

# 1 Gauss formula

Let  $L$  be a smooth  $n$ -dimensional submanifold in a Riemannian manifold  $M$ . Let us denote by  $T_p L$  and  $N_p L$  the tangent and normal spaces of  $L$  at  $p$ .

Recall that *second fundamental form*  $\mathbb{I}$  at  $p$  is a symmetric quadratic form on  $T_p L$  with values in  $N_p L$ . It is uniquely defined by the following identity

$$\mathbb{I}(v, v) := \gamma_v''(0),$$

where  $v \in T_p L$  and  $\gamma_v$  an  $L$ -geodesic that starts at  $p$  with initial velocity vector  $v$ .

Given  $p \in L$ , denote by  $\text{Sc}(p)$ ,  $H(p)$ , and  $\mathcal{K}(p)$  the scalar curvature, the mean curvature vector, and the average value of  $|\mathbb{I}(u, u)|^2$  at  $p$  respectively.

Let us denote by  $\widetilde{\text{Sc}}$  the *outer scalar curvature* of  $L$  in  $M$ ; that is, if  $e_1, \dots, e_n$  is an orthonormal basis of  $T_p L$ , then

$$\widetilde{\text{Sc}}(p) = 2 \cdot \sum_{i < j} K_{ij},$$

where  $K_{ij}$  denotes sectional curvature of  $M$  in the direction spanned by  $e_i$  and  $e_j$ .

The following claim gives a better version of [4, 5.B].

**Formula.** *The following identity*

$$\text{Sc} - \widetilde{\text{Sc}} = \frac{3}{2} \cdot |H|^2 - \frac{n \cdot (n+2)}{2} \cdot \mathcal{K}$$

*holds for any smooth  $n$ -dimensional submanifold  $L$  in a Riemannian manifold.*

*Proof.* Without loss of generality, we can assume that the ambient manifold is flat; in particular  $\widetilde{\text{Sc}} = 0$ .

Assume  $\text{codim } L = 1$ . Choose  $p \in L$ ; denote by  $k_1, \dots, k_n$  the principal curvatures of  $L$  at  $p$ . Note that

$$|H|^2 = \sum_i k_i^2 + 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

Further,

$$n \cdot (n+2) \cdot \mathcal{K} = 3 \cdot \sum_i k_i^2 + 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

The last identity follows since  $\mathcal{K}$  is the average value of  $(\sum_i k_i \cdot x_i^2)^2$  on  $\mathbb{S}^{n-1}$ . One has to take into account that 3 and 1 are the average values of  $n \cdot (n+2) \cdot x_i^4$  and  $n \cdot (n+2) \cdot x_i^2 \cdot x_j^2$  for  $i \neq j$  on the unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ . Here we assume that  $(x_1, \dots, x_n)$  are the standard coordinates in  $\mathbb{R}^n$ .

By Gauss formula,

$$\text{Sc} = 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

It remains to rewrite this formula using the expressions for  $|H|^2$  and  $\mathcal{K}$ .

If  $\text{codim } L = k > 1$ , then the second fundamental form can be presented as a direct sum of  $k$  real-valued quadratic forms  $\mathbb{I}_1 \oplus \dots \oplus \mathbb{I}_k$ ; that is,

$$\mathbb{I} = e_1 \cdot \mathbb{I}_1 + \dots + e_k \cdot \mathbb{I}_k,$$

where  $e_1, \dots, e_k$  is an orthonormal basis of  $N_p L$ . Denote by  $\text{Sc}_i$ ,  $H_i$ , and  $\mathcal{K}_i$  the values associated with  $\mathbb{I}_i$ . From above, we get

$$\text{Sc}_i = \frac{3}{2} \cdot |H_i|^2 - \frac{n \cdot (n+2)}{2} \cdot \mathcal{K}_i$$

for each  $i$ .

Note that

$$\begin{aligned} \text{Sc} &= \text{Sc}_1 + \dots + \text{Sc}_k, \\ |H|^2 &= |H_1|^2 + \dots + |H_k|^2, \\ \mathcal{K} &= \mathcal{K}_1 + \dots + \mathcal{K}_k. \end{aligned}$$

Whence the general case follows.  $\square$

## 2 Embeddings into sphere

The obtained formula shows that some results in [2–4] are exact. In this and the following section, we will list of some of them.

Let us denote by  $\mathbb{T}^n$  the  $n$ -dimensional torus — the smooth manifold diffeomorphic to the product of  $n$  circles. The next statement follows from the formula since any Riemannian metric on the torus has nonpositive scalar curvature at some point.

**Theorem.** *Suppose  $\iota: \mathbb{T}^n \hookrightarrow \mathbb{S}^q$  is a smooth immersion. Then*

$$\mathcal{K}(p) \geq 2 \cdot \frac{n-1}{n+2}$$

*at some point  $p \in \mathbb{T}^n$ .*

*In particular, there is a tangent direction of  $\mathbb{T}^n$  with normal curvature at least*

$$\kappa_n = \sqrt{2 \cdot \frac{n-1}{n+2}}.$$

It was shown [4] that there is an isometric embedding of the torus  $\mathbb{T}^n$  with a flat metric that has normal curvature  $\kappa_n$  in any direction at any point. In particular, the above bound on normal curvature is optimal. The compression lemma [3], implies that *any closed smooth manifold is diffeomorphic to a submanifold with normal curvatures at most  $\sqrt{2}$  in the unit sphere of sufficiently large dimension. Moreover, the induced Riemannian metric can be chosen to be proportional to any given metric  $g$ .* Applying the theorem, we get the following.

**Corollary.** *The bound  $\sqrt{2}$  is optimal.*

### 3 Embeddings into ball

Let us denote by  $\mathbb{B}^q$  the unit ball in  $q$ -dimensional Euclidean space.

**Question.** Suppose  $\iota: \mathbb{T}^n \rightarrow \mathbb{B}^q$  is a smooth immersion. Is it true that  $\mathcal{K}(p) \geq 3 \cdot \frac{n}{n+2}$  at some point  $p \in \mathbb{T}^n$ ?

If true, then for  $q \gg n^2 \gg 1$ , the optimal asymptotic lower bound on normal curvatures is  $\sqrt{3}$ . Playing a bit with the formulas below seems to give an asymptotic lower bound  $\sqrt{8/3}$ ; it is quite close to  $\sqrt{3}$  and can be improved a bit further, but the optimal bound requires an extra idea.

Below we answer the question in three cases:  $n = 2$ ,  $n = 4$ , and if the induced metric is flat. Since  $2 \cdot \frac{n-1}{n+2} + 1 = 3 \cdot \frac{n}{n+2}$ , the theorem in the previous section answers the question if the image of  $\iota$  lies in the boundary of the ball.

The following lemma was essentially proved by István Fáry [1]; see also the survey of Serge Tabachnikov [5].

**Lemma.** Let  $\iota: \mathbb{T}^n \rightarrow \mathbb{B}^q$  be a smooth immersion. Then the average value of  $|H|$  is at least  $n$ .

*Proof.* Consider the function  $u: p \mapsto \frac{1}{2} \cdot |\iota(p)|^2$  on  $\mathbb{T}^n$ . Note that  $\Delta u = n + \langle H, \iota \rangle$ . It follows that the average value of  $\langle H, \iota \rangle$  is  $-n$ . Since  $|\iota| \leq 1$ , we get the result.  $\square$

The lemma and formula imply the following.

**Proposition.** Let  $L$  be a flat closed  $n$ -dimensional submanifold in  $\mathbb{B}^q$ . Then the average value of  $\mathcal{K}$  on  $L$  is at least  $3 \cdot \frac{n}{n+2}$ .

**Theorem.** Suppose  $\iota: \mathbb{T}^2 \rightarrow \mathbb{B}^q$  is a smooth immersion. Then the average value of  $\mathcal{K}$  on  $\mathbb{T}^2$  is at least  $\frac{3}{2}$ .

*Proof.* By the lemma, the average value of  $|H|^2$  is at least 4. Applying the formula and Gauss–Bonnet, we get the result.  $\square$

**Theorem.** Suppose  $\iota: \mathbb{T}^4 \rightarrow \mathbb{B}^q$  is a smooth immersion. Then  $\mathcal{K}(p) \geq 2$  for some point  $p \in \mathbb{T}^4$ .

Let  $g$  be a Riemannian metric on  $\mathbb{T}^n$ . Suppose  $n \geq 3$ , and  $u: \mathbb{T}^n \rightarrow \mathbb{R}$  is a positive function. The scalar curvature of the metric  $u^{\frac{4}{n-2}} \cdot g$  can be expressed as

$$\left( \text{Sc} \cdot u - \frac{4 \cdot (n-1)}{n-2} \cdot \Delta u \right) \cdot u^{\frac{n-2}{n+2}}.$$

Recall that any Riemannian metric  $g$  on  $\mathbb{T}^n$  has nonpositive scalar curvature at some point. Therefore we get the following.

**Claim.** Let  $g$  be a Riemannian metric on  $\mathbb{T}^n$ . Then, for any positive smooth function  $u$  on  $\mathbb{T}^n$ , the function

$$\text{Sc} \cdot u - \frac{4 \cdot (n-1)}{n-2} \cdot \Delta u$$

returns a nonpositive value at some point

*Proof of the theorem.* Consider the function  $u: p \mapsto \exp(-|\iota(p)|^2)$ .

We will apply the following formula

$$\Delta(\varphi \circ f) = \varphi' \cdot \Delta f + \varphi'' \cdot |\nabla f|^2$$

to  $f: p \mapsto \frac{1}{2} \cdot |\iota(p)|^2$  and  $\varphi: x \mapsto \exp(-2 \cdot x)$ ; so  $u = \varphi \circ f$ .

Let  $\alpha = \angle(H, \iota)$ , then  $\Delta f = 4 + |H| \cdot |\iota| \cdot \cos \alpha$  and  $|\nabla f| \leq |\iota| \cdot \sin \alpha$ . Therefore

$$\Delta u \leq u \cdot [-2 \cdot (4 + |H| \cdot |\iota| \cdot \cos \alpha) + (2 \cdot |\iota| \cdot \sin \alpha)^2].$$

Applying the claim for  $n = 4$ , we get that

$$\text{Sc} \cdot u - 6 \cdot \Delta u$$

returns a negative value at some point  $p \in \mathbb{T}^4$ . Applying the formula, we get

$$12 \cdot \mathcal{K}(p) \geq \frac{3}{2} \cdot [(|H(p)| - 4)^2 + 16].$$

Whence the statement follows.  $\square$

## References

- [1] I. Fáry. “Sur certaines inégalités géométriques”. *Acta Sci. Math.* 12 (1950), 117–124.
- [2] M. Gromov. *Curvature, Kolmogorov Diameter, Hilbert Rational Designs and Overtwisted Immersions*. 2022.
- [3] M. Gromov. *Isometric Immersions with Controlled Curvatures*. 2022.
- [4] M. Gromov. *Scalar Curvature, Injectivity Radius and Immersions with Small Second Fundamental Forms*. 2022.
- [5] S. Tabachnikov. “The tale of a geometric inequality.” *MASS selecta: teaching and learning advanced undergraduate mathematics*. 2003, 257–262.