ON THE TOTAL CURVATURE OF MINIMIZING GEODESICS ON CONVEX SURFACES

NINA LEBEDEVA AND ANTON PETRUNIN

Abstract. We give a universal upper bound for the total curvature of minimizing geodesic on a convex surface in the Euclidean space.

1. Introduction

Recall that the *total curvature* of a curve $\gamma\colon[0,\ell]\to\mathbb{R}^3$ (briefly $\operatorname{TotCurv}\gamma$) is defined as supremum of sum of exterior angles for the broken lines inscribed in γ. If γ is smooth and equipped with the natural parameter, then

$$\operatorname{TotCurv}\gamma=\int\limits_0^\ell\kappa(t)\cdot dt,$$

where $\kappa(t)=|\ddot\gamma(t)|$ is the curvature of γ at t.

By convex surface in the Euclidean 3-space $\mathbb{R}^3$ we understand the boundary of closed convex set with nonempty interior.

1.1. Main theorem. *Let Σ be a convex surface in $\mathbb{R}^3$ and γ be a minimizing geodesic in Σ then*

$$\operatorname{TotCurv}\gamma\leqslant\omega,$$

where ω is a universal real constant.

This question was formulated in [1], [2] and [3], but we have learned it from Dmitry Burago only few years ago.

Let us briefly discuss the related results.

- ⋄ In [4], Liberman gives a bound on the total curvature of short geodesic in terms of the ratio diameter and inradius of K. In the proof he use now so called Liberman's lemma 2.1 discussed below. This statement was rediscovered in [3].
- ⋄ In [5], Usov gives the optimal bound for total curvature of geodesic on the graph of ℓ-Lipscitz convex function. Namely, he proves that if $f\colon\mathbb{R}^2\to\mathbb{R}$ is ℓ-Lipschitz and convex then any geodesic in its graph
$$\Gamma_f=\left\{\,(x,y,z)\in\mathbb{R}^3\;\middle|\;z=f(x,y)\,\right\}$$
has total curvature at most $2\cdot\ell$. This statement was also rediscovered in [3]. Yet an amusing generalization of Usov's result is given by Berg in [6].
- ⋄ In [7], Pogorelov conjectured that any the spherical image of geodesic on convex surface has to be contructable. It is easy to see that the length of spherical image of geodesic can not be smaller than its total curvature, so this conjecture (if it would be true) would be stronger than Liberman's theorem. Counterexamples were found indepenently by Milka in [8], Usov in [9] and yet much later rediscovered by Pach in [2].

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◇ In [3], Bárány, Kuperberg, and Zamfirescu have constructed a corkscrew minimizing geodesic on a closed hypersurface; that is a minimizing geodesic which twists around given line arbitrary many times. In the same paper they also constructed a minimizing geodesic on a convex surface in $\mathbb{R}^3$ with total curvature bigger that $2\cdot\pi$. (Note that $2\cdot\pi$ is the optimal bound for the analogous problem in the plane.)

Plan of the proof. We prove is divided in three steps.

First we prove a sequence of propositions which alow us to consider only special case of surfaces and curves. Namely we show that we can assume that

(i) *(Proposition 3.1).* The surface Σ is C^∞-smooth.
(ii) *(Proposition 5.1).* The z-component of $\dot\gamma$ is positive,
(iii) *(Proposition 9.3).* The surface Σ is formed by a graph $z = f(x, y)$ of a smooth convex function $f\colon \mathbb{R}^2 \to \mathbb{R}$.

On the second step we give number of geometric inequalities which relate the angles between $\dot\gamma(t)$ and with the coordinate axis.

The last step in the proof is purely algebraic, here we combine the obtained inequalities to give a universal bound on the total curvature of γ_n which satisfies the conditions (i)–(i).

2. Preliminaries

Let Σ be a convex hypersurface in the Euclidean space.

Given a point $p \in \Sigma$, we will denote by n_p the outer normal vector of Σ at p; the map $\Sigma \to \mathbb{S}^2$ defined as $p \mapsto n_p$ sometimes is called *Gauss map.*

Fix a points $z \notin \Sigma$. Given a point $p \in \Sigma$, we say that p lies on light (dark) side from z if if $\langle z - p, n_p\rangle \leqslant 0$ (correspondingly $\langle z - p, n_p\rangle \geqslant 0$). If $\langle z - p, n_p\rangle = 0$ we say that p lies on the horizon from p. Note that if z lies inside of Σ then all points on Σ lie on the dark side from z.

Let γ be a space curve parametrized by length. Fix a point $z \notin \gamma$. Let us define *Liberman's development* of γ with respect to z as the unit-speed plane cure $\tilde\gamma_z$ such that the direction $\tilde\gamma_z(t)$ changes counterclockwise as t changes and $|\tilde\gamma_p(t)| = |\gamma(t) - z|$ for any t.

The Liberman's development $\tilde\gamma_z$ is called convex concave at $\tilde\gamma_z(t)$ if there the curvelinear triangle ???

2.1. Liberman lemma. *Let Σ be a convex surface in the Euclidean space $z \notin \Sigma$ and γ be a unit-speed geodesic in Σ. Then the development $\tilde\gamma_z$ is locally convex (concave) at the points on dark (light) side of Σ with respect to z.*

Assume $\gamma\colon [0, \ell] \to \Sigma$ is a unit-speed curve in the space.

The vector $\ddot\gamma(t)$ is the curvature vector of γ at t. The total curvature of γ can be defined as

$$\operatorname{TotCurv}\gamma \stackrel{\text{def}}{=\!=} \int\limits_0^\ell |\ddot\gamma(t)|\cdot dt.$$

The total curvature of $\tilde\gamma_z$ is called the total curvature of γ in the direction of z and denoted as $\operatorname{TotCurv}_z\gamma$ Given a point z, let us define the total curvature of γ in the direction of z as

$$\operatorname{TotCurv}_z\gamma \stackrel{\text{def}}{=\!=} \int\limits_0^\ell \left|\langle\ddot\gamma(t), \tfrac{z-\gamma(t)}{|z-\gamma(t)|}\rangle\right|\cdot dt.$$

2.2. Key Lemma. *Let $\gamma\colon [0, \ell] \to \Sigma$ be a geodesic on the convex surface in the Euclidean space and $u \in \mathbb{S}^2$. Assume that $0 = t_0 < t_1 < \dots < t_n = \ell$ be the values* *such that each arcs* $\gamma|_{[t_{i-1},t_i]}$ *alternating light and dark side of* Σ *with respect to* u. *Set* $\alpha_i = \measuredangle(\dot\gamma(t_i), u)$ *Then*

$$\operatorname{TotCurv}_u \gamma = |\sum_i (-1)^i \alpha_i|.$$

Moreover, if $1 < i < n$ *and* Ω_i *denotes the domain of* Σ *bounded by the arc* $\gamma|_{[t_{i-1},t_i]}$ *and the* u*-horizon then*

$$|\alpha_i - \alpha_{i-1}| \leqslant \operatorname{curv}\Omega_i,$$

where $\operatorname{curv}\Omega_i$ *denotes the total curvature of* Ω_i. *In particular,*

$$\operatorname{TotCurv}_u \gamma \leqslant 4{\cdot}\pi + \sum_i \operatorname{curv}\Omega_i.$$

Remarks. Clearly $\operatorname{TotCurv}_z \gamma \leqslant \operatorname{TotCurv}\gamma$ for any curve γ in Σ.

On the other hand given few points z_i which do not lie in one plane one can estimate $\operatorname{TotCurv}\gamma$ in terms of $\operatorname{TotCurv}_{z_i}\gamma$ the distances between z_i and the maximal distance to γ.

Let $N = N(\Sigma, \gamma, u)$ be the maximal integer such that at most N of the domains Ω_i intersect at one point. Note that from [3], it follows that the value N can take arbitrary large value. The number N can be estimated through the maximal rotation number of subarcs of γ with respect to the lines. In particular the total curvature of geodesic γ can be bounded in terms of maximal rotation number of subarcs of γ around the lines. The later was claimed in [3] without a proof.

Then

$$\sum_{i=2}^{n-1} \operatorname{curv}\Omega_i \leqslant N{\cdot}\operatorname{curv}\Sigma \leqslant 4{\cdot}N{\cdot}\pi.$$

Therefore, we get an estimate

$$\operatorname{TotCurv}_u \gamma \leqslant 4{\cdot}N{\cdot}\pi + |\alpha_0 - \alpha_1| + |\alpha_{n-1} - \alpha_n| \leqslant (4{\cdot}N + 2){\cdot}\pi.$$

Since the same holds for any vector u, we can taking avarage we get

$$\operatorname{TotCurv}\gamma \leqslant 3{\cdot}(4{\cdot}N + 2){\cdot}\pi.$$

3. Smoothing

3.1. Proposition. *Let* $K_1, K_2, \dots, K_\infty$ *be closed convex sets in* $\mathbb{R}^3$ *and* $K_n \to K_\infty$ *in the sense of Hausdorff. Then for any minimizing geodesic* γ_∞ *in the surface of* K_∞ *there is a a sequence of minimizing geodesics* γ_n *in the surface of* K_n *such that* $\gamma_n \to \gamma_\infty$.

Proof. Assume γ is a minimizing geodesic on a convex surface Σ. Assume γ parametrized by its length $[0, \ell]$.

Fix a subinterval $[a, b] \subset [0, \ell]$ such that $0 < a$ and $b < \ell$. Set $p = \gamma(a)$ and $q = \gamma(b)$.

Assume Σ_n be a sequence of smooth convex surfaces converging to Σ. and $p_n, q_n \in \Sigma_n$ be a two sequences of points which converge to p and q correspondingly.

Denote by γ_n a minimizing geodesic from p_n to q_n in Σ_n. Note that γ_n converges to $\gamma|_{[a,b]}$ as $n \to \infty$. □

4. Length and diameter

Let $\varepsilon > 0$. A curve $\gamma\colon [a,b] \to \mathbb{R}^3$ will be called ε-straight if

$$\operatorname{length}\gamma \leqslant e^{\varepsilon}\cdot|\gamma(b) - \gamma(a)|$$

4.1. Lemma. *Given $\varepsilon > 0$ there is $\delta > 0$ (any $\delta < (1 - e^{-\varepsilon})/2$ will do) such that in any minimizing geodesic of length ℓ on a convex surface Σ in $\mathbb{R}^3$ there an ε-straight arc of length at least $\delta\cdot\ell$;*

Proof. Set $\alpha = \arccos e^{-\varepsilon}$. Let N be the maximal number of points in $\mathbb{S}^2$ which lie on distance at least $2\cdot\alpha$ from each other.

Let $\gamma\colon [0,\ell] \to \Sigma$ be a minimizing geodesic parametrized by its length.

Given a value $t \in [0,\ell]$, set t' to be the maximal value in $[0,\ell]$ such that the interval $[t,t']$ i ε-straight.

Consider the maximal sequence $0 = t_0 < t_1 < \dots < t_n < \ell$ such that $t_{i+1} = t_i'$.

Denote by ν_i the outer unit normal vector to Σ at $\gamma(t_i)$. Note that $\measuredangle(\nu_i,\nu_j) > 2\cdot\alpha$ for all i and j. It follows that the sequence (t_i) terminates after at most N steps. Therefore any $\delta < \frac{1}{N+1}$ does the job. $\square$

4.2. Lemma. *Assume γ is a minimizing geodesic on a convex surface in $\mathbb{R}^3$. Then*

$$\operatorname{length}\gamma < 4\cdot \operatorname{diam}\gamma.$$

Proof. Assume contrary; that is, there is convex surface $\Sigma \subset \mathbb{R}^3$ and a geodesic $\gamma\colon [0,4] \to \Sigma$ is parametrized by its length with $\operatorname{diam}\gamma \leqslant 1$.

Denote by ν_0, ν_2 and ν_4 the outer unit normal vectors to Σ at $\gamma(0)$, $\gamma(2)$ and $\gamma(4)$ correspondingly.

Note that $\measuredangle(\nu_0,\nu_2), \measuredangle(\nu_2,\nu_4) \geqslant \frac{2}{3}\cdot\pi$ and $\measuredangle(\nu_0,\nu_2) > \frac{2}{3}\cdot\pi$, a contradiction. $\square$

5. Reduction to a monotonic case

In this section we show that to prove the Main theorem, it is sufficient to consider only the geodesics which go almost in one direction. The following proposition will be applied to $\varepsilon = \frac{\pi}{4}$; in this case one can take $\delta = 10^{-10}$.

5.1. Proposition. *Given $\varepsilon > 0$ there is $\delta > 0$ such that the following statement holds.*

If $\gamma\colon [0,\ell] \to \Sigma$ is a minimizing geodeisc on a smooth strongly convex surface Σ in $\mathbb{R}^3$ then there is an interval $[a,b] \subset [0,\ell]$ such that

$$\operatorname{TotCurv}(\gamma|_{[a,b]}) > \delta\cdot \operatorname{TotCurv}\gamma.$$

and

$$\measuredangle(\dot\gamma(t),\boldsymbol{k}) < \varepsilon$$

for any $t \in [a,b]$ and a fixed unit vector $\boldsymbol{k}$.

Proof. Applying rescaling, we can assume that $\operatorname{diam}\gamma = 3$. By Lemma 4.2 $\operatorname{length}\gamma_n < 12$. Therefore we can subdivide γ into 12 arcs $\gamma_1,\dots,\gamma_{12}$ such that for each n there is a point $p_n \in K$ which lies on the distance at least 1 from γ_n and $\operatorname{length}\gamma_n \leqslant 1$. Choose an arc $\gamma' = \gamma_n$ with the maximal total curvature and set $p' = p_n$. Clearly

$$\operatorname{TotCurv}\gamma' \geqslant \tfrac{1}{12}\cdot \operatorname{TotCurv}\gamma.$$

Applying Liberman's Lemma to γ' with the reference point p' we get that

$$\operatorname{TotCurv}_{p'}\gamma' < \pi + 1 < 5.$$

Choose an integer $N > \frac{2}{\varepsilon}$. Note that we can divide γ' into N arcs $\gamma'_1, \dots, \gamma'_N$ so that

$$\operatorname{TotCurv}_{p'} \gamma'_n \leqslant \tfrac{5}{N}$$

for each n. Choose among these arcs the one with maximal total curvature, denote it further by γ''. Clearly

$$\operatorname{TotCurv} \gamma'' > \tfrac{\varepsilon}{10^3} \cdot \operatorname{TotCurv} \gamma'.$$

Fix a parameter t of γ'' and denote by α the angle between $\dot\gamma''(t)$ and $p - \gamma''(t)$. If $\alpha < \frac{\varepsilon}{2}$ or $\alpha > \pi - \frac{\varepsilon}{2}$, then the problem is solved.

Otherwise applying Lemma 4.1 we get a nondegenerate (say equilateral) triangle $\triangle a_1a_2a_3$ in K_n of the size comparable to $\operatorname{diam}\gamma'$ and on the distance comparable to $\operatorname{diam}\gamma'$ from any point of γ'', say side of triangle can be taken to be $\frac{\varepsilon^2}{1000} \cdot \operatorname{diam}\bar\gamma$ and the distance to any point can be assumed to be between $\operatorname{diam}\bar\gamma$ and $2 \cdot \operatorname{diam}\bar\gamma$

Apply the construction to each vertex of the triangle. We pass to an arc of $\hat\gamma$ such that the angle between $\dot\gamma(t)$ and $a_i - \gamma(t)$ and the distance $|\gamma(t) - a_i|$ are nearly constant for each i. The later imply that $\dot\gamma$ is nearly constant. $\square$

6. Elevating geodesics

In this section we fix notations which will be used further without additional explanation.

Fix a (x, y, z)-coordinates on the Euclidean space; denote by $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ the standard basis.

The lines parallel to the z-axis will be called *vertical*; the lines and planes parallel to (x, y)-plane will be called *horizontal*.

6.1. Definition. *A smooth curve $\gamma\colon [0, \ell] \to \mathbb{R}^3$ is called* elevating *if both ends $\gamma(0)$ and $\gamma(\ell)$ lie on the z-axiz and $\langle \dot\gamma(t), \boldsymbol{k}\rangle > 0$ for all t.*

According to Proposition 5.1, it is sufficient to prove Main theorem only for elevating geodesics.

$(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$-frame. Let Σ be a convex surface and $\gamma\colon [0, \ell] \to \Sigma$ is an elevating minimizing geodesic with unit-speed parametrization.

Given $t \in [0, \ell]$, consider the oriented orthonormal frame $\lambda(t), \mu(t), \nu(t)$ such that $\nu(t)$ is the outer normal to Σ at $\gamma(t)$, the vector $\mu(t)$ is horizontal and therefore the vector $\lambda(t)$ lies in the plane spanned by $\nu(t)$ and the z-axis. We assume in addition that $\langle \lambda, \boldsymbol{k}\rangle \geqslant 0$.

Since $\langle \dot\gamma(t), \boldsymbol{k}\rangle > 0$, $\nu(t)$ can not be vertical and therefore the frame (λ, μ, ν) is uniquely defined for any $t \in [0, \ell]$.

Angle functions. Set

$$\varphi(t) = \measuredangle(\boldsymbol{k}, \dot\gamma(t)), \qquad \psi(t) = \tfrac{\pi}{2} - \measuredangle(\boldsymbol{k}, \nu(t)), \qquad \alpha(t) = \tfrac{\pi}{2} - \measuredangle(\mu(t), \dot\gamma(t)),$$

From the above definitions it follows that $|\alpha(t)|, |\psi(t)| \leqslant \frac{\pi}{2}$ and for each t there is a right spherical triangle with legs $|\alpha(t)|, |\psi(t)|$ and hypotenuse $\varphi(t)$. In particular $\cos\alpha \cdot \cos\psi = \cos\varphi$. Whence we get the following.

6.2. Claim. *For any t we have*

$$\varphi(t) \geqslant |\psi(t)| \quad \text{and} \quad \varphi(t) \geqslant |\alpha(t)|$$

In particular, $\varphi_n \geqslant |\psi_n|$ and $\varphi_n \geqslant |\alpha_n|$ for any n.

Applying Liberman's Lemma in the direction $\boldsymbol{k}$ we also get the following.

6.3. Claim. *If an arc $\gamma|_{[a,b]}$ lies in the dark side for $\boldsymbol{k}$ then the function φ is nondecreasing in $[a, b]$.*

7. Plane sections

Assume γ is curve on a smooth strictly convex surface Σ in $\mathbb{R}^3$. Consider a plane L passing through two points of γ, say $p=\gamma(a)$ and $q=\gamma(b)$ with $a<b$. Let $L_\pm$ be a half-planes in L bounded by the line trough p and q. Set $\sigma_\pm=\Sigma\cap L_\pm$; note that $\sigma_\pm$ are a smooth convex plane curve connecting p to q in Σ.

7.1. Observation. *If γ is a minimizing geodesic in the convex surface $\Sigma\subset\mathbb{R}^3$ and a, b and $\sigma_\pm$ as above then*

$$\operatorname{length}\sigma_\pm\geqslant\operatorname{length}(\gamma|_{[a,b]}).$$

Based on this observation we give couple of estimates on elevating minimizing geodesics.

7.2. Propostion. *Assume $\gamma\colon[0,\ell]\to\Sigma$ is elevating minimizing geodesic in the convex surface $\Sigma\subset\mathbb{R}^3$. Assume that for a subsegment $[a,b]\subset[0,\ell]$ the following conditions hold*

(i) The points $\gamma(a)$ and $\gamma(b)$ lie in a one half-plane with boundary line formed by the z-axis and the arc $\gamma|_{[a,b]}$ goes around the z-axis at least once.

(ii) $\gamma(a)$ lies above the horizontal plane through $\frac12{\cdot}(\gamma(0)+\gamma(\ell))$.

Then $\gamma(b)$ lies on the dark side of Σ with respect to $\boldsymbol{k}$.

Proof. We apply the observation above to the plane containing z-axis and $\gamma(b)$ and perform straightforward computations.

We can assume that $\gamma(0)$ is the origin of the (x,y,z)-coordinate system and both points $p=\gamma(a)$ and $q=\gamma(b)$ lie in the (x,z)-coordinate half-plane with $x\geqslant0$, denoted by Π. We can assume that $\sigma_+\subset\Pi$. Let $(x_p,0,z_p)$ and $(x_q,0,z_q)$ be the coordinates of p and q.

From the assumptions $z_p<z_q<2{\cdot}z_p$. From convexity of the curve $\Pi\cap\Sigma$ we get

$$\operatorname{length}\sigma_+\leqslant\sqrt{(z_q-z_p)^2+x_p^2}$$

On the other hand, since $\gamma|_{[a,b]}$ goes around z-axis at least once, we get

$$\operatorname{length}\gamma|_{[a,b]}\geqslant\sqrt{(z_q-z_p)^2+(x_p+x_q)^2}.$$

These two estimates contradict Observation 7.1. □

7.3. Corollary. *If Σ, γ, ℓ, a and b as in the Proposition and the arc $\gamma|_{[a,b]}$ goes around the z-axis at least twice then the arc $\gamma|_{[b,\ell]}$ lies on the dark side with respect to $\boldsymbol{k}$.*

Proof. Fix $b'\in[b,\ell]$. Note that one can find $a'\in[a,b]$ such that the the assumtions of Proposition 7.2 hold for the interval $[a',b']$. Applying Proposition we get the result. □

7.4. Propostion. *Assume $\gamma\colon[0,\ell]\to\Sigma$ is elevating minimizing geodesic in the convex surface $\Sigma\subset\mathbb{R}^3$. Assume that the arc $\gamma|_{[b,\ell]}$ lies in the dark side of Σ with respect to $\boldsymbol{k}$. Set $\varphi(t)=\measuredangle(\boldsymbol{k},\dot\gamma(t))$ and $\psi(t)=\frac{\pi}{2}-\measuredangle(\boldsymbol{k},\nu(t))$. If $b\leqslant s<t\leqslant\ell$ and the point $\gamma(s)$ lies in the plane Π through $\gamma(t)$ spanned by $\nu(t)$ and $\lambda(t)$ then*

$$\varphi(s)\leqslant\psi(t).$$

Proof. We apply the observation to the plane Π and $p=\gamma(s)$ and $q=\gamma(t)$.

Let z_p and z_q be the z-coordinates of p and q.

Since $\gamma|_{[s,t]}$ lies in the dark side, its Liberman's development $\tilde\gamma|_{[s,t]}$ with respect to $\boldsymbol{k}$ is concave. In particular

$$\operatorname{length}(\gamma|_{[s,t]}) = \operatorname{length}(\tilde\gamma|_{[s,t]}) \geqslant \tfrac{z_q - z_p}{\cos\varphi(s)}.$$

On the other hand, convexity of σ_+ imply that

$$\operatorname{length}\sigma_+ \leqslant \tfrac{z_q - z_p}{\cos\psi(t)}.$$

It remains to apply Observation 7.1. □

8. *s*-pairs

Let $\Sigma \subset \mathbb{R}^3$ be a strongly convex surface and $\gamma\colon [0,\ell] \to \Sigma$ be an elevating minimizing geodesic.

After rotating (x,y)-plane if necessary, we can assume that the border of shadow in the directions of x-axis, say ω_x, is a smooth curve and γ intersects them transversely.

Let $t_1 < t_2 < \dots < t_k$ be the time moments in $[0,\ell]$ at which γ crossing ω_x. Note that

$$\mu(t_n) = s_n \cdot e_x \ \text{ for some } \ s_n = \pm 1.$$

Set

$$\varphi_n = \varphi(t_n) \qquad \psi_n = \psi(t_n) \qquad \alpha_n = \alpha(t_n)$$

We say that a pair of indexes $i < j$ forms an $\boldsymbol{s}$*-pair* if

$$\sum_{n=i}^{j} s_n = 0 \ \text{ and } \ \sum_{n=i}^{j'} s_n > 0$$

if $i < j' < j$.

Note that for any index i appears in at most one $\boldsymbol{s}$-pair and for any $\boldsymbol{s}$-pair (i,j) we have

- ◇ $s_i = 1$; that is, i-th braket has to be openning.
- ◇ $s_j = -1$; that is, j-th braket has to be closing.

In particular,

$$s_i \cdot \alpha_i + s_j \cdot \alpha_j = \alpha_i - \alpha_j.$$

Bracket interpretation. If you exchange "+1" and "−1" in $\boldsymbol{s}$ by "(" and ")" correspondingly then (i,j) is an $\boldsymbol{s}$-pair if and only if the i-th bracket forms a pair with j-bracket.

Embedded disc interpretation. Assume (i,j) is an $\boldsymbol{s}$-pair. Note that in this case there is an arc of $\omega_{\boldsymbol{i}}$ from $\gamma(t_i)$ to $\gamma(t_j)$ with monotonic z-coordinate. Moreover this arc, say σ together with $\gamma|[t_i,t_j]$ bounds an immesed disc in Σ. That is there is an immesion $\iota\colon \mathbb{D} \to \Sigma$ such that the closed curve $\iota|_{\partial\mathbb{D}}$ is formed by joint of σ and $\gamma|[t_i,t_j]$.

The proof can be guessed from the diagram. It shows a lift of γ in the universal cover of strip of Σ between horizontal planes through $\gamma(t_i)$ and $\gamma(t_j)$; the solid vertical lines correspond are lifts of σ and the dashed lines corresponds to the lifts of the other component of $\omega_{\boldsymbol{i}}$ bethween the planes.

We say that q is the depth of an $\boldsymbol{s}$-pair (i,j) (briefly $q = \operatorname{depth}_{\boldsymbol{s}}(i,j)$) if q is the maximal number such that theis q-long nested sequence of $\boldsymbol{s}$-pairs starting with (i,j); that is a sequence of $\boldsymbol{s}$-pairs $(i,j) = (i_1,j_1), (i_2,j_2), \dots, (i_q,j_q)$ such that

$$i = i_1 < \dots < i_q < j_q < \dots < j_1 = j.$$

We say that q is the depth of an $\boldsymbol{s}$-pair (i,j) (briefly $q=\operatorname{depth}_{\boldsymbol{s}}(i,j)$) if q is the maximal number such that theis q-long nested sequence of $\boldsymbol{s}$-pairs starting with (i,j); that is a sequence of $\boldsymbol{s}$-pairs $(i,j)=(i_1,j_1),(i_2,j_2),\dots,(i_q,j_q)$ such that

$$i=i_1<\dots<i_q<j_q<\dots<j_1=j.$$

Note that the $\boldsymbol{s}$-pair of the same depth do not overlap; that is if for two distinct $\boldsymbol{s}$-pairs (i,j) and (i',j'), we have $\operatorname{depth}(i,j)=\operatorname{depth}(i',j')$ then either $i<j<i'<j'$ or $i'<j'<i<j$.

The following proposition follow directly from the definitions above.

8.1. Proposition. *Let (i,j) be an $\boldsymbol{s}$-pair. Then the arcs $\gamma|_{[t_i,t_j]}$ and an arc of $\omega_{\boldsymbol{i}}$ bound an immesed disc in Σ which lies between horizontal planes through $\gamma(t_i)$ and $\gamma(t_j)$. Moreover the maximal multiplicity of the disc is at most* $\operatorname{depth}_{\boldsymbol{s}}(i,j)$.

8.2. Corollary. *Let us denote by S_q the subset of indiexes $\{1,\dots,k\}$ which are the parts of $\boldsymbol{s}$-pairs with depth q. Then*

$$\sum_{n\in S_q} s_n\cdot\alpha_n\leqslant 4\cdot\pi\cdot q.$$

Proof. For each n denote by K_n the integral of Gauss curvature of the part of surface Σ which lies below horizontal plane through $\gamma(t_n)$. Note that

$$0\leqslant K_1\leqslant\ldots\leqslant K_k\leqslant 4\cdot\pi.$$

By Proposition 8.1 and the Key Lemma, we get

$$s_i\cdot\alpha_i+s_j\cdot\alpha_j=\alpha_i-\alpha_j\leqslant q\cdot(K_j-K_i)$$

The statement follows since the $\boldsymbol{s}$-pairs with the same depth do not overlap. □

9. Geometric growth

9.1. Claim. *Let $\gamma\colon[0,\ell]\to\Sigma$ be elevating minimizing geodesic in a strongly convex surface Σ and t_n, s_n, φ_n as defined above.*

Assume there are indexes $k_0<k_1$ such that we have the point $\gamma(t_{k_0})$ lies above the horizontal plane through $\tfrac12\cdot(\gamma(0)+\gamma(\ell))$ and

$$\Big|\sum_{n=k_0}^{k_1}s_n\Big|\geqslant 5.$$

Then for any pair of indexes $j>i>k_1$, such that

$$\Big|\sum_{n=i}^{j}s_n\Big|\geqslant 5$$

we have

$$\varphi_j>2\cdot\varphi_i.$$

Proof. Note that the arc $\gamma|_{[t_{k_0},t_{k_1}]}$ goes around the z-axis twice. By Corollary 7.3, the arc $\gamma|_{t_{k_1},\ell}$ lies in the dark side of Σ with respect to $\boldsymbol{k}$. By Claim ???, the function φ increase in the interval $[t_{k_1},\ell]$.

□

Set $R=\{1,\dots,k\}\backslash(S_1\cup\dots\cup S_5)$; this is the set of indexes which do not appear in any $\boldsymbol{s}$-pairs as well as the indexes which appera in $\boldsymbol{s}$-pairs of depth at least 6.

According to ???, the theorem will follow if we can find an universal upper bound on

$$\Big|\sum_{n\in R} s_n\cdot\alpha_n\Big|.$$

Further let us subdivide R into $R_1,\dots R_5$, by setting

$$R_r = \{\, n\in R \mid s_1+\dots+s_n\equiv r \pmod 5 \,\}.$$

Consider yet two more angle functions.

◇ Let $\varphi(t)=\measuredangle(\dot\gamma(t),\boldsymbol{k})$. Set $\varphi_n=\varphi(t_n)$.

◇ Let $\psi(t)$ be the signed angle between $\nu(t)$ and (x,y)-plane. Set $\psi_n=\psi(t_n)$.

Note that

$$\varphi(t)\geqslant|\psi(t)| \ \text{ and } \ \varphi(t)\geqslant|\alpha(t)|$$

for any t. In particular

$$\varphi_n\geqslant|\psi_n| \ \text{ and } \ \varphi_n\geqslant|\alpha_n|$$

for any n. It follows that

$$\Big|\sum_{n\in R} s_n\cdot\alpha_n\Big|\leqslant\sum_{n\in R}\varphi_n.$$

Given $n\in R_r$, denote by n' the lest index in R_r bigger than n. Our aim is to show that $\varphi_{n'}\geqslant 2\cdot\varphi_n$. In this case

Since $\varphi_n\leqslant\pi$, we will get

$$\sum_{n\in R_r}\varphi_n\leqslant 2\cdot\pi.$$

Horizontal rotation. Note that $\mu(t)$ is horizontal for any t.

Define the rotation $\rho[a,b]$ of the interval $[a,b]\subset[0,\ell]$ as the algebraic rotation of $\mu(t)$ around the origin in (x,y)-plane; say it can be defined by the formula

$$\rho_{[a,b]}=\int\limits_a^b\langle\mu(t),\mathrm{J}(\dot\mu(t))\rangle\cdot dt,$$

where $\mathrm{J}\colon\mathbb{R}^2\to\mathbb{R}^2$ denotes the rotation by angle $\frac{\pi}{2}$ around the origin.

Note that

$$\rho[t_i,t_{i+1}]=\tfrac{\pi}{2}\cdot(s_i+s_{i+1}).$$

9.2. Claim. *Assume that*

$$\gamma_z(t_i)\geqslant\tfrac12\cdot(\gamma_z(\ell)+\gamma_z(0)) \quad \textit{and} \quad \left|\sum_{n=i}^{j}s_n\right|=6.$$

Then $\psi(t)>0$ for any $t\geqslant t_j$.

Note that the claim implies that from t_j, the geodesic γ lies on a graph $z=f(x,y)$ of a concave function $f\colon\mathbb{R}^2\to\mathbb{R}$ and forms a minimizing geodesic in this graph. Indeed fix $\varepsilon>0$ such that $\psi(t)>\varepsilon$ for any $t\geqslant t_j$. Consider the set W which lies under all the supporting planes such that its outer normal vector forms angle at most $\frac{\pi}{2}-\varepsilon$ with the vertical direction. Note that the set W forms a subgraph $z\leqslant f(x,y)$ of a concave $(\operatorname{ctg}\varepsilon)$-Lipschitz function $f\colon\mathbb{R}^2\to\mathbb{R}$ and all the points of γ lie the graph $z=f(x,y)$

Proof. Assume contrary, that is $\varphi(t)\leqslant 0$ for some $t\geqslant t_j$. Let us draw the half-plane Π_+ through $\gamma(t)$ bounded by the z-axis. Denote by Π_- the opposite half-plane for Π_+ .

Note that there are two values $t_+<t_-$ in $[t_i,t_j]$ such that $\gamma(t_\pm)\in\Pi_\pm$.

Consider the sub-arc σ of $\Sigma\cap\Pi$ from $\gamma(t)$ to $\gamma(t')$. Since γ is minimizing we have that

$$\operatorname{length}\gamma|_{[t',t]}\leqslant\operatorname{length}\sigma.$$

The later contradicts straightforward estimates. Namely assume a and a_+ be the distances from $\gamma(t)$ and $\gamma(t_+)$ to the z-axis. Further set $b=\langle\gamma(t)-\gamma(0),\boldsymbol{k}\rangle$ and $b_+=\langle\gamma(t)-\gamma(0),\boldsymbol{k}\rangle$. By the assumptions we have $b\leqslant 2{\cdot}b_+$; it follows that $a_+\leqslant a\leqslant 2{\cdot}a_+$ and therefore

$$(\operatorname{length}\sigma)^2\leqslant(b-b_+)^2+a_+^2.$$

On the other hand, since $t_+<t_-<t$ we get that

$$(\operatorname{length}\gamma|_{[t_+,t]})^2\geqslant(b-b_+)^2+(a+a_+)^2.$$

$\square$

Note that the last claim imply the following.

9.3. Proposition. *Assume Main Theorem does not hold; that is, there is a sequence of convex surfaces Σ_n and a sequence of minimizing geodesic γ_n in Σ_n such that*

$$\operatorname{TotCurv}\gamma_n\to\infty\quad as\quad n\to\infty.$$

Then we can make in addition one of the following assumtions:

(i) Σ_n is a graph $z=f_n(x,y)$ of a smooth convex function $f_n\colon\mathbb{R}^2\to\mathbb{R}$ and $\dot\gamma_z(t)>0$ for any $t\in[0,\ell]$.

(ii) $\left|\sum_{n=i}^j s_n\right|<10$ for any $i<j$.

In particular, from now on $\psi(t)>0$ for any $t\in[0,\ell]$. Note also that by Liberman's lemma $\varphi(t)$ is a nondecreasing function on $[0,\ell]$. The two cases (i) and (ii) will be done separetely. The case (i) is more involved.

9.4. Claim. *Let $[a,b]\subset[0,\ell]$ and $\rho[a,b]\geqslant 3{\cdot}\pi$. Then $\psi(t)\geqslant\varphi(a)$ for any $t\geqslant b$.*

9.5. Claim. *Assume $\psi(t)\geqslant\varepsilon>0$ for any $t\in[a,b]$. Then $\alpha(b)-\alpha(a)>\varepsilon{\cdot}\rho[a,b]$. In particular, either $|\alpha(a)|>\frac12{\cdot}\varepsilon{\cdot}\rho[a,b]$ or $|\alpha(b)|>\frac12{\cdot}\varepsilon{\cdot}\rho[a,b]$.*

Note that above two claims imply the following.

9.6. Proposition. *Assume γ as in Proposition 9.3 and for some $i<j$ we have*

$$\left|\sum_{n=i}^{j}s_i\right|=5$$

Then $\varphi_j>2{\cdot}\varphi_i$.

10. An estimate for graphs

10.1. Proposition. *There is c constant ω' ($\omega'=$??? will do) such that if γ is an elevating minimizing geodesic on a graph $z=f(x,y)$ of a concave function f then*

$$\operatorname{TotCurv}\gamma\leqslant\omega'.$$

Proof. We can assume that γ cross the $\boldsymbol{i}$ horizon $\omega_{\boldsymbol{i}}$ tranfersally. Let $t_1<\dots<t_k$ be the values of parameter at which γ cross $\omega_{\boldsymbol{i}}$ and $s_1,\dots,s_k$ the signs as in ...

Recall that S_q denotes the subset of indexes $\{1,\dots,k\}$ which appear in $\boldsymbol{s}$-pair with depph q. According to Corollary 8.2,

$$\left|\sum_{n\in S_q}s_n{\cdot}\alpha_n\right|\leqslant 4{\cdot}q{\cdot}\pi.$$

In particular,

$$\left|\sum_{n\in S_1\cup\dots\cup S_5} s_n\cdot\alpha_n\right| \leqslant 40\cdot\pi.$$

Set $R = \{1,\dots,k\}\backslash(S_1\cup\dots\cup S_5)$; this is the set of indexes which appear in $\boldsymbol{s}$-pairs with depth at least 6 as well as those which do not appear in any $\boldsymbol{s}$-pair.

According to ???

$$\left|\sum_{n\in R} s_n\cdot\alpha_n\right| \leqslant \sum_{n\in R}\varphi_n.$$

To estimate the last sum will use the results in Section 9. First let us subdivide R into 5 subsets $R_1,\dots,R_5$, by setting $n\in R_m$ if $m\equiv n \pmod 5$.

Given $n\in R_m$, denote by n' the least index in R_m which is larger n; n' is defined for any $n\in R_m$ exept the largest one. According to ??? $\varphi_{n'} > 2\cdot\varphi_n$. Since φ_n is nondecreasing in n, we get

$$\sum_{n\in R_m}\varphi_n \leqslant 2\cdot\varphi_k.$$

It follows that

$$\sum_{n\in R}\varphi_n \leqslant 10\cdot\varphi_k < 5\cdot\pi.$$

According to Liberman's lemma

$$\begin{aligned}\operatorname{TotCurv}_{\boldsymbol{i}}\gamma &\leqslant 4\cdot\pi + 2\cdot[s_1\cdot\alpha_1+\dots+s_k\cdot\alpha_k] \leqslant\\ &\leqslant 100\cdot\pi.\end{aligned}$$

After rotationg (x,y) plane we can assume that

$$\operatorname{TotCurv}\gamma \leqslant 10\cdot\operatorname{TotCurv}_{\boldsymbol{i}}\gamma.$$

Hence the result follows.

11. Final proof assembling

Assume there is a minimizing geodesic $\gamma\colon[0,\ell]\to\Sigma$ in a convex surface $\Sigma\subset\mathbb{R}^3$ such that

$$\operatorname{TotCurv}\gamma = \omega.$$

According to ??? we can assume that Σ is strongly convex.

According to ???, we can pass to an elevating arc, of γ for some (x,y,z)-coordinate system with total curvature $> \frac{\omega}{10^6}$. Rename this arc by γ and let us use the notations in Section 6.

Let us subdivide γ into three arcs lower middle and upper arcs γ_-, γ_0 and γ_+ the the following way.

Note that according to ??? γ_+ lies on a graph of concvae function. By Proposition 9.3, we get

$$\operatorname{TotCurv}\gamma_+ \leqslant 100\cdot\pi. \qquad ❶$$

Similarly γ_- lies on a graph of convex function and the same proposition implies

$$\operatorname{TotCurv}\gamma_- \leqslant 100\cdot\pi. \qquad ❷$$

It remains to estimate $\operatorname{TotCurv}\gamma_0$.

Note that any $\boldsymbol{s}$ pair on γ_0 has depth at most ???. Therefore ???

It remains to note that the total number of indexes which do not appear in any $\boldsymbol{s}$ on γ_0 is at most ???. Hence

$$\sum_{t_n\in ???} s_n\cdot\alpha_n \leqslant ???$$

By Liberman's lemma, we get

$$\operatorname{TotCurv}_{\boldsymbol{i}} \gamma_0 \leqslant ???$$

Since we can rotate (x, y)-plane arbitrary, we get

$$\operatorname{TotCurv} \gamma_0 \leqslant ???$$

Together with ❶ and ❷, the later implies that

$$\operatorname{TotCurv} \gamma \leqslant ???.$$

□

References

[1] P. K. Agarwal, S. Har-Peled, M. Sharir, and K. R. Varadarajan. "Approximating shortest paths on a convex polytope in three dimensions". *J. ACM* 44.4 (1997), pp. 567–584.

[2] J. Pach. "Folding and turning along geodesics in a convex surface". *Geombinatorics* 7.2 (1997), pp. 61–65.

[3] I. Bárány, K. Kuperberg, and T. Zamfirescu. "Total curvature and spiralling shortest paths". *Discrete Comput. Geom.* 30.2 (2003). U.S.-Hungarian Workshops on Discrete Geometry and Convexity (Budapest, 1999/Auburn, AL, 2000), pp. 167–176.

[4] J. Liberman. "Geodesic lines on convex surfaces". *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 32 (1941), pp. 310–313.

[5] V. V. Usov. "The length of the spherical image of a geodesic on a convex surface". *Sibirsk. Mat. Ž.* 17.1 (1976), pp. 233–236.

[6] I. D. Berg. "An estimate on the total curvature of a geodesic in Euclidean 3-space-with-boundary". *Geom. Dedicata* 13.1 (1982), pp. 1–6.

[7] A. V. Pogorelov. *Extrinsic geometry of convex surfaces*. Translated from the Russian by Israel Program for Scientific Translations, Translations of Mathematical Monographs, Vol. 35. American Mathematical Society, Providence, R.I., 1973, pp. vi+669.

[8] A. D. Milka. "A shortest path with nonrectifiable spherical representation. I". *Ukrain. Geometr. Sb.* 16 (1974), pp. 35–52, ii.

[9] V. V. Usov. "The three-dimensional swerve of curves on convex surfaces". *Sibirsk. Mat. Ž.* 17.6 (1976), pp. 1427–1430, 1440.

N. Lebedeva
Math. Dept. St. Petersburg State University, Universitetsky pr., 28, Stary Peterhof, 198504, Russia.
Steklov Institute, 27 Fontanka, St. Petersburg, 191023, Russia.
E-mail address: lebed@pdmi.ras.ru

A. Petrunin
Math. Dept. PSU, University Park, PA 16802, USA
E-mail address: petrunin@math.psu.edu