Introduction to topology

MATH 429, Spring 2022

Anton Petrunin

Contents

Lecture 1

Metric spaces

In this chapter we discuss *metric spaces* — a motivating example that will guide us toward the definition of *topological spaces* — the main object of topology.

Examples of metric spaces were considered for thousands of years, but the first general definition was given only in 1906 by Maurice Fréchet.

A Definition

In the following definition we grab together the most important properties of the intuitive notion of *distance*.

1.1. Definition. *Let $\mathcal{X}$ be a nonempty set with a function that returns a real number, denoted as $|x-y|$, for any pair $x, y \in \mathcal{X}$. Assume that the following conditions are satisfied for any $x, y, z \in \mathcal{X}$:*

(a) Positiveness:

$$|x - y| \geqslant 0.$$

(b) Identity of indiscernibles:

$$x = y \quad \textit{if and only if} \quad |x - y| = 0.$$

(c) Symmetry:

$$|x - y| = |y - x|.$$

(d) Triangle inequality:

$$|x - y| + |y - z| \geqslant |x - z|.$$

In this case, we say that $\mathcal{X}$ is a metric space *and the function*

$$(x, y) \mapsto |x - y|$$

is called a metric*.*

The elements of $\mathcal{X}$ are called points *of the metric space. Given two points $x, y \in \mathcal{X}$, the value $|x - y|$ is called* distance *from x to y.*

Note that for two points in a metric space the difference between points $x - y$ may have no meaning, but $|x - y|$ always has the meaning defined above.

Typically, we consider only one metric on set, but if few metrics are needed, we can distinguish them by an index, say $|x - y|_\bullet$ or $|x - y|_{239}$. If we need to emphasize that the distance is taken in the metric space $\mathcal{X}$ we write $|x - y|_\mathcal{X}$ instead of $|x - y|$.

B Examples

Let us give a few examples of metric spaces.

- *Discrete space.* Let $\mathcal{X}$ be an arbitrary set. For any $x, y \in \mathcal{X}$, set $|x - y| = 0$ if $x = y$ and $|x - y| = 1$ otherwise. This metric is called *discrete metric* on $\mathcal{X}$ and the obtained metric space is called *discrete*.

- *Real line.* Set of all real numbers ($\mathbb{R}$) with metric defined as $|x - y|$.

- *Metrics on the plane.* Let us denote by $\mathbb{R}^2$ the set of all pairs (x_1, y_2) of real numbers. Consider two points $p_1 = (x_1, y_2)$ and $p_2 = (x_2, y_2)$ in $\mathbb{R}^2$. One can equip $\mathbb{R}^2$ with the following metrics:

  - *Euclidean metric,* denoted by

  $$|p_1 - p_2|_2 = \sqrt{(x_1 - x_2)^2 + (y_2 - y_2)^2}.$$

  - *Manhattan metric,* denoted by $|* - *|_1$ and defined as

  $$|p_1 - p_2|_1 = |x_1 - x_2| + |y_2 - y_2|.$$

  - *Maximum metric,* denoted by $|* - *|_\infty$ and defined as

  $$|p_1 - p_2|_\infty = \max\{|x_1 - x_2|, |y_2 - y_2|\}.$$

1.2. Exercise. *Prove that (a)* $|{*}-{*}|_2$*; (b)* $|{*}-{*}|_1$ *and (c)* $|{*}-{*}|_\infty$ *are metrics on* $\mathbb{R}^2$*.*

1.3. Exercise. *Show that*

$$|x-y|_\natural = (x-y)^2$$

is not a metric on $\mathbb{R}$*.*

1.4. Exercise. *Show that if* $(x,y)\mapsto|x-y|$ *is a metric then so is*

$$(x,y)\mapsto|x-y|_{\max}=\max\{1,|x-y|\}.$$

C Continuous maps

Recall that a real-to-real function f is called *continuous* if for any $x\in\mathbb{R}$ and any $\varepsilon>0$ there exists $\delta>0$ such that $|f(x)-f(y)|<\varepsilon$, whenever $|x-y|<\delta$.

This definition can be used for the functions defined on Euclidean space if $|x-y|$ denotes the Euclidean distance $|x-y|_2$ between the points x and y. It admits the following straightforward generalization to *metric spaces*:

1.5. Definition. *A function* $f\colon\mathcal{X}\to\mathcal{Y}$ *between metric spaces is called* continuous *if for any* $x\in\mathcal{X}$ *and any* $\varepsilon>0$ *there exists* $\delta>0$ *such that* $|f(x)-f(y)|_{\mathcal{Y}}<\varepsilon$*, for any* $y\in\mathcal{X}$ *such that* $|x-y|_{\mathcal{X}}<\delta$*.*

1.6. Exercise. *Let* $\mathcal{X}$ *be a metric space and* $z\in\mathcal{X}$ *be a fixed point. Show that the function*

$$f(x)\stackrel{\text{def}}{=\!=}|x-z|$$

is continuous.

1.7. Exercise. *Let* $\mathcal{X}$*,* $\mathcal{Y}$ *and* $\mathcal{Z}$ *be metric spaces. Assume that the functions* $f\colon\mathcal{X}\to\mathcal{Y}$ *and* $g\colon\mathcal{Y}\to\mathcal{Z}$ *are continuous, and*

$$h=g\circ f\colon\mathcal{X}\to\mathcal{Z}$$

is its composition; that is, $h(x)=g(f(x))$ *for any* $x\in\mathcal{X}$*. Show that* $h\colon\mathcal{X}\to\mathcal{Z}$ *is continuous at any point.*

1.8. Exercise. *Show that any distance-preserving map is continuous.*

More precisely, assume that $f\colon \mathcal{X} \to \mathcal{Y}$ is a map between metric space such that

$$|x - x'|_{\mathcal{X}} = |f(x) - f(x')|_{\mathcal{Y}}$$

for any $x, x' \in \mathcal{X}$ then f is continuous.

1.9. Exercise. *Let $\mathcal{X}$ be a discrete metric space (defined in 1B) and $\mathcal{Y}$ be arbitrary metric space. Show that for any function $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous.*

D Balls

Let $\mathcal{X}$ be a metric space, x is a point in $\mathcal{X}$ and r is a positive real number. The set of points in $\mathcal{X}$ which lies on the distance smaller than r is called *ball of radius r centered at x*. It is denoted as $\mathrm{B}(x, r)$ or $\mathrm{B}(x, r)_{\mathcal{X}}$ if we need to emphasize that it is taken in the space $\mathcal{X}$.

The ball $\mathrm{B}(x, r)$ is also called r-neighborhood of x.

1.10. Exercise. *Sketch the unit balls for the metrics $|{*} - {*}|_1$, $|{*} - {*}|_2$ and $|{*} - {*}|_\infty$ defined right before Exercise 1.2.*

1.11. Exercise. *Assume $\mathrm{B}(x, r)$ and $\mathrm{B}(y, R)$ is a pair of balls in a metric space and $\mathrm{B}(x, r) \subsetneq \mathrm{B}(y, R)$. Show that $r < 2 \cdot R$.*

Give an example of a metric space and a pair of balls as above such that $r > R$.

Let us reformulate the definition of continuous map (1.5) using the introduced notion of ball.

1.12. Definition. *A function $f\colon \mathcal{X} \to \mathcal{Y}$ between metric spaces is called* continuous *if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$f(\mathrm{B}(x, \delta)_{\mathcal{X}}) \subset \mathrm{B}(f(x), \varepsilon)_{\mathcal{Y}}.$$

1.13. Exercise. *Prove the equivalence of definitions 1.5 and 1.12.*

E Open sets

1.14. Definition. *A subset V in a metric space $\mathcal{X}$ is called* open *if for any $x \in V$ there is $\varepsilon > 0$ such that $\mathrm{B}(x, \varepsilon) \subset V$.*

In other words, V is open if, together with each point, V contains its ε-neighborhood for some $\varepsilon > 0$. For example, any set in a discrete metric space is open since together with any point it contains its 1-neighborhood. Further the set of positive real numbers

$$(0, \infty) = \{\, x \in \mathbb{R} \mid x > 0 \,\}$$

is open since together with each point $x > 0$ it contains its x-neighborhood. On the other hand, the set of nonnegative reals

$$[0, \infty) = \{\, x \in \mathbb{R} \mid x \geqslant 0 \,\}$$

is not open since there are negative numbers in any neighborhood of 0.

1.15. Exercise. *Show that any ball in a metric space is open.*

1.16. Exercise. *Show that any open set in a metric space is a union of balls.*

1.17. Exercise. *Show that the union of an arbitrary collection of open sets is open.*

1.18. Exercise. *Show that the intersection of two open sets is open.*

1.19. Exercise. *Give an example of metric space $\mathcal{X}$ and an infinite sequence of open sets $V_1, V_2, \dots$ such that the intersection*

$$\bigcap_n V_n$$

is not open.

1.20. Exercise. *Show that the metrics $|{*}-{*}|_1$, $|{*}-{*}|_2$ and $|{*}-{*}|_\infty$ (defined in 1B) give rise to the same open sets in $\mathbb{R}^2$. That is, if $V \subset \mathbb{R}^2$ is open for one of these metrics then it is open for the others.*

F Gateway to topology

The following result is the main gateway to topology. It says that continuous maps can be defined entirely in terms of open sets.

1.21. Proposition. *A function $f\colon \mathcal{X} \to \mathcal{Y}$ between two metric spaces is continuous if and only if for any open set $W \subset \mathcal{Y}$ its inverse images*

$$f^{-1}(W) = \{\, x \in \mathcal{X} \mid f(x) \in W \,\}$$

is open.

Note that proposition says nothing about the images of open sets. In fact, before going into proof it would be useful to solve the following exercise.

1.22. Exercise. *Give an example of a continuous $f\colon \mathbb{R}\to\mathbb{R}$ and an open set $V\subset\mathbb{R}$ such that the image $f(V)\subset\mathbb{R}$ is not open.*

The formulation of the proposition contains "if and only if" and the proof breaks into two parts "if"-part and "only if"-part.

Proof; "only-if" part. Let $W\subset\mathcal{Y}$ be an open set and $V=f^{-1}(W)$. Choose $x\in V$; note that so $f(x)\in W$.

Since W is open,

$$\text{❶}\qquad \mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}\subset W$$

for some $\varepsilon>0$.

Since f is continuous, by Definition 1.12, there is $\delta>0$ such that

$$f(B(x,\delta)_{\mathcal{X}})\subset \mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}.$$

It follows that together with any point $x\in V$, the set V contains $B(x,\delta)$; that is, V is open.

"If" part. Fix $x\in\mathcal{X}$ and $\varepsilon>0$. According to Exercise 1.15,

$$W=\mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}$$

is an open set in $\mathcal{Y}$. Therefore its inverse image $f^{-1}(W)$ is open.

Clearly $x\in f^{-1}(W)$. By the definition of open set (1.14)

$$\mathrm{B}(x,\delta)_{\mathcal{X}}\subset f^{-1}(W)$$

for some $\delta>0$. Or equivalently

$$f(\mathrm{B}(x,\delta)_{\mathcal{X}})\subset W=\mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}.$$

Hence the "if"-part follows. □

G Limits

1.23. Definition. *Let $(x_n)=x_1,x_2,\dots$ be a sequence of points in a metric space $\mathcal{X}$. We say the sequence x_n converges to a point $x\in\mathcal{X}$ if*

$$|x-x_n|_{\mathcal{X}}\to 0\quad \textit{as}\quad n\to\infty.$$

In this case, we say that the sequence (x_n) *is a converging sequence and* x *is its limit; the latter will be written as*

$$x = \lim_{n\to\infty} x_n$$

Note that we defined the convergence of points in a metric space using the convergence of real numbers $d_n = |x - x_n|_{\mathcal{X}}$, which we assume to be known.

1.24. Exercise. *Let* $f\colon \mathcal{X} \to \mathcal{Y}$ *be a function between metric spaces. Show that* f *is continuous if and only if for any converging sequence* (x_n) *in* $\mathcal{X}$ *the sequence* $y_n = f(x_n)$ *is converging in* $\mathcal{Y}$ *and*

$$f(x_\infty) = y_\infty,$$

if $x_n \to x_\infty$ *and* $y_n \to y_\infty$ *as* $n \to \infty$.

H Closed sets

Let A be a set in a metric space $\mathcal{X}$. A point $x \in \mathcal{X}$ is a *limit point* of A if there is a sequence $x_n \in A$ such that $x_n \to x$ as $n \to \infty$.

The set of all limit points of A is called the *closure* of A and denoted as $\bar{A}$. Note that $\bar{A} \supset A$; indeed, any point $x \in A$ is a limit point of the constant sequence $x_n = x$.

If $\bar{A} = A$ then the set is called *closed*.

1.25. Exercise. *Show that closure of any set in metric space is a closed set; that is,* $\bar{\bar{A}} = \bar{A}$.

1.26. Exercise. *Show that a subset* A *in a metric space* $\mathcal{X}$ *is closed if and only if its complement* $\mathcal{X} \backslash A$ *is open.*

Lecture 2

Topological spaces

In the previous chapter we defined open sets in metric spaces and showed that continuity could be defined using only the notion of open sets. Now we will state the most important properties of these open sets as axioms. It will give us a definition of *topological space* as a set with a distinguished class of subsets called *open sets*.

The *topological properties* are loosely defined as properties which survive under arbitrary continuous deformation. They were studied since 19th century. The first definition of topological spaces was given by Felix Hausdorff in 1914. In 1922, the definition was generalized slightly by Kazimierz Kuratowski; his definition is given below.

A Definitions

We are about to define *abstract open sets* without referring to metric spaces. The exercises 1.17 and 1.18 motivate this definition.

2.1. Definition. *Suppose $\mathcal{X}$ is a set with a distinguished type of subsets, called* open sets *such that*

(i) The empty set and $\mathcal{X}$ are open.

(ii) The union of any collection of open sets is an open set. That is, if V_α is open for any α the index set $\mathcal{I}$ then the set

$$W = \bigcup_{\alpha \in \mathcal{I}} V_\alpha = \{\, x \in \mathcal{X} \mid x \in V_\alpha \quad \textit{for some} \quad \alpha \,\}$$

is open

(iii) The intersection of two open sets is an open set. That is, if V_1 and V_2 are open, then the intersection $W = V_1 \cap V_2$ is open.

In this case, $\mathcal{X}$ is called topological space*.*

The collection of all open sets in $\mathcal{X}$ is called a topology *on $\mathcal{X}$ and denoted as $\mathcal{O}_{\mathcal{X}}$; so instead of saying V* is an open set in the topological space $\mathcal{X}$*, we might write $V \in \mathcal{O}_{\mathcal{X}}$.*

From (iii) it follows that the intersection of a finite collection of open sets is open. That is, if V_1, V_2, ..., V_n are open then the intersection

$$W = V_1 \cap V_2 \cap \dots \cap V_n$$

is open. This can be proved by applying induction on n since

$$V_1 \cap \dots \cap V_{n-1} \cap V_n = (V_1 \cap \dots V_{n-1}) \cap V_n.$$

B Examples

The so-called *connected two-point space* is a simple but nontrivial example of topological space. This space consists of two points

$$\mathcal{X} = \{a, b\}$$

and it has three open sets:

$$\varnothing, \quad \{a\} \quad \text{and} \quad \{a, b\}.$$

It is instructive to check that this is indeed a topology.

Further, for any set $\mathcal{X}$, we can always define the following topologies:

- The discrete topology — the topology consisting of all subsets of a set $\mathcal{X}$.
- The concrete topology — the topology consisting of just whole space $\mathcal{X}$ and the empty set, $\varnothing$.
- The cofinite topology — the topology consisting of the empty set, $\varnothing$ and the complements to finite sets.

2.2. Exercise. *Show that $\varnothing$, $\mathbb{R}$ and the intervals $[a, \infty)$, (a, ∞) for all $a \in \mathbb{R}$ define a topology on the real line $\mathbb{R}$. (The obtained space will be denoted by $\mathbb{R}_{\geqslant}$.)*

Let $\mathscr{W}$ and $\mathscr{S}$ be two topologies on one set. Suppose $\mathscr{W} \subset \mathscr{S}$; that is, any open set in $\mathscr{W}$-topology is open in $\mathscr{S}$-topology. In this case we say that $\mathscr{W}$ is weaker than $\mathscr{S}$, or, equivalently, $\mathscr{S}$ is stronger than $\mathscr{W}$.

2.3. Exercise. *Let $\mathscr{W}$ and $\mathscr{S}$ be two topologies on one set. Suppose that for any point x and any $S \in \mathscr{S}$ such that $S \ni x$, there is $W \in \mathscr{W}$ such that $S \supset W \ni x$. Show that $\mathscr{W}$ is weaker than $\mathscr{S}$.*

C Continuous maps

The following definition is motivated by Proposition 1.21.

2.4. Definition. *A function between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called* continuous *if for any open set W in $\mathcal{Y}$, its inverse image $f^{-1}(W)$ is open in $\mathcal{X}$. That is, if W is an open subset in $\mathcal{Y}$ then the set*

$$V = f^{-1}(W) = \{\, x \in X \mid f(x) \in W \,\}$$

is an open subset $\mathcal{X}$

2.5. Exercise. *Let $\mathbb{R}$ be the real line with the standard topology and $\mathcal{X}$ be the connected two-point space described above.*

(a) Construct a nonconstant continuous function $\mathbb{R} \to \mathcal{X}$

(b) Show that any continuous function $\mathcal{X} \to \mathbb{R}$ is constant.

Recall that $\mathbb{R}_{\geqslant}$ be the real line with topology defined in 2.2.

2.6. Exercise. *Show that a function $f\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing if and only if it defines a continuous map $\mathbb{R}_{\geqslant} \to \mathbb{R}_{\geqslant}$.*

D Metrizable spaces

According to exercises 1.17 and 1.18 any metric space is a topological space if one defines open sets as in the definition 1.14. As it follows from Exercise 1.20, different metrics might define the same topology.

The topological spaces which can be obtained this way are called *metrizable*. For example, discrete topology is the topology induced by the discrete metric (defined in 1B). Some examples of topological spaces mentioned above are not metrizable.

2.7. Exercise. *Assume $\mathcal{X}$ is a concrete space containing at least two points. Show that $\mathcal{X}$ is not metrizable.*

2.8. Exercise. *Assume an infinite set $\mathcal{X}$ equipped with the cofinite topology. Show that $\mathcal{X}$ is not metrizable.*

2.9. Exercise. *Show that $\mathbb{R}_{\geqslant}$ is not metrizable; it is defined in 2.2.*

2.10. Exercise. *Show that finite topological space is metrizable if and only if it is discrete. In particular, connected two-point space is not metrizable.*