature of Σ_t at $\iota_t(x)$. Note that the value $(Lf)(x)$ is the derivative of the function $t \mapsto H_t(x)$ at $t = 0$.

Therefore the condition $(*)$ means that we can push Σ into one of its sides so that its mean curvature does not increase in the first order. Since Σ is area minimizing, such push can be obtained by increasing the pressure on one side of Σ.

(Read further if you are not convinced.) $\square$

Formal end of proof. Denote by $\delta(f)$ the second variation of area of Σ_t; that is, consider the area function $a(t) = \operatorname{area}\Sigma_t$ and set $\delta(f) = a''(0)$. Direct calculations show that

$$\begin{aligned}\delta(f) &= \int\limits_{\Sigma} \left(-[\mathrm{Ric}(\nu_x) + \kappa_x^2]\cdot f^2(x) + |\nabla f(x)|^2\right)\cdot d_x \operatorname{area} = \\ &= \int\limits_{\Sigma} (Lf)(x)\cdot f(x)\cdot d_x \operatorname{area}.\end{aligned}$$

Since Σ is area minimizing we get

$$(**) \qquad \delta(f) \geqslant 0$$

for any f.

Choose a function f that minimize $\delta(f)$ among all the functions such that $\int_\Sigma f^2(x)\cdot d_x \operatorname{area} = 1$. Note that f an eigenfunction for the linear operator L; in particular f is smooth. Denote by λ the eigenvalue of f; by $(**)$, $\lambda \geqslant 0$.

Show that $f(x) > 0$ at any x. Since $Lf = \lambda\cdot f$, the inequalities $(*)$ follow. $\square$

The problem is due to Mikhael Gromov and Blaine Lawson [see 104]. Earlier, in [105], Shing-Tung Yau and Richard Schoen showed that the same assumptions imply existence of conformal factor on Σ that makes it positively curved. Both statement are used the same way to proof that $\mathbb{T}^3$ does not admit a metric with positive scalar curvature.

Both statements admit straightforward generalization to higher dimensions and they can be used to show non existence metric with positive scalar curvature on $\mathbb{T}^m$ with $m \leqslant 7$. For $m = 8$, the proof stops to work since in this dimension the area minimizing hypersurfaces might have singularities. For example, any domain in the cone in $\mathbb{R}^8$ defined by the identity

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2$$

is area minimizing among the hypersurfaces with the same boundary.

No approximation. Fix an increasing function $\varphi\colon (0,r) \to \mathbb{R}$ such that

$$\varphi'' + (n-1)\cdot(\varphi')^2 + C = 0.$$

If $\operatorname{Ric}_{g_n} \geqslant C$, then the function $x \mapsto \varphi(|q-x|_{g_n})$ is subharmonic. Therefore for arbitrary array of points q_i and positive reals λ_i the function $f_n \colon M_n \to \mathbb{R}$ defined by the formula

$$f(x) = \sum_i \lambda_i \cdot \varphi(|q_i - x|_M)$$

is subharmonic. In particular f_n cannot admit a local minimums in M_n.

Passing to the limit as $n \to \infty$, we get that any function $f \colon \mathbb{R}^m \to \mathbb{R}$ of the form

$$f(x) = \sum_i \lambda_i \cdot \varphi(|q_i - x|_{\ell_p})$$

does not admit a local minimums in $\mathbb{R}^m$.

Let e_i be the standard basis of $\mathbb{R}^m$. If $p < 2$, consider the sum

$$f(x) = \sum \varphi(|q - x|_{\ell_p}),$$

where $q = \pm\varepsilon\cdot e_i$ for all singes and i's. Straightforward calculation show that if $\varepsilon > 0$ is small, then f has strict local minimum at 0.

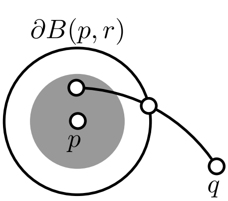
If $p > 2$, one has to take the same sum for $p = \sum_i \pm\varepsilon\cdot e_i$ for all choices of signs. In both case we arrive to a contradiction. $\square$

The argument given here is very close to the proof of Abresch–Gromoll inequality [see 106]. The solution admits a straightforward generalization which imples that if an m-dimensional Finsler manifold F is a Gromov–Hausdorff limit of m-dimensional Riemannian manifolds with uniform lower bound on Ricci curvature, then F has to be Riemannian.

An alternative solution of this problem can be build on almost splitting theorem proved by Jeff Cheeger and Tobias Colding [see 107].

Area of spheres. Fix $r_0 > 0$. Given $r > r_0$, choose a point q on the distance $2\cdot r$ from p.

Note that any minimizing geodesic from q to a point in $B(p, r_0)$ has to cross $\partial B(p, r)$. By volume comparison, we get that



$$\operatorname{vol} B(p, r_0) \leqslant C_m \cdot r_0 \cdot \operatorname{area} \partial B(p, r),$$

where C_m is a constant depending only on the dimension $m = \dim M$ ($C_m = 10^m$ will do). $\square$

Applying the coarea formula, we get that volume growth of M is at least linear and in particular it has infinite volume. The latter was proved independently by Eugenio Calabi and Shing-Tung Yau [see 108, 109].

Flat coordinate planes. Fix $\varepsilon > 0$ such that there is unique geodesic between any two points on distance $< \varepsilon$ from the origin of $\mathbb{R}^3$.

Consider three points a, b and c on the coordinate lines that are ε-close to the origin. The following observation is the key to the proof.

$(*)$ *There is a solid flat geodesic triangle in* $(\mathbb{R}^3, g)$ *with vertex at* a, b *and* c.

Indeed, note that parallel translation along the coordinate lines preserves the directions into coordinate planes. In particular the angles between coordinate planes in $(\mathbb{R}^3, g)$ are constant. It follows that the angles of the triangle $[abc]$ coincide with its *model angles*; that is, the angles in the plane triangle with the same sides.

Both curvature conditions imply that the triangle $[abc]$ bounds a solid flat geodesic triangle in $(\mathbb{R}^3, g)$.

Use the family of constructed flat triangles to show that at any x point in the $\frac{\varepsilon}{10}$-neighborhood of the origin the sectional curvature vanish in an open set of sectional directions. The latter implies that the curvature is identically zero in this neighborhood.

Move the origin and apply the same argument locally. This way we get that the curvature is identically zero everywhere. $\square$

This problem is based on a lemma discovered by Sergei Buyalo in [see Lemma 5.8 in 110 and also 111].

Two-convexity. *Morse-style solution.* Equip $\mathbb{R}^4$ with coordinates (x, y, z, t).

Consider a generic linear function $\ell\colon \mathbb{R}^4 \to \mathbb{R}$ that is close to the sum of coordinates $x+y+z+t$. Note that ℓ has non-degenerate critical points on ∂K and all its critical values are different.

Consider the sets

$$W_s = \left\{\, w \in \mathbb{R}^4 \backslash K \;\middle|\; \ell(w) < s \,\right\}.$$

Note that W_{-1000} contains a closed curve, say α that is contactable in $\mathbb{R}^4\backslash K$, but not constructible in W_{-1000}.

Set s_0 to be the infimum of the values s such that the α is contactable in W_s.

Note that s_0 is a critical value of ℓ on ∂K; denote by p_0 the corresponding critical point. By 2-convexity of $\mathbb{R}^4\backslash K$, the index of p_0 has to be at most 1. On the other hand, since the disk hangs at this point, its index has to be at least 2, a contradiction. $\square$

Alexandrov-style proof. Assume that the complement of K is two-convex.

Note that two-convexity is preserved under linear transformation. Apply a linear transformation of $\mathbb{R}^4$ that makes the coordinate planes Π_1 and Π_2 not orthogonal.

According to the main result in [112], $W = \mathbb{R}^4\backslash(\operatorname{Int} K)$ has nonpositive curvature in the sense of Alexandrov. In particular the universal metric cover $\tilde{W}$ of W is a CAT[0] space.

By rescaling $\tilde{W}$ and passing to the limit we obtain that universal Riemannian cover Z of $\mathbb{R}^4$ branching in the planes Π_1 and Π_2 is a CAT[0] space.

Note that Z is isometric to the Euclidean cone over universal cover Σ of $\mathbb{S}^3$ branching in two great circles $\Gamma_i = \mathbb{S}^3 \cap \Pi_i$ that are not orthogonal. The shortest path in $\mathbb{S}^3$ between Γ_1 and Γ_2 traveled 4 times back and forth is shorter than $2{\cdot}\pi$ and it lifts to closed geodesic in Σ. It follows that Σ is not CAT[1] and therefore Z is not CAT[0], a contradiction. □

The Morse-style proof is based on the idea of Mikhael Gromov [see 49, §$\frac{1}{2}$], where two-convexity was introduced.

Note that the 1-neighborhood of these two planes has two-convex complement W in the sense of the second definition; that is, if a closed curve γ lies in the plane Π and contactable in W then it is contactable in $\Pi \cap W$. Clearly the boundary of this neighborhood is not smooth and as it follows from the problem, it cannot be smoothed in the class of two-convex sets.

Two-convexity also shows up in comparison geometry — the maximal open flat sets in the manifolds of nonnegative or nonpositive curvature are two convex [see 111].

Chapter 4

Curvature free differential geometry

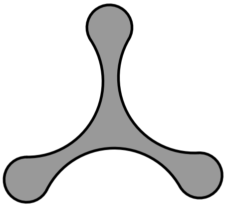
The reader should be familiar with the notions of smooth manifolds, Riemannian metrics and symplectic forms.

Distant involution

Construct a Riemannian metric g on $\mathbb{S}^3$ and an involution $\iota\colon \mathbb{S}^3 \to \to \mathbb{S}^3$ such that $\mathrm{vol}(\mathbb{S}^3, g)$ *is arbitrary small and*

$$|x - \iota(x)|_g > 1$$

for any $x \in \mathbb{S}^3$.



Semisolution. Given $\varepsilon > 0$, construct a disk Δ in the plane with

$$\operatorname{length} \partial\Delta < 10 \quad \text{and} \quad \operatorname{area} \Delta < \varepsilon$$

that admits an continuous involution ι such that

$$|\iota(x) - x| \geqslant 1$$

for any $x \in \partial\Delta$.

An example of Δ can be guessed from the picture; the invoultion ι makes a length preserving half turn of its boundary $\partial\Delta$.

Take the product $\Delta \times \Delta \subset \mathbb{R}^4$; it is homeomorphic to the 4-ball. Note that

$$\mathrm{vol}_3[\partial(\Delta \times \Delta)] = 2 \cdot \operatorname{area} \Delta \cdot \operatorname{length} \partial\Delta < 20 \cdot \varepsilon.$$

The boundary $\partial(\Delta \times \Delta)$ homeomorphic to $\mathbb{S}^3$ and the restriction of the involution $(x, y) \mapsto (\iota(x), \iota(y))$ has the needed property.

It remains to slightly smooth $\partial(\Delta \times \Delta)$. $\square$

This example is given by Christopher Croke [see 113].

It is instructive to show that for $\mathbb{S}^2$ such thing is not possible. Note also, that according to Gromov's systolic inequality [see 89], the involution ι above cannot be made isometric.

Besikovitch inequality

Let g be a Riemannian metric on an m-dimensional cube Q such that any curve connecting opposite faces has length at least 1. Prove that

$$\operatorname{vol}(Q, g) \geqslant 1,$$

and the equality holds if and only if (Q, g) is isometric to the unit cube.

Minimal foliation^{+}

The minimal surfaces in Riemannian manifolds are defined on page 33.

Consider the product of spheres $\mathbb{S}^2 \times \mathbb{S}^2$ equipped with a Riemannian metric g that is C^∞-close to the product metric. Prove that there is a conformally equivalent metric $\lambda \cdot g$ and a re-parametrization of $\mathbb{S}^2 \times \mathbb{S}^2$ such that for any $x \in \mathbb{S}^2$, the spheres $x \times \mathbb{S}^2$ and $\mathbb{S}^2 \times x$ are minimal surfaces in $(\mathbb{S}^2 \times \mathbb{S}^2, \lambda \cdot g)$.

The expected solution requires pseudo-holomorphic curves introduced by Mikhael Gromov [see 114].

Volume and convexity^{+}

A function f defined on a Riemannian manifold is called convex if for any geodesic γ, the composition $f \circ \gamma$ is a convex real-to-real function.

Let M be a complete Riemannian manifold that admits a non-constant convex function. Prove that M has infinite volume.

The expected solution uses Liouville's theorem on phase volume. It implies in particular, that the geodesic flow on the unit tangent bundle of a Riemannian manifold preserves the volume.

Sasaki metric

Let (M,g) be a Riemannian manifold. The Sasaki metric is a natural choice of Riemannian metric $\hat{g}$ on the total space of the tangent bundle $\tau\colon \mathrm{T}M \to M$. It is uniquely defined by the following properties:

- ⋄ The map $\tau\colon (\mathrm{T}M,\hat{g}) \to (M,g)$ is a Riemannian submersion.
- ⋄ The metric on each tangent space $\mathrm{T}_p \subset \mathrm{T}M$ is the Euclidean metric induced by g.
- ⋄ Assume $\gamma(t)$ is a curve in M and $v(t) \in \mathrm{T}_{\gamma(t)}$ is a parallel vector field along γ. Note that $v(t)$ forms a curve in $\mathrm{T}M$. For the Sasaki metric, we have $v'(t) \perp \mathrm{T}_{\gamma(t)}$ for any t; that is, the curve $v(t)$ normally crosses the tangent spaces $\mathrm{T}_{\gamma(t)} \subset \mathrm{T}M$.

In other words, we identify the tangent space $\mathrm{T}_u[\mathrm{T}M]$ for any $u \in \mathrm{T}_pM$ with the direct sum of vertical and horizontal subspaces $\mathrm{T}_pM \oplus \oplus \mathrm{T}_pM$. The projection of this splitting is defined by the differential $d\tau\colon \mathrm{TT}M \to \mathrm{T}M$ and we assume that the velocity of a curve in $\mathrm{T}M$ formed by a parallel field along a curve in M is horizontal. Then $\mathrm{T}_u[\mathrm{T}M]$ is equipped with the metric $\hat{g}$ defined as

$$\hat{g}(X,Y) = g(X^V,Y^V) + g(X^H,Y^H),$$

where $X^V, X^H \in \mathrm{T}_pM$ denote the vertical and horizontal components of $X \in \mathrm{T}_u[\mathrm{T}M]$.

🕮 *Let g be a Riemannian metric on the sphere $\mathbb{S}^2$. Consider the tangent bundle $\mathrm{T}\mathbb{S}^2$ equipped with the induced Sasaki metric $\hat{g}$. Show that the space $(\mathrm{T}\mathbb{S}^2,\hat{g})$ lies at bounded distance to the ray $\mathbb{R}_+ = [0,\infty)$ in the sense of Gromov–Hausdorff.*

Two-systole

🕮 *Given a big real number L, construct a Riemannian metric g on the 3-dimensional torus $\mathbb{T}^3$ such that* $\operatorname{vol}(\mathbb{T}^3,g) = 1$ *and*

$$\operatorname{area} S \geqslant L$$

for any closed surface S that does not bound in $\mathbb{T}^3$.

According to Gromov's systolic inequality [see 89], the volume of $(\mathbb{T}^3,g)$ can be bounded below in terms of its *1-systole* defined to be the shortest length of a noncontractible closed curve in $(\mathbb{T}^3,g)$. The lower bound on the area of S in the problem is called the 2-systole of $(\mathbb{T}^3,g)$.

The problem implies that the Gromov's systolic inequality does not have a direct 2-dimensional analog.

Normal exponential map°

Let (M,g) be a Riemannian manifold; denote by $\mathrm{T}M$ the tangent bundle over M and by $\mathrm{T}_p = \mathrm{T}_pM$ the tangent space at the point p.

Given a vector $v \in \mathrm{T}_pM$ denote by γ_v the geodesic in (M,g) such that $\gamma(0) = p$ and $\gamma'(0) = v$. The map $\exp\colon \mathrm{T}M \to M$ defined by $v \mapsto \gamma_v(1)$ is called the exponential map.

The restriction of $\exp|_{\mathrm{T}_p}$ is called the *exponential map at p* and is denoted as $\exp_p$.

Given a smooth immersion $L \to M$; denote by $\mathrm{N}L$ the normal bundle over L. The restriction $\exp|_{\mathrm{N}L}$ is called the *normal exponential map* of L and is denoted by $\exp_L$.

Let M, L be complete connected Riemannian manifolds. Assume L is immersed into M. Show that the image of the normal exponential map of L is dense in M.

Symplectic squeezing in the torus

Let

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$$

be the standard symplectic form on $\mathbb{R}^4$. Assume $\mathbb{Z}^2$ is the integer lattice in the (x_1, y_1) coordinate plane of $\mathbb{R}^4$.

Show that an arbitrary bounded domain $\Omega \subset (\mathbb{R}^4, \omega)$ admits a symplectic embedding into the quotient space $(\mathbb{R}^4, \omega)/\mathbb{Z}^2$.

Diffeomorphism test°

Let M and N be complete m-dimensional simply connected Riemannian manifolds. Assume $f\colon M \to N$ is a smooth map such that

$$|df(v)| \geqslant |v|$$

for any tangent vector v of M. Show that f is a diffeomorphism.

Volume of tubular neighborhoods⁺

Assume M and M' are isometric closed smooth submanifolds in a Euclidean space. Show that for all small $r > 0$ we have

$$\operatorname{vol} B(M, r) = \operatorname{vol} B(M', r),$$

where $B(M, r)$ denotes the r-neighborhood of M.

disk*

🕮 *Given a big real number L, construct a Riemannian metric g on the disk $\mathbb{D}$ with*

$$\operatorname{diam}(\mathbb{D}, g) \leqslant 1 \quad \textit{and} \quad \operatorname{length} \partial\mathbb{D} \leqslant 1$$

such that the boundary curve in $\mathbb{D}$ is not contractible in the class of closed curves with g-length less than L.

Shortening homotopy

🕮 *Let M be a compact Riemannian manifold with diameter D and $p \in M$. Assume that for some $L > D$, there are no geodesic loops based at p in M with length in the interval $(L-D, L+D]$. Show that for any path γ_0 in (M, g) starting at p, there is a homotopy γ_t rel. to its endpoints such that*

a) $\operatorname{length} \gamma_1 < L$*;*

b) $\operatorname{length} \gamma_t \leqslant \operatorname{length} \gamma_0 + 2{\cdot}D$ *for any $t \in [0,1]$.*

It is not easy at all to find an example of a manifold satisfying the above condition for some L; they can be found among the Zoll spheres by Florent Balachev, Christopher Croke and Mikhail Katz [see 115].

Convex hypersurface

Recall that a subset K of Riemannian manifold is called *convex* if every minimizing geodesic connecting two points in K lies completely in K.

🕮 *Let M be a totally geodesic hypersurface in a closed Riemannian m-dimensional manifold W. Assume that the injectivity radius of M is at least* 1 *and M forms a convex set in W.*

Show that the maximal distance from M to the points of W can be bounded below by a positive constant ε_m that depends only on the dimension m (in fact, $\varepsilon_m = \frac{2}{m+3}$ will do).

Note that we did not make any assumption on the injectivity radius of W.

Almost constant function

The unit tangent bundle $\mathrm{U}M$ over a closed Riemannian manifold M admits a natural choice of volume. Let us equip $\mathrm{U}M$ with the probability measure that is proportional to the volume.

We say that a unit-speed geodesic $\gamma\colon \mathbb{R} \to M$ is *random* if $\gamma'(0)$ takes a random value in $\mathrm{U}M$.

Assume $\varepsilon > 0$ is given. Show that there is a positive integer m such that for any closed m-dimensional Riemannian manifold M and any smooth 1-Lipschitz function $f\colon M \to \mathbb{R}$ the following holds.

For a random unit-speed geodesic γ in M the event

$$|f\circ\gamma(0) - f\circ\gamma(1)| > \varepsilon$$

happens with probability at most ε.

Semisolutions

Besikovitch inequality. Without loss of generality, we may assume that $Q = [0,1]^m$. Set

$$A_i = \{\, (x_1, \dots, x_m) \in Q \mid x_i = 0 \,\}.$$

Consider the functions $f_i\colon Q \to \mathbb{R}$ defined as

$$f_i(x) = \min\{1, \operatorname{dist}_{A_i}(x)\}$$

Note that each f_i is 1-Lipschitz, in particular $|\nabla f_i| \leqslant 1$ almost everywhere.

Consider the map

$$\boldsymbol{f}\colon x \mapsto (f_1(x), \dots, f_m(x)).$$

Note that it maps Q to itself and, moreover, it maps each face of Q to itself. It follows that the restriction $\boldsymbol{f}|_{\partial Q}\colon \partial Q \to \partial Q$ has degree one and therefore $\boldsymbol{f}\colon Q \to Q$ is onto.

Assume h is the canonical metric on the cube Q. Denote by J the Jacobian of the map $f\colon (Q,g) \to (Q,h)$. Note that

$$|\mathrm{J}(x)| = |\nabla_x f_1 \wedge \dots \wedge \nabla_x f_m| \leqslant 1.$$

By the area formula, we get

$$\begin{aligned} \operatorname{vol}(Q,g) &\geqslant \int_Q |\mathrm{J}(x)| \cdot d_x \operatorname{vol}_g \geqslant \\ &\geqslant \operatorname{vol}(Q,h) = \\ &= 1 \end{aligned}$$

In the case of equality we have that $\langle\nabla_x f_i,\nabla_x f_j\rangle=0$ for $i\ne j$ and $|\nabla_x f_i|=1$ for almost all x. It follows then that the map

$$\boldsymbol{f}\colon (Q,g)\to(Q,h)$$

is an isometry. $\square$

This inequality was proved by Abram Besikovitch [see 116]. It has a number of applications in Riemannian geometry. For example using this inequality it is easy to solve the following problem.

Assume a metric g on $\mathbb{R}^m$ coincides with the Euclidean one outside of a bounded set K; assume further that any geodesic that comes into K goes out from K the same way the Euclidean geodesic would. Show that g is flat.

The same proof gives a ruf version of this inequality for any metric on the cube. Here is one of its applications suggested by Stephan Stadler.

Suppose X is contractible metric space with vanishing $(n+1)$-dimensional Hausdorff measure. Assume $\Delta_1,\Delta_2\subset X$ are two embedded n-disks having the same boundary. Show that $\Delta_1=\Delta_2$.

Minimal foliation. The proof is based on the observation that a self-dual harmonic 2-form on $(\mathbb{S}^2\times\mathbb{S}^2,g)$ without zeros defines a symplectic structure.

Note that there is a self-dual harmonic 2-form on $(\mathbb{S}^2\times\mathbb{S}^2,g)$; that is, a 2-form ω such that $d\omega=0$ and $\star\omega=\omega$, where $\star$ denotes the Hodge star operator. Indeed, fix a generic harmonic form φ. Note that the form $\star\varphi$ is also harmonic. Since $\star(\star\varphi)=\varphi$, the form $\omega=\varphi+\star\varphi$ does the job.

Fix $p\in\mathbb{S}^2\times\mathbb{S}^2$. We can use g_p to identify the tangent space T_p and the cotangent space T_p^*. There is a g_p-orthonormal basis e_1,e_2,e_3,e_4 on T_p such that

$$\omega_p=\lambda_p{\cdot}e_1\wedge e_2+\lambda_p'{\cdot}e_3\wedge e_4.$$

Note that

$$\star\omega_p=\lambda_p'{\cdot}e_1\wedge e_2+\lambda_p{\cdot}e_3\wedge e_4.$$

Since $\star\omega_p=\omega_p$, we have $\lambda_p=\lambda_p'$.

Consider the rotation $\mathrm{J}_p\colon\mathrm{T}_p\to\mathrm{T}_p$ defined by

$$e_1\mapsto -e_2,\quad e_2\mapsto e_1,\quad e_3\mapsto -e_4,\quad e_4\mapsto e_3.$$

Note that

$$\mathrm{J}_p \circ \mathrm{J}_p = -\,\mathrm{id} \quad \text{and} \quad \omega(X,Y) = \lambda_p \cdot g(X, \mathrm{J}_p Y)$$

for any two tangent vectors $X, Y \in \mathrm{T}_p$.

Consider the canonical symplectic form ω_0 on $\mathbb{S}^2 \times \mathbb{S}^2$; that is, the sum of the pullbacks of the volume form on $\mathbb{S}^2$ by the two coordinate projections $\mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{S}^2$. Note that for the canonical metric on $\mathbb{S}^2 \times \mathbb{S}^2$, the form ω_0 is harmonic and self-dual. Since g is close to the standard metric, we can assume that ω is close to ω_0. In particular $\lambda_p \neq 0$ for any $p \in \mathbb{S}^2 \times \mathbb{S}^2$.

It follows that J is a pseudo-complex structure for the symplectic form ω on $\mathbb{S}^2 \times \mathbb{S}^2$. The Riemannian metric $g' = \lambda \cdot g$ is conformal to g and $\omega(X,Y) = g'(X, \mathrm{J}Y)$ for any two tangent vectors X, Y at one point. In this case the J-holomorphic curves are minimal with respect to g'; in fact, they are area minimizing in its homology class.

It remains to re-parametrize $\mathbb{S}^2 \times \mathbb{S}^2$ so that vertical and horizontal spheres will form pseudo-holomorphic curves in the homology classes of $x \times \mathbb{S}^2$ and $\mathbb{S}^2 \times y$. $\square$

For general metrics the form ω might vanish at some points. If the metric is generic, then it happens on disjoint circles [see 117].

Volume and convexity. We use the idea from the proof of the Poincaré recurrence theorem.

Let M be a complete Riemannian manifold that admits a convex function f. Denote by $\tau\colon \mathrm{U}M \to M$ the unit tangent bundle over M. Consider the function $F\colon \mathrm{U}M \to \mathbb{R}$ defined as $F(u) = f \circ \tau(u)$.

Note that there is a nonempty bounded open set $\Omega \subset \mathrm{U}M$ such that $df(u) > \varepsilon$ for any $u \in \Omega$ and some fixed $\varepsilon > 0$.

Denote by φ^t the geodesic flow for time t on $\mathrm{U}M$. By Liouville's theorem on phase volume, we have

$$(*) \qquad \operatorname{vol}[\varphi^t(\Omega)] = \operatorname{vol}\Omega$$

for any t.

Given $u \in \mathrm{U}M$, consider the function $h_u(t) = F \circ \varphi^t(u)$. Since f is convex, so is h_u. Therefore $h_u'(t) > \varepsilon$ for any $t \geqslant 0$ and $u \in \Omega$.

It follows that there is an infinite sequence of times

$$0 = t_0 < t_1 < t_2 < \dots$$

such that

$$h_v(t_{i-1}) < h_u(t_i)$$

for any $u, v \in \Omega$ and i. In particular, we have

$$\varphi^{t_i}(\Omega) \cap \varphi^{t_j}(\Omega) = \varnothing$$

for $i \neq j$. By $(*)$, the latter implies that $\mathrm{vol}(\mathrm{U}M) = \infty$. Hence

$$\mathrm{vol}\, M = \infty. \qquad \square$$

The problem is due to Richard Bishop and Barrett O'Neill [see 118], it was generalized by Shing-Tung Yau [see 119].

Sasaki metric. Fix a point $p \in \mathbb{S}^2$. Note that any rotation of the tangent space $\mathrm{T}_p = \mathrm{T}_p(\mathbb{S}^2, g)$ appear as a holonomy of some loop at p; moreover the length of such loop can be bounded by some constant, say ℓ.

Indeed, fix a smooth homotopy $\gamma_t \colon [0,1] \to \mathbb{S}^2$, $t \in [0,1]$ of loops based at p that sweeps out $\mathbb{S}^2$. By the Gauss–Bonnet formula, the total curvature of $(\mathbb{S}^2, g)$ is $4\cdot\pi$. It follows that any rotation of T_p appears as the holonomy of γ_t for some t. Therefore one can take

$$\ell = \max\{\, \mathrm{length}\, \gamma_t \mid t \in [0,1]\,\}.$$

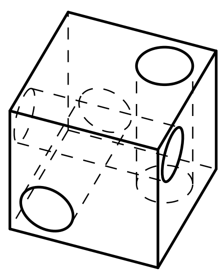
Denote by d the diameter of $(\mathbb{S}^2, g)$. From above follows that for any two unit tangent vectors $v \in \mathrm{T}_p$ and $w \in T_q$ there is a path $\gamma \colon [0,1] \to \mathbb{S}^2$ from p to q such that

$$\mathrm{length}\, \gamma \leqslant \ell + d$$

and w is the parallel transport of v along γ.

In particular, the diameter of the set of all vectors of fixed magnitude in $(\mathrm{T}\mathbb{S}^2, \hat{g})$ has diameter at most $\ell + d$. Therefore the map $\mathrm{T}\mathbb{S}^2 \to [0, \infty)$ defined as $v \mapsto |v|$ preserves the distance up to an error of $\ell + d$. Hence the result follows. $\square$

Two-systole. Consider the unit cube with three not intersecting cylindrical tunnels between the pairs of opposite faces. In each tunnel, shrink the metric long-wise and expand it cross-wise while keeping the volume the same.



More precisely, assume (x, y, z) is the coordinate system on the cylindrical tunnel $\mathbb{D} \times [0,1]$. Then the new metric is defined as

$$g = \varphi\cdot[(dx)^2 + (dy)^2] + \tfrac{1}{\varphi^2}\cdot(dz)^2,$$

where $\varphi = \varphi(x, y)$ is a positive smooth function on $\mathbb{D}$ taking huge values around the center and equal to 1 near the boundary of $\mathbb{D}$.

Gluing the opposite faces of the cube, we obtain a 3-dimensional torus with a smooth Riemannian metric.

Since the surface S does not bound in $\mathbb{T}^3=\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1$, one of the three coordinates projections $\mathbb{T}^3\to\mathbb{T}^2=\mathbb{S}^1\times\mathbb{S}^1$ induces a map of non-zero degree $S\to\mathbb{T}^2$. It follows that

$$\operatorname{area} S\geqslant \operatorname{area}(\mathbb{D},\varphi\cdot[(dx)^2+(dy)^2]).$$

For the right choice of the function φ, the right hand side can be made larger than the given number L. Hence the statement follows. $\square$

I learned this problem from Dmirti Burago.

Normal exponential map. Assume there are $p\in M$, $\varepsilon>0$, such that the image of the normal exponential map to L does not intersect the ball $B(p,\varepsilon)_M$; that is, no geodesic normal to L crosses the ball.

Fix a positive real number R such that $B(p,R)_M\cap L\neq\varnothing$. The sectional curvature of M in the ball $B(p,R)$ is bounded below by some constant, say K.

Given $q\in L$, denote by $v_q\in \mathrm{T}_qM$ the direction of a minimizing geodesic $[qp]$. Note that $v_q\notin \mathrm{N}_qL$. Moreover there is $\delta=\delta(\varepsilon,K,R)>0$ such that for any point $q\in B(p,R)_M\cap L$, and any normal vector $n\in \mathrm{N}_qL$, we have

$$\measuredangle(v_q,n)>\delta.$$

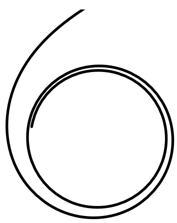
Otherwise the geodesic in the direction of n would cross $B(p,\varepsilon)_M$.

It follows that starting at any point $q\in B(p,R)_M\cap L$ one can construct a unit-speed curve γ in L such that

$$|p-\gamma(t)|\leqslant|p-q|-t\cdot\sin\delta.$$

Following γ for enuf time brings us to p; that is, $p\in L$, a contradiction. $\square$

The problem was suggested by Alexander Lytchak.



From the picture, you should guess an example of an immersion such that one point does not lie in the image of the corresponding normal exponential map. It might be interesting to understand what type of subsets can be avoided by such images.

Symplectic squeezing in the torus. The embedding will be given as a composition of a linear symplectomorphism λ with the quotient map $\varphi\colon\mathbb{R}^4\to\mathbb{T}^2\times\mathbb{R}^2$ by the integer (x_1,y_1)-lattice.

The composition $\varphi\circ\lambda$ will preserve the symplectic structure; it remains to find λ such that the restriction $\varphi\circ\lambda|_\Omega$ is injective.

Without loss of generality, we can assume that Ω is a ball centered at the origin. Choose an oriented 2-dimensional subspace V of $\mathbb{R}^4$ such that the integral of ω over $\Omega \cap V$ is a positive number smaller than $\frac{\pi}{4}$.

Note that there is a linear symplectomorphism λ that maps planes parallel to V to planes parallel to the (x_1, y_1)-plane, and that maps the disk $V \cap \Omega$ to a round disk. It follows that the intersection of $\lambda(\Omega)$ with any plane parallel to the (x_1, y_1)-plane is a disk of radius at most $\frac{1}{2}$. In particular $\varphi \circ \lambda|_\Omega$ is injective. $\square$

This construction was given by Larry Guth [see 120] and attributed to Leonid Polterovich.

Note that according to the Gromov's non-squeezing theorem [see 114], an analogous statement with $\mathbb{C} \times \mathbb{D}$ as the target space does not hold, here $\mathbb{D} \subset \mathbb{C}$ is the open unit disk with the induced symplectic structure. In particular, it shows that the projection of $\lambda(\Omega)$ as above to the (x_1, y_1)-plane cannot be made arbitrary small.

Diffeomorphism test. Note that the map f is an open immersion.

Let h be the pullback metric on M for $f\colon M \to N$. Clearly $h \geqslant g$. In particular (M, h) is complete and the map $f\colon (M, h) \to N$ is a local isometry.

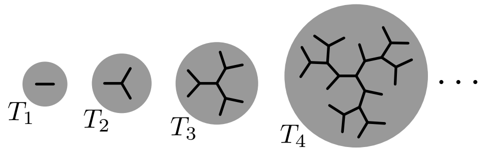
Note that any local isometry between complete connected Riemannian manifolds of the same dimension if a covering map. Since N is simply connected, the result follows. $\square$

Volume of tubular neighborhoods. This problem is a direct corollary of the so called *tube formula* given by Hermann Weyl [see 121]. It expresses the volume of the r-neighborhood of M as a polynomial $p(r)$; the coefficients of p, up to a multiplicative constant, are integrals along M of some quantities called the *Lipschitz–Killing curvatures* — these are certain scalars which can be expressed in terms of the curvature tensor at the given point.

Disk. The following claim is the key step in the proof.

$(*)$ *Given a positive integer n there is a binary tree T_n embedded into the disk $\mathbb{D}$ such that any null-homotopy of $\partial\mathbb{D}$ passes thru a curve that intersects n different edges.*

The proof of the claim can be done by induction on n; the base is trivial. Assuming we constructed T_{n-1}, the tree T_n can be obtained by identifying three endpoints of three copies of T_{n-1}.



Fix $\varepsilon = \frac{1}{10}$ and a large integer n. Let us construct a metric on the disk $\mathbb{D}$ with the embedded tree T_n as in $(*)$ such that its diameter and the length of its boundary are less than 1 and the distance between any two edges of T_n without a common vertex is at least ε.

Fix a Riemannian metric g on the cylinder $\mathbb{S}^1 \times [0,1]$ such that

- ◇ The ε-neighborhoods of the boundary components have product metrics.
- ◇ Any vertical segment $x \times [0,1]$ has length $\frac{1}{2}$.
- ◇ One of the boundary component has length ε.
- ◇ The other boundary component has length $2 \cdot m \cdot \varepsilon$, where m is the number of edges in the tree T_n.

Equip T_n with a length-metric so that each edge has length ε. Glue by a piecewise isometry the cylinder $(\mathbb{S}^1 \times [0,1], g)$ along its long boundary component to the tree T_n in such a way that the resulting space is homeomorphic to a disk and the obtained embedding of T_n in $\mathbb{D}$ is the same as in the claim $(*)$.

By $(*)$ and the construction, for any null-homotopy of the boundary the least length it exceeds $\frac{\varepsilon}{10} \cdot n = \frac{1}{100} \cdot n$. The obtained metric is not Riemannian, but is easy to smooth while keeping this property.

Since n is large the result follows. □

This example was constructed by Sidney Frankel and Mikhail Katz [see 122].

Shortening homotopy. Set

$$p = \gamma_0(0) \quad \text{and} \quad \ell_0 = \operatorname{length} \gamma_0.$$

By a compactness argument, there exists $\delta > 0$ such that no geodesic loop based at p has length in the interval $(L - D, L + D + \delta]$.

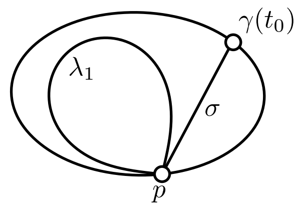
Assume $\ell_0 \geqslant L + \delta$. Choose $t_0 \in [0,1]$ such that

$$\operatorname{length}\left(\gamma_0|_{[0,t_0]}\right) = L + \delta$$

Let σ be a minimizing geodesic from $\gamma(t_0)$ to p. Note that γ_0 is homotopic to the concatenation

$$\gamma_0' = \gamma_0|_{[0,t_0]} * \sigma * \bar{\sigma} * \gamma|_{[t_0,1]},$$

where $\bar{\sigma}$ denotes the backward parametrization of σ.



Applying a curve shortening process to the loop $\lambda_0 = \gamma|_{[0,t_0]} * \sigma$, we get a homotopy λ_t rel. to its endpoints from the loop λ_0 to a geodesic loop λ_1 at p. From above,

$$\operatorname{length} \lambda_1 \leqslant L - D.$$

The concatenation $\gamma_t = \lambda_t * \bar{\sigma} * \gamma|_{[t_0,1]}$ is a homotopy from γ_0' to another curve γ_1. From the construction it is clear that

$$\begin{aligned}\operatorname{length} \gamma_t &\leqslant \operatorname{length} \gamma_0 + 2 \cdot \operatorname{length} \sigma \leqslant \\ &\leqslant \operatorname{length} \gamma_0 + 2 \cdot D\end{aligned}$$

for any $t \in [0,1]$ and

$$\begin{aligned}\operatorname{length} \gamma_1 &= \operatorname{length} \lambda_1 + \operatorname{length} \sigma + \operatorname{length} \gamma|_{[t_0,1]} \leqslant \\ &\leqslant L - D + D + \operatorname{length} \gamma - (L + \delta) = \\ &= \ell_0 - \delta.\end{aligned}$$

Repeating the procedure enuf times we get we get curves $\gamma_2, \dots, \gamma_n$ connected by the needed homotopies so that $\ell_{i+1} \leqslant \ell_i - \delta$ and $\ell_n < L + \delta$, where $\ell_i = \operatorname{length} \gamma_i$.

If $\ell_n \leqslant L$, we are done. Otherwise repeat the argument once more for $\delta' = \ell_n - L$. □

The problem is due to Alexander Nabutovsky and Regina Rotman [see 123].

Convex hypersurface. First let us define the *cone construction* of maps into M.

Let Δ' be a simplex and Δ be its facet opposite to the vertex v. Assume $f\colon \Delta \to M$ is a map such that $f(\Delta) \subset B(x,1)_M$ for some $x \in M$. Given $w \in \Delta$, let $\gamma_w\colon [0,1] \to M$ be the minimizing geodesic path from x to $f(w)$. Since the injectivity radius of M is at least 1, the path γ_w is uniquely defined. The map $f'\colon \Delta' \to M$ defined as

$$f'\colon (1-t)\cdot v + t \cdot w \mapsto \gamma_w(t)$$

is called the *cone over* f with vertex x.

One may start with a map $f_0\colon \Delta_0 \to M$ and iterate the cone construction for the vertices $x_1, \dots x_k$, to get a sequence of maps $f_i\colon \Delta_i \to M$ as long as $f_{i-1}(\Delta_{i-1}) \subset B(x_i, 1)$. Straightforward application of the triangle inequality shows that the latter conditions hold if $f_0(\Delta_0) \subset B(x_i, s)$ for each i and $s < \frac{2}{2+k}$.

Now we go back to the solution of the problem.

Fix a fine triangulation of W so that M becomes a sub-complex of W. We can assume that the diameter of each simplex in τ is less than any given $\varepsilon > 0$. Furthermore, we can assume that all the vertices of τ can be colored into $m+2$ colors $(0,\dots,m+1)$ in such a way that the vertices of each simplex have different colors; the latter can be achieved by passing to the barycentric subdivision of τ. Denote by τ_i the maximal i-dimensional sub-complex of τ with all the vertices colored by $0,\dots,i$.

Let h be the maximal distance from points in W to M. For each vertex v in τ choose a point $v' \in M$ at distance $\leqslant h$. Note that if v and w are vertices of one simplex, then

$$|v' - w'|_M < 2{\cdot}h + \varepsilon.$$

Assume $\frac{2}{m+3} > h$. Fix a positive value $\varepsilon < \frac{2}{m+3} - h$ and use it in the construction of the triangulation τ above. Applying the iterated cone construction for each simplex of τ we get an extension of the map $v \mapsto v'$ defined on τ_0 to $\tau_1, \dots \tau_{m+1}$. According to the above estimates, the cone constructions are defined at each of the needed $m+1$ iterations.

This way we get to a retraction $W \to M$. It follows that the fundamental class of M vanishes in the homology ring of M, a contradiction. □

This problem is a stripped version of the bound on filling radius given by Mikhael Gromov [see 89].

Almost constant function. Given a positive integer m, denote by δ_m the expected value of $|x_1|$ for the random unit vector $\boldsymbol{x} = (x_1,\dots,x_m) \in \mathbb{R}^m$ with respect to the uniform distribution.

Observe that $\delta_m \to 0$ as $m \to \infty$. Indeed, from symmetry and Bunyakovsky inequality we get

$$\tfrac{1}{m} = \tfrac{1}{m}{\cdot}\mathrm{E}(|\boldsymbol{x}|^2) = \mathrm{E}(x_1^2) \geqslant \mathrm{E}(|x_1|)^2 = \delta_m^2.$$

Since f is 1-Lipschitz,

$$\mathrm{E}(|df(w)|) \leqslant \delta_m$$

for a random vector w in $\mathrm{U}M$.

Note that

$$\begin{aligned}|f\circ\gamma(1) - f\circ\gamma(0)| &= \left|\int_0^1 df(\gamma'(t)){\cdot}dt\right| \leqslant \\ &\leqslant \int_0^1 |df(\gamma'(t))|\,{\cdot}dt.\end{aligned}$$

Assume $\gamma'(0)$ takes random value in $\mathrm{U}M$. By Liouville's theorem on phase volume, the same holds for $\gamma'(t)$ for any fixed t. Therefore

$$\mathrm{E}(|f\circ\gamma(1)-f\circ\gamma(0)|)\leqslant \mathrm{E}\left(\int\limits_0^1 |df(\gamma'(t))|\cdot dt\right)\leqslant \delta_m.$$

By Markov's inequality, the probability of the event

$$|f\circ\gamma(1)-f\circ\gamma(0)|>\varepsilon$$

is at most $\frac{\delta_m}{\varepsilon}$. Hence the result follows. $\square$

I learned this problem from Mikhael Gromov. It gives an example in the Riemannian world of the so called *concentration of measure phenomenon* [see 124, 125].

Chapter 5

Metric geometry

In this chapter, we consider metric spaces. All the necessary material could be found in the first three chapters of the textbook [126].

Let us fix a few standard notations.

- ⋄ The distance between two points x and y in a metric space X will be denoted as

$$\operatorname{dist}_x(y), \quad |x-y| \quad \text{or} \quad |x-y|_X,$$

  the latter notation is used to emphasize that x and y belong to the space X.
- ⋄ A metric space X is called *length-metric space* if for any two points $x, y \in X$ and any $\varepsilon > 0$. The points x and y can be connected by a curve α with

$$\operatorname{length}\alpha < |x-y|_X + \varepsilon.$$

  In this case we say the metric on X is a *length-metric.*

Embedding of a compact

Prove that any compact metric space is isometric to a subset of a compact length-metric space.

Semisolution. Let K be a compact metric space. Denote by $\mathcal{B}(K, \mathbb{R})$ the space of real-valued bounded functions on K equipped with sup-norm; that is,

$$|f| = \sup\{\, |f(x)| \mid x \in K \,\}.$$

Note that the map $K \to \mathcal{B}(K, \mathbb{R})$, defined by $x \mapsto \operatorname{dist}_x$ is a distance preserving embedding. Indeed, by the triangle inequality we have

$$|\operatorname{dist}_x(z) - \operatorname{dist}_y(z)| \leqslant |x - y|_K$$

for any $z \in K$ and the equality holds for $z = x$.

In other words, we can and will consider K as a subspace of $\mathcal{B}(K, \mathbb{R})$.

Denote by W the linear convex hull of K in $\mathcal{B}(K, \mathbb{R})$; that is, W is the intersection of all closed convex subsets containing K. Clearly W is a complete subspace of $\mathcal{B}(K, \mathbb{R})$.

Since K is compact we can choose a finite ε-net K_ε in K. The set K_ε lies in a finite dimensional subspace; therefore its convex hull W_ε is compact. Note that W lies in the ε-neighborhood of W_ε. Therefore, W admits a compact ε-net for any $\varepsilon > 0$. That is, W is totally bounded and complete and therefore compact.

Note that line segments in W are geodesics for the metric induced by the sup-norm. In particular W is a compact length-metric space as required. □

The map $x \mapsto \operatorname{dist}_x$ is called the *Kuratowski embedding*, it was constructed in [127]. Essentially the same map was described by Maurice Fréchet [in 128, this is the paper where metric spaces were introduced].

The problem also follows directly from a theorem of John Isbell, stating that *injective envelope* of compact metric space is compact; injective envelope is an analog of convex hull in the category of metric spaces [see 2.11 in 129].

The following related problem is open even for three-point sets. This problem is known in folklore for long time [I know it since 1993], but in print it was mentioned only recently [see 130 after Theorem 2.10, 131 and Question 2.17 in 132].

Is it true that any compact subset of a complete CAT(0) *length-space lies in a convex compact set?*

Non-contracting map°

A map $f\colon X \to Y$ between metric spaces is called *distance non-contracting* if

$$|f(x) - f(x')|_Y \geqslant |x - x'|_X$$

for any two points $x, x' \in X$.

Let K be a compact metric space and

$$f\colon K \to K$$

be a distance non-contracting map. Prove that f is an isometry.

Finite-whole extension

A map $f\colon X\to Y$ between metric spaces is called *non-expanding* if

$$|f(x)-f(x')|_Y\leqslant|x-x'|_X$$

for any two points $x,x'\in X$.

Let X and Y be metric spaces, Y compact, $A\subset X$ and $f\colon A\to Y$ be a non-expanding map. Assume that for any finite set $F\subset X$ there is a non-expanding map $F\to Y$ that agrees with f in $F\cap A$. Show that there is a non-expanding map $X\to Y$ that agrees with f on A.

Horo-compactification°

Let X be a metric space. Denote by $C(X,\mathbb{R})$ the space of continuous functions $X\to\mathbb{R}$ equipped with the *compact-open topology*; that is, for any compact set $K\subset X$ and any open set $U\subset\mathbb{R}$ the set of all continuous functions $f\colon X\to\mathbb{R}$ such that $f(K)\subset U$ is declared to be open.

Fix a point $x_0\in X$. Given a point $z\in X$, let $f_z\in C(X,\mathbb{R})$ be the function defined as

$$f_z(x)=\mathrm{dist}_z(x)-\mathrm{dist}_z(x_0).$$

Let $F_X\colon X\to C(X,\mathbb{R})$ be the map defined as $F_X\colon z\mapsto f_z$.

Denote by $\bar X$ the closure of $F_X(X)$ in $C(X,\mathbb{R})$; note that $\bar X$ is compact. That is, if F_X is an embedding, then $\bar X$ is a compactification of X, which is called the *horo-compactification*. In this case, the complement $\partial_\infty X=\bar X\backslash F_X(X)$ is called the *horo-absolute* of X.

The construction above is due to Mikhael Gromov [see 133].

Construct a proper metric space X such that

$$F_X\colon X\to C(X,\mathbb{R})$$

is not an embedding. Show that there are no such examples among proper length-metric spaces.

Approximation of the ball by a sphere

Construct a sequence of Riemannian metrics on $\mathbb{S}^3$ converging in the sense of Gromov–Hausdorff to the unit ball in $\mathbb{R}^3$.

Macroscopic dimension$^\circ$

Let X be a locally compact metric space and $a > 0$.

Following Mikhael Gromov [see 134], we say that the *macroscopic dimension* of X at the scale a is the least integer m such that there is a continuous map f from X to an m-dimensional simplicial complex K with

$$\operatorname{diam}[f^{-1}\{k\}] < a$$

for any point $k \in K$.

Equivalently, the macroscopic dimension of X on scale a can be defined as the least integer m such that X admits an open covering with diameter of each set less than a and such that each point in X is covered by at most $m + 1$ sets in the cover.

Let M be a simply connected Riemannian manifold with the following property: any closed curve is null-homotopic in its own 1-neighborhood. Prove that the macroscopic dimension of M at the scale 100 *is at most* 1.

No Lipschitz embedding*

Construct a length-metric d on $\mathbb{R}^3$, such that the space $(\mathbb{R}^3, d)$ does not admit a locally Lipschitz embedding into the 3-dimensional Euclidean space.

Sub-Riemannian sphere$^+$

Let us define sub-Riemannian metric.

Fix a Riemannian manifold (M, g). Assume that in the tangent bundle $\mathrm{T}M$ a choice of sub-bundle H is given.

Let us call the sub-bundle H *horizontal distribution*. The tangent vectors in H will be called *horizontal*. A piecewise smooth curve will be called *horizontal* if all its tangent vectors are horizontal.

The sub-Riemannian distance between any two points x and y is defined as the infimum of lengths of horizontal curves connecting x to y.

Alternatively, the distance can be defined as a limit of Riemannian distances for the metrics

$$g_\lambda(X, Y) = g(X^H, Y^H) + \lambda{\cdot}g(X^V, Y^V)$$

as $\lambda \to \infty$, where X^H denotes the horizontal part of X; that is, the orthogonal projection of X to H and X^V denotes the vertical part of X; so, $X^V + X^H = X$.

In addition we need to add an additional condition to ensure the following properties

- ◇ The sub-Riemannian metric induce the original topology on the manifold. In particular, if M is connected, then the distance cannot take infinite values.
- ◇ Any curve in M can be arbitrary well approximated by a horizontal curve with the same endpoints.

The most common condition of this type is the so called *complete non-integrability*; it means that for any $x \in M$, one can choose a basis in its tangent space T_xM from the vectors of the following type

$$A(x), \quad [A,B](x), \quad [A,[B,C]](x), \quad [A,[B,[C,D]]](x), \dots$$

where $[*,*]$ denotes the Lie bracket and the vector fields $A, B, C, D, \dots$ are horizontal.

🕮 *Prove that any sub-Riemannian metric on $\mathbb{S}^m$ is isometric to the intrinsic metric of a hypersurface in $\mathbb{R}^{m+1}$.*

It will be hard to solve the problem without knowing proof of Nash–Kuiper theorem on length preserving C^1-embeddings. The original papers of John Nash and Nicolaas Kuiper [see 135, 136] are very readable.

Length-preserving map$^+$

A continuous map $f\colon X \to Y$ between metric spaces is called *length-preserving* if it preserves the length of curves; that is, for any curve α in X we have

$$\operatorname{length}(f \circ \alpha) = \operatorname{length} \alpha.$$

🕮 *Show that there is no length-preserving map $\mathbb{R}^2 \to \mathbb{R}$.*

The expected solution use Rademacher's theorem on differentiability of Lipschitz functions [see 137].

Fixed segment

🕮 *Let $\rho(x,y) = \|x-y\|$ be a metric on $\mathbb{R}^m$ induced by a norm $\|*\|$.*

Assume that $f\colon (\mathbb{R}^m,\rho) \to (\mathbb{R}^m,\rho)$ is an isometry that fixes two distinct points a and b. Show that f fixes the line segment between a and b.

Evidently f maps the line segment $[ab]$ to a minimizing geodesic connecting a to b in $(\mathbb{R}^m,\rho)$. However, in general there might be many minimizing geodesics connecting a to b in $(\mathbb{R}^m,\rho)$. The problem states that $[ab]$ is mapped to itself.

Pogorelov's construction°

Let μ be a regular centrally symmetric finite measure on $\mathbb{S}^2$ which is positive on every open set. Given two points $x, y \in \mathbb{S}^2$, set

$$\rho(x, y) = \mu[B(x, \tfrac{\pi}{2}) \backslash B(y, \tfrac{\pi}{2})].$$

Show that ρ is a length-metric on $\mathbb{S}^2$ and moreover, the geodesics in $(\mathbb{S}^2, \rho)$ run along the great circles of $\mathbb{S}^2$.

Straight geodesics

Let ρ be a length-metric on $\mathbb{R}^m$ bi-Lipschitz equivalent to the canonical metric. Assume that every geodesic γ in $(\mathbb{R}^d, \rho)$ is affine*; that is, $\gamma(t) = v + w{\cdot}t$ for some $v, w \in \mathbb{R}^m$.*

Show that ρ is induced by a norm on $\mathbb{R}^m$.

Hyperbolic space

Recall that a map $f\colon X \to Y$ between metric spaces is called bi-Lipschitz if there if a constant $\varepsilon > 0$ such that

$$\varepsilon{\cdot}|x - y|_X \leqslant |f(x) - f(y)|_Y \leqslant \tfrac{1}{\varepsilon}{\cdot}|x - y|_X.$$

for any $x, y \in X$.

Construct a bi-Lipschitz map from the hyperbolic 3-space to the product of two hyperbolic planes.

Quasi-isometry of a Euclidean space$^+$

A map $f\colon X \to Y$ between metric spaces is called a *quasi-isometry* if there is a real constant $C > 1$ such that

$$\tfrac{1}{C}{\cdot}|x - x'|_X - C \leqslant |f(x) - f(x')|_Y \leqslant C{\cdot}|x - x'|_X + C$$

for any $x, x' \in X$ and $f(X)$ is a *C-net* in Y; that is, for any $y \in Y$ there is $x \in X$ such that $|f(x) - y|_Y \leqslant C$.

Note that a quasi-isometry is not assumed to be continuous, for example any map between compact metric spaces is a quasi-isometry.

Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a quasi-isometry. Show that there is a (bi-Lipschitz) homeomorphism $h\colon \mathbb{R}^m \to \mathbb{R}^m$ on a bounded distance from f; that is, there is a real constant C such that

$$|f(x) - h(x)| \leqslant C$$

for any $x \in \mathbb{R}^m$.

The expected solution requires the so called *gluing theorem*, a corollary of the theorem proved by Laurence Siebenmann [see 138]. It states that if $V_1, V_2 \subset \mathbb{R}^m$ are open and the two embedding $f_1\colon V_1 \to \mathbb{R}^m$ and $f_2\colon V_2 \to \mathbb{R}^m$ are sufficiently close to each other on the overlap $U = V_1 \cap V_2$, then there is an embedding f defined on an open set W' which is slightly smaller than $W = V_1 \cup V_2$ and such that f is sufficiently close to each f_1 and f_2 at the points where they are defined.

The bi-Lipschitz version requires an analogous statement in the category of bi-Lipschitz embeddings; it was proved by Dennis Sullivan [see 139].

Family of sets with no section°

Construct a family of closed sets $C_t \subset \mathbb{S}^1$, $t \in [0,1]$ *that is continuous in the Hausdorff topology, but does not admit a* section. *That is, there is no path* $c\colon [0,1] \to \mathbb{S}^1$ *such that* $c(t) \in C_t$ *for all* t.

Spaces with isometric balls

Construct a pair of locally compact length-metric spaces X *and* Y *that are not isometric, but for some fixed points* $x_0 \in X$, $y_0 \in Y$ *and any radius* R *the ball* $B(x_0, R)_X$ *is isometric to the ball* $B(y_0, R)_Y$.

Average distance°

Show that for any compact length-metric space X *there is unique number* $\ell = \ell(X)$ *such that for any finite collection of points there is a point average distance* ℓ *from the collection; that is, for any* $x_1, \dots, x_n \in X$ *there is* $z \in X$ *such that*

$$\tfrac{1}{n} \cdot \sum_i |x_i - z|_X = \ell.$$

Semisolutions

Non-contracting map. Given any pair of point $x_0, y_0 \in K$, consider two sequences $x_0, x_1, \dots$ and $y_0, y_1, \dots$ such that and $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ for each n.

Since K is compact, we can choose an increasing sequence of integers n_k such that both sequences $(x_{n_i})_{i=1}^{\infty}$ and $(y_{n_i})_{i=1}^{\infty}$ converge. In particular, both are Cauchy sequences; that is,

$$|x_{n_i} - x_{n_j}|_K, |y_{n_i} - y_{n_j}|_K \to 0 \quad \text{as} \quad \min\{i, j\} \to \infty.$$

Since f is non-contracting, we get

$$|x_0 - x_{|n_i - n_j|}| \leqslant |x_{n_i} - x_{n_j}|.$$

It follows that there is a sequence $m_i \to \infty$ such that

$$(*) \qquad x_{m_i} \to x_0 \quad \text{and} \quad y_{m_i} \to y_0 \quad \text{as} \quad i \to \infty.$$

Set

$$\ell_n = |x_n - y_n|_K.$$

Since f is non-contracting, the sequence (ℓ_n) is non-decreasing.

By $(*)$, $\ell_{m_i} \to \ell_0$ as $m_i \to \infty$. It follows that (ℓ_n) is a constant sequence.

In particular

$$|x_0 - y_0|_K = \ell_0 = \ell_1 = |f(x_0) - f(y_0)|_K$$

for any pair of points (x_0, y_0) in K. That is, f is distance preserving, in particular injective.

From $(*)$, we also get that $f(K)$ is everywhere dense. Since K is compact $f\colon K \to K$ is surjective. Hence the result follows. $\square$

This is a basic lemma in the introduction to Gromov–Hausdorff distance [see 7.3.30 in 126]. I learned this proof from Travis Morrison, a student in my MASS class at Penn State, Fall 2011.

As an easy corollary one can get that any surjective non-expanding map from a compact metric space to itself is an isometry. The following problem due to Aleksander Całka [see 140]; is closely related but more involved.

Show that any local isometry from a connected compact metric space to itself is a homeomorphism.

Finite-whole extension. Consider the space Y^X of all maps $X \to Y$ equipped with the product topology.

Given a finite set $F \in X$; denote by $\mathfrak{C}_F$ the set of maps $h \in Y^X$ such that its restriction $h|_F$ is short and the restriction $h|_{A \cap F}$ agrees with $f\colon A \to Y$. By assumption, the sets $\mathfrak{C}_F \subset Y^X$ are closed and nonempty.

Note that for any finite collection of finite sets $F_1, \dots, F_n \subset X$ we have

$$\mathfrak{C}_{F_1} \cap \dots \cap \mathfrak{C}_{F_n} \supset \mathfrak{C}_{F_1 \cup \dots \cup F_n}.$$

In particular, the intersection is nonempty.

According to Tikhonov's theorem [see 141, and the references there in], Y^X is compact. By the finite intersection propery, the intersection $\bigcap_F \mathfrak{C}_F$ with F ranging along all finite subsets of X is nonempty. It remains to note that any map $h \in \bigcap_F \mathfrak{C}_F$ solves the problem. □

This observation was used by Stephan Stadler and me [see 55].

Horo-compactification. For the first part of the problem, take X to be the set of non-negative integers with the metric ρ defined as

$$\rho(m, n) = m + n$$

for $m \neq n$.

The second part is proved by contradiction. Assume X is proper length space and F_X is not an embedding. That is, there is a sequence of points $z_1, z_2, \dots$ and a point z_∞, such that $f_{z_n} \to f_{z_\infty}$ in $C(X, \mathbb{R})$ as $n \to \infty$, while $|z_n - z_\infty|_X > \varepsilon$ for some fixed $\varepsilon > 0$ and all n.

Note that any pair of points $x, y \in X$ can be connected by a minimizing geodesic $[xy]$. Choose $\bar{z}_n$ on a geodesic $[z_\infty z_n]$ such that $|z_\infty - \bar{z}_n| = \varepsilon$. Note that

$$f_{z_n}(z_\infty) - f_{z_n}(\bar{z}_n) = \varepsilon$$

and

$$f_{z_\infty}(z_\infty) - f_{z_n}(\bar{z}_n) = -\varepsilon$$

for all n.

Since X is proper, we can pass to a subsequence of z_n so that the sequence $\bar{z}_n$ converges; denote its limit by $\bar{z}_\infty$. From the above identities follows that

$$f_{z_n}(\bar{z}_\infty) \not\to f_{z_\infty}(\bar{z}_\infty) \quad \text{or} \quad f_{z_n}(z_\infty) \not\to f_{z_\infty}(z_\infty),$$

a contradiction. □

I learned this problem from Linus Kramer and Alexander Lytchak; the example was also mentioned in the lectures of Anders Karlsson and attributed to Uri Bader [see 2.3 in 142].

Approximation of the ball by a sphere. Make fine burrows in the standard 3-ball without changing its topology, but at the same time come sufficiently close to any point in the ball.

Consider the doubling of the obtained ball along its boundary. The obtained space is homeomorphic to $\mathbb{S}^3$. Note that the burrows can be made so that the obtained space is sufficiently close to the original ball in the Gromov–Hausdorff metric.

It remains to smooth the obtained space slightly to get a genuine Riemannian metric with the needed property. $\square$

This construction is a stripped version of the theorem of Steven Ferry and Boris Okun [see 143]. The theorem states that Riemannian metrics on a fixed smooth closed manifold M with $\dim M \geqslant 3$ can approximate a given compact length-metric space X if and only if there is a continuous map $M \to X$ which is surjective on the fundamental groups.

The two-dimensional case is quite different. There is no sequence of Riemannian metrics on $\mathbb{S}^2$ converging to the unit disk in the sense of Gromov–Hausdorff. In fact, if X is a limit of $(\mathbb{S}^2, g_n)$, then any point $x_0 \in X$ either admits a neighborhood homeomorphic to $\mathbb{R}^2$ or is a cut point; that is, $X\backslash\{x_0\}$ is disconnected [see 3.32 in 65].

Macroscopic dimension. The following claim resembles Besikovitch inequality; it is key to the proof.

$(*)$ *Let a be a positive real number. Assume that a closed curve γ in a metric space X can be sudivided into 4 arcs α, β, α', and β' in such a way that*

- $\diamond$ *$|x - x'| > a$ for any $x \in \alpha$ and $x' \in \alpha'$ and*
- $\diamond$ *$|y - y'| > a$ for any $y \in \beta$ and $y' \in \beta'$.*

Then γ is not contractable in its $\frac{a}{2}$-neighborhood.

To prove $(*)$, consider two functions defined on X as

$$w_1(x) = \min\{\, a, \operatorname{dist}_\alpha(x)\,\}$$
$$w_2(x) = \min\{\, a, \operatorname{dist}_\beta(x)\,\}$$

and the map $\boldsymbol{w}\colon X \to [0,a] \times [0,a]$, defined as

$$\boldsymbol{w}\colon x \mapsto (w_1(x), w_2(x)).$$

Note that

$$\begin{aligned} \boldsymbol{w}(\alpha) &= 0 \times [0,a], & \boldsymbol{w}(\beta) &= [0,a] \times 0,\\ \boldsymbol{w}(\alpha') &= a \times [0,a], & \boldsymbol{w}(\beta') &= [0,a] \times a, \end{aligned}$$

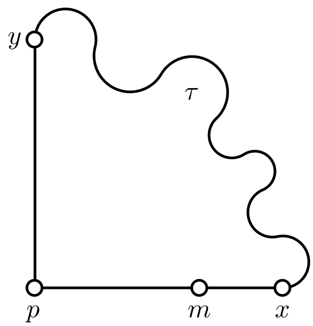
Therefore, the composition $\boldsymbol{w} \circ \gamma$ is a degree 1 map

$$\mathbb{S}^1 \to \partial([0,a] \times [0,a]).$$

It follows that if $h\colon \mathbb{D}\to X$ shrinks γ then there is a point $z\in\mathbb{D}$ such that $\boldsymbol{w}\circ h(z)=(\tfrac{a}{2},\tfrac{a}{2})$. Therefore $h(z)$ lies at distance at least $\tfrac{a}{2}$ from α, β, α', β' and therefore from γ. Hence the claim $(*)$ follows.

Fix a point $p\in M$. Let us cover M with the connected components of the inverse images $\operatorname{dist}_p^{-1}((n-1,n+1))$ for all integers n. Clearly any point in M is covered by at most two of these components. It remains to show that each of these components has diameter less than 100.

Assume the contrary; let x and y be two points in one connected component and $|x-y|_M\geqslant 100$. Connect x to y with a curve τ in the component. Consider the closed curve σ formed by τ and two geodesics $[px]$, $[py]$.

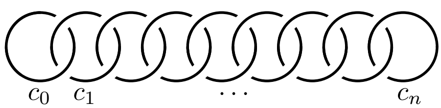


Note that $|p-x|>40$. Therefore there is a point m on $[px]$ such that $|m-x|=20$.

By the triangle inequality, the subsdivision of σ into the arcs $[pm]$, $[mx]$, τ and $[yp]$ satisfy the conditions of the claim $(*)$ for $a=10$. Hence the statement follows. $\square$

The problem was discussed in a talk by Nikita Zinoviev around 2004.

No Lipschitz embedding. Consider a chain of circles $c_0,\dots,c_n$ in $\mathbb{R}^3$; that is, c_i and c_{i-1} are linked for each i.



Assume that $\mathbb{R}^3$ is equipped with a length-metric ρ, such that the total length of the circles is ℓ and U is an open bounded set containing all the circles c_i. Note that for any L-Lipschitz embedding $f\colon (U,\rho)\to\mathbb{R}^3$ the distance from $f(c_0)$ to $f(c_n)$ is less than $L{\cdot}\ell$.

The ρ-distance from c_0 to c_n might be much larger than $L{\cdot}\ell$. Indeed, fix a line segment $[ab]$ in $\mathbb{R}^3$. Modify the length-metric on $\mathbb{R}^3$ in a small neighborhood of $[ab]$ so that there is a chain (c_i) of circles as above, that goes from a to b such that (1) the total length, say ℓ, of all the circles c_i is arbitrary small, but (2) the obtained metric ρ is arbitrary close to the canonical one, say

$$\big|\rho(x,y)-|x-y|\big|<\varepsilon$$

for any two points $x, y \in \mathbb{R}^3$ and fixed in advanced small $\varepsilon > 0$. The construction of ρ is done by shrinking the length of each circle and expanding the length in the normal directions to the circles in a small neighborhood. The latter is made in order to make impossible to use the circles c_i as a shortcut; that is, one spends more time to go from one circle to another than the time one saves by going along the circle.

Set $a_n = (0, \frac{1}{n}, 0)$ and $b_n = (1, \frac{1}{n}, 0)$. Note that the line segments $[a_n b_n]$ are disjoint and converging to $[a_\infty b_\infty]$, where $a_\infty = (0, 0, 0)$ and $b_\infty = (1, 0, 0)$.

Apply the above construction in non-overlapping convex neighborhoods of $[a_n b_n]$ for sequences ε_n and ℓ_n converging to zero very fast.

The obtained length-metric ρ is still close to the canonical metric on $\mathbb{R}^3$, but it does not admit a locally Lipschitz homeomorphism to $\mathbb{R}^3$. Indeed, assume such homeomorphism h exists. Fix a bounded open set U containing $[a_\infty b_\infty]$; note that the restriction $h|_U$ is L-Lipschitz for some L. From the above construction, we get

$$\begin{aligned}|h(a_\infty) - h(b_\infty)| \leqslant |h(a_n) - h(b_n)|+&\\ + |h(a_\infty) - h(a_n)| + |h(b_n) - h(b_\infty)| &\leqslant \\ \leqslant L\cdot\ell_n + \tfrac{2}{n} + 100\cdot\varepsilon_n&\end{aligned}$$

for any positive integer n. The right hand side converges to 0 as $n \to \infty$. Therefore

$$h(a_\infty) = h(b_\infty),$$

a contradiction. □

The problem is due to Dmitri Burago, Sergei Ivanov and David Shoenthal [see 144].

It is expected that any metric on $\mathbb{R}^2$ admits locally Lipschitz embeddings into the Euclidean plane. Also, it seems feasible that any metric on $\mathbb{R}^3$ admits a locally Lipschitz embedding into $\mathbb{R}^4$.

Note that any metric on the cube in $\mathbb{R}^3$ admits a proper locally Lipschitz map to the unit cube with the canonical metric of degree 1. Moreover one can make this map injective on any finite set of points. It is instructive to visualize this map for the metric of the solution.

Sub-Riemannian sphere. If d is a sub-Riemannian metric on $\mathbb{S}^m$, then there is a non-decreasing sequence of Riemannian metric tensors $g_0 < g_1 < \dots$ such that their induced metrics $d_1 < d_2 < \dots$ converge to d. The metric g_0 can be assumed to be the metric of a round sphere, so it is induced by an embedding $h_0 \colon \mathbb{S}^m \to \mathbb{R}^{m+1}$.

Applying the construction as in Nash–Kuiper theorem, one can produce a sequence of smooth embeddings $h_n \colon \mathbb{S}^m \to \mathbb{R}^{m+1}$ with the induced metrics g'_n such that $|g'_n - g_n| \to 0$. In particular, if we denote by d'_n the metric corresponding to g'_n, then $d'_n \to d$ an $n \to \infty$.

From the same construction follows that if one chooses $\varepsilon_n > 0$, depending on h_n, then we can assume that

$$|h_{n+1}(x) - h_n(x)| < \varepsilon_n$$

for any $x \in \mathbb{S}^m$.

Let us introduce two conditions on the values ε_n, called *weak* and *strong.*

The weak condition states that $\varepsilon_n < \frac{1}{2}\cdot\varepsilon_{n-1}$ for any n. This ensures that the sequence of maps h_n converges pointwise; denote its limit by h_∞.

Denote by $\bar{d}$ the length-metric induced by h_∞. Note that $\bar{d} \leqslant d$. The strong condition on ε_n will ensure that actually $\bar{d} = d$.

Fix n and assume that h_n and therefore ε_{n-1} are constructed already. Set $\Sigma = h_n(\mathbb{S}^m)$ and let Σ_r be the tubular r-neighborhood of Σ. Equip Σ and Σ_r with the induced length-metrics. Since Σ is a smooth hypersurface, we can choose $r_n \in (0, \varepsilon_{n-1}]$ so that the inclusion $\Sigma \hookrightarrow \Sigma_{r_n}$ preserves the distance up to the error $\frac{1}{2^n}$. Then the strong condition states that $\varepsilon_n < \frac{1}{2}\cdot r_n$, which is evidently stronger than the weak condition $\varepsilon_n < \frac{1}{2}\cdot\varepsilon_{n-1}$ above.

Note that if the sequence h_n is constructed with the described choice of ε_n, then $|h_\infty(x) - h_n(x)| < r_n$ for any $x \in \mathbb{S}^m$. Therefore

$$\bar{d}(x, y) + 2\cdot r_n + \tfrac{1}{2^n} \geqslant d'_n(x, y)$$

for any n and $x, y \in \mathbb{S}^m$; hence $\bar{d} \geqslant d$ as required. $\square$

The problem on this list was first discovered by Enrico Le Donne [see 145]. A similar construction is described in the lecture notes by Allan Yashinski and me [see 146] which are aimed for undergraduate students. Yet the results in [147] are closely relevant.

The construction in the Nash–Kuiper embedding theorem can be used to prove some seemingly irrelevant statements. Here is one example based on the observation that Weyl curvature tensor vanishes on hypersurfaces in the Euclidean space.

Let M be a Riemannian manifold diffeomorphic to the m-sphere. Show that there is a Riemannian manifold M' arbitrary close to M in the Lipschitz metric and whose Weyl curvature tensor is identically 0.

Length-preserving map. Assume contrary; let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a length-preserving map.

Note that f is Lipschitz. Therefore by Rademacher's theorem [see 137], the differential $d_x f$ is defined for almost all x.

Fix a unit vector u. Given $x\in\mathbb{R}^2$, consider the path $\alpha_x(t)=x+t\cdot u$ defined for $t\in[0,1]$. Note that

$$\alpha_x'(t)=(d_{\alpha_x(t)}f)(u)$$

holds for almost all x and t. It follows that

$$\operatorname{length}(f\circ\alpha_x)=\int\limits_0^1|(d_{\alpha_x(t)}f)(u)|\cdot dt$$

for almost all x.

Therefore $|d_xf(v)|=|v|$ for almost all $x,v\in\mathbb{R}^2$. In particular there is $x\in\mathbb{R}^2$ such that the differential d_xf is defined and

$$|d_xf(e_1)|=|e_1|,\quad |d_xf(e_2)|=|e_2|,\quad |d_xf(e_1+e_2)|=|e_1+e_2|$$

for a basis (e_1,e_2) of $\mathbb{R}^2$. It follows that d_xf has rank 2, a contradiction. □

The idea above can also be used to solve the following problem.

🕮 *Assume ρ is a metric on $\mathbb{R}^2$ that is induced by a norm. Show that $(\mathbb{R}^2,\rho)$ admits a length-preserving map to $\mathbb{R}^3$ if and only if $(\mathbb{R}^2,\rho)$ is isometric to the Euclidean plane.*

Fixed segment. Note that it is sufficient to show that if

$$f(a)=a\quad\text{and}\quad f(b)=b$$

for some $a,b\in\mathbb{R}^m$, then

$$f(\tfrac{a+b}{2})=\tfrac12\cdot(f(a)+f(b)).$$

Without loss of generality, we can assume that $b+a=0$.

Set $f_0=f$. Consider the sequence of isometries $f_0,\,f_1,\dots$ recursively defined as

$$f_{n+1}(x)=-f_n^{-1}(-f_n(x))$$

for all n.

Note that for all n we have $f_n(a)=a$, $f_n(b)=b$ and

$$|f_{n+1}(0)|=2\cdot|f_n(0)|.$$

Therefore if $f(0)\neq0$, then $|f_n(0)|\to\infty$ as $n\to\infty$.

On the other hand, since f_n is isometry and $f(a)=a$, we also have $|f_n(0)|\leqslant 2\cdot|a|$, a contradiction. □

The idea of the proof is due to Jussi Väisälä's [see 148]. The problem is the main step in the proof of Mazur–Ulam [see 149], which states that any isometry of $(\mathbb{R}^m, \rho)$ is an affine map.

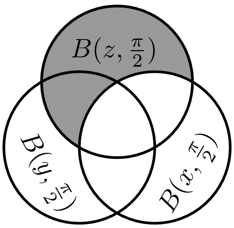
Pogorelov's construction. Positivity and symmetry of ρ is evident.

The triangle inequality follows since

$$(*)\qquad [B(x,\tfrac{\pi}{2})\backslash B(y,\tfrac{\pi}{2})] \cup [B(y,\tfrac{\pi}{2})\backslash B(z,\tfrac{\pi}{2})] \supseteq B(x,\tfrac{\pi}{2})\backslash B(z,\tfrac{\pi}{2})$$

and $B(x,\frac{\pi}{2})\backslash B(y,\frac{\pi}{2})$ does not overlap $B(y,\frac{\pi}{2})\backslash B(z,\frac{\pi}{2})$.

Note that we get equality in $(*)$ if and only if y lies on the great circle arc from x to z. Therefore the second statement follows. $\square$



This construction was given by Aleksei Pogorelov [see 150]. It is closely related to the construction given by David Hilbert in [see 151] which was the motivating example for his 4th problem.

Straight geodesics. From the uniqueness of the straight segment between two given points in $\mathbb{R}^m$, it follows that any straight line in $\mathbb{R}^m$ is a geodesic in $(\mathbb{R}^m, \rho)$.

Set

$$\|\boldsymbol{v}\|_{\boldsymbol{x}} = \rho(\boldsymbol{x}, (\boldsymbol{x}+\boldsymbol{v})).$$

Note that

$$\|\lambda\cdot\boldsymbol{v}\|_{\boldsymbol{x}} = |\lambda|\cdot\|\boldsymbol{v}\|_{\boldsymbol{x}}$$

for any $\boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$.

Denote by $|x-y|$ the Euclidean distance between the points x and y. Since ρ is bi-Lipschitz to $|*-*|$, applying the triangle inequality twice for the points $\boldsymbol{x}$, $\boldsymbol{x}+\lambda\cdot\boldsymbol{v}$, $\boldsymbol{x}'$ and $\boldsymbol{x}'+\lambda\cdot\boldsymbol{v}$, we get

$$\big|\|\lambda\cdot\boldsymbol{v}\|_{\boldsymbol{x}} - \|\lambda\cdot\boldsymbol{v}\|_{\boldsymbol{x}'}\big| \leqslant C\cdot|\boldsymbol{x}-\boldsymbol{x}'|$$

for any $\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{v} \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$ and a fixed real constant C.

Passing to the limit as $\lambda \to \infty$, we obtain that $\|\boldsymbol{v}\|_{\boldsymbol{x}}$ does not depend on $\boldsymbol{x}$; hence the result follows. $\square$

This idea is due to Thomas Foertsch and Viktor Schroeder [see 152]. A more general statement was proved by Petra Hitzelberger and Alexander Lytchak [see 153]. Namely they show that if any pair of points in a geodesic metric space X can be separated by an *affine function*, then X is isometric to a convex subset of a normed vector space. (A function $f\colon X \to \mathbb{R}$ is called affine if for any geodesic γ in X, the composition $f\circ\gamma$ is affine.)

Hyperbolic space. The hyperbolic plane $\mathbb{H}^2$ is isometric to $(\mathbb{R}^2, g)$, where

$$g(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & e^x \end{pmatrix}.$$

The same way, the hyperbolic space $\mathbb{H}^3$ can be viewed as $(\mathbb{R}^3, h)$, where

$$h(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^x & 0 \\ 0 & 0 & e^x \end{pmatrix}.$$

In the described coordinates, consider the projections $\mathbb{H}^3 \to \mathbb{H}^2$ defined as $\varphi\colon (x,y,z) \mapsto (x,y)$ and $\psi\colon (x,y,z) \mapsto (x,z)$. Note that

$$\begin{aligned} &|\varphi(p) - \varphi(q)|_{\mathbb{H}^2}, \\ &|\psi(p) - \psi(q)|_{\mathbb{H}^2} \leqslant |p-q|_{\mathbb{H}^3} \leqslant \\ &\qquad\qquad\leqslant |\varphi(p) - \varphi(q)|_{\mathbb{H}^2} + |\psi(p) - \psi(q)|_{\mathbb{H}^2} \end{aligned}$$

for any two points $p, q \in \mathbb{H}^3$. In particular, the map $\mathbb{H}^3 \to \mathbb{H}^2 \times \mathbb{H}^2$ defined as $p \mapsto (\varphi(p), \psi(p))$ is $2^{\mp 1}$-bi-Lipschitz. $\square$

We used that horo-spheres in the hyperbolic space are isometric to the Euclidean plane. This observation was previously made by Nikolai Lobachevsky [see 34 in 154]. The same observation is used in the following construction discovered by Károly Böröczky [see 155 and also 156].

📖 *Construct a tessellation of the hyperbolic plane with one polygonal tile of arbitrarily small area and/or diameter.*

Quasi-isometry of a Euclidean space. Fix two constants $M \geqslant 1$ and $A \geqslant 0$. A map $f\colon X \to Y$ between metric spaces X and Y such that for any $x, y \in X$, we have

$$\tfrac{1}{M} \cdot |x-y| - A \leqslant |f(x) - f(y)| \leqslant M \cdot |x-y| + A$$

and any point in Y lies on the distance at most A from a point in the image $f(X)$ will be called (M, A)-quasi-isometry.

Note that $(M, 0)$-quasi-isometry is a $[\frac{1}{M}, M]$-bi-Lipschitz map. Moreover, if $f_n\colon \mathbb{R}^m \to \mathbb{R}^m$ is a $(M, \frac{1}{n})$-quasi-isometry for each n, then any partial limit of f_n as $n \to \infty$ is a $[\frac{1}{M}, M]$-bi-Lipschitz map.

It follows that given $M \geqslant 1$ and $\varepsilon > 0$ there is $\delta > 0$ such that for any (M, δ)-quasi-isometry $f\colon \mathbb{R}^m \to \mathbb{R}^m$ and any $p \in \mathbb{R}^m$ there is an $[\frac{1}{M}, M]$-bi-Lipschitz map $h\colon B(p,1) \to \mathbb{R}^m$ such that

$$|f(x) - h(x)| < \varepsilon$$

for any $x \in B(p,1)$.

Applying rescaling, we can get the following equivalent formulation. Given $M \geqslant 1$, $A \geqslant 0$ and $\varepsilon > 0$ there is big enuf $R > 0$ such that for any (M,A)-quasi-isometry $f\colon \mathbb{R}^m \to \mathbb{R}^m$ and any $p \in \mathbb{R}^m$ there is a $[\frac{1}{M}, M]$-bi-Lipschitz map $h\colon B(p,R) \to \mathbb{R}^m$ such that

$$|f(x) - h(x)| < \varepsilon{\cdot}R$$

for any $x \in B(p,R)$.

Cover $\mathbb{R}^m$ by balls $B(p_n,R)$, construct a $[\frac{1}{M}, M]$-bi-Lipschitz map $h_n\colon B(p_n,R) \to \mathbb{R}^m$ close to the restrictions $f|_{B(p_n,R)}$ for each n.

The maps h_n are $2{\cdot}\varepsilon{\cdot}R$ close to each other on the overlaps of their domains of definition. This makes possible to deform slightly each h_n so that they agree on the overlaps. This can be done by Siebenmann's Theorem [see 138]. If instead you apply Sullivan's theorem [see 139], you get a bi-Lipschitz homeomorphism $h\colon \mathbb{R}^m \to \mathbb{R}^m$. □

The problem was suggested by Dmitri Burago.

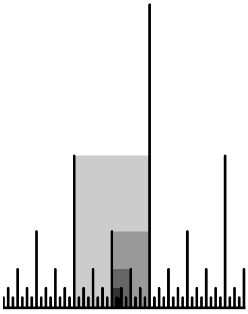
Family of sets with no section. Given $t \in (0,1]$ consider the real interval $\tilde{C}_t = [\frac{1}{t} + t, \frac{1}{t} + 1]$. Denote by C_t the image of $\tilde{C}_t$ under the covering map $\pi\colon \mathbb{R} \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

Set $C_0 = \mathbb{S}^1$. Note that the Hausdorff distance from C_0 to C_t is $\frac{t}{2}$. Therefore $\{C_t\}_{t\in[0,1]}$ is a family of compact subsets in $\mathbb{S}^1$ that is continuous in the sense of Hausdorff.

Assume there is a continuous section $c(t) \in C_t$, for $t \in [0,1]$. Since π is a covering map, we can lift the path c to a path $\tilde{c}\colon [0,1] \to \mathbb{R}$ such that $\tilde{c}(t) \in \tilde{C}_t$ for all t. In particular $\tilde{c}(t) \to \infty$ as $t \to 0$, a contradiction. □

The problem was suggested by Stephan Stadler. Here is a simpler, closely related problem.

📖 *Show that any Hausdorff continuous family of compact sets in* $\mathbb{R}$ *admits a continuous section.*



Spaces with isometric balls. The needed examples can be constructed from the upper half-plane by cutting it along a "dyadic comb" shown on the diagram and equipping the obtained space with the intrinsic metric induced from the ℓ_∞-norm on the plane. Few concentric balls in this metric are shown on the diagram.

First let us describe the comb precisely. Fix an infinite sequence $a_0, a_1, \ldots$ of zeros and ones. Given an integer k cut the upper half-plane along the line segment from $(k,0)$ to $(k,2^{m+1})$ if m is the maximal number such that

$$k \equiv a_0 + 2{\cdot}a_1 + \dots + 2^{m-1}{\cdot}a_{m-1} \pmod{2^m};$$

If the equality holds for all m, cut the half-plane along the vertical half-line from $(k,0)$.

Note that all the obtained spaces, independently from the sequence (a_n), meet the conditions of the problem for the point $x_0 = (\tfrac{1}{2},0)$.

Note yet that the resulting spaces for two sequences (a_n) and (a'_n) are isometric only in the following two cases

◇ if $a_n = a'_n$ for all large n, or

◇ if $a_n = 1 - a'_n$ for all large n.

It remains to produce two sequences that do not have these identities for all large n; two random sequences of zeros and ones will do the job with probability one. □

Average distance. Arguing by contradiction assume there is no such number. Then, by 1-dimesional Helly's theorem, there is a pair of point-arrays $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_m\}$ such that for their average distance functions $f(z) = \tfrac{1}{n}\cdot\sum_i |x_i - z|$ and $h(z) = \tfrac{1}{m}\cdot\sum_j |y_j - z|$, we have

$$f(p) > h(q)$$

for any two points p, q.

Note that

$$\tfrac{1}{m}\cdot\sum_j f(y_j) = \tfrac{1}{m\cdot n}\cdot\sum_{i,j}|x_i - y_j| = \tfrac{1}{n}\cdot\sum_i h(x_i),$$

a contradiction.

To prove uniqueness, it is sufficient to approximate a constant function by average distance functions. □

This is a result of Oliver Gross [see 157].

Chapter 6

Actions and coverings

Bounded orbit

Recall that a metric space is called *proper* if all its bounded closed sets are compact.

🕮 *Let X be a proper metric space and $\iota\colon X\to X$ an isometry. Assume that for some $x\in X$, the sequence $x_n=\iota^n(x)$, $n\in\mathbb{Z}$ has a converging subsequence. Prove that x_n is bounded.*

Semisolution. Note that we can assume that the orbit $\{x_n\}$ is dense in X; otherwise we can pass to the closure of the orbit. In particular, we can choose a finite number of positive integer values $n_1,\dots,n_k$ such that the set of points $\{x_{n_1},\dots,x_{n_k}\}$ is a 1-net for the ball $B(x_0,10)$; that is, for any $x\in B(x_0,10)$ there is x_{n_i} such that

$$|x-x_{n_i}|<1.$$

Assume $x_m\in B(x_0,1)$ for some m. Then

$$B(x_m,10)=f^m(B(x_0,10))\supset B(x_0,1).$$

In particular, $\{x_{m+n_1},\dots,x_{m+n_k}\}$ is a 1-net for the ball $B(x_0,1)$ Therefore $x_{m+n_i}\in B(x_0,1)$ for some $i\in\{1,\dots,k\}$.

Set $N=\max_i\{n_i\}$. Applying the above observation inductively, we get that from any string $x_{i+1},\dots x_{i+N}$ at least one point lies in $B(x_0,1)$. In particular, the N balls

$$B(x_1,10),\dots,B(x_N,10)$$

cover whole X. Hence the result follows. □

The problem is due to Aleksander Całka's [see 158].

Finite action

Show that for any nontrivial continuous action of a finite group on the unit sphere there is an orbit that does not lie in the interior of a hemisphere.

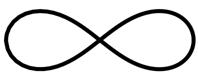
Covers of the figure eight

Given a covering

$$f\colon \tilde{X} \to X$$

of the length-metric space X, one can consider the induced length-metric on $\tilde{X}$ defining length of curve α in X as the length of the composition $f \circ \alpha$; the obtained metric space $\tilde{X}$ is called the *metric covering* of X.

Let us define the *figure eight* as the length-metric space obtained by gluing together all four ends of two unit segments.



Show that any compact length-metric space is a Gromov–Hausdorff limit of a sequence of metric covers

$$(\widetilde{\Phi}_n, \tilde{d}/n) \to (\Phi, d/n),$$

where (Φ, d) *denotes the figure eight.*

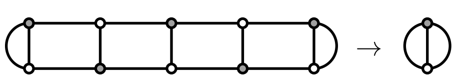
Diameter of m-fold cover*

The metric covering is defined in the previous problem.

Let X be a length-metric space and $\tilde{X}$ be its m-fold metric covering. Show that

$$\operatorname{diam}\tilde{X} \leqslant m\cdot \operatorname{diam} X.$$

From the diagram below you could guess an example of 5-fold cover with the diameter of the total space being exactly 5 times the diameter of the target.



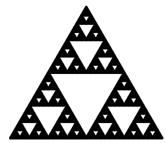
Symmetric square$^{\circ}$

Let X be a topological space. Note that $X \times X$ admits a natural $\mathbb{Z}_2$-action generated by the involution $(x, y) \mapsto (y, x)$. The quotient space $X \times X/\mathbb{Z}_2$ is called *symmetric square* of X.

Show that the symmetric square of any path connected topological space has commutative fundamental group.

Sierpiński gasket$^{\circ}$

To construct Sierpiński gasket, start with a solid equilateral triangle, subdivide it into four smaller congruent equilateral triangles and remove the interior of the central one. Repeat this procedure recursively for each of the remaining solid triangles.



Find the homeomorphism group of the Sierpiński gasket.

Lattices in a Lie group

Let L and M be two discrete subgroups of a connected Lie group G and h be a left invariant metric on G. Equip the groups L and M with the metrics induced from G. Assume $L\backslash G$ and $M\backslash G$ are compact and

$$\operatorname{vol}(L\backslash(G, h)) = \operatorname{vol}(M\backslash(G, h)).$$

Prove that there is a bi-Lipschitz one-to-one mapping $f \colon L \to M$, not necessarily a homomorphism.

Piecewise Euclidean quotient

Note that the quotient of the Euclidean space by a finite subgroup of $\mathrm{SO}(m)$ is a *polyhedral space* as is defined on page 112; on the same page you can find the definition of piecewise linear homeomorphism.

Let Γ be a finite subgroup of $\mathrm{SO}(m)$. Denote by P the quotient $\mathbb{R}^m/\Gamma$ equipped with the induced polyhedral metric. Assume P admits a piecewise linear homeomorphism to $\mathbb{R}^m$. Show that Γ is generated by rotations around subspaces of codimension 2.

The action of the symmetric group S_m on $\mathbb{C}^m = \mathbb{R}^{2 \cdot m}$ by permuting the complex coordinates provides a remarkable example. The homeomorphism $\mathbb{C}^m/S_m \to \mathbb{C}^m$ can be given by symmetric polynomials on $\mathbb{C}^m$; that is, $(z_1, \dots, z_m) \mapsto (a_0, \dots, a_{m-1})$ if

$$(z - z_1) \cdots (z - z_m) = a_0 + a_1 \cdot z + \dots + a_{m-1} \cdot z^{m-1} + z^m.$$

This homeomorphism is isotopic to a piecewise linear homeomorphism.

Subgroups of a free group

Show that every finitely generated subgroup of a free group is an intersection of subgroups of finite index.

Short generators°

Let M be a compact Riemannian manifold and $p \in M$. Show that the fundamental group $\pi_1(M, p)$ is generated by the homotopy classes of the loops with length at most $2\cdot \operatorname{diam} M$.

Number of generators

Let M be a complete connected Riemannian manifold with nonnegative sectional curvature. Show that the minimal number of generators of the fundamental group $\pi_1 M$ can be bounded above in terms of the dimension of M.

Equation in a Lie group°

Assume G is a compact connected Lie group and n is a positive integer. Show that given a collection of elements $g_1, \dots, g_n \in G$ the equation

$$x\cdot g_1\cdot x\cdot g_2 \cdots x\cdot g_n = 1$$

has a solution $x \in G$.

Quotient of Hilbert space*

Construct a free action by affine isometries on the Hilbert space whose quotient space is isometric to the sphere $\mathbb{S}^3$.

Semisolutions

Finite action. Without loss of generality, we may assume that the action is generated by a nontrivial homeomorphism

$$a\colon \mathbb{S}^m \to \mathbb{S}^m$$

with prime order p.

Assume the contrary; that is, any a-orbit lies in an open hemisphere. Then

$$h(x) = \sum_{n=1}^{p} a^n \cdot x \neq 0$$

for any $x \in \mathbb{S}^m$; here we consider $\mathbb{S}^m$ as the unit sphere in $\mathbb{R}^{m+1}$.

Consider the map $f\colon \mathbb{S}^m \to \mathbb{S}^m$ defined as $f(x) = \frac{h(x)}{|h(x)|}$. Note that

◇ if $a(x) = x$, then $f(x) = x$;

◇ $f(x) = f \circ a(x)$ for any $x \in \mathbb{S}^m$.

Note further that f is homotopic to the identity; in particular

$$(*) \qquad \deg f = 1.$$

The homotopy can be constructed as $(x,t) \mapsto \gamma_x(t)$, where γ_x is the minimizing geodesic path in $\mathbb{S}^m$ from x to $f(x)$. By construction, $|x - f(x)|_{\mathbb{S}^m} < \frac{\pi}{2}$; therefore γ_x is uniquely defined.

Fix $x \in \mathbb{S}^m$ such that $a(x) \neq x$. Note that the group acts without fixed points on the inverse image $W = f^{-1}(V)$ of a small open neighborhood $V \ni x$. Therefore the quotient map $\theta\colon W \to W' = W/\mathbb{Z}_p$ is a p-fold covering. From (6), the restriction $f|_W$ factors thru θ; that is, there is $f'\colon W' \to V$ such that $f|_W = f' \circ \theta$.

Assume $p \neq 2$. Note that f' and θ have well defined degrees and

$$\deg f \equiv \deg \theta \cdot \deg f' \pmod p$$

Since θ is a p-fold covering, we have $\deg \theta \equiv 0 \pmod p$. Therefore

$$(**) \qquad \deg f \equiv 0 \pmod p.$$

Finally observe that $(*)$ contradicts $(**)$.

In the case $p = 2$ the same proof works, but the degrees have to be considered modulo 2. □

Along the same lines one can get a lower bound for the maximal diameter of the orbits for any nontrivial action of a finite group on a Riemannian manifold.

Applying the problem to the conjugate actions, one gets that if a fixed point set of a finite group acting on a sphere has nonempty interior, then the action is trivial. The same holds for any connected manifold. All this was proved by Max Newman [see 159].

The following problem from [160] can be solved using Newman's theorem.

Assume h is a homeomorphism of a connected manifold M such that each h-orbit is finite. Show that h has finite order.

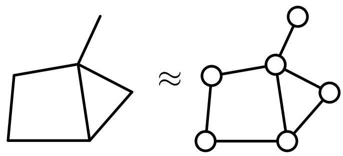
Covers of the figure eight. First note that any compact length-metric space K can be approximated by finite metric graphs.

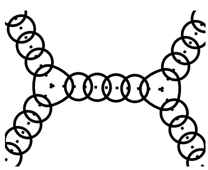
Indeed, fix a finite ε-net F in K. For each pair $x, y \in F$ choose a chain of points $x = x_0, x_1 \dots x_n = y$ such that $|x_i - x_{i-1}|_K < \varepsilon$ for each i and

$$|x - y|_K = |x_0 - x_1|_K + \dots + |x_{n-1} - x_n|_K.$$

Denote by F' the union of all these chains with F; Consider the metric graph with F' as the set of vertexes where every pair of vertexes v and w such that $|v - w|_K < \varepsilon$ is connected by an edge of length $|v - w|_K$. Note that the obtained metric graph is ε close to K in the sense of Gromov–Hausdorff.

Further, any finite metric graph is a limit of cubic[1] metric graphs Γ_n such that the length of each edge is a multiple of $\frac{1}{n}$. A construction can be guessed from the diagram.





It remains to approximate Γ_n by finite coverings of $(\Phi, d/n)$. Guess this part from the picture; it shows the needed covering of the figure eight for the doted cubic graph. □

The same idea works if instead of the figure eight, we have any compact length-metric space X that admits a map $X \to \Phi$ inducing an onto-homomorphism of fundamental groups. Such spaces X can be found among compact hyperbolic manifolds of any dimension $\geqslant 2$. All this is due to Vedrin Sahovic [see 161].

A similar idea was used later to show that any finitely presented group can appear as a fundamental group of the underlying space of a 3-dimensional hyperbolic orbifold [see 162].

Diameter of m-fold cover. Fix points $\tilde{p}, \tilde{q} \in \tilde{M}$. Let $\tilde{\gamma}\colon [0,1] \to \tilde{M}$ be a minimizing geodesic path from $\tilde{p}$ to $\tilde{q}$.

We need to show that

$$\operatorname{length} \tilde{\gamma} \leqslant m \cdot \operatorname{diam} M.$$

Suppose the contrary.

Denote by p, q and γ the projections to M of $\tilde{p}, \tilde{q}$ and $\tilde{\gamma}$ correspondingly. Represent γ as the concatenation of m paths of equal length,

$$\gamma = \gamma_1 * \dots * \gamma_m,$$

[1] A graph is cubic if the degree of each vertex is 3.

so
$$\operatorname{length}\gamma_i = \tfrac{1}{m}\cdot\operatorname{length}\gamma > \operatorname{diam} M.$$

Let σ_i be a minimizing geodesic in M connecting the endpoints of γ_i. Note that
$$\operatorname{length}\sigma_i \leqslant \operatorname{diam} M < \operatorname{length}\gamma_i.$$

Consider $m+1$ paths $\alpha_0, \dots, \alpha_m$ defined as the concatenations
$$\alpha_i = \sigma_1 * \dots * \sigma_i * \gamma_{i+1} * \dots * \gamma_m.$$

Let $\tilde\alpha_0, \dots, \tilde\alpha_m$ be their liftings with $\tilde q$ as their endpoint.

The staring points of $\tilde\alpha_i$ lie in the m inverse images of p. Therefore two curves, α_i and α_j for $i<j$, have the same starting point in $\tilde M$.

Note that the concatenation
$$\beta = \gamma_1 * \dots * \gamma_i * \sigma_{i+1} * \dots * \sigma_j * \gamma_{j+1} * \dots * \gamma_m.$$
admits a lift $\tilde\beta$ that connects $\tilde p$ to $\tilde q$ in $\tilde M$. Clearly $\operatorname{length}\tilde\beta < \operatorname{length}\gamma$, a contradiction. $\square$

The question was asked by Alexander Nabutovsky and answered by Sergei Ivanov [see 163].

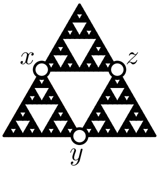
Symmetric square. Let $\Gamma = \pi_1 X$ and $\Delta = \pi_1((X\times X)/\mathbb{Z}_2)$. Consider the homomorphism $\varphi\colon \Gamma\times\Gamma\to\Delta$ induced by the quotient map $X\times X\to (X\times X)/\mathbb{Z}_2$.

Note that $\varphi(\alpha,1)=\varphi(1,\alpha)$ for any $\alpha\in\Gamma$ and the restrictions $\varphi|_{\Gamma\times\{1\}}$ and $\varphi|_{\{1\}\times\Gamma}$ are onto.

It remains to note that
$$\varphi(\alpha,1)\varphi(1,\beta)=\varphi(1,\beta)\varphi(\alpha,1)$$
for any α and β in Γ. $\square$

The problem was suggested by Rostislav Matveyev.

Sierpiński gasket. Denote the Sierpiński gasket by $\triangle$.



Let us show that any homeomorphism of $\triangle$ is also an isometry. Therefore its homeomorphism group is the symmetric group S_3.

Let $\{x,y,z\}$ be a 3-point set in $\triangle$ such that $\triangle\backslash\{x,y,z\}$ has 3 connected components. Note that there is unique choice for the set $\{x,y,z\}$ and it is formed by the midpoints of the long sides.

It follows that any homeomorphism of $\triangle$ permutes the set $\{x,y,z\}$.

Applying a similar argument recursively to the smaller triangles, we get that this permutation uniquely describes the homeomorphism. □

The problem was suggested by Bruce Kleiner. Note that the homeomorphism group of the Sierpiński carpet is much more interesting.

Latices in a Lie group. Denote by V_ℓ and W_m the Voronoi domains for each $\ell \in L$ and $m \in M$ correspondingly; that is,

$$V_\ell = \{\, g \in G \mid |g - \ell|_G \leqslant |g - \ell'|_G \text{ for any } \ell' \in L \,\}$$

$$W_m = \{\, g \in G \mid |g - m|_G \leqslant |g - m'|_G \text{ for any } m' \in M \,\}$$

Note that for any $\ell \in L$ and $m \in M$ we have

$$(*) \qquad \begin{aligned} \operatorname{vol} V_\ell &= \operatorname{vol}(L\backslash(G,h)) = \\ &= \operatorname{vol}(M\backslash(G,h)) = \\ &= \operatorname{vol} W_m. \end{aligned}$$

Consider the bipartite graph Γ with the parts L and M such that $\ell \in L$ is adjacent to $m \in M$ if and only if $V_\ell \cap W_m \neq \varnothing$.

By $(*)$ the graph Γ satisfies the condition of the marriage theorem — any subset in L has at least that many neighbors in M and the other way around [see 164]. Therefore there is a bijection $f\colon L \to M$ such that

$$V_\ell \cap W_{f(\ell)} \neq \varnothing$$

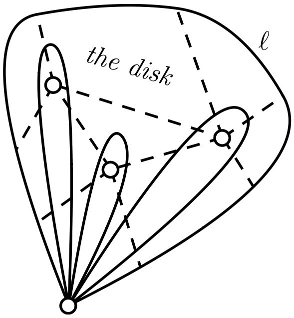
for any $\ell \in L$.

It remains to observe that f is bi-Lipschitz. □

The problem is due to Dmitri Burago and Bruce Kleiner [see 165]. For a finitely generated group G it is not known if G and $G \times \mathbb{Z}_2$ can fail to be bi-Lipschitz. (The groups are assumed to be equipped with the word metric.)

Piecewise Euclidean quotient. Note that the group Γ is the holonomy group of the quotient space $P = \mathbb{R}^m/\Gamma$. More precisely, one can identify $\mathbb{R}^m$ with the tangent space of a regular point x_0 of P in such a way that for any $\gamma \in \Gamma$ there is a loop ℓ based at x_0 that runs in the regular locus of P and has the holonomy γ.

Fix γ and ℓ as above. Since P is simply connected, we can shrink ℓ by a disk. By general position argument we can assume that the disk only pass thru simplices of codimension 0, 1 and 2 and intersects the simplices of codimension 2 transversely.

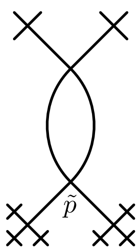


In other words, ℓ can be presented as a product of loops such that each loop goes around a single simplex of codimension 2 and comes back. The holonomy for each of these loops is a rotation around a hyperplane. Hence the result follows. □

The converse of the problem also holds; it was proved by Christian Lange [see 166]; his proof is based on earlier results of Marina Mikhailova [see 167].

Note that the cone over the spherical suspension over the Poincaré sphere is homeomorphic to $\mathbb{R}^5$ and it is the quotient of $\mathbb{R}^5$ by the binary icosahedral group, which is a subgroup of $\mathrm{SO}(5)$ of order 120. Therefore, if one exchanges "piecewise linear homeomorphism" for "homeomorphism" in the formulation, then the answer is different; a complete classification of such actions is given in [166].

Subgroups of a free group. The proof exploits the fact that free groups are the fundamental groups of graphs.



$\bar B(\tilde p, 2+\frac12)$

Let F be a free group and G be a finitely generated subgroup in F. We need to show that G is an intersection of subgroups of finite index in F. Without loss of generality we can assume that F has a finite number of generators, denote it by m.

Let W be the wedge sum of m circles, so $\pi_1(W,p) = F$. Equip W with the length-metric such that each circle has unit length.

Pass to the metric cover $\tilde W$ of W such that $\pi_1(\tilde W,\tilde p) = G$ for a lift $\tilde p$ of p.

Fix a sufficiently large integer n and consider the doubling of the closed ball $\bar B(\tilde p, n+\frac12)$ along its boundary. Let us denote the obtained doubling by Z_n and set $G_n = \pi_1(Z_n,\tilde p)$.

Note that Z_n is a metric covering of W; it makes possible to consider G_n as a subgroup of F. By construction, Z_n is compact; therefore G_n has finite index in F.

It remains to show that

$$G = \bigcap_{n>k} G_n,$$

where k is the maximal word length in the generating set of G. □

Originally the problem was solved by Marshall Hall [see 164]. The proof presented here is close to the solution of John Stalings [see 168 and also 169].

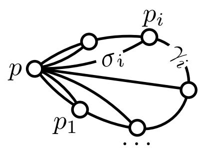
The same idea can be used to solve many other problems; here are some examples.

Show that subgroups of free groups are free.

Show that two elements u and v of a free group commute if and only if they are both powers of the some element w.

Short generators. Choose a length minimizing loop γ that represents a given element $a \in \pi_1 M$.

Fix $\varepsilon > 0$. Represent γ as a concatenation



$$\gamma = \gamma_1 * \dots * \gamma_n$$

of paths and

$$\operatorname{length} \gamma_i < \varepsilon$$

for each i.

Denote by $p = p_0, p_1, \dots, p_n = p$ the endpoints of these arcs. Connect p to p_i by a minimizing geodesic σ_i. Note that γ is homotopic to a product of loops

$$\alpha_i = \sigma_{i-1} * \gamma_i * \bar{\sigma}_i,$$

where $\bar{\sigma}_i$ denotes the path σ_i traveled backwards. In particular,

$$\operatorname{length} \alpha_i < 2 \cdot \operatorname{diam} M + \varepsilon$$

for each i.

Note that given $\ell > 0$, there are only finitely many elements of the fundamental group that can be realized by loops at p with length shorter than ℓ. It follows that for the right choice of $\varepsilon > 0$, any loop α_i is homotopic to a loop of length at most $2 \cdot \operatorname{diam} M$. Hence the result follows. □

The statement is due to Mikhael Gromov [see Proposition 3.22 in 65].

Number of generators. Consider the universal Riemannian cover $\tilde{M}$ of M. Note that $\tilde{M}$ is non-negatively curved and $\pi_1 M$ acts by isometries on $\tilde{M}$.

Fix $p \in \tilde{M}$. Given $a \in \pi_1 M$, set

$$|a| = |p - a \cdot p|_{\tilde{M}}.$$

Consider the so called *short basis* in $\pi_1 M$; that is, a sequence of elements $a_1, a_2, \dots \in \pi_1 M$ defined the following way:

(i) Choose $a_1 \in \pi_1 M$ so that $|a_1|$ takes the minimal value.
(ii) Choose $a_2 \in \pi_1 M \backslash \langle a_1 \rangle$ so that $|a_2|$ takes the minimal value.
(iii) Choose $a_3 \in \pi_1 M \backslash \langle a_1, a_2 \rangle$ so that $|a_3|$ takes the minimal value.
(iv) and so on.

Note that the sequence terminates at the n-th step if $a_1, \dots, a_n$ generate $\pi_1 M$. By construction, we have

$$|a_j \cdot a_i^{-1}| \geqslant |a_j| \geqslant |a_i|$$

for any $j > i$. Set $p_i = a_i \cdot p$. Note that

$$\begin{aligned}|p_j - p_i|_{\tilde M} &= |a_j \cdot a_i^{-1}| \geqslant \\ &\geqslant |a_j| = \\ &= |p_j - p|_{\tilde M} \geqslant \\ &\geqslant |a_i| = \\ &= |p_i - p|_{\tilde M}.\end{aligned}$$

By the Toponogov comparison theorem we get

$$\measuredangle[p\,{}^{p_i}_{p_j}] \geqslant \tfrac{\pi}{3}.$$

That is, the directions from p to all p_i make an angle of at least $\frac{\pi}{3}$ with each other.

Therefore the number of points p_i can be bounded in terms of the dimension of M. Hence the result follows. □

The *short basis construction*, as well as the result above are due to Mikhael Gromov [see 15].

Equation in a Lie group. We will assume that G is equipped with a bi-invariant metric. In particular geodesics starting from $1 \in G$ are given by homomorphisms $\mathbb{R} \to G$.

Consider the map $\varphi\colon G \to G$ defined by

$$\varphi(x) = x \cdot g_1 \cdot x \cdot g_2 \cdots x \cdot g_n.$$

We need to show that φ is onto. Note that it is sufficient to show that φ has non zero degree.

The map φ is homotopic to the map $\psi\colon x \mapsto x^n$. Therefore it is sufficient to show that

$$(*) \qquad \deg \psi \neq 0$$

Note that the claim $(*)$ follows from $(**)$.

(**) *For any $x \in G$ the differential*

$$d_x\psi\colon \mathrm{T}_xG \to \mathrm{T}_{x^n}G$$

does not revert orientation.

Indeed, connect 1 to a given point $y \in G$ by a geodesic path γ, so $\gamma(0) = 1$ and $\gamma(1) = y$. Since $\gamma\colon \mathbb{R} \to G$ is a homomorphism, $\psi(x) = y$ for $x = \gamma(\frac{1}{n})$. In particular the inverse image $\psi^{-1}\{y\}$ is nonempty for any $y \in G$.

By (**), for a regular value y, each point in the inverse image $\psi^{-1}\{y\}$ conributes 1 to the degree of ψ. Hence (*) follows.

It remains to prove (**). Given an element $g \in G$, denote by $L_g, R_g\colon G \to G$ its left and right shifts; that is, $L_g(x) = g{\cdot}x$ and $R_g(x) = x{\cdot}g$.

Identify the tangent spaces T_xG and $\mathrm{T}_{x^n}G$ with the Lie algebra $\mathfrak{g} = \mathrm{T}_eG$ using $dR_x\colon \mathfrak{g} \to \mathrm{T}_xG$ and $dR_x^n\colon \mathfrak{g} \to \mathrm{T}_{x^n}G$ correspondingly. Then for any $V \in \mathfrak{g}$, we have

$$d_x\psi(V) = V + \mathrm{Ad}_x(V) + \dots + \mathrm{Ad}_x^{n-1}(V),$$

where $\mathrm{Ad}_x = d_e(L_x \circ R_{x^{-1}})\colon \mathfrak{g} \to \mathfrak{g}$. Since the metric on G is bi-invariant, we have that Ad_x is an isometry of $\mathfrak{g}$. It remains to note that the linear transformation

$$V \mapsto V + T(V) + \dots + T^{n-1}(V)$$

can not revert orientation for any isometric linear transformation T of the Euclidean space. The last statement is an exercise in linear algebra. □

The idea of this solution is due to Murray Gerstenhaber and Oscar Rothaus [see 170]. In fact, the degree of g is n^k, where k is the rank of G [see 171].

Quotient of Hilbert space. We consider $\mathbb{S}^3$ as the set of unit quaternions, in particular it has a group structure.

Let $\mathbb{H}$ be the set of paths in $\mathbb{S}^3$ starting at 1 and of *class* $W^{1,2}$; that is, its velocity is square-integrable. The point-wise multiplication of paths defines a group structure on $\mathbb{H}$. Denote by Ω the subset of all loops in $\mathbb{H}$.

It remains to equip $\mathbb{H}$ with the structure of a Hilbert space so that the right action of Ω on $\mathbb{H}$ is isometric and the quotient is isometric to $\mathbb{S}^3$.

We will prove the statement for any connected Lie group G with a bi-invariant metric, in particular for $G = \mathbb{S}^3$. Denote by $\mathfrak{g} = \mathrm{T}_1G$ the

Lie algebra of G. Equip G with a bi-invariant metric and let $\langle *,*\rangle_{\mathfrak{g}}$ be the corresponding scalar product in $\mathfrak{g}$.

Consider the Hilbert space $\mathbb{H}$ of all L^2-functions $f\colon [0,1]\to\mathfrak{g}$ with the scalar product defined by

$$\langle f,g\rangle = \int\limits_{[0,1]} \langle f(t),g(t)\rangle_{\mathfrak{g}}\cdot dt.$$

Construction of the quotient map $\varphi\colon \mathbb{H}\to G$. Given $v\in\mathfrak{g}$ denote by $\tilde v$ the corresponding right invariant tangent field on G.

Given $f\colon [0,1]\to\mathfrak{g}$ in $\mathbb{H}$, consider the path

$$\Gamma_f\colon [0,1]\to G$$

with $\Gamma_f(0)=1$ and $\Gamma'_f(t)=\tilde f(t)$ for any t.

The map $\varphi\colon \mathbb{H}\to G$ is the evaluation map $\varphi\colon f\mapsto \Gamma_f(1)$. Since G is connected, φ is onto.

Group structure on $\mathbb{H}$. Note that the functional $f\mapsto\Gamma_f$ is an injective map from $\mathbb{H}$ to the space of paths in G starting at 1.

Recall that $\mathrm{Ad}_\alpha\colon \mathfrak{g}\to\mathfrak{g}$ denotes the adjoint transformation for $\alpha\in G$; that is, $\mathrm{Ad}_\alpha = d_1\,\mathrm{Inn}_\alpha$, where $\mathrm{Inn}_\alpha\colon x\mapsto \alpha\cdot x\cdot\alpha^{-1}$ is the inner automorphism of G. Note that Ad_α preserves the scalar product on $\mathfrak{g}$.

Consider the multiplication $\star$ on $\mathbb{H}$ by

$$(h\star f)(t) = h(t) + \mathrm{Ad}_{\Gamma_h(t)}[f(t)]. \tag{$*$}$$

Note that

$$\Gamma_{h\star f}(t) = \Gamma_h(t)\cdot\Gamma_f(t)$$

for any $t\in[0,1]$. In particular, $(\mathbb{H},\star)$ is a group with neutral element 0.

From $(*)$, we get

$$(h\star f)(t)-(h\star g)(t) = \mathrm{Ad}_{\Gamma_h(t)}(f(t)-g(t))$$

and therefore

$$|(f\star h)(t)-(g\star h)(t)| = |f(t)-g(t)|$$

for any t. It follows that for any fixed h, the transformation $f\mapsto h\star f$ is an affine isometry of $\mathbb{H}$.

The set $\Omega=\varphi^{-1}\{1\}$ is a subgroup of $(\mathbb{H},\star)$; it can be viewed as the group of $W^{1,2}$-loops in G. It remains to note that $\varphi\colon \mathbb{H}\to G$ is the quotient map for the right action of Ω on $\mathbb{H}$. $\square$

Aleternative solution. Again, we will prove the statement for any connected Lie group G with a bi-invariant metric.

Denote by G^n the direct product of n copies of G. Consider the map $\varphi_n\colon G^n \to G$ defined by

$$\varphi_n\colon (\alpha_1, \dots, \alpha_n) \mapsto \alpha_1 \cdots \alpha_n.$$

Note that φ_n is the quotient map for the G^{n-1}-action on G^n defined by

$$(\beta_1, \dots, \beta_{n-1})\cdot(\alpha_1, \dots, \alpha_n) = (\alpha_1\cdot\beta_1^{-1}, \beta_1\cdot\alpha_2\cdot\beta_2^{-1}, \dots, \beta_{n-1}\cdot\alpha_n).$$

Denote by ρ_n the product metric on G^n rescaled with factor $\sqrt{n}$. Note that the quotient $(G^n, \rho_n)/G^{n-1}$ is isometric to $G = (G, \rho_1)$.

As $n \to \infty$ the curvature of (G^n, ρ_n) converges to zero and its injectivity radius goes to infinity. Therefore passing to the ultra-limit of G^n as $n \to \infty$ we get the Hilbert space. It remains to observe that the limit action has the required property. $\square$

This construction is given by Chuu-Lian Terng and Gudlaugur Thorbergsson [see section 4 in 172]; the alternative solution was suggested by Alexander Lytchak.

Instead of the group Ω, one could consider the subgroup Ω_H of paths $\gamma\colon [0,1] \to G$ such that the pairs $(\gamma(0), \gamma(1))$ belong to a given subgroup $H < G \times G$. In this case the quotient $\mathbb{H}/\Omega_H$ is isometric to the *double quotient* $G/\!\!/H$; that is, the quotient of the action on G defined by $(h_1, h_2)\cdot g = h_1\cdot g\cdot h_2^{-1}$ for $(h_1, h_2) \in H < G \times G$.

Chapter 7

Topology

In this chapter we consider geometrical problems with strong topological flavor. A typical introductory course in topology, say [173], contains all the necessary material.

Isotropy

Recall that an isotopy is a continuous one parameter family of embeddings.

Let K_1 and K_2 be homeomorphic closed subsets of the coordinate subspace $\mathbb{R}^m$ in $\mathbb{R}^{2\cdot m}$. Show that there is a homeomorphism

$$h\colon \mathbb{R}^{2\cdot m} \to \mathbb{R}^{2\cdot m}$$

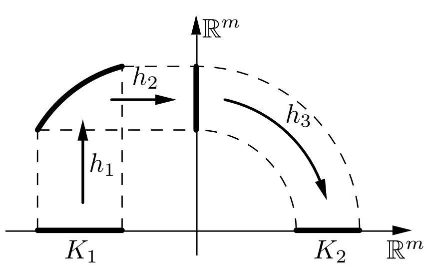
such that $K_2 = h(K_1)$. Moreover, h can be chosen to be isotopic to the identity map.

Semisolution. Fix a homeomorphism $\varphi\colon K_1 \to K_2$.

By the Tietze extension theorem, the homeomorphisms $\varphi\colon K_1 \to K_2$ and $\varphi^{-1}\colon K_2 \to K_1$ can be extended to continuous maps; denote these maps by $f\colon \mathbb{R}^m \to \mathbb{R}^m$ and $g\colon \mathbb{R}^m \to \mathbb{R}^m$ correspondingly.

Consider the homeomorphisms $h_1, h_2, h_3\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ defined the following way

$$\begin{aligned}
h_1(x,y) &= (x, y + f(x)),\\
h_2(x,y) &= (x - g(y), y),\\
h_3(x,y) &= (y, -x).
\end{aligned}$$



It remains to observe that each homeomorphism h_i is isotopic to the identity map and

$$K_2 = h_3 \circ h_2 \circ h_1(K_1). \qquad \square$$

This construction is called *Klee's trick*; it is due to Victor Klee [see 174]. This trick is used in the five-line proof of the Jordan separation theorem by Patrick Doyle [see 175]; a proof of the separation theorem for the embeddings $\mathbb{S}^n \hookrightarrow \mathbb{S}^{n+1}$ can be built with the same idea [see 176].

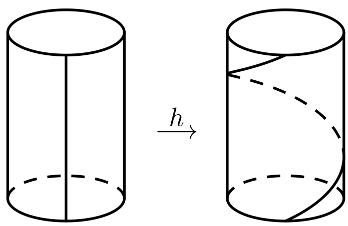
The problem "Monotonic homotopy" on page 125 looks similar.

Immersed disks

Two immersions f_1 and f_2 of the disk $\mathbb{D}$ into the plane will be called *essentially different* if there is no diffeomorphism $h\colon \mathbb{D} \to \mathbb{D}$ such that $f_1 = f_2 \circ h$.

Construct two essentially different smooth immersions of the disk into the plane that coincide near the boundary.

Positive Dehn twist



Let Σ be a surface and

$$\gamma\colon \mathbb{R}/\mathbb{Z} \to \Sigma$$

be a non-contractible closed simple curve. Let U_γ be a neighborhood of γ that admits a parametrization

$$\iota\colon \mathbb{R}/\mathbb{Z} \times (0,1) \to U_\gamma.$$

A *Dehn twist* along γ is a homeomorphism $h\colon \Sigma \to \Sigma$ that is the identity outside of U_γ and such that

$$\iota^{-1} \circ h \circ \iota\colon (x, y) \mapsto (x + y, y).$$

If Σ is oriented and ι is orientation preserving, then the Dehn twist described above is called *positive*.

Let Σ be a compact oriented surface with nonempty boundary. Prove that any composition of positive Dehn twists of Σ is not homotopic to the identity rel. boundary.

In other words, any product of positive Dehn twists represents a nontrivial class in the mapping class group of Σ.

Conic neighborhood

Let p be a point in a topological space X. We say that an open neighborhood $U \ni p$ is *conic* if there is a homeomorphism from a cone to U that sends the vertex to p.

Show that any two conic neighborhoods of one point are homeomorphic to each other.

Note that two cones $\mathrm{Cone}(\Sigma_1)$ and $\mathrm{Cone}(\Sigma_2)$ might be homeomorphic while Σ_1 and Σ_2 are not; existence of such examples follow from the double suspension theorem.

Unknots°

Prove that the set of smooth embeddings $f\colon \mathbb{S}^1 \to \mathbb{R}^3$ equipped with the C^0-topology forms a connected space.

Stabilization

Construct two compact subsets $K_1, K_2 \subset \mathbb{R}^2$ such that K_1 is not homeomorphic to K_2, but $K_1 \times [0,1]$ is homeomorphic to $K_2 \times [0,1]$.

Homeomorphism of a cube

Let $\square^m$ be a cube in $\mathbb{R}^m$ and $h\colon \square^m \to \square^m$ be a homeomorphism that sends each face of $\square^m$ to itself. Extend h to a homeomorphism $f\colon \mathbb{R}^m \to \mathbb{R}^m$ that coincides with the identity map outside of a bounded set.

Finite topological space°

Given a finite topological space F construct a finite simplicial complex K that admits a weak homotopy equivalence $K \to F$.

Dense homeomorphism°

Denote by $\mathcal{H}$ be the set of all orientation preserving homeomorphisms $\mathbb{S}^2 \to \mathbb{S}^2$ equipped with the C^0-metric. Show that there is a homeomorphism $h \in \mathcal{H}$ such that its conjugations $a \circ h \circ a^{-1}$ for all $a \in \mathcal{H}$ form a dense set in $\mathcal{H}$.

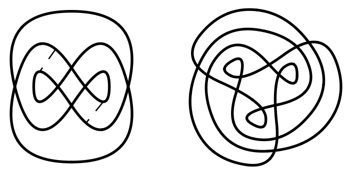
Simple path°

Let p and q be distinct points in a Hausdorff topological space X. Assume p and q are connected by a path. Show that they can be connected by a simple path; that is, there is an injective continuous map $\beta\colon [0,1] \to X$ such that $\beta(0) = p$ and $\beta(1) = q$.

Path in a surface°

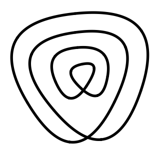
Show that any path with distinct ends in a surface is homotopic (relative to the ends) to a simple path.

Semisolutions



Immersed disks. Both circles on the picture bound essentially different disks.

On the first diagram, the dashed lines and the solid lines together bound three embedded disks; gluing these disks along the dashed lines gives the first immersion. The reflection of this immersion across the vertical line of symmetry gives another immersion which is essentially different. □

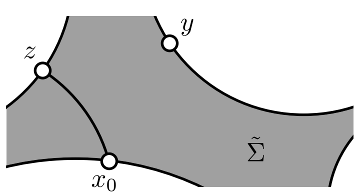


It is a good exercise to count the essentially different disks in the second example. (The answer is 5.)

The existence of examples of that type is generally attributed to John Milnor [see 177].

An easier problem would be to construct two essentially different immersions of annuli with the same boundary curves; a solution is shown on the picture [for more details and references see 178].

Positive Dehn twist. Consider the universal covering $f\colon \tilde\Sigma \to \Sigma$. The surface $\tilde\Sigma$ comes with the orientation induced from Σ.



Fix a point x_0 on the boundary $\partial\tilde\Sigma$. Given two other points y and z in $\partial\tilde\Sigma$ we will write $z \succ y$ if y lies on the right side from some simple curve from x_0 to z in $\tilde\Sigma$. Note that $\succ$ defines a linear order on $\partial\tilde\Sigma\backslash\{x_0\}$. We will write $z \succeq y$ if $z \succ y$ or $z = y$.

Note that any homeomorphism $h\colon \Sigma \to \Sigma$ that is identity on the boundary lifts to the unique homeomorphism $\tilde{h}\colon \tilde{\Sigma} \to \tilde{\Sigma}$ such that $\tilde{h}(x_0) = x_0$. The following claim is the key step in the proof.

($*$) *Assume h is a positive Dehn twist along a closed curve γ. Then $y \succeq \tilde{h}(y)$ for any $y \in \partial\tilde{\Sigma}\backslash\{x_0\}$ and $y_0 \succ \tilde{h}(y_0)$ for some $y_0 \in \partial\tilde{\Sigma}\backslash\{x_0\}$.*

Note that the property in ($*$) is a homotopy invariant and it survives under compositions of maps. Therefore the problem follows from ($*$).

If Σ is not an annulus, then by the uniformization theorem, we can assume that Σ has a hyperbolic metric with geodesic boundary; the lifted metric on $\tilde{\Sigma}$ has the same properties. Furthermore, we can assume that (1) γ is a closed geodesic, (2) the parametrization $\iota\colon \mathbb{R}/\mathbb{Z}\times(0,1) \to U_\gamma$ from the definition of Dehn twist is rotationally symmetric and (3) for any $u \in \mathbb{R}/\mathbb{Z}$ the arc $\iota(u\times(0,1))$ is a geodesic perpendicular to γ.

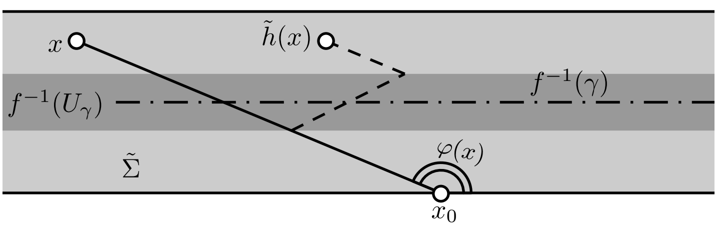
Consider the polar coordinates (φ, ρ) on $\tilde{\Sigma}$ with the origin at x_0; since x_0 lies on the boundary, the angle coordinate φ is defined in $[0, \pi]$. By construction of Dehn twist, we get

$$\varphi(x) \geqslant \varphi \circ \tilde{h}(x)$$

for any $x \neq x_0$ and if the geodesic $[x_0x]$ crosses $f^{-1}(U_\gamma)$ then

$$\varphi(x) > \varphi \circ \tilde{h}(x).$$

In particular, if x lies on the boundary then $\tilde{h}(x)$ lies on the right side from the geodesic $[x_0x]$; hence the claim ($*$) follows.



If Σ is an annulus, then the same argument works except we have to choose a flat metric on Σ. In this case $\tilde{\Sigma}$ is a strip between two parallel lines in the plane, see the diagram. $\square$

The problem was suggested by Rostislav Matveyev. It is instructive to solve the following problem.

Construct a composition of positive Dehn twists on a compact oriented surface without boundary which is homotopic to the identity.

Conic neighborhood. Let V and W be two conic neighborhoods of p. Without loss of generality, we may assume that $V \Subset W$; that is, the closure of V lies in W.

We will need to construct a sequence of embeddings $f_n : V \to W$ such that

- ◇ For any compact set $K \subset V$ there is a positive integer $n = n_K$ such that $f_n(k) = f_m(k)$ for any $k \in K$ and $m, n \geqslant n_K$.
- ◇ For any point $w \in W$ there is a point $v \in V$ such that $f_n(v) = w$ for all large n.

Note that once such sequence is constructed, $f : V \to W$ defined as $f(v) = f_n(v)$ for all large values of n gives the needed homeomorphism.

The sequence f_n can be constructed recursively

$$f_{n+1} = \Psi_n \circ f_n \circ \Phi_n,$$

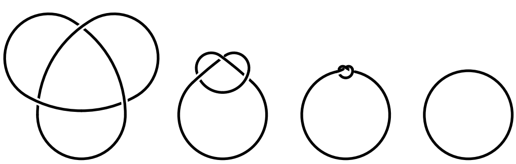
where $\Phi_n : V \to V$ and $\Psi_n : W \to W$ are homeomorphisms of the form

$$\Phi_n(x) = \varphi_n(x) * x \quad \text{and} \quad \Phi_n(x) = \psi_n(x) \star x,$$

where $\varphi_n : V \to \mathbb{R}_+$, $\psi_n : W \to \mathbb{R}_+$ are suitable continuous functions; "$*$" and "$\star$" denote the *multiplication* in the cone structures of V and W correspondingly. □

The problem is due to Kyung Whan Kwun [see 179].

Unknots.

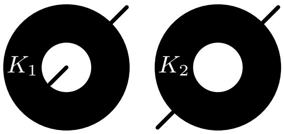


Observe that it is possible to draw an arbitrary tight knot while keeping it smoothly embedded all the time including the last moment. □

This problem was suggested by Greg Kuperberg.

Stabilization. The example can be guessed from the diagram.

The two sets K_1 and K_2 are subspaces of the plane, each one is a closed annulus with two attached line segments. In K_1 one segment is attached from inside and the other from outside and in K_2 both segments are attached from outside.



The product spaces $K_1 \times [0,1]$ and $K_2 \times [0,1]$ are solid tori with attached rectangles. A homeomorphism $K_1 \times [0,1] \to K_2 \times [0,1]$ can be constructed by twisting a part of one solid torus.

To prove the nonexistence of a homeomorphism $K_1 \to K_2$ consider the sets of cut points $V_i \subset K_i$ and the sets $W_i \subset K_i$ of points that admit a punctured simply connected neighborhood. Note that the set V_i is the union of the attached line segments and W_i is the boundary of the annulus without points where the segments are attached. Note that $V_i \cup W_i = \partial K_i$; in particular, a homeomorphism $K_1 \to K_2$ (if exists) sends ∂K_1 to ∂K_2.

Finally note that each ∂K_i has two connected components and V_1 intersects both components of ∂K_1 while V_2 lies in one component of ∂K_2. Hence $K_1 \not\cong K_2$. $\square$

I learned this problem from Maria Goluzina around 1988.

Homeomorphism of a cube. Let us extend the homeomorphism h to $\mathbb{R}^m$ by reflecting the cube across its facets. We get a homeomorphism $\tilde{h}\colon \mathbb{R}^m \to \mathbb{R}^m$ such that $\tilde{h}(x) = h(x)$ for any $x \in \square^m$ and

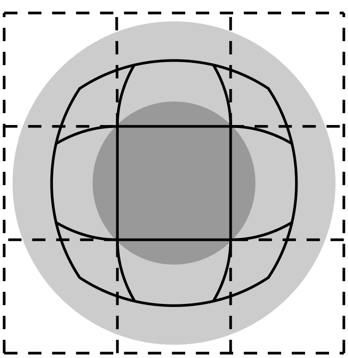
$$\tilde{h} \circ \gamma = \gamma \circ \tilde{h},$$

where γ is any reflection with respect to the facets of the cube.

Without loss of generality, we may assume that the cube $\square^m$ is inscribed in the unit sphere centered at the origin of $\mathbb{R}^m$. In this case $\tilde{h}$ has *displacement* at most 2; that is,

$$|\tilde{h}(x) - x| \leqslant 2$$

for any $x \in \mathbb{R}^m$.



Fix a smooth increasing concave function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}$ such that

$$\varphi(r) = r$$

for any $r \leqslant 1$ and

$$\sup\{\varphi(r)\} = 2.$$

Equip $\mathbb{R}^m$ with the polar coordinates (u, r), where $u \in \mathbb{S}^{m-1}$ and $r \geqslant 0$. Consider the open embedding $\Phi\colon \mathbb{R}^m \hookrightarrow \mathbb{R}^m$ defined as $\Phi(u, r) = (u, \varphi(r))$.

Set

$$f(x) = \begin{cases} x & \text{if } |x| \geqslant 2 \\ \Phi \circ \tilde{h} \circ \Phi^{-1}(x) & \text{if } |x| < 2 \end{cases}$$

It remains to observe that $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is a solution. □

This problem is stripped from a proof of Robion Kirby [see 180]. The condition that each face is mapped to itself can be removed and instead of homeomorphism one could take any embedding close to the identity.

An interesting twist to this idea was given by Dennis Sullivan [see 139]. Instead of the discrete group of motions of the Euclidean space, he uses a discrete group of motions of the hyperbolic space in the conformal disk model.

To see the idea, note that the construction of $\tilde{h}$ can be done for a Coxeter polytope in the hyperbolic space instead of a cube. Then the constructed map $\tilde{h}$ coincides with the identity on the absolute and therefore the last "shrinking" step in the proof above is not needed. Moreover, if the original homeomorphism is bi-Lipschitz, then the Sullivan construction produces a bi-Lipschitz homeomorphism — this is the main advantage.

Finite topological space. Given a point $p \in F$, denote by O_p the minimal open set in F containing p. Note that we can assume that F is a connected T_0-space; in particular, $O_p = O_q$ if and only if $p = q$.

Let us write $p \preccurlyeq q$ if $O_p \subset O_q$. The relation $\preccurlyeq$ is a partial order on F.

Let us construct a simplicial complex K by taking F as the set of vertices and declaring a collection of vertices to be a simplex if it can be linearly ordered with respect to $\preccurlyeq$.

Given $k \in K$, consider the minimal simplex $(f_0, \dots, f_m) \ni k$; we can assume that $f_0 \preccurlyeq \dots \preccurlyeq f_m$. Set $h\colon k \mapsto f_0$; it defines a map $K \to F$.

It remains to check that h is continuous and induces isomorphisms for all the homotopy groups. □

In a similar fashion, one can construct a finite topological space F for any given simplicial complex K such that there is a weak homotopy equivalence $K \to F$. Both constructions are due to Pavel Alexandrov [see 181, 182].

Dense homeomorphism. Note that there is countable set of homeomorphisms $h_1, h_2, \dots$ that is dense in $\mathcal{H}$ such that each h_n fixes all the points outside an open round disk, say D_n.

Choose a countable disjoint collection of round disks D'_n. Consider the homeomorphism $h\colon \mathbb{S}^2 \to \mathbb{S}^2$ that fixes all the points outside of $\bigcup_n D'_n$ and for each n, the restriction $h|_{D'_n}$ is conjugate to $h_n|_{D_n}$.

Note that for large n, the homeomorphism h is conjugate to a homeomorphism close to h_n. Therefore h is a solution. $\square$

The problem was mentioned by Frederic Le Rox [see 183] on a problem section at a conference in Oberwolfach, where he also conjectured that this is not true for the area-preserving homeomorphisms. An affirmative answer to this conjecture was given by Daniel Dore, Andrew Hanlon and Sobhan Seyfaddini [see 184, 185]. In particular it implies the following seemingly evident but nontrivial statement.

Given $\varepsilon > 0$ there is $\delta > 0$ such that

$$\Omega \cap h(\Omega) \neq \varnothing$$

for any topological disk $\Omega \subset \mathbb{S}^2$ with area at least ε and any area-preserving homeomorphism $h\colon \mathbb{S}^2 \to \mathbb{S}^2$ with displacement at most δ; that is, $|h(x) - x|_{\mathbb{S}^2} < \delta$ for any $x \in \mathbb{S}^2$.

Simple path. We will give two solutions, the first one is elementary and the second one is involved.

First solution. Let α be a path connecting p to q.

Passing to a subinterval if necessary, we can assume that $\alpha(t) \neq p, q$ for $t \neq 0, 1$.

An open set Ω in $(0, 1)$ will be called *suitable* if for any connected component (a, b) of Ω we have $\alpha(a) = \alpha(b)$. Since the union of nested suitable sets is suitable, we can find a maximal suitable set $\hat{\Omega}$.

Define $\beta(t) = \alpha(a)$ for any t in a connected component $(a, b) \subset \Omega$. Note that β is continuous and monotonic; that is, for any $x \in [0, 1]$ the set $\beta^{-1}\{\beta(x)\}$ is connected.

It remains to re-parametrize β to make it injective. In other words we need to construct a non-decreasing surjective function $\tau\colon [0, 1] \to$ $\to [0, 1]$ such that $\tau(t_1) = \tau(t_2)$ if and only if there is a connected component (a, b) such that $t_1, t_2 \in [a, b]$. The construction is similar to the construction of devil's staircase. $\square$

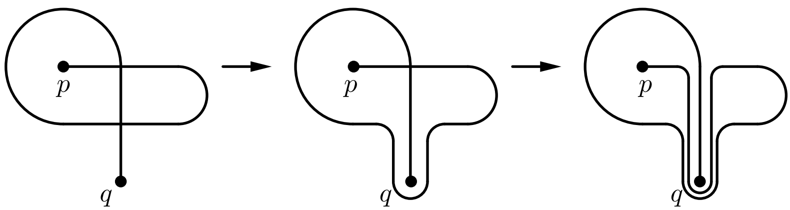
Second solution. Note that one can assume that X coincides with the image of α. In particular, X is a connected, locally connected, compact Hausdorff space.

Any such space admits a length-metric. This statement is not at all trivial; it was conjectured by Karl Menger [see 186] and proved independently by R. H. Bing [see 187, 188] and Edwin Moise [see 189].

It remains to consider a geodesic path from p to q. □

The problem was inspired by a lemma proved by Alexander Lytchak and Stefan Wenger [see 7.13 in 190].

Path in a surface. Denote the surface by Σ, assume the path runs from p to q. The following picture suggests an idea for an induction proof on the number of self-crossings.



To do the proof formally, let us present the path as a concatenation $\alpha*\beta$ of two paths such that α is simple and β does not pass thru p. We can assume that $\beta\colon[0,1]\to\Sigma$ is smooth.

Choose a smooth time depending vector field V_t on Σ such that

$$V_t(\beta(t))=\beta'(t)\quad\text{and}\quad V_t(p)=0$$

for any $t\in[0,1]$.

Consider the flow $\Phi^t\colon\Sigma\to\Sigma$ along V_t; that is,

$$\Phi^0(x)=x\quad\text{and}\quad\tfrac{d}{dt}(\Phi^t(x))=V_t(\Phi^t(x))$$

for any $t\in[0,1]$ and $x\in\Sigma$. The map $\Phi^1:\Sigma\to\Sigma$ is a diffeomorphism; in particular Φ^1 sends the simple path α to a simple path $\alpha_1=\Phi^1\circ\alpha$. By construction $\alpha_1(1)=q$. Since $V_t(p)=0$ for any t, we have $\alpha_1(0)=p$. That is, the path α_1 runs from p to q.

It remains to show that α_1 is homotopic to $\alpha*\beta$ relative to the ends. Set $\alpha_\tau=\Phi^\tau\circ\alpha$ and denote by β_τ the path running along β from $\beta(\tau)$ to q; that is,

$$\beta_\tau(t)=\beta(\tau+\tfrac{1}{1-\tau}\cdot t).$$

The concatenation $\alpha_\tau*\beta_\tau$ provides a homotopy from $\alpha*\beta$ to $\alpha_1*\beta_1$. Since β_1 is a constant path, $\alpha*\beta$ is homotopic to α_1. Hence the statement follows. □

This is a stripped version of the problem suggested by Jarosław Kędra [see 2]; it was used by Michael Khanevsky [see Lemma 3 in 191].

Chapter 8

Piecewise linear geometry

A *polyhedral space* is a complete length-metric space that admits a locally finite triangulation such that each simplex is isometric to a simplex in a Euclidean space. By a *triangulation* of a polyhedral space we always understand it as a triangulation as above.

A point in a polyhedral space is called *regular* if it has a neighborhood isometric to an open set in a Euclidean space; otherwise it is called *singular*.

If we replace the Euclidean spaces by the unit spheres or the hyperbolic spaces, we arrive to the definition of *spherical* and *hyperbolic polyhedral spaces* correspondingly.

The term *piecewise* typically mean that there is a triangulation with some property on each triangle. For example, if P and Q are polyhedral spaces, then

◇ a map $f\colon P\to Q$ is called *piecewise distance preserving* if there is a triangulation $\mathcal{T}$ of P such that for any simplex $\Delta\in\mathcal{T}$ the restriction $f|_\Delta$ is distance preserving,

◇ a map $h\colon P\to Q$ is called *piecewise linear* if both spaces P and Q admit triangulations such that each simplex of P is mapped to a simplex of Q by an affine map. In particular, a *piecewise linear homeomorphism* is a piecewise linear map which is a homeomorphism.

Spherical arm lemma

Recall that a polygon without self intersections is called *simple*.

Let $A=[a_1\dots a_n]$ and $B=[b_1\dots b_n]$ be two simple spherical polygons with equal corresponding sides. Assume A lies in a hemisphere and $\measuredangle a_i\geqslant\measuredangle b_i$ for each i. Show that A is congruent to B.

Semisolution. Let us cut the polygon A from the sphere and glue instead the polygon B. Denote by Σ the obtained spherical polyhedral space. Note that

- $\diamond$ Σ is homeomorphic $\mathbb{S}^2$.
- $\diamond$ Σ has curvature $\geqslant 1$ in the sense of Alexandrov; that is, the total angle around each singular point is less than $2{\cdot}\pi$.
- $\diamond$ All the singular points of Σ lie outside of an isometric copy of a hemisphere $\mathbb{S}^2_+ \subset \Sigma$

Denote by n the number of singular points in Σ. It is sufficient to show that $n = 0$.

Assume the contrary; that is, $n \geqslant 1$. We can assume that n takes the minimal possible value.

Clearly $n > 1$; that is, Σ can not have a single singular point. Therefore we can choose two singular points $p, q \in \Sigma$. Cut Σ along a geodesic $[pq]$. The hole can be patched so that we obtain a new polyhedral space Σ' of the same type but with $n - 1$ singular points. Since n is minimal, we arrive to a contradiction

Namely, if the total angles around p and q are $2{\cdot}\pi - \alpha$ and $2{\cdot}\pi - \beta$ correspondingly, consider the spherical triangle $\triangle$ with base $|p - q|_\Sigma$ and adjusted angles $\frac{\alpha}{2}$, $\frac{\beta}{2}$. The needed patch is obtained by doubling $\triangle$ along its lateral sides. $\square$

Alternative end of proof. By the Alexandrov embedding theorem, Σ is isometric to the surface of a convex polyhedron P in the unit 3-dimensional sphere $\mathbb{S}^3$. The center of the hemisphere has to lie in a facet, say F of P. It remains to note that F contains the equator and therefore P has to be a hemisphere in $\mathbb{S}^3$ or an intersection of two hemispheres. In both cases its surface is isometric to $\mathbb{S}^2$. $\square$

The problem is due to Victor Zalgaller [see 192]; the result of Victor Toponogov in [193] gives a smooth analog of this statement. The patch construction above was introduced by Aleksandr Alexandrov in his proof of convex embeddability of polyhedra [see 194, VI, §7]. The alternative end of proof is taken from [111].

Triangulation of 3-sphere

Construct a triangulation of $\mathbb{S}^3$ *with* 100 *vertices such that any two vertices are connected by an edge.*

Folding problem

Let P be a compact 2*-dimensional polyhedral space. Construct a piecewise distance preserving map $f\colon P \to \mathbb{R}^2$.*

Piecewise distance preserving extension

Prove that any 1-Lipschitz map from a finite subset $F \subset \mathbb{R}^2$ to $\mathbb{R}^2$ can be extended to a piecewise distance preserving map $\mathbb{R}^2 \to \mathbb{R}^2$.

Closed polyhedral surface

Construct a closed polyhedral surface Σ in $\mathbb{R}^3$ with nonpositive curvature; that is, the total angle around each vertex of Σ is at least $2{\cdot}\pi$.

Minimal polyhedral disk

By a polyhedral disk in $\mathbb{R}^3$ we understand a triangulation of a plane polygon P with a map $P \to \mathbb{R}^3$ that is affine on each triangle. The area of the polyhedral disk is defined as the sum of areas of the images of the triangles in the triangulation.

Consider the class of polyhedral disks glued from n triangles in $\mathbb{R}^3$ with fixed broken line as the boundary. Let Σ_n be a disk of minimal area in this class. Show that Σ_n is saddle*; that is, a plane can not cut all the edges coming from one of the interior vertices of Σ_n.*

Coherent triangulation°

A triangulation of a convex polygon is called coherent if there is a convex function that is linear on each triangle and changes its gradient on every edge of the triangulation.

Find a non-coherent triangulation of a triangle.

Sphere with one edge*

Given a polyhedral space P, denote by P_s the set of its singular points.

Construct spherical polyhedral space P that is homeomorphic to $\mathbb{S}^3$ and such that P_s is formed by a knotted circle.

In addition the total length of P_s can be made arbitrary large and the angle around P_s can be made strictly less than $2{\cdot}\pi$.

Triangulation of a torus

Show that the torus does not admit a triangulation such that one vertex has 5 edges, one has 7 edges and all other vertices have 6 edges.

No simple geodesics°

Construct a convex polyhedron P whose surface does not have a closed simple geodesic.

Semisolutions

Triangulation of 3-sphere. Choose 100 distinct points $p_1 \dots, p_{100}$ on the *moment curve*

$$\gamma\colon t \mapsto (t, t^2, t^3, t^4)$$

in $\mathbb{R}^4$. Denote by P the convex hull of $\{p_1, \dots, p_{100}\}$.

The surface of P is homeomorphic to $\mathbb{S}^2$. Therefore it is sufficient to show that any two vertices of P are connected by an edge. The latter follows from the following claim.

$(*)$ *For any two points p and q on γ there is a hyperplane H in $\mathbb{R}^4$ which intersects γ only at p and q and leaves γ on one side.*

To prove the claim, assume that $p = \gamma(t_1)$ and $q = \gamma(t_2)$. Consider the polynomial

$$f(t) = a + b{\cdot}t + c{\cdot}t^2 + d{\cdot}t^3 + t^4 = (t - t_1)^2{\cdot}(t - t_2)^2.$$

Clearly $f(t) \geqslant 0$ and the equality holds only at t_1 and t_2. It follows that the affine function $\ell\colon \mathbb{R}^4 \to \mathbb{R}$ defined as

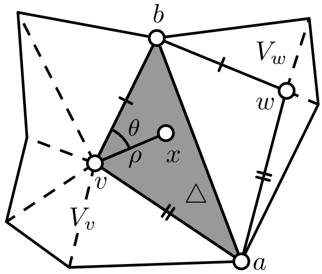
$$\ell\colon (w, x, y, z) \mapsto a + b{\cdot}w + c{\cdot}x + d{\cdot}y + z$$

is nonnegative at the points of γ and vanish only at p and q. Therefore the zero set of ℓ is the required hyperplane H in $(*)$. $\square$

The polyhedron P above is an example of the so called *cyclic polytopes.*

Folding problem. Given a triangulation of P, consider the Voronoi domain V_v for each vertex v; that is, V_v is the set of all points in P closer to v than to any other vertex. Note that the triangulation can be subdivided if necessary so that the Voronoi domain of each vertex is isometric to a convex subset in the cone with the vertex corresponding to the tip.

The boundaries of all the Voronoi domains form a graph with straight edges. Let us triangulate P so that each triangle has one of those edges as the base and the opposite vertex is the center of an adjusted Voronoi domain; such a vertex will be called the *main* vertex of the triangle.



Fix a solid triangle $\triangle = [vab]$ in the constructed triangulation; let v be its main vertex. Given a point $x \in \triangle$, set

$$\rho(x) = |x - v|$$

and

$$\theta(x) = \min\{\measuredangle[v\,{}^a_x], \measuredangle[v\,{}^b_x]\}.$$

Let us map x to the point with polar coordinates $(\rho(x), \theta(x))$ in the plane.

Note that for each triangle $\triangle$, the constructed map $\triangle \to \mathbb{R}^2$ is piecewise distance preserving. It remains to check that these maps agree on the common sides of the triangles. $\square$

This construction was given by Victor Zalgaller [see 195]. Svetlana Krat generalized the statement to higher dimensions [see 196].

Piecewise distance preserving extension. Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be two collections of points in $\mathbb{R}^2$ such that

$$|a_i - a_j| \geqslant |b_i - b_j|$$

for all pairs i, j. We need to construct a piecewise distance preserving map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(a_i) = b_i$ for each i.

Assume that the problem is already solved for $n < m$; let us do the case $n = m$. By assumption, there is a piecewise liner length-preserving map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(a_i) = b_i$ for each $i > 1$.

Consider the set

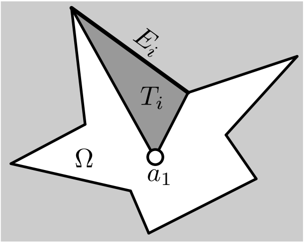
$$\Omega = \left\{\, x \in \mathbb{R}^2 \;\middle|\; |f(x) - b_1| > |x - a_1| \,\right\}.$$

Since $|a_i - a_1| \geqslant |b_i - b_1|$, we get $a_i \notin \Omega$ for $i > 1$.

Note that we can assume that the map f and therefore the set Ω are bounded. Indeed, let $\square$ be a square containing all the points b_i. There is a piecewise isometric map $h\colon \mathbb{R}^2 \to \square$ obtained by folding plane along the lines of the grid defined by $\square$. Then the composition $h \circ f$ is bounded and it satisfies all the properties of f described above.

If $\Omega = \varnothing$, then $f(a_1) = b_1$; that is, f is a solution. It remains to consider the case $\Omega \neq \varnothing$.

Note that Ω is star-shaped with respect to a_1. Indeed, if $x \in \Omega$, then $|a_1 - x| < |b_1 - f(x)|$. If $y \in [a_1 x]$ then $|a_1 - y| + |y - x| = |a_1 - x|$ and since f is length-preserving we get $|f(x) - f(y)| \leqslant |x - y|$. By the triangle inequality, $|a_1 - y| < |b_1 - f(y)|$; that is, $y \in \Omega$.

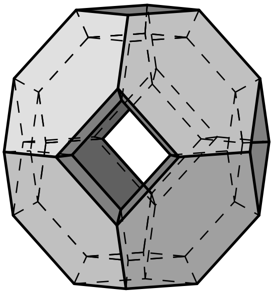


The boundary $\partial\Omega$ can be subdivided into a finite collection of line segments $\{E_i\}$ so that f maps rigidly each E_i. Note that $|f(x)-b_1| = |x-a_1|$ for any $x \in E_i$. Denote by T_i the triangle with base E_i and vertex a_1. From above there is a rigid motion m_i of T_i such that $m_i(x) = f(x)$ for any $x \in E_i$ and $m_i(a_1) = b_1$. Let us redefine the map f in Ω by sending x to $m_i(x)$ for $x \in T_i$.

The maps m_i agree on the common sides of triangles T_i. Therefore we have produced a new piecewise isometric map $f'\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying all the requirements. □

The same proof works in all dimensions.

The statement was proved by Ulrich Brehm and rediscovered by Arseniy Akopyan and Alexey Tarasov [see 197, 198 and also the section 2 in 146].



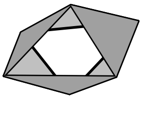
Closed polyhedral surface. An example can be constructed by drilling a polyhedral cave from your favorite convex polyhedron. On the diagram you see the result of this construction for the octahedron.

Choose a convex polyhedron K. We can assume that the interior of K contains the origin $0 \in \mathbb{R}^3$. Remove from K the interior of $K' = \frac{5}{6}\cdot K$.

Note that one can drill from each vertex of K a polyhedral tunnel to the corresponding vertex of K' so that the surface Σ of the obtained non-convex polytope is a solution. □

The problem suggested by Jarosław Kędra.

The construction above produces a surface of genus at least 3. One can also construct a polyhedral surface in $\mathbb{R}^3$ which is isometric to a flat torus. The existence of such torus follows from very general result of Burago and Zalgaller [see 199]. They show in particular that any 1-Lipschitz smooth embedding of the flat torus in $\mathbb{R}^3$ can be approximated by a piecewise distance preserving embedding.



The following construction is more direct; it is a bent version of the so called *Schwarz boots* [see 200]. Construct an isometric piecewise linear embedding of a cylinder $\mathbb{S}^1 \times [0,a]$ from six triangles like in the diagram in such a way that the planes thru the boundary triangles meet at an angle of $\frac{\pi}{n}$ for a positive integer n. It remains to reflect the obtained surface several times with respect to the planes thru the boundary triangles.

The following related problem was proposed by Brian Bowditch; a solution can be built with the construction of Joel Hass [see 201].

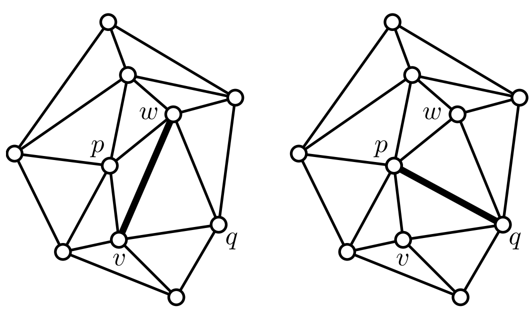
Construct a polyhedral metric on the 3-sphere such that the total angle around any edge of its triangulation is at least $2{\cdot}\pi$.

Minimal polyhedral disk. Arguing by contradiction, assume a polyhedral disk Σ minimizing the area is not saddle; that is, there is an interior vertex v of Σ such that all the edges from v can be cut with a plane.

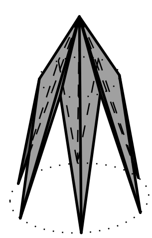
Note that we can move v in such a way that the lengths of all its edges decrease.

Since the area is minimal, this deformation does not decrease the area. Taking the derivative of the total area along this deformation implies that Σ contains two adjusted non-coplanar triangles $[pvw]$ and $[qvw]$ such that

$$\measuredangle[p\,{}^{v}_{w}]+\measuredangle[q\,{}^{v}_{w}]>\pi.$$

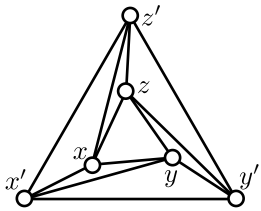


In this case replacing the triangles $[pvw]$ and $[qvw]$ by the triangles $[vpq]$ and $[wpq]$ leads to a polyhedral surface with smaller area. That is, Σ is not area minimizing, a contradiction. □



For a general polyhedral surface, a deformation decreasing the lengths of all edges may not decrease the area. Moreover, the surface that minimizes the area among all surfaces with a fixed triangulation might not be saddle; the symmetric tent shown on the diagram provides an example [see 202 for more details].

Coherent triangulation. An example is shown on the diagram. The triangulation of the triangle $[x'y'z']$ has a homothetic triangle $[xyz]$ and the edges $[xx']$, $[yy']$, $[zz']$, $[yx']$, $[zy']$, $[xz']$.



Assume this triangulation is coherent; let f be the corresponding piecewise linear convex function. Without loss of generality we can assume that f vanishes on the boundary of the big triangle.

From the convexity of f at the edges $[x'y]$, $[y'z]$ and $[z'x]$, we get

$$f(x) > f(y) > f(z) > f(x),$$

a contradiction. □

The problem is discussed in the book of Israel Gelfand, Mikhail Kapranov and Andrei Zelevinsky [see 7C in 203]. The given example is closely related to the so called *Schönhardt polyhedron*, an example of a non-convex polyhedron which does not admit a triangulation [see 204].

Sphere with one edge. An example P can be found among polyhedral spaces with an isometric $\mathbb{S}^1$-action with geodesic orbits. (Equivalently the cone over P admits a complex structure; that is, one can cut simplexes from $\mathbb{C}^2$ and glue the cone from them so that the complex structures agree on the gluing.)

Let us identify $\mathbb{S}^3$ with the unit sphere in the hyperplane Π described by $x + y + z = 0$ of $\mathbb{C}^3$. The symmetric group S_3 acts on $\mathbb{S}^3$ by permuting the coordinates. Take $P = \mathbb{S}^3/S_3$.

Note that P is a spherical polyhedral space. Moreover, P is the underlying space of an orbifold whose isotopy groups are either trivial or $\mathbb{Z}_2$. In particular P is a 3-manifold. Clearly P is compact and simply connected, in particular it is homeomorphic to the 3-sphere. (The later can be also seen by parametrizing P using the symmetric polynomials $u = xy + yz + zx$ and $v = xyz$.)

Multiplications by unit complex numbers give an $\mathbb{S}^1$-action on $\mathbb{S}^3$ which commutes with the S_3-action. The singular set P_s of P is the image of the orbit $\mathbb{S}^1 \cdot p$ where p is a point fixed by an odd permutation of S_3. In particular P_s is a circle.

Note that the subgroup of even permutations $\mathbb{Z}_3 \lhd S_3$ acts freely on $\mathbb{S}^3$. The quotient space $\mathbb{S}^3/\mathbb{Z}_3$ is the double cover of P branching in P_s. That is, a double cover of the sphere P branching in the knot P_s is not simply connected. Therefore P_s is a nontrivial knot.

(In fact P_s is a trefoil and in the (u,v) coordinates it can be written as $u^3 = v^2$.) □

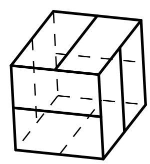
This construction is given by Dmitri Panov [see 205].

Note that the quotient space $P' = P/\mathbb{S}^1$ is isometric to the doubling of a triangle in $\mathbb{C}\mathrm{P}^1 = \mathbb{S}^3/\mathbb{S}^1$ with the angles $\frac{\pi}{2}$, $\frac{\pi}{2}$ and $\frac{\pi}{3}$. Starting with other triangles one may produce P with isometric $\mathbb{S}^1$ and arbitrary torus knot as the singular set. It can also produce arbitrary long singular sets. In these examples, the cone over P can be holomorphically parametrized by $\mathbb{C}^2$ in such a way that its singular set becomes an algebraic curve $u^p = v^q$ in some (u,v)-coordinates of $\mathbb{C}^2$. Here is a related problem.

🕮 *Construct a complex orbifold with the underlying space homeomorphic to* $\mathbb{C}\mathrm{P}^2$.

The solution of the problem gives the polyhedral metric on $\mathbb{C}\mathrm{P}^2$ with nonnegative curvature in the sense of Alexandrov. It is not known whether the canonical metric on $\mathbb{C}\mathrm{P}^2$ can be approximated by such polyhedral metrics or not.

I do not know if such knots exist in Euclidean polyhedral spaces, but there are links. For example, the Borromean rings can appear as the singular set of a Euclidean polyhedral metric on $\mathbb{S}^3$. It can be obtained by gluing each face of a cube to itself along the reflections with respect to the middle lines shown on the picture. This construction is due to William Thurston [see 206]



Triangulation of a torus. Assume contrary; let τ be a trainagulation of the torus with the vertex z_5 meeting 5 triangles, vertex z_7 meeting 7 triangles and every other vertex meeting 6 triangles.

Let us equip the torus with the flat metric such that each triangle is equilateral. The metric will have two singular cone points z_5 and z_7. The total angle around z_5 is $\frac{5}{3}\cdot\pi$ and the total angle around z_7 is $\frac{7}{3}\cdot\pi$. Note the following.

$(*)$ *The holonomy group of the obtained polyhedral metric on the torus is generated by the rotation by* $\frac{\pi}{3}$.

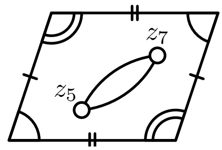
Indeed, since parallel translation along any loop preserves the directions of the sides of any triangle; it can only permute it cyclically, which corresponds to rotations by multiple of $\frac{\pi}{3}$. On the other hand, the holonomy of the loop which surrounds z_5 is a rotation by $\frac{\pi}{3}$.

Consider a closed geodesic γ_1 minimizing the length among all not null-homotopic circles. Let γ_2 be another closed geodesic that minimize the length and is not homotopic to any power of γ_1.

Note that γ_1 and γ_2 intersect at a single point. Otherwise one could shorten one of them keeping the defining property.

Note that γ_i does not contain z_5. In fact no geodesic can pass thru any singular point with total angle smaller than $2{\cdot}\pi$.

Assume γ_i passes thru z_7. Then by $(*)$, one of the two angles cut by γ_i at z_7 equals π. It follows that one can push γ_i aside so that it does not longer pass thru z_7, but remains to be a closed geodesic with the same length.

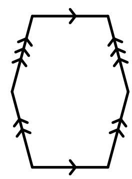


Cut $\mathbb{T}^2$ along γ_1 and γ_2. In the obtained quadrilateral, connect z_5 to z_7 by a minimizing geodesic and cut along it. This way we obtain an annulus Ω with flat metric.

Note that a neighborhood of the first boundary component is a parallelogram — it has equal opposite sides and its angles add up to $2{\cdot}\pi$. In particular Ω admits an isometric immersion into the plane.

The second boundary component has to be mapped to a diangle with straight sides and angles $\frac{\pi}{3}$. Such diangle does not exist in the plane, a contradiction. □

The problem was originally discovered and solved by Stanislav Jendroľ and Ernest Jucovič [see 207], their proof is combinatorial. The solution described above was given by Rostislav Matveyev in his lectures [see 208]. A complex-analytic proof was found by Ivan Izmestiev, Robert Kusner, Günter Rote, Boris Springborn and John Sullivan [see 209].



There are flat metrics on the torus with only two singular points with total angles $\frac{5}{3}{\cdot}\pi$ and $\frac{7}{3}{\cdot}\pi$. Such an example can be obtained by identifying the hexagon on the picture according to the arrows. However, the holonomy group of the obtained torus is generated by the rotation by $\frac{\pi}{6}$. In particular, the observation $(*)$ is essential in the proof.

The same argument shows that the holonomy group of a flat torus with exactly two singular points with total angle $2{\cdot}(1\pm\frac{1}{n}){\cdot}\pi$ has more than n elements. In the solution we did the case $n=6$.

If one denotes by v_m the number of vertexes in a triangulation of the torus with m incoming edges, then by Euler's formula, we get

$$(**)\qquad \sum_m (m-6){\cdot}v_m = 0.$$

Note that this equation says nothing about v_6. It turns out that for almost any sequence $v_3, v_4, \dots$ satisfying $(**)$ one can adjust v_6 so that it corresponds to a triangulation of the torus — the sequence

$$0, 0, 1, v_6, 1, 0, 0, \ldots$$

discussed in the problem is the only exception.

The following problem is harder. Recall that the curvature of a point s in a polyhedral surface is defined as $2\cdot\pi - \theta$, where θ denotes the total angle around s. Note that all regular points in a polyhedral surface have zero curvature.

Let Σ be a spherical polyhedral space homeomorphic to the 2-sphere and $\omega_1, \ldots, \omega_n$ be the curvatures of its singular points. Set

$$\delta_i = \min\left\{\, \left|\tfrac{\omega_i}{2} - 2\cdot k\cdot\pi\right| \;\middle|\; k \in \mathbb{Z} \,\right\}.$$

Show that there is a closed polygonal curve in the unit sphere with sides $\delta_1, \ldots, \delta_n$.

This problem was stated and solved by Gabirele Mondello and Dmitri Panov [see 210]. The solution requires another holonomy group — it assigns an element of the double cover of $\mathrm{SO}(3)$ (which is $\mathrm{SU}(2) = \mathbb{S}^3$) to any loop in Σ that avoids singularities.

No simple geodesics. The curvature of a vertex on the surface of a convex polyhedron is defined as $2\cdot\pi - \theta$, where θ is the total angle around the vertex.

By the Gauss–Bonnet formula, a simple closed geodesic cuts the surface into two disks each with total curvature $2\cdot\pi$. Therefore it is sufficient to construct a convex polyhedron with curvatures of the vertices $\omega_1, \ldots, \omega_n$ such that $2\cdot\pi$ cannot be obtained as sum of some of the ω_i.

An example of that type can be found among the tetrahedra. $\square$

The problem is due to Gregory Galperin [see 211] and rediscovered by Dmitry Fuchs and Serge Tabachnikov [see 20.8 in 7]. The following problem is closely related.

Assume that the surface of convex polyhedron P contains arbitrary long closed simple geodesics. Show that P is an isosceles tetrahedron*; that is, a tetrahedron with equal opposite edges.*

The latter statement was proved by Vladimir Protasov [see 212 and also 213 and 214].

Chapter 9

Discrete geometry

In this chapter we consider geometrical problems with strong combinatoric flavor. No special prerequisite is needed.

Round circles in 3-sphere

Suppose that $\mathcal{C}$ is a finite collection of pairwise linked round circles in the unit 3-sphere. Prove that there is an isotopy of $\mathcal{C}$ that moves all of them into great circles.

Semisolution. For each circle in $\mathcal{C}$ consider the plane containing it. Note that the circles are linked if and only if the corresponding planes intersect at a single point inside the unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$.

Take the intersection of the planes with the sphere of radius $R \geqslant 1$, rescale and pass to the limit as $R \to \infty$. This way we get the needed isotopy. □

This problem was discussed by Genevieve Walsh [see 215]. The same idea was used by Michael Freedman and Richard Skora to show that any link made from pairwise not linked round circles is trivial; in particular, Borromean rings can not be realized by round circles [see Lemma 3.2 in 216].

Box in a box

Assume that a rectangular parallelepiped with sides a, b, c lies inside another rectangular parallelepiped with sides a', b', c'. Show that

$$a' + b' + c' \geqslant a + b + c.$$

Harnack's circles

Prove that a smooth algebraic curve of degree d in $\mathbb{R}\mathrm{P}^2$ consists of at most $n = \frac{1}{2}\cdot(d^2 - 3\cdot d + 4)$ connected components.

Two points on each line

Construct a set in the Euclidean plane that intersects each line at exactly 2 points.

Balls without gaps

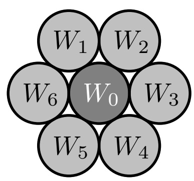
Let $B_1, \dots, B_n$ be balls of raduses $r_1, \dots, r_n$ in a Euclidean space. Assume that no hyperplane divides the balls into two non-empty sets without intersecting at least one of the balls. Show that the balls $B_1, \dots, B_n$ can be covered by a ball of radius $r = r_1 + \dots + r_n$.

Covering lemma

Let $\{B_i\}_{i\in F}$ be any finite collection of balls in m-dimensional Euclidean space. Show that there is a subcollection of pairwise disjoint balls $\{B_i\}_{i\in G}$, $G \subset F$ such that

$$\mathrm{vol}\left(\bigcup_{i\in F} B_i\right) \leqslant 3^m\cdot \mathrm{vol}\left(\bigcup_{i\in G} B_i\right).$$

Kissing number°



Let W_0 be a convex body in $\mathbb{R}^m$. We say that k is the *kissing number* of W_0 (briefly $k = \operatorname{kiss} W_0$) if k is the maximal integer such that there are k bodies $W_1, \dots, W_k$ such that (1) each W_i is congruent to W_0, (2) $W_i \cap W_0 \neq \varnothing$ for each i and (3) no pair W_i, W_j has common interior points.

As you may guess from the diagram, the kissing number of the round disk in the plane is 6.

Show that for any convex body W_0 in $\mathbb{R}^m$

$$\operatorname{kiss} W_0 \geqslant \operatorname{kiss} B,$$

where B denotes the unit ball in $\mathbb{R}^m$.

Monotonic homotopy

Let F be a finite set and $h_0, h_1\colon F \to \mathbb{R}^m$ be two maps. Consider $\mathbb{R}^m$ as a subspace of $\mathbb{R}^{2\cdot m}$. Show that there is a homotopy $h_t\colon F \to \to \mathbb{R}^{2\cdot m}$ from h_0 to h_1 such that the function

$$t \mapsto |h_t(x) - h_t(y)|$$

is monotonic for any pair $x, y \in F$.

Cube

Half of the vertices of an m-dimensional cube are colored in white and the other half in black. Show that the cube has at least 2^{m-1} edges connecting vertices of different colors.

Geodesic loop

Show that the surface of a cube in $\mathbb{R}^3$ does not admit a geodesic loop with a vertex as the base point.

Right and acute triangles

Let $x_1, \dots, x_n \in \mathbb{R}^m$ be a collection of points such that any triangle $[x_i x_j x_k]$ is right or acute. Show that $n \leqslant 2^m$.

Right-angled polyhedron^{+}

A polyhedron is called *right-angled* if all its dihedral angles are right.

Show that in all sufficiently large dimensions, there is no compact convex hyperbolic right-angled polyhedron.

Let us give a short summary of the Dehn–Sommerville equations which can help to solve this problem.

Assume P is a *simple* Euclidean m-dimensional polyhedron; that is, exactly m facets meet at each vertex of P. Denote by f_k the number of k-dimensional faces of P; the array of integers $(f_0, \dots f_m)$ is called the f-vector of P.

Fix an ordering of the vertices $v_1, \dots, v_{f_0}$ of P so that for some linear function ℓ, we have $\ell(v_i) < \ell(v_j) \;\Leftrightarrow\; i < j$. The *index* of the vertex v_i is defined as the number of edges $[v_i v_j]$ of P such that $i > j$. The number of vertices of index k will be denoted as h_k. The array of integers $(h_0, \dots h_m)$ is called the h-vector of P. Clearly $h_0 = h_m = 1$ and

$$(*) \qquad h_k \geqslant 0 \quad \text{for all} \quad k.$$

Each k-face of P contains a unique vertex that maximizes ℓ; if the vertex has index i, then $i \geqslant k$ and then it is the maximal vertex of exactly $\frac{i!}{k!\cdot(i-k)!}$ faces of dimension k. This observation can be packed in the following polynomial identity

$$\sum_k h_k\cdot(t+1)^k = \sum_k f_k\cdot t^k.$$

Note that the identity above implies that the h-vector does not depend on the choice of ℓ. In particular, the h vector is the same for the reversed order; that is,

$$(**) \qquad h_k = h_{m-k}$$

for any k.

The identities $(**)$ are called the Dehn–Sommerville equations. It gives the complete list of linear equations for h-vectors (and therefore f-vectors) of simple polyhedrons.

Note that the Dehn–Sommerville equations $(**)$ as well as the inequalities $(*)$ can be rewritten in terms of f-vectors.

Semisolutions

Box in a box. Let Π be a parallelepiped with dimensions a, b and c. Denote by $v(r)$ the volume of the r-neighborhood of Π,

Note that for all positive r we have

$$(*) \qquad v_\Pi(r) = w_3(\Pi) + w_2(\Pi)\cdot r + w_1(\Pi)\cdot r^2 + w_0(\Pi)\cdot r^3,$$

where

- ◇ $w_0(\Pi) = \frac{4}{3}\cdot\pi$ is the volume of the unit ball,
- ◇ $w_1(\Pi) = \pi\cdot(a+b+c)$,
- ◇ $w_2(\Pi) = 2\cdot(a\cdot b + b\cdot c + c\cdot a)$ is the surface area of Π,
- ◇ $w_3(\Pi) = a\cdot b\cdot c$ is the volume of Π,

Assume Π' be another parallelepiped with dimensions a', b' and c'. If $\Pi \subset \Pi'$, then $v_\Pi(r) \leqslant v_{\Pi'}(r)$ for any r. For $r \to \infty$, these inequalities imply

$$a + b + c \leqslant a' + b' + c'. \qquad \square$$

Alternative proof. Note that the average length of the projection of Π to a line is $\mathrm{Const}\cdot(a+b+c)$ for some $\mathrm{Const}>0$. (In fact $\mathrm{Const}=\frac{1}{2}$, but we will not need it.)

Since $\Pi\subset\Pi'$, the average length of the projection of Π can not exceed the average length of the projection of Π'. Hence the statement follows. □

The problem was discussed by Alexander Shen [see 217].

A formula analogous to $(*)$ holds for an arbitrary convex body B of arbitrary dimension m. It was discovered by Jakob Steiner [see 218]. The coefficient $w_i(B)$ in the polynomial with different normalization constants appear under different names, most commonly *intrinsic volumes* and *quermassintegrals*. Up to a normalization constant they can also be defined as the average area of the projections of B to the i-dimensional planes. In particular, if B' and B are convex bodies such that $B'\subset B$, then $w_i(B')\leqslant w_i(B)$ for any i. This generalizes our problem quite a bit. Further generalizations lead to the theory of *mixed volumes* [see 219].

The equality $w_1(\Pi)=\pi\cdot(a+b+c)$ still holds for all parallelepipeds, not only rectangular ones. In particular, if one parallelepiped lies inside another then sum of all edges of the first one cannot exceed the sum for the second.

Harnack's circles. Let $\sigma\subset\mathbb{RP}^2$ be a algebraic curve of degree d. Consider the complexification $\Sigma\subset\mathbb{CP}^2$ of σ. Without loss of generality, we may assume that Σ is regular.

Note that all regular complex algebraic curves of degree d in $\mathbb{CP}^2$ are isotopic to each other in the class of regular algebraic curves of degree d. Indeed, the set of equations of degree d that correspond to singular curves have real codimesion 2. Therefore the set of equations of degree d that correspond to regular curves is connected. In particular one can construct an isotopy from one regular curve to any other by changing continuously the parameters of the equations.

In particular it follows that all regular complex algebraic curves of degree d in $\mathbb{CP}^2$ have the same genus, denote it by g. Perturbing a singular curve formed by d lines in $\mathbb{CP}^2$, we can see that

$$g=\tfrac{1}{2}\cdot(d-1)\cdot(d-2).$$

The real curve σ forms the fixed point set in Σ by the complex conjugation. In particular σ divides Σ into two symmetric surfaces with boundary formed by σ. It follows that each connected component of σ adds one to the genus of Σ. Hence the result follows. □

The inequality was originally proved by Axel Harnack using a different method [see 220]. The idea to use complexification is due to

Felix Klein [see 221]. This problem is a background for the Hilbert's 16th problem.

Two points on each line. Take any complete ordering of the set of all lines so that each beginning interval has cardinality less than continuum.

Assume we have a set of points X of cardinality less than continuum such that each line intersects X in at most 2 points.

Choose the least line ℓ in the ordering that intersects X in 0 or 1 point. Note that the set of all lines intersecting X at two points has cardinality less than continuum. Therefore we can choose a point on ℓ and add it to X so that the remaining lines are not overloaded.

It remains to apply well ordering principle. □

This problem has an endless list of variations. The following problem looks similar but far more involved; a solution follows from the proof of Paul Monsky that a square cannot be cut into triangles with equal areas [see 222].

Subdivide the plane into three everywhere dense sets A, B and C such that each line meets exactly two of these sets.

Balls without gaps. Assume the mass of each ball is proportional to its radius. Denote by z the center of mass of the balls. It is sufficient to show the following.

$(*)$ *The ball $B(z,r)$ contains all $B_1,\dots,B_n$.*

Assume this is not the case. Then there is a line ℓ thru z, such that the orthogonal projection of some ball B_i to ℓ does not lie completely inside the projection of B. (This projection reduces the problem to the one-dimensional case.)

Note that the projection of all balls $B_1,\dots,B_n$ has to be connected and it contains a line segment longer than r on one side from z. In this case, the center of mass of the balls projects inside of this segment, a contradiction. □

The statement was conjectured by Paul Erdős. The solution was given by Adolph and Ruth Goodmans [see 223 and also 224].

Covering lemma. The required collection $\{B_i\}_{i\in G}$ is constructed using the *greedy algorithm.* We choose the balls one by one; on each step we take the largest ball that does not intersect those which we have chosen already.

Note that each ball in the original collection $\{B_i\}_{i\in F}$ intersects a ball in $\{B_i\}_{i\in G}$ with larger radius. Therefore

$$\bigcup_{i\in F} B_i \subset \bigcup_{i\in G} 3{\cdot}B_i, \tag{$*$}$$

where $3{\cdot}B_i$ denotes the ball with the same center as B_i and with radius three times larger. Hence the statement follows. □

The constant 3^n is not optimal. The optimal constant is at least 2^n, but its value is not known and maybe no one is willing to know.

The inclusion $(*)$ is called the *Vitali covering lemma*. The following statement is called the *Besikovitch covering lemma*; it has a similar proof.

For any positive integer m there is a positive integer M such that any finite collection of balls $\{B_i\}_{i\in F}$ in the m-dimensional Euclidean space contains a subcollection $\{B_i\}_{i\in G}$ such that center of any ball in $\{B_i\}_{i\in F}$ lies inside one of the balls from $\{B_i\}_{i\in G}$ and the collection $\{B_i\}_{i\in G}$ can be subdivided into M subcollections of pairwise disjoint balls.

Both lemmas were used to prove the so called *covering theorems* in measure theory, which state that "undesirable sets" have vanishing measure. Their applications overlap but aren't identical, *Vitali covering theorem* works for nice measures in arbitrary metric spaces while *Besikovitch covering theorem* work in nice metric spaces with arbitrary Borel measures.

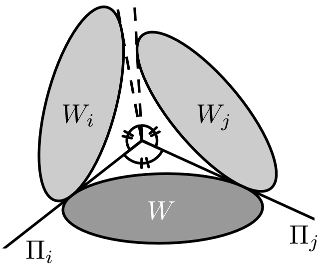
More precisely, Vitali works in arbitrary metric spaces with a *doubling measure* μ; the latter means that

$$\mu B(x, 2{\cdot}r) \leqslant C{\cdot}\mu B(x, r)$$

for some fixed constant C and any ball $B(x, r)$ in the metric space. On the other hand, Besikovitch works for all Borel measures in the so called *directionally limited* metric spaces [see 2.8.9 in 225]; these include Alexandrov spaces with curvature bounded below.

Kissing number. Fix the dimension m. Set $n = \operatorname{kiss} B$. Let $B_1, \dots, B_n$ be copies of the ball B that touch B and don't have common interior points. For each B_i consider the vector v_i from the center of B to the center of B_i. Note that $\measuredangle(v_i, v_j) \geqslant \frac{\pi}{3}$ if $i \ne j$.

For each i, consider the supporting hyperplane Π_i of W with outer normal vector v_i. Denote by W_i the reflection of W with respect to Π_i.



Note that W_i and W_j have no common interior points if $i \neq j$; the latter gives the needed inequality. $\square$

The proof is given by Charles Halberg, Eugene Levin and Ernst Straus [see 226]. It is not known if the same inequality holds for the orientation-preserving version of the kissing number.

Monotonic homotopy. Note that we can assume that $h_0(F)$ and $h_1(F)$ both lie in the coordinate m-spaces of $\mathbb{R}^{2\cdot m} = \mathbb{R}^m \times \mathbb{R}^m$; that is, $h_0(F) \subset \mathbb{R}^m \times \{0\}$ and $h_1(F) \subset \{0\} \times \mathbb{R}^m$.

Direct calculations show that the following homotopy is monotonic

$$h_t(x) = \left(h_0(x)\cdot \cos \tfrac{\pi\cdot t}{2}\,,\ h_1(x)\cdot \sin \tfrac{\pi\cdot t}{2}\right). \qquad \square$$

This homotopy was discovered by Ralph Alexander [see 227]. It has a number of applications, one of the most beautiful is the given by Károly Bezdek and Robert Connelly in the proof of Kneser–Poulsen and Klee–Wagon conjectures in the two-dimensional case [see 228].

The dimension $2\cdot m$ is optimal; that is, for any positive integer m, there are two maps $h_0, h_1\colon F \to \mathbb{R}^m$ that cannot be connected by a monotonic homotopy $h_t\colon F \to \mathbb{R}^{2\cdot m-1}$. The latter was shown by Maria Belk and Robert Connelly [see 229]

Cube. Consider the cube $[-1,1]^m \subset \mathbb{R}^m$. Any vertex of this cube has the form $\boldsymbol{q} = (q_1, \dots, q_m)$, where $q_i = \pm 1$.

For each vertex $\boldsymbol{q}$, consider the intersection of the corresponding hyperoctant with the unit sphere; that is, the set

$$V_{\boldsymbol{q}} = \left\{\, (x_1, \dots, x_m) \in \mathbb{S}^{m-1} \;\middle|\; q_i\cdot x_i \geqslant 0 \text{ for each } i \,\right\}.$$

Let $\mathcal{A} \subset \mathbb{S}^{m-1}$ be the union of all the sets $V_{\boldsymbol{q}}$ for black $\boldsymbol{q}$. Note that

$$\operatorname{vol}_{m-1}\mathcal{A} = \tfrac{1}{2}\cdot \operatorname{vol}_{m-1}\mathbb{S}^{m-1}.$$

By the spherical isoperimetric inequality,

$$\operatorname{vol}_{m-2}\partial\mathcal{A} \geqslant \operatorname{vol}_{m-2}\mathbb{S}^{m-2}.$$

It remains to observe that

$$\operatorname{vol}_{m-2}\partial\mathcal{A} = \tfrac{k}{2^{m-1}}\cdot \operatorname{vol}_{m-2}\mathbb{S}^{m-2},$$

where k is the number of edges of the cube with one black end and the other in white. $\square$

The problem was suggested by Greg Kuperberg.

Geodesic loop. Assume such loop exists; denote it by γ and let v be its base vertex.

Denote by ξ and ζ the directions of exit and the entrance of the loop. Let α be the angle between ξ and ζ measured in the tangent cone to the surface of the cube at v.

Note that $\alpha = \frac{\pi}{2}$. It can be seen from the Gauss–Bonnet formula since each vertex of the cube has curvature $\frac{\pi}{2}$. Alternatively, it can be proved by unfolding γ on the plane.

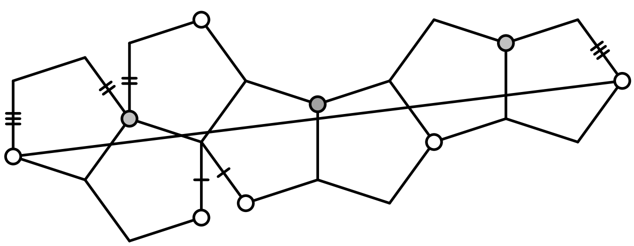
It follows that there is a rotational symmetry of the cube with order 3 that fixes v and sends ξ to ζ. The latter leads to a contradiction. $\square$

The same idea can be used to solve the following harder problems.

Show the same for the surface of the higher dimensional cube.

Show the same for the surface of the tetrahedron, octahedron and icosahedron.

For the dodecahedron such loop exists; its development is shown on the diagram. The vertices of a cube inscribed in the dodecahedron are circled.



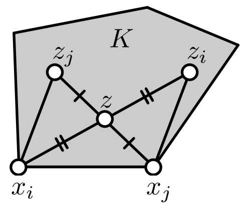
The problem suggested by Jarosław Kędra.

Right and acute triangles. Denote by K the convex hull of $\{x_1, \dots \dots, x_n\}$. Without loss of generality we can assume that K is m-dimensional. Note that for any distinct points x_i and x_j and any interior point z in K we have

$$(*) \qquad \measuredangle[x_i\,{}^{x_j}_{z}] < \tfrac{\pi}{2}.$$

Indeed, if $(*)$ does not hold, then $\langle x_j - x_i, z - x_i\rangle < 0$. Since $z \in K$ we have $\langle x_j - x_i, x_k - x_i\rangle < 0$ for some vertex x_k. That is, $\measuredangle[x_i\,{}^{x_j}_{x_k}] > \frac{\pi}{2}$, a contradiction.

Denote by h_i the homothety with center x_i and coefficient $\frac{1}{2}$. Set $K_i = h_i(K)$.



Let us show that K_i and K_j have no common interior points. Assume the contrary; that is,

$$z = h_i(z_i) = h_j(z_j);$$

for some interior points z_i and z_j in K. Note that

$$\measuredangle[x_i\,{}^{x_j}_{z_j}] + \measuredangle[x_j\,{}^{x_i}_{z_i}] = \pi,$$

which contradicts (∗).

Note that $K_i \subset K$ for any i; it follows that

$$\begin{aligned}\tfrac{n}{2^m}\cdot \operatorname{vol} K = \sum_{i=1}^{n} \operatorname{vol} K_i \leqslant \\ \leqslant \operatorname{vol} K.\end{aligned}$$

Hence the result follows. □

The problem was posted by Paul Erdős and solved by Ludwig Danzer and Branko Grünbaum [see 230, 231].

Grigori Perelman noticed that the same proof works for a similar problem in Alexandrov spaces [see 232]; the later led to interesting connections with the crystallographic groups [see 233].

Surprisingly, the maximal number of points that make only acute triangles grows exponentially with m as well. The latter was shown by Paul Erdős and Zoltán Füredi [see 234] using the *probabilistic method*. Later, an elementary constructive argument was found and improved by Dmitriy Zakharov, `grizzly` (an anonymous mathematician), Balázs Gerencsér and Viktor Harangi [see 235–237]; the current lower bound is $2^{m-1}+1$, which is exponentially optimal.

Right-angled polyhedron. Let P be a right-angled hyperbolic polyhedron of dimension m. Note that P is simple; that is, exactly m facets meet at each vertex of P.

From the projective model of the hyperbolic plane, one can see that for any simple compact hyperbolic polyhedron there is a simple Euclidean polyhedron with the same combinatorics. In particular Dehn–Sommerville equations hold for P.

Denote by $(f_0,\dots f_m)$ and $(h_0,\dots h_m)$ the f- and h-vectors of P. Recall that $h_i \geqslant 0$ for any i and $h_0 = h_m = 1$. By the Dehn–Sommerville equations, we get

$$(*) \qquad f_2 > \tfrac{m-2}{4}\cdot f_1.$$

Since P is hyperbolic, each 2-dimensional face of P has at least 5 sides. It follows that

$$f_2 \leqslant \tfrac{m-1}{5}\cdot f_1.$$

The latter contradicts $(*)$ for $m \geqslant 6$. $\square$

The above proof is the core of the proof of nonexistence of compact hyperbolic Coxeter's polyhedrons of large dimensions given by Ernest Vinberg [see 238, 239].

Playing a bit more with the same inequalities, one gets nonexistence of right-angled hyperbolic polyhedrons, in all dimensions starting from 5. In the 4-dimensional case, there is a regular right-angled hyperbolic polyhedron with *120-cells* — a 4-dimensional uncle of the dodecahedra.

The following question is open.

Let m be a large integer. Can a cocompact group of isomtries on the m-dimensional Lobachevsky space be generated by finite order elements (for example by central symmetries)?

Index

Bibliography

[1] P. Winkler. *Mathematical puzzles: a connoisseur's collection.* A K Peters, Ltd., Natick, MA, 2004.

[2] A. Petrunin. *One-step problems in geometry.* URL: `http://mathoverflow.net/q/8247`.

[3] D. Hilbert and S. Cohn-Vossen. *Geometry and the imagination.* Translated by P. Neményi. Chelsea Publishing Company, New York, N. Y., 1952.

[4] V. A. Toponogov. *Differential geometry of curves and surfaces.* A concise guide, With the editorial assistance of Vladimir Y. Rovenski. Birkhäuser Boston, Inc., Boston, MA, 2006.

[5] P. G. Tait. "Note on the circles of curvature of a plane curve." *Proc. Edinb. Math. Soc.* 14 (1896), p. 26.

[6] A. Kneser. „Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie." *Heinrich Weber Festschrift.* 1912.

[7] D. Fuchs and S. Tabachnikov. *Mathematical omnibus.* Thirty lectures on classic mathematics. American Mathematical Society, Providence, RI, 2007.

[8] W. Blaschke. *Kreis und Kugel.* Verlag von Veit & Comp., Leipzig, 1916.

[9] Г. Пестов и В. Ионин. «О наибольшем круге, вложенном в замкнутую кривую». *Докл. АН СССР* 127 (1959), с. 1170—1172.

[10] K. Pankrashkin. "An inequality for the maximum curvature through a geometric flow". *Arch. Math. (Basel)* 105.3 (2015), pp. 297–300.

[11] D. Panov and A. Petrunin. "Ramification conjecture and Hirzebruch's property of line arrangements". *Compos. Math.* 152.12 (2016), pp. 2443–2460.

[12] R. H. Bing. "Some aspects of the topology of 3-manifolds related to the Poincaré conjecture". *Lectures on modern mathematics, Vol. II.* Wiley, New York, 1964, pp. 93–128.

[13] В. Н. Лагунов. «О наибольшем шаре, вложенном в замкнутую поверхность, II». *Сибирский математический журнал* 2.6 (1961), с. 874—883.

[14] S. Alexander and R. Bishop. "Thin Riemannian manifolds with boundary". *Math. Ann.* 311.1 (1998), pp. 55–70.

[15] M. Gromov. "Almost flat manifolds." *J. Differ. Geom.* 13 (1978), pp. 231–241.

[16] P. Buser and D. Gromoll. "On the almost negatively curved 3-sphere". *Geometry and analysis on manifolds (Katata/Kyoto, 1987).* Vol. 1339. Lecture Notes in Math. Springer, Berlin, 1988, pp. 78–85.

[17] I. Fáry. "Sur certaines inégalités geométriques." *Acta Sci. Math.* 12 (1950), pp. 117–124.

[18] S. Tabachnikov. "The tale of a geometric inequality." *MASS selecta: teaching and learning advanced undergraduate mathematics.* Providence, RI: American Mathematical Society (AMS), 2003, pp. 257–262.

[19] J. Lagarias and T. Richardson. "Convexity and the average curvature of plane curves." *Geom. Dedicata* 67.1 (1997), pp. 1–30.

[20] А. И. Назаров и Ф. В. Петров. «О гипотезе С. Л. Табачникова». *Алгебра и анализ* 19.1 (2007), с. 177—193.

[21] I. Fáry. "Sur la courbure totale d'une courbe gauche faisant un nœud". *Bull. Soc. Math. France* 77 (1949), pp. 128–138.

[22] J. Milnor. "On the total curvature of knots." *Ann. Math. (2)* 52 (1950), pp. 248–257.

[23] S. Alexander and R. Bishop. "The Fary-Milnor theorem in Hadamard manifolds". *Proc. Amer. Math. Soc.* 126.11 (1998), pp. 3427–3436.

[24] T. Ekholm, B. White, and D. Wienholtz. "Embeddedness of minimal surfaces with total boundary curvature at most 4π." *Ann. Math. (2)* 155.1 (2002), pp. 209–234.

[25] S. Tabachnikov. "Supporting cords of convex sets. Problem 91-2 in Mathematical Entertainments". *Mathematical Intelligencer* 13.1 (1991), p. 33.

[26] S. Tabachnikov. "The (un)equal tangents problem." *Am. Math. Mon.* 119.5 (2012), pp. 398–405.

[27] Z. E. Brady. *Is it possible to capture a sphere in a knot?* URL: `http://mathoverflow.net/q/8091`.

[28] W. K. Hayman. "Research problems in function theory: new problems". *Proceedings of the Symposium on Complex Analysis (Univ. Kent, Canterbury, 1973).* Cambridge Univ. Press, London, 1974, 155–180. London Math. Soc. Lecture Note Ser., No. 12.

[29] M. Mateljević. "Isoperimetric inequality, F. Gehring's problem on linked curves and capacity". *Filomat* 29.3 (2015), pp. 629–650.

[30] J. Cantarella, J. H. G. Fu, R. Kusner, J. Sullivan, and N. C. Wrinkle. "Criticality for the Gehring link problem". *Geom. Topol.* 10 (2006), pp. 2055–2116.

[31] M. Edelstein and B. Schwarz. "On the length of linked curves." *Isr. J. Math.* 23 (1976), pp. 94–95.

[32] M. D. Kirszbraun. „Über die zusammenziehende und Lipschitzsche Transformationen." *Fundam. Math.* 22 (1934), S. 77–108.

[33] F. A. Valentine. "On the extension of a vector function so as to preserve a Lipschitz condition." *Bull. Am. Math. Soc.* 49 (1943), pp. 100–108.

[34] L. Danzer, B. Grünbaum, and V. Klee. "Helly's theorem and its relatives". *Proc. Sympos. Pure Math., Vol. VII.* Amer. Math. Soc., Providence, R.I., 1963, pp. 101–180.

[35] B. Knaster. "Un continu dont tout sous-continu est indécomposable." *Fundam. Math.* 3 (1922), pp. 247–286.

[36] K. Yoneyama. "Theory of continuous set of points." *Tohoku Math. J.* 12 (1918), pp. 43–158.

[37] L. Antoine. "Sur l'homeomorphisme de deux figures et leurs voisinages". *J. Math. Pures Appl.* 4 (1921), pp. 221–325.

[38] S. Eilenberg and O. G. Harrold Jr. "Continua of finite linear measure. I". *Amer. J. Math.* 65 (1943), pp. 137–146.

[39] K. J. Falconer. *The geometry of fractal sets.* Vol. 85. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1986.

[40] B. Kirchheim, E. Spadaro, and L. Székelyhidi. "Equidimensional isometric maps". *Comment. Math. Helv.* 90.4 (2015), pp. 761–798.

[41] T. Zamfirescu. "Baire categories in convexity." *Atti Semin. Mat. Fis. Univ. Modena* 39.1 (1991), pp. 139–164.

[42] А. Д. Милка. «Кратчайшие линии на выпуклых поверхностях». *Доклады АН СССР* 248.1 (1979), с. 34—36.

[43] S. Cohn-Vossen. „Totalkrümmung und geodätische Linien auf einfachzusammenhängenden offenen vollständigen Flächenstücken". *Матем. сб.* 1(43).2 (1936), S. 139–164.

[44] А. Д. Александров и В. В. Стрельцов. «Изопериметрическая задача и оценки длины кривой на поверхности». *Двумерные многообразия ограниченной кривизны. Часть II. Сборник статей по внутренней геометрии поверхностей.* М.–Л.: Наука, 1965, с. 67—80.

[45] V. Bangert. "Geodesics and totally convex sets on surfaces." *Invent. Math.* 63 (1981), pp. 507–517.

[46] I. D. Berg. "An estimate on the total curvature of a geodesic in Euclidean 3-space- with-boundary." *Geom. Dedicata* 13 (1982), pp. 1–6.

[47] В. В. Усов. «О длине сферического изображения геодезической на выпуклой поверхности.» *Сибирский математический журнал* 17.1 (1976), с. 233—236.

[48] И. М. Либерман. «Геодезические линии на выпуклых поверхностях». *ДАН СССР* 32 (1941), с. 310—313.

[49] M. Gromov. "Sign and geometric meaning of curvature." *Rend. Semin. Mat. Fis. Milano* 61 (1991), pp. 9–123.

[50] F. Rodriguez Hertz. *On the geodesic flow of surfaces of nonpositive curvature.* arXiv: `0301010 [math.DS]`.

[51] R. Schoen and S.-T. Yau. "On univalent harmonic maps between surfaces". *Invent. Math.* 44.3 (1978), pp. 265–278.

[52] S. Bernstein. «Sur un théoréme de géométrie et son application aux équations aux dérivées partielles du type elliptique.» *Сообщения Харьковского математического общества* 15.1 (1915). Russian translation in «Успехах математических наук», вып. VIII (1941), 75—81 и в С. Н. Бернштейн, Собрание сочинений. Т. 3. (1960) с. 251—258; German translation in Math. Ztschr., 26 (1927), 551–558., с. 38—45.

[53] С. З. Шефель. «О внутренней геометрии седловых поверхностей». *Сибирский метематический журнал* 5 (1964), с. 1382—1396.

[54] S. Alexander, Kapovitch V., and A. Petrunin. *Invitation to Alexandrov geometry: CAT[0] spaces.* arXiv: `1701.03483 [math.DG]`.

[55] A. Petrunin and Stadler S. *Monotonicity of saddle maps.* arXiv: `1707.02367 [math.DG]`.

[56] D. Panov. "Parabolic curves and gradient mappings". *Proc. Steklov Inst. Math.* 2(221) (1998), pp. 261–278.

[57] S. Stadler. *The structure of minimal surfaces in CAT(0) spaces.* arXiv: `1808.06410 [math.DG]`.

[58] S. Brendle and P. K. Hung. *Area bounds for minimal surfaces that pass through a prescribed point in a ball.* arXiv: `1607.04631 [math.DG]`.

[59] H. Alexander and R. Osserman. "Area bounds for various classes of surfaces." *Am. J. Math.* 97 (1975), pp. 753–769.

[60] H. Alexander, D. Hoffman, and R. Osserman. "Area estimates for submanifolds of Euclidean space". *Symposia Mathematica, Vol. XIV*. Academic Press, London, 1974, pp. 445–455.

[61] J. O'Rourke. *Why is the half-torus rigid?* URL: `http://mathoverflow.net/q/77760`.

[62] E. Rembs. „Verbiegungen höherer Ordnung und ebene Flächenrinnen." *Math. Z.* 36 (1932), S. 110–121.

[63] Н. В. Ефимов. «Качественные вопросы теории деформаций поверхностей». *УМН* 3.2(24) (1948), с. 47—158.

[64] I. Kh. Sabitov. "On infinitesimal bendings of troughs of revolution. I." *Math. USSR, Sb.* 27 (1977), pp. 103–117.

[65] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* 3rd printing. Basel: Birkhäuser, 2007.

[66] A. Petrunin. *Two discs with no parallel tangent planes.* URL: `http://mathoverflow.net/q/17486`.

[67] P. Pushkar. *A generalization of Cauchy's mean value theorem.* URL: `http://mathoverflow.net/q/16335`.

[68] J. Cheeger and D. G. Ebin. *Comparison theorems in Riemannian geometry.* Revised reprint of the 1975 original. AMS Chelsea Publishing, Providence, RI, 2008.

[69] F. Fang, S. Mendonça, and X. Rong. "A connectedness principle in the geometry of positive curvature." *Commun. Anal. Geom.* 13.4 (2005), pp. 671–695.

[70] B. Wilking. "Torus actions on manifolds of positive sectional curvature." *Acta Math.* 191.2 (2003), pp. 259–297.

[71] S. Alexander, V. Kapovitch, and Petrunin A. *Alexandrov geometry.* URL: `www.math.psu.edu/petrunin`.

[72] R. E. Greene and H. Wu. "On the subharmonicity and plurisubharmonicity of geodesically convex functions." *Indiana Univ. Math. J.* 22 (1973), pp. 641–653.

[73] S. Alexander. "Locally convex hypersurfaces of negatively curved spaces." *Proc. Am. Math. Soc.* 64 (1977), pp. 321–325.

[74] J.-H. Eschenburg. "Local convexity and nonnegative curvature — Gromov's proof of the sphere theorem." *Invent. Math.* 84 (1986), pp. 507–522.

[75] B. Andrews. "Contraction of convex hypersurfaces in Riemannian spaces." *J. Differ. Geom.* 39.2 (1994), pp. 407–431.

[76] F. Almgren. "Optimal isoperimetric inequalities." *Indiana Univ. Math. J.* 35 (1986), pp. 451–547.

[77] M. Gromov. "Isoperimetric inequalities in Riemannian manifolds". V. Milman and G. Schechtman. *Asymptotic theory of finite dimensional normed spaces.* Vol. 1200. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986, pp. 114–129.

[78] A. Weinstein. "Positively curved n-manifolds in $\mathbb{R}^{n+2}$." *J. Differ. Geom.* 4 (1970), pp. 1–4.

[79] M. Micallef and J. Moore. "Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes." *Ann. Math. (2)* 127.1 (1988), pp. 199–227.

[80] C. Böhm and B. Wilking. "Manifolds with positive curvature operators are space forms." *Ann. Math. (2)* 167.3 (2008), pp. 1079–1097.

[81] J. L. Synge. "On the connectivity of spaces of positive curvature." *Q. J. Math., Oxf. Ser.* 7 (1936), pp. 316–320.

[82] T. Frankel. "On the fundamental group of a compact minimal submanifolds." *Ann. Math. (2)* 83 (1966), pp. 68–73.

[83] S. Bochner. "Vector fields and Ricci curvature." *Bull. Am. Math. Soc.* 52 (1946), pp. 776–797.

[84] W.-Y. Hsiang and B. Kleiner. "On the topology of positively curved 4-manifolds with symmetry." *J. Differ. Geom.* 29.3 (1989), pp. 615–621.

[85] K. Grove. "Geometry of, and via, symmetries". *Conformal, Riemannian and Lagrangian geometry (Knoxville, TN, 2000).* Vol. 27. Univ. Lecture Ser. Amer. Math. Soc., Providence, RI, 2002, pp. 31–53.

[86] K. Grove and B. Wilking. "A knot characterization and 1-connected nonnegatively curved 4-manifolds with circle symmetry." *Geom. Topol.* 18.5 (2014), pp. 3091–3110.

[87] C. Croke. "Lower bounds on the energy of maps." *Duke Math. J.* 55 (1987), pp. 901–908.

[88] B. White. "Infima of energy functionals in homotopy classes of mappings." *J. Differ. Geom.* 23 (1986), pp. 127–142.

[89] M. Gromov. "Filling Riemannian manifolds." *J. Differ. Geom.* 18 (1983), pp. 1–147.

[90] C. Croke. "A sharp four dimensional isoperimetric inequality." *Comment. Math. Helv.* 59 (1984), pp. 187–192.

[91] C. Croke. "Some isoperimetric inequalities and eigenvalue estimates." *Ann. Sci. Éc. Norm. Supér. (4)* 13 (1980), pp. 419–435.

[92] E. Hopf. "Closed surfaces without conjugate points". *Proc. Nat. Acad. Sci. U. S. A.* 34 (1948), pp. 47–51.

[93] L. W. Green. "A theorem of E. Hopf". *Michigan Math. J.* 5 (1958), pp. 31–34.

[94] D. Gromoll, W. Klingenberg, and W. Meyer. *Riemannsche Geometrie im Grossen*. Lecture Notes in Mathematics, No. 55. Springer-Verlag, Berlin-New York, 1968.

[95] J. Cheeger. "Some examples of manifolds of nonnegative curvature." *J. Differ. Geom.* 8 (1973), pp. 623–628.

[96] S. Aloff and N. Wallach. "An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures." *Bull. Am. Math. Soc.* 81 (1975), pp. 93–97.

[97] D. Gromoll and W. Meyer. "An exotic sphere with nonnegative sectional curvature." *Ann. Math. (2)* 100 (1974), pp. 401–406.

[98] J.-H. Eschenburg. "New examples of manifolds with strictly positive curvature." *Invent. Math.* 66 (1982), pp. 469–480.

[99] Ya. V. Bazaikin. "On a family of 13-dimensional closed Riemannian manifolds of positive curvature". *Siberian Math. J.* 37.6 (1996), pp. 1068–1085.

[100] P. Petersen, F. Wilhelm, and S. Zhu. "Spaces on and beyond the boundary of existence". *J. Geom. Anal.* 5.3 (1995), pp. 419–426.

[101] V. Kapovitch. "Restrictions on collapsing with a lower sectional curvature bound". *Math. Z.* 249.3 (2005), pp. 519–539.

[102] А. Д. Милка. «Многомерные пространства с многогранной метрикой неотрицательной кривизны I». *Украинский геометрический сборник* 5–6 (1968), с. 103—114.

[103] G. Perelman. "Proof of the soul conjecture of Cheeger and Gromoll." *J. Differ. Geom.* 40.1 (1994), pp. 209–212.

[104] M. Gromov and B. Lawson. "Positive scalar curvature and the Dirac operator on complete Riemannian manifolds." *Publ. Math., Inst. Hautes Étud. Sci.* 58 (1983), pp. 83–196.

[105] R. Schoen and S.-T. Yau. "Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature." *Ann. Math. (2)* 110 (1979), pp. 127–142.

[106] U. Abresch and D. Gromoll. "On complete manifolds with nonnegative Ricci curvature." *J. Am. Math. Soc.* 3.2 (1990), pp. 355–374.

[107] J. Cheeger and T. Colding. "Lower bounds on Ricci curvature and the almost rigidity of warped products." *Ann. Math. (2)* 144.1 (1996), pp. 189–237.

[108] E. Calabi. "On manifolds with non-negative Ricci curvature II". *Notices AMS* 22 (1975), A205.

[109] S.-T. Yau. "Some function-theoretic properties of complete Riemannian manifold and their applications to geometry." *Indiana Univ. Math. J.* 25 (1976), pp. 659–670.

[110] S. Buyalo. "Volume and the fundamental group of a manifold of nonpositive curvature." *Math. USSR, Sb.* 50 (1985), pp. 137–150.

[111] D. Panov and A. Petrunin. "Sweeping out sectional curvature." *Geom. Topol.* 18.2 (2014), pp. 617–631.

[112] S. Alexander, D. Berg, and R. Bishop. "Geometric curvature bounds in Riemannian manifolds with boundary." *Trans. Am. Math. Soc.* 339.2 (1993), pp. 703–716.

[113] C. Croke. "Small volume on big n-spheres." *Proc. Am. Math. Soc.* 136.2 (2008), pp. 715–717.

[114] M. Gromov. "Pseudo holomorphic curves in symplectic manifolds." *Invent. Math.* 82 (1985), pp. 307–347.

[115] F. Balacheff, C. Croke, and M. Katz. "A Zoll counterexample to a geodesic length conjecture." *Geom. Funct. Anal.* 19.1 (2009), pp. 1–10.

[116] A. S. Besicovitch. "On two problems of Loewner." *J. Lond. Math. Soc.* 27 (1952), pp. 141–144.

[117] K. Honda. "Transversality theorems for harmonic forms." *Rocky Mt. J. Math.* 34.2 (2004), pp. 629–664.

[118] R. Bishop and B. O'Neill. "Manifolds of negative curvature." *Trans. Am. Math. Soc.* 145 (1969), pp. 1–49.

[119] S.-T. Yau. "Non-existence of continuous convex functions on certain Riemannian manifolds." *Math. Ann.* 207 (1974), pp. 269–270.

[120] L. Guth. "Symplectic embeddings of polydisks." *Invent. Math.* 172.3 (2008), pp. 477–489.

[121] H. Weyl. "On the volume of tubes." *Am. J. Math.* 61 (1939), pp. 461–472.

[122] S. Frankel and M. Katz. "The Morse landscape of a Riemannian disk." *Ann. Inst. Fourier* 43.2 (1993), pp. 503–507.

[123] A. Nabutovsky and R. Rotman. "Length of geodesics and quantitative Morse theory on loop spaces." *Geom. Funct. Anal.* 23.1 (2013), pp. 367–414.

[124] V. Milman and G. Schechtman. *Asymptotic theory of finite-dimensional normed spaces.* Vol. 1200. Lecture Notes in Mathematics. With an appendix by M. Gromov. Springer-Verlag, Berlin, 1986.

[125] M. Ledoux. *The concentration of measure phenomenon.* Vol. 89. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2001.

[126] D. Burago, Yu. Burago, and S. Ivanov. *A course in metric geometry.* Providence, RI: American Mathematical Society (AMS), 2001.

[127] C. Kuratowski. "Quelques problèmes concernant les espaces métriques nonseparables." *Fundam. Math.* 25 (1935), pp. 534–545.

[128] M. Fréchet. "Les ensembles abstraits et le calcul fonctionnel." *Rend. Circ. Mat. Palermo* 30 (1910), pp. 1–26.

[129] J. R. Isbell. "Six theorems about injective metric spaces". *Comment. Math. Helv.* 39 (1964), pp. 65–76.

[130] E. Kopecká and S. Reich. "Nonexpansive retracts in Banach spaces". *Fixed point theory and its applications.* Vol. 77. Banach Center Publ. Polish Acad. Sci. Inst. Math., Warsaw, 2007, pp. 161–174.

[131] Petrunin A. *Convex hull in CAT(0).* MathOverflow.

[132] B. Duchesne. *Groups acting on spaces of non-positive curvature.* arXiv: `arXiv:1603.04573v2 [math.DG]`.

[133] M. Gromov. "Hyperbolic manifolds, groups and actions". *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference.* Vol. 97. Ann. of Math. Stud. Princeton Univ. Press, Princeton, N.J., 1981, pp. 183–213.

[134] M. Gromov. "Positive curvature, macroscopic dimension, spectral gaps and higher signatures". *Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993).* Vol. 132. Progr. Math. Birkhäuser Boston, Boston, MA, 1996, pp. 1–213.

[135] J. Nash. "C^1 isometric imbeddings." *Ann. Math. (2)* 60 (1954), pp. 383–396.

[136] N. Kuiper. "On C^1-isometric imbeddings. I, II." *Nederl. Akad. Wet., Proc., Ser. A* 58 (1955), pp. 545–556, 683–689.

[137] H. Rademacher. „Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variablen und über die Transformation der Doppelintegrale. I, II." *Math. Ann.* 79 (1920), S. 340–359.

[138] L. C. Siebenmann. “Deformation of homeomorphisms on stratified sets.” *Comment. Math. Helv.* 47 (1972), pp. 123–136.

[139] D. Sullivan. “Hyperbolic geometry and homeomorphisms”. *Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977).* Academic Press, New York-London, 1979, pp. 543–555.

[140] A. Całka. “On local isometries of finitely compact metric spaces”. *Pacific J. Math.* 103.2 (1982), pp. 337–345.

[141] D. Wright. “Tychonoff’s theorem”. *Proc. Amer. Math. Soc.* 120.3 (1994), pp. 985–987.

[142] A. Karlsson. *Ergodic theorems for noncommuting random products.* URL: http://www.unige.ch/math/folks/karlsson/.

[143] S. Ferry and B. Okun. “Approximating topological metrics by Riemannian metrics.” *Proc. Am. Math. Soc.* 123.6 (1995), pp. 1865–1872.

[144] D. Burago, S. Ivanov, and D. Shoenthal. “Two counterexamples in low-dimensional length geometry.” *St. Petersbg. Math. J.* 19.1 (2008), pp. 33–43.

[145] E. Le Donne. “Lipschitz and path isometric embeddings of metric spaces”. *Geom. Dedicata* 166 (2013), pp. 47–66.

[146] A. Petrunin and A. Yashinski. “Piecewise isometric mappings”. *St. Petersburg Math. J.* 27.1 (2016), pp. 155–175.

[147] A. Petrunin. “On intrinsic isometries to Euclidean space.” *St. Petersbg. Math. J.* 22.5 (2011), pp. 803–812.

[148] J. Väisälä. “A proof of the Mazur–Ulam theorem.” *Am. Math. Mon.* 110.7 (2003), pp. 633–635.

[149] S. Mazur and S. Ulam. “Sur les transformations isométriques d’espaces vectoriels, normés.” *C. R. Acad. Sci., Paris* 194 (1932), pp. 946–948.

[150] A. Pogorelov. *Hilbert’s fourth problem.* V. H. Winston & Sons, Washington, D.C.; A Halsted Press Book, John Wiley & Sons, New York-Toronto, Ont.-London, 1979.

[151] D. Hilbert. „Ueber die gerade Linie als kürzeste Verbindung zweier Punkte.“ *Math. Ann.* 46 (1895), S. 91–96.

[152] T. Foertsch and V. Schroeder. “Minkowski versus Euclidean rank for products of metric spaces.” *Adv. Geom.* 2.2 (2002), pp. 123–131.

[153] P. Hitzelberger and A. Lytchak. “Spaces with many affine functions.” *Proc. Am. Math. Soc.* 135.7 (2007), pp. 2263–2271.

[154] N. I. Lobachevsky. *Geometrische Untersuchungen zur Theorie der Parallellinien.* Berlin: F. Fincke, 1840.

[155] K. Böröczky. “Gömbkitöltések allandó görbületű terekben I.” *Mat. Lapok* 25 (1977), pp. 265–306.

[156] C. Radin. “Orbits of orbs: sphere packing meets Penrose tilings”. *Amer. Math. Monthly* 111.2 (2004), pp. 137–149.

[157] O. Gross. “The rendezvous value of metric space”. *Advances in game theory.* Princeton Univ. Press, Princeton, N.J., 1964, pp. 49–53.

[158] A. Całka. “On conditions under which isometries have bounded orbits.” *Colloq. Math.* 48 (1984), pp. 219–227.

[159] M. H. A. Newman. “A theorem on periodic transformations of spaces.” *Q. J. Math., Oxf. Ser.* 2 (1931), pp. 1–8.

[160] D. Montgomery. “Pointwise periodic homeomorphisms.” *Am. J. Math.* 59 (1937), pp. 118–120.

[161] V. Šahović. “Approximations of Riemannian manifolds with linear curvature constraints.” Thesis. Univ. Münster, 2009.

[162] D. Panov and A. Petrunin. “Telescopic actions.” *Geom. Funct. Anal.* 22.6 (2012), pp. 1814–1831.

[163] A. Petrunin. *Diameter of m-fold cover.* URL: `http://mathoverflow.net/q/7732`.

[164] P. Hall. “On representatives of subsets”. *J. London Math. Soc* 10.1 (1935), pp. 26–30.

[165] D. Burago and B. Kleiner. “Rectifying separated nets.” *Geom. Funct. Anal.* 12.1 (2002), pp. 80–92.

[166] C. Lange. *When is the underlying space of an orbifold a topological manifold.* arXiv: `1307.4875 [math.GN]`.

[167] М. А. Михайлова. «О факторпространстве по действию конечной группы, порожденной псевдоотражениями». *Изв. АН СССР. Сер. матем.* 48.1 (1984), с. 104—126.

[168] J. Stallings. “Topology of finite graphs.” *Invent. Math.* 71 (1983), pp. 551–565.

[169] H. Wilton. *In Memoriam J. R. Stallings — Topology of Finite Graphs.* URL: `https://ldtopology.wordpress.com/2008/12/01/`.

[170] M. Gerstenhaber and O. S. Rothaus. “The solution of sets of equations in groups”. *Proc. Nat. Acad. Sci. U.S.A.* 48 (1962), pp. 1531–1533.

[171] H. Hopf. „Über den Rang geschlossener Liescher Gruppen“. *Comment. Math. Helv.* 13 (1940), S. 119–143.

[172] C.-L. Terng and G. Thorbergsson. “Submanifold geometry in symmetric spaces”. *J. Differential Geom.* 42.3 (1995), pp. 665–718.

[173] C. Kosniowski. *A first course in algebraic topology.* Cambridge University Press, Cambridge-New York, 1980.

[174] V. Klee. “Some topological properties of convex sets”. *Trans. Amer. Math. Soc.* 78 (1955), pp. 30–45.

[175] P. H. Doyle. “Plane separation”. *Proc. Cambridge Philos. Soc.* 64 (1968), p. 291.

[176] D. Ramras. *Klee’s trick — more applications.* URL: `https://mathoverflow.net/q/273003`.

[177] D. Bennequin. “Exemples d’immersions du disque dans le plan qui ne sont pas projections de plongements dans l’espace.” *C. R. Acad. Sci., Paris, Sér. A* 281 (1975), pp. 81–84.

[178] D. Eppstein and E. Mumford. “Self-overlapping curves revisited”. *Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms.* SIAM, Philadelphia, PA, 2009, pp. 160–169.

[179] K. W. Kwun. “Uniqueness of the open cone neighborhood”. *Proc. Amer. Math. Soc.* 15 (1964), pp. 476–479.

[180] R. C. Kirby. “Stable homeomorphisms and the annulus conjecture.” *Ann. Math. (2)* 89 (1969), pp. 575–582.

[181] P. Alexandroff. „Diskrete Räume.“ *Матем. сб.* 2 (1937), S. 501–519.

[182] M. C. McCord. “Singular homology groups and homotopy groups of finite topological spaces.” *Duke Math. J.* 33 (1966), pp. 465–474.

[183] “Geometric group theory, hyperbolic dynamics and symplectic geometry”. *Oberwolfach Rep.* 9.3 (2012). Abstracts from the workshop held July 15–21, 2012, pp. 2139–2203.

[184] D. Dore and A. Hanlon. “Area preserving maps on S^2: a lower bound on the C^0-norm using symplectic spectral invariants”. *Electron. Res. Announc. Math. Sci.* 20 (2013), pp. 97–102.

[185] S. Seyfaddini. “The displaced disks problem via symplectic topology”. *C. R. Math. Acad. Sci. Paris* 351.21-22 (2013), pp. 841–843.

[186] K. Menger. „Untersuchungen über allgemeine Metrik“. *Math. Ann.* 100.1 (1928), S. 75–163.

[187] R. H. Bing. “A convex metric for a locally connected continuum”. *Bull. Amer. Math. Soc.* 55 (1949), pp. 812–819.

[188] R. H. Bing. “Partitioning continuous curves”. *Bull. Amer. Math. Soc.* 58 (1952), pp. 536–556.

[189] E. E. Moise. “Grille decomposition and convexification theorems for compact metric locally connected continua”. *Bull. Amer. Math. Soc.* 55 (1949), pp. 1111–1121.

[190] A. Lytchak and S. Wenger. *Intrinsic structure of minimal discs in metric spaces.* arXiv: `1602.06755 [math.DG]`.

[191] M. Khanevsky. “Hofer’s length spectrum of symplectic surfaces”. *J. Mod. Dyn.* 9 (2015), pp. 219–235.

[192] В. А. Залгаллер. «О деформациях многоугольника на сфере». *УМН* 11.5(71) (1956), с. 177—178.

[193] В. А. Топоногов. «Оценка длины замкнутой геодезической на выпуклой поверхности». *Докл. АН СССР* 124.2 (1959), с. 282—284.

[194] А. Д. Александров. *Внутренняя геометрия выпуклых поверхностей.* ОГИЗ, М.-Л., 1948.

[195] В. А. Залгаллер. «Изометричекие вложения полиэдров». *Доклады АН СССР* 123 (1958), с. 599—601.

[196] S. Krat. “Approximation Problems in Length Geometry”. Ph.D. thesis. Pennsylvania State University, 2005.

[197] U. Brehm. “Extensions of distance reducing mappings to piecewise congruent mappings on $\mathbb{R}^m$.” *J. Geom.* 16 (1981), pp. 187–193.

[198] A. Akopyan and A. Tarasov. “A constructive proof of Kirszbraun’s theorem.” *Math. Notes* 84.5 (2008), pp. 725–728.

[199] Yu. D. Burago and V. A. Zalgaller. “Isometric piecewise-linear embeddings of two-dimensional manifolds with a polyhedral metric into $\mathbb{R}^3$”. *St. Petersburg Math. J.* 7.3 (1996), pp. 369–385.

[200] H. A. Schwarz. “Sur une définition erronée de l’aire d’une surface courbe”. *Gesammelte Mathematische Abhandlungen* 1 (1890), pp. 309–311.

[201] J. Hass. “Bounded 3-manifolds admit negatively curved metrics with concave boundary”. *J. Differential Geom.* 40.3 (1994), pp. 449–459.

[202] A. Petrunin. “Area minimizing polyhedral surfaces are saddle”. *Amer. Math. Monthly* 122.3 (2015), pp. 264–267.

[203] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants.* Modern Birkhäuser Classics. Reprint of the 1994 edition. Birkhäuser Boston, Inc., Boston, MA, 2008.

[204] E. Schönhardt. „Über die Zerlegung von Dreieckspolyedern in Tetraeder“. *Math. Ann.* 98.1 (1928), S. 309–312.

[205] D. Panov. “Polyhedral Kähler manifolds.” *Geom. Topol.* 13.4 (2009), pp. 2205–2252.

[206] W. Thurston. *Three-dimensional geometry and topology. Vol. 1.* Vol. 35. Princeton Mathematical Series. Edited by Silvio Levy. Princeton University Press, Princeton, NJ, 1997.

[207] S. Jendroľ and E. Jucovič. “On the toroidal analogue of Eberhard’s theorem”. *Proc. London Math. Soc. (3)* 25 (1972), pp. 385–398.

[208] R. Matveev. “Surfaces with polyhedral metrics”. International Mathematical Summer School for Students 2011. Jacobs University, Bremen.

[209] I. Izmestiev, R. Kusner, G. Rote, B. Springborn, and J. Sullivan. “There is no triangulation of the torus with vertex degrees 5,6,...,6,7 and related results: geometric proofs for combinatorial theorems.” *Geom. Dedicata* 166 (2013), pp. 15–29.

[210] G. Mondello and D. Panov. *Spherical metrics with conical singularities on a 2-sphere: angle constraints.* arXiv: `1505.01994 [math.DG]`.

[211] G. Galperin. “Convex polyhedra without simple closed geodesics.” *Regul. Chaotic Dyn.* 8.1 (2003), pp. 45–58.

[212] В. Ю. Протасов. «О числе замкнутых геодезических на многограннике». *УМН* 63.5(383) (2008), с. 197—198.

[213] A. Akopyan and A. Petrunin. *Long geodesics on convex surfaces.* arXiv: `1702.05172 [math.DG]`.

[214] J. Itoh, J. Rouyer, and C. Vîlcu. *Polyhedra with simple dense geodesics.* arXiv: `1704.05011 [math.DG]`.

[215] G. Walsh. "Great circle links in the three-sphere". Thesis (Ph.D.)–University of California, Davis. 2003.

[216] M. Freedman and R. Skora. "Strange actions of groups on spheres". *J. Differential Geom.* 25.1 (1987), pp. 75–98.

[217] A. Shen. "Unexpected proofs. Boxes in a Train". *Math. Intelligencer* 21.3 (1999), pp. 48–50.

[218] J. Steiner. "Über parallele flächen". *Monatsber. Preuss. Akad. Wiss* 2 (1840), pp. 114–118.

[219] Yu. Burago and V. Zalgaller. *Geometric inequalities.* Berlin etc.: Springer-Verlag, 1988.

[220] A. Harnack. „Ueber die Vieltheiligkeit der ebenen algebraischen Curven". *Math. Ann.* 10.2 (1876), S. 189–198.

[221] F. Klein. „Ueber den Verlauf der Abel'schen Integrale bei den Curven vierten Grades". *Math. Ann.* 10.3 (1876), S. 365–397.

[222] P. Monsky. "On dividing a square into triangles." *Am. Math. Mon.* 77 (1970), pp. 161–164.

[223] A. W. Goodman and R. E. Goodman. "A circle covering theorem". *Amer. Math. Monthly* 52 (1945), pp. 494–498.

[224] H. Hadwiger. "Nonseparable convex systems". *Amer. Math. Monthly* 54 (1947), pp. 583–585.

[225] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[226] C. Halberg, E. Levin, and E. G. Straus. "On contiguous congruent sets in Euclidean space." *Proc. Am. Math. Soc.* 10 (1959), pp. 335–344.

[227] R. Alexander. "Lipschitzian mappings and total mean curvature of polyhedral surfaces. I." *Trans. Am. Math. Soc.* 288 (1985), pp. 661–678.

[228] K. Bezdek and R. Connelly. "Pushing disks apart — the Kneser–Poulsen conjecture in the plane." *J. Reine Angew. Math.* 553 (2002), pp. 221–236.

[229] M. Belk and R. Connelly. *Making contractions continuous: a problem related to the Kneser–Poulsen conjecture.* URL: `math.bard.edu/~mbelk/`.

[230] P. Erdős. "Some unsolved problems". *Michigan Math. J.* 4 (1957), pp. 291–300.

[231] L. Danzer und B. Grünbaum. „Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee". *Math. Z.* 79 (1962), S. 95–99.

[232] G. Perelman. "Spaces with curvature bounded below". *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994).* Birkhäuser, Basel, 1995, pp. 517–525.

[233] N. Lebedeva. "Alexandrov spaces with maximal number of extremal points". *Geom. Topol.* 19.3 (2015), pp. 1493–1521.

[234] P. Erdős and Z. Füredi. "The greatest angle among n points in the d-dimensional Euclidean space". *Combinatorial mathematics (Marseille-Luminy, 1981).* Vol. 75. North-Holland Math. Stud. North-Holland, Amsterdam, 1983, pp. 275–283.

[235] D. Zakharov. "Acute sets". *Discrete & Computational Geometry* (2017), pp. 1–6.

[236] `grizzly`. *Улучшено (?) решение Эрдёша по остроугольным треугольникам.* URL: `http://dxdy.ru/post1222167.html`.

[237] B. Gerencsér and V. Harangi. "Acute sets of exponentially optimal size". *Discrete & Computational Geometry* (2018), pp. 1–6.

[238] Э. Б. Винберг. «Дискретные группы отражений в пространствах Лобачевского большой размерности». *Модули и алгебраические группы. 2*. Л.: Наука, 1983, с. 62—68.

[239] Э. Б. Винберг. «Отсутствие кристаллографических групп отражений в пространствах Лобачевского большой размерности». *Функц. анализ и его прил.* 15.2 (1981), с. 67—68.