Introduction to topology

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Here is very preliminary noncomplete lectures from my introductory course in topology at PennState. I used several sources, inducing notes of Allen Hatcher [1], lectures of Sergei Ivanov [2], the textbook of Czes Kosniowski [3], and the textbook of Oleg Viro, Oleg Ivanov, Nikita Netsvetaev, and Viatcheslav Kharlamov [4].

Hope that someone will find it useful for something.

Anton Petrunin

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Chapter 1

Metric spaces

In this chapter we discuss *metric spaces* — a motivating example that will guide us toward the definition of our main object of study — *topological spaces.*

Examples of metric spaces were considered for thousands of years, but the first general definition was given only in 1906 by Maurice Fréchet [5].

A Definition

In the following definition we grab together the most important properties of the intuitive notion of *distance.*

1.1. Definition. *Let $\mathcal{X}$ be a set with a function that returns a real number, denoted as $|x-y|$, for any pair $x, y \in \mathcal{X}$. Assume that the following conditions are satisfied for any $x, y, z \in \mathcal{X}$:*

(a) $|x-y| \geqslant 0.$

(b) $x = y$ *if and only if* $|x-y| = 0.$

(c) $|x-y| = |y-x|.$

(d) $|x-y| + |y-z| \geqslant |x-z|$*; this property is called the* triangle inequality*.*

In this case, we say that $\mathcal{X}$ is a metric space *and the function*

$$(x, y) \mapsto |x-y|$$

is called a m e t r i c.

The elements of $\mathcal{X}$ are called p o i n t s *of the metric space. Given two points $x, y \in \mathcal{X}$, the value $|x-y|$ is called* d i s t a n c e *from x to y.*

Note that for two points in a metric space the difference between points $x-y$ may have no meaning, but $|x-y|$ always has the meaning defined above.

Typically, we consider only one metric on set, but if few metrics are needed, we can distinguish them by an index, say $|x-y|_\bullet$ or $|x-y|_{239}$. If we need to emphasize that the distance is taken in the metric space $\mathcal{X}$ we write $|x-y|_{\mathcal{X}}$ instead of $|x-y|$.

1.2. Exercise. *Prove that (a) $|{*}-{*}|_1$; (b) $|{*}-{*}|_2$ and (c) $|{*}-{*}|_\infty$ are metrics on $\mathbb{R}^2$.*

1.3. Exercise. *Show that*

$$|x-y|_\natural = (x-y)^2$$

is not a metric on $\mathbb{R}$.

1.4. Exercise. *Show that if $(x,y) \mapsto |x-y|$ is a metric, then so is*

$$(x,y) \mapsto |x-y|_{\max} = \max\{1, |x-y|\}.$$

B Examples

Let us give a few examples of metric spaces.

- D i s c r e t e s p a c e. Let $\mathcal{X}$ be an arbitrary set. For any $x, y \in \mathcal{X}$, set $|x-y| = 0$ if $x = y$ and $|x-y| = 1$ otherwise. This metric is called d i s c r e t e m e t r i c on $\mathcal{X}$ and the obtained metric space is called d i s c r e t e.

- R e a l l i n e. The set $\mathbb{R}$ of all real numbers with metric defined as $|x-y|$. (Unless it is stated othewise, the real line $\mathbb{R}$ will be considered with this metric.)

- M e t r i c s o n t h e p l a n e. Let us denote by $\mathbb{R}^2$ the set of all pairs (x,y) of real numbers. Consider two points $p = (x_p, y_p)$ and $q = (x_q, y_q)$ in $\mathbb{R}^2$. One can equip $\mathbb{R}^2$ with the following metrics:

– Euclidean metric,

$$|p - q|_2 = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}.$$

(Unless it is stated othewise, the real line $\mathbb{R}^2$ will be considered with the Euclidean metric.)

– Manhattan metric,

$$|p - q|_1 = |x_p - x_q| + |y_p - y_q|.$$

– Maximum metric,

$$|p - q|_\infty = \max\{|x_p - x_q|, |y_p - y_q|\}.$$

C Subspaces

Any subset $\mathcal{A}$ of metric space $\mathcal{X}$ forms a metric space on its own; it is called subspace of $\mathcal{X}$. This construction produces many more examples of metric spaces. For example, the disc

$$\mathbb{D}^2 = \{\, (x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1 \,\}$$

and the circle

$$\mathbb{S}^1 = \{\, (x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1 \,\},$$

are metric spaces with metrics taken from the Euclidean plane. Similarly, the interval $[0, 1)$ is a metric space with metric taken from $\mathbb{R}$.

D Continuous maps

Recall that a real-to-real function f is called *continuous* if *for any $x \in \mathbb{R}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$, whenever $|x - y| < \delta$.* It admits the following straightforward generalization to metric spaces:

1.5. Definition. *A map $f\colon \mathcal{X} \to \mathcal{Y}$ between metric spaces is called continuous if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)|_{\mathcal{Y}} < \varepsilon$, for any $y \in \mathcal{X}$ such that $|x - y|_{\mathcal{X}} < \delta$.*

1.6. Exercise. *Let $\mathcal{X}$ be a metric space and $z \in \mathcal{X}$ be a fixed point. Show that the function*

$$f(x) := |x - z|_{\mathcal{X}}$$

is continuous.

1.7. Exercise. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be metric spaces. Assume that the maps $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ are continuous, and*

$$h = g \circ f\colon \mathcal{X} \to \mathcal{Z}$$

is their composition; that is, $h(x) = g(f(x))$ for any $x \in \mathcal{X}$. Show that $h\colon \mathcal{X} \to \mathcal{Z}$ is continuous at any point.

1.8. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a distance-preserving map between metric spaces; that is,*

$$|x - x'|_{\mathcal{X}} = |f(x) - f(x')|_{\mathcal{Y}}$$

for any $x, x' \in \mathcal{X}$.

(a) *Show that f is continuous.*

(b) *Show that f is injective; that is, if $x \neq x'$ then $f(x) \neq f(x')$.*

1.9. Exercise. *Let $\mathcal{X}$ be a discrete metric space (defined in 1B) and $\mathcal{Y}$ be arbitrary metric space. Show that for map $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous.*

1.10. Advanced exercise. *Construct a continuous function*

$$f\colon [0,1] \to [0,1]$$

that takes every value in $[0,1]$ an infinite number of times.

E Balls

Let x be a point in a metric space $\mathcal{X}$, and $r > 0$. The set of points in $\mathcal{X}$ which lies on the distance smaller than r is called open ball of radius r centered at x. It is denoted as $\mathrm{B}(x,r)$ or $\mathrm{B}(x,r)_{\mathcal{X}}$; the latter notation is used if we need to emphasize that it is taken in the space $\mathcal{X}$.[1]

The ball $\mathrm{B}(x,r)$ is also called r-neighborhood of x.

Analogously we may define closed balls

$$\bar{\mathrm{B}}[x,r] = \bar{\mathrm{B}}[x,r]_{\mathcal{X}} = \{\, y \in \mathcal{X} \mid |x - y| \leqslant r \,\}.$$

[1] Many authors use the notations $\mathrm{B}_r(x)$ and $\mathrm{B}_r(x)_{\mathcal{X}}$ as well.

1.11. Exercise. *Sketch the unit balls for the metrics* $|*-*|_1$, $|*-*|_2$ *and* $|*-*|_\infty$ *defined in 1B.*

1.12. Exercise. *Assume* $\mathrm{B}(x,r)$ *and* $\mathrm{B}(y,R)$ *is a pair of balls in a metric space and* $\mathrm{B}(x,r) \subsetneq \mathrm{B}(y,R)$. *Show that* $r < 2{\cdot}R$.

Give an example of a metric space and a pair of balls as above such that $r > R$.

Let us reformulate the definition of continuous map (1.5) using the introduced notion of ball.

1.13. Definition. *A map* $f\colon \mathcal{X} \to \mathcal{Y}$ *between metric spaces is called continuous if for any* $x \in \mathcal{X}$ *and any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$f(\mathrm{B}(x,\delta)_{\mathcal{X}}) \subset \mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}.$$

1.14. Exercise. *Prove the equivalence of definitions 1.5 and 1.13.*

F Open sets

1.15. Definition. *A subset* V *in a metric space* $\mathcal{X}$ *is called open if for any* $x \in V$ *there is* $\varepsilon > 0$ *such that* $\mathrm{B}(x,\varepsilon) \subset V$.

In other words, V is open if, together with each point, V contains its ε-neighborhood for some $\varepsilon > 0$. For example, any set in a discrete metric space is open since together with any point it contains its 1-neighborhood. Further the set of positive real numbers

$$(0,\infty) = \{\, x \in \mathbb{R} \,|\, x > 0 \,\}$$

is open subset of $\mathbb{R}$; indeed, for any $x > 0$ its x-neighborhood lies in $(0,\infty)$. On the other hand, the set of nonnegative reals

$$[0,\infty) = \{\, x \in \mathbb{R} \,|\, x \geqslant 0 \,\}$$

is not open since there are negative numbers in any neighborhood of 0.

1.16. Exercise. *Show that any open ball in a metric space is open.*[2]

1.17. Exercise. *Show that the union of an arbitrary collection of open sets is open.*

[2]In other words, show that for any $y \in \mathrm{B}(x,r)$ there is $\varepsilon > 0$ such that $\mathrm{B}(y,\varepsilon) \subset$ $\subset \mathrm{B}(x,r)$.

1.18. Exercise. *Show that the intersection of two open sets is open.*

1.19. Exercise. *Show that a set in a metric space is open if and only if it is a union of balls.*

1.20. Exercise. *Give an example of metric space $\mathcal{X}$ and an infinite sequence of open sets $V_1, V_2, \ldots$ such that their intersection $V_1 \cap V_2 \cap \ldots$ is not open.*

1.21. Exercise. *Show that the metrics $|*-*|_1$, $|*-*|_2$ and $|*-*|_\infty$ (defined in 1B) give rise to the same open sets in $\mathbb{R}^2$. That is, if $V \subset \mathbb{R}^2$ is open for one of these metrics, then it is open for the others.*

G Gateway to topology

The following result is the main gateway to topology. It says that continuous maps can be defined entirely in terms of open sets.

1.22. Proposition. *A map $f\colon \mathcal{X} \to \mathcal{Y}$ between two metric spaces is continuous if and only if inverse image of any open set is open; that is, for any open set $W \subset \mathcal{Y}$ its inverse image*

$$f^{-1}(W) = \{\, x \in \mathcal{X} \mid f(x) \in W \,\}$$

is open.

The following exercise emphasizes that the proposition says nothing about the images of open sets; it is instructive to solve it before going into the proof (see also 4B).

1.23. Exercise. *Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and an open set $V \subset \mathbb{R}$ such that the image $f(V) \subset \mathbb{R}$ is not open.*

Proof; only-if part. Let $W \subset \mathcal{Y}$ be an open set and $V = f^{-1}(W)$. Choose $x \in V$; note that so $f(x) \in W$.

Since W is open,

$$\text{❶} \qquad \mathrm{B}(f(x), \varepsilon)_{\mathcal{Y}} \subset W$$

for some $\varepsilon > 0$.

Since f is continuous, by Definition 1.13, there is $\delta > 0$ such that

$$f(\mathrm{B}(x, \delta)_{\mathcal{X}}) \subset \mathrm{B}(f(x), \varepsilon)_{\mathcal{Y}}.$$

It follows that together with any point $x \in V$, the set V contains $\mathrm{B}(x, \delta)$; that is, V is open.

If part. Fix $x \in \mathcal{X}$ and $\varepsilon > 0$. According to Exercise 1.16,

$$W = \mathrm{B}(f(x), \varepsilon)_{\mathcal{Y}}$$

is an open set in $\mathcal{Y}$. Therefore its inverse image $f^{-1}(W)$ is open.

Clearly $x \in f^{-1}(W)$. By the definition of open set (1.15)

$$\mathrm{B}(x, \delta)_{\mathcal{X}} \subset f^{-1}(W)$$

for some $\delta > 0$. Or equivalently

$$f(\mathrm{B}(x, \delta)_{\mathcal{X}}) \subset W = \mathrm{B}(f(x), \varepsilon)_{\mathcal{Y}}.$$

Hence the if part follows. □

H Limits

1.24. Definition. *Let $x_1, x_2, \dots$ be a sequence of points in a metric space $\mathcal{X}$. We say that the sequence $x_1, x_2, \dots$ converges to a point $x_\infty \in \mathcal{X}$ if*

$$|x_\infty - x_n|_{\mathcal{X}} \to 0 \quad \text{as} \quad n \to \infty.$$

In this case, we say that the sequence $x_1, x_2, \dots$ is converging and x_∞ is its limit; it can be expressed by $x_n \to x_\infty$ as $n \to \infty$ or

$$x_\infty = \lim_{n \to \infty} x_n.$$

Note that we defined the convergence of points in a metric space using the convergence of real numbers $d_n = |x_\infty - x_n|_{\mathcal{X}}$, which we assume to be known.

1.25. Exercise. *Show that any sequence of points in a metric space has at most one limit.*

1.26. Exercise. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a map between metric spaces. Show that f is continuous if and only if the following condition holds:*

- *If $x_n \to x_\infty$ as $n \to \infty$ in $\mathcal{X}$, then the sequence $y_n = f(x_n)$ converges to $y_\infty = f(x_\infty)$ as $n \to \infty$ in $\mathcal{Y}$.*

I Closed sets

Let A be a set in a metric space $\mathcal{X}$. A point $x \in \mathcal{X}$ is a limit point of A if there is a sequence $x_n \in A$ such that $x_n \to x$ as $n \to \infty$.[3]

The set of all limit points of A is called the closure of A and denoted as $\bar{A}$. Note that $\bar{A} \supset A$; indeed, any point $x \in A$ is a limit point of the constant sequence $x_n = x$.

If $\bar{A} = A$, then the set A is called closed.

1.27. Exercise. *Give an example of a subset $A \subset \mathbb{R}$ that is neither closed nor open.*

1.28. Exercise. *Show that closure of any set in metric space is a closed set; that is, $\bar{\bar{A}} = \bar{A}$.*

1.29. Exercise. *Show that a subset Q in a metric space $\mathcal{X}$ is closed if and only if its complement $V = \mathcal{X} \setminus Q$ is open.*

[3]Sometimes limit points are defined, assuming in addition that $x_n \neq x$ for any n — we do *not* follow this convention.

Chapter 2

Topological spaces

The topological properties are loosely defined as properties which survive under arbitrary continuous deformation. They were studied since 19th century. The first definition of topological spaces was given by Felix Hausdorff in 1914. In 1922, the definition was generalized slightly by Kazimierz Kuratowski; his definition is given below.

Recall that in the previous chapter we defined open sets in metric spaces and showed that continuity could be defined using only the notion of open sets.

In this chapter we collect key properties of open sets and state them as axioms. It will give us a definition of *topological space* as a set with a distinguished class of subsets called *open sets.*

A Definitions

We are about to define *abstract open sets* without referring to metric spaces; this definition is bases on two properties in exercises 1.17 and 1.18.

2.1. Definition. *Suppose $\mathcal{X}$ is a set with a distinguished class of subsets, called open sets such that*

(a) The empty set $\varnothing$ and the whole $\mathcal{X}$ are open.

(b) The union of any collection of open sets is an open set. That is, if V_α is open for any α the index set $\mathcal{I}$, then the set

$$W = \bigcup_{\alpha\in\mathcal{I}} V_\alpha = \{\, x \in \mathcal{X} \mid x \in V_\alpha \quad \textit{for some} \quad \alpha \in \mathcal{I} \,\}$$

is open.

(c) The intersection of two open sets is an open set. That is, if V_1 and V_2 are open, then the intersection $W = V_1 \cap V_2$ is open.

In this case, $\mathcal{X}$ is called topological space. The collection of all open sets in $\mathcal{X}$ is called a topology on $\mathcal{X}$.

Usually we consider set with one topology, therefore it is acceptable to use the same notation for the set and the corresponding topological space. Rarely we will need to consider different topologies, say $\mathscr{T}_1$ and $\mathscr{T}_2$, on the same set $\mathcal{X}$; in this case, the corresponding topological spaces will be denoted by $(\mathcal{X}, \mathscr{T}_1)$ and $(\mathcal{X}, \mathscr{T}_2)$.

From (2.1c) it follows that the intersection of a finite collection of open sets is open. That is, if V_1, V_2, $\dots, V_n$ are open, then the intersection

$$W = V_1 \cap \dots \cap V_n$$

is open. The latter is proved by induction on n using the identity

$$V_1 \cap \dots \cap V_n = (V_1 \cap \dots \cap V_{n-1}) \cap V_n.$$

B Examples

According to exercises 1.17 and 1.18 any metric space is a topological space if one defines open sets as in the definition 1.15. For example, the real line $\mathbb{R}$ comes with natural metric which defines a topology on $\mathbb{R}$; if not stated otherwise, the real line $\mathbb{R}$ will be considered with this topology.

As it follows from Exercise 1.21, different metrics on one set might define the same topology.

A topological space is called metrizable if its topology can be defined by a metric — these examples are most important.

The so-called connected two-point space is a simple but non-trivial example of topological space. This space consists of two points

$$\mathcal{X} = \{a, b\}$$

and it has three open sets:

$$\varnothing, \quad \{a\} \quad \text{and} \quad \{a, b\}.$$

It is instructive to check that this is indeed a topology.

2.2. Exercise. *Show that finite topological space is metrizable if and only if it is discrete. In particular, connected two-point space is not metrizable.*

For any set $\mathcal{X}$, we can define the following topologies:

- The *discrete topology* — the topology consisting of all subsets of a set $\mathcal{X}$.
- The *concrete topology* — the topology consisting of just the whole set $\mathcal{X}$ and the empty set, $\varnothing$.
- The *cofinite topology* — the topology consisting of the empty set, $\varnothing$ and the complements to finite sets.

2.3. Exercise. *Assume an infinite set $\mathcal{X}$ equipped with the cofinite topology. Show that $\mathcal{X}$ is not metrizable.*

C Comparison of topologies

Let $\mathscr{W}$ and $\mathscr{S}$ be two topologies on one set. Suppose $\mathscr{W} \subset \mathscr{S}$; that is, any open set in $\mathscr{W}$-topology is open in $\mathscr{S}$-topology. In this case, we say that $\mathscr{W}$ is *weaker* than $\mathscr{S}$, or, equivalently, $\mathscr{S}$ is *stronger* than $\mathscr{W}$.

Note that on any set, the concrete topology is the weakest and discrete topology is the strongest.

2.4. Exercise. *Let $\mathscr{W}$ and $\mathscr{S}$ be two topologies on one set. Suppose that for any point x and any $W \in \mathscr{W}$ such that $W \ni x$, there is $S \in \mathscr{S}$ such that $W \supset S \ni x$. Show that $\mathscr{W}$ is weaker than $\mathscr{S}$.*

D Continuous maps

Our next challenge is to reformulate definitions from the previous chapter using only open sets. Continuous maps are first in the line. The following definition is motivated by Proposition 1.22.

2.5. Definition. *A map between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called continuous if the inverse image of any open set is open. That is, if W is an open subset in $\mathcal{Y}$, then its inverse image*

$$V = f^{-1}(W) = \{\, x \in X \mid f(x) \in W \,\}$$

is an open subset in $\mathcal{X}$

2.6. Exercise. *Let $\mathbb{R}$ be the real line with the standard topology and $\mathcal{X} = \{a, b\}$ be the connected two-point space described in 2B — it has only three open sets: $\varnothing$, $\{a\}$, and $\{a, b\}$.*

(a) *Construct a nonconstant continuous map $\mathbb{R} \to \mathcal{X}$.*

(b) *Show that any continuous function $\mathcal{X} \to \mathbb{R}$ is constant.*

2.7. Exercise. *Show that the composition of continuous maps is continuous.*

2.8. Exercise. *Let $\mathscr{T}$ be a collection of subsets in $\mathbb{R}$ that consists of $\varnothing$, $\mathbb{R}$ and the intervals $[a,\infty)$, (a,∞) for all $a \in \mathbb{R}$.*

(a) *Show that $\mathscr{T}$ is a topology on $\mathbb{R}$.*

(b) *Show that the topological space $(\mathbb{R}, \mathscr{T})$ is not metrizable.*

(c) *Show that a function $f\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing if and only if it defines a continuous map $(\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T})$.*

Chapter 3

Subsets

A Closed sets

The definitions of open and closed sets are mirror-symmetric to each other. There is no particular reason why we define topological space using open sets — we could use closed sets instead. In fact, closed sets were considered before open sets — the former were introdiced by Georg Cantor in 1884 [6], and the latter by René Baire in 1899 [7].

Let $\mathcal{X}$ be a topological space. A subset $K \subset \mathcal{X}$ is called closed if its complement $\mathcal{X} \setminus K$ is open.

Sometimes it is easier to use closed sets; for example, the cofinite topology can be defined by declaring that the whole space and all its finite sets are closed.

From the definition of topological spaces the following properties of closed sets follow.

3.1. Proposition. *Let $\mathcal{X}$ be a topological space.*

(a) The empty set and $\mathcal{X}$ are closed.

(b) The intersection of any collection of closed sets is a closed set. That is, if K_α is open for any α in the index set $\mathcal{I}$, then the set

$$Q = \bigcap_{\alpha \in \mathcal{I}} K_\alpha = \{\, x \in \mathcal{X} \mid x \in K_\alpha \quad \textit{for any} \quad \alpha \in \mathcal{I} \,\}$$

is closed.

(c) The union of two closed sets (or any finite collection of closed sets) is closed. That is, if K_1 and K_2 are closed, then the union $Q = K_1 \cup K_2$ is closed.

The following proposition is completely analogous to the original definition of continuous maps via open sets (2.5).

3.2. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. A map $f\colon \mathcal{X} \to \to \mathcal{Y}$ is continuous if and only if any closed set Q has closed inverse image $f^{-1}(Q)$.*

Proof. In the proof, we will use following set-theoretical identity. Suppose $A \subset \mathcal{Y}$ and $B = \mathcal{Y} \setminus A$ (or, equvalently, $A = \mathcal{Y} \setminus B$). Then

$$❶ \qquad f^{-1}(B) = \mathcal{X} \setminus f^{-1}(A)$$

for any map $f\colon \mathcal{X} \to \mathcal{Y}$. This identity is tautological, to prove it observe that both sides can be spelled as

$$\{\, x \in \mathcal{X} \mid f(x) \notin A \,\}.$$

Only-if part. Let $B \subset \mathcal{Y}$ be a closed set. Then $A = \mathcal{Y} \setminus B$ is open. Since f is continuous, $f^{-1}(A)$ is open. By ❶, $f^{-1}(B)$ is the complement of $f^{-1}(A)$ in $\mathcal{X}$. Hence $f^{-1}(B)$ is closed.

If part. Fix an open set B, its complement $A = \mathcal{Y} \setminus B$ is closed. Therefore $f^{-1}(A)$ is closed. By ❶, $f^{-1}(B)$ is a complement of $f^{-1}(A)$ in $\mathcal{X}$. Hence $f^{-1}(B)$ is open.

The statement follows since B is an arbitrary open set. □

3.3. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map between topological spaces. Assume $A, B \subset \mathcal{X}$ are closed sets such that $A \cup B = \mathcal{X}$. Show that f is continuous if and only if so are the restrictions $f|_A$ and $f|_B$.*

B Interior and closure

Let A be an arbitrary subset in a topological space $\mathcal{X}$.

The union of all open subsets of A is called the interior of A and denoted as $\mathring{A}$.

Note that $\mathring{A}$ is open. Indeed, it is defined as a union of open sets and such union has to be open by definition of topology (2.1). So we can say that $\mathring{A}$ is the *maximal* open set in A, as any open subset of A lies in $\mathring{A}$.

The intersection of all closed subsets containing A is called the closure of A and denoted as $\bar{A}$.

The set $\bar{A}$ is closed. Indeed, it is defined as an intersection of closed sets and such intersection has to be closed by 3.1. In other words, $\bar{A}$

is the *minimal* closed set that contains A, as any closed subset of A contains $\bar{A}$.

3.4. Exercise. *Assume A is a subset of a topological space $\mathcal{X}$; consider its complement $B = \mathcal{X} \setminus A$. Show that*

$$\bar{B} = \mathcal{X} \setminus \mathring{A}.$$

The following exercises are a based on the so-called *Kuratowski's problem.*

3.5. Exercise. *Show that the following holds for any set A of a topological space:*

$$\mathring{\mathring{A}} = \mathring{A} \subset A \subset \bar{A} = \bar{\bar{A}}$$

3.6. Exercise. *Let A and B be subsets of a topological space. Suppose that $A \subset B$. Show that*

$$\bar{A} \subset \bar{B} \quad \textit{and} \quad \mathring{A} \subset \mathring{B}.$$

3.7. Exercise. *Show that*

$$\bar{\mathring{Q}} \subset Q$$

for any closed set Q.

3.8. Exercise. *Show that*

$$\mathring{\bar{V}} \supset V$$

for any open set V.

3.9. Advanced exercise. *Give an example of a topological space $\mathcal{X}$ with a subset A such that all the following 7 subsets are distinct:*

$$\bar{\mathring{\bar{A}}},\ \mathring{\bar{A}},\ \bar{A},\ A,\ \mathring{A},\ \bar{\mathring{A}},\ \mathring{\bar{\mathring{A}}}.$$

Show that given a set A one can get at most 7 by repeatedly applying the set operations of closure and interior.

C Boundary

Let A be an arbitrary subset in a topological space $\mathcal{X}$. The boundary of A (briefly ∂A) is defined as the complement

$$\partial A = \bar{A} \setminus \mathring{A}.$$

3.10. Exercise. *Show that the boundary of any set is closed.*

3.11. Exercise. *Show that the set A is closed if and only if $\partial A \subset A$.*

3.12. Advanced exercise. *Find three disjoint open sets on the real line that have the same nonempty boundary.*

D Neighborhoods

Let x be a point in a topological space $\mathcal{X}$. A neighborhood of x is any open set N containing x. In topology, neighborhoods often replace the notion of balls (the latter can be used only in metric spaces).

3.13. Exercise. *Let A be a set in a topological space $\mathcal{X}$. Show that $x \in \partial A$ if and only if any neighborhood of x contains points in A and its complement $\mathcal{X} \setminus A$.*

Let A and B be subsets of a topological space $\mathcal{X}$. The set A is said to be dense in B if $\bar{A} \supset B$.

3.14. Exercise. *Show that A is dense in B if and only if any neighborhood of any point in B intersects A.*

3.15. Exercise. *Classify topological spaces (up to homeomorphism) containing a unique nowhere dense subset.*

E Limits

3.16. Definition. *Suppose x_n is a sequence of points in a topological space $\mathcal{X}$. We say that x_n converges to a point $x_\infty \in \mathcal{X}$ (briefly $x_n \to x_\infty$ as $n \to \infty$) if for any neighborhood N of x_∞, we have that $x_n \in N$ for all sufficiently large n.*

3.17. Exercise. *Prove that above definition agrees with 1.24. In other words, a sequence of points $x_1, x_2, \dots$ in a metric space converges* *to a point x_∞ in the sense of definition 1.24 if and only if it converges in the sense of definition 3.16.*

3.18. Exercise. *Show that in a space with concrete topology any sequence converges to any point. In particular, a sequence might have different limits.*

3.19. Exercise. *Show that a convergent sequence of points in a topological space is also convergent for every weaker topology.*

The following exercise shows that in general, converging sequences do *not* provide an adequate description of topology. In other words, an analog of 1.26 does not hold.(The so-called nets provide an appropriate generalization of sequences that works well in topological spaces, but we are not going to consider them in the sequel.)

Recall that a set is called countable if it admits a bijection to some subset of the set of natural numbers. In particular, all finite sets are countable.

3.20. Advanced exercise. *Let $\mathcal{X}$ be $\mathbb{R}$ with the so-called cocountable topology; its closed sets are either countable or the whole $\mathbb{R}$.*

(a) Construct a map $f\colon \mathcal{X} \to \mathcal{X}$ that is not continuous.

(b) Describe all converging sequences in $\mathcal{X}$.

(c) Show that if the sequence x_n converges to x_∞ in $\mathcal{X}$ then for any map $f\colon \mathcal{X} \to \mathcal{X}$ the sequence $y_n = f(x_n)$ converges to $y_\infty = f(x_\infty)$.

Chapter 4

Maps

A Homeomorphisms

4.1. Definition. *A bijection $f\colon \mathcal{X} \to \mathcal{Y}$ between topological spaces is called* h o m e o m o r p h i s m *if f and its inverse $f^{-1}\colon \mathcal{Y} \to \mathcal{X}$ are continuous.*[1]

Topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are called h o m e o m o r p h i c *(briefly, $\mathcal{X} \simeq \mathcal{Y}$) if there is a homeomorphism $f\colon \mathcal{X} \to \mathcal{Y}$.*

4.2. Exercise. *Show that any homomorphism is a continuous bijection.*

Give an example of continuous bijection between topological spaces that is not a homeomorphism.

4.3. Exercise. *Show that $x \mapsto e^x$ is a homeomorphism $\mathbb{R} \to (0, \infty)$.*

4.4. Exercise. *Construct a homeomorphism $f\colon \mathbb{R} \to (0, 1)$.*

4.5. Exercise. *Show that $\simeq$ is an equivalence relation; that is, for any topological spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ we have the following:*

(a) $\mathcal{X} \simeq \mathcal{X}$;

(b) *if* $\mathcal{X} \simeq \mathcal{Y}$, *then* $\mathcal{Y} \simeq \mathcal{X}$;

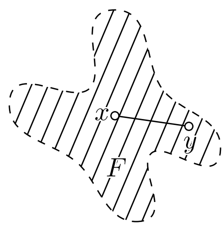
(c) *if* $\mathcal{X} \simeq \mathcal{Y}$ *and* $\mathcal{Y} \simeq \mathcal{Z}$, *then* $\mathcal{X} \simeq \mathcal{Z}$.

---

[1] The term *homomorphism* from abstract algebra looks similar and it has similar meaning but should not to be confused with *homeomorphism*.

4.6. Advanced exercise. *Prove that the complement of a circle in the Euclidean space is homeomorphic to the Euclidean space without line ℓ and a point $p \notin \ell$.*

Recall that a figure F is called s t a r - s h a p e d if there exists a point $p \in F$ such that for all $x \in F$ the line segment px lies in F.



4.7. Advanced exercise. *Any nonempty open star-shaped set in the plane is homeomorphic to the open disc.*

4.8. Advanced exercise. *Show that the complements of two countable dense subsets of the plane are homeomorphic.*

B Closed and open maps

4.9. Definition. *A map between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called* o p e n *if, for any open set $V \subset \mathcal{X}$, the image $f(V)$ is open in $\mathcal{Y}$.*

A map between topological spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is called c l o s e d *if, for any closed set $Q \subset \mathcal{X}$, the image $f(Q)$ is closed in $\mathcal{Y}$.*

Note that homeomorphism can be defined as a *continuous open bijection.*

4.10. Exercise. *Show that a bijective map between topological spaces is closed if and only if it is open.*

4.11. Exercise. *Give an example of a map $f\colon \mathcal{X} \to \mathcal{Y}$ between two topological spaces such that*

(a) f is continuous and open, but not closed,

(b) f is continuous and closed, but not open,

(c) f is closed and open, but not continuous.

4.12. Advanced exercise. *Construct a function $\mathbb{R} \to \mathbb{R}$ that is closed and open, but not continuous.*

Chapter 5

Constructions

In this chapter we will discuss a few constructions that produce new topological spaces from the given ones.

A Induced topology

5.1. Proposition. *Let A be a subset of a topological space $\mathcal{Y}$. Then all subsets $V \subset A$ such that $V = A \cap W$ for some open set W in $\mathcal{Y}$ form a topology on A.*

The described topology is called induced topology on A.

A subset A in a topological space $\mathcal{Y}$ equipped with the induced topology is called a subspace of $\mathcal{Y}$. It is straightforward to check that this notion agrees with the notion introduced in 1C; that is, if $\mathcal{Y}$ is a metric space, then any subset $A \subset \mathcal{Y}$ comes with metric and the topology defined by this metric coincides with the induced topology on A.

A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called embedding if f defines a homeomorphism from space $\mathcal{X}$ to the subspace $f(\mathcal{X})$ in $\mathcal{Y}$.

Proof. We need to check the conditions in 2.1.

First, the whole set A and the empty set are included; indeed, $\varnothing = A \cap \varnothing$ and $A = A \cap \mathcal{Y}$.

Assume $\{V_\alpha\}$ is a collection of open sets in A; here α runs in some index set, say $\mathcal{I}$. In other words, for each V_α there is an open set $W_\alpha \subset \mathcal{Y}$ such that $V_\alpha = A \cap W_\alpha$. Note that

$$\bigcup_\alpha V_\alpha = A \cap \left(\bigcup_\alpha W_\alpha\right).$$

Since the union of $\{W_\alpha\}$ is open in $\mathcal{Y}$ (2.1b), the union of $\{V_\alpha\}$ is open in the induced topology on A.

Assume V_1 and V_2 are open in A; that is, $V_1 = A \cap W_1$ and $V_2 = = A \cap W_2$ for some open sets $W_1, W_2 \subset \mathcal{Y}$. Note that

$$V_1 \cap V_2 = A \cap (W_1 \cap W_2).$$

Since the intersection $W_1 \cap W_2$ is open in $\mathcal{Y}$ (2.1c), the intersection $V_1 \cap V_2$ is open in A. $\square$

B Product space

Recall that $\mathcal{X} \times \mathcal{Y}$ denotes the set of all pairs (x, y) such that $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Suppose that the sets $\mathcal{X}$ and $\mathcal{Y}$ are equipped with topologies. Let us construct the product topology on $\mathcal{X} \times \mathcal{Y}$ by declaring that a set is open in $\mathcal{X} \times \mathcal{Y}$ if it can be presented as a union of sets of the following type: $V \times W$ for open sets $V \subset \mathcal{X}$ and $W \subset \mathcal{Y}$. In other words, a subset U is open in $\mathcal{X} \times \mathcal{Y}$ if and only if there are collections of open sets $V_\alpha \subset \mathcal{X}$ and $W_\alpha \subset \mathcal{Y}$ such that

$$U = \bigcup_\alpha V_\alpha \times W_\alpha,$$

here α runs in some index set.

By default, we assume that $\mathcal{X} \times \mathcal{Y}$ is equipped with the product topology; in this case $\mathcal{X} \times \mathcal{Y}$ is called product space.

5.2. Proposition. *The product topology is indeed a topology.*

Proof. Parts *(a)* and *(b)* in 2.1 are evident. It remains to check *(c)*. Consider two sets

$$U = \bigcup_\alpha V_\alpha \times W_\alpha \quad \text{and} \quad U' = \bigcup_\beta V'_\beta \times W'_\beta.$$

where α and β run in some index sets, say $\mathcal{I}$ and $\mathcal{J}$ respectively. We need to show that $U \cap U'$ can be presented as a union of products of open sets; the latter follows from the next set-theoretical identity

$$U \cap U' = \bigcup_{\alpha,\beta} (V_\alpha \cup V'_\beta) \times (W_\alpha \cup W'_\beta). \qquad ❶$$

Checking ❶ is straightforward. Indeed, $(x, y) \in U \cap U'$ means that $(x, y) \in U$ and $(x, y) \in U'$; the latter means that $x \in V_\alpha$, $y \in W_\alpha$ and

$x \in V'_\beta$, $y \in W'_\beta$ for *some* α and β. In other words, $x \in V_\alpha \cap V'_\beta$ and $y \in W_\alpha \cap W'_\beta$ for *some* α and β; the latter means that (x, y) belongs to the right-hand side in ❶. $\square$

Recall that an embedding is defined in 5A.

C Base

5.3. Definition. *A collection $\mathscr{B}$ of open sets in a topological space $\mathcal{X}$ is called its b a s e if every open set in $\mathcal{X}$ is a union of sets in $\mathscr{B}$.*

The definition is motivated by the fact that *open balls form a base of metric space* (1.19).

A base completely defines its topology, but typically a topology has many different bases. On metric spaces, for example, the set of all balls with rational radiuses is a base; another example is the set of all balls with radiuses smaller than 1.

In many cases, it is convenient to describe topology by specifying its base. For example, the product topology on $\mathcal{X} \times \mathcal{Y}$ can be redefined as a *topology with a base formed by all products $V \times W$, where V is open in $\mathcal{X}$, and W is open in $\mathcal{Y}$.*

5.4. Exercise. *Let $\mathscr{B}$ be a base for the topology on $\mathcal{Y}$. Show that a map $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if and only if $f^{-1}(B)$ is open for any set B in $\mathscr{B}$.*

5.5. Exercise. *Let $\mathscr{B}$ be a collection of open sets in a topological space $\mathcal{X}$. Show that $\mathscr{B}$ is a base in $\mathcal{X}$ if and only if any point $x \in \mathcal{X}$ and any neighborhood $N \ni x$ there is $B \in \mathscr{B}$ such that $x \in B \subset N$.*

5.6. Proposition. *Let $\mathscr{B}$ be a set of subsets in some set $\mathcal{X}$. Show that $\mathscr{B}$ is a base of some topology on $\mathcal{X}$ if and only if it satisfies the following conditions:*

(a) $\mathscr{B}$ c o v e r s $\mathcal{X}$; that is, every point $x \in \mathcal{X}$ lies in some set $B \in \mathscr{B}$.

(b) For each pair of sets $B_1, B_2 \in \mathscr{B}$ and each point $x \in B_1 \cap B_2$ there exists a set $B \in \mathscr{B}$ such that $x \in B \subset B_1 \cap B_2$.

Proof. Denote by $\mathscr{O}$ the set of all unions of sets in $\mathscr{B}$. We need to show that $\mathscr{O}$ is a topology on $\mathcal{X}$.

Evidently, the union of any collection of sets in $\mathscr{O}$ is in $\mathscr{O}$. Further, $\mathcal{X}$ is in $\mathscr{O}$ by *(a)*. The empty set is in $\mathscr{O}$ since it is a union of the empty collection.

It remins to check 2.1c; suppose

$$O = \bigcup_{\alpha} B_\alpha \quad \text{and} \quad O' = \bigcup_{\beta} B'_\beta,$$

where α and β run in some index sets, and B_α, $B'_\beta \in \mathscr{B}$ for any α and β. Then $x \in O \cap O'$ if and only if for some α and β we have $x \in B_\alpha$ and $x \in B'_\beta$. By *(b)*, we can choose $B \in \mathscr{B}$ so that $x \in B \subset B_\alpha \cap B'_\beta$. Since $B_\alpha \cap B'_\beta \subset O \cap O'$, it follows that

for any $x \in O \cap O'$ there is $B_x \in \mathscr{B}$ such that $x \in B \subset O \cap O'$.

Observe that

$$O \cap O' = \bigcup_{x \in O \cap O'} B_x.$$

It follows that $O \cap O' \in \mathscr{O}$ if $O, O' \in \mathscr{O}$. $\square$

D Prebase

Suppose $\mathscr{P}$ is a collection of subsets in $\mathcal{X}$ that covers the whole space; that is, $\mathcal{X}$ is a union of all sets in $\mathscr{P}$. By 5.6, the set of all finite intersections of sets in $\mathscr{P}$ is a base for *some* topology on $\mathcal{X}$. The set $\mathscr{P}$ is called prebase for this topology (also known as subbase).

5.7. Exercise. *Let $\mathscr{P}$ be a prebase for the topology on $\mathcal{Y}$. Show that a map $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if and only if $f^{-1}(P)$ is open for any set P in $\mathscr{P}$.*

There are almost no restrictions on prebase — we may start with any collection $\mathscr{P}$ of subsets of $\mathcal{X}$ that covers the whole $\mathcal{X}$ and define a topology by declaring that *$\mathscr{P}$ is a prebase for the topology.* It defines the weakest topology on $\mathcal{X}$ such that every set of $\mathscr{P}$ is open.

For example, the product topology on $\mathcal{X} \times \mathcal{Y}$ can be redefined as a *topology with prebase formed by all products $\mathcal{X} \times W$ and $V \times \mathcal{Y}$, where V is open in $\mathcal{X}$ and W is open in $\mathcal{Y}$.*

5.8. Exercise. *Given a map $f\colon \mathcal{X} \to \mathcal{Y}$, consider the map $F\colon \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ defined by $F\colon x \mapsto (x, f(x))$. Show that f is continuous if and only if F is an embedding.*

E Family of maps

Note that the product topology on $\mathcal{X} \times \mathcal{Y}$ is the weakest topology such that the following two projections are continuous: $\mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ and $\mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ defined by $(x, y) \mapsto x$ and $(x, y) \mapsto y$ respectively.

Indeed, these projections are continuous if the inverse images of all opens sets in $\mathcal{X}$ and $\mathcal{Y}$ are open in $\mathcal{X} \times \mathcal{Y}$. In other words the topology on $\mathcal{X} \times \mathcal{Y}$ must contain all sets of the form $V \times \mathcal{Y}$ and $\mathcal{X} \times W$ for open sets $V \subset \mathcal{X}$ and $W \subset \mathcal{Y}$. These sets form a prebase in $\mathcal{X} \times \mathcal{Y}$ and its topology is the weakest topology that contains these sets.

More generally, given a collection of maps $f_\alpha \colon S \to \mathcal{Y}_\alpha$ from a set S to topological spaces $\mathcal{Y}_\alpha$, we can introduce topology on S by stating that the inverse images $f_\alpha^{-1}(W_\alpha)$ for open sets $W_\alpha \subset \mathcal{Y}_\alpha$ form its prebase. It defines topology on S induced by the maps f_α; it is weakest topology on S that makes all maps f_α to be continuous.

This construction produces a topology on the source space; by that reason it is also called initial topology. In 9A we will discuss final topology — an analogous construction that moves to topology from source to target of a map.

For example, this construction can be used to define topology on infinite product of spaces, as the induced topology for all its projections. This topology is called product topology, or Tychonoff topology.

5.9. Advanced exericse. *Let $\mathcal{O}$ be the topology on $\mathbb{R}$ induced by the maps $x \mapsto (\cos(t{\cdot}x), \sin(t{\cdot}x))$ for all $t \in \mathbb{R}$. Show that the space $(\mathbb{R}, \mathcal{O})$ is not metrizable.*

Chapter 6

Compactness

A Definition

We will denote by $\{V_\alpha\} = \{V_\alpha\}_{\alpha \in \mathcal{I}}$ a collection of sets, where α runs in an arbitrary index set $\mathcal{I}$.

6.1. Definition. *A collection $\{V_\alpha\}$ of open subsets in a topological space $\mathcal{X}$ is called its open cover if it covers the whole $\mathcal{X}$; that is, if any $x \in \mathcal{X}$ belongs to some V_α.*

More generally, $\{V_\alpha\}$ is an open cover of a subset $S \subset \mathcal{X}$ if any $s \in S$ belongs to some V_α.

A subset of $\{V_\alpha\}$ that is also a cover is called its subcover.

6.2. Exercise. *Let $\{V_\alpha\}$ be an open cover of a topological space $\mathcal{X}$. Show that $W \subset \mathcal{X}$ is open if and only if for any α the intersection $W \cap V_\alpha$ is open.*

6.3. Definition. *A topological space $\mathcal{X}$ is called compact if any cover $\{V_\alpha\}$ of $\mathcal{X}$ contains a finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$.*

Analogously, a subset S in a topological space $\mathcal{X}$ is called compact if any cover of S contains a finite subcover of S.

6.4. Exercise. *Show that a subset S of a topological space is compact if and only if S equipped with induced topology is a compact space.*

Clearly, *any finite topological space is compact.* In fact, the role of compact spaces in topology reminds the role of finite sets in the set theory. The next exercise provides a source of examples of infinite compact spaces. More interesting examples are given in Section 6C.

6.5. Exercise. *Any set equipped with cofinite topology is compact.*

6.6. Exercise. *Let S be an unbounded subset of the real line; that is, for any $c \in \mathbb{R}$ there is $s \in S$ such that $|s| > c$. Show that S is not compact.*

6.7. Exercise. *Let S be a subset of $\mathbb{R}$. Assume S is not closed. Show that S is not compact.*

6.8. Exercise. *Construct a topological space with two compact sets such that their intersection is not compact.*

B Finite intersection property

6.9. Proposition. *Show that space $\mathcal{X}$ is compact if for any collection of closed sets $\{Q_\alpha\}$ in $\mathcal{X}$ such that*

$$\bigcap_\alpha Q_\alpha = \varnothing$$

There is a finite collection $\{Q_{\alpha_1}, \dots, Q_{\alpha_n}\}$ such that

$$Q_{\alpha_1} \cap \dots \cap Q_{\alpha_n} = \varnothing.$$

The condition in the above proposition is called finite intersection property; it redefines compactness via closed sets.

Proof. Consider the complements $V_\alpha = \mathcal{X} \setminus Q_\alpha$. Note that

$$\bigcup_\alpha V_\alpha = \mathcal{X} \setminus \left(\bigcap_\alpha Q_\alpha\right) = \mathcal{X};$$

that is, $\{V_\alpha\}$ is a cover of $\mathcal{X}$.

Choose a finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$; so, $V_{\alpha_1} \cup \dots \cup V_{\alpha_n} = \mathcal{X}$. Observe that

$$Q_{\alpha_1} \cap \dots \cap Q_{\alpha_n} = \mathcal{X} \setminus (V_{\alpha_1} \cup \dots \cup V_{\alpha_n}) = \varnothing. \qquad \square$$

6.10. Exercise. *Let $Q_1 \supset Q_2 \supset \dots$ be a nested sequence of closed nonempty sets in a compact space $\mathcal{K}$. Show that there is a point $q \in \mathcal{K}$ such that $q \in Q_n$ for any n.*

C Real interval

6.11. Theorem. *Any closed interval $[a, b]$ is a compact subset of the real line.*

Proof. Set $a_0 = a$ and $b_0 = b$, so $[a, b] = [a_0, b_0]$.

Arguing by contradiction, assume that there is an open cover $\{V_\alpha\}$ of $[a_0, b_0]$ that has no finite subcovers.

Note that $\{V_\alpha\}$ is also a cover for two intervals

$$[a_0, \tfrac{a_0+b_0}{2}] \quad \text{and} \quad [\tfrac{a_0+b_0}{2}, b_0].$$

Assume $\{V_\alpha\}$ has a finite subcover of each of these two subintervals. Then these two subcovers together give a finite cover of $[a, b]$. It follows that $\{V_\alpha\}$ has no finite subcovers of *at least one* of these subintervals; denote it by $[a_1, b_1]$; so either $a_1 = a_0$ and $b_1 = \frac{a_0+b_0}{2}$ or $a_1 = \frac{a_0+b_0}{2}$ and $b_1 = b_0$.

Continuing in this manner we get a sequence of intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset \dots$$

such that no finite collection of sets from $\{V_\alpha\}$ covers any of the intervals $[a_n, b_n]$. In particular,

❶ $$[a_n, b_n] \not\subset V_\alpha \quad \text{for any } n \text{ and } \alpha.$$

Observe that

$$\begin{aligned} a_0 \leqslant a_1 \leqslant \dots \\ \dots \leqslant b_1 \leqslant b_0, \\ b_n - a_n = \tfrac{b-a}{2^n}. \end{aligned}$$

Denote by x the least upper bound of $\{a_n\}$. Note that $x \in [a_n, b_n]$ for any n.[1]

Since $\{V_\alpha\}$ is a cover, we can choose $V_\alpha \ni x$. Since V_α is open, it contains the interval $(x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$. Choose a large n so that $\frac{b-a}{2^n} < \varepsilon$. Clearly, $V_\alpha \supset (x - \varepsilon, x + \varepsilon) \supset [a_n, b_n]$; the latter contradicts ❶. □

[1] In fact, $a_n \to x$ and $b_n \to x$ as $n \to \infty$, but we will not use it directly.

D Images

6.12. Proposition. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map between topological spaces and K is a compact set in $\mathcal{X}$. Then the image $Q = f(K)$ is compact in $\mathcal{Y}$.*

Proof. Choose an open cover $\{W_\alpha\}$ of Q. Since f is continuous, $V_\alpha = f^{-1}(W_\alpha)$ is open for each α. Note that $\{V_\alpha\}$ covers K.

Since K is compact, there is a finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$. It remains to observe that $\{W_{\alpha_1}, \dots, W_{\alpha_n}\}$ covers Q. $\square$

6.13. Exercise. *Show that the circle $\mathbb{S}^1$ is compact.*

E Closed subsets

6.14. Proposition. *A closed set in a compact space is compact.*

Proof. Let Q be a closed set in a compact space $\mathcal{K}$. Since Q is closed, its complement $W = \mathcal{K} \setminus Q$ is open.

Consider an open cover $\{V_\alpha\}_{\alpha \in \mathcal{I}}$ of Q. Add to it W; that is, consider the collection of sets that includes W and all V_α for $\alpha \in \mathcal{I}$. Note that we get an open cover of $\mathcal{K}$. Indeed, W covers all points in the complement of Q and any point of Q is covered by some V_α.

Since $\mathcal{K}$ is compact, we can choose a finite subcover, say $\{W, V_{\alpha_1}, \dots \dots, V_{\alpha_n}\}$ — without loss of generality, we can assume that it includes W. Observe that $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$ is a cover of Q, hence the result. $\square$

In the proof, we add an extra open set to the cover, used it, and took it away. This type of reasoning is useful in all branches of mathematics; sometimes it is called 17 camels trick[2] [8].

6.15. Exercise. *Show that any closed bounded subset of the real line is compact.*

[2]The name comes from the following mathematical parable: *A father left 17 camels to his three sons and, according to the will, the eldest son should be given half of all camels, the middle son the 1/3 part, and the youngest son the 1/9. It was impossible to follow his will until a wise man appeared. He added his own camel, the oldest son took* $18/2 = 9$ *camels, the second son took* $18/3 = 6$ *camels, and the third son* $18/9 = 2$ *camels, the wise man took his camel and went away.*

F Product spaces

6.16. Theorem. *Assume $\mathcal{X}$ and $\mathcal{Y}$ are compact topological spaces. Then their product space $\mathcal{X} \times \mathcal{Y}$ is compact.*

The following exercise provides a partial converse.

6.17. Exercise. *Suppose that a product space $\mathcal{X} \times \mathcal{Y}$ is nonempty and compact. Show that its factors $\mathcal{X}$ and $\mathcal{Y}$ are compact.*

In the proof, we will need the following definition.

6.18. Definition. *Let $\{V_\alpha\}$ and $\{W_\beta\}$ be two covers of a topological space $\mathcal{X}$. We say that $\{V_\alpha\}$ is inscribed in $\{W_\beta\}$ if for any α there is β such that $V_\alpha \subset W_\beta$.*

6.19. Exercise. *Let $\mathscr{B}$ be a base in a topological space $\mathcal{X}$. Show that for any cover $\{V_\alpha\}$ of $\mathcal{X}$ there is an inscribed cover of sets in $\mathscr{B}$.*

Suppose that $\{V_\alpha\}$ is inscribed in $\{W_\beta\}$. If $\{V_\alpha\}$ has a finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$. Then for each α_i we can choose β_i such that $V_{\alpha_i} \subset W_{\beta_i}$. Note that $\{W_{\beta_1}, \dots, W_{\beta_n}\}$ is a finite subcover of $\{W_\beta\}$. It proves the following:

6.20. Observation. *A space $\mathcal{X}$ is compact if and only if any cover of $\mathcal{X}$ has a finite inscribed cover.*

It is instructive to solve the following exercise before reading the proof of 6.16.

6.21. Exercise. *Find a flaw in the following argument.*

Fake proof of 6.16. Fix an open cover $\{U_\beta\}$ of $\mathcal{X} \times \mathcal{Y}$. Consider all product sets $V_\alpha \times W_\alpha$ such that $V_\alpha \times W_\alpha \subset U_\beta$ for some β (as before, V_α and W_α are open in $\mathcal{X}$ and $\mathcal{Y}$ respectively). Note that $\{V_\alpha \times W_\alpha\}$ is a cover of $\mathcal{X} \times \mathcal{Y}$ that is inscribed in $\{U_\beta\}$. By the observation, it is sufficient to find a finite subcover of $\{V_\alpha \times W_\alpha\}$.

Note that $\{V_\alpha\}$ is a cover of $\mathcal{X}$. Since $\mathcal{X}$ is compact, we can choose its finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$. Similarly, $\{W_\alpha\}$ is a cover of $\mathcal{Y}$. So we can choose its finite subcover $\{W_{\alpha'_1}, \dots, W_{\alpha'_m}\}$.

Finally observe that

$$\{V_{\alpha_1} \times W_{\alpha_1}, \dots, V_{\alpha_n} \times W_{\alpha_n}, V_{\alpha'_1} \times W_{\alpha'_1}, \dots, V_{\alpha'_m} \times W_{\alpha'_m}\}$$

is a finite cover of $\mathcal{X} \times \mathcal{Y}$. □

Proof of 6.16. Recall that by definition of product topology, any open set in $\mathcal{X} \times \mathcal{Y}$ is a union of product sets $V_\alpha \times W_\alpha$, where V_α is open in $\mathcal{X}$ and W_α is open in $\mathcal{Y}$.

Fix an open cover $\{U_\beta\}$ of $\mathcal{X}\times\mathcal{Y}$. Consider all product sets $V_\alpha\times W_\alpha$ such that $V_\alpha \times W_\alpha \subset U_\beta$ for some β (as before, V_α and W_α are open in $\mathcal{X}$ and $\mathcal{Y}$ respectively). Note that $\{V_\alpha \times W_\alpha\}$ is a cover of $\mathcal{X} \times \mathcal{Y}$ that is inscribed in $\{U_\beta\}$. By the observation, it is sufficient to find a finite subcover of $\{V_\alpha \times W_\alpha\}$.

Fix $x \in \mathcal{X}$. Note that the subspace $\{x\}\times\mathcal{Y}$ is homeomorphic to $\mathcal{Y}$; see 5.8. In particular, the set $\{x\} \times \mathcal{Y}$ is compact. Therefore, $\{x\} \times \mathcal{Y}$ has a finite cover $\{V_{\alpha_1} \times W_{\alpha_1}, \dots, V_{\alpha_n} \times W_{\alpha_n}\}$; that is,

$$(V_{\alpha_1} \times W_{\alpha_1}) \cup \dots \cup (V_{\alpha_n} \times W_{\alpha_n}) \supset \{x\} \times \mathcal{Y}$$

Consider the set

$$N_x = V_{\alpha_1} \cap \dots \cap V_{\alpha_n};$$

note that N_x is open in $\mathcal{X}$. Since $N_x \subset V_{\alpha_i}$ for any i, we have

$$N_x \times \mathcal{Y} \subset (V_{\alpha_1} \times W_{\alpha_1}) \cup \dots \cup (V_{\alpha_n} \times W_{\alpha_n}).$$

Therefore

❷ *every point $x \in \mathcal{X}$ admits an open neighborhood N_x such that $N_x\times\mathcal{Y}$ can be covered by finitely many product sets from $\{V_\alpha \times W_\alpha\}$*

The sets $\{N_x\}_{x\in\mathcal{X}}$ form a cover of $\mathcal{X}$. Since $\mathcal{X}$ is compact, there is a finite subcover $\{N_{x_1}, \dots, N_{x_m}\}$. Note that

$$\mathcal{X} \times \mathcal{Y} = (N_{x_1} \times \mathcal{Y}) \cup \dots \cup (N_{x_m} \times \mathcal{Y});$$

that is, $\mathcal{X}\times\mathcal{Y}$ can be covered by a finite set of sets from $\{N_x\times\mathcal{Y}\}_{x\in\mathcal{X}}$. Applying ❷, we get that $\mathcal{X} \times \mathcal{Y}$ can be covered by finite number of product sets from $\{V_\alpha \times W_\alpha\}$. $\square$

6.22. Advanced exercise. *Let $f\colon \mathcal{X} \to \mathcal{K}$ be a map between topological spaces. Assume $\mathcal{K}$ is compact. Show that f is continuous if and only if its graph $\Gamma = \{\, (x, f(x)) \,|\, x \in \mathcal{X} \,\}$ is a closed set in $\mathcal{X} \times \mathcal{K}$.*

Chapter 7

Compactness of metric spaces

Recall that any metric space has natural topology. In particular we may talk about compact metric spaces. In this chapter we discuss specific properties of compact metric spaces.

A Lebesgue number

The following lemma is a very useful tool.

7.1. Lebesgue number. *Let $\{V_\alpha\}$ be an open cover of a compact metric space $\mathcal{M}$. Then there is $\varepsilon > 0$ such that for every $x \in \mathcal{M}$ there is α such that $V_\alpha \supset \mathrm{B}(x, \varepsilon)$.*

The number ε in the lemma is called Lebesgue number of the cover.

Proof. Given a point $p \in \mathcal{M}$ we can choose $r = r(p) > 0$ such that the ball $\mathrm{B}(p, 2{\cdot}r)$ lies in V_α for some α. Observe that all balls $\mathrm{B}(p, r(p))$ form an open covering of $\mathcal{M}$. Since $\mathcal{M}$ is compact, we can choose a finite subcover $\{\mathrm{B}(p_1, r_1), \dots, \mathrm{B}(p_n, r_n)\}$.

Let $\varepsilon = \min\{r_1, \dots, r_n\}$. For any $p \in \mathcal{M}$ we can choose a ball $\mathrm{B}(p_i, r_i) \ni p$. Observe that $\mathrm{B}(p, \varepsilon) \subset \mathrm{B}(p_i, 2{\cdot}r_i)$. Since $\mathrm{B}(p_i, 2{\cdot}r_i)$ lies in some V_{α_i}, so is $\mathrm{B}(p, \varepsilon)$. ☐

7.2. Exercise. *Construct a noncompact metric space $\mathcal{M}$ such that* 1 *is a Lebesgue number for any of cover of $\mathcal{M}$.*

B Compactness $\Rightarrow$ sequential compactness

A topological space is called sequentially compact if any its sequence has a converging subsequence. For general topological spaces sequential compactness does not imply compactness, and the other way around. The theorem above states that these two notions are equivalent for metric spaces.

7.3. Exercise. *Show that product of two sequentially compact spaces is sequentially compact.*

7.4. Proposition. *A metric space $\mathcal{M}$ is compact if and only if it is sequentially compact.*

In this section, we prove the only-if part. The if part requires deeper diving into metric spaces; it will be done in 7E after proving auxilary statements in the following two sections.

Proof of the only-if part in 7.4. Choose a sequence $x_1, x_2, \dots \in \mathcal{M}$.

Note that a point $p \in \mathcal{M}$ is a limit of subsequence of x_n if and only if for any $\varepsilon > 0$, the ball $\mathrm{B}(p, \varepsilon)$ contains infinite number of elements of x_n. Indeed, if this property holds, then we can choose i_1 such that $x_{i_1} \in \mathrm{B}(p, 1)$, further $i_2 > i_1$ such that $x_{i_2} \in \mathrm{B}(p, \frac{1}{2})$ and so on; on n^{th} step we get $i_n > i_{n-1}$ such that $x_{i_n} \in \mathrm{B}(p, \frac{1}{n})$. The obtained subsequence $x_{i_1}, x_{i_2}, \dots$ converges to p.

Assume the sequence x_n has no converging subsequence. Then for any point p there is $\varepsilon_p > 0$ such that $\mathrm{B}(p, \varepsilon_p)$ contains only finitely many elements of x_n. Note that $\mathrm{B}(p, \varepsilon_p)$ for all p forms a cover of $\mathcal{M}$. Since the sequence is infinite, this cover does not have a fine subcover. That is, if a sequence x_n has no converging subsequence, then $\mathcal{M}$ is not compact. $\square$

A sequence $x_1, x_2, \dots$ of points in a metric space is called Cauchy if for any $\varepsilon > 0$ there is n such that $|x_i - x_j| < \varepsilon$ for all $i, j > n$. It is easy to prove that any converging sequence is Cauchy, the converse does not hold in general. A metric space $\mathcal{M}$ is called complete if any Cauchy sequence in $\mathcal{M}$ converges to a point in $\mathcal{M}$.

For example, as it follows from Cauchy test, the real line $\mathbb{R}$ with stndard metric is a complete space. On the other hand, an open interval $(0, 1)$ forms a noncomplete subspace of $\mathbb{R}$; indeed, the sequence $x_n = \frac{1}{2 \cdot n}$ is a Cauchy, it also converges to zero in $\mathbb{R}$ which not a point of the subspace.

7.5. Exercise. *Show that any compact metric space is complete.*

C Nets and separability

Let $\mathcal{M}$ be a metric space. A subset $A \subset \mathcal{M}$ is called ε-net of $\mathcal{M}$ if for any $p \in \mathcal{M}$ there is $a \in A$ such that $|p - a|_{\mathcal{M}} < \varepsilon$.

7.6. Lemma. *Let $\mathcal{M}$ be a sequentially compact metric space. Then for any $\varepsilon > 0$ there is a finite ε-net in $\mathcal{M}$.*

Proof. Choose $\varepsilon > 0$. Consider the following recursive procedure.

Choose a point x_1 in $\mathcal{M}$. Further, choose a point x_2 so that $|x_1 - - x_2| > \varepsilon$. Further, choose a point x_3 so that $|x_1 - x_3| > \varepsilon$ and $|x_2 - x_3| > \varepsilon$; and so on. On the n^{th} step we choose a point x_n such that $|x_i - x_n| > \varepsilon$ for all $i < n$.

Suppose that the procedure terminates at some n; that is, there is no point x_n such that $|x_i - x_n| > \varepsilon$ for all $i < n$. In this case, the set $\{x_1, \dots, x_{n-1}\}$ is an ε-net in $\mathcal{M}$ — the lemma is proved.

If the procedure does not terminate, we get an infinite sequence of points $X_1, x_2, \dots$ such that $|x_i - x_j| > \varepsilon$ for all $i \neq j$. Any of its subsequence has the same property; in particular non of its subsequences converge — a contradiction. □

A topological space is called separable if it contains a countable dense subset.

7.7. Corollary. *Sequentially compact metric spaces are separable.*

Proof. Let $\mathcal{M}$ be a sequentially compact metric space.

By 7.6, for each positive integer n, we can choose a finite ε-net $N_n \subset \mathcal{M}$. It remains to observe that the union $N_1 \cup N_2 \cup \dots$ is a countable everywhere dense set. □

D Countable base

7.8. Proposition. *Any sequentially compact metric space has a countable base.*

Topological spaces that admit a countable base are called second-countable. So the proposition states that *any sequentially compact metric space is second-countable.*

Proof. Let $\mathcal{M}$ be a sequentially compact metric space. By 7.7, we can choose a countable dense subset $A \subset \mathcal{M}$.

Consider the set of all balls $\mathrm{B}(a, \frac{1}{n})$ for $a \in A$ and positive integers n. Note that this set is countable; it remains to show that it forms a base in $\mathcal{M}$.

Let x be a point in an open set V. Then $\mathrm{B}(x,\varepsilon) \subset V$ for some $\varepsilon > 0$. Choose n so that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since A is everywhere dense, we can choose $a \in A$ so that $|a - x| < \frac{1}{n}$. By the triangle inequality, $x \in \mathrm{B}(a, \frac{1}{n}) \subset \mathrm{B}(x,\varepsilon)$; in particular,

$$x \in \mathrm{B}(a, \tfrac{1}{n}) \subset V.$$

It remains to apply 5.6. □

7.9. Lemma. *Let $\mathcal{X}$ be a topological space with a countable base. Then any open cover of $\mathcal{X}$ has a countable subcover.*

Proof. Choose an open cover $\{V_\alpha\}$.

Note that it is sufficient to show that there is a countable open cover that is inscribed in $\{V_\alpha\}$. Indeed, suppose $\{W_1, W_2, \dots\}$ is an open subcover inscribed in $\{V_\alpha\}$. Then for any W_i we can choose $V_{\alpha_i} \supset W_i$; evidently $\{V_{\alpha_1}, V_{\alpha_2}, \dots\}$ is a countable subcover of $\{V_\alpha\}$.

Let $\{B_1, B_2, \dots\}$ be a countable base of $\mathcal{X}$. By 5.5, for any $x \in \mathcal{X}$ we can choose $i = i(x)$ such that $x \in B_i \subset V_\alpha$ for some α. Denote by S all integers that appear as $i(x)$ for some x. Then $\{B_i\}_{i \in S}$ is a countable open cover that is inscribed in $\{V_\alpha\}$. □

E Sequential compactness ⇒ compactness

Proof of the if part in 7.4. Choose an open cover of $\mathcal{M}$. By 7.9, we can assume that the cover is countable; denote it by $\{V_1, V_2, \dots\}$.

Assume $\{V_1, V_2, \dots\}$ does not have a finite subcover. Then we can choose a sequence of points $x_1, x_2, \dots \in \mathcal{M}$ such that

$$x_n \notin V_1 \cup \dots \cup V_n$$

for any n.

Since $\mathcal{M}$ is sequentially compact, its subsequence has a limit, say x; we have that $x \in V_n$ for some n. It follows that $x_i \in V_n$ for an infinite set of indices i. By construction, $x_i \notin V_n$ for all $i > n$ — a contradiction. □

Chapter 8

Hausdorff spaces

A Definition

8.1. Definition. *A topological space $\mathcal{X}$ is called* Hausdorff *if for each pair of distinct points $x, y \in \mathcal{X}$ there are disjoint neighborhoods $V \ni x$ and $W \ni y$.*

8.2. Observation. *Any metrizable space is Hausdorff.*

Proof. Assume that topology on the space $\mathcal{X}$ is induced by a metric $|* - *|$.

If the points $x, y \in \mathcal{X}$ are distinct then $|x - y| > 0$. By triangle inequality $\mathrm{B}(x, \frac{r}{2}) \cap \mathrm{B}(y, \frac{r}{2}) = \varnothing$. Hence the statement follows $\square$

Recall that a sequence of points $x_1, x_2, \ldots$ in a topological space $\mathcal{X}$ converges to a point $x \in \mathcal{X}$ if for any neighborhood $V \ni x$ we have $x_n \in V$ for all, but finitely many n.

Note that in general, a sequence of points in a topological space might have different limits. For example, consider the real line with the cofinite topology and a sequence $x_1, x_2, \ldots$ such that $x_m \neq x_n$ for $m \neq n$. Note that the sequence converges to *every* $x \in \mathbb{R}$. Indeed, a complement of any neighborhood $V \ni x$ is a finite set; therefore $x_n \in V$ for all, but finitely many indexes n.

8.3. Exercise. *Show that any converging sequence in Hausdorff space has a unique limit.*

8.4. Exercise. *Show that a topological space $\mathcal{X}$ is Hausdorff if and only if the diagonal*

$$\Delta = \{\, (x, x) \in \mathcal{X} \times \mathcal{X} \,\}$$

is a closed set in the product space $\mathcal{X} \times \mathcal{X}$.

B Observations

8.5. Observation. *Any one-point set in a Hausdorff space is closed.*

If every one-point space in a topological space is closed then the space is called T_1-space or sometimes Tikhonov space. Therefore the observation above states that *any Hausdorff space is T_1.*

Proof. Let $\mathcal{X}$ be a Hausdorff space and $x \in \mathcal{X}$. By 8.1, given a point $y \neq x$, there are disjoint open sets $V_y \ni x$ and $W_y \ni y$. In particular $W_y \not\ni x$.

Note that

$$\mathcal{X} \setminus \{x\} = \bigcup_{y \neq x} W_y.$$

It follows that $\mathcal{X} \setminus \{x\}$ is open, and therefore $\{x\}$ is closed. □

8.6. Observation. *Any subspace of Hausdorff space is Hausdorff.*

Proof. Choose two points x, y in a subspace A of a Hausdorff space $\mathcal{X}$. Since $\mathcal{X}$ is Hausdorff, we can choose neighborhoods $V \ni x$ ad $W \ni y$ such that $V \cap W = \varnothing$. Then $A \cap V$ and $A \cap W$ are neighborhoods of x and y in A. Clearly,

$$(A \cap V) \cap (A \cap W) \subset V \cap W = \varnothing.$$

Whence the observation follows. □

C Games with compactness

8.7. Proposition. *Any compact subset of a Hausdorff space is closed.*

Noote that any one-point set is compact. Therefore the proposition generalizes Observation 8.5. The proof is similar but requires an extra step. It is instructive to solve the following exercise before reading the proof.

8.8. Exercise. *Describe a topological space $\mathcal{X}$ with a nonclosed, but compact subset K.*

The proof of proposition is beased on the following theorem.

8.9. Theorem. *Let $\mathcal{X}$ be a Hausdorff space and $K \subset X$ be a compact subset. Then for any point $y \notin K$ there are open sets $V \supset K$ and $W \ni y$ such that $V \cap W = \varnothing$*

Proof of 8.7 modulo 8.9. For $y \notin K$, let us denote by W_y the open set provided by 8.9; in particular, $W_y \ni y$ and $W_y \cap K = \varnothing$. Note that

$$\mathcal{X} \setminus K = \bigcup_{y \notin K} W_y.$$

It follows that $\mathcal{X} \setminus K$ is open; therefore, K is closed. □

Proof of 8.9. By definition of Hausdorff space, for any point $x \in K$ there is a pair of disjoint openset $V_x \ni x$ and $W_x \ni y$. Note that sets $\{V_x\}_{x \in K}$ forms cover of K. Since K is compact we can choose a finite subcover $\{V_{x_1}, \dots, V_{x_n}\}$. Set

$$V = V_{x_1} \cup \dots \cup V_{x_n}$$
$$W = W_{x_1} \cap \dots \cap W_{x_n}$$

It remains to observe that V and W are open, $y \in W$, $K \subset V$, and

$$V \cap W \subset \bigcup_i (V_{x_i} \cap W_{x_i}) = \varnothing.$$ □

8.10. Exercise. *Let $\mathcal{X}$ be a Hausdorff space and $K, L \subset X$ be two compact subsets. Assume $K \cap L = \varnothing$, show that there are open sets $V \supset K$ and $W \supset L$ such that $V \cap W = \varnothing$*

Chapter 9

More constructions

A Moving topology by a map

Recall that in 5E, we defined a natural way to move a topology from target space to the source of a map.

Namely, suppose $f\colon \mathcal{X} \to \mathcal{Y}$ be a map between two sets. Assume $\mathcal{Y}$ is equipped with a topology. Declare a subset $V \subset \mathcal{X}$ to be open in the induced topology for f if there is an open subset $W \subset \mathcal{Y}$ such that $V = f^{-1}(W)$.

The following exercise describes an analogous construction that moves a topology from source to target. It can be solved by checking the conditions in 2.1 as we did in 5A.

9.1. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map between two sets. Assume $\mathcal{X}$ is equipped with a topology. Declare a subset $W \subset \mathcal{Y}$ to be open if the subset $V = f^{-1}(W)$ is open in $\mathcal{X}$. Show that it defines a topology on $\mathcal{X}$.*

The constructed topology on $\mathcal{Y}$ is called final topology.

9.2. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map between topological spaces.*

(a) Show that the initial topology on $\mathcal{X}$ is weaker than its own topology.

(b) Show that the final topology on $\mathcal{Y}$ is stronger than its own topology.

9.3. Exercise. *Let $g\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map.*

(a) Suppose $\mathcal{X}$ is equipped with the initial topology. Show that a map $f\colon \mathcal{W} \to \mathcal{X}$ is continuous if and only if the composition $f \circ g\colon \mathcal{W} \to \mathcal{Y}$ is continuous.

(b) Suppose $\mathcal{Y}$ is equipped with the final topology. Show that a map $h\colon \mathcal{Y} \to \mathcal{Z}$ is continuous if and only if the composition $h \circ f\colon \mathcal{X} \to \mathcal{Z}$ is continuous.

The initial topology is used mostly for injective maps; in this case, it is nearly the same as induced topology. Similarly, final topology is mostly used for surjective maps. This particular case of the construction is called quotient topology; it is discussed in the following section.

9.4. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous surjective map. Assume f is (a) open or (b) closed. Show that $\mathcal{Y}$ is equipped with the quotient topology.*

B Quotient topology

Let $\sim$ be an equivalence relation on a topological space $\mathcal{X}$; that is, for any $x, y, z \in \mathcal{X}$ the following conditions hold:

- $x \sim x$;
- if $x \sim y$, then $y \sim x$;
- if $x \sim y$ and $y \sim z$, then $x \sim z$.

Recall that the set

$$[x] = \{\, y \in \mathcal{X} \mid y \sim x \,\}$$

is called the equivalence class of x. The set of all equivalence classes in $\mathcal{X}$ will be denoted by $\mathcal{X}/{\sim}$.

Observe that $x \mapsto [x]$ defines a surjective map $\mathcal{X} \to \mathcal{X}/{\sim}$. The corresponding final topology on $\mathcal{X}/{\sim}$ is called quotient topology on $\mathcal{X}/{\sim}$. By default, $\mathcal{X}/{\sim}$ is equipped with the quotient topology in this case, it is called quotient space.

The following exercise ties equivalence relations with maps.

9.5. Exercise. *Show that an arbitrary map $f\colon \mathcal{X} \to \mathcal{Y}$ defines the following equivalence relation on $\mathcal{X}$:*

$$x \sim x' \quad \textit{if and only if} \quad f(x) = f(x').$$

Moreover,

$$y = f(x) \quad \textit{if and only if} \quad [x] = f^{-1}\{f(x)\}.$$

Given a subset A in a topological space $\mathcal{X}$, the space $\mathcal{X}/A$ is defined as the quotient space $\mathcal{X}/{\sim}$ for the minimal equivalence relation such that $a \sim b$ for any $a, b \in A$. For example the quotient space $[0,1]/{\sim}$ discussed above can be also denoted by $[0,1]/\{0,1\}$ — it is the interval $[0,1]$ with identified two-element subset $\{0,1\}$.

9.6. Exercise. *Describe the quotient space* $[0,1]/(0,1)$*, where* $[0,1]$ *and* $(0,1)$ *are real intervals; that is, list the points and the open sets of the quotient space.*

C Compact-to-Hausdorff maps

9.7. Observation. *Let* $f\colon \mathcal{K} \to \mathcal{Y}$ *be a continuous map. Assume that the space* $\mathcal{K}$ *is compact, and* $\mathcal{Y}$ *is Hausdorff. Then* f *is a closed map.*

If in addition, the map f *is onto, then* $\mathcal{Y}$ *is equipped with the quotient topology induced by* f*.*

9.8. Corollary. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proof of 9.7. Since $\mathcal{K}$ is compact, any closed subset $Q \subset \mathcal{K}$ is compact (6.14). Since the image of a compact set is compact we have that $f(Q)$ is a compact subset of $\mathcal{Y}$. Since $\mathcal{Y}$ is Hausdorff $f(Q)$ is closed (8.7). Hence the first statement follows.

The second statement follows from 9.4. □

Recall that $\mathbb{D}$ denotes the unit disc and $\mathbb{S}^1$ denotes the unit circle; that is,

$$\mathbb{D} = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 \leqslant 1 \right\},$$
$$\mathbb{S}^1 = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1 \right\}.$$

9.9. Exercise. *Show that* $\mathbb{S}^1$ *is homeomorphic to the quotient space* $[0,1]/\{0,1\}$*. (In other words,* $\mathbb{S}^1$ *is homeomorphic to the unit interval with glued ends.)*

9.10. Exercise. *Show that the quotient space* $\mathbb{D}/\mathbb{S}^1$ *is homeomorphic to the unit sphere*

$$\mathbb{S}^2 = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \right\}.$$

D Orbit spaces

9.11. Definition. *Let $\mathcal{X}$ be a topological space and G be a group. Suppose that $(g, x) \mapsto g{\cdot}x$ is a map $G \times \mathcal{X} \to \mathcal{X}$ such that*

(a) $1{\cdot}x = x$ *for any* $x \in \mathcal{X}$, *here* 1 *denotes the identity element of* G;

(b) $g{\cdot}(h{\cdot}x) = (g{\cdot}h){\cdot}x$ *for any* $g, h \in G$ *and* $x \in \mathcal{X}$;

(c) for any $g \in G$, *the map* $x \mapsto g{\cdot}x$ *is continuous.*

Then we say that G acts on $\mathcal{X}$, or $\mathcal{X}$ is a G-space.

In this case, the set

$$G{\cdot}x := \{\, g{\cdot}x \mid g \in G \,\}$$

is called the G-orbit of x (or, briefly, orbit).

Note that *(b)* implies that the expression

$$g{\cdot}h{\cdot}x$$

makes sense; that is, it does not depend on parentheses.

9.12. Exercise. *Suppose that a group G acts on a topological space $\mathcal{X}$. Show that for any $g \in G$, the map $x \mapsto g{\cdot}x$ defines a homeomorphism $\mathcal{X} \to \mathcal{X}$.*

Suppose that a group G acts on a topological space $\mathcal{X}$. Set $x \sim y$ if there is $g \in G$ such that $y = g{\cdot}x$.

Observe that $\sim$ is an equivalence relation on $\mathcal{X}$. Indeed, $x \sim x$ since $x = 1{\cdot}x$. Further, if $y = g{\cdot}x$, then by *(a)* and *(b)* we get

$$x = 1{\cdot}x = g^{-1}{\cdot}g{\cdot}x = g^{-1}{\cdot}y;$$

since $g^{-1} \in G$ we get that $x \sim y \implies y \sim x$. Finally, suppose $x \sim y$ and $y \sim z$; that is, $y = g{\cdot}x$ and $z = h{\cdot}y$ for some $g, h \in G$. By *(b)*, $z = (h{\cdot}g){\cdot}x$; therefore $x \sim z$.

For the described equivalence relation, the quotient space $\mathcal{X}/{\sim}$ can be also denoted by $\mathcal{X}/G$; it is called quotient of $\mathcal{X}$ by the action of G.

Note that $[x] = G{\cdot}x$; that is, the orbit of x coincides with its equivalence class. By that reason $\mathcal{X}/G$ is also called orbit space.

9.13. Exercise. *Suppose a group G acts on a topological space $\mathcal{X}$ and $f \colon \mathcal{X} \to \mathcal{X}/G$ is the quotient map.*

(a) Show that f is open.

(b) Assume G is finite. Show that f is closed.

Chapter 10

Connected spaces

A Definitions

A subset of topological space is called c l o p e n if it is closed and open at the same time.

10.1. Definition. *A topological space $\mathcal{X}$ is called c o n n e c t e d if it has exactly two clopen sets $\varnothing$ and the whole space $\mathcal{X}$.*

According to our definition, *the empty space is not connected.* Not everyone agrees with this convention.

Suppose V is a p r o p e r clopen subset in a topological space $\mathcal{X}$; that is, $V \neq \varnothing$ and $V \neq \mathcal{X}$. Note that its complement $W = \mathcal{X} \setminus V$ is also a proper clopen subset. In particular, there are two open sets $V, W \subset \mathcal{X}$ such that $V \neq \varnothing$, $W \neq \varnothing$, $V \cup W = \mathcal{X}$ and $V \cap W = \varnothing$.

A subset of topological space is called the connected or disconnected if so is the corresponding subspace. Spelling the notion of subspace we get the following definition.

10.2. Definition. *A subset A of a topological space is called d i s c o n n e c t e d if it is empty or there are two open sets V and W such that*

$$V \cap W \cap A = \varnothing, \quad V \cap A \neq \varnothing, \quad W \cap A \neq \varnothing, \quad \textit{and} \quad V \cup W \supset A.$$

Otherwise, we say that A is c o n n e c t e d.

A pair of open sets V and W as in the definition will be called o p e n s p l i t t i n g of A. So we can say that *a nonempty set A is disconnected if and only if it admits an open splitting.*

10.3. Proposition. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map between topological spaces. Suppose $A \subset \mathcal{X}$ is a connected set. Then the image $f(A)$ is a connected set in $\mathcal{Y}$.*

Proof. We can assume that $B \neq \varnothing$; otherwise the statement is trivial.

Assume that $B = f(A)$ is disconnected. Choose an open splitting V and W of B; that is,

❶ $V \cap W \cap B = \varnothing, \quad V \cap B \neq \varnothing, \quad W \cap B \neq \varnothing, \quad \text{and} \quad V \cup W \supset B.$

Since f is continuous, $V' = f^{-1}(V)$ and $W' = f^{-1}(W)$ are open sets in $\mathcal{X}$. Note that ❶ implies that V' and W' is an open splitting of A; that is,

$$V' \cap W' \cap A = \varnothing, \quad V' \cap A \neq \varnothing, \quad W' \cap A \neq \varnothing, \quad \text{and} \quad V' \cup W' \supset A.$$

Therefore A is disconnected — a contradiction. □

10.4. Exercise. *Let $\mathcal{X}$ be a connected space. Show that the quotient space $\mathcal{X}/\!\sim$ is connected for any equivalence relation $\sim$ on $\mathcal{X}$.*

10.5. Proposition. *Assume $\{A_\alpha\}_{\alpha\in\mathcal{I}}$ is a collection of connected subsets of a topological space. Suppose that $\bigcap_\alpha A_\alpha \neq \varnothing$. Then*

$$A = \bigcup_\alpha A_\alpha$$

is connected.

Proof. Assume that A is disconnected; choose its open splitting V, W. Since $\bigcap_\alpha A_\alpha \neq \varnothing$, we can fix $p \in \bigcap_\alpha A_\alpha$. Without loss of generality, we can assume that $p \in V$.

In particular, $V \cap A_\alpha \neq \varnothing$ for any α. Since A_α is connected, we have that $W \cap A_\alpha = \varnothing$ for each α; otherwise V and W form an open splitting of A_α. Therefore

$$\begin{aligned} W \cap A &= W \cap \left(\bigcup_\alpha A_\alpha\right) \\ &= \bigcup_\alpha (W \cap A_\alpha) \\ &= \varnothing, \end{aligned}$$

a contradiction. □

10.6. Exercise. *Let A be a connected set in a topological space $\mathcal{X}$. Suppose that $A \subset B \subset \bar{A}$. Show that B is connected.*

B Real interval

10.7. Proposition. *The real interval* $[0, 1]$ *is connected.*

Proof. Assume contrary; let V and W be an open splitting of $[0, 1]$. Fix a $a_0 \in V$ and $b_0 \in W$; without loss of generality, we can assume that $a_0 < b_0$.

Let us construct a nested sequence of closed intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset \dots$$

such that

$$❷ \qquad b_n - a_n = \tfrac{1}{2^n}(b_0 - a_0), \quad a_n \in V, \quad \text{and} \quad b_n \in W$$

for any n.

The construction is recursive. Assume $[a_{n-1}, b_{n-1}]$ is already constructed. Set $c = \frac{1}{2}\cdot(a_{n-1} + b_{n-1})$. If $c \in V$, then set $a_n = c$ and $b_n = b_{n-1}$; if $c \in W$, then set $a_n = a_{n-1}$ and $b_n = c$. In both cases, ❷ holds.

The sequence a_n is nondecreasing and bounded above by b_0. In particular, the sequence a_n converges; denote its limit by x. Since $b_n - a_n = \frac{1}{2^n}\cdot(b_0 - a_0)$, the sequence b_n also converges to x. The point x has to belong to V or W. Since both V and W are open, one of them contains a_n and b_n for all large n — a contradiction. □

10.8. Exercise. *Show that the real line* $\mathbb{R}$ *is a connected space.*

C Connected components

Let x be a point in a topological space $\mathcal{X}$. The intersection of all clopen sets containing x is called connected component of x. Note that the space $\mathcal{X}$ is connected if and only if $\mathcal{X}$ is a connected component of some (and therefore any) point in $\mathcal{X}$.

10.9. Exercise. *Show that any connected component is a closed set.*

Construct an example of topological space $\mathcal{X}$ *and a point* $x \in \mathcal{X}$ *such that the connected component of* x *is not open.*

10.10. Exercise. *Show that two connected components either coincide or disjoint.*

10.11. Exercise. *Suppose that a space* $\mathcal{X}$ *has a finite number of connected components. Show that each connected component of* $\mathcal{X}$ *is open.*

D Cut points

Evidently, number of connected components is a topological invariant; that is, *if two spaces are homeomorphic, then they have the same number of connected components.* In particular, connected space is not homeomorphic to a disconnected space.

Let us describe a more refined way to apply this observation. Suppose $\mathcal{X}$ is a connected space, a point $x \in \mathcal{X}$ is called a cut point if removing x from $\mathcal{X}$ produces a disconnected space; that is, the subset $\mathcal{X} \setminus \{x\}$ is disconnected.

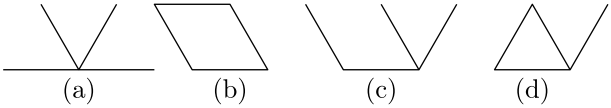
Note that if $f\colon \mathcal{X} \to \mathcal{Y}$ is a homeomorphism, then a point $x \in \mathcal{X}$ is a cut point of $\mathcal{X}$ if and only if $y = f(x)$ is a cut point of $\mathcal{Y}$. Indeed, the restriction of f defines a homeomorphism $\mathcal{X} \setminus \{x\} \to \mathcal{Y} \setminus \{y\}$. In particular, we get that the spaces $\mathcal{X} \setminus \{x\}$ and $\mathcal{Y} \setminus \{y\}$ have the same number of connected components.

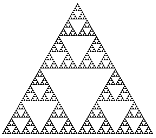
These observations can be used to solve the following exercises.

10.12. Exercise. *Show that the circle* $\mathbb{S}^1$ *is not homeomorphic to the line segment* $[0,1]$.

10.13. Exercise. *Show that the plane* $\mathbb{R}^2$ *is not homeomorphic to the real line* $\mathbb{R}$.

10.14. Exercise. *Show that no two of the following four closed connected sets in the plane are not homeomorphic. (Each set is a union of four line segments.)*





Sierpiński gasket is constructed the following way: start with a solid equilateral triangle, subdivide it into four smaller congruent equilateral triangles and remove the interior of the central one. Repeat this procedure recursively for each of the remaining solid triangles.

10.15. Advanced exercise.

(a) Prove that Sierpiński triangle is connected.

(b) Describe all the homeomorphisms from the Sierpiński triangle to it self.

Chapter 11

Path-connected spaces

A Definitions

Let $\mathcal{X}$ be a topological space. A continuous map $f\colon [0,1] \to \mathcal{X}$ is called path. If $x = f(0)$ and $y = f(1)$ we say that f is a path from x to y.

A space $\mathcal{X}$ is called path-connected if it is nonempty and any two points in $\mathcal{X}$ can be connected by a path; that is, for any $x, y \in \mathcal{X}$ there is a path f from x to y.

11.1. Exericise. *Show that any convex set in a Euclidean space is path-connected.*

11.2. Exericise. *Show that the connected two-point space $\mathcal{X} = \{a, b\}$ (defined in 2B) is path-connected.*

11.3. Theorem. *Any path-connected space is connected; the converse does not hold.*

Proof; main part. Let $\mathcal{X}$ be a path-connected space.

By Proposition 10.7, the unit interval $[0,1]$ is connected. By Proposition 10.3, for any path $f\colon [0,1] \to \mathcal{X}$ the image $f([0,1])$ is connected.

Fix $x \in \mathcal{X}$. Since $\mathcal{X}$ is path-connected,

$$\mathcal{X} = \bigcup_{f} f([0,1]),$$

where the union is taken for all paths f starting from x. It remains to apply 10.5.

Second part. We need to present an example of a space that connected, but not path-connected.

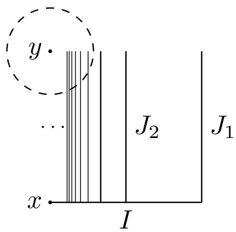
Denote by I the closed line segment from $(0,0)$ to $(1,0)$ in the plane. Further, denote by J_n the closed line segment from $(\frac{1}{n},0)$ to $(\frac{1}{n},1)$. Consider the union of all these segments

$$W = I \cup J_1 \cup J_2 \cup \dots$$

and set

$$W' = W \cup \{y\},$$

where $y=(0,1)$. The space W' is called flea and comb; the set W is called comb, and the point y is called flea.



Note that $W \subset W' \subset \bar{W}$. Therefore, by 10.6, W' is connected.

It remains to show that W' is not path-connected. Assume the contrary. Let f be a path from $x=(0,0)$ to $y=(0,1)$.

Note that $f^{-1}(\{y\})$ is closed subset of compact space $[0,1]$. Therefore $f^{-1}(\{y\})$ is compact. In particular, the set $f^{-1}(\{y\})$ has the minimal element, denote it by b. Note that $b>0$, $f(b)=y$ and $f(t)\neq y$ for any $t<b$.

Choose positive $\varepsilon<1$. Since f is continuous, there is $a<b$ such that $|f(t)-y|<\varepsilon$ for any $t\in[a,b]$. Note that $f(a)\in J_n$ for some n.

Denote by N the intersection of ε-neighborhood of y with the comb. note that the intersection of J_n with ε-neighborhood of y forms a connected component of N. By 10.3, $f(t)\in J_n$ for any $t\in[a,b]$; in particular, $f(b)\neq y$ — a contradiction. □

11.4. Advanced exercise. *Recall that $\mathbb{Q}$ denotes the set of rational numbers. Consider the following sets in the plane:*

$$A = \left\{\, (x,y)\in\mathbb{R}^2 \,\middle|\, x,y\in\mathbb{Q} \,\right\} \quad \textit{and} \quad B = \left\{\, (x,y)\in\mathbb{R}^2 \,\middle|\, x,y\notin\mathbb{Q} \,\right\}.$$

Show that $A\cup B$ is path connected.

B Operations on paths

Given a path $f\colon [0,1]\to\mathcal{X}$ one can consider the time-reversed path $\bar{f}$, defined by

$$\bar{f}(t) = f(1-t).$$

Note that $\bar{f}$ is continuous since f is.

Let f and h be paths in the topological space $\mathcal{X}$. If $f(1) = h(0)$ we can join these two paths into one $g\colon [0,1] \to \mathcal{X}$ defined as

$$g(t) = \left[\begin{aligned} f(2{\cdot}t) &\quad \text{if } t \leqslant \tfrac{1}{2} \\ h(2{\cdot}t - 1) &\quad \text{if } t \geqslant \tfrac{1}{2} \end{aligned}\right.$$

The path g is called the product (or concatenation) of paths f and h, briefly it is denoted as $g = f * h$.

By construction, the restrictions $f * h|_{[0,\frac{1}{2}]}$ and $f * h|_{[\frac{1}{2},1]}$ are continuous. Therefore, 3.3 implies that $f * h$ is continuous; in other words $f * h$ is indeed a path.

Consider the following relation on the set of points of topological space: $x \sim y \Leftrightarrow$ there is a path from x to y.

11.5. Exercise. *Show that $\sim$ is an equivalence relation; that is, for any points x, y, and z in a topological space we have*

(a) $x \sim x$.

(b) If $x \sim y$, then $y \sim x$

(c) If $x \sim y$ and $y \sim z$, then $x \sim z$.

The equivalence class of point x for the equivalence relation $\sim$ is called path-connected component of x.

11.6. Exercise. *Assume every path-connected component in a topological space $\mathcal{X}$ is closed. Show that $\mathcal{X}$ is connected if and only if $\mathcal{X}$ is path-connected.*

11.7. Exercise. *Show that the image of a path-connected set under a continuous map is path-connected.*

11.8. Exercise. *Show that the product of path-connected spaces is path-connected.*

C Open sets of Euclidean space

The following theorem provides a class of topological spaces for which connectedness implies path-connectedness.

11.9. Theorem. *An open set in a Euclidean space $\mathbb{R}^n$ is path-connected if and only if it is connected.*

Proof. The only-if part follows from 11.3; it remains to prove the if part.

Let $\Omega \subset \mathbb{R}^n$ be an open subset. Choose a point $p \in \Omega$; denote by $P \subset \Omega$ the path-connected component of p.

Let us show that for any point $q \in \Omega$ there is $\varepsilon > 0$ such that either $\mathrm{B}(q, \varepsilon) \subset P$, or $\mathrm{B}(q, \varepsilon) \cap P = \varnothing$.

Indeed, since Ω is open, we can choose $\varepsilon > 0$ such that the ball $\mathrm{B}(q, \varepsilon)$ lies in Ω. Note that $\mathrm{B}(q, \varepsilon)$ is convex, in particular path-connected. Therefore if $\mathrm{B}(q, \varepsilon) \cap P \neq \varnothing$, then $\mathrm{B}(q, \varepsilon) \subset P$.

It follows that P and its complement $\Omega \setminus P$ are open. Since Ω is connected, we get that $\Omega \setminus P = \varnothing$ — hence the result. □

A topological space $\mathcal{X}$ is called l o c a l l y p a t h - c o n n e c t e d if any point $p \in \mathcal{X}$ and its neighborhood V there is a path-connected open set W such that $V \supset W \ni p$.

Note that Euclidean spaceis locally path-connected; it follows since any open ball in a Euclidean space is path-connected. Therefore the following exercise generalizes the theorem above.

11.10. Exercise. *Show that an open set in a locally path-connected space is path-connected if and only if it is connected.*

Appendix A

Semisolutions

1.2. Check all the conditions in Definition 1.1. Further we discuss the triangle inequality — the remaining conditions are nearly evident.

Let $a = (x_a, y_a)$, $b = (x_b, y_b)$, and $c = (x_c, y_c)$. Set

$$\begin{aligned} x_1 &= x_b - x_a, & y_1 &= y_b - y_a, \\ x_2 &= x_c - x_b, & y_2 &= y_c - y_b. \end{aligned}$$

(a). The inequality

$$|a - c|_1 \leqslant |a - b|_1 + |b - c|_1$$

can be written as

$$|x_1 + x_2| + |y_1 + y_2| \leqslant |x_1| + |y_1| + |x_2| + |y_2|.$$

The latter follows since $|x_1 + x_2| \leqslant |x_1| + |x_2|$ and $|y_1 + y_2| \leqslant |y_1| + |y_2|$.

(b). The inequality

$$❶ \qquad |a - c|_2 \leqslant |a - b|_2 + |b - c|_2$$

can be written as

$$\begin{aligned} &\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \leqslant \\ &\leqslant \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}. \end{aligned}$$

Take the square of the left and the right-hand sides, simplify, take the square again and simplify again. You should get the following inequality:

$$0 \leqslant (x_1 \cdot y_2 - x_2 \cdot y_1)^2,$$

which is equivalent to ❶ and evidently true.

(c). The inequality

$$|a - c|_\infty \leqslant |a - b|_\infty + |b - c|_\infty$$

can be written as

$$❷ \qquad \begin{aligned} &\max\{|x_1 + x_2|, |y_1 + y_2|\} \leqslant \\ &\leqslant \max\{|x_1|, |y_1|\} + \max\{|x_2|, |y_2|\}. \end{aligned}$$

Without loss of generality, we may assume that

$$\max\{|x_1 + x_2|, |y_1 + y_2|\} = |x_1 + x_2|.$$

Further,

$$\begin{aligned} &|x_1 + x_2| \leqslant |x_1| + |x_2| \leqslant \\ &\leqslant \max\{|x_1|, |y_1|\} + \max\{|x_2|, |y_2|\}. \end{aligned}$$

Hence ❷ follows.

1.3. Check the triangle inequality for 0, $\frac{1}{2}$, and 1.

1.4. Check the conditions in 1.1.

1.6. Show that the triangle inequality implies that $|f(x) - f(y)| < \varepsilon$, if $|x - y|_{\mathcal{X}} < \varepsilon$; make a conclusion.

1.7. Fix $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $f(x) = y$.

Fix $\varepsilon > 0$. Since g is continuous at y, there is a positive value δ_1 such that

$$d_{\mathcal{Z}}(g(y'), g(y)) < \varepsilon \quad \text{if} \quad d_{\mathcal{Y}}(y', y) < \delta_1.$$

Since f is continuous at x, there is $\delta_2 > 0$ such that

$$d_{\mathcal{Y}}(f(x'), f(x)) < \delta_1 \quad \text{if} \quad d_{\mathcal{X}}(x', x) < \delta_2.$$

Since $f(x) = y$, we get that

$$d_{\mathcal{Z}}(h(x'), h(x)) < \varepsilon \quad \text{if} \quad d_{\mathcal{X}}(x', x) < \delta_2.$$

Hence the result.

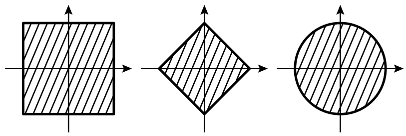
1.8. *(a)* Show that the triangle inequality implies that $|f(x)-f(y)|_{\mathcal{Y}}<\varepsilon$, if $|x-y|_{\mathcal{X}}<\varepsilon$; make a conclusion.

(b). Apply 1.1*b*.

1.9. Show and use that in 1.5 one can take $\delta=1$ for any $\varepsilon>0$.

1.10. Learn about space-filling curves and think.

1.11. Figure out which is which.



1.12. Apply triangle inequality. For the second part, describe the balls $\mathrm{B}(2,3)$ and $\mathrm{B}(0,4)$ in $[0,\infty)$.

1.14. Spell the definitions.

1.16. If $y\in\mathrm{B}(x,R)$, then $r=R-|x-y|>0$. Use triangle inequality to show that $\mathrm{B}(y,r)\subset$ $\subset\mathrm{B}(x,R)$. Make a conclusion.

1.17. Apply 1.15.

1.18. Apply 1.15.

1.19. The if part follows from 1.16 and 1.17. It remains to prove the only-if part.

Let V be an open set. By 1.15, for any $x\in V$ there is $r_x>0$ such that $\mathrm{B}(x,r_x)\subset V$. Observe that

$$V=\bigcup_{x\in V}\mathrm{B}(x,r_x).$$

1.20. Consider that open segments $(-\varepsilon,\varepsilon)$ for all $\varepsilon>0$ in $\mathbb{R}$. Note that

$$\{0\}=\bigcap_{\varepsilon>0}(-\varepsilon,\varepsilon)$$

and the one-point set $\{0\}$ is not open.

1.21. Show and use that

$$\mathrm{B}(x,r)_1\subset\mathrm{B}(x,r)_2\subset\mathrm{B}(x,r)_\infty\subset\mathrm{B}(x,2{\cdot}r)_1;$$

here $\mathrm{B}(x,r)_1$, $\mathrm{B}(x,r)_2$, and $\mathrm{B}(x,r)_\infty$ denote the balls in the metrics $|*-*|_1$, $|*-*|_2$, and $|*-*|_\infty$ respectively.

1.23. Look at the image of $\mathbb{R}$ for the function $x\mapsto|x|$.

1.25. Assume the contrary; that is, a sequence $x_1,x_2,\dots$ has two limits y and z. Set $r=|y-z|$. Note that $\mathrm{B}(y,\frac{r}{2})$ contains all but finitely many elements of the sequence $x_1,x_2,\dots$; the same holds for $\mathrm{B}(z,\frac{r}{2})$. Observe that $\mathrm{B}(y,\frac{r}{2})\cap\mathrm{B}(z,\frac{r}{2})=\varnothing$ and arrive at a contradiction.

1.26. Suppose f is not continuous. Note that it means that there is a point x_∞ and $\varepsilon>0$ such that there is a point $x_n\in\mathrm{B}(x_\infty,\frac{1}{n})$ such that $|f(x_n)-f(x_\infty)|>\varepsilon$ in particular, $y_n=f(x_n)$ does not converge to $y_\infty=f(x_\infty)$. It proves the if part of the exercise.

To prove the only-if part, suppose that there is a sequence $x_n\to x_\infty$ such that $y_n\not\to y_\infty$ as $n\to\infty$. Note that in this case we can pass to a subseqence so that $x_n\in\mathrm{B}(x_\infty,\frac{1}{n})$ and $|y_n-y_\infty|>\varepsilon$ for some fixed $\varepsilon>0$. From above f is not continuous.

1.27. Show that the semiopen interval $[0,1)$ is neither open nor closed in $\mathbb{R}$.

1.28. Choose a point $z\in\bar{\bar{A}}$. It means that, there is a sequence $y_1,y_2,\dots\in\bar{A}$ such that $y_n\to z$ as $n\to\infty$. The latter means that for each y_i there is a sequence $x_{i,1},x_{i,2},\dots\in A$ such that $x_{i,n}\to y_i$ as $n\to\infty$. Try to choose a sequence of integers m_n such that $x_{n,m_n}\to z$ as $n\to\infty$. Make a conclusion.

1.29. Show that Q is closed if $x\in Q$ if and only if $\mathrm{B}(x,\varepsilon)\cap Q\neq\varnothing$ for any $\varepsilon>0$. Show that the latter is equivalent to $y\in V$ if and only if $\mathrm{B}(y,\varepsilon)\subset V$ for some $\varepsilon>0$. Make a conclusion.

2.2. Let $\mathcal{F}$ be a finite metric space. Observe that there is $\varepsilon>0$ such that $|x-y|>\varepsilon$ for any two distinct points $x,y\in\mathcal{F}$. It follows that $\{x\}=\mathrm{B}(x,\varepsilon)$ for any $x\in\mathcal{F}$; in particular, each one-point set is open. By 2.1*b*, any set in $\mathcal{F}$ is open.

2.3. Let $\mathcal{X}$ be an infinite set with cofinite topology. Show that any two nonempty open sets in $\mathcal{X}$ have nonempty intersection. Show that the latter does not hold for open balls in a metric space with at least two points.

2.4. Choose $W\in\mathscr{W}$. By assumption, for any $w\in W$ there is $S_w\in\mathscr{S}$ such that $W\supset S_w\ni w$. Observe that

$$W=\bigcup_{w\in W}S_w.$$

It follows that $\mathscr{W}\subset\mathscr{S}$; in other words, any $\mathscr{W}$-open set is $\mathscr{S}$-open.

2.6; *(a).* Consider the function defined by

$$f(x) = \begin{cases} a & \text{if } x < 0, \\ b & \text{if } x \geqslant 0. \end{cases}$$

(b). Suppose $f\colon \mathcal{X} \to \mathbb{R}$ is nonconstant; that is, $f(a) \neq f(b)$. Note that $W = \mathbb{R} \setminus \{f(a)\}$ is an open set containing b. Assume f is continuous. Then $\{b\} = f^{-1}(W)$ is an open set — a contradiction.

2.7. Apply the definitions.

2.8; *(a).*Check the conditions in 2.1.

(b). Show that that every two nonempty set in $\mathbb{R}_{\geqslant}$ intersect. Show that the latter statement does not hold in a metric space with at least two points. Make a conclusion.

(c). To do the only-if part check the condition in 2.5.

To do the if part, suppose f is not nondecreasing; that is, we can find $x < y$ such that $f(x) > f(y)$. Note that the inverse image $V = f^{-1}([f(x), \infty))$ contains x but does not contain y. Show that V is not open in $\mathbb{R}_{\geqslant}$, so $f\colon \mathbb{R}_{\geqslant} \to \mathbb{R}_{\geqslant}$ is not continuous.

3.3. Set $a = f|_A$ and $b = f|_B$; choose a closed set $Q \subset \mathcal{Y}$. By 3.2, $a^{-1}(Q)$ and $b^{-1}(Q)$ are closed. Note that $a^{-1}(Q) = f^{-1}(Q) \cap A$ and $b^{-1}(Q) = f^{-1}(Q) \cap B$. Apply 3.1$c$ to show that $f^{-1}(Q)$ is closed. Further, apply 3.2.

3.4. Suppose V is an open subset and Q is its complement. Recall that Q is closed; see 3A. Show and use that $V \subset A$ if and only if $Q \supset B$.

3.5. Each of four statements can be deduced by spelling the needed definition.

3.6. Use 3.5 together with the definitions of closure and interior.

3.7. Observe that $\mathring{Q} \subset Q$, and apply 3.6.

3.8. Observe that $\bar{V} \supset V$, and apply 3.6.

3.9. Try to choose a subset A in $\mathbb{R}$ so that it meets the following conditions:

- A and $\mathbb{R} \setminus A$ contain isolated points,
- A and $\mathbb{R} \setminus A$ contain intervals,
- A and $\mathbb{R} \setminus A$ are dense in some interval.

For the second part, show and use the following

$$\bar{\bar{A}} = \bar{A}, \quad \mathring{\mathring{A}} = \mathring{A}, \quad \mathring{\bar{\mathring{\bar{A}}}} = \mathring{\bar{A}}, \quad \text{and} \quad \bar{\mathring{\bar{\mathring{A}}}} = \bar{\mathring{A}}.$$

3.10–3.11. Apply the definitions of boundary and closed set.

3.12. Read about the Catnor set and think.

3.13. Let A be a subset of a topological space $\mathcal{X}$. Denote by B its complement; that is $B = \mathcal{X} \setminus A$. Show and use that the statement is equivalent to the following

$$\mathcal{X} \setminus \partial A = \mathring{A} \cup \mathring{B}.$$

3.14. Apply the definitions of neighborhood and dense set.

3.15. In other words, we need to classify spaces such that any nonempty subset is everywhere dense. Observe that every point of the space lies in every nonempty open set. Conclude that the space has concrete topology. Show that any such space has this property.

3.17. Note that any ball $\mathrm{B}(x_\infty, \varepsilon)$ is a neighbohood of x_∞. Observe that it implies the only-if part.

To prove the if part, show and use that for any neighbohood N of x_∞ there is $\varepsilon > 0$ such that $\mathrm{B}(x_\infty, \varepsilon) \subset N$.

3.18–3.20. Apply the definition of the convergence.

4.2. Let $f\colon \mathcal{X} \to \mathcal{Y}$ is a homeomorphism. Recall that by deinition of inverse, we have $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$ for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. It remains to show that the existence of $f^{-1}\colon \mathcal{Y} \to \mathcal{X}$ implies that f is a bijection $\mathcal{X} \leftrightarrow \mathcal{Y}$.

For the second part, try to find such bijection for the subspaces $A = [0,1) \cup \{2\}$ and $B = [0,1]$ of $\mathbb{R}$.

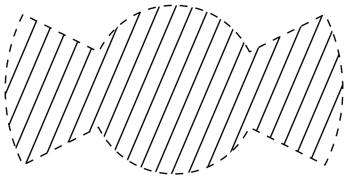
4.3. Observe that $y \mapsto \ln y$ is inverse of $x \mapsto e^x$, and show that both functions are continuous. (You can use that differentiable functions are continuous.)

4.4. Try to build the needed function from $x \mapsto \arctan x$ or $x \mapsto e^{-e^x}$

4.5. Apply the definitions of homeomorphism and 2.7.

4.6. Learn about inversion and try to apply it.

4.7. Let Ω be an open star-shaped set with respect to the origin. It is not hard to prove the statement if can be described by the inequality $r < f_n(\theta)$ in the polar (r, θ)-coordinates, were $f_n : \mathbb{S}^1 \to \mathbb{R}$ is a continuous function But general star-shaped set, for example one on the diagram is problematic.



To do the general case, show that Ω can be presented as a union of a nested sequence of open sets $\Omega_0 \subset \Omega_1 \subset \dots$ such that for each Ω_n can be described by the inequality $r < f_n(\theta)$ in the polar (r, θ)-coordinates with continuous $f_n : \mathbb{S}^1 \to \mathbb{R}$. We can assume that Ω_0 is a round disc around the origin.

Further construct a sequence of homeomorphisms $\varphi_n : \Omega_{n-1} \to \Omega_n$ such that the compositions $\Phi_n = \varphi_n \circ \dots \circ \varphi_1 : \Omega_0 \to \Omega_n$ stabilises for each $x \in \Omega_0$; that is, $\Phi_n(x)$ is a fixed point for all sufficiently large n. Set

$$\Phi(x) = \lim_{n\to\infty} \Phi_n(x),$$

and show that Φ defines the needed homeomorphism $\Omega_0 \leftrightarrow \Omega$.

4.8. $\Phi_n : \mathbb{R}^2 \leftrightarrow \mathbb{R}^2$ Suppose that the sets are $P = \{p_1, p_2, \dots\}$ and $Q = \{q_1, q_2, \dots\}$. Try to construct a sequence of homeomorphisms $\Phi_n : \mathbb{R}^2 \leftrightarrow \mathbb{R}^2$ such that Φ_n converge to a homeomorphism $\Phi : \mathbb{R}^2 \leftrightarrow \mathbb{R}^2$ and for any n we have $\Phi_n(\{p_1, \dots, p_n\}) \subset Q$ and $\Phi_n^{-1}(\{q_1, \dots, q_n\}) \subset \subset P$.

4.10. Apply the definitions.

4.11. Try the maps between two-point spaces with appropriate topologies.

4.12. Consider a map that vanish on the Cantor set and sends each remaining interval homeomorpically to the whole $\mathbb{R}$.

5.4; *only-if part.* Apply that a base is a collection of open sets.

If part. Choose an open set $W \subset \mathcal{Y}$. By 5.3,

$$W = \bigcup_\alpha B_\alpha,$$

for some collection $\{B_\alpha\}$ of sets in the base. Then

$$f^{-1}(W) = \bigcup_\alpha f^{-1}(B_\alpha).$$

By the assumption $f^{-1}(B_\alpha)$ is open for any α; it remains to apply 2.1b.

5.5; *if part.* Choose an open set N. For any $x \in N$ choose an element of base B_x such that $x \in B_x \subset N$. Observe that

$$N = \bigcup_{x\in N} B_x.$$

Only-if part. Suppose that $\mathscr{B}$ is a base. Then

$$N = \bigcup_\alpha B_\alpha,$$

where $B_\alpha \in \mathscr{B}$ for each α. Then for any $x \in N$ there is α such that $B_\alpha \ni x$; in this case, $x \in B_\alpha \subset N$.

5.7; *only-if part.* Apply that a prebase is a collection of open sets.

If part. Show that for any finite collection of sets $P_1, \dots, P_n$ in the prebase the inverse image $f^{-1}(P_1 \cap \dots \cap P_n)$ is open. Further apply 5.5.

5.8. To show that the map F is continuous, apply 5.7 to the prebase described before the exercise. Further, show and use that projection $G : (x, f(x)) \to x$ is a continuous left inverse; that is $G(F(x)) = x$ for any x.

5.9. Show that *every two disjoint closed sets of a metric space have disjoint open neighborhoods*; that is, for any two closed sets A and B there are open sets $V \supset A$ and $W \supset B$ such that $V \cap W = \varnothing$. (Topological spaces that share this propery are called n o r m a l; so you need to show that *any metrizable space is normal.*)

Observe that arithmetic progression is a closed set in the initial topology. Construct two disjoint arithmetic progressions that do not admit disjoint open neighborhoods.

6.2. Use 2.1b and 2.1c.

6.4. Spell the definitions.

6.5. We may assume that the space is nonempty; otherwise there is nothing to prove. Choose a nonempty set V_0 from the covering. Its complement is a vinete set, say $\{x_1, \dots, x_n\}$. For each x_i choose a set $V_i \ni x_i$ from the covering. Observe that $\{V_0, \dots, V_0\}$ is a subcover.

6.6. Consider covering of S by intervals $(-c, c)$ for all $c > 0$.

6.7. Choose a point $s \in \bar{S} \setminus S$ and consider the cover by intervals $(-\infty, s-\varepsilon)$ and $(s+\varepsilon, +\infty)$ for all $\varepsilon > 0$.

6.8. Choose a noncompact space $\mathcal{X}$ and consider topology on the union $\mathcal{X} \cup \{a, b\}$ that includes $\mathcal{X} \cup \{a, b\}$, $\mathcal{X} \cup \{a\}$, $\mathcal{X} \cup \{b\}$ and all open sets in $\mathcal{X}$. Observe that the sets $\mathcal{X} \cup \{a\}$ and $\mathcal{X} \cup \{b\}$ are compact, but their intersection is not.

6.10. Apply the finite intersection property.

6.13. Show that $\mathbb{S}^1$ is an image of closed interval under a continuous map, and apply 6.12.

6.15. Apply 6.14 and 6.11.

6.17. Apply 6.12 to the projections $\mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ and $\mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$.

6.19. Apply 5.4.

6.21. Show by example that the obtained collection $\{V_{\alpha_1} \times W_{\alpha_1}, \dots, V_{\alpha_n} \times W_{\alpha_n}, V_{\alpha'_1} \times$ $\times W_{\alpha'_1}, \dots, V_{\alpha'_m} \times W_{\alpha'_m}\}$ might not cover the whole $\mathcal{X} \times \mathcal{Y}$.

6.22. By 3.2, it is sufficient to show that any closed set $A \subset \mathcal{K}$ has closed inverse image $B = f^{-1}(A) \subset \mathcal{X}$.

Observe that the set $C = \Gamma \cap (\mathcal{X} \times A)$ is closed, so its complement U can be presented as a union $\bigcup_\alpha V_\alpha \times W_\alpha$.

Suppse B is not closed, choose a point $p \in$ $\in \bar{B} \backslash B$. Note that $\{p\} \times \mathcal{K}$ is a compact set in U. Argue as in 6.16 to prove that there is an open set $N_p \ni p$ such that $N_p \times \mathcal{K} \subset U$. Arrive at a contradiction.

Remark. The following function $f \colon \mathbb{R} \to \mathbb{R}$ has closed graph, but is not continuous:

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It shows that compactness of $\mathcal{K}$ is a necessary assumption.

7.2. Consider an infinite set of points with discrete metric.

7.3. Suppose that a sequence x_n converges to x_∞ in $\mathcal{X}$ and y_n converges to y_∞ in $\mathcal{Y}$. Show and use that (x_n, y_n) converges to (x_∞, y_∞) in $\mathcal{X} \times \mathcal{Y}$ as $n \to \infty$.

7.5. By 7.4, any sequence has a converging subseqence; denote by x the limit of this subsequence. Show that if the sequence is Cauchy, then it converges to x.

8.3. Arguing by contradiction, assume a sequence has two limits x and y. Since the space is Hausdorff we can choose disjoint neighbohoods $V \ni x$ and $W \ni y$. Since the sequence converges to x, the set V contains all but finitely many elements of the sequence. The same holds for W — a contradiction.

8.4. The set Δ is closed if and only if its complement $U = (\mathcal{X} \times \mathcal{X}) \setminus \Delta$ is open. Show and use that the latter means that there is a family $\{(V_\alpha, W_\alpha)\}$ of disjoint pairs of open sets in $\mathcal{X}$ such that

$$U = \bigcup_\alpha V_\alpha \times W_\alpha.$$

8.8. Look at the subsets of a concrete space.

8.10. By 8.9, for any $y \in L$ there is a pair of open sets V_y, W_y such that $V_y \supset K$ and $W_y \ni y$ such that $V_y \cap W_y = \varnothing$. Mimic the proof of 8.9 using these pairs.

9.1. Check the conditions in 2.1 directly.

9.2.

9.3.

9.4. Since f is continuous, $V = f^{-1}(W)$ is open for any open set $W \subset \mathcal{Y}$. It remains to show that if V is open so is $W \subset \mathcal{Y}$.

Note that $W = f(V)$. If f is open, then $W = f(V)$ is open as well.

Set $A = \mathcal{X} \setminus V$ and $B = \mathcal{Y} \setminus W$. Since V is open A is closed. Since f is surjective, $B = f(A)$. Since f is a closed map, $B = f(A)$ is closed as well. Therefore, $W = \mathcal{Y} \setminus B$ is open.

9.5. Check the conditions, in the definitions of equivalence relation and equivalence class.

9.6. It has tree points $a = [0]$, $b = [\frac{1}{2}]$, and $c = [1]$ and the open sets are

$$\varnothing,\ \{a, b\},\ \{b, c\},\ \{a, b, c\}.$$

9.9. Apply 9.7 to the map $[0, 1] \to \mathbb{S}^1$ defined by $t \mapsto (\cos(2 \cdot \pi \cdot t), \sin(2 \cdot \pi \cdot t))$.

9.10. Apply 9.7 to the map $\mathbb{D} \to \mathbb{R}^3$ that is written from polar to spherical coordinates as

$$(r, \theta) \mapsto (1, \theta, \pi \cdot r).$$

9.12. Show and use that $y \mapsto g^{-1} \cdot y$ is inverse of $x \mapsto g \cdot x$.

9.13 Let $f\colon \mathcal{X} \to \mathcal{X}/G$ is the quotient map. Show that for any set $V \subset \mathcal{X}$ we have

$$f^{-1}\circ f(V) = \bigcup_{g\in G} g\cdot V.$$

(a). Apply this formula and 9.12 to show that if V is open then so is $f^{-1}\circ f(V)$. Finally apply the definition of quotient topology.

(b). Apply this formula and 9.12 to show that if G is finite and V is closed then so is $f^{-1}\circ f(V)$.

10.4. Apply 10.3.

10.6. Show that any open splitting of B splits A as well.

10.8. Apply 10.7, 10.3, and 10.5.

10.9. Connected component is an intersection of clopen sets; in particular it is closed.

Consider the following subspace of real line $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$. Show that the one-point set $\{0\}$ is a connected component in A and it is not open in A.

10.10. Check that being in one connected component defines an equivalence relation on points of topological space.

10.11. Use 10.9 and 10.10.

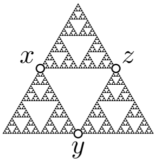
10.12. Show that $\mathbb{S}^1$ has no cut points, but $[0,1]$ has.

10.13. Show that $\mathbb{R}^2$ has no cut points, but $\mathbb{R}$ has.

10.14. Count cut points and noncut points for each space.

10.15; *(a)* Let T_n be a the union of all sides of the 3^n thriangles after n^{th} iteration. Note that the sequence is nested; that is, $T_0 \subset T_1 \subset \dots$ Use induction to show that each T_i is connected. Conclude that the union $T = T_0 \cup T_1 \cup \dots$ is connected. Finally, show that Sierpiński triangle is the closure of T and apply 10.6.

(b). Denote the Sierpiński triangle by $\triangle$.



Let $\{x, y, z\}$ be a 3-point set in $\triangle$ such that $\triangle \setminus \{x, y, z\}$ has 3 connected components. Show and use that there is a unique choice for the set $\{x, y, z\}$ and it is formed by the midpoints of the original triangle.

11.1. Let p and q be points in a convex set F. Observe that the linear path

$$f(t) = (1-t)\cdot p + t\cdot q$$

lies in F.

11.2. Recall that $\{a\}$ is an open set in $\mathcal{X}$. Show and use that $f\colon [0,1] \to \mathcal{X}$ defined by

$$f(t) = \begin{cases} a & \text{if } t < 1, \\ b & \text{if } t = 1 \end{cases}$$

is continuous map.

11.4. Show and use that for any rational numbers a and b, the line $y = a\cdot x + b$ lies in $A \cup B$.

11.5. For *(a)*, use the constant path $f(t) = x$. For *(b)*: if f is a path from x to y, then $\bar{f}$ is from y to x. For *(c)*, suppose f is a path from x to y and g is a path from y to z. Observe that $f * g$ is a path from x to z.

11.6.

11.7. Show and use that for any continuous map φ and any path f, the composition $\varphi\circ f$ is a path.

11.8. Suppose that f and g are paths in $\mathcal{X}$ and $\mathcal{Y}$ respectively. Show and use that $t \mapsto (f(t), g(t))$ is a path in $\mathcal{X} \times \mathcal{Y}$.

11.10. Mimic the proof of 11.9.

Appendix B

Jordan curve theorem

B.1. Theorem. *Suppose $\Gamma \subset \mathbb{R}^2$ is a closed set homeomorphic to $\mathbb{R}$. Then $\mathbb{R}^2 \setminus \Gamma$ has at least two connected components.*

Note that the assumption that Γ is closed is necessary; indeed a finite open interval I of a line in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}$, but its complement $\mathbb{R}^2 \setminus I$ is connected.

The theorem follows from B.2, B.5, and B.7.

B.2. Proposition. *Suppose $\Gamma \subset \mathbb{R}^2$ is a closed set such that the complement $X = \mathbb{R}^2 \setminus \Gamma$ is connected. Let us identify $\mathbb{R}^2$ with the (x, y)-plane in $\mathbb{R}^3$. Then the complement $Y = \mathbb{R}^3 \setminus \Gamma$ is simply-connected.*

The proof is based on the following partial case of Van Kampen theorem.

B.3. Exercise. *Suppose that V and W are open simply-connected subsets of topological space $\mathcal{X}$ such that $\mathcal{X} = V \cup W$, and the set $V \cap W$ is path-connected. Show that $\mathcal{X}$ is simply-connected.*

Conclude that sphere $\mathbb{S}^2$ is simply-connected.

Proof. Denote by A (respectively B) the sets that include Γ and the points below (respectively above) Γ; that is,

$$A = \{\,(x, y, z) \mid (x, y) \in \Gamma \text{ and } z \leqslant 0\,\},$$
$$B = \{\,(x, y, z) \mid (x, y) \in \Gamma \text{ and } z \geqslant 0\,\}.$$

Consider their complements $V = \mathbb{R}^3 \setminus A$ and $W = \mathbb{R}^3 \setminus B$. Note that $Y = V \cup W$.

The sets V and W are simply-connected. Indeed, the horizontal plane $z = 1$ is a deformation retract of V; a retraction can be defined by $(x, y, z) \mapsto (x, y, 1)$ and the following homotopy shows that it is homotopic to the identity map:

$$h_t(x, y, z) = (x, y, (1 - t)\cdot z + t).$$

The plane is contractible, in particular simply-connected; therefore so is V. Similarly one proves that W is simply-connected.

Since X is open and connected set in $\mathbb{R}^2$. By 11.9, X is path-connected. Further, note that $V \cup W = X \times \mathbb{R}$. Therefore $V \cup W$ is path-connected as well.

Summarizing, V and W are open simply-connected sets, $Y = V \cup W$, and $V \cap W$ is path-connected. Applying B.3, we get that Y is simply-connected. □

B.4. Exercise. *Observe that the proposition above does not hold without assuming that* Γ *is close. Spot the place in the proof that brakes in this case.*

B.5. Proposition. *Suppose* $\Gamma \subset \mathbb{R}^2$ *is a closed subset homeomorphic to* $\mathbb{R}$*. Then there is a homeomorphism* $\mathbb{R}^3 \to \mathbb{R}^3$ *that maps* Γ *the* z*-axis.*

B.6. Technical lemma. *Suppose* $\Gamma \subset \mathbb{R}^2$ *is a closed subset homeomorphic to* $\mathbb{R}$ *and* $h\colon \mathbb{R} \to \Gamma$ *is a homeomorphism. Then there is a function* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *such that* $f \circ h(t) = t$ *for any* $t \in \mathbb{R}$*.*

This lemma follows directly from the so-called Tietze–Urysohn extension theorem, but we sketch a more elementary proof.

Assume first that h is L-Lipschitz; that is, $|h(t_0) - h(t_1)| \leqslant L{\cdot}|t_0 - t_1|$ for any $t_0, t_1 \in \mathbb{R}$. In this case it is easy to see that the function

$$f(x) = \sup_{t \in \mathbb{R}} \{\, t - L{\cdot}|h(t) - x| \,\}$$

meets the conditions in the lemma. To do the general case one has to be more inventive with the choice of f, but the idea is very the same.

Sketch of proof. Consider the function $\Phi\colon \mathbb{R} \times [0, \infty) \to [0, \infty)$ defined by

$$\Phi(t, r) := \sup_{s \in \mathbb{R}} \{\, |s - t|{\cdot}(1 + r - |h(s) - h(t)|) \,\}.$$

Note that Φ is continuous.

Moreover, if $r \geqslant |h(t) - h(s)|$, then

$$|s - t| \leqslant \Phi(t, r)$$

for any $s, t \in \mathbb{R}$. It follows that $f \circ h(t) = t$ where

$$f(p) := \sup_{t \in \mathbb{R}} \{\, t - \Phi(t, |p - h(t)|) \,\}.$$

It remains to observe that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous. □

The following proof uses the so-called Klee trick which is quite useful in many topological problems.

Proof of B.5. Let $h\colon t\mapsto(a(t),b(t))$ be a homeomorphism $\mathbb{R}\to\Gamma$. By B.6, there is a function $f\colon\mathbb{R}^2\to\mathbb{R}$ such that

$$f(a(t),b(t))=f\circ h(t)=t$$

for any $t\in\mathbb{R}$.

Note that the map

$$F\colon(x,y,z)\mapsto(x,y,z+f(x,y))$$

is a homeomorphism. Indeed, this map is continuous and its inverse

$$F^{-1}\colon(x,y,z)\mapsto(x,y,z-f(x,y))$$

is continuous as well.

Similarly, the map

$$G\colon(x,y,z)\mapsto(x-a(z),y-b(z),z)$$

is a homeomorphism as well. Indeed, G is continuous and it has inverse

$$G^{-1}\colon(x,y,z)\mapsto(x+a(z),y+b(z),z)$$

that is continuous as well.

It follows that the composition $G\circ F\colon\mathbb{R}^3\to\mathbb{R}^3$ is a homeomorphism. Since $f(a(t),b(t))=t$,

$$G\circ F(a(t),b(t),0)=G(a(t),b(t),t)=(0,0,t).$$

It follows that $G\circ F$ sends Γ to the z-axis as required. □

B.7. Exercise. *Show that the complement of the z-axis in $\mathbb{R}^3$ is not simply-connected.*

B.8. Theorem. *Let $J\subset\mathbb{S}^2$ be a subset homeomorphic to $\mathbb{S}^1$. Then $\mathbb{S}^2\setminus J$ has at least two connected components.*

This theorem is a partial case of famous Jordan's theorem; it is known for simple formulation and annoyingly tricky proofs. The presented proof is found by Patrick Doyle [9]; it is among the shortest proofs, but it uses quite a bit of topology.

Proof. Remove a point p from J to get a closed line $\Gamma=J\setminus\{p\}$ in $\mathbb{S}^2\setminus\{p\}\simeq\mathbb{R}^2$. It remains to apply B.1. □

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