Mildly curved submanifolds in a ball

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Abstract

Suppose M is a closed submanifold in a Euclidean ball of large dimension. We give an optimal bound on the normal curvatures of M that guarantee that it is a sphere. The border cases consist of Veronese embeddings of four projective planes.

1. Introduction. Let $M\subset\mathbb{R}^d$ be a closed smooth n-dimensional submanifold. Assume d is large and M lies in an r-ball. *What can we say about the normal curvatures of M?*

First note that the curvatures cannot be smaller than $\frac1r$ at all points. Moreover, *the average value of $|H|$ must be at least $n\cdot\frac1r$*; here H denotes the mean curvature vector [2, 28.2.5], [8, 3.1]. This statement is a straightforward generalization of the result of István Fáry [3, 12] about closed curves in a ball.

On the other hand, the n-dimensional torus can be embedded into an r-ball with all normal curvatures $\sqrt{3\cdot n/(n+2)}\cdot\frac1r$. This embedding was found by Michael Gromov [6, 2.A], [5, 1.1.A]. This bound is optimal; that is, any smooth n-dimensional torus in an r-ball has normal curvature at least $\sqrt{3\cdot n/(n+2)}\cdot\frac1r$ at some point [8]. Gromov's examples easily imply the following: any closed smooth manifold M admits a smooth embedding into an r-ball of sufficiently large dimension with normal curvatures less than $\sqrt3\cdot\frac1r$ [6, 1.D], [5, 1.1.C]. *But what happens between $\frac1r$ and $\sqrt3\cdot\frac1r$?*

In this note, we consider embeddings in an r-ball with normal curvatures at most $\frac{2}{\sqrt3}\cdot\frac1r$. We show that if the inequality is strict, then the manifold must be homeomorphic to a sphere (see § 2). For the nonstrict inequality, in addition to spheres we get real, complex, quaternionic, and octonionic planes mapped by the corresponding Veronese embedding up to rescaling (see § 3).

2. Sphere theorem. *Let M be a closed smooth n-dimensional submanifold in a closed r-ball in $\mathbb{R}^d$. Suppose that the normal curvatures of M are strictly less than $\frac{2}{\sqrt3}\cdot\frac1r$. Then M is homeomorphic to the n-sphere.*

Proof. Denote the r-ball by $\mathbb{B}^d$. We can assume that $r=\frac{1}{\sqrt3}$; therefore, the normal curvatures of M are smaller than 2.

Choose a unit-speed geodesic $\gamma\colon[0,\frac\pi2]\to M$; let $x=\gamma(0)$ and $y=\gamma(\frac\pi2)$. By the assumption, the curvature of γ in $\mathbb{R}^d$ is less than 2. Applying Schur's bow lemma, we get $|x-y|>1$.

Let Π be the perpendicular bisector to $[x,y]$. Since the curvature of γ is smaller than 2,

$$\measuredangle(\gamma'(t_0),\gamma'(t))<2\cdot|t-t_0|,\quad\text{and}\quad\langle\gamma'(t_0),\gamma'(t)\rangle>\cos(2\cdot|t-t_0|)$$

if $|t-t_0|>0$. Therefore

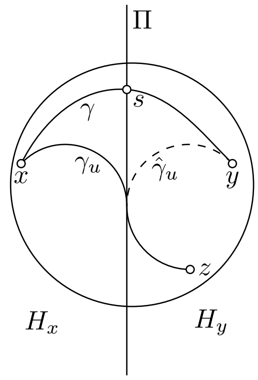
$$\langle y-x,\gamma'(t_0)\rangle > \int\limits_0^{\frac{\pi}{2}} \cos(2{\cdot}|t-t_0|){\cdot}dt \geqslant 0.$$

In particular, the function $f\colon[0,\frac{\pi}{2}]\to\mathbb{R}$ defined by $f\colon t\mapsto\langle y-x,\gamma(t)\rangle$ has positive derivative. Therefore, γ intersects Π transversely at a single point; denote it by s.

Choose a unit vector $u\in \mathrm{T}_x$; let $\gamma_u\colon[0,\frac{\pi}{2}]\to M$ be the unit-speed geodesic that starts from x in the direction u, and let $z=\gamma_u(\frac{\pi}{2})$. The argument above shows that $|x-z|>1$.

Denote by H_x and H_y the closed half-spaces bounded by Π that contain x and y respectively. Assume $z\in H_x$, then we have $|y-z|\geqslant|x-z|>1$. Since $|x-y|>1$, the triangle $[xyz]$ has all sides larger than 1, which is impossible since $x,y,z\in\mathbb{B}^d$. Therefore, γ_v meets Π before in $\frac{\pi}{2}$; denote by $r(u)$ be the first such time moment.

Let us show that the function $u\mapsto r(u)$ is smooth. In other words, γ_u intersects Π transversely at time $r(u)$. Assume this is not the case, so γ_u is tangent to Π at $r(u)$. Let $\hat\gamma_u$ be the concatenation of the reflection of $\gamma_u|_{[0,r(u)]}$ across Π and $\gamma_u|_{[r(u),\frac{\pi}{2}]}$. Note that $\hat\gamma_u$ is C^1-smooth and it is C^∞-smooth everywhere except $r(u)$. Therefore, Schur's bow lemma is applicable to $\hat\gamma_u$, and hence, $|y-z|>1$. Again, all sides of triangle $[xyz]$ are larger than 1; hence, it cannot lie in $\mathbb{B}^d$ — a contradiction.



It follows that the set

$$V_x=\{\,t{\cdot}v\in\mathrm{T}_x\;:\;|u|=1,\quad 0\leqslant t\leqslant a(v),\,\}$$

is diffeomorphic to the closed n-disc. Denote by W_x the connected component of x in $M\cap H_x$.

From the Gauss formula [9, Lemma 5], the sectional curvatures of M are less than 4. In particular, the exponential map $\exp_x\colon\mathrm{T}_x\to M$ is a local diffeomorphism in the $\frac{\pi}{2}$-ball centered at the origin of T_x.

It follows that $\exp_x\colon V_x\to W_x$ is a local diffeomorphism; in particular, W_x is a smooth manifold with boundary. Since V_x is simply connected, $\exp_x$ defines a diffeomorphism $V_x\to W_x$. In particular, W_x is a closed topological n-disc and ∂W_x is a smooth hypersurface in M.

Let us swap the roles of x and y, and repeat the construction. We get another closed topological n-disc $W_y\subset M$ bounded by a smooth hypersurface ∂W_y.

Observe that ∂W_x intersects ∂W_y at s. Furthermore, both ∂W_x and ∂W_y are connected components of s in $M\cap\Pi$. Therefore, $\partial W_x=\partial W_y$. That is, M can be obtained by gluing two n-discs by a diffeomorphism between their boundaries. Hence M is homeomorphic to the n-sphere. $\square$

3. Veronese embeddings. The real, complex, quaternionic, and octonionic projective planes will be denoted by $\mathbb{RP}^2$, $\mathbb{CP}^2$, $\mathbb{HP}^2$, and $\mathbb{OP}^2$, respectively. We assume that each of these planes is equipped with the canonical metric; it has closed geodesics of length π and therefore its sectional curvature is in the range $[1, 4]$.

Proposition. *There are smooth isometric embeddings*

$$\mathbb{R}\mathrm{P}^2 \hookrightarrow \mathbb{R}^5, \qquad \mathbb{C}\mathrm{P}^2 \hookrightarrow \mathbb{R}^8, \qquad \mathbb{H}\mathrm{P}^2 \hookrightarrow \mathbb{R}^{14}, \qquad \mathbb{O}\mathrm{P}^2 \hookrightarrow \mathbb{R}^{26}$$

that maps each geodesic to a round circle.

This proposition can be extracted from two theorems in [11, § 2]. The embeddings provided by the proposition will be called Veronese embeddings. From the proof below it will follow that the properties in the proposition uniquely describe Veronese embeddings up to isometric inclusion.

The Veronese embeddings have a very explicit algebraic description and many nice geometric properties. In particular, these embeddings are equivariant, their images lie in spheres of radius $1/\sqrt{3}$, and they are minimal submanifolds in these spheres. All of this is discussed in the cited paper by Kunio Sakamoto.

Rigidity theorem. *Let $\mathbb{B}^d$ be $\frac{1}{\sqrt{3}}$ ball in $\mathbb{R}^d$. Suppose a closed manifold M admits a smooth embedding f with normal curvatures at most 2 in $\mathbb{B}^d$ for some d. If M is not diffeomorphic to a sphere, then it is isometric to $\mathbb{R}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^2$, or $\mathbb{O}\mathrm{P}^2$. Moreover, f can be written as one of the following compositions*

$$\begin{aligned} &\mathbb{R}\mathrm{P}^2 \hookrightarrow \mathbb{R}^5 \hookrightarrow \mathbb{R}^d, && \mathbb{C}\mathrm{P}^2 \hookrightarrow \mathbb{R}^8 \hookrightarrow \mathbb{R}^d,\\ &\mathbb{H}\mathrm{P}^2 \hookrightarrow \mathbb{R}^{14} \hookrightarrow \mathbb{R}^d, && \mathbb{O}\mathrm{P}^2 \hookrightarrow \mathbb{R}^{26} \hookrightarrow \mathbb{R}^d; \end{aligned}$$

in each composition the first map is the Veronese embedding, and the second map is an isometric inclusion.

The proof is an application of the following theorem. In a weaker form, it was proved by Detlef Gromoll and Karsten Grove [4]; the final step was made by Burkhard Wilking [13].

Gromoll–Grove–Wilking theorem. *Let M be a compact Riemannian manifold with sectional curvature at least 1 and diameter at least $\frac{\pi}{2}$. If M is not homeomorphic to a sphere, then it is locally isometric to a rank-one symmetric space.*

Proof of the rigidity theorem. Assume M is not a sphere.

Choose a unit-speed geodesic $\gamma\colon [0, \frac{\pi}{2}] \to M$; let $x = \gamma(0)$ and $y = \gamma(\frac{\pi}{2})$. The argument in our sphere theorem implies that $|x - y| = 1$. The rigidity case in the bow lemma implies that γ is a half-circle of curvature 2. Since any two points in M can be connected by a geodesic, we get

- ◇ the diameter of M is 1,
- ◇ the intrinsic diameter and injectivity radius of M are equal to $\frac{\pi}{2}$,
- ◇ all geodesics in M are circles of curvature 2.

Furthermore, for x and y as above, there is another point $z \in M$ such that $|x - z| = |y - z| = 1$. If not, then again, the argument in our sphere theorem would imply that M is a sphere. But since $x, y, z \in \mathbb{B}^d$, the equalities $|x - y| = |y - z| = |x - z| = 1$ imply that $x \in \partial\mathbb{B}^d$.

The choice of $x \in M$ was arbitrary. Therefore, M lies in the sphere $\partial\mathbb{B}^d$ of radius $r = 1/\sqrt{3}$. This sphere has sectional curvature $1/r^2 = 3$; the normal curvatures of M in the sphere are $\sqrt{2^2 - 1/r^2} = 1$. By the Gauss formula [9, Lemma 5], the sectional curvatures of M are at least $3 - 2{\cdot}1^2 = 1$.

By the Gromoll–Grove–Wilking theorem, the universal cover $\tilde{M}$ of M is isometric to a rank-one symmetric space. Let us show that $\tilde{M}$ is either $\mathbb{S}^2$, $\mathbb{C}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^2$, or $\mathbb{O}\mathrm{P}^2$. Indeed, from above, the diameter of $\tilde{M}$ must be multiple of $\frac{\pi}{2}$ and it must have exactly 3 points on distance $\frac{\pi}{2}$; it excludes all other choices. Since the injectivity radius of M is $\frac{\pi}{2}$, it has to be isometric to $\mathbb{R}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^2$, or $\mathbb{O}\mathrm{P}^2$.

We can assume that $d > 26$; denote by $v\colon M \hookrightarrow \mathbb{R}^d$ the corresponding Veronese embedding; that is, v is a composition of Veronese embedding and isometric inclusion $M \hookrightarrow \mathbb{R}^m \hookrightarrow \mathbb{R}^d$, where $m = 5$, 8, 14, or 26 for $\mathbb{R}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^2$, or $\mathbb{O}\mathrm{P}^2$, respectively.

Recall that the second fundamental form is a bilinear symmetric form on tangent space with values in the normal space. Assume that there is a motion ι of $\mathbb{R}^d$ such that for some point $p \in M$ we have

$$f(p) = \iota \circ v(p), \qquad \mathrm{T}_{f(p)}[f(M)] = \mathrm{T}_{f(p)}[\iota \circ v(M)],$$

and the second fundamental forms of f and v at p coincide. Then $f(M) = \iota \circ v(M)$. Indeed, since every geodesic is mapped to a round circle, the image of a geodesic in direction $w \in \mathrm{T}_p$ is completely described by $\mathrm{I\!I}(w, w)$. And these circles sweep the image of the whole M.

Recall that extrinsic curvature tensor Φ is defined as

$$\Phi(X, Y, V, W) = \langle \mathrm{I\!I}(X, Y), \mathrm{I\!I}(V, W)\rangle;$$

see [7]. Note that it describes the second fundamental form $\mathrm{I\!I}$ up to rotation of the normal space. Therefore, once we show that the extrinsic curvature tensor coincides for $f(M)$ and $v(M)$ at one point, we get that $f(M)$ and $v(M)$ are congruent.

The tensor Φ can be written as

$$\Phi(X, Y, V, W) = E(X, Y, V, W) + \tfrac{1}{3}{\cdot}(\mathrm{Rm}(X, V, Y, W) + \mathrm{Rm}(X, W, Y, V))$$

where E is the total symmetrization of Φ; that is,

$$E(X, Y, V, W) = \tfrac{1}{3}{\cdot}(\Phi(X, Y, V, W) + \Phi(Y, V, X, W) + \Phi(V, X, Y, W)),$$

and

$$\mathrm{Rm}(X, Y, V, W) = \Phi(X, V, Y, W) - \Phi(X, W, Y, V)$$

is the Riemannian curvature tensor of M.

Since both embeddings v and f are isometric, they induce the same Riemannian curvature tensor on M. It remains to show that the E-tensors are the same. But

$$f(X) = E(X, X, X, X) = |\mathrm{I\!I}(X, X)|^2$$

is a homogeneous polynomial of degree 4 on the tangent space and it describes E completely. All normal curvatures are equal to 2, so $\mathrm{I\!I}(X, X) = 2{\cdot}|X|^2$ and $E(X, X, X, X) = 4{\cdot}|X|^4$ for both embeddings and for any X. This finishes the proof. $\square$

4. Final remarks. This note was motivated by the following question [10].

Question. *Is it true that the Veronese embedding minimizes the maximal normal curvature among all smooth embeddings of $\mathbb{R}\mathrm{P}^n$ into r-balls in Euclidean space of large dimension?*

The same question can also be asked about $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$. A keen reader might have noticed that the case $n = 2$ is already solved.

I suspect that the answer to the following question is "yes".

Question. *Let M be as in our sphere theorem; does it have to be diffeomorphic to the standard n-sphere?*

If, in addition, M lies in the boundary of the r-ball, then by the Gauss formula [9, Lemma 5], its sectional curvature is strictly quarter-pinched; in this case, it *is* diffeomorphic to a sphere [1].

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