Week 1: Olympiad Warm-Up

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1. In a far-away country, every shop is owned by one of three companies. One day, a new law was instituted to fight the monopoly. If at the end of any given day one of the companies owns more than half of all the shops and the number of shops it owns is divisible by 5, then the company must close all but 1/5 of its shops. Is it possible that three days after the law takes into effect the number of shops owned by each company decreased?

2. Find the smallest positive integer n such that $n^2 + 20n + 19$ is divisible by 2019.

3. Let ABCD be a trapezoid with bases AD and BC. Let $|AB| = |BD|$, M be the midpoint of CD, and O be the point of intersection of AC and BM. Show that the triangle BOC is isosceles. (**Hint**: Extend BM to a point N such that $|BM| = |MN|$)

4. 2019 grasshoppers sit on a line. On every move, any of the grasshoppers can jump over any other grasshopper so that the distance between the two remains the same. Given that the grasshoppers can jump only to the right, they figured out a way so that, after some number of jumps, there are at least two of them less than 1 mm apart. Show that they can still find a way to have at least two grasshoppers less than 1 mm apart if they can only jump to the left from their starting position.

5. Let K be a point inside of the isosceles triangle ABC such that $|AB| = |BC| = |CK|$ and $\angle KAC = 30$ degrees. Find the measure of $\angle AKB$.

6. Given an $n \times n$ checkered board with a black square in the bottom left corner, show that it is impossible to place n rooks with no rook attacking any other rook such that there is an odd number of rooks on white squares.