1 The formula

Let L be a smooth n-dimensional submanifold in a Riemannian manifold M. Let us denote by T_pL and N_pL the tangent and normal spaces of L at p.

Recall that *second fundamental form* $\mathrm{I\!I}$ at p is a symmetric quadratic form on T_pL with values in N_pL. It is uniquely defined by the following identity

$$\mathrm{I\!I}(\mathrm{v},\mathrm{v}) := \gamma_{\mathrm{v}}''(0),$$

where $\mathrm{v} \in \mathrm{T}_pL$ and γ_{v} an L-geodesic that starts at p with initial velocity vector v.

Given $p \in L$, denote by $\mathrm{Sc}(p)$, $H(p)$, and $\text{Ӿ}(p)$ the scalar curvature, the mean curvature vector, and the average value of $|\mathrm{I\!I}(\mathrm{u},\mathrm{u})|^2$ at p respectively.

Let us denote by $\widetilde{\mathrm{Sc}}$ the *outer scalar curvature* of L in M; that is, if $e_1,\dots,e_n$ is an orthonormal basis of T_pL, then

$$\widetilde{\mathrm{Sc}}(p) = 2\cdot\sum_{i<j} K_{ij},$$

where K_{ij} denotes sectional curvature of M in the direction spanned by e_i and e_j.

The following claim gives a better version of [4, 5.B].

Formula. *The following identity*

$$\mathrm{Sc} - \widetilde{\mathrm{Sc}} = \tfrac{3}{2}\cdot|H|^2 - \tfrac{n\cdot(n+2)}{2}\cdot\text{Ӿ}$$

holds for any smooth n-dimensional submanifold L in a Riemannian manifold.

Proof. Without loss of generality, we can assume that the ambient manifold is flat; in particular $\widetilde{\mathrm{Sc}} = 0$.

Assume $\operatorname{codim} L = 1$. Choose $p \in L$; denote by $k_1,\dots,k_n$ the principal curvatures of L at p. Note that

$$|H|^2 = \sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

Further,

$$n\cdot(n+2)\cdot\text{Ӿ} = 3\cdot\sum_i k_i^2 + 2\cdot\sum_{i<j} k_i\cdot k_j.$$

The last identity follows since Ӿ is the average value of $\left(\sum_i k_i\cdot x_i^2\right)^2$ on $\mathbb{S}^{n-1}$. One has to take into account that 3 and 1 are the average values of $n\cdot(n+2)\cdot x_i^4$ and $n\cdot(n+2)\cdot x_i^2\cdot x_j^2$ for $i \ne j$ on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Here we assume that $(x_1,\dots,x_n)$ are the standard coordinates in $\mathbb{R}^n$.

By Gauss formula,

$$\mathrm{Sc} = 2\cdot\sum_{i<j} k_i\cdot k_j.$$

It remains to rewrite this formula using the expressions for $|H|^2$ and ꓘK.

If $\operatorname{codim} L = k > 1$, then the second fundamental form can be presented as a direct sum of k real-valued quadratic forms $\mathrm{I\!I}_1 \oplus \dots \oplus \mathrm{I\!I}_k$; that is,

$$\mathrm{I\!I} = e_1\cdot\mathrm{I\!I}_1 + \dots + e_k\cdot\mathrm{I\!I}_k,$$

where $e_1, \dots, e_k$ is an orthonormal basis of $\mathrm{N}_p L$. Denote by Sc_i, H_i, and ꓘK_i the values associated with $\mathrm{I\!I}_i$. From above, we get

$$\mathrm{Sc}_i = \tfrac{3}{2}\cdot|H_i|^2 - \tfrac{n\cdot(n+2)}{2}\cdot\text{ꓘK}_i$$

for each i.

Note that

$$\begin{aligned}\mathrm{Sc} &= \mathrm{Sc}_1 + \dots + \mathrm{Sc}_k,\\ |H^2| &= |H_1|^2 + \dots + |H_k|^2,\\ \text{ꓘK} &= \text{ꓘK}_1 + \dots + \text{ꓘK}_k.\end{aligned}$$

Whence the general case follows. $\square$

2 Embeddings into sphere

The obtained formula shows that some results in [2–4] are exact. In this and the following section, we will list of some of them.

Let us denote by $\mathbb{T}^n$ the n-dimensional torus — the smooth manifold diffeomorphic to the product of n circles. The next statement follows from the formula since any Riemannian metric on the torus has nonpositive scalar curvature at some point.

Theorem. *Suppose $\iota\colon \mathbb{T}^n \looparrowright \mathbb{S}^q$ is a smooth immersion. Then*

$$\text{ꓘK}(p) \geqslant \tfrac{2\cdot(n-1)}{n+2}$$

at some point $p \in \mathbb{T}^n$.

In particular, there is a tangent direction of $\mathbb{T}^n$ with normal curvature at least

$$\kappa_n = \sqrt{\tfrac{2\cdot(n-1)}{n+2}}.$$

It was shown [4] that there is an isometric embedding of the torus $\mathbb{T}^n$ with a flat metric that has normal curvature κ_n in any direction at any point. In particular, the above bound on normal curvature is optimal. The compression lemma [3], implies that *any closed smooth manifold is diffeomorphic to a submanifold with normal curvatures at most $\sqrt{2}$ in the unit sphere of sufficiently large dimension. Moreover, the induced Riemannian metric can be chosen to be proportional to any given metric g.* Applying the theorem, we get the following.

Corollary. *The bound $\sqrt{2}$ is optimal.*

3 Embeddings into ball

Let us denote by $\mathbb{B}^q$ the unit ball in q-dimensional Euclidean space.

Question. *Suppose $\iota\colon \mathbb{T}^n \to \mathbb{B}^q$ is a smooth immersion. Is it true that $\mathcal{K}(p) \geqslant \frac{3\cdot n}{n+2}$ at some point $p \in \mathbb{T}^n$?*

If true, then for large $q \gg n^2 \gg 1$, the optimal asymptotic lower bound on normal curvatures is $\sqrt{3}$. Playing a bit with the formulas below seems to give an asymptotic lower bound $\sqrt{8/3}$; it is quite close to $\sqrt{3}$ and can be improved a bit further, but the optimal bound requires an extra idea.

Below we answer the question in three cases: $n = 2$, $n = 4$, and if the induced metric is flat. The following lemma was essentially proved by István Fáry [1]; see also the survey of Serge Tabachnikov [5].

Lemma. *Let $\iota\colon \mathbb{T}^n \to \mathbb{B}^q$ be a smooth immersion. Then the average value of $|H|$ is at least n.*

Proof. Consider the function $u\colon p \mapsto \frac{1}{2}\cdot|\iota(p)|^2$ on $\mathbb{T}^n$. Note that $\Delta u = n + \langle H, \iota\rangle$. It follows that the average value of $\langle H, \iota\rangle$ is $-n$. Since $|\iota| \leqslant 1$, we get the result. $\square$

The lemma and formula imply the following.

Proposition. *Let L be a flat closed n-dimensional submanifold in $\mathbb{B}^q$. Then the average value of $\mathcal{K}$ on L is at least $\frac{3\cdot n}{n+2}$.*

Theorem. *Suppose $\iota\colon \mathbb{T}^2 \to \mathbb{B}^q$ is a smooth immersion. Then the average value of $\mathcal{K}$ on $\mathbb{T}^2$ is at least $\frac{3}{2}$.*

Proof. By the lemma, the average value of $|H|^2$ is at least 4. Applying the formula and Gauss–Bonnet, we get the result. $\square$

Theorem. *Suppose $\iota\colon \mathbb{T}^4 \to \mathbb{B}^q$ is a smooth immersion. Then $\mathcal{K}(p) \geqslant 2$ for some point $p \in \mathbb{T}^4$.*

Let g be a Riemannian metric on $\mathbb{T}^n$. Suppose $n \geqslant 3$, and $u\colon \mathbb{T}^n \to \mathbb{R}$ is a positive function. The scalar curvature of the metric $u^{\frac{4}{n-2}}\cdot g$ can be expressed as

$$\left(\mathrm{Sc}\cdot u - \tfrac{4\cdot(n-1)}{n-2}\cdot\Delta u\right)\cdot u^{\frac{n-2}{n+2}}.$$

Recall that *any Riemannian metric g on $\mathbb{T}^n$ has nonpositive scalar curvature at some point.* Therefore we get the following.

Claim. *Let g be a Riemannian metric on $\mathbb{T}^n$. Then, for any positive smooth function u on $\mathbb{T}^n$, the function*

$$\mathrm{Sc}\cdot u - \tfrac{4\cdot(n-1)}{n-2}\cdot\Delta u$$

returns a nonpositive value at some point

Proof of the theorem. Consider the function $u\colon p \mapsto \exp(-|\iota(p)|^2)$.

We will apply the following formula

$$\Delta(\varphi\circ f) = \varphi'\cdot\Delta f + \varphi''\cdot|\nabla f|^2$$

to $f\colon p\mapsto \frac{1}{2}\cdot|\iota(p)|^2$ and $\varphi\colon x\mapsto \exp(-2\cdot x)$; so $u=\varphi\circ f$.

Let $\alpha=\measuredangle(H,\iota)$, then $\Delta f=4{+}|H|\cdot|\iota|\cdot\cos\alpha$ and $|\nabla f|\leqslant|\iota|\cdot\sin\alpha$. Therefore

$$\Delta u\leqslant u\cdot[-2\cdot(4+|H|\cdot|\iota|\cdot\cos\alpha)+(2\cdot|\iota|\cdot\sin\alpha)^2].$$

Applying the claim for $n=4$, we get that

$$\mathrm{Sc}\cdot u-6\cdot\Delta u$$

returns a negative value at some point $p\in\mathbb{T}^4$. Applying the formula, we get

$$12\cdot\text{Ӿ}K(p)\geqslant\tfrac{3}{2}\cdot[(|H(p)|-4)^2+16].$$

Whence the statement follows. $\square$

References

[1] I. Fáry. “Sur certaines inégalités géométriques”. *Acta Sci. Math.* 12 (1950), 117–124.

[2] M. Gromov. *Curvature, Kolmogorov Diameter, Hilbert Rational Designs and Overtwisted Immersions.* 2022.

[3] M. Gromov. *Isometric Immersions with Controlled Curvatures.* 2022.

[4] M. Gromov. *Scalar Curvature, Injectivity Radius and Immersions with Small Second Fundamental Forms.* 2022.

[5] S. Tabachnikov. “The tale of a geometric inequality.” *MASS selecta: teaching and learning advanced undergraduate mathematics.* 2003, 257–262.