Normal curvatures of torii

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Abstract

We give an optimal lower bound on principle curvatures of immersed n-torus in a Euclidean ball of large dimension.

1 Gauss formula

Let L be a smooth n-dimensional submanifold in a Riemannian manifold M. Let us denote by T_pL and N_pL the tangent and normal spaces of L at p.

Recall that *second fundamental form* $\mathrm{I\!I}$ at p is a symmetric quadratic form on T_pL with values in N_pL. It is uniquely defined by the following identity

$$\mathrm{I\!I}(\mathrm{v},\mathrm{v}) := \gamma_{\mathrm{v}}''(0),$$

where $\mathrm{v} \in \mathrm{T}_pL$ and γ_{v} an L-geodesic that starts at p with initial velocity vector v.

Given $p \in L$, denote by $\mathrm{Sc}(p)$, $H(p)$, and $\varkappa(p)$ the scalar curvature, the mean curvature vector, and the average value of $|\mathrm{I\!I}(\mathrm{u},\mathrm{u})|^2$ at p respectively.

Let us denote by $\widetilde{\mathrm{Sc}}$ the *outer scalar curvature* of L in M; that is, if $e_1,\dots,e_n$ is an orthonormal basis of T_pL, then

$$\widetilde{\mathrm{Sc}}(p) = 2\cdot\sum_{i<j} K_{ij},$$

where K_{ij} denotes sectional curvature of M in the direction spanned by e_i and e_j.

The following formula is closely related to [5, 5.B].

1.1. Formula. *The following identity*

$$\mathrm{Sc} - \widetilde{\mathrm{Sc}} = \tfrac{3}{2}\cdot|H|^2 - \tfrac{n\cdot(n+2)}{2}\cdot\varkappa$$

holds for any smooth n-dimensional submanifold L in a Riemannian manifold.

Proof. Without loss of generality, we can assume that the ambient manifold is flat; in particular $\widetilde{\mathrm{Sc}} = 0$.

Assume $\operatorname{codim} L = 1$. Choose $p \in L$; denote by $k_1,\dots,k_n$ the principal curvatures of L at p. Note that

$$|H|^2 = \sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

Further,

$$n\cdot(n+2)\cdot\varkappa = 3\cdot\sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

The last identity follows since Ӿ is the average value of $\left(\sum_i k_i\cdot x_i^2\right)^2$ on the unit sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n$. One has to take into account that the following functions have unit average values: $\frac13\cdot n\cdot(n+2)\cdot x_i^4$ and $n\cdot(n+2)\cdot x_i^2\cdot x_j^2$ for $i\ne j$. Here we assume that $(x_1,\dots,x_n)$ are the standard coordinates in $\mathbb{R}^n$.

By Gauss formula,

$$\mathrm{Sc}=2\cdot\sum_{i<j}k_i\cdot k_j.$$

It remains to rewrite this formula using the expressions for $|H|^2$ and Ӿ.

If $\operatorname{codim}L=k>1$, then the second fundamental form can be presented as a direct sum of k real-valued quadratic forms $\mathrm{I\!I}_1\oplus\dots\oplus\mathrm{I\!I}_k$; that is,

$$\mathrm{I\!I}=e_1\cdot\mathrm{I\!I}_1+\dots+e_k\cdot\mathrm{I\!I}_k,$$

where $e_1,\dots,e_k$ is an orthonormal basis of N_pL. Denote by Sc_i, H_i, and Ӿ_i the values associated with $\mathrm{I\!I}_i$. From above, we get

$$\mathrm{Sc}_i=\tfrac32\cdot|H_i|^2-\tfrac{n\cdot(n+2)}{2}\cdot\text{Ӿ}_i$$

for each i.

Note that

$$\begin{aligned}\mathrm{Sc}&=\mathrm{Sc}_1+\dots+\mathrm{Sc}_k,\\ |H^2|&=|H_1|^2+\dots+|H_k|^2,\\ \text{Ӿ}&=\text{Ӿ}_1+\dots+\text{Ӿ}_k.\end{aligned}$$

Hence the general case follows. □

2 Embeddings into sphere

The obtained formula shows that some results in [3–5] are exact. In this and the following section, we will list of some of them.

Let us denote by $\mathbb{T}^n$ the n-dimensional torus — the smooth manifold diffeomorphic to the product of n circles. The next statement follows from the formula since any Riemannian metric on the torus has nonpositive scalar curvature at some point.

2.1. Theorem. *Suppose $\iota\colon\mathbb{T}^n\looparrowright\mathbb{S}^q$ is a smooth immersion. Then*

$$\text{Ӿ}(p)\geqslant 2\cdot\tfrac{n-1}{n+2}$$

at some point $p\in\mathbb{T}^n$.

In particular, there is a tangent direction of $\mathbb{T}^n$ with normal curvature at least

$$\kappa_n=\sqrt{2\cdot\tfrac{n-1}{n+2}}.$$

It was shown [5] that there is an isometric embedding of the torus $\mathbb{T}^n$ with a flat metric that has normal curvature κ_n in any direction at any point. In particular, the above bound on normal curvature is optimal. The compression lemma [4], implies that *any closed smooth manifold is diffeomorphic to a submanifold with normal curvatures at most $\sqrt2$ in the unit sphere of sufficiently* *large dimension. Moreover, the induced Riemannian metric can be chosen to be proportional to any given metric g.* Applying the theorem, we get the following.

2.2. Corollary. *The bound $\sqrt{2}$ is optimal.*

3 Embeddings into ball

Let us denote by $\mathbb{B}^q$ the unit ball in q-dimensional Euclidean space. The following lemma was essentially proved by István Fáry [2]; see also the survey of Serge Tabachnikov [6].

3.1. Lemma. *Let $\iota\colon \mathbb{T}^n \looparrowright \mathbb{B}^q$ be a smooth immersion. Then the average value of $|H|$ is at least n.*

Proof. Consider the function $u\colon p \mapsto \frac{1}{2}\cdot|\iota(p)|^2$ on $\mathbb{T}^n$. Note that

$$\Delta u = n + \langle H, \iota\rangle.$$

It follows that the average value of $\langle H, \iota\rangle$ is $-n$. Since $|\iota| \leqslant 1$, we get the result. □

Since $\mathrm{Sc} = \frac{3}{2}\cdot|H|^2 - \frac{n\cdot(n+2)}{2}\cdot \text{ꓘ}$ (see 1.1), the lemma implies the following.

3.2. Proposition. *Let L be a flat closed n-dimensional manifold that is isometrically immersed in $\mathbb{B}^q$. Then the average value of ꓘ on L is at least $3\cdot\frac{n}{n+2}$.*

3.3. Theorem. *Suppose $\iota\colon \mathbb{T}^2 \to \mathbb{B}^q$ is a smooth immersion. Then the average value of ꓘ on $\mathbb{T}^2$ is at least $\frac{3}{2}$.*

Proof. By 3.1, the average value of $|H|^2$ is at least 4. Applying the formula and Gauss–Bonnet, we get the result. □

3.4. Theorem. *Let $\iota\colon \mathbb{T}^n \looparrowright \mathbb{B}^q$ be a smooth immersion. Then its maximal normal curvature is at least $\sqrt{3\cdot\frac{n}{n+2}}$.*

Frame of the proof. The case $n = 2$ follows from 3.3; so we can assume that $n \geqslant 3$. We will prove that *if the normal curvatures at most 2, then*

❶ $$\text{ꓘ} \geqslant 3\cdot\tfrac{n}{n+2}$$

at some point. Clearly it implies 3.4. The following lemma tells how we are going to use the assumtion on normal curvatures. In case $n \leqslant 4$, the proof below implies ❶ without this assumption.

3.5. Lemma. *Let $\iota\colon M \looparrowright \mathbb{B}^q$ be a smooth immersion with normal curvatures at most 2. Given $p \in M$, denote by $\beta(p)$ the angle between $\iota(p)$ and the normal space at $\iota(p)$. Then*

$$|\iota| \leqslant \cos\beta.$$

Proof. Assume the inequality does not hold at p. Shoot a geodesic in M that runs from p at angle $\beta(p)$ from $\iota(p)$. Since this geodesic has curvature at most 2, it will leave $\mathbb{B}^q$ in time π — a contradiction. □

Let g be a Riemannian metric on $\mathbb{T}^n$. Suppose $n \geqslant 3$, and $u\colon \mathbb{T}^n \to \mathbb{R}$ is a positive function. Here is the well-known formula for the scalar scalar curvature of the metric $u^{\frac{4}{n-2}}\cdot g$:

$$\left(\mathrm{Sc}\cdot u - 4\cdot\tfrac{n-1}{n-2}\cdot\Delta u\right)\cdot u^{\frac{n-2}{n+2}};$$

see for example [1, 6.3]. Recall that *any Riemannian metric g on $\mathbb{T}^n$ has nonpositive scalar curvature at some point.* Hence we get the following.

3.6. Claim. *Let g be a Riemannian metric on $\mathbb{T}^n$. Then, for any positive smooth function u on $\mathbb{T}^n$, the function*

$$\mathrm{Sc}\cdot u - 4\cdot\tfrac{n-1}{n-2}\cdot\Delta u$$

returns a nonpositive value at some point.

This claim plays central role in the following proof.

Proof of 3.4. Consider the function $u\colon p \mapsto \exp(-\frac{k}{2}\cdot|\iota(p)|^2)$.

We will apply the following formula

$$\Delta(\varphi\circ f) = \varphi'\cdot\Delta f + \varphi''\cdot|\nabla f|^2$$

to $f\colon p \mapsto \frac{1}{2}\cdot|\iota(p)|^2$ and $\varphi\colon x \mapsto \exp(-k\cdot x)$; so $u = \varphi\circ f$.

Set $\alpha = \measuredangle(H,\iota)$ and β as in 3.5. Note that

$$❷\qquad \beta \leqslant \alpha \leqslant \pi - \beta.$$

Observe that

$$\Delta f = n + |H|\cdot|\iota|\cdot\cos\alpha,\quad |\nabla f| = |\iota|\cdot\sin\beta,\quad \varphi' = -k\cdot\varphi,\quad \varphi'' = k^2\cdot\varphi.$$

Therefore

$$\Delta u = u\cdot[-k\cdot n - k\cdot|H|\cdot|\iota|\cdot\cos\alpha + k^2\cdot|\iota|^2\cdot(\sin\beta)^2].$$

By 1.1,

$$\mathrm{Sc} = -\tfrac{n\cdot(n+2)}{2}\cdot \text{Ӂ}K + \tfrac{3}{2}\cdot|H|^2.$$

By 3.6, the function

$$\begin{aligned}\mathrm{Sc}\cdot u &- 4\cdot\tfrac{n-1}{n-2}\cdot\Delta u = \\ &= u\cdot\Big[-\tfrac{n\cdot(n+2)}{2}\cdot \text{Ӂ}K + \tfrac{3}{2}\cdot|H|^2 + 4\cdot\tfrac{n-1}{n-2}\cdot k\cdot|H|\cdot|\iota|\cdot\cos\alpha - \\ &\quad + 4\cdot\tfrac{n-1}{n-2}\cdot(k\cdot n - k^2\cdot|\iota|^2\cdot(\sin\beta)^2)\Big]\end{aligned}$$

returns a nonpositive value at some point $p\in\mathbb{T}^n$.

Choose

$$k = \tfrac{3}{4}\cdot\tfrac{n-2}{n-1}\cdot n, \quad\text{so}\quad n = \tfrac{4}{3}\cdot\tfrac{n-1}{n-2}\cdot k.$$

At the point p, we have

$$\begin{aligned}\tfrac{n\cdot(n+2)}{2}\cdot \text{Ӂ}K &\geqslant \tfrac{3}{2}\cdot(|H| + n\cdot|\iota|\cdot\cos\alpha)^2 - \tfrac{3}{2}\cdot n^2\cdot|\iota|^2\cdot\cos^2\alpha + \\ &\quad + 3\cdot n^2 - \tfrac{9}{4}\cdot\tfrac{n-2}{n-1}\cdot n^2\cdot|\iota|^2\cdot(\sin\beta)^2 \geqslant \\ &\geqslant \tfrac{3}{2}\cdot n^2.\end{aligned}$$

The last inequality follows since $\cos^2\beta + \sin^2\alpha \leqslant 1$ and $|\iota|^2 + \sin^2\beta \leqslant 1$; see ❷ and 3.5. Hence ❶ follows. $\square$

References

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