Lesson 5: Invariants and Geometric Constructions

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Problem 1.
2004 pluses and 2005 minuses are written on the board. Each move consists of erasing two signs and writing one instead: if the erased signs were the same you write a plus, otherwise a minus. At some point, there will be only one sign left. What sign will it be?

Problem 2.
Each second three integers a, b, c are changed to the integers $a+b-c, a+c-b, b+c-a$. The initial set of integers was $2000, 2002, 2003$. Can it at some point become $2001, 2002, 2003$?

Problem 3.
The number 8^{2019} is written on the board. At each step it is replaced by the sum of its digits, until a 1-digit number is left. What is that one-digit number?

Problem 4.
A group of children is standing in a circle, and each child has an even number of candies. Every minute all children simultaneously give half of their candies to their neighbor on the right. If after such operation someone has an odd number of candies, they get one extra candy from the teacher. Show that at some point all children will have the same number of candies.

Problem 5.
Let M be the midpoint of the side AB of $\triangle ABC$. Consider the line through M which is parallel to AC, and suppose it intersects BC at point P. Show that P is the midpoint of BC. *Hint: consider the line through P parallel to AB and its intersection with AC.*

Problem 6.
Given a line ℓ and a point P not lying on ℓ, construct a line through P which makes a 60° angle with ℓ.