Week 3: 01125 Fundamental Topological Concepts and Metric Spaces

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24th of January 2020

3.21

Let S be a point set with more than one element equipped with the discrete topology.

1)

Show that a topological space M is connected if and only if every continuous mapping $f : M \to S$ is constant.

Proof. "$\Rightarrow$"
Assume M is connected. We now choose an $n \in f(M)$, and define A and B such that:

$$A = \{m \in M \mid f(m) = n\},\ B = \{m \in M \mid f(m) \neq n\}$$

Then $A = f^{-1}(\{n\})$ and $B = f^{-1}(S\backslash\{n\})$ where both n and S\n are open subsets of S, and therefore both A and B are open in M. We also know that:

$$A \cap B = \emptyset \quad \wedge \quad M = A \cup B$$

From this it follows that:

$$B = M\backslash A$$

This means A is both open and closed. We also know that A is non-empty, since $n \in f(M)$. According to Lemma 3.8.1 it follows directly that A = M. In turn the function $f : M \to S$ is a constant function with value n.

"$\Leftarrow$"
Let any continuous mapping $f : M \to S$ be constant, and assume that M is not connected. Now, cf. Lemma 3.8.1, there exists a non-empty, open, closed, and proper subset $U \subset M$ which implies that $M\backslash U$ is also non-empty, open, and closed in M.
Let's now consider the the function, $h : M \to S$:

$$h(x) = \begin{cases} 0, & x \in U \\ 1, & x \in M\backslash U \end{cases}$$

Cf. Def. 2.4.12 and since $h^{-1}(\{0\}) = U$ is open, and the same holds for $h^{-1}(\{1\}) = M\backslash U$, h is continuous. Since both U and $M\backslash U$ is non-empty, it is clear that h is not constant. This is a obvious contradiction of our initial assumptions. In other words, any continuous mapping $f : M \to S$ cannot be constant, if M is disconnected. Thus for any continuous function f, if f is constant it implies that M is connected.

$\square$

2)

Let $\{W_i \mid i \in I\}$ be a family of connected subsets in a topological space M, such that for every pair of sets W_i and W_j from the family it holds that $W_i \cap W_j \neq \emptyset$. Show that the union $\cup_{i\in I} W_i$ is a connected subset in M.

Proof by contradiction. Let's assume $\cup_{i\in I} W_i$ is not connected. This means there exists $U \subseteq M$, and $V \subseteq M$ non-empty, open and disjoint sets s.t.:

$$\cup_{i\in I} W_i = U \cup V$$

Now let's define a function $g : U \cup V \to \{1, 0\}$ s.t.:

$$g(x) = \begin{cases} 0, & x \in U \\ 1, & x \in V \end{cases}$$

Cf. Def. 2.4.12 and since $g^{-1}(\{0\}) = U$ is open, and the same holds for $g^{-1}(\{1\}) = V$, g is continuous. Let's choose an arbitrary $x_0 \in U$. Then there must exist an i s.t. $x_0 \in W_i$. Since g is continuous and W_i is connected, then $g(W_i)$ must cf. theorem 3.8.4 also be connected. So $g(W_i) = \{0\}$ must be true. We further know that for any $j \in I$ we have that $W_i \cap W_j \neq \emptyset$, this together with theorem 3.8.4 implies that $g(W_j) = \{0\}$. Since we chose j to be any element in I, it must hold for all elements in I. This means that g must be constant and in turn $V = \emptyset$. This is a contradiction and thus the union $\cup_{i\in I} W_i$ must be a connected subset in M.

$\square$

3.27

Let $f : X \to Y$ be a continuous map between metric spaces (X, d_X) and (Y, d_Y). Prove that if $S \subset X$ is a pathwise connected subset in X, then the image $f(S) \subset Y$ under f is a pathwise connected subset in Y.

Proof. We define $f(x), f(y) \in f(S)$, where $x, y \in S$ and a continuous map g.

$$g : [0,1] \to S \; s.t. \; g(0) = x \; \wedge \; g(1) = y$$

We know a continuous map like g exists due to S being pathwise connected.(Def 3.8.5)

Now we define a composite map h.

$$h : [0,1] \to f(S), \quad where \quad h = f \circ g$$

We know h is a continuous map due to theorem 2.4.13. and,

$$h(0) = f(g(0)) = f(x)$$
$$h(1) = f(g(1)) = f(y)$$

Both $f(x), f(y) \in f(S)$ and therefore the image $f(S) \subset Y$ under f complies with all criteria in Def 3.8.5 and is therefore a pathwise connected subset in Y. □

4.3

Let $E = C^{\infty}([0, 2\pi], \mathbb{R})$ be the vector space of differentiable functions $f : [0, 2\pi] \to \mathbb{R}$ of class C^{∞}.

For $f \in E$ we set:

$$\|f\|_0 = sup\{\ |f(x)|\ |x \in [0, 2\pi]\}$$

$$\|f\|_1 = sup\{\ |f(x)| + |f'(x)||x \in [0, 2\pi]\}$$

1)

Show that $\|f\|_0$ and $\|f\|_1$ are norms in E.

Proof. In order for $\|f\|_0$ and $\|f\|_1$ to be norms in E, they must satisfy NORM 1, NORM 2, and NORM 3 listed on page 84.

NORM 1 (Positive definite):
For $\|f\|_0$:
The supremum of a set of absolute values will satisfy that $\|f\|_0 \geq 0$ for all $f \in E$. The supremum will only be 0, if all values in the corresponding set of non-negative values are 0.

For $\|f\|_1$:
The argument for $\|f\|_1$ is exactly as for $\|f\|_0$.

NORM 2 (Uniform scaling):
For $\|f\|_0$:
$\|\alpha f\|_0 = sup\{\ |\alpha f(x)|\ |x \in [0, 2\pi]\} = sup\{\ |\alpha||f(x)|\ |x \in [0, 2\pi]\} =$
$|\alpha| sup\{\ |f(x)|\ |x \in [0, 2\pi]\} = |\alpha|\, \|f\|_0$

Clearly $\|\cdot\|_0$ satisfies uniform scaling.

For $\|f\|_1$:
$\|\alpha f\|_1 = sup\{\ |\alpha f(x)| + |\alpha f'(x)|\ |x \in [0, 2\pi]\} =$
$sup\{\ |\alpha||f(x)| + |\alpha||f'(x)|\ |x \in [0, 2\pi]\} =$
$|\alpha| sup\{\ |f(x)| + |f'(x)|\ |x \in [0, 2\pi]\} = |\alpha|\, \|f\|_1$

Clearly $\|\cdot\|_1$ satisfies uniform scaling.

NORM 3 (Triangle inequality):
For $\|f\|_0$:
$\|f+g\|_0 = sup\{\ |(f+g)(x)|\ |x \in [0,2\pi]\} =$
$sup\{\ |f(x)+g(x)|\ |x \in [0,2\pi]\} \leq sup\{\ |f(x)|+|g(x)|\ |x \in [0,2\pi]\} \leq$
$sup\{\ |f(x)|\ |x \in [0,2\pi]\} + sup\{\ |g(x)|\ |x \in [0,2\pi]\} =$
$\|f\|_0 + \|g\|_0$
So $\|\cdot\|_0$ satisfies the triangle inequality.

For $\|f\|_1$:
$\|f+g\|_1 = sup\{\ |(f+g)(x)|+|(f'+g')(x)|\ |x \in [0,2\pi]\} =$
$sup\{\ |f(x)+g(x)|+|f'(x)+g'(x)|\ |x \in [0,2\pi]\} \leq$
$sup\{\ |f(x)|+|g(x)|+|f'(x)|+|g'(x)|\ |x \in [0,2\pi]\} \leq$
$sup\{\ |f(x)|+|f'(x)|\ |x \in [0,2\pi]\} + sup\{\ |g(x)|+|g'(x)|\ |x \in [0,2\pi]\} =$
$\|f\|_1 + \|g\|_1$
So $\|\cdot\|_1$ satisfies the triangle inequality.

$\square$

2)

Define the linear mapping $D : E \to E$ by associating to $f \in E$ the derivative $f' \in E$ of f, i.e.

$$D(f) = f' \quad for \quad f \in E.$$

Show that for every $n \in \mathbb{N}$ there exists a function $f_n \in E$ for which $\|f_n\|_0 = 1$ and $\|D(f_n)\|_0 = n$.
Utilize this to show that $D : E \to E$ is not continuous, when E is equipped with the norm $\|\cdot\|_0$.

Proof. The following function is chosen,

$$f_n(x) = sin(nx),$$
$$D(f_n(x)) = n \cdot cos(nx).$$

The norm of f_n and $D(f_n)$ is found,

$$\|f_n\|_0 = sup\{|sin(nx)| \mid x \in [0, 2\pi]\} = 1$$
$$\|D(f_n)\|_0 = sup\{|n \cdot cos(nx)| \mid x \in [0, 2\pi]\} = n$$

We see that supremum of $|sin(nx)|$ will equal 1, for all $n \in \mathbb{N}$.
$sup\{|cos(nx)|\}$ will as sinus also equal 1 for all $n \in \mathbb{N}$ and hence $sup\{|n \cdot cos(nx)|\}$ will equal n.

We now proceed with a proof by contradiction.

Consider the normed vector space $\|\cdot\|_0$. Assume that $D : E \to E$ is continuous in E. By theorem 4.2.4.(4) we know,

$$\exists k \in \mathbb{R} : \|D(f_n)\|_0 \leq k \cdot \|f_n\|_0, \ \forall f \in E, \ \forall n \in \mathbb{N}.$$

This must hold for the whole space. Hence we will investigate if it holds for the chosen function, $f_n(x) = sin(nx)$.

$$\|D(sin(nx))\|_0 \leq k \cdot \|sin(nx)\|_0 \Rightarrow$$
$$n \leq k$$

Now pick $n = k + 1$. Then we get $k + 1 \leq k$, which is a contradiction. Since $D : E \to E$ does not comply with theorem 4.2.4.(4) we can through the same theorem (4.2.4.(1)) conclude that $D : E \to E$ is not continuous equipped with the norm $\|\cdot\|_0$. $\square$

3)

Show that $D : E_1 \to E_0$ is continuous when E_1 is E equipped with the norm $\| \cdot \|_1$, and E_0 is E equipped with the norm $\| \cdot \|_0$.

Proof. We will again use theorem 4.2.4.(4) that says,

$$\exists k \in \mathbb{R} : \|T(x)\|_W \leq k \cdot \|x\|_V \,, \ \forall x \in V.$$

In our case this translates to,

$$\exists k \in \mathbb{R} : \|D(f)\|_0 \leq k \cdot \|f\|_1 \,, \ \forall f \in E \quad \Rightarrow$$
$$sup\{|f'(x)| \mid x \in [0, 2\pi]\} \leq k \cdot sup\{|f(x)| + |f'(x)| \mid x \in [0, 2\pi]\}$$

Now we see that $0 \leq |f(x)|$ and therefore it will always be the case that $sup\{|f'(x)| \mid x \in [0, 2\pi]\} \leq sup\{|f(x)| + |f'(x)| \mid x \in [0, 2\pi]\}$. Hence we choose $k = 1$.
We can now cf. theorem 4.2.4. conclude that $D : E_1 \to E_0$ is continuous.

□