Ex1 Consider the mass-spring-damper system in the notes 5.1, 5.2, p 82, with force $\{F_t : t \geq 0\}$ being white noise with a given intensity $S_{FF}(\omega) = \sigma^2$.
Write the system in the standard form $dX_t = AX_t\,dt + G\,dB_t$.

- $$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \qquad G = \sigma \cdot \begin{bmatrix} 0 \\ 1/m \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma/m \end{bmatrix}$$

Ex2 Using the general form, simulate the system on the time interval $t \in [0, 1000]$ using the Euler method. Take system parameters $m = 1$ kg, $k = \frac{1}{2}$ N/m, $c = 0{,}2$ Ns/m and $\sigma^2 = 100\ N^2 s$. Let the system start at rest at $t=0$. Use a time step of $\Delta t = 0.01$ s.

See R.

Ex3 Estimate from your simulation the steady-state variance of position Q_t, of velocity V_t, and the covariance between the two. Compare with the solution of the algebraic Lypanov equation governing the variance.

- From the Sylvester equation (see wiki) we know

$$AX + XB = C \iff (I_m \otimes A + B^T \otimes I_n)\,\text{vec}\,X = \text{vec}\,C$$

where vec(·) is the vectorization operator. Hence the Lyapunov equation, eq 5.27, can be rewritten as follows

$$A\Sigma + \Sigma A^T + GG^T = 0 \quad \Rightarrow$$

$$A\Sigma + \Sigma A^T = -GG^T \quad \Rightarrow$$

$$(I_n \otimes A + A \otimes I_n)\,\text{vec}\,\Sigma = -\text{vec}(GG^T) \quad \Rightarrow$$

$$\text{vec}\,\Sigma = -(I_n \otimes A + A \otimes I_n)^{-1}\,\text{vec}(GG^T)$$

Hence we can solve for the analytical covariance matrix Σ

$$\Sigma = \begin{bmatrix} 500 & 0 \\ 0 & 250 \end{bmatrix}$$

See the rest in R

Ex4 The kinetic energy is $\frac{1}{2} m V_t^2$ while the potential energy is $\frac{1}{2} k Q^2$. In steady-state, what is the expected kinetic and potential energy?

- we first calculate the potential energy

$$\begin{aligned} \mathbb{E}[\tfrac{1}{2} k Q_t^2] &= \tfrac{1}{2} \cdot \tfrac{1}{2}\, \mathbb{E}[Q_t^2] \\ &= \tfrac{1}{4}\left(\mathbb{V}[Q_t] + \mathbb{E}[Q_t]^2\right) \\ &= \tfrac{1}{4}\, \mathbb{V}[Q_t] \end{aligned}$$

$$\left(\begin{aligned} \mathbb{V}[Q] &= \mathbb{E}[Q^2] - \mathbb{E}[Q]^2 \\ \mathbb{E}[Q^2] &= \mathbb{V}[Q] + \mathbb{E}[Q]^2 \end{aligned}\right)$$

$(\mathbb{E}[Q] = 0$ in steady state$)$

from the previous exercise we know $\mathbb{V}[Q_t] = 500$ so

$$\mathbb{E}[\tfrac{1}{2} k Q_t^2] = 125$$

Next the kinetic energy

$$\begin{aligned} \mathbb{E}[\tfrac{1}{2} m V_t^2] &= \tfrac{1}{2} \cdot 1 \cdot \mathbb{E}[V_t^2] \\ &= \tfrac{1}{2}\, \mathbb{V}[V_t] \end{aligned}$$

From the previous exercise we know $\mathbb{V}[V_t] = 250$ so

$$\mathbb{E}[\tfrac{1}{2} m V_t^2] = 125$$

This makes sense because the kinetic and potential energy should be equal in the steady state.

Ex 5 For the simulation, compute and plot the empirical acf of $\{Q_t\}$ up to lag 50. Add to the plot the theoretical acf.

- We find the theoretical acf on page 111 as eq. 5.28.

$$\rho(t) = \Sigma \exp(A^T t), \quad \text{for } t \geq 0$$

Hence in our case

$$\rho(t) = \begin{bmatrix} 500 & 0 \\ 0 & 250 \end{bmatrix} \exp\left(\begin{bmatrix} 0 & -1/2 \\ 1 & -1/5 \end{bmatrix} t \right)$$

The matrix exponential is possible to calculate analytically but it is very nasty. Hence we will not write it here but only calculate it numerically in R using the function and package expm. See the rest in R.

Ex 6. Plot as a function of the frequency ω, the amplitude and phase of the frequency response from the noise to the position. Plot also the theoretical variance spectrum of the position.

- On page 96 the noise is substituted with a force but it could just as well have been a noise term. The frequency response from noise to position (in the book from external force to internal state, $U_t \to X_t$), is given as

$$H(\omega) = (i\omega \cdot I - A)^{-1} G,$$

for linear systems.
Hence for our system

$$H(\omega) = \frac{100}{2i\omega - 10\omega^2 + 5} \begin{bmatrix} 1 \\ i\omega \end{bmatrix}$$

where the position is the first entry and the velocity is the second.

On page 104, eq 5.14 the variance spectrum is given as

$$S_X(\omega) = H(-\omega) \cdot S_U(\omega) \cdot H^T(\omega)$$

where S_U is the white noise spectrum. It is in this exercise given as $S_U(\omega) = S_{FF}(\omega) = \sigma^2$, so

$$\begin{aligned} S_X(\omega) &= H(-\omega) \cdot \sigma^2 \cdot H^T(\omega) \\ &= H(\omega) \cdot 100 \cdot H^T(\omega) \\ &= \begin{bmatrix} \dfrac{-10^6}{-100\omega^4 + 96\omega^2 - 25} & \dfrac{-10^6 \, i\omega}{-100\omega^4 + 96\omega^2 - 25} \\ \dfrac{10^6}{-100\omega^4 + 96\omega^2 - 25} & \dfrac{-10^6 \omega^2}{-100\omega^4 + 96\omega^2 - 25} \end{bmatrix} \end{aligned}$$

where the variance spectrum for the position is given is $S_X(\omega)[1,1]$ i.e.

$$S_Q(\omega) = S_X(\omega)[1,1] = \frac{-10^6}{-100\omega^4 + 96\omega^2 - 25}$$

See plots in R.

Variance in a scalar linear system

Consider the scalar liner system

$$\dot{X}_t = aX_t + gU_t, \quad X_0 = x$$

where $\{U_t : t \geq 0\}$ is Gaussian white noise, i.e. the formal derivative of standard Brownian motion.

Ex 7. Write up the mean $\mathbb{E}[X_t]$ as a function of time

- We know that

$$\dot{X} = AX + GU$$

$$\Updownarrow$$

$$X_t - X_0 = \int_0^t \dot{X}_s \, ds = \int_0^t AX_s + GU_s \, ds$$

$$= \int_0^t AX_s \, ds + \int_0^t GU_s \, ds$$

$$= \int_0^t AX_s \, ds + GB_t$$ [because U is white noise and it is the derivative of brownian motion]

Hence we also know that

$$\frac{d}{dt}\mathbb{E}[X_t] = A\,\mathbb{E}[X_t] + G\,\mathbb{E}[B_t]$$

$$= A\,\mathbb{E}[X_t]$$

Hence we know that $\mathbb{E}[X]$ is just the deterministic solution to the differential equation.

$$\mathbb{E}[X_t] = xe^{at}$$

Ex8 Write up the differential Lyapunov equation governing the variance $Var[X_t]$, and solve it.

- The differential Lyapunov equation is given as eq (5.21) in the book and rephrased here in scalar form.

$$\frac{d\Sigma(t)}{dt} = a\,\Sigma(t) + \Sigma(t)\,a + g^2$$

$$= 2a\,\Sigma(t) + g^2$$

We solve the differential equation with $\Sigma(0) = c$

$$\Sigma(t) = e^{2at}\left(c + \frac{g^2}{2a}\right) - \frac{g^2}{2a}$$

using the formula

$$\frac{dx}{dt} = ax + b\,,\quad x(0) = c \Rightarrow x(t) = e^{at}\left(c + \frac{b}{a}\right) - \frac{b}{a}$$

Ex9 Assume that the system is stable. What is the steady-state variance, $\lim_{t \to \infty} \text{Var}[X_t]$

- The stationary variance is given by equation 5.27 and here given for the scalar case.

$$2a\Sigma + g^2 = 0. \quad \Rightarrow \quad \Sigma = -\frac{g^2}{2a}$$

Ex10 Varify that the steady-state variance is a equilibrium point of the Lyapunov equation.

- An equilibrium point is a point where no change will happen i.e.

$$\frac{d\Sigma}{dt} = 0$$

We check

$$\begin{aligned} \frac{d\Sigma}{dt} &= 2a\Sigma + g^2 \\ &= 2a(-g^2/2a) + g^2 \\ &= 0 \end{aligned}$$