Exercises in Orthodox Geometry

Anton Petrunin

arXiv:0906.0290 [math.HO]

Contents

Instead of Introduction

I am collecting these problems for fun, but they might be used to improve the problem solving skills in geometry. Every problem has a short elegant solution — this gives a hint which was not available when it was solved for the first time.

How to read it. Open at a random chapter, make sure you like the practice problem.

If yes try to solve a random problem in the chapter. A semisolution is given in the end of the chapter, but you expected to think before reading it, otherwise it might not help.

Acknowledgments. I want to thank everyone who helped me; here is an incomplete list: Stephanie Alexander, Christopher Croke, Bogdan Georgiev, Jouni Luukkainen, Alexander Lytchak, Rostislav Matveyev, Peter Petersen, Idzhad Sabitov, Serge Tabachnikov. Let me also thank the students in my classes and everyone who took part in the discussion of this list on math*overflow* [see 1].

Very special thanks for everyone who shared the problems with me.

Some problems are marked by $\circ$, $*$, $+$ or $\sharp$.

$\circ$ — easy problem;

$*$ — the solution requires at least two ideas;

$+$ — the solution requires knowledge of a theorem;

$\sharp$ — there are interesting solutions based on different ideas.

Chapter 1

Curves

Recall that a *curve* is a continuous map defined on real interval and a *closed curve* is a continuous map defined on a circle. If the map is injective then the curve is called *simple*.

We assume that the reader is familiar with related definitions including length of curve and its curvature. The necessary material is covered in the first couple of lectures of a standard introduction to differential geometry, see [2, §26–27] or [3, Chapter 1].

We give a practice problem with a solution, after that you are on your own.

Spiral

The following problem states that if you drive on the plane and turn the steering wheel to the right all the time, then you will not be able to come back to the same place.

Assume γ is a smooth regular plane curve with strictly monotonic curvature. Show that γ has no self-intersections.

Solution. The trick is to show that the osculating circles of γ are nested.

Without loss of generality we may assume that the curve is parametrized by its length and its curvature decreases.

Let $z(t)$ be the center of osculating circle at $\gamma(t)$ and $r(t)$ is its radius. Note that

$$z(t) = \gamma(t) + \tfrac{\ddot\gamma(t)}{|\ddot\gamma(t)|^2}, \qquad\qquad r(t) = \tfrac{1}{|\ddot\gamma(t)|}.$$

Straightforward calculations show that

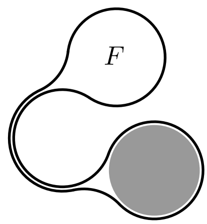
$$|\dot{z}(t)| \leqslant \dot{r}(t).$$

Dentote by D_t the osculating disc of γ at $\gamma(t)$; it has center at $z(t)$ and radius $r(t)$. It follows that $D_{t_1} \supset D_{t_0}$ for $t_1 > t_0$. Hence the result follows. □

This problem gives a continuous analog of the Leibniz's test for alternating series. It was considered by Peter Tait in [4] and later rediscovered by Adolf Kneser in [5]; see also [6].

It is instructive to check that 3-dimensional analog does not hold.

The moon in the puddle



A smooth closed simple plane curve with curvature less than 1 *bounds a figure* F. *Prove that* F *contains a disc of radius* 1.

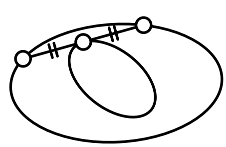
A spring in a tin

Let α *be a closed smooth immersed curve inside a unit disc. Prove that the average absolute curvature of* α *is at least* 1, *with equality if and only if* α *is the unit circle possibly traversed more than once.*

A curve in a sphere

Show that if a closed curve on the unit sphere intersects every equator then it has length at least $2{\cdot}\pi$.

Oval in oval



Consider two closed smooth strictly convex planar curves, one inside the other. Show that there is a chord of the outer curve, which is tangent to the inner curve at its midpoint.

Capture a sphere in a knot*

The following formulation use the notion of smooth isotopy of knots; that is, one parameter of embeddings

$$f_t\colon \mathbb{S}^1 \to \mathbb{R}^3, \quad t \in [0,1]$$

such that the map $[0,1]\times\mathbb{S}^1\to\mathbb{R}^3$ is smooth.

Show that one can not capture a sphere in a knot.

More precisely, let B be the closed unit ball in $\mathbb{R}^3$ and $f\colon\mathbb{S}^1\to\to\mathbb{R}^3\backslash B$ be a knot. Show that there is a smooth isotopy

$$f_t\colon\mathbb{S}^1\to\mathbb{R}^3\backslash B,\quad t\in[0,1],$$

such that $f_0=f$, the length of f_t does not increase in t and $f_1(\mathbb{S}^1)$ can be separated from B by a plane.

Linked circles

Suppose that two linked simple closed curves in $\mathbb{R}^3$ lie at a distance at least 1 from each other. Show that the length of each curve is at least $2\cdot\pi$.

Surrounded area

Consider two simple closed plane curves $\gamma_1,\gamma_2\colon\mathbb{S}^1\to\mathbb{R}^2$. Assume

$$|\gamma_1(v)-\gamma_1(w)|\leqslant|\gamma_2(v)-\gamma_2(w)|$$

for any $v,w\in\mathbb{S}^1$. Show that the area surrounded by γ_1 does not exceed the area surrounded by γ_2.

Crooked circle

Construct a bounded open disc in $\mathbb{R}^2$ such that its boundary contains no simple curve.

Rectifiable curve

For the following problem we need the notion of *Hausdorff measure.* Fix a compact set $X\subset\mathbb{R}^2$ and $\alpha>0$. Given $\delta>0$ consider the value

$$h(\delta)=\inf\left\{\sum_i(\operatorname{diam}X_i)^\alpha\right\}$$

where the infimum is taken for all coverings of X by $\{X_i\}$ such that $\operatorname{diam}X_i<\delta$ for each i.

Note that the function $\delta\mapsto h(\delta)$ is not decreasing in δ. In particular, the there is a (possibly infinite) limit, say h, of $h(\delta)$ as $\delta\to 0$. This value h is called α-dimensional Hausdorff measure of X.

Let $X\subset\mathbb{R}^2$ be a compact connected set with finite 1-dimensional Hausdorff measure. Show that X is an image of rectifiable curve.

Typical convex functions

Recall that *G-delta set* is defined as a countable intersection of open sets. According to *Baire category theorem*, in a complete metric spaces, any intersection of countable collection of dense open set has to be dense.

In particular, in a complete metric spaces, the intersection of a finite or countable collection of G-delta dense sets is also G-delta dense. The later means that G-delta dense sets contains *most* of the points of a complete metric space. This is the meaning of the word *most* used in the following problem.

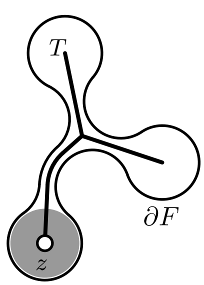
🕮 *Consider the space of convex* 1*-Lipschitz functions defined on* $[0,1]$*, equipped with the metric induced by sup-norm.*

Show that most of these functions have vanishing the second derivative at every point where it is defined.

Semisolutions

The moon in the puddle. In the proof we will use *cut locus* of F with respect to its boundary[1]; it will be further denoted as T. The cut locus can be defined as the closure of the set of points $x \in F$ such that there are two or more points in ∂F which minimize distance to x.

For each point $x \in T$, consider the subset $X \subset \partial F$ which lies on the minimal distance from F. If X is not connected then we say that x is a *cut point*; equivalently it means that for any sufficiently small neighborhood $U \ni x$, the complement $U \backslash T$ has at least two connected components. If X is connected then we say that x is a *focal point*; equivalently it means that the osculating circle to ∂F at any point of X centered at x.



The trick is to show that T contains a focal point, say z. Since ∂F has curvature of at most 1, the radius of any osculating circle has radius at lest 1. Hence z lies on the distance at least 1 from ∂F and the statement will follow.

Note that after a small perturbation of ∂F we may assume that T is a graph embedded in F with finite number of edges.

Note that T is a deformation retract of F. The retraction $F \to T$ can be obtained the following

[1] Also called *medial axis*.

way: (1) given a point $x \in F \setminus T$, consider be the (necessary unique) point $\hat{x} \in \partial F$ which minimize the distance $|x - \hat{x}|$ and (2) move x along the extension of the line segment $[\hat{x}x]$ behind x until it hits T.

In particular, T is a tree. Therefore T has an end vertex say z. The point z is focal since there is arbitrary small neighborhood U of z such that the complement $U \setminus T$ is connected. □

The problem discussed by German Pestov and Vladimir Ionin in [7]. An other solution via curve shortening flow was given by Konstantin Pankrashkin in [8]. The statement still holds if the curve fails to be smooth at one point. A spherical version of the later statement was used by Dmitri Panov and me in [9].

The 3-dimensional analog of this statement does not hold. Namely, there is a smooth embedding $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ with all the principle curvatures between -1 and 1 such that it does not surround a ball of radius 1. Such example can be obtained by fattening a nontrivial contractible 2-complex in $\mathbb{R}^3$ [the Bing's house constructed in 10, will do the job]. This problem is discussed by Vladimir Lagunov in [11] and it was generalized to Riemannian manifolds with boundary by Stephanie Alexander and Richard Bishop [see 12].

A similar argument shows that for any point $p \in (\mathbb{S}^2, g)$ there is a minimizing geodesic $[pq]$ with conjugate ends. On the other hand, for $(\mathbb{S}^3, g)$ this is not true. Related examples discussed after the hint for "Almost flat manifold", page 45.

A spring in a tin. To solve this problem, you should imagine that you travel on a train along the curve $\alpha(t)$ and watch the position of the center of the disc in the frame of your wagon.

Denote by ℓ the length of α. Equip the plane with complex coordinates so that 0 is the center of the unit disc. We can assume that α equipped with ℓ-periodic parametrization by length.

Consider the curve $\beta(t) = t - \frac{\alpha(t)}{\dot{\alpha}(t)}$. Note that

$$\beta(t + \ell) = \beta(t) + \ell$$

for any t. In particular

$$\operatorname{length}(\beta|_{[0,\ell]}) \geqslant |\beta(\ell) - \beta(0)| = \ell.$$

Note that

$$\begin{aligned} |\dot{\beta}(t)| &= |\tfrac{\alpha(t) \cdot \ddot{\alpha}(t)}{\dot{\alpha}(t)^2}| \leqslant \\ &\leqslant |\ddot{\alpha}(t)|. \end{aligned}$$

Since $|\ddot{\alpha}(t)|$ is the curvature of α at t, we get the result. □

The statement was originally proved by István Fáry in [13]; number of different proofs are discussed by Serge Tabachnikov in [14], see also [19.5 in 15].

If instead of a disc, we have a region bounded by closed convex curve γ, then it is still true that the average curvature of α is at least as big as average curvature of γ. The proof was given by Jeffrey Lagarias and Thomas Richardson in [16], see also [17].

A curve in a sphere. Let us present two solutions, both by contradiction. We assume that α is a closed curve in $\mathbb{S}^2$ of length $2\cdot\ell$ which intersects each equator.

A solution with the Crofton formula. Note that we can assume that α is a broken line.

Given a unit vector u denote by e_u the equator with pole at u. Let $k(u)$ the number of intersections of the α and e_u.

Note that for almost all $u\in\mathbb{S}^2$, the value $k(u)$ is even. Since each equator intersects α, we get $k(u)\geqslant 2$ for almost all u.

Then we get

$$\begin{aligned}2\cdot\ell &= \tfrac14\cdot\int\limits_{\mathbb{S}^2} k(u)\cdot d_u\,\mathrm{area}\geqslant\\ &\geqslant \tfrac12\cdot\mathrm{area}\,\mathbb{S}^2=\\ &=2\cdot\pi.\end{aligned}$$

The first identity above is called *Crofton formula.* Prove this formula first for a curve formed by one geodesic segment, summing up we get it for broken lines and by approximation it holds for all curves. □

A solution by symmetry. Let $\check\alpha$ be a sub-arc of α of length ℓ, with endpoints p and q. Let z be the midpoint of a minimizing geodesic $[pq]$ in $\mathbb{S}^2$.

Let r be a point of intersection of α with the equator with pole at z. Without loss of generality we may assume that $r\in\check\alpha$.

The arc $\check\alpha$ together with its reflection in the point z form a closed curve of length $2\cdot\ell$ that passes through r and its antipodal point r^*. Therefore

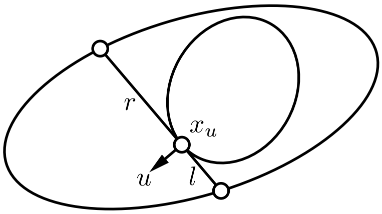
$$\ell=\mathrm{length}\,\check\alpha\geqslant|r-r^*|_{\mathbb{S}^2}=\pi.$$

Here $|r-r^*|_{\mathbb{S}^2}$ denotes the angle metric in the sphere $\mathbb{S}^2$. □

The problem was suggested by Nikolai Nadirashvili. It is nearly equivalent to the following:

◇ Show that total curvature of any closed smooth regular space curve is at least $2\cdot\pi$.

A way more advanced problem is to show that any embedded circle of total curvature at most $4{\cdot}\pi$ is unknot. It was solved independently by István Fáry [in 18] and John Milnor [in 19]. Later many interesting generalizations and refinements were found including a generalization to singular spaces by Stephanie Alexander and Richard Bishop [in 20] and the theorem on embedded minimal disc proved by Tobias Ekholm, Brian White and Daniel Wienholtz [in 21].



Oval in oval. Choose the a chord which minimizes (or maximizes) the ratio, in which it divides the bigger oval. If the chord is not divided into equal parts, then you can rotate it slightly to decrease the ratio. Hence the problem follows. $\square$

Alternative solution. Given a unit vector u, denote by x_u the point on the inner curve with outer normal vector u. Draw a chord of outer curve which is tangent to the inner curve at x_u; denote by $r = r(u)$ and $l = l(u)$ the lengths of this chord at the right and left from x_u.

Arguing by contradiction, assume $r(u) \neq l(u)$ for any $u \in \mathbb{S}^1$. Since the functions r and l are continuous, we can assume that

$$(*) \qquad r(u) > l(u) \quad \text{for any} \quad u \in \mathbb{S}^1.$$

Prove that each of the following two integrals

$$\tfrac{1}{2}{\cdot}\int\limits_{\mathbb{S}^1} r^2(u){\cdot}du \quad \text{and} \quad \tfrac{1}{2}{\cdot}\int\limits_{\mathbb{S}^1} l^2(u){\cdot}du$$

gives the area between the curves. In particular, the integrals are equal to each other. The latter contradicts $(*)$. $\square$

This is a problem of Serge Tabachnikov [see 22]. A closely related, so called *equal tangents problem* is discussed by the same author in [23].

Capture a sphere in a knot. We can assume that the knot is given by a diagram on the sphere.

Fix a Möbius transformation $m\colon \mathbb{S}^2 \to \mathbb{S}^2$ which close to identity and not a rotation. Denote by u the conformal factor of m. Since the area is preserved, we get

$$\frac{1}{\operatorname{area}\mathbb{S}^2}{\cdot}\int\limits_{\mathbb{S}^2} u^2 = 1.$$

Therefore,

$$\frac{1}{\operatorname{area}\mathbb{S}^2}\cdot\int\limits_{\mathbb{S}^2} u < 1.$$

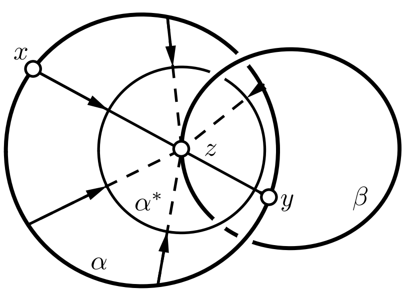
It follows that after a suitable rotation of $\mathbb{S}^2$, the map m decrease the length of the knot.

Iterate this construction and pass to the limit as $m \to \mathrm{id}$. This way you get a continuous one parameter family of Möbius transformations which moves the knot in a hemisphere. Therefore it allows the ball to escape. □

This is a problem of Zarathustra Brady, the given solution is based on the idea of David Eppstein [see 24].

Linked circles. Denote the linked circles by α and β.

Fix a point $x \in \alpha$. Note that one can find another point $y \in \alpha$ such that the interval $[xy]$ intersects β, say at the point z. Otherwise we can move each point of α along the line segment to x — this deformation of α will not cross β; the latter contradicts that α and β are linked.



Consider the curve α^* which is the central projection of α from z onto the unit sphere around z. Clearly

$$\operatorname{length}\alpha \geqslant \operatorname{length}\alpha^*.$$

Note that α^* passes through two antipodal points of the sphere, one corresponds to x and the other to y. Therefore

$$\operatorname{length}\alpha^* \geqslant 2\cdot\pi.$$

Hence the result follows. □

This is the simplest case of so called *Gehring's problem*. The solution above was given by Michael Edelstein and Binyamin Schwarz in [25]; later the same solution was rediscovered few times.

Surrounded area. Let C_1 and C_2 be the compact regions bounded by γ_1 and γ_2 correspondingly.

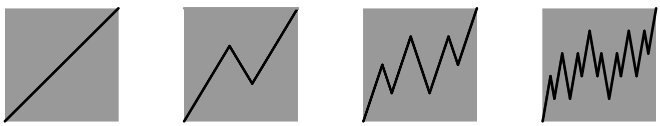
By Kirszbraun theorem, any 1-Lipschitz map $X \to \mathbb{R}^2$ defined on $X \subset \mathbb{R}^2$ can be extended to a 1-Lipschitz map on the whole $\mathbb{R}^2$. In particular, there is a 1-Lipschitz map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\gamma_2(v)) = f(\gamma_1(v))$ for any $v \in \mathbb{S}^1$.

Note that $f(C_2) \supset C_1$. Whence the statement follows. □

The Kirszbraun theorem appears in his thesis [see 26] and rediscovered later by Frederick Valentine in [27]. An interesting survey is given by Ludwig Danzer, Branko Grünbaum and Victor Klee in [28].

Crooked circle. A continuous function $f\colon [0,1] \to [0,1]$ will be called ε-crooked if $f(0) = 0$, $f(1) = 1$ and for any segment $[a,b] \subset [0,1]$ one can choose $a \leqslant x \leqslant y \leqslant b$ such that

$$|f(y) - f(a)| \leqslant \varepsilon \quad \text{and} \quad |f(x) - f(b)| \leqslant \varepsilon.$$



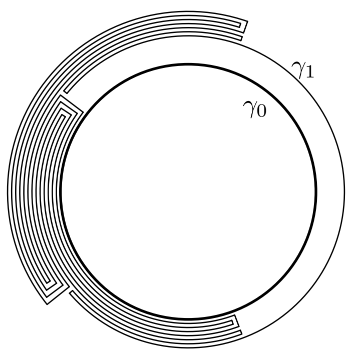
$\varepsilon = \frac{1}{2}$ $\qquad$ $\varepsilon = \frac{1}{3}$ $\qquad$ $\varepsilon = \frac{1}{4}$ $\qquad$ $\varepsilon = \frac{1}{5}$

A sequence of $\frac{1}{n}$-crooked maps can be constructed recursively. Guess the construction from the diagram.

Now, start with the unit circle, $\gamma_0(t) = (\cos\frac{t}{2\cdot\pi}, \sin\frac{t}{2\cdot\pi})$. Fix a sequence of positive numbers ε_n which converges to zero very fast. Construct recursively a sequence of simple closed curves $\gamma_n\colon [0,1] \to \mathbb{R}^2$. Such that γ_{n+1} runs outside of the disc bounded by γ_n and

$$|\gamma_{n+1}(t) - \gamma_n \circ f_n(t)| < \varepsilon_n,$$

for some ε_n-crooked function f_n. (On the diagram you see an attempt to draw the first iteration.)



Denote by D the union of all discs bounded by γ_n. Clearly D is homeomorphic to an open disc. For the right choice of the sequence ε_n, the set D is bounded and by construction its boundary contains no simple curves. $\square$

The proof use so called on *pseudo-arc* constructed by Bronisław Knaster in [29]. The construction is similar to the construction of the Cantor set. Here are few similar problems:

- ◇ *Construct three disjoint non-empty open sets in* $\mathbb{R}$ *which have the same boundary.*
- ◇ *Construct three open discs in* $\mathbb{R}^2$ *which have the same boundary.* (These discs are called *lakes of Wada*; it is described by Kunizô Yoneyama in [30].)

- ◇ *Construct a Cantor set in $\mathbb{R}^3$ with non simply connected complement.* (This example was is called *Antoine's necklace*; it is constructed in [31].)
- ◇ *Construct an open set in $\mathbb{R}^3$ with fundamental group isomorphic to the additive group of rational numbers.*

More advanced examples include *Whitehead manifold*, *Dogbone space*, *Casson handle*; see also the problem "Conic neighborhood" on page 91.

Rectifiable curve. The 1-dimensional Hausdorff measure will be denoted as $\mathcal{H}_1$.

Set $L = \mathcal{H}_1(K)$. Without loss of generality, we may assume that K has diameter 1.

Since K is connected, we get

$$\mathcal{H}_1(B(x,\varepsilon)\cap K) \geqslant \varepsilon \tag{$*$}$$

for any $x \in K$ and $0 < \varepsilon < \frac{1}{2}$.

Let $x_1, \dots, x_n$ be a maximal set of points in K such that

$$|x_i - x_j| \geqslant \varepsilon$$

for all $i \neq j$. From $(*)$ we have $n \leqslant 2\cdot L/\varepsilon$.

Note that there is a tree T_ε with the vertices $x_1, \dots, x_n$ and straight edges with length at most $2\cdot\varepsilon$ each. Therefore the total length of T_ε is below $2\cdot n\cdot\varepsilon \leqslant 4\cdot L$. By construction T_ε is ε-close to K in the Hausdorff metric.

Note that there is a closed curve γ_ε which image is T_ε and its length twice the length twice the total length of T_ε; that is,

$$\operatorname{length}\gamma_\varepsilon \leqslant 8\cdot L.$$

Passing to a partial limit of γ_ε as $\varepsilon \to 0$, we get the needed curve. □

This is a problem of Kenneth Falconer [see Exercise 3.5 in 32].

Typical convex functions. Denote by $\mathfrak{F}$ the space of all convex 1-Lipschitz functions defined on $[0,1]$ with the sup-norm. Note that $\mathfrak{F}$ is a complete metric space.

Note that if a function $f \in \mathfrak{F}$ has nonzero second derivative at some point then there is $\varepsilon > 0$ such that

$$|f - g| > \frac{\varepsilon}{n^{100}}$$

for any piecewise linear function $g \in \mathfrak{F}$ which is made from at most n linear functions.

Fix a countable set of piecewise linear convex functions $g_1, g_2, \dots$ which is dense in $\mathfrak{F}$. Denote by n_i the number of linear intervals of g_i. For any positive integer k, consider the set $\Omega_k \subset \mathfrak{F}$ defined as

$$\Omega_k = \left\{\ |f - g_i| < \frac{1}{k \cdot n_i^{100}}\ \middle|\ \text{for some } i\ \right\}.$$

From above we get that if the second derivative of f does not vanishing at some point then $f \notin \Omega_k$ for large k.

Note that Ω_k is open and everywhere dense in $\mathfrak{F}$. Therefore

$$\Omega = \bigcap_k \Omega_k$$

is a G-delta dense set. Hence the statement follows. □

It worth to note that the second derivative of a convex function is defined almost everywhere.

This problem states that typical convex functions is very far from what we used to work with.

This is a typical answer for such question — typically we do not see the typical objects. For example, according to the result of Bernd Kirchheim, Emanuele Spadaro and László Székelyhidi proved in [33] a typical 1-Lipschitz maps from the plane to itself preserves the length of all curves. The same way one could show that the typical open discs in the plane contain no simple curves in their boundary, although the construction of a concrete example is not trivial; see "Crooked circle", page 7.

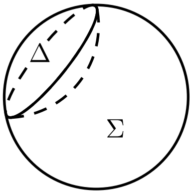
More problems of that type are surveyed by Tudor Zamfirescu in [34].

Chapter 2

Surfaces

We assume that the reader is familiar with smooth surfaces and the related definitions including intrinsic metric, geodesics, convex and saddle surfaces and different types of curvature. An introductory course in differential geometry should cover all necessary background material, say [2, §28–29] or [3].

Convex hat



Let Σ be a smooth closed convex surface in $\mathbb{R}^3$ and Π be a plane which cuts from Σ a disc Δ. Assume that the reflection of Δ in Π lies inside Σ. Show that Δ is convex *in the intrinsic metric of Σ; that is, if the ends of a minimizing geodesic in Σ lie in Δ, then whole geodesic lies in Δ.*

Solution. Let γ be a minimizing geodesic with the ends in Δ.

Assume $\gamma\backslash\Delta \neq \varnothing$. Denote by $\hat{\gamma}$ the curve formed by $\gamma\cap\Delta$ and the reflection on $\gamma\backslash\Delta$ in Π. Note

$$\operatorname{length}\hat{\gamma} = \operatorname{length}\gamma$$

and $\hat{\gamma}$ runs partly along Σ and partly outside of Σ, but does not get inside Σ.

Denote by $\bar{\gamma}$ the closest point projection of $\hat{\gamma}$ on Σ. Since Σ is convex, the closest point projection shrinks the length of γ. Therefore the curve $\bar{\gamma}$ lies in Σ has the same ends as γ.

It remains to note that

$$\operatorname{length}\bar{\gamma} < \operatorname{length}\gamma;$$

the latter leads to a contradiction. $\square$

Unbended geodesic

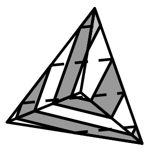
🕮 *Let Σ be a smooth closed strictly convex surface in $\mathbb{R}^3$ and $\gamma\colon [0,\ell] \to$ $\to \Sigma$ be a unit-speed minimizing geodesic in Σ. Set $p = \gamma(0)$, $q = \gamma(\ell)$ and*

$$p_t = \gamma(t) - t{\cdot}\dot\gamma(t),$$

where $\dot\gamma(t)$ denotes the velocity vector of γ at t.

Show that for any $t \in (0,\ell)$, one cannot see *q from p_t; that is, the line segment $[p_t q]$ intersects Σ at a point distinct from q.*

Long geodesic



Recall that a closed curve called *simple* if it is given by an injective map from the circle to the space.

🕮 *Assume that the surface of convex body B in $\mathbb{R}^3$ admits an arbitrary long simple closed geodesic. Show that B is a tetrahedron with equal opposite sides.*

Let us mention couple of theorems about intrinsic metric on convex surfaces which should help to solve this problem. These theorems can be proved easily for smooth or polyhedral surfaces and then the general case can be done by approximation. A very short but comprehensive introduction to the subject was written by Alexander Alexandrov in [35].

On a convex surface (not necessary smooth) one could define so called *curvature measure* which we denote further by κ. It is the (necessary unique) non-negative measure such that for any triangle $\triangle$, we have

$$\kappa(\triangle) = \alpha + \beta + \gamma - \pi,$$

where α, β and γ are the angles of $\triangle$, measured in the intrinsic metric of the surface.

For curvature measure, an analog of Gauss–Bonnet formula holds; in particular

$$\kappa(\Sigma) = 4{\cdot}\pi$$

for any closed surface Σ.

Further, given a triangle $\triangle$ in a metric space, its model triangle $\tilde\triangle$ is defined as a triangle in the plane with the same side lengths. The angles $\tilde\alpha$, $\tilde\beta$ and $\tilde\gamma$ of the model triangle are called *model angle* of triangle. The comparison theorem states that for any triangle in a surface with non-negative curvature measure its model angles do not exceed the actual angles; that is,

$$\tilde\alpha \leqslant \alpha, \quad \tilde\beta \leqslant \beta, \quad \tilde\gamma \leqslant \gamma.$$

The same holds for the area; that is,

$$\operatorname{area}\tilde{\triangle} \leqslant \operatorname{area}\triangle.$$

Geodesics for birds

Let $\gamma\colon [a,b] \to \mathbb{R}^m$ be a curve. The *total curvature* of γ is defined as the least upper bound of sum of external angles for broken lines inscribed in γ. Namely, it is

$$\sup\left\{\ \sum_{i=1}^{n-1}\alpha_i\ \middle|\ a = t_0 < t_1 < \dots < t_n = b\ \right\},$$

where $\alpha_i = \pi - \measuredangle[\gamma(t_i)\,{}^{\gamma(t_{i-1})}_{\gamma(t_{i+1})}]$.

If γ is smooth and parametrized by the arc length, then its total curvature equals to

$$\int\limits_a^b |\ddot\gamma(t)|\cdot dt.$$

The *geodesics* in the following problem are defined as the curves locally minimizing the length; that is, a sufficiently short arc of the curve containing the given value of parameter is length minimizing.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a ℓ-Lipschitz function. Let $W \subset \mathbb{R}^3$ be the epigraph of f; that is,

$$W = \left\{\ (x,y,z) \in \mathbb{R}^3\ \middle|\ z \geqslant f(x,y)\ \right\}.$$

Equip W with the induced intrinsic metric.

Show that any geodesic in W has total curvature at most $2\cdot\ell$.

Simple geodesic

Let Σ be a complete unbounded convex surface in $\mathbb{R}^3$. Show that there is a two-sided infinite geodesic in Σ with no self-intersections.

Immersed surface

Let Σ be an connected immersed surface in $\mathbb{R}^3$ with strictly positive Gauss curvature and nonempty boundary $\partial\Sigma$. Assume $\partial\Sigma$ lies in a plane Π and whole Σ lies on one side from Π. Prove that Σ is an embedded disc.

Periodic asymptote

🕮 *Let Σ be a closed smooth surface with non-positive curvature and γ be a geodesic in Σ. Assume that γ is not periodic and the curvature of Σ vanish at every point of γ. Show that γ does not have a periodic asymptote; that is, there is no periodic geodesic δ such that the distance from $\gamma(t)$ to δ converges to 0 as $t \to \infty$.*

Saddle surface

Recall that a smooth surface Σ in $\mathbb{R}^3$ is called *saddle* at point p if its principle curvatures at this point have opposite signs. We say that Σ is *saddle* if it saddle at all points.

🕮 *Let Σ be a saddle surface in $\mathbb{R}^3$ homeomorphic to a disc. Assume that orthogonal projection to (x, y)-plane maps the boundary of Σ injectively to convex closed curve. Show that the orthogonal projection to (x, y)-plane is injective on whole Σ.*

In particular, Σ is a graph $z = f(x, y)$ for a function f defined on a convex figure in the (x, y)-plane.

Asymptotic line

The saddle surfaces are defined in the previous problem.

Recall that *asymptotic line* on the smooth surface $\Sigma \subset \mathbb{R}^3$ is a curve always tangent to an *asymptotic direction* of Σ; that is, a direction with vanishing normal curvature.

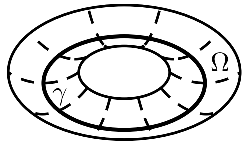
🕮 *Let $\Sigma \subset \mathbb{R}^3$ be the graph $z = f(x, y)$ of smooth function f and γ be a closed smooth asymptotic line in Σ. Assume Σ is saddle in a neighborhood of γ. Show that the projection of γ to the (x, y)-plane cannot be star-shaped.*

A minimal surface

Recall that a smooth surface in $\mathbb{R}^3$ is called *minimal* if its mean curvature vanish at all points. The *mean curvature* is defined as the sum of the principle curvatures at the point.

🕮 *Let Σ be a minimal surface in $\mathbb{R}^3$ which has boundary on a unit sphere. Assume Σ passes through the center of the sphere. Show that the area of Σ is at least π.*

Round gutter*



A round gutter is the surface shown on the picture.

Formally: consider torus T; that is, a surface generated by revolving a circle in $\mathbb{R}^3$ about an axis coplanar with the circle. Let $\gamma \subset T$ be one of the circles in T which locally separates positive and negative curvature on T; a plane containing γ it tangent to T at all points of γ. Let Ω be an neighborhood of γ in T. The surface Ω will be called *round gutter* and the circle γ will be called its *main latitude.*

🕮 *Let $\Omega \subset \mathbb{R}^3$ is a round gutter with main latitude γ. Assume $\iota\colon \Omega \to \mathbb{R}^3$ is a smooth length-preserving embedding which is sufficiently close to the identity. Show that γ and $\iota(\gamma)$ are congruent; that is, there is a motion of $\mathbb{R}^3$ which sends γ to $\iota(\gamma)$*

Non-contractible geodesics

🕮 *Give an example of a non-flat metric on the 2-torus such that it has no contractible geodesics.*

The last problem of Poincaré*

🕮 *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an area preserving homeomorphism such that*

$$f(z) = \left[\begin{array}{lll} z-i & \text{if} & \operatorname{Re}(z) \leqslant -1, \\ z+i & \text{if} & \operatorname{Re}(z) \geqslant 1. \end{array}\right.$$

and $f(z+i) = f(z)+i$ for any $z \in \mathbb{C}$.

Show that f has a fixed point.

Two discs

🕮 *Let Σ_1 and Σ_2 be two smoothly embedded open discs in $\mathbb{R}^3$ which have a common closed smooth curve γ. Show that there is a pair of points $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$ with parallel tangent planes.*

Semisolutions

Unbended geodesic. Denote by W the closed unbounded set formed by Σ and its exterior points. Here is the key observation: *the concate-*

nation of the line segment $[p_t\gamma(t)]$ and the arc $\gamma|_{[t,\ell]}$ forms a minimizing geodesic in the intrinsic metric induced on W. Try to prove it before reading further.

Let Π_t be the tangent plane to Σ at $\gamma(t)$; set $\alpha(t) = p_t$. Note that $\dot\alpha(t)$ is perpendicular to Π_t and it points to the half-space opposite from Σ.

It follows that for any $x \in \Sigma$ the the function

$$t \mapsto |x - p_t| \quad \text{and therefore} \quad t \mapsto |x - p_t|_W$$

are non-decreasing, where $|x - p_t|_W$ stays for the intrinsic distance from x to p_t in W.

On the other hand, by construction

$$|q - p_t|_W \leqslant |q - p|_\Sigma;$$

therefore, from above

$$|q - p_t|_W = |q - p|_\Sigma$$

for any t.

It follows that the concatenation, say γ_t, of the line segment $[p_t\gamma(t)]$ and the arc $\gamma|_{[t,\ell]}$ forms a minimizing geodesic from p_t to q in the intrinsic metric of W.

If q is visible from p_t for some t then the line segment $[qp_t]$ intersects Σ only at q. From above, γ_t coincides with the line segment $[qp_t]$. On the other hand $\gamma(t)$ lies on γ_t, a contradiction. $\square$

This observation was used by Anatoliy Milka to prove a comparison theorem for convex surfaces [see 36].

Long geodesic. By cutting the surface ∂B along a sufficiently long closed simple geodesic, we get two discs with. The key step is to show that each of these discs is long and thin.

Choose one of the discs, say D; equip it with the intrinsic metric further denoted by $|{*} - {*}|_D$.

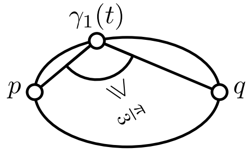
Since ∂B has non-negative curvature in the sense of Alexandrov, so is D. Choose a pair of points p and q on ∂D which maximize the distance. Clearly,

$$|x - p|_D, |x - q|_D \leqslant |p - q|_D$$

for any other point $x \in D$. By comparison,

$$\measuredangle[x\,{}^p_q] \geqslant \tfrac{\pi}{3}. \qquad (*)$$

The points p and q divide ∂D into two arcs, say γ_1 and γ_2; let us parametrize them by arc-length from p to q. Then by $(*)$



$$\tfrac{d}{dt}\left(|x-\gamma_i(t)|_D - |x-\gamma_i(t)|_D\right) \geqslant \tfrac12.$$

In particular

$$|p-q|_D \geqslant \tfrac18\cdot \operatorname{length}[\partial D].$$

That is, if the geodesic was long then D has large diameter.

Choose two points $x\in\gamma_1$ and $y\in\gamma_2$ sufficiently close to p such that $|x-q|_D=|y-q|_D$. By comparison

$$\operatorname{area}\tilde\triangle qxy \leqslant \operatorname{area}\triangle qxy \leqslant \operatorname{area}\partial B.$$

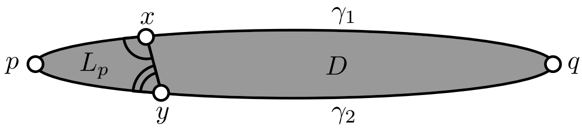
It follows that

$$(**)\qquad \begin{aligned}|x-y|_D &\leqslant \frac{2\cdot\operatorname{area}[\tilde\triangle(xyq)]}{|q-x|_D}\leqslant\\ &\leqslant \frac{100\cdot\operatorname{area}[\partial B]}{\operatorname{length}\partial D}.\end{aligned}$$

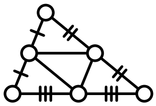
Cut from D the loon L_p with p along a minimizing geodesic $[xy]$. Note that the curvature of L_p is $\alpha+\beta$, where α and β the angles as on the diagram. By comparison these $\alpha\geqslant\tilde\measuredangle(x\,{}^p_y)$ and $\beta\geqslant\tilde\measuredangle(y\,{}^p_x)$. Therefore curvature of L_p is at least $\pi-\tilde\measuredangle(p\,{}^x_y)$. In particular, if $|x-y|_D$ much less then $|p-x|_D+|p-y|_D$ then the curvature of L_p is almost π.

Fix $\varepsilon>0$. By $(*)$, if length$[\partial D]$ is long enough, we can find choose a loop L_p with diameter at most ε, such that curvature L_p is at least $\pi-\varepsilon$.

Using the same construction for p and q in the disc D, and for the other disc, we get four loons in ∂B each of diameter at most ε and each with curvature at least $\pi-\varepsilon$.



By Gauss–Bonnet formula, each lens has total curvature at least $\pi-\varepsilon$ and the total curvature of ∂B is $4\cdot\pi$. Since $\varepsilon>0$ is arbitrary, we get that there are 4 points in Σ, each with curvature π and the remaining part of Σ is flat.



It remains to show that any surface with this property is isometric to the surface of a tetrahedron with equal opposite edges. To do this cut Σ along three geodesics which connect one singular point to the remaining three, develop the obtained flat surface on the plane and think; also look at the diagram. $\square$

The problem was suggested by Arseniy Akopyan.

Geodesics for birds. Consider a geodesic

$$\gamma\colon t\mapsto (x(t),y(t),z(t))$$

in W; assume it is defined in the interval $\mathbb{I}\subset\mathbb{R}$. Let us denote by φ the total curvature; it is a measure on $\mathbb{I}$. We need to estimate $\varphi(\mathbb{I})$.

Denote by $s=s(t)$ the natural parameter of the plane curve

$$t\mapsto (x(t),y(t)).$$

Note that the function $f\colon s\mapsto z$ is concave. Indeed, if this is not the case then one can shorten γ by pushing it up in arbitrary small neighborhood of some interior value t_0 in $\mathbb{I}$. In particular, γ is not locally length minimizing in W, a contradiction.

Given a semi-open interval $\mathbb{J}=(a,b]\subset\mathbb{I}$, set

$$\mu(\mathbb{J})=f^+(a)-f^+(b),$$

where f^+ denotes right derivatives. The function μ extends to a measure which could be also written as

$$\mu=\tfrac{d^2z}{ds^2}\cdot ds.$$

if $\frac{dz^2}{d^2s}$ understood in the sense of distribution.

Note that $|\frac{dz}{ds}|\leqslant\ell$. In particular, $\mu(\mathbb{I})\leqslant 2\cdot\ell$.

Further note that $\varphi\leqslant\sqrt{1+\ell^2}\cdot\mu$. In particular,

$$\varphi(\mathbb{I})\leqslant 2\cdot\ell\cdot\sqrt{1+\ell^2}.$$

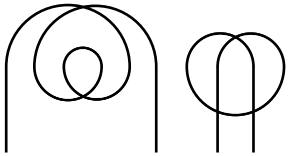
A straightforward improvement of these estimates gives

$$\varphi(\mathbb{I})\leqslant 2\cdot\ell.$$ $\square$

This bound is optimal, check a both side infinite geodesic on the graph of

$$f(x,y)=-\ell\cdot\sqrt{x^2+y^2}.$$

The problem is due to David Berg [see 37] the same bound for convex ℓ-Lipschitz surfaces was proved earlier by Vladimir Usov in [38]. The main observation (the concavity of the function $s\mapsto z$) is called *Liberman's lemma*; it was used yet earlier to bound the total curvature of a geodesic on a convex surface [see 39].



Simple geodesic. Look at two combinatoric types of self intersections shown on the diagram. One of them can and the other can not appear as self intersections of geodesic on an unbounded convex surface. Try to determine which is which before reading further.

Let γ be a two-sided infinite geodesic in Σ. The following is the key statement in the proof.

Claim. *The geodesic γ contains at most one simple loop.*

To prove the claim use the following observations.

◇ The integral curvature ω of Σ cannot exceed $2{\cdot}\pi$.

◇ If φ is the angle at the base of a simple geodesic loop then the integral curvature surrounded by the loop equals to $\pi + \varphi$; in particular there are no obtuse loops.

Once the claim is proved, note that, if a geodesic γ has a self-intersection, then it contains a simple loop. From above there is only one such loop; it cuts a disc from Σ and can go around it either clockwise or counterclockwise. This way we divide all the self-intersecting geodesics into two sets which we will call *clockwise* and *counterclockwise.*

Note that the geodesic $t \mapsto \gamma(t)$ is clockwise if and only if $t \mapsto \gamma(-t)$ is counterclockwise. The sets of clockwise and counterclockwise are open and the space of geodesics is connected. It follows that there are geodesics which are, neither clockwise, nor counterclockwise; by the definition, these geodesics have no self-intersections. □

The problem is due to Stephan Cohn-Vossen, [40, Satz 9]; generalizations were obtained in [41] by Vladimir Streltsov and Alexandr Alexandrov and in [42] by Victor Bangert.

Immersed surface. Let ℓ be a linear function which vanishes on Π and is positive on Σ.

Let z_0 be a point of maximum of ℓ on Σ; set $s_0 = \ell(z_0)$. Given $s < s_0$, denote by Σ_s the connected component of z_0 in $\Sigma \cap \ell^{-1}([s, s_0])$. Note that for all s sufficiently close to s_0 we have

◇ Σ_s is an embedded disc;

◇ $\partial\Sigma_s$ is convex plane curve.

Applying open-closed argument, we get that the same holds for all $s \in [0, s_0)$.

Since Σ is connected, $\Sigma_0 = \Sigma$. Hence the result follows. □

This problem is discussed in the lectures of Mikhael Gromov [see §$\frac{1}{2}$ in 43].

Periodic asymptote. Assume the contrary.

Passing to a finite cover, we can ensure that the asymptote has no self intersections. In this case the restriction $\gamma|_{[a,\infty)}$ has no self-intersections, if a is large enough.

Cut Σ along $\gamma([a,\infty))$ and then cut from the obtained surface an infinite triangle $\triangle$. The triangle $\triangle$ should have two sides formed by both sides of cuts along γ; let us denote these sides of $\triangle$ by γ_- and γ_+. Note that

$$(*) \qquad \operatorname{area}\triangle < \operatorname{area}\Sigma < \infty$$

and both sides $\gamma_\pm$ form infinite minimizing geodesics in $\triangle$.

Consider the Busemann function f for γ_+; denote by $\ell(t)$ the length of the level curve $f^{-1}(t)$. Let $-\kappa(t)$ be the total curvature of the sup-level set $f^{-1}([t,\infty))$. From Gauss–Bonnet formula,

$$(**) \qquad \ell'(t) = \kappa(t).$$

The level curve $f^{-1}(t)$ can be parametrized by a unit-speed curve, say $\theta_t\colon [0,\ell(t)] \to \triangle$. By coarea formula we have

$$\kappa'(t) = -\int\limits_0^{\ell(t)} K_{\theta_t(\tau)}\cdot d\tau,$$

where K_x denotes the Gauss curvature of Σ at the point x. Since $K_{\theta_t(0)} = K_{\theta_t(\ell_t)} = 0$ and the surface is smooth, there is a constant C such that $|K_{\theta_t(\tau)}| \leqslant C\cdot\ell(t)^2$ for all t, τ. Therefore

$$({}^{*}_{**}) \qquad \kappa'(t) \leqslant C\cdot\ell(t)^3$$

Together, $(**)$ and $({}^{*}_{**})$ imply that there is $\varepsilon > 0$ such that

$$\ell(t) \geqslant \frac{\varepsilon}{t-a}$$

for any large t. By the coarea formula we get

$$\operatorname{area}\triangle = \int\limits_a^{\infty} \ell(t) = \infty;$$

the latter contradicts $(*)$. $\square$

I've learned the problem from Dmitri Burago and Sergei Ivanov, it is originated from a discussion with Keith Burns, Michael Brin and Yakov Pesin.

Here is its motivation. Assume Σ be a closed surface with non-positive curvature which is not flat. The space Γ of all unit-speed geodesics $\gamma\colon \mathbb{R} \to \Sigma$ can be identified with the unit tangent bundle $\mathrm{U}\Sigma$. In particular Γ comes with a natural choice of measure. Denote by $\Gamma_0 \subset \Gamma$ the set of geodesics which run in the set of zero curvature all the time. It is expected that Γ_0 has vanishing measure. In all known examples Γ_0 contains only periodic geodesics in only finitely many homotopy classes [read more in 44].

Saddle surface. Denote by Σ° the interior of Σ. Fix a plane Π. Note that the intersection $\Pi \cap \Sigma^\circ$ locally looks like a curve or two curves intersecting transversally; in the latter case Π is tangent to Σ° at the cross-point.

Further note that $\Pi \cap \Sigma^\circ$ has no cycle. Otherwise Σ fails to be saddle at the point of the disc surrounded by the cycle which maximize the distance to Π.

Note that if Σ is not a graph then there is a point $p \in \Sigma$ with vertical tangent plane; denote it by Π. Note that the intersection $\Pi \cap \Sigma$ has cross-point at p.

Since the boundary of Σ projects injectively to a closed convex curve in (x, y)-plane, the intersection of $\Pi \cap \partial\Sigma$ has at most 2 points — these are the only endpoints of $\Pi \cap \Sigma$.

It follows that the connected component of p in $\Pi\cap\Sigma$ is a tree with a vertex of degree 4 at p and at most two end-points, a contradiction. $\square$

The proof above is based of the observation that for any plane Π, each connected component of $\Pi \cap \Sigma$ is simply connected. One can define saddle surfaces as arbitrary (non necessarily smooth) surface which satisfies this condition. The geometry of these surfaces is far from being understood, Samuil Shefel has number of beautiful results about them, see [45] and references there in.

Asymptotic line. Arguing by contradiction, assume that the projection $\bar\gamma$ of γ on (x, y)-plane is star shaped with respect to the origin.

Consider the function

$$h(t) = (d_{\bar\gamma(t)} f)(\gamma(t)).$$

Prove that $h'(t) \neq 0$. In particular $h(t)$ is a strictly monotonic function of $\mathbb{S}^1$, a contradiction. $\square$

The problem is discussed by Dmitri Panov in [46].

A minimal surface. Without loss of generality we may assume that the sphere is centered at the origin of $\mathbb{R}^3$.

Consider the restriction h of the function $x \mapsto |x|^2$ to the surface Σ. Prove that $\Delta_\Sigma h \leqslant 4$ and apply the divergence theorem for $\nabla_\Sigma h$. It follows that the function

$$f\colon r \mapsto \frac{\operatorname{area}(\Sigma \cap B(0,r))}{r^2}$$

is non-decreasing in the interval $(0,1)$. Hence the result follows. □

We described a partial case of so called *monotonicity formula.*

The same argument shows that if 0 is a double point of Σ then $\operatorname{area}\Sigma \geqslant 2{\cdot}\pi$. This observation was used in the proof that the minimal disc bounded by a simple closed curve with total curvature $\leqslant 4{\cdot}\pi$ is necessarily embedded. It was proved by ; an amusing simplification and generalization was obtained by Stephan Stadler. This result also implies that any embedded circle of total curvature at most $4{\cdot}\pi$ is unknot. The latter was proved inependently by István Fáry [in 18] and John Milnor [in 19].

Note that if we assume in addition that the surface is a disc, then the statement holds for any saddle surface. Indeed, denote by S_r the sphere of radius r concentrated with the unit sphere. Then according to the problem “A curve in a sphere” [page 6],

$$\operatorname{length}(\Sigma \cap S_r) \geqslant 2{\cdot}\pi{\cdot}r.$$

Then the coarea formula leads to the solution.

On the other hand there are saddle surfaces homeomorphic to the cylinder that may have arbitrary small area in the ball.

If Σ does not pass through the center and we only know the distance, say r, from the center to Σ, then the optimal bound is $\pi{\cdot}(1-r^2)$. It was conjectured for about 40 years and proved by Simon Brendle and Pei-Ken Hung in [47]; their proof is based on a similar idea and quite elementary. Earlier Herbert Alexander, David Hoffman and Robert Osserman proved it in two cases (1) if Σ is homeomorphic to a disc and (2) for arbitrary area minimizing surfaces, any dimension and codimension [see 48, 49].

Round gutter. Let K be the convex hull of $\Omega' = \iota(\Omega)$. Consider the boundary curve γ' of $\partial K \cap \Omega'$ in Ω'.

First note that the Gauss curvature of Ω' has to vanish at the points of γ'; in other words, $\gamma' = \iota(\gamma)$. Indeed since γ' lies on convex part, the Gauss curvature at the points of γ' has to be non-negative. On the other hand γ' bounds a flat disc in ∂K; therefore its integral intrinsic curvature has to be $2{\cdot}\pi$. If the Gauss curvature is positive at some point of γ', then total intrinsic curvature of γ' has to be $< 2{\cdot}\pi$, a contradiction.

Prove that γ' is an asymptotic line. (Hint: assume that the asymptotic direction goes transversely to $\gamma'(t)$ and conclude $\gamma(t) \notin \partial K$.)

Without loss of generality, we can assume that the length of γ is $2{\cdot}\pi$ and its intrinsic curvature is 1 at all points. Therefore, as the space curve, γ' has to be a curve with constant curvature 1 and it should be closed. Any such curve is congruent to a unit circle. □

It is not known if Ω' is congruent to Ω.

The solution presented above is based on my answer to the question of Joseph O'Rourke [see 50]. Here are some related statements.

- ◇ A half-torus is second order rigid; this was proved by Eduard Rembs in [51], see also [52, p. 135].
- ◇ Any second order rigid surface does not admit analytic deformation [proved by Nikolay Efimov, see 52, p. 121] and for the surfaces of revolution, the assumption of analyticity can be removed [proved by Idzhad Sabitov, see 53].

Non-contractible geodesics. Take a torus of revolution T. It has a family *meridians* — the family of circles which form closed geodesics.

Note that a geodesic on T is either a meridian or it intersects meridians transversally. No closed curve of these types can be contractible. □

I learned this problem from the book of Mikhael Gromov [see 54], where it is attributed to Y. Colin de Verdière. I am not not know any generic metric of that type.

The last problem of Poincaré. Set

$$\begin{aligned} H_+ &= \{\, z \in \mathbb{C} \mid \operatorname{Re}(z) \geqslant 1 \,\}, \\ H_- &= \{\, z \in \mathbb{C} \mid \operatorname{Re}(z) \leqslant -1 \,\}. \end{aligned}$$

Assume f has no fixed points; in other words the image of the map

$$\varphi\colon z \mapsto f(z) - z$$

lies in $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

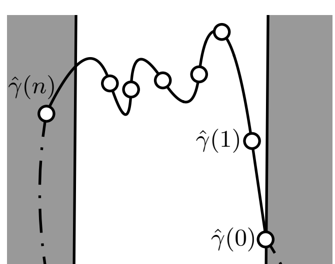
Fix $\varepsilon > 0$ such that $|f(z) - z| > \varepsilon$ for any $z \in \mathbb{C}$. Note that the map

$$\check f\colon z \mapsto f(z) + \varepsilon$$

is area preserving and has no fixed points.

Prove that for some positive integer n, there is a curve

$$\check\gamma\colon [0, n] \to \mathbb{C}$$



which starts in H_-, ends in H_+ and $\check f\circ\check\gamma(t)=\check\gamma(t+1)$ for any $t\in[0,n-1]$.

Repeat the same construction for the function

$$\hat f(z)=f(z)-\varepsilon$$

and obtain a curve

$$\hat\gamma\colon [0,m]\to\mathbb{C}$$

starting in H_+ and ending in H_-.

Connect $\check\gamma(n)$ to $\hat\gamma(0)$ by a curve in H_+ and $\hat\gamma(m)$ to $\check\gamma(0)$ by a curve in H_-. Denote by σ the obtained loop.

Prove that

◇ The loop $\varphi\circ\sigma$ has to be null-homotopic in $\mathbb{C}^*$.

◇ The loop $\varphi\circ\sigma$ is a generator of $\pi_1\mathbb{C}^*$.

These two statements contradict each other. □

The question was asked by Henri Poincaré [see 55] and answered by George Birkhoff in [56].

Two discs. Choose a continuous map $h\colon\Sigma_1\to\Sigma_2$ which is identical on γ. Let us prove that for some $p_1\in\Sigma_1$ and $p_2=h(p_1)\in\Sigma_2$ the tangent plane $\mathrm{T}_{p_1}\Sigma_1$ is parallel to the tangent plane $\mathrm{T}_{p_2}\Sigma_2$; this is stronger than required.

Arguing by contradiction, assume that such point does not exist. Then for each $p\in\Sigma_1$ there is unique line $\ell_p\ni p$ which is parallel to each of the tangent planes $\mathrm{T}_p\Sigma_1$ and $\mathrm{T}_{h(p)}\Sigma_2$.

Note that the lines ℓ_p form a tangent line distribution over Σ_1 and ℓ_p is tangent to γ at any $p\in\gamma$.

Let Δ be the disc in Σ_1 bounded by γ. Consider the doubling of Δ along γ; it is diffeomorphic to $\mathbb{S}^2$. The line distribution ℓ lifts to a line distribution on the doubling; the latter contradicts the hairy ball theorem. □

This proof was suggested nearly simultaneously by Steven Sivek and Damiano Testa [see 57].

Note that the same proof works in case Σ_i are oriented open surfaces such that γ cuts a compact domain in each Σ_i.

There are examples of three disks Σ_1, Σ_2 and Σ_3 with a common closed curve γ such that there no triple of points $p_i\in\Sigma_i$ with parallel tangent planes. Such examples can be found among ruled surfaces [see 58].

Chapter 3

Comparison geometry

In this chapter we consider Riemannian manifolds with curvature bounds.

This chapter is the very demanding; we assume that the reader is familiar with the main definitions in the subject, including Jacobi fields, Shape operator and second fundamental form, equations of Riccati and Jacobi, comparison theorems and Morse theory. The classical book [59] covers all the necessary material.

Geodesic immersion*

Let M be a simply connected positively curved Riemannian manifold and $\iota\colon N \looparrowright M$ be a totally geodesic immersion. Assume that

$$\dim N > \tfrac{1}{2} \cdot \dim M.$$

Prove that ι is an embedding.

Semisolution. Set $n = \dim N$, $m = \dim M$.

Fix a smooth increasing strictly concave function φ. Consider the function $f = \varphi \circ \mathrm{dist}_N$.

Note that if f is smooth at some point $x \in M$ then the Hessian of f at x (briefly $\mathrm{Hess}_x f$) has at least $n+1$ negative eigenvalues.

Moreover, at any point $x \notin \iota(N)$ the same holds in the barrier sense. That is, there is a smooth function h defined on M such that $h(x) = f(x)$, $h(y) \geqslant f(y)$ for any y and $\mathrm{Hess}_x h$ has at least $n+1$ negative eigenvalues.

Use that $m < 2{\cdot}n$ and the described property to prove the following analog of Morse lemma for f.

Claim. *Given $x \notin \iota(N)$ there is a neighborhood $U \ni x$ such that the set*

$$U_- = \{\, z \in U \mid f(z) < f(x) \,\}$$

is simply connected.

Since M is simply connected, any closed curve in $\iota(N)$ can be contracted by a disc, say $s_0 \colon \mathbb{D} \to M$.

Applying the claim, one can construct an f-decreasing homotopy starting at s_0. That is, there is an homotopy $s_t \colon \mathbb{D} \to M$, $t \in [0,1]$ such that $s_t(\partial\mathbb{D}) \subset \iota(N)$ for any t and $s_1(\mathbb{D}) \subset \iota(N)$. It follows that $\iota(N)$ is simply connected.

Finally note that if $\iota \colon N \to M$ has a self-intersection, then the image $\iota(N)$ is not simply connected. Hence the result follows. $\square$

The statement was proved by Fuquan Fang, Sérgio Mendonça and Xiaochun Rong in [60]. The main idea was discovered by Burkhard Wilking [see 61].

Geodesic hypersurface

Recall that a submanifold of a Riemannian manifold is called *totally geodesic* if any geodesic on the submanifold with its induced Riemannian metric is also a geodesic on the ambient Riemannian manifold.

Assume a compact connected positively curved manifold M has a totally geodesic embedded hypersurface. Show that M or its double cover is homeomorphic to the sphere.

If convex, then embedded

Let M be a complete simply connected Riemannian manifold with non-positive curvature and dimension at least 3. Prove that any immersed locally convex compact hypersurface in M is embedded.

Immersed ball*

Prove that any immersed locally convex hypersurface $\iota \colon \Sigma \looparrowright M$ in a compact positively curved manifold M of dimension $m \geqslant 3$, is the boundary of an immersed ball. That is, there is an immersion of a closed ball $f \colon \bar{B}^m \looparrowright M$ and a diffeomorphism $h \colon \Sigma \to \partial\bar{B}^m$ such that $\iota = f \circ h$.

Minimal surface in the sphere

A smooth n-dimensional surface Σ in an m-dimensional Riemannian manifold M is called minimal if it locally minimized the n-dimensional area; that is, sufficiently small regions of Σ do not admit area decreasing deformations with fixed boundary.

The minimal surfaces can be also defined via mean curvature vector as follows. Let $\mathrm{T} = \mathrm{T}\,\Sigma$ and $\mathrm{N} = \mathrm{N}\,\Sigma$ correspondingly tangent and normal bundle. Let s denotes the second fundamental form of Σ; it is a quadratic from on T with values in N, see the remark after problem "Hypercurve" below. Let e_i is an orthonormal basis for T_x, set

$$H_x = \sum_i s(e_i, e_i) \in \mathrm{N}_x.$$

H_x is the mean curvature vector at $x \in \Sigma$. We say that Σ is *minimal* if $H \equiv 0$.

Let Σ be a closed n-dimensional minimal surface in $\mathbb{S}^m$. Prove that $\mathrm{vol}_n\,\Sigma \geqslant \mathrm{vol}_n\,\mathbb{S}^n$.

Hypercurve

The Riemannian curvature tensor R can be viewed as an operator $\mathbf{R}$ on the space of tangent bi-vectors $\bigwedge^2\mathrm{T}$; it is uniquely defined by identity

$$\langle \mathbf{R}(X \wedge Y), V \wedge W\rangle = \langle R(X,Y)V, W\rangle.$$

The operator $\mathbf{R}\colon \bigwedge^2\mathrm{T} \to \bigwedge^2\mathrm{T}$ is called *curvature operator* and it is said to be *positive definite* if $\langle \mathbf{R}(\varphi), \varphi\rangle > 0$ for all non zero bi-vector $\varphi \in \bigwedge^2\mathrm{T}$.

Let $M^m \hookrightarrow \mathbb{R}^{m+2}$ be a closed smooth m-dimensional submanifold and let g be the induced Riemannian metric on M^m. Assume that sectional curvature of g is positive. Prove that the curvature operator of g is positive definite.

The second fundamental form for manifolds of arbitrary codimension which we are about to describe might help to solve this problem.

Assume M is a smooth submanifold in $\mathbb{R}^m$. Given a point $p \in M$ denote by T_p and $\mathrm{N}_p = \mathrm{T}_p^\perp$ the tangent and normal spaces of M at p. The *second fundamental form* of M at p is defined as

$$s(X,Y) = (\nabla_X Y)^\perp,$$

where $(\nabla_X Y)^\perp$ denotes the orthogonal projection of covariant derivative $\nabla_X Y$ onto the normal bundle.

The curvature tensor of M can be found from the second fundamental form using the following formula

$$\langle R(X,Y)Y,X\rangle = \langle s(X,X), s(Y,Y)\rangle - \langle s(X,Y), s(X,Y)\rangle,$$

which is direct generalization of the formula for Gauss curvature of surface.

Horosphere

We say that a Riemannian manifold has negatively pinched sectional curvature, if its sectional curvature at any point in any sectional direction lies in $[-a^2, -b^2]$, for fixed constants $a > b > 0$.

Let M be a complete Riemannian manifold and γ is a ray in M; that is, $\gamma\colon [0,\infty) \to M$ is a minimizing unit-speed geodesic.

The *Busemann function* $b_\gamma\colon M \to \mathbb{R}$ is defined by

$$b_\gamma(p) = \lim_{t\to\infty} \left(|p - \gamma(t)|_M - t\right).$$

From the triangle inequality, the expression under the limit is non-increasing in t; therefore the limit above is defined for any p.

A *horosphere* in M is defined as a level set of a Busemann function in M.

We say that a complete Riemannian manifold M has *polynomial volume growth* if for some (and therefore any) $p \in M$, we have

$$\operatorname{vol} B(p,r)_M \leqslant C\cdot(r^k + 1),$$

where $B(p,r)_M$ denotes the ball in M and C, k are real constants.

Let M be a complete simply connected manifold with negatively pinched sectional curvature and $\Sigma \subset M$ be an horosphere in M. Show that Σ with the induced intrinsic metric has polynomial volume growth.

Minimal spheres

Recall that two subsets A and B in a metric space X are called *equidistant* if the distance function $\operatorname{dist}_A\colon X \to \mathbb{R}$ is constant on B and dist_B is constant on A.

The minimal surfaces are defined on page 32.

Show that a 4-dimensional compact positively curved Riemannian manifold cannot contain infinite number of mutually equidistant minimal 2-spheres.

Positive curvature and symmetry^{+}

Assume $\mathbb{S}^1$ *acts isometrically on a* 4*-dimensional positively curved closed Riemannian manifold. Show that the action has at most* 3 *isolated fixed points.*

The following statement might be useful.

◇ If (M, g) is a Riemannian manifold with sectional curvature $\geqslant 1$ which admits a continuous isometric action of $\mathbb{S}^1$, then $A = (M, g)/\mathbb{S}^1$ is an Alexandrov space with curvature $\geqslant 1$; that is, the conclusion of Toponogov comparison theorem holds in A.

For more on Alexandrov Geometry read our book [62].

Energy minimizer

Let F be a smooth map from a closed Riemannian manifold M to a Riemannian manifold N. Then energy functional of F is defined as

$$E(F) = \int\limits_M |d_x F|^2 \cdot d_x \operatorname{vol}_M .$$

If $(a_{i,j})$ denote the components of the differential $d_x F$ written in the orthonormal bases of the tangent spaces $\mathrm{T}_x M$ and $\mathrm{T}_{F(x)} N$, then

$$|d_x F|^2 = \sum_{i,j} a_{i,j}^2.$$

Show that the identity map on $\mathbb{R}\mathrm{P}^m$ *is energy minimizing in its homotopy class. Here we assume that* $\mathbb{R}\mathrm{P}^m$ *is equipped with canonical metric.*

Curvature vs. injectivity radius^{+}

Let (M, g) *be a closed Riemannian* m*-dimensional manifold. Assume average of sectional curvatures of* (M, g) *is* 1*. Show that the injectivity radius of* (M, g) *is at most* π*.*

A solution use that geodesic flow on the tangent bundle to a Riemannian manifold preserves the volume form; this is a corollary of Liouville's theorem.

Almost flat manifold

Nil-manifolds form the minimal class of manifolds which includes a point, and has the following property: the total space of any principle $\mathbb{S}^1$-bundle over a nil-manifold is a nil-manifold.

The nil-manifolds can be also defined as the quotients of a connected nilpotent Lie group by a lattice.

A compact Riemannian manifold M is called ε-flat if its sectional curvature at all points in all directions lie in the interval $[-\varepsilon, \varepsilon]$.

The main theorem of Gromov in [63], states that for any positive integer n there is $\varepsilon > 0$ such that any ε-flat compact n-dimensional manifold with diameter at most 1 admits a finite cover by a nil-manifold. A more detailed proof can be found in [64] and a more precise statement can be found in [65].

Given $\varepsilon > 0$ construct a compact Riemannian manifold M of sufficiently large dimension which admits a Riemannian metric with diameter $\leqslant 1$ and sectional curvature $|K| < \varepsilon$, but does not admit a finite covering by a nil-manifold.

Approximation of a quotient

Let (M, g) be a compact Riemannian manifold and G is a compact Lie group acting by isometries on (M, g). Construct a sequence of metrics g_n on a fixed manifold N such that (N, g_n) converges to the quotient space $(M, g)/G$ in the sense of Gromov–Hausdorff.

Polar points$^\sharp$

Let M be a compact Riemannian manifold with sectional curvature at least 1 and the dimension at least 2. Prove that for any point $p \in M$ there is a point $p^* \in M$ such that

$$|p - x|_M + |x - p^*|_M \leqslant \pi$$

for any $x \in M$.

Isometric section*

Let M and W be compact Riemannian manifolds, $\dim W > \dim M$ *and $s\colon W \to M$ be a Riemannian submersion. Assume that W has positive sectional curvature. Show that s does not admit an isometric section; that is, there is no isometric embedding $\iota\colon M \hookrightarrow W$ such that $s \circ \iota(p) = p$ for any $p \in M$.*

Warped product

Let (M,g) and (N,h) be Riemannian manifolds and f be a smooth positive function defined on M. Consider the product manifold $W = M\times N$. Given a tangent vector $X\in \mathrm{T}_{(p,q)}W = \mathrm{T}_pM\times \mathrm{T}_pN$ denote by $X_M\in \mathrm{T}M$ and $X_N\in \mathrm{T}N$ its projections. Let us equip W with the Riemannian metric defined as

$$s(X,Y) = g(X_M,Y_M)+f^2\cdot h(X_N,Y_N).$$

The obtained Riemannian manifold (W,s) is called *warped product* of M and N with respect to $f\colon M\to\mathbb{R}$; it can be written as

$$(W,g)=(N,h)\times_f(M,g).$$

🕮 *Assume M is an oriented 3-dimensional Riemannian manifold with positive scalar curvature and $\Sigma\subset M$ is an oriented smooth hypersurface which is area minimizing in its homology class.*

Show that there is a positive smooth function $f\colon\Sigma\to\mathbb{R}$ such that the warped product $\mathbb{S}^1\times_f\Sigma$ has positive scalar curvature; here Σ is equipped with the Riemannian metric induced from M.

No approximation$^\sharp$

🕮 *Prove that if $p\ne 2$, then $\mathbb{R}^m$ equipped with the metric induced by the ℓ^p-norm cannot be a Gromov–Hausdorff limit of m-dimensional Riemannian manifolds (M_n,g_n) with $\mathrm{Ric}_{g_n}\geqslant C$ for some fixed real constant C.*

Area of spheres

🕮 *Let M be a complete non-compact Riemannian manifold with non-negative Ricci curvature and $p\in M$. Then there is $\varepsilon>0$ such that*

$$\operatorname{area}[\partial B(p,r)]>\varepsilon$$

for all sufficiently large r.

Curvature hollow

🕮 *Construct a Riemannian metric on $\mathbb{R}^3$ which is Euclidean outside of an open bounded set Ω and with negative scalar curvature in Ω.*

Flat coordinate planes

Let g be a Riemannian metric on $\mathbb{R}^3$, such that the coordinate planes $x = 0$, $y = 0$ and $z = 0$ are flat and totally geodesic. Assume the sectional curvature of g is either non-negative or non-positive. Show that in both cases g is flat.

Two-convexity$^\sharp$

An open subset V with smooth boundary in the Euclidean space is called *two-convex* if at most one principle curvatures in the outward direction to V is negative.

The two-convexity of V is equivalent to the following property: assume a closed curve γ lies in V and in the plane Π, if γ is contactable in V then it is contactable in $\Pi \cap V$.

Let K be a closed set bounded by a smooth surface in $\mathbb{R}^4$. Assume K contains two coordinate planes

$$\{(x, y, 0, 0) \in \mathbb{R}^4\} \quad \textit{and} \quad \{(0, 0, z, t) \in \mathbb{R}^4\}$$

in its interior and also belongs to the closed 1-neighborhood of these two planes.

Show that the complement of K is not two-convex.

Semisolutions

Geodesic hypersurface. Assume Σ is a totally geodesic embedded hypersurface in M. Without loss of generality, we can assume that Σ is connected.[1]

The complement $M\backslash\Sigma$ has one or two connected components. First let us show that if the number of connected components is two, then M is homeomorphic to sphere.

By cutting M along Σ we get two manifolds, say M_1 and M_2, with geodesic boundaries. Denote by f_1 and f_2 the distance functions to the boundary on M_1 and M_2. Prove that f_i is strictly convex in the interior of M_i.

Smooth the functions f_i keeping them convex; this can be done by applying Greene–Wu Theorem [see Theorem 2 in 66]. After the smoothing, each f_i has single critical point which is its maximum.

Applying Morse lemma, we get that each manifold M_i is homeomorphic to a ball; hence M is homeomorphic to the sphere.

[1]In fact, by Frankel's theorem [see page 3] Σ is connected.

If $M\backslash\Sigma$ is connected, passing to a double cover of M, we reduce the problem to the case which already has been considered. $\square$

The problem was suggested by Peter Petersen.

If convex, then embedded. Observe first that any closed embedded locally convex hypersurface in a non-positively curved simply connected complete manifold bounds a convex region.

Let Σ be an immersed locally convex hypersurface in M. Set

$$m = \dim \Sigma = \dim M - 1$$

Given a point in p on Σ denote by p_r the point on distance r from p which lies on the geodesic starting from p in the outer normal direction to Σ. For fixed $r \geqslant 0$, the points p_r sweep an immersed locally convex hypersurface which we denote by Σ_r.

Fix $z \in \Sigma$. Denote by S_r the sphere of radius r centered at z. Note that S_r is diffeomorphic to m-dimensional sphere.

Denote by d the diameter of Σ. Note that for all $r > 0$ any point on Σ_r lies on a distance at most d from S_r. Conclude that for large r the closest point projection $\varphi_r \colon \Sigma_r \to S_r$ is an immersion.

Since Σ is connected and $m \geqslant 2$, it follows that φ_r is a diffeomorphism for all large r.

By the observation above, Σ_r bounds a convex region for all large r. By an open-closed argument, the same holds for all $r \geqslant 0$. Hence the result follows. $\square$

The problem is due to Stephanie Alexander [see 67].

Immersed ball. Equip Σ with the induced intrinsic metric. Denote by κ the lower bound for principle curvatures of Σ. Note that we can assume that $\kappa > 0$.

Fix sufficiently small $\varepsilon = \varepsilon(M, \kappa) > 0$. Given $p \in \Sigma$ denote by $\Delta(p)$ the ε-ball in Σ centered at p. Consider the lift $\tilde{h}_p \colon \Delta(p) \to \mathrm{T}_{h(p)}$ along the exponential map $\exp_{h(p)} \colon \mathrm{T}_{h(p)} \to M$. More precisely:

1. Connect each point $q \in \Delta(p) \subset \Sigma$ to p by a minimizing geodesic path $\gamma_q \colon [0,1] \to \Sigma$
2. Consider the lifting $\tilde{\gamma}_q$ in $\mathrm{T}_{h(p)}$; that is, the curve such that $\tilde{\gamma}_q(0) = 0$ and $\exp_{h(p)} \circ \tilde{\gamma}_q(t) = \gamma_q(t)$ for any $t \in [0,1]$.
3. Set $\tilde{h}(q) = \tilde{\gamma}_q(1)$.

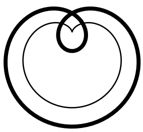
Show that all the hypersurfaces $\tilde{h}_p(\Delta(p)) \subset \mathrm{T}_{h(p)}$ has principle curvatures at least $\frac{\kappa}{2}$.

Use the same idea as in the solution of "Immersed surface" [page 18] to show that one can fix $\varepsilon = \varepsilon(M, \kappa) > 0$ such that the restriction of $\tilde{h}_p|_{\Delta(p)}$ is injective. Conclude that the restriction $h|_{\Delta(p)}$ is injective for any $p \in \Sigma$.

Now consider locally equidistant surfaces Σ_t in the inward direction for small t. The principle curvatures of Σ_t remain at least κ in the barrier sense. By the same argument as above, any ε-ball in Σ_t is embedded.

Applying open-closed argument we get a one parameter family of locally convex locally equidistant surfaces Σ_t for t in the maximal interval $[0, a]$, where the surface Σ_a degenerates to a point, say p.

To construct the immersion $\partial \bar{B}^m \looparrowright M$, take the point p as the image of the center $\bar{B}^m$ and take the surfaces Σ_t as the restrictions of the embedding to the spheres; the existence of the immersion follows from the Morse lemma. $\square$



As you see from the picture, the analogous statement does not hold in the two-dimensional case.

The proof presented above was indicated in the lectures of Mikhael Gromov [see 43] and written rigorously by Jost Eschenburg in [68].

A variation of Gromov's proof was obtained independently by Ben Andrews in [69]. Instead of equidistant deformation, he uses a so called *inverse mean curvature flow*; this way one has to perform some calculations to show that convexity survives in the flow, but one does not have to worry about non-smoothness of the hypersurfaces Σ_t.

Minimal surface in the sphere. Fix a geodesic n-dimensional sphere $\tilde{\Sigma}$ in $\mathbb{S}^m$.

Given $r \in (0, \frac{\pi}{2}]$, denote by U_r and $\tilde{U}_r$ the tubular r-neighborhood of Σ and $\tilde{\Sigma}$ in $\mathbb{S}^m$ correspondingly.

Prove that $U_{\frac{\pi}{2}} \supset \mathbb{S}^m$. Then it follows that

$$(*) \qquad U_{\frac{\pi}{2}} = \tilde{U}_{\frac{\pi}{2}} = \mathbb{S}^m.$$

Prove that for any $x \in \partial U_r$ we have

$$H_r(x) \geqslant \tilde{H}_r,$$

where $H_r(x)$ denotes the mean curvature of ∂U_r at a point x and $\tilde{H}_r$ is the mean curvature of $\partial \tilde{U}_r$, the latter is the same at all points.

Set

$$a(r) = \mathrm{vol}_{m-1}\, \partial U_r, \qquad \tilde{a}(r) = \mathrm{vol}_{m-1}\, \partial \tilde{U}_r,$$
$$v(r) = \mathrm{vol}_m\, U_r, \qquad \tilde{v}(r) = \mathrm{vol}_m\, \tilde{U}_r.$$

by the coarea formula,

$$\tfrac{d}{dr} v(r) \overset{a.e.}{=\!=} a(r), \qquad \tfrac{d}{dr} \tilde{v}(r) = \tilde{a}(r).$$

Note that

$$\begin{aligned}\tfrac{d}{dr}a(r) &\leqslant \int\limits_{\partial U_r} H_r(x)\cdot d_x \operatorname{vol}_{m-1} \leqslant \\ &\leqslant a(r)\cdot\tilde H_r\end{aligned}$$

and

$$\tfrac{d}{dr}\tilde a(r) = \tilde a(r)\cdot\tilde H_r.$$

It follows that

$$\frac{v''(r)}{v(r)} \leqslant \frac{\tilde v''(r)}{\tilde v(r)}$$

for almost all r. Therefore

$$v(r) \leqslant \frac{\operatorname{area}\Sigma}{\operatorname{area}\tilde\Sigma}\cdot\tilde v(r)$$

for any $r > 0$.

According to $(*)$,

$$v(\tfrac{\pi}{2}) = \tilde v(\tfrac{\pi}{2}) = \operatorname{vol}\mathbb{S}^m.$$

Whence the result follows. □

This problem is the geometric part of the isoperimetric inequality proved by Frederick Almgren in [70]. The argument is similar to the proof of isometric inequality for manifolds with positive Ricci curvature given by Mikhael Gromov in [71].

Hypercurve. Fix $p \in M$. Denote by s the second fundamental form of M at p. Recall that s is a symmetric bi-linear form on the tangent space T_pM of M with values in the normal space N_pM to M, see page 3.

Note that the normal space N_pM is two-dimensional.

Prove that if the sectional curvature of M is positive, then

$$(*) \qquad \langle s(X,X), s(Y,Y)\rangle > 0$$

for any pair of nonzero vectors $X, Y \in \mathrm{T}_pM$.

Show that $(*)$ implies that there is an orthonormal basis e_1, e_2 in N_pM such that the real-valued quadratic forms

$$s_1(X,X) = \langle s(X,X), e_1\rangle, \qquad s_2(X,X) = \langle s(X,X), e_2\rangle$$

are positive definite.

Note that the curvature operators R_1 and R_2, defined by the following identity

$$R_i(X \wedge Y), V \wedge W\rangle = s_i(X,W)\cdot s_i(Y,V) - s_i(X,V)\cdot s_i(Y,W),$$

are positive. Finally, note that $R_1 + R_2$ is the curvature operator of M at p. $\square$

The problem is due to Alan Weinstein [see 72]. Note that from [73]/[74] it follows that that the universal cover of M is homeomorphic/diffeomorphic to a standard sphere.

Horosphere. Set $m = \dim \Sigma = \dim M - 1$.

Let $b\colon M \to \mathbb{R}$ be the Busemann function such that $\Sigma = b^{-1}(\{0\})$. Set $\Sigma_r = b^{-1}(\{r\})$, so $\Sigma_0 = \Sigma$.

Let us equip each Σ_r with induced Riemannian metric. Note that all Σ_r have bounded curvature. In particular, the unit ball in Σ_r has volume bounded above by a universal constant, say v_0.

Given $x \in \Sigma$ denote by γ_x the unit-speed geodesic such that $\gamma_x(0) = x$ and $b(\gamma_x(t)) = t$ for any t. Consider the map $\varphi_r\colon \Sigma \to \Sigma_r$ defined as $\varphi_r\colon x \mapsto \gamma_x(r)$.

Notice that φ_r is a bi-Lipschitz map with the Lipschitz constants $e^{a\cdot r}$ and $e^{b\cdot r}$. In particular, the ball of radius R in Σ is mapped by φ_r to a ball of radius $e^{a\cdot r}\cdot R$ in Σ_r. Therefore

$$\mathrm{vol}_m B(x,R)_\Sigma \leqslant e^{m\cdot b\cdot r}\cdot \mathrm{vol}_m B(x, e^{a\cdot r}\cdot R)_{\Sigma_r}$$

for any $R, r > 0$. Taking $e^{a\cdot r}\cdot R = 1$, we get

$$\mathrm{vol}_m B(x,R)_\Sigma \leqslant v_0 \cdot R^{m\cdot \frac{b}{a}}$$

for any $R \geqslant 1$. Hence the statement follows. $\square$

The problem was suggested by Vitali Kapovitch.

There are examples of horospheres as above with degree of polynomial growth higher than m. For example, consider the horosphere Σ in the the complex hyperbolic space of real dimension 4. Clearly $m = \dim \Sigma = 3$, but the degree of its volume growth is 4.

In this case Σ is isometric to the Heisenberg group defined below with a left-invariant metric. It instructive to show that any such metric has volume growth of degree 4.

Heisenberg group is the group of 3×3 upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

under the operation of matrix multiplication.

Minimal spheres. Choose a pair of sufficiently close minimal spheres Σ and Σ', say assume that the distance a between Σ and Σ' is strictly smaller than the injectivity radius of the manifold. Note that in this case there is a bijection $\Sigma \to \Sigma'$, which will be denoted by $p \mapsto p'$ such that the distance $|p - p'| = a$ for any $p \in \Sigma$.

Let $\iota_p \colon \mathrm{T}_p \to \mathrm{T}_{p'}$ be the parallel translation along the (necessary unique) minimizing geodesic from p to p'. Use the hairy ball theorem to show that there is a pair (p, p') such that $\iota_p(\mathrm{T}_p\Sigma) = \mathrm{T}_{p'}\Sigma'$.

Consider pairs of unit-speed geodesics α and α' in Σ and Σ' which start at p and p' correspondingly and go in the parallel directions, say ν and ν'. Set $\ell_\nu(t) = |\alpha(t) - \alpha'(t)|$.

Use the second variation formula together with the lower bound on Ricci curvature to show that $\ell_\nu''(0)$ has negative average for all tangent directions ν to Σ at p. In particular $\ell_\nu''(0) < 0$ for a pair α and α' as above. It follows that there are points $v \in \Sigma$ near p and $v' \in \Sigma'$ near p' such that

$$|v - v'| < |p - p'|,$$

a contradiction. □

Likely, any compact positively curved 4-dimensional manifold cannot contain a pair of equidistant spheres. The argument above implies that the distance between such a pair has to exceed the injectivity radius of the manifold.

The problem was suggested by Dmitri Burago. Here is a short list of classical problems with use second variation formula in similar fission:

- ⋄ Synge's problem [see 75].
  - ∘ *Any compact even-dimensional orientable manifold with positive sectional curvature is simply connected.*
- ⋄ Frankel's problems [see 76].
  - ∘ *Any two compact minimal hypersurfaces in a Riemannian manifold with positive Ricci curvature must intersect.*
  - ∘ *Assume Σ_1 and Σ_2 be two compact geodesic submanifolds in a manifold with positive sectional curvature M and*

    $$\dim \Sigma_1 + \dim \Sigma_2 \geqslant \dim M.$$

    Show that $\Sigma_1 \cap \Sigma_2 \neq \varnothing$.
- ⋄ Bochner's problem [see 77].

- *Let (M,g) be a closed Riemannian manifold with negative Ricci curvature. Prove that (M,g) does not admit an isometric $\mathbb{S}^1$-action.*

The problem "Geodesic immersion" [page 30] can be considered as further development of the idea.

Positive curvature and symmetry. Let M be a 4-dimensional Riemannian manifold with isometric $\mathbb{S}^1$-action. Consider the quotient space $X = M/\mathbb{S}^1$. Note that X is a positively curved 3-dimensional Alexandrov space. In particular the angle $\measuredangle[x\,{}^{y}_{z}]$ between any two geodesics $[xy]$ and $[xz]$ is defined and for any non-degenerate triangle $[xyz]$ formed by the minimizing geodesics $[xy]$, $[yz]$ and $[zx]$ in X we have

$$(*)\qquad \measuredangle[x\,{}^{y}_{z}] + \measuredangle[y\,{}^{z}_{x}] + \measuredangle[z\,{}^{x}_{y}] > \pi.$$

Assume $p \in X$ corresponds to a fixed point of $\mathbb{S}^1$-action. Show that for any three geodesics $[px]$, $[py]$ and $[pz]$ in X we have

$$(**)\qquad \measuredangle[p\,{}^{x}_{y}] + \measuredangle[p\,{}^{y}_{z}] + \measuredangle[p\,{}^{z}_{x}] \leqslant \pi.$$

and

$$({}^{*}_{**})\qquad \measuredangle[p\,{}^{x}_{y}], \measuredangle[p\,{}^{y}_{z}], \measuredangle[p\,{}^{z}_{x}] \leqslant \tfrac{\pi}{2}.$$

Arguing by contradiction, assume that there are 4 fixed points q_1, q_2, q_3 and q_4. Connect each pair $q_i \neq q_j$ by a minimizing geodesic $[q_i q_j]$.

Denote by ω the sum of all 12 angles of the type $\measuredangle[q_i\,{}^{q_j}_{q_k}]$. By $({}^{*}_{**})$, each triangle $[q_i q_j q_k]$ is non-degenerate. Therefore by $(*)$, we have

$$\omega > 4\cdot\pi.$$

Applying $(**)$ at each vertex q_i, we have

$$\omega \leqslant 4\cdot\pi,$$

a contradiction. □

The problem is due to Wu-Yi Hsiang and Bruce Kleiner [see 78]. The connection of this proof to Alexandrov geometry was noticed by Karsten Grove in [79]. An interesting new twist of the idea is given by Karsten Grove and Burkhard Wilking in [80].

Energy minimizer. Denote by $\mathcal{U}$ the unit tangent bundle over $\mathbb{R}\mathrm{P}^m$ and by $\mathcal{L}$ the space of projective lines in $\ell\colon \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^m$. The spaces $\mathcal{U}$ and $\mathcal{L}$ have dimensions $2\cdot m - 1$ and $2\cdot(m-1)$ correspondingly.

According to Liouville's theorem, the identity

$$\int_{\mathcal{U}} f(v)\cdot d_v \operatorname{vol}_{2\cdot m-1} = \int_{\mathcal{L}} d_\ell \operatorname{vol}_{2\cdot(m-1)} \cdot \int_{\mathbb{RP}^1} f(\ell'(t))\cdot dt$$

holds for any integrable function $f\colon \mathcal{U}\to\mathbb{R}$.

Let $F\colon \mathbb{RP}^m \to \mathbb{RP}^m$ be a smooth map. Note that up to a multiplicative constant, the energy of F can be expressed the following way

$$\int_{\mathcal{U}} |dF(v)|^2\cdot d_v \operatorname{vol}_{2m-1} = \int_{\mathcal{L}} d_\ell \operatorname{vol}_{2\cdot(m-1)} \cdot \int_{\mathbb{RP}^1} |[d(F\circ\ell)](t)|^2\cdot dt.$$

The result follows since

$$\int_{\mathbb{RP}^1} |[d(F\circ\ell)](t)|^2\cdot dt \geqslant \pi$$

for any line $\ell\colon \mathbb{RP}^1 \to \mathbb{RP}^m$. □

The problem is due to Christopher Croke [see 81]. He uses the same idea to show that the identity map on $\mathbb{CP}^m$ is energy minimizing in its homotopy class. For $\mathbb{S}^m$, an analogous statement does not hold if $m \geqslant 3$. In fact, if a closed Riemannian manifold M has dimension at least 3 and $\pi_1 M = \pi_2 M = 0$, then the identity map on M is homotopic to a map with arbitrary small energy; the latter was shown by Brian White in [82].

The same idea is used to prove Loewner's inequality on the volume in of $\mathbb{RP}^m$ with metric conformally equivalent to the canonical one [see 83]. Among the other applications, the sharp isoperimetric inequality for 4-dimensional Hadamard manifolds; it was proved by Christopher Croke in [84], see also [85].

Curvature vs. injectivity radius. We will show that if the injectivity radius of the manifold (M,g) is at least π, then the average of sectional curvatures on (M,g) is at most 1. This is equivalent to the problem.

Fix a point $p\in M$ and two orthonormal vectors $U,V\in \mathrm{T}_pM$. Consider the geodesic γ in M such that $\dot\gamma(0)=U$.

Set $U_t = \dot\gamma(t)\in \mathrm{T}_{\gamma(t)}$ and let $V_t\in \mathrm{T}_{\gamma(t)}$ be the parallel translation of $V=V_0$ along γ.

Consider the field $W_t = \sin t\cdot V_t$ on γ. Set

$$\begin{aligned}\gamma_\tau(t) &= \exp_{\gamma(t)}(\tau\cdot W_t),\\ \ell(\tau) &= \operatorname{length}(\gamma_\tau|_{[0,\pi]}),\\ q(U,V) &= \ell''(0).\end{aligned}$$

Note that

$$(*)\qquad q(U,V)=\int\limits_0^\pi[(\cos t)^2-K(U_t,V_t)\cdot(\sin t)^2]\cdot dt,$$

where $K(U,V)$ is the sectional curvature in the direction spanned by U and V.

Since any geodesics of length π is minimizing, we get $q(U,V)\geqslant 0$ for any pair of orthonormal vectors U and V. It follows that average value of the right hand side in $(*)$ is non-negative.

By Liouville's theorem, while taking the average of $(*)$, we can switch the order of integrals; therefore

$$0\leqslant\tfrac{\pi}{2}\cdot(1-\bar K),$$

where $\bar K$ denotes the average of sectional curvatures on (M,g). Hence the result follows. □

The problem illustrates the idea of Eberhard Hopf [see 86] which was developed further by Leon Green in [87]. Hopf used it to show that a metric on torus without conjugate points must be flat and Green showed that average of sectional curvature on closed manifold without conjugate points can not be positive.

More applications of Liouville's theorem discussed in the comments the solution of "Energy minimizer", page 44.

Almost flat manifold. First prove that for given $\varepsilon>0$, there is big enough m and $m\times m$ integer matrix A such that all its eigenvalues are ε-close to 1.

Consider $(m+1)$-dimensional manifold S obtained from $\mathbb{T}^m\times[0,1]$ by gluing $\mathbb{T}^m\times 0$ to $\mathbb{T}^m\times 1$ along the map given by A.

Show that S does not admit a finite cover by a nill-manifold.

Assuming that ε is small, show that S admits a metric with curvature and diameter sufficiently small. □

This example was constructed by Galina Guzhvina in [88].

It is expected that for small enough $\varepsilon>0$, a Riemannian manifold (M,g) of any dimension with $\operatorname{diam}(M,g)\leqslant 1$ and $|K_g|\leqslant\varepsilon$ cannot be simply connected, here K_g denotes the sectional curvature of g.

The latter does not hold with the condition $K_g\leqslant\varepsilon$ instead. In fact, for any $\varepsilon>0$, there is a metric g on $\mathbb{S}^3$ with $K_g\leqslant\varepsilon$ and $\operatorname{diam}(\mathbb{S}^3,g)\leqslant 1$. This example was originally constructed by Mikhael Gromov in [63]; a simplified proof was given by Peter Buser and Detlef Gromoll in [89].

Approximation of a quotient. Note that G admits an embedding into a compact connected Lie group H, say we can assume $H = \mathrm{SO}(n)$, for large enough n.

Fix a $\kappa \leqslant 0$ such that the curvature bound of (M, g) is bounded below by κ.

The bi-invariant metric h on H is non-negatively curved. Therefore for any positive integer n the product $(H, \frac{1}{n}\cdot h) \times (M, g)$ is a Riemannian manifold with curvature bounded below by κ.

The diagonal action of G on $(H, \frac{1}{n}\cdot h)\times(M, g)$ is isometric and free. Therefore the quotient $(H, \frac{1}{n}\cdot h)\times(M, g)/G$ is a Riemannian manifold, say (N, g_n). By O'Nail's formula, (N, g_n) has curvature bounded below by κ.

Clearly, (N, g_n) converge to $(M, g)/G$ as $n \to \infty$. $\square$

This construction is called *Cheeger's trick*, although it was used before Cheeger; say in [90], it was used to show that Berger's spheres have positive curvature. This trick is used to construct most of the known examples of positively and non-negatively curved manifolds [see 91–95].

The quotient space $(M, g)/G$ has finite dimension and curvature bounded below in the sense of Alexandrov. It is expected that not all spaces with this property admit approximation by Riemannian manifolds with curvature bounded below, some partial results are discussed in [96, 97].

Polar points. Fix a unit-speed geodesic γ which starts at p; that is, $\gamma(0) = p$. Set $p^* = \gamma(\pi)$.

Prove that p^* is a solution. $\square$

Alternative proof. Assume the contrary; that is, for any $x \in M$ there is a point x' such that

$$|x - x'|_g + |p - x'|_g > \pi.$$

Show that there is a continuous map $x \mapsto x'$ such that the above inequality holds for any x.

Fix sufficiently small $\varepsilon > 0$. Prove that the set $W_\varepsilon = M\backslash B(p, \varepsilon)$ is homeomorphic to a ball and the map $x \mapsto x'$ sends W_ε into itself.

By Brouwer's fixed-point theorem, $x = x'$ for some x. In this case

$$\begin{aligned}|x - x'|_g + |p - x'|_g = |p - x'|_g &\leqslant \\ &\leqslant \pi,\end{aligned}$$

a contradiction. $\square$

The problem is due to Anatoliy Milka [see 98].

Isometric section. Arguing by contradiction, assume there is an isometric section $\iota\colon M \to W$. It makes possible to treat M as a submanifold in W.

Given $p \in M$, denote by N_p^1 the unit normal space to M at p. Given $v \in \mathrm{N}_p^1$ and real value k, set

$$p^{k\cdot v} = s \circ \exp_p(k\cdot v).$$

Note that

$$(*) \qquad p^{0\cdot v} = p \quad \text{for any} \quad p \in M \quad \text{and} \quad v \in \mathrm{N}_p^1.$$

Fix sufficiently small $\delta > 0$. By Rauch comparison, if $w \in \mathrm{N}_q^1$ is the parallel translation of $v \in \mathrm{N}_q^1$ along a minimizing geodesic from p to q in M, then

$$(**) \qquad |p^{k\cdot v} - q^{k\cdot w}|_M < |p - q|_M$$

assuming $|k| \leqslant \delta$. The same comparison implies that

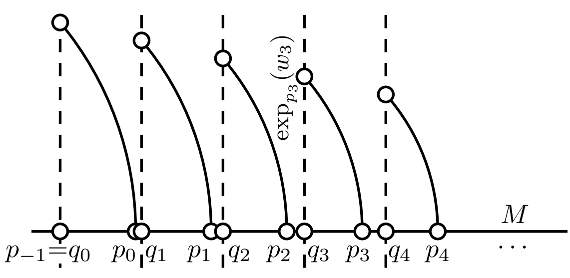
$$({}^{\,*}_{**}) \qquad |p^{k\cdot v} - q^{k'\cdot w}|_M^2 < |p - q|_M^2 + (k - k')^2$$

assuming $|k|, |k'| \leqslant \delta$.

Choose p and $v \in \mathrm{N}_p^1$ so that $r = |p - p^{\delta\cdot v}|$ takes the maximal possible value. From $(**)$ it follows that $r > 0$.

Let γ be the extension of the unit-speed minimizing geodesic from p_v to p; denote by v_t the parallel translation of v to $\gamma(t)$ along γ.

We can choose the parameter of γ so that $p = \gamma(0)$, $p^v = \gamma(-r)$. Set $p_n = \gamma(n\cdot r)$, so $p = p_0$ and $p^v = p_{-1}$. Fix large integer N and set $w_n = (1 - \frac{n}{N})\cdot v_{n\cdot r}$ and $q_n = p_n^{w_n}$.



From $({}^{\,*}_{**})$, there is a constant C independent of N such that

$$|q_k - q_{k+1}| < r + \tfrac{C}{N^2}\cdot\delta^2.$$

Therefore

$$|q_{k+1} - p_{k+1}| > |q_k - p_k| - \tfrac{C}{N^2}\cdot\delta^2.$$

By induction, we get

$$|q_N - p_N| > r - \tfrac{C}{N}\cdot\delta^2.$$

Since N is large we get

$$|q_N - p_N| > 0.$$

By ($*$) we get $q_N = p_N^0 = p_N$, a contradiction. $\square$

This is the core of the solution of Soul conjecture by Grigori Perelman [see 99].

Warped product. Given $x \in \Sigma$, denote by ν_x the normal vector to Σ at x which agrees with the orientations of Σ and M. Denote by κ_x the non-negative principle curvature of Σ at x; since Σ is minimal the other principle curvature has to be $-\kappa_x$.

Consider the warped product $W = \mathbb{S}^1 \times_f \Sigma$ for some positive smooth function $f\colon \Sigma \to \mathbb{R}$. Assume that a point $y \in W$ projects to a point $x \in \Sigma$. Straightforward computations show that

$$\begin{aligned}\mathrm{Sc}_W(y) &= \mathrm{Sc}_\Sigma(x) - 2\cdot\frac{\Delta f(x)}{f(x)} = \\ &= \mathrm{Sc}_M(x) - 2\cdot\mathrm{Ric}(\nu_x) - 2\cdot\kappa_x^2 - 2\cdot\frac{\Delta f(x)}{f(x)}.\end{aligned}$$

Consider linear operator L on the space of smooth functions on Σ defined as

$$(Lf)(x) = -[\mathrm{Ric}(\nu_x) + \kappa_x^2]\cdot f(x) - (\Delta f)(x)$$

It is sufficient to find a smooth function f on Σ such that

$$(*) \qquad f(x) > 0 \quad\text{and}\quad (Lf)(x) \geqslant 0$$

for any $x \in \Sigma$.

Fix a smooth function $f\colon \Sigma \to \mathbb{R}$. Extend the field $f(x)\cdot\nu_x$ on Σ to a smooth field, say v, on whole M. Denote by ι_t the flow along v for time t and set $\Sigma_t = \iota_t(\Sigma)$.

Informal end of proof. Denote by $H_t(x)$ the mean curvature of Σ_t at $\iota_t(x)$. Note that the value $(Lf)(x)$ is the derivative of the function $t \mapsto H_t(x)$ at $t = 0$.

Therefore the condition ($*$) means that we can push Σ into one of its sides so that its mean curvature does not increase in the first order. Since Σ is area minimizing, the existence of such push follows; read further if you are not convinced. $\square$

Formal end of proof. Denote by $\delta(f)$ the second variation of area of Σ_t; that is, consider the area function $a(t) = \operatorname{area}\Sigma_t$ and set $\delta(f) = a''(0)$. Direct calculations show that

$$\begin{aligned}\delta(f) &= \int\limits_\Sigma \left(-[\operatorname{Ric}(\nu_x) + \kappa_x^2]\cdot f^2(x) + |\nabla f(x)|^2\right)\cdot d_x \operatorname{area} = \\ &= \int\limits_\Sigma (Lf)(x)\cdot f(x)\cdot d_x \operatorname{area}.\end{aligned}$$

Since Σ is area minimizing we get

$$(**) \qquad \delta(f) \geqslant 0$$

for any f.

Choose a function f which minimize $\delta(f)$ among all the functions such that $\int_\Sigma f^2(x)\cdot d_x \operatorname{area} = 1$. Note that f an eigenfunction for the linear operator L; in particular f is smooth. Denote by λ the eigenvalue of f; by $(**)$, $\lambda \geqslant 0$.

Show that $f(x) > 0$ at any x. Since $Lf = \lambda\cdot f$, the inequalities $(*)$ follow. □

The problem is due to Mikhael Gromov and Blaine Lawson [see 100]. Earlier, Shing-Tung Yau and Richard Schoen showed that the same assumptions imply existence of conformal factor on Σ which makes it positively curved. Both statement are used to proof that $\mathbb{T}^3$ does not admit a metric with positive scalar curvature; the original proof is given in [101].

Both statements admit straightforward generalization to higher dimensions and they can be used to show non existence metric with positive scalar curvature on $\mathbb{T}^m$ with $m \leqslant 7$. For $m = 8$, the proof stops to work since in this dimension the area minimizing hypersurfaces might have singularities. For example, any domain in the cone in $\mathbb{R}^8$ defined by the identity

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2$$

is area minimizing among the hypersurfaces with the same boundary.

No approximation. Fix an increasing function $\varphi\colon (0, r) \to \mathbb{R}$ such that

$$\varphi'' + (n-1)\cdot(\varphi')^2 + C = 0.$$

Note that if $\operatorname{Ric}_{g_n} \geqslant C$, then the function $x \mapsto \varphi(|p - x|_{g_n})$ is subharmonic. Therefore for arbitrary array of points p_i and positive reals λ_i the function $f_n\colon M_n \to \mathbb{R}$ defined by the formula

$$f(x) = \sum_i \lambda_i\cdot\varphi(|p_i - x|_M)$$

is subharmonic. In particular f_n cannot admit a local minima in M_n.

Passing the limit as $n \to \infty$, we get that any function $f\colon \mathbb{R}^m \to \mathbb{R}$ of the form

$$f(x) = \sum_i \lambda_i \cdot \varphi(|p_i - x|_{\ell_p})$$

does not admit a local minima in $\mathbb{R}^m$.

Arrive to a contradiction by showing that if $p \neq 2$, then there is an array points p_i and positive reals λ_i such that the function

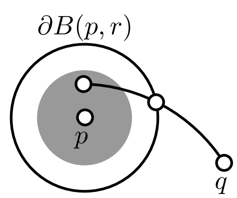
$$f(x) = \sum_i \lambda_i \cdot \varphi(|p_i - x|_{\ell_p})$$

has strict local minimum. □

The argument given here is very close to the proof of Abresch–Gromoll inequality in [102]. An alternative solution of this problem can be build on almost splitting theorem proved by Jeff Cheeger and Tobias Colding in [103].

Area of spheres. Fix $r_0 > 0$. Given $r > r_0$, choose a point q on the distance $2{\cdot}r$ from p.

Note that any minimizing geodesic from q to a point in $B(p, r_0)$ has to cross $\partial B(p, r)$. By volume comparison, we get that



$$\operatorname{vol} B(p, r_0) \leqslant C_m \cdot r_0 \cdot \operatorname{area} \partial B(p, r),$$

where C_m is a constant depending only on the dimension $m = \dim M$; say, $C_m = 10^m$ will do.

□

Applying the coarea formula, we get that volume growth of M is at least linear and in particular it has infinite volume. The latter was proved independently by Eugenio Calabi and Shing-Tung Yau [see 104, 105].

Curvature hollow. Construct a metric that the connected sum $\mathbb{R}^3 \# \mathbb{S}^2 \times \mathbb{S}^1$ which is flat outside a compact set and has non positive scalar curvature. Further, note that such metric can be constructed in such a way that it has a closed geodesic γ with trivial holonomy and with constant negative curvature in its a tubular neighborhood.

Let us cut tubular neighborhood $V \simeq \mathbb{D}^2 \times \mathbb{S}^1$ of γ and glue in the product $W \simeq \mathbb{S}^1 \times \mathbb{D}^2$ with the swapped factors. Note that after this surgery we get back $\mathbb{R}^3$.

It remains to construct a metric g on W with negative scalar curvature which is identical to the original metric on V near its boundary.

The needed space (W,g) can be found among wrap products $\mathbb{S}^1\times_f\mathbb{D}^2$ [see page 36]. $\square$

This construction was given by Joachim Lohkamp in [106], he describes there yet an other equally simple construction. In fact, his constructions produce $\mathbb{S}^1$-invariant hollows with negative Ricci curvature.

On the other hand, there are no hollows with positive scalar curvature and negative sectional curvature. The former is equivalent to the positive mass conjecture [see 107, and the references therein] and the latter is an easy exercise.

Flat coordinate planes. Fix $\varepsilon>0$ such that there is unique geodesic between any two points on distance $<\varepsilon$ from the origin of $\mathbb{R}^3$.

Consider three points a, b and c on the coordinate lines which are ε-close to the origin.

Prove that the angles of the triangle $[abc]$ coincide with its model angles. It follows that there is a flat geodesic triangle in $(\mathbb{R}^3,g)$ with vertex at a, b and c.

Use the family of constructed flat triangles to show that at any x point in the $\frac{\varepsilon}{10}$-neighborhood of the origin the sectional curvature vanish in an open set of sectional directions. The latter implies that the curvature is identically zero in this neighborhood.

Moving the origin and apply the same argument we get that the curvature is identically zero everywhere. $\square$

This problem appears in the paper of Dmitri Panov and me [see 108]; it is based on a lemma discovered by Sergei Buyalo in [Lemma 5.8 in 109].

Two-convexity. *Morse-style solution.* Equip $\mathbb{R}^4$ with coordinates (x,y,z,t).

Consider a generic linear function $\ell\colon\mathbb{R}^4\to\mathbb{R}$ which is close to the sum of coordinates $x+y+z+t$. Note that ℓ has non-degenerate critical points on ∂K and all its critical values are different.

Consider the sets

$$W_s=\left\{\,w\in\mathbb{R}^4\backslash K\;\middle|\;\ell(w)<s\,\right\}.$$

Note that W_{-1000} contains a closed curve, say α, which is contactable in $\mathbb{R}^4\backslash K$, but not constructible in the set W_{-1000}.

Set s_0 to be the infimum of the values s such that the α is contactable in W_s.

Note that s_0 is a critical value of ℓ on ∂K; denote by p_0 the corresponding critical point. By 2-convexity of $\mathbb{R}^4\backslash K$, the index of p_0 has to be at most 1. On the other hand, since the disc hangs at this point, its index has to be at least 2, a contradiction. □

Alexandrov-style proof. Fix a constant metric g on $\mathbb{R}^4$. According to the main result of Alexander Bishop and Berg in [110], $W_g = (\mathbb{R}^4\backslash(\operatorname{Int} K), g)$ has non-positive curvature in the sense of Alexandrov. In particular the universal cover of $\tilde{W}_g$ of W_g is a CAT[0] space.

By rescaling g and passing to the limit we obtain that universal Riemannian cover Z_g of $(\mathbb{R}^4, g)$ branching in the coordinate planes is a CAT[0] space. Show that Z_g is CAT[0] space if and only if the two planes are orthogonal with respect to g; the latter leads to a contradiction. □

The Morse-style proof was essentially given by Mikhael Gromov in [43, §$\frac{1}{2}$], where two-convexity was introduced.

Note that the 1-neighborhood of these two planes has two-convex complement W in the sense of the second definition; that is, if a closed curve γ lies in the plane Π and contactable in W then it is contactable in $\Pi \cap W$. Clearly the boundary of this neighborhood is not smooth and as it follows from above, it cannot be smoothed in the class of two-convex sets.

Yet an other place where two-convexity shows up — the zero curvature set in the manifolds of nonnegative or nonpositive curvature is two-convex [see 108].

Chapter 4

Curvature free differential geometry

The reader should be familiar with the notion of smooth manifold, Riemannian metric and symplectic form.

Besikovitch inequality

Let g be a Riemannian metric on an m-dimensional cube $Q = [0,1]^m$ such that any curve connecting opposite faces has length at least 1. Prove that $\operatorname{vol}(Q,g) \geqslant 1$, and equality holds if and only if (Q,g) is isometric to the unit cube.

Semisolution. Set

$$A_i = \{\, (x_1, x_2, \dots, x_m) \in [0,1]^m \mid x_i = 0 \,\}.$$

Consider functions $f_i\colon [0,1]^m \to \mathbb{R}$ defined by $f_i(x) = \operatorname{dist}_{A_i}(x)$. Note that the map $\boldsymbol{f}\colon ([0,1]^m, g) \to \mathbb{R}^m$ defined as

$$\boldsymbol{f}\colon x \mapsto (f_1(x), f_2(x), \dots, f_m(x))$$

is Lipschitz.

Prove that Jacobian of $\boldsymbol{f}$ is at most 1 and $\boldsymbol{f}([0,1]^m) \supset [0,1]^m$. Therefore

$$\operatorname{vol}(Q,g) \geqslant \operatorname{vol}([0,1]^m) = 1.$$

The equality case is left for the reader. □

The inequality was proved by Abram Besikovitch in [111]. It has number applications in Riemannian geometry. For example using this inequality it is easy to solve the following problem.

◇ *Assume a metric g on $\mathbb{R}^m$ coincides with Euclidean outside of a bounded set K; assume further that any geodesic which comes into K goes out from K the same way as if the metric would be Euclidean everywhere. Show that g is flat.*

Minimal foliation$^+$

The minimal surface in Riemannian manifolds are defined on page 32.

🕮 *Consider $\mathbb{S}^2 \times \mathbb{S}^2$ equipped with a Riemannian metric g which is C^∞-close to the product metric. Prove that there is a conformally equivalent metric $\lambda\cdot g$ and re-parametrization of $\mathbb{S}^2 \times \mathbb{S}^2$ such that each sphere $x \times \mathbb{S}^2$ and $\mathbb{S}^2 \times y$ forms a minimal surface in $(\mathbb{S}^2 \times \mathbb{S}^2, \lambda\cdot g)$.*

The expected solution requires pseudo-holomorphic curves introduced by Mikhael Gromov in [112].

Volume and convexity$^+$

A function f defined on Riemannian manifold is called convex if for any geodesic γ, the composition $f \circ \gamma$ is a convex real-to-real function.

🕮 *Let M be a complete Riemannian manifold which admits a non-constant convex function. Prove that M has an infinite volume.*

The Liouville's theorem should help to solve this problem; it states that geodesic flow on the tangent bundle to a Riemannian manifold preserves the volume form.

Sasaki metric

Let (M, g) be a Riemannian manifold. The Sasaki metric is the most natural choice of metric on the tangent space $\mathrm{T}M$. It is uniquely defined by the following properties:

(i) The natural projection $\tau\colon \mathrm{T}M \to M$ is a Riemannian submersion.

(ii) The metric on each tangent space $\mathrm{T}_p \subset \mathrm{T}M$ is the Euclidean metric induced by g.

(iii) Assume $\gamma(t)$ is a curve in M and $v(t) \in \mathrm{T}_{\gamma(t)}$ is a parallel vector field along γ. Note that $v(t)$ forms a curve in $\mathrm{T}M$. For the Sasaki metric, we have the curve $\dot v(t)$ is perpendicular to the fiber $\mathrm{T}_{\gamma(t)}$ for any t.

A more constructive way to describe Sasaki metric is given by identifying $\mathrm{T}_u[\mathrm{T}M]$ for any $u \in \mathrm{T}_pM$ with the direct sum of so called vertical and horizontal vectors $\mathrm{T}_pM \oplus \mathrm{T}_pM$. The projection of this splitting defined by the differential of τ and the Levi-Civita connection. Then $\mathrm{T}_u[\mathrm{T}M]$ is equipped with the metric defined as

$$\hat{g}(X,Y) = g(X^V,Y^V) + g(X^H,Y^H),$$

where $X^V, X^H \in \mathrm{T}_pM$ denotes the vertical and horizontal components of $X \in \mathrm{T}_u[\mathrm{T}M]$.

Consider the tangent bundle $\mathrm{T}\mathbb{S}^2$ *equipped with Sasaki metric* $\hat{g}$ *induced by a Riemannian metric* g *on* $\mathbb{S}^2$*. Show that, in the sense of Gromov–Hausdorff, the space* $(\mathrm{T}\mathbb{S}^2, \hat{g})$ *lies on bounded distance to the ray* $\mathbb{R}_+$*.*

Distant involution

Construct a Riemannian metric g *on* $\mathbb{S}^3$ *and an involution* $\iota\colon \mathbb{S}^3 \to \mathbb{S}^3$ *such that* $\mathrm{vol}(\mathbb{S}^3, g)$ *is arbitrary small and*

$$|x - \iota(x)|_g > 1$$

for any $x \in \mathbb{S}^3$*.*

Two-systole

Construct a Riemannian metric g *on the 3-dimensional torus* $\mathbb{T}^3$ *such that* $\mathrm{vol}(\mathbb{T}^3, g) = 1$ *and*

$$\operatorname{area} S \geqslant 1000$$

for any closed surface S *in* $\mathbb{T}^3$ *which does not bound.*

Normal exponential map°

Let (M,g) be a Riemannian manifold; denote by $\mathrm{T}M$ the tangent bundle over M and by $\mathrm{T}_p = \mathrm{T}_pM$ the tangent space at point $p \in M$.

Given a vector $v \in \mathrm{T}_pM$ denote by γ_v the geodesic in (M,g) such that $\gamma(0) = p$ and $\gamma'(0) = v$. The map $\exp\colon \mathrm{T}M \to M$ defined by $v \mapsto \gamma_v(1)$ is called exponential map.

The restriction of exp to the T_p is called *exponential map at* p and denoted as $\exp_p$.

Given a smooth submanifold $S \subset M$; denote by $\mathrm{N}S$ the normal bundle over S. The restriction of exp to $\mathrm{N}S$ is called *normal exponential map* of S and denoted as $\exp_S$.

Let M, N *be complete connected Riemannian manifolds. Assume* N *is immersed into* M*. Show that the image of the normal exponential map of* N *is dense in* M*.*

Symplectic squeezing in the torus

Let

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$$

be the standard symplectic form on $\mathbb{R}^4$. *Assume* $\mathbb{Z}^2$ *is the integer lattice in* (x_1, y_1) *coordinate plane of* $\mathbb{R}^4$.

Show that an arbitrary bounded domain $\Omega \subset (\mathbb{R}^4, \omega)$ *admits a symplectic embedding into* $(\mathbb{R}^4, \omega)/\mathbb{Z}^2$.

Diffeomorphism test°

Let M *and* N *be complete* m*-dimensional simply connected Riemannian manifolds. Assume* $f\colon M \to N$ *is a smooth map such that*

$$|df(v)| \geqslant |v|$$

for any tangent vector v *of* M. *Show that* f *is a diffeomorphism.*

Volume of tubular neighborhoods

Assume M *and* M' *be isometric closed smooth submanifolds in* $\mathbb{R}^m$. *Show that for all small* r *we have*

$$\operatorname{vol} B(M, r) = \operatorname{vol} B(M', r),$$

where $B(M, r)$ *denotes the* r*-neighborhood of* M.

Disc*

Given a big real number L, *construct a Riemannian metric* g *on the disc* $\mathbb{D}$ *with*

$$\operatorname{diam}(\mathbb{D}, g) \leqslant 1 \quad \text{and} \quad \operatorname{length} \partial\mathbb{D} \leqslant 1$$

such that any null-homotopy of the boundary in $(\mathbb{D}, g)$ *has a curve of length at least* L.

Shortening homotopy

Let M *be a compact Riemannian manifold with diameter* D. *Assume that for some* $L > D$, *there are no geodesic loops in* M *with length in the interval* $(L - D, L + D]$. *Show that for any path* γ_0 *in* (M, g) *there is a homotopy* γ_t *rel. to the ends such that*

a) $\operatorname{length} \gamma_1 < L$;

b) $\operatorname{length} \gamma_t \leqslant \operatorname{length} \gamma_0 + 2{\cdot}D$ *for any* $t \in [0, 1]$.

Convex hypersurface

Recall that a subset K of Riemannian manifold is called *convex* if every minimizing geodesic connecting two points in K completely lies in K.

Let M be a totally geodesic hypersurface in a closed Riemannian m-dimensional manifold W. Assume that the injectivity radius of M is at least 1 *and it forms a convex set in W. Show that there is a point in W which lies on a distance at least $\frac{1}{2\cdot(m+1)}$ from M.*

Almost constant function

Assume $\varepsilon > 0$ is given. Show that there is a positive integer m such that for any closed m-dimensional Riemannian manifold M and any smooth 1*-Lipschitz function $f\colon M \to \mathbb{R}$ the following holds.*

◇ *For a random unit-speed geodesic γ in M the event*

$$|f\circ\gamma(0) - f\circ\gamma(1)| > \varepsilon$$

happens with probability at most ε.

Here random *means that $\gamma'(0)$ takes the random value in the unit tangent bundle of M for the natural choice of probability distribution.*

Bounded geometry

Denote by $\mathcal{R}$ the space of all Riemannian metrics on $\mathbb{S}^5$ with absolute value of sectional curvature $\leqslant 1$, and injectivity radius $\geqslant 1$.

It is easy to see that any metric $g_1 \in \mathcal{R}$ can be connected to the canonical metric g_0 on $\mathbb{S}^5$ by a continuous family of metrics $g_t \in \mathcal{R}$ where $t \in [0,1]$. In fact, the one parameter family of metrics g_t can be found among the metrics of the type

$$g_t = a(t)\cdot g_0 + b(t)\cdot g_1,$$

where $a, b\colon [0,1] \to \mathbb{R}$ are smooth functions such that $a(0) = 1 = b(1)$ and $a(1-s) = 0 = b(s)$ for $s \leqslant \frac{1}{3}$. In order to keep the bounds on the curvature and injectivity radius, the functions a and b have to take huge values in the middle of interval.

Fix a fast growing function, say

$$\Phi(x) = 1000^{1000\cdot(x+1000)}.$$

Show that there is a metric g_1 such that for any family g_t as above

$$\max_{t\in[0,1]}\{\mathrm{vol}(g_t)\} > \Phi(\mathrm{vol}(g_1)).$$

The expected solution requires Novikov theorem on the algorithmic undecidability of the problem of recognition of the sphere $\mathbb{S}^m$ for $m \geqslant 5$. A detailed proof of this theorem can be found in [113].

Semisolutions

Minimal foliation. First show that there is a self-dual harmonic 2-form on $(\mathbb{S}^2\times\mathbb{S}^2, g)$; that is, a 2-form ω such that $d\omega = 0$ and $\star\omega = \omega$, where $\star$ denotes the Hodge star operator.

Fix $p \in \mathbb{S}^2 \times \mathbb{S}^2$. Use the identity $\star\omega_p = \omega_p$ to show that there is a real number λ_p and the isometry $\mathrm{J}_p\colon \mathrm{T}_p \to \mathrm{T}_p$ such that $\mathrm{J}_p\circ\mathrm{J}_p = -\mathrm{id}$ and $\omega(X,Y) = \lambda_p{\cdot}g(X, \mathrm{J}_pY)$ for any $X, Y \in \mathrm{T}_p$.

Consider the canonical symplectic form ω_0 on $\mathbb{S}^2\times\mathbb{S}^2$; that is, the sum of pullbacks of the volume forms on $\mathbb{S}^2$ for the two projections $\mathbb{S}^2\times\mathbb{S}^2 \to \mathbb{S}^2$. Note that for the canonical metric on $\mathbb{S}^2\times\mathbb{S}^2$, the form ω_0 is harmonic and self-dual. Since g is close to the standard metric, we can assume that ω is close to ω_0. In particular $\lambda_p \neq 0$ for any $p \in \mathbb{S}^2 \times \mathbb{S}^2$.

It follows that ω defines symplectic structure on $\mathbb{S}^2\times\mathbb{S}^2$ and J is its pseudo-complex structure. It remains to take the re-parametrization of $\mathbb{S}^2 \times \mathbb{S}^2$ so that vertical and horizontal spheres will form pseudo-holomorphic curves in the homology classes of $x \times \mathbb{S}^2$ and $\mathbb{S}^2 \times y$. $\square$

For general metric the form ω might vanish at some points; if the metric is generic, then it happens on disjoint circles [see 114].

Volume and convexity. Assume the contrary; that is, there is a complete Riemannian manifold M with finite volume which admits a convex function f.

Denote by $\tau\colon \mathrm{U}M \to M$ the unit tangent bundle over M. Clearly $\mathrm{vol}(\mathrm{U}M)$ is finite.

Note that there is a nonempty bounded open set $\Omega \subset \mathrm{U}M$ such that $df(u) > \varepsilon$ for any $u \in \Omega$ and some fixed $\varepsilon > 0$.

Denote by φ^t the geodesic flow on $\mathrm{U}M$. By Liouville's theorem

$$\mathrm{vol}[\varphi^t(\Omega)] = \mathrm{vol}\,\Omega$$

for any t.

Given $u \in \Omega$, consider the function $h\colon t \mapsto f\circ\tau\circ\varphi^t(u)$. Note that $h'(t) > \varepsilon$ for any $t \geqslant 0$. It follows that there is an infinite sequence of positive reals $t_1, t_2, \ldots$ such that

$$\varphi^{t_i}(\Omega) \cap \varphi^{t_j}(\Omega) = \varnothing$$

if $i \neq j$. The latter implies that $\mathrm{vol}(\mathrm{U}M) = \infty$, a contradiction. $\square$

The idea in the proof is the same as in Poincaré recurrence theorem.

The problem is due to Richard Bishop and Barrett O'Neill [see 115], it was generalized by Shing-Tung Yau in [116].

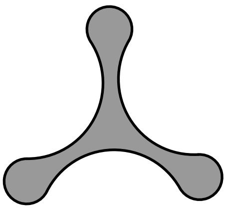
Sasaki metric. Show that there is a constant ℓ such that for any two unit tangent vectors $v \in \mathrm{T}_p\mathbb{S}^2$ and $w \in T_q\mathbb{S}^2$ there is a path $\gamma\colon [0,1] \to \mathbb{S}^2$ from p to q such that

$$\operatorname{length}\gamma \leqslant \ell$$

and w is the parallel transformation of v along γ.

Note that once it is proved, it follows that diameter of the set of all vectors of fixed length in $\mathrm{T}\mathbb{S}^2$ has diameter at most ℓ; in particular the map $\mathrm{T}\mathbb{S}^2 \to [0,\infty)$ defined as $v \mapsto |v|$ preserves the distance with the maximal error ℓ. Hence the result follows.

Distant involution. Given $\varepsilon > 0$, construct a disc Δ in the plane with



$$\operatorname{length}\partial\Delta < 10 \quad \text{and} \quad \operatorname{area}\Delta < \varepsilon$$

which admits an continuous involution ι such that

$$|\iota(x) - x| \geqslant 1$$

for any $x \in \partial\Delta$. An example of Δ can be guessed from the picture.

Take the product $\Delta \times \Delta \subset \mathbb{R}^4$; it is homeomorphic to the 4-ball. Note that

$$\operatorname{vol}_3[\partial(\Delta\times\Delta)] = 2\cdot \operatorname{area}\Delta \cdot \operatorname{length}\partial\Delta < 20\cdot\varepsilon.$$

The boundary $\partial(\Delta\times\Delta)$ homeomorphic to $\mathbb{S}^3$ and the restriction of the involution $(x,y) \mapsto (\iota(x), \iota(y))$ has the needed property.

It remains to smooth the boundary $\partial(\Delta\times\Delta)$. □

This example is given by Christopher Croke in [117].

It is instructive to show that for $\mathbb{S}^2$ such thing is not possible.

By systolic inequality [see 83], the involution ι above cannot be made isometric.

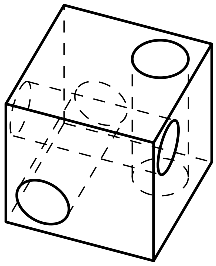
Two-systole. Consider the unit cube with three not intersecting cylindrical tunnels between the pairs of opposite faces.

In each tunnel, shrink the metric long-wise and expand it cross-wise while keeping the volume the same.

More precisely, if (x,y,z) is the coordinate system on the cylindrical tunnel $\mathbb{D}\times[0,1]$ so that (x,y)-plane is parallel to the base then the new metric is

$$g = \varphi\cdot(dx)^2 + \varphi\cdot(dy)^2 + \frac{1}{\varphi^2}\cdot(dz)^2,$$

where $\varphi = \varphi(x, y)$ is a positive smooth function on $\mathbb{D}$ which takes huge values around the center and equals to 1 near the boundary of $\mathbb{D}$.



Gluing the opposite faces of the cube, we obtain a 3-dimensional torus with a smooth Riemannian metric. It remains to show that for the right choice of function φ, this construction gives the needed example. $\square$

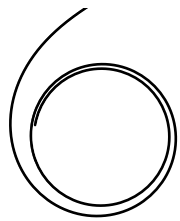
I learned this problem from Dmirti Burago.

Normal exponential map. Assume the contrary; that is, there is a point $p \in M$ such that the image of normal exponential map to N does not touch ε-neighborhood of p.

Show that given $R > 0$ there is $\delta > 0$ such that if $x \in N$ and $|p - x|_M < R$, then there is a unit-speed curve in N which moves to p with velocity at least δ. (In fact, the value δ depends on ε, R and the curvature bounds in $B(p, R)$.)

Following this curve for sufficient time brings us to p; that is, $p \in N$, a contradiction. $\square$

The problem was suggested by Alexander Lytchak.



From the picture, you should guess an example of immersion such that one point does not lie in the image of the corresponding normal exponential. It might be interesting to see in more details which sets can be avoided by such images.

Symplectic squeezing in the torus. The embedding will be given as a composition of a linear symplectomorphism λ with the quotient map $\varphi\colon \mathbb{R}^4 \to \mathbb{T}^2 \times \mathbb{R}^2$ by the integer (x_1, y_1)-lattice. The composition $\varphi \circ \lambda$ always preserves the symplectic structure; it remains to find λ such that the restriction $\varphi \circ \lambda|_\Omega$ is injective.

Without loss of generality, we can assume that Ω is a ball centered at the origin. Choose an oriented 2-dimensional subspace V subspace of $\mathbb{R}^4$ such that the integral of ω over $\Omega \cap V$ is small positive number, say $\frac{\pi}{4}$.

Note that there is a linear symplectomorphism λ which maps planes parallel to V to planes parallel to the (x_1, y_1)-plane, and that maps the disk $V \cap \Omega$ to a disk. It follows that the intersection of $\lambda(\Omega)$ with any plane parallel to the (x_1, y_1)-plane is a disk of radius at most $\frac{1}{2}$. In particular $\varphi \circ \lambda|_\Omega$ is injective. $\square$

This construction is given by Larry Guth in [118] and attributed to Leonid Polterovich.

Note that according to the Gromov's non-squeezing theorem [see 112], an analogous statement with $\mathbb{C}\times\mathbb{D}$ as the target does not hold, here $\mathbb{D}\subset\mathbb{C}$ is the open disc with the induced symplectic structure. In particular, it shows that the projection of $\lambda(\Omega)$ as above to (x_1,y_1)-plane cannot be made arbitrary small.

Diffeomorphism test. Since N is simply connected, it is sufficient to show that $f\colon M\to N$ is a covering map.

Note that f is an open immersion. Let h be the pullback metric on M for $f\colon M\to N$. Clearly $h\geqslant g$. In particular (M,h) is complete and the map $f\colon (M,h)\to N$ is a local isometry.

It remains to prove that any local isometry between complete connected Riemannian manifolds of the same dimension if a covering map. □

Volume of tubular neighborhoods. Let us denote by $\mathrm{N}M$ and $\mathrm{T}M$ the normal and tangent bundle of M in $\mathbb{R}^m$.

Consider the normal exponential map $\exp_M\colon \mathrm{N}M\to\mathbb{R}^m$ [defined on page 55] and denote by J_V its Jacobian at $V\in\mathrm{N}_p=\mathrm{N}_pM$. Note that for all small $r>0$, we have

$$(*)\qquad \operatorname{vol}B(M,r)=\int\limits_M d_p\operatorname{vol}_m\cdot\int\limits_{B(0,r)_{\mathrm{N}_p}} J_V\cdot d_V\operatorname{vol}_{n-m},$$

where $B(0,r)_{\mathrm{N}_p}$ denotes the ball in N_p

Set $m=\dim M$. Given $p\in M$, denote by $s_p\colon \mathrm{T}_p\times\mathrm{T}_p\to\mathrm{N}_p$ the second fundamental form of M [defined on page 32]. Recall that the curvature tensor of M at p can be expressed the following way

$$R_p(X\wedge Y),V\wedge W\rangle=\langle s_p(X,W),s_p(Y,V)\rangle-\langle s_p(X,V),s_p(Y,W)\rangle.$$

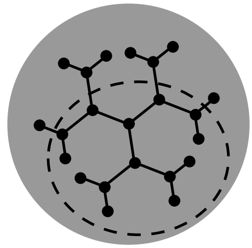
Given $V\in\mathrm{N}_pM$, express J_V in terms of $|V|$, the dimension m and the real-valued quadratic form $s_V(X,Y)=\langle s_p(X,Y),V\rangle$. Show that for small r the integral

$$v(r)=\int\limits_{B(0,r)_{\mathrm{N}_pM}} J_V\cdot d_V\operatorname{vol}_{n-m}$$

is a polynomial of r and its coefficients can be expressed in terms of the curvature tensor R_p.

It follows that the right hand side in $(*)$ can be expressed in terms of curvature tensor of M. The problem follows since the curvature tensor can be expressed in terms of metric tensor of M. □

The formula for volume of tubular neighborhood was given by Hermann Weyl in [119].



Disc. Show that given a positive integer n one can construct a tree T embedded into the disc such that any homotopy of the boundary of the disc to a point pass through a curve which intersects n different edges. (For the tree on the diagram $n = 3$.)

Fix small $\varepsilon > 0$, say $\varepsilon = \frac{1}{10}$. Consider the disc with embedded tree T as above. We will construct a metric on the disc with diameter and length of its boundary below 1 such that the distance between any two edges of T of without common vertex is at least ε.

To construct such a metric, fix a metric on the cylinder $\mathbb{S}^1 \times [0, 1]$ such that

- ◇ The ε-neighborhoods of the boundary components have product metrics.
- ◇ Any vertical segment $x \times [0, 1]$ has length $\frac{1}{2}$.
- ◇ One of the boundary component has length ε.
- ◇ The other boundary component has length $2{\cdot}m{\cdot}\varepsilon$, where m is the number of edges in the tree T.

Equip T with a length-metric so that each edge has length ε and glue the long boundary component of the cylinder to T by piecewise isometry so that the resulting space is homeomorphic to disc and the tree corresponds to it-self.

According to the first construction, for any null-homotopy of the boundary the least length is at least $n{\cdot}\frac{\varepsilon}{10}$. The obtained metric is not Riemannian, but is easy to smooth. Since n is arbitrary the result follows. □

This example was constructed by Sidney Frankel and Mikhail Katz in [120].

Shortening homotopy. Set

$$p = \gamma_0(0) \quad \text{and} \quad \ell_0 = \operatorname{length} \gamma_0.$$

By compactness argument, there exists $\delta > 0$ such that no geodesic loops based at p with has length in the interval $(L - D, L + D + \delta]$.

Assume $\ell_0 \geqslant L + \delta$. Choose $t_0 \in [0, 1]$ such that

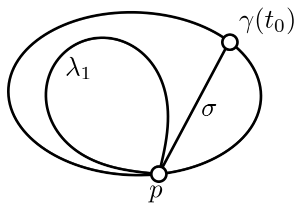
$$\operatorname{length}\left(\gamma_0|_{[0,t_0]}\right) = L + \delta$$

Let σ be a minimizing geodesic from $\gamma(t_0)$ to p. Note that γ_0 is homotopic to the concatenation

$$\gamma_0' = \gamma_0|_{[0,t_0]} * \sigma * \bar\sigma * \gamma|_{[t_0,1]},$$

where $\bar\sigma$ denotes the backward parametrization of σ.

Consider the loop λ_0 at p formed by the concatenation of $\gamma|_{[0,t_0]}$ and σ. Applying a curve shortening process to λ_0, we get a homotopy λ_t rel. its ends from the loop λ_0 to a geodesic loop λ_1 at p. From above,



$$\operatorname{length}\lambda_1 \leqslant L - D.$$

The concatenation $\gamma_t = \lambda_t * \bar\sigma * \gamma|_{[t_0,1]}$ is a homotopy from γ_0' to an other curve γ_1. From the construction it is clear that

$$\begin{aligned}\operatorname{length}\gamma_t &\leqslant \operatorname{length}\gamma_0 + 2\cdot\operatorname{length}\sigma \leqslant\\ &\leqslant \operatorname{length}\gamma_0 + 2\cdot D\end{aligned}$$

for any $t \in [0,1]$ and

$$\begin{aligned}\operatorname{length}\gamma_1 &= \operatorname{length}\lambda_1 + \operatorname{length}\sigma + \operatorname{length}\gamma|_{[t_0,1]} \leqslant\\ &\leqslant L - D + D + \operatorname{length}\gamma - (L+\delta) =\\ &= \ell_0 - \delta.\end{aligned}$$

Repeating the procedure few times we get we get curves $\gamma_2, \gamma_3, \dots, \gamma_n$ connected by the needed homotopies so that $\ell_{i+1} \leqslant \ell_i - \delta$ and $\ell_n < L + \delta$, where $\ell_i = \operatorname{length}\gamma_i$.

If $\ell_n \leqslant L$, we are done. Otherwise repeat the argument once more for $\delta' = \ell_n - L$. □

The problem is due to Alexander Nabutovsky and Regina Rotman [see 121].

It is not at all easy to find an example of a manifold which satisfy the above condition for some L; they are found among the Zoll spheres by Florent Balachev, Christopher Croke and Mikhail Katz [see 122].

Convex hypersurface. Let us define the *cone construction*. Let Δ be a simplex and Δ_v facet of Δ opposite to the vertex v. Assume $f_v\colon \Delta_v \to M$ is a map and $x \in M$ such that $f_v(\Delta_v) \subset B(x,1)$. Given $w \in \Delta_v$ denote by γ_w the (necessary unique) minimizing geodesic path from x to $f_v(w)$. Then the map $f\colon \Delta \to M$ defined as

$$f\colon (1-t)\cdot v + t\cdot w \mapsto \gamma_w(t)$$

is called *cone over f_v* with vertex x.

Let h be the maximal distance from points in W to M.

Fix a fine triangulation of W so that M becomes a sub-complex. Say, let us assume that the diameter of each simplex in τ is less than ε. We can assume that τ is a barycentric subdivision of an other triangulation, so all the vertices of τ can be colored into colors $(0, \dots, m+1)$

in such a way that the vertices of each simplex get different colors. Denote by τ_i the maximal i-dimensional sub-complex of τ with all the vertices colored by $0, \dots, i$.

For each vertex v in τ choose a point $v' \in M$ on the distance $\leqslant h$. Note that if v and w are the vertices of one simplex, then

$$|v' - w'|_M < 2\cdot h + \varepsilon.$$

If $\frac{1}{2\cdot(m+1)} > h$, fix $\varepsilon < \frac{1}{2\cdot(m+1)} - h$. Let us extend the map $v \mapsto v'$ to a continuous map $W \to M$. The map is already defined on τ_0. Using the cone construction we can extend it to τ_1; we can do this since the distance between vertices in one simplex are below injectivity radius of M. Repeat the cone construction recursively, to extend the map to $\tau_2, \dots, \tau_{m+1} = \tau$; the distance estimates are needed in this construction.

It follows that fundamental class of M vanish in the homology ring of M, a contradiction. $\square$

This problem is a stripped version of the bound on filling radius given by Mikhael Gromov in [83].

Almost constant function. Given a positive integer m, denote by δ_m the expected value $|x_1|$ of a unit vector $\boldsymbol{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ with respect to the uniform distribution.

Observe that $\delta_m \to 0$ as $m \to \infty$.

Equip the unit tangent bundle $\mathrm{U}M$ of M with the natural probability measure. Since f is 1-Lipschitz, for a random vector v in $\mathrm{U}M$, the expected value of $|df(w)|$ is at most δ_m.

Note that

$$\begin{aligned}|f\circ\gamma(1) - f\circ\gamma(0)| &= \left|\int_0^1 df(\dot\gamma(t))\cdot dt\right| \leqslant \\ &\leqslant \int_0^1 |df(\dot\gamma(t))|\cdot dt.\end{aligned}$$

Assume $\dot\gamma(0)$ takes random value in $\mathrm{U}M$. By Liouville's theorem, the same holds for $\dot\gamma(t)$ for any fixed t. Therefore the expected value of

$$\int_0^1 |df(\dot\gamma(t))|\cdot dt$$

is at most δ_m.

By Markov's inequality, the probability of the event

$$|f\circ\gamma(1) - f\circ\gamma(0)| > \varepsilon$$

is at most $\frac{\delta_m}{\varepsilon}$. Hence the result follows. $\square$

I learned the problem from Mikhael Gromov. It gives an example in the Riemannian world of so called *concentration of measure phenomenon* [see 123, 124].

Bounded geometry. Show that there is an algorithm to estimate the Gromov–Hausdorff distance between two Riemannian manifolds given in any reasonable way.

Show that if two manifolds with bounded curvature are sufficiently close to each other then they are diffeomorphic.

Now assume contrary; that is, for any metric $g_1 \in \mathcal{R}$ there is a path g_t in $\mathcal{R}$ connecting the canonical metric g_0 to g_1 such that

$$\operatorname{vol} g_t \leqslant \Phi(\operatorname{vol} g_1).$$

If a 5-dimensional Riemannian manifold M with curvature between ± 1 is diffeomorphic to $\mathbb{S}^5$ then it can be described by a metric g_1 in $\mathcal{R}$. Let g_t be the path as above.

Construct a finite set $F \subset \mathcal{R}$ which is sufficiently dense in the set of metrics in $\mathcal{R}$ with volume at most $\Phi(\operatorname{vol} g_1)$. The path g_t as above can be approximated by a sequence of metrics from F.

Using the algorithms above one can list all such sequences. It implies existence of algorithm which recognize $\mathbb{S}^5$ among 5-dimensional manifolds. The later contradicts Novikov theorem. $\square$

Instead of function Φ one can take any Turing computable function. The problem and number of its generalizations are due to Alexander Nabutovsky [see 125].

Chapter 5

Metric geometry

In this chapter, we consider metric spaces. All the necessary material could be found in the first three chapters of the textbook [126].

Let us fix few standard notations.

◇ The distance between two points x and y in a metric space X will be denoted as
$$|x-y|_X.$$

◇ A metric space X is called *length-metric space* if for any $\varepsilon > 0$ two points $x, y \in X$ can be connected by curve α such that
$$\operatorname{length}\alpha < |x-y|_X + \varepsilon.$$
In this case the metric on X is called *length-metric.*

Embedding of a compact

Prove that any compact metric space is isometric to a subset of a compact length-metric spaces.

Semisolution. Let K be a compact metric space. Denote by $B(K)$ the space of bounded functions on K equipped with sup-norm; that is,
$$|f| = \sup_{x\in K} |f(x)|.$$

Note that the map $\varphi\colon K \to B(K)$, defied by $x \mapsto \operatorname{dist}_x$ is a distance preserving embedding.

Denote by W the linear convex hull of the image $\varphi(K) \subset B(K)$ with the metric induced from $B(K)$. It remains to show that W forms a compact length-metric space. □

The map φ is called *Kuratowski embedding*, it was constructed in [127], although essentially the same map was described by Maurice Fréchet in the same paper he introduced metric spaces [see 128].

Non-contracting map°

Let K be a compact metric space and

$$f\colon K \to K$$

be a non-contracting map. Prove that f is an isometry.

Horocompactification°

Let X be a metric space. Denote by $C(X,\mathbb{R})$ the space of continuous real-valued functions equipped with the compact-open topology.

Fix a point $z_0 \in X$. Given a point $z \in X$, let $f_z \in C(X,\mathbb{R})$ be the function defined as

$$f_z(x) = |z - x|_X - |z - x_0|_X.$$

Let $F_X\colon X \to C(X,\mathbb{R})$ be the map defined as $F_X\colon z \mapsto f_z$.

Denote by $\bar{X}$ the closure of $F_X(X)$ in $C(X,\mathbb{R})$; note that $\bar{X}$ is compact. That is, if F_X is an embedding, then $\bar{X}$ forms a compactification of X, which is called *horocompactification*. The complement $\partial_\infty X = \bar{X} \backslash F_X(X)$ is called *horoabsolute* of X.

This construction was introduced by Mikhael Gromov in [129].

Construct a proper metric space X such that F_X is not an embedding. Show that there are no such examples among proper length-metric spaces.

A ball and a sphere

Construct a sequence of Riemannian metrics on $\mathbb{S}^3$ which converges in the sense of Gromov–Hausdorff to the unit ball in $\mathbb{R}^3$.

Macro-dimension°

Let X be a locally compact metric space, m is an integer and $a > 0$. We say that the macro-dimension of X at the scale a is at most m if there is a continuous map f from X to an m-dimensional simplicial complex K such that for any $k \in K$ the inverse image $f^{-1}(\{k\})$ has diameter less than a.

If macro-dimension of X at the scale a is at most m, but not at most $m-1$, we say that m is the *macro-dimension* of X at the scale a.

Equivalently, the macro-dimension of X on scale a can be defined as the least integer m such that X admits an open covering with diameter of each set less than a and such that each point in X is covered by at most $m+1$ sets in the cover.

Let M be a simply connected Riemannian manifold with the following property: any closed curve is null-homotopic in its own 1-neighborhood. Prove that the macro-dimension of M on the scale 100 *is at most* 1.

No Lipschitz embedding*

Construct a length-metric d on $\mathbb{R}^3$, such that for any open set $U \subset \mathbb{R}^3$, there is no Lipschitz embeddings $(U,d) \to \mathbb{R}^3$, where $\mathbb{R}^3$ equipped with the canonical metric.

Sub-Riemannian sphere+

Let us define sub-Riemannian metric.

Fix a Riemannian manifold (M,g). Assume that in the tangent bundle $\mathrm{T}M$ a choice of sub-bundle H is given; the sub-bundle H which will be called *horizontal distribution*. The tangent vectors which lie in H will be called *horizontal*. A piecewise smooth curve will be called *horizontal* if all its tangent vectors are horizontal.

The sub-Riemannian distance between points x and y is defined as infimum of lengths of horizontal curves connecting x to y.

Alternatively, the distance can be defined as a limit of Riemannian distances for the metrics

$$g_\lambda(X,Y) = g(X^H,Y^H) + \lambda{\cdot}g(X^V,Y^V)$$

as $\lambda \to \infty$, where X^H denotes the horizontal part of X; that is, the orthogonal projection of X to H and X^V denotes the vertical part of X; so, $X^V + X^H = X$.

One usually adds a condition which ensure that any curve in M can be arbitrary well approximated by a horizontal curve with the same endpoints. (In particular this ensures that the distance will not take infinite values.) The most common condition is so called *complete non-integrability*; it means that for any $x \in M$, one can choose a basis in T_xM from the vectors of the following type: $A(x)$,

$[A,B](x)$, $[A,[B,C]](x)$, $[A,[B,[C,D]]](x), \dots$ where all vector fields $A, B, C, D, \dots$ are horizontal.

🕮 *Prove that any sub-Riemannian metric on the $\mathbb{S}^m$ is isometric to the intrinsic metric of a hypersurface in $\mathbb{R}^{m+1}$.*

It will be hard to solve the problem without knowing proof of Nash–Kuiper theorem on length preserving C^1-embeddings. The original papers of John Nash and Nicolaas Kuiper [see 130, 131] are very readable.

Length-preserving map$^+$

A continuous map $f\colon X \to Y$ between metric spaces is called *length-preserving* if it preserves the length of curves; that is, for any curve α in X we have

$$\operatorname{length}(f \circ \alpha) = \operatorname{length} \alpha.$$

🕮 *Show that there is no length-preserving map $\mathbb{R}^2 \to \mathbb{R}$.*

The expected solution use Rademacher's theorem on differentiability of Lipschitz functions [see 132].

Fixed segment

🕮 *Let $\rho(x,y) = \|x - y\|$ be a metric on $\mathbb{R}^m$ induced by a norm $\|{*}\|$.*

Assume that $f\colon (\mathbb{R}^m, \rho) \to (\mathbb{R}^m, \rho)$ is an isometry which fixes two distinct points. Show that f fixes the line segment between them.

Pogorelov's construction$^\circ$

🕮 *Let μ be a regular centrally symmetric finite measure on $\mathbb{S}^2$ which is positive on every open set. Given two points $x, y \in \mathbb{S}^2$, set*

$$\rho(x,y) = \mu[B(x, \tfrac{\pi}{2}) \backslash B(y, \tfrac{\pi}{2})].$$

Show that ρ is a length-metric on $\mathbb{S}^2$ and moreover, geodesics in this metric formed by arcs of great circles.

Straight geodesics

🕮 *Let ρ be a length-metric on $\mathbb{R}^m$, which is bi-Lipschitz equivalent to the canonical metric. Assume that every geodesic γ in $(\mathbb{R}^d, \rho)$ is linear (that is, $\gamma(t) = v + w{\cdot}t$ for some $v, w \in \mathbb{R}^m$). Show that ρ is induced by a norm on $\mathbb{R}^m$.*

Hyperbolic space

A map $f\colon X\to Y$ between metric spaces is called a *quasi-isometry* if there is a positive real constant C such that $f(X)$ is a C-net in Y and

$$\tfrac{1}{C}\cdot|x-y|_X-C\leqslant|f(x)-f(y)|_Y\leqslant C\cdot|x-y|_X+C.$$

Note that a quasi-isometry is not assumed to be continuous, for example any map between compact metric spaces is a quasi-isometry.

Construct a quasi-isometry from the hyperbolic 3*-space to a subset of the product of two hyperbolic planes.*

A homeomorphism near quasi-isometry^{+}

The quasi-isometry is defined few lines above.

Let $f\colon\mathbb{R}^m\to\mathbb{R}^m$ be a quasi-isometry. Show that there is a (bi-Lipschitz) homeomorphism $h\colon\mathbb{R}^m\to\mathbb{R}^m$ on a bounded distance from f; that is, there is

$$|f(x)-h(x)|\leqslant C$$

for any $x\in\mathbb{R}^m$ and a real constant C.

The expected solution requires a corollary of the theorem of Laurence Siebenmann [133]. It states that if $V_1,V_2\subset\mathbb{R}^m$ are open and the two embedding $f_1\colon V_1\to\mathbb{R}^m$ and $f_2\colon V_2\to\mathbb{R}^m$ are sufficiently close to each other on the overlap $U=V_1\cap V_2$, then there is an embedding f defined on an open set W' which is slightly smaller than $W=V_1\cup V_2$ and such that f is sufficiently close to each f_1 and f_2 at the points where they are defined.

The bi-Lipschitz version requires an analogous statement in the category of bi-Lipschitz embeddings; it was proved by Dennis Sullivan [see 134]. This result can be used to prove that the homeomorphism is bi-Lipschitz.

A family of sets with no section

Construct a family of closed sets $C_t\subset\mathbb{S}^1$, $t\in[0,1]$ which is continuous in the Hausdorff topology, but does not admit a section. *That is, there is no continuous map $c\colon[0,1]\to\mathbb{S}^1$ such that $c(t)\in C_t$ for any t.*

Spaces with isometric balls

🕮 *Construct a pair of locally compact length-metric spaces X and Y which are not isometric, but for some fixed points $x_0 \in X$, $y_0 \in Y$ and any radius R the ball $B(x_0, R)$ in X is isometric to the ball $B(y_0, R)$ in Y.*

Semisolutions

Non-contracting map. Given any pair of point $x_0, y_0 \in K$, consider two sequences $x_0, x_1, \dots$ and $y_0, y_1, \dots$ such that and $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ for each n.

Since K is compact, we can choose an increasing sequence of integers n_k such that both sequences $(x_{n_i})_{i=1}^{\infty}$ and $(y_{n_i})_{i=1}^{\infty}$ converge. In particular, both of these sequences are Cauchy; that is,

$$|x_{n_i} - x_{n_j}|_K, |y_{n_i} - y_{n_j}|_K \to 0 \quad \text{as} \quad \min\{i, j\} \to \infty.$$

Since f is non-contracting, we get

$$|x_0 - x_{|n_i - n_j|}| \leqslant |x_{n_i} - x_{n_j}|.$$

It follows that there is a sequence $m_i \to \infty$ such that

$$(*) \qquad x_{m_i} \to x_0 \quad \text{and} \quad y_{m_i} \to y_0 \quad \text{as} \quad i \to \infty.$$

Set

$$\ell_n = |x_n - y_n|_K.$$

Since f is non-contracting, (ℓ_n) is a non-decreasing sequence.

By $(*)$, $\ell_{m_i} \to \ell_0$ as $m_i \to \infty$. It follows that (ℓ_n) is a constant sequence.

In particular

$$|x_0 - y_0|_K = \ell_0 = \ell_1 = |f(x_0) - f(y_0)|_K$$

for any pair of points (x_0, y_0) in K. That is, f is distance preserving, in particular injective.

From $(*)$, we also get that $f(K)$ is everywhere dense. Since K is compact $f\colon K \to K$ is surjective. Hence the result follows. □

This is a basic lemma in the introduction to Gromov–Hausdorff distance [see 7.3.30 in 126]. The proof presented here is not standard, it was given by Travis Morrison, a students in my MASS class at Penn State, Fall 2011.

Horocompactification. For the first part of the problem, take X to be the set of non-negative integers with the metric ρ defined as

$$\rho(m,n) = m + n$$

for $m \neq n$.

Now assume X is proper length space and F_X is not an embedding. Then there is a sequence of points $z_1, z_2, \dots$ and a point z_∞, such that $f_{z_n} \to f_{z_\infty}$ in $C(X,\mathbb{R})$ as $n \to \infty$, while $|z_n - z_\infty|_X > \varepsilon$ for some fixed $\varepsilon > 0$ and all n.

Note that any pair of points in X can be connected by a minimizing geodesic. Choose $\bar z_n$ on a geodesic $[z_\infty z_n]$ such that $|z_\infty - \bar z_n| = \varepsilon$. Note that

$$f_{z_n}(z_\infty) - f_{z_n}(\bar z_n) = \varepsilon$$

and

$$f_{z_\infty}(z_\infty) - f_{z_n}(\bar z_n) = -\varepsilon$$

for any n.

Since X is proper, we can pass to a subsequence of z_n so that the sequence $\bar z_n$ converges, say to $\bar z_\infty$. From the identities above, it follows that

$$f_{z_n}(\bar z_\infty) \not\to f_{z_\infty}(\bar z_\infty) \quad \text{or} \quad f_{z_n}(z_\infty) \not\to f_{z_\infty}(z_\infty),$$

a contradiction. □

I learned this problem from Linus Kramer and Alexander Lytchak; the example was also mentioned in the lectures of Anders Karlsson and attributed to Uri Bader [see 2.3 in 135].

A ball near a sphere. Make fine burrows in the standard 3-ball which do not change its topology, but at the same time a come sufficiently close to any point in the ball.

Consider the doubling of obtained ball along its boundary. Clearly the obtained space is homeomorphic to $\mathbb{S}^3$. Prove that the burrows can be made so that it is sufficiently close to the original ball in the Gromov–Hausdorff metric.

It remains to smooth the obtained space slightly to get a genuine Riemannian metric with needed property. □

This construction is a stripped version of the theorem of Steven Ferry and Boris Okunin [see 136]. The theorem states that Riemannian metrics on a smooth closed manifold M with $\dim M \geqslant 3$ can approximate given compact length-metric space X if and only if there is a continuous map $M \to X$ which is surjective on the fundamental groups.

The two-dimensional case is quite different. There is no sequence of Riemannian metrics on $\mathbb{S}^2$ which converge to the unit disc in the sense of Gromov–Hausdorff. In fact, if X is a limit of $(\mathbb{S}^2, g_n)$, then any point $x_0 \in X$ either admits a neighborhood homeomorphic to $\mathbb{R}^2$ or it is a cut point; that is, $X \backslash \{x_0\}$ is disconnected [see 3.32 in 54].

Macro-dimension. Choose a point $p \in M$, denote by f the distance function from p.

Let us cover M by the connected components of the inverse images $f^{-1}((n-1, n+1))$. Clearly any point in M is covered by at most two such components. It remains to show that each of these components has diameter less than 100.

Assume the contrary; let x and y be two points in such connected component and $|x-y|_M \geqslant 100$. Connect x to y by a curve τ in the component. Consider the closed curve σ formed by two geodesics $[px]$, $[py]$ and τ.

Prove that σ can be divided into 4 arcs α, β, γ and δ in such a way that the minimal distance from α to γ as well as the minimal distance from β to δ is at least 10.

Use the last statement to show that σ cannot be shrank by a disc it its 1-neighborhood; the latter contradicts the assumption. $\square$

The problem was discussed in a talk by Nikita Zinoviev around 2004.

No Lipschitz embedding. Consider a chain of disjoint circles $c_0, \dots, c_n$ in $\mathbb{R}^3$; that is, c_i and c_{i-1} are linked for each i.

Assume that $\mathbb{R}^3$ is equipped with a length-metric ρ, such that the total length of the circles is ℓ and U is an open set containing all the circles c_i. Note that for any Lipschitz homeomorphism $f\colon (U, \rho) \to \mathbb{R}^3$ the distance from $f(c_0)$ to $f(c_n)$ is less than ℓ.

Let us show that the ρ-distance from c_0 to c_n might be much larger than ℓ. Fix a line segment $[ab]$ in $\mathbb{R}^3$. Modify the length-metric on $\mathbb{R}^3$ in arbitrary small neighborhood of $[ab]$ so that there is a chain (c_i) of circles as above, which goes from a to b such that (1) the total length, say ℓ, of (c_i) is arbitrary small, but (2) the obtained metric ρ is arbitrary close to the canonical, say

$$\big|\rho(x,y) - |x-y|\big| < \varepsilon$$

for any two points $x, y \in \mathbb{R}^3$ and fixed in advanced small value $\varepsilon > 0$. The construction of ρ is done by shrinking the length of each circle and expanding the length in the normal directions to the circles in their small neighborhood. The latter is made in order to make impossible to use the circles c_i as a shortcut; that is, one spends more time to go from one circle to an other than saves on going along the circle.

Set $a_n = (0, \frac{1}{n}, 0)$ and $b_n = (1, \frac{1}{n}, 0)$. Note that the line segments $[a_n b_n]$ are disjoint and converging to $[a_\infty b_\infty]$ where $a_\infty = (0,0,0)$ and $b_\infty = (1,0,0)$.

Apply the above construction in non-overlapping convex neighborhoods of $[a_n b_n]$ and for a sequences ε_n and ℓ_n which converge to zero very fast.

The obtained length-metric ρ is still close to the canonical, but for any open set U containing $[a_\infty b_\infty]$ the space (U, ρ) does not admit a Lipschitz homeomorphism to $\mathbb{R}^3$. Indeed, if such homeomorphism h exists, then from the above construction, we get

$$\begin{aligned}|h(a_\infty) - f(b_\infty)| &\leqslant |h(a_n) - f(b_n)| + \\ &\qquad + |h(a_\infty) - f(a_n)| + |h(b_n) - f(b_\infty)| \leqslant \\ &\leqslant \ell_n + \tfrac{2}{n} + 100{\cdot}\varepsilon_n.\end{aligned}$$

The right hand side converges to 0 as $n \to \infty$. Therefore

$$h(a_\infty) = f(b_\infty),$$

a contradiction.

It remains to performs similar construction countably many times so a bad segment as $[a_\infty b_\infty]$ above appears in any open set of $\mathbb{R}^3$. $\square$

The problem is due Dmitri Burago, Sergei Ivanov and David Shoenthal [see 137].

Sub-Riemannian sphere. Prove that there is a non-decreasing sequence of Riemannian metric tensors $g_0 \leqslant g_1 \leqslant \ldots$ such that the induced metrics converge to the given sub-Riemannian metrics. The metric g_0 can be assumed to be a metric on round sphere.

Applying the construction as in Nash–Kuiper theorem, one can produce a sequence of smooth embeddings $h_n \colon \mathbb{S}^m \to \mathbb{R}^{m+1}$ with the induced metrics g_n' such that $|g_n' - g_n| \to 0$.

Moreover, assume we assign a positive real number $\varepsilon(h)$ for any smooth embedding $h \colon \mathbb{S}^m \to \mathbb{R}^{m+1}$. Then we can assume that

$$|h_{n+1}(x) - h_n(x)| < \varepsilon(h_n)$$

for any $x \in \mathbb{S}^m$ and n.

Show that for a right choice of function $\varepsilon(h_n)$, the sequence h_n converges, say to h_∞, and the metric induced by h_∞ coincides with the given sub-Riemannian metric. □

The problem appeared on this list first rediscovered by Enrico Le Donne [see 138]. Similar construction described in the lecture notes by Allan Yashinski and me [see 139] which is aimed for undergraduate students. Yet the results in [140] are closely relevant.

The same idea can be used to prove the following.

- ⋄ Let M be a Riemannian manifold diffeomorphic to the n-sphere. Show that there is a Riemannian manifold M' arbitrary close to M in Lipschitz metric and vanishing Weyl curvature tensor.

Length-preserving map. Recall that according to Rademacher's theorem [132], any Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable almost everywhere.

Assume there is a length-preserving map $f\colon \mathbb{R}^2 \to \mathbb{R}$.

Note that f is Lipschitz. Therefore by Rademacher's theorem, f is differentiable almost everywhere.

Fix a unit vector u. Prove that, for almost all x, the length of the path $\alpha\colon t \mapsto x + t{\cdot}u$ with $t \in [0,1]$ can be expressed as the integral

$$\int\limits_0^1 (d_{\alpha(t)}f)(u){\cdot}dt.$$

It follows that $|d_x f(v)| = |v|$ for almost all $x, v \in \mathbb{R}^2$; in particular $d_x f$ is defined and has rank 2 at some point x, a contradiction. □

The idea above can be also used to solve the following problem.

- ⋄ *Assume ρ is a metric on $\mathbb{R}^2$ which is induced by a norm. Show that $(\mathbb{R}^2, \rho)$ admits a length-preserving map to $\mathbb{R}^3$ if and only if $(\mathbb{R}^2, \rho)$ is isometric to the Euclidean plane.*

Fixed segment. Note that it is sufficient to show that if

$$f(a) = a \quad\text{and}\quad f(b) = b$$

for some $a, b \in \mathbb{R}^m$, then

$$f(\tfrac{a+b}{2}) = \tfrac12{\cdot}(f(a) + f(b)).$$

(This statement is not trivial since in general metric midpoint of a and b in $(\mathbb{R}^m, \rho)$ are not defined uniquely.)

Without loss of generality, we can assume that $b + a = 0$.

Set $f_0 = f$. Consider the recursively defined sequence of isometries $f_0, f_1, \dots$ defined recursively

$$f_{n+1}(x) = -f_n^{-1}(-f_n(x)).$$

Note that $f_n(a) = a$ and $f_n(b) = b$ for any n and

$$|f_{n+1}(0)| = 2\cdot|f_n(0)|.$$

Therefore if $f(0) \neq 0$, then $|f_n(0)| \to \infty$ as $n \to \infty$. On the other hand, since f_n is isometry and $f(a) = a$, we get $|f_n(0)| \leqslant 2\cdot|a|$, a contradiction. □

The problem is a stripped version of Mazur–Ulam theorem proved in [141]; it states that any isometry of $(\mathbb{R}^m, \rho)$ to itself is an affine map.

The idea in the proof is due to Jussi Väisälä's in [see 142].

Pogorelov's construction. Positivity and symmetry of ρ is evident.

The triangle inequality follows since

$$(*)\qquad [B(x,\tfrac{\pi}{2})\backslash B(y,\tfrac{\pi}{2})] \cup [B(y,\tfrac{\pi}{2})\backslash B(z,\tfrac{\pi}{2})] \supseteq B(x,\tfrac{\pi}{2})\backslash B(z,\tfrac{\pi}{2})$$

and $B(x,\frac{\pi}{2})\backslash B(y,\frac{\pi}{2})$ does not overlap $B(y,\frac{\pi}{2})\backslash B(z,\frac{\pi}{2})$.

Note that we get equality in $(*)$ if and only if y lies on the great circle arc from x to z. Therefore the second statement follows. □

This construction was given by Aleksei Pogorelov in [143]. It is closely related to the construction given by David Hilbert in [144] which was the motivating example of his 4th problem [see 145].

Straight geodesics. From the uniqueness of straight segment between given points in $\mathbb{R}^m$, it follows that any straight line in $\mathbb{R}^m$ forms a geodesic in $(\mathbb{R}^m, \rho)$.

Set

$$\|\boldsymbol{v}\|_{\boldsymbol{x}} = \rho(\boldsymbol{x}, (\boldsymbol{x}+\boldsymbol{v})).$$

Note that

$$\|\lambda\cdot\boldsymbol{v}\|_{\boldsymbol{x}} = |\lambda|\cdot\|\boldsymbol{v}\|_{\boldsymbol{x}}$$

for any $\boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$.

Prove that

$$\|\lambda\cdot\boldsymbol{v}\|_{\boldsymbol{x}} - \|\lambda\cdot\boldsymbol{v}\|_{\boldsymbol{x'}} \leqslant C\cdot|\boldsymbol{x} - \boldsymbol{x'}|$$

for any $\boldsymbol{x}, \boldsymbol{x'}, \boldsymbol{v} \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$ and a fixed real constant C.

Passing to the limit as $\lambda \to \infty$, we get $\|\boldsymbol{v}\|_{\boldsymbol{x}}$ does not depend on $\boldsymbol{x}$; hence the result follows. □

The idea is due to Thomas Foertsch and Viktor Schroeder [see 146]. A more general statement was proved by Petra Hitzelberger and Alexander Lytchak in [147]. Namely they show that if any pair of points in a geodesic metric space X can be separated by an affine function, then X is isometric to a convex subset in a normed vector space.

Hyperbolic space. Note that 2-dimensional hyperbolic space can be viewed as $(\mathbb{R}^2, g)$, where

$$g(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & e^x \end{pmatrix}.$$

The same way 3-dimensional hyperbolic space can be viewed as $(\mathbb{R}^3, h)$, where where

$$h(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^x & 0 \\ 0 & 0 & e^x \end{pmatrix}.$$

Prove that the map $\mathbb{R}^3 \to \mathbb{R}^4$ defined as

$$(x,y,z) \mapsto (x,y,x,z)$$

is a quasi-isometry from $(\mathbb{R}^3, h)$ to its image in $(\mathbb{R}^2, g) \times (\mathbb{R}^2, g)$. $\square$

In the proof we used that horosphere in the hyperbolic space is isometric to the Euclidean plane. This observation appears already in the book of Nikolai Lobachevsky [see 34 in 148].

A homeomorphism near quasi-isometry. Fix two constants $M \geqslant 1$ and $A \geqslant 0$. A map $f\colon X \to Y$ between metric spaces X and Y such that for any $x, y \in X$, we have

$$\tfrac{1}{M}\cdot|x-y| - A \leqslant |f(x) - f(y)| \leqslant M\cdot|x-y| + A$$

and any point in Y lies on the distance at most A from a point in the image $f(X)$ will be called (M,A)-quasi-isometry.

Note that $(M,0)$-quasi-isometry is a $[\tfrac{1}{M}, M]$-bi-Lipschitz map. Moreover, if $f_n\colon \mathbb{R}^m \to \mathbb{R}^m$ is a $(M, \tfrac{1}{n})$-quasi-isometry for each n, then any partial limit of f_n as $n \to \infty$ is a $[\tfrac{1}{M}, M]$-bi-Lipschitz map.

It follows that given $M \geqslant 1$ and $\varepsilon > 0$ there is $\delta > 0$ such that for any (M,δ)-quasi-isometry $f\colon \mathbb{R}^m \to \mathbb{R}^m$ and any $p \in \mathbb{R}^m$ there is an $[\tfrac{1}{M}, M]$-bi-Lipschitz map $h\colon B(p,1) \to \mathbb{R}^m$ such that

$$|f(x) - h(x)| < \varepsilon$$

for any $x \in B(p,1)$.

Applying rescaling, we can get the following equivalent formulation. Given $M \geqslant 1$, $A \geqslant 0$ and $\varepsilon > 0$ there is big enough $R > 0$ such that for any (M,A)-quasi-isometry $f\colon \mathbb{R}^m \to \mathbb{R}^m$ and any $p \in \mathbb{R}^m$ there is a $[\tfrac{1}{M}, M]$-bi-Lipschitz map $h\colon B(p,R) \to \mathbb{R}^m$ such that

$$|f(x) - h(x)| < \varepsilon\cdot R$$

for any $x \in B(p,R)$.

Now cover $\mathbb{R}^m$ by balls $B(p_n,R)$, construct a $[\frac{1}{M},M]$-bi-Lipschitz map $h_n\colon B(p_n,R)\to\mathbb{R}^m$ for each n.

The maps h_n are $2\cdot\varepsilon\cdot R$ close to each other on the overlaps of their domains of definition. This makes possible to deform slightly each h_n so that they agree on the overlaps. This can be done by Siebenmann's Theorem [133]. If instead you apply Sullivan's theorem [134], you get a bi-Lipschitz homeomorphism $h\colon\mathbb{R}^m\to\mathbb{R}^m$. □

The problem was suggested by Dmitri Burago.

A family of sets with no section. Identify $\mathbb{S}^1$ with $[0,1]/(0\sim 1)$. Given $t\in[0,\frac12]$, Consider the set $K_t\subset\mathbb{S}^1$ formed by all possible sums $\sum_{n=1}^{\infty} a_n\cdot t^n$, where a_i is 0 or 1.

Note that K_t is a Cantor set and $K_{\frac12}=\mathbb{S}^1$.

Denote by $\rho_\alpha\colon\mathbb{S}^1\to\mathbb{S}^1$ the rotation by angle α. Set

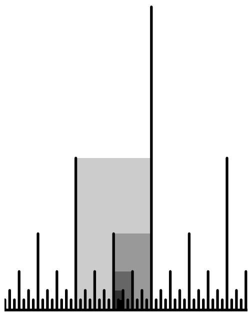
$$Z_t=\left[\begin{array}{ll} Z_t=\rho_{\frac{1}{1-2\cdot t}}(K_t) & \text{if } t\in[0,\tfrac12),\\ Z_t=\mathbb{S}^1 & \text{if } t=\tfrac12.\end{array}\right.$$

Prove that the family of sets Z_t is a continuous in the Hausdorff topology and it does not have a section. □

The problem is suggested by Stephan Stadler.

It is instructive to check that any Hausdorff continuous family of closed sets in $\mathbb{R}$ admits a continuous section.

Spaces with isometric balls. Take the upper half-plane and cut it along a "dyadic comb" shown on the diagram. Equip the obtained space with the intrinsic metric induced from the ℓ_∞-norm on the plane. (Few concentric balls in this metric are shown on the diagram.)



To construct a comb we have to make infinite number of decisions — for each tooth size we have to decide on which side from the origin will be closest tooth of that size.

Show that the needed spaces X and Y can be obtained this way for a pair of different combs.

Chapter 6

Actions and coverings

In this chapter we consider isometric actions and coverings.

We often use the following construction. Given a covering

$$f\colon \tilde X \to X$$

of the length-metric space X, one can consider the induced length-metric on $\tilde X$ defining length of curve α in X as the length of the composition $f\circ\alpha$; the obtained metric space $\tilde X$ is called *metric covering* of X.

Bounded orbit

Recall that a metric space is called *proper* if all its bounded closed sets are compact.

Let X be a proper metric space and $\iota\colon X \to X$ is an isometry. Assume that for some $x \in X$, the sequence $x_n = \iota^n(x)$, $n \in \mathbb{Z}$ has a converging subsequence. Prove that x_n is bounded.

Semisolution. Note that we can assume that the orbit $x_n = \iota^n(x)$ is dense in X; otherwise pass to the closure of this orbit. In particular, we can choose a finite number of positive integer values $n_1, n_2, \dots, n_k$ such that the points x_{n_1}, $x_{n_2}, \dots, x_{n_k}$ form a $\frac{1}{10}$-net in the ball $B(x_0, 10)$, that is, for any $x \in B(x_0, 10)$ there is x_{n_i} such that

$$|x - x_{n_i}| < \tfrac{1}{10}.$$

Prove that if $x_m \in B(x_0, 1)$, then $x_{m+n_i} \in B(x_0, 1)$ for some $i \in$ $\in \{1, \dots, k\}$.

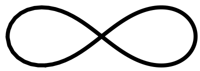
Set $N = \max_i\{n_i\}$. Note that among any N elements in a row $x_{i+1}, \dots x_{i+N}$ there is at least one in $B(x_0, 1)$. In particular, N isometric copies of $B(x_0, 1)$ cover whole X. Hence the result follows. □

The problem is due to Aleksander Całka's [see 149].

Finite action

Show that for any nontrivial continuous action of a finite group on the unit sphere there is an orbit which does not lie in the interior of a hemisphere.

Covers of figure eight



Let us define *figure eight* as the length-metric space which is obtained by gluing together all four ends of two unit segments.

Prove that any compact length-metric spaces K can be presented as a Gromov–Hausdorff limit of a sequence of metric covers

$$(\widetilde{\Phi}_n, \tilde{d}/n) \to (\Phi, d/n),$$

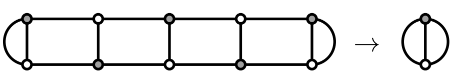
where (Φ, d) denotes the figure eight.

Diameter of m-fold cover*

Let X be a length-metric space and $\tilde{X}$ be a connected m-fold cover of X equipped with the induced intrinsic metric. Show that

$$\operatorname{diam} \tilde{X} \leqslant m \cdot \operatorname{diam} X.$$

From the diagram below you could guess an example of 5-fold cover with diameter of the total space exactly 5 times diameter of the target.

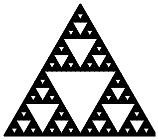


Symmetric square°

Let X be a topological space. Note that $X \times X$ admits a natural $\mathbb{Z}_2$-action by $(x, y) \mapsto (y, x)$. The quotient space $X \times X/\mathbb{Z}_2$ is called *symmetric square* of X.

Show that symmetric square of any path connected topological space has commutative the fundamental group.

Sierpiński gasket°



To construct Sierpiński gasket, start with a solid equilateral triangle, subdivide it into four smaller congruent equilateral triangles and remove the interior of the central one. Repeat this procedure recursively for each of the remaining solid triangles.

Find the homeomorphism group of the Sierpiński gasket.

Lattices in a Lie group

Let L and M be two discrete subgroups of a connected Lie group G and h be a left invariant metric on G. Equip the groups L and M with the induced left invariant metric from G. Assume $L\backslash G$ and $M\backslash G$ are compact and moreover

$$\operatorname{vol}(L\backslash(G,h)) = \operatorname{vol}(M\backslash(G,h)).$$

Prove that there is a bi-Lipschitz one-to-one mapping (not necessarily a homomorphism)

$$f\colon L \to M.$$

Piecewise Euclidean quotient

Note that the quotient of Euclidean space by a finite subgroup is a *polyhedral space* as it defined on page 97; on the same page you find the definition of piecewise linear homeomorphism.

Let Γ be a finite subgroup of $\mathrm{SO}(m)$. Denote by P the quotient $\mathbb{R}^m/\Gamma$ equipped with induced polyhedral metric. Assume P admits a piecewise linear homeomorphism to $\mathbb{R}^m$. Show that Γ is generated by rotations around subspaces of codimension 2.

Subgroups of the free group

Show that every finitely generated subgroup of the free group is an intersection of subgroups of finite index.

Lengths of generators of the fundamental group°

Let M be a compact Riemannian manifold and $p \in M$. Show that the fundamental group $\pi_1(M,p)$ is generated by the homotopy classes of loops with length at most $2\cdot\operatorname{diam} M$.

Number of generators

📖 *Let M be a complete connected Riemannian manifold with non-negative sectional curvature. Show that the minimal number of generators of the fundamental group $\pi_1 M$ can be bounded above in terms of the dimension of M.*

Equations in the group°

📖 *Assume G is a compact connected Lie group and n is a positive integer. Show that given a collection of elements $g_1, g_2 \dots, g_n \in G$ the equation*

$$x{\cdot}g_1{\cdot}x{\cdot}g_2 \cdots x{\cdot}g_n = 1$$

has a solution $x \in G$.

Semisolutions

Finite action. Without loss of generality, we may assume that the action is generated by a nontrivial homeomorphism

$$a\colon \mathbb{S}^m \to \mathbb{S}^m$$

and $a^p = \mathrm{id}_{\mathbb{S}^m}$ for some prime p.

Assume that any a-orbit lies in an open hemisphere. Then

$$h(x) = \sum_{n=1}^{p} a^n{\cdot}x \neq 0$$

for any $x \in \mathbb{S}^m$; here we consider $\mathbb{S}^m$ as the unit sphere in $\mathbb{R}^{m+1}$.

Consider the map $f\colon \mathbb{S}^m \to \mathbb{S}^m$ defined as $f(x) = \frac{h(x)}{|h(x)|}$. Note that

(i) if $a(x) = x$, then $f(x) = x$;

(ii) $f(x) = f \circ a(x)$ for any $x \in \mathbb{S}^m$.

Prove that f is homotopic to the identity; in particular

$$\deg f = 1. \qquad (*)$$

Fix $x \in \mathbb{S}^m$ such that $a(x) \neq x$. Show that a acts without fixed points on the inverse image $W = f^{-1}(V)$ of a small open neighborhood $V \ni x$. Therefore the quotient map $\theta\colon W \to W' = W/\mathbb{Z}_p$ is a p-fold covering. From (ii), the restriction $f|_W$ factors through θ; that is, there is $f'\colon W' \to V$ such that $f|_W = f' \circ \theta$.

Assume $p \neq 2$. Show that f' and θ have well defined degrees and

$$\deg f \equiv \deg \theta \cdot \deg f' \pmod{p}$$

Since θ is a p-fold covering, we have $\deg \theta \equiv 0 \pmod{p}$. Therefore

$$(**) \qquad \deg f \equiv 0 \pmod{p}.$$

Finally observe that $(*)$ contradicts $(**)$.

In the case $p = 2$ the same proof works, but the degrees have to be defined only modulo 2. □

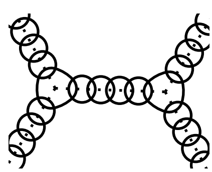
Along the same lines one can get a lower bound for the maximal diameter of orbit for any nontrivial actions of finite groups on a Riemannian manifold.

Applying the problem to the conjugate actions, one gets that if a fixed point set of a finite group acting on a sphere has nonempty interior, then the action is trivial. The same holds for any connected manifold; it was proved by Max Newman in [150].

The Newman's theorem was used by Deane Montgomery in [151] to show that *if h is a homeomorphism of a connected manifold M such that each h-orbit is finite, then $h^n = \mathrm{id}_M$ for some positive integer n.*

Covers of figure eight. First show that any compact metric space can be presented as a limit of a sequence of finite metric graphs Γ_n. Further, show that one can assume each vertex of Γ_n has degree 3 and the length of each edge in Γ_n is multiple of $\frac{1}{n}$.

It remains to approximate Γ_n by finite coverings of $(\Phi, d/n)$. Guess this part from the picture; it shows the needed approximation of the doted graph. □



The same idea works if instead of figure eight, we have any compact length-metric space X which admits a map $X \to \Phi$ which is surjective on fundamental groups. Such spaces X can be found among compact hyperbolic manifolds of any dimension $\geqslant 2$. All this due to Vedrin Sahovic [see 152].

A similar idea was used later to show that any group can appear as a fundamental group of underlying space of 3-dimensional hyperbolic orbifold [see 153].

Diameter of m-fold cover. Fix points $\tilde p, \tilde q \in \tilde M$. Let $\tilde\gamma \colon [0,1] \to \tilde M$ be a minimizing geodesic path from $\tilde p$ to $\tilde q$.

We need to show that

$$\operatorname{length} \tilde\gamma \leqslant m \cdot \operatorname{diam} M.$$

Suppose the contrary.

Denote by p, q and γ the projections to M of $\tilde{p}, \tilde{q}$ and $\tilde{\gamma}$. Represent γ as the concatenation of m paths of equal length,

$$\gamma = \gamma_1 * \ldots * \gamma_m,$$

so

$$\operatorname{length} \gamma_i = \frac{\operatorname{length} \gamma}{m} > \operatorname{diam} M.$$

Let σ_i be a minimizing geodesic in M connecting the endpoints of γ_i. Note that

$$\operatorname{length} \sigma_i \leqslant \operatorname{diam} M < \operatorname{length} \gamma_i.$$

Consider $m+1$ paths $\alpha_0, \ldots, \alpha_m$ defined as the concatenations

$$\alpha_i = \sigma_1 * \ldots * \sigma_i * \gamma_{i+1} * \ldots * \gamma_m.$$

Consider their liftings $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_m$ with $\tilde{q}$ as the endpoint. Note that two curves, say α_i and α_j for $i < j$, have the same starting point in $\tilde{M}$.

Consider the concatenation

$$\beta = \gamma_1 * \ldots * \gamma_i * \sigma_{i+1} * \ldots * \sigma_j * \gamma_{j+1} * \ldots * \gamma_m.$$

Prove that there is lift $\tilde{\beta}$ of β which connects $\tilde{p}$ to $\tilde{q}$ in $\tilde{M}$. Clearly $\operatorname{length} \beta < \operatorname{length} \gamma$, a contradiction. □

The question was asked by Alexander Nabutovsky and answered by Sergei Ivanov [see 154].

Symmetric square. Let $\Gamma = \pi_1 X$ and $\Delta = \pi_1((X \times X)/\mathbb{Z}_2)$. Consider the homomorphism $\varphi\colon \Gamma \times \Gamma \to \Delta$ induced by the projection $X \times X \to (X \times X)/\mathbb{Z}_2$.

Note that $\varphi(\alpha, 1) = \varphi(1, \alpha)$ for any $\alpha \in \Gamma$. Show that the restrictions $\varphi|_{\Gamma \times \{1\}}$ and $\varphi|_{\{1\} \times \Gamma}$ are onto.

It remains to note that

$$\varphi(\alpha, 1)\varphi(1, \beta) = \varphi(1, \beta)\varphi(\alpha, 1)$$

for any α and β in Γ. □

The problem was suggested by Rostislav Matveyev.

Sierpiński gasket. Denote the Sierpiński gasket by $\triangle$.

Let us show that any homeomorphism of $\triangle$ is also its isometry. Therefore the group homeomorphisms is the symmetric group S_3.

Let $\{x, y, z\}$ be a 3-point set in $\triangle$ such that $\triangle\backslash\{x, y, z\}$ has 3 connected components. Prove that there is unique choice for the set $\{x, y, z\}$ and it is formed by the midpoints of its big sides.

It follows that any homeomorphism of $\triangle$ permutes the set $\{x, y, z\}$.

A similar argument shows that this permutation uniquely describes the homeomorphism. $\square$

The problem was suggested by Bruce Kleiner. The homeomorphism group of Sierpiński carpet is much bigger, it is instructive to describe this group.

Latices in a Lie group. Denote by V_ℓ and W_m the Voronoi domain of for each $\ell \in L$ and $m \in M$ correspondingly; that is,

$$V_\ell = \{\, g \in G \,|\, |g - \ell|_G \leqslant |g - \ell'|_G \text{ for any } \ell' \in L \,\}$$

$$W_m = \{\, g \in G \,|\, |g - m|_G \leqslant |g - m'|_G \text{ for any } m' \in M \,\}$$

Note that for any $\ell \in L$ and $m \in M$ we have

$$(*)\qquad \begin{aligned} \operatorname{vol} V_\ell &= \operatorname{vol}(L\backslash(G, h)) = \\ &= \operatorname{vol}(M\backslash(G, h)) = \\ &= \operatorname{vol} W_m. \end{aligned}$$

Consider the bipartite graph Γ with vertices formed by the elements of L and M such that $\ell \in L$ is adjacent to $m \in M$ if and only if $V_\ell \cap W_m \neq \varnothing$.

By $(*)$ the graph Γ satisfies the condition in the marriage theorem — any subset in L has at least that many neighbors in M and the other way around. Therefore there is a bijection $f\colon L \to M$ such that

$$V_\ell \cap W_{f(\ell)} \neq \varnothing$$

for any $\ell \in L$.

It remains to notice that f is bi-Lipschitz. $\square$

The problem is due to Dmitri Burago and Bruce Kleiner [see 155]. For a finitely generated group G it is not known if G and $G \times \mathbb{Z}_2$ can fail to be bi-Lipschitz. (The groups are assumed to be equipped with word metric.)

Piecewise Euclidean quotient. Note that the group Γ serves as holonomy group of the quotient space $P = \mathbb{R}^m/\Gamma$ with the induced polyhedral metric. More precisely, one can identify $\mathbb{R}^m$ with the tangent space of a regular point x_0 of P in such a way that for any $\gamma \in \Gamma$ there is a loop ℓ in P which pass only through regular points and has the holonomy γ.

Fix $\gamma \in \Gamma$. Let ℓ be the corresponding loop. Since P is simply connected, we can shrink ℓ by a disc. By general position argument we can assume that the disc only pass through simplices of codimension 0, 1 and 2 and intersect the simplices of codimension 2 transversely.

In other words, ℓ can be presented as a product of loops such that each loop goes around a single simplex of codimension 2 and comes back. The holonomy for each of these loops is a rotation around a hyperplane. Hence the result follows. $\square$

The converse to the problem also holds; it was proved by Christian Lange in [156], his proof based earlier results of Marina Mikhailova [see 157].

Note that the cone over spherical suspension over Poincaré sphere is homeomorphic to $\mathbb{R}^5$ and it is quotient of $\mathbb{R}^5$ by a finite subgroup of $\mathrm{SO}(5)$. Therefore, if one exchanges "piecewise linear homeomorphism" to "homeomorphism" in the formulation, then the answer is different; a complete classification of such actions is given in [156].

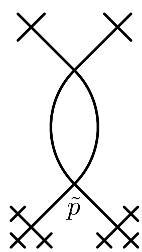
Subgroups of free group. Let F be a free group; note that we can assume that F has finite number, say m, generators.

Let W be the wedge sum of m circles, so $\pi_1(W,p) = F$. Equip W with the length-metric such that each circle has unit length.

Fix a finitely generated subgroup G in F.

Pass to the metric cover $\tilde{W}$ of W such that $\pi_1(\tilde{W},\tilde{p}) = G$ for a lift $\tilde{p}$ of p.

Fix sufficiently large integer n and consider doubling of the closed ball $\bar{B}(\tilde{p}, n+\frac{1}{2})$ along its boundary. Let us denote the obtained doubling by Z_n and set $G_n = \pi(Z_n,\tilde{p})$.



$\bar{B}(\tilde{p}, 2+\frac{1}{2})$

Prove that Z_n is a metric covering of W; it makes possible to consider G_n as a subgroup of F. By construction, Z_n is compact; therefore G_n has finite index in F.

It remains to show that

$$G = \bigcap_{n>k} G_n,$$

where k is the maximal length of word in the generating set of G. $\square$

Originally the problem was solved by Marshall Hall in [158]. The proof presented here is close to the solution of John Stalings in [159]; see also [160].

The same idea can be used to solve many other problems; here are some examples.

◇ *Show that subgroups of free groups are free.*

◇ *Show that two elements of the free groups u and v commute if and only if they are both powers of the some element w.*

Lengths of generators of the fundamental group. Choose a length minimizing loop γ which represents a given element $a \in \pi_1 M$.

Fix $\varepsilon > 0$. Represent γ as a concatenation

$$\gamma = \gamma_1 * \ldots * \gamma_n$$

of paths with length $\gamma_i < \varepsilon$ for each i.

Denote by $p = p_0, p_1, \ldots, p_n = p$ the endpoints of these arcs. Connect p to p_i by a minimizing geodesic σ_i. Note that γ is homotopic to a product of loops

$$\alpha_i = \sigma_{i-1} * \gamma_i * \sigma_{i-1}$$

and length $\alpha_i < 2 \cdot \operatorname{diam} M + \varepsilon$ for each i.

It remains to show that for sufficiently small $\varepsilon > 0$ any loop with length less than $2 \cdot \operatorname{diam} M + \varepsilon$ is homotopic to a loop with length at most $2 \cdot \operatorname{diam} M$. $\square$

The statement is due to Mikhael Gromov [Prop. 3.22 in 54].

Number of generators. Consider the universal Riemannian cover $\tilde{M}$ of M. Note that $\tilde{M}$ is non-negatively curved and $\pi_1 M$ acts by isometries on $\tilde{M}$.

Fix $p \in \tilde{M}$. Given $a \in \pi_1 M$, set

$$|a| = |p - a \cdot p|_{\tilde{M}}.$$

Consider the so called *short basis* in $\pi_1 M$; that is, a sequence of elements $a_1, a_2, \dots \in \pi_1 M$ defined the following way:

(i) Choose $a_1 \in \pi_1 M$ so that $|a_1|$ takes the minimal value.
(ii) Choose $a_2 \in \pi_1 M \backslash \langle a_1 \rangle$ so that $|a_2|$ takes the minimal value.
(iii) Choose $a_3 \in \pi_1 M \backslash \langle a_1, a_2 \rangle$ so that $|a_2|$ takes the minimal value.
(iv) and so on.

Note that the sequence terminates at n-th step if $(a_1, \ldots, a_n)$ forms a generating system. By construction, we have

$$|a_j \cdot a_i^{-1}| \geqslant |a_j| \geqslant |a_i|$$

for any $j > i$. Set $p_i = a_i \cdot p$. Note that

$$\begin{aligned} |p_j - p_i|_{\tilde{M}} &= |a_j \cdot a_i^{-1}| \geqslant \\ &\geqslant |a_j| = \\ &= |p_j - p|_{\tilde{M}} \geqslant \\ &\geqslant |a_i| = \\ &= |p_i - p|_{\tilde{M}}. \end{aligned}$$

By Toponogov comparison theorem we get

$$\measuredangle[p\,{}^{p_i}_{p_j}] \geqslant \tfrac{\pi}{3}.$$

That is, the directions from p to all p_i lie on the angle at least $\frac{\pi}{3}$ from each other.

Therefore the number of points p_i can be bounded in terms of the dimension of M. Hence the result follows. $\square$

This construction introduced by Mikhael Gromov [see 63].

Equations in the group. Consider the map $f\colon G \to G$ defined as

$$f(x) = x{\cdot}g_1{\cdot}x{\cdot}g_2 \cdots x{\cdot}g_n.$$

Note that f is homotopic to the map

$$h\colon x \mapsto x^n.$$

Show that the map $h\colon G \to G$ has nonzero degree.

It follows that both maps f, h are onto and the result follows. $\square$

The idea of this solution is due to Murray Gerstenhaber and Oscar Rothaus [see 161]. In fact the degree of g is n^k, where k is the rank of G; the latter was proved by Heinz Hopf [see 162].

Chapter 7

Topology

In this chapter we consider geometrical problems with strong topological flavor. A typical introductory course in topology, say [163], contains all the necessary material.

Isotropy

Recall that an isotopy is a continuous one parameter family of embeddings.

Let K_1 and K_2 be homeomorphic compact subsets of the coordinate subspace $\mathbb{R}^m$ in $\mathbb{R}^{2\cdot m}$. Show that there is a homeomorphism

$$h\colon \mathbb{R}^{2\cdot m}\to\mathbb{R}^{2\cdot m}$$

such that $K_2 = h(K_1)$. Moreover, h can be chosen to be isotopic to the identity map.

Semisolution. Fix a homeomorphism $\varphi\colon K_1\to K_2$.

By Tietze extension theorem, the homeomorphisms $\varphi\colon K_1\to K_2$ and $\varphi^{-1}\colon K_2\to K_1$ can be extended to a continuous maps, say $f\colon\mathbb{R}^m\to\mathbb{R}^m$ and $g\colon\mathbb{R}^m\to\mathbb{R}^m$ correspondingly.

Consider the homeomorphisms $h_1,h_2,h_3\colon\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m\times\mathbb{R}^m$ defined the following way

$$\begin{aligned}
h_1(x,y)&=(x,y+f(x)),\\
h_2(x,y)&=(x-g(y),y),\\
h_3(x,y)&=(y,-x).
\end{aligned}$$

It remains to prove that each homeomorphism h_i is isotopic to the identity map and we have $K_2=h(K_1)$ for

$$h=h_3\circ h_2\circ h_1.$$

□

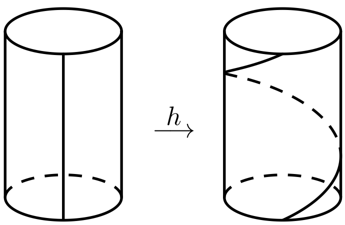
The problem is due to Victor Klee [see 164]. The problem "Monotonic homotopy" on page 106 is closely related.

Immersed disks

Two immersions $f_1, f_2\colon D \looparrowright \mathbb{R}^2$ are called *essentially different* if there is no diffeomorphism $h\colon D \to D$ such that $f_1 = f_2 \circ h$.

📖 *Construct two essentially different smooth immersions of the disk into the plane which coincide near the boundary.*

Positive Dehn twist



Let Σ be a surface and

$$\gamma\colon \mathbb{R}/\mathbb{Z} \to \Sigma$$

be non-contractible closed simple curve. Let U_γ be a neighborhood of γ which admits a parametrization

$$\iota\colon \mathbb{R}/\mathbb{Z} \times (0,1) \to U_\gamma.$$

Dehn twist along γ is a homeomorphism $h\colon \Sigma \to \Sigma$ which is identity outside of U_γ and such that

$$\iota^{-1} \circ h \circ \iota\colon (x, y) \mapsto (x + y, y).$$

If Σ is oriented and ι is orientation preserving, then the Dehn twist described above is called *positive.*

📖 *Let Σ be an compact oriented surface with non empty boundary. Prove that any composition of positive Dehn twists of Σ is not homotopic to identity rel. boundary.*

In other words, any product of positive Dehn twists is nontrivial in the mapping class group of Σ.

Function with no critical points

📖 *Given an integer $m \geqslant 2$, construct a smooth function $f\colon \mathbb{R}^m \to \mathbb{R}$ with no critical points in the unit ball B^m such that the restriction $f|_{B^m}$ does not factor through a linear function; that is, $f|_{B^m}$ cannot be presented as a composition $\ell \circ \varphi$, where $\ell\colon \mathbb{R}^m \to \mathbb{R}$ is a linear function and $\varphi\colon B^m \to \mathbb{R}^m$ is a smooth embedding.*

Conic neighborhood

Let p be a point in a topological space X. We say that an open neighborhood $U \ni p$ is *conic* if there is a homeomorphism from a cone to U which sends its vertex to p.

Show that any two conic neighborhoods of one point are homeomorphic to each other.

Unknots

Prove that the set of smooth embeddings $f\colon \mathbb{S}^1 \to \mathbb{R}^3$ equipped with the C^0-topology forms a connected space.

Stabilization

Construct two compact subsets $K_1, K_2 \subset \mathbb{R}^2$ such that K_1 is not homeomorphic to K_2, but $K_1 \times [0,1]$ is homeomorphic to $K_2 \times [0,1]$.

Homeomorphism of cube

Let $\square^m$ be a cube in $\mathbb{R}^m$ and $h\colon \square^m \to \square^m$ be a homeomorphism which sends each face of $\square^m$ to itself. Extend h to a homeomorphism $f\colon \mathbb{R}^m \to \mathbb{R}^m$ which coincides with the identity map outside of a bounded set.

Finite topological space°

Given a finite topological space F construct a finite simplicial complex K which admits a weak homotopy equivalence $K \to F$.

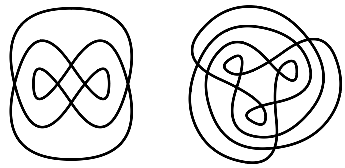
Dense homeomorphism°

Let $\mathcal{H}$ be the set of all homeomorphisms $\mathbb{S}^2 \to \mathbb{S}^2$ equipped with the C^0-metric. Show that there is a homeomorphism $h \in \mathcal{H}$ such that its conjugations $a \circ h \circ a^{-1}$ for all $a \in \mathcal{H}$ form a dense set in $\mathcal{H}$.

Simple path°

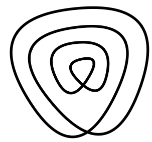
Let p and q be distinct points in Hausdorff topological space X. Assume p and q are connected by a path. Show that they can be connected by a simple path; that is, there is an injective continuous map $\beta\colon [0,1] \to X$ such that $\beta(0) = p$ and $\beta(1) = q$.

Semisolutions



Immersed disks. Both circles on the picture bound essentially different discs.

It is a good exercise to count the discs in these examples. (The answers are 2 and 5 correspondingly.) □

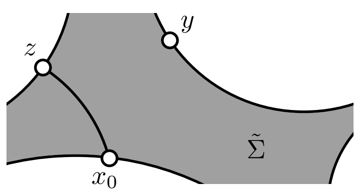


The existence of examples of that type is generally attributed to John Milnor [see 165].

An easier problem would be to construct two essentially different immersions of annuli with the same boundary curves; a solution is shown on the picture.

Positive Dehn twist. Consider the universal covering $f\colon \tilde\Sigma \to \Sigma$. The surface $\tilde\Sigma$ comes with the orientation induced from Σ.

Fix a point x_0 on the boundary $\partial\tilde\Sigma$. Given two other points y and z in $\partial\tilde\Sigma$ we will write $z \succ y$ if y lies on the right side from one simple curve from x_0 to z in $\tilde\Sigma$. Note that $\succ$ defines a linear order on $\partial\tilde\Sigma\backslash\{x_0\}$. We will write $z \succeq y$ if $z \succ y$ or $z = y$.



Note that any homeomorphism $h\colon \Sigma \to \Sigma$ which is identity on the boundary lifts to the unique homeomorphism $\tilde h\colon \tilde\Sigma \to \tilde\Sigma$ such that $\tilde h(x_0) = x_0$.

Claim. Assume h is a positive Dehn twist about closed curve γ. Then $y \succeq \tilde h(y)$ for any $y \in \partial\tilde\Sigma\backslash\{x_0\}$ and $y_0 \succ \tilde h(y_0)$ for some $y_0 \in \partial\tilde\Sigma\backslash\{x_0\}$.

Note that the property in the claim is a homotopy invariant and it survives under compositions of maps. Therefore the problem follows from the claim.

If Σ is not an annulus, then by uniformization theorem, we can assume that Σ has hyperbolic metric and geodesic boundary; the lifted metric on $\tilde\Sigma$ has the same properties. Further we can assume that (1) γ is a closed geodesic, (2) the parametrization $\iota\colon \mathbb{R}/\mathbb{Z} \times (0,1) \to U_\gamma$ from the definition of Dehn twist is rotationally symmetric and (3) for any $u \in \mathbb{R}/\mathbb{Z}$ the arc $\iota(u \times (0,1))$ is a geodesic perpendicular to γ.

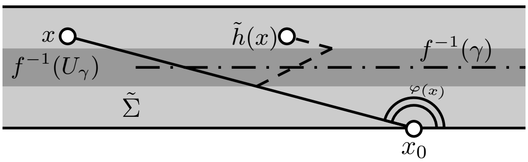
Consider the polar coordinates (φ, ρ) on $\tilde{\Sigma}$ with the origin at x_0; since x_0 lies on the boundary, the angle coordinate φ is defined in $[0, \pi]$. Show that

$$\varphi(x) \geqslant \varphi \circ \tilde{h}(x)$$

for any $x \neq x_0$ and if the geodesic $[x_0 x]$ crosses $f^{-1}(U_\gamma)$ then

$$\varphi(x) > \varphi \circ \tilde{h}(x).$$

In particular, if x lies on the boundary then $\tilde{h}(x)$ lies on the right side from the geodesic $[x_0 x]$; hence the claim follows.

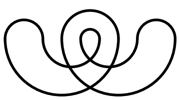


If Σ is an annulus, then the same argument works except we have to choose a flat metric on Σ. In this case $\tilde{\Sigma}$ is a strip between two parallel lines in the plane, see the diagram. □

The problem was suggested by Rostislav Matveyev.

The statement does not hold for surfaces without boundary. It is instructive to find a counterexample.

Function with no critical points. Construct an immersion $\psi \colon B^m \looparrowright \mathbb{R}^m$ such that



$$\ell \circ \varphi \neq \ell \circ \psi$$

for any embedding $\varphi \colon B^m \to \mathbb{R}^m$. The two-dimensional case can be guessed from the picture.

It remains to note that the composition $f = \ell \circ \psi$ has no critical points. □

The problem was suggested by Petr Pushkar.

Conic neighborhood. Let V and W be two conic neighborhoods of p. Without loss of generality, we may assume that the closure of V lies in W.

We will need to construct a sequence of embeddings $f_n \colon V \to W$ such that

(i) For any compact set $K \subset V$ there is a positive integer $n = n_K$ such that $f_n(k) = f_m(k)$ for any $k \in K$ and $m \geqslant n$.

(ii) For any point $w \in W$ there is a point $v \in V$ such that $f_n(v) = w$ for all large n.

Note that once such sequence is constructed, $f\colon V\to W$ defined as $f(v)=f_n(v)$ for all large values of n gives the needed homeomorphism.

The sequence f_n can be constructed recursively, setting

$$f_{n+1}=\Psi_n\circ f_n\circ\Phi_n,$$

where $\Phi_n\colon V\to V$ and $\Psi_n\colon W\to W$ are homeomorphisms of the form

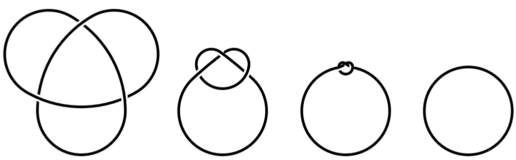
$$\Phi_n(x)=\varphi_n(x)\cdot x\quad \Phi_n(x)=\psi_n(x)\cdot x,$$

where $\varphi_n\colon V\to\mathbb{R}_+$, $\psi_n\colon W\to\mathbb{R}_+$ are suitable continuous functions and "$\cdot$" denotes the "multiplication" in the cone structures of V and W correspondingly. $\square$

The problem is due to Kyung Whan Kwun [see 166].

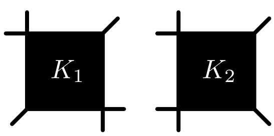
Note that for two cones $\mathrm{Cone}(\Sigma_1)$ and $\mathrm{Cone}(\Sigma_2)$ might be homeomorphic while Σ_1 and Σ_2 are not.

Unknots.



Observe that it is possible to draw tight arbitrary knot while keeping it smoothly embedded all the time including the last moment. $\square$

This problem was suggested by Greg Kuperberg.



Simple stabilization. The example can be guessed from the diagram. $\square$

I learned this problem in my analysis class taught by Maria Goluzina.

Homeomorphism of cube. Let us extend the homeomorphism h to whole $\mathbb{R}^m$ by reflecting the cube in its facets. We get a homeomorphism say $\tilde h\colon\mathbb{R}^m\to\mathbb{R}^m$ such that $\tilde h(x)=h(x)$ for any $x\in\square^m$ and

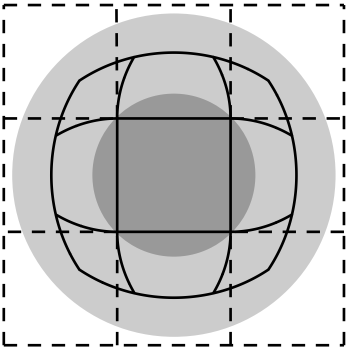
$$\tilde h\circ\gamma=\gamma\circ\tilde h,$$

where γ is a reflection through the facets of the cube.

Without loss of generality, we may assume that the cube $\square^m$ is inscribed in the unit sphere centered at the origin of $\mathbb{R}^m$. In this case $\tilde h$ has *displacement* at most 2; that is,

$$|\tilde h(x)-x|\leqslant 2$$

for any $x\in\mathbb{R}^m$.



Fix a smooth increasing concave function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that

$$\varphi(r) = r$$

for any $r \leqslant 1$ and

$$\sup\{\varphi(r)\} = 2.$$

Equip $\mathbb{R}^m$ with the polar coordinates (u, r), where $u \in \mathbb{S}^{m-1}$ and $r \geqslant 0$. Consider the open embedding $\Phi\colon \mathbb{R}^m \hookrightarrow \mathbb{R}^m$ defined as $\Phi(u, r) = (u, \varphi(r))$.

Set

$$f(x) = \left[\begin{array}{ll} x & \text{if } |x| \geqslant 2 \\ \Phi \circ \tilde{h} \circ \Phi^{-1}(x) & \text{if } |x| < 2 \end{array}\right.$$

It remains to show that $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is a solution. $\square$

The problem is a stripped from a proof of Robion Kirby [see 167]. The condition that face is mapped to face can be removed and instead of homeomorphism one can take an embedding which is close enough to the identity.

An interesting twist to this idea was given by Dennis Sullivan in [134]. Instead of the discrete group of motions of Euclidean space, he use a discrete group of motions of hyperbolic space in the conformal disk model. Say, assume we repeat the same argument if instead of cube we have a Coxeter polytope in the hyperbolic space. Then the constructed map coincides with the identity on the absolute and therefore the last "shrinking" step in the proof above is not needed. Moreover, if the original homeomorphism is bi-Lipschitz, then the construction also produce a bi-Lipschitz homeomorphism — this is the advantage.

Finite topological space. Given a point $p \in F$, denote by O_p the minimal open set in F containing p. Note that we can assume that F connected and is T_0; that is, $O_p = O_q$ if and only if $p = q$.

Let us write $p \preccurlyeq q$ if $O_p \subset O_q$. The relation $\preccurlyeq$ is a partial order on F.

Let us construct a simplicial complex K by taking F as the set of its vertices and saying that a collection of vertices form a simplex if they can form an increasing sequence with respect to $\preccurlyeq$.

Given $k \in K$, consider the minimal simplex $(f_0, \dots, f_m) \ni k$; we can assume that $f_0 \preccurlyeq \dots \preccurlyeq f_m$. Set $h\colon k \mapsto f_0$; it defines a map $K \to F$.

It remains to check that h is continuous and induces an isomorphism of all the homotopy groups. □

In a similar fashion, one can construct a finite topological space F for given simplicial complex K such that there is a weak homotopy equivalence $K \to F$. Both constructions are due to Pavel Alexandrov, [see 168, 169].

Dense homeomorphism. Note that there is countable set of homeomorphisms $h_1, h_2, \dots$ which is dense in $\mathcal{H}$ such that each h_n fix all the points outside an open round disc, say D_n.

Choose a countable disjoint collection of round discs D'_n and consider the homeomorphism $h\colon \mathbb{S}^2 \to \mathbb{S}^2$ which fix all the points outside of $\bigcup_n D'_n$ and for each n, the restriction $h|_{D'_n}$ is conjugate to $h_n|_{D_n}$.

Show that h solves the problem. □

The problem was mentioned by Frederic Le Rox [see 170].

Simple path. Let α be a path connecting p to q.

Passing to a subinterval if necessary, we can assume that $\alpha(t) \neq p, q$ for $t \neq 0, 1$.

An open set in $[0,1]$ will be called *suitable* if for any connected component (a,b) of Ω we have $\alpha(a) = \alpha(b)$. Show that there is a maximal suitable open set Ω; that Ω is suitable and it is not a subset of any other suitable set.

Define $\beta(t) = \alpha(a)$ for any t in a connected component $(a,b) \subset \Omega$.

It remains to reparametrize β to make it injective. □

The problem inspired by a Lemma 7.13 proved by Alexander Lytchak and Stefan Wenger in [171]

A more involved solution goes the following way: Note that one can assume that X coincides with the image of α. In particular it is connected, locally connected and compact.

Any such space admits a length-metric. This statement was conjectured by Karl Menger in [172] and proved independently by R. H. Bing [see 173, 174] and Edwin Moise [see 175].

It remains to consider a geodesic path from p to q.

Chapter 8

Piecewise linear geometry

A *polyhedral space* is complete length-metric space which admits a locally finite triangulation such that each simplex is isometric to a simplex in a Euclidean space. By *triangulation* of polyhedral space we always understand triangulation as above.

A point in a polyhedral space is called *regular* if it has a neighborhood isometric to an open set in a Euclidean space; otherwise it called *singular*.

If above we exchange the Euclidean spaces to the unit spheres or the hyperbolic spaces, we arrive to the definition of *spherical* and correspondingly *hyperbolic polyhedral spaces*.

The term *piecewise* typically mean that there is a triangulation with some property on each triangle. For example, if P and Q are polyhedral spaces, then

- ◇ a map $f\colon P\to Q$ is called *piecewise distance preserving* if there is a triangulation $\mathcal{T}$ of P such that at any simplex $\Delta\in\mathcal{T}$ the restriction $f|_\Delta$ is distance preserving,
- ◇ a map $h\colon P\to Q$ is called *piecewise linear* if both spaces P and Q admit triangulations such that each simplex of P is mapped to a simplex of Q by an affine map. In particular, a *piecewise linear homeomorphism* is a piecewise linear map which is a homeomorphism.

Spherical arm lemma

Let $A=a_1a_2\dots a_n$ and $B=b_1b_2\dots b_n$ be two simple spherical polygons with equal corresponding sides. Assume A lies in a hemisphere and $\measuredangle a_i\geqslant\measuredangle b_i$ for each i. Show that A is congruent to B.

Semisolution. Let us cut the polygon A from the sphere and glue instead the polygon B. Denote by Σ the obtained spherical polyhedral space. Note that

- ⋄ Σ is homeomorphic $\mathbb{S}^2$.
- ⋄ Σ has curvature $\geqslant 1$ in the sense of Alexandrov; that is, the total angle around each singular point is less than $2\cdot\pi$.
- ⋄ All the singular points of Σ lie outside of an isometric copy of a hemisphere $\mathbb{S}^2_+ \subset \Sigma$

Denote by n the number of singular points in Σ. It is sufficient to show that $n = 0$.

Assume the contrary; that is $n \geqslant 1$. We will arrive to a contradiction applying induction on n. The base case $n = 1$ is trivial; that is, Σ cannot have single singular point.

Now assume Σ has $n > 1$ singular points. Choose two singular points p, q, cut Σ along a geodesic $[pq]$. Show that the hole hole can be patched so that we obtain a new polyhedral space Σ' of the same type but wit $n - 1$ singular points. (The needed patch is obtained by doubling a spherical triangle along two sides.)

By induction hypothesis Σ' does not exist. Hence the result follow. □

The problem is due to Victor Zalgaller [see 176]; the result of Victor Toponogov in [177] gives a smooth analog of this statement. The patch construction above was introduced by Aleksandr Alexandrov in his proof of convex embeddability of polyhedrons [see 178, VI, §7].

Here is an alternative end of proof from [108]: By Alexandrov embedding theorem, Σ is isometric to the surface of convex polyhedron P in the unit 3-dimensional sphere $\mathbb{S}^3$. The center of hemisphere has to lie in a facet, say F of P. It remains to note that F contains the equator and therefore P has to be hemisphere in $\mathbb{S}^3$ or intersection of two hemispheres. In both cases its surface is isometric to $\mathbb{S}^2$.

Triangulation of 3-sphere

Construct a triangulation of $\mathbb{S}^3$ such with 100 vertices such that any two vertices are connected by an edge.

Folding problem

Let P be a compact 2-dimensional polyhedral space. Construct a piecewise distance preserving map $f\colon P \to \mathbb{R}^2$.

Piecewise linear extension

Prove that any 1-Lipschitz map from a finite subset $F \subset \mathbb{R}^2$ to $\mathbb{R}^2$ can be extended to a piecewise distance preserving map $\mathbb{R}^2 \to \mathbb{R}^2$.

Closed polyhedral surface

Construct a closed polyhedral surface in $\mathbb{R}^3$ with nonpositive curvature; that is, the total angle around each vertex is at least $2{\cdot}\pi$.

Minimal polyhedral disc

By a polyhedral disc in $\mathbb{R}^3$ we understand a triangulation of a plane polygon with a map in $\mathbb{R}^3$ which is affine on each triangle. The area of the polyhedral disc is defined as the sum of areas of the images of the triangles in the triangulation.

Consider the class of polyhedral discs glued from n triangles in $\mathbb{R}^3$ with fixed broken line as the boundary. Let Σ_n be a disc of minimal area in this class. Show that Σ_n is saddle*; that is, a plane can not cut all the edges coming from one of the interior vertices of Σ_n.*

Coherent triangulation°

A triangulation of a convex polygon is called coherent if there is a convex function which is linear on each triangle and changes the gradient on every edge of the triangulation.

Find a non-coherent triangulation of a triangle.

A sphere with one edge*

Given a spherical polyhedral space P, denote by P_s the subset of its singular points.

Construct spherical polyhedral space P which is homeomorphic to $\mathbb{S}^3$ and such that P_s is formed by a knotted circle. Show that in such an example the total length of P_s can be arbitrary large and the angle around P_s can be made strictly less than $2{\cdot}\pi$.

Triangulation of a torus

Show that the torus does not admit a triangulation such that one vertex has 5 edges, one has 7 edges and all other vertices have 6 edges.

No simple geodesics°

Construct a convex polyhedron P whose surface does not have a closed simple geodesic.

Semisolutions

Triangulation of 3-sphere. Choose 100 distinct points $x_1, x_2, \dots$ $\dots, x_{100}$ on the *moment curve*

$$\gamma\colon t \mapsto (t, t^2, t^3, t^4)$$

in $\mathbb{R}^4$. Let P be the convex hull of $\{x_1, x_2, \dots, x_{100}\}$.

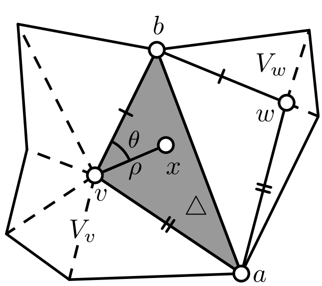
Prove that for any two points x_i and x_j there is a hyperplane H in $\mathbb{R}^4$ which pass through x_i and x_i and leaves γ on one side. The latter statement implies that any two vertices x_i and x_j of P are connected by an edge.

The statement follows since the surface of P is homeomorphic to $\mathbb{S}^2$. $\square$

The polyhedron P above is an example of so called *cyclic polytopes.*

Folding problem. Given a triangulation of P consider the Voronoi domain V_v for each vertex v. Prove that the triangulation can be subdivided if necessary so that Voronoi domain of each vertex is isometric to a convex subset in the cone with vertex corresponding to the tip.

Note that the boundaries of all the Voronoi domains form a graph with straight edges. One can triangulate P so that each triangle has such edge as the base and the opposite vertex is the center of an adjusted Voronoi domain; such a vertex will be called *main* vertex of the triangle.



Fix a solid triangle $\triangle = [vab]$ in the constructed triangulation; let v be its main vertex. Given a point $x \in \triangle$, set

$$\rho(x) = |x - v|$$

and

$$\theta(x) = \min\{\measuredangle[v\,{}^{a}_{x}], \measuredangle[v\,{}^{b}_{x}]\}.$$

Map x to the plane the point with polar coordinates $(\rho(x), \theta(x))$.

Note that for each triangle $\triangle$, the constructed map $\triangle \to \mathbb{R}^2$ is piecewise distance preserving. It remains to check that these maps agree on the common sides of the triangles. $\square$

This construction was given by Victor Zalgaller in [179], see also [139]. Svetlana Krat generalized the statement to the higher dimensions [see 180].

Piecewise linear extension. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two collections of points in $\mathbb{R}^2$ such that $|a_i - a_j| \geqslant |b_i - b_j|$ for all pairs i, j. We need to construct a piecewise distance preserving map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(a_i) = b_i$ for each i.

Assume that the problem is already solved if $n < m$; let us do the case $n = m$. By assumption, there is a piecewise liner length-preserving map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(a_i) = b_i$ for each $i > 1$. Consider the set

$$\Omega = \left\{\, x \in \mathbb{R}^2 \;\middle|\; |f(x) - b_1| > |x - a_1| \,\right\}.$$

If $\Omega = \varnothing$, then $f(a_1) = b_1$; that is, f is a solution.

Assume $\Omega \neq \varnothing$. Prove that Ω is the interior of a polygon which is star-shaped with respect to a_1. Redefine the map f inside Ω so that it remains piecewise distance preserving and $f(a_1) = b_1$. $\square$

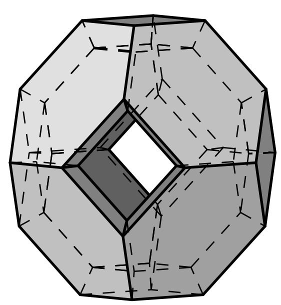
The same proof works in all dimensions.

The statement was proved by Ulrich Brehm in [181] and rediscovered by Arseniy Akopyan and Alexey Tarasov in [182], see also [139]. The idea in the proof is the same as in the proof of Kirszbraun's theorem given in [27].

Closed polyhedral surface. Start with you favorite convex polyhedron K. Assume that the interior of K contains the origin $0 \in \mathbb{R}^3$. Remove from K the interior of $K' = \frac{5}{6}\cdot K$.

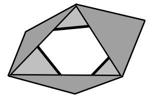
Note that one can drill from each vertex of K a polyhedral tunnel to the corresponding vertex K' so that the surface of obtained non-convex polytope is a solution.

(On the diagram you see the result of this construction for the octahedron.) $\square$



The construction above produce a surface of genus at least 3. One can also construct a polyhedral surface in $\mathbb{R}^3$ which is isometric to a flat torus. It follows from very general result of Burago and Zalgaller [see 183]. They show in particular that any

1-Lipschitz smooth embedding of flat torus in $\mathbb{R}^3$ can be approximated by piecewise linear isometric embedding.



The following construction is more direct; it is a bent version of so called *Schwarz boot* [see 184]. Construct an isometric piecewise linear embedding of cylinder as shown on the diagram such that the planes thru the boundary triangles meet at the angle $\frac{\pi}{n}$ for a positive integer n. It remains to reflect the obtained surface several times in the planes through the boundary triangles.

The problem suggested by Jarosław Kędra.

Minimal polyhedral disc. Arguing by contradiction, assume a polyhedral disc Σ_n minimize the area but not saddle.

Prove that one can move one of the vertices of Σ_n in such a way that the lengths of all edges starting at this vertex decrease.

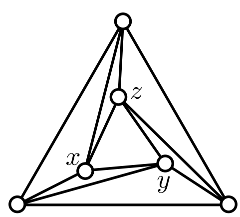
Prove that if, by this deformation, the area does not decease, then there are two adjusted triangles in the triangulation, say $[pxy]$ and $[qxy]$ such that

$$\measuredangle[p\,{}^x_y] + \measuredangle[q\,{}^x_y] > \pi.$$

Finally show that in this case exchanging triangles $[pxy]$ and $[qxy]$ to the triangles $[xpq]$ and $[ypq]$ leads to a polyhedral surface with smaller area. That is, Σ_n is not area minimizing, a contradiction. $\square$

This problem is discussed in [185].

For general polyhedral surface, the deformation which decrease the lengths of all edges may not decrease the area. Moreover, the surface which minimize the area among all surfaces with fixed triangulation might be not saddle; try to construct such example.



Coherent triangulation. An example shown on the diagram.

Assume it is coherent and f is the corresponding piecewise linear convex function. Without loss of generality we can assume that f vanish on the boundary of big triangle. Arrive to a contradiction by showing

$$f(x) > f(y) > f(z) > f(x).$$

$\square$

The problem was discussed by Israel Gelfand, Mikhail Kapranov and Andrei Zelevinsky in [186, p. 7C]. The given example is closely related to so called *Schönhardt polyhedron*, an example of non-convex polyhedron which does not admit a triangulation [see 187].

A sphere with one edge. An example, say P, can be found among the spherical polyhedral spaces which admit an isometric $\mathbb{S}^1$-action with geodesic orbits.

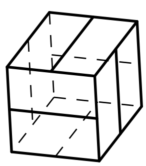
Fix large relatively prime integers $p > q$. Consider the triangle Δ with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and say $\pi\cdot(1-\frac{1}{p})$ in the sphere of radius $\frac{1}{2}$. Denote by $\hat\Delta$ the doubling of Δ along its boundary. Note that $\hat\Delta$ is homeomorphic to $\mathbb{S}^2$, it has 3 singular points with total angles $2\cdot\frac{\pi}{p}$, $2\cdot\frac{\pi}{q}$ and $2\cdot\pi\cdot(1-\frac{1}{p})$.

Consider $\mathbb{S}^1$-action on $\mathbb{S}^3 \subset \mathbb{C}^2$ by the diagonal matrices $\left(\begin{smallmatrix} z^p & 0 \\ 0 & z^q \end{smallmatrix}\right)$, $z \in \mathbb{S}^1 \subset \mathbb{C}$. Construct a spherical polyhedral metric ρ on $\mathbb{S}^3$ such that the $\mathbb{S}^1$-orbits become geodesics and the quotient $(\mathbb{S}^3, \rho)/\mathbb{S}^1$ is isometric to $\hat\Delta$.

In the constructed example the singular points with total angles $2\cdot\frac{\pi}{p}$ and $2\cdot\frac{\pi}{q}$ should correspond to the points with isotropy groups $\mathbb{Z}/p$ and $\mathbb{Z}/q$ of the action. The points in $P = (\mathbb{S}^3, d)$ on the orbits over these points will be regular points of P. The singular locus $P^\star$ of P will be formed by the orbit corresponding to the remaining singular point of $\hat\Delta$. By construction,

- ⋄ $P^\star$ is a closed geodesic with angle $2\cdot\pi\cdot(1-\frac{1}{p})$ around it.
- ⋄ $P^\star$ forms a (p,q)-torus knot in the ambient $\mathbb{S}^3$. □

The construction given by Dmitri Panov in [188]. The cone K over P is a polyhedral space with natural complex structure; that is, one can cut simplices from $\mathbb{C}^2$ and the glue the cone from them in such a way that complex structures will agree along the gluings. Moreover the cone K can be holomorphically parametrized by $\mathbb{C}^2$ in such a way that its singular set becomes an algebraic curve $z^p = w^q$ in some (z,w)-coordinates of $\mathbb{C}^2$.



It would be interesting to understand what types of knots can appear this way; the given construction produces only torus knots. We do not know if such knots exist for Euclidean polyhedral spaces, but there are links. For example, the Borromean rings can appear as the singular set of a Euclidean polyhedral metrics on $\mathbb{S}^3$. It can be obtained by gluing each face of cube to it self along the reflections in the middle lines shown on the picture. This construction is due to William Thurston [see 189]

Triangulation of a torus. Let us equip the torus with the flat metric such that each triangle is equilateral. The metric will have two singular cone points, the first corresponds to the vertex v_5 with 5 triangles, the total angle around this point is $\frac{5}{3}\cdot\pi$ and the second corresponds to the vertex v_7 with 7 triangles, the total angle around this point is $\frac{7}{3}\cdot\pi$.

Prove the following.

Observation. The holonomy group of this metric is generated by rotation by $\frac{\pi}{3}$.

Consider a closed geodesic γ_1 which minimize the length of all circles which are not null-homotopic. Let γ_2 be an other closed geodesic which minimize the length and is not homotopic to any power of γ_1.

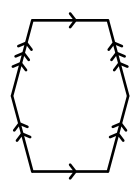
Show that γ_1 and γ_2 intersect at a single point.

Show that γ_i cannot pass v_5.

Apply the observation above to show that if γ_i pass through v_7, then the measure of one of two angles which γ_i cuts at v_7 equals to π. Use the latter statement to show that one can push γ_i aside so it does not longer pass through v_7, but remains a closed geodesic.

Cut $\mathbb{T}^2$ along γ_1 and γ_2. In the obtained quadrilateral, connect v_5 to v_7 by a minimizing geodesic and cut along it. This way we obtain an annulus with flat metric. Look at the neighborhood of the boundary components and show that the annulus can and cannot be isometrically immersed into the plane; this is a contradiction. $\square$

There are flat metrics on the torus with only two singular points which have the total angles $\frac{5}{3}\cdot\pi$ and $\frac{7}{3}\cdot\pi$. Such example can be obtained by identifying the hexagon on the picture according to the arrows. But the holonomy group of the obtained torus is generated by the rotation by angle $\frac{\pi}{6}$. In particular, the observation is necessary in the proof.



The same argument shows that holonomy group of flat torus with exactly two singular points with total angle $2\cdot(1 \pm \frac{1}{n})\cdot\pi$ has more than n elements. In the solution we did the case $n = 6$.

The problem was originally discovered and solved by Stanislav Jendrol' and Ernest Jucovič, in [190], their proof is combinatorial. The solution described above was given by Rostislav Matveyev in his lectures [see 191]. A complex-analytic proof was found by Ivan Izmestiev, Robert Kusner, Günter Rote, Boris Springborn and John Sullivan in [192].

No simple geodesics. The curvature of a vertex on the surface of a convex polyhedron is defined as the $2\cdot\pi - \theta$, where θ is the total angle around the vertex.

Notice that a simple closed geodesic cuts the surface into two discs with total curvature $2\cdot\pi$ each. Therefore it is sufficient to construct a convex polyhedron with curvatures of the vertices $\omega_1, \omega_2, \ldots, \omega_n$ such that $2\cdot\pi$ cannot be obtained as sum of some of ω_i. An example of that type can be found among 3-simplexes. $\square$

The problem is due to Gregory Galperin [see 193] and rediscovered by Dmitry Fuchs and Serge Tabachnikov [see 20.8 in 15].

Chapter 9

Discrete geometry

In this chapter we consider geometrical problems with strong combinatoric flavor. No special prerequisite is needed.

Round circles in $\mathbb{S}^3$

Suppose that you have a finite collection of pairwise linked round circles in the unit 3-sphere, not necessarily all of the same radius. Prove that there is an isotopy of the collection of circles which moves all of them into great circles.

Semisolution. For each circle consider the plane containing it. Note that the circles are linked if and only if the corresponding planes intersect at a single point inside the unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$.

Take the intersection of the planes with the sphere of radius $R \geqslant 1$, rescale and pass to the limit as $R \to \infty$. This way we get needed isotopy. $\square$

The problem was discussed by Genevieve Walsh in [194].

Box in a box

Assume that a parallelepiped with sizes a, b, c lies inside another parallelepiped with sizes a', b', c'. Show that

$$a' + b' + c' \geqslant a + b + c.$$

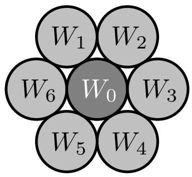
Harnack's circles

Prove that a smooth algebraic curve of degree d in $\mathbb{R}\mathrm{P}^2$ consists of at most $n = \frac{1}{2}\cdot(d^2 - 3\cdot d + 4)$ connected components.

Two points on each line

Construct a set in the Euclidean plane, which intersects each line at exactly 2 points.

Kissing number°



Let W_0 be a convex body in $\mathbb{R}^m$. We say that k is the *kissing number* of W_0 (briefly $k = \operatorname{kiss} W_0$) if k the maximal integer such that there are k bodies $W_1, W_2, \dots, W_k$ such that (1) each W_i is congruent to W_0, (2) $W_i \cap W_0 \neq \varnothing$ for each i and (3) no pair W_i, W_j has common interior points.

As you may guess from the diagram, the kissing number of round disc in the plane is 6.

Show that for any convex body W_0 in $\mathbb{R}^m$

$$\operatorname{kiss} W_0 \geqslant \operatorname{kiss} B,$$

where B denotes the unit ball in $\mathbb{R}^m$.

Monotonic homotopy

Let F be a finite set and $h_0, h_1 \colon F \to \mathbb{R}^m$ be two maps. Consider $\mathbb{R}^m$ as a subspace of $\mathbb{R}^{2\cdot m}$. Show that there is a homotopy $h_t \colon F \to \mathbb{R}^{2\cdot m}$ from h_0 to h_1 such that the function

$$t \mapsto |h_t(x) - h_t(y)|$$

is monotonic for any $x, y \in F$.

Cube

Half of the vertices of an m-dimensional cube are colored in white and the other half in black. Show that the cube has at least 2^{m-1} edges which connect the vertices of different colors.

Geodesic loop

Show that the surface of cube in $\mathbb{R}^3$ does not admit a geodesic loop with the base point at a vertex.

Right and acute triangles

Let $x_1, \dots, x_n \in \mathbb{R}^m$ be a collection of points such that any triangle $[x_i x_j x_k]$ is right or acute. Show that $n \leqslant 2^m$.

Right-angled polyhedron^{+}

A polyhedron is called *right-angled* if all its dihedral angles are right.

Show that in all sufficiently large dimensions, there is no compact convex hyperbolic right-angled polyhedron.

Let us give a short summary of Dehn–Sommerville equations which can help you to solve this problem.

Assume P is a *simple* Euclidean m-dimensional polyhedron; that is, every vertex of P exactly m facets are meeting. Denote by f_k the number of k-dimensional faces of P; the array of integers $(f_0, f_1, \dots f_m)$ is called f-vector of P.

Fix an order of the vertices $v_1, v_2, \dots v_{f_0}$ of P so that for some linear function ℓ, we have $\ell(v_i) > \ell(v_j) \;\Leftrightarrow\; i < j$. The *index* of the vertex v_i is defined as the number of edges $[v_i v_j]$ such that $i < j$. The number of vertices of given index k will be denoted as h_k. The array of integers $(h_0, h_1, \dots h_m)$ is called h-vector of P. Clearly $h_0 = h_m = 1$ and $h_k \geqslant 0$ for all k.

Each k-face of P contains unique vertex which maximize ℓ; if the vertex has index i, then $i \geqslant k$ and then it is the maximal vertex for exactly $\frac{i!}{k!\cdot(i-k)!}$ faces of dimension k. This observation can be packed in the following polynomial identity

$$\sum_k h_k \cdot (t+1)^k = \sum_k f_k \cdot t^k.$$

Note that the identity above implies that h-vector does not depend on the choice of order of the vertices. In particular, the h vector is the same for the reversed order; that is

$$h_k = h_{m-k}$$

for any k. These identities are called Dehn–Sommerville equations. It gives the complete list of linear equations for h-vectors (and therefore f-vectors) of simple polyhedrons.

Balls without gaps

Let $B_1, \dots, B_n$ be the balls of radii $r_1, \dots, r_n$ in a Euclidean space. Assume that no hyperplane divides the balls into two non-empty sets without intersecting at least one of the balls. Show that the balls $B_1, \dots, B_n$ can be covered by a ball of radius $r = r_1 + \dots + r_n$.

Semisolutions

Box in a box. Let Π be a parallelepiped with dimensions a, b and c. Denote by $v(r)$ the volume of r-neighborhoods of Π,

Note that for all positive r we have

$$v(r) = w_3 + w_2{\cdot}r + w_1{\cdot}r^2 + w_0{\cdot}r^3, \qquad (*)$$

where

◇ $w_0 = \frac{4}{3}{\cdot}\pi$ is the volume of unit ball,
◇ $w_1 = \pi{\cdot}(a + b + c)$,
◇ $w_2 = 2{\cdot}(a{\cdot}b + b{\cdot}c + c{\cdot}a)$ is the surface area of Π,
◇ $w_3 = a{\cdot}b{\cdot}c$ is the volume of Π,

Assume Π' be an other parallelepiped with dimensions a', b' and c'. For the volume $v'(r)$ the volume of r-neighborhoods of Π' we have a formula similar $(*)$.

If $\Pi \subset \Pi'$, then $v(r) \leqslant v'(r)$ for any r. For $r \to \infty$, these inequalities imply

$$a + b + c \leqslant a' + b' + c'. \qquad \square$$

The problem was discussed by Alexander Shen in [195].

A formula analogous to $(*)$ holds for arbitrary convex body B in arbitrary dimension m. The coefficient $w_i(B)$ in the polynomial with different normalization constants appear under different names most commonly *intrinsic volume* and *quermassintegral*. They also can be defined as the average of area of projections of B to the i-dimensional planes. In particular if B' and B are convex bodies such that $B' \subset B$, then $w_i(B') \leqslant w_i(B)$ for any i. This generalize our problem quite a bit. Further generalizations lead to so called *mixed volumes* [see 196].

Harnack's circles. Let $\sigma \subset \mathbb{R}\mathrm{P}^2$ be a algebraic curve of degree d. Consider the complexification $\Sigma \subset \mathbb{C}\mathrm{P}^2$ of σ. Without loss of generality, we may assume that Σ is regular.

Prove that all regular complex algebraic curves of degree d in $\mathbb{R}\mathrm{P}^2$ are homeomorphic to each other; denote by g their genus. Perturbing a singular curve formed by d lines in $\mathbb{C}\mathrm{P}^2$, we get that

$$g = \tfrac{1}{2}{\cdot}(d - 1){\cdot}(d - 2).$$

The real curve σ forms the fixed point set in Σ by the complex conjugation. Use it to show that $\Sigma\backslash\sigma$ has at most 2 connected components.

It follows that each connected component of σ adds at least one to the genus of Σ. Hence the result follows. □

The inequality was originally proved by Axel Harnack using a different method [see 197]. The idea to use complexification is due to Felix Klein [see 198]. This problem formed the background to Hilbert's 16th problem [see 145].

Two points on each line. Take any complete ordering of the set of all lines so that each beginning interval has cardinality less than continuum.

Assume we have a set of points X such that each line intersects X at most 2 points and cardinality of X is less than continuum.

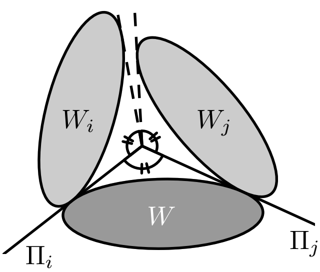
Choose the least line ℓ in the ordering which intersect X by 0 or 1 point. Note that the set of all lines intersecting X at two points has cardinality less than continuum. Therefore we can choose a point on ℓ and add it to X so that the remaining lines are not overloaded.

It remains to apply well ordering principle. □

This problem has endless list of variations. The following problem look similar but far more involved; a solution follows from the proof that a square cannot be cut into triangles of equal area given given by Paul Monsky in [199].

◇ *Subdivide the plane into three everywhere dense sets A, B and C such that each line meets exactly two of these sets.*

Kissing number. Fix m and set $n = \operatorname{kiss} B$. Let $B_1, B_2, \dots, B_n$ the copies of the ball B which touch B and have no common interior points. For each B_i consider the vector v_i from the center of B to the center of B_i. Note that $\measuredangle(v_i, v_j) \geqslant \frac{\pi}{3}$ if $i \neq j$.



For each i, consider supporting hyperplane Π_i to W with outer normal vector v_i. Denote by W_i the reflection of W in Π_i.

Prove that W_i and W_j have no common interior points if $i \neq j$; the latter gives the needed inequality. □

The proof is given by Charles Halberg, Eugene Levin and Ernst Straus in [200]. It is not known if the same inequality holds for the orientation-preserving version of kissing number.

Monotonic homotopy. Note that we can assume that $h_0(F)$ and $h_1(F)$ both lie in the coordinate m-spaces of $\mathbb{R}^{2\cdot m} = \mathbb{R}^m \times \mathbb{R}^m$; that is, $h_0(F) \subset \mathbb{R}^m \times \{0\}$ and $h_1(F) \subset \{0\} \times \mathbb{R}^m$.

Show that the following homotopy is monotonic

$$h_t(x) = \left(h_0(x)\cdot \cos \tfrac{\pi\cdot t}{2}\,,\, h_1(x)\cdot \sin \tfrac{\pi\cdot t}{2}\right).$$

□

This homotopy was discovered by Ralph Alexander in [201]. It has number of applications, one of the most beautiful is the given by Károly Bezdek and Robert Connelly [202] in their proof of Kneser–Poulsen and Klee–Wagon conjectures in dimension 2.

The dimension $2\cdot m$ is optimal; that is, for any positive integer m, there are two maps $h_0, h_1\colon F \to \mathbb{R}^m$ which cannot be connected by a monotonic homotopy $h_t\colon F \to \mathbb{R}^{2\cdot m-1}$. The latter was shown by Maria Belk and Robert Connelly in [203]

Cube. Consider the cube $[-1,1]^m \subset \mathbb{R}^m$. Any vertex this cube has the form $\boldsymbol{q} = (q_1, q_2, \dots, q_m)$, where $q_i = \pm 1$.

For each vertex $\boldsymbol{q}$, consider the intersection of the corresponding octant with the unit sphere; that is, the set

$$V_{\boldsymbol{q}} = \left\{\ (x_1, x_2, \dots, x_m) \in \mathbb{S}^{m-1}\ \middle|\ q_i\cdot x_i \geqslant 0 \text{ for each } i\ \right\}.$$

Consider the set $\mathcal{A} \subset \mathbb{S}^{m-1}$ formed by the union of all the sets $V_{\boldsymbol{q}}$ for black $\boldsymbol{q}$. Note that

$$\operatorname{vol}_{m-1} \mathcal{A} = \tfrac{1}{2}\cdot \operatorname{vol}_{m-1} \mathbb{S}^{m-1}$$

and

$$\operatorname{vol}_{m-2} \partial\mathcal{A} = \tfrac{k}{2^{m-1}}\cdot \operatorname{vol}_{m-2} \mathbb{S}^{m-2},$$

where k is the number of edges of the cube with one end black and the other in white.

It remains to show that

$$\operatorname{vol}_{m-2} \partial\mathcal{A} \geqslant \operatorname{vol}_{m-2} \mathbb{S}^{m-2}.$$

The latter follows from the isoperimetric inequality for $\mathbb{S}^m$. □

The problem was suggested by Greg Kuperberg.

Geodesic loop. Assume such loop exists; denote it by γ and let v be its base vertex.

Denote by ξ and ζ the directions of exit and the entrance of the loop. Let α be the angle between ξ and ζ measured in the tangent cone to the surface of cube at v.

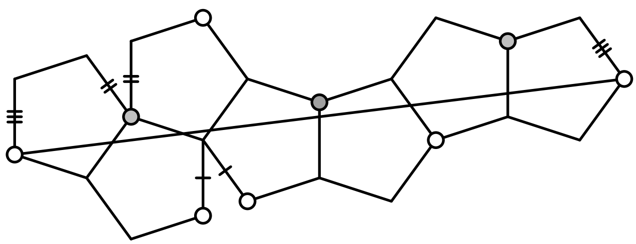
Prove that $\alpha = \frac{\pi}{2}$. To do this you can use Gauss–Bonnet theorem; alternatively, you may look at the unfolding of γ on the plane.

It follows that there is a rotational symmetry of cube with order 3 which fix v and sends ξ to ζ. The later leads to a contradiction. $\square$

For the surface of a higher dimensional cube, the same idea provides existence of symmetry of the cube which fix v and swaps ξ and ζ. From this point one can arrive to a contradiction, but it is harder.

The same statement holds for tetrahedron, octahedron and icosahedron. In this case α is a multiple of $\frac{\pi}{3}$; it implies existence of rotational symmetry of which fix v and sends ξ to ζ and leads to a contradiction the same way.

For the dodecahedron such loop exists; its development shown on the diagram. (The vertices of a cube inscribed in the dodecahedron are circled.)



The problem suggested by Jarosław Kędra.

Right and acute triangles. Denote by K the convex hull of $\{x_1, \dots , x_n\}$. Without loss of generality we can assume that K is non-degenerate polytope. Note that for any distinct x_i and x_j and any interior point z in K we have

$$\measuredangle[x_i\,{}^{x_j}_{z}] < \tfrac{\pi}{2}.$$

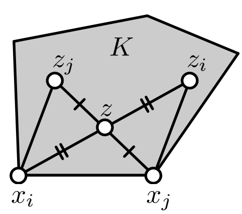
Denote by h_i the homothety with center at x_i and coefficient $\frac{1}{2}$. Set $K_i = h_i(K)$.

Let us show that K_i and K_j have no common interior point. Assume contrary; that is,

$$z = h_i(z_i) = h_j(z_j);$$

for some interior points z_i and z_j in K. Note that

$$\measuredangle[x_i\,{}^{x_j}_{z_i}] + \measuredangle[x_j\,{}^{x_i}_{z_j}] = \pi,$$



which contradicts the claim.

Note that $K_i \subset K$ for any i; it follows that

$$\tfrac{n}{2^m} \cdot \operatorname{vol} K = \sum_{i=1}^{n} \operatorname{vol} K_i \leqslant$$
$$\leqslant \operatorname{vol} K.$$

Hence the result follows. □

The problem was posted by Paul Erdős in [204] and solved by Ludwig Danzer and Branko Grünbaum in [205]. Under the name *Angels in Space*, this problem appears in the *connoisseur's collection* [206] of Peter Winkler.

Grigori Perelman noticed that the same proof works for a similar problem for Alexandrov space [see 207]; the later led to interesting connections to the crystallographic groups [see 208].

The upper bound for the number of points with only acute triangles grows exponentially with m; the later was shown by Paul Erdős and Zoltán Füredi in [209]; the proof use so called *probabilistic method.*

Right-angled polyhedron. Let P be a right-angled hyperbolic polyhedron of dimension m. Note that P is simple; that is, exactly m facets meet at each vertex of P.

From the projective model of hyperbolic plane, one can see that for any simple compact hyperbolic polyhedron there is a simple Euclidean polyhedron with the same combinatorics. In particular Dehn–Sommerville equations hold for P.

Denote by $(f_0, f_1, \dots f_m)$ and $(h_0, h_1, \dots h_m)$ the f- and h-vectors of P. Recall that $h_i \geqslant 0$ for any i and $h_0 = h_m = 1$. By Dehn–Sommerville equations, we get

$$(*) \qquad f_2 > \tfrac{m-2}{4} \cdot f_1.$$

Since P is hyperbolic, each 2-dimensional face of P has at least 5 sides. It follows that

$$f_2 \leqslant \tfrac{m-1}{5} \cdot f_1.$$

The latter contradicts $(*)$ for $m \geqslant 6$. □

The proof above is the core of proof of nonexistence of compact hyperbolic Coxeter's polyhedra of large dimensions given by Ernest Vinberg in [210], see also [211].

Playing a bit more with the same inequalities, one gets nonexistence of right-angled hyperbolic polyhedra, in all dimensions starting from 5. In 4-dimensional case, an example of a bonded right-angled hyperbolic polyhedron can be found among regular *120-cells* — the 4-dimensional brothers of dodecahedra.

Balls without gaps. Assume that each ball has the mass proportional to its radius. Denote by z the center of mass of the balls.

Show that all balls lie in the ball $B(z, r)$. □

The statement was conjectured by Paul Erdős. The solution is given by Adolph and Ruth Goodmans in [212]. A variation was given later by Hugo Hardwiger in [213].

Index

Bibliography

[1] A. Petrunin. *One-step problems in geometry.* URL: `http://mathoverflow.net/q/8247`.

[2] D. Hilbert and S. Cohn-Vossen. *Geometry and the imagination.* Translated by P. Neményi. Chelsea Publishing Company, New York, N. Y., 1952.

[3] V. A. Toponogov. *Differential geometry of curves and surfaces.* A concise guide, With the editorial assistance of Vladimir Y. Rovenski. Birkhäuser Boston, Inc., Boston, MA, 2006.

[4] P. G. Tait. "Note on the circles of curvature of a plane curve." *Proc. Edinb. Math. Soc.* 14 (1896), p. 26.

[5] A. Kneser. „Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie.“ *Heinrich Weber Festschrift.* 1912.

[6] V. Ovsienko and S. Tabachnikov. *Projective differential geometry old and new. From the Schwarzian derivative to the cohomology of diffeomorphism groups.* Cambridge: Cambridge University Press, 2005.

[7] Г. Пестов и В. Ионин. «О наибольшем круге, вложенном в замкнутую кривую». *Докл. АН СССР* 127 (1959), с. 1170—1172.

[8] K. Pankrashkin. "An inequality for the maximum curvature through a geometric flow". *Arch. Math. (Basel)* 105.3 (2015), pp. 297–300.

[9] D. Panov and A. Petrunin. *Ramification conjecture and Hirzebruch's property of line arrangements.* arXiv: `1312.6856 [math.GT]`.

[10] R. H. Bing. "Some aspects of the topology of 3-manifolds related to the Poincaré conjecture". *Lectures on modern mathematics, Vol. II.* Wiley, New York, 1964, pp. 93–128.

[11] В. Н. Лагунов. «О наибольшем шаре, вложенном в замкнутую поверхность, II». *Сибирский математический журнал* 2.6 (1961), с. 874—883.

[12] Stephanie B. Alexander and Richard L. Bishop. "Thin Riemannian manifolds with boundary". *Math. Ann.* 311.1 (1998), pp. 55–70.

[13] I. Fáry. "Sur certaines inégalités geométriques." *Acta Sci. Math.* 12 (1950), pp. 117–124.

[14] S. Tabachnikov. "The tale of a geometric inequality." *MASS selecta: teaching and learning advanced undergraduate mathematics.* Providence, RI: American Mathematical Society (AMS), 2003, pp. 257–262.

[15] D. Fuchs and S. Tabachnikov. *Mathematical omnibus.* Thirty lectures on classic mathematics. American Mathematical Society, Providence, RI, 2007.

[16] J. Lagarias and T. Richardson. "Convexity and the average curvature of plane curves." *Geom. Dedicata* 67.1 (1997), pp. 1–30.

[17] А. И. Назаров и Ф. В. Петров. «О гипотезе С. Л. Табачникова». *Алгебра и анализ* 19.1 (2007), с. 177—193.

[18] I. Fáry. "Sur la courbure totale d'une courbe gauche faisant un nœud". *Bull. Soc. Math. France* 77 (1949), pp. 128–138.

[19] J. Milnor. "On the total curvature of knots." *Ann. Math. (2)* 52 (1950), pp. 248–257.

[20] S. Alexander and R. Bishop. "The Fary-Milnor theorem in Hadamard manifolds". *Proc. Amer. Math. Soc.* 126.11 (1998), pp. 3427–3436.

[21] T. Ekholm, B. White, and D. Wienholtz. "Embeddedness of minimal surfaces with total boundary curvature at most 4π." *Ann. Math. (2)* 155.1 (2002), pp. 209–234.

[22] S. Tabachnikov. "Supporting cords of convex sets. Problem 91-2 in Mathematical Entertainments". *Mathematical Intelligencer* 13.1 (1991), p. 33.

[23] S. Tabachnikov. "The (un)equal tangents problem." *Am. Math. Mon.* 119.5 (2012), pp. 398–405.

[24] Z. E. Brady. *Is it possible to capture a sphere in a knot?* URL: `http://mathoverflow.net/q/8091`.

[25] M. Edelstein and B. Schwarz. "On the length of linked curves." *Isr. J. Math.* 23 (1976), pp. 94–95.

[26] M. D. Kirszbraun. „Über die zusammenziehende und Lipschitzsche Transformationen.“ *Fundam. Math.* 22 (1934), S. 77–108.

[27] F. A. Valentine. "On the extension of a vector function so as to preserve a Lipschitz condition." *Bull. Am. Math. Soc.* 49 (1943), pp. 100–108.

[28] L. Danzer, B. Grünbaum, and V. Klee. "Helly's theorem and its relatives". *Proc. Sympos. Pure Math., Vol. VII.* Amer. Math. Soc., Providence, R.I., 1963, pp. 101–180.

[29] B. Knaster. "Un continu dont tout sous-continu est indécomposable." *Fundam. Math.* 3 (1922), pp. 247–286.

[30] K. Yoneyama. "Theory of continuous set of points." *Tohoku Math. J.* 12 (1918), pp. 43–158.

[31] L. Antoine. "Sur l'homeomorphisme de deux figures et leurs voisinages". *J. Math. Pures Appl.* 4 (1921), pp. 221–325.

[32] K. J. Falconer. *The geometry of fractal sets.* Vol. 85. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1986.

[33] B. Kirchheim, E. Spadaro, and L. Székelyhidi. "Equidimensional isometric maps". *Comment. Math. Helv.* 90.4 (2015), pp. 761–798.

[34] T. Zamfirescu. "Baire categories in convexity." *Atti Semin. Mat. Fis. Univ. Modena* 39.1 (1991), pp. 139–164.

[35] А. Д. Александров. «Внутренняя геометрия произвольной выпуклой поверхности». *Доклады Академии наук СССР* 32.7 (1941), с. 467—470.

[36] А. Д. Милка. «Кратчайшие линии на выпуклых поверхностях». *Доклады АН СССР* 248.1 (1979), с. 34—36.

[37] I. D. Berg. "An estimate on the total curvature of a geodesic in Euclidean 3-space- with-boundary." *Geom. Dedicata* 13 (1982), pp. 1–6.

[38] В. В. Усов. «О длине сферического изображения геодезической на выпуклой поверхности.» *Сибирский математический журнал* 17.1 (1976), с. 233—236.

[39] И. М. Либерман. «Геодезические линии на выпуклых поверхностях». *ДАН СССР* 32 (1941), с. 310—313.

[40] S. Cohn-Vossen. „Totalkrümmung und geodätische Linien auf einfachzusammenhängenden offenen vollständigen Flächenstücken“. *Матем. сб.* 1(43).2 (1936), S. 139–164.

[41] А. Д. Александров и В. В. Стрельцов. «Изопериметрическая задача и оценки длины кривой на поверхности». *Двумерные многообразия ограниченной кривизны. Часть II. Сборник статей по внутренней геометрии поверхностей.* М.–Л.: Наука, 1965, с. 67—80.

[42] V. Bangert. "Geodesics and totally convex sets on surfaces." *Invent. Math.* 63 (1981), pp. 507–517.

[43] M. Gromov. "Sign and geometric meaning of curvature." *Rend. Semin. Mat. Fis. Milano* 61 (1991), pp. 9–123.

[44] F. Rodriguez Hertz. *On the geodesic flow of surfaces of nonpositive curvature.* arXiv: `0301010 [math.DS]`.

[45] С. З. Шефель. «О внутренней геометрии седловых поверхностей». *Сибирский метематический журнал* 5 (1964), с. 1382—1396.

[46] D. Panov. "Parabolic curves and gradient mappings". *Tr. Mat. Inst. Steklova* 221 (1998), pp. 271–288.

[47] P. K. Brendle S. Hung. *Area bounds for minimal surfaces that pass through a prescribed point in a ball.* arXiv: `1607.04631 [math.DG]`.

[48] H. Alexander and R. Osserman. "Area bounds for various classes of surfaces." *Am. J. Math.* 97 (1975), pp. 753–769.

[49] H. Alexander, D. Hoffman, and R. Osserman. "Area estimates for submanifolds of Euclidean space". *Symposia Mathematica, Vol. XIV*. Academic Press, London, 1974, pp. 445–455.

[50] J. O'Rourke. *Why is the half-torus rigid?* URL: `http://mathoverflow.net/q/77760`.

[51] E. Rembs. „Verbiegungen höherer Ordnung und ebene Flächenrinnen." *Math. Z.* 36 (1932), S. 110–121.

[52] Н. В. Ефимов. «Качественные вопросы теории деформаций поверхностей». *УМН* 3.2(24) (1948), с. 47—158.

[53] I. Kh. Sabitov. "On infinitesimal bendings of troughs of revolution. I." *Math. USSR, Sb.* 27 (1977), pp. 103–117.

[54] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* 3rd printing. Basel: Birkhäuser, 2007.

[55] H. Poincaré. "Sur un théorème de géometrie". *Rend. Circ. Mat. Palermo* 33 (1912), pp. 375–407.

[56] G. D. Birkhoff. "Proof of *Poincaré* 's geometric theorem." *Trans. Am. Math. Soc.* 14 (1913), pp. 14–22.

[57] A. Petrunin. *Two discs with no parallel tangent planes.* URL: `http://mathoverflow.net/q/17486`.

[58] P. Pushkar. *A generalization of Cauchy's mean value theorem.* URL: `http://mathoverflow.net/q/16335`.

[59] J. Cheeger and D. G. Ebin. *Comparison theorems in Riemannian geometry.* Revised reprint of the 1975 original. AMS Chelsea Publishing, Providence, RI, 2008.

[60] F. Fang, S. Mendonça, and X. Rong. "A connectedness principle in the geometry of positive curvature." *Commun. Anal. Geom.* 13.4 (2005), pp. 671–695.

[61] B. Wilking. "Torus actions on manifolds of positive sectional curvature." *Acta Math.* 191.2 (2003), pp. 259–297.

[62] S. Alexander, V. Kapovitch, and Petrunin A. *Alexandrov geometry.* URL: `www.math.psu.edu/petrunin`.

[63] M. Gromov. "Almost flat manifolds." *J. Differ. Geom.* 13 (1978), pp. 231–241.

[64] P. Buser and H. Karcher. *Gromov's almost flat manifolds.* Vol. 81. Astérisque. Société Mathématique de France, Paris, 1981.

[65] E. Ruh. "Almost flat manifolds." *J. Differ. Geom.* 17 (1982), pp. 1–14.

[66] R. E. Greene and H. Wu. "On the subharmonicity and plurisubharmonicity of geodesically convex functions." *Indiana Univ. Math. J.* 22 (1973), pp. 641–653.

[67] S. Alexander. "Locally convex hypersurfaces of negatively curved spaces." *Proc. Am. Math. Soc.* 64 (1977), pp. 321–325.

[68] J.-H. Eschenburg. "Local convexity and nonnegative curvature — Gromov's proof of the sphere theorem." *Invent. Math.* 84 (1986), pp. 507–522.

[69] B. Andrews. "Contraction of convex hypersurfaces in Riemannian spaces." *J. Differ. Geom.* 39.2 (1994), pp. 407–431.

[70] F. Almgren. "Optimal isoperimetric inequalities." *Indiana Univ. Math. J.* 35 (1986), pp. 451–547.

[71] M. Gromov. "Isoperimetric inequalities in Riemannian manifolds". V. Milman and G. Schechtman. *Asymptotic theory of finite dimensional normed spaces.* Vol. 1200. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986, pp. 114–129.

[72] A. Weinstein. "Positively curved n-manifolds in $\mathbb{R}^{n+2}$." *J. Differ. Geom.* 4 (1970), pp. 1–4.

[73] M. Micallef and J. Moore. "Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes." *Ann. Math. (2)* 127.1 (1988), pp. 199–227.

[74] C. Böhm and B. Wilking. "Manifolds with positive curvature operators are space forms." *Ann. Math. (2)* 167.3 (2008), pp. 1079–1097.

[75] J. L. Synge. "On the connectivity of spaces of positive curvature." *Q. J. Math., Oxf. Ser.* 7 (1936), pp. 316–320.

[76] T. Frankel. "On the fundamental group of a compact minimal submanifolds." *Ann. Math. (2)* 83 (1966), pp. 68–73.

[77] S. Bochner. "Vector fields and Ricci curvature." *Bull. Am. Math. Soc.* 52 (1946), pp. 776–797.

[78] W.-Y. Hsiang and B. Kleiner. "On the topology of positively curved 4-manifolds with symmetry." *J. Differ. Geom.* 29.3 (1989), pp. 615–621.

[79] K. Grove. "Geometry of, and via, symmetries". *Conformal, Riemannian and Lagrangian geometry (Knoxville, TN, 2000).* Vol. 27. Univ. Lecture Ser. Amer. Math. Soc., Providence, RI, 2002, pp. 31–53.

[80] K. Grove and B. Wilking. "A knot characterization and 1-connected nonnegatively curved 4-manifolds with circle symmetry." *Geom. Topol.* 18.5 (2014), pp. 3091–3110.

[81] C. Croke. "Lower bounds on the energy of maps." *Duke Math. J.* 55 (1987), pp. 901–908.

[82] B. White. "Infima of energy functionals in homotopy classes of mappings." *J. Differ. Geom.* 23 (1986), pp. 127–142.

[83] M. Gromov. "Filling Riemannian manifolds." *J. Differ. Geom.* 18 (1983), pp. 1–147.

[84] C. Croke. "A sharp four dimensional isoperimetric inequality." *Comment. Math. Helv.* 59 (1984), pp. 187–192.

[85] C. Croke. "Some isoperimetric inequalities and eigenvalue estimates." *Ann. Sci. Éc. Norm. Supér. (4)* 13 (1980), pp. 419–435.

[86] E. Hopf. "Closed surfaces without conjugate points". *Proc. Nat. Acad. Sci. U. S. A.* 34 (1948), pp. 47–51.

[87] L. W. Green. "A theorem of E. Hopf". *Michigan Math. J.* 5 (1958), pp. 31–34.

[88] G. Guzhvina. *Gromov's pinching constant.* arXiv: `0804.0201 [math.DG]`.

[89] P. Buser and D. Gromoll. "On the almost negatively curved 3-sphere". *Geometry and analysis on manifolds (Katata/Kyoto, 1987).* Vol. 1339. Lecture Notes in Math. Springer, Berlin, 1988, pp. 78–85.

[90] D. Gromoll, W. Klingenberg, and W. Meyer. *Riemannsche Geometrie im Grossen.* Lecture Notes in Mathematics, No. 55. Springer-Verlag, Berlin-New York, 1968.

[91] J. Cheeger. "Some examples of manifolds of nonnegative curvature." *J. Differ. Geom.* 8 (1973), pp. 623–628.

[92] S. Aloff and N. Wallach. “An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures.” *Bull. Am. Math. Soc.* 81 (1975), pp. 93–97.

[93] D. Gromoll and W. Meyer. “An exotic sphere with nonnegative sectional curvature.” *Ann. Math. (2)* 100 (1974), pp. 401–406.

[94] J.-H. Eschenburg. “New examples of manifolds with strictly positive curvature.” *Invent. Math.* 66 (1982), pp. 469–480.

[95] Ya. V. Bazaikin. “On a family of 13-dimensional closed Riemannian manifolds of positive curvature”. *Siberian Math. J.* 37.6 (1996), pp. 1068–1085.

[96] P. Petersen, F. Wilhelm, and S. Zhu. “Spaces on and beyond the boundary of existence”. *J. Geom. Anal.* 5.3 (1995), pp. 419–426.

[97] V. Kapovitch. “Restrictions on collapsing with a lower sectional curvature bound”. *Math. Z.* 249.3 (2005), pp. 519–539.

[98] А. Д. Милка. «Многомерные пространства с многогранной метрикой неотрицательной кривизны I». *Украинский геометрический сборник* 5–6 (1968), с. 103—114.

[99] G. Perelman. “Proof of the soul conjecture of Cheeger and Gromoll.” *J. Differ. Geom.* 40.1 (1994), pp. 209–212.

[100] M. Gromov and B. Lawson. “Positive scalar curvature and the Dirac operator on complete Riemannian manifolds.” *Publ. Math., Inst. Hautes Étud. Sci.* 58 (1983), pp. 83–196.

[101] R. Schoen and S.-T. Yau. “Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature.” *Ann. Math. (2)* 110 (1979), pp. 127–142.

[102] U. Abresch and D. Gromoll. “On complete manifolds with nonnegative Ricci curvature.” *J. Am. Math. Soc.* 3.2 (1990), pp. 355–374.

[103] J. Cheeger and T. Colding. “Lower bounds on Ricci curvature and the almost rigidity of warped products.” *Ann. Math. (2)* 144.1 (1996), pp. 189–237.

[104] E. Calabi. “On manifolds with non-negative Ricci curvature II”. *Notices AMS* 22 (1975), A205.

[105] S.-T. Yau. “Some function-theoretic properties of complete Riemannian manifold and their applications to geometry.” *Indiana Univ. Math. J.* 25 (1976), pp. 659–670.

[106] J. Lohkamp. “Metrics of negative Ricci curvature.” *Ann. Math. (2)* 140.3 (1994), pp. 655–683.

[107] Edward Witten. “A new proof of the positive energy theorem”. *Comm. Math. Phys.* 80.3 (1981), pp. 381–402.

[108] D. Panov and A. Petrunin. “Sweeping out sectional curvature.” *Geom. Topol.* 18.2 (2014), pp. 617–631.

[109] S. Buyalo. “Volume and the fundamental group of a manifold of nonpositive curvature.” *Math. USSR, Sb.* 50 (1985), pp. 137–150.

[110] S. Alexander, D. Berg, and R. Bishop. “Geometric curvature bounds in Riemannian manifolds with boundary.” *Trans. Am. Math. Soc.* 339.2 (1993), pp. 703–716.

[111] A. S. Besicovitch. “On two problems of Loewner.” *J. Lond. Math. Soc.* 27 (1952), pp. 141–144.

[112] M. Gromov. “Pseudo holomorphic curves in symplectic manifolds.” *Invent. Math.* 82 (1985), pp. 307–347.

[113] A. Nabutovsky. “Einstein structures: Existence versus uniqueness.” *Geom. Funct. Anal.* 5.1 (1995), pp. 76–91.

[114] K. Honda. “Transversality theorems for harmonic forms.” *Rocky Mt. J. Math.* 34.2 (2004), pp. 629–664.

[115] R. Bishop and B. O’Neill. “Manifolds of negative curvature.” *Trans. Am. Math. Soc.* 145 (1969), pp. 1–49.

[116] S.-T. Yau. "Non-existence of continuous convex functions on certain Riemannian manifolds." *Math. Ann.* 207 (1974), pp. 269–270.

[117] C. Croke. "Small volume on big n-spheres." *Proc. Am. Math. Soc.* 136.2 (2008), pp. 715–717.

[118] L. Guth. "Symplectic embeddings of polydisks." *Invent. Math.* 172.3 (2008), pp. 477–489.

[119] H. Weyl. "On the volume of tubes." *Am. J. Math.* 61 (1939), pp. 461–472.

[120] S. Frankel and M. Katz. "The Morse landscape of a Riemannian disk." *Ann. Inst. Fourier* 43.2 (1993), pp. 503–507.

[121] A. Nabutovsky and R. Rotman. "Length of geodesics and quantitative Morse theory on loop spaces." *Geom. Funct. Anal.* 23.1 (2013), pp. 367–414.

[122] F. Balacheff, C. Croke, and M. Katz. "A Zoll counterexample to a geodesic length conjecture." *Geom. Funct. Anal.* 19.1 (2009), pp. 1–10.

[123] V. Milman and G. Schechtman. *Asymptotic theory of finite-dimensional normed spaces*. Vol. 1200. Lecture Notes in Mathematics. With an appendix by M. Gromov. Springer-Verlag, Berlin, 1986.

[124] M. Ledoux. *The concentration of measure phenomenon*. Vol. 89. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2001.

[125] A. Nabutovsky. "Disconnectedness of sublevel sets of some Riemannian functionals." *Geom. Funct. Anal.* 6.4 (1996), pp. 703–725.

[126] D. Burago, Yu. Burago, and S. Ivanov. *A course in metric geometry*. Providence, RI: American Mathematical Society (AMS), 2001.

[127] C. Kuratowski. "Quelques problèmes concernant les espaces métriques nonseparables." *Fundam. Math.* 25 (1935), pp. 534–545.

[128] M. Fréchet. "Les ensembles abstraits et le calcul fonctionnel." *Rend. Circ. Mat. Palermo* 30 (1910), pp. 1–26.

[129] M. Gromov. "Hyperbolic manifolds, groups and actions". *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference*. Vol. 97. Ann. of Math. Stud. Princeton Univ. Press, Princeton, N.J., 1981, pp. 183–213.

[130] J. Nash. "C^1 isometric imbeddings." *Ann. Math. (2)* 60 (1954), pp. 383–396.

[131] N. Kuiper. "On C^1-isometric imbeddings. I, II." *Nederl. Akad. Wet., Proc., Ser. A* 58 (1955), pp. 545–556, 683–689.

[132] H. Rademacher. „Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variablen und über die Transformation der Doppelintegrale. I, II.“ *Math. Ann.* 79 (1920), S. 340–359.

[133] L. C. Siebenmann. "Deformation of homeomorphisms on stratified sets." *Comment. Math. Helv.* 47 (1972), pp. 123–136.

[134] D. Sullivan. "Hyperbolic geometry and homeomorphisms". *Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977)*. Academic Press, New York-London, 1979, pp. 543–555.

[135] A. Karlsson. *Ergodic theorems for noncommuting random products*. URL: `http://www.unige.ch/math/folks/karlsson/`.

[136] S. Ferry and B. Okun. "Approximating topological metrics by Riemannian metrics." *Proc. Am. Math. Soc.* 123.6 (1995), pp. 1865–1872.

[137] D. Burago, S. Ivanov, and D. Shoenthal. "Two counterexamples in low-dimensional length geometry." *St. Petersbg. Math. J.* 19.1 (2008), pp. 33–43.

[138] E. Le Donne. "Lipschitz and path isometric embeddings of metric spaces". *Geom. Dedicata* 166 (2013), pp. 47–66.

[139] A. Petrunin and A. Yashinski. "Piecewise isometric mappings". *St. Petersburg Math. J.* 27.1 (2016), pp. 155–175.

[140] A. Petrunin. "On intrinsic isometries to Euclidean space." *St. Petersbg. Math. J.* 22.5 (2011), pp. 803–812.

[141] S. Mazur and S. Ulam. "Sur les transformations isométriques d'espaces vectoriels, normés." *C. R. Acad. Sci., Paris* 194 (1932), pp. 946–948.

[142] J. Väisälä. "A proof of the Mazur–Ulam theorem." *Am. Math. Mon.* 110.7 (2003), pp. 633–635.

[143] A. Pogorelov. *Hilbert's fourth problem.* V. H. Winston & Sons, Washington, D.C.; A Halsted Press Book, John Wiley & Sons, New York-Toronto, Ont.-London, 1979.

[144] D. Hilbert. „Ueber die gerade Linie als kürzeste Verbindung zweier Punkte." *Math. Ann.* 46 (1895), S. 91–96.

[145] D. Hilbert. "Mathematical problems". *Bull. Amer. Math. Soc.* 8.10 (1902), pp. 437–479.

[146] T. Foertsch and V. Schroeder. "Minkowski versus Euclidean rank for products of metric spaces." *Adv. Geom.* 2.2 (2002), pp. 123–131.

[147] P. Hitzelberger and A. Lytchak. "Spaces with many affine functions." *Proc. Am. Math. Soc.* 135.7 (2007), pp. 2263–2271.

[148] N. I. Lobachevsky. *Geometrische Untersuchungen zur Theorie der Parallellinien.* Berlin: F. Fincke, 1840.

[149] A. Całka. "On conditions under which isometries have bounded orbits." *Colloq. Math.* 48 (1984), pp. 219–227.

[150] M. H. A. Newman. "A theorem on periodic transformations of spaces." *Q. J. Math., Oxf. Ser.* 2 (1931), pp. 1–8.

[151] D. Montgomery. "Pointwise periodic homeomorphisms." *Am. J. Math.* 59 (1937), pp. 118–120.

[152] V. Šahović. "Approximations of Riemannian manifolds with linear curvature constraints." Thesis. Univ. Münster, 2009.

[153] D. Panov and A. Petrunin. "Telescopic actions." *Geom. Funct. Anal.* 22.6 (2012), pp. 1814–1831.

[154] A. Petrunin. *Diameter of m-fold cover.* URL: `http://mathoverflow.net/q/7732`.

[155] D. Burago and B. Kleiner. "Rectifying separated nets." *Geom. Funct. Anal.* 12.1 (2002), pp. 80–92.

[156] C. Lange. *When is the underlying space of an orbifold a topological manifold.* arXiv: `1307.4875 [math.GN]`.

[157] М. А. Михайлова. «О факторпространстве по действию конечной группы, порожденной псевдоотражениями». *Изв. АН СССР. Сер. матем.* 48.1 (1984), с. 104—126.

[158] M. Hall. "Subgroups of finite index in free groups." *Can. J. Math.* 1 (1949), pp. 187–190.

[159] J. Stallings. "Topology of finite graphs." *Invent. Math.* 71 (1983), pp. 551–565.

[160] H. Wilton. *In Memoriam J. R. Stallings — Topology of Finite Graphs.* URL: `https://ldtopology.wordpress.com/2008/12/01/`.

[161] M. Gerstenhaber and O. S. Rothaus. "The solution of sets of equations in groups". *Proc. Nat. Acad. Sci. U.S.A.* 48 (1962), pp. 1531–1533.

[162] H. Hopf. „Über den Rang geschlossener Liescher Gruppen". *Comment. Math. Helv.* 13 (1940), S. 119–143.

[163] C. Kosniowski. *A first course in algebraic topology.* Cambridge University Press, Cambridge-New York, 1980.

[164] V. Klee. "Some topological properties of convex sets". *Trans. Amer. Math. Soc.* 78 (1955), pp. 30–45.

[165] D. Bennequin. "Exemples d'immersions du disque dans le plan qui ne sont pas projections de plongements dans l'espace." *C. R. Acad. Sci., Paris, Sér. A* 281 (1975), pp. 81–84.

[166] K. W. Kwun. "Uniqueness of the open cone neighborhood". *Proc. Amer. Math. Soc.* 15 (1964), pp. 476–479.

[167] R. C. Kirby. "Stable homeomorphisms and the annulus conjecture." *Ann. Math. (2)* 89 (1969), pp. 575–582.

[168] P. Alexandroff. „Diskrete Räume." *Матем. сб.* 2 (1937), S. 501–519.

[169] M. C. McCord. "Singular homology groups and homotopy groups of finite topological spaces." *Duke Math. J.* 33 (1966), pp. 465–474.

[170] "Geometric group theory, hyperbolic dynamics and symplectic geometry". *Oberwolfach Rep.* 9.3 (2012). Abstracts from the workshop held July 15–21, 2012, pp. 2139–2203.

[171] A. Lytchak and S. Wenger. *Intrinsic structure of minimal discs in metric spaces.* arXiv: `1602.06755 [math.DG]`.

[172] K. Menger. „Untersuchungen über allgemeine Metrik". *Math. Ann.* 100.1 (1928), S. 75–163.

[173] R. H. Bing. "A convex metric for a locally connected continuum". *Bull. Amer. Math. Soc.* 55 (1949), pp. 812–819.

[174] R. H. Bing. "Partitioning continuous curves". *Bull. Amer. Math. Soc.* 58 (1952), pp. 536–556.

[175] E. E. Moise. "Grille decomposition and convexification theorems for compact metric locally connected continua". *Bull. Amer. Math. Soc.* 55 (1949), pp. 1111–1121.

[176] В. А. Залгаллер. «О деформациях многоугольника на сфере». *УМН* 11.5(71) (1956), с. 177—178.

[177] В. А. Топоногов. «Оценка длины замкнутой геодезической на выпуклой поверхности». *Докл. АН СССР* 124.2 (1959), с. 282—284.

[178] А. Д. Александров. *Внутренняя геометрия выпуклых поверхностей.* ОГИЗ, М.-Л., 1948.

[179] В. А. Залгаллер. «Изометричекие вложения полиэдров». *Доклады АН СССР* 123 (1958), с. 599—601.

[180] S. Krat. "Approximation Problems in Length Geometry". Ph.D. thesis. Pennsylvania State University, 2005.

[181] U. Brehm. "Extensions of distance reducing mappings to piecewise congruent mappings on $\mathbb{R}^m$." *J. Geom.* 16 (1981), pp. 187–193.

[182] A. Akopyan and A. Tarasov. "A constructive proof of Kirszbraun's theorem." *Math. Notes* 84.5 (2008), pp. 725–728.

[183] Yu. D. Burago and V. A. Zalgaller. "Isometric piecewise-linear embeddings of two-dimensional manifolds with a polyhedral metric into $\mathbb{R}^3$". *St. Petersburg Math. J.* 7.3 (1996), pp. 369–385.

[184] H. A. Schwarz. "Sur une définition erronée de l'aire d'une surface courbe". *Gesammelte Mathematische Abhandlungen* 1 (1890), pp. 309–311.

[185] A. Petrunin. "Area minimizing polyhedral surfaces are saddle". *Amer. Math. Monthly* 122.3 (2015), pp. 264–267.

[186] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants.* Modern Birkhäuser Classics. Reprint of the 1994 edition. Birkhäuser Boston, Inc., Boston, MA, 2008.

[187] E. Schönhardt. „Über die Zerlegung von Dreieckspolyedern in Tetraeder". *Math. Ann.* 98.1 (1928), S. 309–312.

[188] D. Panov. "Polyhedral Kähler manifolds." *Geom. Topol.* 13.4 (2009), pp. 2205–2252.

[189] W. Thurston. *Three-dimensional geometry and topology. Vol. 1.* Vol. 35. Princeton Mathematical Series. Edited by Silvio Levy. Princeton University Press, Princeton, NJ, 1997.

[190] S. Jendrol' and E. Jucovič. "On the toroidal analogue of Eberhard's theorem". *Proc. London Math. Soc. (3)* 25 (1972), pp. 385–398.

[191] R. Matveev. "Surfaces with polyhedral metrics". International Mathematical Summer School for Students 2011. Jacobs University, Bremen.

[192] I. Izmestiev, R. Kusner, G. Rote, B. Springborn, and J. Sullivan. "There is no triangulation of the torus with vertex degrees 5,6,...,6,7 and related results: geometric proofs for combinatorial theorems." *Geom. Dedicata* 166 (2013), pp. 15–29.

[193] G. Galperin. "Convex polyhedra without simple closed geodesics." *Regul. Chaotic Dyn.* 8.1 (2003), pp. 45–58.

[194] G. Walsh. "Great circle links in the three-sphere". Thesis (Ph.D.)–University of California, Davis. 2003.

[195] A. Shen. "Unexpected proofs. Boxes in a Train". *Math. Intelligencer* 21.3 (1999), pp. 48–50.

[196] Yu. Burago and V. Zalgaller. *Geometric inequalities.* Berlin etc.: Springer-Verlag, 1988.

[197] A. Harnack. „Ueber die Vieltheiligkeit der ebenen algebraischen Curven". *Math. Ann.* 10.2 (1876), S. 189–198.

[198] F. Klein. „Ueber den Verlauf der Abel'schen Integrale bei den Curven vierten Grades". *Math. Ann.* 10.3 (1876), S. 365–397.

[199] P. Monsky. "On dividing a square into triangles." *Am. Math. Mon.* 77 (1970), pp. 161–164.

[200] C. Halberg, E. Levin, and E. G. Straus. "On contiguous congruent sets in Euclidean space." *Proc. Am. Math. Soc.* 10 (1959), pp. 335–344.

[201] R. Alexander. "Lipschitzian mappings and total mean curvature of polyhedral surfaces. I." *Trans. Am. Math. Soc.* 288 (1985), pp. 661–678.

[202] K. Bezdek and R. Connelly. "Pushing disks apart — the Kneser–Poulsen conjecture in the plane." *J. Reine Angew. Math.* 553 (2002), pp. 221–236.

[203] M. Belk and R. Connelly. *Making contractions continuous: a problem related to the Kneser–Poulsen conjecture.* URL: `math.bard.edu/~mbelk/`.

[204] P. Erdős. "Some unsolved problems". *Michigan Math. J.* 4 (1957), pp. 291–300.

[205] L. Danzer und B. Grünbaum. „Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee". *Math. Z.* 79 (1962), S. 95–99.

[206] P. Winkler. *Mathematical puzzles: a connoisseur's collection.* A K Peters, Ltd., Natick, MA, 2004.

[207] G. Perelman. "Spaces with curvature bounded below". *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994).* Birkhäuser, Basel, 1995, pp. 517–525.

[208] N. Lebedeva. "Alexandrov spaces with maximal number of extremal points". *Geom. Topol.* 19.3 (2015), pp. 1493–1521.

[209] P. Erdős and Z. Füredi. "The greatest angle among n points in the d-dimensional Euclidean space". *Combinatorial mathematics (Marseille-Luminy, 1981).* Vol. 75. North-Holland Math. Stud. North-Holland, Amsterdam, 1983, pp. 275–283.

[210] Э. Б. Винберг. «Дискретные группы отражений в пространствах Лобачевского большой размерности». *Модули и алгебраические группы. 2.* Л.: Наука, 1983, с. 62—68.

[211] Э. Б. Винберг. «Отсутствие кристаллографических групп отражений в пространствах Лобачевского большой размерности». *Функц. анализ и его прил.* 15.2 (1981), с. 67—68.

[212] A. W. Goodman and R. E. Goodman. "A circle covering theorem". *Amer. Math. Monthly* 52 (1945), pp. 494–498.

[213] H. Hadwiger. "Nonseparable convex systems". *Amer. Math. Monthly* 54 (1947), pp. 583–585.