Q1: Verify that $Y_t = \sinh B_t$ satisfies the Itô SDE

$$dY_t = \frac{1}{2} Y_t \, dt + \sqrt{1 + Y_t^2} \, dB_t$$

- We first see that $h(x,t) = \sinh x$. Next we state Itô's lemma from thm 7.3.1

$$dY_t = \frac{d}{dt} h(x,t) \cdot dt + \frac{d}{dx} h(x,t) \, dB_t + \frac{1}{2} \frac{d^2}{dx^2} h(x,t) \, dB_t^2$$

We know from the quadratic variation of brownian motion that $dB_t^2 = dt$ and the derivatives equal.

$$\frac{d}{dt} h(x,t) = 0 \quad , \quad \frac{d}{dx} h(x,t) = \cosh(x) \quad , \quad \frac{d^2}{dx^2} h(x,t) = \sinh(x)$$

So

$$\begin{aligned} dY_t &= 0 \, dt + \cosh(B_t) \, dB_t + \frac{1}{2} \sinh(B_t) \, dt \\ &= \frac{1}{2} \sinh(B_t) \, dt + \cosh(B_t) \, dB_t \end{aligned}$$

we find the inverse of $Y_t = \sinh(B_t) \Rightarrow B_t = \operatorname{arcsinh}(Y_t)$

We insert

$$\begin{aligned} dY_t &= \frac{1}{2} \sinh(\operatorname{arcsinh}(Y_t)) \, dt + \cosh(\operatorname{arcsinh}(Y_t)) \, dB_t \\ &= \underline{\underline{\frac{1}{2} Y_t \, dt + \sqrt{1 + Y_t} \, dB_t}} \end{aligned}$$

Consider the Itô SDE governing $\{X_t\}$, the abundance of bacteria in a population

$$dX_t = X_t(1 - X_t)\,dt + \sigma X_t\,dB_t$$

Q2 Using Itô's lemma to perform a coordinate transformation: Identify a Lamperti transform h, i.e. find a transformed coordinate $Y_t = h(X_t)$ such that the Itô equation for $\{Y_t\}$ has additive noise. Write up this Itô's equation.

On page 157 we are given the Lamperti transformation as

$$h(x) = \int^x \frac{1}{g(v)}\,dv \qquad (Y_t = h(X_t))$$

for a scalar SDE of the form

$$dX_t = f(X_t)\,dt + g(X_t)\,dB_t$$

then

$$\frac{d}{dt}h(x,t) = 0, \quad \frac{d}{dx} = \frac{1}{g(x)}, \quad \frac{d^2}{dx^2} = -\frac{g'(x)}{g^2(x)}$$

Itô's lemma then yields

$$dY_t = \frac{d}{dt}h(x,t)\,dt + \frac{d}{dx}h(x,t)\,dX_t + \frac{1}{2}\frac{d^2}{dx^2}h(x,t)\,dX_t^2$$

$$(f(X_t)dt + g(X_t)dB_t)^2 = f(X_t)^2dt^2 + 2f()g()dt\,dB_t + g(X_t)^2B_t^2 = g(X_t)^2B_t^2 \quad (\text{Because } dt^2 = 0,\ dt\,dB_t = 0)$$

$$= 0 + \frac{1}{g(x)}\left(f(X_t)dt + g(X_t)dB_t\right) - \frac{1}{2}g'(X_t)\,dt$$

$$= \left[\frac{f(h^{-1}(Y_t))}{g(h^{-1}(Y_t))} - \frac{1}{2}g'(h^{-1}(Y_t))\right]dt + dB_t$$

We now see that for our case $g(x) = \sigma x$ and $f(x) = x(1-x)$ so

$$h(x) = \int^x \frac{1}{\sigma v}\,dv, \quad \frac{d}{dx}h(x) = \frac{1}{\sigma x}, \quad \frac{d^2}{dx^2}h(x) = \frac{\sigma}{\sigma^2 x^2}$$

We get from maple that $h^{-1}(Y_t) = \exp(Y_t\sigma)$. We insert and get

$$dY_t = \left(\frac{1}{\sigma} - \frac{1}{\sigma}e^{Y_t\sigma} - \frac{\sigma}{2}\right)dt + dB_t$$

Consider the SDE

$$dX_t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X_t \, dt + \sigma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} dB_t \qquad (1)$$

with the initial condition $X_0 = x$. Here $X_t \in \mathbb{R}^2$ and $\{B_t : t \geq 0\}$ is two dimensional Brownian motion.

Q6 Verify that

$$\mathbb{E}[X_t] = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x \, , \qquad \forall t$$

· We know Itô processes are martingale and hence (1) is martingale. It then follows that the expectation of X_t is just the solution to the deterministic part of (1), i.e.

$$\frac{d}{dt} X_t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X_t \quad \Rightarrow \quad X_t = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

see solution of a linear ODE.

Q7 Write $S_t = \|X_t\|^2$ as an Itô process and find $\mathbb{E}[S_t]$ as a function of t.

- We are given the function $h(X) = \|X_t\|^2 = \left(\sqrt{x_1^2 + x_2^2}\right)^2 = x_1^2 + x_2^2$

We use the multivariat Itô's lemma from thm 7.3.1 to find dS_t.

$$dY_t = \dot{h}\, dt + \left(\nabla h\, F_t + \frac{1}{2} \operatorname{tr} G_t^T Hh\, G_t\right) dt + \nabla h\, G_t\, dB_t$$

We calculate

$$\dot{h} = 0 \quad , \quad \nabla h = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \quad , \quad Hh = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$F_t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X_t \,, \quad G_t = \sigma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We insert

$$dS_t = \left(2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2} \operatorname{tr}\left(\begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \right)\right) dt$$

$$+ 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} dB_t \qquad \left(dB_t = \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}_t, \text{ it is given in Q6}\right)$$

$$= 2\sigma^2 dt + 2\sigma\left(x_1 dB_1 + x_2 dB_2\right) \qquad (1)$$

We know (1) is a martingale because it is a Itô process so

$$\mathbb{E}[S_t] = \int 2\sigma^2 dt = 2\sigma^2 t$$

Q8 Pose and solve the differential Lyapunov equation governing the variance-covariance matrix of X_t.

The differential Lyapunov equation is given in eq 5.21 and again in ex 5.8. Here we state it again.

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A^T + GG^T \qquad (1)$$

with initial condition $\Sigma(0) = Q = Q^T$, governing the variance-covariance matrix of a linear system $dX_t = AX_t dt + G dB_t$.

We now must solve the system

$$AS + SA^T + GG^T = 0 \qquad (2)$$

to solve (1) by

$$\Sigma(t) = S - e^{At}(S - Q)e^{A^T t}$$

We can use Sylvesters equation to solve (2)

$$AS + SA^T = -GG^T \Rightarrow (I \otimes A + A \otimes I)\,\mathrm{vec}(X) = \mathrm{vec}(-GG^T)$$

We have $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$ so we try to Solve it with maple.

We though see that the determinant of $(I \otimes A + A \otimes I)$ is 0 so the system is singular and can't be solved. The reason why the Sylvester approach does not work is that it relays on the system having a steady state and our does not. We hence must use a different approach. In the solution they guess it but we can also just realize that

$$= \mathbb{E}[X_t X_t^T] = \mathbb{E}[\mathrm{tr}(X_t X_t^T)] = \mathrm{tr}(\mathbb{E}[X_t X_t^T]) = \mathrm{tr}\,\Sigma_t$$

We also know that $\mathbb{E}[S_t] - \mathbb{E}[X_t X_t^T] = 2\sigma^2 t$ (from Q7)

and the off diagonal is zero so

$$\Sigma_t = \begin{bmatrix} \sigma^2 t & 0 \\ 0 & \sigma^2 t \end{bmatrix}$$