

Gromov's torii are optimal

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Abstract

We give an optimal bound on normal curvatures of immersed n -torus in a Euclidean ball of large dimension.

1 Introduction

Let us denote by \mathbb{B}^q the closed unit ball in \mathbb{R}^q centered at the origin. Further, \mathbb{T}^n will denote the n -dimensional torus — the smooth manifold diffeomorphic to the product of n circles.

This note is inspired by examples of embeddings $\mathbb{T}^n \hookrightarrow \mathbb{B}^q$ for large q with constant normal curvatures $K_n = \sqrt{3 \cdot n / (n + 2)}$. In other words, any geodesic in the torus has constant curvature K_n as a curve in \mathbb{R}^q . These examples were found by Michael Gromov among geodesic subtorii in Clifford's torii [5, 2.A], [4, 1.1.A.]. In particular, Gromov's torii have flat induced metric. (Recall that Clifford's torus is a product of m circles of radius $1/\sqrt{m}$ in $\mathbb{R}^{2 \cdot m}$; its normal curvatures lie in the range $[1, \sqrt{m}]$.)

Gromov's examples lead to the following surprising facts: *any closed smooth manifold L admits a smooth embedding into \mathbb{B}^q for large q with normal curvatures less than $\sqrt{3}$; moreover, the induced Riemannian metric on L can be chosen to be proportional to any given metric g ; see [5, 1.D] and [4, 1.1.C].*

The next theorem implies that Gromov's torii have the best upper bound on normal curvatures; in particular, the $\sqrt{3}$ -bound is optimal.

1.1. Theorem. *Suppose \mathbb{T}^n is smoothly immersed in \mathbb{B}^q . Then its maximal normal curvature is at least*

$$\sqrt{3 \cdot \frac{n}{n+2}}.$$

To make the statement more exact, we need one more notation. Assume that L is a smooth n -dimensional manifold immersed in \mathbb{R}^q ; we will always assume that L is equipped with induced Riemannian metric. Let us denote by T_x and N_x the tangent and normal spaces of L at x .

Recall that *second fundamental form* \mathbb{I} at x is a symmetric quadratic form on T_x with values in N_x . It is uniquely defined by the identity $\mathbb{I}(v, v) \equiv \gamma_v''(0)$, where $v \in T_x$ and γ_v is an L -geodesic that starts at x with initial velocity vector v .

Given $x \in L$, denote by $\mathcal{K}(x)$ the average value of $|\mathbb{I}(u, u)|^2$ for $u \in T_x$ such that $|u| = 1$. Since $K(u) = |\mathbb{I}(u, u)|$ is the normal curvature in the direction u , we have that $\mathcal{K}(x)$ is the average value $K^2(u)$. (The Cyrillic zhe \mathcal{K} is used since it resembles K^2 .)

1.2. Theorem. Suppose \mathbb{T}^n is smoothly immersed in \mathbb{B}^q . Let us equip \mathbb{T}^n with the induced Riemannian metric; so we can take average values with respect to the induced volume.

- (a) If $n = 2$, then the average value of λK is at least $\frac{3}{2}$.
- (b) If the metric on \mathbb{T}^n is flat, then the average value of λK is at least $3 \cdot \frac{n}{n+2}$.
- (c) If the image of \mathbb{T}^n lies in $\partial \mathbb{B}^q$, then $\lambda K \geq 3 \cdot \frac{n}{n+2}$ at some point of \mathbb{T}^n .
- (d) If $n \leq 4$, then $\lambda K \geq 3 \cdot \frac{n}{n+2}$ at some point of \mathbb{T}^n .
- (e) If the normal curvatures of \mathbb{T}^n do not exceed 2, then $\lambda K \geq 3 \cdot \frac{n}{n+2}$ at some point of \mathbb{T}^n .

Note that part (e) is a stronger version of 1.1. The remaining statements (a)–(d) are stronger versions of 1.1 in some partial cases. All this follows since the normal curvature in some direction at x is at least $\sqrt{\lambda K(x)}$.

All proofs use our version of the Gauss formula; see below. The proofs of (c)–(e) use in addition that *torus does not admit a metric with positive scalar curvature* [7, Corollary A].

1.3. Open question. Is it true that for any smooth immersion $\mathbb{T}^n \looparrowright \mathbb{B}^q$ the inequality $\lambda K \geq 3 \cdot \frac{n}{n+2}$ holds at some point?

1.4. Open question. Suppose $\mathbb{RP}^n \looparrowright \mathbb{B}^q$ is a smooth immersion. Is it true that its normal curvature is at least $\sqrt{2 \cdot n / (n+1)}$ in some direction?

The last question asks if the *Veronese embedding is optimal*. Recall that the Veronese embedding $\mathbb{RP}^n \hookrightarrow \mathbb{B}^q$ has all normal curvatures $\sqrt{2 \cdot n / (n+1)}$; here $q = \frac{1}{2} \cdot (n+1)(n+2)$. An analogous question for immersions into unit spheres is open as well [9]. You may also ask for the optimal constant for your favorite closed manifold.

2 Gauss formula

Recall that L is a smooth n -dimensional manifold immersed in \mathbb{R}^q . Given $p \in L$, denote by $\text{Sc}(p)$ and $H(p)$ the scalar curvature, the mean curvature vector at p .

The following version of the Gauss formula plays a central role in all proofs; it is used instead of the formula in [6, 5.B].

2.1. Gauss formula. The following identity

$$\text{Sc} = \frac{3}{2} \cdot |H|^2 - \frac{n \cdot (n+2)}{2} \cdot \lambda K$$

holds for any smooth n -dimensional immersed manifold in a Euclidean space.

Proof. Choose a point $p \in L$.

Assume $\text{codim } L = 1$. Denote by k_1, \dots, k_n the principal curvatures of L at p . Note that

$$|H|^2 = \sum_i k_i^2 + 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

Further,

$$n \cdot (n+2) \cdot \lambda K = 3 \cdot \sum_i k_i^2 + 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

The last identity follows since \mathcal{K} is the average value of $(\sum_i k_i \cdot x_i^2)^2$ on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n = T_p$; here (x_1, \dots, x_n) are the standard coordinates in \mathbb{R}^n . One has to take into account that the following functions have unit average values: $\frac{1}{3} \cdot n \cdot (n+2) \cdot x_i^4$ and $n \cdot (n+2) \cdot x_i^2 \cdot x_j^2$ for $i \neq j$.

By the standard Gauss formula,

$$\text{Sc} = 2 \cdot \sum_{i < j} k_i \cdot k_j.$$

It remains to rewrite the right-hand side using the expressions for $|H|^2$ and \mathcal{K} .

If $\text{codim } L = k > 1$, then the second fundamental form at p can be presented as a direct sum of k real-valued quadratic forms $\mathbb{I}_1 \oplus \dots \oplus \mathbb{I}_k$; that is,

$$\mathbb{I} = e_1 \cdot \mathbb{I}_1 + \dots + e_k \cdot \mathbb{I}_k,$$

where e_1, \dots, e_k is an orthonormal basis of N_p . Denote by Sc_i , H_i , and \mathcal{K}_i the values associated with \mathbb{I}_i . From above, we get

$$\text{Sc}_i = \frac{3}{2} \cdot |H_i|^2 - \frac{n \cdot (n+2)}{2} \cdot \mathcal{K}_i$$

for each i .

Note that

$$\text{Sc} = \sum_i \text{Sc}_i, \quad |H|^2 = \sum_i |H_i|^2, \quad \text{and} \quad \mathcal{K} = \sum_i \mathcal{K}_i.$$

Hence the general case follows. \square

Remark. A more direct proof of this formula can be obtained using the so-called *extrinsic curvature tensor* which is defined by $\Phi(x, y, v, w) := \langle \mathbb{I}(x, y), \mathbb{I}(v, w) \rangle$; the necessary properties of this tensor are discussed in [8]. As a bonus, one gets an explicit expression for the second fundamental forms of all Gromov's torii.

3 Special cases

The following statement appears in the book of Yuri Burago and Viktor Zalgaller [2, Theorem 28.2.5]; it generalizes the result of István Fáry about average curvature of a curve in the unit ball [3, 12], but the proof is essentially the same.

3.1. Lemma. *Let L be a closed n -dimension manifold that is smoothly immersed in \mathbb{B}^q . Then the average value of $|H|$ on L is at least n .*

Proof. Consider the function $u: x \mapsto \frac{1}{2} \cdot |x|^2$ on L . Note that

$$(\Delta u)(x) = n + \langle H(x), x \rangle.$$

It follows that the average value of $\langle H(x), x \rangle$ is $-n$. Since $|x| \leq 1$, we get the result. \square

Proof of 1.2a. By 3.1, the average value of $|H|^2$ is at least 4. Further, by the Gauss–Bonnet formula, scalar curvature (which is twice the Gauss curvature in this case) has zero average. Therefore 2.1 implies the statement. \square

Proof of 1.2b. By 3.1, the average value of $|H|^2$ is at least n^2 . Since $\text{Sc} \equiv 0$, it remains to apply 2.1. \square

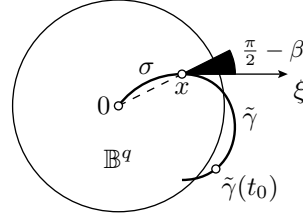
Proof of 1.2c. Since the image lies in the unit sphere, we have that $|H|^2$ is at least n^2 at each point. Since \mathbb{T}^n does not admit a metric with positive scalar curvature [7, Corollary A], we have $\text{Sc}(x) \leq 0$ at some point x . It remains to apply 2.1 at x . \square

4 Main case

The following lemma is an easy corollary of the bow lemma of Axel Schur [10, 11]. It tells how we use the bound on normal curvatures in 1.2e. If $n \leq 4$, then the proof works without this assumption.

4.1. Lemma. *Let L be a manifold smoothly immersed in \mathbb{B}^q . Suppose its normal curvatures are at most 2. Given $x \in L$, denote by $\beta = \beta(x)$ the angle between vector x and the normal space N_x . Then $|x| \leq \cos \beta$.*

Proof. Let ξ be a tangent direction at x such that $\angle(x, \xi) = \frac{\pi}{2} - \beta$. In the plane spanned by x and ξ , choose a unit-speed circle arc σ from 0 to x that comes to x in the direction opposite to ξ ; extend σ after x by a unit-speed semicircle $\tilde{\gamma}$ with curvature 2 in such a way that the concatenation $\sigma * \tilde{\gamma}$ is an arc of a C^1 -smooth convex plane curve; see the figure.



Observe that if $|x| > \cos \beta$, then $\tilde{\gamma}$ leaves \mathbb{B}^q ; that is, $|\tilde{\gamma}(t_0)| > 1$ for some t_0 .

Let γ be the unit-speed geodesic in L that runs from x in the direction ξ . Note that curvatures of $\sigma * \gamma$ do not exceed the curvatures of $\sigma * \tilde{\gamma}$ at the corresponding points. Applying the bow lemma for $\sigma * \gamma$ and $\sigma * \tilde{\gamma}$, we get $|\gamma(t_0)| \geq |\tilde{\gamma}(t_0)|$. It follows that L does not lie in \mathbb{B}^q — a contradiction. \square

Let g be a Riemannian metric on \mathbb{T}^n . Suppose $n \geq 3$, and $u: \mathbb{T}^n \rightarrow \mathbb{R}$ is a smooth positive function. Recall that

$$\left(\text{Sc} \cdot u - 4 \cdot \frac{n-1}{n-2} \cdot \Delta u \right) \cdot u^{\frac{n-2}{n+2}}$$

is the scalar curvature of the metric $u^{\frac{4}{n-2}} \cdot g$; see for example [1, 6.3]. Since *any Riemannian metric on \mathbb{T}^n has nonpositive scalar curvature at some point* [7, Corollary A], we get the following.

4.2. Claim. *For any Riemannian metric on \mathbb{T}^n and any positive smooth function $u: \mathbb{T}^n \rightarrow \mathbb{R}$, the function*

$$\text{Sc} \cdot u - 4 \cdot \frac{n-1}{n-2} \cdot \Delta u$$

returns a nonpositive value at some point.

Proof of 1.2d and 1.2e. The case $n = 2$ follows from 1.2a; so we can assume that $n \geq 3$. Consider the function $u: x \mapsto \exp(-\frac{k}{2} \cdot |x|^2)$ on the torus.

We will apply the following formula

$$\Delta(\varphi \circ f) = (\varphi' \circ f) \cdot \Delta f + (\varphi'' \circ f) \cdot |\nabla f|^2$$

to $f: x \mapsto \frac{1}{2} \cdot |x|^2$ and $\varphi: y \mapsto \exp(-k \cdot y)$; so $u = \varphi \circ f$.

Set $r(x) = |x|$, $\alpha(x) = \angle(H(x), x)$, and $\beta(x)$ as in 4.1. Note that

$$\textcircled{1} \quad \beta \leq \alpha \leq \pi - \beta.$$

Observe that

$$\Delta f = |H| \cdot r \cdot \cos \alpha + n, \quad |\nabla f| = r \cdot \sin \beta, \quad \varphi' = -k \cdot \varphi, \quad \varphi'' = k^2 \cdot \varphi.$$

Therefore

$$\Delta u = u \cdot [-k \cdot |H| \cdot r \cdot \cos \alpha - k \cdot n + k^2 \cdot r^2 \cdot \sin^2 \beta].$$

Recall that $\text{Sc} = -\frac{n \cdot (n+2)}{2} \cdot \mathcal{K} + \frac{3}{2} \cdot |H|^2$; see 2.1. By 4.2, the function

$$\begin{aligned} \text{Sc} \cdot u - 4 \cdot \frac{n-1}{n-2} \cdot \Delta u = u \cdot \left[-\frac{n \cdot (n+2)}{2} \cdot \mathcal{K} + \frac{3}{2} \cdot |H|^2 + 4 \cdot \frac{n-1}{n-2} \cdot k \cdot |H| \cdot r \cdot \cos \alpha + \right. \\ \left. + 4 \cdot \frac{n-1}{n-2} \cdot (k \cdot n - k^2 \cdot r^2 \cdot \sin^2 \beta) \right] \end{aligned}$$

returns a nonpositive value at some point $x \in \mathbb{T}^n$.

Choose

$$k = \frac{3}{4} \cdot \frac{n-2}{n-1} \cdot n, \quad \text{so} \quad n = \frac{4}{3} \cdot \frac{n-1}{n-2} \cdot k.$$

At the point x , we have

$$\textcircled{2} \quad \begin{aligned} \frac{n \cdot (n+2)}{2} \cdot \mathcal{K} &\geq \frac{3}{2} \cdot (|H| + n \cdot r \cdot \cos \alpha)^2 - \frac{3}{2} \cdot n^2 \cdot r^2 \cdot \cos^2 \alpha + \\ &+ 3 \cdot n^2 - \frac{9}{4} \cdot \frac{n-2}{n-1} \cdot n^2 \cdot r^2 \cdot \sin^2 \beta \geq \frac{3}{2} \cdot n^2. \end{aligned}$$

Indeed, by $\textcircled{1}$, $\cos^2 \alpha + \sin^2 \beta \leq 1$. If $n \leq 4$, then $\frac{3}{2} \geq \frac{9}{4} \cdot \frac{n-2}{n-1}$; therefore the last inequality follows, and it proves 1.2d.

Further, if $n \geq 5$, then for the last inequality in $\textcircled{2}$ we need to use in addition that $r^2 + \sin^2 \beta \leq 1$ which follows from 4.1. Hence 1.2e follows. \square

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