

# Lower bound on normal curvatures of torus

Anton Petrunin

## Abstract

We give an optimal lower bound on normal curvatures of immersed  $n$ -torus in a Euclidean ball of large dimension.

## 1 Introduction

This note is inspired by examples of isometric embeddings of  $n$ -torus  $\mathbb{T}^n$  into the unit ball  $\mathbb{B}^q$  of  $q$ -dimensional Euclidean space for all large  $q$  such that all its normal curvatures identically equal to  $\sqrt{3 \cdot n/(n+2)}$  and the induced metric is flat. These examples are geodesic subtorii in Clifford torii; they were constructed by Michael Gromov; see [5, 2.A] and [4, 1.1.A.].

The existence of such examples implies the following surprising results: *any closed smooth manifold admits a smooth embedding into  $\mathbb{B}^q$  for large  $q$  with normal curvatures less than  $\sqrt{3}$ . Moreover, the induced Riemannian metric can be chosen to be proportional to any given metric  $g$ ; see [5, 1.D] and [4, 1.1.C].*

Our theorem implies that the bounds above are optimal.

**1.1. Theorem.** *Suppose  $\mathbb{T}^n$  is smoothly immersed in  $\mathbb{B}^q$ . Then its maximal normal curvature is at least*

$$\sqrt{3 \cdot \frac{n}{n+2}}.$$

To make the statement more exact, we need some notations. Assume that a smooth  $n$ -dimensional manifold  $L$  is immersed in the Euclidean space  $\mathbb{R}^q$ ; we will always assume that  $L$  is equipped with induced Riemannian metric. Let us denote by  $T_x$  and  $N_x$  the tangent and normal spaces of  $L$  at  $x$ .

Recall that *second fundamental form*  $\mathbb{I}$  at  $x$  is a symmetric quadratic form on  $T_x$  with values in  $N_x$ . It is uniquely defined by the following identity

$$\mathbb{I}(v, v) \equiv \gamma_v''(0),$$

where  $v \in T_x$  and  $\gamma_v$  an  $L$ -geodesic that starts at  $x$  with initial velocity vector  $v$ .

Given  $x \in L$ , denote by  $\mathcal{K}(x)$  the average value of  $|\mathbb{I}(u, u)|^2$  for  $u \in T_x$  such that  $|u| = 1$ ; in other words,  $\mathcal{K}(x)$  is the average value of squared normal curvatures at  $x$ . (The letter  $\mathcal{K}$  should resemble squared  $K$ .)

**1.2. Theorem.** *Suppose  $\mathbb{T}^n$  is smoothly immersed in  $\mathbb{B}^q$ . Let us equip  $\mathbb{T}^n$  with the induced Riemannian metric; so we can take average values with respect to the induced volume.*

- (a) *If  $n = 2$ , then the average value of  $\mathcal{K}$  is at least  $\frac{3}{2}$ .*
- (b) *If the metric on  $\mathbb{T}^n$  is flat, then the average value of  $\mathcal{K}$  is at least  $3 \cdot \frac{n}{n+2}$ .*
- (c) *If the image of  $\mathbb{T}^n$  lies in  $\mathbb{S}^{q-1} = \partial\mathbb{B}^q$ , then  $\mathcal{K} \geq 3 \cdot \frac{n}{n+2}$  at some point.*

- (d) If  $n \leq 4$ , then  $\mathcal{K} \geq 3 \cdot \frac{n}{n+2}$  at some point of  $\mathbb{T}^n$ .  
(e) If the normal curvatures of  $\mathbb{T}^n$  do not exceed 2, then  $\mathcal{K} \geq 3 \cdot \frac{n}{n+2}$  at some point of  $\mathbb{T}^n$ .

Note that part (e) is a stronger version of 1.1. The remaining statements (a)–(d) are stronger versions of 1.1 in some partial cases. All this follows since the normal curvature in some direction at  $x$  is at least  $\sqrt{\mathcal{K}(x)}$ .

All proofs use our version of the Gauss formula; see below. The proofs of (c)–(e) use in addition that *torus does not admit a metric with positive scalar curvature* [7, Corollary A].

**1.3. Open question.** *Is it true that for any smooth immersion  $\mathbb{T}^n \looparrowright \mathbb{B}^q$  the inequality  $\mathcal{K} \geq 3 \cdot \frac{n}{n+2}$  holds at some point?*

## 2 Gauss formula

Recall that  $L$  is a smooth  $n$ -dimensional manifold immersed in  $\mathbb{R}^q$ . Given  $p \in L$ , denote by  $\text{Sc}(p)$  and  $H(p)$  the scalar curvature, the mean curvature vector at  $p$ .

The following version of the Gauss formula plays a central role in all proofs; it is used instead of the formula in [6, 5.B].

**2.1. Gauss formula.** *The following identity*

$$\text{Sc} = \frac{3}{2} \cdot |H|^2 - \frac{n \cdot (n+2)}{2} \cdot \mathcal{K}$$

*holds for any smooth  $n$ -dimensional immersed manifold in a Euclidean space.*

*Proof.* Assume  $\text{codim } L = 1$ . Choose  $p \in L$ ; denote by  $k_1, \dots, k_n$  the principal curvatures of  $L$  at  $p$ . Note that

$$|H|^2 = \sum_i k_i^2 + 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

Further,

$$n \cdot (n+2) \cdot \mathcal{K} = 3 \cdot \sum_i k_i^2 + 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

The last identity follows since  $\mathcal{K}$  is the average value of  $(\sum_i k_i \cdot x_i^2)^2$  on the unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ ; here  $(x_1, \dots, x_n)$  are the standard coordinates in  $\mathbb{R}^n$ . One has to take into account that the following functions have unit average values:  $\frac{1}{3} \cdot n \cdot (n+2) \cdot x_i^4$  and  $n \cdot (n+2) \cdot x_i^2 \cdot x_j^2$  for  $i \neq j$ .

By the standard Gauss formula,

$$\text{Sc} = 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

It remains to rewrite the right-hand side using the expressions for  $|H|^2$  and  $\mathcal{K}$ .

If  $\text{codim } L = k > 1$ , then the second fundamental form can be presented as a direct sum of  $k$  real-valued quadratic forms  $\mathbb{I}_1 \oplus \dots \oplus \mathbb{I}_k$ ; that is,

$$\mathbb{I} = e_1 \cdot \mathbb{I}_1 + \dots + e_k \cdot \mathbb{I}_k,$$

where  $e_1, \dots, e_k$  is an orthonormal basis of  $N_p$ . Denote by  $\text{Sc}_i$ ,  $H_i$ , and  $\mathcal{K}_i$  the values associated with  $\mathbb{I}_i$ . From above, we get

$$\text{Sc}_i = \frac{3}{2} \cdot |H_i|^2 - \frac{n \cdot (n+2)}{2} \cdot \mathcal{K}_i$$

for each  $i$ .

Note that

$$\text{Sc} = \sum_i \text{Sc}_i, \quad |H|^2 = \sum_i |H_i|^2, \quad \text{and} \quad \mathcal{K} = \sum_i \mathcal{K}_i.$$

Hence the general case follows.  $\square$

*Remark.* A more direct proof of this formula can be obtained using the so called *extrinsic curvature tensor* that is defined by  $\Phi(x, y, v, w) := \langle \mathbb{I}(x, y), \mathbb{I}(v, w) \rangle$ ; the necessary properties of this tensor are discussed in [8].

### 3 Special cases

Recall that  $\mathbb{T}^n$  is the  $n$ -dimensional torus — the smooth manifold diffeomorphic to the product of  $n$  circles. Further,  $\mathbb{B}^q$  is the unit ball centered at the origin of the  $q$ -dimensional Euclidean space  $\mathbb{R}^q$ .

The following statement appears in the book of Yuri Burago and Viktor Zalgaller [2, Theorem 28.2.5]; it generalizes the result of István Fáry about average curvature of a curve in the unit ball [3, 11], but the proof is essentially the same.

**3.1. Lemma.** *Let  $L$  be a closed  $n$ -dimension manifold that is smoothly immersed in  $\mathbb{B}^q$ . Then the average value of  $|H|$  on  $L$  is at least  $n$ .*

*Proof.* Consider the function  $u: x \mapsto \frac{1}{2} \cdot |x|^2$  on  $L$ . Note that

$$(\Delta u)(x) = n + \langle H(x), x \rangle.$$

It follows that the average value of  $\langle H(x), x \rangle$  is  $-n$ . Since  $|x| \leq 1$ , we get the result.  $\square$

*Proof of 1.2a.* By 3.1, the average value of  $|H|^2$  is at least 4. Further, by the Gauss–Bonnet formula, scalar curvature (which is twice the Gauss curvature in this case) has zero average. Therefore 2.1 implies the statement.  $\square$

*Proof of 1.2b.* By 3.1, the average value of  $|H|^2$  is at least  $n^2$ . Since  $\text{Sc} \equiv 0$ , it remains to apply 2.1.  $\square$

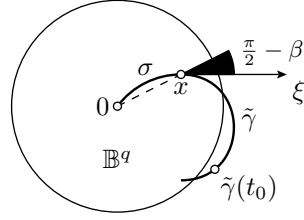
*Proof of 1.2c.* Since the image lies in the unit sphere, we have that  $|H|^2$  is at least  $n^2$  at each point. Since  $\mathbb{T}^n$  does not admit a metric with positive scalar curvature [7, Corollary A], we have  $\text{Sc}(x) \leq 0$  at some point  $x$ . It remains to apply 2.1 at  $x$ .  $\square$

## 4 Main case

The following lemma is an easy corollary of the bow lemma of Axel Schur [9, 10]. It tells how we use the bound on normal curvatures in 1.2e. If  $n \leq 4$ , then the proof works without this assumption.

**4.1. Lemma.** *Let  $L$  be a manifold smoothly immersed in  $\mathbb{B}^q$ . Suppose its normal curvatures are at most 2. Given  $x \in L$ , denote by  $\beta$  the angle between vector  $x$  and the normal space  $N_x$ . Then  $|x| \leq \cos \beta$ .*

*Proof.* Let  $\xi$  be a tangent direction at  $x$  such that  $\angle(x, \xi) = \frac{\pi}{2} - \beta$ . In the plane spanned by  $x$  and  $\xi$ , choose a unit-speed circle arc  $\sigma$  from 0 to  $x$  that comes to  $x$  in the direction opposite to  $\xi$ ; extend  $\sigma$  after  $x$  by a unit-speed semicircle  $\tilde{\gamma}$  with curvature 2 in such a way that the concatenation  $\sigma * \tilde{\gamma}$  is an arc of a smooth convex plane curve; see the figure.



Observe that if  $|x| > \cos \beta$ , then  $\tilde{\gamma}$  leaves  $\mathbb{B}^q$ ; that is,  $|\tilde{\gamma}(t_0)| > 1$  for some  $t_0$ . Let  $\gamma$  be the unit-speed geodesic in  $L$  that runs from  $x$  in the direction  $\xi$ . Note that curvatures of  $\sigma * \gamma$  do not exceed the curvatures of  $\sigma * \tilde{\gamma}$  at the corresponding points. Applying the bow lemma for  $\sigma * \gamma$  and  $\sigma * \tilde{\gamma}$ , we get  $|\gamma(t_0)| \geq |\tilde{\gamma}(t_0)|$ . It follows that  $L$  does not lie in  $\mathbb{B}^q$  — a contradiction.  $\square$

Let  $g$  be a Riemannian metric on  $\mathbb{T}^n$ . Suppose  $n \geq 3$ , and  $u: \mathbb{T}^n \rightarrow \mathbb{R}$  is a smooth positive function. Recall that

$$\left( \text{Sc} \cdot u - 4 \cdot \frac{n-1}{n-2} \cdot \Delta u \right) \cdot u^{\frac{n-2}{n+2}}$$

is the scalar curvature of the metric  $u^{\frac{4}{n-2}} \cdot g$ ; see for example [1, 6.3]. Since any Riemannian metric  $g$  on  $\mathbb{T}^n$  has nonpositive scalar curvature at some point [7, Corollary A], we get the following.

**4.2. Claim.** *Let  $g$  be a Riemannian metric on  $\mathbb{T}^n$ . Then, for any positive smooth function  $u$  on  $\mathbb{T}^n$ , the function*

$$\text{Sc} \cdot u - 4 \cdot \frac{n-1}{n-2} \cdot \Delta u$$

*returns a nonpositive value at some point.*

*Proof of 1.2d and 1.2e.* The case  $n = 2$  follows from 1.2a; so we can assume that  $n \geq 3$ . Consider the function  $u: x \mapsto \exp(-\frac{k}{2} \cdot |x|^2)$  on the torus.

We will apply the following formula

$$\Delta(\varphi \circ f) = \varphi' \cdot \Delta f + \varphi'' \cdot |\nabla f|^2$$

to  $f: x \mapsto \frac{1}{2} \cdot |x|^2$  and  $\varphi: y \mapsto \exp(-k \cdot y)$ ; so  $u = \varphi \circ f$ .

Set  $r(x) = |x|$ ,  $\alpha(x) = \angle(H(x), x)$ , and  $\beta(x)$  as in 4.1. Note that

$$\textcircled{1} \quad \beta \leq \alpha \leq \pi - \beta.$$

Observe that

$$\Delta f = |H| \cdot r \cdot \cos \alpha + n, \quad |\nabla f| = r \cdot \sin \beta, \quad \varphi' = -k \cdot \varphi, \quad \varphi'' = k^2 \cdot \varphi.$$

Therefore

$$\Delta u = u \cdot [-k \cdot |H| \cdot r \cdot \cos \alpha - k \cdot n + k^2 \cdot r^2 \cdot \sin^2 \beta].$$

Recall that  $\text{Sc} = -\frac{n \cdot (n+2)}{2} \cdot \mathcal{K} + \frac{3}{2} \cdot |H|^2$ ; see 2.1. By 4.2, the function

$$\begin{aligned} \text{Sc} \cdot u - 4 \cdot \frac{n-1}{n-2} \cdot \Delta u = u \cdot & \left[ -\frac{n \cdot (n+2)}{2} \cdot \mathcal{K} + \frac{3}{2} \cdot |H|^2 + 4 \cdot \frac{n-1}{n-2} \cdot k \cdot |H| \cdot r \cdot \cos \alpha + \right. \\ & \left. + 4 \cdot \frac{n-1}{n-2} \cdot (k \cdot n - k^2 \cdot r^2 \cdot \sin^2 \beta) \right] \end{aligned}$$

returns a nonpositive value at some point  $x \in \mathbb{T}^n$ .

Choose

$$k = \frac{3}{4} \cdot \frac{n-2}{n-1} \cdot n, \quad \text{so} \quad n = \frac{4}{3} \cdot \frac{n-1}{n-2} \cdot k.$$

At the point  $x$ , we have

$$\begin{aligned} \textcircled{2} \quad \frac{n \cdot (n+2)}{2} \cdot \mathcal{K} & \geq \frac{3}{2} \cdot (|H| + n \cdot r \cdot \cos \alpha)^2 - \frac{3}{2} \cdot n^2 \cdot r^2 \cdot \cos^2 \alpha + \\ & + 3 \cdot n^2 - \frac{9}{4} \cdot \frac{n-2}{n-1} \cdot n^2 \cdot r^2 \cdot \sin^2 \beta \geq \frac{3}{2} \cdot n^2. \end{aligned}$$

By  $\textcircled{1}$ ,  $\cos^2 \beta + \sin^2 \alpha \leq 1$ . If  $n \leq 4$ , then  $\frac{3}{2} \geq \frac{9}{4} \cdot \frac{n-2}{n-1}$ ; therefore the last inequality follows, and it proves 1.2d.

If  $n \geq 5$ , then for the last inequality in  $\textcircled{2}$  we need to use in addition that  $r^2 + \sin^2 \beta \leq 1$  which follows 4.1. Hence 1.2e follows.  $\square$

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