Ex1

On page 70 we are given the mean and covariance matrix of a finite dimensional brownian motion.

$$\bar{B} = (B_{t_1}, B_{t_2}, B_{t_3}, \dots, B_{t_n})$$

$$\mathbb{E}[\bar{B}] = (0, 0, 0, \dots, 0)$$

$$\mathbb{E}[\bar{B}^T \bar{B}] = \mathrm{Cov}[\bar{B}] = \begin{bmatrix} t_1 & t_1 & \dots & t_1 \\ t_1 & t_2 & \dots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \dots & t_n \end{bmatrix}$$

this is also confirmed by the simulation.

Ex 4: Show that $\{B_t^2 - t\}$ is a martingale.

We define $Y_t = B_t^2 - t$ and now we want to show

$$\mathbb{E}[Y_t \mid \mathcal{F}_s] = Y_s \quad \text{for} \quad s < t$$

c.f. s. 83 [in the text. not as thm, Def or lemma].

We note that

$$\mathbb{E}[Y_t \mid \mathcal{F}_s] = \mathbb{E}[B_t^2 \mid \mathcal{F}_s] - \mathbb{E}[t \mid \mathcal{F}_s]$$
$$= \mathbb{E}[B_t^2 \mid \mathcal{F}_s] - t$$

(because t is a const.)

We now work with $\mathbb{E}[B_t^2 \mid \mathcal{F}_s]$

$$\mathbb{E}[B_t^2 \mid \mathcal{F}_s] = \mathbb{E}\left[((B_t - B_s) + B_s)^2 \mid \mathcal{F}_s\right]$$
$$= \mathbb{E}\left[(B_t - B_s)^2 \mid \mathcal{F}_s\right] + \mathbb{E}[B_s^2 \mid \mathcal{F}_s] + \mathbb{E}\left[2((B_t - B_s)B_s) \mid \mathcal{F}_s\right]$$
$$= \mathbb{E}\left[(B_t - B_s)^2\right] + B_s^2 + 0$$
$$= (t - s) + B_s^2$$

[Variance of the increment is by def $(t-s)$ because B_s is known under $\mathcal{F}_s$]

Thus

$$\mathbb{E}[Y_t \mid \mathcal{F}_s] = (t-s) + B_s^2 - t = B_s^2 - s = Y_s$$

$$\mathbb{E}[2((B_t - B_s)B_s \mid \mathcal{F}_s] = 2B_s\, \mathbb{E}[B_t - B_s \mid \mathcal{F}_s]$$

(Because B_s is a constant under $\mathcal{F}_s$, i.e. we know it under $\mathcal{F}_s$)

$$= 2B_s(0)$$

Because we know $\mathbb{E}[B_t - B_s \mid \mathcal{F}_s]$ is an increment from B_s to B_t which in expection is 0.

Ex5: (Ex20 in the book) Let X be a random variable such that $\mathbb{E}[X] < \infty$, and let $\{\mathcal{F}_t : t \geq 0\}$ be a filtration. Show that the process $\{M_t : t \geq 0\}$ given by

$$M_t = \mathbb{E}[X \mid \mathcal{F}_t]$$

is a martingale, wrt $\{\mathcal{F}_t : t \geq 0\}$.

By def 4.5.1. a martingale M_t is defined by the following 3 properties.

1) The process M_t is adapted to the filtration $\mathcal{F}_t$
2) For all times $t \geq 0$, $\mathbb{E}[M_t] < \infty$
3) $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$, whenever $t \geq s \geq 0$.

We start with property 3). We have

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_t] \mid \mathcal{F}_s] \qquad t \geq s$$

from the def. of filtrations we know $\mathcal{F}_s \subseteq \mathcal{F}_t$ and hence the law of iterated expectations hold ($\mathbb{E}[X] = \mathbb{E}[X \mid Y]$)

$$\mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \mathbb{E}[X \mid \mathcal{F}_s] = M_s$$

Now 2)

$$\mathbb{E}[M_t] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_t]] = \mathbb{E}[X] < \infty$$ (use iterated expec and def of X)

Lastly property 1) follows from how we defined M_t.

Hence M_t is a martingale.

Ex 6 (Ex 4.11) Show that if $\{M_t : t \geq 0\}$ is a martingale such that $\mathbb{E}[|M_t|^2] < \infty$ for all t, then the increments

$$M_t - M_s \text{ and } M_v - M_u$$

are uncorrelated whenever $0 \leq s \leq t \leq u \leq v$.

- We first see that given the filtration $\mathcal{F}_t$

$$\begin{aligned}\mathbb{E}[M_v - M_u \mid \mathcal{F}_t] &= \mathbb{E}[M_v \mid \mathcal{F}_t] - \mathbb{E}[M_u \mid \mathcal{F}_t] \\ &= M_t - M_t \qquad \text{(Martingale prop)} \\ &= 0\end{aligned}$$

Next we can use tower property and get

$$\begin{aligned}\mathbb{E}[(M_t - M_s)(M_v - M_u)] &= \mathbb{E}[\mathbb{E}[(M_t - M_s)(M_v - M_u) \mid \mathcal{F}_t]] \\ &= \mathbb{E}[(M_t - M_s)\mathbb{E}[M_v - M_u \mid \mathcal{F}_t]] \\ &= \mathbb{E}[(M_t - M_s)\cdot 0] \\ &= 0\end{aligned}$$

Next, show that the variance of increments is additive:

$$V[M_u - M_s] = V[M_u - M_t] + V[M_t - M_s]$$

- Variance of the sum of two random variables is given by

$$V[X+Y] = V[X] + V[Y] + Cov[X,Y]$$

So

$$\begin{aligned}V[M_u - M_s] &= V[(M_u - M_t) + (M_t - M_s)] \\ &= V[M_u - M_t] + V[M_t - M_s] + Cov[(M_u - M_t), (M_t - M_s)] \\ &= V[M_u - M_t] + V[M_t - M_s] \quad \text{(because } \overset{\uparrow}{} 0 \text{, c.f. first part of this exercise)}\end{aligned}$$

Show that the variance is increasing

- We let $0 \leq s \leq t$, then

$$V[M_t] = V[M_s] + V[M_t - M_s] \geq V[M_s]$$