

# On the total curvature of minimizing geodesics on convex surfaces

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## Abstract

We give a universal upper bound for the total curvature of minimizing geodesic on a convex surface in the Euclidean space.

## 1 Introduction

In this note we give an affirmative answer to the question asked by Dmitry Burago; later we found the same question in [1], [2] and [3]. Namely, we prove the following.

**1.1. Main theorem.** *The total curvature of a minimizing geodesic on a convex surface in  $\mathbb{R}^3$  can not exceed  $1000^{1000}$ .*

The value  $2\cdot\pi$  is the optimal bound for the analogous problem in the plane. The total curvature of minimizing geodesic on a convex surface in  $\mathbb{R}^3$  can exceed  $2\cdot\pi$  and the optimal bound is expected to be slightly bigger than  $2\cdot\pi$ . The former example was constructed by Bárány, Kuperberg, and Zamfirescu in [3].

Let us list other related results.

- ◊ In [4], Liberman gives a bound on the total curvature of short geodesic on the surface of convex body  $K$  in terms of the ratio diameter and inradius of  $K$ . In the proof he use an analog of Lemma 3.1 discussed below.
- ◊ In [5], Usov gives the optimal bound for the total curvature of geodesics on the graphs of  $\ell$ -Lipschitz convex function. Namely, he proves that if  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  is  $\ell$ -Lipschitz and convex then any geodesic in its graph

$$\Gamma_f = \{ (x, y, z) \in \mathbb{R}^3 \mid z = f(x, y) \}$$

has total curvature at most  $2\cdot\ell$ . An amusing generalization of Usov's result is given by Berg in [6].

- ◊ In [7], Pogorelov conjectured that the spherical image of geodesic on convex surface has to be (locally) rectifiable. It is easy to see that the length of spherical image of geodesic can not be smaller than its total curvature, so this conjecture (if it would be true) would be stronger than Liberman's theorem. Counterexamples to the different forms of this conjecture were found by Zalgaller in [8], Milka in [9] and Usov in [10]; these results were partly rediscovered later by Pach in [2].

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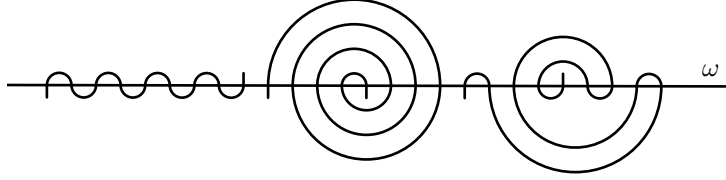
- ◊ In [3], Bárány, Kuperberg and Zamfirescu have constructed a corkscrew minimizing geodesic on a closed convex surface; that is a minimizing geodesic which twists around a given line arbitrary many times. They also rediscovered the results of Liberman and Usov mentioned above.

**Idea of the proof.** First we show that it is sufficient to estimate total curvature for the minimizing geodesics with almost constant velocity vector, say  $\dot{\gamma}(t) \approx \mathbf{i}$ .

To estimate total curvature in this case it is sufficient to estimate the integral

$$\int \langle \ddot{\gamma}(t), \mathbf{j} \rangle \cdot dt$$

for a vector  $\mathbf{j} \perp \mathbf{i}$ . To understand the idea of this estimate, imagine that the surface is lightened in the direction of  $\mathbf{j}$ , so it is divided into the dark and bright sides by a curve  $\omega$ . On the diagram you see different combinatorics in which  $\gamma$  meets  $\omega$ .



In the first case the total curvature is estimated by integral of Gauss curvature of the regions squeezed between  $\gamma$  and  $\omega$ . The latter follows from the Tongue Lemma 4.2, which is the hart of our proof.

The second case might look impossible, but the corkscrew geodesic constructed in [3] can meets  $\omega$  in this order. Here we calculate the total curvature for each twists and show that the obtained sequence of numbers grows geometrically from some place in the middle to the ends. At the ends the integral of the full twists can not be bigger than  $2 \cdot \pi$  — this is enough to estimate the total curvature of the whole geodesic.

The last case is a mixture of first two and it is done by mixing both techniques.

## 2 Preliminaries

**Total curvature.** Recall that the *total curvature* of a curve  $\gamma: [0, \ell] \rightarrow \mathbb{R}^3$  (briefly  $\text{TotCurv } \gamma$ ) is defined as supremum of sum of exterior angles for the polygonal lines inscribed in  $\gamma$ .

Note that for a polygonal line  $\sigma$ , its total curvature coincides with the sum of its exterior angles.

If  $\gamma$  is a smooth curve with unit-speed parametrization, then

$$\text{TotCurv } \gamma = \int_0^\ell \kappa(t) \cdot dt,$$

where  $\kappa(t) = |\ddot{\gamma}(t)|$  is the curvature of  $\gamma$  at  $t$ .

**2.1. Proposition.** Assume  $\gamma_n: \mathbb{I} \rightarrow \mathbb{R}^3$  is a sequence of curves converging pointwise to a curve  $\gamma_\infty: \mathbb{I} \rightarrow \mathbb{R}^3$ . Then

$$\liminf_{n \rightarrow \infty} \text{TotCurv } \gamma_n \geq \text{TotCurv } \gamma_\infty.$$

*Proof.* Fix a polygonal line  $\sigma_\infty$  inscribed in  $\gamma_\infty$ . Let  $\gamma_\infty(t_0), \dots, \gamma_\infty(t_k)$  be its vertices as they appear on  $\gamma_\infty$ . Consider the polygonal line  $\sigma_n$  inscribed in  $\gamma_n$  with the vertices  $\gamma_n(t_0), \dots, \gamma_n(t_k)$ . Note that

$$\text{TotCurv } \sigma_n \rightarrow \text{TotCurv } \sigma_\infty \text{ as } n \rightarrow \infty.$$

By the definition of total curvature,

$$\text{TotCurv } \sigma_n \leq \text{TotCurv } \gamma_n.$$

The statement follows since the broken line  $\sigma_\infty$  can be chosen in such a way that  $\text{TotCurv } \sigma_\infty$  is arbitrary close to  $\text{TotCurv } \gamma_\infty$ .  $\square$

**Convergence of sets.** Given a closed set  $\Sigma \subset \mathbb{R}^3$ , denote by  $\text{dist}_\Sigma$  the distance function from  $\Sigma$ ; that is

$$\text{dist}_\Sigma x = \inf \{ |x - y| \mid y \in \Sigma \}.$$

We say that a sequence of closed sets  $\Sigma_n \subset \mathbb{R}^3$  *converges* to the closed set  $\Sigma_\infty \subset \mathbb{R}^3$  if for any  $x \in \mathbb{R}^3$ , we have  $\text{dist}_{\Sigma_n} x \rightarrow \text{dist}_{\Sigma_\infty} x$  as  $n \rightarrow \infty$ .

**Convex surfaces.** By *convex surface*  $\Sigma$  in the Euclidean 3-space  $\mathbb{R}^3$  we understand the boundary of a closed convex set  $K$  with nonempty interior. If  $K$  is compact we say that the  $\Sigma$  is *closed*.

Assume  $\Sigma$  is smooth. If at every point both of  $\Sigma$  the principle curvatures are positive, we say that  $\Sigma$  is *strongly convex*.

**2.2. Proposition.** Assume  $\Sigma_n$  be a sequence of convex surfaces which converge to a convex surface  $\Sigma_\infty$ . Then for any minimizing geodesic  $\gamma_\infty$  in  $\Sigma_\infty$  there is a sequence of minimizing geodesics  $\gamma_n$  in  $\Sigma_n$  which pointwise converge to  $\gamma_\infty$  as  $n \rightarrow \infty$ .

*Proof.* Assume  $\gamma_\infty: [0, \ell] \rightarrow \Sigma_\infty$  is parametrized by its arc-length.

Fix a sub-interval  $[a, b] \subset (0, \ell)$ . Set  $p_\infty = \gamma_\infty(a)$  and  $q_\infty = \gamma_\infty(b)$ . Let  $p_n, q_n \in \Sigma_n$  be a two sequences of points which converge to  $p_\infty$  and  $q_\infty$  correspondingly.

Denote by  $\gamma_n$  a minimizing geodesic from  $p_n$  to  $q_n$  in  $\Sigma_n$ . Note that  $\gamma_n$  converges to  $\gamma_\infty|_{[a, b]}$  as  $n \rightarrow \infty$ .

Taking the sub-interval  $[a, b]$  closer and closer to  $[0, \ell]$  and applying diagonal procedure, we get the result.  $\square$

### 3 Liberman's lemma

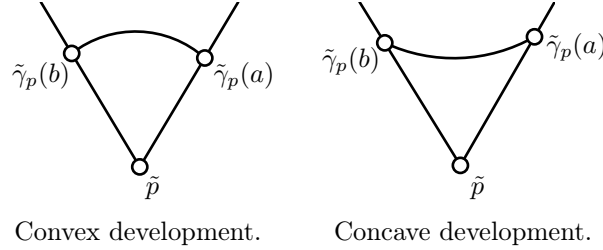
In this section we give a slight generalization of the construction given by Liberman in [4]; see also [11] and [12].

**Development.** Let  $\gamma: [0, \ell] \rightarrow \mathbb{R}^3$  be a curve parametrized by arc-length and a point  $p$  does not lie on  $\gamma$ .

Assume  $\tilde{\gamma}_p: [0, \ell] \rightarrow \mathbb{R}^2$  is a plane curve parametrized by arc-length and  $\tilde{p}$  is a point in the plane such that

$$|\tilde{p} - \tilde{\gamma}(t)| = |p - \gamma(t)|$$

for any  $t \in [0, \ell]$ . Moreover, the direction from  $\tilde{p}$  to  $\tilde{\gamma}(t)$  changes monotonically (clockwise or counterclockwise). Then  $\tilde{\gamma}_p$  is called *development* of  $\gamma$  with respect to  $p$ .



We say that the development  $\tilde{\gamma}_p$  is *convex* (*concave*) in the interval  $[a, b]$  if the arc  $\tilde{\gamma}_p|_{[a, b]}$  cuts from the solid angle  $\angle \tilde{p} \tilde{\gamma}_p^{(a)} \tilde{\gamma}_p^{(b)}$  convex bounded (correspondingly unbounded) domain.

We say that  $\tilde{\gamma}_p$  is *locally convex* (*concave*) in the interval  $[a, b]$  if any point  $x \in [a, b]$  admits a closed neighborhood  $[a', b']$  in  $[a, b]$  such that  $\tilde{\gamma}_p$  is convex (correspondingly concave) in the interval  $[a', b']$ .

If we pass to the limit of this construction as  $p$  moves to infinity along a half-line in the direction of a unit vector  $-\mathbf{u}$  then the limit curve is called development of  $\gamma$  in the direction  $\mathbf{u}$  and denoted as  $\tilde{\gamma}_{\mathbf{u}}$ .

We can define the development  $\tilde{\gamma}_{\mathbf{u}}$  directly: (1) the development  $\tilde{\gamma}_{\mathbf{u}}: [0, \ell] \rightarrow \mathbb{R}^2$  is parametrized by arc-length, (2) for a fixed unit vector  $\tilde{\mathbf{u}} \in \mathbb{R}^2$ , we have

$$\langle \tilde{\mathbf{u}}, \tilde{\gamma}_{\mathbf{u}}(t) \rangle = \langle \mathbf{u}, \gamma(t) \rangle$$

for any  $t \in [0, \ell]$  and (3) the projection of  $\tilde{\gamma}_{\mathbf{u}}(t)$  to the line normal to  $\tilde{\mathbf{u}}$  is monotonic in  $t$ .

We can assume that  $\tilde{\mathbf{u}}$  is the vertical vector in the coordinate plane. In this case we say that  $\tilde{\gamma}_{\mathbf{u}}$  is concave (convex) in the interval  $[a, b]$  if the lune bounded by arc  $\tilde{\gamma}_{\mathbf{u}}|_{[a, b]}$  and the segment  $[\tilde{\gamma}_{\mathbf{u}}(a)\tilde{\gamma}_{\mathbf{u}}(b)]$  is convex and lies above (correspondingly below) the line segment  $[\tilde{\gamma}_{\mathbf{u}}(a)\tilde{\gamma}_{\mathbf{u}}(b)]$ .

**Dark and bright sides.** Let  $\Sigma \subset \mathbb{R}^3$  be a convex surface,  $p \in \Sigma$  and  $z \neq p$ .

We say that  $p$  lies on the *dark* (*bright*) side of  $\Sigma$  with from  $z$  if non of the points  $p + t \cdot (p - z)$  lie inside of  $\Sigma$  for  $t > 0$  (correspondingly for  $t < 0$ ). The intersection of dark and bright side is called horizon of  $z$ ; it is denoted by  $\omega_z$ .

Note that if  $z$  lies inside  $\Sigma$  then all the points on  $\Sigma$  lie on the dark side from  $z$  and its horizon  $\omega_z$  is empty.

If  $\Sigma$  is smooth we can define the outer normal vector  $\nu_p$  to  $\Sigma$  at  $p$ . In this case  $p$  lies on dark (bright) side of  $\Sigma$  from  $z$  if and only if  $\langle p - z, \nu_p \rangle \geq 0$  (correspondingly  $\langle p - z, \nu_p \rangle \leq 0$ ). If in addition  $\Sigma$  is closed and strongly convex then the horizon is empty for  $z$  inside  $\Sigma$  and it is formed by a closed smooth curve for  $z$  outside  $\Sigma$ .

We could also define bright/dark sides and horizon in the limit case, as  $p$  escapes to infinity along a half-line in direction  $-\mathbf{u}$ .

The latter can be defined directly. We say that a point  $p \in \Sigma$  lies on dark (bright) side for the unit vector  $\mathbf{u}$  if non of the points  $p + \mathbf{u} \cdot t$  lie inside of  $\Sigma$  for  $t > 0$ , (correspondingly  $t < 0$ ). As before the intersection of bright and dark side is called horizon of  $\mathbf{u}$  and it is denoted as  $\omega_{\mathbf{u}}$ .

In the smooth case the latter means that  $\langle \nu_p, \mathbf{u} \rangle \geq 0$  (correspondingly  $\langle \nu_p, \mathbf{u} \rangle \leq 0$ ). If in addition  $\Sigma$  is closed strongly convex then  $\omega_{\mathbf{u}}$  is a closed smooth curve.

**3.1. Liberman's Lemma.** *Assume  $\gamma$  be a geodesic on the convex surface  $\Sigma \subset \mathbb{R}^3$ . Then for any point  $z \notin \Sigma$  the development  $\tilde{\gamma}_z$  is locally convex (locally concave) if  $\gamma$  lies on dark (correspondingly bright) side of  $\Sigma$  from  $z$ .*

*Similarly for any unit vector  $\mathbf{u}$ , the development  $\tilde{\gamma}_{\mathbf{u}}$  is locally convex (locally concave) if  $\gamma$  lies on dark (correspondingly bright) side of  $\Sigma$  for  $\mathbf{u}$ .*

Note that for any space curve  $\gamma$  and any unit vector  $\mathbf{u}$  we have

$$\text{TotCurv } \tilde{\gamma}_{\mathbf{u}} \leq \text{TotCurv } \gamma.$$

On the other hand total curvature of few developments gives an estimate for the total curvature of the original curve. For example, if  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  is the orthonormal basis then

$$\text{TotCurv } \gamma \leq \text{TotCurv } \tilde{\gamma}_{\mathbf{i}} + \text{TotCurv } \tilde{\gamma}_{\mathbf{j}} + \text{TotCurv } \tilde{\gamma}_{\mathbf{k}}.$$

If  $\gamma$  lies completely on the dark (or bright) side for direction  $\mathbf{u}$  then by Liberman's lemma we get

$$\text{TotCurv } \tilde{\gamma}_{\mathbf{u}} \leq \pi.$$

It follows that if  $\gamma$  cross the horizons  $\omega_{\mathbf{i}}, \omega_{\mathbf{j}}$  and  $\omega_{\mathbf{k}}$  at most  $N$  times then

$$\begin{aligned} \text{TotCurv } \gamma &\leq \text{TotCurv } \tilde{\gamma}_{\mathbf{i}} + \text{TotCurv } \tilde{\gamma}_{\mathbf{j}} + \text{TotCurv } \tilde{\gamma}_{\mathbf{k}} \leq \\ &\leq (N+1) \cdot \pi. \end{aligned}$$

Therefore, to violate Main Theorem  $\gamma$  has to cross the horizons  $\omega_{\mathbf{i}}, \omega_{\mathbf{j}}$  and  $\omega_{\mathbf{k}}$  huge number of times.

## 4 Curvature of development

Let  $\Sigma \subset \mathbb{R}^3$  be a closed smooth strongly convex surface and  $\gamma: [0, \ell] \rightarrow \Sigma$  be a unit-speed geodesic. Assume that for some unit vector  $\mathbf{u}$ , the geodesic  $\gamma$  cross the horizon  $\omega_{\mathbf{u}}$  transversely at  $t_0 < \dots < t_k$ . Set  $\alpha_i = \angle(\dot{\gamma}(t_i), \mathbf{u}) - \frac{\pi}{2}$  for each  $i$ . Note that  $|\alpha_i| \leq \frac{\pi}{2}$ .

The values  $t_i$  and  $\alpha_i$  will be called correspondingly *meeting moments* and *meeting angles* of the geodesic  $\gamma$  with the horizon  $\omega_{\mathbf{u}}$ .

Let us introduce new notation

$$\text{TotCurv}_{\mathbf{u}} \gamma \stackrel{\text{def}}{=} \text{TotCurv } \tilde{\gamma}_{\mathbf{u}}.$$

From Liberman's lemma 3.1, we get the following.

**4.1. Corollary.** *Let  $\Sigma \subset \mathbb{R}^3$  be a strongly convex smooth surface,  $\gamma: [0, \ell] \rightarrow \Sigma$  be a unit-speed geodesic and  $\mathbf{u}$  is a unit vector. Assume that  $\gamma$  cross the horizon  $\omega_{\mathbf{u}}$  transversely and  $t_0 < \dots < t_k$  be its meeting moments and  $\alpha_0, \dots, \alpha_k$  be its meeting angles with the horizon  $\omega_{\mathbf{u}}$ . Then*

$$\text{TotCurv}_{\mathbf{u}} \gamma \leq 3 \cdot \pi + 2 \cdot |\alpha_0 - \alpha_1 + \dots + (-1)^k \cdot \alpha_k|.$$

As you will see further, in order to find the needed estimate the total curvature of geodesic we will get an upper bound for the sum

$$|\alpha_0 - \alpha_1 + \dots + (-1)^k \cdot \alpha_k|.$$

Finding such an upper bound is the most important ingredient in the proof of the Main Theorem.

*Proof.* By Liberman's lemma,

$$\text{TotCurv}_{\mathbf{u}}(\gamma|_{[t_{i-1}, t_i]}) = \pm(\alpha_{i-1} - \alpha_i)$$

where the sign is “+” if  $\gamma|_{[t_i, t_{i+1}]}$  lies on the dark side and “−” if it lies on the bright side from  $\mathbf{u}$ . Summing all this up we get

$$\text{TotCurv}_{\mathbf{u}}(\gamma|_{[t_0, t_k]}) = |\alpha_0 - 2 \cdot \alpha_1 + \dots + (-1)^{k-1} \cdot 2 \cdot \alpha_{k-1} + (-1)^k \cdot \alpha_k|.$$

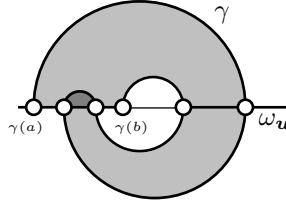
By Liberman's lemma we also have

$$\text{TotCurv}_{\mathbf{u}}(\gamma|_{[0, t_0]}), \text{TotCurv}_{\mathbf{u}}(\gamma|_{[t_k, \ell]}) \leq \pi$$

Since  $\alpha_0, \alpha_k \leq \frac{\pi}{2}$ , the statement follows.  $\square$

If  $\Sigma$  is a surface in  $\mathbb{R}^3$  and  $p \in \Sigma$  we denote by  $K_p$  the Gauss curvature of  $\Sigma$  at  $p$ .

Assume  $a, b$  be the meeting moments of minimizing geodesic  $\gamma$  with  $\omega_{\mathbf{u}}$ . The arc  $\gamma|_{[a, b]}$  will be called  $\omega_{\mathbf{u}}$ -tongue if there is an immersed disc  $\iota: \mathbb{D} \rightarrow \Sigma$  such that the closed curve  $\iota|_{\partial \mathbb{D}}$  is formed by joint of the arc  $\gamma|_{[a, b]}$  and an arc of  $\omega_{\mathbf{u}}$ . In this case the immersion  $\iota$  is called the *disc of the tongue*.



**4.2. Tongue Lemma.** *Let  $\mathbf{u}$  be a unit vector,  $\gamma: [a, b] \rightarrow \Sigma$  be a minimizing geodesic and  $\omega_{\mathbf{u}}$ -tongue on the strongly convex surface  $\Sigma \subset \mathbb{R}^3$  and  $\iota: \mathbb{D} \rightarrow \Sigma$  be its disc.*

*Then*

$$\int_{\mathbb{D}} K_{\iota(x)} \cdot d_{\iota(x)} \text{ area}_{\Sigma}$$

*takes one of the value*

$$\alpha - \beta, -\alpha + \beta, \pi - \alpha - \beta, \pi + \alpha + \beta \pmod{2 \cdot \pi}.$$

*In particular*

$$|\alpha - \beta| \leq \int_{\mathbb{D}} K_{\iota(x)} \cdot d_{\iota(x)} \text{ area}_{\Sigma}.$$

**1**

If in addition the image  $\iota(\mathbb{D})$  lies completely in the dark or bright side for  $\mathbf{u}$  then

$$\text{TotCurv } \gamma \leq \int_{\iota(\mathbb{D})} K_p \cdot d_p \text{ area}_\Sigma. \quad \textcircled{2}$$

*Proof.* Since  $\gamma$  is a geodesic, the parallel translation along  $\gamma$  sends  $\dot{\gamma}(a)$  to  $\dot{\gamma}(b)$ .

Note also that  $\mathbf{u}$  belongs to the tangent plane to  $\Sigma$  at any point on the horizon  $\omega_{\mathbf{u}}$ ; in particular the  $\mathbf{u}$  extends to a parallel tangent vector field on  $\omega_{\mathbf{u}}$ .

It follows that parallel translation along  $\iota|_{\partial\mathbb{D}}$  rotates the tangent plane by angle

$$\pm(\frac{\pi}{2} + \alpha) \pm (\frac{\pi}{2} + \beta)$$

To prove the main statement of the lemma, it remains to apply Gauss–Bonnet formula.

Denote by  $R$  the right hand side in  $\textcircled{1}$ . Note that  $R \geq 0$  and  $|\alpha|, |\beta| \leq \frac{\pi}{2}$ . From the main statement of the lemma it follows then that the minimal possible value for  $R$  is  $|\alpha - \beta|$ .

To prove  $\textcircled{2}$ , note that in this case  $\iota$  is an embedding. Further note that the spherical image of the dark side of  $\Sigma$  is hemisphere. Therefore  $2 \cdot \pi$  is the integral of Gauss curvature along the dark side. It follows that

$$\int_{\mathbb{D}} K_{\iota(x)} \cdot d_{\iota(x)} \text{ area}_\Sigma = \int_{\iota(\mathbb{D})} K_p \cdot d_p \text{ area}_\Sigma < 2 \cdot \pi.$$

By Liberman’s lemma the statement follows.  $\square$

## 5 Almost straight arcs

Let  $\varepsilon > 0$ . A curve  $\gamma: [a, b] \rightarrow \mathbb{R}^3$  will be called  $\varepsilon$ -straight if

$$(1 - \varepsilon) \cdot \text{length } \gamma \leq |\gamma(b) - \gamma(a)|.$$

**5.1. Lemma.** *Assume  $\varepsilon > 0$  and  $n$  is a positive integer such that  $n \cdot \varepsilon > 2$ . Then any minimizing geodesic on a convex surface  $\Sigma$  in  $\mathbb{R}^3$  can be subdivided into  $\varepsilon$ -straight arcs  $\gamma_1, \dots, \gamma_n$ .*

*Proof.* Let  $\vartheta \in (0, \pi)$  be such that

$$1 - \cos \vartheta = \varepsilon.$$

Assume two points  $p$  and  $q$  lie on the convex surface  $\Sigma$ . Denote by  $\nu_p$  and  $\nu_q$  the outer normal vectors at  $p$  and  $q$  correspondingly. Note that if

$$\angle(\nu_p, \nu_q) \leq 2 \cdot \vartheta,$$

then any minimizing geodesic from  $p$  to  $q$  on  $\Sigma$  is  $\varepsilon$ -straight.

Let  $\gamma: [0, \ell] \rightarrow \Sigma$  be a minimizing geodesic parametrized by its arc-length.

Assume  $\gamma_{[t, \ell]}$  is not  $\varepsilon$ -straight. Set  $t'$  to be the maximal value in  $[t, \ell]$  such that the interval  $[t, t']$  is  $\varepsilon$ -straight. Consider a sequence  $0 = t_0 < t_1 < \dots <$

$t_n < \ell$  such that  $t_{i+1} = t'_i$  for each  $i$ . Denote by  $\nu_i$  the outer unit normal vector to  $\Sigma$  at  $\gamma(t_i)$ . From above we get

$$\angle(\nu_i, \nu_j) \geq 2 \cdot \vartheta$$

for all  $i \neq j$ . In other words, the open balls  $B_\vartheta(\nu_i)$  do not overlap in  $\mathbb{S}^2$ .

It remains to note that

$$\text{area}[B_\vartheta(\nu_i)] = 2 \cdot \pi \cdot \varepsilon \quad \text{and} \quad \text{area } \mathbb{S}^2 = 4 \cdot \pi.$$

Hence the result follows.  $\square$

**5.2. Corollary.** *Assume  $\gamma: [0, \ell] \rightarrow \Sigma$  is a unit-speed minimizing geodesic on the convex surface  $\Sigma$  in  $\mathbb{R}^3$ . Then  $\text{diam } \gamma \geq \frac{\ell}{10}$ .*

*Proof.* Apply Lemma 5.1 for  $\varepsilon = \frac{1}{2}$ .  $\square$

## 6 An arc in almost constant direction

**6.1. Proposition.** *For any  $\varepsilon > 0$  there is  $\delta > 0$  such that the following holds.*

*If  $\gamma: [a, b] \rightarrow \Sigma$  is a minimizing geodesic on a smooth strongly convex surface  $\Sigma$  in  $\mathbb{R}^3$  then there is an interval  $[a', b'] \subset [a, b]$  such that*

$$\text{TotCurv}(\gamma|_{[a', b']}) > \delta \cdot \text{TotCurv } \gamma.$$

and

$$\angle(\dot{\gamma}(t), \mathbf{u}) < \varepsilon$$

for a fixed unit vector  $\mathbf{u}$  and any  $t \in [a', b']$ .

Moreover, if  $\varepsilon = \frac{1}{10}$  then one can assume  $\delta = \frac{1}{100^{100}}$ .

In the proof we will need the following two lemmas.

**6.2. Lemma.** *For any  $\varepsilon$  there is  $\delta > 0$  such that the following holds.*

*Assume  $\gamma$  is a curve,  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be two vectors in  $\mathbb{R}^3$  and  $0 \leq \alpha_1, \alpha_2 \leq \pi$  be such that*

$$\begin{aligned} \varepsilon &< \angle(\mathbf{v}_1, \mathbf{v}_2) < \pi - \varepsilon \\ \alpha_i - \delta &< \angle(\mathbf{v}_i, \dot{\gamma}(t)) < \alpha_i + \delta \end{aligned}$$

then there is a vector  $\mathbf{u}$  such that  $\angle(\mathbf{u}, \dot{\gamma}(t)) < \varepsilon$ .

Moreover if  $\varepsilon < \frac{1}{10}$  then one can take  $\delta = \varepsilon^{10}$ .

The proof of the lemma above is straightforward computation; we omit it.

**6.3. Lemma.** *For any  $\varepsilon > 0$  there is  $\delta > 0$  such that the following holds.*

*Let  $\gamma: [a, b] \rightarrow \Sigma$  be an  $\delta$ -straight minimizing geodesic on a smooth strongly convex surface  $\Sigma$  in  $\mathbb{R}^3$ . Set  $\mathbf{v}_\gamma = \gamma(b) - \gamma(a)$ . Then there is a sub-interval  $[a', b']$  in  $[a, b]$  such that*

$$\text{TotCurv}(\gamma|_{[a', b']}) \geq \delta \cdot \text{TotCurv } \gamma.$$



and

$$\alpha - \varepsilon \leq \angle(\dot{\gamma}(t), \mathbf{v}_\gamma) \leq \alpha + \varepsilon$$

for some fixed  $\alpha$  and any  $t \in [a', b']$ .

Moreover if  $\varepsilon < \frac{1}{10}$  one can take  $\delta = \varepsilon^{10}$ .

*Proof.* Without loss of generality we can assume that  $a = 0$ ,  $b = 2$  and

$$\text{TotCurv}(\gamma|_{[1,2]}) \geq \frac{1}{2} \cdot \text{TotCurv } \gamma.$$

Set  $p = \gamma(0)$ . Let  $\vartheta \in (0, \pi)$  be such that  $1 - \cos \vartheta = \delta$ . Note that

$$\angle(\mathbf{v}_\gamma, \gamma(t) - p) \leq \angle(\tilde{\gamma}_p(1) - \tilde{p}, \tilde{\gamma}_p(2) - \tilde{p}) < \vartheta \quad \textbf{①}$$

for any  $t \geq 1$ .

By Liberman's lemma

$$\text{TotCurv}_p(\gamma|_{[1,2]}) < \pi + \vartheta.$$

Assume  $N = \lceil \frac{\pi}{\vartheta} + 1 \rceil$ . Then we can subdivide  $\gamma|_{[1,2]}$  into  $N$  arcs  $\gamma_1, \gamma_2, \dots, \gamma_N$  such that

$$\text{TotCurv}_p(\gamma_n) \leq \vartheta \quad \textbf{②}$$

for each  $n$ .

From ① and ②, it follows that for each  $n$ , there is  $\alpha_n$  such that

$$\alpha_n - \vartheta \leq \angle(\dot{\gamma}_n(t), \mathbf{v}_\gamma) \leq \alpha_n + \vartheta.$$

The arc  $\gamma_n$  with the maximal total curvature will solve the proposition.

It remains to choose  $\delta$  so that  $\vartheta(\delta) < \frac{\varepsilon}{100}$ . □

*Proof of Proposition 6.1.* Set  $\gamma_0 = \gamma$ .

Fix  $\delta > 0$ , set  $n = \lceil \frac{2}{\delta} \rceil$ . By Lemma 5.1, the geodesic  $\gamma_0$  can be subdivided into  $n$  arcs which are  $\delta$ -straight. Let us choose the arc  $\gamma'_0$  with the maximal total curvature. Assuming  $\delta < \frac{1}{10}$  we get

$$\text{TotCurv } \gamma'_0 \geq \frac{\delta}{10} \cdot \text{TotCurv } \gamma_0.$$

Let  $\alpha_1$  be the angle and  $\gamma_1$  be the arc in  $\gamma'_0$  provided by Lemma 6.3. In particular

$$\begin{aligned} \text{TotCurv } \gamma_1 &\geq \delta \cdot \text{TotCurv } \gamma'_0 \geq \\ &\geq \frac{\delta^2}{10} \cdot \text{TotCurv } \gamma_0. \end{aligned}$$

If  $\alpha_1 \leq \frac{\varepsilon}{2}$  or  $\alpha_1 \geq \pi - \frac{\varepsilon}{2}$  and  $\delta$  is small enough then statement holds for the arc  $\gamma_1$  and the vector  $\mathbf{u} = \pm \mathbf{v}_{\gamma'_0}$ .

Otherwise let us repeat the above construction for  $\gamma_1$ . Namely, apply Lemma 5.1 to the geodesic  $\gamma_1$  and denote by  $\gamma'_1$  the  $\delta$ -straight arc with maximal total curvature. If  $\delta$  is small, we get

$$\frac{\varepsilon}{3} < \angle(\mathbf{v}_{\gamma'_1}, \mathbf{v}_{\gamma'_0}) < \pi - \frac{\varepsilon}{3} \quad \textbf{③}$$

Again, we get

$$\text{TotCurv } \gamma'_1 \geq \frac{\delta}{10} \cdot \text{TotCurv } \gamma_1 \geq \frac{\delta^3}{100} \cdot \text{TotCurv } \gamma_0.$$

Further apply Lemma 6.3 to  $\gamma'_1$ . Denote by  $\gamma_2$  and  $\alpha_2$  the angle and the sub-arc of  $\gamma'_1$ . Again

$$\text{TotCurv } \gamma_2 \geq \frac{\delta^4}{100} \cdot \text{TotCurv } \gamma_0.$$

The curve  $\gamma_2$  runs under nearly constant angle to  $\mathbf{v}_{\gamma'_0}$  and  $\mathbf{v}_{\gamma'_1}$ . The inequality ③ makes possible to apply Lemma 6.2. Hence the main statement in the proposition follows.

Straightforward computations prove the last statement.  $\square$

## 7 Drifting geodesics

In this section we fix notations which will be used further without additional explanation.

Fix a  $(x, y, z)$ -coordinates on the Euclidean space; denote by  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$  the standard basis.

A plane parallel to say  $(y, z)$ -coordinate plane will be called  $(y, z)$ -plane.

**7.1. Definition.** A smooth curve  $\gamma: [0, \ell] \rightarrow \mathbb{R}^3$  is called  $\mathbf{i}$ -drifting if both ends  $\gamma(0)$  and  $\gamma(\ell)$  lie on the  $x$ -axis and  $\angle(\dot{\gamma}(t), \mathbf{i}) < \frac{1}{10}$  for all  $t$ .

**$(\lambda, \mu, \nu)$ -frame.** Let  $\Sigma$  be a convex surface and  $\gamma: [0, \ell] \rightarrow \Sigma$  be an  $\mathbf{i}$ -drifting minimizing geodesic with unit-speed parametrization.

Given  $t \in [0, \ell]$ , consider the oriented orthonormal frame  $\lambda(t), \mu(t), \nu(t)$  such that  $\nu(t)$  is the outer normal to  $\Sigma$  at  $\gamma(t)$ , the vector  $\mu(t)$  lies in  $(y, z)$ -plane and therefore the vector  $\lambda(t)$  lies in the plane spanned by  $\nu(t)$  and the  $x$ -axis. We assume in addition that  $\langle \lambda, \mathbf{i} \rangle \geq 0$ .

Since  $\langle \dot{\gamma}(t), \mathbf{i} \rangle > 0$ , we have  $\nu(t) \neq \mathbf{i}$ . Therefore the frame  $(\lambda, \mu, \nu)$  is uniquely defined for any  $t \in [0, \ell]$ .

**Angle functions.** Set

$$\varphi(t) = \angle(\mathbf{i}, \dot{\gamma}(t)), \quad \psi(t) = \frac{\pi}{2} - \angle(\mathbf{i}, \nu(t)), \quad \vartheta(t) = \frac{\pi}{2} - \angle(\mu(t), \dot{\gamma}(t)),$$

From the above definitions it follows that  $|\vartheta(t)|, |\psi(t)| \leq \frac{\pi}{2}$  and for each  $t$  there is a right spherical triangle with legs  $|\vartheta(t)|, |\psi(t)|$  and hypotenuse  $\varphi(t)$ . In particular  $\cos \vartheta \cdot \cos \psi = \cos \varphi$ . Whence we get the following.

**7.2. Claim.** For any  $t$  we have

$$\varphi(t) \geq |\psi(t)| \quad \text{and} \quad \varphi(t) \geq |\vartheta(t)|$$

Applying Liberman's Lemma in the direction  $\mathbf{i}$  we also get the following.

**7.3. Claim.** If an arc  $\gamma|_{[a, b]}$  lies in the dark (bright) side for  $\mathbf{i}$  then the angle function  $\varphi$  is non-decreasing (correspondingly non-increasing) in  $[a, b]$ .

## 8 Plane sections

Assume  $\gamma$  is a curve on a smooth strongly convex surface  $\Sigma$  in  $\mathbb{R}^3$ . Consider a plane  $L$  passing through two points of  $\gamma$ , say  $p = \gamma(a)$  and  $q = \gamma(b)$  with  $a < b$ . Let  $L_{\pm}$  be a half-planes in  $L$  bounded by the line through  $p$  and  $q$ . Set  $\sigma_{\pm} = \Sigma \cap L_{\pm}$ .

**8.1. Observation.** *If  $\gamma$  is a minimizing geodesic in the smooth strongly convex surface  $\Sigma \subset \mathbb{R}^3$  and  $a, b$  and  $\sigma_{\pm}$  as above then*

$$\text{length } \sigma_{\pm} \geq \text{length}(\gamma|_{[a,b]}).$$

To prove the observation, it is sufficient to note that  $\sigma_{\pm}$  are smooth convex plane curve connecting  $p$  to  $q$  in  $\Sigma$ .

Based on this observation we give couple of estimates on drifting minimizing geodesics.

Let  $\gamma: [a, b] \rightarrow \mathbb{R}^3$  be a curve and  $\ell$  be a line which does not pass through points of  $\gamma$ . Assume  $\varphi: [a, b] \rightarrow \mathbb{R}$  be a continuous azimuth angle of  $\gamma$  in the cylindrical coordinates with the axis  $\ell$ . If

$$|\varphi(b) - \varphi(a)| \geq 2 \cdot n \cdot \pi,$$

we will say that  $\gamma$  goes around the line  $\ell$  at least  $n$  times.

**8.2. Proposition.** *Assume  $\gamma: [0, \ell] \rightarrow \Sigma$  is an  $i$ -drifting minimizing geodesic in the convex surface  $\Sigma \subset \mathbb{R}^3$ , a sub-segment  $[a, b] \subset [0, \ell]$  and the following conditions hold*

- (i) *The points  $\gamma(a)$  and  $\gamma(b)$  lie in a half-plane with boundary line formed by the  $x$ -axis and the arc  $\gamma|_{[a,b]}$  goes around the  $x$ -axis at least once.*
- (ii) *The  $x$ -coordinate of  $\gamma(a)$  is larger than the  $x$ -coordinate of  $\frac{1}{2} \cdot (\gamma(0) + \gamma(\ell))$ . Then  $\gamma(b)$  lies on the dark side for  $i$ .*

*Proof.* Let us apply Observation 8.1 to the plane containing  $x$ -axis,  $\gamma(a)$  and  $\gamma(b)$ .

We can assume that  $\gamma(0)$  is the origin of the  $(x, y, z)$ -coordinate system and both points  $p = \gamma(a)$  and  $q = \gamma(b)$  lie in the  $(x, z)$ -coordinate half-plane with  $x \geq 0$ , denoted by  $\Pi$ . We can assume that  $\sigma_+ \subset \Pi$ . Let  $(x_p, 0, z_p)$  and  $(x_q, 0, z_q)$  be the coordinates of  $p$  and  $q$ .

From the assumptions we get  $x_p < x_q < 2 \cdot x_p$ .

Assume contrary, then  $\gamma(b)$  lies on the bright side for  $i$ . Then from convexity of the curve  $\Pi \cap \Sigma$  we get

$$\text{length } \sigma_+ \leq \sqrt{(x_q - x_p)^2 + z_p^2}.$$

On the other hand, since  $\gamma|_{[a,b]}$  goes around  $x$ -axis at least once, we get

$$\text{length } \gamma|_{[a,b]} \geq \sqrt{(x_q - x_p)^2 + (z_p + z_q)^2}.$$

These two estimates contradict Observation 8.1. □

**8.3. Corollary.** *If  $\Sigma, \gamma, \ell, a$  and  $b$  as in the Proposition 8.2 and the arc  $\gamma|_{[a,b]}$  goes around the  $x$ -axis at least twice then the arc  $\gamma|_{[b,\ell]}$  lies on the dark side with respect to  $i$ .*

*Proof.* Fix  $b' \in [b, \ell]$ . Note that one can find  $a' \in [a, b]$  such that the assumptions of Proposition 8.2 hold for the interval  $[a', b']$ . Applying the proposition we get the result.  $\square$

**8.4. Proposition.** *Assume  $\gamma: [0, \ell] \rightarrow \Sigma$  is an  $\mathbf{i}$ -drifting minimizing geodesic in the convex surface  $\Sigma \subset \mathbb{R}^3$ . Assume that the arc  $\gamma|_{[b, \ell]}$  lies on the dark side of  $\Sigma$  with respect to  $\mathbf{i}$ . If  $b \leq s < t \leq \ell$  and the point  $\gamma(s)$  lies in the plane  $\Pi$  through  $\gamma(t)$  spanned by  $\nu(t)$  and  $\lambda(t)$  then*

$$\varphi(s) \leq \psi(t).$$

*Proof.* Let us apply Observation 8.1 to the plane  $\Pi$  and  $p = \gamma(s)$  and  $q = \gamma(t)$ .

Let  $x_p$  and  $x_q$  be the  $x$ -coordinates of  $p$  and  $q$ .

Since  $\gamma|_{[s, t]}$  lies in the dark side, its Liberman's development  $\tilde{\gamma}|_{[s, t]}$  with respect to  $\mathbf{i}$  is concave. In particular

$$\text{length}(\gamma|_{[s, t]}) = \text{length}(\tilde{\gamma}|_{[s, t]}) \geq \frac{x_q - x_p}{\cos \varphi(s)}.$$

On the other hand, convexity of  $\sigma_+$  imply that

$$\text{length } \sigma_+ \leq \frac{x_q - x_p}{\cos \psi(t)}.$$

It remains to apply Observation 8.1.  $\square$

## 9 $s$ -pairs

Let  $\Sigma \subset \mathbb{R}^3$  be a strongly convex surface and  $\gamma: [0, \ell] \rightarrow \Sigma$  be an  $\mathbf{i}$ -drifting minimizing geodesic.

Assume that  $\gamma$  intersects the horizon  $\omega_j$  transversely.

Let  $t_0 < t_1 < \dots < t_k$  be the meeting moments of  $\gamma$  with  $\omega_j$ . Set

$$\varphi_n = \varphi(t_n) \quad \psi_n = \psi(t_n) \quad \vartheta_n = \vartheta(t_n).$$

Note that  $\vartheta_n = \pm \alpha_n$  so we can say  $s_n \cdot \vartheta_n = (-1)^m \cdot \alpha_n$  for some sequence of signs  $s_i = \pm 1$ . In particular

$$\alpha_0 - \alpha_1 + \dots + (-1)^k \cdot \alpha_k = s_0 \cdot \vartheta_0 + s_1 \cdot \vartheta_1 + \dots + s_k \cdot \vartheta_k.$$

Note that for the right choice of orientation, if  $s_n = +1$  then  $\nu_{\gamma(t)}$  moves clockwise in  $\mathbb{S}^2$  at  $t_n$  and if  $s_n = -1$  then it moves counterclockwise.

We say that a pair of indexes  $i < j$  forms an  $s$ -pair if

$$\sum_{n=i}^j s_n = 0 \quad \text{and} \quad \sum_{n=i}^{j'} s_n > 0$$

if  $i < j' < j$ .

If you exchange “+1” and “-1” in  $s$  by “(” and “)” correspondingly then  $(i, j)$  is an  $s$ -pair if and only if the  $i$ -th bracket forms a pair with  $j$ -bracket.

Note that any index  $i$  appears in at most one  $s$ -pair and for any  $s$ -pair  $(i, j)$  we have

- ◇  $s_i = 1$ ; that is,  $i$ -th bracket has to be opening.
- ◇  $s_j = -1$ ; that is,  $j$ -th bracket has to be closing.

In particular,

$$\begin{aligned} s_i \cdot \vartheta_i + s_j \cdot \vartheta_j &= \vartheta_i - \vartheta_j = \\ &= (-1)^i \cdot \alpha_i + (-1)^j \cdot \alpha_j. \end{aligned}$$

**Tongue interpretation.** Assume  $(i, j)$  is an  $s$ -pair. Note that in this case there is an arc of  $\omega_j$  from  $\gamma(t_i)$  to  $\gamma(t_j)$  with monotonic  $x$ -coordinate. Moreover a disc of the tone has this arc in the boundary.

The proof can be guessed from the diagram. It shows a lift of  $\gamma$  in the universal cover of strip of  $\Sigma$  between  $(y, z)$ -planes containing  $\gamma|_{[t_i, t_j]}$ ; the solid horizontal lines correspond are lifts of  $\omega_j$ .

We say that  $q$  is the depth of an  $s$ -pair  $(i, j)$  (briefly  $q = \text{depth}_s(i, j)$ ) if  $q$  is the maximal number such that there is  $q$ -long nested sequence of  $s$ -pairs starting with  $(i, j)$ . For example the  $s$ -pair on the diagram has depth 5.

More precisely, the depth of  $(i, j)$  is the maximal number  $q$  for which there is a sequence of  $s$ -pairs  $(i, j) = (i_1, j_1), (i_2, j_2), \dots, (i_q, j_q)$  such that

$$i = i_1 < \dots < i_q < j_q < \dots < j_1 = j.$$

Note that the  $s$ -pairs of the same depth do not overlap; that is if  $\text{depth}(i, j) = \text{depth}(i', j')$  for two distinct  $s$ -pairs  $(i, j)$  and  $(i', j')$  then either  $i < j < i' < j'$  or  $i' < j' < i < j$ .

The following proposition follow directly from the discussion above.

**9.1. Proposition.** *Let  $(i, j)$  be an  $s$ -pair. Then the arcs  $\gamma|_{[t_i, t_j]}$  and an arc of  $\omega_j$  bound an immersed disc in  $\Sigma$  which lies between  $(y, z)$ -planes through  $\gamma(t_i)$  and  $\gamma(t_j)$ . Moreover the maximal multiplicity of the disc is at most  $\text{depth}_s(i, j)$ .*

**9.2. Corollary.** *Let us denote by  $S_q$  the subset of indexes  $\{1, \dots, k\}$  which are the parts of  $s$ -pairs with depth  $q$ . Then*

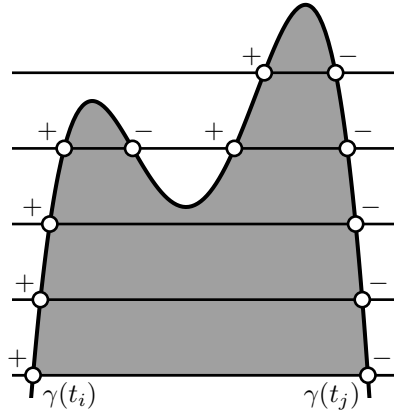
$$\left| \sum_{n \in S_q} (-1)^n \cdot \alpha_n \right| = \left| \sum_{n \in S_q} s_n \cdot \vartheta_n \right| \leq 4 \cdot \pi \cdot q.$$

*Proof.* For each  $n$  denote by  $K_n$  the integral of Gauss curvature of the part of surface  $\Sigma$  with the  $x$ -coordinate less then the  $x$ -coordinate of  $\gamma(t_n)$ . Note that

$$0 \leq K_1 \leq \dots \leq K_k \leq 4 \cdot \pi.$$

By Proposition 9.1 and the Tongue Lemma, we get

$$s_i \cdot \vartheta_i + s_j \cdot \vartheta_j = \vartheta_i - \vartheta_j \leq q \cdot (K_j - K_i)$$



The statement follows since the  $s$ -pairs with the same depth do not overlap.  $\square$

**9.3. Corollary.** *Assume*

$$q = \max_{1 \leq i < j \leq k} \left\{ \left| \sum_{n=i}^j s_n \right| \right\}$$

*Then*

$$\left| \sum_{n=1}^k s_n \cdot \vartheta_n \right| \leq 2 \cdot q \cdot \left(q + \frac{3}{2}\right) \cdot \pi.$$

*Proof.* Denote by  $S$  the set of all indexes which appear in some  $s$ -pair.

Note that depth of any  $s$ -pair is at most  $q$ . That is,

$$S = S_1 \cup \dots \cup S_q.$$

By Corollary 9.2,

$$\left| \sum_{n \in S} s_n \cdot \vartheta_n \right| \leq 2 \cdot q \cdot (q + 1) \cdot \pi. \quad \bullet$$

Set  $R = \{1, \dots, k\} \setminus S$ ; this is the set of indexes which do not appear in an  $s$ -pair.

Given  $r$ , set  $i \in Q_r$  if

$$\sum_{n=1}^i s_n = r.$$

Note that  $Q_r \neq \emptyset$  for at most  $q$  values of  $r$  and in each set  $Q_r$  there are at most 2 indexes which do not appear in an  $s$ -pair; that is  $Q_r \cap R$  has at most two indexes for each  $r$ .

Since  $|a_n| < \frac{\pi}{2}$ , we get

$$\left| \sum_{n \in R} s_n \cdot \vartheta_n \right| \leq q \cdot \pi.$$

The latter inequality together with  $\bullet$  implies the statement in the corollary.  $\square$

## 10 Geometric growth

**10.1. Claim.** *Assume  $\psi(t) > \varepsilon$  for  $t \in [t_i, t_{i+1}]$  and  $s_i = s_{i+1}$ . Then*

$$|\vartheta_{i+1} - \vartheta_i| > \pi \cdot \sin \varepsilon.$$

*Proof.* Note that the arc  $\gamma|_{[t_i, t_{i+1}]}$  is a tongue with embedded disc  $\iota: \mathbb{D}^2 \rightarrow \Sigma$ . Since  $\psi(t) > \varepsilon$ , the spherical image  $\nu \circ \iota(\mathbb{D}^2)$  of  $\iota(\mathbb{D}^2)$  lies in a half-disc of radius  $\frac{\pi}{2} - \varepsilon$  in  $\mathbb{S}^2$ . Note that

$$K(\iota(\mathbb{D}^2)) = \text{area}(\nu \circ \iota(\mathbb{D}^2)) < \pi \cdot (1 - \sin \varepsilon).$$

It remains to apply Tongue Lemma 4.2.  $\square$

**10.2. Claim.** *Assume  $\gamma$  lies on the dark side for  $i$ . Then for any pair of indexes  $j > i$ , such that*

$$\left| \sum_{n=i}^j s_n \right| > 5$$

*we have*

$$\varphi_j > \frac{3}{2} \cdot \varphi_i.$$

*Proof.* By Claim 7.3, we may assume that

$$\sum_{n=i}^j s_n = 6$$

Let  $j'$  be the least index such that

$$\left| \sum_{n=i}^{j'} s_n \right| = 5.$$

Note that for any  $b > t_j$  there is  $a \in [t_i, t_j]$  such that interval  $[a, b]$  satisfies the assumptions of Proposition 8.4. In particular  $\psi(b) > \varphi_i$  for any  $b > t_j$ . Applying Claim 10.1, we get that  $|\vartheta_j| > \frac{\pi}{2} \cdot \varphi_i$  or  $|\vartheta_{j'}| > \frac{\pi}{2} \cdot \varphi_i$ . By Claim 7.3,  $\varphi_n$  is non-decreasing, and  $\varphi_n \geq |\vartheta_n|$  for any  $n$ , in both cases we get

$$\varphi_j > \frac{\pi}{2} \cdot \varphi_i.$$

Hence the result follows.  $\square$

**10.3. Proposition.** *If  $\gamma$  is an  $i$ -drifting minimizing geodesic on the dark side for  $i$  then*

$$\text{TotCurv}_j \gamma \leq 100 \cdot \pi.$$

*Proof.* We can assume that  $\gamma$  cross the  $j$ -horizon  $\omega_j$  transversely. Let  $t_0 < \dots < t_k$  be the meeting moments of  $\gamma$  with  $\omega_j$  and  $s_0, \dots, s_k$  the signs.

Recall that  $S_q$  denotes the subset of indexes  $\{1, \dots, k\}$  which appear in  $s$ -pair with depth  $q$ . According to Corollary 9.2,

$$\left| \sum_{n \in S_q} s_n \cdot \vartheta_n \right| \leq 4 \cdot q \cdot \pi.$$

In particular,

$$\left| \sum_{n \in S_1 \cup \dots \cup S_5} s_n \cdot \vartheta_n \right| \leq 40 \cdot \pi.$$

Set  $R = \{1, \dots, k\} \setminus (S_1 \cup \dots \cup S_5)$ ; this is the set of indexes which appear in  $s$ -pairs with depth at least 6 as well as those which do not appear in any  $s$ -pair.

According to Claim 7.2,

$$\left| \sum_{n \in R} s_n \cdot \vartheta_n \right| \leq \sum_{n \in R} |\vartheta_n| \leq \sum_{n \in R} \varphi_n.$$

To estimate the last sum will use the results in Section 10. First let us subdivide  $R$  into 5 subsets  $R_1, \dots, R_5$ , by setting  $n \in R_m$  if  $m \equiv n \pmod{5}$ .

Given  $n \in R_m$ , denote by  $n'$  the least index in  $R_m$  which is larger  $n$ ;  $n'$  is defined for any  $n \in R_m$  except the largest one. According to Claim 10.2,  $\varphi_{n'} > \frac{3}{2} \cdot \varphi_n$ ; that is, the sequence  $(\varphi_n)_{n \in R_m}$  grows faster than the geometric progression with coefficient  $\frac{3}{2}$ . Since  $\varphi_n$  is non-decreasing in  $n$ , we get

$$\sum_{n \in R_m} \varphi_n < 3 \cdot \varphi_k.$$

It follows that

$$\sum_{n \in R} \varphi_n < 15 \cdot \varphi_k \leq \frac{15}{2} \cdot \pi.$$

By Corollary 4.1,

$$\begin{aligned} \text{TotCurv}_j \gamma &\leq 2 \cdot \pi + 2 \cdot [\alpha_0 - \alpha_1 + \dots + (-1)^k \cdot \alpha_k] < \\ &< 100 \cdot \pi. \end{aligned}$$

□

## 11 Assembling of the proof

Assume  $\gamma: [0, \ell] \rightarrow \Sigma$  is a minimizing geodesic in a convex surface  $\Sigma \subset \mathbb{R}^3$ .

According to propositions 2.1 and 2.2 we can assume that  $\Sigma$  is closed, strongly convex and smooth and the geodesic  $\gamma$  has finite length.

According to Proposition 6.1, we can pass to an  $i$ -drifting arc  $\gamma'$  of  $\gamma$  for some  $(x, y, z)$ -coordinate system such that

$$\text{TotCurv} \gamma' > \frac{1}{100^{100}} \cdot \text{TotCurv} \gamma. \quad \bullet$$

We will use the notations in Section 7 for  $\gamma'$ .

Rotating  $(y, z)$ -coordinate plane we can ensure that

$$\text{TotCurv} \gamma' \leq 10 \cdot \text{TotCurv}_j \gamma'$$

and that  $\gamma'$  cross the horizon  $\omega_j$  transversally.

By Corollary 8.3, we can subdivide  $\gamma'$  into at most three arcs:

- ◊ *Left arc*  $\gamma'_-$  which lies on the bright side for  $i$ ,
- ◊ *Middle arc*  $\gamma'_0$  which rotates around  $x$ -axis at most 4 times.
- ◊ *Right arc*  $\gamma'_+$  which lies on the dark side for  $i$ .

Indeed, choose an arc  $\gamma'|_{[a, b]}$  on the right from the  $(y, z)$ -plane through  $\frac{1}{2} \cdot (\gamma'(0) + \gamma'(\ell))$  which rotates around  $x$ -axis 2 times and assume that  $b$  takes the minimal possible value. Note that if  $\gamma'(s)$  lies on  $(y, z)$ -plane through  $\frac{1}{2} \cdot (\gamma'(0) + \gamma'(\ell))$  then  $[s, b] \supset [a, b]$  and any sub-arc of  $[s, b]$  rotates around  $x$ -axis at most 2 times.



By Corollary 8.3, we can take  $\gamma'_+ = \gamma'|_{[b,\ell]}$ ; in case if there is no such arc  $[a,b]$ , we assume that  $\gamma'_+$  is not presented. Repeat the construction reverting the direction of  $x$ -axis; we get the left arc  $\gamma'_-$ . The remaining arc is assumed to be  $\gamma'_0$ ; note that any sub-arc of  $\gamma'_0$  is divided by the  $(y,z)$ -plane through  $\frac{1}{2} \cdot (\gamma'(0) + \gamma'(\ell))$  into two each of which rotate around  $x$ -axis at most 2 times. Therefore the number of rotations of any arc in  $\gamma'_0$  is at most 4.

Let us estimate the total curvature of  $\gamma'_-$ ,  $\gamma'_0$  and  $\gamma'_+$  separately.

By Proposition 10.3, we get

$$\text{TotCurv}_j \gamma'_+ \leq 100 \cdot \pi. \quad \textcircled{2}$$

Similarly

$$\text{TotCurv}_j \gamma'_- \leq 100 \cdot \pi. \quad \textcircled{3}$$

By Corollary 9.3,

$$\text{TotCurv}_j \gamma'_0 \leq 100 \cdot \pi. \quad \textcircled{4}$$

Together with  $\textcircled{2}$ ,  $\textcircled{3}$  and  $\textcircled{4}$  the latter implies that

$$\text{TotCurv}_j \gamma' \leq 300 \cdot \pi.$$

From  $\textcircled{1}$  the result follows.  $\square$

## 12 Final remarks

Note that our main theorem does not hold if one removes word *minimizing* from its formulation. The minimizing geodesics have distinguishing properties in the class of all geodesics on a convex hypersurfaces. We want to mention some of them although they did not help us in the proof.

The first of these properties was essentially discovered by Anatolii Milka in [13]. To keep things simpler, we give a formulation only in the smooth case.

**12.1. Stretching lemma.** *Let  $\gamma: [0, \ell] \rightarrow \Sigma$  be a unit-speed minimizing geodesic from  $p$  to  $q$  on smooth convex hypersurface  $\Sigma \subset \mathbb{R}^m$  and  $p_t = \gamma(t) - \dot{\gamma}(t) \cdot t$ . Then for any  $t \in [0, \ell]$ , the point  $q$  lies on the dark side from  $p_t$ .*

**12.2. Convex hat lemma.** *Let  $\Sigma \subset \mathbb{R}^m$  be a closed convex hypersurface and  $\Pi$  be a hyperplane which cuts  $\Sigma$  into two parts  $\Sigma_1$  and  $\Sigma_2$ . Assume that the reflection of  $\Sigma_1$  in  $\Pi$  lies inside  $\Sigma$ . Then  $\Sigma_1$  forms a convex set in  $\Sigma$ ; that is, any minimizing geodesic in  $\Sigma$  with the ends in  $\Sigma_1$  lies completely in  $\Sigma_1$ .*

The proofs of both statements can be found in [14].

It is ashamed to confess, but we were not able to generalize the main theorem to higher dimensions. Namely we can not give an answer to the following question.

**12.3. Open question.** *Is it true that for any positive integer  $m$ , there is a constant  $C_m$  such that the total curvature of arbitrary minimizing geodesic on a convex hypersurface in  $\mathbb{R}^m$  does not exceed  $C_m$ ?*

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