ON THE TOTAL CURVATURE OF MINIMIZING GEODESICS ON CONVEX SURFACES

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Abstract. We give a universal upper bound for the total curvature of minimizing geodesic on a convex surface in the Euclidean space.

1. Introduction

In this note we give an affirmative answer to the question asked by Dmitry Burago; the same question was also stated in [1], [2] and [3]. Namely, we prove the following.

1.1. Main theorem. *The total curvature of a minimizing geodesic on a convex surface in $\mathbb{R}^3$ can not exceed 10^{100}.*

The optimal upper bound is expected to be slightly bigger than $2{\cdot}\pi$. The value $2{\cdot}\pi$ is the optimal bound for the analogous problem in the plane. A minimizing geodesic on a convex surface in $\mathbb{R}^3$ with total curvature slightly bigger that $2{\cdot}\pi$ was constructed by Bárány, Kuperberg, and Zamfirescu in [3].

Let us list other related results.

- ◇ In [4], Liberman gives a bound on the total curvature of short geodesic in terms of the ratio diameter and inradius of K. In the proof he use now so called Liberman's lemma, see 3.1 below.
- ◇ In [5], Usov gives the optimal bound for total curvature of geodesic on the graph of ℓ-Lipscitz convex function. Namely, he proves that if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is ℓ-Lipschitz and convex then any geodesic in its graph

  $$\Gamma_f = \left\{\, (x, y, z) \in \mathbb{R}^3 \,\middle|\, z = f(x, y) \,\right\}$$

  has total curvature at most $2{\cdot}\ell$. Yet an amusing generalization of Usov's result is given by Berg in [6].
- ◇ In [7], Pogorelov conjectured that any the spherical image of geodesic on convex surface has to be contructable. It is easy to see that the length of spherical image of geodesic can not be smaller than its total curvature, so this conjecture (if it would be true) would be stronger than Liberman's theorem. Counterexamples were found indepenently by Milka in [8], Usov in [9] and yet much later rediscovered by Pach in [2].
- ◇ In [3], Bárány, Kuperberg, and Zamfirescu have constructed a corkscrew minimizing geodesic on a closed hypersurface; that is a minimizing geodesic which twists around given line arbitrary many times. They also rediscovered the results of Liberman and Usov mentioned above.

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2. Preliminaries

Semicontinuity of total curvature. Recall that the *total curvature* of a curve $\gamma\colon[0,\ell]\to\mathbb{R}^3$ (briefly $\operatorname{TotCurv}\gamma$) is defined as supremum of sum of exterior angles for the polygonal lines inscribed in γ.

Note that for a polygonal line σ, its total curvature coinside with the sum of its exterior angles.

If γ is a smooth curve with unit-speed parametrization, then

$$\operatorname{TotCurv}\gamma=\int\limits_0^\ell\kappa(t)\cdot dt,$$

where $\kappa(t)=|\ddot\gamma(t)|$ is the curvature of γ at t.

2.1. Proposition. *Assume $\gamma_n\colon\mathbb{I}\to\mathbb{R}^3$ is a sequence of curves converging pointwise to a curve $\gamma_\infty\colon\mathbb{I}\to\mathbb{R}^3$. Then*

$$\liminf_{n\to\infty}\operatorname{TotCurv}\gamma_n\geqslant\operatorname{TotCurv}\gamma_\infty.$$

Proof. Choose a polygonal line σ_∞ inscribed in γ_∞ with total curvature sufficiently close to $\operatorname{TotCurv}\gamma_\infty$. Let $\gamma_\infty(t_0),\dots,\gamma_\infty(t_k)$ for $t_0<\dots<t_k$ be the vertices of σ_∞. Consider the polygonal lines σ_n inscribed in γ_n with the vertices $\gamma_n(t_0),\dots,\gamma_n(t_k)$. Note that

$$\operatorname{TotCurv}\sigma_n\to\operatorname{TotCurv}\sigma_\infty.$$

By the definition of total curvature

$$\operatorname{TotCurv}\sigma_n\leqslant\operatorname{TotCurv}\gamma_n.$$

The statement follows since the broken line σ_∞ can be chosen in such a way that $\operatorname{TotCurv}\sigma_n$ is arbitrary close to $\operatorname{TotCurv}\gamma_n$. □

Convergence. Given a closed set $\Sigma\subset\mathbb{R}^3$, denote by $\operatorname{dist}_\Sigma$ the distance function from Σ; that is

$$\operatorname{dist}_\Sigma(x)=\inf\{\,|x-y|\mid y\in\Sigma\,\}.$$

We say that a sequence of closed sets $\Sigma_n\subset\mathbb{R}^3$ converges to the closed set $\Sigma_\infty\subset\mathbb{R}^3$ if for any $x\in\mathbb{R}^3$, we have $\operatorname{dist}_{\Sigma_n}(x)\to\operatorname{dist}_{\Sigma_\infty}(x)$ as $n\to\infty$.

Convex surfaces. By convex surface in the Euclidean 3-space $\mathbb{R}^3$ we understand the boundary of closed convex set with nonempty interior.

2.2. Proposition. *Assume Σ_n be a sequence of convex surfaces which converge to a convex surface Σ_∞. Then for any minimizing geodesic γ_∞ in Σ_∞ there is a sequence of minimizing geodesics γ_n in Σ_n which pointwise converge to γ_∞ as $n\to\infty$.*

Proof. Assume $\gamma_\infty\colon[0,\ell]\to\Sigma_\infty$ is parametrized by its arc length.

Fix a subinterval $[a,b]\subset[0,\ell]$ such that $0<a$ and $b<\ell$. Set $p_\infty=\gamma_\infty(a)$ and $q_\infty=\gamma_\infty(b)$.

Let $p_n,q_n\in\Sigma_n$ be a two sequences of points which converge to p_∞ and q_∞ correspondingly.

Denote by γ_n a minimizing geodesic from p_n to q_n in Σ_n. Note that γ_n converges to $\gamma_\infty|_{[a,b]}$ as $n\to\infty$.

Taking the subinterval $[a,b]$ closer and closer to $[0,\ell]$ and applying diagonal procedure, we get the result. □

3. Liberman's lemma.

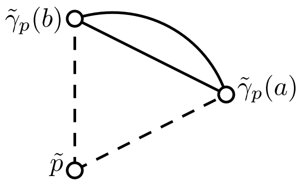
In this section we give a slight generalization of the construction given by Liberman in [4].

Development. Let $\gamma\colon [0,\ell]\to\mathbb{R}^3$ be a curve parametrized by length and a point p does not lie on γ.

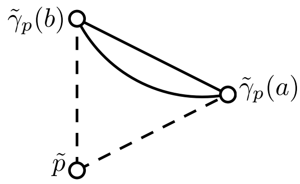
Assume $\tilde\gamma_p\colon [0,\ell]\to\mathbb{R}^2$ is a plane curve parametrized by length and $\tilde p$ is a point in the plane such that

$$|\tilde p-\tilde\gamma(t)|=|\tilde p-\tilde\gamma(t)|$$

for any $t\in[0,\ell]$ and the direction from $\tilde p$ to $\tilde\gamma(t)$ changes counterclockwise. Then $\tilde\gamma_p$ is called *development* of γ with respect to p.



Convex development.



Concave development.

We say that the development $\tilde\gamma_p$ is convex (concave) in the interval $[a,b]$ if the lune bounded by arc $\tilde\gamma_p|_{[a,b]}$ and the segment $[\tilde\gamma_p(a)\tilde\gamma_p(b)]$ is convex and lies on with $\tilde p$ on the opposite side from the line $(\tilde\gamma_p(a)\tilde\gamma_p(b))$ (correspondingly the same side from the line $(\tilde\gamma_p(a)\tilde\gamma_p(b))$).

We say that $\tilde\gamma_p$ is locally convex (concave) in the interval $[a,b]$ if any point $x\in[a,b]$ admits a closed neighborhood $[a',b']$ in $[a,b]$ such that $\tilde\gamma_p$ is convex (correspondingly concave) in the interval $[a',b']$.

If we pass to the limit of this construction as p moves to infinity along a half-line in the derection of a unit vector $\boldsymbol{u}$ then the limit curve is called development of γ in direction $\boldsymbol{u}$ and denoted as $\tilde\gamma_{\boldsymbol{u}}$.

The development $\tilde\gamma_{\boldsymbol{u}}$ could be also defined directly. Namely the curve $\tilde\gamma_{\boldsymbol{u}}\colon[0,\ell]\to\mathbb{R}^2$ is parametrized by length and and for a fixed unit vector $\tilde{\boldsymbol{u}}\in\mathbb{R}^2$, we have

$$\langle\tilde{\boldsymbol{u}},\tilde\gamma_{\boldsymbol{u}}(t)\rangle=\langle\boldsymbol{u},\gamma(t)\rangle$$

for any $t\in[0,\ell]$ and the projection of $\tilde\gamma_{\boldsymbol{u}}(t)$ to the line normal to $\tilde{\boldsymbol{u}}$ is monotonic in t.

We can assume that $\tilde{\boldsymbol{u}}$ is the vertical vector in the coordinate plane. We say that $\tilde\gamma_{\boldsymbol{u}}$ is concave (convex) in the interval $[a,b]$ if the lune bounded by arc $\tilde\gamma_{\boldsymbol{u}}|_{[a,b]}$ and the segment $[\tilde\gamma_{\boldsymbol{u}}(a)\tilde\gamma_{\boldsymbol{u}}(b)]$ is convex and lies above (correspondingly below) the line segment $[\tilde\gamma_{\boldsymbol{u}}(a)\tilde\gamma_{\boldsymbol{u}}(b)]$.

Dark and light sides. Let $\Sigma\subset\mathbb{R}^3$ be a smooth convex surface and $z\notin\Sigma$ and $p\in\Sigma$.

We say that p lies on the dark (light) side of Σ with from z if non of the points $p+t\cdot(p-z)$ lie inside of Σ for $t>0$ (correspondingly for $t<0$). The intersection of dark and light side is called horizon of z; it is denoted by ω_z.

Note that if z lies inside Σ then all the points on Σ lies on dark side from z.

If Σ is smooth we can define the outer normal vector ν_p to Σ at p. In this case p lies on dark (light) side of Σ with from z if and only if $\langle p-z,\nu_p\rangle\geqslant 0$ (correspondingly for $\langle p-z,\nu_p\rangle\leqslant 0$). If in addition Σ is strongly convex then the horizon is formed by a collection of disjoint smooth curves.

We could also define light/dark sides and horizon in the limit case, as p escapes to infinity along a half-line in derection $\boldsymbol{u}$. Let us also define it derectly, we say that a point $p\in\Sigma$ lies on dark (light) side for the unit vector u if non of the points

$p+\boldsymbol{u}\cdot t$ lie inside of Σ for $t>0$, (correspondingly $t<0$). As before the intersection of light and dark side is called horizon of $\boldsymbol{u}$ and it is denoted as $\omega_{\boldsymbol{u}}$.

In the smooth case the later means that $\langle\nu_p,u\rangle\geqslant 0$ (correspondingly $\langle\nu_p,u\rangle\leqslant 0$). If Σ is strongly convex then $\omega_{\boldsymbol{u}}$ is locally a smooth curve.

3.1. Liberman's Lemma. *Assume γ be a geodesic on convex surface $\Sigma\subset\mathbb{R}^3$. Then for any point $z\notin\Sigma$ the development $\tilde\gamma_z$ is locally convex (locally concve) if γ lies on dark (correspondingly light) side of Σ from z.*

Similarly for any unit vector $\boldsymbol{u}$, the development $\tilde\gamma_{\boldsymbol{u}}$ is locally convex (locally concave) if γ lies on dark (correspondingly light) side of Σ for $\boldsymbol{u}$.

Note that for any space curve γ and any unit vector $\boldsymbol{u}$ we have

$$\operatorname{TotCurv}\tilde\gamma_{\boldsymbol{u}}\leqslant\operatorname{TotCurv}\gamma.$$

On the other hand total curvature of few developments gives an estimate for the total curvature of the original curve. For example, if $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ is the orthonormal basis then

$$\operatorname{TotCurv}\gamma\leqslant\operatorname{TotCurv}\tilde\gamma_{\boldsymbol{i}}+\operatorname{TotCurv}\tilde\gamma_{\boldsymbol{j}}+\operatorname{TotCurv}\tilde\gamma_{\boldsymbol{k}}.$$

If γ lies on one dark or light side for direction $\boldsymbol{u}$ then by Liberman's lemma we get

$$\operatorname{TotCurv}\tilde\gamma_{\boldsymbol{u}}\leqslant\pi.$$

It follows that is γ cross the horisons $\omega_{\boldsymbol{i}}$, $\omega_{\boldsymbol{j}}$ and $\omega_{\boldsymbol{k}}$ at most N times then

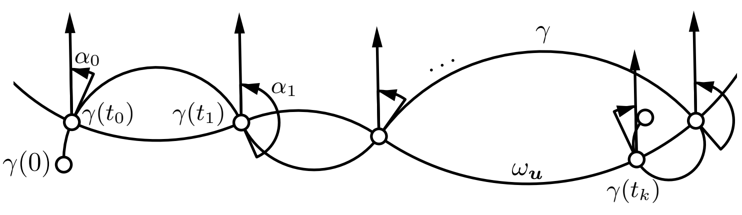
$$\begin{aligned}\operatorname{TotCurv}\gamma&\leqslant\operatorname{TotCurv}\tilde\gamma_{\boldsymbol{i}}+\operatorname{TotCurv}\tilde\gamma_{\boldsymbol{j}}+\operatorname{TotCurv}\tilde\gamma_{\boldsymbol{k}}\leqslant\\&\leqslant(N+1)\cdot\pi.\end{aligned}$$

Therefore if γ violetes Main Theorem then it has to cross the horisons $\omega_{\boldsymbol{i}}$, $\omega_{\boldsymbol{j}}$ and $\omega_{\boldsymbol{k}}$ more that 10^9 times.

4. Curvature of development

Let $\Sigma\subset\mathbb{R}^3$ be a strongly convex smooth surface and $\gamma\colon[0,\ell]\to\Sigma$ be a unit-speed geodesic. Assume that for some unit vector $\boldsymbol{u}$, the γ cross the horizon $\omega_{\boldsymbol{u}}$ transversely at $t_1<\dots t_k$. Set $\alpha_i=\pm\measuredangle(\dot\gamma(t_i),\boldsymbol{u})$ for each i, we can assume that $\alpha_i\in(-\pi,\pi)$ and the sign is taken so that $[\dot\gamma(t_i),\boldsymbol{u}]=\sin\alpha_i\cdot\nu_{\gamma(t_i)}$, where $[*,*$ denotes vector product and ν_p is the outer normal vector to Σ at the point $p\in\Sigma$.

The values t_i will be called *meeting moments* and the angles α_i will be called *meeting angles* of the geodesic γ with the horizon $\omega_{\boldsymbol{u}}$. The arc $\gamma|_{[t_{i-1},t_i]}$ will be called $\omega_{\boldsymbol{u}}$*-archway.*



Let us introduce new notation

$$\operatorname{TotCurv}_{\boldsymbol{u}}\gamma\stackrel{\text{def}}{=\!=}\operatorname{TotCurv}\tilde\gamma_{\boldsymbol{u}}.$$

From Liberman's lemma 3.1, we get the following.

4.1. Corollary. *Let $\Sigma\subset\mathbb{R}^3$ be a strongly convex smooth surface and $\gamma\colon[0,\ell]\to\Sigma$ be a unit-speed geodesic. Assume that for some unit vector $\boldsymbol{u}$, the γ cross the horizon $\omega_{\boldsymbol{u}}$ transversely and $t_1<\dots t_k$ be its meeting moments and $\alpha_1,\dots\alpha_k$ be its meeting angles with the horizon $\omega_{\boldsymbol{u}}$. Then*

$$\operatorname{TotCurv}_{\boldsymbol{u}}\gamma\leqslant 4\cdot\pi+2\cdot\big||\alpha_1|-|\alpha_2|+\dots-(-1)^k\cdot|\alpha_k|\big|$$

To find the needed estimate the total curvature of geodesic we will get an upper bound for

$$\big||\alpha_1| - |\alpha_2| + \dots - (-1)^k{\cdot}|\alpha_k|\big|.$$

Most of the remaining part of paper devoted to finding such an upper bound.

Proof. By Liberman's lemma,

$$\operatorname{TotCurv}_{\boldsymbol{u}}(\gamma|_{[t_{i-1},t_i]}) = \pm(|\alpha_{i-1}| - |\alpha_i)|$$

where the sign is "+" if the archway $[t_i, t_{i+1}]$ lies on the dark side and "−" if it lies on the light side from u. Summing all this up we get

$$\operatorname{TotCurv}_{\boldsymbol{u}}(\gamma|_{[t_1,t_k]}) = \big||\alpha_1| - 2{\cdot}|\alpha_2| + \dots + (-1)^k{\cdot}2{\cdot}|\alpha_{k-1}| - (-1)^k{\cdot}|\alpha_k|\big|.$$

By Liberman's lemma we also have

$$\operatorname{TotCurv}_{\boldsymbol{u}}(\gamma|_{[0,t_1]}), \operatorname{TotCurv}_{\boldsymbol{u}}(\gamma|_{[t_k,\ell]}) \leqslant \pi$$

Since $|\alpha_k| \leqslant \pi$, the statement follows. □

If Σ is a surface in $\mathbb{R}^3$ and $p \in \Sigma$ we denote by K_p the Gauss curvature of Σ at p.

4.2. Archway Lemma. *Let $\boldsymbol{u}$ be a unit vector, $\gamma\colon [a, b] \to \Sigma$ be a minimizing geodesic on the strongly convex surface $\Sigma \subset \mathbb{R}^3$. Assume that $\gamma(a), \gamma(b) \in \omega_{\boldsymbol{u}}$, α and β are the meeting angles at a and b correspondingly and there is an immersion of disc $\iota\colon \mathbb{D} \looparrowright \Sigma$ with the boundary curve $\iota|_{\partial\mathbb{D}}$ formed by γ and an arc of $\omega_{\boldsymbol{u}}$.*

Then

$$\int\limits_{\mathbb{D}} K_{\iota(x)}{\cdot}d_{\iota(x)}\,\mathrm{area}_\Sigma = \pm(\alpha - \beta) \pmod{2{\cdot}\pi} \qquad ❶$$

The sign above is "+" if γ goes contrclockwise in $\mathbb{D}$ and "−" otherwise. In particular

$$\big||\alpha| - |\beta|\big| \leqslant \int\limits_{\mathbb{D}} K_{\iota(x)}{\cdot}d_{\iota(x)}\,\mathrm{area}_\Sigma. \qquad ❷$$

In particular if γ is an archway; that is it lies completely in the dark or light side or $\boldsymbol{u}$ then

$$\operatorname{TotCurv}\gamma \leqslant \int\limits_{\iota(\mathbb{D})} K_p{\cdot}d_p\,\mathrm{area}_\Sigma. \qquad ❸$$

Proof. Since γ is a geodesic, the parallel translation along γ maps $\dot\gamma(a)$ to $\dot\gamma(b)$.

Note also that $\boldsymbol{u}$ belongs to the tangent plane to Σ at any point $p \in \omega_{\boldsymbol{u}}$; in particular the $\boldsymbol{u}$ extends to a parallel tangent vector field on $\omega_{\boldsymbol{u}}$.

It follows that parallel translation along $\iota|_{\partial\mathbb{D}}$ rotates the tangent plane by angle $\alpha - \beta$; To prove ❶ remains to apply Gauss–Bonnet formula.

Denote by R the right hand side in ❷. Note that $R \geqslant 0$ and $|\alpha|, |\beta| \leqslant \pi$. From the main statement of lemma it follows then that the minimal possible value for R is $\big||\alpha| - |\beta|\big|$.

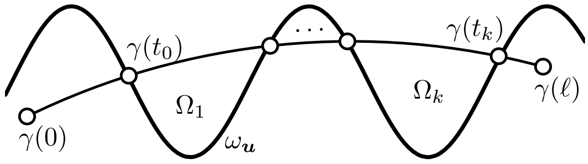
To prove ❸, note that in this case ι is an embedding. Therefore

$$\int\limits_{\mathbb{D}} K_{\iota(x)}{\cdot}d_{\iota(x)}\,\mathrm{area}_\Sigma = \int\limits_{\iota(\mathbb{D})} K_p{\cdot}d_p\,\mathrm{area}_\Sigma.$$

Whence the statement follows from Liberman's lemma. □

Snakes and spirals. Let us show how one can use the Archway Lemma in the simplest case and explains the difficulty in the general case.

Assume γ cross the horizon $\omega_{\boldsymbol{u}}$ as shown on the picture.



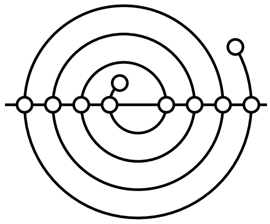
That is, $t_0 < \dots < t_k$ are the $\boldsymbol{u}$-crossing moments and the points $\gamma(t_0),\dots,\gamma(t_k)$ appear along $\omega_{\boldsymbol{u}}$ in the same order.

In this case we can chose nonoverlaping discs $\Omega_1,\dots,\Omega_k$ which γ cuts from light and dark side alternatively. Then by Archway lemma,

$$\begin{aligned}\operatorname{TotCurv}_{\boldsymbol{u}}\gamma &= \operatorname{TotCurv}_{\boldsymbol{u}}(\gamma|_{[0,t_0]}) + \operatorname{TotCurv}_{\boldsymbol{u}}(\gamma|_{[t_0,t_1]}) + \dots\\ &\quad\dots + \operatorname{TotCurv}_{\boldsymbol{u}}(\gamma|_{[t_{k-1},t_k]}) + \operatorname{TotCurv}_{\boldsymbol{u}}(\gamma|_{[t_k,\ell]}) \leqslant\\ &\leqslant \pi + K(\Omega_1) + \dots + K(\Omega_k) + \pi \leqslant\\ &\leqslant 6{\cdot}\pi.\end{aligned}$$

The last inequality holds since the discs Ω_i do not overlap. Therefore by Gauss–Bonnet,

$$K(\Omega_1) + \dots + K(\Omega_k) \leqslant K(\Sigma) \leqslant 4{\cdot}\pi.$$



If the γ cross $\omega_{\boldsymbol{u}}$ in a different order, say as the spiral shown on the picture then the idea above is not directly applicable. Indeed, no matter how we choose the discs Ω_i for the archways they will always overlap.

5. Length and diameter

Let $\varepsilon > 0$. A curve $\gamma\colon [a,b] \to \mathbb{R}^3$ will be called ε-straight if

$$(1-\varepsilon){\cdot}\operatorname{length}\gamma < |\gamma(b) - \gamma(a)|$$

5.1. Lemma. *Assume $\varepsilon > 0$ and n is a positive integer such that $n{\cdot}\varepsilon > 2$. Then any in any minimizing geodesic on a convex surface Σ in $\mathbb{R}^3$ can be sudivided into n ε-straight arcs.*

Proof. Let $\alpha \in (0,\pi)$ be such that

$$1 - \cos\alpha = \varepsilon.$$

Assume two points p and q lie on the convex surface Σ. Denote by ν_p and ν_q the outer normal vectors at p and q correspondingly. Note that any minimizing geodesic geodesic from p to q on Σ is ε-straight if

$$\measuredangle(\nu_p,\nu_q) \leqslant 2{\cdot}\alpha.$$

Note that there are at most $\frac{2}{1-e^{-\varepsilon}}$ points in $\mathbb{S}^2$ which lie on distance at least $2{\cdot}\alpha$ from each other.

Let $\gamma\colon [0,\ell] \to \Sigma$ be a minimizing geodesic parametrized by its length.

Given a value $t \in [0,\ell]$, set t' to be the minimal value in $[t,\ell]$ such that the interval $[t,t']$ is not ε-straight.

Consider a sequence $0 = t_0 < t_1 < \dots < t_n < \ell$ such that $t_{i+1} = t_i'$ for each i. Note that non of intervals $[t_i,t_j]$ for $j > i$ is ε-straight.

Denote by ν_i the outer unit normal vector to Σ at $\gamma(t_i)$. It follows that

$$\measuredangle(\nu_i,\nu_j) \geqslant 2{\cdot}\alpha$$

for all i and j. In other words, the open balls $\mathrm{B}_\alpha(\nu_i)$ in $\mathbb{S}^2$ do not intersect with each other.

It remains to note that

$$\operatorname{area} \mathrm{B}_\alpha(\nu_i) = 2\cdot\pi\cdot\varepsilon \quad\text{and}\quad \operatorname{area} \mathbb{S}^2 = 4\cdot\pi.$$

Hence the result follows. □

5.2. Lemma. *Assume γ is a minimizing geodesic on a convex surface in $\mathbb{R}^3$. Then*

$$\operatorname{length}\gamma < 4\cdot \operatorname{diam}\gamma.$$

Proof. Assume contrary; that is, there is convex surface $\Sigma \subset \mathbb{R}^3$ and a geodesic $\gamma\colon [0,4] \to \Sigma$ is parametrized by its length with $\operatorname{diam}\gamma \leqslant 1$.

Denote by ν_0, ν_2 and ν_4 the outer unit normal vectors to Σ at $\gamma(0)$, $\gamma(2)$ and $\gamma(4)$ correspondingly.

Note that $\measuredangle(\nu_0,\nu_2), \measuredangle(\nu_2,\nu_4) \geqslant \frac{2}{3}\cdot\pi$ and $\measuredangle(\nu_0,\nu_2) > \frac{2}{3}\cdot\pi$, a contradiction. □

6. Reduction to a monotonic case

6.1. Proposition. *For any $\varepsilon > 0$ there is $\delta > 0$ such that the following holds.*

If $\gamma\colon [a,b] \to \Sigma$ is a minimizing geodeisc on a smooth strongly convex surface Σ in $\mathbb{R}^3$ then there is an interval $[a',b'] \subset [a,b]$ such that

$$\operatorname{TotCurv}(\gamma|_{[a',b']}) > \delta\cdot \operatorname{TotCurv}\gamma.$$

and

$$\measuredangle(\dot\gamma(t),\boldsymbol{u}) < \varepsilon$$

for any $t \in [a,b]$ and a fixed unit vector $\boldsymbol{u}$.

Moreover, if $\varepsilon = \frac{1}{10}$ then one can assume $\delta = \frac{1}{100^{100}}$.

In the proof we will need the following two lemmas.

6.2. Lemma. *For any ε there is $\delta > 0$ such that the following holds.*

Assume γ is a curve, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be two vectors in $\mathbb{R}^3$ and $0 \leqslant \alpha_1, \alpha_2 \leqslant \pi$ be such that such that

$$\varepsilon < \measuredangle(\boldsymbol{v}_1,\boldsymbol{v}_2) < \pi - \varepsilon$$
$$\alpha_i - \delta < \measuredangle(\boldsymbol{v}_i,\dot\gamma(t)) < \alpha_i + \delta$$

then there is a vector $\boldsymbol{u}$ such that $\measuredangle(\boldsymbol{u},\dot\gamma(t)) < \varepsilon$.

Moreover if $\varepsilon < \frac{1}{10}$ one can take $\delta = \varepsilon^{10}$.

The proof is straightforward computation; we omit it.

6.3. Lemma. *For any $\varepsilon > 0$ there is $\delta > 0$ such that the following holds.*

Let $\gamma\colon [a,b] \to \Sigma$ be an δ-straight minimizing geodesic on a smooth strongly convex surface Σ in $\mathbb{R}^3$. Set $\boldsymbol{v}_\gamma = \gamma(b) - \gamma(a)$. Then there in a subinterval $[a',b']$ in $[a,b]$ such that

$$\operatorname{TotCurv}(\gamma|_{[a',b']}) \geqslant \delta\cdot \operatorname{TotCurv}\gamma.$$

and

$$\alpha - \varepsilon \leqslant \measuredangle(\dot\gamma(t),\boldsymbol{v}_\gamma) \leqslant \alpha + \varepsilon$$

for some fixed α and any $t \in [a',b']$ we have.

Moreover if $\varepsilon < \frac{1}{10}$ one can take $\delta = \varepsilon^{10}$.

Proof. Without loss of generality we can assume that $a = 0$, $b = 2$ and

$$\operatorname{TotCurv}(\gamma|_{[1,2]}) \geqslant \tfrac{1}{2}\cdot \operatorname{TotCurv}\gamma.$$

Set $p = \gamma(0)$. Let $\theta \in (0,\pi)$ be such that $1 - \cos\theta = \delta$. Note that

$$\measuredangle(\boldsymbol{v}_\gamma, \gamma(t) - p) \leqslant \measuredangle(\tilde\gamma_p(1) - \tilde p, \tilde\gamma_p(2) - \tilde p) < \theta \tag{1}$$

for any $t \geqslant 1$.

By Liberman's lemma

$$\operatorname{TotCurv}_p(\gamma|_{[1,2]}) < \pi + \theta.$$

Assume $N = \lceil\frac{\pi}{\theta} + 1\rceil$. Then we can subdivide $\gamma|_{[1,2]}$ into N arcs such that $\gamma_1, \gamma_2, \dots, \gamma_N$ such that

$$\operatorname{TotCurv}_p(\gamma_n) \leqslant \theta \qquad ❷$$

for each n.

From ❶ and ❷, it follows that for each n, there is α_n such that

$$\alpha_n - \theta \leqslant \measuredangle(\dot\gamma_n(t), \boldsymbol{v}_\gamma) \leqslant \alpha_n + \theta.$$

The arc γ_n with the maximal total curvature will solves the proposition.

It remains to choose δ so that $\theta(\delta) < \frac{\varepsilon}{100}$. □

Proof of Proposition 6.1. Set $\gamma_0 = \gamma$ and $\lambda_0 = \operatorname{TotCurv}\gamma_0$.

Fix $\delta > 0$, set $n = \lceil\frac{2}{\delta}\rceil$. By Lemma 5.1, the geodesic γ_0 can be subdivided into n arcs which are δ-straight. Let us choose the arc γ_0' with the maximal total curvature. Assuming $\delta < \frac{1}{10}$ we get

$$\operatorname{TotCurv}\gamma_0' \geqslant \tfrac{\delta}{10}\cdot\lambda_0.$$

Let α_1 be the angle and γ_1 be the arc in γ_0' provided by Lemma 6.3. In particular

$$\begin{aligned}\operatorname{TotCurv}\gamma_1 &\geqslant \delta\cdot\operatorname{TotCurv}\gamma_0' \geqslant\\ &\geqslant \tfrac{\delta^2}{10}\cdot\lambda_0\end{aligned}$$

If $\alpha_1 \leqslant \frac{\varepsilon}{2}$ or $\alpha_1 \geqslant \pi - \frac{\varepsilon}{2}$ and δ is small enough then statement holds for the arc γ_1 and the vector $\boldsymbol{u} = \pm\boldsymbol{v}_{\gamma_0'}$.

Otherwise let us repeat the above construction for γ_1. Namely, apply Lemma 5.1 to the geodesic γ_1 and denote by γ_1' the δ-straight arc with maximal total curvature. If δ is small, we get

$$\tfrac{\varepsilon}{3} < \measuredangle(\boldsymbol{v}_{\gamma_1'}, \boldsymbol{v}_{\gamma_0'}) < \pi - \tfrac{\varepsilon}{3} \qquad ❸$$

Again, we get

$$\operatorname{TotCurv}\gamma_1' \geqslant \tfrac{\delta}{10}\cdot\operatorname{TotCurv}\gamma_1 \geqslant \tfrac{\delta^3}{100}\cdot\lambda_0$$

Further apply Lemma 6.3 to γ_1'. Denote by γ_2 and α_2 the angle and the subarc of γ_1'. Again

$$\operatorname{TotCurv}\gamma_2 \geqslant \tfrac{\delta^4}{100}\cdot\lambda_0$$

The curve γ_2 runs under nearly constant angle to $\boldsymbol{v}_{\gamma_0'}$ and $\boldsymbol{v}_{\gamma_1'}$. The inequality ❸ makes possible to apply Lemma 6.2. Hence the main statement in the proposition follows.

Straightforward computations prove the last statement. □

7. Elevating geodesics

In this section we fix notations which will be used further without additional explanation.

Fix a (x, y, z)-coordinates on the Euclidean space; denote by $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ the standard basis.

The lines parallel to the z-axis will be called *vertical*; the lines and planes parallel to (x, y)-plane will be called *horizontal*.

7.1. Definition. *A smooth curve $\gamma\colon [0, \ell] \to \mathbb{R}^3$ is called* elevating *if both ends $\gamma(0)$ and $\gamma(\ell)$ lie on the z-axiz and $\langle\dot\gamma(t), \boldsymbol{k}\rangle > 0$ for all t.*

$(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$-frame. Let Σ be a convex surface and $\gamma\colon [0, \ell] \to \Sigma$ is an elevating minimizing geodesic with unit-speed parametrization.

Given $t \in [0, \ell]$, consider the oriented orthonormal frame $\lambda(t), \mu(t), \nu(t)$ such that $\nu(t)$ is the outer normal to Σ at $\gamma(t)$, the vector $\mu(t)$ is horizontal and therefore the vector $\lambda(t)$ lies in the plane spanned by $\nu(t)$ and the z-axis. We assume in addition that $\langle \lambda, \boldsymbol{k} \rangle \geqslant 0$.

Since $\langle \dot\gamma(t), \boldsymbol{k} \rangle > 0$, $\nu(t)$ can not be vertical and therefore the frame (λ, μ, ν) is uniquely defined for any $t \in [0, \ell]$.

Angle functions. Set

$$\varphi(t) = \measuredangle(\boldsymbol{k}, \dot\gamma(t)), \qquad \psi(t) = \tfrac{\pi}{2} - \measuredangle(\boldsymbol{k}, \nu(t)), \qquad \alpha(t) = \tfrac{\pi}{2} - \measuredangle(\mu(t), \dot\gamma(t)),$$

From the above definitions it follows that $|\alpha(t)|, |\psi(t)| \leqslant \frac{\pi}{2}$ and for each t there is a right spherical triangle with legs $|\alpha(t)|, |\psi(t)|$ and hypotenuse $\varphi(t)$. In particular $\cos\alpha \cdot \cos\psi = \cos\varphi$. Whence we get the following.

7.2. Claim. *For any t we have*

$$\varphi(t) \geqslant |\psi(t)| \quad \textit{and} \quad \varphi(t) \geqslant |\alpha(t)|$$

Applying Liberman's Lemma in the direction $\boldsymbol{k}$ we also get the following.

7.3. Claim. *If an arc $\gamma|_{[a,b]}$ lies in the dark (light) side for $\boldsymbol{k}$ then the function φ is nondecreasing (correspondingly nonincreasing) in $[a, b]$.*

8. Plane sections

Assume γ is a curve on a smooth strictly convex surface Σ in $\mathbb{R}^3$. Consider a plane L passing through two points of γ, say $p = \gamma(a)$ and $q = \gamma(b)$ with $a < b$. Let $L_\pm$ be a half-planes in L bounded by the line trough p and q. Set $\sigma_\pm = \Sigma \cap L_\pm$; note that $\sigma_\pm$ are a smooth convex plane curve connecting p to q in Σ.

8.1. Observation. *If γ is a minimizing geodesic in the convex surface $\Sigma \subset \mathbb{R}^3$ and a, b and $\sigma_\pm$ as above then*

$$\operatorname{length}\sigma_\pm \geqslant \operatorname{length}(\gamma|_{[a,b]}).$$

Based on this observation we give couple of estimates on elevating minimizing geodesics.

8.2. Propostion. *Assume $\gamma\colon [0, \ell] \to \Sigma$ is an elevating minimizing geodesic in the convex surface $\Sigma \subset \mathbb{R}^3$. Assume that for a subsegment $[a, b] \subset [0, \ell]$ the following conditions hold*

(i) The points $\gamma(a)$ and $\gamma(b)$ lie in a half-plane with boundary line formed by the z-axis and the arc $\gamma|_{[a,b]}$ goes around the z-axis at least once.

(ii) $\gamma(a)$ lies above the horizontal plane through $\frac{1}{2}\cdot(\gamma(0) + \gamma(\ell))$.

Then $\gamma(b)$ lies on the dark side of Σ with respect to $\boldsymbol{k}$.

Proof. Let us apply Observation 8.1 to the plane containing z-axis, $\gamma(a)$ and $\gamma(b)$.

We can assume that $\gamma(0)$ is the origin of the (x, y, z)-coordinate system and both points $p = \gamma(a)$ and $q = \gamma(b)$ lie in the (x, z)-coordinate half-plane with $x \geqslant 0$, denoted by Π. We can assume that $\sigma_+ \subset \Pi$. Let $(x_p, 0, z_p)$ and $(x_q, 0, z_q)$ be the coordinates of p and q.

From the assumptions we get $z_p < z_q < 2\cdot z_p$.

By convexity of the curve $\Pi \cap \Sigma$ we get

$$\operatorname{length}\sigma_+ \leqslant \sqrt{(z_q - z_p)^2 + x_p^2}$$

On the other hand, since $\gamma|_{[a,b]}$ goes around z-axis at least once, we get

$$\operatorname{length}\gamma|_{[a,b]} \geqslant \sqrt{(z_q - z_p)^2 + (x_p + x_q)^2}.$$

These two estimates contradict Observation 8.1. □

8.3. Corollary. *If Σ, γ, ℓ, a and b as in the Proposition and the arc $\gamma|_{[a,b]}$ goes around the z-axis at least twice then the arc $\gamma|_{[b,\ell]}$ lies on the dark side with respect to $\boldsymbol{k}$.*

Proof. Fix $b' \in [b,\ell]$. Note that one can find $a' \in [a,b]$ such that the the assumtions of Proposition 8.2 hold for the interval $[a',b']$. Applying Proposition we get the result. □

8.4. Propostion. *Assume $\gamma\colon [0,\ell] \to \Sigma$ is elevating minimizing geodesic in the convex surface $\Sigma \subset \mathbb{R}^3$. Assume that the arc $\gamma|_{[b,\ell]}$ lies in the dark side of Σ with respect to $\boldsymbol{k}$. Set $\varphi(t) = \measuredangle(\boldsymbol{k}, \dot\gamma(t))$ and $\psi(t) = \frac{\pi}{2} - \measuredangle(\boldsymbol{k}, \nu(t))$. If $b \leqslant s < t \leqslant \ell$ and the point $\gamma(s)$ lies in the plane Π through $\gamma(t)$ spanned by $\nu(t)$ and $\lambda(t)$ then*

$$\varphi(s) \leqslant \psi(t).$$

Proof. Let us apply Observation 8.1 to the plane Π and $p = \gamma(s)$ and $q = \gamma(t)$.

Let z_p and z_q be the z-coordinates of p and q.

Since $\gamma|_{[s,t]}$ lies in the dark side, its Liberman's development $\tilde\gamma|_{[s,t]}$ with respect to $\boldsymbol{k}$ is concave. In particular

$$\operatorname{length}(\gamma|_{[s,t]}) = \operatorname{length}(\tilde\gamma|_{[s,t]}) \geqslant \tfrac{z_q - z_p}{\cos\varphi(s)}.$$

On the other hand, convexity of σ_+ imply that

$$\operatorname{length}\sigma_+ \leqslant \tfrac{z_q - z_p}{\cos\psi(t)}.$$

It remains to apply Observation 8.1. □

9. s-pairs

Let $\Sigma \subset \mathbb{R}^3$ be a strongly convex surface and $\gamma\colon [0,\ell] \to \Sigma$ be an elevating minimizing geodesic.

After rotating (x,y)-plane if necessary, we can assume that the border of shadow in the directions of x-axis, say ω_x, is a smooth curve and γ intersects them transversely.

Let $t_1 < t_2 < \dots < t_k$ be the time moments in $[0,\ell]$ at which γ crossing ω_x. Note that

$$\mu(t_n) = s_n \cdot e_x \quad \text{for some} \quad s_n = \pm 1.$$

Set

$$\varphi_n = \varphi(t_n) \qquad \psi_n = \psi(t_n) \qquad \alpha_n = \alpha(t_n)$$

We say that a pair of indexes $i < j$ forms an ***s**-pair* if

$$\sum_{n=i}^{j} s_n = 0 \quad \text{and} \quad \sum_{n=i}^{j'} s_n > 0$$

if $i < j' < j$.

Note that for any index i appears in at most one $\boldsymbol{s}$-pair and for any $\boldsymbol{s}$-pair (i,j) we have

- ◇ $s_i = 1$; that is, i-th braket has to be openning.
- ◇ $s_j = -1$; that is, j-th braket has to be closing.

In particular,

$$s_i{\cdot}\alpha_i + s_j{\cdot}\alpha_j = \alpha_i - \alpha_j.$$

Bracket interpretation. If you exchange "+1" and "−1" in $\boldsymbol{s}$ by "(" and ")" correspondingly then (i,j) is an $\boldsymbol{s}$-pair if and only if the i-th bracket forms a pair with j-bracket.

Embedded disc interpretation. Assume (i,j) is an $\boldsymbol{s}$-pair. Note that in this case there is an arc of $\omega_{\boldsymbol{i}}$ from $\gamma(t_i)$ to $\gamma(t_j)$ with monotonic z-coordinate. Moreover this arc, say σ together with $\gamma|[t_i,t_j]$ bounds an immesed disc in Σ. That is there is an immesion $\iota\colon \mathbb{D}\to\Sigma$ such that the closed curve $\iota|_{\partial\mathbb{D}}$ is formed by joint of σ and $\gamma|[t_i,t_j]$.

The proof can be guessed from the diagram. It shows a lift of γ in the universal cover of strip of Σ between horizontal planes through $\gamma(t_i)$ and $\gamma(t_j)$; the solid vertical lines correspond are lifts of σ and the dashed lines corresponds to the lifts of the other component of $\omega_{\boldsymbol{i}}$ bethween the planes.

We say that q is the depth of an $\boldsymbol{s}$-pair (i,j) (briefly $q=\operatorname{depth}_{\boldsymbol{s}}(i,j)$) if q is the maximal number such that theis q-long nested sequence of $\boldsymbol{s}$-pairs starting with (i,j); that is a sequence of $\boldsymbol{s}$-pairs $(i,j)=(i_1,j_1),(i_2,j_2),\dots,(i_q,j_q)$ such that

$$i=i_1<\dots<i_q<j_q<\dots<j_1=j.$$

We say that q is the depth of an $\boldsymbol{s}$-pair (i,j) (briefly $q=\operatorname{depth}_{\boldsymbol{s}}(i,j)$) if q is the maximal number such that theis q-long nested sequence of $\boldsymbol{s}$-pairs starting with (i,j); that is a sequence of $\boldsymbol{s}$-pairs $(i,j)=(i_1,j_1),(i_2,j_2),\dots,(i_q,j_q)$ such that

$$i=i_1<\dots<i_q<j_q<\dots<j_1=j.$$

Note that the $\boldsymbol{s}$-pair of the same depth do not overlap; that is if for two distinct $\boldsymbol{s}$-pairs (i,j) and (i',j'), we have $\operatorname{depth}(i,j)=\operatorname{depth}(i',j')$ then either $i<j<i'<j'$ or $i'<j'<i<j$.

The following proposition follow directly from the definitions above.

9.1. Proposition. *Let (i,j) be an $\boldsymbol{s}$-pair. Then the arcs $\gamma|_{[t_i,t_j]}$ and an arc of $\omega_{\boldsymbol{i}}$ bound an immesed disc in Σ which lies between horizontal planes through $\gamma(t_i)$ and $\gamma(t_j)$. Moreover the maximal multiplicity of the disc is at most* $\operatorname{depth}_{\boldsymbol{s}}(i,j)$.

9.2. Corollary. *Let us denote by S_q the subset of indiexes $\{1,\dots,k\}$ which are the parts of $\boldsymbol{s}$-pairs with depth q. Then*

$$\sum_{n\in S_q} s_n{\cdot}\alpha_n \leqslant 4{\cdot}\pi{\cdot}q.$$

Proof. For each n denote by K_n the integral of Gauss curvature of the part of surface Σ which lies below horizontal plane through $\gamma(t_n)$. Note that

$$0\leqslant K_1\leqslant\ldots\leqslant K_k\leqslant 4{\cdot}\pi.$$

By Proposition 9.1 and the Key Lemma, we get

$$s_i{\cdot}\alpha_i + s_j{\cdot}\alpha_j = \alpha_i - \alpha_j \leqslant q{\cdot}(K_j-K_i)$$

The statement follows since the $\boldsymbol{s}$-pairs with the same depth do not overlap. □

9.3. Corollary. *Assume*

$$q=\max_{1\leqslant i<j\leqslant k}\left\{\left|\sum_{n=i}^{j} s_n\right|\right\}$$

Then

$$\left|\sum_{n=1}^{k} s_n{\cdot}\alpha_n\right| \leqslant 2{\cdot}q{\cdot}(q+\tfrac{3}{2}){\cdot}\pi.$$

Proof. Denote by S the set of all indexes which appear in some $\boldsymbol{s}$-pair.

Note that depth of any $\boldsymbol{s}$ pair is at most q. That is,

$$S = S_1 \cup \dots \cup S_q.$$

By Corollary 9.2,

$$\sum_{n\in S} s_n{\cdot}\alpha_n \leqslant 2{\cdot}q{\cdot}(q+1){\cdot}\pi. \qquad ❶$$

Set $R = \{1, dots, k\}\backslash S$; this is the set of indexes which do not appear in an $\boldsymbol{s}$-pair.

Given r, set $i \in Q_r$ if

$$\sum_{n=1}^{i} s_n = r.$$

Note that $Q_r \neq \varnothing$ for at most q values of r and in each set Q_r there are at most 2 indexes which do not appear in an $\boldsymbol{s}$-pair; that is $Q_r \cap R$ has at most two indexes for each r.

Sine $|a_n| < \frac{\pi}{2}$, we get

$$\left|\sum_{n\in R} s_n{\cdot}\alpha_n\right| \leqslant q{\cdot}\pi.$$

The later inequality together with ❶ implies the statement in the corollary. □

10. Geometric growth

10.1. Claim. *Assume $\psi(t) > \varepsilon$ for ant $t \in [t_i, t_j]$ and $s_i + \dots + s_j = 2$ Then*

$$|\alpha_j - \alpha_i| > \pi{\cdot}\sin\varepsilon.$$

10.2. Claim. *Let γ be elevating minimizing geodesic on a graph $z = f(x, y)$ of a concave function. Then for any pair of indexes $j > i$, such that*

$$|\sum_{n=i}^{j} s_n| \geqslant 5$$

we have

$$\varphi_j > \tfrac{3}{2}{\cdot}\varphi_i.$$

Proof. Without loss of generality, we may assume that

$$\sum_{n=i}^{j} s_n = 6$$

Let j' be the least index such that

$$|\sum_{n=i}^{j'} s_n| = 5.$$

Note that for any $b > t_j$ there is $a \in [t_i, t_j]$ such that intrval $[a, b]$ satisfies the assumptions of Proposition 8.4. In particular $\psi(b) > \varphi_i$ for any $b > t_j$. Applying

Claim 10.1, we get that $|\alpha_j| > \frac{\pi}{2}\cdot\varphi_i$ or $|\alpha_{j'}| > \frac{\pi}{2}\cdot\varphi_i$. Since φ_n is nondecreasing, and $\varphi_n \geqslant |\alpha_n|$ for any n, in both cases we get

$$\varphi_j > \tfrac{\pi}{2}\cdot\varphi_i.$$

□

11. An estimate for graphs

11.1. Proposition. *There is c constant ω' ($\omega' = 10\cdot\pi$ will do) such that if γ is an elevating minimizing geodesic on a graph $z = f(x,y)$ of a concave function f then*

$$\operatorname{TotCurv}_{\boldsymbol{i}}\gamma \leqslant \omega'.$$

Proof. We can assume that γ cross the $\boldsymbol{i}$ horizon $\omega_{\boldsymbol{i}}$ tranfersally. Let $t_1 < \cdots < t_k$ be the values of parameter at which γ cross $\omega_{\boldsymbol{i}}$ and $s_1,\dots,s_k$ the signs as in ...

Recall that S_q denotes the subset of indexes $\{1,\dots,k\}$ which appear in $\boldsymbol{s}$-pair with depph q. According to Corollary 9.2,

$$\left|\sum_{n\in S_q} s_n\cdot\alpha_n\right| \leqslant 4\cdot q\cdot\pi.$$

In particular,

$$\left|\sum_{n\in S_1\cup\dots\cup S_5} s_n\cdot\alpha_n\right| \leqslant 40\cdot\pi.$$

Set $R = \{1,\dots,k\}\backslash(S_1\cup\dots\cup S_5)$; this is the set of indexes which appear in $\boldsymbol{s}$-pairs with depth at least 6 as well as those which do not appear in any $\boldsymbol{s}$-pair.

According to ???

$$\left|\sum_{n\in R} s_n\cdot\alpha_n\right| \leqslant \sum_{n\in R}\varphi_n.$$

To estimate the last sum will use the results in Section 10. First let us subdivide R into 5 subsets $R_1,\dots,R_5$, by setting $n\in R_m$ if $m\equiv n \pmod 5$.

Given $n\in R_m$, denote by n' the least index in R_m which is larger n; n' is defined for any $n\in R_m$ exept the largest one. According to ??? $\varphi_{n'} > 2\cdot\varphi_n$. Since φ_n is nondecreasing in n, we get

$$\sum_{n\in R_m}\varphi_n \leqslant 2\cdot\varphi_k.$$

It follows that

$$\sum_{n\in R}\varphi_n \leqslant 10\cdot\varphi_k < 5\cdot\pi.$$

According to Liberman's lemma

$$\begin{aligned}\operatorname{TotCurv}_{\boldsymbol{i}}\gamma &\leqslant 4\cdot\pi + 2\cdot[s_1\cdot\alpha_1+\dots+s_k\cdot\alpha_k] \leqslant\\ &\leqslant 100\cdot\pi.\end{aligned}$$

□

12. Final assembling of the proof

Assume there is a minimizing geodesic $\gamma\colon[0,\ell]\to\Sigma$ in a convex surface $\Sigma\subset\mathbb{R}^3$ such that

$$\operatorname{TotCurv}\gamma=\omega.$$

According to ??? we can assume that Σ is strongly convex.

According to ???, we can pass to an elevating arc, of γ for some (x,y,z)-coordinate system with total curvature $>\frac{\omega}{10^6}$. Rename this arc by γ and let us use the notations in Section 7.

Rotating (x,y)-coordinate plane we can ensure that

$$\operatorname{TotCurv}\gamma\leqslant 10\cdot\operatorname{TotCurv}_{\boldsymbol{i}}\gamma$$

and that γ cross the horizon $\omega_{\boldsymbol{i}}$ transversally.

Let us subdivide γ into three arcs lower middle and upper arcs γ_-, γ_0 and γ_+ the the following way.

Note that according to ??? γ_+ lies on a graph of concvae function. By Proposition **??**, we get

$$\operatorname{TotCurv}_{\boldsymbol{i}}\gamma_+\leqslant 100\cdot\pi. \qquad ❶$$

Similarly γ_- lies on a graph of convex function and the same proposition implies

$$\operatorname{TotCurv}_{\boldsymbol{i}}\gamma_-\leqslant 100\cdot\pi. \qquad ❷$$

By Corollary 9.3,

$$\operatorname{TotCurv}_{\boldsymbol{i}}\gamma_0\leqslant 100\cdot\pi.$$

$$\operatorname{TotCurv}\gamma_0\leqslant ???$$

Together with ❶ and ❷, the later implies that

$$\operatorname{TotCurv}_{\boldsymbol{i}}\gamma\leqslant 300\cdot\pi.$$

□

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