Homework 7: Quadratic equations VI

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1 Reading

Solution 1 (H5.1)**.**
In all parts, we will use Vieta's theorem, which tells us that $x_0 + x_1 = -8/3$ and $x_0x_1 = -1/3$. Then we will try to rewrite the expressions in terms of $x_0 + x_1$ and x_0x_1.
a)

$$\begin{aligned} x_0x_1^4 + x_1x_0^4 &= x_0x_1(x_0^3 + x_1^3) = x_0x_1((x_0 + x_1)^3 - 3x_0x_1^2 - 3x_0^2x_1) \\ &= x_0x_1((x_0 + x_1)^3 - 3x_0x_1(x_0 + x_1)) = -\frac{1}{3}\left(-\frac{8^3}{3^3} - 3 \cdot \frac{1}{3} \cdot \frac{8}{3}\right) = \frac{584}{81} \end{aligned}$$

b)

$$\begin{aligned} x_0^4 + x_1^4 &= (x_0 + x_1)^4 - 4x_0x_1^3 - 6x_0^2x_1^2 - 4x_0^3x_1 \\ &= (x_0 + x_1)^4 - 4x_0x_1((x_0 + x_1)^2 - 2x_0x_1) - 6(x_0x_1)^2 \\ &= \frac{8^4}{3^4} + 4 \cdot \frac{1}{3}\left(\frac{8^2}{3^2} + 2 \cdot \frac{1}{3}\right) - 6\frac{1^2}{3^2} = \frac{4882}{81} \end{aligned}$$

c) Notice that 0 is not a root of our equation, so dividing by x_0 and x_1 is valid.

$$\frac{x_0}{x_1} + \frac{x_1}{x_0} = \frac{x_0^2 + x_1^2}{x_0x_1} = \frac{(x_0 + x_1)^2 - 2x_0x_1}{x_0x_1} = -3\left(\frac{8^2}{3^2} + 2 \cdot \frac{1}{3}\right) = -\frac{70}{3}$$

Solution 2 (H5.2)**.**
Let O be the common center of the incircle and the circumcircle of $ABCD$. Let P, Q, R, S be the points at which the incircle is tangent to the sides AB, BC, CD, DA respectively. Then OP is the altitude in the isosceles $\triangle AOB$, which means that $AP = BP$. But we also know that $BP = BQ$. Applying the same logic to each side we get $AP = PB = BQ = QC = CR = RD = DS = SA$. But this implies that $AB = BC = CD = DA$, so $ABCD$ is a rhombus. But we know that in a rhombus $\angle A = \angle C$ and $\angle B = \angle D$, and the condition of being inscribed means that $\angle A + \angle C = \angle B + \angle D = 180°$. This implies that $\angle A = \angle B = \angle C = \angle D = 90°$, which means that our rhombus is in fact a square.

2 Homework

Problem 1.
The quadratic equation $x^2 + bx + c$ has two distinct real roots. Is it possible that after each coefficient was increased by 1, each root also increased by 1?

Problem 2.
Let $ABCD$ be a convex quadrilateral. Show that the quadrilateral created by the four angle bisectors of $ABCD$ is inscribed.