

# Lesson 6: Invariants and Geometric Constructions

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## 1 From Last Time

### Problem 1.

- a) Consider an  $n \times m$  table filled with integers. With one operation, you are allowed to take any row or column and negate every number in that row/column. Show that it is possible to make sure every row and column has nonnegative sum using such operations.
- b) Same problem with real numbers in the table, not integers.

### Problem 2.

Consider  $n$  segments on the plane with  $2n$  distinct endpoints. The following process is performed: if two segments  $AB$  and  $CD$  intersect, we replace them by segments  $AD$  and  $BC$ . Show that eventually no two segments will intersect.

## 2 New Problems

### Problem 1 (Intercept Theorem).

- a) Consider two rays  $r, \ell$  out of point  $O$  and distinct points  $A_1, \dots, A_n$  on  $r$  such that

$$A_1A_2 = A_2A_3 = \dots = A_{n-1}A_n$$

Show that if  $B_1, \dots, B_n$  are points on  $\ell$  such that the lines  $A_iB_i$  are parallel to each other for all  $1 \leq i \leq n$ , then

$$B_1B_2 = B_2B_3 = \dots = B_{n-1}B_n$$

*Hint: Use L6.5.*

- b) With  $O, r, \ell$  as in part a), let  $A, B, C$  be points on  $r$  such that  $AB/BC$  is an integer. Show that if  $A', B', C'$  are points on  $\ell$  such that  $AA' \parallel BB' \parallel CC'$ , then  $A'B'/B'C' = AB/BC$ .
- c) Same as part b), except  $AB/BC$  is rational.

### Problem 2.

Let  $AB$  be a given segment, and  $n$  be a positive integer. Use straightedge and compass to split  $AB$  into  $n$  equal parts. You may assume the result of problem 1a).

*Hint: Construct a random auxiliary line  $\ell$  through  $A$  and points  $A_0, \dots, A_n$  on  $\ell$  such that  $A_0 = A$  and*

$$A_0A_1 = A_1A_2 = \dots = A_{n-1}A_n$$

*Now use 1a).*

### Problem 3.

Let  $ABCD$  be an arbitrary quadrilateral. If  $M, N, P, Q$  are the midpoints of  $AB, BC, CD, DA$  respectively, show that  $MNPQ$  is a parallelogram.