Gromov's torii are optimal

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Abstract

We give an optimal lower bound on normal curvatures of immersed n-torus in a Euclidean ball of large dimension.

1 Introduction

Let us denote by $\mathbb{B}^q$ the unit ball in $\mathbb{R}^q$ centered at the origin. Further, $\mathbb{T}^n$ will denote the n-dimensional torus — the smooth manifold diffeomorphic to the product of n circles.

This note is inspired by examples of embeddings $\mathbb{T}^n \hookrightarrow \mathbb{B}^q$ for large q with constant normal curvatures $K_n = \sqrt{3{\cdot}n/(n+2)}$. In other words, any geodesic in the torus has constant curvature K_n as a curve in $\mathbb{R}^q$. These examples are found among geodesic subtorii in Clifford's torii; they were constructed by Michael Gromov; see [5, 2.A] and [4, 1.1.A.].

The existence of such torii implies the following surprising results: *any closed smooth manifold admits a smooth embedding into $\mathbb{B}^q$ for large q with normal curvatures less than $\sqrt{3}$. Moreover, the induced Riemannian metric can be chosen to be proportional to any given metric g;* see [5, 1.D] and [4, 1.1.C].

The next theorem implies that Gromov's torii have the best upper bound on normal curvatures; in particular the $\sqrt{3}$-bound is optimal.

1.1. Theorem. *Suppose $\mathbb{T}^n$ is smoothly immersed in $\mathbb{B}^q$. Then its maximal normal curvature is at least*

$$\sqrt{3{\cdot}\tfrac{n}{n+2}}.$$

To make the statement more exact, we need one more notation. Assume that L is a smooth n-dimensional manifold immersed in $\mathbb{R}^q$; we will always assume that L is equipped with induced Riemannian metric. Let us denote by T_x and N_x the tangent and normal spaces of L at x.

Recall that *second fundamental form* $\mathrm{I\!I}$ at x is a symmetric quadratic form on T_x with values in N_x. It is uniquely defined by the following identity

$$\mathrm{I\!I}(\mathrm{v},\mathrm{v}) \equiv \gamma_{\mathrm{v}}''(0),$$

where $\mathrm{v} \in \mathrm{T}_x$ and γ_{v} is an L-geodesic that starts at x with initial velocity vector v.

Given $x \in L$, denote by $\text{Ж}(x)$ the average value of $|\mathrm{I\!I}(\mathrm{u},\mathrm{u})|^2$ for $\mathrm{u} \in \mathrm{T}_x$ such that $|\mathrm{u}| = 1$. In other words, if $K(\mathrm{u})$ denotes normal curvature in the direction u, then $\text{Ж}(x)$ is the average value $K^2(\mathrm{u})$. (The cyrillic zhe Ж is used since it resembles squared K.)

1.2. Theorem. *Suppose $\mathbb{T}^n$ is smoothly immersed in $\mathbb{B}^q$. Let us equip $\mathbb{T}^n$ with the induced Riemannian metric; so we can take average values with respect to the induced volume.*

(a) *If $n = 2$, then the average value of $\mathcal{K}$ is at least $\frac{3}{2}$.*
(b) *If the metric on $\mathbb{T}^n$ is flat, then the average value of $\mathcal{K}$ is at least $3\cdot\frac{n}{n+2}$.*
(c) *If the image of $\mathbb{T}^n$ lies in $\partial\mathbb{B}^q$, then $\mathcal{K} \geqslant 3\cdot\frac{n}{n+2}$ at some point of $\mathbb{T}^n$.*
(d) *If $n \leqslant 4$, then $\mathcal{K} \geqslant 3\cdot\frac{n}{n+2}$ at some point of $\mathbb{T}^n$.*
(e) *If the normal curvatures of $\mathbb{T}^n$ do not exceed 2, then $\mathcal{K} \geqslant 3\cdot\frac{n}{n+2}$ at some point of $\mathbb{T}^n$.*

Note that part *(e)* is a stronger version of 1.1. The remaining statements *(a)–(d)* are stronger versions of 1.1 in some partial cases. All this follows since the normal curvature in some direction at x is at least $\sqrt{\mathcal{K}(x)}$.

All proofs use our version of the Gauss formula; see below. The proofs of *(c)–(e)* use in addition that *torus does not admit a metric with positive scalar curvature* [7, Corollary A].

1.3. Open question. *Is it true that for any smooth immersion $\mathbb{T}^n \looparrowright \mathbb{B}^q$ the inequality $\mathcal{K} \geqslant 3\cdot\frac{n}{n+2}$ holds at some point?*

2 Gauss formula

Recall that L is a smooth n-dimensional manifold immersed in $\mathbb{R}^q$. Given $p \in L$, denote by $\mathrm{Sc}(p)$ and $H(p)$ the scalar curvature, the mean curvature vector at p.

The following version of the Gauss formula plays a central role in all proofs; it is used instead of the formula in [6, 5.B].

2.1. Gauss formula. *The following identity*

$$\mathrm{Sc} = \tfrac{3}{2}\cdot|H|^2 - \tfrac{n\cdot(n+2)}{2}\cdot\mathcal{K}$$

holds for any smooth n-dimensional immersed manifold in a Euclidean space.

Proof. Assume $\operatorname{codim} L = 1$. Choose $p \in L$; denote by $k_1, \dots, k_n$ the principal curvatures of L at p. Note that

$$|H|^2 = \sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

Further,

$$n\cdot(n+2)\cdot\mathcal{K} = 3\cdot\sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

The last identity follows since $\mathcal{K}$ is the average value of $\left(\sum_i k_i\cdot x_i^2\right)^2$ on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n = \mathrm{T}_p$; here $(x_1, \dots, x_n)$ are the standard coordinates in $\mathbb{R}^n$. One has to take into account that the following functions have unit average values: $\frac{1}{3}\cdot n\cdot(n+2)\cdot x_i^4$ and $n\cdot(n+2)\cdot x_i^2\cdot x_j^2$ for $i \neq j$.

By the standard Gauss formula,

$$\mathrm{Sc} = 2\cdot\sum_{i<j} k_i\cdot k_j.$$

It remains to rewrite the right-hand side using the expressions for $|H|^2$ and $\varkappa$.

If $\operatorname{codim} L = k > 1$, then the second fundamental form can be presented as a direct sum of k real-valued quadratic forms $\mathrm{I\!I}_1 \oplus \dots \oplus \mathrm{I\!I}_k$; that is,

$$\mathrm{I\!I} = e_1 \cdot \mathrm{I\!I}_1 + \dots + e_k \cdot \mathrm{I\!I}_k,$$

where $e_1, \dots, e_k$ is an orthonormal basis of N_p. Denote by Sc_i, H_i, and $\varkappa_i$ the values associated with $\mathrm{I\!I}_i$. From above, we get

$$\mathrm{Sc}_i = \tfrac{3}{2} \cdot |H_i|^2 - \tfrac{n \cdot (n+2)}{2} \cdot \varkappa_i$$

for each i.

Note that

$$\mathrm{Sc} = \sum_i \mathrm{Sc}_i, \qquad |H^2| = \sum_i |H_i|^2, \quad \text{and} \quad \varkappa = \sum_i \varkappa_i.$$

Hence the general case follows. □

Remark. A more direct proof of this formula can be obtained using the so-called *extrinsic curvature tensor* which is defined by $\Phi(\mathrm{X}, \mathrm{Y}, \mathrm{V}, \mathrm{W}) := \langle \mathrm{I\!I}(\mathrm{X}, \mathrm{Y}), \mathrm{I\!I}(\mathrm{V}, \mathrm{W}) \rangle$; the necessary properties of this tensor are discussed in [8]. As a bonus, one gets explicit expression for the second fundamental forms of all Gromov's torii.

3 Special cases

The following statement appears in the book of Yuri Burago and Viktor Zalgaller [2, Theorem 28.2.5]; it generalizes the result of István Fáry about average curvature of a curve in the unit ball [3, 11], but the proof is essentially the same.

3.1. Lemma. *Let L be a closed n-dimension manifold that is smoothly immersed in $\mathbb{B}^q$. Then the average value of $|H|$ on L is at least n.*

Proof. Consider the function $u\colon x \mapsto \frac{1}{2} \cdot |x|^2$ on L. Note that

$$(\Delta u)(x) = n + \langle H(x), x \rangle.$$

It follows that the average value of $\langle H(x), x \rangle$ is $-n$. Since $|x| \leqslant 1$, we get the result. □

Proof of 1.2a. By 3.1, the average value of $|H|^2$ is at least 4. Further, by the Gauss–Bonnet formula, scalar curvature (which is twice the Gauss curvature in this case) has zero average. Therefore 2.1 implies the statement. □

Proof of 1.2b. By 3.1, the average value of $|H|^2$ is at least n^2. Since $\mathrm{Sc} \equiv 0$, it remains to apply 2.1. □

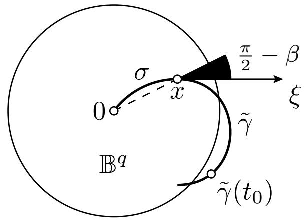
Proof of 1.2c. Since the image lies in the unit sphere, we have that $|H|^2$ is at least n^2 at each point. Since $\mathbb{T}^n$ does not admit a metric with positive scalar curvature [7, Corollary A], we have $\mathrm{Sc}(x) \leqslant 0$ at some point x. It remains to apply 2.1 at x. □

4 Main case

The following lemma is an easy corollary of the bow lemma of Axel Schur [9, 10]. It tells how we use the bound on normal curvatures in 1.2*e*. If $n \leqslant 4$, then the proof works without this assumption.

4.1. Lemma. *Let L be a manifold smoothly immersed in $\mathbb{B}^q$. Suppose its normal curvatures are at most 2. Given $x \in L$, denote by $\beta = \beta(x)$ the angle between vector x and the normal space N_x. Then $|x| \leqslant \cos\beta$.*

Proof. Let ξ be a tangent direction at x such that $\measuredangle(x,\xi) = \frac{\pi}{2} - \beta$. In the plane spanned by x and ξ, choose a unit-speed circle arc σ from 0 to x that comes to x in the direction opposite to ξ; extend σ after x by a unit-speed semicircle $\tilde\gamma$ with curvature 2 in such a way that the concatenation $\sigma * \tilde\gamma$ is an arc of a smooth convex plane curve; see the figure.



Observe that if $|x| > \cos\beta$, then $\tilde\gamma$ leaves $\mathbb{B}^q$; that is, $|\tilde\gamma(t_0)| > 1$ for some t_0.

Let γ be the unit-speed geodesic in L that runs from x in the direction ξ. Note that curvatures of $\sigma * \gamma$ do not exceed the curvatures of $\sigma * \tilde\gamma$ at the corresponding points. Applying the bow lemma for $\sigma * \gamma$ and $\sigma * \tilde\gamma$, we get $|\gamma(t_0)| \geqslant |\tilde\gamma(t_0)|$. It follows that L does not lie in $\mathbb{B}^q$ — a contradiction. ☐

Let g be a Riemannian metric on $\mathbb{T}^n$. Suppose $n \geqslant 3$, and $u\colon \mathbb{T}^n \to \mathbb{R}$ is a smooth positive function. Recall that

$$\left(\mathrm{Sc}\cdot u - 4\cdot\tfrac{n-1}{n-2}\cdot\Delta u\right)\cdot u^{\frac{n-2}{n+2}}$$

is the scalar curvature of the metric $u^{\frac{4}{n-2}}\cdot g$; see for example [1, 6.3]. Since *any Riemannian metric on $\mathbb{T}^n$ has nonpositive scalar curvature at some point* [7, Corollary A], we get the following.

4.2. Claim. *For any Riemannian metric on $\mathbb{T}^n$ and any positive smooth function $u\colon \mathbb{T}^n \to \mathbb{R}$, the function*

$$\mathrm{Sc}\cdot u - 4\cdot\tfrac{n-1}{n-2}\cdot\Delta u$$

returns a nonpositive value at some point.

Proof of 1.2d and 1.2e. The case $n = 2$ follows from 1.2*a*; so we can assume that $n \geqslant 3$. Consider the function $u\colon x \mapsto \exp(-\frac{k}{2}\cdot|x|^2)$ on the torus.

We will apply the following formula

$$\Delta(\varphi \circ f) = \varphi'\cdot\Delta f + \varphi''\cdot|\nabla f|^2$$

to $f\colon x \mapsto \frac{1}{2}\cdot|x|^2$ and $\varphi\colon y \mapsto \exp(-k\cdot y)$; so $u = \varphi \circ f$.

Set $r(x) = |x|$, $\alpha(x) = \measuredangle(H(x), x)$, and $\beta(x)$ as in 4.1. Note that

$$\text{❶} \qquad \beta \leqslant \alpha \leqslant \pi - \beta.$$

Observe that

$$\Delta f = |H|\cdot r\cdot\cos\alpha + n, \quad |\nabla f| = r\cdot\sin\beta, \quad \varphi' = -k\cdot\varphi, \quad \varphi'' = k^2\cdot\varphi.$$

Therefore

$$\Delta u = u\cdot[-k\cdot|H|\cdot r\cdot \cos\alpha - k\cdot n + k^2\cdot r^2\cdot \sin^2\beta].$$

Recall that $\mathrm{Sc} = -\frac{n\cdot(n+2)}{2}\cdot Ж\!K + \frac{3}{2}\cdot|H|^2$; see 2.1. By 4.2, the function

$$\begin{aligned}\mathrm{Sc}\cdot u - 4\cdot\tfrac{n-1}{n-2}\cdot\Delta u = u\cdot\Big[&-\tfrac{n\cdot(n+2)}{2}\cdot Ж\!K + \tfrac{3}{2}\cdot|H|^2 + 4\cdot\tfrac{n-1}{n-2}\cdot k\cdot|H|\cdot r\cdot\cos\alpha +\\ &+ 4\cdot\tfrac{n-1}{n-2}\cdot(k\cdot n - k^2\cdot r^2\cdot\sin^2\beta)\Big]\end{aligned}$$

returns a nonpositive value at some point $x\in\mathbb{T}^n$.

Choose

$$k = \tfrac{3}{4}\cdot\tfrac{n-2}{n-1}\cdot n, \quad \text{so} \quad n = \tfrac{4}{3}\cdot\tfrac{n-1}{n-2}\cdot k.$$

At the point x, we have

$$\begin{aligned}\tfrac{n\cdot(n+2)}{2}\cdot Ж\!K \geqslant \tfrac{3}{2}\cdot(|H| + n\cdot r\cdot\cos\alpha)^2 - \tfrac{3}{2}\cdot n^2\cdot r^2\cdot\cos^2\alpha +&\\ + 3\cdot n^2 - \tfrac{9}{4}\cdot\tfrac{n-2}{n-1}\cdot n^2\cdot r^2\cdot\sin^2\beta \geqslant \tfrac{3}{2}\cdot n^2.&\end{aligned} \qquad ❷$$

Indeed, by ❶, $\cos^2\alpha + \sin^2\beta \leqslant 1$. If $n\leqslant 4$, then $\frac{3}{2}\geqslant\frac{9}{4}\cdot\frac{n-2}{n-1}$; therefore the last inequality follows, and it proves 1.2*d*.

Further, if $n\geqslant 5$, then for the last inequality in ❷ we need to use in addition that $r^2+\sin^2\beta\leqslant 1$ which follows 4.1. Hence 1.2*e* follows. □

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