Lesson 5, problem 3. Divisibility and remainders

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Problem 1.
The number 8^{2019} is written on the board. At each step it is replaced by the sum of its digits, until a 1-digit number is left. What is that one-digit number?

Solution 1.
Let's first prove a lemma:

Lemma 1 (Divisibility rule by 9)**.**
Any nonnegative integer A has the same remainder modulo 9 (i.e., has the same remainder after dividing by 9) as does the sum of its digits.

Proof. Let's write $A = \overline{a_n a_{n-1} ... a_1 a_0}$, where bar denotes that $a_n, ..., a_0$ are digits in A. For example, if $a_2 = 9, a_1 = 3, a_0 = 7$, then $\overline{a_2 a_1 a_0} = 937$. Note that

$$A = 10^n a_n + 10^{n-1} a_{n-1} + ... + 10^1 a_1 + 10^0 a_0$$

Looking at the difference $A - (a_n + a_{n-1} + ... + a_0)$, we see that the difference is divisible by 9:

$$\begin{aligned} A - (a_n + a_{n-1} + ... + a_0) &= 10^n a_n + 10^{n-1} a_{n-1} + ... + 10^1 a_1 + 10^0 a_0 \\ &- (a_n + a_{n-1} + ... + a_1 + a_0) \\ &= \underbrace{99...9}_{n-1} a_n + \underbrace{99...9}_{n-2} a_{n-1} + ... + 99 a_2 + 9 a_1 \\ &= 9 * (\underbrace{11...1}_{n-1} a_n + \underbrace{11...1}_{n-2} a_{n-1} + ... + 11 a_2 + a_1) \end{aligned}$$

If A and $(a_n + a_{n-1} + ... + a_0)$ had different remainders modulo 9, their difference would have nonzero remainder modulo 9, and thus would not be divisible by 9. Therefore, A and $(a_n + a_{n-1} + ... + a_0)$ have the same remainder modulo 9. $\square$

Note that, in particular, this means that if the sum of the digits of A is divisible by 9 (remainder is 0), then A itself is divisible by 9. This statement is probably more familiar to you!

Let's return to the original problem. We can see an invariant: **The remainder of our number modulo 9**. Indeed, we just proved that the remainder stays the same when a number is substituted by the sum of its digits! This means that if we find the remainder modulo 9 of the original number 8^{2019}, the remainder of the one-digit number left on the board would be the same.

To finish the proof, we'll need one more thing. Suppose two numbers A and B have remainders a and b modulo 9. Then the product AB has the same remainder modulo 9 as ab. Indeed, we can write:

$$AB = (9k + a)(9n + b) = \underbrace{81kn + 9kb + 9an}_{\textit{divisible by } 9} + ab$$

We now need to find the remainder of 8^{2019} modulo 9. 8 has remainder 8 modulo 9 (obviously). 8^2 has remainder 1 (check!). Then, by the fact above, 8^3 has remainder 8 modulo 9. Similarly, 8^4 has remainder 8 and so on. We can see that remainder of 8^n will be 1 if n is even, and 8 if n is odd. Thus, 8^{2019} has remainder 8, and so the remaining one-digit number's remainder is also 8. The only such number is 8.