

Lower bound on normal curvatures of torus

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Abstract

We give an optimal lower bound on normal curvatures of immersed n -torus in a Euclidean ball of large dimension.

1 Introduction

This note is inspired by examples of isometric embeddings of n -torus \mathbb{T}^n into the unit ball \mathbb{B}^q of q -dimensional Euclidean space for all large q such that all its normal curvatures identically equal to $\sqrt{3 \cdot n / (n + 2)}$ and the induced metric is flat. These examples are geodesic subtorii in Clifford torii; they were constructed by Michael Gromov; see [5, 2.A] and [4, 1.1.A.].

The existence of such examples implies the following surprising results: *any closed smooth manifold admits a smooth embedding into \mathbb{B}^q for large q with normal curvatures less than $\sqrt{3}$. Moreover, the induced Riemannian metric can be chosen to be proportional to any given metric g ; see [5, 1.D] and [4, 1.1.C].*

Our theorem implies that the bounds above are optimal.

1.1. Theorem. *Suppose \mathbb{T}^n is smoothly immersed in \mathbb{B}^q . Then its maximal normal curvature is at least*

$$\sqrt{3 \cdot \frac{n}{n+2}}.$$

To make the statement more exact, we need some notations. Assume that a smooth n -dimensional manifold L is immersed in the Euclidean space \mathbb{R}^q ; we will always assume that L is equipped with induced Riemannian metric. Let us denote by T_x and N_x the tangent and normal spaces of L at x .

Recall that *second fundamental form* \mathbb{I} at x is a symmetric quadratic form on T_x with values in N_x . It is uniquely defined by the following identity

$$\mathbb{I}(v, v) \equiv \gamma_v''(0),$$

where $v \in T_x$ and γ_v an L -geodesic that starts at x with initial velocity vector v .

Given $x \in L$, denote by $\mathcal{K}(x)$ the average value of $|\mathbb{I}(u, u)|^2$ for $u \in T_x$ such that $|u| = 1$; in other words, $\mathcal{K}(x)$ is the average value of squared normal curvatures at x . (The letter \mathcal{K} should resemble squared K .)

1.2. Theorem. *Suppose \mathbb{T}^n is smoothly immersed in \mathbb{B}^q . Let us equip \mathbb{T}^n with the induced Riemannian metric; so we can take average values with respect to the induced volume.*

- (a) *If $n = 2$, then the average value of \mathcal{K} is at least $\frac{3}{2}$.*
- (b) *If the metric on \mathbb{T}^n is flat, then the average value of \mathcal{K} is at least $3 \cdot \frac{n}{n+2}$.*
- (c) *If the image of \mathbb{T}^n lies in $\mathbb{S}^{q-1} = \partial\mathbb{B}^q$, then $\mathcal{K} \geq 3 \cdot \frac{n}{n+2}$ at some point.*

- (d) If $n \leq 4$, then $\mathcal{K} \geq 3 \cdot \frac{n}{n+2}$ at some point of \mathbb{T}^n .
(e) If the normal curvatures of \mathbb{T}^n do not exceed 2, then $\mathcal{K} \geq 3 \cdot \frac{n}{n+2}$ at some point of \mathbb{T}^n .

Note that part (e) is a stronger version of 1.1. The remaining statements (a)–(d) are stronger versions of 1.1 in some partial cases. All this follows since the normal curvature in some direction at x is at least $\sqrt{\mathcal{K}(x)}$.

1.3. Open question. *Is it true that for any smooth immersion $\mathbb{T}^n \looparrowright \mathbb{B}^q$ the inequality $\mathcal{K} \geq 3 \cdot \frac{n}{n+2}$ holds at some point?*

2 Gauss formula

Recall that L is a smooth n -dimensional manifold immersed in \mathbb{R}^q . Given $p \in L$, denote by $\text{Sc}(p)$ and $H(p)$ the scalar curvature, the mean curvature vector at p .

The following version of the Gauss formula plays a central role in all proofs; it is used instead of the formula in [6, 5.B].

2.1. Gauss formula. *The following identity*

$$\text{Sc} = \frac{3}{2} \cdot |H|^2 - \frac{n \cdot (n+2)}{2} \cdot \mathcal{K}$$

holds for any smooth n -dimensional immersed manifold in a Euclidean space.

Proof. Assume $\text{codim } L = 1$. Choose $p \in L$; denote by k_1, \dots, k_n the principal curvatures of L at p . Note that

$$|H|^2 = \sum_i k_i^2 + 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

Further,

$$n \cdot (n+2) \cdot \mathcal{K} = 3 \cdot \sum_i k_i^2 + 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

The last identity follows since \mathcal{K} is the average value of $(\sum_i k_i \cdot x_i^2)^2$ on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$; here (x_1, \dots, x_n) are the standard coordinates in \mathbb{R}^n . One has to take into account that the following functions have unit average values: $\frac{1}{3} \cdot n \cdot (n+2) \cdot x_i^4$ and $n \cdot (n+2) \cdot x_i^2 \cdot x_j^2$ for $i \neq j$.

By the standard Gauss formula,

$$\text{Sc} = 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

It remains to rewrite the right-hand side using the expressions for $|H|^2$ and \mathcal{K} .

If $\text{codim } L = k > 1$, then the second fundamental form can be presented as a direct sum of k real-valued quadratic forms $\mathbb{I}_1 \oplus \dots \oplus \mathbb{I}_k$; that is,

$$\mathbb{I} = e_1 \cdot \mathbb{I}_1 + \dots + e_k \cdot \mathbb{I}_k,$$

where e_1, \dots, e_k is an orthonormal basis of N_p . Denote by Sc_i , H_i , and \mathcal{K}_i the values associated with \mathbb{I}_i . From above, we get

$$\text{Sc}_i = \frac{3}{2} \cdot |H_i|^2 - \frac{n \cdot (n+2)}{2} \cdot \mathcal{K}_i$$

for each i .

Note that

$$\text{Sc} = \sum_i \text{Sc}_i, \quad |H^2| = \sum_i |H_i|^2, \quad \text{and} \quad \lambda K = \sum_i \lambda K_i.$$

Hence the general case follows. \square

Remark. A more direct proof of this formula can be obtained using the so called *extrinsic curvature tensor* that is defined by $\Phi(x, y, v, w) := \langle \mathbb{I}(x, y), \mathbb{I}(v, w) \rangle$; the necessary properties of this tensor are discussed in [8].

3 Special cases

Recall that \mathbb{T}^n is the n -dimensional torus — the smooth manifold diffeomorphic to the product of n circles. Further, \mathbb{B}^q is the unit ball centered at the origin of the q -dimensional Euclidean space \mathbb{R}^q .

The following statement appears in the book of Yuri Burago and Viktor Zalgaller [2, Theorem 28.2.5]; it generalizes the result of István Fáry about average curvature of a curve in the unit ball [3, 11], but the proof is essentially the same.

3.1. Lemma. *Let L be a closed n -dimension manifold that is smoothly immersed in \mathbb{B}^q . Then the average value of $|H|$ on L is at least n .*

Proof. Consider the function $u: x \mapsto \frac{1}{2} \cdot |x|^2$ on L . Note that

$$(\Delta u)(x) = n + \langle H(x), x \rangle.$$

It follows that the average value of $\langle H(x), x \rangle$ is $-n$. Since $|x| \leq 1$, we get the result. \square

Proof of 1.2a. By 3.1, the average value of $|H|^2$ is at least 4. Further, by the Gauss–Bonnet formula, scalar curvature (which is twice the Gauss curvature in this case) has zero average. Therefore 2.1 implies the statement. \square

Proof of 1.2b. By 3.1, the average value of $|H|^2$ is at least n^2 . Since $\text{Sc} \equiv 0$, it remains to apply 2.1. \square

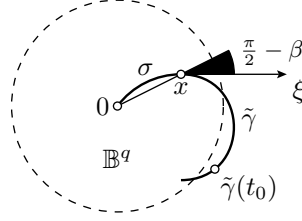
Proof of 1.2c. Since the image lies in the unit sphere, we have that $|H|^2$ is at least n^2 at each point. Since \mathbb{T}^n does not admit a metric with positive scalar curvature [7, Cor. A], we have $\text{Sc}(x) \leq 0$ at some point x . It remains to apply 2.1 at x . \square

4 Main case

The following lemma is an easy corollary of the bow lemma of Axel Schur [9, 10]. It tells how we use the assumption in 1.2e that normal curvatures do not exceed 2. If $n \leq 4$, then the proof works without this assumption.

4.1. Lemma. *Let L be a manifold smoothly immersed in \mathbb{B}^q . Suppose its normal curvatures are at most 2. Given $x \in L$, denote by β the angle between vector x and the normal space N_x . Then $|x| \leq \cos \beta$.*

Proof. Let ξ be a tangent direction at x such that $\angle(x, \xi) = \frac{\pi}{2} - \beta$. In the plane spanned by x and ξ , choose a unit-speed circle arc σ from 0 to x that comes to x in the direction opposite to ξ ; extend σ after x by a unit-speed semicircle $\tilde{\gamma}$ with curvature 2 in such a way that the concatenation $\sigma * \tilde{\gamma}$ is an arc of a smooth convex plane curve; see the figure.



Observe that if $|x| > \cos \beta$, then $\tilde{\gamma}$ leaves \mathbb{B}^q ; that is, $|\tilde{\gamma}(t_0)| > 1$ for some t_0 .

Let γ be the unit-speed geodesic in L that runs from x in the direction ξ . Note that curvatures of $\sigma * \gamma$ do not exceed the curvatures of $\sigma * \tilde{\gamma}$ at the corresponding points. Applying the bow lemma for $\sigma * \gamma$ and $\sigma * \tilde{\gamma}$, we get $|\gamma(t_0)| \geq |\tilde{\gamma}(t_0)|$. It follows that L does not lie in \mathbb{B}^q — a contradiction. \square

Let g be a Riemannian metric on \mathbb{T}^n . Suppose $n \geq 3$, and $u: \mathbb{T}^n \rightarrow \mathbb{R}$ is a smooth positive function. Recall that

$$\left(\text{Sc} \cdot u - 4 \cdot \frac{n-1}{n-2} \cdot \Delta u \right) \cdot u^{\frac{n-2}{n+2}}$$

is the scalar curvature of the metric $u^{\frac{4}{n-2}} \cdot g$; see for example [1, 6.3]. Since *any* Riemannian metric g on \mathbb{T}^n has nonpositive scalar curvature at some point [7, Cor. A], we get the following.

4.2. Claim. *Let g be a Riemannian metric on \mathbb{T}^n . Then, for any positive smooth function u on \mathbb{T}^n , the function*

$$\text{Sc} \cdot u - 4 \cdot \frac{n-1}{n-2} \cdot \Delta u$$

returns a nonpositive value at some point.

Proof of 1.2d and 1.2e. The case $n = 2$ follows from 1.2a; so we can assume that $n \geq 3$. Consider the function $u: x \mapsto \exp(-\frac{k}{2} \cdot |x|^2)$ on the torus.

We will apply the following formula

$$\Delta(\varphi \circ f) = \varphi' \cdot \Delta f + \varphi'' \cdot |\nabla f|^2$$

to $f: x \mapsto \frac{1}{2} \cdot |x|^2$ and $\varphi: y \mapsto \exp(-k \cdot y)$; so $u = \varphi \circ f$.

Set $r(x) = |x|$, $\alpha(x) = \angle(H(x), x)$, and $\beta(x)$ as in 4.1. Note that

$$\textcircled{1} \quad \beta \leq \alpha \leq \pi - \beta.$$

Observe that

$$\Delta f = |H| \cdot r \cdot \cos \alpha + n, \quad |\nabla f| = r \cdot \sin \beta, \quad \varphi' = -k \cdot \varphi, \quad \varphi'' = k^2 \cdot \varphi.$$

Therefore

$$\Delta u = u \cdot [-k \cdot |H| \cdot r \cdot \cos \alpha - k \cdot n + k^2 \cdot r^2 \cdot \sin^2 \beta].$$

Recall that $\text{Sc} = -\frac{n \cdot (n+2)}{2} \cdot \mathcal{K} + \frac{3}{2} \cdot |H|^2$; see 2.1. By 4.2, the function

$$\begin{aligned} \text{Sc} \cdot u - 4 \cdot \frac{n-1}{n-2} \cdot \Delta u &= \\ &= u \cdot \left[-\frac{n \cdot (n+2)}{2} \cdot \mathcal{K} + \frac{3}{2} \cdot |H|^2 + 4 \cdot \frac{n-1}{n-2} \cdot k \cdot |H| \cdot r \cdot \cos \alpha + \right. \\ &\quad \left. + 4 \cdot \frac{n-1}{n-2} \cdot (k \cdot n - k^2 \cdot r^2 \cdot \sin^2 \beta) \right] \end{aligned}$$

returns a nonpositive value at some point $x \in \mathbb{T}^n$.

Choose

$$k = \frac{3}{4} \cdot \frac{n-2}{n-1} \cdot n, \quad \text{so} \quad n = \frac{4}{3} \cdot \frac{n-1}{n-2} \cdot k.$$

At the point x , we have

$$\begin{aligned} \textcircled{2} \quad \frac{n \cdot (n+2)}{2} \cdot \mathcal{H} &\geq \frac{3}{2} \cdot (|H| + n \cdot r \cdot \cos \alpha)^2 - \frac{3}{2} \cdot n^2 \cdot r^2 \cdot \cos^2 \alpha + \\ &\quad + 3 \cdot n^2 - \frac{9}{4} \cdot \frac{n-2}{n-1} \cdot n^2 \cdot r^2 \cdot \sin^2 \beta \geq \\ &\geq \frac{3}{2} \cdot n^2. \end{aligned}$$

By $\textcircled{1}$, $\cos^2 \beta + \sin^2 \alpha \leq 1$. If $n \leq 4$, then $\frac{3}{2} \geq \frac{9}{4} \cdot \frac{n-2}{n-1}$; therefore the last inequality follows, and it proves 1.2d.

If $n \geq 5$, then for the last inequality in $\textcircled{2}$ we need to use in addition that $r^2 + \sin^2 \beta \leq 1$ which follows 4.1. Hence 1.2e follows. \square

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References

- [1] T. Aubin. *Nonlinear analysis on manifolds. Monge-Ampère equations*. Vol. 252. Grundlehren der mathematischen Wissenschaften. 1982.
- [2] Yu. D. Burago and V. A. Zalgaller. *Geometric inequalities*. Vol. 285. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, 1988.
- [3] I. Fáry. “Sur certaines inégalités géométriques”. *Acta Sci. Math.* 12 (1950), 117–124.
- [4] M. Gromov. *Curvature, Kolmogorov Diameter, Hilbert Rational Designs and Overtwisted Immersions*. 2022.
- [5] M. Gromov. *Isometric immersions with controlled curvatures*. 2022.
- [6] M. Gromov. *Scalar Curvature, Injectivity Radius and Immersions with Small Second Fundamental Forms*. 2022.
- [7] M. Gromov and B. Lawson. “Spin and scalar curvature in the presence of a fundamental group. I”. *Ann. of Math. (2)* 111.2 (1980), 209–230.
- [8] A. Petrunin. “Polyhedral approximations of Riemannian manifolds”. *Turkish J. Math.* 27.1 (2003), 173–187.
- [9] A. Petrunin and S. Zamora Barrera. *What is differential geometry: curves and surfaces*. 2022. arXiv: 2012.11814 [math.HO].
- [10] A. Schur. “Über die Schwarzsche Extremaleigenschaft des Kreises unter den Kurven konstanter Krümmung”. *Math. Ann.* 83.1-2 (1921), 143–148.
- [11] S. Tabachnikov. “The tale of a geometric inequality.” *MASS selecta: teaching and learning advanced undergraduate mathematics*. 2003, 257–262.