Introduction to Dynamical Systems: Assignment 1

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1 Problem 1

We are to explore the technique called *Non-dimensionalization*, which is used to reduce the number of parameters in a dynamical system. To explore this technique we will apply it on the logistic differential equation.

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}) \tag{1}$$

Here P is the size of a population, K is the caring capacity, r is the growth rate and t is time. We are given that $r > 0$ and $K > 0$ but it still leaves us with two infinities of systems to analyze. We will therefore apply the technique of *Non-dimensionalization* to remove the parameters r and K. To do this we are told to rewrite P and t by the following relations.

$$\begin{aligned} P &= P_c p \\ t &= t_c \tau \end{aligned} \tag{2}$$

where P_c and t_c are constants

1.1 Question (a)

We are to show that using (2) to rewrite (1) we obtain the form

$$\frac{dp}{d\tau} = t_c r p \left(1 - \frac{P_c p}{K}\right) \tag{3}$$

We insert (2) in (1)

$$\frac{d(P_c p)}{d(t_c \tau)} = rP_c p(1 - \frac{P_c p}{K})$$

Because P_c and t_c are constants we can pull them out of the differentiation and obtain

$$\begin{aligned} \frac{P_c}{t_c}\frac{d(p)}{d(\tau)} &= rP_c p(1 - \frac{P_c p}{K}) \Rightarrow \\ \frac{d(p)}{d(\tau)} &= \frac{t_c}{P_c} rP_c p(1 - \frac{P_c p}{K}) \Rightarrow \\ \frac{dp}{d\tau} &= t_c r p \left(1 - \frac{P_c p}{K}\right) \end{aligned} \tag{4}$$

Hence we see that (4) is the same as (3).

1.2 Question (b)

Now we are to choose values for P_c and t_c so the system simplifies to

$$\frac{dp}{d\tau} = p(1 - p) \tag{5}$$

We hence choose $t_c = r^{-1}$ and $P_c = K$ and plug them into (4).

$$\frac{dp}{d\tau} = r^{-1} rp\left(1 - \frac{Kp}{K}\right) \Rightarrow$$
$$\frac{dp}{d\tau} = p(1-p) \tag{6}$$

We see by choosing $t_c = r^{-1}$ and $P_c = K$ we obtain (5).

1.3 Question (c)

We will now analyze (6) for $p \geq 0$. To do this we plot (6) in figure 1

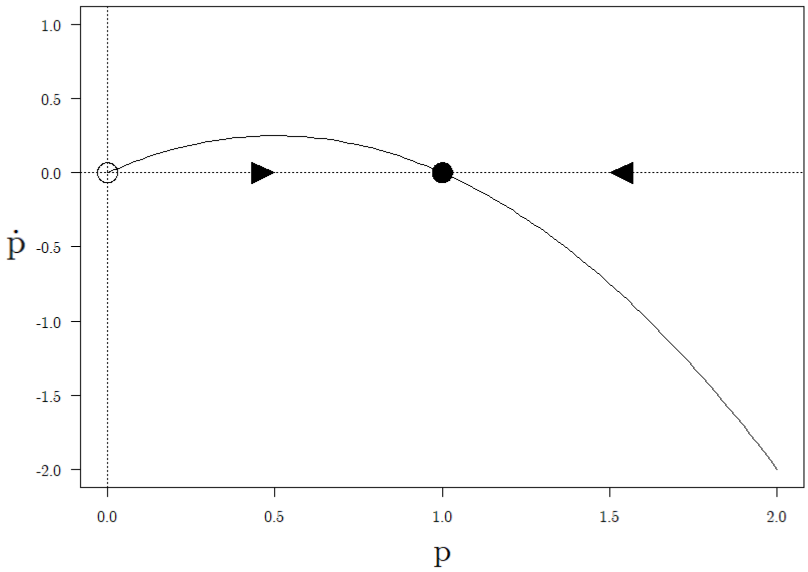

Figure 1 – Illustration of the logistic differential equation.

We see we have two fixed points, one at $p = 0$ and one at $p = 1$. We further see that $p = 0$ is an unstable node and $p = 1$ is a stable node.
We see that starting in $P = 0$ the solution will be trivial $P = 0$. For any other positive P, the solution will converge to $P = 1$.

1.4 Question (d)

We will now consider a larger system.

$$\begin{aligned}\dot{X} &= aX - bXY \\ \dot{Y} &= -dY + cXY\end{aligned} \tag{7}$$

where a, b, c, d all are positive parameters. We are now to show that one can non-dimensionalize to obtain the system

$$\begin{aligned} \frac{dx}{d\tau} &= x - xy \\ \frac{dy}{d\tau} &= -\alpha y + xy \end{aligned} \tag{8}$$

and determine α in terms of (some of) the parameters a, b, c and d. Furthermore we will also apply non-dimensionalization to obtain the system

$$\begin{aligned} \frac{dx}{d\tau} &= \beta x - xy \\ \frac{dy}{d\tau} &= -y + xy \end{aligned} \tag{9}$$

and determine the value of β.

But before doing so we will analyze the units of the different components in the system. We start by analysing the units of the variables in 7.

Variable	Unit
$\dot{X}$	$\frac{[X]}{t}$
$\dot{Y}$	$\frac{[Y]}{t}$
X	$[X]$
Y	$[Y]$

Table 1 – Units of variables in 7

Hence the parameters must have units such that the units of the rhs and the lhs of (7) matches. We hence deduce the units of the parameters given in table 2

Parameter	Unit
a	$\frac{1}{t}$
b	$\frac{1}{[Y]t}$
c	$\frac{1}{[X]t}$
d	$\frac{1}{t}$

Table 2 – Units of parameters in (7)

We must now chose the re-scaling of our variables such that we cancel as many dimensions as possible. We do this by making the units cancel. We start with time and here we have two choices. Both a and b has the inverse units of time. We chose a for now.

$$\tau = at \Rightarrow t = \frac{1}{a}\tau \tag{10}$$

Next up is X. Here only the unit of c is related to X but it also introduces time. We hence both use a and c to cancel all units.

$$x = \frac{a}{c}X \Rightarrow X = \frac{a}{c}x \tag{11}$$

Lastly we have Y which we handle the same way as X except we swap c for b.

$$y = \frac{a}{b}Y \Rightarrow Y = \frac{a}{c}y \tag{12}$$

We now insert in (7).

$$\begin{aligned}
\dot{X} &= aX - bXY \\
\dot{Y} &= -dY + cXY \Rightarrow \\
\frac{a^2}{c}\frac{dx}{d\tau} &= \frac{a^2}{c}x - b\frac{a}{c}\frac{a}{b}xy \\
\frac{a^2}{b}\frac{dy}{d\tau} &= -d\frac{a}{b}y + c\frac{a}{c}\frac{a}{b}xy \Rightarrow \\
\frac{dx}{d\tau} &= x - xy \\
\frac{dy}{d\tau} &= -\frac{d}{a}y + xy
\end{aligned}$$

So we see that when using a as the invers unit of time we get (8) and $\alpha = \frac{d}{a}$. We now use d as the inverse unit of time instead. Hence

$$\tau = dt \Rightarrow t = \frac{1}{d}\tau \tag{13}$$

$$x = \frac{d}{c}X \Rightarrow X = \frac{d}{c}x \tag{14}$$

$$y = \frac{d}{b}Y \Rightarrow Y = \frac{d}{c}y \tag{15}$$

And we non-dimensionalize

$$\begin{aligned}
\dot{X} &= aX - bXY \\
\dot{Y} &= -dY + cXY \Rightarrow \\
\frac{d^2}{c}\frac{dx}{d\tau} &= \frac{d}{c}ax - b\frac{d}{c}\frac{d}{b}xy \\
\frac{d^2}{b}\frac{dy}{d\tau} &= -\frac{d^2}{b}y + c\frac{d}{c}\frac{d}{b}xy \Rightarrow \\
\frac{dx}{d\tau} &= \frac{a}{d}x - xy \\
\frac{dy}{d\tau} &= -y + xy
\end{aligned}$$

We now obtain (17) and $\beta = \frac{a}{d}$

2 Problem 2

The dynamics of piece-wise linear vector fields is an active area of research. In this problem, we look at one such vector field. Let:

$$\begin{bmatrix} 2a & -1 \\ 1 & 0 \end{bmatrix}, \quad |a| < 1 \tag{16}$$

And consider:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} \tag{17}$$

2.1 Question (a)

What kind of equilibrium point is the origin?
We start by computing the trace and determinant of the system matrix, **A**.

$$\tau = 2a, \qquad \delta = 1 \tag{18}$$

We can classify the equilibrium point by use of theorem of section 1.5 25 in [1]. We can immediately tell that the equilibrium point is not a saddle due to the positive determinant but to say more we need the following quantity.

$$\tau^2 - 4\delta = 4a^2 - 4 < 0, \quad |a| < 1 \tag{19}$$

We hence get that:

$$\begin{aligned} 0 < a < 1 \quad &\rightarrow \quad \text{unstable focus} \\ a = 0 \quad &\rightarrow \quad \text{center} \\ -1 < a < 0 \quad &\rightarrow \quad \text{stable focus} \end{aligned}$$

Lastly we examine the direction of rotation by inserting $[0,1]^T$ in (17).

$$\begin{bmatrix} 2a & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a \\ 1 \end{bmatrix} \tag{20}$$

We see that y changes in the anti-clockwise direction independent of a. The following three scenarios can be seen in figure 2.

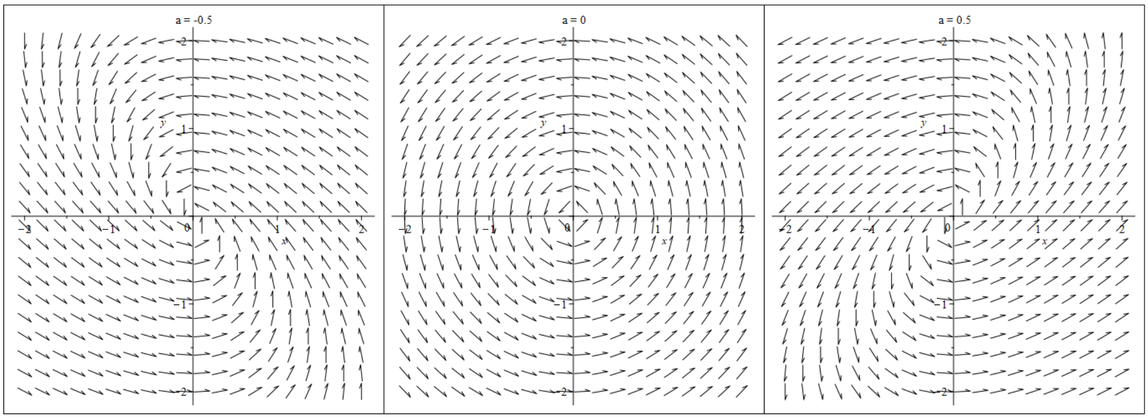

Figure 2 – The phase space for a = -0.5, a = 0 and a = 0.5

2.2 Question (b)

Let $\omega = \sqrt{1-a^2}$, and show that $\lambda = a + i\omega$ is an eigenvalue, and that $\mathbf{w} = (\lambda, 1)$ is a corresponding eigenvector.
We start by determining the characteristic polynomial of the system matrix, $\mathbf{A}$.

$$-(2a-\lambda)\lambda + 1 = \lambda^2 - 2a\lambda + 1 = 0 \tag{21}$$

By insertion of $\lambda = a + i\omega$ and $\omega = \sqrt{1-a^2}$ we get:

$$\begin{aligned} (a + i\sqrt{1-a^2})^2 - 2a(a + i\sqrt{1-a^2}) + 1 &= 0 \\ a^2 - (1-a^2) + 2ai\sqrt{1-a^2} - 2a^2 - 2ai\sqrt{1-a^2} + 1 &= 0 \\ a^2 - 1 + a^2 - 2a^2 + 1 &= 0 \\ 0 &= 0 \end{aligned} \tag{22}$$

Hence λ is in fact an eigenvalue to the system matrix, $\mathbf{A}$. We will now find the corresponding eigenvectors to λ and $\bar{\lambda}$.
We solve the linear equations given by:

$$\begin{bmatrix} 2a-\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \mathbf{w_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2a-\bar{\lambda} & -1 \\ 1 & -\bar{\lambda} \end{bmatrix} \mathbf{w_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{23}$$

We get:

$$\mathbf{w_1} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}, \qquad \mathbf{w_2} = \begin{bmatrix} \bar{\lambda} \\ 1 \end{bmatrix} \tag{24}$$

2.3 Question (c)

Let:

$$\mathbf{P} = \begin{bmatrix} \omega & a \\ 0 & 1 \end{bmatrix} \tag{25}$$

Find the matrix $\mathbf{R}$ that satisfy:

$$\mathbf{A} = \mathbf{P}\mathbf{R}\mathbf{P}^{-1} \tag{26}$$

We start by splitting the eigenvector up into real and imaginary part:

$$\mathbf{w_1} = \mathbf{u} + i\mathbf{v} = \begin{bmatrix} \omega \\ 0 \end{bmatrix} + i \begin{bmatrix} a \\ 1 \end{bmatrix} \tag{27}$$

We know from the theorem section 1.6 in [1], that P is invertible and R has the form:

$$\mathbf{R} = \begin{bmatrix} a & -\omega \\ \omega & a \end{bmatrix} \tag{28}$$

The same result could have been obtained simply by inverting $\mathbf{P}$:

$$\mathbf{R} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{29}$$

2.4 Question (d)

Find $e^{t\mathbf{R}}$.
We use corollary 3 from section 1.3 in [1]. Corollary 3 states that given a matrix, $\mathbf{A}$, on the form:

$$\mathbf{A} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{30}$$

Then

$$e^{\mathbf{A}} = e^{a} \begin{bmatrix} cos(b) & -sin(b) \\ sin(b) & cos(b) \end{bmatrix} \tag{31}$$

Our matrix, $t\mathbf{R}$, is given by:

$$t\mathbf{R} = \begin{bmatrix} ta & -t\omega \\ t\omega & ta \end{bmatrix} \tag{32}$$

Hence:

$$e^{t\mathbf{R}} = e^{ta} \begin{bmatrix} cos(t\omega) & -sin(t\omega) \\ sin(t\omega) & cos(t\omega) \end{bmatrix} \tag{33}$$

2.5 Question (e)

Show that $T(a) = \frac{\pi}{\omega} = \frac{\pi}{\sqrt{1-a^2}}$.
In exercise 2(d) we found the flow in **uv**-coordinates.

$$\mathbf{x_{uv}}(t) = e^{at} \begin{bmatrix} cos(t\omega) & -sin(t\omega) \\ sin(t\omega) & cos(t\omega) \end{bmatrix} \mathbf{x_{0uv}} \tag{34}$$

The **uv**-space has one very nice property, namely that the rotation and scaling is completely separated and hence it would be very easy to show that $T(a) = \frac{\pi}{\omega} = \frac{\pi}{\sqrt{1-a^2}}$. What needs to hold if we prove it in **uv**-space is that it should also hold for **xy**-space. We actually know this will hold because **uv**-space is a linear transformation of **xy**-space and hence any straight line in one space will also be a straight line in the other space. We therefore only need to figure out, how long a half-rotation takes in **uv**-space to determine $T(a)$. The rotation is completely determined by the rotation matrix which rotates with a period of 2π which means a half rotation is given by:

$$\min T > 0 \quad : \quad \omega T = \pi \tag{35}$$

Since we know that $\omega = \sqrt{1-a^2}$ and $|a| < 1$, we know that $\omega > 0$, hence:

$$\min T > 0 \quad : \quad \omega T = \pi \quad \Rightarrow \quad T = \frac{\pi}{\omega} \tag{36}$$

This is also illustrated in figure 3

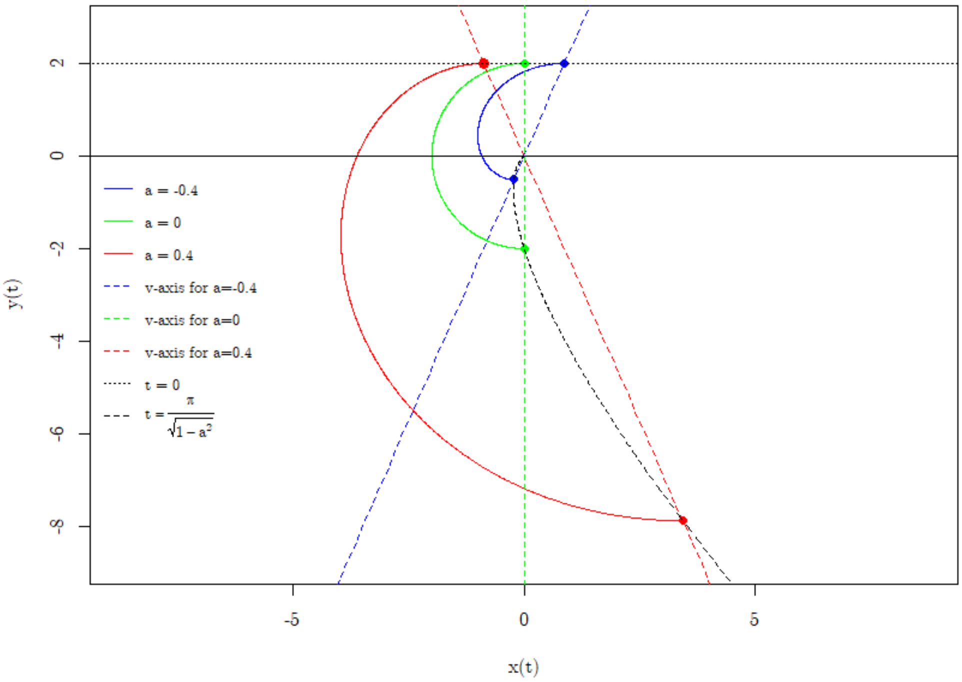

Figure 3 – Half rotations in **uv**-space for a = -0.5, a = 0 and a = 0.5

2.6 Question (f)

Show that the flow ϕ satisfies:

$$\phi_{T(a)}(0, y) = (0, -e^{\frac{\pi a}{\sqrt{1-a^2}}} y) \tag{37}$$

We make use of the corollary from section 1.6 in [1] and the solution from exercise 2(d). We get the solution in **xy**-coordinates to:

$$\phi_{T(a)}(0, y) = \mathbf{P}e^{aT(a)} \begin{bmatrix} cos(T(a)\omega) & -sin(T(a)\omega) \\ sin(T(a)\omega) & cos(T(a)\omega) \end{bmatrix} \mathbf{P}^{-1} \begin{bmatrix} 0 \\ y \end{bmatrix} \tag{38}$$

Inserting $\omega T(a) = \pi$ we get:

$$\phi_{\pi}(0, y) = \mathbf{P}e^{a\frac{\pi}{\omega}} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{P}^{-1} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -e^{a\frac{\pi}{\omega}} y \end{bmatrix} = \begin{bmatrix} 0 \\ -e^{\frac{a\pi}{\sqrt{1-a^2}}} y \end{bmatrix} \tag{39}$$

Which is what we wanted to show.

Let now:

$$\mathbf{B} = \begin{bmatrix} 2b & -1 \\ 1 & 0 \end{bmatrix} \tag{40}$$

With $|b| < 1$, and consider the vector field:

$$\mathbf{f}(x, y) = \begin{cases} \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} & x \leq 0 \\ \mathbf{B}\begin{pmatrix} x \\ y \end{pmatrix} & x \geq 0 \end{cases} \tag{41}$$

The vector field is continuous but not C^1, one can however show that existence and uniqueness still holds, so $\mathbf{f}$ defines a flow ϕ. Notice that the flow of $\mathbf{f}$ carries the positive y-axis onto the negative y-axis in time T(a), and the negative y-axis onto the positive one in time T(b).

2.7 Question (g)

By considering the flow $\phi_{T(b)+T(a)}$ on the positive y-axis, explain why the origin is stable if and only if:

$$\frac{a}{\sqrt{1-a^2}} + \frac{b}{\sqrt{1-b^2}} \leq 0 \tag{42}$$

We notice that $\mathbf{B}$ is equal to $\mathbf{A}$ where $a = b$. We therefore know that the flow are following the same type of path, only governed by the value of a and b. We also know that $T(a)\omega_a = T(b)\omega_b = \pi$. We further notice that for the origin to be stable, it must mean that during the course of a full rotation (from y-axis to y-axis), the flow, $\phi_{T(b)+T(a)}$, must not have taken us farther away from the origin, because if we first have moved further away we can never return due to paths are not able to cross in C^1.

As described in 2(e), the flow will be counter-clockwise, hence the flow initially follows the path governed by a. After it reached the y-axis, the flow will then be governed by b. Specifically, the flow will be:

$$\begin{aligned} \phi_{T(b)+T(a)}(0, y) &= (0, -e^{\frac{\pi b}{\sqrt{1-b^2}}}(-e^{\frac{\pi a}{\sqrt{1-a^2}}}y)) \\ &= (0, e^{\frac{\pi a}{\sqrt{1-a^2}}+\frac{\pi b}{\sqrt{1-b^2}}}y)) \\ &= (0, e^{\pi(\frac{a}{\sqrt{1-a^2}}+\frac{b}{\sqrt{1-b^2}})}y)) \end{aligned}$$

Notice that $\frac{a}{\sqrt{1-a^2}} + \frac{b}{\sqrt{1-b^2}} = 0$ means that we end up exactly where we started, $\frac{a}{\sqrt{1-a^2}} + \frac{b}{\sqrt{1-b^2}} < 0$ means we get closer to the origin, while $\frac{a}{\sqrt{1-a^2}} + \frac{b}{\sqrt{1-b^2}} > 0$ means that will have gotten farther away.

We can therefore conclude, that the origin is stable if and only if (42) holds.

References

[1] L. Perko, *Differential Equations and Dynamical Systems.* Springer, 2001.