Lec2 · SDE · S1

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = [0,1]^2$, $\mathcal{F}$ is the usual Borel-algebra on Ω, and $\mathbb{P}$ being the Lesbesgue measure, i.e. area. For $\omega = (x,y) \in \Omega$ define

$X(\omega) = x$, $Y(\omega) = y$, and $Z(\omega) = x + y$.

Q1:
Varify that X and Y are independent, and that each are uniformly distributed on $[0,1]$.

- From def. 3.4.2 we know that two random variables A and B are independent if $P(A \cap B) = P(A)P(B)$.

We test this.

$$P(X \in [a,b] \times [0,1] \wedge Y \in [0,1] \times [c,d]) = \int_{\Omega} 1_{[a,b]\times[0,1]} 1_{[0,1]\times[c,d]} \, dP(\omega)$$

$$= \int_a^b \int_c^d 1 \, dy\, dx$$

$$= (b-a)(d-c)$$

we assume independent

$$P(X \in [a,b]\times[0,1]) P(Y \in [0,1]\times[c,d]) = \int_{\Omega} 1_{[a,b]\times[0,1]} dP(\omega) \int_{\Omega} 1_{[0,1]\times[c,d]} dP(\omega)$$

$$= \int_a^b 1 \, dx \int_c^d 1 \, dy$$

$$= (b-a)(d-c)$$

Hence independent.

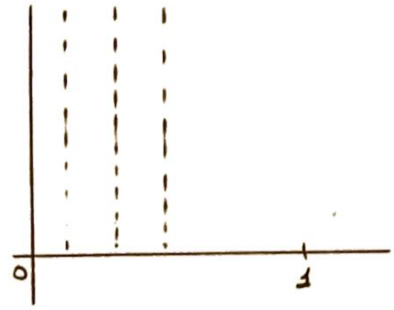
Uniformity follow directly the Lebesgue measure. See the solution.

Lec2 · SDE · S2

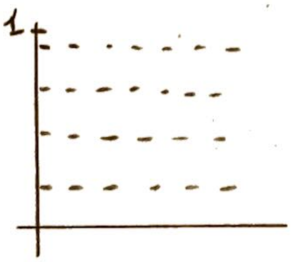
Q2: Sketch level sets for X, Y and Z. Show typical elements in the σ-algebras $\sigma(X)$, $\sigma(Y)$ and $\sigma(Z)$.

- levelsets are defined by the preimage $X^{-1}(1/2) = \{(x,y) : x = 1/2\}$.

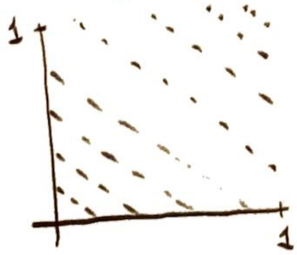
Hence level sets for X are given by vertical lines



For Y they are given as horizontal lines



and for Z it is diagonal lines



The operator $\sigma(\cdot)$ is defined generally in def. 3.4.1 in the book and for the real numbers its all intervals. (hyper rectangles for higher dimensions) Hence see the drawings in the solution for examples.

Q4: Find a function $g: \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}[Z|X] = g(X)$. Verify that this "candidate" conditional expectation satisfies the defining property

$$\int_H g(X(\omega))\, dP(\omega) = \int_H Z(\omega)\, dP(\omega)$$

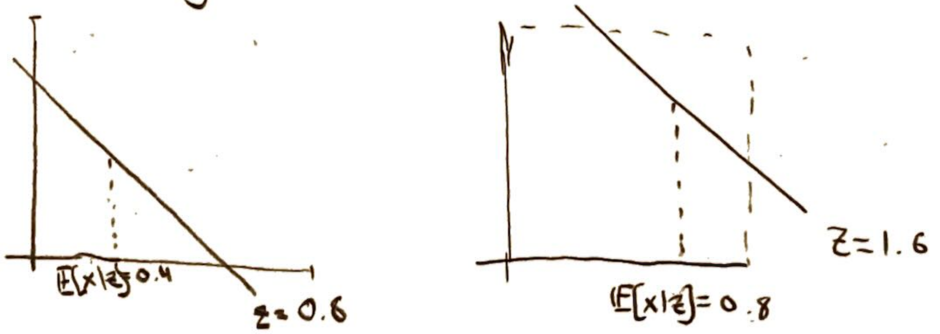
for any $H \in \sigma(X)$.

- See solution.

Q5

A more thorough derivation is done on next page but this is a quick proof.

$g(Z) = \frac{1}{2} Z$. we can check by realizing Z distributes uniformly on the interval $[0,2]$ while X distributes uniformly on $[0,1]$. Hence $g(Z)$ must be the function.



Q5:

Z is the sum of two independent and identical distributed variables. Hence $\mathbb{E}[X|Z] = \mathbb{E}[Y|Z]$.

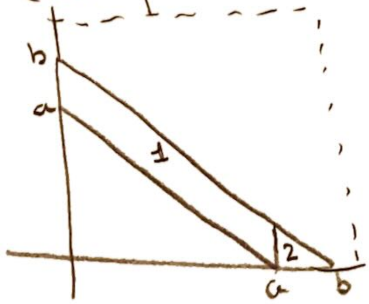
$$\mathbb{E}[X|Z] = \frac{1}{2}\left(\mathbb{E}[X|Z] + \mathbb{E}[Y|Z]\right)$$
$$= \frac{1}{2}\mathbb{E}[X+Y|Z]$$
$$= \frac{1}{2}\mathbb{E}[Z|Z]$$
$$= \frac{1}{2}Z$$

We now need to check if our candidate $g(Z) = \frac{1}{2}Z$ satisfies

$$\int_H g(Z(\omega))\, d\mathbb{P}(\omega) = \int_H X(\omega)\, d\mathbb{P}(\omega)$$

The area H we can parameterize by 4 parts. We have two areas, 1 and 2. We first: we parameterize these by

$(Z<1$

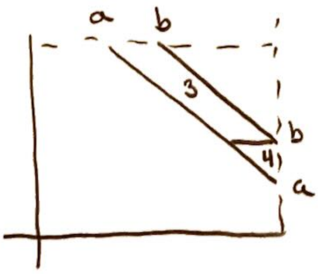


$1: \{(x,y): x\in[0,a], y\in[a-x, b-x]\}$

$2: \{(x,y): x\in[a,b], y\in[0, b-x]\}$

second

$(Z>1)$



$3: \{(x,y): x\in[a+(1-y), b+(1-y)], y\in[b,1]\}$

$4: \{(x,y): x\in[a+(1-y), 1], y\in[a,b]\}$

We can then integrate

$$\int_H g(Z(\omega)) - X(\omega)\, d\mathbb{P}(\omega) = \int_H \frac{y}{2} - \frac{x}{2}\, dxdy$$

$$= \int_1 \frac{y}{2} - \frac{x}{2}\, dxdy + \int_2 \frac{y}{2} - \frac{x}{2}\, dxdy + \int_3 \frac{y}{2} - \frac{x}{2}\, dxdy + \int_4 \frac{y}{2} - \frac{x}{2}\, d$$

$$\underline{\underline{=0}}$$

Hence $g(Z) = \frac{1}{2}Z$ is good.