On Homoclinic Orbits to Center Manifolds in Hamiltonian Systems

A thesis submitted to Imperial College London

for the degree of

Doctor of Philosophy

by

William Giles

Department of Mathematics

Imperial College

180 Queen's Gate, London SW7 2BZ

JANUARY 24, 2015

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Signed: William Giles

Copyright

Abstract

The objects of study in this thesis are Hamiltonian systems of ordinary differential equations possessing homoclinic orbits. A homoclinic orbit is a solution of the system which converges to the same invariant set as time approaches both positive and negative infinity. In our case, the invariant set in question is assumed to be a nonhyperbolic equilibrium state. Such an equilibrium state possesses a center manifold, containing all orbits which remain close to the equilibrium. We are concerned with finding orbits which converge to orbits in the center manifold in both time directions.
We consider firstly the case in which the nonhyperbolic eigenvalues at the equilibrium consist of pairs of nonzero purely imaginary eigenvalues. We study the set of homoclinics to the center manifold by constructing an operator on a suitable function space whose zeros correspond to homoclinics. We use a Lyapunov-Schmidt technique to reduce the problem to that of studying the zero set of a real-valued function defined on the center manifold, which has a critical point at the origin. A formula is found for the Hessian matrix at this critical point, involving the so called scattering matrix. Under nonresonance and nondegeneracy conditions, we characterise the possible Morse indices of the Hessian, permitting an application of the Morse lemma to describe the set of homoclinics. We also consider special cases, including reversible systems.
We then consider a more geometric approach to the problem, allowing us to define a nonlinear analogue of the scattering matrix using stable and unstable foliations of the invariant manifolds. We use this approach to unfold the system in parametrised families - we consider here also the case of a two dimensional center manifold corresponding to zero eigenvalues - bifurcation diagrams are produced for homoclinics

to the origin in this case. The effects of additional reversible structure are again considered.

Table of contents

Abstract 4

1 Introduction 9

2 Preliminaries 17
- 2.1 Hamiltonian systems 17
- 2.2 Reversible systems 19
- 2.3 Homoclinic orbits 19
- 2.4 Invariant foliation theorems 23

3 A Melnikov-type quadratic form. 26
- 3.0.1 Problem setting 28
- 3.0.2 Statement of results 30
- 3.0.3 The Lyapunov-Schmidt reduction 33
- 3.1 Weighted function spaces and Fredholm properties 35
- 3.1.1 The Hessian Matrix 42
- 3.2 The scattering matrix 46
- 3.2.1 Indefiniteness of the Hessian 48
- 3.2.2 A different proof 50
- 3.2.3 Scattering matrix is nonhyperbolic 53
- 3.3 Near-identity scattering matrices 54
- 3.3.1 An example with $\sigma = I$ 54
- 3.3.2 Perturbation result 57
- 3.4 All indefinite signatures are possible 58
- 3.5 Special case 65
- 3.5.1 The reversible Hamiltonian case 65
- 3.6 Discussion 67

4 A type of scattering map 70
- 4.1 Finding semi-orbits with a jump in energy 72
- 4.1.1 $X^{\pm}$ are symplectic and transverse to the leaves of the foliations. 74
- 4.2 Reversible systems 78

5 Applications of the scattering technique **83**
5.1 Unfolding the saddle center loop 83
5.1.1 Expansion of the bifurcation function. 84
5.1.2 Bifurcation diagram. 86
5.2 A homoclinic to a bifurcating equilibrium 87
5.2.1 Setup 87
5.2.2 Bifurcation curves 92

Discussion **101**

A Codimension calculation **106**

References **110**

Acknowledgements

There are many people who have helped me write this thesis, in many different ways.

Firstly I would like to thank my supervisors Jeroen Lamb and Dmitry Turaev for their unfailing support, encouragement and generosity throughout my time as a PhD student. I have learned a great deal from both of you and I am extremely grateful.

I thank all of my colleagues from the dynamical systems group at Imperial college, in particular Kevin Webster for the help and advice he has given me. My fellow PhD students I thank for their wise words and good humour, especially Mark Callaway. Also, long live Nerv.

I am hugely grateful to my parents, my brother Tom and my nephew Finn, for their unconditional support. Finally I thank Sam Marsh for her patience and kindness, and for believing in me.

Chapter 1

Introduction

In this thesis we study Hamiltonian systems of ordinary differential equations which possess homoclinic loops to nonhyperbolic equilibria. A homoclinic loop is an orbit of the system which converges to the same invariant set in both forward and backward time directions. Homoclinic orbits play a key role in understanding the existence and nature of chaotic behaviour in dynamical systems, since they are often accompanied by intricate recurrent structures which lead to extreme sensitivity of the system to very slight changes [69]. For a recent survey of progress in the field of homoclinic (and heteroclinic) bifurcation theory, see the article [39] by Homburg and Sandstede.

Close to nonhyperbolic equilibria there are collections of solutions characterised by their slow asymptotic behaviour. These solutions form the so called center manifold of the equilibrium. Given the existence of a homoclinic loop to the equilibrium, it is natural to ask whether there are necessarily other orbits close to this loop which converge to other solutions in the center manifold. If the system additionally depends on parameters, we can ask questions regarding bifurcation of homoclinic orbits, that is, how these connections are created or destroyed as the parameters are varied.

From a geometric point of view, such solutions define the intersection of the center stable and center unstable manifolds of the equilibrium. The Hamiltonian (and, at times, reversible) nature of the systems we consider means that these invariant manifolds themselves possess structure, and so their intersections cannot be arbitrary. As such, our question is really to discover how these properties of the system manifest themselves, and what restrictions they place on the possible homoclinic intersections.

Following this introduction, the chapters of this thesis are organised as follows;

In chapter 2 we take the chance to briefly recap some definitions and fix notation for the rest of the thesis, as well as describing some consequences of Hamiltonian and reversible structure which are relevant for our consideration of homoclinics.

The contribution of the thesis begins in chapter 3, where we consider the case of an elliptic-hyperbolic equilibrium, that is, an equilibrium which has a number of pairs of nonzero purely imaginary eigenvalues and at least one further hyperbolic degree of freedom. We seek homoclinics close to the assumed principal homoclinic $\gamma(t)$ as zeros of an operator between weighted function spaces, and use the Lyapunov-Schmidt technique to derive a finite dimensional reduced problem. This leads us to studying the zero set of a real valued function, which has a critical point at the origin. We then take a singularity theoretic approach to the analysis of this reduced equation, studying the possible signatures of the Hessian matrix at the singularity, this being the most basic information which could yield a description of the zero set in a neighbourhood of the origin, for example by the Morse lemma. As it turns out we can express the Hessian matrix in terms of the Hessian of the Hamiltonian restricted to the center manifold, and another matrix which is derived from the dynamics of the variational equation along the homoclinic. We refer to this second matrix, which is symplectic, as the scattering matrix. Using this representation of the Hessian we prove 2 results (theorems 3 and 4) about its possible signatures. The second of these theorems describes the case in which the system is reversible and the homoclinic $\gamma(t)$ is symmetric.

In chapter 4 we construct a map from the center manifold to itself using the invariant foliations of the center stable and center unstable foliations, together with another application of the Lyapunov-Schmidt reduction. This nonlinear map S inherits the symplectic character of the flow and is useful in finding homoclinics to the center manifold; a homoclinic orbit corresponds to an intersection of a level set in the center manifold with its image under S. In the reversible case this map also inherits reversibility.

We then use this tool in chapter 5 to study unfoldings of systems with homoclinic orbits to nonhyperbolic equilibria. We firstly unfold the Morse singularity studied in chapter 3 in the simplest case of a single pair of imaginary eigenvalues. We then consider a reversible Hamiltonian system with a symmetric homoclinic to an equilibrium undergoing a reversible transcritical bifurcation. Using the reversibility of the scattering map S, we describe all one-round homoclinics to the origin, by finding the intersections of the zero level set with its image by S. This completes a theorem from [82], by describing the bifurcation of nonsymmetric homoclinics as well as symmetric ones.

A short discussion follows, outlining some interesting unanswered questions and possible future research directions related to the work in this thesis.

In the remaining part of this introductory section, we discuss some background for the problems and techniques studied in this thesis, making reference to related work in the literature, to guide the interested reader in their further investigations.

The techniques employed in this thesis are, for the most part, analytical in nature. They are based around extensions of the so called *Melnikov* theory. Melnikov[1] [63] developed a criterion for studying the splitting of homoclinic loops to hyperbolic equilibria in planar systems subject to periodic forcing. His technique provided a way of verifying the existence of chaotic dynamics. In the periodically forced system, the hyperbolic equilbrium of the unperturbed system becomes a hyperbolic fixed point of the Poincaré map (or 'period map'). Melnikov provided a function defined using an integral along the unperturbed homoclinic, which measures the splitting of the invariant manifolds of this periodic orbit to first order in the perturbation parameter. A nondegenerate zero of this function implies a point of transversal intersection of the manifolds, which then, via the Smale-Birkhoff-Shilnikov theorem, implies the existence of a horseshoe for the Poincaré map, and thus chaotic dynamics for the full system. A very readable account of the basic Melnikov approach is given in the textbook [15].

[1] Melnikov's integral was apparently also known to Poincaré

Similar techniques have been generalised to higher dimensional systems by numerous authors. In the Hamiltonian context, most work is focussed on the near-integrable case, for instance [2][2], [36], [72], [19]. In a completely integrable system, the existence of many conserved quantities usually forces intersections of invariant manifolds to be higher dimensional than in a general system. The idea is to measure the splitting of the manifolds under perturbation using the values of conserved quantities of the unperturbed system. The problem can be studied in the context of maps as well as flows, see the survey [22] and references therein. Further references are found in chapter 3.

However, the Melnikov type methods in principal do not require such additional geometric structure, and can be applied in general systems. One approach to the problem, originating (to the best of my knowledge) with Chow, Hale and Mallet-Paret [16], is functional analytic. Orbits in a tubular neighbourhood of the principal homoclinic, which we call $\gamma(t)$, are viewed as small perturbations of $\gamma(t)$;

$$\tilde{\gamma}(t) = \gamma(t) + v(t). \tag{1.1}$$

For such a curve $\tilde{\gamma}(t)$ to define a homoclinic to the equilibrium (or, as in this work, to the center manifold), a suitable $v(t)$ is found as a zero of a related operator defined on a suitably chosen function space. This operator equation can then be simplified to a finite dimensional equation by a reduction procedure based on an application of the implicit function theorem, referred to as Lyapunov-Schmidt reduction. Since we generally make no assumption of near-integrability in this thesis, the geometry of the problem is less restricted - as a consequence the Melnikov theory we employ in this thesis has most in common with this Lyapunov-Schmidt approach, also developed in papers by Gruendler [29] who studied loops to hyperbolic equilibria in general systems, Palmer [70] who considered periodic forcing (see also [5]), and latterly Yagasaki [86], who studied periodic perturbations of Hamiltonian systems with elliptic-hyperbolic equilibria, whose invariant manifolds are permitted to intersect in a degenerate man-

[2]Due to Arnold's contribution in this paper, some authors refer to the *Poincaré-Melnikov-Arnold* method

ner. Another author who made use of the Lyapunov-Schmidt technique in homoclinic and heteroclinic bifurcation theory is Lin [57, 58] whose methodology was further developed by Sandstede [73].
Others have used the Lyapunov-Schmidt reduction in conjunction with the variational structure of Hamiltonian systems to study homoclinics in a similar perturbative setting, see the papers of Ambrosetti & Badiale [4], Berti & Bolle [6] and Berti & Carminati [6]. The problem of finding homoclinics near a given homoclinic manifold (that is, a manifold of trajectories each of which is a homoclinic - usually occurring due to some symmetry) is reduced to that of finding critical points of a reduced version of the action functional, defined on a finite dimensional space whose topology is governed by that of the homoclinic manifold. Topological arguments are then applied to provide estimates on the number of critical points this reduced functional will have. In the planar periodically forced case, Angenent [1] used a variational approach to provide estimates on the size of the splitting of a loop.

In the book [31] by Haller and the papers [32, 34, 33, 30, 50, 51, 49, 52, 62] by Haller, Wiggins, Kovacic and others, homoclinic and heteroclinic behaviour in near integrable systems is studied in the context of *resonances*. Resonances are relationships between frequencies of angular variables which lead to the existence of invariant sets in phase space (for example, a circle or torus of fixed points). In applications one sometimes encounters systems which can be written as a a small perturbation of an integrable system which contains resonant structures connected by manifolds of homoclinic or heteroclinic orbits. The resonant invariant sets are found to be contained in normally hyperbolic invariant manifolds, which persist when the system is perturbed slightly (often in the systems studied here, the perturbation is dissipative, so the conservative nature of the unperturbed system is broken, e.g [49], [52]). By the theory of Fenichel, these normally hyperbolic invariant manifolds possess their own stable and unstable manifolds, and the authors use this theory, together with the geometry of the unperturbed phase space to derive conditions under which chains of unperturbed connecting orbits can be ‘glued’ after perturbation to produce multipulse homoclinics and heteroclinic chains - the methods applied here fall into the category of ‘geometric singular perturbation theory’ - for a summary see [42], [43].

Surviving homoclinic orbits to the invariant manifolds are often found to have properties similar to those studied by Shilnikov [78], leading to the existence of nearby chaotic dynamics. A variant of the Melnikov method is also developed in [9] for detecting homoclinic orbits with multiple excursions from a hyperbolic manifold in near integrable systems

One interesting property possessed by many important examples of Hamiltonian systems is *reversibility*, characterised by the existence of a transformation R with $R^2 = Id$, which maps orbits of the system onto other orbits, while reversing the direction in which the orbits proceed as time progresses. This means that orbits are either *symmetric*, meaning that they are fixed set-wise by R, or occur in pairs. The canonical example is a so called *natural* Hamiltonian, of the form;

$$H(q,p) = \frac{p^2}{2} + V(q).$$

These are ubiquitous in classical mechanics; the p^2 term corresponds to kinetic energy while the function V of the position variables describes the potential energy of the system - the vector field generated by H is reversible with repect to the transformation $R(q,p) := (q,-p)$. Precise definitions regarding reversibility are given in chapter 2. Although reversible structure is distinct from being Hamiltonian, vector fields on even-dimensional phase spaces which are reversible, with a reversing symmetry whose fixed point space has half the total dimension, enjoy many properties similar to those of Hamiltonian vector fields, although there are some important differences, see for instance the paper [23] by Devaney. For one example of a fundamental similarity in the context of homoclinic behaviour, Fiedler and Vanderbauwhede prove in [81] that symmetric orbits in reversible systems which are homoclinic to hyperbolic fixed points and satisfy a nondegeneracy condition, are persistent under sufficiently small parameter variations, and accompanied by a family of periodic orbits which accumulates on the homoclinic itself. The same result is also shown to hold for nondegenerate homoclinics in conservative (and hence Hamiltonian) systems. A survey article on homoclinic behaviour in reversible systems is [11]. All of the systems considered in this thesis are Hamiltonian and thus reversibility, when it appears, is only considered as an additional feature. This means that we cannot 'compare' Hamiltonian structure with reversibility directly in our results. However, in chapter 3, when we study a certain quadratic form whose zero set yields information regarding the homoclinic connections we seek, we are able to observe that the extra hypothesis of being reversible can be quite restrictive, in that much of the structure which is possible in a purely Hamiltonian system is ruled out in the reversible Hamiltonian case. We also note that these differences only emerge in higher dimensional systems.

Another common motivation for studying homoclinic solutions of ODEs actually comes from PDEs. An important class of solutions to partial differential equations are the so called travelling waves. A travelling wave is a solution $v(x, t)$ of a PDE of the form $v(x, t) = \psi(\xi)$ where the variable $\xi := x - ct$ combines the spatial and temporal variables, with c representing the speed of the wave. Substitution of this ansatz for v into the PDE leads to a system of ODEs in the independent variable ξ whose solutions describe profiles for waves, with a homoclinic solution describing a spatially localised moving pulse. A well known example of this theory comes from the study of water waves; the PDE in question is a version of the Korteweg de Vries (KdV) equation, and the travelling waves ODEs inherit Hamiltonian and reversible structure from the PDE, see [12] and references therein. Homoclinic solutions to nonhyperbolic equilibria of the kind considered in this thesis, when encountered in this context can describe so called 'embedded solitons', see [13]. In section 5.2 we study the bifurcations from a homoclinic orbit to an equilibrium in a reversible Hamiltonian system undergoing a transcritical bifurcation. This is motivated in part by considerations from a paper by Wagenknecht and Champneys [82], (see also [83]), where the same scenario is considered, but only symmetric (with respect to the reversing symmetry) bifurcating homoclinic solutions are sought. We add detail to the bifurcation diagrams found in those references by considering also the nonsymmetric solutions, taking into account the Hamiltonian character of the equations. We are also able to relate these results to numerical studies of the KdV system, mentioned above.

Whilst on this subject, we briefly mention one motivation for the approach taken in chapter 3 of this thesis; the study of travelling waves in lattice systems. One way to view lattice systems is to think of a partial differential equation where the spatial variable has been discretised, resulting in an infinite set of coupled ODEs, indexed by the spatial coordinate. In the lattice context, as a consequence of the discrete spatial variable, following the usual ansatz procedure described above leads not to a system of ODEs but instead to a system of so called 'advance delay equations' [3]. Initial value problems for advance delay equations tend to be ill-posed, meaning that many of the flow-based techniques available in the ODE context are not applicable. However, the fundamental techniques employed in chapter 3 in the study of the variational equation (namely the link between exponential dichotomies and Fredholm operators) have been proven to still hold in the advance delay context, see [59], [76]. As a consequence, I believe that the results derived in that chapter could also be applied in the advance delay case, to study travelling waves in lattice systems. However, at the time of writing I have not proven that such a generalisation is valid and as such it represents future work.

[3] also called 'functional differential equations of mixed type'

Chapter 2

Preliminaries

The purpose of this section is to set notation, recall some facts and provide some useful references regarding Hamiltonian and reversible systems and homoclinic orbits.

2.1 Hamiltonian systems

In this report, we consider Hamiltonian dynamics described by an ordinary differential equation

$$\dot{u} = X_H(u, \mu) \tag{2.1}$$

where $u \in \mathbb{R}^{2n}$, and the parameter $\mu \in \mathbb{R}^k$. The right hand side of (2.1) is a *Hamiltonian vector field* which is defined using a smooth (at least C^2) function $H : \mathbb{R}^{2n} \to \mathbb{R}$ which we call the *Hamiltonian*, and a skew-symmetric nondegenerate bilinear form ω referred to as the *symplectic form*. The vector field is then defined via the relation

$$\omega(X_H(u), v) = DH(u) \cdot v \quad \text{for each } u, v \in \mathbb{R}^{2n}. \tag{2.2}$$

Letting $\mathbb{J}$ be the matrix satisfying $\omega(u, v) = \langle u, \mathbb{J}v \rangle$ for every $u, v \in \mathbb{R}^{2n}$, we have that

$X_H(u) = \mathbb{J}\nabla H(u)$. Note that we can always find a basis in which $\mathbb{J}$ takes the form

$$\mathbb{J} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix.

For a subspace $V \in \mathbb{R}^{2n}$, we write $V^\omega := \{u \in \mathbb{R}^{2n} : \omega(v, u) = 0 \quad \forall v \in V\}$, and we call V^ω the *symplectic orthogonal complement* of V. The subspace V is called *isotropic* if $V \subset V^\omega$, *coisotropic* if $V^\omega \subset V$, *Lagrangian* if $V^\omega = V$ and *symplectic* if $V^\omega \cap V = \{0\}$. A submanifold is called isotropic, coisotropic, Lagrangian or symplectic if its tangent space at every point satisfies the corresponding condition as a subspace of $\mathbb{R}^{2n}$.

Let us consider a nonhyperbolic equilibrium state, that is, a zero of the Hamiltonian vector field at which the spectrum of the linearisation $DX_H(0)$ consists of $2l$ eigenvalues (counted with multiplicity) with zero real part for some $l > 0$, and $2(n-l)$ eigenvalues λ_i, whose real parts are bounded away from zero. Note that because of the Hamiltonian structure, if the matrix $DX_H(0)$ has a zero eigenvalue then it occurs with even multiplicity, so that the dimension of the *center subspace*, that is the direct sum of generalised eigenspaces associated to eigenvalues with zero real part, is even dimensional. This structure also dictates that of the remaining hyperbolic eigenvalues, $n-l$ will have positive real part and $n-l$ will have negative real part. The standard invariant manifold theorems (see eg. [15], [77]) guarantee the existence of a $2l$ dimensional center manifold $W^c(0)$, as well as $n-l$ dimensional stable and unstable manifolds $W^s(0)$ and $W^u(0)$, and $n+l$ dimensional center stable and center unstable manifolds $W^{cs}(0)$ and $W^{cu}(0)$. In a Hamiltonian system, the center manifold is symplectic, while $W^s(0)$ and $W^u(0)$ are isotropic, and $W^{cu}(0)$, $W^{cs}(0)$ are coisotropic see [65]. Also, the restriction of the vector field X_H to the center manifold also defines a Hamiltonian system.

The following textbooks represent good general introductory references for Hamiltonian systems [64], [3], [60].

2.2 Reversible systems

A vector field $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called *reversible* if there exists a linear involution R (i.e $R^2 = Id$) with $\dim(Fix(R)) = \dim\{x \in \mathbb{R}^{2n} : Rx = x\} = 2n$ such that

$$f(Rx) = -Rf(x)$$

A consequence of this definition is that $R \circ \phi^t \circ R = \phi^{-t}$, where ϕ denotes the flow of f. An orbit $\mathfrak{O}(x) = \{\phi^t(x) : t \in \mathbb{R}\}$ is called *symmetric* if $R\mathfrak{O}(x) = \mathfrak{O}(x)$. We can deduce that if p is a symmetric equilibrium with both hyperbolic and nonhyperbolic eigenvalues, then $RW^s(p) = W^u(p)$, and, although the center stable and center unstable manifolds are in general not unique, for any choice of $W^{cs}(p)$, we can take $RW^{cs}(p) = W^{cu}(p)$, which means that, since $W^c(p) = W^{cs}(p) \cap W^{cu}(p)$ we get $RW^c(p) = W^c(p)$. Also, the restriction $f|_{W^c(p)}$ of f to the center manifold is a reversible vector field [41].

We will always demand that the reversing symmetry R acts *anti-symplectically*, meaning that $R\mathbb{J} = -\mathbb{J}R$. This has the consequence that R preserves the level sets of the Hamiltonian. This is the usual scenario in a reversible Hamiltonian system, but there are cases in which it is of interest to consider a reversing symmetry which acts symplectically, and there are consequences for the dynamics and bifurcations, see for example [8]. Under that assumption that R acts antisymplectically, it is possible to make a symplectic change of coordinates so that R is represented by an orthogonal matrix; the idea is that R and $\mathbb{J}$ generate a finite group, and the desired coordinate change comes from averaging the inner product over this finite group, see appendix B of [40].

2.3 Homoclinic orbits

For the purposes of this thesis, an orbit will be called *homoclinic* if it has the same invariant set as its alpha and omega limit set.

Hence, a homoclinic orbit lies in an intersection of invariant manifolds. Whenever we assume the existence of a homoclinic $\gamma(t)$ to a nonhyperbolic equilibrium at the origin, we require that it satisfies a nondegeneracy condition of the form

$$\dim(T_{\gamma(0)}W^{cu}(0) \cap T_{\gamma(0)}W^{s}(0)) = \dim(T_{\gamma(0)}W^{cs}(0) \cap T_{\gamma(0)}W^{u}(0)) = 1.$$

Of course, the existence of the homoclinic $\gamma(t)$ which is contained in the intersection of W^u and W^s implies that the dimension of this intersection is at least one (the vector field direction), so this condition means that this dimension is minimal: there is no further degeneracy leading to a higher dimensional intersection. This assumption of nondegeneracy allows us to write the tangent space at $\gamma(0)$ according to the following decomposition (see also [44], [83]),

$$\mathbb{R}^{2n} = \operatorname{span}\{\dot{\gamma}(0)\} \oplus Y^s \oplus Y^u \oplus Y^c \oplus Z := \operatorname{span}\{\dot{\gamma}(0)\} \oplus \Sigma$$

where Y^s, Y^u and Y^c are subspaces satisfying

$$\begin{aligned}
\operatorname{span}\{\dot{\gamma}(0)\} \oplus Y^c &= T_{\gamma(0)}W^{cu}(0) \cap T_{\gamma(0)}W^{cs}(0) \\
\operatorname{span}\{\dot{\gamma}(0)\} \oplus Y^s &= T_{\gamma(0)}W^{s}(0) \\
\operatorname{span}\{\dot{\gamma}(0)\} \oplus Y^u &= T_{\gamma(0)}W^{u}(0).
\end{aligned}$$

Let us explore the implications of Hamiltonian and reversible structure on the subspaces involved in this decomposition. The linearised flow along the homoclinic orbit defines the variational equation

$$\dot{x}(t) = DX_H(\gamma(t))x(t).$$

We note that the evolution of this nonautonomous linear system preserves the symplectic form.

We note that by counting dimensions, $\dim Y^c = 2l$, where (as previously) $2l$ is the number of nonhyperbolic eigenvalues at the equilibrium, and $\dim Z = 1$. The subspaces in this decomposition can also be related to the projections which make up *exponential trichotomies* on $\mathbb{R}^{\pm}$. Recall that a linear nonautonomous equation $\dot{x}(t) = A(t)x(t)$ with transition matrix $\Phi(\cdot,\cdot)$ has an exponential trichotomy on $\mathbb{R}^+$

if there exists a family of projections $P_s^+(t)$, $P_c^+(t)$, $P_u^+(t)$, $t \in \mathbb{R}^+$, with $P_s^+(t) + P_c^+(t) + P_u^+(t) \equiv Id$, and constants $\alpha_s < -\alpha_c < 0 < \alpha_c < \alpha_u$ and $K > 0$ such that

$$\Phi(t,\tau)P_i^+(\tau) = P_i^+(t)\Phi(t,\tau), \;\; i = s, c, u$$

and

$$\begin{gathered}\|\Phi(t,\tau)P_s^+(\tau)\| \leq Ke^{\alpha_s(t-\tau)}\;, \|\Phi(t,\tau)P_c^+(\tau)\| \leq Ke^{\alpha_c(t-\tau)},\; t \geq \tau \\ \|\Phi(t,\tau)P_c^+(\tau)\| \leq Ke^{-\alpha_c(t-\tau)}\;, \|\Phi(t,\tau)P_u^+(\tau)\| \leq Ke^{\alpha_u(t-\tau)},\; \tau \geq t.\end{gathered}$$

The definition for a trichotomy on $\mathbb{R}^-$ is similar. It is proven in [44] (lemma 2.1) that the variational equation along $\gamma(t)$ has exponential trichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. In fact, the images of these projections can be interpreted as the tangent spaces of the invariant manifolds, so that we have

$$\mathcal{R}(P_s^+(0)) = T_{\gamma(0)}W^s, \;\; \mathcal{R}(P_u^-(0)) = T_{\gamma(0)}W^u$$

and in fact,

$$\mathcal{R}(P_c^+(0)) \cap \Sigma = \mathcal{R}(P_c^-(0)) \cap \Sigma = Y^c.$$

These facts are useful in proving the following proposition.

Proposition 2.3.1. *Letting $\mathcal{E}_0$ denote the energy level of the origin, we have*

$$(\mathrm{span}\{\dot{\gamma}(0)\} \oplus Y^s \oplus Y^u \oplus Y^c) = T_{\gamma(0)}\mathcal{E}_0$$

Proof. The normal to the energy surface is given by $\nabla H(\gamma(0))$. Using the compatibility of the symplectic form with the inner product,

$$\langle \nabla H(\gamma(0)), v\rangle = \omega(\dot{\gamma}(0), v).$$

Clearly if $v \in \mathrm{span}\{\dot{\gamma}(0)\}$, then this gives $\langle \nabla H(\gamma(0)), v\rangle = 0$. For $v \in Y^s$, v is an initial condition of the variational equation along $\gamma(t)$, which, as a consequence of the exponential trichotomy, decays exponentially fast to zero in forward time.

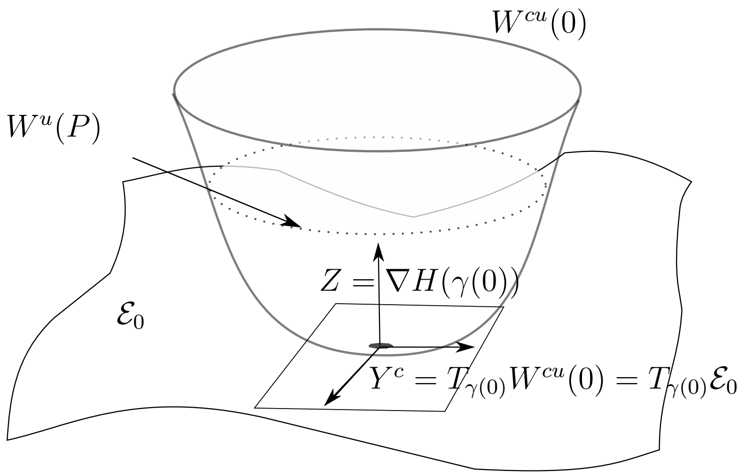


Figure 2.1: *The picture inside a section Σ in the 2 d.o.f case. The dashed ellipse is the unstable manifold of a periodic orbit P from the center manifold. Note that in this lowest dimensional case, $Y^s = Y^u = \{0\}$.*

Hence, letting $\Phi(t, 0)$ denote the solution operator for the variational equation, since Φ preserves ω, we have

$$0 = \lim_{t\to\infty} \omega(\Phi(t, 0)\dot{\gamma}(0), \Phi(t, 0)v) = \omega(\dot{\gamma}(0), v) = \langle \nabla H(\gamma(0)), v\rangle . \tag{2.3}$$

The same conclusion is reached for $v \in Y^u$ by letting time tend to negative infinity. For $v \in Y^c$, we see from the exponential trichotomy that any growth of $\Phi(t, 0)v$ (in either time direction) is dominated by the faster exponential convergence of $\Phi(t, 0)\dot{\gamma}(0)$ and so (2.3) holds again. □

This proposition implies that we can choose $Z = \text{span}\{\nabla H(\gamma(0))\}$. This in particular means that Z is foliated by the level curves of H. The geometry of this proposition is illustrated in the lowest-dimensional case in figure 2.1. Arguments of this form can also be used to deduce that Y^c is symplectic, Y^u and Y^s are isotropic, and $Y^s \oplus Y^u$ is symplectic, which means in particular that we can choose $Y^u = \mathbb{J}Y^s$. Note also that the symplectic orthogonal complement Z^ω of Z is equal to Σ. This knowledge allows us to prove;

Proposition 2.3.2.

$$\dim(T_{\gamma(0)}W^{cu} \cap T_{\gamma(0)}W^{s}) = 1 \tag{2.4}$$

$$\Rightarrow \dim(T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^{u}) = 1 \tag{2.5}$$

Proof. We note that assumption (2.4) implies that $Y^c \cap Y^s = \{0\}$. Since Y^c is symplectic, $\mathbb{J}Y^c = Y^c$, and we have chosen $\mathbb{J}Y^s = Y^u$, so

$$Y^c \cap Y^u = \mathbb{J}Y^c \cap \mathbb{J}Y^s = \mathbb{J}(Y^c \cap Y^s) = \{0\}.$$

This implies (2.5): if there were an element $v \in (T_{\gamma(0)}W^{cs}(0) \cap T_{\gamma(0)}W^u) \setminus \text{span}\{\dot{\gamma}(0)\}$, then by definition v would be in $Y^c \cap Y^u$. □

So, assuming nondegenerate intersection of W^{cu} with W^s along $\gamma(t)$ also gives nondegeneracy for W^{cs} with W^u, and this argument can be reversed. The same situation is true in reversible systems with a symmetric homoclinic, simply because of the symmetry relations between the invariant manifolds; applying R to in the intersection in equation 2.4 gives the intersection in equation 2.5.

In the case that the system is reversible, and $\gamma(t)$ is symmetric, then it intersects $\text{Fix}(R)$ in a single point (see eg [81]), WLOG $\gamma(0)$. If the inner product $\langle\cdot,\cdot\rangle$ is R-invariant (as can be achieved via a symplectic transformation, as mentioned previously), we can further choose that $Y^s \perp \text{span}\{\dot{\gamma}(0)\}$. We get also $Y^u \perp \text{span}\{\dot{\gamma}(0)\}$ because we chose $\mathbb{J}Y^s = Y^u$, and $Y^s \perp^\omega \text{span}\{\dot{\gamma}(0)\}$, and now also $RY^s = Y^u$. By construction $RY^c = Y^c$.

2.4 Invariant foliation theorems

In this section we state a theorem about the existence of invariant foliations of the center stable and center unstable manifold. We use these foliations in chapter 4 to construct a map which will help us identify homoclinic orbits to the center manifold.

Theorem 1 ([77])**.** *Let the system be C^r- smooth, $1 \leq r < \infty$* [1]*. In a small neigh-*

[1] If the system is C^∞, then any finite degree of smoothness can be achieved for the invariant

bourhood of the origin 0 *there exists an* $(n+l)$*-dimensional invariant center stable manifold* $W^{cs}_{loc}(0) = \{z = \psi^{cs}(x,y)\}$ *of class* C^r*, which contains* 0 *and which is tangent to the subspace* $\{z = 0\}$ *at* 0*. The manifold* $W^{cs}_{loc}(0)$ *contains all the trajectories which stay in a small neighbourhood of* 0 *for all positive times. Though the center stable manifold is not defined uniquely, for any two manifolds* $W^{cs}_1(0)$*,* $W^{cs}_2(0)$*, the functions* ψ^{cs}_1*,* ψ^{cs}_2 *have the same derivatives to order* r *at each point whose trajectory stays in a small neighbourhood of* 0 *for all* $t \geq 0$*.*
On $W^{cs}_{loc}(0)$ *there exists a* C^{r-1}*-smooth invariant foliation* $\mathcal{F}^{ss}$ *with* C^r *smooth leaves transverse to the center manifold* $W^c(0)$*; for each point whose forward trajectory stays in a small neighbourhood of* 0*, the corresponding leaf is uniquely defined by the system.*

The foliation mentioned in this theorem decomposes the centre stable manifold into submanifolds (the 'leaves') parametrised by points on the centre manifold itself. A point $v \in W^{cs}_{loc}(0)$ lies in the leaf $l^{ss}(x)$ above $x \in W^c_{loc}(0)$ if its orbit converges to the orbit of x at a given exponential rate, in our case;

$$\|\phi_t(v) - \phi_t(x)\| \leq Ke^{-\alpha t}, \text{ as } t \to \infty,$$

where $0 < \alpha < |\Re\{\lambda_1\}|$, with λ_1 the leading hyperbolic eigenvalue at the origin as defined above.

Note also that according to the theorem, the stable leaves above points in the centre manifold are only uniquely defined for those points whose orbits remain in a small neighbourhood of the origin for all positive time. We can modify the dynamics on the center manifold in a neighbourhood of the origin, using a smooth cutoff function, so that all orbits remain in a given neighbourhood for all time. Letting $f(q,p)$: be a C^∞ cutoff function such that $f(q,p) \equiv 1$ for $\sqrt{q^2+p^2} \leq C_1$ and $f(q,p) \equiv 0$ for $\sqrt{q^2+p^2} \geq C_1 + C_2$. We write

$$H^c_f(q,p) = H^c(f(q,p)q, f(q,p)p)$$

for the modified Hamiltonian. For the modified Hamiltonian, the dynamics within the

manifolds in a sufficiently small neighbourhood of zero, although in general not C^∞, see also [80] ball of radius C_1 are unchanged. Outside of the ball of radius $C_1 + C_2$, the dynamics are stationary, and in between, the smooth decrease in value of the cutoff function causes orbits to slow down, interpolating between the original and the stationary dynamics. Since the dynamics on the center manifold are now stationary outside of a neighbourhood of the origin, all trajectories beginning in this neighbourhood must remain there forever. This means that the stable and unstable foliations are uniquely defined for all points, which will be of considerable use in what follows. Of course, we can choose the neighbourhood so that the equilibrium itself and any orbits of interest which really do remain sufficiently close to 0 (e.g small homoclinic or periodic orbits) are unaffected. Thus any orbits which we find which are homoclinic to these trajectories are legitimate orbits of the original system, and any others can be disregarded, since they are of little interest to us. In this sense the modified dynamics are easier from a technical perspective, but still contain all features relevant to our study.

Chapter 3

A Melnikov-type quadratic form.

In this chapter we investigate the intersection of center stable and center unstable manifolds belonging to a nonhyperbolic equilibrium, whose stable and unstable manifolds intersect along a homoclinic loop. Orbits lying in this intersection converge to orbits in the center manifold in both positive and negative time. More specifically, we locate intersections corresponding to 1-round homoclinics by deriving a real valued function whose zeros correspond to these points of intersection. This function has a critical point at the origin, and in order to apply basic singularity theory to analyse the zero set close to the origin, we study the eigenvalues of the Hessian matrix at the critical point. This matrix inherits structure from the Hamiltonian character of the system, meaning that its eigenvalues are not arbitrary. Their structure is investigated using the 'scattering matrix' of the linearised variational equation along the homoclinic to the equilibrium, an approach also employed in [87], [88]. A similar object also appears in [47].

The lowest dimensional example of an elliptic-hyperbolic equilibrium is the saddle-center in a two degree of freedom system. Here a neighbourhood of the equilibium in the center manifold is filled with a Lyapunov family of periodic orbits, parametrised by the value of the Hamiltonian. Lerman [56] proved the generic existence of 4 transverse homoclinics to each periodic orbit sufficiently close to the equilibrium (see also

Grotta-Ragazzo [28]), implying the existence of complex dynamics (horseshoes) in each of these energy levels. This result was generalised for systems with any number of hyperbolic degrees of freedom in [47]. The genericity condition in [47] is that the scattering matrix is not equal to a rotation, and this result can be derived from our main theorem, see the discussion following the statement of that theorem. Looking beyond the linear dynamics of the variational equation, in the two degrees of freedom case, Yagasaki [87] showed that horsehoes are still present if the scattering matrix *is* a rotation, but there is some expansion and contraction coming from the higher order terms in the expansion of the flow along the homoclinic. This higher order analysis goes further than the linear analysis conducted in this chapter, although [87] does impose an additional restriction, namely that the homoclinic loop in contained in an invariant plane. Multi-round homoclinics are also found in unfoldings in [46], and in [66], [14] where reversible Hamiltonian systems are studied. For higher dimensional center manifolds, homoclinics to invariant tori in small perturbations of completely integrable Hamiltonian systems have been found in [48] and [20].

In [88], Yagasaki derived the same quadratic form studied here with 1 hyperbolic degree of freedom, under the additional hypothesis that the homoclinic loop is contained in an invariant plane, by another variant of the Melnikov method. The focus in [88] is on the existence of heteroclinic chains between invariant tori in the center manifold. Under extra (mild) hypotheses necessary for KAM type results to yield the existence of a family of invariant tori in the center manifold, Yagasaki proves that when expressed in polar coordinates, a zero of the quadratic form at which the partial derivatives in each of the angular variables are nonzero corresponds to a transversal intersection of invariant manifolds of two (not necessarily distinct) tori. This nondegeneracy condition on the angular derivatives at the zero means that when the quadratic form is restricted to points in the center manifold belonging the same invariant torus, the zero is isolated, which is used to derive the transversality of the intersection of invariant manifolds. The existence of chains of heteroclinic orbits shadowed by real 'diffusing' orbits is then shown for some examples. By contrast, our results do not assume that the homoclinic is contained in an invariant plane, and describe general properties of the quadratic form, as derived from the structure it inherits from the

Hamiltonian system, rather than the detailed examination of the zero set and its dynamical implications in concrete examples satisfying extra hypotheses. The results in this chapter provide less detailed information about dynamical behaviour than some of those mentioned in this introduction, but they may provide a first step towards a more systematic approach; the knowledge of the possible structure present in our reduced function at the linear level could be extended to develop normal forms for problems of this type.

The organisation of the chapter is as follows. In the remainder of section 3 we describe the set up and our assumptions, and outline the Lyapunov-Schmidt reduction. Section 3.1 establishes the necessary results for the reduction, and begins the study of the Hessian matrix. In section 3.2 we introduce the scattering matrix, and derive the formula for the Hessian matrix featuring in theorem 2. We also prove the first part of theorem 3, which states that the Hessian matrix cannot be positive or negative definite. We then prove in section 3.3, using a result from [68], that any symplectic matrix which is sufficiently close to the identity can be realised as the scattering matrix of a system which satisfies our assumptions and use this result in section 3.4 to demonstrate that the Hessian matrix can have any indefinite signature, using a theorem from [67]. We then consider in section 3.5 the special case in which the system is additionally reversible with respect to a time-reversing symmetry, as is common in examples coming from classical mechanics.

3.0.1 Problem setting

The system is defined by the ordinary differential equations

$$\dot{u} = X_H(u). \tag{3.1}$$

on $\mathbb{R}^{2n}$, as described in chapter 2. We assume that the origin is an elliptic-hyperbolic equlibrium of system (3.1), that is;

Assumption 1. *The spectrum of the linearisation $DX_H(0)$ consists of $2l$ distinct eigenvalues with zero real part, $\pm i\omega_j$, $j \in \{1, ..., l\}$, and $2(n-l)$ eigenvalues λ_i, whose* *real parts are bounded away from zero;* $0 < \alpha < |\Re(\lambda_i)|$ $i \in \{1, .., 2(n-l)\}$.

The equilibrium possesses $(n-l)$-dimensional stable and unstable manifolds W^s and W^u, which are assumed to intersect along a homoclinic loop $\gamma(t)$, namely;

Assumption 2. *There exists an orbit* $\Gamma = \{\gamma(t) : t \in \mathbb{R}\}$ *such that* $\Gamma \subset W^s \cap W^u$.

We denote by $E^u(DX_H(0))$, $E^s(DX_H(0))$ the unstable and stable eigenspaces of the linearisation at the origin. The centre subspace, corresponding to the purely imaginary eigenvalues, which is symplectic, will be denoted $E^c(DX_H(0))$, or simply E^c when the context is clear. Under these assumptions, the equilibrium possesses a $2l$ dimensional center manifold, which is symplectic. The center manifold may not be unique, but any center manifold will be tangent at the origin to $E^c(DX_H(0))$, yielding the same linearisation, and the same result in our context. The restriction of H to the center manifold defines a Hamiltonian system with l degrees of freedom and an elliptic critical point at the origin. Writing the tangent space at the equilibrium according to the symplectic splitting (see [65]);

$$\mathbb{R}^{2n} = E^c \oplus (E^u \oplus E^s)$$

we have

$$\mathbb{J} = \begin{pmatrix} \mathbb{J}_1 & 0 \\ 0 & \mathbb{J}_2 \end{pmatrix}.$$

Choosing a symplectic basis on E^c such that $\mathbb{J}_1 = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, since $\mathbb{J}_1 D^2H(0)|_{E^c}$ has distinct purely imaginary eigenvalues, we can (and do) make a symplectic change of coordinates in E^c (see [85]) which brings $D^2H(0)|_{E^c}$ to the form

$$D^2H(0)|_{E^c} = \begin{pmatrix} \omega_1 & & & & & \\ & \ddots & & & & \\ & & \omega_l & & & \\ & & & \omega_1 & & \\ & & & & \ddots & \\ & & & & & \omega_l \end{pmatrix} =: \mathrm{diag}(\omega_1, ..., \omega_l, \omega_1, ...\omega_l).$$

The origin also possesses $(n+l)$-dimensional center-stable and center-unstable manifolds W^{cs} and W^{cu}. The orbits we seek, which converge to the center manifold in forward and backward time, are contained in the intersection $W^{cs} \cap W^{cu}$.We make the following assumption on the invariant manifolds;

Assumption 3. $\dim(T_{\gamma(0)}W^{cu} \cap T_{\gamma(0)}W^{s}) = \dim(T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^{u}) = 1.$

Of course, the existence of the homoclinic $\gamma(t)$ which is contained in the intersection of W^u and W^s implies that the dimension of the intersection in assumption 3 is at least one, so this assumption means that this dimension is minimal: there is no further degeneracy leading to a higher dimensional intersection. As observed in chapter 2, assuming this degeneracy between W^{cu} and W^s implies the same condition for W^{cs} and W^u.

3.0.2 Statement of results

Theorem 2. *Under the assumptions 1, 2 and 3, homoclinic orbits to the center manifold of the origin are given by zeros of a function* $\mathfrak{g} : \mathbb{R}^{2l} \to \mathbb{R}$*. The function* $\mathfrak{g}$ *has the property that* $\nabla \mathfrak{g}(0) = 0$*, and the Hessian is given by*

$$D^2\mathfrak{g} = \sigma^T D^2 H(0)|_{E^c}\sigma - D^2 H(0)|_{E^c}$$

where σ *is the symplectic scattering matrix (see section 3.2).*

The zeros of the function $\mathfrak{g}$ correspond to intersections of W^{cs} and W^{cu}. These manifolds are foliated by the strong-stable and, resp., strong-unstable leaves of the points in W^c: if a forward orbit of a point $M \in W^c$ stays in a small neighbourhood of the equilibrium at the origin, then its strong-stable leaf $l^{ss}(M)$ consists of all points whose forward orbits tend to the forward orbit of M exponentially with a rate at least $e^{-\alpha t}$, the same for the strong-unstable leaf $l^{uu}(M)$ and backward orbits. We prove in Theorem 2 that $\mathfrak{g}(M) = 0$ if and only if there exists a point $\bar{M} \in W^c$ such that $l^{uu}(M)$ has a point of intersection with $l^{ss}(\bar{M})$, and the orbit of this intersection point is close to the homoclinic loop Γ when it goes from a small neighbourhood of M to a small neighbourhood of $\bar{M}$. This orbit is homoclinic to W^c (and corresponds to a solution of system (3.1) which is bounded and uniformly close to $\gamma(t)$ for all t)

if both the backward orbit of M and the forward orbit of $\bar{M}$ are bounded and stay close to the origin.

As mentioned in the introduction to this chapter, an equivalent quadratic form is derived in [88] in the case of 1 hyperbolic degree of freedom (i.e $n = l+1$ in our notation), under some extra assumptions (described in the introduction). It would seem that the methods from the current paper combined with those from [88] allow one to prove the existence of heteroclinic chains and accompanying diffusion behaviour in a much larger class of far-from-integrable Hamiltonian systems.

The following theorem is the main result of this chapter.

Theorem 3. *Under assumptions 1, 2, 3,*

(i) $D^2\mathfrak{g}$ *can be neither positive nor negative definite.*

(ii) *All indefinite signatures for* $D^2\mathfrak{g}$ *can be realised by systems satisfying the assumptions. Furthermore, they can be realised in systems which are a small* C^1 *perturbation of a completely integrable system.*

The first part of the theorem says that as long as the critical point of $\mathfrak{g}$ is Morse, the homoclinic $\gamma(t)$ is never an isolated intersection point of the center stable and center unstable manifolds - a situation which in the general (non-Hamiltonian) case could arise. In the case $l = 1$, we find agreement with a result from [47]; the existence of one positive and one negative eigenvalue leads to a degenerate hyperbola (a 'cross') for the zero set of $\mathfrak{g}$, which intersects each sufficiently small periodic orbit surrounding the origin in 4 places, leading to 4 homoclinics - the genericity condition from [47], which requires the scattering matrix to have some expansion and contraction, is equivalent to that of $D^2\mathfrak{g}$ being invertible. In the two degree of freedom case, it is possible to understand the geometry of the zero set (the 'cross' mentioned above) by considering a three dimensional section Σ transverse to the flow at $\gamma(0)$ as in the picture below, figure 3.1. The traces of $W^{cu}(0)$ and $W^{cs}(0)$ in the section are tangent to the energy level $\mathcal{E}_0$ as discussed in the chapter 2. These traces are composed of the traces of the unstable and stable manifolds of all of the periodic orbits present in the center manifold. Since the Hamiltonian on the center manifold has an extremum at the equilibrium, conservation of energy implies that both of the traces lie completely 'on one side' of this energy level in the section. The traces of the stable and unstable manifolds of a periodic orbit define two circles lying in the same energy level in the section. These circles have the same area (by symplecticity of the flow), and the genericity condition guarantees that they do not coincide, leading to a situation as illustrated in figure 3.1.

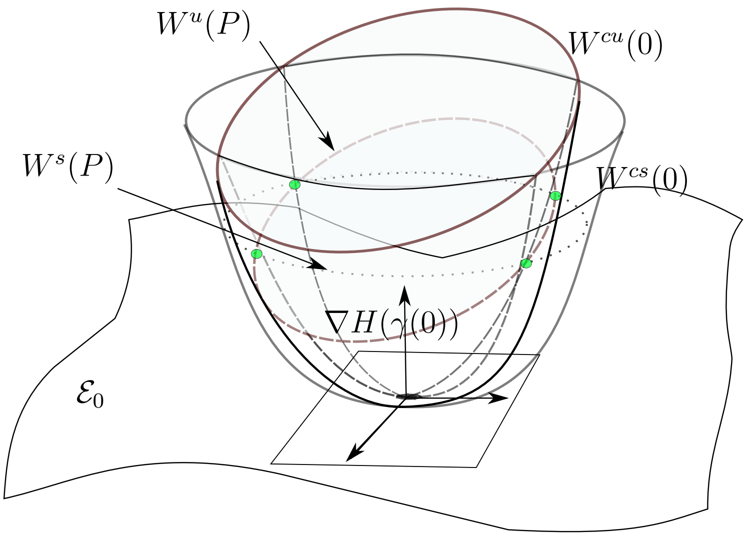


Figure 3.1: *Inside a section Σ in the 2 d.o.f case. The green blobs are at the points of intersection of the invariant manifolds of the periodic orbit P, corresponding to homoclinic orbits. Running over all of the sufficiently small periodic orbits, these points draw out the cross shape of the intersection of $W^{cu}(0)$ and $W^{cs}(0)$, which corresponds to $\mathfrak{g}^{-1}(0)$.*

The rest of the theorem says that in general there is no further restriction on the singularity. In section 3.5 we will consider also the case in which the vector field is reversible;

Assumption 4. *Letting R be a linear involution which acts antisymplectically, that is $R^2 = I$ and $R\mathbb{J} = -\mathbb{J}R$,*

(i) X_H is R-reversible: $X_H(Ru) = -RX_H(u)$.

(ii) The homoclinic $\gamma(t)$ is $R-$ symmetric: writing $\Gamma = \{\gamma(t) : t \in \mathbb{R}\}$, we have $R\Gamma = \Gamma$.

In this case we find;

Theorem 4. *Under assumptions 1, 2, 3, 4, the signature of $D^2\mathfrak{g}$ is (l, l).*

3.0.3 The Lyapunov-Schmidt reduction

Returning now to system (3.1), that is,

$$\dot{u} = X_H(u)$$

with the homoclinic orbit $\gamma(t)$, we seek homoclinic orbits $\tilde{\gamma}(t)$ as perturbations of $\gamma(t)$, by first writing

$$\tilde{\gamma}(t) = \gamma(t) + x(t).$$

Substituting this into (3.1) and rearranging for $x(t)$ brings us to the equation

$$\dot{x}(t) = X_H(\gamma(t) + x(t)) - X_H(\gamma(t)) \tag{3.2}$$

We then define an operator F by

$$F(x) := \dot{x}(t) - X_H(\gamma(t) + x(t)) - X_H(\gamma(t))$$

so that zeros of F correspond to solutions of (3.2). By choosing an appropriate domain $\mathcal{X}$ and target space $\mathcal{Y}$ for F, we can search for solutions $x(t)$ which satisfy prescribed conditions on their asymptotic behaviour, which corresponds to finding homoclinic solutions with desired features. Clearly, $F(0) = 0$. Taking a Frechet derivative of F at 0 leads us to the operator

$$DF(0)x(t) := Lx(t) = \dot{x}(t) - DX_H(\gamma(t))x(t)$$

so that zeros of L are solutions of the *variational equation*

$$\dot{x}(t) = DX_H(\gamma(t))x(t). \tag{3.3}$$

Note that one solution (which, since $\gamma(t)$ lies in the intersection of the stable and unstable manifolds of the equilibrium, decays exponentially fast in both forward and backward time) of (3.3) is given by $\dot{\gamma}(t)$. A crucial point, discussed in more detail in the following section, is that L is a *Fredholm* operator. This means by definition that $\ker(L) \subset \mathcal{X}$ is finite dimensional, and the range $\mathcal{R}(L) \subset \mathcal{Y}$ is of finite codimension. The index of L is then the integer $\mathrm{ind}(L) = \dim\ker(L) - \mathrm{codim}(\mathcal{R}(L))$. This will allow us to perform a Lyapunov-Schmidt reduction of the map F at zero. The procedure is as follows; we decompose $\mathcal{X}$ and $\mathcal{Y}$ in the following way

$$\begin{aligned} \mathcal{X} &= \ker(L) \oplus \mathcal{M} \\ \mathcal{Y} &= \mathcal{N} \oplus \mathcal{R}(L) \end{aligned}$$

and now look for solutions of the following equivalent system, where the variable $x = k + w$ is split according to the decomposition of $\mathcal{X}$ and P is the projection onto $\mathcal{R}(L)$ in $\mathcal{Y}$ with $\ker(P) = \mathcal{N}$;

$$\begin{cases} PF(k+w) & = 0 \\ (I-P)F(k+w) & = 0. \end{cases} \tag{3.4}$$

The advantage of this construction is that the derivative in the first component of the system with repect to w, $D_w PF(v)|_{(0)}$, is invertible, and so we can use the implicit function theorem to locally solve this first equation. This allows us to write $v \in \mathcal{X}$ as $k + w(k)$, where $w : \ker(L) \to \mathcal{M}$ is such that

$$PF(k+\sigma) = 0 \Leftrightarrow \sigma = w(k)$$

in a neighbourhood of $x = 0$. We note also that $D_k w(0) = 0$; differentiating the top component of (3.4) with respect to k at zero leads to

$$L(D_k w(0)) = 0,$$

and since $D_k w(0) \in \mathcal{M}$, we can invert L, yielding $D_k w(0) = 0$. We are then only required to find zeros of the map defined by

$$(I - P)F(k + w(k)) : \ker(L) \to \mathcal{N},$$

which we denote by $\mathcal{G}(k)$. Zeros of $\mathcal{G}$ then correspond to zeros of the full system. Furthermore, $\mathcal{G}$ has a critical point at the origin:

$$D\mathcal{G}(0) = (I - P)DF(k)(D_k w(k))|_{k=0} = 0.$$

So, as long as this singularity is nondegenerate[1] we can locally describe the zero set of $\mathcal{G}$ by classifying the critical point at the origin and appealing to the Morse lemma, which gives us a normal form for the quadratic part of $\mathcal{G}$.

3.1 Weighted function spaces and Fredholm properties

For $\beta \in \mathbb{R}$, we define the Banach space

$$C^1_\beta(\mathbb{R}, \mathbb{R}^{2n}) = \{x : \mathbb{R} \to \mathbb{R}^{2n} \text{ with } \sup_{t\in\mathbb{R}} \|e^{\beta|t|}x(t)\| < \infty, \sup_{t\in\mathbb{R}} \|e^{\beta|t|}\dot{x}(t)\| < \infty\}.$$

We will seek our solutions in a space with a negative weight; we allow only solutions x for which $x(t)$ and $\dot{x}(t)$ do not grow too fast. We will require the following result;

Lemma 3.1.1. *There exists a $\beta \in (0, \alpha)$ such that a solution $x(t)$ of the equation $F(x) = 0$ gives rise to an orbit $\tilde{\gamma}(t) = \gamma(t) + x(t)$ which remains in a tubular neighbourhood of $\gamma(t)$ and is homoclinic to the centre manifold of the origin if and only if $x(t) \in C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n})$, and $x(t)$ is sufficiently small in norm.*

[1] in the sense that is it a Morse singularity: the Hessian matrix is invertible

Proof. If $\tilde{\gamma}(t) \in W^{cu}(0) \cap W^{cs}(0)$, then it approaches an orbit $\eta(t)$ in the center manifold;

$$\|\tilde{\gamma}(t) - \eta(t)\| \to 0 \text{ as } t \to \infty$$

but since $\gamma(t) \to 0$ exponentially fast, we have

$$\|x(t) - \eta(t)\| \to 0 \text{ as } t \to \infty \tag{3.5}$$

Since $\eta(t)$ is contained in the center manifold, $\eta \in C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n})$ for any $\beta \in (0, \alpha)$, and hence we can use (3.5) to conclude that $x \in C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n})$. Moreover, if $\tilde{\gamma}(t)$ lies in a small tubular neighbourhood of $\gamma(t)$ then the norm of $x(t)$ is necessarily small.

Conversely, assume that $x(t) \in C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n})$ for some $\beta > 0$, is small in norm and $F(x) = 0$. Then $\tilde{\gamma}(t)$ defines a trajectory which stays in a small tubular neighbourhood of $\gamma(t)$. We'll show that if β is sufficiently small, then $\tilde{\gamma}(t)$ approaches the centre manifold in both forward and backward time. In a neighbourhood U of the origin, the system (3.1) is topologically equivalent to a system of the form

$$\begin{cases} \dot{u} = Cu + g(u, V(u)) \\ \dot{v} = Sv \end{cases} \tag{3.6}$$

see [53], where the top component corresponds to the restriction of the system to the center manifold, and the second component corresponds to the saddle behaviour, and is linear: S is a matrix with no purely imaginary eigenvalues. If the norm of $x(t)$ is small enough, then $\tilde{\gamma}(t)$ is contained in a tubular neighbourhood of $\gamma(t)$ which in contained in U for $|t|$ large enough. Let $\beta^* > 0$ bound the absolute values of real parts of the spectrum of S away from zero. If $x(t) \in C^1_{-\beta^*}(\mathbb{R}, \mathbb{R}^{2n})$ then no part of $x(t)$ asymptotically grows fast enough for $t > 0$ to be contained in the unstable subspace, and hence the second component of $x(t) = (x_u(t), x_v(t))$ approaches zero. Hence, $x(t)$ approaches a solution of the system restricted to the center manifold. The same argument applies in negative time, mutatis mutandis. Changing back to our original system via the topological conjugacy may change the value of β^*, call the new value $\beta^{**} > 0$, but the convergence of $\tilde{\gamma}(t)$ to the center manifold is preserved, and hence we can take $\beta = \beta^{**}$.

$\square$

This result justifies the use of an exponentially weighted norm on the domain of F to capture all of the solutions which do not grow faster than a given exponential factor. For a weight function, we choose a smooth function which closely approximates the absolute value function. If we were working in Sobolev spaces, we could choose the absolute value function itself, but in our context it is necessary to take a continuously differentiable weight, because the derivative of the weight appears in the expression for the adjoint operator, and we require this term to be continuous. Letting $\phi(t) \in C^1(\mathbb{R}, \mathbb{R})$ be such that

$$\begin{cases} \phi(t) = |t| \text{ for } t \in (-\infty, -1] \cup [1, \infty) \\ \sup_{t \in [-1,1]} |\phi(t) - |t|| << 1 \\ \phi(t) > 0 \text{ for } t \in \mathbb{R}, \end{cases}$$

we now consider the weighted inner product

$$< u, v >_\delta \; = \int_{\mathbb{R}} e^{-2\delta\phi(t)} < u(t), v(t) > dt$$

which is defined for any $u, v \in C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n})$ with $0 < \beta < \delta$. We are hence free to choose β, δ satisfying the following condition.

Assumption 5. *The constants satisfy* $0 < \beta < \delta < \alpha$*, and* $\delta - \alpha < \beta - \delta$*, where* α *is as defined in assumption 1.*

We calculate an expression for the adjoint L^* with respect to the weighted inner product as follows;

$$\begin{aligned} \langle Lu, v \rangle_\delta &= \int_{\mathbb{R}} e^{-2\delta\phi(t)} \langle \dot{u}(t) - DX_H(\gamma(t))u(t), v(t) \rangle \, dt \\ &= \int_{\mathbb{R}} \left\langle \dot{u}(t), e^{-2\delta\phi(t)} v(t) \right\rangle - e^{-2\delta\phi(t)} \langle DX_H(\gamma(t))u(t), v(t) \rangle \, dt \end{aligned}$$

$$= \int_{\mathbb{R}} - \left\langle u(t), \frac{d}{dt}(e^{-2\delta\phi(t)} v(t)) \right\rangle - e^{-2\delta\phi(t)} \left\langle u(t), DX_H(\gamma(t))^* v(t) \right\rangle dt$$
$$= \int_{\mathbb{R}} -e^{-2\delta\phi(t)} \left\langle u(t), \frac{d}{dt} v(t) - 2\delta\dot{\phi}(t) v(t) \right\rangle - e^{-2\delta\phi(t)} \left\langle u(t), DX_H(\gamma(t))^* v(t) \right\rangle dt,$$

We conclude from this line that

$$L^* = -\frac{d}{dt} + 2\delta\dot{\phi}(t) - DX_H(\gamma(t))^*$$

we refer to $L^* u = 0$ as the *adjoint variational equation.*

Lemma 3.1.2. $DF(0,0) := L : C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n}) \to C^0_{-\beta}(\mathbb{R}, \mathbb{R}^{2n})$ *is a Fredholm operator of index* $2l$*. Furthermore,* $y(t) \in \mathcal{R}(L)$ *if and only if*

$$\int_{\mathbb{R}} e^{-2\delta\phi(t)} \langle y(t), \psi(t) \rangle \, dt = 0, \text{ for every } \psi \in C^1_{-\beta} \text{ solving } L^*\psi = 0.$$

To prove lemma 3.1.2 we will make use of a conjugacy beetween L and a 'shifted' version of L on a differently weighted function space. We observe that $L = DF(0,0) : C^1_{-\beta} \to C^0_{-\beta}$ is conjugate to the shifted operator $L_\delta : C^1_{\delta-\beta} \to C^0_{\delta-\beta}$ given by

$$L_\delta u(t) = \frac{du}{dt} - \delta\dot{\phi}(t) u(t) - DX_{\tilde{H}}(\gamma(t)) u(t),$$

The conjugacy is given by the isomorphism $v(t) \mapsto e^{-\delta\phi(t)} v(t)$ which maps from $C^1_{-\beta}$ into $C^1_{\delta-\beta}$, which is endowed with the unweighted inner product. The utility of this conjugacy stems from the fact that the limits

$$\lim_{t \to \pm\infty} (\delta\dot{\phi}(t) I + DX_H(\gamma(t))) \tag{3.7}$$

are now hyperbolic, since the imaginary eigenvalues of $DX_H(0)$ are now shifted, to the right of the imaginary axis in negative time and to the left in positive time. We will make use of the following theorem:

Theorem 5 (Palmer, [70]). *Let $A(t)$ be an $n \times n$ matrix function bounded and continuous on $\mathbb{R}$ and such that*

$$\lim_{t\to-\infty} A(t) = A_{-\infty}, \quad \lim_{t\to\infty} A(t) = A_{\infty}$$

exist and are hyperbolic. Then

$$B : C^1(\mathbb{R}, \mathbb{R}^{2n}) \to C^0(\mathbb{R}, \mathbb{R}^{2n})$$
$$Bx = \dot{x}(t) - A(t)x(t)$$

is Fredholm, and $y \in \mathcal{R}(B)$ if and only if

$$\int_{\mathbb{R}} \langle y(t), \psi(t)\rangle \, dt = 0, \ \textit{for every bounded } \psi \textit{ solving } \dot{\psi}(t) = -A^*(t)\psi(t).$$

Furthermore, if $A_{-\infty}$, A_{∞} have a_- and a_+ unstable eigenvalues respectively, then

$$\operatorname{ind}(L) = a_- - a_+.$$

Proof of lemma 3.1.2. We first consider our shifted operator defined on the larger function space $C^1(\mathbb{R}, \mathbb{R}^{2n})$ of bounded continuous functions, as in the statement of theorem 5. Call this operator $\hat{L}_\delta$. Applying theorem 5 to $\hat{L}_\delta$ tells us that the index of $\hat{L}_\delta = 2l$. Firstly this means that $\ker(\hat{L}_\delta) < \infty$. This remains true for L_δ, since $\ker(\hat{L}_\delta) = \ker(L_\delta)$: any bounded solutions decay at a rate of at least $e^{\delta-\alpha}$ in negative time and $e^{-(\alpha+\delta)}$ in positive time (as can be seen by looking at the spectrum of the limit matrices in (3.7)), and so, in particular, faster than $e^{\beta-\delta}$ in both time directions, as a consequence of assumption 5. Hence, these solutions lie in $C^1_{\delta-\beta}$.

The application of theorem 5 also gives $\mathcal{R}(\hat{L}_\delta) = \ker(\hat{L}^*_\delta)^\perp$. We find that $\ker(\hat{L}^*_\delta) = \ker(L^*_\delta)$ for the same reasons as in the previous paragraph, and so

$$\ker(L^*_\delta)^\perp = \mathcal{R}(\hat{L}_\delta) \cap C^0_{\delta-\beta} = \mathcal{R}(L_\delta)$$

is it clear from these considerations that $\operatorname{ind}(\hat{L}_\delta) = \operatorname{ind}(L_\delta)$.

Finally, applying the inverse of the conjugacy brings us back to the original operator L, preserving the required properties. □

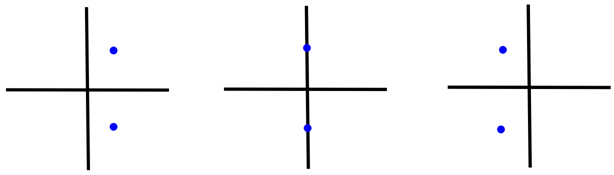


Figure 3.2: Eigenvalues cross the imaginary axis from right to left, as time progresses through $\mathbb{R}$, inducing a positive Fredholm index.

We note that assumption 3 implies that $\dot{\gamma}(t)$ is the only solution (up to a scalar multiple) of the variational equation which decays at an exponential rate[2]. This also implies that the only (again, up to a scalar multiple) exponentially decaying solution of the adjoint variational equation (with respect to the unweighted inner product) is given by $\mathbb{J}\dot{\gamma}(t) = \nabla H(\gamma(t))$.

Lemma 3.1.3. $\ker(L^*) = \text{span}\{e^{2\delta\phi(t)}\nabla H(\gamma(t))\}$.

Proof. If $L^*u = 0$ with $u(t) \in C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n})$, then

$$\begin{aligned} 0 = L^*u &= e^{-2\delta\phi(t)}L^*u \\ &= -e^{-2\delta\phi(t)}\frac{du}{dt} + 2\delta\dot{\phi}(t)e^{-2\delta\phi(t)}u(t) - DX_{\tilde{H}}(\gamma(t),0)^*e^{-2\delta\phi(t)}u(t) \\ &= -\frac{d}{dt}(e^{-2\delta\phi(t)}u(t)) - DX_{\tilde{H}}(\gamma(t),0)^*e^{-2\delta\phi(t)}u(t) \end{aligned}$$

The expression on the right hand side here is the adjoint variational equation with respect to the unweighted inner product. Now, $e^{-2\delta\phi(t)}u(t)$ is an exponentially decaying solution of the unweighted adjoint variational equation, and hence $e^{-2\delta\phi(t)}u(t) \in \text{span}\{\nabla H(\gamma(t))\}$, meaning that $u(t) \in \text{span}\{e^{2\delta\phi(t)}\nabla H(\gamma(t))\}$.

Similarly, if $v(t) \in \text{span}\{\nabla H(\gamma(t))\}$, then v(t) solves

$$-\frac{d}{dt}v(t) - DX_{\tilde{H}}(\gamma(t),0)^*v(t) = 0$$

[2]in fact, $\dot{\gamma}(t) \in C^1_{\alpha}(\mathbb{R}, \mathbb{R}^{2n})$

while $e^{2\delta\phi(t)}v(t) \in C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n})$ and

$$\begin{aligned}
-\frac{d}{dt}(e^{2\delta\phi(t)v(t)}) + 2\delta\dot{\phi}(t)e^{2\delta\phi(t)}v(t) - e^{2\delta\phi(t)}DX_{\tilde{H}}(\gamma(t),0)^*v(t) = \\
= -2\delta\dot{\phi}(t)e^{2\delta\phi(t)}v(t) - e^{2\delta\phi(t)}\frac{d}{dt}v(t) + 2\delta\dot{\phi}(t)e^{2\delta\phi(t)}v(t) \\
- e^{2\delta\phi(t)}DX_{\tilde{H}}(\gamma(t),0)^*v(t) \\
= e^{2\delta\phi(t)}(L^*(v(t))) = 0
\end{aligned}$$

□

Hence $\mathcal{R}(L)^\perp$ is one dimensional, and so $\dim(\ker(L)) = 1 + \mathrm{ind}(L) = 2l + 1$.

As a check, we observe that if $\psi(t) \in C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n})$ is a solution of the adjoint variational equation, and $f \in \mathcal{R}(L)$, that is, $f(t) = \dot{x}(t) - DX_H(q(t),0)x(t)$ for some $x(t) \in C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n})$ then

$$\begin{aligned}
\langle \psi(t), f(t) \rangle_\delta &= \int_{\mathbb{R}} e^{-2\delta\phi(t)} \Big\langle \psi(t), \dot{x}(t) - DX_H(t)(\gamma(t),0)x(t) \Big\rangle dt \\
&= \int_{\mathbb{R}} e^{-2\delta\phi(t)} \Big\langle \psi(t), \dot{x}(t) \Big\rangle - \Big\langle DX_H(\gamma(t),0)^*\psi(t), x(t) \Big\rangle dt \\
&= \int_{\mathbb{R}} e^{-2\delta\phi(t)} \Big\langle \psi(t), \dot{x}(t) \Big\rangle + \Big\langle \dot{\psi}(t) - 2\delta\dot{\phi}(t)\psi(t), x(t) \Big\rangle dt \\
&= \int_{\mathbb{R}} e^{-2\delta\phi(t)} \left(\frac{d}{dt} \Big\langle \psi(t), x(t) \Big\rangle - 2\delta\dot{\phi}(t) \Big\langle \psi(t), x(t) \Big\rangle \right) dt \\
&= \int_{\mathbb{R}} \frac{d}{dt} \left(e^{-2\delta\phi(t)} \Big\langle \psi(t), x(t) \Big\rangle \right) dt \\
&= \left[e^{-2\delta\phi(t)} \Big\langle \psi(t), x(t) \Big\rangle \right]_{-\infty}^{\infty} = 0
\end{aligned}$$

When we construct a reduced map by Lyapunov-Schmidt reduction, we project onto $\ker(L^*)$ by taking the weighted inner product with this unique exponentially decaying solution. The exponential factors in the weight and the solution will then cancel, leaving us with an expression which involves an unweighted inner product.

The results from this section facilitate a Lyapunov-Schmidt reduction of the map

F at zero according to a decomposition of the following form;

$$C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n}) = \ker(L) \oplus \mathcal{M}$$

$$C^0_{-\beta}(\mathbb{R}, \mathbb{R}^{2n}) = \ker(L^*) \oplus \mathcal{R}(L)$$

Performing the reduction as described in section 3.0.3 leads to the reduced map

$$\mathcal{G}(k) := (I - P)F(k + w(k)) : \ker(L) \to \ker(L^*)$$

so $\mathcal{G}$ maps from a $2l+1$ dimensional space into a 1 dimensional space, as a consequence of the positive Fredholm index of L, and $\mathcal{G}$ has a critical point at the origin. This proves the first part of theorem 2.

Remark 3.1.4. *See also the paper [75] by Sandstede and Scheel for similar results regarding Fredholm indices of operators on weighted spaces derived from variational equations along connecting orbits, and [84] by Wechselberger for another similar application of the Lyapunov-Schmidt reduction on weighted spaces, for finding canard solutions.*

3.1.1 The Hessian Matrix

We now study this critical point of the reduced map $\mathcal{G}(k)$ by investigating the Hessian matrix. For the calculations, we now let k_i, $i \in \{1, ..., 2l+1\}$ be a chosen basis of $\ker(L)$, with $k_1 = \dot{\gamma}(t)$, and we write $\mathfrak{g}(\beta_1, ..., \beta_{2l+1}) := \mathcal{G}(\beta_1 k_1, ..., \beta_{2l+1} k_{2l+1})$, so that

$$\mathfrak{g}(\beta) = \int_{\mathbb{R}} e^{-2\delta\phi(t)} \Big\langle e^{2\delta\phi(t)} \nabla H(\gamma(t)), \dot{\gamma}(t) + \Sigma_i \beta_i \dot{k}_i(t) \\ + \dot{w}(\beta)(t) - X_H(\gamma(t) + \Sigma_i \beta_i k_i(t) + w(\beta)(t)) \Big\rangle dt$$

The following lemma provides a formula for the derivatives of $\mathfrak{g}(0)$. The proof is the same in essence as the one in [29] (theorem 5), in which a homoclinic orbit to a hyperbolic equilibrium is studied. We include the proof here for completeness.

Lemma 3.1.5.

$$\frac{\partial \mathfrak{g}}{\partial \beta_i}(0) = 0 \tag{3.8a}$$

$$\frac{\partial^2 \mathfrak{g}}{\partial \beta_i \partial \beta_j}(0) = \int_{\mathbb{R}} \left\langle \dot{\gamma}(t), D_x^3 H(\gamma(t))(k_i(t), k_j(t) \right\rangle dt \tag{3.8b}$$

Proof. The first equation simply states that the reduced map has a singularity at the origin, which is true for any map produced in this way via the Lyapunov-Schmidt reduction, as discussed in section 3.0.3. As for the second, differentiating $\mathfrak{g}$ twice and evaluating at $\beta = 0$ gives:

$$\begin{aligned}\frac{\partial^2 \mathfrak{g}}{\partial \beta_i \partial \beta_j} = &\int_{\mathbb{R}} \left\langle \nabla H(\gamma(t)), \frac{\partial^2 \dot{w}(0)}{\partial \beta_i \partial \beta_j} - DX_H(\gamma(t)) \frac{\partial^2 w(0)}{\partial \beta_i \partial \beta_j} \right\rangle dt \\ &- \int_{\mathbb{R}} \left\langle \nabla H(\gamma(t)), D^2 X_H(\gamma(t))(k_i(t), k_j(t)) \right\rangle dt\end{aligned}$$

and the first term is zero for each (i, j), since $\frac{\partial^2 \dot{w}(0)}{\partial \beta_i \partial \beta_j} - DX_H(\gamma(t)) \frac{\partial^2 w(0)}{\partial \beta_i \partial \beta_j}$ lies in the range of L. The final step is to recall that X_H can be written as $\mathbb{J}\nabla H$, and that $\dot{\gamma}(t) = \mathbb{J}\nabla H(\gamma(t))$. Applying the isometry $\mathbb{J}$ in both sides of the inner product and using these facts yields (3.8b). $\square$

In fact, we can restrict our attention to finding zeros of $\mathfrak{g}$ with its first argument (the coefficient of $\dot{\gamma}(t)$) fixed at zero. Considering the direct sum decomposition

$$C^1_{-\beta}(\mathbb{R}, \mathbb{R}^{2n}) = \ker(L) \oplus \mathcal{M},$$

we can choose $\mathcal{M}$ to be $\ker(L)^{\perp}$, the orthogonal complement with respect to the weighted inner product $\langle u, v \rangle_{\delta}$, which can be constructed due to the finite dimensionality of $\ker(L)$. This being done, and having chosen an orthogonal basis $\{\dot{\gamma}(t), k_2(t), ..., k_{2l-1}(t)\}$ for $\ker(L)$, we have that $k + w(k)$ satisfies

$$\int_{\mathbb{R}} e^{-2\delta\phi(t)} < \dot{\gamma}(t), (k + w(k))(t) > dt = 0 \Leftrightarrow k \in \operatorname{span}\{k_2(t), ..., k_{2l-1}(t)\}$$

since $w : \ker(L) \to \ker(L)^{\perp}$. We now show that all geometrically distinct homoclinics can be found by considering $\mathfrak{g}$ with the coefficient of $\dot{\gamma}(t)$ fixed at zero. We do this by proving:

Proposition 3.1.6. *Every solution*

$$\tilde{\gamma}(t) = \gamma(t) + (k + w(k))(t)$$

with k sufficiently small, can also be expressed as

$$\tilde{\gamma}(t) = \gamma(t+\alpha) + (k^* + w(k^*))(t+\alpha) \tag{3.9}$$

with $k^* \in \operatorname{span}\{k_2(t), ..., k_{2l-1}(t)\}$.

In other words, the homoclinics obtained with nonzero coefficients of $\dot{\gamma}(t)$ are only time translations of those obtained with the coefficient of $\dot{\gamma}(t)$ set to zero. The following proof uses ideas from [45].

Proof. We apply the implicit function theorem to the functional

$$P : C^1_{-\beta} \times \mathbb{R} \to \mathbb{R}, \;\; P(x,\alpha) := \int_{\mathbb{R}} e^{-2\delta\phi(t+\alpha)} < x(t) - \gamma(t+\alpha), \dot{\gamma}(t+\alpha) > dt.$$

We observe that

1. $P(\gamma, 0) = 0$.

2. $D_\alpha P(x,\alpha)|_{(\gamma,0)} = -\int_{\mathbb{R}} e^{-2\delta\phi(t)} < \dot{\gamma}(t), \dot{\gamma}(t) > dt \neq 0$.

So we can apply the IFT and write

$$P(x,\alpha) = 0 \Leftrightarrow \alpha = \alpha^*(x)$$

for (x,α) in a neighbourhood of $(\gamma, 0)$. Now, since in the expression of our homoclinic $\tilde{\gamma}(t)$, k is sufficiently small, we have that $\tilde{\gamma}$ is close to γ, and so we can write

$$0 = P(\tilde{\gamma}, \alpha^*(\tilde{\gamma})) = \int_{\mathbb{R}} e^{-2\delta\phi(t+\alpha^*(\tilde{\gamma}))} < \tilde{\gamma}(t) - \gamma(t+\alpha^*(\tilde{\gamma})), \dot{\gamma}(t+\alpha^*(\tilde{\gamma})) > dt$$

$$= \int_{\mathbb{R}} e^{-2\delta\phi(t)} < \tilde{\gamma}(t - \alpha^*(\tilde{\gamma})) - \gamma(t), \dot{\gamma}(t) > dt \tag{3.10}$$

So, the term $z^*(t) = \tilde{\gamma}(t - \alpha^*(\tilde{\gamma})) - \gamma(t)$ is small, and

$$\tilde{\gamma}(t) = \gamma(t + \alpha^*(\tilde{\gamma})) + z^*(t + \alpha^*(\tilde{\gamma}))$$

so that $z^* = k^* + w(k^*)$, and by (3.10) we have $k^* \in \text{span}\{k_2(t), ...k_{2l-1}(t)\}$. Hence, we have found the k^* from equation (3.9), so the claim is proved. □

Lemma 3.1.7. *For $i, j \in \{2, ..., 2l+1\}$, we have*

$$\begin{aligned} \frac{\partial^2 \mathfrak{g}}{\partial k_i \partial k_j} &= \int_{\mathbb{R}} \langle \dot{\gamma}(t), d_x^3 H(\gamma(t))(k_i(t), k_j(t) \rangle \, dt \\ &= \int_{\mathbb{R}} \frac{d}{dt} \langle k_i(t), d_x^2 H(\gamma(t))(k_j) \rangle \, dt \end{aligned}$$

Proof. We observe that the integrand here can be written as

$$\langle \dot{\gamma}(t), d_x^3 H(\gamma(t))(k_i(t), k_j(t)) \rangle = \frac{d}{dt} \langle k_i(t), d_x^2 H(\gamma(t))(k_j) \rangle$$

$$- \left\langle \dot{k}_i(t), d_x^2 H(\gamma(t))(k_j(t)) \right\rangle - \left\langle k_i(t), d_x^2 H(\gamma(t))(\dot{k}_j(t)) \right\rangle$$

But two of the terms on the right hand side here cancel out;

$$\begin{aligned} \left\langle \dot{k}_i(t), d_x^2 H(\gamma(t))(k_j(t)) \right\rangle &= \langle \mathbb{J} d_x^2 H(\gamma(t))(k_i(t)), d_x^2 H(\gamma(t))(k_j(t)) \rangle \\ &= - \, \omega(d_x^2 H(\gamma(t))(k_i(t)), d_x^2 H(\gamma(t))(k_j(t))) \end{aligned}$$

and, since $d_x^2 H(\gamma(t))$ is symmetric,

$$\begin{aligned} \left\langle k_i(t), d_x^2 H(\gamma(t))(\dot{k}_j(t)) \right\rangle &= \left\langle \dot{k}_j(t), d_x^2 H(\gamma(t))(k_i(t)) \right\rangle \\ &= - \, \omega(d_x^2 H(\gamma(t))(k_j(t)), d_x^2 H(\gamma(t))(k_i(t))) \\ &= \omega(d_x^2 H(\gamma(t))(k_i(t)), d_x^2 H(\gamma(t))(k_j(t))) \end{aligned}$$

since the symplectic form is skew-symmetric. Note that when $i = j$, both terms are zero. □

Remark 3.1.8. *See also [7], where similar calculations are performed in a different bifurcation scenario.*

3.2 The scattering matrix

In order to evaluate the integrals from lemma 3.1.7 which define the elements of the Hessian matrix, we introduce the scattering matrix. This is a linear map defined on the centre subspace of the equilibrium which maps asymptotic initial conditions of the linearised variational equation from this sympectic subspace at negative infinity to their resting places in the same subspace at positive infinity, while accounting for the effects of the asymptotic motion in the center subspace. Since this map is defined using the (linear) Hamiltonian flow, and the space on which it is defined is symplectic, it is represented by a symplectic matrix. It is referred to as the *scattering matrix*, and we call it σ. See also [47], [87] and [88].

Each $k(t) \in \text{span}\{k_2, ..., k_{2l+1}\}$ approaches the orbit of a point in the center subspace as $t \to \pm\infty$;

$$\lim_{t\to\pm\infty} k(t) = \Psi(t)k_{\pm\infty} \text{ with } k_{\pm\infty} \in E^c$$

with $\Psi(\cdot)$ denoting the fundamental matrix of the linear system on the center subspace $\dot{u} = \mathbb{J}D^2H(0)|_{E^C}u(t)$. There is thus a family of $2l$ dimensional symplectic subspaces $Y(t) \subset T_{\gamma(t)}\mathbb{R}^{2n}$ $t \in \mathbb{R}$ spanned by the initial conditions k_t such that $\Phi(s,t)k_t$ lies asymptotically in the center subspace E^c at the equilibrium as $s \to \pm\infty$. Let $\Phi^c(t,s) : Y(t) \to Y(s)$, $s,t \in \mathbb{R}$ denote the restriction of the solution operator for the variational equation to these subspaces. Observing then that we can relate $k_{-\infty}$ to $k_{+\infty}$ via

$$k_{+\infty} = \left(\lim_{t\to\infty} \Psi(-t)\Phi(t,0)\right)\left(\lim_{t\to-\infty} \Psi(-t)\Phi(t,0)\right)^{-1} k_{-\infty}$$

we note that each of the limits in this definition exist:

Proposition 3.2.1. *The limits*

$$\lim_{t\to\pm\infty} \Psi(-t)\Phi^c(t,0)$$

exist and are nonsingular.

Proof. We write

$$\begin{aligned}\dot{y} &= DX_H(0)|_{E^c}y(t) + \left(DX_H(\gamma(t))|_{Y^c(t)} - DX_H(0)|_{E^c}\right)y(t)\\ &=: DX_H(0)|_{E^c}y(t) + M(t)y(t)\end{aligned}$$

noting that $\|M(t)\| < Ce^{-\lambda t}$ for $0 < \lambda < \alpha$[3] and a constant $C \in \mathbb{R}$ as a consequence of the exponential convergence of the homoclinic orbit $\gamma(t)$ to the origin. We find solutions $\tilde{\phi}_j(t)$ such that

$$\lim_{t\to\infty} \tilde{\phi}_j(t)e^{-\lambda_j t} = p_j$$

where $DX_H(0)|_{E^c}p_j = \lambda_j p_j$ for each p_j. The $\tilde{\phi}_j(t)$ are found as fixed points of an operator $T_{t^*,j}$ mapping from the space of bounded continuous functions on the interval $[t^*,\infty)$, $C([t^*,\infty),\mathbb{R}^{2l})$ with the supremum norm $|\cdot|_\infty$, into itself. We show that for t^* sufficiently large, each $T_{t^*,j}$ is a contraction. The $T_{t^*,j}$ are defined by

$$T_{t*,j}(\phi(t)) = e^{\lambda_j t}p_j - \int_t^\infty e^{DX_H(0)|_{E^c}(t-s)}M(s)\phi(s)ds$$

We have:

$$\begin{aligned}\|T_{t^*,j}\phi_1(t) - T_{t^*,j}\phi_2(t)\| &\le |\phi_1(t) - \phi_2(t)|_\infty \int_t^\infty \|e^{DX_H(0)|_{E^c}(t-s)}\|Ce^{-\lambda s}ds\\ &\le |\phi_1(t) - \phi_2(t)|_\infty CC_1\frac{e^{-\lambda t^*}}{\lambda}\end{aligned}$$

so this is a contraction for t^* large enough, for each $j \in \{1, ..., 2l\}$. Using this approach for each j, we can build a fundamental matrix $\tilde{\Phi}(t) = \left(\tilde{\phi}_1(t)\middle| \ldots \middle|\tilde{\phi}_{2l}(t)\right)$ (that is,

[3] with α being smaller than the real parts of the hyperbolic eigenvalues of the linearisation at the origin

using the $\tilde{\phi}_j(t)$ as columns), so that

$$\lim_{t\to\infty} \tilde{\Phi}(t) = \left(e^{tDX_H(0)|_{E^c}} p_1 \middle| \dots \middle| e^{tDX_H(0)|_{E^c}} p_{2l} \right)$$

which implies

$$\lim_{t\to\infty} \Psi(-t)\tilde{\Phi}(t) = P,$$

where $\det(P) \neq 0$. Now we can return to our original fundamental matrix via $\Phi(t,0) = \tilde{\Phi}(t)\tilde{P}$ for a nonsingular matrix $\tilde{P}$. We conclude

$$\lim_{t\to\infty} \Psi(-t)\Phi(t,0) = P\tilde{P}$$

which is nonsingular. A similar argument holds in negative time. $\square$

We then define the scattering matrix $\sigma : E^c \to E^c$ by

$$\sigma := \lim_{t\to\infty} \Psi(-t)\Phi^c(t,-t)\Psi(-t). \tag{3.11}$$

Thus, since $\Psi(t)$ is orthogonal and commutes with $D^2H(0)$, we have

$$\begin{aligned}
&\lim_{t\to\infty} \left\langle D^2H(\gamma(t))k_i(t), k_j(t) \right\rangle - \lim_{t\to-\infty} \left\langle D^2H(\gamma(t))k_i(t), k_j(t) \right\rangle \\
&= \left\langle D^2H(0)\Psi(t)k_{i,+\infty}, \Psi(t)k_{j,+\infty} \right\rangle - \left\langle D^2H(0)\Psi(t)k_{i,-\infty}, \Psi(t)k_{j,-\infty} \right\rangle \\
&= \left\langle D^2H(0)k_{i,+\infty}, k_{j,+\infty} \right\rangle - \left\langle D^2H(0)k_{i,-\infty}, k_{j,-\infty} \right\rangle
\end{aligned}$$

which, together with the expression (3.8b) leads to the following representation of the Hessian, concluding the proof of theorem 2;

$$D^2\mathfrak{g} = \sigma^T D^2H(0)|_{E^c}\sigma - D^2H(0)|_{E^c}. \tag{3.12}$$

3.2.1 Indefiniteness of the Hessian

In this subsection we prove part (i) of theorem 3. The argument uses the minimax principle (see eg. chapter 4 of [17]) and the linear nonsqueezing theorem to show that the most negative and most positive eigenvalues of $\sigma^T D^2H(0)|_{E^c}\sigma$ cannot be closer to zero than those of $D^2H(0)|_{E^c}$, which implies that $D^2\mathfrak{g}$ must be indefinite. Hence, if it is invertible, it can't have the signature $(0, 2l)$ or $(2l, 0)$. Recall, we assume (without loss of generality) that the matrix $D^2H(0)|_{E^c}$ takes the form $D^2H(0)|_{E^c} = \mathrm{diag}(\omega_1, ..., \omega_l, \omega_1, ..., \omega_l)$.

Proof of theorem 3 (i). Seeking a contradiction, we assume that G is positive definite. This implies that the eigenvalues λ_i of the symmetric matrix $\sigma^T D^2H(0)\sigma$ (ordered in increasing size) are larger than those of $D^2H(0)$[4]. That is, they satisfy

$$\begin{cases} \lambda_1, \lambda_2 & > \omega_1 \\ ... & \\ ... & \\ \lambda_{2l-1}, \lambda_{2l} & > \omega_l \end{cases}$$

We now consider the minimax principle for the first eigenvalue λ_1 of $\sigma^T D^2H(0)\sigma$, which states;

$$\omega_1 < \lambda_1 = \min\{\max \left\langle D^2H(0)\sigma v, \sigma v\right\rangle |\|v\| = 1, v \in U,\ U \text{ subspace with } \dim(U) = 1\}. \tag{3.13}$$

The 2 dimensional symplectic eigenspace of $D^2H(0)$ associated with ω_1 is $E_{\omega_1} = \mathrm{span}\{q_j, p_j\}$ for some $j \in \{1, ..., l\}$. Consider now the symplectic subspace $\sigma^{-1}(E_{\omega_1})$. By the linear version of Gromov's nonsqueezing theorem (see eg.[61]), the unit ball in $\mathbb{R}^{2l}$ cannot be mapped into the cylinder $C_r(q_j, p_j) = \{(q, p)|q_j^2 + p_j^2 \leq r^2\}$ for $r^2 < 1$, so either $\|\sigma v\| = 1$ for all $v \in \{\sigma^{-1}(E_{\omega_1})|\|v\| = 1\}$, or there exist $v_+, v_- \in \{\sigma^{-1}(E_{\omega_1})|\|v\| = 1\}$ such that

$$\|\sigma v_+\| > 1,\ \|\sigma v_-\| < 1.$$

In either case, we arrive at a contradiction to the statement (3.13) of the minimax

[4] This fact itself can also be proved using the minimax principle

principle: in the former we can take any v from $\{\sigma^{-1}(E_{\omega_1})|\|v\|=1\}$ to get $\lambda_1=\omega_1$, and in the latter we can take v_- if $\omega_1>0$ or v_+ if $\omega_1<0$ to arrive at $\lambda_1<\omega_1$.

If we assume instead that G is negative definite, we can consider the minimax principle for the largest eigenvalue λ_{2l}, which in this case will give

$$\omega_l > \lambda_{2l} = \max\{\min \left\langle D^2H(0)\sigma v, \sigma v\right\rangle |\|v\|=1, v\in U,\ U \text{ subspace with } \dim(U)=1\}. \tag{3.14}$$

A similar argument to the one above then yields $\omega_l \le \lambda_{2l}$, the required contradiction.

□

In the case of the smallest eigenvalue, the 'max' in the minimax principle is redundant (likewise for the 'min' for the largest eigenvalue). For other eigenvalues however, these elements come into play, meaning that in general the argument cannot be repeated to rule out other signatures.

3.2.2 A different proof

For completeness, we mention also now a second method for proving the indefiniteness of the Hessian matrix, based on the structure of the invariant subspaces of σ. We consider first the case in which the scattering matrix is hyperbolic - in this case, the stable and unstable subspaces of σ are transverse Lagrangian subspaces, W^s_σ, W^u_σ. The idea is first to prove that there exist transformations which brings W^s_σ and W^u_σ to a position of nontrivial intersection, while leaving the Hamiltonian invariant. This fact can then be used to yield a second proof of theorem 3 (i), again by contradiction.

We assume;

Assumption 6. $W^s_\sigma \pitchfork \{\mathbb{R}^l\times\{0\}\}\in\mathbb{R}^{2l}$, $W^u_\sigma \pitchfork \{\{0\}\times\mathbb{R}^l\}\in\mathbb{R}^{2l}$.

This transversality assumption means that the stable and unstable subspaces can be represented by symmetric matrices $\mathcal{A}$, $\mathcal{B}$ [18] via

$$W^s_\sigma(0)=\{(q,p)|(\mathcal{A}p,p)\},\ W^u_\sigma(0)=\{(q,p)|(q,\mathcal{B}q)\}$$

We write

$$\mathcal{A} = \begin{pmatrix} \alpha & a^T \\ a & A \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} \beta & b^T \\ b & B \end{pmatrix}$$

with α, $\beta \in \mathbb{R}$, a,$b \in \mathbb{R}^{l-1}$ and A, $B \in \mathbb{R}^{(l-1)\times(l-1)}$. We make the following nondegeneracy assumption

Assumption 7. *The matrices A, B, $(A - B^{-1})$, $(B - A^{-1})$ are invertible, and $a^T(A - B^{-1})^{-1}B^{-1}b \neq 0$.*

Definition 3.2.2. *A symplectic rotation is a real symplectic matrix $R_\theta = [r_{i,j}] \in Sp(2n, \mathbb{R})$ with $\theta = (\theta_1, ..., \theta_n) \in \mathbb{R}^n$ such that for each $i \in \{1, ..., n\}$,*

$$\begin{pmatrix} r_{i,i} & r_{i,n+i} \\ r_{n+i,i} & r_{n+i,n+i} \end{pmatrix} = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}$$

and $r_{i,j} = 0$ otherwise.

So R_θ acts by a rotation through an angle θ_i in each pair of conjugate directions (x_i, x_{n+i}). Now letting $\Theta = (\theta, 0, ..., 0)$, $\Psi = (\psi, 0, ..., 0) \in \mathbb{R}^l$ and using our notation for a symplectic rotation, we have the following useful proposition;

Proposition 3.2.3. *There exists a pair of angles (θ, ψ) such that*

$$R_\Theta W^s_\sigma \cap R_\Psi W^u_\sigma \neq \emptyset. \tag{3.15}$$

Proof. The symplectic rotations R_Θ and R_Ψ act only in the (q_1, p_1) plane. Writing a point as $(q, p) = (q_1, q_2, p_1, p_2)$ with q_1, $p_1 \in \mathbb{R}$ and q_2, $p_2 \in \mathbb{R}^{(l-1)}$, and writing $R_\Phi(q, p) = (q_{1,\phi}, q_2, p_{1,\phi}, p_2)$, the condition (3.15) is equivalent to finding θ,ψ such that;

$$\begin{cases} q_{1,\theta} & = \alpha p_{1,\theta} + a^T p_2 \\ q_2 & = p_{1,\theta} a + A p_2 \\ p_{1,\psi} & = \beta q_{1,\psi} + b^T q_2 \\ p_2 & = q_{1,\psi} b + B q_2 \end{cases} \tag{3.16}$$

Since (by assumption) A is invertible, we can multiply the second equation in (3.16) by A^{-1} and add it to the fourth, yielding

$$(B - A^{-1})q_2 = -p_{1,\theta}A^{-1}a - q_{1,\psi}b$$

Similarly, using the inverti bility of B and the other two equations gives

$$(A - B^{-1})p_2 = -p_{1,\theta}a - q_{1,\psi}B^{-1}b$$

Writing $C_1 = (A - B^{-1})^{-1}$, $C_2 = (B - A^{-1})^{-1}$, $D = (A - B^{-1})^{-1}B^{-1}$, we can derive

$$\begin{cases} b^T q_2 & = -p_{1,\theta}b^T D^T a - q_{1,\psi}b^T C_2 b \\ a^T p_2 & = -p_{1,\theta}a^T C_1 a - q_{1,\psi}a^T Db. \end{cases} \tag{3.17}$$

Writing now $\nu := a^T Db$ and substituting back into (3.16), we arrive at equations of the form

$$\begin{pmatrix} 1 & -\alpha'\nu \\ 0 & \nu \end{pmatrix} \begin{pmatrix} q_{1,\theta} \\ p_{1,\theta} \end{pmatrix} = \begin{pmatrix} \nu & 0 \\ -\beta'\nu & 1 \end{pmatrix} \begin{pmatrix} q_{1,\psi} \\ p_{1,\psi} \end{pmatrix} \tag{3.18}$$

Since (by assumption) $\nu \neq 0$, we can invert the matrix on the RHS to arrive at

$$\begin{pmatrix} \frac{1}{\nu} & -\alpha' \\ \beta' & \nu(1 - \alpha'\beta') \end{pmatrix} \begin{pmatrix} q_{1,\theta} \\ p_{1,\theta} \end{pmatrix} = \begin{pmatrix} q_{1,\psi} \\ p_{1,\psi} \end{pmatrix} \tag{3.19}$$

Thinking geometrically, we can find solutions of (3.19) as intersections of a circle centered at the origin with its image under the matrix on the left hand side. Notice that the determinant of this matrix is equal to 1, so the matrix preserves area and so intersections must occur (see figure 3.3). □

Second proof of theorem 3 (i), in the case when σ is hyperbolic. Assuming again for contradiction that G is positive definite, we observe that the quadratic form given by $D^2H(0)$ is strictly increasing along σ orbits. This has the consequence that

$$\left\langle D^2H(0)v, v \right\rangle > 0 \text{ for } v \in W^u_\sigma, \quad \left\langle D^2H(0)v, v \right\rangle < 0 \text{ for } v \in W^s_\sigma. \tag{3.20}$$

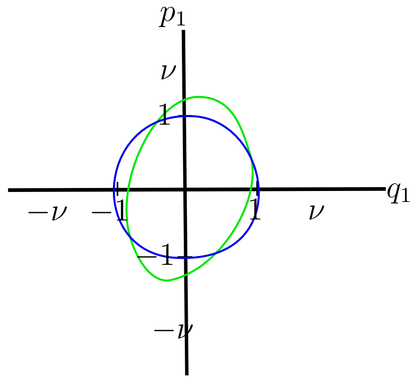


Figure 3.3: *Geometry of solutions of (3.19).*

This follows from the fact that orbits in W^s_σ approach the origin in forward time, meaning that the value of the quadratic form must increase from below zero along these orbits, and a similar argument applies for W^u_σ (the form must decrease down to zero for orbits under σ^{-1} here). We now invoke proposition 3.2.3. Since the quadratic form $D^2H(0)$ is invariant under the symplectic rotations R_Θ and R_Ψ, we arrive at a contradiction to 3.20. If G were negative definite, we would simply have the opposite inequalities in 3.20, so the same contradiction arises. $\square$

Remark 3.2.4. *If we had at this point assumed the existence of a reversing involution R, then (as we will see later) σ would be R reversible, so that R itself would map the stable subspace onto the unstable one, leaving the Hamiltonian unchanged, meaning there would be no need to consider the symplectic rotations of the proposition above. This approach can be applied to prove that in the reversible case, the signature of R is necessarily (l, l), which we will prove slightly differently later on.*

We note also that in the case of nonhyperbolic σ, we can also use the dynamics of points under σ to obtain a contradiction.

3.2.3 Scattering matrix is nonhyperbolic

Second proof of theorem 3 (i), when σ is nonhyperbolic. Let us assume that σ has a pair (or more) of purely imaginary eigenvalues. Then it has a least one periodic or quasiperiodic orbit $\sigma^n(x_0)$, $n \in \mathbb{Z}$. If G is positive definite, then

$$\left\langle D^2H(0)\sigma^{n+1}(x_0), \sigma^{n+1}(x_0)\right\rangle > \left\langle D^2H(0)\sigma^n(x_0), \sigma^n(x_0)\right\rangle \text{ for every n.}$$

That is, the value of the quadratic form defined by $D^2H(0)$ is strictly increasing along the σ orbit of x_0. However, since this orbit eventually enters an arbitrarily small neighbourhood of x_0, we can use continuity of the quadratic form to obtain a contradiction. If G were negative definite, the quadratic form would decrease along orbits so we can obtain a similar contradiction. $\square$

3.3 Near-identity scattering matrices

Remark 3.3.1. *For our considerations, the scattering matrix σ is only determined up to left multiplication by a symplectic rotation, since*

$$\left\langle D^2H(0)|_{E^c} R_\theta \sigma k_l, R_\theta \sigma k_m\right\rangle = \left\langle R_\theta D^2H(0)|_{E^c} \sigma k_l, R_\theta \sigma k_m\right\rangle = \left\langle D^2H(0)|_{E^c} \sigma k_l, \sigma k_m\right\rangle$$

Hence, considering the form (3.12) of the Hessian of our reduced function, we see that two scattering matrices σ and $R_\theta\sigma$ are equivalent in the sense that they yield the same Hessian matrix.

Recall also our standing nondegeneracy assumption on the invariant manifolds, assumption 3. Since it can be shown that all of the spaces appearing in the statement of assumption 3 are in fact contained in the tangent space of the level set $T_{\gamma(0)}\mathcal{E}$, which has dimension $2n-1$, and since $\dim(T_{\gamma(0)}W^{cu} \oplus T_{\gamma(0)}W^s) = \dim(T_{\gamma(0)}W^{cs} \oplus T_{\gamma(0)}W^u) = 2n$, we see that in fact the assumption amounts to a transversal intersection inside the level set $\mathcal{E}$.

3.3.1 An example with $\sigma = I$

We now show that there exist systems in which the scattering matrix is given by the identity. Once we have this, we are able to apply a perturbation result due to Alishah and Lopes Diaz [68] to conclude that any symplectic matrix which is sufficiently close to the identity can be realised as the scattering matrix of a system, and furthermore that this system satisfies our assumptions.

We consider a C^3 Hamiltonian H_0 of the form;

$$\begin{aligned} H_0(q_1, ..., q_l, p_1, ..., p_l, x_1, ..., x_{n-l}, y_1, ..., y_{n-l}) = & \\ h_c(q_1, ..., q_l, p_1, ..., p_l) \quad & + h_s(x_1, ..., x_{n-l}, y_1, ..., y_{n-l}) \end{aligned} \tag{3.21}$$

Where the quadratic part of h_c is $h_{c,2} = \sum_{i+1}^{l} \frac{\omega_i}{2}(q_i^2 + p_i^2)$, with each $\omega_i \in \mathbb{R}$ distinct and such that the $(n-l)$ degree of freedom Hamiltonian vector field given by h_s has a hyperbolic equilibrium at the origin with a nondegenerate homoclinic orbit $\gamma_0(t)$, that is, a homoclinic along which the intersection of the tangent spaces to the stable and unstable manifolds is one dimensional. Furthermore, notice that the orbit $\gamma_0(t)$ is contained in the subspace $\{(\mathbf{q}, \mathbf{p}) = \mathbf{0}\}$.

We note furthermore that the Hamiltonian H_0 can be chosen to be completely integrable. For instance, we could take

$$h_c(q_1, ..., q_l, p_1, ..., p_l) = \sum_{i=1}^{l} \frac{\omega_i}{2}(q_i^2 + p_i^2) \tag{3.22}$$

$$h_s(x_1, ..., x_{n-l}, y_1, ..., y_{n-l}) = \frac{y_1^2}{2} - \frac{x_1^2}{2} + \frac{x_1^3}{3} + \sum_{i=2}^{n-l} \frac{\alpha_i}{2}(y_i^2 - x_i^2) \tag{3.23}$$

with $\alpha_i \in \mathbb{R}$. This leads to a system which has a homoclinic loop in the (x_1, y_1) plane given by $\gamma_0(t) = (g(t), \dot{g}(t))$, $g(t) = \frac{3}{2}\text{sech}^2(\frac{t}{2})$, $t \in \mathbb{R}$, and n first integrals H_0, $\xi_1, ...\xi_l$, $\eta_2, ...\eta_{n-l}$ where $\xi_i = \frac{\omega_i}{2}(q_i^2 + p_i^2)$ and $\eta_i = \frac{\alpha_i}{2}(y_i^2 - x_i^2)$. These first integrals commute with respect to the standard Poisson bracket $\{f_1, f_2\}(\cdot) = \omega(X_{f_1}(\cdot), X_{f_2}(\cdot))$.

Proposition 3.3.2. *The scattering matrix of the orbit $\gamma_0(t)$ in the system given by X_{H_0} as defined above, is the identity.*

Proof. In this case the variational equation along $\gamma_0(t)$ takes the form

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \mathbb{J}_l D^2 h_c(0) \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} \quad , \quad \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{pmatrix} = \mathbb{J}_{n-l} D^2 h_s(\gamma_0(t)) \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} .$$

The $2l$ dimensional $(\mathbf{q}, \mathbf{p})$ subsystem has constant coefficients and the fundamental matrix is a symplectic rotation, $R_{t\omega}$, where $\omega = (\omega_{\mathbf{1}}, ..., \omega_{\mathbf{l}})$. The $(\mathbf{x}, \mathbf{y})$ subsystem has only one bounded solution on $\mathbb{R}$ (as a consequence of the nondegeneracy of $\gamma_0(t)$); it is given by $\dot{\gamma_0}(t)$. For the scattering matrix, we find $\lim_{t\to\infty} R_{-t\omega}R_{2t\omega}R_{-t\omega} = I$. $\square$

Remark 3.3.3. *In the case that the integrable Hamiltonian also includes coupling terms between the center and the saddle variables, we expect to find that the scattering matrix is no longer equal to the identity, but a symplectic rotation acting in each pair of conjugate variables (see remark 12 in [48]). Since $D^2H(0)|_{E^c}$ is invariant under such transformations, the effects of this rotation can be ignored, as they will not affect the quadratic form $D^2\mathfrak{g}$. As a consequence, the results in the following sections pertaining to the realisations of the signatures should be directly applicable in a wider class of near integrable systems than merely the perturbations of H_0 studied here.*

It is also straightforward to see from the 'product' structure of the system and the nondegeneracy of $\gamma_0(t)$ that the vector field X_{H_0} satisfies the transversality assumption 3 on the invariant manifolds. In order to apply a perturbation method for constructing other scattering matrices, we will consider perturbing the flow close to $\gamma_0(t)$ on a finite time interval $[-T, T]$, for some $T > 0$. We write the scattering matrix as a composition of symplectic matrices which represent the linear flow of the $(\mathbf{q}, \mathbf{p})$ part of the variational equation on $[-\infty, -T]$, $[-T, T]$ and $[T, \infty]$ respectively;

$$\sigma = \lim_{t\to\infty} \Psi(-t)\Phi^c(t, T)\Phi^c(T, -T)\Phi^c(-T, -t)\Psi(-t) \tag{3.24}$$

Using the result of Alishah and Lopes Diaz (theorem 6) below, we see that through a localised perturbation of the flow on the segment $[-T, T]$ we can modify the $\Phi^c(T, -T)$ term to become $A\Phi^c(T, -T)$, with A symplectic and sufficiently close to the identity, without altering the other factors, or violating the nondegeneracy assumptions on the invariant manifolds.

We can choose T such that $R_{T\omega}$ (and hence also $R_{-T\omega}$) is arbitrarily close to I. Applying theorem 6 with perturbation $\tilde{A} = R_{T\omega}AR_{-T\omega}$, we find that after the perturbation we have $\sigma = A$.

Any symplectic matrix which is sufficiently close to the identity can be expressed in terms of a Hamiltonian matrix (i.e a matrix of the form $\mathbb{J}B$ with B symmetric - we denote the space of such matrices by $\mathfrak{sp}(2l, \mathbb{R})$), via the exponential map (see eg. [18])

$$\exp : \mathfrak{sp}(2l, \mathbb{R}) \to Sp(2n, \mathbb{R})$$
$$\exp(\mathbb{J}B) = Id + \mathbb{J}B + \frac{(\mathbb{J}B)^2}{2} + \frac{(\mathbb{J}B)^3}{3!} + ...$$

Since the exponential map is a diffeomorphism when restricted to a neighbourhood of zero (see eg. [60]), and we have shown that we can realise any symplectic matrix in a sufficiently small neighbourhood of the identity as a scattering matrix, we can equivalently consider $\exp(\mathbb{J}B)$ with any sufficiently small symmetric matrix B. In what follows the 'smallness' for our considerations can be achieved by a rescaling, so that we are able to consider arbitrary symmetric B.

3.3.2 Perturbation result

Following [68], we define a particular neighbourhood of a Hamiltonian $H \in C^2(\mathbb{R}^{2n})$. For $\varepsilon > 0$ and $D \subset \mathbb{R}^{2n}$, we write

$$B_\varepsilon(H, D) = \{H' \in C^2(\mathbb{R}^{2n}) : \|H' - H\|_{C^2} < \varepsilon, X_{H'} = X_H \text{ on } D\}.$$

We fix two points z, z' which lie on the same orbit of the flow given by X_H, in the level set $\mathcal{E}$ of H, $z' = \phi_H^T(z)$. Letting Σ, Σ' denote sections at z, z' transverse to the flow in the level set[5] $\mathcal{E}$. The linearised Poincaré map $D_z P_H$ is the derivative of the Poincaré map at the point z, and maps symplectically from $T_z\Sigma$ to $T_{z'}\Sigma'$. We then have the following theorem, which is a special case of theorem 2.2 from [68],

Theorem 6 (Theorem 2.2, [68]). *Let $\varepsilon > 0$, $H \in C^2(\mathbb{R}^{2n})$ with an orbit segment Γ starting at z. Then, there is a $\delta > 0$ such that for every tubular neighbourhood W of*

[5] That is $T_z\mathcal{E} = \mathbb{R}X_H(p) \oplus \Sigma$

Γ,

$$\{AD_zP_H : A \in Sp(2(n-1), \mathbb{R}) : \|A - I\|_1 < \delta\} \subset \{D_zP_{H'} : H' \in B_\varepsilon(H, D)\}$$

where $D = (M \setminus W) \cup \Gamma$.

The matrix 1-norm here is defined by $\|M\|_1 = \max_j \sum_i |m_{i,j}|$. Noting that the fundamental matrix for the variational equation, mapping between the tangent spaces at two points on an orbit is in fact the same object as the linear Poincaré map (both are obtained by linearising the flow map), this applies to our situation as outlined above; if we consider sections at $T_{\gamma_0(-T)}\mathcal{E}$ and $T_{\gamma_0(T)}\mathcal{E}$ which contain the symplectic subspaces $\{(\mathbf{x}, \mathbf{y}) = \mathbf{0}\}$, we can obtain all sufficiently near-identity symplectic matrices between these subspaces as a result of the theorem. Furthermore, since our unperturbed Hamiltonian H_0 is C^3, we see from the proofs in [68] that we can realise the perturbed linear flows with perturbed Hamiltonians which are also C^3, although they are only guaranteed to be close to H_0 in the C^2 topology. This still results in a C^1 perturbation of the flow which can be arbitrarily small, meaning that the tranversality assumption 3 will not be violated. Since the perturbed Hamiltonians agree with H_0 outside of the tubular neighbourhood, all of the assumptions about the equilbrium will also be satisfied.

3.4 All indefinite signatures are possible

In light of the previous section, we now investigate the case in which the scattering matrix is a near identity symplectic transformation, which can be expressed as the flow along a Hamiltonian vector field. This means that we can write

$$\sigma = \exp(-\varepsilon \mathbb{J} B) = I - \varepsilon \mathbb{J} B + h.o.t \tag{3.25}$$

with $\varepsilon << 1$ and B an arbitrary symmetric matrix. Substituting the form (3.25) into the expression (3.12) yields;

$$\frac{1}{\varepsilon}\frac{\partial^2 \mathfrak{g}}{\partial \beta_i \partial \beta_j} = B\mathbb{J}D^2H(0) - D^2H(0)\mathbb{J}B + \mathcal{O}(\varepsilon)$$

Since the eigenvalues of a matrix depend continuously on its entries, for sufficiently small ε, the Hessian of $\mathfrak{g}$ has the same signature as $B\mathbb{J}D^2H(0) - D^2H(0)\mathbb{J}B$. Our goal then, is to determine the possible signatures of this matrix and apply the Morse lemma to gain a local picture of the set of homoclinic connections to the centre manifold. As a first observation, a simple calculation tells us that the trace is zero, which rules out the possibility that the matrix could be sign-definite, in agreement with theorem 3 (i). For a deeper investigation, we begin by defining the map

$$\begin{aligned} \chi_A : \mathrm{Sym}(\mathbb{R}^{2n\times 2n}) &\to \mathrm{Sym}(\mathbb{R}^{2n\times 2n}) \\ \chi_A(B) &= B\mathbb{J}A - A\mathbb{J}B \end{aligned}$$

Where $\mathrm{Sym}(\mathbb{R}^{2n\times 2n})$ denotes the symmetric $2n \times 2n$ matrices with real entries. We can now express the set of matrices that we are studying as $\mathcal{R}(\chi_{D^2H(0)|_{W^c}})$. To gain a characterisation of this range, we endow $\mathrm{Sym}(\mathbb{R}^{2n\times 2n})$ with the inner product

$$\langle M_1, M_2 \rangle = \mathrm{tr}(M_1 M_2) \tag{3.26}$$

that is, the inner product of M_1 and M_2 is the trace of their ordinary matrix product. This allows us to write $\mathcal{R}(\chi_{D^2H(0)|_{W^c}}) = \ker(\chi^*_{D^2H(0)|_{W^c}})^{\perp}$, where both the adjoint and the orthogonal complement are taken with respect to (3.26). We calculate the adjoint as follows

$$\begin{aligned} \mathrm{tr}(\chi_A(B)M) &= \mathrm{tr}((B\mathbb{J}A - A\mathbb{J}B)M) \\ &= \mathrm{tr}(B\mathbb{J}AM) - \mathrm{tr}(A\mathbb{J}BM) \\ &= \mathrm{tr}(B\mathbb{J}AM) - \mathrm{tr}(BMA\mathbb{J}) \\ &= \mathrm{tr}(B(\mathbb{J}AM - MA\mathbb{J})) \end{aligned}$$

Hence,

$$\chi_A^*(M) = \mathbb{J}AM - MA\mathbb{J}$$

Recall that in our coordinates the second derivative of the Hamiltonian restricted to the center subspace takes the diagonal form

$$D^2H(0)|_{E^c} = \mathrm{diag}(\omega_1, \dots, \omega_n, \omega_1, \dots, \omega_n)$$

We also use the notation $\mathrm{diag}(M)$, for $M \in \mathbb{R}^{2n\times 2n}$, to denote the vector which contains the diagonal elements of M.

Lemma 3.4.1. *If* $\omega_1^2 \neq \omega_2^2 \neq \ldots \neq \omega_n^2$, *then*

(i) $\ker(\chi^*_{D^2H(0)}) = \{\mathrm{diag}(a_1, \dots, a_n, a_1, \dots, a_n)|\, a_i \in \mathbb{R}\}$

(ii) $\mathcal{R}(\chi_{D^2H(0)}) = \{M \in \mathrm{Sym}(\mathbb{R}^{2n\times 2n})|\, \mathrm{diag}(M) = (g_1, \dots, g_n, -g_1, \dots, -g_n),\ \ g_i \in \mathbb{R}\}$

Proof. (i) We use an induction argument on n. The statement is easily verified for $n = 1$. Assuming the case $n = i$, we now consider $n = i + 1$. We write $K \in \mathrm{Sym}(\mathbb{R}^{2(i+1)\times 2(i+1)})$ as

$$K = \begin{pmatrix} & & k^1_{1,i+1} & & & k^2_{1,i+1} \\ & K^1 & \vdots & & K^2 & \vdots \\ k^1_{i+1,1} & \dots & k^1_{i+1,i+1} & k^2_{i+1,1} & \dots & k^2_{i+1,i+1} \\ & & k^2_{i+1,1} & & & k^3_{1,i+1} \\ & (K^2)^T & \vdots & & K^3 & \vdots \\ k^2_{1,i+1} & \dots & k^2_{i+1,i+1} & k^3_{i+1,1} & \dots & k^3_{i+1,i+1} \end{pmatrix}$$

with each K^j being an $i\times i$ matrix, with K^1 and K^3 symmetric, and also writing A^i for the matrix $\mathrm{diag}(\omega_1, \dots, \omega_i)$, we arrive at

$$KD^2H(0)|_{W^c}\mathbb{J} =$$

$$
\begin{pmatrix}
 & & -\omega_{i+1}k^2_{1,i+1} & & & \omega_{i+1}k^1_{1,i+1} \\
 & -K^2A^i & \vdots & & K^1A^i & \vdots \\
-\omega_1 k^2_{i+1,1} & \dots & -\omega_{i+1}k^2_{i+1,i+1} & \omega_1 k^1_{i+1,1} & \dots & \omega_{i+1}k^1_{i+1,i+1} \\
 & & -\omega_{i+1}k^3_{1,i+1} & & & \omega_{i+1}k^2_{i+1,i} \\
 & -K^3A^i & \vdots & & (K^2)^TA^i & \vdots \\
-\omega_1 k^3_{1,i+1} & \dots & -\omega_{i+1}k^3_{i+1,i+1} & \omega_1 k^2_{1,i+1} & \dots & \omega_{i+1}k^2_{i+1,i+1}
\end{pmatrix}
$$

and

$$
\mathbb{J}D^2H(0)|_{W^c}K =
$$

$$
\begin{pmatrix}
 & & \omega_1 k^2_{i+1,1} & & & \omega k^3_{1,i+1} \\
 & A^i(K^2)^T & \vdots & & A^iK^3 & \vdots \\
\omega_{i+1}k^2_{1,i+1} & \dots & \omega_{i+1}k^2_{i+1,i+1} & \omega_{i+1}k^3_{1,i+1} & \dots & \omega_{i+1}k^3_{i+1,i+1} \\
 & & -\omega_1 k^1_{1,i+1} & & & -\omega_1 k^2_{1,i+1} \\
 & -A^iK^1 & \vdots & & -A^iK^2 & \vdots \\
-\omega_{i+1}k^1_{i+1,1} & \dots & -\omega_{i+1}k^1_{i+1,i+1} & -\omega_{i+1}k^2_{i+1,1} & \dots & -\omega_{i+1}k^2_{i+1,i+1}
\end{pmatrix}
$$

Equating these matrices, we find that the block components are equal if and only if the matrix

$$
\begin{pmatrix} K^1 & K^2 \\ (K^2)^T & K^3 \end{pmatrix} \in \mathrm{Sym}(\mathbb{R}^{2i \times 2i})
$$

lies in the kernel for the i-dimensional case. By the induction hypothesis, this matrix thus has the form given in (i). Equating the remaining components gives firstly

$$
\begin{aligned}
-\omega_{i+1}k^2_{i+1,i+1} &= \omega_{i+1}k^2_{i+1,i+1} & &\Rightarrow k^2_{i+1,i+1} = 0 \\
\omega_{i+1}k^1_{i+1,i+1} &= \omega_{i+1}k^3_{i+1,i+1} & &\Rightarrow k^1_{i+1,i+1} = k^3_{i+1,i+1}
\end{aligned}
$$

Furthermore, we obtain a collection of pairs of simultaneous linear equations, one example being

$$
\begin{pmatrix} w_{i+1} & -\omega_1 \\ -\omega_1 & \omega_{i+1} \end{pmatrix} \begin{pmatrix} k^1_{1,i+1} \\ k^3_{1,i+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

If $\omega_{i+1}^2 - \omega_1^2 \neq 0$, we thus obtain that $k_{1,i+1}^1 = k_{1,i+1}^3 = 0$. Accounting for all components in a similar way tells us that provided $\omega_{i+1}^2 \neq \omega_k^2$ for $k \in \{1, \ldots, i\}$, we must have all other components equal to zero. Thus the only degree of freedom is in choosing the value of $k_{i+1,i+1}^1 = k_{i+1,i+1}^3$, and so K itself is of the form given in (i). This concludes the induction step and thus the proof of (i).

(ii) This follows easily from (i), using the characterisation

$$\mathcal{R}(\chi_{D^2H(0)|_{W^c}}) = \ker(\chi^*_{D^2H(0)|_{W^c}})^{\perp}$$

. □

In this section we prove the following theorem:

Theorem 7. *The first order approximation to the Hessian*

$$D^2H(0)\mathbb{J}B - B\mathbb{J}D^2H(0)$$

can take any signature except $(2l, 0)$ *or* $(0, 2l)$.

The proof is based upon an application of a theorem from [67]. Before stating the theorem we introduce some notation

Definition 3.4.2. *For two vectors* $(a_1, \ldots, a_n)$ *and* $(b_1, \ldots, b_n)$ *in* $\mathbb{R}^n$*, the expression*

$$(a_1, \ldots, a_n) \prec (b_1, \ldots, b_n)$$

will mean that when the elements are renumbered so that

$$a_1 \geq \ldots \geq a_n, \text{ and } b_1 \geq \ldots \geq b_n,$$

then

$$a_1 + \ldots + a_k \leq b_1 + \ldots + b_k \;\; (k = 1, \ldots \;\; n-1) \tag{3.27}$$

$$a_1 + \ldots + a_n = b_1 + \ldots + b_n. \tag{3.28}$$

Theorem 8 (L. Mirsky, [67]). *Let $\omega_1, \ldots, \omega_n$, $a_1, \ldots, a_n$ be real numbers. Then*

$$(a_1, \ldots, a_n) \prec (\omega_1, \ldots, \omega_n)$$

is the necessary and sufficient condition for the existence of a real symmetric $n \times n$ matrix with $\omega_1, \ldots, \omega_n$ as its eigenvalues and $a_1, \ldots, a_n$, in that order, as its diagonal elements.

We now use this criterion to prove theorem 7. The idea of the proof will be to demonstrate that taking the vector g given by

$$(g_1, \ldots, g_l, g_{l+1}, \ldots, g_{2l}) = (1, \ldots, 1, -1, \ldots, -1),$$

and any $m \in \{1, \ldots, 2l-1\}$, we can demonstrate a vector $b \in \mathbb{R}^{2l}$ with m positive and $(2l-m)$ negative elements, satisfying

$$(g_1, \ldots, g_{2l}) \prec (b_1, \ldots, b_{2l}).$$

Appealing to theorem 8 will then provide us with a matrix in $G \in \mathrm{Sym}(\mathbb{R}^{2l \times 2l})$ whose diagonal elements are given by g (and hence $G \in \mathcal{R}(\chi_{D^2H(0)})$), whose eigenvalues are $b_1, \ldots, b_{2l}$, and hence G has signature $(m, 2l-m)$.

Proof of theorem 7. Choose any $m \in \{1, \ldots, 2l-1\}$, and write

$$b = \Big(\underbrace{2l-m, 1, \ldots, 1}_{m \text{ elements}}, \underbrace{\frac{-(2l-1)}{(2l-m)}, \frac{-(2l-1)}{(2l-m)}, \ldots, \frac{-(2l-1)}{(2l-m)}}_{(2l-m) \text{ elements}} \Big)$$

As explained above, the theorem will be proved if we can demonstrate that $g \prec b$ (with g as defined above). Firstly, we note that the elements of g and b are already numbered in the appropriate nonincreasing order, and that

$$g_1 + \ldots + g_{2l} = b_1 + \ldots + b_{2l} = 0.$$

To prove that (3.27) is satisfied, we consider the cases $m > l$ and $m \le l$ separately.

Case (1a): $m > l$, $k \in \{1, \dots, l\}$. For k in this range, the inequalities in (3.27) take the form

$$\begin{aligned} k \ &\leq (2l - m) + (k - 1) \\ \Leftrightarrow 0 \ &\leq 2l - m - 1 \end{aligned}$$

which is true since $m \in \{1, \dots, 2l - 1\}$.
Case (1b): $m > l$, $k \in \{l + 1, \dots, m\}$ Here (3.27) becomes

$$2l - k \ \leq \ (2l - m) + (k - 1)$$

so

$$-k \ \leq \ (k - 1) - m$$

and since $l + 1 \leq k \leq m$, this means

$$l + 1 - m - 1 \leq k - m - 1$$

so we need $l - m \geq -k$. But $m \leq (2l - 1)$ so

$$\begin{aligned} l - m &\geq l - (2l - 1) \\ &\geq -l - 1 \\ &\geq -k. \end{aligned}$$

Case (1c): $m > l$, $k \in \{m + 1, \dots, 2l\}$. We now have

$$2l - k \leq (2l - 1) - (k - m)\frac{(2l - 1)}{(2l - m)}$$

and since $(2l - m) > 0$ this simplifies to

$$km \leq k + 2l(m - 1).$$

Assuming for contradiction that $km > k + 2l(m-1)$ leads to

$$m - 1 > \frac{2l}{k}(m-1)$$

but since $\frac{2l}{k} \geq 1$, this is our required contradiction.
Case (2a): $m \leq l$, $k \in \{1, \ldots, m\}$ This is the same as case (1a).
Case (2b): $m \leq l$, $k \in \{m, \ldots, l\}$ We now need to show

$$k \leq (2l-1) - (k-m)\frac{(2l-1)}{(2l-m)}.$$

This simplifies to

$$\frac{k}{2l}(4l - m - 1) \leq (2l - 1)$$

and since $\frac{k}{2l} \leq \frac{1}{2}$ and $(4l - m - 1) \leq (4l - 2)$, this is true.
Case (2c): $m \leq l$, $k \in \{l+1, \ldots, 2l\}$ This is the same as case (1c). $\square$

3.5 Special case

3.5.1 The reversible Hamiltonian case

Let us now assume further that our Hamiltonian system is reversible with respect to a linear involution which acts antisymplectically $R : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, and also that the homoclinic to the equilibrium $\gamma(t)$ is *symmetric*, as described by assumption 4. This implies that R and $DX_H(0)$ share the same invariant subspaces, and in particular the restriction of $\mathbb{J}DX_H(0)$ to the center subspace E^c is reversible with the respect to the restriction of R to E^c. By a symplectic change of coordinates in E^c which amounts to averaging the inner product over the finite group generated by R and $\mathbb{J}$, we are able to assume without loss of generality that $\mathbb{J}$ takes its standard form $\mathbb{J} = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ and R is orthogonal, (see for instance appendix B of [40]). Since $R^2 = I$, this means that R is symmetric. In what follows we sometimes write R for the restriction of R to E^c, when the context is clear.

In this section we prove theorem 4. First, we assemble some properties of the scattering matrix and the Hessian.

Lemma 3.5.1. *Under assumption 4,*

(i) The scattering matrix σ satisfies $\sigma \circ R \circ \sigma = R$.

(ii) $D^2\mathfrak{g} \circ (R \circ \sigma) = -(R \circ \sigma)^T \circ D^2\mathfrak{g}$.

Proof. (i.) The scattering matrix is defined as $\lim_{t\to\infty} \Psi(-t)\Phi^c(t,-t)\Psi(-t)$. As a consequence of assumption 4 we have $R\Phi(t,-t) = \Phi(-t,t)R$ and since the dynamics in the centre subspace of the equilibrium are reversible, we also have $R\Psi(-t) = \Psi(t)R$. Furthermore, the family of subspaces $Y^c(t) \subset T_{\gamma(t)}\mathbb{R}^{2n}$ satisfy $RY^c(t) = Y^c(-t)$ which leads to $P^c(t) = RP^c(-t)R$ where $P^c(t)$ is our projection onto $Y^c(t)$. Combining these relations and applying them to the definition of σ yields $R \circ \sigma = \sigma^{-1} \circ R$ and hence the result. Regarding part (iii), we already have the expression $-D^2\mathfrak{g} = D^2H(0)|_{E^c} - \sigma^T(D^2H(0)|_{E^c})\sigma$, so that

$$-D^2\mathfrak{g} \circ (R \circ \sigma) = (D^2H(0)|_{E^c})R\sigma - \sigma^T(D^2H(0)|_{E^c})\sigma R\sigma.$$

Since the linearisation $\mathbb{J}D^2H(0)|_{E^c}$ is reversible, and since R acts antisymplectically, this implies that $D^2H(0)$ commutes with R. Using this fact and (i) brings us to

$$-D^2\mathfrak{g} \circ (R \circ \sigma) = R(D^2H(0)|_{E^c})\sigma - \sigma^T R(D^2H(0)|_{E^c}).$$

The claim follows using $R^T = R$ and (i) again. □

The idea in what follows is to choose a basis of $\ker(L)$ in which $D^2\mathfrak{g}$ becomes $R\circ\sigma$ reversible, thus implying a symmetry of the spectrum, which gives the (l,l) signature. Looking at (ii), we see that $D^2\mathfrak{g}$ is $(R \circ \sigma)$ reversible if $(R \circ \sigma)$ is symmetric. Since $(R \circ \sigma)$ is an involution, this is the same as being orthogonal.

Proof of theorem 4 part (b). Define a new inner product by

$$[x,y] = \frac{1}{2}\left(\langle x,y\rangle + \langle (R\circ\sigma)x, (R\circ\sigma)y\rangle\right)$$

$$= \left\langle \frac{1}{2}(I + (R \circ \sigma)^T (R \circ \sigma))x, y \right\rangle.$$

Note that

$$[(R \circ \sigma)x, (R \circ \sigma)y] = [x, y]. \tag{3.29}$$

Since $\frac{1}{2}(I + (R \circ \sigma)^T(R \circ \sigma))$ is symmetric and positive definite, it has a uniquely defined symmetric square root so we can write

$$\frac{1}{2}(I + (R \circ \sigma)^T (R \circ \sigma)) = S^T S$$

and hence

$$[x, y] = \langle Sx, Sy \rangle$$

So, the new inner product is just the old one but in the new basis given by applying S to the old basis. Looking at (3.29) tells us that in this basis, $R \circ \sigma$ is an isometry, and hence represented by an orthogonal matrix. So, in this basis we have the relation

$$D^2\mathfrak{g} \circ (R \circ \sigma) = -(R \circ \sigma) \circ D^2\mathfrak{g}$$

which is what we wanted, and so the signature of $D^2\mathfrak{g}$ must be (l, l), since $D^2\mathfrak{g}$ is related to $-D^2\mathfrak{g}$ by a similarity transform. $\square$

3.6 Discussion

The results from this chapter show that, under a weak assumption of nondegeneracy (which I believe to be generic), there are always homoclinic connections to the center manifold other than $\gamma(t)$ itself. This can be seen as a weak generalisation of the result of Lerman and Koltsova from [47] concerning the case of a single pair of purely imaginary eigenvalues to the case of a center manifold of arbitrary dimension. In [47], under the same transversality hypothesis on $\gamma(t)$ as here, and with the assumption that the scattering matrix is not a pure rotation (which is implied by the nondegeneracy of the critical point of $\mathfrak{g}$, as assumed here), four homoclinics to each sufficiently small periodic orbit in the center manifold are found. This implies in particular that the intersection of the invariant manifolds corresponding to the homoclinic $\gamma(t)$ is not isolated, a fact implied by our theorem 3 in the context of an arbitraily high dimensional center manifold.

There are a number of further questions which would represent interesting future work, regarding the nature of the elements in $\mathfrak{g}^{-1}(0)$. For example, which elements represent homoclinics to the same invariant sets within the center manifold, and which represent heteroclinics? To begin this investigation, we could firstly make slightly stronger hypotheses on the equilibrium; namely a stronger nonresonance assumption in the center subspace and an isoenergetic nondegeneracy condition, in order to apply KAM results [71], giving a Cantor set of positive measure filled with invariant tori in the center manifold around the origin.

A closer inspection of the way in which σ maps this collection of tori onto itself should allow us to distinguish homoclinic and heteroclinic orbits to these invariant tori in W^c. This is in the same spirit as the study by Yagasaki [88], who also confirms (under restrictive hypotheses on $\gamma(t)$, as mentioned in the introduction to this chapter) that a zero of $\mathfrak{g}$ at which the derivatives in the angular directions (in symplectic polar coordinates) are nonzero, corresponds to a transversal intersection of stable and unstable manifolds of a pair of tori.

If the system is reversible then a heteroclinic orbit between two tori is accompanied by a second orbit going the other way between them, closing a heteroclinic chain. With the help of a version of the lambda lemma (see eg [25]), it should then be possible to deduce the existence of shift dynamics.

Since the proportion of the center manifold occupied by KAM tori approaches 1 in sufficiently small neighbourhoods of the origin [71], it would seem plausible that (at least in the positive definite case), the set $\mathfrak{g}^{-1}(0)$ may intersect a KAM torus in every level set close enough to the origin. With reversibility, this would seemingly be enough to conclude the existence of chaotic dynamics in each of these level sets, by the argument sketched above.

It is also possible that some of the conditions and assumptions imposed in this section could be relaxed. For instance, the distinctness of the eigenvalues in assumption 1 may not be necessary, although the linear algebra in section 3.4 would require modification. We could also allow the 'center' eigenvalues to have a small real part, making the equilibrium into a saddle-focus. In principle the Lyapunov-Schmidt approach could still be applied (as the weighted norm can still be used to separate these degrees of freedom with very slow growth), but the details of the linear analysis conducted here in the study of the signatures would be affected by the lack of invariance under symplectic rotations of $D^2H(0)$ in the center subspace. By unfolding the system in a parameterised family (the Lyapunov-Schmidt reduction extends to this situation naturally), the loop $\gamma(t)$ could be broken, and the effects on the zero set could also be studied.

Chapter 4

A type of scattering map

In this chapter we allow our system to depend on parameters, and use the implicit function theorem and the invariant foliations of the center stable and center unstable manifolds to construct a symplectic map S which allows us to find homoclinics to the center manifold as zeros of the bifurcation function $\xi(x, \mu) = H(S(x, \mu), \mu) - H(x, \mu)$. That is, we find the nonlinear analogue of the quadratic form we studied previously.

The application of the implicit function theorem here achieves essentially the same thing as that performed in the previous section (in the Lyapunov-Schmidt reduction) when we were working in function spaces - the difference here is that we make use of the geometry of the invariant foliations. As mentioned previously, the tools used in the previous chapter (Fredholm theory, Lyapunov-Schmidt reduction) are available even in some scenarios in which the invariant foliations are not known to exist, such as the advance-delay equation context.

The name *scattering map* is chosen primarily because it plays the same role as the scattering matrix did in the linear theory of the previous chapter, and secondly since, as alluded to in the introduction, it resembles the construction of the scattering map which appears in the literature, see for instance [27, 21] in the more general context of normally hyperbolic invariant manifolds. The purpose of our map however is not quite the same as that which appears in these works. We use our scattering map to detect intersections of stable and unstable fibres, that is, homoclinic intersections. The scattering map constructed in these other papers however, is used to detect orbits in a neighbourhood of a given homoclinic intersection which follow certain itineraries. To achieve this, a finite-time analog of the scattering map (called the transition map) is used, along with the so-called 'inner map', which is given by the dynamics inside the normally hyperbolic invariant manifold. Pseudo orbits are found which alternately follow the dynamics of the transition map (corresponding to fast excursions) and some power of the inner map (corresponding to periods spent close to the normally hyperbolic invariant manifold). Shadowing results are then used to infer the existence of real orbits close to the pseudo orbits. In these works, a standing assumption is that the stable and unstable manifolds of the normally hyperbolic invariant manifold intersect transversally - in our situation this is not true for our given homoclinic orbit γ; as a consequence of the conservation of energy the center stable and center unstable manifolds cannot intersect transversally along our homoclinic orbit. However, if a nondegenerate zero of ξ is found, then the restriction of our scattering map to the zero set of ξ in a neighbourhood of this point would agree with the definition in these other works. Despite these differences, we persist in calling our map a scattering map, at least for the duration of this thesis. Most similar to our approach (in the sense that it deals with center manifolds) is the paper [26] by Garcia, where a scattering map is constructed for the dynamics of a Poincaré map in a periodically forced Hamiltonian system. Many of the considerations are similar, but again, in our continuous time case, the transversality assumptions are not satisfied. In the context of one-round homoclinics, the paper [44] also uses the approach of finding pairs of semi-orbits with a jump in a certain subspace of a Poincaré section, in a reversible system without Hamiltonian sructure.

Once our map S is defined, we can find zeros of ξ by finding intersections of level sets of the Hamiltonian on the center manifold with their image by the scattering map. This is especially useful in the case of a 2D center manifold, as we will see in section 5.2.

4.1 Finding semi-orbits with a jump in energy

Considering system 2.1 once more, that is

$$\dot{u} = X_H(u, \mu) \qquad \mu \in \mathbb{R}^k$$

we again assume that the origin has $2l$ eigenvalues with zero real part and that there exists a homoclinic loop $\gamma(t)$:

Assumption 8. *The spectrum of the linearisation $DX_H(0)$ consists of $2l$ distinct eigenvalues with zero real part, and $2(n-l)$ eigenvalues λ_i, whose real parts are bounded away from zero;* $0 < \alpha < |\Re(\lambda_i)|$ $i \in \{1, .., 2(n-l)\}$.

Note that this assumption allows for the existence of zero eigenvalues as well as pairs of nonzero purely imaginary ones. We also assume that the Hamiltonian on the center manifold in a neighbourhood of the origin has been modified by a smooth cutoff function as described in chapter 2 in order to keep all orbits bounded.

Assumption 9. *When $\mu = 0$ there exists an orbit $\Gamma = \{\gamma(t) : t \in \mathbb{R}\}$ such that $\Gamma \subset W^s \cap W^u$.*

The homoclinic must also be nondegenerate, so we again make assumption 3, namely

$$\dim(T_{\gamma(0)}W^{cu} \cap T_{\gamma(0)}W^s) = \dim(T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^u) = 1.$$

Firstly we find pairs of initial conditions $x^- \in W^{cu}(0) \cap \Sigma$, $x^+ \in W^{cs}(0) \cap \Sigma$, which differ only by a jump in a designated subspace of our transverse section Σ. Recall, as discussed in chapter 2, that the section Σ at $\gamma(0)$ is decomposed as

$$\Sigma = Y^c \oplus Y^s \oplus Y^u \oplus Z$$

where $Z = \text{span}\{\nabla H(\gamma(t))\}$. Choosing coordinates according to this splitting, the images in Σ of the local center unstable and center stable manifolds, mapped by the flow along $\gamma(t)$ in forward and backward time respectively, are graphs of functions h^{cu}, h^{cs};

$$h^{cu} : Y^c \oplus Y^u \times \mathbb{R}^m \to Y^s \oplus Z$$
$$h^{cs} : Y^c \oplus Y^s \times \mathbb{R}^m \to Y^u \oplus Z$$

satisfying $D_y h^{cs}(0,0,0) = D_y h^{cu}(0,0,0) = 0$. So we have

$$\Sigma \cap W^{cu}(0,\mu) = \{(y_c, h_s^{cu}(y_c, y_u, \mu), y_u, h_z^{cu}(y_c, y_u, \mu))\}$$
$$\Sigma \cap W^{cs}(0,\mu) = \{(y_c, y_s, h_u^{cs}(y_c, y_s, \mu), h_z^{cs}(y_c, y_s, \mu))\}$$

and we want to solve

$$\begin{cases} y_c = y_c \\ y_s = h_s^{cu}(y_c, y_u, \mu) \\ y_u = h_u^{cs}(y_c, y_s, \mu) \end{cases} \tag{4.1}$$

simultaneously for $(y_u(y_c,\mu), y_s(y_c,\mu))$, so that we can take

$$v^-(y_c,\mu) = (y_c, h_s^{cu}(y_c, y_u(y_c,\mu),\mu), y_u(y_c,\mu), h_z^{cu}(y_c, y_u(y_c,\mu),\mu)),$$
$$v^+(y_c,\mu) = (y_c, y_s(y_c,\mu), h_u^{cs}(y_c, y_s(y_c,\mu),\mu), h_z^{cs}(y_c, y_s(y_c,\mu),\mu))$$

so that the initial conditions

$$x^-(y_c,\mu) := \gamma(0) + v^-(y_c,\mu)$$
$$x^+(y_c,\mu) := \gamma(0) + v^+(y_c,\mu)$$

lead to the desired semi- orbits with a jump equal to

$$h_z^{cu}(y_c, y_u(y_c,\mu),\mu) - h_z^{cs}(y_c, y_s(y_c,\mu),\mu) \in Z.$$

Defining

$$\mathcal{F} : Y^c \oplus Y^s \oplus Y^u \times \mathbb{R}^m \to Y^s \oplus Y^u$$
$$\mathcal{F}(y_c, y_s, y_u, \mu) = \begin{pmatrix} y_s - h_s^{cu}(y_c, y_u, \mu) \\ y_u - h_u^{cs}(y_c, y_s, \mu) \end{pmatrix}$$

we observe that $\mathcal{F}(0,0,0,0) = (0,0)$ and $D_{(y_s,y_u)}\mathcal{F}(0,0,0,0) = Id$, meaning that we can apply the implicit function theorem to solve (4.7) in a neighbourhood of zero, yielding the pairs of initial conditions we seek.

4.1.1 $X^{\pm}$ are symplectic and transverse to the leaves of the foliations.

The sets of initial conditions found above define manifolds X^-, X^+ in Σ. We show here that these manifolds are transverse to the leaves of the unstable (resp. stable) foliations of the center unstable (resp. center stable) manifolds. This fact will allow us to define diffeomorphisms from X^- and X^+ to the center manifold using the flow and the unstable (resp. stable) projections.

We will then show that $X^{\pm}$ carry a symplectic structure, and that the diffeomorphisms described in the previous paragraph are symplectic. These diffeomorphisms are key in defining the scattering map.

Definition 4.1.1.

$$X_\mu^{\pm} = \{x \in \Sigma : x = x^{\pm}(y_c, \mu) \text{ for some } y_c \in Y^c\} \subset W_\mu^{cu,cs}(0) \cap \Sigma$$

Proposition 4.1.2. *$X_\mu^{\pm}$ are symplectic surfaces. That is, $\omega|_{X_\mu^{\pm}}$ is nondegenerate.*

Proof. We focus on X^-, since the argument for X^+ is similar. The intersection $W^{cu} \cap \Sigma$ is given locally by the graph of a function

$$h^{cu} : Y^c \oplus Y^u \times \mathbb{R}^m \to Y^s \oplus Z$$

where $D_1 h^{cu}(0,0) = 0$ since $W_{(0,0)}^{cu} \cap \Sigma$ is tangent to $Y^c \oplus Y^u$ at $\gamma(0)$. Now, X^- can be seen as the graph of a restricted version $\tilde{h}^{cu}$ of h^{cu} defined using the implicit function constructed in section 4.1;

$$\begin{aligned} \tilde{h}^{cu} : \ Y^c \times \mathbb{R}^m &\to \quad Y^u \oplus Y^s \oplus Z. \\ \tilde{h}^{cu}(y_c, \mu) = &\quad h^{cu}(y_c, y_u(y_c, \mu), \mu) \end{aligned}$$

So, X^- is a smooth surface, and the tangent space $T_{\gamma(0)}X_0^-$ is given by

$$T_{\gamma(0)}X_0^- = \{y_u = D_{y_c}y_u(0) \cdot y_c\} \subset Y^c \oplus Y^u.$$

Now, since $\omega|_{Y^c}$ is nondegenerate, for any $y_c \in Y^c$ there is a $y_c^* \in Y^c$ such that $\omega(y_c, y_c^*) \neq 0$, so, taking $y_c + D_{y_c}y_u(0) \cdot y_c \in T_{\gamma(0)}X_0^-$ and using both that $Y^u \perp^\omega Y^c$ and $\omega|_{Y^u} = 0$, we find;

$$\omega(y_c + D_{y_c}y_u(0) \cdot y_c, y_c^* + D_{y_c}y_u(0) \cdot y_c^*) = \omega(y_c, y_c^*) \neq 0.$$

Hence $\omega|_{X^-}$ is nondegenerate at the origin, and hence in a neighbourhood of the origin, when $\mu = 0$, and so by continuous dependence, also in a neighbourhood of the origin in X_μ^- for sufficiently small parameter values.

□

Proposition 4.1.3. *X_μ^- and X_μ^+ are transverse to the leaves of the unstable (resp.) stable foliations in a neighbourhood of zero in Σ.*

Proof. We prove the statement for X_μ^-, since X_μ^+ is similar. As the statement is local, it suffices to prove that X_0^- is transverse in $W_0^{cu}(0)$ to the leaf containing the origin in Σ, which corresponds to $\gamma(0)$. Once this is established, the continuity of the foliation and the smooth dependence on parameters yields the full statement. The tangent space to the intersection of this leaf with Σ is exactly Y^u. As seen in the proof of proposition 4.1.2, the tangent space to X^- contains Y^c. Hence, the sum of these tangent spaces spans $Y^c \oplus Y^u$, which is exactly the tangent space to $\Sigma \cap W^{cu}(0)$. □

Proposition 4.1.4. *The restricted projections $\pi^u : \phi^{-T}(X^-) \to W^c$ and $\pi^s : \phi^T(X^+) \to W^c$ are symplectomorphisms.*

Proof. The flow ϕ preserves the transversality to the leaves of the foliation, and so the smoothness and invertibility are given by the standard results from [77], as discussed in chapter 2.

We have the following argument for symplecticity: (largely borrowed from [26]): Let $z \in \phi^{-T}(X^-)$ and $v_1,\ v_2 \in T_z\phi^{-T}(X^-)$. We need to show that

$$\omega_z(v_1, v_2) = \omega_{\pi^u(z)}(D\pi^u(z)v_1, D\pi^u(z)v_2). \tag{4.2}$$

Note that for every t,

$$\omega_z(v_1, v_2) = \omega_{\phi^{-t}(z)}(D\phi^{-t}(z)v_1, D\phi^{-t}(z)v_2)$$

and

$$\begin{aligned}
\omega_{\pi^u(z)}(D\pi^u(z)v_1, D\pi^u(z)v_2) &= \omega_{\phi^{-t}(\pi^u(z))}(D\phi^{-t}(\pi^u(z))D\pi^u(z)v_1, D\phi^{-t}(\pi^u(z))D\pi^u(z)v_2) \\
&= \omega_{\phi^{-t}(\pi^u(z))}(D(\phi^{-t} \circ \pi^u)(z)v_1, D(\phi^{-t} \circ \pi^u)(z)v_2) \\
&= \omega_{\phi^{-t}(\pi^u(z))}(D(\pi^u \circ \phi^{-t})(z)v_1, D(\pi^u \circ \phi^{-t})(z)v_2)
\end{aligned}$$

but letting $t \to \infty$, $\phi^{-t}(z) \to \phi^{-t}(\pi^u(z))$, and $D(\pi^u \circ \phi^{-t})(z)v_i \to D\phi^{-t}(z)v_1$, so (4.2) follows. $\square$

We now write

$$\begin{aligned}
S^- &:= \phi^{-T} \circ (\pi^u)^{-1} \\
S^+ &:= \pi^s \circ \phi^T.
\end{aligned}$$

So that $X^-_\mu = \mathrm{Im}(S^-_\mu)$ and $X^+_\mu = \mathrm{Im}((S^+_\mu)^{-1})$. As established in section 4.1, each point $S^-_\mu(x) \in X^-_\mu$ is equal to $x^-(y_c, \mu)$ for a unique y_c, and so there exists a unique $x^+(y_c, \mu) \in X^+_\mu$ such that

$$x^-(y_c, \mu) - x^+(y_c, \mu) =: \mathcal{Z}(S^-(x, \mu), \mu) \in Z,$$

We can now define our mapping S by

$$S(x,\mu) := S^+\Big(S^-(x,\mu) - \mathcal{Z}(S^-(x,\mu),\mu)),\mu\Big) \tag{4.3}$$

With this definition in place, it is clear that the condition for the existence of a homoclinic orbit connecting the center manifold to itself, which approaches the orbit of the point $x \in W^c$ as time tends to negative infinity, is

$$\xi(x,\mu) := H(S(x,\mu),\mu) - H(x,\mu) = 0.$$

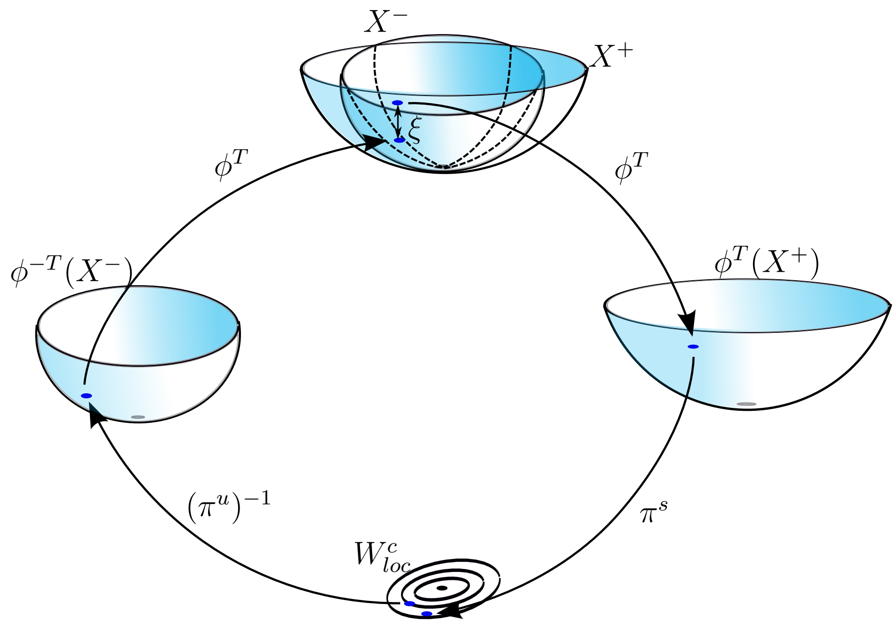


Figure 4.1: *Schematic showing the construction of S*

Lemma 4.1.5. *For each μ, $S(\cdot,\mu)$ is symplectic.*

Proof. Suppressing parameter dependence for brevity, and using symplecticity of S^+ and S^-, we calculate;

$$\begin{aligned}
&\omega|_{W^c}(DS(x)v_1, DS(x)v_2) = \\
&= \ \omega|_{W^c}(DS^+(S^-(x) - \mathcal{Z}(S^-(x))) \cdot (DS^-(x)v_1 - D\mathcal{Z}(S^-(x))v_1), \\
&\qquad\qquad DS^+(S^-(x) - \mathcal{Z}(S^-(x))) \cdot (DS^-(x)v_2 - D\mathcal{Z}(S^-(x))v_2))
\end{aligned}$$

$$
\begin{aligned}
&= \omega|_{X^+}(DS^-(x)v_1 - D\mathcal{Z}(S^-(x))v_1, DS^-(x)v_2 - D\mathcal{Z}(S^-(x))v_2) \\
&= \omega|_{X^-}(DS^-(x)v_1, DS^-(x)v_2) - \omega_{\gamma(0)}(\underbrace{D\mathcal{Z}(S^-(x))v_1}_{\in Z}, \underbrace{DS^-(x)v_2}_{\in \Sigma}) \\
&\qquad - \omega_{\omega(0)}(\underbrace{DS^-(x)v_1}_{\in \Sigma}, \underbrace{D\mathcal{Z}(S^-(x))v_2}_{\in Z}) + \omega_{\gamma(0)}(\underbrace{D\mathcal{Z}(S^-(x))v_1}_{\in Z}, \underbrace{D\mathcal{Z}(S^-(x))v_2}_{\in Z}) \\
&= \omega|_{W^c}(v_1, v_2),
\end{aligned}
$$

where we have used that $Z^\omega = \Sigma$ and that Z is isotropic. □

Remark 4.1.6. *It should also be possible to prove that when the flow mapping is* exact *symplectic, then the the scattering map inherits this property. This is proven for the scattering map constructed in [21], [27].*

4.2 Reversible systems

In this section we consider the case in which the vector field is also reversible with a linear reversing involution R. If a symmetric homoclinic $\gamma(t)$ to a symmetric equilibrium state is considered (that is, if assumption 4 is in place again), we prove that the scattering map is reversible.

In this context we can choose the center stable and unstable manifolds so that $RW^{cu}(0) = W^{cs}(0)$, implying that $RW^c(0) = W^c(0)$. The restriction of the vector field to the center manifold is reversible. We also choose an R-invariant inner product, see chapter 2.

The goal of this section is to prove that the scattering map S is reversible, that is, $R \circ S = S^{-1} \circ R$.

Proposition 4.2.1. *Under the additional assumption 4*

$$\pi^s \circ R = R \circ \pi^u$$

Proof. By definition,

$$(\pi^s \circ R)(v) = x \Leftrightarrow \|\phi^t(Rv) - \phi^t(x)\| \leq Ke^{-\alpha t},\ t \to \infty \tag{4.4}$$

$$(R \circ \pi^u)(v) = x \Leftrightarrow \|\phi^t(v) - \phi^t(Rx)\| \leq Ke^{\alpha t},\ t \to -\infty. \tag{4.5}$$

Taking (4.4), and using firstly that $\phi^t \circ R = R \circ \phi^{-t}$ and secondly that the distance $\|\cdot\|$ is R-invariant (because we chose an R-invariant inner product), we obtain

$$\begin{aligned}(4.4) \Rightarrow & \|R\phi^{-t}(v) - \phi^t(x)\| \leq Ke^{-\alpha t} \\ \Rightarrow & \|\phi^t(v) - R\phi^t(x)\| \leq Ke^{-\alpha t}.\end{aligned}$$

Now sending $t \mapsto -t$ yields (4.5). These steps can be reversed, completing the argument. □

Proposition 4.2.2. *Under the additional assumption 4,*

$$\mathcal{Z}(y, \mu) = -\mathcal{Z}(Ry, \mu)$$

Remark 4.2.3. *Note that this property of the jump is also proved in [44].*

Proof. By considering on the one hand the system

$$\begin{cases} Ry_c = Ry_c \\ y_s = h_s^{cu}(Ry_c, y_u) \\ y_u = h_u^{cs}(Ry_c, y_s), \end{cases} \tag{4.6}$$

which is uniquely solvable for $(y_s(Ry_c), y_u(Ry_c))$ with Ry_c in a neighbourhood of zero as in the previous section, and on the other hand applying R to the original equations

$$\begin{cases} Ry_c = Ry_c \\ Ry_u(y_c) = Rh_u^{cs}(y_c, y_s(y_c)) \\ Ry_s(y_c) = Rh_s^{cu}(y_c, y_u(y_c)) \end{cases} \tag{4.7}$$

since R maps the trace of W^{cs} onto the trace of W^{cu} in Σ and vice versa, by uniqueness of the solution to this system we get

$$(y_s(Ry_c), y_u(Ry_c)) = (Ry_u(y_c), Ry_s(y_c)).$$

This then yields as a consequence of the graph structure

$$\begin{aligned} Rh_z^{cu}(y_c, y_u(y_c)) &= h_z^{cs}(Ry_c, y_s(Ry_c)), \\ Rh_z^{cs}(y_c, y_s(y_c)) &= h_z^{cu}(Ry_c, y_u(Ry_c)). \end{aligned}$$

Hence, using these facts and also that $\mathcal{Z}(y,\mu) \in Fix(R)$ (which follows from $R\mathbb{J} = -\mathbb{J}R$, since $\dot{\gamma}(0) \in Fix(-R)$ and $Z = \mathrm{span}\{\nabla H(\gamma(0))\}$), we get

$$\begin{aligned} \mathcal{Z}(y,\mu) &= h_z^{cu}(y_c, y_u(y_c), \mu) - h_z^{cs}(y_c, y_s(y_c), \mu) \\ &= Rh_z^{cu}(y_c, y_u(y_c), \mu) - Rh_z^{cs}(y_c, y_s(y_c), \mu) \\ &= h_z^{cs}(Ry_c, y_s(Ry_c), \mu) - h_z^{cu}(Ry_c, y_u(Ry_c), \mu) \\ &= -\mathcal{Z}(Ry, \mu). \end{aligned}$$

□

Lemma 4.2.4. *Under the additional assumption 4,*

$$R \circ S = S^{-1} \circ R$$

Proof. Noting that proposition 4.2.1 yields also $R \circ (\pi^u)^{-1} = (\pi^s)^{-1} \circ R$, we have

$$\begin{aligned} (R \circ S)(x) &= R \circ (\pi^s \circ \phi^T)\left((\phi^T \circ (\pi^u)^{-1})(x) + \mathcal{Z}((\phi^T \circ (\pi^u)^{-1})(x))\right) \\ &= (\pi^u \circ \phi^{-T})R\left((\phi^T \circ (\pi^u)^{-1})(x) + \mathcal{Z}((\phi^T \circ (\pi^u)^{-1})(x))\right) \\ &= (\pi^u \circ \phi^{-T})\left((\phi^{-T} \circ (\pi^s)^{-1})(Rx) + \mathcal{Z}((\phi^T \circ (\pi^u)^{-1})(x))\right) \\ &= (\pi^u \circ \phi^{-T})\left((\phi^{-T} \circ (\pi^s)^{-1})(Rx) - \mathcal{Z}((\phi^{-T} \circ (\pi^s)^{-1})(Rx))\right) \\ &= (S^{-1} \circ R)(x) \end{aligned}$$

□

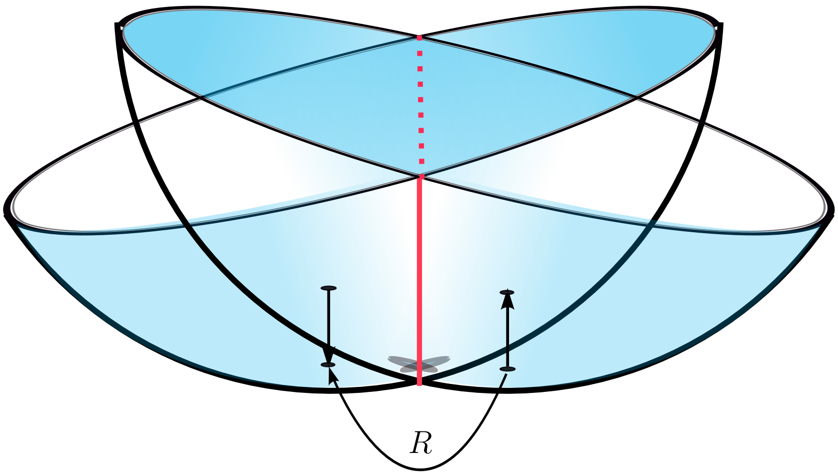


Figure 4.2: *Schematic showing the geometry behind proposition 4.2.2*

We can also characterise which homoclinics found using the scattering map are symmetric. Recall that to find a symmetric homoclinic, we need only to find an intersection of the center unstable manifold with $\text{Fix}(R) \subset \Sigma$. By reversibility, this corresponds to an intersection with the center stable manifold, and the corresponding orbit is symmetric.

Proposition 4.2.5. *A zero of the bifurcation function* ξ *corresponds to a symmetric orbit if and only if*

$$S(x) = Rx$$

Proof. Firstly, if x is a zero and the corresponding orbit is symmetric, then $S^-(x) \in Fix(R)$, and $S(x) = S^+(S^-(x))$, so we have

$$\begin{aligned}(R \circ S)(x) &= R \circ (\pi^s \circ \phi^T) \circ S^-(x) \\ &= (\pi^u \circ \phi^{-T}) \circ RS^-(x) \\ &= (S^-)^{-1} \circ S^-(x) = x\end{aligned}$$

and the result follows. Conversely, if x is a zero and $S(x) = Rx$, then in a similar manner we get $S^+(S^-(x)) = Rx$ giving

$$S^+S^-(x) = Rx \Rightarrow S^-(x) = (\phi^{-T} \circ (\pi^s)^{-1})(Rx)$$

$$\Rightarrow RS^-(x) = (\phi^T \circ (\pi^u)^{-1})(x) = S^-(x)$$

so $S^-(x) \in Fix(R)$ and so the orbit is symmetric. $\square$

Chapter 5

Applications of the scattering technique

In this chapter we use the tools constructed in chapters 3 and 4 to study parametrised Hamiltonian vector fields which, when the parameter is set to zero, have homoclinic loops to nonhyperbolic equilibria. As before, we are aiming to find the homoclinics to the center manifold, but we now extend our search to look for these orbits at parameter values in a neighbourhood of zero.

5.1 Unfolding the saddle center loop

In this section we consider unfolding the singularity studied in the previous chapters, in the simplest case of a two-dimensional center manifold. We obtain a bifurcation diagram for one-homoclinic orbits to the center manifold. A 'one-homoclinic' orbit is a homoclinic which passes only once through a tubular neighbourhood of the original homoclinic. Other unfoldings in the literature (Lerman and Koltsova [46], Champneys [14], Grotta-Ragazzo [28]) have focussed on the existence of multi-round homoclinics to the equilibrium, and nearby multi-round periodic orbits. However, in the final subsection of the unpublished work of Lamb and Koltsova [54], the same structure which we see in our bifurcation diagram figure 5.1.2 is described for analytic vector fields in four dimensions.

Recall that the condition for a homoclinic to the center manifold of the equilibrium is

$$\xi(x,\mu) = H(S(x),\mu) - H(x,\mu) = 0. \tag{5.1}$$

where S is the scattering map and H is the Hamiltonian function restricted to the center manifold. This function has a singular point at the origin. The codimension of this singularity is equal to the dimension of the center manifold (a derivation of this fact using our Lyapunov-Schmidt methods can be found in the appendix A). Since in this section we consider the case of a two dimensional center manifold, we form an expansion of (5.1) around the origin in a two parameter family, that is, $\mu = (\mu_1, \mu_2)$. By studying the lowest order terms we are able to draw a bifurcation diagram for one-homoclinic orbits to the center manifold.

5.1.1 Expansion of the bifurcation function.

Write the linear part L of $S(x,\mu)$ at zero according to its symplectic polar decomposition (see eg. [18])

$$L = UP = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \gamma & \delta \end{pmatrix}$$

where the matrix P is symplectic and positive definite as well as symmetric. Noting that the quadratic part H_2 of $H|_{W^c}$ is rotationally symmetric, we can remove the rotation matrix U:

$$\begin{aligned} H_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + UPx, \mu\right) &= H_2\left(U\left(\begin{pmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \end{pmatrix} + Px\right)\right) \\ &= H_2\left(\begin{pmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \end{pmatrix} + Px\right) \end{aligned}$$

Since P is symmetric and positive definite, a further rotation brings it to diagonal form $P = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{smallmatrix}\right)$ without altering the form of H_2. Rechoosing our parameters, an expansion of S around zero looks like

$$S(x,\mu) = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} + \begin{pmatrix}\alpha & 0\\ 0 & \frac{1}{\alpha}\end{pmatrix} x + \mathcal{O}(\|x,\mu\|^2)$$

We now make an extra assumption, which is equivalent to saying $\alpha \neq 1$.

Assumption 10. *The derivative $D_x S(0,0)$ is not equal to a rotation.*

Using this, and noting that the symplectic diagonalisation of $D^2H(0)|_{E^c}$ can be performed in a continuous way in a parametrised family as long as H depends continuously on the parameter, see [24], this means that the lowest order terms of the bifurcation equation (the quadratic terms) look like, writing $x = (q,p)$,

$$\begin{aligned}\xi_2((q,p),\mu) &= a(\mu)\left(q^2 + p^2 - (\mu_1 + \alpha q)^2 - (\mu_2 + \frac{1}{\alpha}p)^2\right)\\ &= a(\mu)\left((1-\alpha^2)\left(q - \frac{\mu_1\alpha}{(1-\alpha^2)}\right)^2 + (1-\frac{1}{\alpha^2})\left(p - \frac{\mu_2}{\alpha(1-\frac{1}{\alpha^2})}\right)^2\right)\\ &\qquad - a(\mu) f(\mu_1,\mu_2,\alpha)\end{aligned}$$

Where

$$f(\mu_1,\mu_2,\alpha) = \frac{1}{\alpha^2-1}\left(\alpha^2\mu_2^2 - \mu_1^2\right). \tag{5.2}$$

Since $a(0) \neq 0$, also $a(\mu) \neq 0$ for small enough μ. So, the nondegenerate critical point found at the origin when $\mu = 0$ persists for small μ, and by the Morse lemma the function is locally determined by its quadratic part. The zero set is given by a two-part hyperbola, except at the values of μ where f is equal to zero, that is, $\mu_2 = \pm\frac{\mu_1}{\alpha}$. At these points, the two parts of the hyperbola intersect transversally. A bifurcation diagram is given below.

5.1.2 Bifurcation diagram.

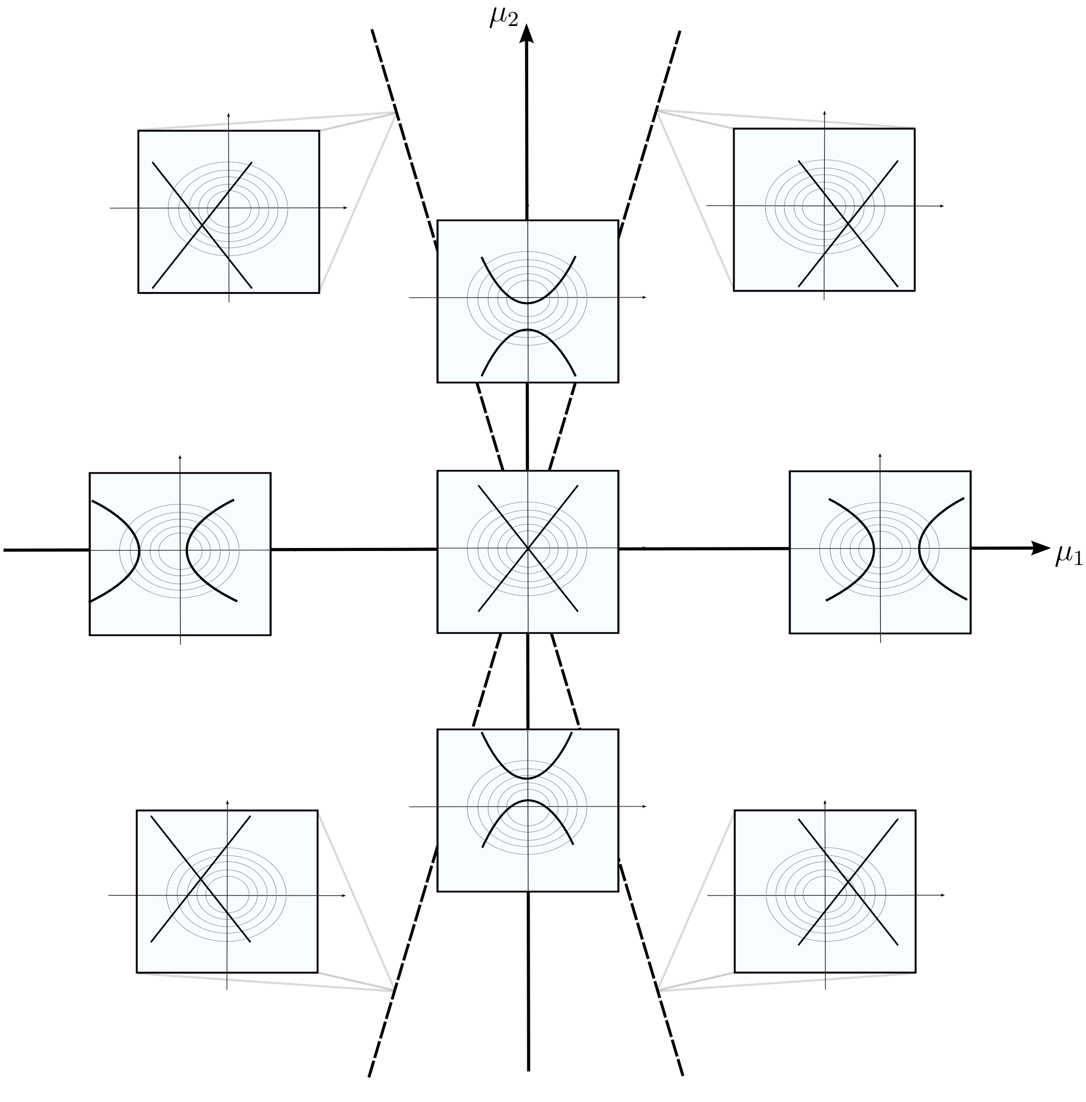


Figure 5.1: *Bifurcation diagram for 1-homoclinics to the center manifold. Pictured in each box is the zero set of* ξ *(in black) in the center manifold at the corresponding value of* (μ_1, μ_2)*. The dashed lines represent points at which the zero set is degenerate, that is, where* $f = 0$*, as defined in equation (5.2). The circles represent the Lyapunov family of periodic orbits in the center manifold.*

5.2 A homoclinic to a bifurcating equilibrium

As a second application, we consider a reversible Hamiltonian system in which the nonhyperbolic equilibrium at the origin possesses a single pair of purely imaginary eigenvalues, which, upon varying a single parameter through zero, pass through zero and become real. We also demand, (as in [82] where purely reversible systems are studied, see also the thesis [83]) that the origin remains an equilibrium for *all small values* of the unfolding parameter. This coupled with the eigenvalue assumptions described above mean that the equilibrium will generically undergo a *transcritical bifurcation*. In this scenario, a second equilibrium passes through the origin as the single parameter passes through zero. For the parameter values at which the origin is a saddle, a small homoclinic emerges in the center manifold.

We find all the homoclinic solutions to the origin for parameter values in a neighbourhood of those at which the homoclinic to the degenerate equilibrium exists.

Note that in one section of [82], a reversible Hamiltonian system is considered, but only homoclinic orbits which are symmetric with respect to the reversing symmetry are sought. It is however observed numerically for an example system (derived from a reduction of the KdV equations), that in a certain region of phase space the only bifurcating homoclinics are indeed symmetric. We investigate how this may occur by studying the possible bifurcation diagrams. See the remark 5.2.1 at the end of the section for a short discussion of the case in which the assumption of a fixed origin in the unfolding is removed.

5.2.1 Setup

We thus consider again system (2.1), with parameter $(\mu, \nu) \in \mathbb{R}^2$;

$$\dot{u} = X_H(u, \mu, \nu)$$

and we again assume the existence of a nondegenerate homoclinic loop.

Assumption 11. *When $\mu = \nu = 0$ there exists an orbit $\Gamma = \{\gamma(t) : t \in \mathbb{R}\}$ such that $\Gamma \subset W^s \cap W^u$, and* $\dim(T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^u) = 1$.

We assume the existence of a linear, antisymplectic reversing symmetry, which fixes Γ that is, assumption 4. The assumption on the equilibrium this time is;

Assumption 12. *We have $X_H(0,\mu,\nu) = 0$ for (μ,ν) in a neighbourhood of the origin. The spectrum of the linearisation $DX_H(0,0,0)$ consists of a double nonsemisimple eigenvalue zero, and $2(n-1)$ eigenvalues λ_i, whose real parts are bounded away from zero; $0 < \alpha < |\Re(\lambda_i)|$ $i \in \{1,..,2(n-1)\}$.*

We now describe the behaviour in the (two dimensional) center manifold in more detail. Under our assumptions we find that the formal normal form in the center manifold at the equilibrium (see [55] and [10]) is

$$H^c(x,y,\mu,\nu) = a(\mu,\nu)y^2 + \sum_{i=2}^{4} b_i(\mu,\nu)x^i + \text{ h.o.t}$$

where, by our assumptions, $b_2(0,0) = 0$. Considering the generic case $b_3(0,0) \neq 0$, we can rescale x, y, t and the parameters so that the normal form becomes

$$H^c(x,y,\mu,\nu) = y^2 - \mu x^2 + x^3 + g(\mu,\nu)x^4 + \text{ h.o.t}, \tag{5.3}$$

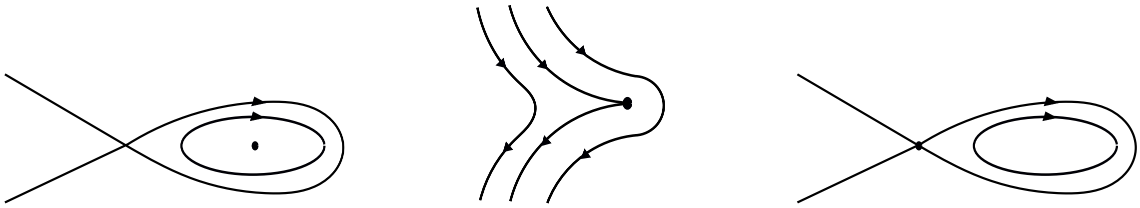
Considering the unfolding of the fast-decaying reversible homoclinic to the equilibrium, we note that the existence of such an orbit corresponds to the image of the origin under the map S^- lying in Fix(R) inside X^-. Since X^- is two dimensional, and $X^- \cap$ Fix(R) is one dimensional, a generic unfolding requires one further parameter, our parameter ν, so that ξ is a function of $\mathbf{x} \in W^c$, and the parameters μ, ν. We make the following transversality assumption on the unfolding.

Assumption 13.

$$\partial_\nu \xi(0,0,0,0) \neq 0$$

This allows us to set $\xi(0,0,0,\nu) = \nu$, for ν sufficiently small so that the line $\nu = 0$ in the parameter plane is the line along which there exists a symmetric homoclinic orbit in the intersection of the strong stable and strong unstable manifolds of the origin.

The normal form (5.3) exhibits the transcritical bifurcation, illustrated in table 5.2.1.



$\mu < 0$ $\mu = 0$ $\mu > 0$

The origin is denoted by '•'

Table 5.1: The transcritical bifurcation

We note that this is the bifurcation occurring in an example system derived from a reduction of the KdV equations, which is studied numerically in [82].

So, to find homoclinics to the origin, we seek the solutions of

$$\xi(\mathbf{x}, \mu, \nu) = H(S(\mathbf{x}, \mu, \nu), \mu, \nu) - H(\mathbf{x}, \mu, \nu)$$

with $\mathbf{x} = (x, y)^T$ lying in $H^{-1}(0, \mu, \nu)$, the level set of the equilibrium with the small homoclinic. We refer to this level set as 'the fish', due to its appearance. We note that S is symplectic and reversible.

Note that symmetric homoclinics are given by $\mathbf{x}$ such that $S(\mathbf{x}) = R\mathbf{x}$, and since RS is an involution (since S is R-reversible) with a $1D$ fixed point space, there is a curve in the center manifold of points such that $S(\mathbf{x}) = Rx$, and by assumption, it passes through 0 when $\mu = \nu = 0$.

If we make assumptions on the fixed point space Fix(RS), we can infer other facts about the image $S(H^{-1}(0, \mu, \nu))$. Let us first assume;

Assumption 14. *When $\nu = 0$, the intersection of Fix(RS) with the line Fix(R) at the origin is transversal.*

This assumption implies that for μ in a neighbourhood of zero, the only point of intersection between Fix(RS) and $H^{-1}(0, \mu, \nu)$ is the origin. The other case is discussed at the end of this section. Note that by assumption (13), the line Fix(RS) moves linearly to leading order in ν. This means that this curve of symmetric solutions lies in the center manifold like the picture below (figure 5.2) for small $\mu, \nu > 0$. Looking at this picture, we see the two points on $H^{-1}(0, \mu, \nu)$ at which $R\mathbf{x} = S(\mathbf{x})$ and we infer that the image $S(H^{-1}(0, \mu, \nu))$ at these parameter values looks like this:

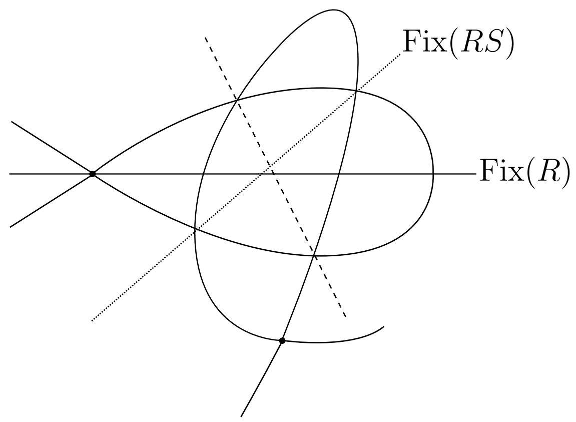


Figure 5.2: The fish and its image under S.

so there is a further pair of asymmetric solutions, which are necessarily related by symmetry - one is the image of the other under the map RS. Now consider increasing ν, until the upper asymmetric point meets the upper symmetric solution. Importantly, since they are related by symmetry, the second asymmetric solution must also move toward the symmetric point, and merge with it at the same time. This results in a reversible pitchfork bifurcation of homoclinic orbits. The moment of bifurcation is illustrated in figure 5.3.

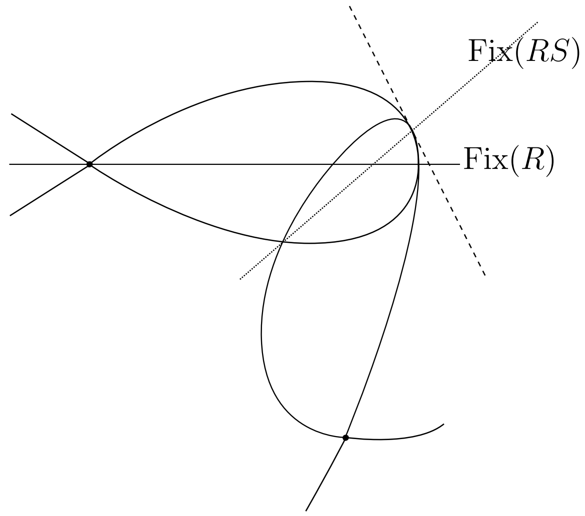


Figure 5.3: The moment of the pitchfork bifurcation.

Continuing to increase ν, we are left with just the two symmetric homoclinics until they also merge and disappear, which happens when the line Fix(RS) becomes tangent to $H^{-1}(0, \mu, \nu)$ and passes through it. This is a fold bifurcation of homoclinic orbits. Figure 5.4 shows the moment of bifurcation.

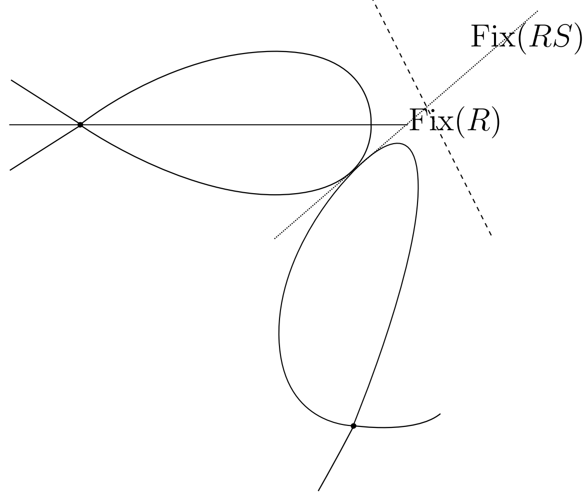


Figure 5.4: The moment of the fold bifurcation.

5.2.2 Bifurcation curves

Let us derive approximate expressions for the bifurcation curves for the pitchfork and fold bifurcations in the μ, ν plane.

Firstly, the fold. The condition for this bifurcation is that $\mathrm{Fix}(RS)$ is tangent to $H^{-1}(0, \mu, \nu)$.

The fish has equation

$$y^2 = \mu x^2 - x^3 + g(\mu, \nu)x^4 + ...$$

and we approximate $\mathrm{Fix}(RS)$ by the line

$$\nu = a_+ x + b_+ y + h.o.t$$

Tangency is the same as

$$\begin{pmatrix} 2y \\ 2\mu x - 3x^2 + \mathcal{O}(x^3) \end{pmatrix} \cdot \begin{pmatrix} a_+ + \mathcal{O}(x, y) \\ b_+ + \mathcal{O}(x, y) \end{pmatrix} = 0$$

which becomes, after squaring both sides of the equation and substituting for y^2 using the equation of the fish;

$$b_+^2(3x - 2\mu)^2 + \mathcal{O}(x^2) = 4a_+^2(\mu - x) + \mathcal{O}(x^2).$$

Writing $\mu\bar{x} = x$, we find

$$\begin{aligned} \mu(1 - \bar{x}) &= \mathcal{O}(\mu^2) \\ \Rightarrow \quad (\bar{x} - 1) &= \mathcal{O}(\mu) \end{aligned}$$

and so $x = \mu + \mathcal{O}(\mu^2)$. Looking again at the tangency equation and substituting our expression for x (and using the same scaling trick, via $\mu^2\bar{y} = y$) leads to

$$y = \frac{b_+}{2a_+}\mu^2 + \mathcal{O}(\mu^3).$$

To balance the fish equation, we write $x = \mu + C\mu^2 + ...$, plug this into the fish along with our new expression for y, and derive $C = g(0,0) - \frac{b_+^2}{4a_+^2}$. So, writing out the equation for Fix(RS) from above including all of the relevant terms, namely

$$\nu = a_+x + b_+y + a'_+\mu x + a''_+\nu x + a'''_+x^2 + \text{ h.o.t} \tag{5.4}$$

that is,

$$\begin{aligned}\nu(1 - a''_+x) &= a_+x + b_+y + a'_+\mu x + a''_+\nu x + a'''_+x^2 + \text{ h.o.t} \\ &= (1 + a''_+x + (a''_+x)^2 + ...)(a_+x + b_+y + a'_+\mu x + a''_+\nu x + a'''_+x^2 + \text{ h.o.t}) \\ &= a_+x + (a''_+a_+ + a'''_+)x^2 + a'_+\mu x + b_+y + \text{ h.o.t}\end{aligned}$$

so that substituting the expressions for x and y from above, we find that on the bifurcation curve the parameters satisfy

$$\nu_F = a_+\mu + \left(a_+g(0,0) - a''_+a_+ + a'''_+ + a'_+ - \frac{b_+^2}{4a_+} + \frac{b_+^2}{2a_+}\right)\mu^2 + \mathcal{O}(\mu^3)$$

Now for the pitchfork bifurcation. As we have observed, the two asymmetric solutions $x_1(\mu,\nu), x_2(\mu,\nu)$ form a pair - one is the image of the other under RS; $x_2 = RSx_1$ say. As a consequence, their difference $x_1 - RSx_1$ (illustrated by the line with longer dashes in the schematic figures) is parallel to Fix($-RD_xS(0,0)$) to leading order, i.e $x_1(\mu,\nu) - RSx_1(\mu,\nu) = v + \mathcal{O}(\|x_1(\mu,\nu)\|^2)$, where $v \in$ Fix($-RD_xS(0,0)$).

The pair of nonsymmetric solutions thus collide (on a symmetric solution) when this line between them becomes tangent to the fish. In other words, when the line Fix(RS) intersects a point on the fish whose tangent vector is parallel to $x_1 - RSx_1$. So, writing

$$\text{Fix}(-RDS_x(0,0)) = \{a_-x + b_-y = 0\},$$

we arrive at the condition

$$\begin{pmatrix} 2y \\ 2\mu x - 3x^2 + \mathcal{O}(x^3) \end{pmatrix} \cdot \begin{pmatrix} a_- + ... \\ b_- + ... \end{pmatrix} = 0.$$

Using the same method as before, we derive this time:

$$x_{PF} = \mu - \frac{b_-^2}{4a_-^2}\mu^2 + \mathcal{O}(\mu^3)$$
$$y_{PF} = \frac{b_-}{2a_-}\mu^2 + \mathcal{O}(\mu^3).$$

Substituting these expressions into equation 5.4 as before yields that the parameter value ν_{PF} is given by

$$\nu_{PF} = a_+\mu + \left(a_+g(0,0) - \frac{a_+b_-^2}{4a_-^2} + a_+''a_+ + a_+''' + a_+' + \frac{b_+b_-}{2a_-}\right)\mu^2 + \mathcal{O}(\mu^3)$$

We can check that for each μ, the pitchfork occurs before the fold, as ν is moved away from zero. We firstly observe that we can arrange for $b_- = b_+ = 1$, and consider the case $a_+ < 0$, and $a_- < 0$, so that $a_+a_- > 0$. In this case, comparing the equations of the bifurcation curves we find that for the pitchfork to happen first, we require

$$\frac{1}{4a_+} < -\frac{a_+}{4a_-^2} + \frac{1}{2a_-}$$

Multiplying by $2a_-$, this is equivalent to

$$\frac{a_-}{2a_+} > 1 - \frac{a_+}{2a_-}$$

which in turn is equivalent to

$$0 < (a_+ - a_-)^2$$

which of course is true. The same argument also works if we allow $a_- < 0$, and also for $a_+ > 0$ with $a_- > 0$ or $a_- > 0$.

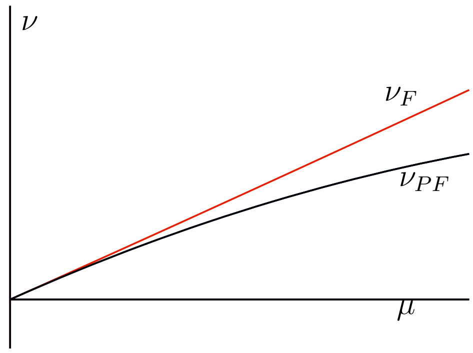


Figure 5.5: The bifurcation curves with $a_+ = 3$, $a_- = -1$.

For parameter values $\mu > 0$, $\nu < 0$, we have a situation as illustrated in figure 5.6. Since only an intersection of the unstable manifold with the stable manifold corresponds to a homoclinic loop, we note that only one of the four intersections here gives a homoclinic orbit to the origin.

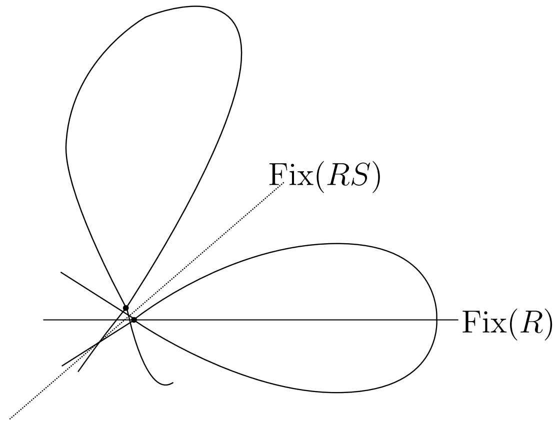


Figure 5.6: Intersections when $\mu > 0$, $\nu < 0$.

We summarise these findings with the following bifurcation diagram and theorem.

Theorem 9. *Under assumptions 4, 11, 12, 13, 14, the parameter plane is divided as in figure 5.7, where;*

- *In region I there are four 1-homoclinics to the origin - two symmetric ones and one asymmetric pair.*

- *Along the curve ν_{PF} there is a pitchfork bifurcation of 1-homoclinic orbits, at which the asymmetric pair merges into one of the symmetric orbits.*
- *In the wedge-shaped region II there are two 1-homoclinic orbits to the origin - both are symmetric.*
- *Along the curve ν_F there is a fold bifurcation of 1-homoclinic orbits.*
- *In the region III there are no 1-homoclinics to the origin.*
- *Along the μ-axis, four 1-homoclinic orbits merge into a single one. On this axis there exists a single 1-homoclinic orbit lying in the intersection of the strong stable and unstable manifolds of the origin (a 'fast decaying' homoclinic).*
- *In region V there exists a single symmetric 1-homoclinic to the origin.*
- *When $\mu = 0$ and $\nu < 0$, there exists a single symmetric 1-homoclinic to the origin.*
- *When $\nu = 0$, $\mu < 0$, there exists a single symmetric 1-homoclinic to the origin.*
- *In region IV there are no 1-homoclinics to the origin.*

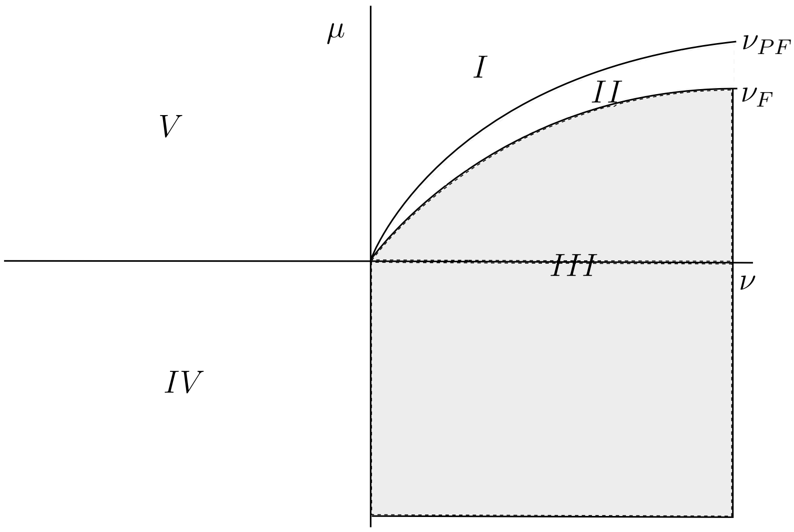


Figure 5.7: Bifurcation diagram for 1-homoclinics to the origin. The homoclinics present in each region are described in theorem 9.

This diagram adds detail to the one provided in [82], (included below for comparison in figure 5.8), which describes only symmetric homoclinics, and thus regions I and II are treated as the same, with the fold curve representing the only bifurcation in the $\mu > 0$, $\nu > 0$ plane.

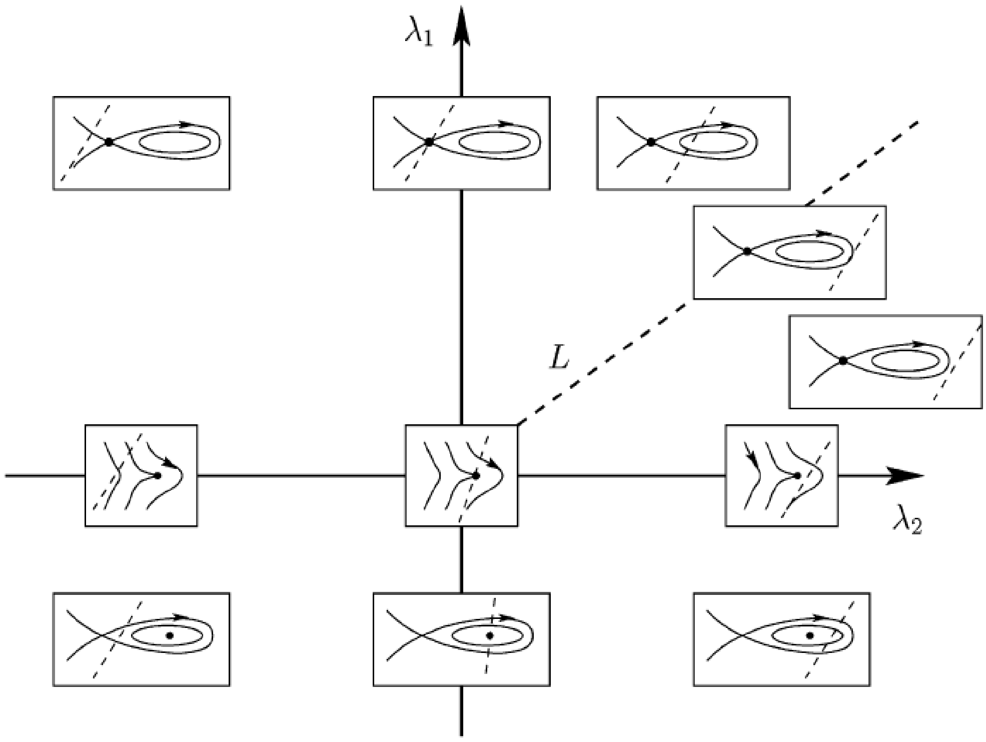


Figure 5.8: From reference [82] Bifurcation diag for 1-homoclinics near $\gamma(t)$. The curve L in the parameter plane corresponds to our curve ν_F, and in each panel, the dashed line corresponds to our $Fix(RS)$, so that the intersections with the fish lead to symmetric homoclinic orbits.

There is another case to be considered; when assumption 14 does not hold. If the line Fix(RS) lies tangent to Fix(R) in the center manifold, then it will intersect the fish's nose when $\mu > 0$, $\nu = 0$. We observe that in this case, for ν values on both sides of zero, there are two intersections between the fish and Fix(RS). Furthermore, for ν on one side of zero these two intersections are the only homoclinics to the origin present (as a special case, if $S = \nu\mathbf{v} + Id$ then the symmetric homoclinics are the only ones present for ν on *both* sides of zero). As ν moves further from the origin, this pair of symmetric homoclinics will collide and disappear in a saddle node bifurcation. This bifurcation pattern is precisely the one observed numerically in the paper [82] (figure 12) for a system obtained from the travelling wave equations for a KdV PDE, which is reversible and Hamiltonian. In [82] it is mentioned *"Note that we do not attempt to prove any of the results for [the KdV system] rigourously. This would amount to proving that the equation fulfills the non-degeneracy conditions imposed for the general analysis. In particular regarding Hypothesis 6 this is a major difficulty."*. The hypothesis 6 mentioned here is the exact equivalent of our assumption 14. Noting that by theorem 9, if assumption 14 holds then the existence of two symmetric homoclinics without the presence of an asymmetric pair on one side of $\nu = 0$ when $\mu > 0$ is ruled out, we must conclude that for the KdV example this transversality condition is not satisfied. In the tangent case, for ν on the other side of zero there are in general two symmetric solutions and a further pair of asymmetric solutions, which I would expect to follow a similar bifurcation pattern to the one displayed in the positive quadrant in figure 5.7, although we do not analyse this region here. We note that no numerical results for the KdV system are given for this region in [82].

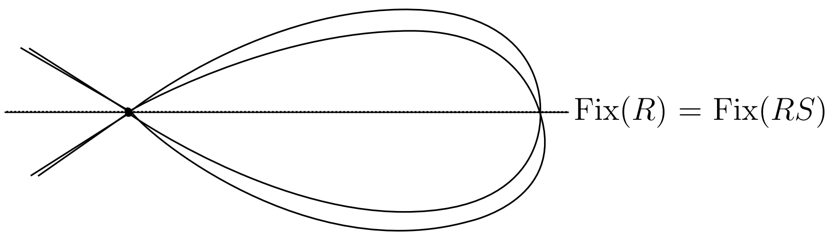


Figure 5.9: The fish and its image when Fix(RS) is tangent to Fix(R), $\mu > 0$, $\nu = 0$.

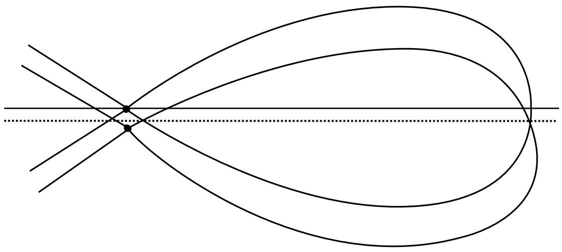


Figure 5.10: The fish and it's image when Fix(RS) is tangent to Fix(R), $\mu > 0$, $\nu < 0$. The two symmetric intersections are the only ones present.

Note that while our theorem only describes 1-round homoclinics, multiround solutions may also be present. The KdV system mentioned here is also an example in the paper [74] by Sandstede, where the emergence of N-homoclinic orbits is studied as a parameter corresponding to our ν is varied, in a region corresponding to $\mu > 0$, fixed. The movement of the homoclinic $\gamma(t)$ through the *strong* stable/unstable manifolds of the equilibrium as ν is varied here is referred to as an orbit-flip bifurcation. Sandstede finds that for the KdV system N-round orbits, for any $N > 1$ are present for ν in a neighbourhood of zero, $\nu \neq 0$. In particular the prediction of 2-pulses from this paper is supported by numerical evidence in [12], which also deals with N-round orbits in the KdV system. In more general reversible Hamiltonian systems experiencing the orbit-flip bifurcation, Sandstede finds in [74] that either N-round orbits for any $N > 1$ emerge as described above, or that none at all exist - these two cases are distingushed by the sign of a certain constant, which is determined by the behaviour of the derivative of the Hamiltonian function in the eigenspaces at the equilibrium.

Remark 5.2.1 (Saddle-center case)**.** *Because we assumed that the origin remains an equilibrium for all parameter values, we have not considered the general case here. Generically, unfolding a pair of zero eigenvalues will cause the equilibrium at the origin to undergo a saddle center bifurcation. In this situation, the normal form on the center manifold to leading order is (see [35]);*

$$H^c(q, p, \bar{\lambda}) = \frac{a}{2}p^2 + \frac{b}{6}q^3 - \tilde{\lambda}q \tag{5.5}$$

where (q, p) *are symplectic coordinates on the center manifold. We may assume that*

$a, b > 0$: if both were negative, the criticality of the bifurcation would simply be reversed, and if a, b had different signs, we could change $q \mapsto -q$ and $(b, \tilde{\lambda}) \mapsto (-b, -\tilde{\lambda})$ without changing the Hamiltonian. We are interested in the parameter range $\tilde{\lambda} > 0$, so we can write $\tilde{\lambda} = \lambda^2$, and then take $\bar{q} = q + \sqrt{\frac{2}{b}}\lambda$, $\bar{p} = p$ which brings the system on the center manifold to the form

$$\begin{cases} \dot{\bar{q}} = a\bar{p} \\ \dot{\bar{p}} = \sqrt{2b}\lambda\bar{q} - \frac{b}{2}\bar{q}^2 \end{cases}.$$

The phase portraits for this system are illustrated below.

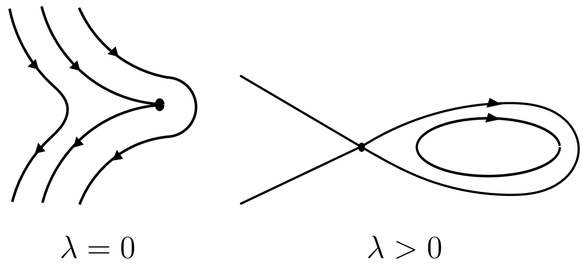


Table 5.2: The saddle-center bifurcation. The origin is denoted by '•'

In this parameter range, the system is now of the same form as the one we studied in this section, and thus our theorem should apply here. Even without performing this transformation, the same techniques could be used to study the bifurcations of a homoclinic loop in this scenario, without restricting to $\lambda > 0$.

Discussion

The work in this thesis poses a number of interesting questions which could be addressed in future research. In this section we briefly discuss a number of these, along with some tentative strategies for their resolution.

In chapter 3 we proved that any of the possible signatures for the Hessian matrix at the singularity can be realised in systems which are a C^1-small perturbation of a completely integrable system. This was done using the symplectic version of Frank's lemma found in [68]. However, it is probable that this closeness can be proven to hold in finer topologies, maybe C^r with r being the smoothness of the flow itself, or even closeness in the analytic category. In the near-integrable case, the so-called Melnikov potential (see the discussion below) has been calculated for concrete analytic examples in [48], and it would be interesting to provide comparable results via the scattering matrix approach employed here. On the same theme, it would be desirable to prove (probably in a similar way, making a modification of the flow in a tubular neighbourhood of the homoclinic) that the nonlinear scattering map from chapter 4 can be any near-identity symplectic map. Or possibly, any Hamiltonian symplectomorphism at all, using a homotopy argument of some sort.

Regarding the results in chapter 3 on the signatures of the Hessian at our singularity, it is clear that a much more detailed picture regarding the dynamic character of the elements of the zero set is desirable. With only the signature in hand, we could apply the Morse lemma to get an idea of the geometry of the zero set, but since in general the coordinate change involved will not be symplectic or preserve tori or any other dynamical features, it seems that this would not be a good way to prove general statements about homoclinics to particular invariant sets (like tori) or heteroclinics between them. To the best of my knowledge, there is no variant of the Morse lemma which preserves symplectic structure. However, it would be interesting to see what kind of normal form we can find for the reduced function $\mathfrak{g}$ using only coordinate changes which preserve the important structure - these would be the symplectic rotations.

One advantage of the Lyapunov-Schmidt approach is that since it provides a formula for the reduced function in terms of integrals which can be approximated numerically, it can be implemented in concrete examples. It would be interesting to numerically approximate the Hessian at the critical point in some far from integrable Hamiltonian systems appearing in applications.

There is another possible avenue of approach which may yield stronger results regarding the existence of homoclinics to invariant tori, namely, there may be a gradient structure to $\mathfrak{g}$ when it is restricted to the torus. In the case of a near-integrable system, it has been demonstrated (see [19, 48, 20]) that homoclinic orbits to an invariant torus in the center manifold of an elliptic hyperbolic equilibrium correspond to critical points of a scalar 'potential' function defined on the torus. This function is referred to in the literature as the *Melnikov potential*, since its gradient vector is precisely the Melnikov function, measuring the splitting of the invariant manifolds of the torus. Under the additional generic assumption that the Melnikov potential is a Morse function (that is, all of its critical points are nondegenerate), Morse theory can be applied to give a lower bound on the number of critical points the function will have, and hence the number of homoclinic orbits to the torus. Let us briefly observe via the following heuristic argument how the scattering map approach can be used in the near integrable case to show the existence of the gradient structure.

We thus consider a Hamiltonian $H = H_0 + \varepsilon H_1$, with H_0 completely integrable and with an elliptic-hyperbolic equilibrium at zero with a homoclinic to the origin. The center manifold of the unperturbed system is filled with invariant tori. It would seem that near integrability in this case yields a 'near identity' situation for the scattering map - or at least, near rotation, since in the integrable case, the invariant manifolds of a torus coincide, meaning that the scattering map maps the torus onto itself. I believe that symplecticity implies that the linear part of the scattering map must be a symplectic rotation, so that we can express S as a rotation plus small corrections, and so we can approximate $S(\varepsilon, \cdot)$ by a time shift via $S(\varepsilon, x) = R_\theta \exp(\varepsilon X_G) + \mathcal{O}(\varepsilon^2)$. Let the action angle variables which define the tori in the unperturbed center manifold be denoted by $(\eta_1, ..., \eta_l, \phi_1, ..., \phi_l)$, and the actions and angles given after the perturbation by the KAM theorem by $(I_1(\varepsilon), ..., I_l(\varepsilon), \psi_1(\varepsilon), ..., \psi_l(\varepsilon))$, and note that $I_i(\varepsilon) = \eta_i + \mathcal{O}(\varepsilon)$. The condition for a homoclinic starting from the orbit of a point $x \in \mathbb{T}_I$ in a persistent invariant torus is

$$\xi(\varepsilon, x) = \begin{pmatrix} I_1(\varepsilon, S(\varepsilon, x)) - I_1(\varepsilon, x) \\ \vdots \\ I_l(\varepsilon, S(\varepsilon, x)) - I_l(\varepsilon, x) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.6}$$

Substituting $S(\varepsilon, x) = R_\theta \exp(\varepsilon X_G) + \mathcal{O}(\varepsilon^2)$, we get

$$\begin{aligned} \xi(\varepsilon, x) &= \begin{pmatrix} -\varepsilon\{G, I_1(\varepsilon)\}(x) + \mathcal{O}(\varepsilon^2) \\ \vdots \\ -\varepsilon\{G, I_l(\varepsilon)\}(x) + \mathcal{O}(\varepsilon^2) \end{pmatrix} \\ &= \begin{pmatrix} -\varepsilon\{G, \eta_1\}(x) + \mathcal{O}(\varepsilon^2) \\ \vdots \\ -\varepsilon\{G, \eta_l\}(x) + \mathcal{O}(\varepsilon^2) \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon \frac{\partial G}{\partial \phi_1} + \mathcal{O}(\varepsilon^2) \\ \vdots \\ \varepsilon \frac{\partial G}{\partial \phi_l} + \mathcal{O}(\varepsilon^2) \end{pmatrix} \\ &= \varepsilon \nabla (G|_{\mathbb{T}_\mathbb{I}}) x + \mathcal{O}(\varepsilon^2). \end{aligned}$$

This shows that the first order approximation to $\xi(\varepsilon, x)$ is given by the gradient of a scalar function defined on the torus - the restriction of the function G, which defines the Hamiltonian flow approximation to the scattering map. It can also be shown that nondegeneracy of a critical point here means that the intersection of invariant manifolds is transversal. Thinking far from integrability, the missing part here is the step in which we write S using a time shift along a Hamiltonian flow - we needed closeness to a rotation here. It may however still be true that S can be written in this way in general. In any case, the invariant manifolds of a torus are still Lagrangian, which may be the key to the gradient structure, rather than integrability. It would be very interesting to see whether this gradient structure is still present in some form.

We could consider the same problem as in section 5.1 with additional reversibility. There are regions in the bifurcation diagram figure 5.1.2 in which the invariant manifolds of families of periodic orbits intersect. In particular, the unstable manifolds of certain periodic orbits have points of transversal intersection with the stable manifold of the whole family of periodic orbits. In the reversible case the periodic orbits are all symmetric, and some of the corresponding homoclinic connections are symmetric. We could now consider adding a small reversible *non-conservative* perturbation. The transverse intersections will persist, but now the lack of a conserved quantity means that the dynamics can be quite different - we are in the context of the paper [38] by Homburg and Lamb. It would appear that theorems from [38] allow us to conclude that there exist sheets of periodic orbits accumulating on the homoclinic orbits. In addition, we could look for 'transition chains' of heteroclinic orbits between periodic orbits (which were impossible in the unperturbed Hamiltonian system), which would likely lead to trajectories exhibiting 'diffusive' behaviour.

One interesting direction which we have neglected entirely in thesis is the study of multiround homoclinic orbits. Results on the existence of multiround homoclinics in families unfolding a loop to a saddle-center equilibrium can be found in [28, 46, 46, 66]. In the context of section 5.2, there are also results [74] detailing the emergence of multiround homoclinics to the equilibrium via a homoclinic 'flip' bifurcation which occurs along the curve in our parameter plane on which there exists a homoclinic contained in the strong stable and strong unstable manifolds of the equilibrium. However, these multirounds are built from segments of orbit which are close to the principal homoclinic, and as such they do not involve the other homoclinics which we've seen are also present (those which follows orbits in the center manifold for periods of time), so there may be even more multiround dynamics present when these are also considered.

The KdV system mentioned in section 5.2 is also used as an example in [37], because there are regions in parameter space in which the symmetric homoclinic orbits to the origin form symmetric 'bellows'. In [37], it is shown that the symmetric bellows configuration is accompanied by shift dynamics. In fact, since we have shown here that there are more homoclinic orbits to the saddle, it seems we can use results from the paper [79] by Turaev and Shilnikov to conclude that the shift dynamics are actually more complicated, involving a larger number of symbols.

Appendix A

Codimension calculation

We show that the destruction of the principal homoclinic loop can be detected using a slight modification of the Lyapunov-Schmidt reduction used previously. It becomes clear that the existence of the loop is a codimension $2l$ phenomenon, where $2l$ is the dimension of the center manifold.

As before, we will consider the space of continuously differentiable functions with an exponentially weighted norm, but this time the sign of the weight will be positive - this means we limit our search for solutions to consider only those functions which decay faster than a given exponential rate. Choosing the weight according to the conditions on the spectrum of the linearisation at the equilibrium will let us use this to find solutions which live in the intersection of the stable and unstable manifolds of the equilibrium.

For $\beta \in \mathbb{R}$, we again define the Banach space

$$C^1_\beta(\mathbb{R}, \mathbb{R}^{2n}) = \{x : \mathbb{R} \to \mathbb{R}^{2n} \text{ with } \sup_{t\in\mathbb{R}} \|e^{\beta|t|}x(t)\| < \infty, \sup_{t\in\mathbb{R}} \|e^{\beta|t|}\dot{x}(t)\| < \infty\}.$$

Recall that $0 < \delta < \alpha$, where α bounds the real parts of hyperbolic eigenvalues away from zero, and δ is the constant appearing in the exponential weight used in section 3.1. Trajectories which lie in the intersection of the stable and unstable manifolds live in the space C^1_α, whereas those in $(W^{cu}(0) \cap W^{cs}(0)) \setminus (W^u(0) \cap W^s)$ decay subexponentially and hence don't live in any space with a positive exponential weight. So, choosing a value β for the weight such that $0 < \delta < \beta < \alpha$ means firstly that we capture solutions lying in the right invariant manifolds, and secondly that we can use a weighted inner product given by

$$< u, v >_{-\delta} = \int_{\mathbb{R}} e^{2\delta\phi(t)} < u(t), v(t) > dt.$$

Assumption 15. $0 < \delta < \beta < \alpha$

Note that the weight in the inner product here is -1 multiplied by the weight from section 3.1. This has the consequence that if we repeat our investigations of the Fredholm properties of the linearised operator $DF(0)$, considered on our new choice of space C^1_β, (with $0 < \delta < \beta < \alpha$), we discover that $DF(0) : C^1_\beta \to C^0_\beta$ is conjugate via the isomorphism $v(t) \to e^{\delta\phi(t)}v(t)$ (mapping from C^1_β into $C^1_{\beta-\delta}$, considered with the unweighted inner product), to the operator

$$L_{-\delta}u(t) = \frac{d}{dt}u(t) + \delta\dot{\phi}(t)u(t) - DX_H(\gamma(t))u(t).$$

The limits are again hyperbolic, but this time the $2l$ purely imaginary eigenvalues of $DX_H(0)$ have been shifted to the *left* of the imaginary axis in negative time, and to the *right* in positive time, i.e they cross the imaginary axis in the opposite direction to the case considered in section 3.1. Looking at Palmer's theorem again, we see that this will induce a Fredholm index of the opposite sign, that is:

Lemma A.0.2. *Fredholm index of $DF(0) : C^1_\beta \to C^0_\beta$ is $-2l$.*

We note that, due to our assumption

Assumption 16. $\dim(T_{\gamma(0)}W^{cu} \cap T_{\gamma(0)}W^s) = \dim(T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^u) = 1,$

the only solution of the linear variational equation living in C^1_β is given by $\dot{\gamma}(t)$, so the kernel of $DF(0)$ is one dimensional - spanned by this solution. Its easy to see (by considering the uniqueness in the application of the implicit function theorem in the Lyapunov-Schmidt reduction) that this single kernel dimension corresponds to the family of solutions given by considering time translations of $\gamma(t)$. Since the Fredholm index of $DF(0)$ is $2l$, the kernel of the adjoint, which now consists of solutions to

$$\dot{u}(t) = -2\delta\dot{\phi}(t)u(t) - DX_H(\gamma(t))^* u(t)$$

is $2l+1$ dimensional. We thus end up at a reduced equation of the form

$$\mathfrak{f}(k) : \mathbb{R} \to \mathbb{R}^{2l+1}$$

At this point we can consider introducing parameters $\mu_1, ..., \mu_{2l}$, and view our Hamiltonian in a parametrised family

$$H : \mathbb{R}^{2n} \times \mathbb{R}^{2l}, \;\; H(u, \mu_1, ...\mu_{2l}) = H(u) + \sum_{i=1}^{2l} \mu_i H_i(u)$$

The reduction procedure yields a family of reduced functions

$$\mathfrak{f}(k, \mu) : \mathbb{R} \times \mathbb{R}^{2l} \to \mathbb{R}^{2l+1}.$$

However, we can reduce further - firstly, since the variable k corresponds to a curve of zeros, all only realising a single geometrically distinct orbit, we can fix $k = 0$, and thereby reduce the dimension of the domain of $\mathfrak{f}$ by one. Secondly, one solution of the adjoint variational equation is given by $e^{-2\delta\phi(t)}\nabla H(\gamma(t))$, and so one component of $\mathfrak{f}$ (without loss of generality $\mathfrak{f}_1$) is given by

$$\begin{aligned}
\mathfrak{f}_1(0, \mu) &= \int_{\mathbb{R}} \left\langle \nabla H(\gamma(t)), \sum_{i=1}^{2l} \mu_i X_{H_i}(\gamma(t)) \right\rangle dt \\
&= \int_{\mathbb{R}} -\sum_{i=1}^{2l} \{H_i, H\}(\phi^t(\gamma(0)))dt \\
&= -\int_{\mathbb{R}} \sum_{i=1}^{2l} \frac{d}{dt} \left(H_i \circ \phi^t(\gamma(0))\right) dt \\
&= \sum_{i=1}^{2l} \left(\lim_{t\to-\infty} H_i(\gamma(t)) - \lim_{t\to\infty} H_i(\gamma(t)) \right) = 0.
\end{aligned}$$

Hence, since $\mathfrak{f}_1 \equiv 0$, we can consider solving only $(\mathfrak{f}_2, ..., \mathfrak{f}_{2l}) = (0, ..., 0)$. This means that we are left with the further-reduced equation

$$\tilde{\mathfrak{f}}(\mu) = (\mathfrak{f}_2, ..., \mathfrak{f}_{2l}).$$

We have that $\tilde{\mathfrak{f}}(0) = 0$, and so, assuming that the unfolding we consider is transversal in the sense that

$$\frac{d}{d\mu_i}\mathfrak{f}(0) \neq 0 \text{ for each } i \in \{1, ..., 2l\},$$

we see that the homoclinic orbit occurs if and only if $(\mu_1, ..., \mu_{2l}) = (0, ..., 0)$ so we conclude that the existence of the homoclinic loop to the equilibrium is a codimension $2l$ phenomenon.

References

[1] S Angenent. A Variational Interpretation of Melnikovs Function and Exponentially Small Separatrix Splitting. *Symplectic Geometry: LMS Lecture Note Series 192*, 1990.

[2] V.I Arnold. Instability of dynamical systems with several degrees of freedom. *Soviet. Math. Dokl.*, 5:581–585, 1964.

[3] V.I Arnold. *Mathematical Methods of Classical Mechanics*. Springer, 2nd edition, 1997.

[4] Marino Badiale and Antonio Ambrosetti. Homoclinics: Poincare-Melnikov type results via a variational approach. *Annales de L'I.H.P., Section C*, 15(2):233–252, 1998.

[5] Flaviano Battelli and Claudio Lazzari. Exponential dichotomies, heteroclinic orbits, and Melnikov functions. *Journal of Differential Equations*, 86(2):342–366, August 1990.

[6] Massimiliano Berti and Philippe Bolle. Homoclinics and chaotic behaviour for Perturbed Second Order Systems (*). *Annali di Matematica pura ed applicata*, 176(4):323–378, 1999.

[7] David Blázquez-Sanz and Kazuyuki Yagasaki. Analytic and algebraic conditions for bifurcations of homoclinic orbits I : Saddle equilibria. *Journal of Differential Equations*, 253(11):2916–2950, 2012.

[8] Claudio A Buzzi and Jeroen S W Lamb. Reversible Hamiltonian Liapounov center theorem. *Discrete and Continuous Dynamical Systems-Series B*, 5(1):51–66, 2005.

[9] Roberto Camassa, Gregor Kovacic, and Siu-Kei Tin. A Melnikov Method for Homoclinic Orbits with Many Pulses. *Archive for Rational Mechanics and Analysis*, 143:105–193, September 1998.

[10] H.W. Capel, J.S.W. Lamb, and J.A.G Roberts. Conditions for local (reversing) symmetries in dynamical systems. *Physica A*, 197:379–422, 1993.

[11] A R Champneys. Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics. *Physica D: Nonlinear Phenomena*, 112:158–186, 1998.

[12] A R Champneys and Mark D. Groves. A global investigation of solitary-wave solutions to a two-parameter model for water waves. *Journal of Fluid Mechanics*, 342:199–229, July 1997.

[13] A R Champneys, B A Malomed, J Yang, and D J Kaup. Embedded solitons: solitary waves in resonance with the linear spectrum. *Physica D: Nonlinear Phenomena*, pages 340–354, 2001.

[14] Alan R. Champneys and Jörg Härterich. Cascades of homoclinic orbits to a saddle-centre for reversible and perturbed Hamiltonian systems. *Dynamics and Stability of Systems*, 15(3):231–252, September 2000.

[15] C Chicone. *Ordinary Differential Equations with Applications*. Springer, 2nd edition, 2006.

[16] Shui-Nee Chow, Jack K. Hale, and John Mallet-Paret. An example of bifurcation to homoclinic orbits. *Journal of Differential Equations*, 37(3):351–373, September 1980.

[17] R Courant and D Hilbert. Methods of Mathematical Physics Volume 1, 1989.

[18] Maurice A De Gosson. *Symplectic Geometry and Quantum Mechanics.* Birkhauser Verlag, 2006.

[19] A. Delshams and P. Gutiérrez. Splitting Potential and the Poincaré-Melnikov Method for Whiskered Tori in Hamiltonian Systems. *Journal of Nonlinear Science*, 10(4):433–476, August 2000.

[20] A. Delshams, P. Gutiérrez, O. Koltsova, and J. R. Pacha. Transverse intersections between invariant manifolds of doubly hyperbolic invariant tori, via the Poincaré-Melnikov method. *Regular and Chaotic Dynamics*, 15(2-3):222–236, April 2010.

[21] Amadeu Delshams, Rafael De La Llave, and Tere M Seara. Geometric properties of the scattering map of a normally hyperbolic invariant manifold. *Advances in Mathematics*, 217:1096–1153, 2007.

[22] Amadeu Delshams and Rafael Ram. Splitting of separatrices in Hamiltonian systems and symplectic maps. *C. Simó, editor, Hamiltonian Systems with Three or More Degrees of Freedom, volume 533 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 39–54, 1999.

[23] Robert L Devaney. Reversible diffeomorphisms and flows. *Transactions of the American Mathematical Society*, 218:89 – 113, 1976.

[24] L H Eliasson. Normal forms for Hamiltonian systems with Poisson commuting integrals - elliptic case. *Comment. Math. Helvetici*, 65:4–35, 1990.

[25] E Fontich and P Martín. Differentiable invariant manifolds for partially hyperbolic tori and a lambda lemma. *Nonlinearity*, 13:1561–1593, 2000.

[26] Antonio Garcia. Transition tori near an elliptic-fixed point. *Discrete and Continuous Dynamical Systems*, 6(2):381–392, January 2000.

[27] Marian Gidea, Amadeu Delshams, and Pablo Roldan. Transition map and shadowing lemma for normally hyperbolic invariant manifolds. *arXiv:1204.1507 [math.DS]*, 2012.

[28] Clodoaldo Grotta-Ragazzo. Irregular dynamics and homoclinic orbits to Hamiltonian saddle centers. *Communications on Pure and Applied Mathematics*, 50(2):105–147, February 1997.

[29] J Gruendler. Homoclinic solutions for autonomous dynamical systems in arbitrary dimension. *SIAM journal on mathematical analysis*, 23(3):702–721, 1992.

[30] G Haller. Universal Homoclinic Bifurcations and Chaos near Double Resonances. *Journal of Statistical Physics*, 86:1011–1051, 1997.

[31] G Haller. *Chaos Near Resonance*. Springer, 1999.

[32] G Haller and S Wiggins. Orbits homoclinic to resonances : the Hamiltonian case. *Physica D*, 66:298–346, 1993.

[33] G Haller and S Wiggins. Multi-pulse jumping orbits and homoclinic trees in a modal truncation of the damped-forced nonlinear Schrodinger equation. *Physica D*, 85:311–347, 1995.

[34] G Haller and S Wiggins. N-Pulse Homoclinic Orbits in Perturbations of Resonant Hamiltonian Systems. *Archive for Rational Mechanics and Analysis*, 130:25–101, 1995.

[35] Heinz Hanssmann. *Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems*. Springer, 2007.

[36] Philip J. Holmes and Jerrold E. Marsden. Melnikovs method and Arnold diffusion for perturbations of integrable Hamiltonian systems. *Journal of Mathematical Physics*, 23(4):669–675, April 1982.

[37] Ale Jan Homburg and J. Knobloch. Multiple homoclinic orbits in conservative and reversible systems. 358(4):1715–1740, 2005.

[38] Ale Jan Homburg and Jeroen S. W. Lamb. Symmetric homoclinic tangles in reversible systems. *Ergodic Theory and Dynamical Systems*, 26(06):1769–1789, November 2006.

[39] Ale Jan Homburg and Björn Sandstede. Homoclinic and Heteroclinic Bifurcations in Vector Fields. In *Handbook of Dynamical Systems III*, pages 379–524. Elsevier, 2010.

[40] I Hoveijn, J S W Lamb, and R M Roberts. Normal forms and unfoldings of linear systems in eigenspaces of (anti)-automorphisms of order two. *Journal of Differential Equations*, 190:182–213, 2003.

[41] Gerard Iooss and M Adelmeyer. *Topics in bifurcation theory and applications. Advanced series in nonlinear dynamics vol.3.* World Scientific, 1992.

[42] Christopher K R T Jones. Geometric Singular Perturbation Theory. In *Dynamical Systems, Lecture Notes in Mathematics Volume 1609*, pages 44–118. Springer, 1995.

[43] T. J Kaper. An Introduction to Geometric Methods and Dynamical Systems Theory for Singular Perturbation Problems. In *Analyzing Multiscale Phenomena Using Singular Perturbation Methods, in Proceedings of Symposia in Applied Mathematics, Vol 56*, volume 56, pages 85–132. 1999.

[44] J. Knobloch and J Klaus. Bifurcation of homoclinic orbits to a saddle-center in reversible systems. *International Journal of Bifurcation and Chaos*, 13(9):2603–2622, 2003.

[45] J. Knobloch and U Schalk. Homoclinic points near degenerate homoclinics. *Nonlinearity*, 8:1133–1141, 1995.

[46] O Koltsova and L.M. Lerman. Periodic and homoclinic orbits in a two-parameter unfolding of a Hamiltonian system with a homoclinic orbit to a saddle-center. *International Journal of Bifurcation and Chaos*, 05(02), 1995.

[47] O. Koltsova and L.M. Lerman. Families of transverse Poincaré homoclinic orbits in 2N-dimensional Hamiltonian systems close to the system with a loop to a saddle-center. *International Journal of Bifurcation and Chaos*, 6(6):991–1006, 1996.

[48] Oksana Koltsova, L.M. Lerman, A. Delshams, and Pere Gutiérrez. Homoclinic orbits to invariant tori near a homoclinic orbit to centercentersaddle equilibrium. *Physica D: Nonlinear Phenomena*, 201(3-4):268–290, February 2005.

[49] Gregor Kovacic. Dissipative dynamics of orbits homoclinic to a resonance band. *Physics Letters A*, 167:143–150, 1992.

[50] Gregor Kovacic. Hamiltonian dynamics of orbits homoclinic to a resonance band. *Physics Letters A*, 167:137–142, 1992.

[51] Gregor Kovacic. Singular Perturbation Theory for Homoclinic Orbits in a Class of Near-Integrable Hamiltonian Systems. *Journal of Dynamics and Differential Equations*, 5(4):559–597, 1993.

[52] Gregor Kovacic. Singular perturbation theory for Homoclinic orbits in a class of near-integrable disspative systems. *SIAM journal on mathematical analysis*, 26(6):1611–1643, 1995.

[53] Yu Kuznetsov. *Elements of Applied Bifurcation Theory, Second Edition.* Springer, 1998.

[54] Jeroen Lamb and O Koltsova. One-round dynamics near a homoclinic orbit to a reversible saddle-center. *In Preparation.*

[55] J.S.W. Lamb and H.W. Capel. Local bifurcations on the plane with reversing point group symmetry. *Chaos, Solitons & Fractals*, 5(2):271–293, February 1995.

[56] L.M. Lerman. Hamiltonian Systems with Loops of a Separatrix of a Saddle-Center. *Selecta Mathematica Sovietica (English translation)*, 10(3):297–306, 1991.

[57] Xiao-Biao Lin. Using Melnikov's method to solve Silnikov's problems. *Proceedings of the Royal Society of Edinburgh:*, 1990.

[58] Xiao-Biao Lin. Homoclinic Bifurcations with Weakly Expanding Center Manifolds. In *Dynamics Reported Vol 5*, pages 99–189. Springer, 1996.

[59] John Mallet-paret. The Fredholm Alternative for Functional Differential Equations of Mixed Type. *Journal of Dynamics and Differential Equations*, 11(1):1–47, 1999.

[60] Jerrold E. Marsden and Tudor S. Ratiu. Introduction to Mechanics and Symmetry. *Springer*, 1994.

[61] Dusa Mcduff and Dietmar Salamon. *Introduction to Symplectic Topology.* Oxford University Press, 1998.

[62] D.W McLaughlin, E.A Overman, S Wiggins, and C Xiong. Homoclinic Orbits in a Four Dimensional Model of a Perturbed NLS Equation: A Geometric Singular Perturbation Study. In *Dynamics Reported Vol 5*, pages 190–287. Springer-Verlag, 1991.

[63] V.K. Melnikov. On the stability of the center for time periodic perturbations. *Trans. Moscow Math. Soc.*, 12:3–52, 1963.

[64] K. R Meyer and G. R Hall. *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem.* Springer, 1991.

[65] A Mielke. *Hamiltonian and Lagrangian flows on Center Manifolds.* Springer-Verlag, 1991.

[66] A Mielke, P Holmes, and O O'Reilly. Cascades of Homoclinic Orbits to, and Chaos Near, a Hamiltonian Saddle-Center. *Journal of Dynamics and Differential Equations*, 4(1):95–126, 1992.

[67] L. Mirsky. Matrices with prescribed characteristic roots and diagonal elements. *Journal of the London Mathematical Society*, 33:14–21, 1958.

[68] H Najafi Alishah and J Lopes Dias. Realization of tangent perturbations in discrete and continuous time conservative systems. *Discrete and Continuous Dynamical Systems*, 34:5359–5374, 2014.

[69] Jacob Palis and Floris Takens. *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*. Cambridge University Press, 1995.

[70] Kenneth J Palmer. Exponential Dichotomies and Transversal Homoclinic Points. *Journal of Differential Equations*, 256:225–256, 1984.

[71] J Pöschel. Integrability of Hamiltonian Systems on Cantor Sets. *Communications on Pure and Applied Mathematics*, pages 653–695, 1982.

[72] Clark Robinson. Horseshoes for autonomous Hamiltonian systems using the Melnikov integral. *Ergodic Theory and Dynamical Systems*, 8(8*):395–409, December 1988.

[73] Björn Sandstede. *Verzweigungstheorie homokliner Verdopplungen*. PhD thesis, Universität Stuttgart, 1993.

[74] Björn Sandstede. Homoclinic flip bifurcations in conservative reversible systems. *Recent Trends in Dynamical Systems*, pages 107–124, 2013.

[75] Björn Sandstede and A Scheel. Relative Morse indices, Fredholm indices, and group velocities. *Discrete and Continuous Dynamical Systems*, 20(1):139–158, 2008.

[76] Björn Sandstede, Arnd Scheel, and Joerg Haerterich. Exponential dichotomies for linear non-autonomous functional differential equations of mixed type. *Indiana University Mathematics Journal*, 51(5):1081–1109, 2002.

[77] Andrey Shilnikov, Leonid Shilnikov, Dmitry Turaev, and Leon Chua. *Methods of qualitative theory in nonlinear dynamics, Part 1.* World Scientific, 1998.

[78] Leonid Shilnikov. A contribution to the problem of the structure of an extended neighbourhood of a rough equilibrium state of saddle-focus type. *Math. USSR Sbornik*, 10(1):91–102, 1970.

[79] Leonid Shilnikov and Dmitry Turaev. On Hamiltonian systems with homoclinic saddle curves. *Sov. Math. Dokl.*, 39:165–168, 1989.

[80] Sebastian J van Strien. Center manifolds are not C^∞. *Mathematische Zeitschrift*, 166(2):143–145, 1979.

[81] A Vanderbauwhede and B Fiedler. Homoclinic period blow-up in reversible and conservative systems. *Zeitschrift für Angewandte Mathematik und Physik*, 43:292–318, 1992.

[82] T. Wagenknecht and A R Champneys. When gap solitons become embedded solitons: a generic unfolding. *Physica D: Nonlinear Phenomena*, 177(1-4):50–70, March 2003.

[83] Thomas Wagenknecht. *Homoclinic Bifurcations in Reversible Systems.* PhD thesis, Technische Universität Ilmenau, 2003.

[84] Martin Wechselberger. Extending Melnikov theory to invariant manifolds on non-compact domains. *Dynamical Systems*, 17(3):215–233, August 2002.

[85] J Williamson. On the algebraic problem concerning the normal forms of linear dynamical systems. *American Journal of Mathematics*, 58(1):141–163, 1936.

[86] Kazuyuki Yagasaki. The method of Melnikov for perturbations of multi-degree-of-freedom Hamiltonian systems. *Nonlinearity*, 12:799–822, 1999.

[87] Kazuyuki Yagasaki. Horseshoes in Two-Degree-of-Freedom Hamiltonian Systems with Saddle-Centers. *Archive for Rational Mechanics and Analysis*, 154(4):275–296, October 2000.

[88] Kazuyuki Yagasaki. Homoclinic and heteroclinic orbits to invariant tori in multi-degree-of-freedom Hamiltonian systems with saddle-centers. *Nonlinearity*, 18:1331–1350, 2005.