Week 2: 01125 Fundamental Topological Concepts and Metric Spaces

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3.17

Proof by contradiction. We assume $\cap_{n=1}^{\infty} \mathrm{K}_n = \emptyset$. We then know via De Morgan's Laws:

$$\mathrm{M} \backslash (\cap_{n=1}^{\infty} \mathrm{K}_n) = \cup_{n=1}^{\infty} (\mathrm{M} \backslash \mathrm{K}_n)$$
$$\Rightarrow$$
$$\mathrm{M} \backslash \emptyset = \mathrm{M}$$
$$\Leftrightarrow$$
$$\cup_{n=1}^{\infty} (\mathrm{M} \backslash \mathrm{K}_n) = \mathrm{M}$$

$\{\mathrm{M} \backslash \mathrm{K}_n | n \in \mathbf{N}\}$ are all open subsets covering M, in particular covering K_1. K_1 is compact and therefore it has a finite sub-covering:

$$\mathrm{K}_1 \subseteq \cup_{n=1}^{S} (\mathrm{M} \backslash \mathrm{K}_n)$$

Since

$$\mathrm{K}_1 \supseteq \mathrm{K}_2 \supseteq \mathrm{K}_3 \supseteq ... \supseteq \mathrm{K}_S \Leftrightarrow \mathrm{M} \backslash \mathrm{K}_1 \subseteq \mathrm{M} \backslash \mathrm{K}_2 \subseteq ... \subseteq \mathrm{M} \backslash \mathrm{K}_S$$
$$\Rightarrow$$
$$\mathrm{M} \backslash \mathrm{K}_S \supseteq \mathrm{K}_1 \supseteq \mathrm{K}_S$$
$$\Rightarrow$$
$$\mathrm{M} \backslash \mathrm{K}_S \supseteq \mathrm{K}_S$$

Hereby we have a contradiction because for $\mathrm{M} \backslash \mathrm{K}_S \supseteq \mathrm{K}_S$ to be true $\mathrm{M} = \emptyset$, and in the exercise we are told the space is non-empty.

$\square$

3.20(2)

Direct proof. Assume f is uniformly continuous in K, then it holds:

$$\forall \gamma > 0 \ \ \exists \delta \ : \ \forall x, y \quad d(x,y) < \delta \ \Rightarrow \ d(f(x), f(y)) < \gamma$$

We further know that all points, $x \in K$, will lie in some open ball with its center at some x_0 and with radius ϵ, such that:

$$d(x_n, x) < \epsilon$$

We can now choose $\delta = \epsilon$ and from here it follows via the definition of uniform continuity that:

$$d(x_n, x) < \delta = \epsilon \ \Rightarrow \ d(f(x_n), f(x)) < \gamma$$

We now have a finite covering of the image set $f(K) \in Y$ of open balls with centers in $f(x_1), ..., f(x_p)$ and all radius γ. Therefore $f(K)$ must by definition be precompact.

$\square$

3.29

1)

Proof by contradiction. Assume $\exists\, x_0, y_0 \in M$ for which $x_0 \neq y_0$ and $Tx_0 = x_0$ and $Ty_0 = y_0$. Hence it follows

$$d(x_0, y_0) = d(Tx_0, Ty_0)$$

From the definition of a *weak contraction* it follows that

$$d(Tx_0, Ty_0) < d(x_0, y_0)$$

From the assumptions this leads to the contradiction

$$d(x_0, y_0) = d(Tx_0, Ty_0) < d(x_0, y_0)$$

Hence, the assumptions were wrong. It then follows that T does not have two or more fixed points, which implies it has at most one fixed point.

□

2)

Proof by example. Define $T : M \to M$ to be $Tx = x + \frac{1}{x}$ for $x \geq 1$ where $x \in \mathbb{R}$. Let y be another point in the same interval. By the usual metric in R we have that $d(x, y) = |x - y|$. At first we want to show that $Tx = x + \frac{1}{x}$ is a *weak contraction.*

$$d(Tx, Ty) = |Tx - Ty| = |(x + \tfrac{1}{x}) - (y + \tfrac{1}{y})| = |x - y| \cdot |1 - \tfrac{1}{x \cdot y}|$$

Since $x, y \geq 1$ we know that $0 < \frac{1}{x \cdot y} \leq 1$. Then it follows that $0 \leq |1 - \frac{1}{x \cdot y}| < 1$. Then we have that

$$d(Tx, Ty) < d(x, y)\ \forall x, y \in M \text{ for } x \neq y$$

Hence, $Tx = x + \frac{1}{x}$ for $x \geq 1$ for $\mathbb{R}$ is a *weak contraction.* However, since $Tx = x$ would imply that $x = 0$, the *weak contraction* $Tx = x - \frac{1}{x}$ does not have a fixed point.

□