Gromov's tori are optimal

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Abstract

We give an optimal bound on normal curvatures of immersed n-torus in a Euclidean ball of large dimension.

1 Introduction

Let us denote by $\mathbb{B}^q$ the closed unit ball in $\mathbb{R}^q$ centered at the origin. Further, $\mathbb{T}^n$ will denote the n-dimensional torus — the smooth manifold diffeomorphic to the product of n circles.

This note is inspired by examples of embeddings $\mathbb{T}^n \hookrightarrow \mathbb{B}^q$ for large q with constant normal curvatures $K_n = \sqrt{3 \cdot n/(n+2)}$. In other words, any geodesic in the torus has constant curvature K_n as a curve in $\mathbb{R}^q$. These examples were found by Michael Gromov among geodesic subtori in Clifford's tori [5, 2.A], [4, 1.1.A.]. In particular, Gromov's tori have flat induced metrics. (Recall that Clifford's torus is a product of m circles of radius $1/\sqrt{m}$ in $\mathbb{R}^{2\cdot m}$; its normal curvatures lie in the range $[1, \sqrt{m}\,]$.)

Gromov's examples lead to the following surprising facts: *any closed smooth manifold L admits a smooth embedding into $\mathbb{B}^q$ for large q with normal curvatures less than $\sqrt{3}$; moreover, the induced Riemannian metric on L can be chosen to be proportional to any given metric g;* see [5, 1.D] and [4, 1.1.C].

The next theorem implies that Gromov's tori have the best upper bound on normal curvatures; in particular, the $\sqrt{3}$-bound is optimal.

1.1. Theorem. *Suppose $\mathbb{T}^n$ is smoothly immersed in $\mathbb{B}^q$. Then its maximal normal curvature is at least*

$$\sqrt{3 \cdot \tfrac{n}{n+2}}.$$

To make the statement more exact, we need one more notation. Assume that L is a smooth n-dimensional manifold immersed in $\mathbb{R}^q$; we will always assume that L is equipped with the induced Riemannian metric. Let us denote by T_x and N_x the tangent and normal spaces of L at x.

Recall that the *second fundamental form* $\mathrm{I\!I}$ at x is a symmetric quadratic form on T_x with values in N_x. It is uniquely defined by the identity $\mathrm{I\!I}(\mathrm{v}, \mathrm{v}) \equiv \gamma_{\mathrm{v}}''(0)$, where $\mathrm{v} \in \mathrm{T}_x$ and γ_{v} is an L-geodesic that starts at x with initial velocity vector v.

Given $x \in L$, denote by $Ж(x)$ the average value of $|\mathrm{I\!I}(\mathrm{u}, \mathrm{u})|^2$ for $\mathrm{u} \in \mathrm{T}_x$ such that $|\mathrm{u}| = 1$. Since $K(\mathrm{u}) = |\mathrm{I\!I}(\mathrm{u}, \mathrm{u})|$ is the normal curvature in the direction u, we have that $Ж(x)$ is the average value $K^2(\mathrm{u})$. (The Cyrillic zhe $Ж$ is used since it resembles K^2.)

1.2. Theorem. *Suppose $\mathbb{T}^n$ is smoothly immersed in $\mathbb{B}^q$. Let us equip $\mathbb{T}^n$ with the induced Riemannian metric; so we can take average values with respect to the induced volume.*

(a) If $n = 2$, then the average value of ꓘK is at least $\frac{3}{2}$.
(b) If the metric on $\mathbb{T}^n$ is flat, then the average value of ꓘK is at least $3\cdot\frac{n}{n+2}$.
(c) If the image of $\mathbb{T}^n$ lies in $\partial\mathbb{B}^q$, then $\text{ꓘK} \geqslant 3\cdot\frac{n}{n+2}$ at some point of $\mathbb{T}^n$.
(d) If $n \leqslant 4$, then $\text{ꓘK} \geqslant 3\cdot\frac{n}{n+2}$ at some point of $\mathbb{T}^n$.
(e) If the normal curvatures of $\mathbb{T}^n$ do not exceed 2, then $\text{ꓘK} \geqslant 3\cdot\frac{n}{n+2}$ at some point of $\mathbb{T}^n$.

Note that part *(e)* is a stronger version of 1.1. The remaining statements *(a)*–*(d)* are stronger versions of 1.1 in some partial cases. All this follows since the normal curvature in some direction at x is at least $\sqrt{\text{ꓘK}(x)}$.

All proofs use our version of the Gauss formula; see below. The proofs of *(c)*–*(e)* use in addition that *the torus does not admit a metric with positive scalar curvature* [7, Corollary A].

1.3. Open question. *Is it true that for any smooth immersion $\mathbb{T}^n \looparrowright \mathbb{B}^q$ the inequality $\text{ꓘK} \geqslant 3\cdot\frac{n}{n+2}$ holds at some point?*

1.4. Open question. *Suppose $\mathbb{RP}^n \looparrowright \mathbb{B}^q$ is a smooth immersion. Is it true that its normal curvature is at least $\sqrt{2\cdot n/(n+1)}$ in some direction?*

The last question asks if *the Veronese embedding is optimal*. Recall that the Veronese embedding $\mathbb{RP}^n \hookrightarrow \mathbb{B}^q$ has all normal curvatures $\sqrt{2\cdot n/(n+1)}$; here $q = \frac{1}{2}\cdot(n+1)(n+2)$. An analogous question for immersions into unit spheres is open as well [9]. You may also ask for the optimal constant for your favorite closed manifold.

2 Gauss formula

Recall that L is a smooth n-dimensional manifold immersed in $\mathbb{R}^q$. Given $p \in L$, denote by $\mathrm{Sc}(p)$ and $H(p)$ the scalar curvature and the mean curvature vector at p.

The following version of the Gauss formula plays a central role in all proofs; it is used instead of the formula in [6, 5.B].

2.1. Gauss formula. *The following identity*

$$\mathrm{Sc} = \tfrac{3}{2}\cdot|H|^2 - \tfrac{n\cdot(n+2)}{2}\cdot\text{ꓘK}$$

holds for any smooth n-dimensional immersed manifold in a Euclidean space.

Proof. Choose a point $p \in L$.

Assume $\operatorname{codim} L = 1$. Denote by $k_1, \dots, k_n$ the principal curvatures of L at p. Note that

$$|H|^2 = \sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

Further,

$$n\cdot(n+2)\cdot\text{ꓘK} = 3\cdot\sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

The last identity follows since $\text{Ӂ}K$ is the average value of $\big(\sum_i k_i\cdot x_i^2\big)^2$ on the unit sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n=\mathrm{T}_p$; here $(x_1,\dots,x_n)$ are the standard coordinates in $\mathbb{R}^n$. One has to take into account that the following functions have unit average values: $\tfrac13\cdot n\cdot(n+2)\cdot x_i^4$ and $n\cdot(n+2)\cdot x_i^2\cdot x_j^2$ for $i\ne j$.

By the standard Gauss formula,

$$\mathrm{Sc}=2\cdot\sum_{i<j}k_i\cdot k_j.$$

It remains to rewrite the right-hand side using the expressions for $|H|^2$ and $\text{Ӂ}K$.

If $\operatorname{codim}L=k>1$, then the second fundamental form at p can be presented as a direct sum of k real-valued quadratic forms $\mathrm{I\!I}_1\oplus\dots\oplus\mathrm{I\!I}_k$; that is,

$$\mathrm{I\!I}=e_1\cdot\mathrm{I\!I}_1+\dots+e_k\cdot\mathrm{I\!I}_k,$$

where $e_1,\dots,e_k$ is an orthonormal basis of N_p. Denote by Sc_i, H_i, and $\text{Ӂ}K_i$ the values associated with $\mathrm{I\!I}_i$. From above, we get

$$\mathrm{Sc}_i=\tfrac32\cdot|H_i|^2-\tfrac{n\cdot(n+2)}{2}\cdot\text{Ӂ}K_i$$

for each i.

Note that

$$\mathrm{Sc}=\sum_i\mathrm{Sc}_i,\qquad |H^2|=\sum_i|H_i|^2,\quad\text{and}\quad \text{Ӂ}K=\sum_i\text{Ӂ}K_i.$$

Hence the general case follows. □

Remark. A more direct proof of this formula can be obtained using the so-called *extrinsic curvature tensor* which is defined by $\Phi(\mathrm{X},\mathrm{Y},\mathrm{V},\mathrm{W}):=\langle\mathrm{I\!I}(\mathrm{X},\mathrm{Y}),\mathrm{I\!I}(\mathrm{V},\mathrm{W})\rangle$; the necessary properties of this tensor are discussed in [8]. As a bonus, one gets an explicit expression for the second fundamental forms of all Gromov's tori.

3 Special cases

The following statement appears in the book of Yuri Burago and Viktor Zalgaller [2, Theorem 28.2.5]; it generalizes the result of István Fáry about average curvature of a curve in the unit ball [3, 12], but the proof is essentially the same.

3.1. Lemma. *Let L be a closed n-dimensional manifold that is smoothly immersed in $\mathbb{B}^q$. Then the average value of $|H|$ on L is at least n.*

Proof. Consider the function $u\colon x\mapsto\tfrac12\cdot|x|^2$ on L. Note that

$$(\Delta u)(x)=n+\langle H(x),x\rangle.$$

It follows that the average value of $\langle H(x),x\rangle$ is $-n$. Since $|x|\leqslant1$, we get the result. □

Proof of 1.2a. By 3.1, the average value of $|H|^2$ is at least 4. Further, by the Gauss–Bonnet formula, the scalar curvature (which is twice the Gauss curvature in this case) has zero average. Therefore 2.1 implies the statement. □

Proof of 1.2b. By 3.1, the average value of $|H|^2$ is at least n^2. Since $\mathrm{Sc} \equiv 0$, it remains to apply 2.1. $\square$

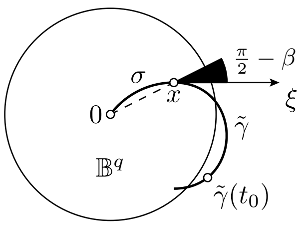
Proof of 1.2c. Since the image lies in the unit sphere, we have that $|H|^2$ is at least n^2 at each point. Since $\mathbb{T}^n$ does not admit a metric with positive scalar curvature [7, Corollary A], we have $\mathrm{Sc}(x) \leqslant 0$ at some point x. It remains to apply 2.1 at x. $\square$

4 Main case

The following lemma is an easy corollary of the bow lemma of Axel Schur [10, 11]. It explains how we use the assumption on normal curvatures in 1.2*e*. If $n \leqslant 4$, then the proof works without this assumption.

4.1. Lemma. *Let L be a manifold smoothly immersed in $\mathbb{B}^q$. Suppose its normal curvatures are at most 2. Given $x \in L$, denote by $\beta = \beta(x)$ the angle between vector x and the normal space N_x. Then $|x| \leqslant \cos\beta$.*

Proof. Let ξ be a tangent direction at x such that $\measuredangle(x, \xi) = \frac{\pi}{2} - \beta$. In the plane spanned by x and ξ, choose a unit-speed circle arc σ from 0 to x that comes to x in the direction opposite to ξ; extend σ after x by a unit-speed semicircle $\tilde\gamma$ with curvature 2 in such a way that the concatenation $\sigma * \tilde\gamma$ is an arc of a C^1-smooth convex plane curve; see the figure.



Observe that if $|x| > \cos\beta$, then $\tilde\gamma$ leaves $\mathbb{B}^q$; that is, $|\tilde\gamma(t_0)| > 1$ for some t_0.

Let γ be the unit-speed geodesic in L that runs from x in the direction ξ. Note that curvatures of $\sigma * \gamma$ do not exceed the curvatures of $\sigma * \tilde\gamma$ at the corresponding points. Applying the bow lemma for $\sigma * \gamma$ and $\sigma * \tilde\gamma$, we get $|\gamma(t_0)| \geqslant |\tilde\gamma(t_0)|$. It follows that L does not lie in $\mathbb{B}^q$ — a contradiction. $\square$

Let g be a Riemannian metric on $\mathbb{T}^n$. Suppose $n \geqslant 3$, and $u\colon \mathbb{T}^n \to \mathbb{R}$ is a smooth positive function. Recall that

$$\left(\mathrm{Sc}\cdot u - 4\cdot\tfrac{n-1}{n-2}\cdot\Delta u\right)\cdot u^{\frac{n-2}{n+2}}$$

is the scalar curvature of the metric $u^{\frac{4}{n-2}}\cdot g$; see for example [1, 6.3]. Since *any Riemannian metric on $\mathbb{T}^n$ has nonpositive scalar curvature at some point* [7, Corollary A], we get the following.

4.2. Claim. *For any Riemannian metric on $\mathbb{T}^n$ and any positive smooth function $u\colon \mathbb{T}^n \to \mathbb{R}$, the function*

$$\mathrm{Sc}\cdot u - 4\cdot\tfrac{n-1}{n-2}\cdot\Delta u$$

returns a nonpositive value at some point.

Proof of 1.2d and 1.2e. The case $n = 2$ follows from 1.2*a*; so we can assume that $n \geqslant 3$. Consider the function $u\colon x \mapsto \exp(-\frac{k}{2}\cdot|x|^2)$ on the torus.

We will apply the following formula

$$\Delta(\varphi\circ f)=(\varphi'\circ f)\cdot\Delta f+(\varphi''\circ f)\cdot|\nabla f|^2$$

to $f\colon x\mapsto\frac12\cdot|x|^2$ and $\varphi\colon y\mapsto\exp(-k\cdot y)$; so $u=\varphi\circ f$.

Set $r(x)=|x|$, $\alpha(x)=\measuredangle(H(x),x)$, and $\beta(x)$ as in 4.1. Note that

❶ $$\beta\leqslant\alpha\leqslant\pi-\beta.$$

Observe that

$$\Delta f=|H|\cdot r\cdot\cos\alpha+n,\quad |\nabla f|=r\cdot\sin\beta,\quad \varphi'=-k\cdot\varphi,\quad \varphi''=k^2\cdot\varphi.$$

Therefore

$$\Delta u=u\cdot[-k\cdot|H|\cdot r\cdot\cos\alpha-k\cdot n+k^2\cdot r^2\cdot\sin^2\beta].$$

Recall that $\mathrm{Sc}=-\frac{n\cdot(n+2)}{2}\cdot\text{ƛ}K+\frac32\cdot|H|^2$; see 2.1. By 4.2, the function

$$\mathrm{Sc}\cdot u-4\cdot\tfrac{n-1}{n-2}\cdot\Delta u=u\cdot\Big[-\tfrac{n\cdot(n+2)}{2}\cdot\text{ƛ}K+\tfrac32\cdot|H|^2+4\cdot\tfrac{n-1}{n-2}\cdot k\cdot|H|\cdot r\cdot\cos\alpha+ \\ +4\cdot\tfrac{n-1}{n-2}\cdot(k\cdot n-k^2\cdot r^2\cdot\sin^2\beta)\Big]$$

returns a nonpositive value at some point $x\in\mathbb{T}^n$.

Choose

$$k=\tfrac34\cdot\tfrac{n-2}{n-1}\cdot n,\quad\text{so}\quad n=\tfrac43\cdot\tfrac{n-1}{n-2}\cdot k.$$

At the point x, we have

❷ $$\tfrac{n\cdot(n+2)}{2}\cdot\text{ƛ}K\geqslant\tfrac32\cdot(|H|+n\cdot r\cdot\cos\alpha)^2-\tfrac32\cdot n^2\cdot r^2\cdot\cos^2\alpha+ \\ +3\cdot n^2-\tfrac94\cdot\tfrac{n-2}{n-1}\cdot n^2\cdot r^2\cdot\sin^2\beta\geqslant\tfrac32\cdot n^2.$$

Indeed, by ❶, $\cos^2\alpha+\sin^2\beta\leqslant1$. If $n\leqslant4$, then $\frac32\geqslant\frac94\cdot\frac{n-2}{n-1}$; therefore the last inequality follows, and it proves 1.2d.

Further, if $n\geqslant5$, then for the last inequality in ❷ we need to use in addition that $r^2+\sin^2\beta\leqslant1$ which follows from 4.1. Hence 1.2e follows. □

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