

1 The formula

Let L be a smooth n -dimensional submanifold in a Riemannian manifold M . Let us denote by $T_p L$ and $N_p L$ the tangent and normal spaces of L at p .

Recall that *second fundamental form* \mathbb{I} at p is a symmetric quadratic form on $T_p L$ with values in $N_p L$. It is uniquely defined by the following identity

$$\mathbb{I}(v, v) := \gamma_v''(0),$$

where $v \in T_p L$ and γ_v an L -geodesic that starts at p with initial velocity vector v .

Given $p \in L$, denote by $\text{Sc}(p)$, $H(p)$, and $\mathcal{K}(p)$ the scalar curvature, the mean curvature vector, and the average value of $|\mathbb{I}(u, u)|^2$ at p respectively.

Let us denote by $\widetilde{\text{Sc}}$ the *outer scalar curvature* of L in M ; that is, if e_1, \dots, e_n is an orthonormal basis of $T_p L$, then

$$\widetilde{\text{Sc}}(p) = 2 \cdot \sum_{i < j} K_{ij},$$

where K_{ij} denotes sectional curvature of M in the direction spanned by e_i and e_j .

The following claim gives a better version of [4, 5.B].

Formula. *The following identity*

$$\text{Sc} - \widetilde{\text{Sc}} = \frac{3}{2} \cdot |H|^2 - \frac{n \cdot (n+2)}{2} \cdot \mathcal{K}$$

holds for any smooth n -dimensional submanifold L in a Riemannian manifold.

Proof. Without loss of generality, we can assume that the ambient manifold is flat; in particular $\widetilde{\text{Sc}} = 0$.

Assume $\text{codim } L = 1$. Choose $p \in L$; denote by k_1, \dots, k_n the principal curvatures of L at p . Note that

$$|H|^2 = \sum_i k_i^2 + 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

Further,

$$n \cdot (n+2) \cdot \mathcal{K} = 3 \cdot \sum_i k_i^2 + 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

The last identity follows since \mathcal{K} is the average value of $(\sum_i k_i \cdot x_i^2)^2$ on \mathbb{S}^{n-1} . One has to take into account that 3 and 1 are the average values of $n \cdot (n+2) \cdot x_i^4$ and $n \cdot (n+2) \cdot x_i^2 \cdot x_j^2$ for $i \neq j$ on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Here we assume that (x_1, \dots, x_n) are the standard coordinates in \mathbb{R}^n .

By Gauss formula,

$$\text{Sc} = 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

It remains to rewrite this formula using the expressions for $|H|^2$ and \mathcal{K} .

If $\text{codim } L = k > 1$, then the second fundamental form can be presented as a direct sum of k real-valued quadratic forms $\mathbb{I}_1 \oplus \dots \oplus \mathbb{I}_k$; that is,

$$\mathbb{I} = e_1 \cdot \mathbb{I}_1 + \dots + e_k \cdot \mathbb{I}_k,$$

where e_1, \dots, e_k is an orthonormal basis of $N_p L$. Denote by Sc_i , H_i , and \mathcal{K}_i the values associated with \mathbb{I}_i . From above, we get

$$\text{Sc}_i = \frac{3}{2} \cdot |H_i|^2 - \frac{n \cdot (n+2)}{2} \cdot \mathcal{K}_i$$

for each i .

Note that

$$\begin{aligned} \text{Sc} &= \text{Sc}_1 + \dots + \text{Sc}_k, \\ |H|^2 &= |H_1|^2 + \dots + |H_k|^2, \\ \mathcal{K} &= \mathcal{K}_1 + \dots + \mathcal{K}_k. \end{aligned}$$

Whence the general case follows. \square

2 Embeddings into sphere

The obtained formula shows that some results in [2–4] are exact. In this and the following section, we will list of some of them.

Let us denote by \mathbb{T}^n the n -dimensional torus — the smooth manifold diffeomorphic to the product of n circles. The next statement follows from the formula since any Riemannian metric on the torus has nonpositive scalar curvature at some point.

Theorem. *Suppose $\iota: \mathbb{T}^n \hookrightarrow \mathbb{S}^q$ is a smooth immersion. Then*

$$\mathcal{K}(p) \geq 2 \cdot \frac{n-1}{n+2}$$

at some point $p \in \mathbb{T}^n$.

In particular, there is a tangent direction of \mathbb{T}^n with normal curvature at least

$$\kappa_n = \sqrt{2 \cdot \frac{n-1}{n+2}}.$$

It was shown [4] that there is an isometric embedding of the torus \mathbb{T}^n with a flat metric that has normal curvature κ_n in any direction at any point. In particular, the above bound on normal curvature is optimal. The compression lemma [3], implies that *any closed smooth manifold is diffeomorphic to a submanifold with normal curvatures at most $\sqrt{2}$ in the unit sphere of sufficiently large dimension. Moreover, the induced Riemannian metric can be chosen to be proportional to any given metric g .* Applying the theorem, we get the following.

Corollary. *The bound $\sqrt{2}$ is optimal.*

3 Embeddings into ball

Let us denote by \mathbb{B}^q the unit ball in q -dimensional Euclidean space.

Question. Suppose $\iota: \mathbb{T}^n \rightarrow \mathbb{B}^q$ is a smooth immersion. Is it true that $\mathcal{HK}(p) \geq 3 \cdot \frac{n}{n+2}$ at some point $p \in \mathbb{T}^n$?

If true, then for $q \gg n^2 \gg 1$, the optimal asymptotic lower bound on normal curvatures is $\sqrt{3}$. Playing a bit with the formulas below seems to give an asymptotic lower bound $\sqrt{8/3}$; it is quite close to $\sqrt{3}$ and can be improved a bit further, but the optimal bound requires an extra idea.

Below we answer the question in three cases: $n = 2$, $n = 4$, and if the induced metric is flat. Since $2 \cdot \frac{n-1}{n+2} + 1 = 3 \cdot \frac{n}{n+2}$, the theorem in the previous section answers the question if the image of ι lies in the boundary of the ball.

The following lemma was essentially proved by István Fáry [1]; see also the survey of Serge Tabachnikov [5].

Lemma. Let $\iota: \mathbb{T}^n \rightarrow \mathbb{B}^q$ be a smooth immersion. Then the average value of $|H|$ is at least n .

Proof. Consider the function $u: p \mapsto \frac{1}{2} \cdot |\iota(p)|^2$ on \mathbb{T}^n . Note that $\Delta u = n + \langle H, \iota \rangle$. It follows that the average value of $\langle H, \iota \rangle$ is $-n$. Since $|\iota| \leq 1$, we get the result. \square

The lemma and formula imply the following.

Proposition. Let L be a flat closed n -dimensional submanifold in \mathbb{B}^q . Then the average value of \mathcal{HK} on L is at least $3 \cdot \frac{n}{n+2}$.

Theorem. Suppose $\iota: \mathbb{T}^2 \rightarrow \mathbb{B}^q$ is a smooth immersion. Then the average value of \mathcal{HK} on \mathbb{T}^2 is at least $\frac{3}{2}$.

Proof. By the lemma, the average value of $|H|^2$ is at least 4. Applying the formula and Gauss–Bonnet, we get the result. \square

Theorem. Suppose $\iota: \mathbb{T}^4 \rightarrow \mathbb{B}^q$ is a smooth immersion. Then $\mathcal{HK}(p) \geq 2$ for some point $p \in \mathbb{T}^4$.

Let g be a Riemannian metric on \mathbb{T}^n . Suppose $n \geq 3$, and $u: \mathbb{T}^n \rightarrow \mathbb{R}$ is a positive function. The scalar curvature of the metric $u^{\frac{4}{n-2}} \cdot g$ can be expressed as

$$\left(\text{Sc} \cdot u - \frac{4 \cdot (n-1)}{n-2} \cdot \Delta u \right) \cdot u^{\frac{n-2}{n+2}}.$$

Recall that any Riemannian metric g on \mathbb{T}^n has nonpositive scalar curvature at some point. Therefore we get the following.

Claim. Let g be a Riemannian metric on \mathbb{T}^n . Then, for any positive smooth function u on \mathbb{T}^n , the function

$$\text{Sc} \cdot u - \frac{4 \cdot (n-1)}{n-2} \cdot \Delta u$$

returns a nonpositive value at some point

Proof of the theorem. Consider the function $u: p \mapsto \exp(-|\iota(p)|^2)$.

We will apply the following formula

$$\Delta(\varphi \circ f) = \varphi' \cdot \Delta f + \varphi'' \cdot |\nabla f|^2$$

to $f: p \mapsto \frac{1}{2} \cdot |\iota(p)|^2$ and $\varphi: x \mapsto \exp(-2 \cdot x)$; so $u = \varphi \circ f$.

Let $\alpha = \angle(H, \iota)$, then $\Delta f = 4 + |H| \cdot |\iota| \cdot \cos \alpha$ and $|\nabla f| \leq |\iota| \cdot \sin \alpha$. Therefore

$$\Delta u \leq u \cdot [-2 \cdot (4 + |H| \cdot |\iota| \cdot \cos \alpha) + (2 \cdot |\iota| \cdot \sin \alpha)^2].$$

Applying the claim for $n = 4$, we get that

$$\text{Sc} \cdot u - 6 \cdot \Delta u$$

returns a negative value at some point $p \in \mathbb{T}^4$. Applying the formula, we get

$$12 \cdot \mathcal{K}(p) \geq \frac{3}{2} \cdot [(|H(p)| - 4)^2 + 16].$$

Whence the statement follows. □

References

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