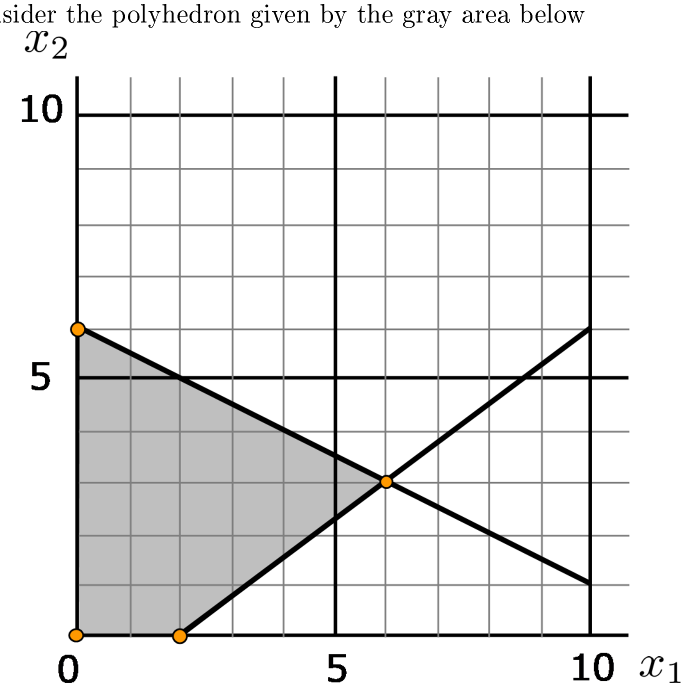
Exercises: Dantzig-Wolfe decomposition

1 Task 1: Minkowski-Weyl

Let X be a bounded polyhedron with extreme points $\bar{x}^p, p \in P$. In the lecture we saw that Minkowski-Weyl's theorem state that X can be represented as:

$$X = \left\{ x \in \mathbb{R}^n : x = \sum_{p \in P} \bar{x}^p, \sum_{p \in P} \lambda^p = 1, \forall p \in P : \lambda^p \geq 0 \right\}$$

Consider the polyhedron given by the gray area below



1.1 Task 1.a

Write up the extreme points of X

1.2 Task 1.b

Write up an LP that maximizes $Z = x_1 + 3x_2$ over the polyhedron X using Minkowski-Weyl's theorem and the extreme points found above.

1.3 Task 1.c

In this sub-task we forget about Minkowski-Weyl. The task is therefore to write up the "normal" constraints that define X and write the LP that maximizes $Z = x_1 + 3x_2$ over the polyhedron X using these constraints.

1.4 Task 1.d

Implement the LPs from task 1.b and 1.c in Julia and verify that both models find the expected solutions.

Note: the purpose of the task is simply to "practice" Minkowski-Weyl's theorem. The LP in 1.b is not "smarter" or better than the LP in 1.*c*.

2 Task 2

In this task we consider a slightly different decomposition of the problem compared to what we did in the lectures. in the model below the constraints marked with A1 are "convexified" and the constraints A2 remains in the master problem.

$$
\begin{array}{lrrcrl}
\max & x & +2y & & & \\
\text{s.t.} & -x & +y & \leq & \frac{5}{2} & (A1) \\
 & x & +y & \leq & \frac{9}{2} & (A1) \\
 & x & -y & \leq & \frac{1}{2} & (A1) \\
 & -x & -y & \leq & -\frac{3}{2} & (A1) \\
 & -x & +y & \leq & 1 & (\mathbf{A1}) \\
 & 3x & +y & \leq & 13 & (A2) \\
 & -x & -3y & \leq & -7 & (A2) \\
 & & x, y & \in & \mathbb{Z} &
\end{array}
$$

1. Draw the new situation in a 2-d coordinate system in the same way as it was done in the lecture.
2. Write up the Dantzig-Wolfe decomposed version of this problem and solve its LP relaxation using a solver. Hint: now there is one more constraint in the representation of X_1 so some extreme points have changed.
3. What happened to the bound provided by the Dantzig-Wolfe decomposition compared to the decomposition we saw in class?
4. What can you say in general about the bound provided by Dantzig-Wolfe decomposition when more or less constraints are included in the definition of X_1?

3 Task 3 (more than two variables)

Now consider the following IP:

$$\max x_1 + x_2 + x_3 + x_4$$

subject to

$$3x_1 - 2x_2 + 2x_3 - x_4 \geq 3 \tag{1}$$
$$x_1 + x_2 + 4x_3 + x_4 \leq 3 \tag{2}$$
$$x_1, x_2, x_3, x_4 \in \{0, 1\} \tag{3}$$

1. Solve the IP and its LP relaxation using a solver.
2. Write up the LP relaxation of the model derived by Dantzig-Wolfe decomposition with constraint (1) being convexified and constraint (2) in the master problem.
3. Solve this LP using a solver and compare to the results from task 3.1.
4. Write up the LP relaxation of the model derived by Dantzig-Wolfee decomposition with constraint (1) in the master problem and constraint (2) being convexified.
5. Solve this LP using a solver and compare to the results from task 3.1 and 3.3.