Introduction to topology

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Lecture 1

Metric spaces

In this chapter we discuss *metric spaces* — a motivating example that will guide us toward the definition of *topological spaces* — the main object of topology.

Examples of metric spaces were considered for thousands of years, but the first general definition was given only in 1906 by Maurice Fréchet.

A Definition

In the following definition we grab together the most important properties of the intuitive notion of *distance.*

1.1. Definition. *Let $\mathcal{X}$ be a nonempty set with a function that returns a real number, denoted as $|x-y|$, for any pair $x, y \in \mathcal{X}$. Assume that the following conditions are satisfied for any $x, y, z \in \mathcal{X}$:*

(a) Positiveness:
$$|x-y| \geqslant 0.$$

(b) Identity of indiscernibles:
$$x = y \quad \textit{if and only if} \quad |x-y| = 0.$$

(c) Symmetry:
$$|x-y| = |y-x|.$$

(d) Triangle inequality:

$$|x - y| + |y - z| \geqslant |x - z|.$$

In this case, we say that $\mathcal{X}$ is a metric space and the function

$$(x, y) \mapsto |x - y|$$

is called a metric.

The elements of $\mathcal{X}$ are called points of the metric space. Given two points $x, y \in \mathcal{X}$, the value $|x - y|$ is called distance from x to y.

Note that for two points in a metric space the difference between points $x - y$ may have no meaning, but $|x - y|$ always has the meaning defined above.

Typically, we consider only one metric on set, but if few metrics are needed, we can distinguish them by an index, say $|x-y|_\bullet$ or $|x-y|_{239}$. If we need to emphasize that the distance is taken in the metric space $\mathcal{X}$ we write $|x - y|_{\mathcal{X}}$ instead of $|x - y|$.

B Examples

Let us give a few examples of metric spaces.

- *Discrete space.* Let $\mathcal{X}$ be an arbitrary set. For any $x, y \in \mathcal{X}$, set $|x - y| = 0$ if $x = y$ and $|x - y| = 1$ otherwise. This metric is called *discrete metric* on $\mathcal{X}$ and the obtained metric space is called *discrete.*

- *Real line.* Set of all real numbers ($\mathbb{R}$) with metric defined as $|x - y|$. (Unless it is stated othewise, the real line $\mathbb{R}$ will be considered with this metric.)

- *Metrics on the plane.* Let us denote by $\mathbb{R}^2$ the set of all pairs (x_1, y_2) of real numbers. Consider two points $p = (x_p, y_p)$ and $q = (x_q, y_q)$ in $\mathbb{R}^2$. One can equip $\mathbb{R}^2$ with the following metrics:

  - *Euclidean metric,* denoted by

  $$|p - q|_2 = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}.$$

  - *Manhattan metric,* denoted by $|* - *|_1$ and defined as

  $$|p - q|_1 = |x_p - x_q| + |y_p - y_q|.$$

– *Maximum metric,* denoted by $|{*}-{*}|_\infty$ and defined as

$$|p-q|_\infty = \max\{|x_p - x_q|, |y_p - y_q|\}.$$

1.2. Exercise. *Prove that (a)* $|{*}-{*}|_2$*; (b)* $|{*}-{*}|_1$ *and (c)* $|{*}-{*}|_\infty$ *are metrics on* $\mathbb{R}^2$*.*

1.3. Exercise. *Show that*

$$|x-y|_\natural = (x-y)^2$$

is not a metric on $\mathbb{R}$*.*

1.4. Exercise. *Show that if* $(x,y)\mapsto|x-y|$ *is a metric, then so is*

$$(x,y)\mapsto |x-y|_{\max} = \max\{1, |x-y|\}.$$

C Subspaces

Any subset $\mathcal{A}$ of metric space $\mathcal{X}$ forms a metric space on its own; it is called *subspace* of $\mathcal{X}$. This construction produces meany more examples of metric spaces. For example, the disc

$$\mathbb{D}^2 = \left\{\,(x,y)\in\mathbb{R}^2 \,\middle|\, x^2+y^2<1\,\right\}$$

and the circle

$$\mathbb{S}^1 = \left\{\,(x,y)\in\mathbb{R}^2 \,\middle|\, x^2+y^2=1\,\right\},$$

are metric spaces with metrics taken from the Euclidean plane. Similarly, the interval $[0,1)$ is a metric space with metric taken from $\mathbb{R}$.

D Continuous maps

Recall that a real-to-real function f is called *continuous* if for any $x\in\mathbb{R}$ and any $\varepsilon>0$ there exists $\delta>0$ such that $|f(x)-f(y)|<\varepsilon$, whenever $|x-y|<\delta$.

This definition can be used for the functions defined on Euclidean space if $|x-y|$ denotes the Euclidean distance $|x-y|_2$ between the points x and y. It admits the following straightforward generalization to *metric spaces*:

1.5. Definition. *A function $f\colon \mathcal{X} \to \mathcal{Y}$ between metric spaces is called continuous if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)|_{\mathcal{Y}} < \varepsilon$, for any $y \in \mathcal{X}$ such that $|x - y|_{\mathcal{X}} < \delta$.*

1.6. Exercise. *Let $\mathcal{X}$ be a metric space and $z \in \mathcal{X}$ be a fixed point. Show that the function*

$$f(x) := |x - z|$$

is continuous.

1.7. Exercise. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be metric spaces. Assume that the functions $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ are continuous, and*

$$h = g \circ f\colon \mathcal{X} \to \mathcal{Z}$$

is its composition; that is, $h(x) = g(f(x))$ for any $x \in \mathcal{X}$. Show that $h\colon \mathcal{X} \to \mathcal{Z}$ is continuous at any point.

1.8. Exercise. *Show that any distance-preserving map is continuous.*

More precisely, if $f\colon \mathcal{X} \to \mathcal{Y}$ is a map between metric space such that

$$|x - x'|_{\mathcal{X}} = |f(x) - f(x')|_{\mathcal{Y}}$$

for any $x, x' \in \mathcal{X}$, then f is continuous.

1.9. Exercise. *Let $\mathcal{X}$ be a discrete metric space (defined in 1B) and $\mathcal{Y}$ be arbitrary metric space. Show that for any function $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous.*

1.10. Advanced exercise. *Construct a continuous function $[0,1] \to \to [0,1]$ that takes every value in $[0,1]$ an infinite number of times.*

E Balls

Let $\mathcal{X}$ be a metric space, x is a point in $\mathcal{X}$ and r is a positive real number. The set of points in $\mathcal{X}$ which lies on the distance smaller than r is called *ball of radius r centered at x*. It is denoted as $\mathrm{B}(x, r)$ or $\mathrm{B}(x, r)_{\mathcal{X}}$ if we need to emphasize that it is taken in the space $\mathcal{X}$.

The ball $\mathrm{B}(x, r)$ is also called r-neighborhood of x.

1.11. Exercise. *Sketch the unit balls for the metrics $|{*} - {*}|_1$, $|{*} - {*}|_2$ and $|{*} - {*}|_\infty$ defined right before Exercise 1.2.*

1.12. Exercise. *Assume $\mathrm{B}(x, r)$ and $\mathrm{B}(y, R)$ is a pair of balls in a metric space and $\mathrm{B}(x, r) \subsetneq \mathrm{B}(y, R)$. Show that $r < 2{\cdot}R$.*

Give an example of a metric space and a pair of balls as above such that $r > R$.

Let us reformulate the definition of continuous map (1.5) using the introduced notion of ball.

1.13. Definition. *A function $f\colon \mathcal{X} \to \mathcal{Y}$ between metric spaces is called continuous if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$f(\mathrm{B}(x,\delta)_{\mathcal{X}}) \subset \mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}.$$

1.14. Exercise. *Prove the equivalence of definitions 1.5 and 1.13.*

F Open sets

1.15. Definition. *A subset V in a metric space $\mathcal{X}$ is called open if for any $x \in V$ there is $\varepsilon > 0$ such that $\mathrm{B}(x,\varepsilon) \subset V$.*

In other words, V is open if, together with each point, V contains its ε-neighborhood for some $\varepsilon > 0$. For example, any set in a discrete metric space is open since together with any point it contains its 1-neighborhood. Further the set of positive real numbers

$$(0,\infty) = \{\, x \in \mathbb{R} \mid x > 0 \,\}$$

is open since together with each point $x > 0$ it contains its x-neighborhood. On the other hand, the set of nonnegative reals

$$[0,\infty) = \{\, x \in \mathbb{R} \mid x \geqslant 0 \,\}$$

is not open since there are negative numbers in any neighborhood of 0.

1.16. Exercise. *Show that any ball in a metric space is open.*

1.17. Exercise. *Show that a set in a metric space is open if and only if it is a union of balls.*

1.18. Exercise. *Show that the union of an arbitrary collection of open sets is open.*

1.19. Exercise. *Show that the intersection of two open sets is open.*

1.20. Exercise. *Give an example of metric space $\mathcal{X}$ and an infinite sequence of open sets $V_1, V_2, \dots$ such that the intersection*

$$\bigcap_n V_n$$

is not open.

1.21. Exercise. *Show that the metrics $|*-*|_1$, $|*-*|_2$ and $|*-*|_\infty$ (defined in 1B) give rise to the same open sets in $\mathbb{R}^2$. That is, if $V\subset\mathbb{R}^2$ is open for one of these metrics, then it is open for the others.*

G Gateway to topology

The following result is the main gateway to topology. It says that continuous maps can be defined entirely in terms of open sets.

1.22. Proposition. *A function $f\colon\mathcal{X}\to\mathcal{Y}$ between two metric spaces is continuous if and only if for any open set $W\subset\mathcal{Y}$ its inverse images*

$$f^{-1}(W)=\{\,x\in\mathcal{X}\mid f(x)\in W\,\}$$

is open.

Note that proposition says nothing about the images of open sets. In fact, before going into proof it would be useful to solve the following exercise.

1.23. Exercise. *Give an example of a continuous $f\colon\mathbb{R}\to\mathbb{R}$ and an open set $V\subset\mathbb{R}$ such that the image $f(V)\subset\mathbb{R}$ is not open.*

The formulation of the proposition contains "if and only if" and the proof breaks into two parts "if"-part and "only if"-part.

Proof; "only-if" part. Let $W\subset\mathcal{Y}$ be an open set and $V=f^{-1}(W)$. Choose $x\in V$; note that so $f(x)\in W$.

Since W is open,

$$\text{❶}\qquad \mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}\subset W$$

for some $\varepsilon>0$.

Since f is continuous, by Definition 1.13, there is $\delta>0$ such that

$$f(B(x,\delta)_{\mathcal{X}})\subset\mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}.$$

It follows that together with any point $x\in V$, the set V contains $B(x,\delta)$; that is, V is open.

"If" part. Fix $x\in\mathcal{X}$ and $\varepsilon>0$. According to Exercise 1.16,

$$W=\mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}$$

is an open set in $\mathcal{Y}$. Therefore its inverse image $f^{-1}(W)$ is open.

Clearly $x \in f^{-1}(W)$. By the definition of open set (1.15)

$$\mathrm{B}(x, \delta)_{\mathcal{X}} \subset f^{-1}(W)$$

for some $\delta > 0$. Or equivalently

$$f(\mathrm{B}(x, \delta)_{\mathcal{X}}) \subset W = \mathrm{B}(f(x), \varepsilon)_{\mathcal{Y}}.$$

Hence the "if"-part follows. $\square$

H Limits

1.24. Definition. *Let $x_1, x_2, \dots$ be a sequence of points in a metric space $\mathcal{X}$. We say the sequence x_n converges to a point $x_\infty \in \mathcal{X}$ if*

$$|x_\infty - x_n|_{\mathcal{X}} \to 0 \quad \textit{as} \quad n \to \infty.$$

In this case, we say that the sequence (x_n) is a converging sequence and x_∞ is its limit; the latter will be written as

$$x_\infty = \lim_{n \to \infty} x_n$$

Note that we defined the convergence of points in a metric space using the convergence of real numbers $d_n = |x_\infty - x_n|_{\mathcal{X}}$, which we assume to be known.

1.25. Exercise. *Show that any sequence of points in a metric space has at most one limit.*

1.26. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a function between metric spaces. Show that f is continuous if and only if the following condition holds:*

- *If $x_n \to x_\infty$ as $n \to \infty$ in $\mathcal{X}$, then the sequence $y_n = f(x_n)$ converges to $y_\infty = f(y_\infty)$ as $n \to \infty$ in $\mathcal{Y}$.*

I Closed sets

Let A be a set in a metric space $\mathcal{X}$. A point $x \in \mathcal{X}$ is a *limit point* of A if there is a sequence $x_n \in A$ such that $x_n \to x$ as $n \to \infty$.[1]

[1] Sometimes limit points are defined, assuming in addition that $x_n \neq x$ for any n — we do *not* follow this convention.

The set of all limit points of A is called the *closure* of A and denoted as $\bar{A}$. Note that $\bar{A} \supset A$; indeed, any point $x \in A$ is a limit point of the constant sequence $x_n = x$.

If $\bar{A} = A$, then the set is called *closed.*

1.27. Exercise. *Show that closure of any set in metric space is a closed set; that is,* $\bar{\bar{A}} = \bar{A}$.

1.28. Exercise. *Show that a subset A in a metric space $\mathcal{X}$ is closed if and only if its complement $\mathcal{X}\backslash A$ is open.*

Lecture 2

Topological spaces

In the previous chapter we defined open sets in metric spaces and showed that continuity could be defined using only the notion of open sets. Now we will state the most important properties of these open sets as axioms. It will give us a definition of *topological space* as a set with a distinguished class of subsets called *open sets*.

The *topological properties* are loosely defined as properties which survive under arbitrary continuous deformation. They were studied since 19th century. The first definition of topological spaces was given by Felix Hausdorff in 1914. In 1922, the definition was generalized slightly by Kazimierz Kuratowski; his definition is given below.

A Definitions

We are about to define *abstract open sets* without referring to metric spaces. The exercises 1.18 and 1.19 motivate this definition.

2.1. Definition. *Suppose $\mathcal{X}$ is a set with a distinguished class of subsets, called* o p e n s e t s *such that*

(a) The empty set $\varnothing$ and the whole $\mathcal{X}$ are open.

(b) The union of any collection of open sets is an open set. That is, if V_α is open for any α the index set $\mathcal{I}$, then the set

$$W = \bigcup_{\alpha\in\mathcal{I}} V_\alpha = \{\, x \in \mathcal{X} \mid x \in V_\alpha \quad \textit{for some} \quad \alpha \in \mathcal{I} \,\}$$

is open.

(c) The intersection of two open sets is an open set. That is, if V_1 and V_2 are open, then the intersection $W = V_1 \cap V_2$ is open.

In this case, $\mathcal{X}$ is called t o p o l o g i c a l s p a c e.

The collection of all open sets in $\mathcal{X}$ is called a t o p o l o g y o n *$\mathcal{X}$ and denoted as $\mathcal{O}_{\mathcal{X}}$; so instead of saying V is an open set in the topological space $\mathcal{X}$, we might write $V \in \mathcal{O}_{\mathcal{X}}$.*

From (iii) it follows that the intersection of a finite collection of open sets is open. That is, if V_1, V_2, $\dots$, V_n are open, then the intersection

$$W = V_1 \cap V_2 \cap \dots \cap V_n$$

is open. This can be proved by applying induction on n since

$$V_1 \cap \dots \cap V_{n-1} \cap V_n = (V_1 \cap \dots V_{n-1}) \cap V_n.$$

B Examples

The so-called *connected two-point space* is a simple but nontrivial example of topological space. This space consists of two points

$$\mathcal{X} = \{a, b\}$$

and it has three open sets:

$$\varnothing, \quad \{a\} \quad \text{and} \quad \{a, b\}.$$

It is instructive to check that this is indeed a topology.

Further, for any set $\mathcal{X}$, we can always define the following topologies:

- The *discrete topology* — the topology consisting of all subsets of a set $\mathcal{X}$.
- The *concrete topology* — the topology consisting of just whole space $\mathcal{X}$ and the empty set, $\varnothing$.
- The *cofinite topology* — the topology consisting of the empty set, $\varnothing$ and the complements to finite sets.

2.2. Exercise. *Show that $\varnothing$, $\mathbb{R}$ and the intervals $[a, \infty)$, (a, ∞) for all $a \in \mathbb{R}$ define a topology on the real line $\mathbb{R}$. (The obtained space will be denoted by $\mathbb{R}_{\geqslant}$.)*

C Comparison of topologies

Let $\mathscr{W}$ and $\mathscr{S}$ be two topologies on one set. Suppose $\mathscr{W} \subset \mathscr{S}$; that is, any open set in $\mathscr{W}$-topology is open in $\mathscr{S}$-topology. In this case we say that $\mathscr{W}$ is *weaker* than $\mathscr{S}$, or, equivalently, $\mathscr{S}$ is *stronger* than $\mathscr{W}$.

2.3. Exercise. *Let $\mathscr{W}$ and $\mathscr{S}$ be two topologies on one set. Suppose that for any point x and any $W \in \mathscr{W}$ such that $W \ni x$, there is $S \in \mathscr{S}$ such that $W \supset S \ni x$. Show that $\mathscr{W}$ is weaker than $\mathscr{S}$.*

D Continuous maps

Our next challenge is to reformulate definitions from the previous chapter using only open sets. Continuous maps are first in the line. The following definition is motivated by Proposition 1.22.

2.4. Definition. *A function between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called continuous if for any open set W in $\mathcal{Y}$, its inverse image $f^{-1}(W)$ is open in $\mathcal{X}$. That is, if W is an open subset in $\mathcal{Y}$, then the set*

$$V = f^{-1}(W) = \{\, x \in X \mid f(x) \in W \,\}$$

is an open subset $\mathcal{X}$

2.5. Exercise. *Let $\mathbb{R}$ be the real line with the standard topology and $\mathcal{X}$ be the connected two-point space described above.*

(a) Construct a nonconstant continuous function $\mathbb{R} \to \mathcal{X}$.

(b) Show that any continuous function $\mathcal{X} \to \mathbb{R}$ is constant.

2.6. Exercise. *Show that a function $f\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing if and only if it defines a continuous map $\mathbb{R}_{\geqslant} \to \mathbb{R}_{\geqslant}$. (The space $\mathbb{R}_{\geqslant}$ is defined in 2.2.)*

E Limits

2.7. Definition. *Suppose x_n is a sequence of points in a topological space $\mathcal{X}$. We say that x_n converges to a point $x_\infty \in \mathcal{X}$ (briefly $x_n \to x_\infty$ as $n \to \infty$) if for any open set $V \ni x_\infty$ there is N such that $x_n \in U$ for any $n \geqslant N$.*

2.8. Exercise. *Prove that above definition agrees with 1.24. In other words, if $x_1, x_2, \dots,$ and x_∞ are points in a metric space, then x_n converges to x_∞ in the sense of definition 1.24 if and only if x_n converges to x_∞ in the sense of definition 2.7.*

2.9. Exercise. *Show that in a space with concrete topology any sequence converges to any point. In particular, the limit point of a sequence is not uniquely defined.*

2.10. Exercise. *Show that a convergent sequence of points in a topological space is also convergent for every weaker topology.*

The following exercise shows that in general, converging sequences do *not* provide an adequate description of topology. In other words, an analog of 1.26 does not hold.[1]

2.11. Advanced exercise. *Let $\mathcal{X}$ be $\mathbb{R}$ with the so-called cocountable topology; its closed sets are either countable or the whole $\mathbb{R}$.*

(a) Construct a map $f\colon \mathcal{X} \to \mathcal{X}$ that is not continuous.

(b) Describe all converging sequences in $\mathcal{X}$.

(c) Show that if the sequence x_n converges to x_∞ in $\mathcal{X}$ then for any map $f\colon \mathcal{X} \to \mathcal{X}$ the sequence $y_n = f(x_n)$ converges to $y_\infty = f(x_\infty)$.

F Metrizable spaces

According to exercises 1.18 and 1.19 any metric space is a topological space if one defines open sets as in the definition 1.15. As it follows from Exercise 1.21, different metrics on one set might define the same topology.

A topological space is called *metrizable* if its topology can be defined by a metric. Let us give few examples of nonmetrizable spaces.

2.12. Exercise. *Show that finite topological space is metrizable if and only if it is discrete. In particular, connected two-point space is not metrizable.*

2.13. Exercise. *Assume an infinite set $\mathcal{X}$ equipped with the cofinite topology. Show that $\mathcal{X}$ is not metrizable.*

[1] The so-called *nets* provide an appropriate analog of sequences that works well in topological spaces, but we are not going to consider them here.

2.14. Exercise. *Show that* $\mathbb{R}_{\geqslant}$ *is not metrizable. (The space* $\mathbb{R}_{\geqslant}$ *is defined in 2.2.)*

Lecture 3

Subsets

A Closed sets

Let $\mathcal{X}$ be a topological space.

A set $K \subset \mathcal{X}$ is called *closed* if its complement $\mathcal{X} \backslash K$ is open.

From the definition of topological spaces the following properties of closed sets follow.

3.1. Proposition. *Let $\mathcal{X}$ be a topological space.*

(i) The empty set and $\mathcal{X}$ are closed.

(ii) The intersection of any collection of closed sets is a closed set. That is, if K_α is open for any α in the index set $\mathcal{I}$, then the set

$$Q = \bigcap_{\alpha \in \mathcal{I}} K_\alpha = \{\, x \in \mathcal{X} \mid x \in K_\alpha \quad \textit{for any} \quad \alpha \in \mathcal{I} \,\}$$

is closed

(iii) The union of two closed sets (or any finite collection of closed sets) is closed. That is, if K_1 and K_2 are closed, then the union $Q = K_1 \cup K_2$ is closed.

The definitions of open and closed sets are mirrow-symmetric to each other. There is no particular reason why we define topological space using open sets — we could use closed sets instead.[1]

Sometimes it is easier to use closed sets; for example, the cofinite topology can be defined by declaring that the whole space and all its finite sets are closed.

[1] In fact, closed sets were considered before open sets — the former were introdiced by Georg Cantor in 1884, and the latter by René Baire in 1899.

The following proposition is completely analogous to the original definition of continuous functions via open sets (4.9).

3.2. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. A function $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if and only if any closed set K has closed inverse image $f^{-1}(K)$.*

Proof. In the proof we will use following set-theoretical identity.

Suppose $A \subset \mathcal{Y}$ and $B = \mathcal{Y}\backslash A$ (equvalently $A = \mathcal{Y}\backslash B$). Then

$$f^{-1}(B) = \mathcal{X}\backslash f^{-1}(A) \qquad ❶$$

for any function $f\colon \mathcal{X} \to \mathcal{Y}$. This identity is tautological, to prove it observe that both sides can be spelled as

$$\{\, x \in \mathcal{X} \mid f(x) \notin A \,\}\,.$$

"Only-if" part. Let $B \subset \mathcal{Y}$ be a closed set. Then $A = \mathcal{Y}\backslash B$ is open. Since f is continuous, $f^{-1}(A)$ is open. By ❶, $f^{-1}(B)$ is the complement of $f^{-1}(A)$ in $\mathcal{X}$. Hence $f^{-1}(B)$ is closed.

"If" part. Fix an open set B, its complement $A = \mathcal{Y}\backslash B$ is closed. Therefore $f^{-1}(A)$ is closed. By ❶, $f^{-1}(B)$ is a complement of $f^{-1}(A)$ in $\mathcal{X}$. Hence $f^{-1}(B)$ is open.

The statement follows since B is an arbitrary open set. □

B Interior and closure

Let A be an arbitrary subset in a topological space $\mathcal{X}$. The union of all open subsets of A is called the *interior* of A and denoted as $\mathring{A}$.

Note that $\mathring{A}$ is open. Indeed, it is defined as a union of open sets and such union has to be open by definition of topology (2.1). So we can say that $\mathring{A}$ is the *maximal* open set in A, as any open subset of A lies in $\mathring{A}$.

In a similar fashion, we define closure. The intersection of all closed subsets containing A is called the *closure* of A and denoted as $\bar{A}$.

The set $\bar{A}$ is closed. Indeed, it is defined as an intersection of closed sets and such intersection has to be closed by Proposition 3.1. In other words, $\bar{A}$ is the minimal closed set that contains A, as any closed subset of A contains $\bar{A}$.

3.3. Exercise. *Assume A is a subset of a topological space $\mathcal{X}$; consider its complement $B = \mathcal{X}\backslash A$. Show that*

$$\bar{B} = \mathcal{X}\backslash\mathring{A}.$$

3.4. Exercise. *Show that the following holds for any set A of a topological space:*

(a) $\mathring{A} \subset A \subset \bar{A}$

(b) $\bar{\bar{A}} = \bar{A}$

(c) $\mathring{\mathring{A}} = \mathring{A}$

3.5. Exercise.

(a) *Give an example of a topological space $\mathcal{X}$ with a closed subset Q such that*
$$\overline{\mathring{Q}} \neq Q.$$

(b) *Show that*
$$\mathring{\overline{\mathring{Q}}} = \mathring{Q}$$
for any closed set Q.

(c) *Give an example of a topological space $\mathcal{X}$ with an open subset V such that*
$$\mathring{\bar{V}} \neq V.$$

(d) *Show that*
$$\overline{\mathring{\bar{V}}} = \bar{V}$$
for any open set V.

(e) *Give an example of a topological space $\mathcal{X}$ with a subset A such that all the following 7 subsets are distinct:*
$$\overline{\mathring{\bar{A}}},\ \mathring{\bar{A}},\ \bar{A},\ A,\ \mathring{A},\ \overline{\mathring{A}},\ \mathring{\overline{\mathring{A}}}.$$

C Boundary

Let A be an arbitrary subset in a topological space $\mathcal{X}$. The *boundary* of A (briefly ∂A) is defined as the complement
$$\partial A = \bar{A} \backslash \mathring{A}.$$

3.6. Exercise. *Show that the boundary of any set is closed.*

3.7. Exercise. *Show that the set A is closed if and only if $\partial A \subset A$.*

3.8. Advanced exercise. *Find three disjoint open sets on the real line that have the same nonempty boundary.*

D Neighborhoods

Let x be a point in a topological space $\mathcal{X}$. A *neighborhood* of x is any open set U containing x. In topology, neighborhoods often replace the notion of ball (the latter can be used only in metric spaces).

3.9. Exercise. *Let A be a set in a topological space $\mathcal{X}$. Show that $x \in \partial A$ if and only if any neighborhood of x contains points in A and its complement $\mathcal{X}\backslash A$.*

Let A and B be subsets of a topological space $\mathcal{X}$. The set A is said to be a *dense in B* if $\bar{A} \supset B$.

3.10. Exercise. *Show that A is dense in B if and only any neighborhood of any point in B intersects A.*

Lecture 4

Maps

A Homeomorphisms

4.1. Definition. *A bijection $f\colon \mathcal{X} \to \mathcal{Y}$ between topological spaces is called homeomorphism if f and its inverse $f^{-1}\colon \mathcal{Y} \to \mathcal{X}$ are continuous.*[1]

Topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are called homeomorphic (briefly, $\mathcal{X} \simeq \mathcal{Y}$) if there is a homeomorphism $f\colon \mathcal{X} \to \mathcal{Y}$.

4.2. Exercise. *Give an example of continuous bijection between topological spaces that is not a homeomorphism.*

4.3. Exercise. *Show that $x \mapsto e^x$ is a homeomorphism $\mathbb{R} \to (0, \infty)$.*

4.4. Exercise. *Construct a homeomorphism $f\colon \mathbb{R} \to (0, 1)$.*

4.5. Exercise. *Show that $\simeq$ is an equivalence relation; that is, for any topological spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ we have the following:*

(a) $\mathcal{X} \simeq \mathcal{X}$;

(b) *if* $\mathcal{X} \simeq \mathcal{Y}$, *then* $\mathcal{Y} \simeq \mathcal{X}$;

(c) *if* $\mathcal{X} \simeq \mathcal{Y}$ *and* $\mathcal{Y} \simeq \mathcal{Z}$, *then* $\mathcal{X} \simeq \mathcal{Z}$.

4.6. Advanced exercise. *Prove that the complement of a circle in the Euclidean space is homeomorphic to the Euclidean space without line ℓ and a point $p \notin \ell$.*

[1] The term *homomorphism* from abstract algebra looks similar and it has similar meaning but should not to be confused with *homeomorphism*.

4.7. Advanced exercise. *Show that any nonempty open star-shaped set in the plane is homeomorphic to the open disc.*

4.8. Advanced exercise. *Show that the complements of two countable dense subsets of the plane are homeomorphic.*

B Closed and open maps

4.9. Definition. *A function between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called open if, for any open set V in $\mathcal{X}$, the image $f(V)$ is open in $\mathcal{Y}$.*

A function between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called closed if, for any open set V in $\mathcal{X}$, the image $f(V)$ is closed in $\mathcal{Y}$.

Note that homeomorphism can be defined as a *continuous open bijection.*

4.10. Exercise. *Show that a bijective map between topological spaces is closed if and only if it is open.*

4.11. Exercise. *Give an example of a map $f\colon \mathcal{X} \to \mathcal{Y}$ between two topological spaces such that*

(a) f is continuous and open, but not closed,

(b) f is continuous and closed, but not open,

(c) f is closed and open, but not continuous.

4.12. Advanced exercise. *Construct two functions $\mathbb{R} \to \mathbb{R}$, one is closed but not continuous, and the other is open but not continuous.*

Lecture 5

Constructions

In this chapter we will discuss a few constructions that produce new topological spaces from the given ones.

A Induced topology

Let A be a subset of a topological space $\mathcal{Y}$. Consider the so-called *induced topology* on A defined the following way: a subset $V \subset A$ is open in A if and only if $V = A \cap W$ for an open set W in $\mathcal{Y}$.

Let us check that induced topology is indeed a topology; in other words, it meets all conditions in 2.1.

First of all the empty set $\varnothing$ is open since $\varnothing = A \cap \varnothing$. Further, $A = A \cap \mathcal{Y}$; therefore A is open in the induced topology.

Assume $\{\, V_\alpha \mid \alpha \in \mathcal{I} \,\}$ is a collection of open sets in A; that is, for each V_α there is a set W_α which is open in $\mathcal{Y}$ and such that $V_\alpha = A \cap W_\alpha$. Note that

$$\bigcup_\alpha V_\alpha = A \cap \left(\bigcup_\alpha W_\alpha\right).$$

Since the union of $\{W_\alpha\}$ is open in $\mathcal{Y}$, the inion of $\{V_\alpha\}$ is open in the induced topology on A.

Assume V_1 and V_2 are open in A; that is, $V_1 = A \cap W_1$ and $V_2 = A \cap W_2$ for some open sets W_1 and W_2 in $\mathcal{Y}$. Note that

$$V_1 \cap V_2 = A \cap (W_1 \cap W_2).$$

Since the intersection $W_1 \cap W_2$ is open in $\mathcal{Y}$, the intersection $V_1 \cap V_2$ is open in A.

A subset A in a topological space $\mathcal{Y}$ equipped with the induced topology is called a *subspace* of $\mathcal{Y}$. It is straightforward to check that this notion agrees with the notion introduced in 1C; that is, if $\mathcal{Y}$ is a metric space, then any subset $A \subset \mathcal{Y}$ comes with metric and the topology defined by this metric coincides with the induced topology on A.

A function $f\colon \mathcal{X} \to \mathcal{Y}$ is called *embedding* if f defines a homeomorphism from space $\mathcal{X}$ to the subspace $f(\mathcal{X})$ in $\mathcal{Y}$.

B Moving topology by a map

The construction in the following exercise moves topology from target space to the source of a map.

5.1. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a function between two sets. Assume $\mathcal{Y}$ is equipped with a topology. Declare a subset $V \subset \mathcal{X}$ to be open if there is an open subset $W \subset \mathcal{Y}$ such that $V = f^{-1}(W)$. Show that it defines a topology on $\mathcal{X}$.*

The constructed topology on $\mathcal{X}$ is called *pullback* topology. It generalizes the notion of induced topology above. Namely, the induced topology on $A \subset \mathcal{Y}$ can be defined as a pullback topology for the inclusion map $\iota\colon A \to \mathcal{Y}$.[1] Indeed, for any $W \subset \mathcal{Y}$ the inverse image $\iota^{-1}(W)$ coincides with the intersection $V = A \cap W$.

The following exercise describes an analogous construction that moves topology from source to target. Both exercises can be solved by checking the conditions in 2.1 as we did in 5A.

5.2. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map between two sets. Assume $\mathcal{X}$ is equipped with a topology. Declare a subset $W \subset \mathcal{Y}$ to be open if the subset $V = f^{-1}(W)$ is open in $\mathcal{X}$. Show that it defines a topology on $\mathcal{X}$.*

The constructed topology on $\mathcal{Y}$ is called *pushforward* topology.

5.3. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map.*

(a) Show that the pullback topology on $\mathcal{X}$ is weaker than its own topology.

(b) Show that the pushforward topology on $\mathcal{Y}$ is stronger than its own topology.

5.4. Exercise. *Let $g\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map.*

[1] The inclusion map $\iota\colon A \to \mathcal{X}$ is defined by $\iota(a) = a$ for any $a \in A$.

(a) *Suppose $\mathcal{X}$ is equipped with the pullback topology. Show that a map $f\colon \mathcal{W} \to \mathcal{X}$ is continuous if and only if the composition $f \circ g\colon \mathcal{W} \to \mathcal{Y}$ is continuous.*

(b) *Suppose $\mathcal{Y}$ is equipped with the pushforward topology. Show that a map $h\colon \mathcal{Y} \to \mathcal{Z}$ is continuous if and only if the composition $h \circ f\colon \mathcal{X} \to \mathcal{Z}$ is continuous.*

The pullback topology is used mostly for injective maps; in this case, it is nearly the same as *induced topology.* Similarly, pushforward topology is mostly used for surjective maps. This particular case of the construction is called *quotient topology*; it is discussed in the following section.

5.5. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous surjective map. Assume f is closed or open. Show that $\mathcal{Y}$ is equipped with the quotient topology.*

C Quotient topology

Let $\sim$ be an *equivalence relation* on a topological space $\mathcal{X}$; that is, for any $x, y, z \in \mathcal{X}$ the following conditions hold:

- $x \sim x$;
- if $x \sim y$, then $y \sim x$;
- if $x \sim y$ and $y \sim z$, then $x \sim z$.

Recall that the set

$$[x] = \{\, y \in \mathcal{X} \mid y \sim x \,\}$$

is called the *equivalence class* of x. The set of all equivalence classes in $\mathcal{X}$ will be denoted by $\mathcal{X}/{\sim}$.

The following exercise ties equivalence relations with maps.

5.6. Exercise. *Show that an arbitrary map $f\colon \mathcal{X} \to \mathcal{Y}$ defines the following equivalence relation on $\mathcal{X}$:*

$$x \sim x' \quad \textit{if and only if} \quad f(x) = f(x').$$

Moreover,

$$y = f(x) \quad \textit{if and only if} \quad [x] = f^{-1}\{f(x)\}.$$

Observe that $x \mapsto [x]$ defines a surjective map $\mathcal{X} \to \mathcal{X}/{\sim}$. The corresponding pushforward topology on $\mathcal{X}/{\sim}$ is called *quotient topology* on $\mathcal{X}/{\sim}$. By default, $\mathcal{X}/{\sim}$ is equipped with the quotient topology in this case, it is called *quotient space.*

D Quotients by subsets

Intuitively, quotient space is the space obtained by gluing equivalent points together. For example, consider the *minimal equivalence relation* on $[0,1]$ such that $0 \sim 1$; that is, $x \sim y$ if and only if one of the following conditions hold $x = y$, or $x = 0$ and $y = 1$, or $x = 1$ and $y = 0$. Then the quotient space $[0,1]/{\sim}$ is homeomorphic to

$$\mathbb{S}^1 = \left\{\, (x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1 \,\right\}.$$

A homeomorphism is induced by the map $[0,1] \to \mathbb{S}^1$

$$f(t) = (\cos(2{\cdot}\pi{\cdot}t), \sin(2{\cdot}\pi{\cdot}t)).$$

The latter statement can be proved directly from the definition of quotient topology, but soon we will prove the following statement that implies this and similar statements effortlessly. So we suggest to wait with proof of this statement.

- Suppose $\mathcal{X}$ is a *compact space* and $\mathcal{Y}$ is *Hausdorff space* and $f\colon \mathcal{X} \to \mathcal{Y}$ is a surjective continuous map. Then $\mathcal{Y}$ is equipped with the quotient topology.

Given a subset A in a topological space $\mathcal{X}$, the space $\mathcal{X}/A$ is defined as the quotient space $\mathcal{X}/{\sim}$ for the minimal equivalence relation such that $a \sim b$ for any $a, b \in A$. For example the quotient space $[0,1]/{\sim}$ discussed above can be also denoted by $[0,1]/\{0,1\}$ — it is the interval $[0,1]$ with identified two-element subset $\{0,1\}$.

5.7. Exercise. *Describe the quotient space $[0,1]/(0,1)$, where $[0,1]$ is the real interval with standard topology; that is, list its points and its open sets.*

E Orbit spaces

5.8. Definition. *Let $\mathcal{X}$ be a topological space and G be a group. Suppose that $(g,x) \mapsto g{\cdot}x$ is a map $G \times \mathcal{X} \to \mathcal{X}$ such that*

(a) $1{\cdot}x = x$ for any $x \in \mathcal{X}$, here 1 denotes the identity element of G;

(b) $g{\cdot}(h{\cdot}x) = (g{\cdot}h){\cdot}x$ for any $g, h \in G$ and $x \in \mathcal{X}$;[2]

[2]This condition means that the expression $g{\cdot}h{\cdot}x$ makes sense; that is, it does not depend on parentheses.

(c) for any $g \in G$, the map $x \mapsto g{\cdot}x$ is continuous.

Then we say that G acts on $\mathcal{X}$, or $\mathcal{X}$ is a G-space (briefly $G \circlearrowright \mathcal{X}$).

In this case, the set

$$G{\cdot}x := \{\, g{\cdot}x \mid g \in G \,\}$$

is called the G-orbit of x (or, briefly, orbit).

5.9. Exercise. *Suppose that a group G acts on a topological space $\mathcal{X}$. Show that for any $g \in G$, the map $x \mapsto g{\cdot}x$ defines a homeomorphism $\mathcal{X} \to \mathcal{X}$.*

Suppose that a group G acts on a topological space $\mathcal{X}$. Set $x \sim y$ if there is $g \in G$ such that $y = g{\cdot}x$.

Observe that $\sim$ is an equivalence relation on $\mathcal{X}$. Indeed, $x \sim x$ since $x = 1{\cdot}x$. Further, if $y = g{\cdot}x$, then

$$x = 1{\cdot}x = g^{-1}{\cdot}g{\cdot}x = g^{-1}{\cdot}y;$$

since $g^{-1} \in G$ we get that $x \sim y \implies y \sim x$. Finally, suppose $x \sim y$ and $y \sim z$; that is, $y = g{\cdot}x$ and $z = h{\cdot}y$ for some $g, h \in G$. Then $z = h{\cdot}g{\cdot}x$; therefore $x \sim z$.

For the described equivalence relation, the quotient space $\mathcal{X}/{\sim}$ can be also denoted by $\mathcal{X}/G$; it is called quotient of $\mathcal{X}$ by the action of G.

Note that $[x] = G{\cdot}x$; that is, the orbit of x coincides with its equivalence class. By that reason $\mathcal{X}/G$ is also called *orbit space.*

5.10. Exercise. *Suppose a group G acts on a topological space $\mathcal{X}$ and $f\colon \mathcal{X} \to \mathcal{X}/G$ is the quotient map.*

(a) Show that f is open.

(b) Assume G is finite. Show that f is closed.

Lecture 6

Product, base, and prebase

A Product space

Recall that $\mathcal{X} \times \mathcal{Y}$ denotes the set of all pairs (x, y) such that $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Suppose that the sets $\mathcal{X}$ and $\mathcal{Y}$ are equipped with topologies. Let us construct the *product topology* on $\mathcal{X} \times \mathcal{Y}$ by declaring that a set is open in $\mathcal{X} \times \mathcal{Y}$ if it can be presented as a union of sets of the following type: $V \times W$ for open sets $V \subset \mathcal{X}$ and $W \subset \mathcal{Y}$. In other words, a subset U is open in $\mathcal{X} \times \mathcal{Y}$ if and only if there are collections of open sets $V_\alpha \subset \mathcal{X}$ and $W_\alpha \subset \mathcal{Y}$ such that

$$U = \bigcup_\alpha V_\alpha \times W_\alpha,$$

here α runs in some index set.

Let us show that it defines a topology on $\mathcal{X} \times \mathcal{Y}$. Parts *(a)* and *(b)* in 2.1 are evident. It remains to check *(c)*. Consider two sets

$$U = \bigcup_\alpha V_\alpha \times W_\alpha \quad \text{and} \quad U' = \bigcup_\beta V'_\beta \times W'_\beta.$$

where α and β run in some index sets, say $\mathcal{I}$ and $\mathcal{J}$ respectively. We need to show that $U \cap U'$ can be presented as a union of products of open sets; the latter follows from the this set-theoretical identity

❶ $$U \cap U' = \bigcup_{\alpha,\beta} (V_\alpha \cup V'_\beta) \times (W_\alpha \cup W'_\beta).$$

Checking ❶ is straightforward. Indeed, $(x, y) \in U \cap U'$ means that $(x, y) \in U$ and $(x, y) \in U'$; the latter means that $x \in V_\alpha$, $y \in W_\alpha$ and $x \in V'_\beta$, $y \in W'_\beta$ for *some* α and β. In other words, $x \in V_\alpha \cap V'_\beta$ and $y \in W_\alpha \cap W'_\beta$ for *some* α and β; the latter means that (x, y) belongs to the right-hand side in ❶.

By default, we assume that $\mathcal{X} \times \mathcal{Y}$ is equipped with the product topology; in this case $\mathcal{X} \times \mathcal{Y}$ is called *product space*;

6.1. Exercise. *Given a map $f\colon \mathcal{X} \to \mathcal{Y}$, consider the map $F\colon \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ defined by $F\colon x \mapsto (x, f(x))$. Show that f is continuous if and only if F is an embedding.*

B Base

6.2. Definition. *A collection $\mathscr{B}$ of open sets in a topological space $\mathcal{X}$ is called its base if every open set in $\mathcal{X}$ is a union of sets in $\mathscr{B}$.*

The definition is motivated by the fact that *open balls form a base of metric space* (1.17).

A base completely defines its topology, but typically a topology has many different bases. On metric spaces, for example, balls with rational radiuses, or balls with radiuses smaller than 1 are bases.

In many cases, it is convenient to describe topology by specifying its base. For example, the product topology on $\mathcal{X} \times \mathcal{Y}$ can be redefined as a *topology with a base formed by all products $V \times W$, where V is open in $\mathcal{X}$, and W is open in $\mathcal{Y}$.*

6.3. Exercise. *Let $\mathscr{B}$ be a base for the topology on $\mathcal{Y}$. Show that a map $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if and only if $f^{-1}(B)$ is open for any set B in $\mathscr{B}$.*

6.4. Exercise. *Let $\mathscr{B}$ be a collection of open sets in a topological space $\mathcal{X}$. Show that $\mathscr{B}$ is a base in $\mathcal{X}$ if and only if any point $x \in \mathcal{X}$ and any neighborhood $N \ni x$ there is $B \in \mathscr{B}$ such that $x \in B \subset N$.*

6.5. Proposition. *Let $\mathscr{B}$ be a set of subsets in some set $\mathcal{X}$. Show that $\mathscr{B}$ is a base of some topology on $\mathcal{X}$ if and only if it satisfies the following conditions:*

(a) $\mathscr{B}$ covers $\mathcal{X}$; that is, every point $x \in \mathcal{X}$ lies in some set $B \in \mathscr{B}$.

(b) For each pair of sets $B_1, B_2 \in \mathscr{B}$ and each point $x \in B_1 \cap B_2$ there exists a set $B \in \mathscr{B}$ such that $x \in B \subset B_1 \cap B_2$.

Proof. Denote by $\mathscr{O}$ the set of all unions of sets in $\mathscr{B}$. We need to show that $\mathscr{O}$ is a topology on $\mathcal{X}$.

Evidently, the union of any collection of sets in $\mathscr{O}$ is in $\mathscr{O}$. Further, $\mathcal{X}$ is in $\mathscr{O}$ by *(a)*. The empty set is in $\mathscr{O}$ since it is a union of the empty collection.

It remains to show that $O \cap O'$ is in $\mathscr{O}$ if O and O' are in $\mathscr{O}$. Equivalently,

❷ *for any $x \in O \cap O'$ there is $B \in \mathscr{B}$ such that $x \in B \subset O \cap O'$.*

Suppose

$$O = \bigcup_{\alpha} B_\alpha \quad \text{and} \quad O' = \bigcup_{\beta} B'_\beta,$$

where α and β run in some index sets, and B_α, $B'_\beta \in \mathscr{B}$ for any α and β. Then $x \in O \cap O'$ if and only if for some α and β we have $x \in B_\alpha$ and $x \in B'_\beta$. By *(b)*, we can choose $B \in \mathscr{B}$ so that $x \in B \subset B_\alpha \cap B'_\beta$. Since $B_\alpha \cap B'_\beta \subset O \cap O'$, ❷ follows. □

C Prebase

Suppose $\mathscr{P}$ is a collection of subsets in $\mathcal{X}$ that coves the whole space; that is, $\mathcal{X}$ is a union of all sets in $\mathscr{P}$. By 6.5, the set of all finite intersections of sets in $\mathscr{P}$ is a base for *some* topology on $\mathcal{X}$. The set $\mathscr{P}$ is called *prebase* for this topology (also known as *subbase*); there are almost no restrictions on prebase — we may start with any collection $\mathscr{P}$ of subsets of $\mathcal{X}$ that covers whole $\mathcal{X}$ and define a topology by declaring that *$\mathscr{P}$ is a prebase for the topology.* It will define the weakest topology on $\mathcal{X}$ such that every set of $\mathscr{P}$ is open.

For example, the product topology on $\mathcal{X} \times \mathcal{Y}$ can be redefined as a *topology with prebase formed by all products $\mathcal{X} \times W$ and $V \times \mathcal{Y}$, where V is open in $\mathcal{X}$ and W is open in $\mathcal{Y}$.*

More generally, given a collection of maps $f_\alpha \colon S \to \mathcal{Y}_\alpha$ from a set S to topological spaces $\mathcal{Y}_\alpha$, we can introduce pullback topology on S by stating that the inverse images $f_\alpha^{-1}(W_\alpha)$ for open sets $W_\alpha \subset \mathcal{Y}_\alpha$ form its prebase. It defines the weakest topology on S that makes all maps f_α to be continuous.