

ON THE TOTAL CURVATURE OF MINIMIZING GEODESICS ON CONVEX SURFACES

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ABSTRACT. We give a universal upper bound for the total curvature of minimizing geodesic on a convex surface in the Euclidean space.

1. INTRODUCTION

In this note we give an affirmative answer to the question asked by Dmitry Burago (the same question was also stated in [1], [2] and [3]). Namely, we prove the following.

1.1. Main theorem. *The total curvature of a minimizing geodesic on a convex surface in \mathbb{R}^3 can not exceed 10^{10} .*

The optimal upper bound is expected to be slightly bigger than $2\cdot\pi$. The value $2\cdot\pi$ is the optimal bound for the analogous problem in the plane. A minimizing geodesic on a convex surface in \mathbb{R}^3 with total curvature slightly bigger than $2\cdot\pi$ was constructed by Bárány, Kuperberg, and Zamfirescu in [3]. Let us list the related results.

- ◊ In [4], Liberman gives a bound on the total curvature of short geodesic in terms of the ratio diameter and inradius of K . In the proof he use now so called Liberman's lemma, see 3.1 below.
- ◊ In [5], Usov gives the optimal bound for total curvature of geodesic on the graph of ℓ -Lipschitz convex function. Namely, he proves that if $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is ℓ -Lipschitz and convex then any geodesic in its graph

$$\Gamma_f = \{ (x, y, z) \in \mathbb{R}^3 \mid z = f(x, y) \}$$

has total curvature at most $2\cdot\ell$. Yet an amusing generalization of Usov's result is given by Berg in [6].

- ◊ In [7], Pogorelov conjectured that any the spherical image of geodesic on convex surface has to be contractible. It is easy to see that the length of spherical image of geodesic can not be smaller than its total curvature, so this conjecture (if it would be true) would be stronger than Liberman's theorem. Counterexamples were found indepenently by Milka in [8], Usov in [9] and yet much later rediscovered by Pach in [2].
- ◊ In [3], Bárány, Kuperberg, and Zamfirescu have constructed a corkscrew minimizing geodesic on a closed hypersurface; that is a minimizing geodesic which twists around given line arbitrary many times. They also rediscovered the results of Liberman and Usov mentioned above.

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2. PRELIMINARIES

Semicontinuity of total curvature. Recall that the *total curvature* of a curve $\gamma: [0, \ell] \rightarrow \mathbb{R}^3$ (briefly $\text{TotCurv } \gamma$) is defined as supremum of sum of exterior angles for the polygonal lines inscribed in γ .

Note that for a polygonal line σ , its total curvature coincide with the sum of its exterior angles.

If γ is a smooth curve with unit-speed parametrization, then

$$\text{TotCurv } \gamma = \int_0^\ell \kappa(t) \cdot dt,$$

where $\kappa(t) = |\ddot{\gamma}(t)|$ is the curvature of γ at t .

2.1. Proposition. *Assume $\gamma_n: \mathbb{I} \rightarrow \mathbb{R}^3$ is a sequence of curves converging pointwise to a curve $\gamma_\infty: \mathbb{I} \rightarrow \mathbb{R}^3$. Then*

$$\liminf_{n \rightarrow \infty} \text{TotCurv } \gamma_n \geq \text{TotCurv } \gamma_\infty.$$

Proof. Choose a polygonal line σ_∞ inscribed in γ_∞ with total curvature sufficiently close to $\text{TotCurv } \gamma_\infty$. Let $\gamma_\infty(t_0), \dots, \gamma_\infty(t_k)$ for $t_0 < \dots < t_k$ be the vertices of σ_∞ . Consider the polygonal lines σ_n inscribed in γ_n with the vertices $\gamma_n(t_0), \dots, \gamma_n(t_k)$. Note that

$$\text{TotCurv } \sigma_n \rightarrow \text{TotCurv } \sigma_\infty.$$

By the definition of total curvature

$$\text{TotCurv } \sigma_n \leq \text{TotCurv } \gamma_n.$$

The statement follows since the broken line σ_∞ can be chosen in such a way that $\text{TotCurv } \sigma_n$ is arbitrary close to $\text{TotCurv } \gamma_n$. \square

Convergence. Given a closed set $\Sigma \subset \mathbb{R}^3$, denote by dist_Σ the distance function from Σ ; that is

$$\text{dist}_\Sigma(x) = \inf \{ |x - y| \mid y \in \Sigma \}.$$

We say that a sequence of closed sets $\Sigma_n \subset \mathbb{R}^3$ converges to the closed set $\Sigma_\infty \subset \mathbb{R}^3$ if for any $x \in \mathbb{R}^3$, we have $\text{dist}_{\Sigma_n}(x) \rightarrow \text{dist}_{\Sigma_\infty}(x)$ as $n \rightarrow \infty$.

Convex surfaces. By convex surface in the Euclidean 3-space \mathbb{R}^3 we understand the boundary of closed convex set with nonempty interior.

2.2. Proposition. *Assume Σ_n be a sequence of convex surfaces which converge to a convex surface Σ_∞ . Then for any minimizing geodesic γ_∞ in Σ_∞ there is a sequence of minimizing geodesics γ_n in Σ_n which pointwise converge to γ_∞ as $n \rightarrow \infty$.*

Proof. Assume $\gamma_\infty: [0, \ell] \rightarrow \Sigma_\infty$ is parametrized by its arc length.

Fix a subinterval $[a, b] \subset [0, \ell]$ such that $0 < a$ and $b < \ell$. Set $p_\infty = \gamma_\infty(a)$ and $q_\infty = \gamma_\infty(b)$.

Let $p_n, q_n \in \Sigma_n$ be a two sequences of points which converge to p_∞ and q_∞ correspondingly.

Denote by γ_n a minimizing geodesic from p_n to q_n in Σ_n . Note that γ_n converges to $\gamma_\infty|_{[a, b]}$ as $n \rightarrow \infty$.

Taking the subinterval $[a, b]$ closer and closer to $[0, \ell]$ and applying diagonal procedure, we get the result. \square

3. LIBERMAN'S LEMMA.

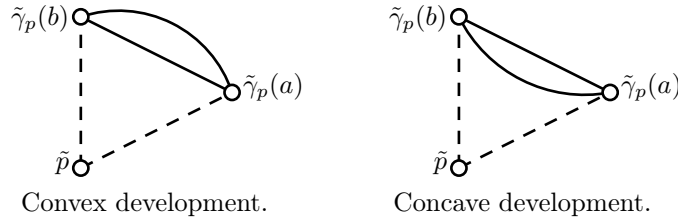
In this section we give a slight generalization of the construction given by Liberman in [4].

Development. Let $\gamma: [0, \ell] \rightarrow \mathbb{R}^3$ be a curve parametrized by length and a point p does not lie on γ .

Assume $\tilde{\gamma}_p: [0, \ell] \rightarrow \mathbb{R}^2$ is a plane curve parametrized by length and \tilde{p} is a point in the plane such that

$$|\tilde{p} - \tilde{\gamma}(t)| = |\tilde{p} - \tilde{\gamma}(t)|$$

for any $t \in [0, \ell]$ and the direction from \tilde{p} to $\tilde{\gamma}(t)$ changes counterclockwise. Then $\tilde{\gamma}_p$ is called *development* of γ with respect to p .



We say that the development $\tilde{\gamma}_p$ is convex (concave) in the interval $[a, b]$ if the lune bounded by arc $\tilde{\gamma}_p|_{[a, b]}$ and the segment $[\tilde{\gamma}_p(a)\tilde{\gamma}_p(b)]$ is convex and lies on with \tilde{p} on the opposite side from the line $(\tilde{\gamma}_p(a)\tilde{\gamma}_p(b))$ (correspondingly the same side from the line $(\tilde{\gamma}_p(a)\tilde{\gamma}_p(b))$).

We say that $\tilde{\gamma}_p$ is locally convex (concave) in the interval $[a, b]$ if any point $x \in [a, b]$ admits a closed neighborhood $[a', b']$ in $[a, b]$ such that $\tilde{\gamma}_p$ is convex (correspondingly concave) in the interval $[a', b']$.

If we pass to the limit of this construction as p moves to infinity along a half-line in the direction of a unit vector \mathbf{u} then the limit curve is called development of γ in direction \mathbf{u} and denoted as $\tilde{\gamma}_{\mathbf{u}}$.

The development $\tilde{\gamma}_{\mathbf{u}}$ could be also defined directly. Namely the curve $\tilde{\gamma}_{\mathbf{u}}: [0, \ell] \rightarrow \mathbb{R}^2$ is parametrized by length and for a fixed unit vector $\tilde{\mathbf{u}} \in \mathbb{R}^2$, we have

$$\langle \tilde{\mathbf{u}}, \tilde{\gamma}_{\mathbf{u}}(t) \rangle = \langle \mathbf{u}, \gamma(t) \rangle$$

for any $t \in [0, \ell]$ and the projection of $\tilde{\gamma}_{\mathbf{u}}(t)$ to the line normal to $\tilde{\mathbf{u}}$ is monotonic in t .

We can assume that $\tilde{\mathbf{u}}$ is the vertical vector in the coordinate plane. We say that $\tilde{\gamma}_{\mathbf{u}}$ is concave (convex) in the interval $[a, b]$ if the lune bounded by arc $\tilde{\gamma}_{\mathbf{u}}|_{[a, b]}$ and the segment $[\tilde{\gamma}_{\mathbf{u}}(a)\tilde{\gamma}_{\mathbf{u}}(b)]$ is convex and lies above (correspondingly below) the line segment $[\tilde{\gamma}_{\mathbf{u}}(a)\tilde{\gamma}_{\mathbf{u}}(b)]$.

Dark and light sides. Let $\Sigma \subset \mathbb{R}^3$ be a smooth convex surface and $z \notin \Sigma$ and $p \in \Sigma$.

We say that p lies on the dark (light) side of Σ with from z if non of the points $p + t \cdot (p - z)$ lie inside of Σ for $t > 0$ (correspondingly for $t < 0$). The intersection of dark and light side is called horizon of z ; it is denoted by ω_z .

Note that if z lies inside Σ then all the points on Σ lies on dark side from z .

If Σ is smooth we can define the outer normal vector ν_p to Σ at p . In this case p lies on dark (light) side of Σ with from z if and only if $\langle p - z, \nu_p \rangle \geq 0$ (correspondingly for $\langle p - z, \nu_p \rangle \leq 0$). If in addition Σ is strongly convex then the horizon is formed by a collection of disjoint smooth curves.

We could also define light/dark sides and horizon in the limit case, as p escapes to infinity along a half-line in direction \mathbf{u} . Let us also define it directly, we say that a point $p \in \Sigma$ lies on dark (light) side for the unit vector \mathbf{u} if non of the points

$p + \mathbf{u} \cdot t$ lie inside of Σ for $t > 0$, (correspondingly $t < 0$). As before the intersection of light and dark side is called horizon of \mathbf{u} and it is denoted as $\omega_{\mathbf{u}}$.

In the smooth case the later means that $\langle \nu_p, \mathbf{u} \rangle \geq 0$ (correspondingly $\langle \nu_p, \mathbf{u} \rangle \leq 0$). If Σ is strongly convex then $\omega_{\mathbf{u}}$ is locally a smooth curve.

3.1. Liberman's Lemma. *Assume γ be a geodesic on convex surface $\Sigma \subset \mathbb{R}^3$. Then for any point $z \notin \Sigma$ the development $\tilde{\gamma}_z$ is locally convex (locally concave) if γ lies on dark (correspondingly light) side of Σ from z .*

Similarly for any unit vector \mathbf{u} , the development $\tilde{\gamma}_{\mathbf{u}}$ is locally convex (locally concave) if γ lies on dark (correspondingly light) side of Σ for \mathbf{u} .

Note that for any space curve γ and any unit vector \mathbf{u} we have

$$\text{TotCurv } \tilde{\gamma}_{\mathbf{u}} \leq \text{TotCurv } \gamma.$$

On the other hand total curvature of few developments gives an estimate for the total curvature of the original curve. For example, if $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is the orthonormal basis then

$$\text{TotCurv } \gamma \leq \text{TotCurv } \tilde{\gamma}_{\mathbf{i}} + \text{TotCurv } \tilde{\gamma}_{\mathbf{j}} + \text{TotCurv } \tilde{\gamma}_{\mathbf{k}}.$$

If γ lies on one dark or light side for direction \mathbf{u} then by Liberman's lemma we get

$$\text{TotCurv } \tilde{\gamma}_{\mathbf{u}} \leq \pi.$$

It follows that is γ cross the horizons $\omega_{\mathbf{i}}$, $\omega_{\mathbf{j}}$ and $\omega_{\mathbf{k}}$ at most N times then

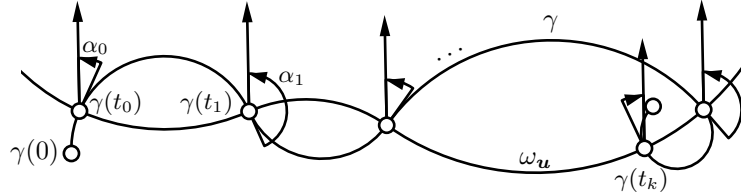
$$\begin{aligned} \text{TotCurv } \gamma &\leq \text{TotCurv } \tilde{\gamma}_{\mathbf{i}} + \text{TotCurv } \tilde{\gamma}_{\mathbf{j}} + \text{TotCurv } \tilde{\gamma}_{\mathbf{k}} \leq \\ &\leq (N+1) \cdot \pi. \end{aligned}$$

Therefore if γ violates Main Theorem then it has to cross the horizons $\omega_{\mathbf{i}}$, $\omega_{\mathbf{j}}$ and $\omega_{\mathbf{k}}$ more that 10^9 times.

4. CURVATURE OF DEVELOPMENT

Let $\Sigma \subset \mathbb{R}^3$ be a strongly convex smooth surface and $\gamma: [0, \ell] \rightarrow \Sigma$ be a unit-speed geodesic. Assume that for some unit vector \mathbf{u} , the γ cross the horizon $\omega_{\mathbf{u}}$ transversely at $t_1 < \dots < t_k$. Set $\alpha_i = \pm \angle(\dot{\gamma}(t_i), \mathbf{u})$ for each i , we can assume that $\alpha_i \in (-\pi, \pi)$ and the sign is taken so that $[\dot{\gamma}(t_i), \mathbf{u}] = \sin \alpha_i \cdot \nu_{\gamma(t_i)}$, where $[\cdot, \cdot]$ denotes vector product and ν_p is the outer normal vector to Σ at the point $p \in \Sigma$.

The values t_i will be called *meeting moments* and the angles α_i will be called *meeting angles* of the geodesic γ with the horizon $\omega_{\mathbf{u}}$. The arc $\gamma|_{[t_{i-1}, t_i]}$ will be called $\omega_{\mathbf{u}}$ -*archway*.



Let us introduce new notation

$$\text{TotCurv}_{\mathbf{u}} \gamma \stackrel{\text{def}}{=} \text{TotCurv } \tilde{\gamma}_{\mathbf{u}}.$$

From Liberman's lemma 3.1, we get the following.

4.1. Corollary. *Let $\Sigma \subset \mathbb{R}^3$ be a strongly convex smooth surface and $\gamma: [0, \ell] \rightarrow \Sigma$ be a unit-speed geodesic. Assume that for some unit vector \mathbf{u} , the γ cross the horizon $\omega_{\mathbf{u}}$ transversely and $t_1 < \dots < t_k$ be its meeting moments and $\alpha_1, \dots, \alpha_k$ be its meeting angles with the horizon $\omega_{\mathbf{u}}$. Then*

$$\text{TotCurv}_{\mathbf{u}} \gamma \leq 4 \cdot \pi + 2 \cdot (|\alpha_1| - |\alpha_2| + \dots - (-1)^k \cdot |\alpha_k|)$$

To find the needed estimate the total curvature of geodesic we will get an upper bound for

$$||\alpha_1| - |\alpha_2| + \cdots - (-1)^k \cdot |\alpha_k||.$$

Most of the remaining part of paper devoted to finding such an upper bound.

Proof. By Liberman's lemma,

$$\text{TotCurv}_{\mathbf{u}}(\gamma|_{[t_{i-1}, t_i]}) = \pm(|\alpha_{i-1}| - |\alpha_i|)$$

where the sign is “+” if the archway $[t_i, t_{i+1}]$ lies on the dark side and “−” if it lies on the light side from \mathbf{u} . Summing all this up we get

$$\text{TotCurv}_{\mathbf{u}}(\gamma|_{[t_1, t_k]}) = ||\alpha_1| - 2 \cdot |\alpha_2| + \cdots + (-1)^k \cdot 2 \cdot |\alpha_{k-1}| - (-1)^k \cdot |\alpha_k||.$$

By Liberman's lemma we also have

$$\text{TotCurv}_{\mathbf{u}}(\gamma|_{[0, t_1]}), \text{TotCurv}_{\mathbf{u}}(\gamma|_{[t_k, \ell]}) \leq \pi$$

Since $|\alpha_k| \leq \pi$, the statement follows. \square

If Σ is a surface in \mathbb{R}^3 and $p \in \Sigma$ we denote by K_p the Gauss curvature of Σ at p .

4.2. Archway Lemma. *Let \mathbf{u} be a unit vector, $\gamma: [a, b] \rightarrow \Sigma$ be a minimizing geodesic on the strongly convex surface $\Sigma \subset \mathbb{R}^3$. Assume that $\gamma(a), \gamma(b) \in \omega_{\mathbf{u}}$, α and β are the meeting angles at a and b correspondingly and there is an immersion of disc $\iota: \mathbb{D} \rightarrow \Sigma$ with the boundary curve $\iota|_{\partial\mathbb{D}}$ formed by γ and an arc of $\omega_{\mathbf{u}}$.*

Then

$$\int_{\mathbb{D}} K_{\iota(x)} \cdot d_{\iota(x)} \text{area}_{\Sigma} = \pm(\alpha - \beta) \pmod{2 \cdot \pi} \quad \textbf{1}$$

The sign above is “+” if γ goes counterclockwise in \mathbb{D} and “−” otherwise. In particular

$$||\alpha| - |\beta|| \leq \int_{\mathbb{D}} K_{\iota(x)} \cdot d_{\iota(x)} \text{area}_{\Sigma}. \quad \textbf{2}$$

In particular if γ is an archway; that is it lies completely in the dark or light side or \mathbf{u} then

$$\text{TotCurv } \gamma \leq \int_{\iota(\mathbb{D})} K_p \cdot d_p \text{area}_{\Sigma}. \quad \textbf{3}$$

Proof. Since γ is a geodesic, the parallel translation along γ maps $\dot{\gamma}(a)$ to $\dot{\gamma}(b)$.

Note also that \mathbf{u} belongs to the tangent plane to Σ at any point $p \in \omega_{\mathbf{u}}$; in particular the \mathbf{u} extends to a parallel tangent vector field on $\omega_{\mathbf{u}}$.

It follows that parallel translation along $\iota|_{\partial\mathbb{D}}$ rotates the tangent plane by angle $\alpha - \beta$; To prove **1** remains to apply Gauss–Bonnet formula.

Denote by R the right hand side in **2**. Note that $R \geq 0$ and $|\alpha|, |\beta| \leq \pi$. From the main statement of lemma it follows then that the minimal possible value for R is $||\alpha| - |\beta||$.

To prove **3**, note that in this case ι is an embedding. Therefore

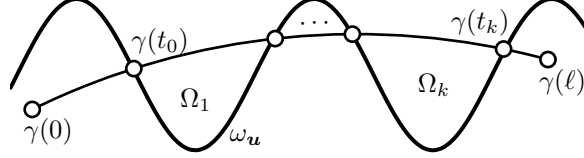
$$\int_{\mathbb{D}} K_{\iota(x)} \cdot d_{\iota(x)} \text{area}_{\Sigma} = \int_{\iota(\mathbb{D})} K_p \cdot d_p \text{area}_{\Sigma}.$$

Whence the statement follows from Liberman's lemma. \square

5. SNAKES AND SPIRALS

Formally speaking, this section is not needed; it only shows how one can use the Archway Lemma in the simplest case and explains the difficulty in the general case.

Assume γ cross the horizon $\omega_{\mathbf{u}}$ as shown on the picture.



That is, $t_0 < \dots < t_k$ are the \mathbf{u} -crossing moments and the points $\gamma(t_0), \dots, \gamma(t_k)$ appear along $\omega_{\mathbf{u}}$ in the same order.

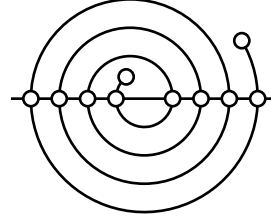
In this case we can chose nonoverlapping discs $\Omega_1, \dots, \Omega_k$ which γ cuts from light and dark side alternatively. Then by Archway lemma,

$$\begin{aligned} \text{TotCurv}_{\mathbf{u}} \gamma &= \text{TotCurv}_{\mathbf{u}}(\gamma|_{[0, t_0]}) + \text{TotCurv}_{\mathbf{u}}(\gamma|_{[t_0, t_1]}) + \dots \\ &\quad \dots + \text{TotCurv}_{\mathbf{u}}(\gamma|_{[t_{k-1}, t_k]}) + \text{TotCurv}_{\mathbf{u}}(\gamma|_{[t_k, \ell]}) \leq \\ &\leq \pi + K(\Omega_1) + \dots + K(\Omega_k) + \pi \leq \\ &\leq 6 \cdot \pi. \end{aligned}$$

The last inequality holds since the discs Ω_i do not overlap. Therefore by Gauss–Bonnet,

$$K(\Omega_1) + \dots + K(\Omega_k) \leq K(\Sigma) \leq 4 \cdot \pi.$$

If the γ cross $\omega_{\mathbf{u}}$ in a different order, say as the spiral shown on the picture then the idea above is not directly applicable. Indeed, no matter how we choose the discs Ω_i for the archways they will always overlap.



6. LENGTH AND DIAMETER

Let $\varepsilon > 0$. A curve $\gamma: [a, b] \rightarrow \mathbb{R}^3$ will be called ε -straight if

$$(1 - \varepsilon) \cdot \text{length } \gamma \leq |\gamma(b) - \gamma(a)|$$

6.1. Lemma. *Assume $\varepsilon > 0$ and n is a positive integer such that $2 \cdot n \cdot \varepsilon > 1$. Then any in any minimizing geodesic on a convex surface Σ in \mathbb{R}^3 can be subdivided into twenty $\frac{1}{10}$ -straight arcs.*

Proof. Assume two points p and q lie on a canovex surface Σ . Denote by ν_p and ν_q the outer normal vectors at p and q correspondingly. Set $\alpha = \angle(\nu_p, \nu_q)$ and $\varepsilon = 1 - \cos \frac{\alpha}{2}$.

Note that any minimizing geodesic from p to q on Σ is ε -straight.

It follows that if two points p and q on Σ can be jointed by a minimizing geodesic which is not $\frac{1}{10}$ -straight then the

Note that there are at most $\frac{2}{1-\varepsilon}$ points in \mathbb{S}^2 which lie on distance at least $2 \cdot \alpha$ from each other.

Let $\gamma: [0, \ell] \rightarrow \Sigma$ be a minimizing geodesic parametrized by its length.

Given a value $t \in [0, \ell]$, set t' to be the maximal value in $[0, \ell]$ such that the interval $[t, t']$ is $\frac{1}{10}$ -straight.

Consider the maximal sequence $0 = t_0 < t_1 < \dots < t_n < \ell$ such that $t_{i+1} = t'_i$. Denote by ν_i the outer unit normal vector to Σ at $\gamma(t_i)$.

Note that $\angle(\nu_i, \nu_j) > 2 \cdot \alpha$ for all i and j . It follows that $n < \frac{2}{1-\varepsilon}$. Hence the problem follows. \square

6.2. Lemma. *Assume γ is a minimizing geodesic on a convex surface in \mathbb{R}^3 . Then $\text{length } \gamma < 4 \cdot \text{diam } \gamma$.*

Proof. Assume contrary; that is, there is convex surface $\Sigma \subset \mathbb{R}^3$ and a geodesic $\gamma: [0, 4] \rightarrow \Sigma$ is parametrized by its length with $\text{diam } \gamma \leq 1$.

Denote by ν_0, ν_2 and ν_4 the outer unit normal vectors to Σ at $\gamma(0), \gamma(2)$ and $\gamma(4)$ correspondingly.

Note that $\angle(\nu_0, \nu_2), \angle(\nu_2, \nu_4) \geq \frac{2}{3} \cdot \pi$ and $\angle(\nu_0, \nu_2) > \frac{2}{3} \cdot \pi$, a contradiction. \square

7. REDUCTION TO A MONOTONIC CASE

7.1. Proposition. *Let $\gamma: [0, \ell] \rightarrow \Sigma$ be a minimizing geodesic on a smooth strongly convex surface Σ in \mathbb{R}^3 . Then there is an interval $[a, b] \subset [0, \ell]$ such that*

$$\text{TotCurv}(\gamma|_{[a,b]}) > \frac{1}{100} \cdot \text{TotCurv } \gamma.$$

and

$$\angle(\dot{\gamma}(t), \mathbf{u}) < \frac{\pi}{3}$$

for any $t \in [a, b]$ and a fixed unit vector \mathbf{u} .

The proof will use the following claim. It states that if a short smooth unit-speed curve runs at nearly the same rate from the vertices of triangle then it runs in nearly the same direction.

7.2. Claim. *Given $\varepsilon > 0$ there is $\delta > 0$ such that the following condition holds.*

Assume γ is a smooth unit-speed curve in the unit ball $B \subset \mathbb{R}^3$. Assume $\text{length } \gamma < \delta$ and B contains an equilateral triangle with vertices x_1, x_2, x_3 with sides ε and for some fixed numbers $\alpha_1, \alpha_2, \alpha_3$ for any i we have either

$$|\gamma(t) - x_i| \leq \frac{\varepsilon}{10} \quad \text{or} \quad \angle(x_i - \gamma(t), \dot{\gamma}(t)) \leq \alpha_i \pm \delta$$

for any t . Then there is a unit vector u such that

$$\angle(\dot{\gamma}(t), u) \leq \frac{\pi}{3}$$

Moreover if $\varepsilon = \frac{1}{100}$ then one can take $\delta = \frac{1}{10000}$.

The proof is straightforward calculations and it is left to the reader.

Proof. Applying rescaling, we can assume that $\text{diam } \gamma = 3$. By Lemma 6.2 $\text{length } \gamma < 12$. Therefore we can subdivide γ into 12 arcs $\gamma_1, \dots, \gamma_{12}$ such that for each n there is a point $p_n \in K$ which lies on the distance at least 1 from γ_n and $\text{length } \gamma_n \leq 1$. Choose an arc $\gamma' = \gamma_n$ with the maximal total curvature and set $p' = p_n$. Clearly

$$\text{TotCurv } \gamma' \geq \frac{1}{12} \cdot \text{TotCurv } \gamma.$$

Applying Liberman's Lemma to γ' with the reference point p' we get that

$$\text{TotCurv } \tilde{\gamma}'_{p'} < \pi + 1 < 5.$$

Choose an integer $N > \frac{2}{\varepsilon}$. Note that we can divide γ' into N arcs $\gamma'_1, \dots, \gamma'_N$ so that

$$\text{TotCurv}_{p'} \gamma'_n \leq \frac{5}{N}$$

for each n . Choose among these arcs the one with maximal total curvature, denote it further by γ'' . Clearly

$$\text{TotCurv } \gamma'' > \frac{\varepsilon}{10^3} \cdot \text{TotCurv } \gamma'.$$

Fix a parameter t of γ'' and denote by α the angle between $\dot{\gamma}''(t)$ and $p - \gamma''(t)$. If $\alpha < \frac{\varepsilon}{2}$ or $\alpha > \pi - \frac{\varepsilon}{2}$, then the problem is solved.

Otherwise applying Lemma 6.1 we get a nondegenerate (say equilateral) triangle $\triangle a_1 a_2 a_3$ in K_n of the size comparable to $\text{diam } \gamma'$ and on the distance comparable to $\text{diam } \gamma'$ from any point of γ'' , say side of triangle can be taken to be $\frac{\varepsilon^2}{1000} \cdot \text{diam } \bar{\gamma}$ and the distance to any point can be assumed to be between $\text{diam } \bar{\gamma}$ and $2 \cdot \text{diam } \bar{\gamma}$.

Apply the construction to each vertex of the triangle. We pass to an arc of $\hat{\gamma}$ such that the angle between $\dot{\gamma}(t)$ and $a_i - \gamma(t)$ and the distance $|\gamma(t) - a_i|$ are nearly constant for each i . The later imply that $\dot{\gamma}$ is nearly constant. \square

8. ELEVATING GEODESICS

In this section we fix notations which will be used further without additional explanation.

Fix a (x, y, z) -coordinates on the Euclidean space; denote by $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ the standard basis.

The lines parallel to the z -axis will be called *vertical*; the lines and planes parallel to (x, y) -plane will be called *horizontal*.

8.1. Definition. A smooth curve $\gamma: [0, \ell] \rightarrow \mathbb{R}^3$ is called *elevating* if both ends $\gamma(0)$ and $\gamma(\ell)$ lie on the z -axis and $\langle \dot{\gamma}(t), \mathbf{k} \rangle > 0$ for all t .

(λ, μ, ν) -frame. Let Σ be a convex surface and $\gamma: [0, \ell] \rightarrow \Sigma$ is an elevating minimizing geodesic with unit-speed parametrization.

Given $t \in [0, \ell]$, consider the oriented orthonormal frame $\lambda(t), \mu(t), \nu(t)$ such that $\nu(t)$ is the outer normal to Σ at $\gamma(t)$, the vector $\mu(t)$ is horizontal and therefore the vector $\lambda(t)$ lies in the plane spanned by $\nu(t)$ and the z -axis. We assume in addition that $\langle \lambda, \mathbf{k} \rangle \geq 0$.

Since $\langle \dot{\gamma}(t), \mathbf{k} \rangle > 0$, $\nu(t)$ can not be vertical and therefore the frame (λ, μ, ν) is uniquely defined for any $t \in [0, \ell]$.

Angle functions. Set

$$\varphi(t) = \angle(\mathbf{k}, \dot{\gamma}(t)), \quad \psi(t) = \frac{\pi}{2} - \angle(\mathbf{k}, \nu(t)), \quad \alpha(t) = \frac{\pi}{2} - \angle(\mu(t), \dot{\gamma}(t)),$$

From the above definitions it follows that $|\alpha(t)|, |\psi(t)| \leq \frac{\pi}{2}$ and for each t there is a right spherical triangle with legs $|\alpha(t)|, |\psi(t)|$ and hypotenuse $\varphi(t)$. In particular $\cos \alpha \cdot \cos \psi = \cos \varphi$. Whence we get the following.

8.2. Claim. For any t we have

$$\varphi(t) \geq |\psi(t)| \quad \text{and} \quad \varphi(t) \geq |\alpha(t)|$$

Applying Liberman's Lemma in the direction \mathbf{k} we also get the following.

8.3. Claim. If an arc $\gamma|_{[a, b]}$ lies in the dark (light) side for \mathbf{k} then the function φ is nondecreasing (correspondingly nonincreasing) in $[a, b]$.

9. PLANE SECTIONS

Assume γ is a curve on a smooth strictly convex surface Σ in \mathbb{R}^3 . Consider a plane L passing through two points of γ , say $p = \gamma(a)$ and $q = \gamma(b)$ with $a < b$. Let L_{\pm} be a half-planes in L bounded by the line through p and q . Set $\sigma_{\pm} = \Sigma \cap L_{\pm}$; note that σ_{\pm} are a smooth convex plane curve connecting p to q in Σ .

9.1. Observation. If γ is a minimizing geodesic in the convex surface $\Sigma \subset \mathbb{R}^3$ and a, b and σ_{\pm} as above then

$$\text{length } \sigma_{\pm} \geq \text{length}(\gamma|_{[a, b]}).$$

Based on this observation we give couple of estimates on elevating minimizing geodesics.

9.2. Propostion. Assume $\gamma: [0, \ell] \rightarrow \Sigma$ is an elevating minimizing geodesic in the convex surface $\Sigma \subset \mathbb{R}^3$. Assume that for a subsegment $[a, b] \subset [0, \ell]$ the following conditions hold

- (i) The points $\gamma(a)$ and $\gamma(b)$ lie in a half-plane with boundary line formed by the z -axis and the arc $\gamma|_{[a, b]}$ goes around the z -axis at least once.
- (ii) $\gamma(a)$ lies above the horizontal plane through $\frac{1}{2} \cdot (\gamma(0) + \gamma(\ell))$.

Then $\gamma(b)$ lies on the dark side of Σ with respect to \mathbf{k} .

Proof. Let us apply Observation 9.1 to the plane containing z -axis, $\gamma(a)$ and $\gamma(b)$.

We can assume that $\gamma(0)$ is the origin of the (x, y, z) -coordinate system and both points $p = \gamma(a)$ and $q = \gamma(b)$ lie in the (x, z) -coordinate half-plane with $x \geq 0$, denoted by Π . We can assume that $\sigma_+ \subset \Pi$. Let $(x_p, 0, z_p)$ and $(x_q, 0, z_q)$ be the coordinates of p and q .

From the assumptions we get $z_p < z_q < 2 \cdot z_p$.

By convexity of the curve $\Pi \cap \Sigma$ we get

$$\text{length } \sigma_+ \leq \sqrt{(z_q - z_p)^2 + x_p^2}$$

On the other hand, since $\gamma|_{[a, b]}$ goes around z -axis at least once, we get

$$\text{length } \gamma|_{[a, b]} \geq \sqrt{(z_q - z_p)^2 + (x_p + x_q)^2}.$$

These two estimates contradict Observation 9.1. \square

9.3. Corollary. If Σ , γ , ℓ , a and b as in the Proposition and the arc $\gamma|_{[a, b]}$ goes around the z -axis at least twice then the arc $\gamma|_{[b, \ell]}$ lies on the dark side with respect to \mathbf{k} .

Proof. Fix $b' \in [b, \ell]$. Note that one can find $a' \in [a, b]$ such that the the assumptions of Proposition 9.2 hold for the interval $[a', b']$. Applying Proposition we get the result. \square

9.4. Propostion. Assume $\gamma: [0, \ell] \rightarrow \Sigma$ is elevating minimizing geodesic in the convex surface $\Sigma \subset \mathbb{R}^3$. Assume that the arc $\gamma|_{[b, \ell]}$ lies in the dark side of Σ with respect to \mathbf{k} . Set $\varphi(t) = \angle(\mathbf{k}, \dot{\gamma}(t))$ and $\psi(t) = \frac{\pi}{2} - \angle(\mathbf{k}, \nu(t))$. If $b \leq s < t \leq \ell$ and the point $\gamma(s)$ lies in the plane Π through $\gamma(t)$ spanned by $\nu(t)$ and $\lambda(t)$ then

$$\varphi(s) \leq \psi(t).$$

Proof. Let us apply Observation 9.1 to the plane Π and $p = \gamma(s)$ and $q = \gamma(t)$.

Let z_p and z_q be the z -coordinates of p and q .

Since $\gamma|_{[s, t]}$ lies in the dark side, its Liberman's development $\tilde{\gamma}|_{[s, t]}$ with respect to \mathbf{k} is concave. In particular

$$\text{length}(\gamma|_{[s, t]}) = \text{length}(\tilde{\gamma}|_{[s, t]}) \geq \frac{z_q - z_p}{\cos \varphi(s)}.$$

On the other hand, convexity of σ_+ imply that

$$\text{length } \sigma_+ \leq \frac{z_q - z_p}{\cos \psi(t)}.$$

It remains to apply Observation 9.1. \square

10. s -PAIRS

Let $\Sigma \subset \mathbb{R}^3$ be a strongly convex surface and $\gamma: [0, \ell] \rightarrow \Sigma$ be an elevating minimizing geodesic.

After rotating (x, y) -plane if necessary, we can assume that the border of shadow in the directions of x -axis, say ω_x , is a smooth curve and γ intersects them transversely.

Let $t_1 < t_2 < \dots < t_k$ be the time moments in $[0, \ell]$ at which γ crossing ω_x . Note that

$$\mu(t_n) = s_n \cdot e_x \quad \text{for some } s_n = \pm 1.$$

Set

$$\varphi_n = \varphi(t_n) \quad \psi_n = \psi(t_n) \quad \alpha_n = \alpha(t_n)$$

We say that a pair of indexes $i < j$ forms an s -pair if

$$\sum_{n=i}^j s_n = 0 \quad \text{and} \quad \sum_{n=i}^{j'} s_n > 0$$

if $i < j' < j$.

Note that for any index i appears in at most one s -pair and for any s -pair (i, j) we have

- ◊ $s_i = 1$; that is, i -th bracket has to be opening.
- ◊ $s_j = -1$; that is, j -th bracket has to be closing.

In particular,

$$s_i \cdot \alpha_i + s_j \cdot \alpha_j = \alpha_i - \alpha_j.$$

Bracket interpretation. If you exchange “+1” and “−1” in s by “(” and “)” correspondingly then (i, j) is an s -pair if and only if the i -th bracket forms a pair with j -bracket.

Embedded disc interpretation. Assume (i, j) is an s -pair. Note that in this case there is an arc of ω_i from $\gamma(t_i)$ to $\gamma(t_j)$ with monotonic z -coordinate. Moreover this arc, say σ together with $\gamma[[t_i, t_j]]$ bounds an immersed disc in Σ . That is there is an immersion $\iota: \mathbb{D} \rightarrow \Sigma$ such that the closed curve $\iota|_{\partial \mathbb{D}}$ is formed by joint of σ and $\gamma[[t_i, t_j]]$.

The proof can be guessed from the diagram. It shows a lift of γ in the universal cover of strip of Σ between horizontal planes through $\gamma(t_i)$ and $\gamma(t_j)$; the solid vertical lines correspond are lifts of σ and the dashed lines corresponds to the lifts of the other component of ω_i between the planes.

We say that q is the depth of an s -pair (i, j) (briefly $q = \text{depth}_s(i, j)$) if q is the maximal number such that their q -long nested sequence of s -pairs starting with (i, j) ; that is a sequence of s -pairs $(i, j) = (i_1, j_1), (i_2, j_2), \dots, (i_q, j_q)$ such that

$$i = i_1 < \dots < i_q < j_q < \dots < j_1 = j.$$

We say that q is the depth of an s -pair (i, j) (briefly $q = \text{depth}_s(i, j)$) if q is the maximal number such that their q -long nested sequence of s -pairs starting with (i, j) ; that is a sequence of s -pairs $(i, j) = (i_1, j_1), (i_2, j_2), \dots, (i_q, j_q)$ such that

$$i = i_1 < \dots < i_q < j_q < \dots < j_1 = j.$$

Note that the s -pair of the same depth do not overlap; that is if for two distinct s -pairs (i, j) and (i', j') , we have $\text{depth}(i, j) = \text{depth}(i', j')$ then either $i < j < i' < j'$ or $i' < j' < i < j$.

The following proposition follow directly from the definitions above.

10.1. Proposition. *Let (i, j) be an \mathbf{s} -pair. Then the arcs $\gamma|_{[t_i, t_j]}$ and an arc of ω_i bound an immersed disc in Σ which lies between horizontal planes through $\gamma(t_i)$ and $\gamma(t_j)$. Moreover the maximal multiplicity of the disc is at most $\text{depth}_{\mathbf{s}}(i, j)$.*

10.2. Corollary. *Let us denote by S_q the subset of indexes $\{1, \dots, k\}$ which are the parts of \mathbf{s} -pairs with depth q . Then*

$$\sum_{n \in S_q} s_n \cdot \alpha_n \leq 4 \cdot \pi \cdot q.$$

Proof. For each n denote by K_n the integral of Gauss curvature of the part of surface Σ which lies below horizontal plane through $\gamma(t_n)$. Note that

$$0 \leq K_1 \leq \dots \leq K_k \leq 4 \cdot \pi.$$

By Proposition 10.1 and the Key Lemma, we get

$$s_i \cdot \alpha_i + s_j \cdot \alpha_j = \alpha_i - \alpha_j \leq q \cdot (K_j - K_i)$$

The statement follows since the \mathbf{s} -pairs with the same depth do not overlap. \square

10.3. Corollary. *Assume*

$$q = \max_{1 \leq i < j \leq k} \left\{ \left| \sum_{n=i}^j s_n \right| \right\}$$

Then

$$\left| \sum_{n=1}^k s_n \cdot \alpha_n \right| \leq 2 \cdot q \cdot \left(q + \frac{3}{2}\right) \cdot \pi.$$

Proof. Denote by S the set of all indexes which appear in some \mathbf{s} -pair.

Note that depth of any \mathbf{s} pair is at most q . That is,

$$S = S_1 \cup \dots \cup S_q.$$

By Corollary 10.2,

$$\sum_{n \in S} s_n \cdot \alpha_n \leq 2 \cdot q \cdot (q + 1) \cdot \pi. \quad \textcircled{4}$$

Set $R = \{1, \dots, k\} \setminus S$; this is the set of indexes which do not appear in an \mathbf{s} -pair.

Given r , set $i \in Q_r$ if

$$\sum_{n=1}^i s_n = r.$$

Note that $Q_r \neq \emptyset$ for at most q values of r and in each set Q_r there are at most 2 indexes which do not appear in an \mathbf{s} -pair; that is $Q_r \cap R$ has at most two indexes for each r .

Sine $|a_n| < \frac{\pi}{2}$, we get

$$\left| \sum_{n \in R} s_n \cdot \alpha_n \right| \leq q \cdot \pi.$$

The later inequality together with $\textcircled{4}$ implies the statement in the corollary. \square

11. GEOMETRIC GROWTH

11.1. Claim. Assume $\psi(t) > \varepsilon$ for ant $t \in [t_i, t_j]$ and $s_i + \dots + s_j = 2$ Then

$$|\alpha_j - \alpha_i| > \pi \cdot \sin \varepsilon.$$

11.2. Claim. Let γ be elevating minimizing geodesic on a graph $z = f(x, y)$ of a concave function. Then for any pair of indexes $j > i$, such that

$$\left| \sum_{n=i}^j s_n \right| \geq 5$$

we have

$$\varphi_j > \frac{3}{2} \cdot \varphi_i.$$

Proof. Without loss of generality, we may assume that

$$\sum_{n=i}^j s_n = 6$$

Let j' be the least index such that

$$\left| \sum_{n=i}^{j'} s_n \right| = 5.$$

Note that for any $b > t_j$ there is $a \in [t_i, t_j]$ such that interval $[a, b]$ satisfies the assumptions of Proposition 9.4. In particular $\psi(b) > \varphi_i$ for any $b > t_j$. Applying Claim 11.1, we get that $|\alpha_j| > \frac{\pi}{2} \cdot \varphi_i$ or $|\alpha_{j'}| > \frac{\pi}{2} \cdot \varphi_i$. Since φ_n is nondecreasing, and $\varphi_n \geq |\alpha_n|$ for any n , in both cases we get

$$\varphi_j > \frac{\pi}{2} \cdot \varphi_i.$$

□

12. AN ESTIMATE FOR GRAPHS

12.1. Proposition. There is a constant ω' ($\omega' = 10 \cdot \pi$ will do) such that if γ is an elevating minimizing geodesic on a graph $z = f(x, y)$ of a concave function f then

$$\text{TotCurv}_i \gamma \leq \omega'.$$

Proof. We can assume that γ cross the i horizon ω_i transversally. Let $t_1 < \dots < t_k$ be the values of parameter at which γ cross ω_i and s_1, \dots, s_k the signs as in ...

Recall that S_q denotes the subset of indexes $\{1, \dots, k\}$ which appear in s -pair with depth q . According to Corollary 10.2,

$$\left| \sum_{n \in S_q} s_n \cdot \alpha_n \right| \leq 4 \cdot q \cdot \pi.$$

In particular,

$$\left| \sum_{n \in S_1 \cup \dots \cup S_5} s_n \cdot \alpha_n \right| \leq 40 \cdot \pi.$$

Set $R = \{1, \dots, k\} \setminus (S_1 \cup \dots \cup S_5)$; this is the set of indexes which appear in s -pairs with depth at least 6 as well as those which do not appear in any s -pair.

According to ???

$$\left| \sum_{n \in R} s_n \cdot \alpha_n \right| \leq \sum_{n \in R} \varphi_n.$$

To estimate the last sum will use the results in Section 11. First let us subdivide R into 5 subsets R_1, \dots, R_5 , by setting $n \in R_m$ if $m \equiv n \pmod{5}$.

Given $n \in R_m$, denote by n' the least index in R_m which is larger n ; n' is defined for any $n \in R_m$ except the largest one. According to ??? $\varphi_{n'} > 2 \cdot \varphi_n$. Since φ_n is nondecreasing in n , we get

$$\sum_{n \in R_m} \varphi_n \leq 2 \cdot \varphi_k.$$

It follows that

$$\sum_{n \in R} \varphi_n \leq 10 \cdot \varphi_k < 5 \cdot \pi.$$

According to Liberman's lemma

$$\begin{aligned} \text{TotCurv}_i \gamma &\leq 4 \cdot \pi + 2 \cdot [s_1 \cdot \alpha_1 + \dots + s_k \cdot \alpha_k] \leq \\ &\leq 100 \cdot \pi. \end{aligned}$$

□

13. FINAL ASSEMBLING

Assume there is a minimizing geodesic $\gamma: [0, \ell] \rightarrow \Sigma$ in a convex surface $\Sigma \subset \mathbb{R}^3$ such that

$$\text{TotCurv} \gamma = \omega.$$

According to ??? we can assume that Σ is strongly convex.

According to ???, we can pass to an elevating arc, of γ for some (x, y, z) -coordinate system with total curvature $> \frac{\omega}{10^6}$. Rename this arc by γ and let us use the notations in Section 8.

Rotating (x, y) -coordinate plane we can ensure that

$$\text{TotCurv} \gamma \leq 10 \cdot \text{TotCurv}_i \gamma$$

and that γ cross the horizon ω_i transversally.

Let us subdivide γ into three arcs lower middle and upper arcs γ_- , γ_0 and γ_+ the the following way.

Note that according to ??? γ_+ lies on a graph of concvae function. By Proposition ??, we get

$$\text{TotCurv}_i \gamma_+ \leq 100 \cdot \pi. \tag{5}$$

Similarly γ_- lies on a graph of convex function and the same proposition implies

$$\text{TotCurv}_i \gamma_- \leq 100 \cdot \pi. \tag{6}$$

By Corollary 10.3,

$$\text{TotCurv}_i \gamma_0 \leq 100 \cdot \pi. \tag{7}$$

$$\text{TotCurv} \gamma_0 \leq ???$$

Together with 5 and 6, the later implies that

$$\text{TotCurv}_i \gamma \leq 300 \cdot \pi.$$

□

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