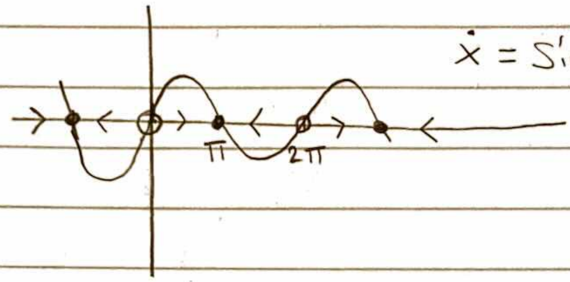
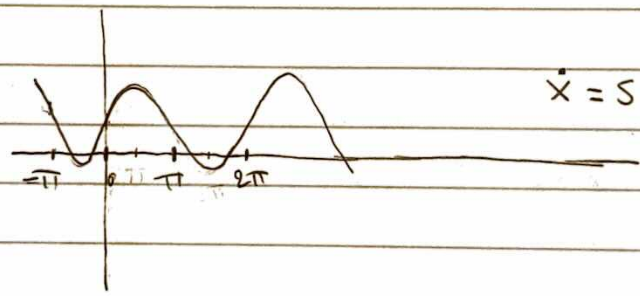
Test exercise: Consider the one-dimensional system

$$\dot{x} = \sin(x) + \mu$$

where μ is a parameter. For a certain value of μ, the phase portrait looks as follow.





So $0 < \mu \leq 1$ because if $\mu > 1$ or $\mu < -1$ would have 0 fixed points and if $-1 \leq \mu \leq 0$ then the two fixed points would lie in $0 \leftrightarrow \pi$.
If $\mu = -1 = 1$ then we would only have 1

So

$$0 < \mu < 1$$

Exercise 1:

a) (a) is a saddle
(b) is a center
(c) is a foci
(d)-(f) are nodes

b) (d) the eigenvalue for the most vertical line is the larger.

(e) the same as (d) except that the difference between the eigenvalues is smaller here than for (d).

(f) Here the "horizontal" eigenvalue is the larges.

c) (d)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \text{Some tilt}$$

(e)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \text{some tilt}$$

(f)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \text{some tilt}$$

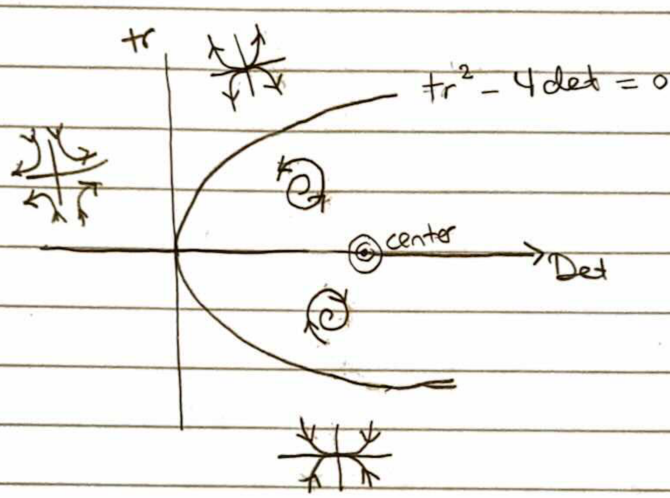
Exercise 2: Consider the 2D linear system $\dot{x} = Ax$ where

$$A = \begin{pmatrix} -1 & a \\ ab & -1 \end{pmatrix}$$

with parameters $a > 0$ and $b \in \mathbb{R}$.

a) Classify the system $a > 0$ and b

$Det(A) = -a^2 b + 1$ $\quad\quad$ $tr(A) = -2$



1) always stable because $tr(A) = -2$ $\forall b \wedge \forall a$

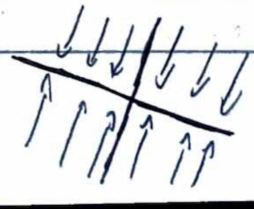
2) When $(a^2 b > 1$ and $b > 0) \rightarrow b > \frac{1}{a^2}$
we have a saddle. $\rightarrow$

3) $(-2)^2 - 4(-a^2 b + 1) = 4 + 4a^2 b - 4 = 4a^2 b$

focus if $b \leq 0$

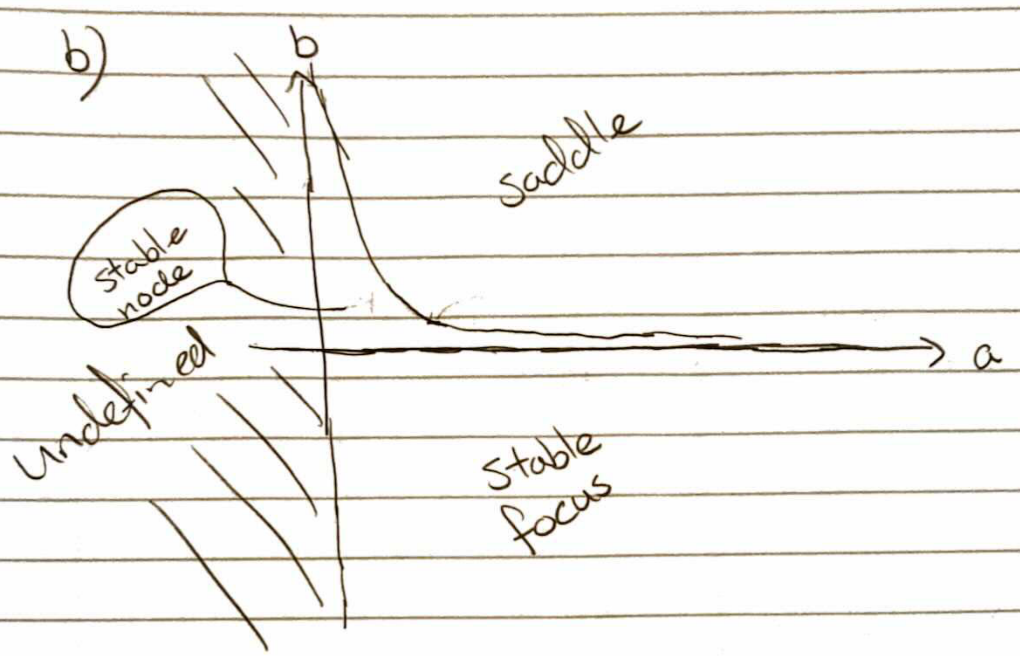
node if $(b \geq 0 \wedge a^2 b < 1) \rightarrow 0 \leq b \wedge b < \frac{1}{a^2}$

4) $a = 1$ $\quad b = 1 \rightarrow det = 0$ makes degenerate



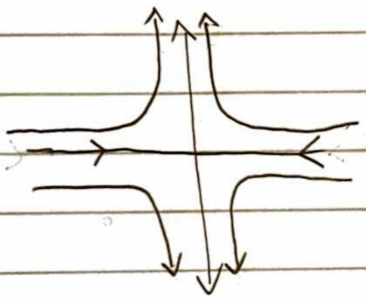
Con't Exercise 2

b)



c)

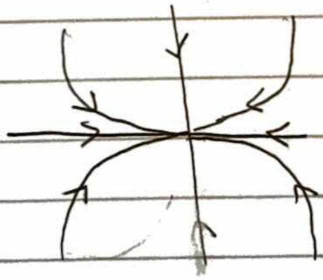
$b = 2 \quad a = b$, saddle



$E^s = \mathbb{R}^1 \qquad E^c = \mathbb{R}^0$

$E^u = \mathbb{R}^1$

$b = \frac{1}{2} \quad a = \frac{1}{2}$, stable node

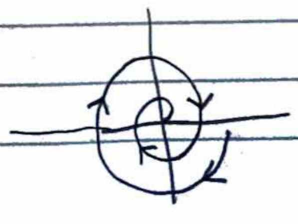


$E^s = \mathbb{R}^2 \qquad E^c = \mathbb{R}^0$

$E^u = \mathbb{R}^0$

$b = -2 \quad a = 2$

$E^s = \mathbb{R}^2 \qquad E^c = \mathbb{R}^0$



$E^u = \mathbb{R}^0$

Exercise 3 Do the following

a) Find two matrices A and B so that

$$e^{A+B} \neq e^{A} e^{B}$$

If $\underline{AB} \neq \underline{BA}$ then $e^{A+B} \neq e^{A} e^{B}$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 10 & 10 \\ 16 & 14 \end{bmatrix} \qquad BA = \begin{bmatrix} 6 & 16 \\ 8 & 18 \end{bmatrix}$$

b) Find a square matrix $A \in \mathbb{R}^{2\times 2}$ where both eigenvalues satisfy $Re(\lambda) \leq 0$ yet there exists a solution of $\dot{x} = Ax$ satisfying $|x(t)| \to \infty$ for $t \to \infty$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \dot{x} = y \\ \dot{y} = 0 \end{matrix} \rightarrow \begin{matrix} x(t) = x_0 + y(t) \\ y(t) = y_0 \end{matrix} \rightarrow \begin{vmatrix} x_0 + y(t) \\ y_0 \end{vmatrix} \rightarrow \infty$$

när $t \to \infty$

c) Determine all $A \in \mathbb{R}^{2\times 2}$ so that

(i) A is semisimple (ii) A is nilpotent

Exercise 4 : Solve the system

$$\dot{x} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix} x$$

Find the stable, unstable and center subspace

eigenvalues of A is $\lambda_1 = 6$, $\lambda_2 = 2i$ and $\lambda_3 = -2i$
therefore we have a unstable manifold in λ_1's direction and a center in λ_2 and λ_3 because $Re(\lambda_2) = Re(\lambda_3) = 0$.

i.e.

$$E^s = \mathbb{R}^0, \quad E^u = \mathbb{R}^1, \quad E^c = \mathbb{R}^2$$