Dantzig-Wolfe lecture 1 solutions

February 6, 2020

1 Task 1:

1.1 Task 1a

Extreme points: (0,0), (2,0), (6,3), (0,6).

1.2 Task 1b

$$\max x_1 + 3x_2$$

subject to

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \lambda_1 + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \lambda_2 + \begin{pmatrix} 6 \\ 3 \end{pmatrix} \lambda_3 + \begin{pmatrix} 0 \\ 6 \end{pmatrix} \lambda_4 \\
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= 1 \\
x_1, x_2 &\geq 0 \\
\lambda_1, \lambda_2, \lambda_3, \lambda_4 &\geq 0
\end{aligned}$$

we can eliminate x_1 and x_2 by substitution to get:

$$\max 1\left(0\lambda_1 + 2\lambda_2 + 6\lambda_3 + 0\lambda_4\right) + 3\left(0\lambda_1 + 0\lambda_2 + 3\lambda_3 + 6\lambda_4\right)$$

subject to

$$\begin{aligned}
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= 1 \\
\lambda_1, \lambda_2, \lambda_3, \lambda_4 &\geq 0
\end{aligned}$$

which we can simplify to

$$\max 2\lambda_2 + 15\lambda_3 + 18\lambda_4$$

subject to

$$\begin{aligned}
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= 1 \\
\lambda_1, \lambda_2, \lambda_3, \lambda_4 &\geq 0
\end{aligned}$$

1.3 Task 1c

$$\max x_1 + 3x_2$$

subject to

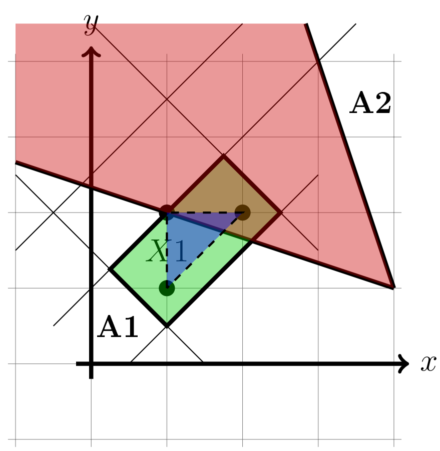
$$\begin{aligned}\frac{1}{2}x_1 + x_2 &\leq 6\\ \frac{3}{4}x_1 - x_2 &\leq \frac{3}{2}\\ x_1, x_2 &\geq 0\end{aligned}$$

1.4 Task 1d

The Julia implementation is available on DTU inside.

2 Task 2

1. The new situation is illustrated on the figure below. Note that the size of A1 and X_1 has shrunk. When $A2$ is intersected with X_1 only a small slice of the solution space remains.



2. The extreme points of the "new" X_1 is $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right\}$.

   New Dantzig-Wolfe decomposition:

$$\begin{array}{rrrcrl} Z^{DW} = \max & 1(1\lambda_1 + 1\lambda_2 + 2\lambda_3) & +2(2\lambda_1 + 1\lambda_2 + 2\lambda_3) & & & \\ \text{s.t.} & 3(1\lambda_1 + 1\lambda_2 + 2\lambda_3) & +(2\lambda_1 + 1\lambda_2 + 2\lambda_3) & \leq & 13 & (A0) \\ & -(1\lambda_1 + 1\lambda_2 + 2\lambda_3) & -3(2\lambda_1 + 1\lambda_2 + 2\lambda_3) & \leq & -7 & (A0) \\ & & \lambda_1 + \lambda_2 + \lambda_3 & = & 1 & \\ & & \lambda_i & \geq & 0 & \forall i = 1, \ldots, 4 \end{array}$$

3. Solving this gives $\lambda_3 = 1$ meaning that the original point (2,2) is optimal and the objective is 6. Thus this LP releaxed model derived by DW-decomposition solves the IP to optimality. We see that we got a better upper bound compared to what we got using the decomposition from class.

4. In general, when more constraints are moved to the set that is "convexified" the better bound one get. When we move constraints away from the "convexified" set we get weaker bounds.

3 Task 3

3.1 Subtask 1:

- Integer solution has objective = 1 and $(x_1, x_2, x_3, x_4) = (1, 0, 0, 0)$
- LP relaxation has objective = 2 and $(x_1, x_2, x_3, x_4) = (1, 0, \frac{1}{3}, \frac{2}{3})$

3.2 Subtask 2:

Extreme points of convexified set:

#	(x_1, x_2, x_3, x_4)
1	(1,0,0,0)
2	(1,1,1,0)
3	(1,0,1,0)
4	(1,0,1,1)

First we rewrite to:

$$\max x_1 + x_2 + x_3 + x_4$$

subject to

$$x_1 + x_2 + 4x_3 + x_4 \leq 3 \tag{1}$$
$$x_1 = 1\lambda_1 + 1\lambda_2 + 1\lambda_3 + 1\lambda_4 \tag{2}$$
$$x_2 = 0\lambda_1 + 1\lambda_2 + 0\lambda_3 + 0\lambda_4 \tag{3}$$
$$x_3 = 0\lambda_1 + 1\lambda_2 + 1\lambda_3 + 1\lambda_4 \tag{4}$$
$$x_4 = 0\lambda_1 + 0\lambda_2 + 0\lambda_3 + 1\lambda_4 \tag{5}$$
$$\sum_{j=1}^{4} \lambda_j = 1 \tag{6}$$
$$x_i \in \{0, 1\} \qquad \forall i \in \{1, 2, 3, 4\} \tag{7}$$
$$\lambda_j \geq 0 \qquad \forall j \in \{1, 2, 3, 4\} \tag{8}$$

LP relax and substitute (we ignore bounds on x_i variables as they are guaranteed to be satisfied given our extreme points)

$$\max (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + (\lambda_2) + (\lambda_2 + \lambda_3 + \lambda_4) + (\lambda_4)$$

subject to

$$(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + (\lambda_2) + 4(\lambda_2 + \lambda_3 + \lambda_4) + (\lambda_4) \leq 3 \tag{9}$$
$$\sum_{j=1}^{4} \lambda_j = 1 \tag{10}$$
$$\lambda_j \geq 0 \forall j \in \{1, 2, 3, 4\} \tag{11}$$

3.3 Subtask 3:

- LP relaxation has objective = 1.8 and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.6, 0.4, 0, 0)$,$(x_1, x_2, x_3, x_4) = (1, 0.4, 0.4, 0)$
- Dantzig-Wolfe decomposition improved the LP relaxation (from 2 to 1.8)

3.4 Subtask 4:

Extreme points of convexified set:

#	(x_1, x_2, x_3, x_4)
1	(0,0,0,0)
2	(0,0,0,1)
3	(0,1,0,0)
4	(0,1,0,1)
5	(1,0,0,0)
6	(1,0,0,1)
7	(1,1,0,0)
8	(1,1,0,1)

First we rewrite to:

$$\max x_1 + x_2 + x_3 + x_4$$

subject to

$$3x_1 - 2x_2 + 2x_3 - x_4 \geq 3 \tag{12}$$
$$x_1 = 0\lambda_1 + 0\lambda_2 + 0\lambda_3 + 0\lambda_4 + 1\lambda_5 + 1\lambda_6 + 1\lambda_7 + 1\lambda_8 \tag{13}$$
$$x_2 = 0\lambda_1 + 0\lambda_2 + 1\lambda_3 + 1\lambda_4 + 0\lambda_5 + 0\lambda_6 + 1\lambda_7 + 1\lambda_8 \tag{14}$$
$$x_3 = 0\lambda_1 + 0\lambda_2 + 0\lambda_3 + 0\lambda_4 + 0\lambda_5 + 0\lambda_6 + 0\lambda_7 + 0\lambda_8 \tag{15}$$
$$x_4 = 0\lambda_1 + 1\lambda_2 + 0\lambda_3 + 1\lambda_4 + 0\lambda_5 + 1\lambda_6 + 0\lambda_7 + 1\lambda_8 \tag{16}$$
$$\sum_{j=1}^{8} \lambda_j = 1 \tag{17}$$
$$x_i \in \{0, 1\} \qquad \forall i \in \{1, 2, 3, 4\} \tag{18}$$
$$\lambda_j \geq 0 \qquad \forall j \in \{1, \dots, 8\} \tag{19}$$

LP relax and substitute (we ignore bounds on x_i variables as they are guaranteed to be satisfied given our extreme points)

$$\max (\lambda_5 + \lambda_6 + \lambda_7 + \lambda_8) + (\lambda_3 + \lambda_4 + \lambda_7 + \lambda_8) + (\lambda_2 + \lambda_4 + \lambda_6 + \lambda_8)$$

subject to

$$3(\lambda_5 + \lambda_6 + \lambda_7 + \lambda_8) - 2(\lambda_3 + \lambda_4 + \lambda_7 + \lambda_8) - (\lambda_2 + \lambda_4 + \lambda_6 + \lambda_8) \geq 3 \tag{20}$$
$$\sum_{j=1}^{8} \lambda_j = 1 \tag{21}$$
$$\lambda_j \geq 0 \forall j \in \{1, \dots, 8\} \tag{22}$$

3.5 Subtask 5:

- LP relaxation has objective $= 1$ and $\lambda_5 = 1$ and $\lambda_j = 0$ for all $j \in \{1, 2, 3, 4, 6, 7, 8\}$,$(x_1, x_2, x_3, x_4) = (1, 0, 0, 0)$
- Dantzig-Wolfe decomposition improved the LP relaxation and the LP relaxed model now finds the integer solution. Conclusion: the decomposition considered in sub-task 4 was “best” in terms of LP bound.