

Veronese minimizes normal curvatures

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Abstract

Suppose M is a closed submanifold in a Euclidean ball of large dimension. We give an optimal bound on the normal curvatures that guarantee that M is a sphere. The border cases consist of Veronese embeddings of the four projective planes.

1. Introduction. Let $M \subset \mathbb{R}^d$ be a closed smooth n -dimensional submanifold. Assume d is large and M lies in an r -ball. *What can we say about the normal curvatures of M ?*

First note that the curvatures cannot be smaller than $\frac{1}{r}$ at all points. Moreover, *the average value of $|H|$ must be at least $n \cdot \frac{1}{r}$* ; here H denotes the mean curvature vector [2, 28.2.5], [8, 3.1]. This statement is a straightforward generalization of the result of István Fáry about closed curves in a ball [3, 12].

On the other hand, the n -dimensional torus can be embedded into an r -ball with all normal curvatures $\sqrt{3 \cdot n / (n + 2)} \cdot \frac{1}{r}$. This embedding was found by Michael Gromov [6, 2.A], [5, 1.1.A]. The bound is optimal; that is, any smooth n -dimensional torus in an r -ball has normal curvature at least $\sqrt{3 \cdot n / (n + 2)} \cdot \frac{1}{r}$ at some point [8]. Gromov's examples easily imply the following: any closed smooth manifold M admits a smooth embedding into an r -ball of sufficiently large dimension with normal curvatures less than $\sqrt{3} \cdot \frac{1}{r}$ [6, 1.D], [5, 1.1.C]. *But what happens between $\frac{1}{r}$ and $\sqrt{3} \cdot \frac{1}{r}$?*

In this note, we consider embeddings in an r -ball with normal curvatures at most $\frac{2}{\sqrt{3}} \cdot \frac{1}{r}$. We show that if the inequality is strict, then the manifold must be homeomorphic to a sphere (see § 2). For the nonstrict inequality, in addition to spheres, we get real, complex, quaternionic, and octonionic planes mapped by rescaled Veronese embeddings (see § 4).

2. Sphere theorem. *Let M be a closed smooth n -dimensional submanifold in a closed r -ball in \mathbb{R}^d . Suppose that the normal curvatures of M are strictly less than $\frac{2}{\sqrt{3}} \cdot \frac{1}{r}$. Then M is homeomorphic to the n -sphere.*

Proof. We can assume that $n \geq 2$; otherwise there is nothing to prove.

Denote the r -ball by \mathbb{B}^d . We can assume that $r = \frac{1}{\sqrt{3}}$; that is, r is the circumradius of an equilateral triangle with side 1. Therefore, the normal curvatures of M are smaller than 2.

Choose a unit-speed geodesic $\gamma: [0, \frac{\pi}{2}] \rightarrow M$; let $x = \gamma(0)$ and $y = \gamma(\frac{\pi}{2})$. By the assumption, the curvature of γ in \mathbb{R}^d is less than 2. Applying Schur's bow lemma, we get $|x - y| > 1$.

Let Π be the perpendicular bisector to $[x, y]$. Since the curvature of γ is smaller than 2,

$$\angle(\gamma'(t_0), \gamma'(t)) < 2 \cdot |t - t_0|, \quad \text{and} \quad \langle \gamma'(t_0), \gamma'(t) \rangle > \cos(2 \cdot |t - t_0|)$$

for $t \neq t_0$. Therefore,

$$\langle y - x, \gamma'(t_0) \rangle > \int_0^{\frac{\pi}{2}} \cos(2 \cdot |t - t_0|) \cdot dt \geq 0.$$

In particular, the derivative of function $f: t \mapsto \langle y - x, \gamma(t) \rangle$ is positive. Therefore, γ intersects Π transversely at a single point; denote it by s .

Choose a unit vector $u \in T_x$; let $\gamma_u: [0, \frac{\pi}{2}] \rightarrow M$ be the unit-speed geodesic that starts from x in the direction u , and let $z = \gamma_u(\frac{\pi}{2})$. The argument above shows that $|x - z| > 1$.

Denote by H_x and H_y the closed half-spaces bounded by Π that contain x and y respectively. Assume $z \in H_x$, then we have $|y - z| \geq |x - z| > 1$. Since $|x - y| > 1$, the triangle $[xyz]$ has all sides larger than 1, which is impossible since $x, y, z \in \mathbb{B}^d$. Therefore, γ_u meets Π before in $\frac{\pi}{2}$; denote by $r(u)$ be the first such time moment.

Let us show that the function $u \mapsto r(u)$ is smooth. In other words, γ_u intersects Π transversely at time $r(u)$. Assume this is not the case, so γ_u is tangent to Π at $r(u)$. Let $\hat{\gamma}_u$ be the concatenation of the reflection of $\gamma_u|_{[0, r(u)]}$ across Π and $\gamma_u|_{[r(u), \frac{\pi}{2}]}$. Note that $\hat{\gamma}_u$ is C^1 -smooth, and it is C^∞ -smooth everywhere except $r(u)$. Therefore, Schur's bow lemma is applicable to $\hat{\gamma}_u$, and hence, $|y - z| > 1$. Again, all sides of triangle $[xyz]$ are larger than 1; hence, it cannot lie in \mathbb{B}^d — a contradiction.

It follows that the set

$$V_x = \{t \cdot u \in T_x : |u| = 1, \quad 0 \leq t \leq r(u), \}$$

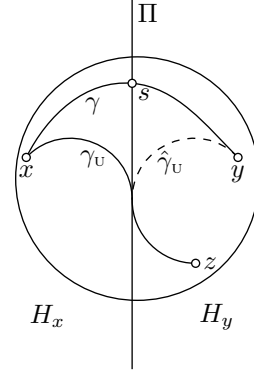
is diffeomorphic to the closed n -disc. Denote by W_x the connected component of x in $M \cap H_x$.

From the Gauss formula [9, Lemma 5], the sectional curvatures of M are less than 4. In particular, the exponential map $\exp_x: T_x \rightarrow M$ is a local diffeomorphism in the $\frac{\pi}{2}$ -ball centered at the origin of T_x .

It follows that $\exp_x: V_x \rightarrow W_x$ is a local diffeomorphism; in particular, W_x is a smooth manifold with boundary. Since V_x is simply connected, \exp_x defines a diffeomorphism $V_x \rightarrow W_x$. In particular, W_x is a closed topological n -disc and ∂W_x is a smooth hypersurface in M .

Let us swap the roles of x and y , and repeat the construction. We get another closed topological n -disc $W_y \subset M$ bounded by a smooth hypersurface ∂W_y .

Observe that ∂W_x intersects ∂W_y at s . Furthermore, both ∂W_x and ∂W_y are connected components of s in $M \cap \Pi$. Therefore, $\partial W_x = \partial W_y$. That is, M can be obtained by gluing two n -discs by a diffeomorphism between their boundaries. Hence M is homeomorphic to the n -sphere. \square



3. Veronese embeddings. The real, complex, quaternionic projective spaces of dimension n , and the octonionic projective plane will be denoted by \mathbb{RP}^n , \mathbb{CP}^n , \mathbb{HP}^n , and \mathbb{OP}^2 respectively. We assume that each of these spaces is

equipped with the canonical metric; in particular, all the spaces have closed geodesics of length π .

Proposition. *There are smooth isometric embeddings*

- ◊ $\mathbb{RP}^n \hookrightarrow \mathbb{R}^d$ for $d \geq n + \frac{1}{2} \cdot n \cdot (n+1)$;
- ◊ $\mathbb{CP}^n \hookrightarrow \mathbb{R}^d$ for $d \geq n + n \cdot (n+1)$;
- ◊ $\mathbb{HP}^n \hookrightarrow \mathbb{R}^d$ for $d \geq n + 2 \cdot n \cdot (n+1)$;
- ◊ $\mathbb{OP}^2 \hookrightarrow \mathbb{R}^d$ for $d \geq 26$;

that map each geodesic to a round circle.

Moreover,

- (i) *all normal curvatures of the images of these embeddings are equal to 2*
- (ii) *the images of these embeddings lie in a sphere of radius $r_n = \sqrt{n/(2 \cdot n + 2)}$ (for \mathbb{OP}^2 , we assume that $n = 2$).*

The proposition can be extracted from two theorems in [11, § 2]. The embeddings provided by the proposition will be called Veronese embeddings. Note that r_n is the circumradius of a regular n -simplex with edge length 1. Note that $r_2 = 1/\sqrt{3}$; therefore, the second part of proposition shows that our sphere theorem has the optimal bound.

The Veronese embeddings have a very explicit algebraic description and many nice geometric properties. In particular, these embeddings are equivariant, and their images are minimal submanifolds in the r_n -spheres. All of this is discussed in the cited paper by Kunio Sakamoto.

The following lemma is also closely related to the result of Kunio Sakamoto; it implies that the properties in the proposition uniquely describe Veronese embeddings up to motion of the ambient space.

Lemma. *Let M and M' be intrinsically isometric smooth submanifolds in \mathbb{R}^d . Suppose that all geodesics in M and M' are closed and each geodesic forms a round circle in \mathbb{R}^d . Then M is congruent to M' ; that is, there is an isometry of \mathbb{R}^d that maps M to M' .*

Proof. Recall that the second fundamental form \mathbb{II} of a submanifold is a bilinear symmetric form on the tangent space with values in the normal space. Assume there is a common point p on M and M' with common tangent space $T_p M = T_p M'$ and such that the second fundamental forms of M and M' at p coincide. Then $M' = M$. Indeed, since every geodesic is mapped to a round circle, the image of a geodesic in direction $u \in T_p$ is completely described by $\mathbb{II}(u, u)$. And these circles sweep the whole M and M' .

Recall that the extrinsic curvature tensor Φ of a submanifold is defined as

$$\Phi(X, Y, v, w) = \langle \mathbb{II}(X, Y), \mathbb{II}(v, w) \rangle,$$

here X, Y, v, w are tangent vectors to the submanifold at some point; see [7]. Note that the Φ -tensor describes the second fundamental form \mathbb{II} up to motion of the ambient space. Therefore, once we show that the Φ -tensors of M and M' coincide at one point, we get that M and M' are congruent.

The tensor Φ can be written as

$$\Phi(X, Y, v, w) = E(X, Y, v, w) + \frac{1}{3} \cdot (\text{Rm}(X, v, Y, w) + \text{Rm}(X, w, Y, v))$$

where E is the total symmetrization of Φ ; that is,

$$E(x, y, v, w) = \frac{1}{3} \cdot (\Phi(x, y, v, w) + \Phi(y, v, x, w) + \Phi(v, x, y, w)),$$

and

$$\text{Rm}(x, y, v, w) = \Phi(x, v, y, w) - \Phi(x, w, y, v)$$

is the Riemannian curvature tensor of M .

Since M is isometric to M' , they have the same Riemannian curvature tensors. It remains to show that the E -tensors are the same. But

$$f(x) = E(x, x, x, x) = |\mathbb{I}(x, x)|^2$$

is a homogeneous polynomial of degree 4 on the tangent space and it describes E completely.

The geodesics in M and M' are closed and have the same length. Since each of these geodesic forms a circle in \mathbb{R}^d , all these circles have the same curvature, say κ . Therefore, $\mathbb{I}(x, x) = \kappa \cdot |x|^2$ and $E(x, x, x, x) = \kappa^2 \cdot |x|^4$ for both submanifolds and for any tangent vector x . This finishes the proof. \square

4. Rigidity theorem. *Let M be a closed smooth n -dimensional submanifold in a closed r -ball in \mathbb{R}^d . Suppose that the normal curvatures of M are at most $\frac{2}{\sqrt{3}} \cdot \frac{1}{r}$. If M is not homeomorphic to a sphere, then up to rescaling, it is congruent to an image of the Veronese embedding of a projective plane \mathbb{RP}^2 , \mathbb{CP}^2 , \mathbb{HP}^2 , or \mathbb{OP}^2 .*

This result is an application of the following theorem; its weaker form was proved by Detlef Gromoll and Karsten Grove [4], and the final step was made by Burkhard Wilking [13].

Gromoll–Grove–Wilking theorem. *Let M be a compact Riemannian manifold with sectional curvature at least 1 and diameter at least $\frac{\pi}{2}$. If M is not homeomorphic to a sphere, then its Riemannian universal cover is isometric to a compact rank-one symmetric space.*

Recall that a compact rank-one symmetric space is isometric to a rescaled copy of one of the following spaces: \mathbb{RP}^n , \mathbb{CP}^n , \mathbb{HP}^n , \mathbb{OP}^2 , and unit spheres \mathbb{S}^n ; see for example, [14, 8.12.2]. As before we assume that these spaces are equipped with the canonical metrics; in particular, all the projective spaces have closed geodesics of length π .

Proof of the rigidity theorem. Assume M is not homeomorphic to a sphere; in this case, $n \geq 2$. As before, \mathbb{B}^d will denote the r -ball in \mathbb{R}^d , and we assume that $r = \frac{1}{\sqrt{3}}$; therefore, the normal curvatures of M are at most 2.

By the proposition in § 3, the images of Veronese embedding satisfy the assumption of the theorem. It remains to show that there are no other embeddings of that type.

Choose a unit-speed geodesic $\gamma: [0, \frac{\pi}{2}] \rightarrow M$; let $x = \gamma(0)$ and $y = \gamma(\frac{\pi}{2})$. The argument in our sphere theorem implies that $|x - y| = 1$. The rigidity case in the bow lemma implies that γ is a half-circle of curvature 2. Since any two points in M can be connected by a geodesic, we get the following.

◊ The diameter of M is 1.

- ◇ The intrinsic diameter and injectivity radius of M are equal to $\frac{\pi}{2}$.
- ◇ All geodesics in M are circles of curvature 2 in \mathbb{R}^d .

Furthermore, for x and y as above, there is another point $z \in M$ such that $|x - z| = |y - z| = 1$. If not, then again, the argument in the sphere theorem would imply that M is a sphere. But since $x, y, z \in \mathbb{B}^d$, the equalities $|x - y| = |y - z| = |x - z| = 1$ imply that $x \in \partial\mathbb{B}^d$.

The choice of $x \in M$ was arbitrary. Therefore, M lies in the sphere $\partial\mathbb{B}^d$ of radius $r = 1/\sqrt{3}$. This sphere has sectional curvature $1/r^2 = 3$; all normal curvatures of M in the sphere are $\kappa = \sqrt{2^2 - 1/r^2} = 1$. By the Gauss formula [9, Lemma 5], the sectional curvatures of M are at least $3 - 2 \cdot \kappa^2 = 1$.

By the Gromoll–Grove–Wilking theorem, the universal cover \tilde{M} of M is isometric to a rank-one symmetric space. Taking into account the injectivity radius and curvature of M , we get that \tilde{M} must be isometric to one of the following spaces $\frac{1}{2} \cdot \mathbb{S}^n$, \mathbb{S}^n , \mathbb{CP}^n , \mathbb{HP}^n for some n , or \mathbb{OP}^2 . Note that the points $x, y, z \in M$ constructed above lie at an intrinsic distance $\frac{\pi}{2}$ from each other. It forbids $\frac{1}{2} \cdot \mathbb{S}^n$ for every n . Furthermore, if $n \geq 3$, then each space \mathbb{S}^n , \mathbb{CP}^n and \mathbb{HP}^n contain 4 points at a distance $\frac{\pi}{2}$ from each other. Since the injectivity radius of M is $\frac{\pi}{2}$, their projections in \tilde{M} must lie at a distance $\frac{\pi}{2}$ from each other as well. It follows that \mathbb{B}^d must contain 4 points at a distance 1 from each other, which is impossible.

Hence, \tilde{M} must be isometric to one of the following spaces \mathbb{S}^2 , \mathbb{CP}^2 , \mathbb{HP}^2 , or \mathbb{OP}^2 . Since the injectivity radius of M is $\frac{\pi}{2}$, it has to be isometric to \mathbb{RP}^2 , \mathbb{CP}^2 , \mathbb{HP}^2 , or \mathbb{OP}^2 .

Denote by $M' \subset \mathbb{R}^d$ the image of the corresponding Veronese embedding provided by the proposition in § 3. Without loss of generality, we can assume that d is large, so M' exists. Applying the lemma in § 3, we get that M is congruent M' — hence the result. \square

5. Final remarks. Recall that the Veronese embeddings map \mathbb{RP}^n , \mathbb{CP}^n , and \mathbb{HP}^n into balls of radius $r_n = \sqrt{n/(2 \cdot n + 2)}$, which is the circumradius of a regular n -simplex with edge length 1. This note is motivated by the following question [10].

Question. *Is it true that the Veronese embedding minimizes the maximal normal curvature among all smooth embeddings of \mathbb{RP}^n into the ball of radius r_n in a Euclidean space of large dimension?*

The same question can be asked about \mathbb{CP}^n and \mathbb{HP}^n . A keen reader might have noticed that the case $n = 2$ is already solved.

Question. *Let M be as in our sphere theorem; does it have to be diffeomorphic to the standard n -sphere?*

I suspect that the answer is *yes*. If, in addition, M lies on the boundary of the r -ball, then by the Gauss formula [9, Lemma 5], M has strictly quarter-pinched curvature; so, it has to be diffeomorphic to a standard sphere [1].

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References

- [1] S. Brendle and R. Schoen. “Manifolds with $1/4$ -pinched curvature are space forms”. *J. Amer. Math. Soc.* 22.1 (2009), 287–307.
- [2] Yu. D. Burago and V. A. Zalgaller. *Geometric inequalities*. Vol. 285. Grundlehren der mathematischen Wissenschaften. 1988.
- [3] I. Fáry. “Sur certaines inégalités géométriques”. *Acta Sci. Math.* 12 (1950), 117–124.
- [4] D. Gromoll and K. Grove. “A generalization of Berger’s rigidity theorem for positively curved manifolds”. *Ann. Sci. École Norm. Sup. (4)* 20.2 (1987), 227–239.
- [5] M. Gromov. “Curvature, Kolmogorov diameter, Hilbert rational designs and overtwisted immersions” (2022). arXiv: 2210.13256 [math.DG].
- [6] M. Gromov. *Isometric immersions with controlled curvatures*. 2022. arXiv: 2212.06122v1 [math.DG].
- [7] A. Petrunin. “Polyhedral approximations of Riemannian manifolds”. *Turkish J. Math.* 27.1 (2003), 173–187.
- [8] A. Petrunin. “Gromov’s tori are optimal”. *Geom. Funct. Anal.* 34.1 (2024), 202–208.
- [9] A. Petrunin. “Tubed embeddings” (2024). arXiv: 2402.16195 [math.DG].
- [10] A. Petrunin. *Normal curvature of Veronese embedding*. MathOverflow. eprint: <https://mathoverflow.net/q/445819>.
- [11] K. Sakamoto. “Planar geodesic immersions”. *Tohoku Math. J. (2)* 29.1 (1977), 25–56.
- [12] S. Tabachnikov. “The tale of a geometric inequality.” *MASS selecta: teaching and learning advanced undergraduate mathematics*. 2003, 257–262.
- [13] B. Wilking. “Index parity of closed geodesics and rigidity of Hopf fibrations”. *Invent. Math.* 144.2 (2001), 281–295.
- [14] J. A. Wolf. *Spaces of constant curvature*. 1967.