Lesson 1: Combinations

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Recall from the lecture that the Pascal's triangle is a way of writing numbers $\binom{n}{k}$ in a shape of a triangle where $\binom{n}{k}$ is the k-th number in the n-th row.

Problem 1.
Show that the k-th number in the n-th row of Pascal's triangle is equal to the number of ways to descend from the top of the triangle to the k-th position in the n-th row while moving down-right or down-left every time.

Problem 2.
Prove the *binomial formula*

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{1}ab^{n-1} + b^n$$

Consequently the numbers $\binom{n}{k}$ are frequently referred to as *binomial coefficients*.

Problem 3.
If the ratios of three binomial coefficients are

$$\binom{n+1}{m+1} : \binom{n+1}{m} : \binom{n+1}{m-1} = 5 : 5 : 3$$

find n, m.

Problem 4.
a) Show that if p is prime and $1 \leq k \leq p$, then

$$p \mid \binom{p}{k}$$

b) Show that if p is prime, then

$$p \mid \binom{2p}{p} - 2$$

Problem 5.
Let $ABCD$ be a cyclic quadrilateral, and let T be the intersection of lines AB and CD. Assume A lies on the segment TB and D lies on the segment TC. Show that $TA \cdot TB = TC \cdot TD$.

Problem 6.
Let BB_1 and CC_1 be altitudes in a triangle $\triangle ABC$. Show that the tangent line at A to the circumcircle of $\triangle ABC$ is parallel to B_1C_1.