Lesson 4: Probability III

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Definition 1 (Random Variable - informal).
A random variable is a quantifiable experiment. That is, an experiment with numerical outcomes. Random variables are usually denoted by capital letters (X, Y, Z, etc.) For example, we could define random variables X = the number of heads after tossing 5 fair coins.

Definition 2 (Random Variable - formal).
Let Ω be a sample space, and $\mathbf{P}$ be a Probability Function on Ω. A **random variable** X is a function $X : \Omega \to \mathbb{R}$.

Definition 3 (Probability Mass Function).
The **probability mass function** of X, denoted $p_X : \mathbb{R} \to [0, 1]$ is defined by:

$$p_X(x) = \mathbf{P}(X = x) = \mathbf{P}(\{X = x\}) = \mathbf{P}(\{\omega \in \Omega : X(\omega) = x\})$$

Definition 4 (Expected Value).
The **expected value** of a random variable X is defined as:

$$\sum_{x \in \mathbb{R}} x p_X(x)$$

1 Introductory problems

Problem 1.
Let X be a discrete random variable on a sample space Ω. Prove that $p_X(x) = 1$.

Problem 2.
You roll an unfair seven-sided die with faces 1, 2, ..., 7, where the probability of landing on each side is proportional to the value on the side. That is, you are twice as likely to roll a 2 as you are to roll a 1, and 7/3 more likely to roll a 7 than a 3. Let X be the value of the face that the die lands on. Find the probability mass function of X and the expected value of X.

Problem 3.
Suppose your friend offers you the following deal: A six-sided die is rolled and you get to wager any number of jolly ranchers on the result being even or odd. If you are correct, you win as many jolly ranchers as you wagered. If you are incorrect, you lose what you wagered. Assuming you just went trick-or-treating and have an infinite supply of jolly ranchers, come up with a strategy with a positive expected value of jolly ranchers won and show that it works.

2 Advanced problems

Problem 4.
Suppose you flip a biased coin that lands heads with probability p. Let X be the number of heads in 3 throws. Find the probability mass function of X, $p(X)$, as well as the expected value of X, $\mathbb{E}(X)$.

Problem 5.

1. Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice (Assuming you don't want a goat)? What is the probability of winning the car if you switch vs if you stick with door No. 1?

2. Now suppose that two of the three doors contain a car. Then after picking a door, say No. 1, the host, who knows what's behind the doors, opens another door, say No. 3, which has a car. Is it advantageous to switch doors or stick with your original pick?

Problem 6.
You are playing a betting game with your friend that involves flipping a fair coin. Every time the coin comes up heads, your friend gives you a dollar. Every time the coin comes up tails, you give your friend a dollar. The rule of the game is that you keep flipping the coin until it comes up heads, and then you stop and the game ends. If we let X be your winnings from the game, find the probability mass function of X.

Extra Hard: Find the expected value of X. If your goal is to make money, should you play this game?