1 The formula

Let L be a smooth n-dimensional submanifold in a Riemannian manifold M. Let us denote by T_pL and N_pL the tangent and normal spaces of L at p.

Recall that second fundamental form $\mathrm{I\!I}$ at p is a symmetric quadratic form on T_pL with values in N_pL. It is uniquely defined by the following identity

$$\mathrm{I\!I}(\mathrm{v},\mathrm{v}) := \gamma_{\mathrm{v}}''(0),$$

where $\mathrm{v} \in \mathrm{T}_pL$ and γ_{v} an L-geodesic that starts at p with initial velocity vector v.

Given $p \in L$, denote by $\mathrm{Sc}(p)$, $H(p)$, and (p) the scalar curvature, the mean curvature vector, and the average value of $|\mathrm{I\!I}(\mathrm{u},\mathrm{u})|^2$ at p respectively.

Let us denote by $\widetilde{\mathrm{Sc}}$ the outer scalar curvature of L in M; that is, if $e_1,\dots,e_n$ is an orthonormal basis of T_pL, then

$$\widetilde{\mathrm{Sc}}(p) = 2\cdot\sum_{i<j} K_{ij},$$

where K_{ij} denotes sectional curvature of M in the direction spanned by e_i and e_j.

The following claim gives a better version of [4, 5.B].

Formula. The following identity

$$\mathrm{Sc} - \widetilde{\mathrm{Sc}} = \tfrac{3}{2}\cdot|H|^2 - \tfrac{n\cdot(n+2)}{2}\cdot$$

holds for any smooth n-dimensional submanifold L in a Riemannian manifold.

Proof. Without loss of generality, we can assume that the ambient manifold is flat; in particular $\widetilde{\mathrm{Sc}} = 0$.

Assume $\operatorname{codim} L = 1$. Choose $p \in L$; denote by $k_1,\dots,k_n$ the principal curvatures of L at p. Note that

$$|H|^2 = \sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

Further,

$$n\cdot(n+2)\cdot = 3\cdot\sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

The last identity follows since is the average value of $\left(\sum_i k_i\cdot x_i^2\right)^2$ on $\mathbb{S}^{n-1}$. One has to take into account that 3 and 1 are the average values of $n\cdot(n+2)\cdot x_i^4$ and $n\cdot(n+2)\cdot x_i^2\cdot x_j^2$ for $i \neq j$ on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Here we assume that $(x_1,\dots,x_n)$ are the standard coordinates in $\mathbb{R}^n$.

By Gauss formula,

$$\mathrm{Sc} = 2\cdot \sum_{i<j} k_i \cdot k_j.$$

It remains to rewrite this formula using the expressions for $|H|^2$ and .

If $\operatorname{codim} L = k > 1$, then the second fundamental form can be presented as a direct sum of k real-valued quadratic forms $\mathrm{I\!I}_1 \oplus \dots \oplus \mathrm{I\!I}_k$; that is,

$$\mathrm{I\!I} = e_1 \cdot \mathrm{I\!I}_1 + \dots + e_k \cdot \mathrm{I\!I}_k,$$

where $e_1, \dots, e_k$ is an orthonormal basis of $\mathrm{N}_p L$. Denote by Sc_i, H_i, and ${}_i$ the values associated with $\mathrm{I\!I}_i$. From above, we get

$$\mathrm{Sc}_i = \tfrac{3}{2} \cdot |H_i|^2 - \tfrac{n\cdot(n+2)}{2} \cdot {}_i$$

for each i.

Note that

$$\begin{aligned} \mathrm{Sc} &= \mathrm{Sc}_1 + \dots + \mathrm{Sc}_k, \\ |H^2| &= |H_1|^2 + \dots + |H_k|^2, \\ &= {}_1 + \dots + {}_k. \end{aligned}$$

Whence the general case follows. □

2 Embeddings into sphere

The obtained formula shows that some results in [2–4] are exact. In this and the following section, we will list of some of them.

Let us denote by $\mathbb{T}^n$ the n-dimensional torus — the smooth manifold diffeomorphic to the product of n circles. The next statement follows from the formula since any Riemannian metric on the torus has nonpositive scalar curvature at some point.

Theorem. Suppose $\iota\colon \mathbb{T}^n \looparrowright \mathbb{S}^q$ is a smooth immersion. Then

$$(p) \geqslant \tfrac{2\cdot(n-1)}{n+2}$$

at some point $p \in \mathbb{T}^n$.

In particular, there is a tangent direction of $\mathbb{T}^n$ with normal curvature at least

$$\kappa_n = \sqrt{\tfrac{2\cdot(n-1)}{n+2}}.$$

It was shown [4] that there is an isometric embedding of the torus $\mathbb{T}^n$ with a flat metric that has normal curvature κ_n in any direction at any point. In particular, the above bound on normal curvature is optimal. The compression lemma [3], implies that any closed smooth manifold is diffeomorphic to a submanifold with normal curvatures at most $\sqrt{2}$ in the unit sphere of sufficiently large dimension. Moreover, the induced Riemannian metric can be chosen to be proportional to any given metric g. Applying the theorem, we get the following.

Corollary. The bound $\sqrt{2}$ is optimal.

3 Embeddings into ball

Let us denote by $\mathbb{B}^q$ the unit ball in q-dimensional Euclidean space.

Question. Suppose $\iota\colon \mathbb{T}^n \to \mathbb{B}^q$ is a smooth immersion. Is it true that $(p) \geqslant \frac{3\cdot n}{n+2}$ at some point $p \in \mathbb{T}^n$?

If true, then for large $q \gg n^2 \gg 1$, the optimal asymptotic lower bound on normal curvatures is $\sqrt{3}$. Playing a bit with the formulas below seems to give an asymptotic lower bound $\sqrt{8/3}$; it is quite close to $\sqrt{3}$ and can be improved a bit further, but the optimal bound requires an extra idea.

Below we answer the question in three cases: $n = 2$, $n = 4$, and if the induced metric is flat. The following lemma was essentially proved by Istvбn Fбry [1]; see also the survey of Serge Tabachnikov [5].

Lemma. Let $\iota\colon \mathbb{T}^n \to \mathbb{B}^q$ be a smooth immersion. Then the average value of $|H|$ is at least n.

Proof. Consider the function $u\colon p \mapsto \frac{1}{2}\cdot|\iota(p)|^2$ on $\mathbb{T}^n$. Note that $\Delta u = n + \langle H, \iota\rangle$. It follows that the average value of $\langle H, \iota\rangle$ is $-n$. Since $|\iota| \leqslant 1$, we get the result. □

The lemma and formula imply the following.

Proposition. Let L be a flat closed n-dimensional submanifold in $\mathbb{B}^q$. Then the average value of on L is at least $\frac{3\cdot n}{n+2}$.

Theorem. Suppose $\iota\colon \mathbb{T}^2 \to \mathbb{B}^q$ is a smooth immersion. Then the average value of on $\mathbb{T}^2$ is at least $\frac{3}{2}$.

Proof. By the lemma, the average value of $|H|^2$ is at least 4. Applying the formula and Gauss–Bonnet, we get the result. □

Theorem. Suppose $\iota\colon \mathbb{T}^4 \to \mathbb{B}^q$ is a smooth immersion. Then $(p) \geqslant 2$ for some point $p \in \mathbb{T}^4$.

Let g be a Riemannian metric on $\mathbb{T}^n$. Suppose $n \geqslant 3$, and $u\colon \mathbb{T}^n \to \mathbb{R}$ is a positive function. The scalar curvature of the metric $u^{\frac{4}{n-2}}\cdot g$ can be expressed as

$$\Big(\mathrm{Sc}\cdot u - \tfrac{4\cdot(n-1)}{n-2}\cdot\Delta u\Big)\cdot u^{\frac{n-2}{n+2}}.$$

Recall that any Riemannian metric g on $\mathbb{T}^n$ has nonpositive scalar curvature at some point. Therefore we get the following.

Claim. Let g be a Riemannian metric on $\mathbb{T}^n$. Then, for any positive smooth function u on $\mathbb{T}^n$, the function

$$\mathrm{Sc}\cdot u - \tfrac{4\cdot(n-1)}{n-2}\cdot\Delta u$$

returns a nonpositive value at some point

Proof of the theorem. Consider the function $u\colon p \mapsto \exp(-|\iota(p)|^2)$.

We will apply the following formula

$$\Delta(\varphi\circ f) = \varphi'\cdot\Delta f + \varphi''\cdot|\nabla f|^2$$

to $f\colon p \mapsto \frac{1}{2}\cdot|\iota(p)|^2$ and $\varphi\colon x \mapsto \exp(-2\cdot x)$; so $u = \varphi \circ f$.

Let $\alpha = \measuredangle(H, \iota)$, then $\Delta f = 4+|H|\cdot|\iota|\cdot\cos\alpha$ and $|\nabla f| \leqslant |\iota|\cdot\sin\alpha$. Therefore

$$\Delta u \leqslant u\cdot[-2\cdot(4 + |H|\cdot|\iota|\cdot\cos\alpha) + (2\cdot|\iota|\cdot\sin\alpha)^2].$$

Applying the claim for $n = 4$, we get that

$$\mathrm{Sc}\cdot u - 6\cdot\Delta u$$

returns a negative value at some point $p \in \mathbb{T}^4$. Applying the formula, we get

$$12\cdot(p) \geqslant \tfrac{3}{2}\cdot[(|H(p)| - 4)^2 + 16].$$

Whence the statement follows. $\square$

References

[1] I. Fбry. "Sur certaines inйgalitйs gйomйtriques". Acta Sci. Math. 12 (1950), 117–124.

[2] M. Gromov. Curvature, Kolmogorov Diameter, Hilbert Rational Designs and Overtwisted Immersions. 2022.

[3] M. Gromov. Isometric Immersions with Controlled Curvatures. 2022.

[4] M. Gromov. Scalar Curvature, Injectivity Radius and Immersions with Small Second Fundamental Forms. 2022.

[5] S. Tabachnikov. "The tale of a geometric inequality." MASS selecta: teaching and learning advanced undergraduate mathematics. 2003, 257–262.