Introduction to topology

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Lecture 1

Metric spaces

In this chapter we discuss *metric spaces* — a motivating example that will guide us toward the definition of *topological spaces* — the main object of topology.

Examples of metric spaces were considered for thousands of years, but the first general definition was given only in 1906 by Maurice Fréchet.

A Definition

In the following definition we grab together the most important properties of intuitive notion of *distance*.

1.1. Definition. *Let $\mathcal{X}$ be a nonempty set with a function which returns a real number, denoted as $|x-y|$, for any pair $x,y\in\mathcal{X}$. Assume that the following conditions are satisfied for any $x,y,z\in\mathcal{X}$:*

(a) Positiveness:

$$|x-y|\geq 0.$$

(b) Identity of indiscernibles:

$$x=y\quad\textit{if and only if}\quad|x-y|=0.$$

(c) Symmetry:

$$|x-y|=|y-x|.$$

(d) Triangle inequality:

$$|x-y|+|y-z|\geq|x-z|.$$

In this case we say that $\mathcal{X}$ *is a* metric space *and the function*

$$(x, y) \mapsto |x - y|$$

is called a metric*.*

The elements of $\mathcal{X}$ *are called* points *of the metric space. Given two points* $x, y \in \mathcal{X}$*, the value* $|x - y|$ *is called* distance *from* x *to* y*.*

Note that for two points in a metric space the difference between points $x-y$ may have no meaning, but $|x-y|$ always has the meaning defined above.

Typically, we consider only one metric on set, but if few metrics are needed, we can distingush them by a index, say $|x - y|_\bullet$ or $|x - y|_{239}$. If we need to emphasize that the distace is taken in the metric space $\mathcal{X}$ we write $|x - y|_\mathcal{X}$ instead of $|x - y|$.

B Examples

Let us give few examples of metric spaces.

- *Discrete sapce.* Let $\mathcal{X}$ be an arbitrary set. For any $x, y \in \mathcal{X}$, set $|x - y| = 0$ if $x = y$ and $|x - y| = 1$ otherwise. This metric is called *discrete metric* on $\mathcal{X}$ and the obtained metric space is called *discrete.*

- *Real line.* Set of all real numbers ($\mathbb{R}$) with metric defined as $|x - y|$.

- *Metrics on the plane.* Let us denote by $\mathbb{R}^2$ the set of all pairs (x_1, y_2) of real numbers. Consider two points $p_1 = (x_1, y_2)$ and $p_2 = (x_2, y_2)$ in $\mathbb{R}^2$. One can equip $\mathbb{R}^2$ with the following metrics:

  - *Euclidean metric,* denoted by

  $$|p_1 - p_2|_2 = \sqrt{(x_1 - x_2)^2 + (y_2 - y_2)^2}.$$

  - *Manhattan metric,* denoted by $|{*} - {*}|_1$ and defined as

  $$|p_1 - p_2|_1 = |x_1 - x_2| + |y_2 - y_2|.$$

  - *Maximum metric,* denoted by $|{*} - {*}|_\infty$ and defined as

  $$|p_1 - p_2|_\infty = \max\{|x_1 - x_2|, |y_2 - y_2|\}.$$

1.2. Exercise. *Prove that (a)* $|{*}-{*}|_2$*; (b)* $|{*}-{*}|_1$ *and (c)* $|{*}-{*}|_\infty$ *are metrics on* $\mathbb{R}^2$*.*

1.3. Exercise. *Show that*

$$|x-y|_\natural = (x-y)^2$$

is not a metric on $\mathbb{R}$*.*

1.4. Exercise. *Show that if* $(x,y)\mapsto |x-y|$ *is a metric then so is*

$$(x,y)\mapsto |x-y|_{\max} = \max\{1,|x-y|\}.$$

C Continuous maps

Recall that a real-to-real function f is called *continuous* if for any $x\in\mathbb{R}$ and any $\varepsilon>0$ there exists $\delta>0$ such that $|f(x)-f(y)|<\varepsilon$, whenever $|x-y|<\delta$.

This definition can be used for the functions defined on Euclidean space if $|x-y|$ denotes the euclidean distance $|x-y|_2$ between the points x and y. It admits the following straightforward generalization to *metric spaces*:

1.5. Definition. *A function* $f\colon\mathcal{X}\to\mathcal{Y}$ *between metric spaces is called* continuous *if for any* $x\in\mathcal{X}$ *and any* $\varepsilon>0$ *there exists* $\delta>0$ *such that* $|f(x)-f(y)|_{\mathcal{Y}}<\varepsilon$*, for any* $y\in\mathcal{X}$ *such that* $|x-y|_{\mathcal{X}}<\delta$*.*

1.6. Exercise. *Let* $\mathcal{X}$ *be a metric space and* $z\in\mathcal{X}$ *be a fixed point. Show that the function*

$$f(x)\stackrel{\text{def}}{=\!=}|x-z|$$

is continuous.

1.7. Exercise. *Let* $\mathcal{X}$*,* $\mathcal{Y}$ *and* $\mathcal{Z}$ *be a metric spaces. Assume that the functions* $f\colon\mathcal{X}\to\mathcal{Y}$ *and* $g\colon\mathcal{Y}\to\mathcal{Z}$ *are continuous, and*

$$h=g\circ f\colon\mathcal{X}\to\mathcal{Z}$$

is its composition; that is, $h(x)=g(f(x))$ *for any* $x\in\mathcal{X}$*. Show that* $h\colon\mathcal{X}\to\mathcal{Z}$ *is continuous at any point.*

1.8. Exercise. *Show that any distance-preserving map is continuous.*

More precisely, assume that $f\colon \mathcal{X} \to \mathcal{Y}$ is a map between metric space such that

$$|x - x'|_{\mathcal{X}} = |f(x) - f(x')|_{\mathcal{Y}}$$

for any $x, x' \in \mathcal{X}$ then f is continuous.

1.9. Exercise. *Let $\mathcal{X}$ be a discrete metric space (defined on page 6) and $\mathcal{Y}$ be arbitrary metric space. Show that for any function $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous.*

D Balls

Let $\mathcal{X}$ be a metric space, x is a point in $\mathcal{X}$ and r is a positive real number. The set of points in $\mathcal{X}$ which lies on the distance smaller than r is called *ball of radius r centered at x*. It is denoted as $\mathrm{B}(x, r)$ or $\mathrm{B}(x, r)_{\mathcal{X}}$ if we need to emphasize that it is taken in the space $\mathcal{X}$.

The ball $\mathrm{B}(x, r)$ is also called r-neighborhood of x.

1.10. Exercise. *Sketch the unit balls for the metrics $|* - *|_1$, $|* - *|_2$ and $|* - *|_\infty$ defined right before Exercise 1.2.*

1.11. Exercise. *Assume $\mathrm{B}(x, r)$ and $\mathrm{B}(y, R)$ is a pair of balls in a metric space and $\mathrm{B}(x, r) \subsetneq \mathrm{B}(y, R)$. Show that $r < 2 \cdot R$.*

Give an example of a metric space and a pair of balls as above such that $r > R$.

Let us reformulate the definition of continuous map (1.5) using the introduced notion of ball.

1.12. Definition. *A function $f\colon \mathcal{X} \to \mathcal{Y}$ between metric spaces is called* continuous *if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$f(\mathrm{B}(x, \delta)_{\mathcal{X}}) \subset \mathrm{B}(f(x), \varepsilon)_{\mathcal{Y}}.$$

1.13. Exercise. *Prove the equivalence of definitions 1.5 and 1.12.*

E Open sets

1.14. Definition. *A subset V in a metric space $\mathcal{X}$ is called* open *if for any $x \in V$ there is $\varepsilon > 0$ such that $\mathrm{B}(x, \varepsilon) \subset V$.*

In other words, V is open if, together with each point, V contains its ε-neighborhood for some $\varepsilon > 0$. For example, any set in a descrete metric space is open since together with any point it contains its 1-neighborhood. Further the set of positive real numbers

$$(0, \infty) = \{\, x \in \mathbb{R} \mid x > 0 \,\}$$

is open since together with each point $x > 0$ it contains its x-neighborhood. On the other hand, the set of nonnegative reals

$$[0, \infty) = \{\, x \in \mathbb{R} \mid x \geq 0 \,\}$$

is not open since there are negative numbers in any neighborhood of 0.

1.15. Exercise. *Show that any ball in a metric space is open.*

1.16. Exercise. *Show that any open set in a metric space is a union of balls.*

1.17. Exercise. *Show that union of arbitrary collection of open sets is open.*

1.18. Exercise. *Show that intersection of two open sets is open.*

1.19. Exercise. *Give an example of metric space $\mathcal{X}$ and an infinite sequence of open sets $V_1, V_2, \ldots$ such that the intesection*

$$\bigcap_n V_n$$

is not open.

1.20. Exercise. *Show that the metrics $|{*}-{*}|_1$, $|{*}-{*}|_2$ and $|{*}-{*}|_\infty$ (defined on page 6) give rise to the same open sets in $\mathbb{R}^2$. That is, if $V \subset \mathbb{R}^2$ is open for one of these metrics then it is open for the others.*

F Gateway to topology

The following result is the main gateway to topology. It says that continuuous maps can be defined entirely in terms of open sets.

1.21. Proposition. *A function $f\colon \mathcal{X} \to \mathcal{Y}$ between two metric spaces is continuous if and only if for any open set $W \subset \mathcal{Y}$ its inverse images*

$$f^{-1}(W) = \{\, x \in \mathcal{X} \mid f(x) \in W \,\}$$

is open.

Note that proposition says nothing about the images of open sets. In fact, before going into proof it would be useful to solve the following exercise.

1.22. Exercise. *Give an example of a continuous $f\colon \mathbb{R}\to\mathbb{R}$ and an open set $V\subset\mathbb{R}$ such that the image $f(V)\subset\mathbb{R}$ is not open.*

The formulation of proposition contains "if and only if" and the proof brakes into two parta "if"-part and "only if"-part.

Proof; "only-if" part. Let $W\subset\mathcal{Y}$ be an open set and $V=f^{-1}(W)$. Choose $x\in V$; note that so $f(x)\in W$.

Since W is open,

❶ $$\mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}\subset W$$

for some $\varepsilon>0$.

Since f is continuous, by Definition 1.12, there is $\delta>0$ such that

$$f(B(x,\delta)_{\mathcal{X}})\subset \mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}.$$

It follows that together with any point $x\in V$, the set V contains $B(x,\delta)$; that is, V is open.

"If" part. Fix $x\in\mathcal{X}$ and $\varepsilon>0$. According to Exercise 1.15,

$$W=\mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}$$

is an open set in $\mathcal{Y}$. Therefore its inverse image $f^{-1}(W)$ is open.

Clearly $x\in f^{-1}(W)$. By the definition of open set (1.14)

$$\mathrm{B}(x,\delta)_{\mathcal{X}}\subset f^{-1}(W)$$

for some $\delta>0$. Or equivalently

$$f(\mathrm{B}(x,\delta)_{\mathcal{X}})\subset W=\mathrm{B}(f(x),\varepsilon)_{\mathcal{Y}}.$$

Hence the "if"-part follows. □

G Limits

1.23. Definition. *Let $(x_n)=x_1,x_2,\dots$ be a sequence of points in a metric space $\mathcal{X}$. We say the the sequence x_n converges to a point $x\in\mathcal{X}$ if*

$$|x-x_n|_{\mathcal{X}}\to 0\quad\textit{as}\quad n\to\infty.$$

In this case we say that the sequence (x_n) is a converging sequence and x is its limit; the latter will be written as

$$x = \lim_{n\to\infty} x_n$$

Note that we defined the convergence of points in a metric space using the convergence of real numbers $d_n = |x - x_n|_{\mathcal{X}}$, which we assume to be known.

1.24. Exercise. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a function between metric spaces. Show that f is continuous if and only if for any converging sequence (x_n) in $\mathcal{X}$ the sequence $y_n = f(x_n)$ is converging in $\mathcal{Y}$ and*

$$f(x_\infty) = y_\infty,$$

if $x_n \to x_\infty$ and $y_n \to y_\infty$ as $n \to \infty$.

H Closed sets

Let A be a set in a metric space $\mathcal{X}$. A point $x \in \mathcal{X}$ is called *limit point* of A if there is a sequence $x_n \in A$ such that $x_n \to x$ as $n \to \infty$.

The set of all limit points of A is called *closure* of A and denoted as $\bar{A}$. Note that $\bar{A} \supset A$; indeed, any point $x \in A$ is a limit point of the constant sequence $x_n = x$.

If $\bar{A} = A$ then the set is called *closed*.

1.25. Exercise. *Show that closure of any set in metric space is a closed set; that is $\bar{\bar{A}} = \bar{A}$.*

1.26. Exercise. *Show that a subset A in a metric space $\mathcal{X}$ is closed if and only if its complement $\mathcal{X}\backslash A$ is open.*