Introduction to topology

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Lecture 1

Metric spaces

In this chapter we discuss metric spaces — a motivating example that will guide us toward the definition of topological spaces — the main object of topology.

Examples of metric spaces were considered for thousands of years, but the first general definition was given only in 1906 by Maurice Fréchet.

A Definition

In the following definition we grab together the most important properties of the intuitive notion of distance.

1.1. Definition. *Let $\mathcal{X}$ be a nonempty set with a function that returns a real number, denoted as $|x-y|$, for any pair $x, y \in \mathcal{X}$. Assume that the following conditions are satisfied for any $x, y, z \in \mathcal{X}$:*

(a) Positiveness:

$$|x-y| \geqslant 0.$$

(b) Identity of indiscernibles:

$$x = y \quad \textit{if and only if} \quad |x-y| = 0.$$

(c) Symmetry:

$$|x-y| = |y-x|.$$

(d) Triangle inequality:

$$|x - y| + |y - z| \geqslant |x - z|.$$

In this case, we say that $\mathcal{X}$ is a metric space and the function

$$(x, y) \mapsto |x - y|$$

is called a metric.

The elements of $\mathcal{X}$ are called points of the metric space. Given two points $x, y \in \mathcal{X}$, the value $|x - y|$ is called distance from x to y.

Note that for two points in a metric space the difference between points $x - y$ may have no meaning, but $|x - y|$ always has the meaning defined above.

Typically, we consider only one metric on set, but if few metrics are needed, we can distinguish them by an index, say $|x-y|_\bullet$ or $|x-y|_{239}$. If we need to emphasize that the distance is taken in the metric space $\mathcal{X}$ we write $|x - y|_{\mathcal{X}}$ instead of $|x - y|$.

B Examples

Let us give a few examples of metric spaces.

- Discrete space. Let $\mathcal{X}$ be an arbitrary set. For any $x, y \in \mathcal{X}$, set $|x - y| = 0$ if $x = y$ and $|x - y| = 1$ otherwise. This metric is called discrete metric on $\mathcal{X}$ and the obtained metric space is called discrete.

- Real line. Set of all real numbers ($\mathbb{R}$) with metric defined as $|x - y|$. (Unless it is stated othewise, the real line $\mathbb{R}$ will be considered with this metric.)

- Metrics on the plane. Let us denote by $\mathbb{R}^2$ the set of all pairs (x, y) of real numbers. Consider two points $p = (x_p, y_p)$ and $q = (x_q, y_q)$ in $\mathbb{R}^2$. One can equip $\mathbb{R}^2$ with the following metrics:

  - Euclidean metric, denoted by $|p - q|_2$ and defined as

    $$|p - q|_2 = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}.$$

  - Manhattan metric,

    $$|p - q|_1 = |x_p - x_q| + |y_p - y_q|.$$

– Maximum metric,

$$|p-q|_\infty = \max\{|x_p - x_q|, |y_p - y_q|\}.$$

1.2. Exercise. *Prove that (a) $|{*}-{*}|_2$; (b) $|{*}-{*}|_1$ and (c) $|{*}-{*}|_\infty$ are metrics on $\mathbb{R}^2$.*

1.3. Exercise. *Show that*

$$|x-y|_\natural = (x-y)^2$$

is not a metric on $\mathbb{R}$.

1.4. Exercise. *Show that if $(x,y) \mapsto |x-y|$ is a metric, then so is*

$$(x,y) \mapsto |x-y|_{\max} = \max\{1, |x-y|\}.$$

C Subspaces

Any subset $\mathcal{A}$ of metric space $\mathcal{X}$ forms a metric space on its own; it is called subspace of $\mathcal{X}$. This construction produces many more examples of metric spaces. For example, the disc

$$\mathbb{D}^2 = \left\{\,(x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 < 1\,\right\}$$

and the circle

$$\mathbb{S}^1 = \left\{\,(x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1\,\right\},$$

are metric spaces with metrics taken from the Euclidean plane. Similarly, the interval $[0,1)$ is a metric space with metric taken from $\mathbb{R}$.

D Continuous maps

Recall that a real-to-real function f is called continuous if for any $x \in \mathbb{R}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$, whenever $|x-y| < \delta$.

This definition can be used for the functions defined on Euclidean space if $|x-y|$ denotes the Euclidean distance $|x-y|_2$ between the points x and y. It admits the following straightforward generalization to metric spaces:

1.5. Definition. *A function $f\colon \mathcal{X} \to \mathcal{Y}$ between metric spaces is called* c o n t i n u o u s *if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)|_{\mathcal{Y}} < \varepsilon$, for any $y \in \mathcal{X}$ such that $|x - y|_{\mathcal{X}} < \delta$.*

1.6. Exercise. *Let $\mathcal{X}$ be a metric space and $z \in \mathcal{X}$ be a fixed point. Show that the function*

$$f(x) := |x - z|$$

is continuous.

1.7. Exercise. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be metric spaces. Assume that the functions $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ are continuous, and*

$$h = g \circ f\colon \mathcal{X} \to \mathcal{Z}$$

is its composition; that is, $h(x) = g(f(x))$ for any $x \in \mathcal{X}$. Show that $h\colon \mathcal{X} \to \mathcal{Z}$ is continuous at any point.

1.8. Exercise. *Show that any distance-preserving map is continuous.*

More precisely, if $f\colon \mathcal{X} \to \mathcal{Y}$ is a map between metric space such that

$$|x - x'|_{\mathcal{X}} = |f(x) - f(x')|_{\mathcal{Y}}$$

for any $x, x' \in \mathcal{X}$, then f is continuous.

1.9. Exercise. *Let $\mathcal{X}$ be a discrete metric space (defined in 1B) and $\mathcal{Y}$ be arbitrary metric space. Show that for any function $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous.*

1.10. Advanced exercise. *Construct a continuous function*

$$f\colon [0,1] \to [0,1]$$

that takes every value in $[0,1]$ an infinite number of times.

E Balls

Let $\mathcal{X}$ be a metric space, x is a point in $\mathcal{X}$ and r is a positive real number. The set of points in $\mathcal{X}$ which lies on the distance smaller than r is called open ball of radius r centered at x. It is denoted as $\mathrm{B}(x,r)$ or $\mathrm{B}(x,r)_{\mathcal{X}}$ if we need to emphasize that it is taken in the space $\mathcal{X}$.

The ball $\mathrm{B}(x,r)$ is also called r-neighborhood of x.

Analogously we may define closed balls

$$\bar{\mathrm{B}}[x,r]=\bar{\mathrm{B}}[x,r]_{\mathcal{X}}=\{\,y\in\mathcal{X}\,|\,|x-y|\leqslant r\,\}.$$

1.11. Exercise. *Sketch the unit balls for the metrics $|{*}-{*}|_1$, $|{*}-{*}|_2$ and $|{*}-{*}|_\infty$ defined right before Exercise 1.2.*

1.12. Exercise. *Assume $\mathrm{B}(x,r)$ and $\mathrm{B}(y,R)$ is a pair of balls in a metric space and $\mathrm{B}(x,r)\subsetneq\mathrm{B}(y,R)$. Show that $r<2{\cdot}R$.*

Give an example of a metric space and a pair of balls as above such that $r>R$.

Let us reformulate the definition of continuous map (1.5) using the introduced notion of ball.

1.13. Definition. *A function $f\colon\mathcal{X}\to\mathcal{Y}$ between metric spaces is called continuous if for any $x\in\mathcal{X}$ and any $\varepsilon>0$ there exists $\delta>0$ such that*

$$f(\mathrm{B}(x,\delta)_{\mathcal{X}})\subset\mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}.$$

1.14. Exercise. *Prove the equivalence of definitions 1.5 and 1.13.*

F Open sets

1.15. Definition. *A subset V in a metric space $\mathcal{X}$ is called open if for any $x\in V$ there is $\varepsilon>0$ such that $\mathrm{B}(x,\varepsilon)\subset V$.*

In other words, V is open if, together with each point, V contains its ε-neighborhood for some $\varepsilon>0$. For example, any set in a discrete metric space is open since together with any point it contains its 1-neighborhood. Further the set of positive real numbers

$$(0,\infty)=\{\,x\in\mathbb{R}\,|\,x>0\,\}$$

is open; indeed, together with each point $x>0$ it contains its x-neighborhood. On the other hand, the set of nonnegative reals

$$[0,\infty)=\{\,x\in\mathbb{R}\,|\,x\geqslant 0\,\}$$

is not open since there are negative numbers in any neighborhood of 0.

1.16. Exercise. *Show that any ball in a metric space is open.*

1.17. Exercise. *Show that a set in a metric space is open if and only if it is a union of balls.*

1.18. Exercise. *Show that the union of an arbitrary collection of open sets is open.*

1.19. Exercise. *Show that the intersection of two open sets is open.*

1.20. Exercise. *Give an example of metric space $\mathcal{X}$ and an infinite sequence of open sets $V_1, V_2, \dots$ such that the intersection*

$$\bigcap_n V_n$$

is not open.

1.21. Exercise. *Show that the metrics $|{*}-{*}|_1$, $|{*}-{*}|_2$ and $|{*}-{*}|_\infty$ (defined in 1B) give rise to the same open sets in $\mathbb{R}^2$. That is, if $V \subset \mathbb{R}^2$ is open for one of these metrics, then it is open for the others.*

G Gateway to topology

The following result is the main gateway to topology. It says that continuous maps can be defined entirely in terms of open sets.

1.22. Proposition. *A function $f\colon \mathcal{X} \to \mathcal{Y}$ between two metric spaces is continuous if and only if for any open set $W \subset \mathcal{Y}$ its inverse images*

$$f^{-1}(W) = \{\, x \in \mathcal{X} \mid f(x) \in W \,\}$$

is open.

Note that proposition says nothing about the images of open sets. In fact, before going into proof it would be useful to solve the following exercise.

1.23. Exercise. *Give an example of a continuous $f\colon \mathbb{R} \to \mathbb{R}$ and an open set $V \subset \mathbb{R}$ such that the image $f(V) \subset \mathbb{R}$ is not open.*

The formulation of the proposition contains "if and only if" and the proof breaks into two parts "if"-part and "only if"-part.

Proof; "only-if" part. Let $W \subset \mathcal{Y}$ be an open set and $V = f^{-1}(W)$. Choose $x \in V$; note that so $f(x) \in W$.

Since W is open,

❶ $$\mathrm{B}(f(x), \varepsilon)_{\mathcal{Y}} \subset W$$

for some $\varepsilon > 0$.

Since f is continuous, by Definition 1.13, there is $\delta > 0$ such that

$$f(B(x,\delta)_{\mathcal{X}}) \subset \mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}.$$

It follows that together with any point $x \in V$, the set V contains $B(x,\delta)$; that is, V is open.

"If" part. Fix $x \in \mathcal{X}$ and $\varepsilon > 0$. According to Exercise 1.16,

$$W = \mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}$$

is an open set in $\mathcal{Y}$. Therefore its inverse image $f^{-1}(W)$ is open.

Clearly $x \in f^{-1}(W)$. By the definition of open set (1.15)

$$\mathrm{B}(x,\delta)_{\mathcal{X}} \subset f^{-1}(W)$$

for some $\delta > 0$. Or equivalently

$$f(\mathrm{B}(x,\delta)_{\mathcal{X}}) \subset W = \mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}.$$

Hence the "if"-part follows. □

H Limits

1.24. Definition. *Let $x_1, x_2, \dots$ be a sequence of points in a metric space $\mathcal{X}$. We say the sequence x_n converges to a point $x_\infty \in \mathcal{X}$ if*

$$|x_\infty - x_n|_{\mathcal{X}} \to 0 \quad \textit{as} \quad n \to \infty.$$

In this case, we say that the sequence (x_n) is a converging sequence and x_∞ is its limit; the latter will be written as

$$x_\infty = \lim_{n\to\infty} x_n$$

Note that we defined the convergence of points in a metric space using the convergence of real numbers $d_n = |x_\infty - x_n|_{\mathcal{X}}$, which we assume to be known.

1.25. Exercise. *Show that any sequence of points in a metric space has at most one limit.*

1.26. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a function between metric spaces. Show that f is continuous if and only if the following condition holds:*

- *If $x_n \to x_\infty$ as $n \to \infty$ in $\mathcal{X}$, then the sequence $y_n = f(x_n)$ converges to $y_\infty = f(y_\infty)$ as $n \to \infty$ in $\mathcal{Y}$.*

I Closed sets

Let A be a set in a metric space $\mathcal{X}$. A point $x \in \mathcal{X}$ is a limit point of A if there is a sequence $x_n \in A$ such that $x_n \to x$ as $n \to \infty$.[1]

The set of all limit points of A is called the closure of A and denoted as $\bar{A}$. Note that $\bar{A} \supset A$; indeed, any point $x \in A$ is a limit point of the constant sequence $x_n = x$.

If $\bar{A} = A$, then the set is called closed.

1.27. Exercise. *Give an example of a subset $A \subset \mathbb{R}$ that is neither closed nor open.*

1.28. Exercise. *Show that closure of any set in metric space is a closed set; that is, $\bar{\bar{A}} = \bar{A}$.*

1.29. Exercise. *Show that a subset Q in a metric space $\mathcal{X}$ is closed if and only if its complement $\mathcal{X}\backslash Q$ is open.*

[1] Sometimes limit points are defined, assuming in addition that $x_n \neq x$ for any n — we do *not* follow this convention.

Lecture 2

Topological spaces

In the previous chapter we defined open sets in metric spaces and showed that continuity could be defined using only the notion of open sets. Now we will state the most important properties of these open sets as axioms. It will give us a definition of topological space as a set with a distinguished class of subsets called open sets.

The topological properties are loosely defined as properties which survive under arbitrary continuous deformation. They were studied since 19th century. The first definition of topological spaces was given by Felix Hausdorff in 1914. In 1922, the definition was generalized slightly by Kazimierz Kuratowski; his definition is given below.

A Definitions

We are about to define abstract open sets without referring to metric spaces. The exercises 1.18 and 1.19 motivate this definition.

2.1. Definition. *Suppose $\mathcal{X}$ is a set with a distinguished class of subsets, called open sets such that*

(a) The empty set $\varnothing$ and the whole $\mathcal{X}$ are open.

(b) The union of any collection of open sets is an open set. That is, if V_α is open for any α the index set $\mathcal{I}$, then the set

$$W = \bigcup_{\alpha\in\mathcal{I}} V_\alpha = \{\, x \in \mathcal{X} \mid x \in V_\alpha \quad \textit{for some} \quad \alpha \in \mathcal{I} \,\}$$

is open.

(c) The intersection of two open sets is an open set. That is, if V_1 and V_2 are open, then the intersection $W = V_1 \cap V_2$ is open.

In this case, $\mathcal{X}$ is called topological space.

The collection of all open sets in $\mathcal{X}$ is called a topology on $\mathcal{X}$ and denoted as $\mathcal{O}_{\mathcal{X}}$; so instead of saying V is an open set in the topological space $\mathcal{X}$, we might write $V \in \mathcal{O}_{\mathcal{X}}$.

From (2.1c) it follows that the intersection of a finite collection of open sets is open. That is, if V_1, V_2, $\dots, V_n$ are open, then the intersection

$$W = V_1 \cap V_2 \cap \dots \cap V_n$$

is open. This can be proved by applying induction on n since

$$V_1 \cap \dots \cap V_{n-1} \cap V_n = (V_1 \cap \dots V_{n-1}) \cap V_n.$$

B Examples

The so-called connected two-point space is a simple but nontrivial example of topological space. This space consists of two points

$$\mathcal{X} = \{a, b\}$$

and it has three open sets:

$$\varnothing, \quad \{a\} \quad \text{and} \quad \{a, b\}.$$

It is instructive to check that this is indeed a topology.

Further, for any set $\mathcal{X}$, we can always define the following topologies:

- The discrete topology — the topology consisting of all subsets of a set $\mathcal{X}$.
- The concrete topology — the topology consisting of just the whole set $\mathcal{X}$ and the empty set, $\varnothing$.
- The cofinite topology — the topology consisting of the empty set, $\varnothing$ and the complements to finite sets.

2.2. Exercise. *Show that $\varnothing$, $\mathbb{R}$ and the intervals $[a, \infty)$, (a, ∞) for all $a \in \mathbb{R}$ define a topology on the real line $\mathbb{R}$. (The obtained space will be denoted by $\mathbb{R}_{\geqslant}$.)*

C Comparison of topologies

Let $\mathscr{W}$ and $\mathscr{S}$ be two topologies on one set. Suppose $\mathscr{W} \subset \mathscr{S}$; that is, any open set in $\mathscr{W}$-topology is open in $\mathscr{S}$-topology. In this case, we say that $\mathscr{W}$ is weaker than $\mathscr{S}$, or, equivalently, $\mathscr{S}$ is stronger than $\mathscr{W}$.

2.3. Exercise. *Let $\mathscr{W}$ and $\mathscr{S}$ be two topologies on one set. Suppose that for any point x and any $W \in \mathscr{W}$ such that $W \ni x$, there is $S \in \mathscr{S}$ such that $W \supset S \ni x$. Show that $\mathscr{W}$ is weaker than $\mathscr{S}$.*

D Continuous maps

Our next challenge is to reformulate definitions from the previous chapter using only open sets. Continuous maps are first in the line. The following definition is motivated by Proposition 1.22.

2.4. Definition. *A function between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called continuous if for any open set W in $\mathcal{Y}$, its inverse image $f^{-1}(W)$ is open in $\mathcal{X}$. That is, if W is an open subset in $\mathcal{Y}$, then the set*

$$V = f^{-1}(W) = \{\, x \in X \mid f(x) \in W \,\}$$

is an open subset $\mathcal{X}$

2.5. Exercise. *Let $\mathbb{R}$ be the real line with the standard topology and $\mathcal{X}$ be the connected two-point space described above.*

(a) Construct a nonconstant continuous function $\mathbb{R} \to \mathcal{X}$.

(b) Show that any continuous function $\mathcal{X} \to \mathbb{R}$ is constant.

2.6. Exercise. *Show that a function $f\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing if and only if it defines a continuous map $\mathbb{R}_\geqslant \to \mathbb{R}_\geqslant$. (The space $\mathbb{R}_\geqslant$ is defined in 2.2.)*

E Limits

2.7. Definition. *Suppose x_n is a sequence of points in a topological space $\mathcal{X}$. We say that x_n converges to a point $x_\infty \in \mathcal{X}$ (briefly $x_n \to x_\infty$ as $n \to \infty$) if for any open set $V \ni x_\infty$ there is N such that $x_n \in U$ for any $n \geqslant N$.*

2.8. Exercise. *Prove that above definition agrees with 1.24. In other words, if $x_1, x_2, \ldots,$ and x_∞ are points in a metric space, then x_n converges to x_∞ in the sense of definition 1.24 if and only if x_n converges to x_∞ in the sense of definition 2.7.*

2.9. Exercise. *Show that in a space with concrete topology any sequence converges to any point. In particular, the limit point of a sequence is not uniquely defined.*

2.10. Exercise. *Show that a convergent sequence of points in a topological space is also convergent for every weaker topology.*

The following exercise shows that in general, converging sequences do not provide an adequate description of topology. In other words, an analog of 1.26 does not hold.[1]

2.11. Advanced exercise. *Let $\mathcal{X}$ be $\mathbb{R}$ with the so-called cocountable topology; its closed sets are either countable or the whole $\mathbb{R}$.*

(a) *Construct a map $f\colon \mathcal{X} \to \mathcal{X}$ that is not continuous.*

(b) *Describe all converging sequences in $\mathcal{X}$.*

(c) *Show that if the sequence x_n converges to x_∞ in $\mathcal{X}$ then for any map $f\colon \mathcal{X} \to \mathcal{X}$ the sequence $y_n = f(x_n)$ converges to $y_\infty = f(x_\infty)$.*

F Metrizable spaces

According to exercises 1.18 and 1.19 any metric space is a topological space if one defines open sets as in the definition 1.15. As it follows from Exercise 1.21, different metrics on one set might define the same topology.

A topological space is called metrizable if its topology can be defined by a metric. Let us give few examples of nonmetrizable spaces.

2.12. Exercise. *Show that finite topological space is metrizable if and only if it is discrete. In particular, connected two-point space is not metrizable.*

2.13. Exercise. *Assume an infinite set $\mathcal{X}$ equipped with the cofinite topology. Show that $\mathcal{X}$ is not metrizable.*

[1] The so-called nets provide an appropriate analog of sequences that works well in topological spaces, but we are not going to consider them here.

2.14. Exercise. *Show that* $\mathbb{R}_{\geqslant}$ *is not metrizable. (The space* $\mathbb{R}_{\geqslant}$ *is defined in 2.2.)*

Lecture 3

Subsets

A Closed sets

Let $\mathcal{X}$ be a topological space.

A set $K \subset \mathcal{X}$ is called closed if its complement $\mathcal{X} \backslash K$ is open.

From the definition of topological spaces the following properties of closed sets follow.

3.1. Proposition. *Let $\mathcal{X}$ be a topological space.*

(i) The empty set and $\mathcal{X}$ are closed.

(ii) The intersection of any collection of closed sets is a closed set. That is, if K_α is open for any α in the index set $\mathcal{I}$, then the set

$$Q = \bigcap_{\alpha \in \mathcal{I}} K_\alpha = \{\, x \in \mathcal{X} \,|\, x \in K_\alpha \quad \textit{for any} \quad \alpha \in \mathcal{I} \,\}$$

is closed

(iii) The union of two closed sets (or any finite collection of closed sets) is closed. That is, if K_1 and K_2 are closed, then the union $Q = K_1 \cup K_2$ is closed.

The definitions of open and closed sets are mirror-symmetric to each other. There is no particular reason why we define topological space using open sets — we could use closed sets instead.[1]

Sometimes it is easier to use closed sets; for example, the cofinite topology can be defined by declaring that the whole space and all its finite sets are closed.

[1]In fact, closed sets were considered before open sets — the former were introdiced by Georg Cantor in 1884, and the latter by René Baire in 1899.

The following proposition is completely analogous to the original definition of continuous functions via open sets (2.4).

3.2. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. A function $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if and only if any closed set K has closed inverse image $f^{-1}(K)$.*

Proof. In the proof, we will use following set-theoretical identity.

Suppose $A \subset \mathcal{Y}$ and $B = \mathcal{Y}\backslash A$ (equvalently $A = \mathcal{Y}\backslash B$). Then

❶ $$f^{-1}(B) = \mathcal{X}\backslash f^{-1}(A)$$

for any function $f\colon \mathcal{X} \to \mathcal{Y}$. This identity is tautological, to prove it observe that both sides can be spelled as

$$\{\, x \in \mathcal{X} \mid f(x) \notin A \,\}.$$

"Only-if" part. Let $B \subset \mathcal{Y}$ be a closed set. Then $A = \mathcal{Y}\backslash B$ is open. Since f is continuous, $f^{-1}(A)$ is open. By ❶, $f^{-1}(B)$ is the complement of $f^{-1}(A)$ in $\mathcal{X}$. Hence $f^{-1}(B)$ is closed.

"If" part. Fix an open set B, its complement $A = \mathcal{Y}\backslash B$ is closed. Therefore $f^{-1}(A)$ is closed. By ❶, $f^{-1}(B)$ is a complement of $f^{-1}(A)$ in $\mathcal{X}$. Hence $f^{-1}(B)$ is open.

The statement follows since B is an arbitrary open set. □

B Interior and closure

Let A be an arbitrary subset in a topological space $\mathcal{X}$. The union of all open subsets of A is called the interior of A and denoted as $\mathring{A}$.

Note that $\mathring{A}$ is open. Indeed, it is defined as a union of open sets and such union has to be open by definition of topology (2.1). So we can say that $\mathring{A}$ is the maximal open set in A, as any open subset of A lies in $\mathring{A}$.

In a similar fashion, we define closure. The intersection of all closed subsets containing A is called the closure of A and denoted as $\bar{A}$.

The set $\bar{A}$ is closed. Indeed, it is defined as an intersection of closed sets and such intersection has to be closed by Proposition 3.1. In other words, $\bar{A}$ is the minimal closed set that contains A, as any closed subset of A contains $\bar{A}$.

3.3. Exercise. *Assume A is a subset of a topological space $\mathcal{X}$; consider its complement $B = \mathcal{X}\backslash A$. Show that*

$$\bar{B} = \mathcal{X}\backslash \mathring{A}.$$

3.4. Exercise. *Show that the following holds for any set A of a topological space:*

(a) $\mathring{A} \subset A \subset \bar{A}$

(b) $\bar{\bar{A}} = \bar{A}$

(c) $\mathring{\mathring{A}} = \mathring{A}$

3.5. Exercise.

(a) *Give an example of a topological space $\mathcal{X}$ with a closed subset Q such that*
$$\bar{\mathring{Q}} \neq Q.$$

(b) *Show that*
$$\mathring{\bar{\mathring{Q}}} = \mathring{Q}$$
for any closed set Q.

(c) *Give an example of a topological space $\mathcal{X}$ with an open subset V such that*
$$\mathring{\bar{V}} \neq V.$$

(d) *Show that*
$$\bar{\mathring{\bar{V}}} = \bar{V}$$
for any open set V.

(e) *Give an example of a topological space $\mathcal{X}$ with a subset A such that all the following 7 subsets are distinct:*
$$\bar{\mathring{\bar{A}}},\ \mathring{\bar{A}},\ \bar{A},\ A,\ \mathring{A},\ \bar{\mathring{A}},\ \mathring{\bar{\mathring{A}}}.$$

C Boundary

Let A be an arbitrary subset in a topological space $\mathcal{X}$. The boundary of A (briefly ∂A) is defined as the complement
$$\partial A = \bar{A} \backslash \mathring{A}.$$

3.6. Exercise. *Show that the boundary of any set is closed.*

3.7. Exercise. *Show that the set A is closed if and only if $\partial A \subset A$.*

3.8. Advanced exercise. *Find three disjoint open sets on the real line that have the same nonempty boundary.*

D Neighborhoods

Let x be a point in a topological space $\mathcal{X}$. A neighborhood of x is any open set U containing x. In topology, neighborhoods often replace the notion of ball (the latter can be used only in metric spaces).

3.9. Exercise. *Let A be a set in a topological space $\mathcal{X}$. Show that $x \in \partial A$ if and only if any neighborhood of x contains points in A and its complement $\mathcal{X} \backslash A$.*

Let A and B be subsets of a topological space $\mathcal{X}$. The set A is said to be a dense in B if $\bar{A} \supset B$.

3.10. Exercise. *Show that A is dense in B if and only any neighborhood of any point in B intersects A.*

Lecture 4

Maps

A Homeomorphisms

4.1. Definition. *A bijection $f: \mathcal{X} \to \mathcal{Y}$ between topological spaces is called* h o m e o m o r p h i s m *if f and its inverse $f^{-1}: \mathcal{Y} \to \mathcal{X}$ are continuous.*[1]

Topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are called h o m e o m o r p h i c *(briefly, $\mathcal{X} \simeq \mathcal{Y}$) if there is a homeomorphism $f: \mathcal{X} \to \mathcal{Y}$.*

4.2. Exercise. *Give an example of continuous bijection between topological spaces that is* n o t *a homeomorphism.*

4.3. Exercise. *Show that $x \mapsto e^x$ is a homeomorphism $\mathbb{R} \to (0, \infty)$.*

4.4. Exercise. *Construct a homeomorphism $f: \mathbb{R} \to (0, 1)$.*

4.5. Exercise. *Show that $\simeq$ is an equivalence relation; that is, for any topological spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ we have the following:*

(a) $\mathcal{X} \simeq \mathcal{X}$;

(b) if $\mathcal{X} \simeq \mathcal{Y}$, then $\mathcal{Y} \simeq \mathcal{X}$;

(c) if $\mathcal{X} \simeq \mathcal{Y}$ and $\mathcal{Y} \simeq \mathcal{Z}$, then $\mathcal{X} \simeq \mathcal{Z}$.

4.6. Advanced exercise. *Prove that the complement of a circle in the Euclidean space is homeomorphic to the Euclidean space without line ℓ and a point $p \notin \ell$.*

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[1]The term h o m o m o r p h i s m from abstract algebra looks similar and it has similar meaning but should not to be confused with h o m e o m o r p h i s m.

4.7. Advanced exercise. *Any nonempty open star-shaped set in the plane is homeomorphic to the open disc.*

4.8. Advanced exercise. *Show that the complements of two countable dense subsets of the plane are homeomorphic.*

B Closed and open maps

4.9. Definition. *A function between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called* open *if, for any open set V in $\mathcal{X}$, the image $f(V)$ is open in $\mathcal{Y}$.*

A function between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called closed *if, for any closed set Q in $\mathcal{X}$, the image $f(Q)$ is closed in $\mathcal{Y}$.*

Note that homeomorphism can be defined as a continuous open bijection.

4.10. Exercise. *Show that a bijective map between topological spaces is closed if and only if it is open.*

4.11. Exercise. *Give an example of a map $f\colon \mathcal{X} \to \mathcal{Y}$ between two topological spaces such that*

(a) f is continuous and open, but not closed,

(b) f is continuous and closed, but not open,

(c) f is closed and open, but not continuous.

4.12. Advanced exercise. *Construct two functions $\mathbb{R} \to \mathbb{R}$, one is closed but not continuous, and the other is open but not continuous.*

Lecture 5

Constructions

In this chapter we will discuss a few constructions that produce new topological spaces from the given ones.

A Induced topology

Let A be a subset of a topological space $\mathcal{Y}$. Consider the so-called induced topology on A defined the following way: a subset $V \subset A$ is open in A if and only if $V = A \cap W$ for an open set W in $\mathcal{Y}$.

Let us check that induced topology is indeed a topology; in other words, it meets all conditions in 2.1.

First of all the empty set $\varnothing$ is open since $\varnothing = A \cap \varnothing$. Further, $A = A \cap \mathcal{Y}$; therefore A is open in the induced topology.

Assume $\{ V_\alpha \mid \alpha \in \mathcal{I} \}$ is a collection of open sets in A; that is, for each V_α there is a set W_α which is open in $\mathcal{Y}$ and such that $V_\alpha = A \cap W_\alpha$. Note that

$$\bigcup_\alpha V_\alpha = A \cap \left(\bigcup_\alpha W_\alpha \right).$$

Since the union of $\{W_\alpha\}$ is open in $\mathcal{Y}$, the union of $\{V_\alpha\}$ is open in the induced topology on A.

Assume V_1 and V_2 are open in A; that is, $V_1 = A \cap W_1$ and $V_2 = A \cap W_2$ for some open sets W_1 and W_2 in $\mathcal{Y}$. Note that

$$V_1 \cap V_2 = A \cap (W_1 \cap W_2).$$

Since the intersection $W_1 \cap W_2$ is open in $\mathcal{Y}$, the intersection $V_1 \cap V_2$ is open in A.

A subset A in a topological space $\mathcal{Y}$ equipped with the induced topology is called a subspace of $\mathcal{Y}$. It is straightforward to check that this notion agrees with the notion introduced in 1C; that is, if $\mathcal{Y}$ is a metric space, then any subset $A \subset \mathcal{Y}$ comes with metric and the topology defined by this metric coincides with the induced topology on A.

A function $f\colon \mathcal{X} \to \mathcal{Y}$ is called embedding if f defines a homeomorphism from space $\mathcal{X}$ to the subspace $f(\mathcal{X})$ in $\mathcal{Y}$.

B Moving topology by a map

The construction in the following exercise moves topology from target space to the source of a map.

5.1. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a function between two sets. Assume $\mathcal{Y}$ is equipped with a topology. Declare a subset $V \subset \mathcal{X}$ to be open if there is an open subset $W \subset \mathcal{Y}$ such that $V = f^{-1}(W)$. Show that it defines a topology on $\mathcal{X}$.*

The constructed topology on $\mathcal{X}$ is called pullback topology. It generalizes the notion of induced topology above. Namely, the induced topology on $A \subset \mathcal{Y}$ can be defined as a pullback topology for the inclusion map $\iota\colon A \to \mathcal{Y}$.[1] Indeed, for any $W \subset \mathcal{Y}$ the inverse image $\iota^{-1}(W)$ coincides with the intersection $V = A \cap W$.

The following exercise describes an analogous construction that moves topology from source to target. Both exercises can be solved by checking the conditions in 2.1 as we did in 5A.

5.2. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map between two sets. Assume $\mathcal{X}$ is equipped with a topology. Declare a subset $W \subset \mathcal{Y}$ to be open if the subset $V = f^{-1}(W)$ is open in $\mathcal{X}$. Show that it defines a topology on $\mathcal{X}$.*

The constructed topology on $\mathcal{Y}$ is called pushforward topology.

5.3. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map.*

(a) Show that the pullback topology on $\mathcal{X}$ is weaker than its own topology.

(b) Show that the pushforward topology on $\mathcal{Y}$ is stronger than its own topology.

5.4. Exercise. *Let $g\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map.*

[1]The inclusion map $\iota\colon A \to \mathcal{X}$ is defined by $\iota(a) = a$ for any $a \in A$.

(a) *Suppose $\mathcal{X}$ is equipped with the pullback topology. Show that a map $f\colon \mathcal{W} \to \mathcal{X}$ is continuous if and only if the composition $f \circ g\colon \mathcal{W} \to \mathcal{Y}$ is continuous.*

(b) *Suppose $\mathcal{Y}$ is equipped with the pushforward topology. Show that a map $h\colon \mathcal{Y} \to \mathcal{Z}$ is continuous if and only if the composition $h \circ f\colon \mathcal{X} \to \mathcal{Z}$ is continuous.*

The pullback topology is used mostly for injective maps; in this case, it is nearly the same as induced topology. Similarly, pushforward topology is mostly used for surjective maps. This particular case of the construction is called quotient topology; it is discussed in the following section.

5.5. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous surjective map. Assume f is closed or open. Show that $\mathcal{Y}$ is equipped with the quotient topology.*

C Quotient topology

Let $\sim$ be an equivalence relation on a topological space $\mathcal{X}$; that is, for any $x, y, z \in \mathcal{X}$ the following conditions hold:

- $x \sim x$;
- if $x \sim y$, then $y \sim x$;
- if $x \sim y$ and $y \sim z$, then $x \sim z$.

Recall that the set

$$[x] = \{\, y \in \mathcal{X} \mid y \sim x \,\}$$

is called the equivalence class of x. The set of all equivalence classes in $\mathcal{X}$ will be denoted by $\mathcal{X}/{\sim}$.

The following exercise ties equivalence relations with maps.

5.6. Exercise. *Show that an arbitrary map $f\colon \mathcal{X} \to \mathcal{Y}$ defines the following equivalence relation on $\mathcal{X}$:*

$$x \sim x' \quad \textit{if and only if} \quad f(x) = f(x').$$

Moreover,

$$y = f(x) \quad \textit{if and only if} \quad [x] = f^{-1}\{f(x)\}.$$

Observe that $x \mapsto [x]$ defines a surjective map $\mathcal{X} \to \mathcal{X}/{\sim}$. The corresponding pushforward topology on $\mathcal{X}/{\sim}$ is called quotient topology on $\mathcal{X}/{\sim}$. By default, $\mathcal{X}/{\sim}$ is equipped with the quotient topology in this case, it is called quotient space.

D Quotients by subsets

Intuitively, quotient space is the space obtained by gluing equivalent points together. For example, consider the minimal equivalence relation on $[0,1]$ such that $0 \sim 1$; that is, $x \sim y$ if and only if one of the following conditions hold $x = y$, or $x = 0$ and $y = 1$, or $x = 1$ and $y = 0$. Then the quotient space $[0,1]/{\sim}$ is homeomorphic to

$$\mathbb{S}^1 = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1 \right\}.$$

A homeomorphism is induced by the map $[0,1] \to \mathbb{S}^1$

$$f(t) = (\cos(2{\cdot}\pi{\cdot}t), \sin(2{\cdot}\pi{\cdot}t)).$$

The latter statement can be proved directly from the definition of quotient topology, but soon we will prove the following statement that implies this and similar statements effortlessly. So we suggest to wait with proof of this statement.

- Suppose $\mathcal{X}$ is a compact space and $\mathcal{Y}$ is Hausdorff space and $f\colon \mathcal{X} \to \mathcal{Y}$ is a surjective continuous map. Then $\mathcal{Y}$ is equipped with the quotient topology.

Given a subset A in a topological space $\mathcal{X}$, the space $\mathcal{X}/A$ is defined as the quotient space $\mathcal{X}/{\sim}$ for the minimal equivalence relation such that $a \sim b$ for any $a, b \in A$. For example the quotient space $[0,1]/{\sim}$ discussed above can be also denoted by $[0,1]/\{0,1\}$ — it is the interval $[0,1]$ with identified two-element subset $\{0,1\}$.

5.7. Exercise. *Describe the quotient space $[0,1]/(0,1)$, where $[0,1]$ is the real interval with standard topology; that is, list its points and its open sets.*

E Orbit spaces

5.8. Definition. *Let $\mathcal{X}$ be a topological space and G be a group. Suppose that $(g,x) \mapsto g{\cdot}x$ is a map $G \times \mathcal{X} \to \mathcal{X}$ such that*

(a) $1{\cdot}x = x$ for any $x \in \mathcal{X}$, here 1 denotes the identity element of G;

(b) $g{\cdot}(h{\cdot}x) = (g{\cdot}h){\cdot}x$ for any $g, h \in G$ and $x \in \mathcal{X}$;[2]

[2] This condition means that the expression $g{\cdot}h{\cdot}x$ makes sense; that is, it does not depend on parentheses.

(c) *for any $g \in G$, the map $x \mapsto g{\cdot}x$ is continuous.*

Then we say that G acts on $\mathcal{X}$, or $\mathcal{X}$ is a G-space (briefly $G \circlearrowright \mathcal{X}$).

In this case, the set

$$G{\cdot}x := \{\, g{\cdot}x \mid g \in G \,\}$$

is called the G-orbit of x (or, briefly, orbit).

5.9. Exercise. *Suppose that a group G acts on a topological space $\mathcal{X}$. Show that for any $g \in G$, the map $x \mapsto g{\cdot}x$ defines a homeomorphism $\mathcal{X} \to \mathcal{X}$.*

Suppose that a group G acts on a topological space $\mathcal{X}$. Set $x \sim y$ if there is $g \in G$ such that $y = g{\cdot}x$.

Observe that $\sim$ is an equivalence relation on $\mathcal{X}$. Indeed, $x \sim x$ since $x = 1{\cdot}x$. Further, if $y = g{\cdot}x$, then

$$x = 1{\cdot}x = g^{-1}{\cdot}g{\cdot}x = g^{-1}{\cdot}y;$$

since $g^{-1} \in G$ we get that $x \sim y \implies y \sim x$. Finally, suppose $x \sim y$ and $y \sim z$; that is, $y = g{\cdot}x$ and $z = h{\cdot}y$ for some $g, h \in G$. Then $z = h{\cdot}g{\cdot}x$; therefore $x \sim z$.

For the described equivalence relation, the quotient space $\mathcal{X}/{\sim}$ can be also denoted by $\mathcal{X}/G$; it is called quotient of $\mathcal{X}$ by the action of G.

Note that $[x] = G{\cdot}x$; that is, the orbit of x coincides with its equivalence class. By that reason $\mathcal{X}/G$ is also called orbit space.

5.10. Exercise. *Suppose a group G acts on a topological space $\mathcal{X}$ and $f\colon \mathcal{X} \to \mathcal{X}/G$ is the quotient map.*

(a) *Show that f is open.*

(b) *Assume G is finite. Show that f is closed.*

Lecture 6

Product, base, and prebase

A Product space

Recall that $\mathcal{X} \times \mathcal{Y}$ denotes the set of all pairs (x, y) such that $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Suppose that the sets $\mathcal{X}$ and $\mathcal{Y}$ are equipped with topologies. Let us construct the product topology on $\mathcal{X} \times \mathcal{Y}$ by declaring that a set is open in $\mathcal{X} \times \mathcal{Y}$ if it can be presented as a union of sets of the following type: $V \times W$ for open sets $V \subset \mathcal{X}$ and $W \subset \mathcal{Y}$. In other words, a subset U is open in $\mathcal{X} \times \mathcal{Y}$ if and only if there are collections of open sets $V_\alpha \subset \mathcal{X}$ and $W_\alpha \subset \mathcal{Y}$ such that

$$U = \bigcup_\alpha V_\alpha \times W_\alpha,$$

here α runs in some index set.

Let us show that it defines a topology on $\mathcal{X} \times \mathcal{Y}$. Parts *(a)* and *(b)* in 2.1 are evident. It remains to check *(c)*. Consider two sets

$$U = \bigcup_\alpha V_\alpha \times W_\alpha \quad \text{and} \quad U' = \bigcup_\beta V'_\beta \times W'_\beta.$$

where α and β run in some index sets, say $\mathcal{I}$ and $\mathcal{J}$ respectively. We need to show that $U \cap U'$ can be presented as a union of products of open sets; the latter follows from the this set-theoretical identity

❶ $$U \cap U' = \bigcup_{\alpha,\beta} (V_\alpha \cup V'_\beta) \times (W_\alpha \cup W'_\beta).$$

Checking ❶ is straightforward. Indeed, $(x,y) \in U \cap U'$ means that $(x,y) \in U$ and $(x,y) \in U'$; the latter means that $x \in V_\alpha$, $y \in W_\alpha$ and $x \in V'_\beta$, $y \in W'_\beta$ for *some* α and β. In other words, $x \in V_\alpha \cap V'_\beta$ and $y \in W_\alpha \cap W'_\beta$ for *some* α and β; the latter means that (x,y) belongs to the right-hand side in ❶.

By default, we assume that $\mathcal{X} \times \mathcal{Y}$ is equipped with the product topology; in this case $\mathcal{X} \times \mathcal{Y}$ is called product space;

6.1. Exercise. *Given a map $f\colon \mathcal{X} \to \mathcal{Y}$, consider the map $F\colon \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ defined by $F\colon x \mapsto (x, f(x))$. Show that f is continuous if and only if F is an embedding.*

B Base

6.2. Definition. *A collection $\mathscr{B}$ of open sets in a topological space $\mathcal{X}$ is called its base if every open set in $\mathcal{X}$ is a union of sets in $\mathscr{B}$.*

The definition is motivated by the fact that *open balls form a base of metric space* (1.17).

A base completely defines its topology, but typically a topology has many different bases. On metric spaces, for example, balls with rational radiuses, or balls with radiuses smaller than 1 are bases.

In many cases, it is convenient to describe topology by specifying its base. For example, the product topology on $\mathcal{X} \times \mathcal{Y}$ can be redefined as a *topology with a base formed by all products $V \times W$, where V is open in $\mathcal{X}$, and W is open in $\mathcal{Y}$.*

6.3. Exercise. *Let $\mathscr{B}$ be a base for the topology on $\mathcal{Y}$. Show that a map $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if and only if $f^{-1}(B)$ is open for any set B in $\mathscr{B}$.*

6.4. Exercise. *Let $\mathscr{B}$ be a collection of open sets in a topological space $\mathcal{X}$. Show that $\mathscr{B}$ is a base in $\mathcal{X}$ if and only if any point $x \in \mathcal{X}$ and any neighborhood $N \ni x$ there is $B \in \mathscr{B}$ such that $x \in B \subset N$.*

6.5. Proposition. *Let $\mathscr{B}$ be a set of subsets in some set $\mathcal{X}$. Show that $\mathscr{B}$ is a base of some topology on $\mathcal{X}$ if and only if it satisfies the following conditions:*

(a) $\mathscr{B}$ covers $\mathcal{X}$; that is, every point $x \in \mathcal{X}$ lies in some set $B \in \mathscr{B}$.

(b) For each pair of sets $B_1, B_2 \in \mathscr{B}$ and each point $x \in B_1 \cap B_2$ there exists a set $B \in \mathscr{B}$ such that $x \in B \subset B_1 \cap B_2$.

Proof. Denote by $\mathscr{O}$ the set of all unions of sets in $\mathscr{B}$. We need to show that $\mathscr{O}$ is a topology on $\mathcal{X}$.

Evidently, the union of any collection of sets in $\mathscr{O}$ is in $\mathscr{O}$. Further, $\mathcal{X}$ is in $\mathscr{O}$ by *(a)*. The empty set is in $\mathscr{O}$ since it is a union of the empty collection.

It remains to show that $O \cap O'$ is in $\mathscr{O}$ if O and O' are in $\mathscr{O}$. Equivalently,

❷ *for any $x \in O \cap O'$ there is $B \in \mathscr{B}$ such that $x \in B \subset O \cap O'$.*

Suppose

$$O = \bigcup_{\alpha} B_\alpha \quad \text{and} \quad O' = \bigcup_{\beta} B'_\beta,$$

where α and β run in some index sets, and B_α, $B'_\beta \in \mathscr{B}$ for any α and β. Then $x \in O \cap O'$ if and only if for some α and β we have $x \in B_\alpha$ and $x \in B'_\beta$. By *(b)*, we can choose $B \in \mathscr{B}$ so that $x \in B \subset B_\alpha \cap B'_\beta$. Since $B_\alpha \cap B'_\beta \subset O \cap O'$, ❷ follows. □

C Prebase

Suppose $\mathscr{P}$ is a collection of subsets in $\mathcal{X}$ that coves the whole space; that is, $\mathcal{X}$ is a union of all sets in $\mathscr{P}$. By 6.5, the set of all finite intersections of sets in $\mathscr{P}$ is a base for *some* topology on $\mathcal{X}$. The set $\mathscr{P}$ is called prebase for this topology (also known as subbase); there are almost no restrictions on prebase — we may start with any collection $\mathscr{P}$ of subsets of $\mathcal{X}$ that covers whole $\mathcal{X}$ and define a topology by declaring that *$\mathscr{P}$ is a prebase for the topology.* It will define the weakest topology on $\mathcal{X}$ such that every set of $\mathscr{P}$ is open.

For example, the product topology on $\mathcal{X} \times \mathcal{Y}$ can be redefined as a *topology with prebase formed by all products $\mathcal{X} \times W$ and $V \times \mathcal{Y}$, where V is open in $\mathcal{X}$ and W is open in $\mathcal{Y}$.*

More generally, given a collection of maps $f_\alpha \colon S \to \mathcal{Y}_\alpha$ from a set S to topological spaces $\mathcal{Y}_\alpha$, we can introduce pullback topology on S by stating that the inverse images $f_\alpha^{-1}(W_\alpha)$ for open sets $W_\alpha \subset \mathcal{Y}_\alpha$ form its prebase. It defines the weakest topology on S that makes all maps f_α to be continuous.

Lecture 7

Compactness

A Definition

Let us denote by $\{V_\alpha\} = \{V_\alpha\}_{\alpha\in\mathcal{I}}$ a collection of sets; we assume that α runs in some index set, say $\mathcal{I}$.

7.1. Definition. *A collection $\{V_\alpha\}$ of open subsets in a topological space $\mathcal{X}$ is called its open cover if it covers whole $\mathcal{X}$; that is, if any $x \in \mathcal{X}$ belongs to some V_α.*

More generally, $\{V_\alpha\}$ is an open cover of a subset $S \subset \mathcal{X}$ is any $s \in S$ belongs to some V_α.

A subset of $\{V_\alpha\}$ that is also a cover is called its subcover.

7.2. Exercise. *Let $\{V_\alpha\}$ be an open cover of topological space $\mathcal{X}$. Show that $W \subset \mathcal{X}$ is open if and only if for any α the intersection $W \cap V_\alpha$ is open.*

7.3. Definition. *A topological space $\mathcal{X}$ is called compact if any cover $\{V_\alpha\}_{\alpha\in\mathcal{I}}$ of $\mathcal{X}$ contains a finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$.*

Analogously, a subset S in a topological space $\mathcal{X}$ is called compact if any cover of S contains a finite subcover of S.

As you will see, compact spaces are particularly nice and simple topological spaces. Clearly, *any finite topological space is compact.* In fact, the role of compact spaces in topology reminds the role of finite sets in set theory. The next exercise provides a source of examples of infinite compact spaces. More interesting examples are given in Section 7C.

7.4. Exercise. *Any space equipped with cofinite topology is compact.*

7.5. Exercise. *Let S be an unbounded subset of the real line; that is, for any $c \in \mathbb{R}$ there is $s \in S$ such that $|s| > c$. Show that S is not compact.*

Hint: Consider covering of S by intervals $(-c, c)$ for all $c > 0$.

7.6. Exercise. *Let B be a bounded subset of $\mathbb{R}$; set $s = \sup B$. Assume $s \notin B$. Show that B is not compact.*

Hint: Consider the covering of B by intervals $(-\infty, x)$ for all $x < s$.

7.7. Exercise. *Let S be a subset of $\mathbb{R}$. Assume S is not closed. Show that S is not compact.*

Hint: Choose a point $s \in \partial S \backslash S$ and consider the cover by intervals $(-\infty, s - \varepsilon)$ and $(s + \varepsilon, +\infty)$ for all $\varepsilon > 0$.

7.8. Exercise. *Show that a subset S of a topological space is compact if and only if S equipped with induced topology is a compact space.*

7.9. Exercise. *Construct a topological space with two compact sets such that their intersection is not compact.*

7.10. Advanced exercise. *Let $f \colon \mathcal{X} \to \mathcal{K}$ be a map between topological spaces. Assume $\mathcal{K}$ is compact. Show that f is continuous if and only if its graph $\{\, (x, f(x)) \mid x \in \mathcal{X} \,\}$ is a closed set in $\mathcal{X} \times \mathcal{K}$.*

B Finite intersection property

The following proposition describes the so-called finite intersection property; it is a useful reformulation of compactness via closed subsets.

7.11. Proposition. *Show that space $\mathcal{X}$ is compact if for any collection of closed sets $\{Q_\alpha\}$ in $\mathcal{X}$ such that*

$$\bigcap_\alpha Q_\alpha = \varnothing$$

There is a finite collection $\{Q_{\alpha_1}, \dots, Q_{\alpha_n}\}$ such that

$$Q_{\alpha_1} \cap \dots \cap Q_{\alpha_n} = \varnothing.$$

Proof. Consider the complements $V_\alpha = \mathcal{X} \backslash Q_\alpha$. Note that

$$\bigcup_\alpha V_\alpha = \mathcal{X} \backslash \left(\bigcap_\alpha Q_\alpha \right) = \mathcal{X};$$

that is, $\{V_\alpha\}$ is a cover of $\mathcal{X}$.

Choose a finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$ and observe that

$$Q_{\alpha_1} \cap \dots \cap Q_{\alpha_n} = \mathcal{X} \backslash (V_{\alpha_1} \cup \dots \cup V_{\alpha_n}) = \varnothing. \qquad \square$$

7.12. Exercise. *Let $Q_1 \supset Q_2 \supset \dots$ be a nested sequence of closed nonempty sets in a compact space $\mathcal{K}$. Show that there is a point $q \in \mathcal{K}$ such that $q \in Q_n$ for any n.*

C Real interval

7.13. Theorem. *Any closed interval $[a, b]$ forms a compact subspace of the real line.*

Proof. Set $a_0 = a$ and $b_0 = b$, so $[a, b] = [a_0, b_0]$.

Arguing by contradiction, assume that there is an open cover $\{V_\alpha\}$ of $[a_0, b_0]$ that has no finite subcovers.

Note that $\{V_\alpha\}$ is also a cover for two intervals

$$[a_0, \tfrac{a_0+b_0}{2}] \quad \text{and} \quad [\tfrac{a_0+b_0}{2}, b_0].$$

If $\{V_\alpha\}$ would have a finite subcover of each of these two subintervals, then these subcovers together would give a finite cover of $[a, b]$. Therefore $\{V_\alpha\}$ has no finite subcovers of *at least one* of these subintervals; denote it by $[a_1, b_1]$; so either $a_1 = a_0$ and $b_1 = \frac{a_0+b_0}{2}$ or $a_1 = \frac{a_0+b_0}{2}$ and $b_1 = b_0$.

Continuing in this manner we get a sequence of intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset \dots$$

such that no finite collection of sets from $\{V_\alpha\}$ covers any of the intervals $[a_n, b_n]$. In particular,

❶ $$[a_n, b_n] \not\subset V_\alpha \quad \text{for any } n \text{ and } \alpha.$$

Observe that

$$\begin{aligned} a_0 \leqslant a_1 \leqslant \dots \\ \dots \leqslant b_1 \leqslant b_0, \\ b_n - a_n = \tfrac{b-a}{2^n}. \end{aligned}$$

Denote by x the least upper bound of $\{a_n\}$. Note that $x \in [a_n, b_n]$ for any n.[1]

[1] In fact, $a_n \to x$ and $b_n \to x$ as $n \to \infty$, but we will not use it directly.

Since $\{V_\alpha\}$ is a cover, we can choose $V_\alpha \ni x$. Since V_α is open, it contains the interval $(x-\varepsilon, x+\varepsilon)$ for some $\varepsilon > 0$. Choose a large n so that $\frac{b-a}{2^n} < \varepsilon$. Clearly, $V_\alpha \supset (x-\varepsilon, x+\varepsilon) \supset [a_n, b_n]$; the latter contradicts ❶. $\square$

D Images

7.14. Proposition. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map between topological spaces and K is a compact set in $\mathcal{X}$. Then the image $Q = f(K)$ is compact in $\mathcal{Y}$.*

Proof. Given an open cover $\{W_\alpha\}$ of Q, we need to find its finite subcover.

Set $V_\alpha = f^{-1}(W_\alpha)$ for each α. Note that $\{V_\alpha\}$ is an open cover of K. Since K is compact, there is a finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$. Observe that $\{W_{\alpha_1}, \dots, W_{\alpha_n}\}$ is a cover of Q. $\square$

7.15. Exercise. *Show that the circle $\mathbb{S}^1$ is compact.*

E Closed subsets

7.16. Proposition. *A closed set in a compact space is compact.*

Proof. Let Q be a closed set in a compact space $\mathcal{K}$. Since Q is closed, its complement $W = \mathcal{K}\backslash Q$ is open.

Consider an open cover $\{V_\alpha\}_{\alpha\in\mathcal{I}}$ of Q. Add to it W; that is, consider the collection of sets that includes W and all V_α for $\alpha \in \mathcal{I}$. Note that we get an open cover of $\mathcal{K}$. Indeed, W covers all points in the complement of Q and any point of Q is covered by some V_α.

Since $\mathcal{K}$ is compact, we can choose a finite subcover, say $\{W, V_{\alpha_1}, \dots, V_{\alpha_n}\}$ — without loss of generality, we can assume that it includes W. Observe that $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$ is a cover of Q, hence the result. $\square$

In the proof, we add an extra open set to the cover, used it, and took it away. This type of reasoning is useful in all branches of mathematics; sometimes it is called 17 camels trick.[2]

[2]The name comes from the following mathematical parable: *A father left 17 camels to his three sons and, according to the will, the eldest son should be given half of all camels, the middle son the 1/3 part, and the youngest son the 1/9. It was impossible to follow his will until a wise man appeared. He added his own camel, the oldest son took* $18/2 = 9$ *camels, the second son took* $18/3 = 6$ *camels, and the third son* $18/9 = 2$ *camels, the wise man took his camel and went away.*

7.17. Exercise. *Show that any closed bounded subset of the real line is compact.*

F Product spaces

7.18. Theorem. *Assume $\mathcal{X}$ and $\mathcal{Y}$ are compact topological spaces. Then their product space $\mathcal{X} \times \mathcal{Y}$ is compact.*

The following exercise provides a partial converse.

7.19. Exercise. *Suppose that a product space $\mathcal{X} \times \mathcal{Y}$ is nonempty and compact. Show that its factors $\mathcal{X}$ and $\mathcal{Y}$ are compact.*

In the proof, we will need the following definition.

7.20. Definition. *Let $\{V_\alpha\}$ and $\{W_\beta\}$ be two covers of a topological space $\mathcal{X}$. We say that $\{V_\alpha\}$ is inscribed in $\{W_\beta\}$ if for any α there is β such that $V_\alpha \subset W_\beta$.*

7.21. Exercise. *Let $\mathscr{B}$ be a base in a topological space $\mathcal{X}$. Show that for any cover $\{V_\alpha\}$ of $\mathcal{X}$ there is an inscribed cover of sets in $\mathscr{B}$.*

Supose that $\{V_\alpha\}$ is inscribed in $\{W_\beta\}$. If $\{V_\alpha\}$ has a finite inscribed cover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$. Then for each α_i we can choose β_i such that $V_{\alpha_i} \subset W_{\beta_i}$. Note that $\{W_{\beta_1}, \dots, W_{\beta_n}\}$ is a finite subcover of $\{W_\beta\}$. It proves the following:

7.22. Observation. *A space $\mathcal{X}$ is compact if and only if any cover of $\mathcal{X}$ has a finite inscribed cover.*

Proof of 7.18. Recall that by definition of product topology, any open set in $\mathcal{X} \times \mathcal{Y}$ is a union of product sets $V_\alpha \times W_\alpha$, where V_α is open in $\mathcal{X}$ and W_α is open in $\mathcal{Y}$.

Fix an open cover $\{U_\beta\}$ of $\mathcal{X} \times \mathcal{Y}$. Consider all product sets $V_\alpha \times W_\alpha$ such that $V_\alpha \times W_\alpha \subset U_\beta$ for some β (as before, V_α and W_α are open in $\mathcal{X}$ and $\mathcal{Y}$ respectively). Note that $\{V_\alpha \times W_\alpha\}$ is a cover of $\mathcal{X} \times \mathcal{Y}$ that is inscribed in $\{U_\beta\}$. By the observation, it is sufficient to find a finite subcover of $\{V_\alpha \times W_\alpha\}$.

Fix $x \in \mathcal{X}$. Note that the subspace $\{x\} \times \mathcal{Y}$ is homeomorphic to $\mathcal{Y}$; in particular, it is compact. Therefore, $\{x\} \times \mathcal{Y}$ has a finite cover $\{V_{\alpha_1} \times W_{\alpha_1}, \dots, V_{\alpha_n} \times W_{\alpha_n}\}$; that is,

$$(V_{\alpha_1} \times W_{\alpha_1}) \cup \dots \cup (V_{\alpha_n} \times W_{\alpha_n}) \supset \{x\} \times \mathcal{Y}$$

Consider the set

$$N_x = V_{\alpha_1} \cap \dots \cap V_{\alpha_n};$$

note that N_x is open in $\mathcal{X}$. Since $N_x \subset V_{\alpha_i}$ for any i, we have

$$N_x \times \mathcal{Y} \subset (V_{\alpha_1} \times W_{\alpha_1}) \cup \dots \cup (V_{\alpha_n} \times W_{\alpha_n}).$$

Therefore

❷ *every point $x \in \mathcal{X}$ admits an open neighborhood N_x such that $N_x \times \mathcal{Y}$ can be covered by finitely many product sets from $\{V_\alpha \times W_\alpha\}$*

The sets $\{N_x\}_{x \in \mathcal{X}}$ form a cover of $\mathcal{X}$. Since $\mathcal{X}$ is compact, there is a finite subcover $\{N_{x_1}, \dots, N_{x_m}\}$. Note that

$$\mathcal{X} \times \mathcal{Y} = (N_{x_1} \times \mathcal{Y}) \cup \dots \cup (N_{x_m} \times \mathcal{Y});$$

that is, $\mathcal{X} \times \mathcal{Y}$ can be covered by a finite set of sets from $\{N_x \times \mathcal{Y}\}_{x \in \mathcal{X}}$. Applying ❷, we get that $\mathcal{X} \times \mathcal{Y}$ can be covered by finite umber of product sets from $\{V_\alpha \times W_\alpha\}$. □

7.23. Exercise. *Find a flaw in the following argument.*

Fake proof of 7.18. Fix an open cover $\{U_\beta\}$ of $\mathcal{X} \times \mathcal{Y}$. Consider all product sets $V_\alpha \times W_\alpha$ such that $V_\alpha \times W_\alpha \subset U_\beta$ for some β (as before, V_α and W_α are open in $\mathcal{X}$ and $\mathcal{Y}$ respectively). Note that $\{V_\alpha \times W_\alpha\}$ is a cover of $\mathcal{X} \times \mathcal{Y}$ that is inscribed in $\{U_\beta\}$. By the observation, it is sufficient to find a finite subcover of $\{V_\alpha \times W_\alpha\}$.

Note that $\{V_\alpha\}$ is a cover of $\mathcal{X}$. Since $\mathcal{X}$ is compact, we can choose its finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$. Similarly, $\{W_\alpha\}$ is a cover of $\mathcal{Y}$. So we can choose its finite subcover $\{W_{\alpha'_1}, \dots, W_{\alpha'_m}\}$.

Finally observe that

$$\{V_{\alpha_1} \times W_{\alpha_1}, \dots, V_{\alpha_n} \times W_{\alpha_n}, V_{\alpha'_1} \times W_{\alpha'_1}, \dots, V_{\alpha'_m} \times W_{\alpha'_m}\}$$

is a finite cover of $\mathcal{X} \times \mathcal{Y}$. □

G Remarks on metric spaces

Recall that any metric space has natural topology. In particular we may talk about compact metric spaces.

7.24. Proposition. *Let $\mathcal{M}$ be a metric space. Then $\mathcal{M}$ is compact if and only if any sequence of points $x_n \in \mathcal{M}$ has a converging subsequence.*

We will prove only the only-if part of this proposition; the if part requires deeper diving into metric spaces.

A topological space is called sequentially compact if any its sequence has a converging subsequence. For general topological spaces sequential compactness does not imply compactness, and the other way around. The theorem above states that these two notions are equivalent for metric spaces.

7.25. Exercise. *Show that product of two sequentially compact spaces is sequentially compact.*

Proof; only-if part. Choose a sequence $x_n \in \mathcal{M}$.

Note that a point $p \in \mathcal{M}$ is a limit of subsequence of x_n if and only if for any $\varepsilon > 0$, the ball $\mathrm{B}(p, \varepsilon)$ contains infinite number of elements of x_n. Indeed, if this property holds, then we can choose i_1 such that $x_{i_1} \in \mathrm{B}(p, 1)$, further $i_2 > i_1$ such that $x_{i_2} \in \mathrm{B}(p, \frac{1}{2})$ and so on; on n^{th} step we get $i_n > i_{n-1}$ such that $x_{i_n} \in \mathrm{B}(p, \frac{1}{n})$. Note that obtained subsequence $x_{i_1}, x_{i_2}, \ldots$ converges to p.

Assume the sequence x_n has no converging subsequence. Then for any point p there is $\varepsilon_p > 0$ such that $\mathrm{B}(p, \varepsilon_p)$ contains only finitely many elements of x_n. Note that $\mathrm{B}(p, \varepsilon_p)$ for all p forms a cover of $\mathcal{M}$. Since the sequence is infinite, this cover does not have a fine subcover. That is, if a sequence x_n has no converging subsequence, then $\mathcal{M}$ is not compact. It proves the only-if part of the proposition. □

7.26. Exercise. *Let $\{V_\alpha\}$ be a cover of a compact metric space $\mathcal{M}$. Show that there is $\varepsilon > 0$ such that for every $x \in \mathcal{M}$ there is α such that $V_\alpha \supset \mathrm{B}(x, \varepsilon)$.*

Hint: Show and use that there is $\varepsilon > 0$ and a finite cover of $\mathcal{M}$ by balls $\mathrm{B}(p_i, r_i)$ such that for each i the ball $\mathrm{B}(p_i, r_i + \varepsilon)$ lies in some V_{α_i}.

The number ε in the exercise is called Lebesgue number and the statement is called Lebesgue lemma. It is a useful tool for compact metric spaces.

7.27. Exercise. *Construct a noncompact metric space $\mathcal{M}$ such that 1 is a Lebesgue number for any of cover of $\mathcal{M}$.*

Lecture 8

Hausdorff spaces

A Definition

8.1. Definition. *A topological space $\mathcal{X}$ is called Hausdorff if for each pair of distinct points $x, y \in \mathcal{X}$ there are disjoint neighborhoods $V \ni x$ and $W \ni y$.*

Recall that a sequence x_n of points in a topological space $\mathcal{X}$ converges to a point $x \in \mathcal{X}$ if for any neighborhood $V \ni x$ we have $x_n \in V$ for all, but finitely many n.

Note that in general, a sequence of points in a topological space might have different limits. For example, consider the real line with the cofinite topology and a sequence x_n such that $x_m \neq x_n$ for $m \neq n$. Note that x_n converges to *every* point in $x \in \mathbb{R}$. Indeed, a complement of any neighborhood $V \ni x$ is a finite set; therefore $x_n \in V$ for all, but finitely many indexes n.

8.2. Exercise. *Show that any converging sequence in Hausdorff space has a unique limit.*

8.3. Exercise. *Show that the topological space $\mathcal{X}$ is Hausdorff if the diagonal*

$$\Delta = \{\, (x, y) \in \mathcal{X} \times \mathcal{X} \mid x = y \,\}$$

is a closed set in the product space $\mathcal{X} \times \mathcal{X}$.

B Observations

8.4. Observation. *Any one-point set in a Hausdorff topological space is closed.*

If every one-point space in a topological space is closed then the space is called T_1-space or sometimes Tikhonov space. Therefore the observation above states that any Hausdorff space is T_1.

Proof. Fix a point x in a Hausdorff topological space $\mathcal{X}$. By definition of Hausdorff space, given a point $y \neq x$, there are disjoint open sets $V_y \ni x$ and $W_y \ni y$. In particular $W_y \not\ni x$.

Note that

$$\mathcal{X}\backslash\{x\} = \bigcup_{y \neq x} W_y.$$

In particular $\mathcal{X}\backslash\{x\}$ is open and therefore $\{x\}$ is closed. □

8.5. Observation. *Any metrizable space is Hausdorff.*

Proof. Assume that topology on the space $\mathcal{X}$ is induced by a metric $|{*} - {*}|$.

If the points $x, y \in \mathcal{X}$ are distinct then $|x - y| > 0$. By triangle inequality $\mathrm{B}(x, \frac{r}{2}) \cap \mathrm{B}(y, \frac{r}{2}) = \varnothing$. Hence the statement follows □

8.6. Observation. *Any subspace of Hausdorff space is Hausdorff.*

C Games with compactness

8.7. Proposition. *Any compact subset of a Hausdorff topological space is closed.*

Since any one-point set is compact, this proposition generalizes Observation 8.4. The proof is similar but requires an extra step. It is instructive to solve the following exercise before reading the proof.

8.8. Exercise. *Describe a topological space $\mathcal{X}$ with a nonclosed, but compact subset K.*

We will prove the following slightly stronger statement.

8.9. Theorem. *Let $\mathcal{X}$ be a Hausdorff space and $K \subset X$ be a compact subset. Then for any point $y \notin K$ there are open sets $V \supset K$ and $W \ni y$ such that $V \cap W = \varnothing$*

Proof of 8.7 modulo 8.9. For $y \notin K$, let us denote by W_y the open set provided by 8.9; in particular, $W_y \ni y$ and $W_y \cap K = \varnothing$. Note that

$$\mathcal{X}\backslash K = \bigcup_{y \notin K} W_y.$$

It follows that $\mathcal{X}\backslash K$ is open; therefore, K is closed. □

Proof. By definition of Hausdorff space, for any point $x \in K$ there is a pair of disjoint openset $V_x \ni x$ and $W_x \ni y$. Note that sets $\{V_x\}_{x \in K}$ forms cover of K. Since K is compact we can pass to a finite subcover $\{V_{x_1}, \dots, V_{x_n}\}$. Set

$$\begin{aligned} V &= V_{x_1} \cup \dots \cup V_{x_n} \\ W &= W_{x_1} \cap \dots \cap W_{x_n} \end{aligned}$$

It remains to observe that $y \in W$, $K \subset V$ and

$$V \cap W \subset \bigcup_i (V_{x_i} \cap W_{x_i}) = \varnothing.$$

□

8.10. Exercise. *Let $\mathcal{X}$ be a Hausdorff space and $K, L \subset X$ be two compact subsets. Assume $K \cap L = \varnothing$, show that there are open sets $V \supset K$ and $W \supset L$ such that $V \cap W = \varnothing$*

D Application to quotients

8.11. Observation. *Let $\mathcal{K}$ be a compact space and $\mathcal{Y}$ is Hausdorff. Then any continuous map $f \colon \mathcal{K} \to \mathcal{Y}$ is closed.*

If in addition, the map f is onto, then $\mathcal{Y}$ is equipped with the quotient topology induced by f.

8.12. Corollary. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proof of 8.11. Since $\mathcal{K}$ is compact, any closed subset $Q \subset \mathcal{K}$ is compact as well (7.16). Since the image of a compact set is compact we have that $f(Q)$ is a compact subset of $\mathcal{Y}$. Since $\mathcal{Y}$ is Hausdorff $f(Q)$ is closed. Hence the first statement follows.

The second statement follows from 5.5. □

Recall that $\mathbb{D}$ denotes the unit disc and $\mathbb{S}^1$ denotes the unit circle; that is,

$$\begin{aligned} \mathbb{D} &= \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 \leqslant 1 \right\}, \\ \mathbb{S}^1 &= \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1 \right\}. \end{aligned}$$

8.13. Exercise. *Show that the quotient space* $[0,1]/\{0,1\}$ *is homeomorphic to $\mathbb{S}^1$.*

8.14. Exercise. *Show that the quotient space $\mathbb{D}/\mathbb{S}^1$ is homeomorphic to the unit sphere*

$$\mathbb{S}^2 = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \right\}.$$

Lecture 9

Connected spaces

A Definitions

A subset of topological space is called clopen if it is closed and open at the same time.

9.1. Definition. *A topological space $\mathcal{X}$ is called connected if it has exactly two clopen sets $\varnothing$ and the whole space $\mathcal{X}$.*

According to our definition, the empty space is not connected. Not everyone agrees with this convention.

Suppose V is a proper clopen subset in a topological space $\mathcal{X}$; that is, $V \neq \varnothing$ and $V \neq \mathcal{X}$. Its complement $W = \mathcal{X} \backslash V$ is also a proper clopen subset. That is, there are two open sets $V, W \subset \mathcal{X}$ such that $V \neq \varnothing$, $W \neq \varnothing$, $V \cup W = \mathcal{X}$ and $V \cap W = \varnothing$. In this case, the pair V and W is called open subdivision of $\mathcal{X}$. Note that a nonempty space is disconnected if and only if it admits an open subdivision.

9.2. Proposition. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous onto map between topological spaces. If $\mathcal{X}$ is connected, then so is $\mathcal{Y}$.*

Proof. We can assume that $\mathcal{X}$, and therefore $\mathcal{Y}$ are nonempty.

Assume contrary; that is, $\mathcal{Y}$ is disconnected. Suppose V and W form an open subdivision of $\mathcal{Y}$.

Since f is continuous, $V' = f^{-1}(V)$ and $W' = f^{-1}(W)$ are open in $\mathcal{X}$. Since the map f is onto $V' \neq \varnothing$ and $W' \neq \varnothing$. Finally observe that $V' \cup W' = f^{-1}(\mathcal{Y}) = \mathcal{X}$ and

$$\begin{aligned} V' \cup W' &= f^{-1}(V \cup W) = f^{-1}(\mathcal{Y}) = \mathcal{X}, \\ V' \cap W' &= f^{-1}(V \cap W) = f^{-1}(\varnothing) = \varnothing. \end{aligned}$$

That is, V' and W' is an open subdivision of $\mathcal{X}$; therefore $\mathcal{X}$ is disconnected — a contradiction. □

A subset of topological space is called the connected or disconnected if so is the corresponding subspace. Spelling the notion of subspace we get the following definition.

9.3. Definition. *Let $\mathcal{X}$ be a topological space.*

A subset $A \subset \mathcal{X}$ is called d i s c o n n e c t e d *if it is empty or there are two disjoint open sets V and W be its open subdivision.*

Otherwise, we say that A is c o n n e c t e d.

9.4. Proposition. *Assume $\{A_\alpha\}_{\alpha\in\mathcal{I}}$ is a collection of connected subsets of a topological space. Suppose that $\bigcap_\alpha A_\alpha \neq \varnothing$. Then*

$$A = \bigcup_\alpha A_\alpha$$

is connected.

Proof. Assume contrary; that is, A is disconnected. Choose its open subdivision V, W. Since $\bigcap_\alpha A_\alpha \neq \varnothing$, we can fix $p \in \bigcap_\alpha A_\alpha$. Without loss of generality, we can assume that $p \in V$.

In particular, $V \cap A_\alpha \neq \varnothing$ for any α. Since A_α is connected, it follows that $W \cap A_\alpha = \varnothing$ for each α. Therefore

$$\begin{aligned} W \cap A &= W \cap \left(\bigcup_\alpha A_\alpha\right) \\ &= \bigcup_\alpha (W \cap A_\alpha) \\ &= \varnothing, \end{aligned}$$

a contradiction. □

B Real interval

9.5. Proposition. *The real interval $[0, 1]$ is connected.*

Proof. Assume contrary; let V and W be an open subdivision of $[0, 1]$. Fix a $a_0 \in V$ and $b_0 \in W$; without loss of generality, we can assume that $a_0 < b_0$.

Let us construct a nested sequence of closed intervals

$$[a_0, b_0] \supset [a_1, b_1] \ldots$$

such that $b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$ and $a_n \in V$ and $b_n \in W$ for any n.

The construction is recursive. Assume $[a_n, b_n]$ is already constructed. Set $c = \frac{1}{2}\cdot(a_n + b_n)$. If $c \in V$, then set $a_{n+1} = c$ and $b_{n+1} = b_n$; if $c \in W$, then set $a_{n+1} = a_n$ and $b_{n+1} = c$.

The sequence a_n is nondecreasing and bounded above by b_0. In particular, the sequence a_n converges; denote its limit by x. Since $b_n - a_n = \frac{1}{2^n}\cdot(b_0 - a_0)$, the sequence b_n also converges to x.

The point x has to belong to V or W. Since both V and W are open, one of them contains a_n and b_n for all large n — a contradiction. □

9.6. Exercise. *Show that the real line is a connected space.*

C Connected components

Let x be a point in a topological space $\mathcal{X}$. The intersection of all clopen sets containing x is called connected component of x. Note that the space $\mathcal{X}$ is connected if $\mathcal{X}$ is the connected component of one (and therefore any) point in $\mathcal{X}$.

As an intersection of closed sets, any connected component has to be closed.

9.7. Exercise. *Construct an example of topological space $\mathcal{X}$ and a point $x \in \mathcal{X}$ such that the connected component of x is not open.*

9.8. Exercise. *Suppose that a space $\mathcal{X}$ has a finite number of connected components. Show that each connected component of $\mathcal{X}$ is open.*

9.9. Exercise. *Show that two connected components either coincide or disjoint. In other words, being in one connected component defines an equivalence relation on points of topological space.*

D Cut points

Evidently, number of connected components is a topological invariant; that is, *if two spaces are homeomorphic, then they have the same number of connected components.* In particular, connected space is not homeomorphic to a disconnected space.

Let us describe a more refined way to apply that connectedness is preserved by homeomorphisms. Suppose $\mathcal{X}$ is a connected space, a point $x \in \mathcal{X}$ is called a cut point if removing x from $\mathcal{X}$ produces a disconnected space $\mathcal{X}\backslash\{x\}$.

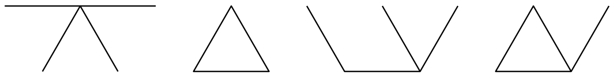
Note that if $f\colon \mathcal{X} \to \mathcal{Y}$ is a homeomorphism, then a point $x \in \mathcal{X}$ is a cut point of $\mathcal{X}$ if and only if $y = f(x)$ is a cut point of $\mathcal{Y}$. Indeed, the restriction of f defines a homeomorphism $\mathcal{X}\backslash\{x\} \to \mathcal{Y}\backslash\{y\}$. Moreover, we get that the spaces $\mathcal{X}\backslash\{x\}$ and $\mathcal{Y}\backslash\{y\}$ have the same number of connected components.

These observations can be used to solve the following exercises.

9.10. Exercise. *Show that the circle $\mathbb{S}^1$ is not homeomorphic to the line segment $[0,1]$.*

9.11. Exercise. *Show that the plane $\mathbb{R}^2$ is not homeomorphic to the real line $\mathbb{R}$.*

9.12. Exercise. *Show that no two of the following four closed connected sets in the plane are homeomorphic. (Each set is a union of line segments.)*



(a) (b) (c) (d)

E Path-connectedness

Let $\mathcal{X}$ be a topological space. A continuous map $f\colon [0,1] \to \mathcal{X}$ is called path. If $x = f(0)$ and $y = f(1)$ we say that f is a path from x to y.

A space $\mathcal{X}$ is called path-connected if it is nonempty and any two points in $\mathcal{X}$ can be connected by a path; that is, for any $x, y \in \mathcal{X}$ there is a path f from x to y.

9.13. Theorem. *Any path-connected space is connected; the converse does not hold.*

Proof; main part. Let $\mathcal{X}$ be a path-connected space.

By Proposition 9.5, the unit interval $[0,1]$ is connected. By Proposition 9.2, for any path $f\colon [0,1] \to \mathcal{X}$ the image $f([0,1])$ is connected.

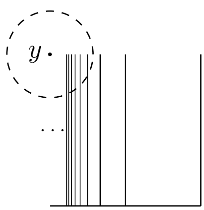
Fix $x \in \mathcal{X}$. Since $\mathcal{X}$ is path-connected,

$$\mathcal{X} = \bigcup_f f([0,1]),$$

where the union is taken for all paths f starting from x. By Proposition 9.4, $\mathcal{X}$ is connected.

Example. It remains to present an example of a space that connected, but not path-connected.

Consider the following segments in the coordinate plane I from $(0, 0)$ to $(1, 0)$ and J_n from $(\frac{1}{n}, 0)$ to $(\frac{1}{n}, 1)$ for each positive integer n. Consider the union of all these segments and the point $y = (0, 1)$



$$W = \{y\} \cup I \cup J_1 \cup J_2 \cup \dots$$

The space W is called flea and comb The union of line segments $I \cup J_1 \cup J_2 \cup \dots$ is called comb, and the point y is called flea.

Let us show that subspace W is not path-connected. Assume the contrary. Let f be a path from $x = (0, 0)$ to $y = (0, 1)$.

Note that $f^{-1}(\{y\})$ is closed subset of compact space $[0, 1]$. Therefore $f^{-1}(\{y\})$ is compact. In particular, the set $f^{-1}(\{y\})$ has the minimal element, denote it by b. Note that $b > 0$; so $f(b) = y$ and $f(t) \neq y$ for any $t < b$.

Choose positive $\varepsilon < 1$. Since f is continuous, there is $a < b$ such that $|f(t) - y| < \varepsilon$ for any $t \in (a, b]$. Note that $f(a) \in J_n$ for some n. Denote by N the intersection of ε-neighborhood of y with the comb. note that the intersection of J_n with ε-neighborhood of y forms a connected component of N. It follows that $f(t) \in J_n$ for any $t \in [a, b]$; in particular, $f(b) \neq y$ — a contradiction. □

9.14. Exericise. *Show that any convex set in a Euclidean space is path-connected.*

F Operations on paths

Given a path $f\colon [0, 1] \to \mathcal{X}$ one can consider the time-reversed path $\bar{f}$. Namely,

$$\bar{f}(t) = f(1 - t).$$

Note that $\bar{f}$ is continuous since f is.

Let $f, h\colon [0, 1] \to \mathcal{X}$ be two paths in the topological space $\mathcal{X}$. If $f(1) = h(0)$ we can join these two paths into one $g\colon [0, 1] \to \mathcal{X}$ defined as

$$g(t) = \left[\begin{array}{rl} f(2{\cdot}t) & \text{if } t \leqslant \frac{1}{2} \\ h(2{\cdot}t - 1) & \text{if } t \geqslant \frac{1}{2} \end{array}\right.$$

The path g is called the product (or concatenation) of paths f and h, briefly it is denoted as $g = f * h$.

In order to show that $f * h$ is indeed a path, we need to check that the defined map $f * h\colon [0,1] \to \mathcal{X}$ is continuous.

Indeed, fix a closed set $C \subset \mathcal{X}$, assume $E = (f * h)^{-1}(C) \subset [0,1]$. Since f is continuous, we get that $E \cap [0, \tfrac{1}{2}]$ is closed. The same way, since h is continuous, we get that $E \cap [\tfrac{1}{2}, 1]$ is closed. Since

$$E = (E \cap [0, \tfrac{1}{2}]) \cup (E \cap [\tfrac{1}{2}, 1]),$$

it follows that E is a closed subset in $[0,1]$. That is, the inverse image of any closed set in $\mathcal{X}$ is closed in $[0,1]$ — by Proposition 3.2, $f * h\colon [0,1] \to \mathcal{X}$ is continuous.

Consider the relation $\sim$ on the set of points of topological space defined as $x \sim y$ if there is a path from x to y.

9.15. Exercise. *Show that $\sim$ is an equivalence relation; that is, for any points x, y, and z in a topological space we have*

(a) $x \sim x$.

(b) *If* $x \sim y$, *then* $y \sim x$

(c) *If* $x \sim y$ *and* $y \sim z$, *then* $x \sim z$.

The equivalence class of point x for the equivalence relation $\sim$ is called path-connected component of x.

9.16. Exercise. *Assume every path-connected component in a topological space $\mathcal{X}$ is closed. Show that $\mathcal{X}$ is connected if and only if $\mathcal{X}$ is path-connected.*

9.17. Exercise. *Show that image of path-connected set under a continuous map is path-connected.*

9.18. Exercise. *Show that the product of path-connected spaces is path-connected.*

G Open sets of Euclidean space

The following theorem provides a class of topological space for which connectedness implies path-connectedness.

9.19. Theorem. *An open set in a Euclidean space $\mathbb{R}^n$ is path-connected if and only if it is connected.*

Proof. The only-if part follows from 9.13; it remains to prove the if part.

Let $\Omega \subset \mathbb{R}^n$ be an open subset. Choose a point $p \in \Omega$; denote by $P \subset \Omega$ the path-connected component of p.

Let us show that for any point $q \in \Omega$ there is $\varepsilon > 0$ such that either $\mathrm{B}(q, \varepsilon) \subset P$, or $\mathrm{B}(q, \varepsilon) \cap P = \varnothing$.

Indeed, since Ω is open, we can choose $\varepsilon > 0$ such that $\mathrm{B}(q, \varepsilon) \subset$ $\subset \Omega$. Note that $\mathrm{B}(q, \varepsilon)$ is convex. Therefore if $\mathrm{B}(q, \varepsilon) \cap P = \varnothing$, then $\mathrm{B}(q, \varepsilon) \subset P$.

It follows that P and its complement $\Omega \backslash P$ are open. Since Ω is connected, we get that $\Omega \backslash P = \varnothing$ — hence the result. $\square$

Lecture 10

Homotopy

A Definitions

Assume $f_t\colon \mathcal{X} \to \mathcal{Y}$ be a one-parameter family of maps, $t \in [0,1]$. If the map $[0,1] \times \mathcal{X} \to \mathcal{Y}$ that is defined as $(t,x) \mapsto f_t(x)$ is continuous, then f_t is called a homotopy of maps from $\mathcal{X}$ to $\mathcal{Y}$.

If $f_t(a)$ is independent of t for any point a in some subset $A \subset \mathcal{X}$, then we say that f_t is a homotopy relative to A.

Two maps $g, h\colon \mathcal{X} \to \mathcal{Y}$ are called homotopic (briefly $g \sim h$) if there is a homotopy $f_t\colon \mathcal{X} \to \mathcal{Y}$ such that $g = f_0$ and $h = f_1$.

Two maps $g, h\colon \mathcal{X} \to \mathcal{Y}$ are called homotopic relative to A (briefly, $g \sim h$ (rel. A)) if there is a homotopy $f_t\colon \mathcal{X} \to \mathcal{Y}$ relative to A such that $g = f_0$ and $h = f_1$.

10.1. Exercise. *Show that "$\sim$" defines equivalence relations on the set of continuous maps between given topological spaces. That is, for any continuous maps $f, g, h\colon \mathcal{X} \to \mathcal{Y}$, we have*

- $f \sim f$,
- *if* $f \sim h$ *then* $h \sim f$, *and*
- *if* $f \sim g$ *and* $g \sim h$ *then* $f \sim h$.

Do the same for and "$\sim$ (rel. A)".

B Contractible spaces

A topological space $\mathcal{X}$ is called contactable if the identity map $\mathrm{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ is homotopic to a constant map; that is there is a homotopy $h_t\colon \mathcal{X} \to \mathcal{X}$ such that $h_0(x) = x$ and $h_1(x) = p$ for some fixed point $p \in \mathcal{X}$ and any t.

10.2. Exercise. *Show that any convex subset of the Euclidean space is contractible.*

10.3. Exercise. *Show that any contractible space is path-connected.*

Later we will show that the circle $\mathbb{S}^1$ is an example of path-connected space that is not contractible.

10.4. Exercise. *Let $\mathcal{X}$ be a contractible space.*

(a) Show that any two continuous maps from a topological space to $\mathcal{X}$ are homotopic.

(b) Show that any two continuous maps from $\mathcal{X}$ to a path-connected space are homotopic.

C Homotopy type

Two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ have the same homotopy type (briefly $\mathcal{X} \sim \mathcal{Y}$) if there are continuous maps $f\colon \mathcal{X} \to \mathcal{Y}$ and $h\colon \mathcal{Y} \to \mathcal{X}$ such that $h \circ f \sim \mathrm{id}_{\mathcal{X}}$ and $f \circ h \sim \mathrm{id}_{\mathcal{Y}}$.

10.5. Exercise. *Show that a topological $\mathcal{X}$ is contractible if and only if it has the same homotopy type with the one-point space.*

10.6. Exercise. *Suppose A is a strong deformation retract of a space $\mathcal{X}$. Show that A and $\mathcal{X}$ have the same homotopy type.*

10.7. Exercise. *Show that the following two closed connected sets in the plane have the same homotopy type. (Each set is a union of line segments.)*

D Homotopy of paths

Suppose that $f_t\colon [0,1] \to \mathcal{X}$ is homotopy of paths relative to its ends; that is, $f_t(0)$ and $f_t(1)$ do not depend on t. Then we say that f_0 is homotopic to f_1, briefly $f_0 \sim f_1$. This is shortcut notation — formally speaking, we had to say homotopic rel. $\{0,1\}$ and write $f_0 \sim f_1$ (rel. $\{0,1\}$),

Since $\sim$ is an equivalence relation (10.1), we can talk about the equivalence class of a path f that will be called its homotopy class; it will be denoted by $[f]$.

E Technical claims

Each of the following claims proved by explicit construction of the needed homotopy. Each time the homotopy constructed as a composition $h \circ s_\tau(t)$, where h is a fixed path and s_τ is a one-parameter family of functions $[0,1] \to [0,1]$. The graphs of s_τ provide more intuitive descriptions of the families; the formulas presented just to make it formally correct.

10.8. Claim. *Suppose f_0 is a path from p to q, and g_0 is a path from q to r. Suppose $f_0 \sim f_1$ and $g_0 \sim g_1$, then*

$$f_0 * g_0 \sim f_1 * g_1.$$

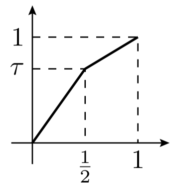
Proof. Choose homotopies f_t from f_0 to f_1, and g_t from g_0 to g_1 and observe that $f_t * g_t$ is a homotopy from $f_0 * g_0$ to $f_1 * g_1$. □

Let us denote by ε_p the constant loop with image p; that is, $\varepsilon_p(t) = p$ for any t.

10.9. Claim. *Suppose f is a path from p to q, then*

$$\varepsilon_p * f \sim f * \varepsilon_q \sim f.$$

Proof. Consider the function



$$s_\tau(t) = \begin{cases} 2{\cdot}\tau{\cdot}t & \text{if } t \leqslant \tfrac12, \\ 2{\cdot}\tau - 1 + 2{\cdot}(1-\tau){\cdot}t & \text{if } t \geqslant \tfrac12. \end{cases}$$

Observe that

$$\begin{aligned} f(s_0(t)) &= \varepsilon_p * f(t), \\ f(s_{\frac12}(t)) &= f(t), \\ f(s_1(t)) &= f(t) * \varepsilon_q \end{aligned}$$

for any t. Since $h_\tau(t) = f(s_\tau(t))$ is a homotopy, the claim follows. □

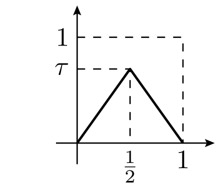
10.10. Exercise. *Suppose that f is a path in a Hausdorff space. Assume $f(0) = p$ and $\varepsilon_p * f = f$. Show that $f = \varepsilon_p$; in particular, $p = q$.*

10.11. Claim. *Suppose f is a path from p to q, then*

$$f * \bar{f} \sim \varepsilon_p \quad \textit{and} \quad \bar{f} * f \sim \varepsilon_q.$$

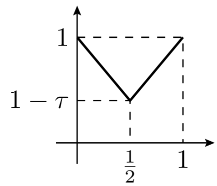
Proof. Consider the function

$$s_\tau(t) = \begin{cases} 2{\cdot}\tau{\cdot}t & \text{if } t \leqslant \tfrac{1}{2}, \\ 1 - 2{\cdot}\tau{\cdot}t & \text{if } t \geqslant \tfrac{1}{2}. \end{cases}$$



Observe that $f(s_1(t)) = f*\bar{f}(t)$ for any t. Therefore $h_\tau(t) = f(s_\tau(t))$ is a homotopy from ε_p to $f * \bar{f}$. It proves the first statement.

To prove the second statement one has to redefine s_τ as



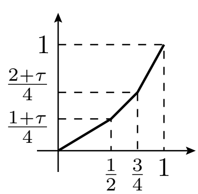
$$s_\tau(t) = \begin{cases} 1 - 2{\cdot}\tau{\cdot}t & \text{if } t \leqslant \tfrac{1}{2}, \\ 1 - 2{\cdot}\tau + 2{\cdot}\tau{\cdot}t & \text{if } t \geqslant \tfrac{1}{2}, \end{cases}$$

and observe that $f(s_1(t)) = \bar{f} * f(t)$ for any t. □

10.12. Claim. *Suppose f, g, and h are paths such that $f(1) = g(0)$ and $g(1) = h(0)$. Then*

$$(f * g) * h \sim f * (g * h).$$

Proof. Consider the function s_τ defined by



$$s_\tau(t) = \begin{cases} \frac{1+\tau}{2}{\cdot}t & \text{if } t \leqslant \tfrac{1}{2}, \\ \frac{\tau-1}{4} + t & \text{if } \tfrac{3}{4} \geqslant t \geqslant \tfrac{1}{2}, \\ \tau - 1 + (2-\tau){\cdot}t & \text{if } t \geqslant \tfrac{3}{4}. \end{cases}$$

Note that $s_1(t) = t$, and therefore

$$f * (g * h)(t) = f * (g * h)(s_1(t))$$

for any t. Further,

$$(f * g) * h(t) = f * (g * h)(s_0(t))$$

for any t. It remains to observe that $f * (g * h)(s_\tau(t))$ is the needed homotopy. $\square$

10.13. Exercise. *Let f and g be paths from p to q. Show that $f \sim g$ if and only if $f * \bar{g} \sim \varepsilon_p$.*

10.14. Advanced exercise. *Let f, g, and h be paths in a Hausdorff space. Suppose that $(f*g)*h = f*(g*h)$ and both sides of the equation are defined. Show that $f = g = h = \varepsilon_p$ for some point p.*

Lecture 11

Fundamental group

A Definition

Let $\mathcal{X}$ be a topological space. A path $f\colon [0,1] \to \mathcal{X}$ is called a loop with base at $p \in \mathcal{X}$ if $f(0) = f(1) = p$;

Note that if f and g are loops based at p, then their products $f * g$, $g * f$ are defined and they are loops based at p as well; see Section 9F. Moreover, the time-reversed paths $\bar{f}$, $\bar{g}$ are also loops based at p.

Recall that $[f]$ denotes the homotopy class of f; for paths and loops homotopy always means homotopy relative to the ends. The multiplication of homotopy classes of loops based at p is defined by

$$[f]\cdot[g] = [f * g];$$

that is, *the product of homotopy classes of loops f and g is the homotopy class of the product $f * g$.*

Observe that the product is well defined; that is, if $[f_0] = [f_1]$ and $[g_0] = [g_1]$, then $[f_0 * g_0] = [f_1 * g_1]$. In other words, if $f_0 \sim f_1$ and $g_0 \sim g_1$ then $f_0 * g_0 \sim f_1 * g_1$. The latter is stated in Claim 10.8.

Denote by $\pi_1(\mathcal{X}, p)$ the set of all homotopy classes of loops at p.

11.1. Theorem. *$\pi_1(\mathcal{X}, p)$ with the introduced multiplication is a group.*

The group $\pi_1(\mathcal{X}, p)$ is called the fundamental group of $\mathcal{X}$ with base point p.

Proof. Recall that ε_p denotes the constant loop at p in $\mathcal{X}$; that is, $\varepsilon_p(t) = p$ for any t. We will show that the homotopy class $[\varepsilon_p]$ is the neutral element of $\pi_1(\mathcal{X}, p)$ and $[\bar{f}] = [f]^{-1}$, where $\bar{f}$ denoted the time-reversed f.

Note that conditions in the definition of group follow from the next three conditions for any loops f, g, and h based at p in $\mathcal{X}$.

(i) $f * \varepsilon_p \sim \varepsilon_p * f \sim f$;

(ii) $f * \bar{f} \sim \bar{f} * f \sim \varepsilon_p$;

(iii) $(f * g) * h \sim f * (g * h)$.

These statements are provided by 10.9, 10.11, and 10.12. ☐

11.2. Exercise. *Suppose that V and W are open subsets of topological space $\mathcal{X}$ such that $\mathcal{X} = V \cup W$, and the set $V \cap W$ is path-connected. Let $p \in V \cap W$. Show that any loop in $\mathcal{X}$ based at p is homotopic to a product of loops in V or W with the same base.*

B Induced homomorphism

Let $\varphi\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map; suppose $\varphi(p) = q$. If f is a loop based at p in $\mathcal{X}$, then $\varphi \circ f$ is a loop based at q in $\mathcal{Y}$.

The following claim implies that the map $f \mapsto \varphi \circ f$ induces a homomorphism

$$\varphi_*\colon [f] \mapsto [\varphi \circ f].$$

11.3. Claim. *Let $\varphi\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map and $\varphi(p) = q$. Suppose that f_0 and f_1 are loops bases at p in $\mathcal{X}$. Then*

(a) $\varphi \circ (f_0 * f_1) = (\varphi \circ f_0) * (\varphi \circ f_1)$,

(b) if $f_0 \sim f_1$, then $\varphi \circ f_0 \sim \varphi \circ f_1$.

Proof; (a). Applying the definition of the product of paths and composition of maps to $\varphi \circ (f_0 * f_1)$ and $(\varphi \circ f_0) * (\varphi \circ f_1)$ we get exactly the same expression:

$$\begin{cases} \varphi \circ f_0(t) & \text{if } t \leqslant \frac{1}{2}, \\ \varphi \circ f_1(t) & \text{if } t \geqslant \frac{1}{2}. \end{cases}$$

Hence *(a)* follows.

(b). Observe that if f_t is a homotopy from f_0 to f_1, then $\varphi \circ f_t$ is a homotopy from $\varphi \circ f_0$ to $\varphi \circ f_1$. Hence *(b)* follows. ☐

11.4. Exercise. *Consider continuous maps $\mathcal{X} \xrightarrow{\varphi} \mathcal{Y} \xrightarrow{\psi} \mathcal{Z}$ between topological spaces. Show that $\psi_* \circ \varphi_* = (\psi \circ \varphi)_*$.*

C Dependence on base point

11.5. Theorem. *Let p and q be two points in a topological space $\mathcal{X}$. Suppose there is a path h from p to q, then the fundamental groups $\pi_1(\mathcal{X}, p)$ and $\pi_1(\mathcal{X}, q)$ are isomorphic.*

Proof. Suppose f is a loop based at p. Note that $\bar{h} * (f * h)$ is a loop at q. Moreover, the map $f \mapsto \bar{h} * (f * h)$ induces a homomorphism $u_h \colon \pi_1(M, p) \to \pi_1(M, q)$.

Indeed, suppose f_t is a homotopy of loops at p. Then $\bar{h} * (f_t * h)$ is a homotopy of loops at q. It follows that the map

$$u_h \colon [f] \mapsto [\bar{h} * (f * h)]$$

is defined; that is, the right-hand side does not depend on the choice of loop f in the homotopy class $[f]$.

Further, if f and g are loops based at p, then 10.9, 10.11, and 10.12 imply that

$$\begin{aligned}(\bar{h} * (f * h)) * (\bar{h} * (g * h)) &\sim \bar{h} * (((f * (h * \bar{h})) * g) * h) \sim \\ &\sim \bar{h} * (((f * \varepsilon_p) * g) * h) \sim \\ &\sim \bar{h} * ((f * g) * h).\end{aligned}$$

Whence the map $u_h \colon \pi_1(M, p) \to \pi_1(M, q)$ is a homomorphism; that is,

$$u_h([f]\cdot[g]) = u_h[f]\cdot u_h[g] \quad \text{for any} \quad [f], [g] \in \pi_1(\mathcal{X}, p).$$

The same argument shows that $u_{\bar{h}} \colon \pi_1(M, q) \to \pi_1(M, p)$ defined by

$$u_{\bar{h}} \colon [k] \mapsto [h * (k * \bar{h})]$$

is a homomorphism. Note that

$$\begin{aligned}h * ((\bar{h} * (f * h)) * \bar{h}) &\sim (h * \bar{h}) * (f * (h * \bar{h})) \sim \\ &\sim \varepsilon_p * (f * \varepsilon_p) \sim \\ &\sim f\end{aligned}$$

for any loop f based at p. Therefore, $u_{\bar{h}}$ is inverse of u_h. The same way we show that u_h is the inverse of $u_{\bar{h}}$. It follows that u_h is an isomorphism. □

According to the theorem, the fundamental group (more precisely its *isomorphism class*) of path-connected space does not depend on its base point. Therefore, for a path-connected space $\mathcal{X}$ we do not need to specify the base point of its fundamental group; so we could write $\pi_1(\mathcal{X})$ instead of $\pi_1(\mathcal{X}, p)$.

11.6. Exercise. *Let $\varphi_t \colon \mathcal{X} \to \mathcal{Y}$ be a homotopy. Suppose that $q_0 = \varphi_0(p)$ and $q_1 = \varphi_1(p)$; consider the path from q_0 to q_1 defined by $h(t) = \varphi_t(p)$. Show that*

$$u_h \circ \varphi_{0*} = \varphi_{1*}.$$

11.7. Exercise. *Suppose that path-connected topological spaces $\mathcal{X}$ and $\mathcal{Y}$ have the same homotopy type. Use 11.6 to show that their fundamental groups are isomorphic.*

11.8. Exercise. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two path-connected topological spaces. Choose points $p \in \mathcal{X}$ and $q \in \mathcal{Y}$. Consider the projections $\varphi \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ and $\psi \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ and their induced homomorphisms $\varphi_* \colon \pi_1(\mathcal{X} \times \mathcal{Y}, (p, q)) \to \pi_1(\mathcal{X}, p)$ and $\psi_* \colon \pi_1(\mathcal{X} \times \mathcal{Y}, (p, q)) \to \pi_1(\mathcal{Y}, q)$. Define $\Phi \colon \pi_1(\mathcal{X} \times \mathcal{Y}, (p, q)) \to \pi_1(\mathcal{X}, p) \times \pi_1(\mathcal{Y}, q)$ by*

$$\Phi \colon \alpha \mapsto (\varphi_*(\alpha), \psi_*(\alpha))$$

for any $\alpha \in \pi_1(\mathcal{X} \times \mathcal{Y}, (p, q))$. Note that the map Φ is a homomorphism.

(a) Show that Φ is a monomorphism; that is, if $\Phi(\alpha) = \Phi(\beta)$ for some $\alpha, \beta \in \pi_1(\mathcal{X} \times \mathcal{Y}, (p, q))$, then $\alpha = \beta$.

(b) Show that Φ is an epimorphism; that is, for any $\gamma \in \pi_1(\mathcal{X}, p) \times \times \pi_1(\mathcal{Y}, q)$ there is $\alpha \in \pi_1(\mathcal{X} \times \mathcal{Y}, (p, q))$ such that $\Phi(\alpha) = \gamma$.

Conclude that $\pi_1(\mathcal{X} \times \mathcal{Y})$ is isomorphic to $\pi_1(\mathcal{X}) \times \pi_1(\mathcal{Y})$.

D Simply-connected spaces

Recall that a group is called trivial if it contains only one element which is necessary the neutral element.

A path connected topological space with trivial fundamental group is called simply-connected.

If the fundamental group $\pi_1(\mathcal{X}, p)$ is trivial, it is common to write $\pi_1(\mathcal{X}, p) = 0$ despite that this equality does not have much sense — in general the group $\pi_1(\mathcal{X}, p)$ is not commutative and so it would be more reasonable to write $\pi_1(\mathcal{X}, p) = \{1\}$, meaning that 1 is the only element of $\pi_1(\mathcal{X}, p)$.

11.9. Exercise. *Suppose that V and W are open simply-connected subsets of topological space $\mathcal{X}$ such that $\mathcal{X} = V \cup W$, and the set $V \cap W$ is path-connected. Show that $\mathcal{X}$ is simply-connected.*

Conclude that sphere $\mathbb{S}^2$ is simply-connected.

Appendix A

Solutions of some exercises

7.13(c). *Prove that the graph of the function* $f\colon [0,1] \to \mathbb{R}$ *is compact if and only if* f *is continuous. Give an example of a discontinuous function* $g : [0,1] \to \mathbb{R}$ *with a graph that is closed but not compact.*

Solution. Denote the graph of f by Γ.

If part. Assume f is continuous.
Note that $F\colon x \mapsto (x, f(x))$ is a continuous function $[0,1] \to \mathbb{R}^2$. Since $[0,1]$ is compact, so is its F-image which is Γ.

Only-if part. Assume Γ is compact. By Theorem 3.3 (page 17 in Kosniowski's book), it is sufficient to show that $f^{-1}(Q)$ is closed for any closed $Q \subset \mathbb{R}$. Note that $P = (\mathbb{R} \times Q) \cap \Gamma$ is closed. Since Γ is compact, so it P.
Observe that $f^{-1}(Q)$ is the projection of P to the first coordinate. Since P is compact, the same holds for its image $f^{-1}(Q)$. Finally, since $f^{-1}(Q)$ compact subset of real line, it is has to be closed (7.7).

Example.

$$g(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Appendix B

Jordan curve theorem

B.1. Theorem. *Suppose $\Gamma \subset \mathbb{R}^2$ is a closed set homeomorphic to $\mathbb{R}$. Then $\mathbb{R}^2 \backslash \Gamma$ has at least two connected components.*

Note that the assumption that Γ is closed is necessary; indeed a finite open interval I of a line in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}$, but its complement $\mathbb{R}^2 \backslash I$ is connected.

The theorem follows from B.2, B.3, and B.5.

B.2. Proposition. *Suppose $\Gamma \subset \mathbb{R}^2$ is a closed set such that the complement $X = \mathbb{R}^2 \backslash \Gamma$ is connected. Let us identify $\mathbb{R}^2$ with the (x, y)-plane in $\mathbb{R}^3$. Then the complement $Y = \mathbb{R}^3 \backslash \Gamma$ is simply-connected.*

Proof. Denote by A (respectively B) the sets that include Γ and the points below (respectively above) Γ; that is,

$$\begin{aligned} A &= \{\, (x, y, z) \mid (x, y) \in \Gamma \text{ and } z \leqslant 0 \,\}, \\ B &= \{\, (x, y, z) \mid (x, y) \in \Gamma \text{ and } z \geqslant 0 \,\}. \end{aligned}$$

Consider their complements $V = \mathbb{R}^3 \backslash A$ and $W = \mathbb{R}^3 \backslash B$. Note that $Y = V \cup W$.

The sets V and W are simply-connected. Indeed, the horizontal plane $z = 1$ is a deformation retract of V; a retraction can be defined by $(x, y, z) \mapsto (x, y, 1)$ and the following homotopy shows that it is homotopic to the identity map:

$$h_t(x, y, z) = (x, y, (1-t)\cdot z + t).$$

The plane is contractible, in particular simply-connected; therefore so is V. Similarly one proves that W is simply-connected.

Since X is open and connected set in $\mathbb{R}^2$. By 9.19, X is path-connected. Further, note that $V \cup W = X \times \mathbb{R}$. Therefore $V \cup W$ is path-connected as well.

Summarizing, V and W are open simply-connected sets, $Y = V \cup \cup W$, and $V \cap W$ is path-connected. Applying 11.9, we get that Y is simply-connected. ☐

B.3. Proposition. *Suppose $\Gamma \subset \mathbb{R}^2$ is a closed subset homeomorphic to $\mathbb{R}$. Then there is a homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ that maps Γ the z-axis.*

B.4. Technical lemma. *Suppose $\Gamma \subset \mathbb{R}^2$ is a closed subset homeomorphic to $\mathbb{R}$ and $h\colon \mathbb{R} \to \Gamma$ is a homeomorphism. Then there is a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f \circ h(t) = t$ for any $t \in \mathbb{R}$.*

This lemma follows directly from the so-called Tietze–Urysohn extension theorem, but we sketch a more elementary proof.

Sketch of proof. Consider the function $\Phi\colon \mathbb{R} \times [0, \infty) \to [0, \infty)$ defined by

$$\Phi(t, r) := \sup_{s \in \mathbb{R}} \{\, |s - t| \cdot (1 + r - |h(s) - h(t)|) \,\}.$$

Note that Φ is continuous.

Moreover, if $r \geqslant |h(t) - h(s)|$, then

$$|s - t| \leqslant \Phi(t, r)$$

for any $s, t \in \mathbb{R}$. It follows that $f \circ h(t) = t$ where

$$f(p) := \sup_{t \in \mathbb{R}} \{\, t - \Phi(t, |p - h(t)|) \,\}.$$

It remains to observe that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous. ☐

The following proof uses the so-called Klee trick which is quite useful in many topological problems.

Proof of B.3. Let $h\colon t \mapsto (a(t), b(t))$ be a homeomorphism $\mathbb{R} \to \Gamma$. By B.4, there is a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(a(t), b(t)) = f \circ h(t) = t$$

for any $t \in \mathbb{R}$.

Note that the map

$$F\colon (x, y, z) \mapsto (x, y, z + f(x, y))$$

is a homeomorphism. Indeed, this map is continuous and its inverse

$$F^{-1}\colon (x,y,z)\mapsto (x,y,z-f(x,y))$$

is continuous as well.

Similarly, the map

$$G\colon (x,y,z)\mapsto (x-a(z),y-b(z),z)$$

is a homeomorphism as well. Indeed, G is continuous and it has inverse

$$G^{-1}\colon (x,y,z)\mapsto (x+a(z),y+b(z),z)$$

that is continuous as well.

It follows that the composition $G\circ F\colon \mathbb{R}^3\to\mathbb{R}^3$ is a homeomorphism. Since $f(a(t),b(t))=t$,

$$G\circ F(a(t),b(t),0)=G(a(t),b(t),t)=(0,0,t).$$

It follows that $G\circ F$ sends Γ to the z-axis as required. $\square$

B.5. Exercise. *Show that the complement of the z-axis in $\mathbb{R}^3$ is not simply-connected.*

B.6. Theorem. *Let $J\subset\mathbb{S}^2$ be a subset homeomorphic to $\mathbb{S}^1$. Then $\mathbb{S}^2\backslash J$ has at least two connected components.*

This theorem is a partial case of famous Jordan's theorem; it is known for simple formulation and annoyingly tricky proofs.

Proof. Remove a point p from J to get a closed line $\Gamma=J\backslash\{p\}$ in $\mathbb{S}^2\backslash\{p\}\simeq\mathbb{R}^2$. It remains to apply B.1. $\square$