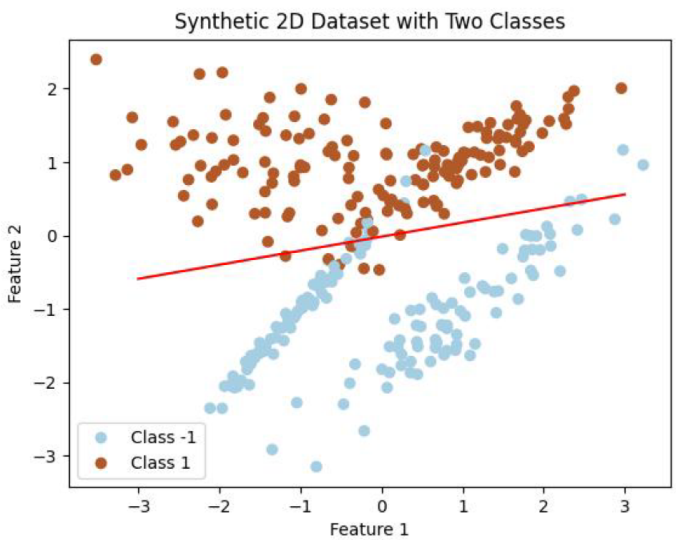
Linear Model for Classification

Assume that we are given points $\left(x^{(i)},\ y^{(i)}\right)$, $i = 1,\ 2,\ \ldots,\ N$. And $y^{(i)} \in \{-1,\ 1\}$. We have a linear model

$$a(x) = w_0 + w_1 x_1 + w_2 x_2,$$

which returns values from $(-\infty,\ \infty)$. We can modify it to return -1 or 1:

$$a(x) = \operatorname{sign}(w_0 + w_1 x_1 + w_2 x_2) = \operatorname{sign} w^T \tilde{x} = \operatorname{sign}\langle w, \tilde{x}\rangle, \qquad \tilde{x} = (1, x_1, x_2).$$



Problem 1. Consider the line through the origin:

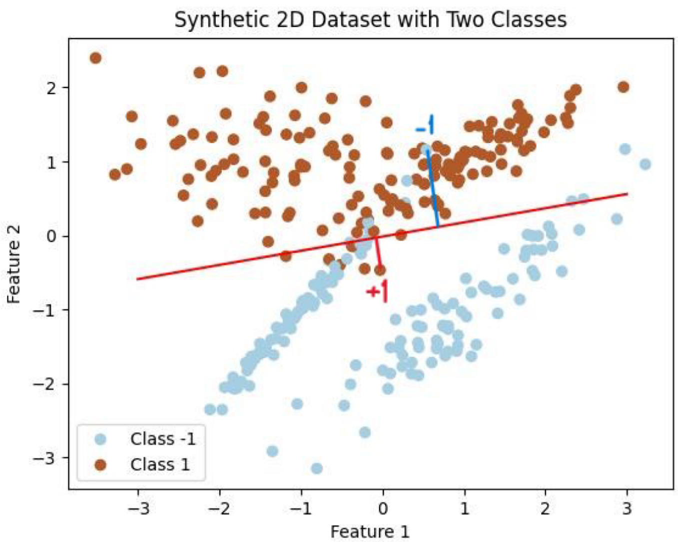
$$w_1 x_1 + w_2 x_2 = 0.$$

Prove that the vector $w = (w_1,\ w_2)^T$ is perpendicular to this line.

Problem 2. Prove that a distance from a point $(x_1,\ x_2)$ above the line to the line is given by

$$d = \frac{\langle w, x\rangle}{|w|} = \frac{w_1 x_1 + w_2 x_2}{\sqrt{w_1^2 + w_2^2}}.$$

Hint: by definition, $\langle \vec{a}, \vec{b}\rangle = |\vec{a}||\vec{b}| \cos\alpha$. Therefore, $\operatorname{Pr}_{\vec{a}} \vec{b} = |\vec{b}| \cos\alpha = \dfrac{\langle \vec{a}, \vec{b}\rangle}{|\vec{a}|}$.



Problem 3. The following value is called **margin**

$$y^{(i)} \left\langle w, x^{(i)} \right\rangle .$$

Prove that for correctly classified point $x^{(i)}$ the margin is positive and for an incorrectly classified point it's negative.

Problem 4. Assuming the separating line has a bias term

$$w_0 + w_1 x_1 + w_2 x_2 = 0,$$

prove that a distance from a point $(x_1,\ x_2)$ above the line to the line is given by

$$d = \frac{\langle w, \tilde{x} \rangle}{|w_{-0}|} = \frac{w_0 + w_1 x_1 + w_2 x_2}{\sqrt{w_1^2 + w_2^2}},$$

where $w = (w_0, w_1, w_2)^T$, $w_{-0} = (w_1, w_2)^T$, $x = (x_1, x_2)^T$, and $\tilde{x} = (1, x_1, x_2)^T$.