# Normal curvatures of torii

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**Abstract**

We give an optimal lower bound on principal curvatures of immersed $n$-torus in a Euclidean ball of large dimension.

## 1 Introduction

This note is inspired by examples of isometric embeddings of $n$-torus $\mathbb{T}^n$ into the unit ball $\mathbb{B}^q$ of $q$-dimensional Euclidean space for all large $q$ such that all it principal curvatures identically equal to $\sqrt{3{\cdot}n/(n+2)}$ and the induced metric is flat. These examples are geodesic subtorii in Clifford torii; they were constructed by Michael Gromov; see [4, 2.A] and [3, 1.1.A.].

The existence of such examples implies the following surprising results: *any closed smooth manifold admits a smooth embedding into $\mathbb{B}^q$ for large $q$ with normal curvatures less than $\sqrt{3}$. Moreover, the induced Riemannian metric can be chosen to be proportional to any given metric $g$; see [4, 1.D] and [3, 1.1.C].*

The following theorem implies that the bound $\sqrt{3{\cdot}n/(n+2)}$ is optimal.

**1.1. Theorem.** *Suppose $\mathbb{T}^n$ is smoothly immersed in $\mathbb{B}^q$. Then its maximal normal curvature is at least $\sqrt{3{\cdot}n/(n+2)}$.*

To make the statement more exact, we need some notations. Assume that a smooth $n$-dimensional manifold $L$ is immersed in the Euclidean space $\mathbb{R}^q$; we will always assume that $L$ is equipped with induced Riemannian metric. Let us denote by $\mathrm{T}_x$ and $\mathrm{N}_x$ the tangent and normal spaces of $L$ at $x$.

Recall that *second fundamental form* $\mathrm{I\!I}$ at $x$ is a symmetric quadratic form on $\mathrm{T}_x$ with values in $\mathrm{N}_x$. It is uniquely defined by the following identity

$$\mathrm{I\!I}(\mathrm{v},\mathrm{v}) \equiv \gamma_{\mathrm{v}}''(0),$$

where $\mathrm{v} \in \mathrm{T}_x$ and $\gamma_{\mathrm{v}}$ an $L$-geodesic that starts at $x$ with initial velocity vector $\mathrm{v}$.

Given $x \in L$, denote by $\reflectbox{K}\!K(x)$ the average value of $|\mathrm{I\!I}(\mathrm{u},\mathrm{u})|^2$ for $\mathrm{u} \in \mathrm{T}_x$ such that $|\mathrm{u}| = 1$; in other words, $\reflectbox{K}\!K(x)$ is the average value of squared normal curvatures at $x$. (The letter $\reflectbox{K}\!K$ suppose to remind that it is squared $K$.)

**1.2. Theorem.** *Suppose $\mathbb{T}^n$ is smoothly immersed in $\mathbb{B}^q$. Let us equip $\mathbb{T}^n$ with the induced Riemannian metric; so we can take average values with respect to the induced volume.*

*(a) If $n = 2$, then the average value of $\reflectbox{K}\!K$ is at least $\frac{3}{2}$.*
*(b) If the metric on $\mathbb{T}^n$ is flat, then the average value of $\reflectbox{K}\!K$ is at least $3{\cdot}\frac{n}{n+2}$.*
*(c) If the image of $\mathbb{T}^n$ lies in $\mathbb{S}^{q-1} = \partial\mathbb{B}^q$, then $\reflectbox{K}\!K \geqslant 3{\cdot}\frac{n}{n+2}$ at some point.*
*(d) If $n \leqslant 4$, then $\reflectbox{K}\!K \geqslant 3{\cdot}\frac{n}{n+2}$ at some point of $\mathbb{T}^n$.*

(e) *If the normal curvatures of $\mathbb{T}^n$ do not exceed 2, then $X\!\!\!K \geqslant 3\cdot\frac{n}{n+2}$ at some point of $\mathbb{T}^n$.*

Note that part *(e)* is a stronger version of 1.1. The remaining statements *(a)–(d)* are stronger versions of 1.1 in some partial cases. All this follows since the normal curvature in some direction at $x$ is at least $\sqrt{X\!\!\!K(x)}$.

**1.3. Open question.** *Is it true that for any smooth immersion $\mathbb{T}^n \looparrowright \mathbb{B}^q$ we have $X\!\!\!K \geqslant 3\cdot\frac{n}{n+2}$ at some point?*

# 2 Gauss formula

Recall that $L$ is a smooth $n$-dimensional manifold immersed in $\mathbb{R}^q$. Given $p \in L$, denote by $\mathrm{Sc}(p)$ and $H(p)$ the scalar curvature, the mean curvature vector.

The following version of the Gauss formula plays a central role in all proofs; it is used instead of the formula in [5, 5.B].

**2.1. Gauss formula.** *The following identity*

$$\mathrm{Sc} = \tfrac{3}{2}\cdot|H|^2 - \tfrac{n\cdot(n+2)}{2}\cdot X\!\!\!K$$

*holds for any smooth $n$-dimensional immersed manifold in a Euclidean space.*

*Proof.* Assume $\operatorname{codim} L = 1$. Choose $p \in L$; denote by $k_1, \dots, k_n$ the principal curvatures of $L$ at $p$. Note that

$$|H|^2 = \sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

Further,

$$n\cdot(n+2)\cdot X\!\!\!K = 3\cdot\sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

The last identity follows since $X\!\!\!K$ is the average value of $\left(\sum_i k_i\cdot x_i^2\right)^2$ on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. One has to take into account that the following functions have unit average values: $\frac{1}{3}\cdot n\cdot(n+2)\cdot x_i^4$ and $n\cdot(n+2)\cdot x_i^2\cdot x_j^2$ for $i \neq j$; here we assume that $(x_1, \dots, x_n)$ are the standard coordinates in $\mathbb{R}^n$.

By the standard Gauss formula,

$$\mathrm{Sc} = 2\cdot\sum_{i<j} k_i\cdot k_j.$$

It remains to rewrite the right-hand side using the expressions for $|H|^2$ and $X\!\!\!K$.

If $\operatorname{codim} L = k > 1$, then the second fundamental form can be presented as a direct sum of $k$ real-valued quadratic forms $\mathrm{I\!I}_1 \oplus \dots \oplus \mathrm{I\!I}_k$; that is,

$$\mathrm{I\!I} = e_1\cdot\mathrm{I\!I}_1 + \dots + e_k\cdot\mathrm{I\!I}_k,$$

where $e_1, \dots, e_k$ is an orthonormal basis of $\mathrm{N}_p$. Denote by $\mathrm{Sc}_i$, $H_i$, and $X\!\!\!K_i$ the values associated with $\mathrm{I\!I}_i$. From above, we get

$$\mathrm{Sc}_i = \tfrac{3}{2}\cdot|H_i|^2 - \tfrac{n\cdot(n+2)}{2}\cdot X\!\!\!K_i$$

for each $i$.

Note that

$$\mathrm{Sc} = \sum_i \mathrm{Sc}_i, \qquad |H^2| = \sum_i |H_i|^2, \quad \text{and} \quad ӜK = \sum_i ӜK_i.$$

Hence the general case follows. □

# 3 Special cases

Recall that $\mathbb{T}^n$ is the $n$-dimensional torus — the smooth manifold diffeomorphic to the product of $n$ circles, and the unit ball centered at the origin of the $q$-dimensional Euclidean space $\mathbb{R}^q$ is denoted by $\mathbb{B}^q$.

The following lemma was essentially proved by István Fáry [2, 9].

**3.1. Lemma.** *Let $L$ be a closed $n$-dimension manifold that is smoothly immersed in $\mathbb{B}^q$. Then the average value of $|H|$ on $L$ is at least $n$.*

*Proof.* Consider the function $u\colon p \mapsto \frac{1}{2}\cdot|\iota(p)|^2$ on $L$. Note that

$$\Delta u = n + \langle H, \iota \rangle.$$

It follows that the average value of $\langle H, \iota\rangle$ is $-n$. Since $|\iota| \leqslant 1$, we get the result. □

*Proof of 1.2a.* By 3.1, the average value of $|H|^2$ is at least 4. Further, by the Gauss–Bonnet formula, scalar curvature (which is twice the Gauss curvature in this case) has zero average. Therefore 2.1 implies the statement. □

*Proof of 1.2b.* By 3.1, the average value of $|H|^2$ is at least $n^2$. Since $\mathrm{Sc} \equiv 0$, it remains to apply 2.1. □

*Proof of 1.2c.* Since the image lies in the unit sphere, we have that $|H|^2$ is at least $n^2$ at each point. Since $\mathbb{T}^n$ does not admit a metric with positive scalar curvature [6, Cor. A], we have $\mathrm{Sc}(x) \leqslant 0$ at some point $x$. It remains to apply 2.1 at $x$. □

# 4 Main case

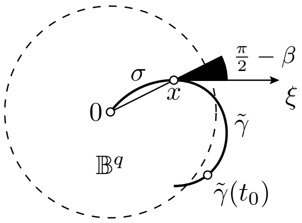
The following lemma is an easy corollary of the bow lemma of Axel Schur [7, 8]. It tells how we use the assumption in 1.2*e* that normal curvatures do not exceed 2. If $n \leqslant 4$, then the proof works without this assumption.

**4.1. Lemma.** *Let $L$ be a manifold smoothly immersed in $\mathbb{B}^q$, and its normal curvatures are at most 2. Given $x \in L$, denote by $\beta$ the angle between vector $x$ and the normal space $\mathrm{N}_x$. Then $|x| \leqslant \cos\beta$.*

*Proof.* Let $\xi$ be a tangent direction at $x$ such that $\measuredangle(x, \xi) = \frac{\pi}{2} - \beta$. In the plane spanned by $x$ and $\xi$, choose a unit-speed circle arc $\sigma$ from 0 to $x$ that comes to $x$ in the direction opposite to $\xi$; extend $\sigma$ after $x$ by a unit-speed semicircle $\tilde\gamma$ with curvature 2 in such a way that the concatenation $\sigma * \tilde\gamma$ is is an arc of a smooth convex plane curve; see the figure.

Observe that if $|x| > \cos\beta$, then $\tilde\gamma$ leaves $\mathbb{B}^q$; that is, $|\tilde\gamma(t_0)| > 1$ for some $t_0$.

Let $\gamma$ be the unit-speed geodesic in $L$ that runs from $x$ in the direction $\xi$. Note that curvatures of $\sigma * \gamma$ do not exceed the curvatures of $\sigma * \tilde\gamma$ at the corresponding points. Applying the bow lemma for $\sigma * \gamma$ and $\sigma * \tilde\gamma$, we get $|\gamma(t_0)| \geqslant |\tilde\gamma(t_0)|$. It follows that $L$ does not lie in $\mathbb{B}^q$ — a contradiction. □




Let $g$ be a Riemannian metric on $\mathbb{T}^n$. Suppose $n \geqslant 3$, and $u\colon \mathbb{T}^n \to \mathbb{R}$ is a smooth positive function. Here is the well-known formula for the scalar curvature of the metric $u^{\frac{4}{n-2}}\cdot g$:

$$\left(\mathrm{Sc}\cdot u - 4\cdot\tfrac{n-1}{n-2}\cdot\Delta u\right)\cdot u^{\frac{n-2}{n+2}};$$

see for example [1, 6.3]. Recall that *any Riemannian metric $g$ on $\mathbb{T}^n$ has nonpositive scalar curvature at some point* [6, Cor. A]. Hence we get the following.

**4.2. Claim.** *Let $g$ be a Riemannian metric on $\mathbb{T}^n$. Then, for any positive smooth function $u$ on $\mathbb{T}^n$, the function*

$$\mathrm{Sc}\cdot u - 4\cdot\tfrac{n-1}{n-2}\cdot\Delta u$$

*returns a nonpositive value at some point.*

This claim plays a central role in the following proof.

*Proof of 1.2d and 1.2e.* The case $n = 2$ follows from 1.2*a*; so we can assume that $n \geqslant 3$. Consider the function $u\colon x \mapsto \exp(-\tfrac{k}{2}\cdot|x|^2)$ on the torus.

We will apply the following formula

$$\Delta(\varphi\circ f) = \varphi'\cdot\Delta f + \varphi''\cdot|\nabla f|^2$$

to $f\colon x \mapsto \tfrac12\cdot|x|^2$ and $\varphi\colon y \mapsto \exp(-k\cdot y)$; so $u = \varphi\circ f$.

Set $r(x) = |x|$, $\alpha(x) = \measuredangle(H(x), x)$ and $\beta(x)$ as in 4.1. Note that

❶ $$\beta \leqslant \alpha \leqslant \pi - \beta.$$

Observe that

$$\Delta f = |H|\cdot r\cdot\cos\alpha + n, \quad |\nabla f| = r\cdot\sin\beta, \quad \varphi' = -k\cdot\varphi, \quad \varphi'' = k^2\cdot\varphi.$$

Therefore

$$\Delta u = u\cdot[-k\cdot|H|\cdot r\cdot\cos\alpha - k\cdot n + k^2\cdot r^2\cdot\sin^2\beta].$$

Recall that $\mathrm{Sc} = -\tfrac{n\cdot(n+2)}{2}\cdot\varkappa + \tfrac32\cdot|H|^2$; see 2.1. By 4.2, the function

$$\begin{aligned}\mathrm{Sc}\cdot u - 4\cdot\tfrac{n-1}{n-2}\cdot\Delta u &= \\ = u\cdot\Big[-\tfrac{n\cdot(n+2)}{2}\cdot\varkappa + \tfrac32\cdot|H|^2 &+ 4\cdot\tfrac{n-1}{n-2}\cdot k\cdot|H|\cdot r\cdot\cos\alpha + \\ &+ 4\cdot\tfrac{n-1}{n-2}\cdot(k\cdot n - k^2\cdot r^2\cdot\sin^2\beta)\Big]\end{aligned}$$

returns a nonpositive value at some point $x \in \mathbb{T}^n$.

Choose

$$k = \tfrac{3}{4}\cdot\tfrac{n-2}{n-1}\cdot n, \quad \text{so} \quad n = \tfrac{4}{3}\cdot\tfrac{n-1}{n-2}\cdot k.$$

At the point $x$, we have

❷
$$\begin{aligned}\tfrac{n\cdot(n+2)}{2}\cdot\mathrm{Sc} \geqslant \tfrac{3}{2}\cdot(|H| + n\cdot r\cdot\cos\alpha)^2 - \tfrac{3}{2}\cdot n^2\cdot r^2\cdot\cos^2\alpha +\\ + 3\cdot n^2 - \tfrac{9}{4}\cdot\tfrac{n-2}{n-1}\cdot n^2\cdot r^2\cdot\sin^2\beta \geqslant\\ \geqslant \tfrac{3}{2}\cdot n^2.\end{aligned}$$

By ❶, $\cos^2\beta+\sin^2\alpha \leqslant 1$. If $n \leqslant 4$, then $\frac{3}{2} \geqslant \frac{9}{4}\cdot\frac{n-2}{n-1}$; therefore the last inequality follows, and it proves 1.2$d$.

If $n \geqslant 5$, then for the last inequality in ❷ we need to use in addition that $r^2 + \sin^2\beta \leqslant 1$ which follows 4.1. Hence 1.2$e$ follows. □

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