Mildly curved submanifolds in a ball

Anton Petrunin

Abstract

Suppose M is a closed submanifold in a Euclidean ball of large dimension. We give an optimal bound on the normal curvatures of M that guarantee that it is a sphere. The border cases consist of Veronese embeddings of four projective planes.

1. Introduction. Let $M\subset\mathbb{R}^d$ be a closed smooth n-dimensional submanifold. Assume d is large and M lies in an r-ball. *What can we say about the normal curvatures of M?*

First note that the curvatures cannot be smaller than $\frac1r$ at all points. Moreover, *the average value of $|H|$ must be at least $n\cdot\frac1r$*; here H denotes the mean curvature vector [2, 28.2.5], [8, 3.1]. This statement is a straightforward generalization of the result of István Fáry [3, 12] about closed curves in a ball.

On the other hand, the n-dimensional torus can be embedded into an r-ball with all normal curvatures $\sqrt{3\cdot n/(n+2)}\cdot\frac1r$. This embedding was found by Michael Gromov [6, 2.A], [5, 1.1.A]. This bound is optimal; that is, any smooth n-dimensional torus in an r-ball has normal curvature at least $\sqrt{3\cdot n/(n+2)}\cdot\frac1r$ at some point [8]. Gromov's examples easily imply the following: any closed smooth manifold M admits a smooth embedding into an r-ball of sufficiently large dimension with normal curvatures less than $\sqrt3\cdot\frac1r$ [6, 1.D], [5, 1.1.C]. *But what happens between $\frac1r$ and $\sqrt3\cdot\frac1r$?*

In this note, we consider embeddings in an r-ball with normal curvatures at most $\frac{2}{\sqrt3}\cdot\frac1r$. We show that if the inequality is strict, then the manifold must be homeomorphic to a sphere (see § 2). For the nonstrict inequality, in addition to spheres we get real, complex, quaternionic, and octonionic planes mapped by the corresponding Veronese embedding up to rescaling (see § 4).

2. Sphere theorem. *Let M be a closed smooth n-dimensional submanifold in a closed r-ball in $\mathbb{R}^d$. Suppose that the normal curvatures of M are strictly less than $\frac{2}{\sqrt3}\cdot\frac1r$. Then M is homeomorphic to the n-sphere.*

Proof. Denote the r-ball by $\mathbb{B}^d$. We can assume that $r=\frac{1}{\sqrt3}$; therefore, the normal curvatures of M are smaller than 2.

Choose a unit-speed geodesic $\gamma\colon[0,\frac\pi2]\to M$; let $x=\gamma(0)$ and $y=\gamma(\frac\pi2)$. By the assumption, the curvature of γ in $\mathbb{R}^d$ is less than 2. Applying Schur's bow lemma, we get $|x-y|>1$.

Let Π be the perpendicular bisector to $[x,y]$. Since the curvature of γ is smaller than 2,

$$\measuredangle(\gamma'(t_0),\gamma'(t))<2\cdot|t-t_0|,\quad\text{and}\quad\langle\gamma'(t_0),\gamma'(t)\rangle>\cos(2\cdot|t-t_0|)$$

if $|t-t_0|>0$. Therefore

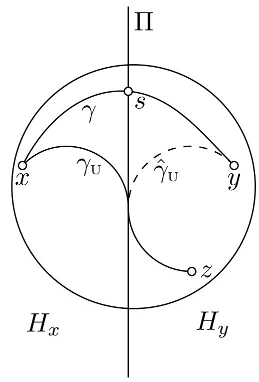
$$\langle y-x,\gamma'(t_0)\rangle>\int\limits_0^{\frac{\pi}{2}}\cos(2{\cdot}|t-t_0|){\cdot}dt\geqslant 0.$$

In particular, the function $f\colon[0,\frac{\pi}{2}]\to\mathbb{R}$ defined by $f\colon t\mapsto\langle y-x,\gamma(t)\rangle$ has positive derivative. Therefore, γ intersects Π transversely at a single point; denote it by s.

Choose a unit vector $\mathrm{u}\in\mathrm{T}_x$; let $\gamma_{\mathrm{u}}\colon[0,\frac{\pi}{2}]\to M$ be the unit-speed geodesic that starts from x in the direction u, and let $z=\gamma_{\mathrm{u}}(\frac{\pi}{2})$. The argument above shows that $|x-z|>1$.

Denote by H_x and H_y the closed half-spaces bounded by Π that contain x and y respectively. Assume $z\in H_x$, then we have $|y-z|\geqslant|x-z|>1$. Since $|x-y|>1$, the triangle $[xyz]$ has all sides larger than 1, which is impossible since $x,y,z\in\mathbb{B}^d$. Therefore, γ_v meets Π before in $\frac{\pi}{2}$; denote by $r(\mathrm{u})$ be the first such time moment.

Let us show that the function $\mathrm{u}\mapsto r(\mathrm{u})$ is smooth. In other words, γ_{u} intersects Π transversely at time $r(\mathrm{u})$. Assume this is not the case, so γ_{u} is tangent to Π at $r(\mathrm{u})$. Let $\hat\gamma_{\mathrm{u}}$ be the concatenation of the reflection of $\gamma_{\mathrm{u}}|_{[0,r(\mathrm{u})]}$ across Π and $\gamma_{\mathrm{u}}|_{[r(\mathrm{u}),\frac{\pi}{2}]}$. Note that $\hat\gamma_{\mathrm{u}}$ is C^1-smooth, and it is C^∞-smooth everywhere except $r(\mathrm{u})$. Therefore, Schur's bow lemma is applicable to $\hat\gamma_{\mathrm{u}}$, and hence, $|y-z|>1$. Again, all sides of triangle $[xyz]$ are larger than 1; hence, it cannot lie in $\mathbb{B}^d$ — a contradiction.



It follows that the set

$$V_x=\{\,t{\cdot}\mathrm{u}\in\mathrm{T}_x\;:\;|\mathrm{u}|=1,\quad 0\leqslant t\leqslant r(\mathrm{u}),\,\}$$

is diffeomorphic to the closed n-disc. Denote by W_x the connected component of x in $M\cap H_x$.

From the Gauss formula [9, Lemma 5], the sectional curvatures of M are less than 4. In particular, the exponential map $\exp_x\colon\mathrm{T}_x\to M$ is a local diffeomorphism in the $\frac{\pi}{2}$-ball centered at the origin of T_x.

It follows that $\exp_x\colon V_x\to W_x$ is a local diffeomorphism; in particular, W_x is a smooth manifold with boundary. Since V_x is simply connected, $\exp_x$ defines a diffeomorphism $V_x\to W_x$. In particular, W_x is a closed topological n-disc and ∂W_x is a smooth hypersurface in M.

Let us swap the roles of x and y, and repeat the construction. We get another closed topological n-disc $W_y\subset M$ bounded by a smooth hypersurface ∂W_y.

Observe that ∂W_x intersects ∂W_y at s. Furthermore, both ∂W_x and ∂W_y are connected components of s in $M\cap\Pi$. Therefore, $\partial W_x=\partial W_y$. That is, M can be obtained by gluing two n-discs by a diffeomorphism between their boundaries. Hence M is homeomorphic to the n-sphere. $\square$

3. Veronese embeddings. Recall a compact rank-one symmetric spaces is isometric to rescaled copy of one the following spaces: unit spheres $\mathbb{S}^n$, real, complex, and quaternionic projective spaces $\mathbb{RP}^n$, $\mathbb{CP}^n$ and $\mathbb{HP}^n$, and octonionic projective plane $\mathbb{O}\mathrm{P}^2$; see for example, [14, 8.12.2]. We assume that each of these spaces is equipped with the canonical metric; so spheres and real projective spaces have constant sectional curvature 1, and the sectional curvature of the remaining projective spaces is in the range $[1, 4]$. In particular, all the projective space have closed geodesics of length π.

Proposition. *There are smooth isometric embeddings*

- $\mathbb{R}\mathrm{P}^n \hookrightarrow \mathbb{R}^d$ *for* $d \geqslant n + \frac{1}{2}\cdot n\cdot(n+1)$*;*
- $\mathbb{C}\mathrm{P}^n \hookrightarrow \mathbb{R}^d$ *for* $d \geqslant n + n\cdot(n+1)$*;*
- $\mathbb{H}\mathrm{P}^n \hookrightarrow \mathbb{R}^d$ *for* $d \geqslant n + 2\cdot n\cdot(n+1)$*;*
- $\mathbb{O}\mathrm{P}^2 \hookrightarrow \mathbb{R}^d$ *for* $d \geqslant 26$*;*

that map each geodesic to a round circle.

In particular, all normal curvatures of the images of these embeddings are equal to 2*. Moreover, the images of these embeddings lie in a sphere of radius* $r = \sqrt{n/(2\cdot n+2)}$ *(for* $\mathbb{O}\mathrm{P}^2$*, we assume that* $n = 2$*).*

The proposition can be extracted from two theorems in [11, § 2]. The embeddings provided by the proposition will be called Veronese embeddings.

The Veronese embeddings have a very explicit algebraic description and many nice geometric properties. In particular, these embeddings are equivariant, and their images are minimal submanifolds in the r-spheres. All of this is discussed in the cited paper by Kunio Sakamoto.

The following lemma is also closely related to the result of Kunio Sakamoto; it says that the properties in the proposition uniquely describe Veronese embeddings up to motion of the ambient space.

Lemma. *Let* M *and* M' *be intrinsically isometric submanifolds in* $\mathbb{R}^d$*. Suppose that all geodesics in* M *and* M' *are closed and each geodesic forms a round circle in* $\mathbb{R}^d$*. Then* M *is congruent to* M'*; that is there is an isometry of* $\mathbb{R}^d$ *that maps* M *to* M'*.*

Proof. Recall that the second fundamental form $\mathrm{I\!I}$ is a bilinear symmetric form on the tangent space with values in the normal space. Assume there is a point $p \in M \cap M'$ such that $\mathrm{T}_p M = \mathrm{T}_p M'$ and the second fundamental forms of M and M' at p coincide. Then $M' = M$. Indeed, since every geodesic is mapped to a round circle, the image of a geodesic in direction $\mathrm{u} \in \mathrm{T}_p$ is completely described by $\mathrm{I\!I}(\mathrm{u}, \mathrm{u})$. And these circles sweep the whole M and M'.

Recall that the extrinsic curvature tensor Φ of a submanifold M is defined as

$$\Phi(\mathrm{x}, \mathrm{y}, \mathrm{v}, \mathrm{w}) = \langle \mathrm{I\!I}(\mathrm{x}, \mathrm{y}), \mathrm{I\!I}(\mathrm{v}, \mathrm{w}) \rangle,$$

here x, y, v, w are tangent vectors to M at some point; see [7]. Note that Φ-tensor describes the second fundamental form $\mathrm{I\!I}$ up to motion of the ambient space. Therefore, once we show that the Φ-tensors of M and M' coincide at one point, we get that M and M' are congruent.

The tensor Φ can be written as

$$\Phi(\mathrm{x}, \mathrm{y}, \mathrm{v}, \mathrm{w}) = E(\mathrm{x}, \mathrm{y}, \mathrm{v}, \mathrm{w}) + \tfrac{1}{3}\cdot(\mathrm{Rm}(\mathrm{x}, \mathrm{v}, \mathrm{y}, \mathrm{w}) + \mathrm{Rm}(\mathrm{x}, \mathrm{w}, \mathrm{y}, \mathrm{v}))$$

where E is the total symmetrization of Φ; that is,

$$E(\mathrm{x}, \mathrm{y}, \mathrm{v}, \mathrm{w}) = \tfrac{1}{3}\cdot(\Phi(\mathrm{x}, \mathrm{y}, \mathrm{v}, \mathrm{w}) + \Phi(\mathrm{y}, \mathrm{v}, \mathrm{x}, \mathrm{w}) + \Phi(\mathrm{v}, \mathrm{x}, \mathrm{y}, \mathrm{w})),$$

and

$$\mathrm{Rm}(\mathrm{X},\mathrm{Y},\mathrm{V},\mathrm{W}) = \Phi(\mathrm{X},\mathrm{V},\mathrm{Y},\mathrm{W}) - \Phi(\mathrm{X},\mathrm{W},\mathrm{Y},\mathrm{V})$$

is the Riemannian curvature tensor of M.

Since M is isometric to M', they have the same Riemannian curvature tensors. It remains to show that the E-tensors are the same. But

$$f(\mathrm{X}) = E(\mathrm{X},\mathrm{X},\mathrm{X},\mathrm{X}) = |\mathrm{I\!I}(\mathrm{X},\mathrm{X})|^2$$

is a homogeneous polynomial of degree 4 on the tangent space and it describes E completely.

The geodesics in M and M' are closed and have the same length. Since each of these geodesic forms a circle in $\mathbb{R}^d$, all these circles have the same curvature, say κ. Therefore, $\mathrm{I\!I}(\mathrm{X},\mathrm{X}) = \kappa\cdot|\mathrm{X}|^2$ and $E(\mathrm{X},\mathrm{X},\mathrm{X},\mathrm{X}) = \kappa^2\cdot|\mathrm{X}|^4$ for both submanifolds and for any tangent vector X. This finishes the proof. $\square$

4. Rigidity theorem. *Let M be a closed smooth n-dimensional submanifold in a closed r-ball in $\mathbb{R}^d$. Suppose that the normal curvatures of M are at most $\frac{2}{\sqrt{3}}\cdot\frac{1}{r}$. If M is not homeomorphic to a sphere, then up to rescaling, it is congruent to an image of the Veronese embedding of a projective plane $\mathbb{R}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^2$, or $\mathbb{O}\mathrm{P}^2$.*

This result is an application of the following theorem; its weaker form was proved by Detlef Gromoll and Karsten Grove [4], and the final step was made by Burkhard Wilking [13].

Gromoll–Grove–Wilking theorem. *Let M be a compact Riemannian manifold with sectional curvature at least 1 and diameter at least $\frac{\pi}{2}$. If M is not homeomorphic to a sphere, then its Riemannian universal cover is isometric to a compact rank-one symmetric space.*

Proof of the rigidity theorem. Assume M is not homeomorphic to a sphere; in this case, $\dim M \geqslant 2$. As before, $\mathbb{B}^d$ will denote the r-ball in $\mathbb{R}^d$, and we assume that $r = \frac{1}{\sqrt{3}}$; therefore, the normal curvatures of M are at most 2.

Choose a unit-speed geodesic $\gamma\colon [0,\frac{\pi}{2}] \to M$; let $x = \gamma(0)$ and $y = \gamma(\frac{\pi}{2})$. The argument in our sphere theorem implies that $|x - y| = 1$. The rigidity case in the bow lemma implies that γ is a half-circle of curvature 2. Since any two points in M can be connected by a geodesic, we get the following.

- ⋄ The diameter of M is 1.
- ⋄ The intrinsic diameter and injectivity radius of M are equal to $\frac{\pi}{2}$.
- ⋄ All geodesics in M are circles of curvature 2 in $\mathbb{R}^d$.

Furthermore, for x and y as above, there is another point $z \in M$ such that $|x - z| = |y - z| = 1$. If not, then again, the argument in the sphere theorem would imply that M is a sphere. But since $x, y, z \in \mathbb{B}^d$, the equalities $|x - y| = |y - z| = |x - z| = 1$ imply that $x \in \partial\mathbb{B}^d$.

The choice of $x \in M$ was arbitrary. Therefore, M lies in the sphere $\partial\mathbb{B}^d$ of radius $r = 1/\sqrt{3}$. This sphere has sectional curvature $1/r^2 = 3$; the normal curvatures of M in the sphere are $\kappa = \sqrt{2^2 - 1/r^2} = 1$. By the Gauss formula [9, Lemma 5], the sectional curvatures of M are at least $3 - 2\cdot\kappa^2 = 1$.

By the Gromoll–Grove–Wilking theorem, the universal cover $\tilde{M}$ of M is isometric to a rank-one symmetric space. Taking into account the injectivity radius and curvature of M, we get that $\tilde{M}$ must be isometric to one of the following spaces $\frac{1}{2}\cdot\mathbb{S}^n$, $\mathbb{S}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ for some n, or $\mathbb{O}\mathrm{P}^2$. Note that the points $x, y, z \in M$ constructed above lie at an intrinsic distance $\frac{\pi}{2}$ from each other. It forbids $\frac{1}{2}\cdot\mathbb{S}^n$ for every n. Furthermore if $n \geqslant 3$, then each space $\mathbb{S}^n$, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$ contain 4 points at a distance $\frac{\pi}{2}$ from each other. Since the injectivity radius of M is $\frac{\pi}{2}$, their projections in M must lie at a distance $\frac{\pi}{2}$ from each other as well. It follows that $\mathbb{B}^d$ must contain 4 points at a distance 1 from each other, which is impossible.

Hence, $\tilde{M}$ must be isometric to one of the following spaces $\mathbb{S}^2$, $\mathbb{C}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^2$, or $\mathbb{O}\mathrm{P}^2$. Since the injectivity radius of M is $\frac{\pi}{2}$, it has to be isometric to $\mathbb{R}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^2$, or $\mathbb{O}\mathrm{P}^2$.

Denote by $M' \subset \mathbb{R}^d$ the image of the corresponding Veronese embedding provided by the proposition. Without loss of generality, we can assume that $d > 26$, so M' exists. Applying the lemma in § 3, we get that M is congruent M' — hence the result. □

5. Final remarks. This note was motivated by the following question [10]. Recall that Veronese embedding maps $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, and $\mathbb{H}\mathrm{P}^n$ into balls of radius $r_n = \sqrt{n/(2\cdot n+2)}$

Question. *Is it true that the Veronese embedding minimizes the maximal normal curvature among all smooth embeddings of $\mathbb{R}\mathrm{P}^n$ into the ball of radius r_n in a Euclidean space of large dimension?*

The same question can also be asked about $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$. A keen reader might have noticed that the case $n = 2$ is already solved.

Question. *Let M be as in our sphere theorem; does it have to be diffeomorphic to the standard n-sphere?*

I suspect that the answer is "yes". If, in addition, M lies in the boundary of the r-ball, then by the Gauss formula [9, Lemma 5], its sectional curvature is strictly quarter-pinched, and in this case, it has to be diffeomorphic to a sphere [1].

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