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REVERSIBLE HAMILTONIAN LIAPUNOV CENTER THEOREM

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Abstract. We study the existence of periodic solutions in the neighbourhood of symmetric (partially) elliptic equilibria in purely reversible Hamiltonian vector fields. These are Hamiltonian vector fields with an involutory reversing symmetry R. We contrast the cases where R acts symplectically and anti-symplectically.

In case R acts anti-symplectically, generically purely imaginary eigenvalues are isolated, and the equilibrium is contained in a local two-dimensional invariant manifold containing symmetric periodic solutions encircling the equilibrium point.

In case R acts symplectically, generically purely imaginary eigenvalues are doubly degenerate, and the equilibrium is contained in two two-dimensional invariant manifolds containing nonsymmetric periodic solutions encircling the equilibrium point. In addition, there exists a three-dimensional invariant surface containing a two-parameter family of symmetric periodic solutions.

1. **Introduction.** It is well known that (purely) reversible and Hamiltonian dynamical systems have many striking features in common, see for instance [18, 17, 13]. Surprisingly, despite the frequent occurrence in applications, only relatively few theoretical studies have focussed on the category of dynamical systems that are both Hamiltonian and reversible at the same time.

In this paper, we discuss the dynamics in the neighbourhood of an equilibrium with some purely imaginary pairs of eigenvalues. We call such an equilibrium (partially) elliptic. Importantly, such types of equilibria occur generically (with codimension zero) in reversible, Hamiltonian, and reversible Hamiltonian dynamical systems. We discuss the existence of periodic solutions in the immediate vicinity of such equilibria. It is well known [1] that generic (partially) elliptic equilibria in Hamiltonian vector fields are contained in two-dimensional manifolds containing one-parameter families of periodic solutions. Such families of periodic solutions are called *Liapunov Center families*. Devaney [7] was the first to prove a Liapunov Center theorem for (purely) reversible vector fields, see also Vanderbauwhede [19] and Sevryuk [18].

In this paper we present a Liapunov Center theorem for purely reversible Hamiltonian vector fields, that are vector fields which are Hamiltonian and reversible at the same time, describing all periodic solutions in the neighbourhood of typical (partially) elliptic equilibria. Our only assumption is that the reversing symmetry respects the symplectic structure in the sense that it is symplectic or anti-symplectic.

It turns out that the cases with symplectic and anti-symplectic reversing symmetry are essentially different. If the reversing symmetry acts anti-symplectically, generically there is a single isolated pair of purely imaginary eigenvalues, in which case one recovers the main conclusion of the Liapunov center theorem in the purely reversible and Hamiltonian contexts: the equilibrium is contained in a two-dimensional local smooth invariant manifold containing a one-parameter family of symmetric periodic solutions encircling the equilibrium point.

However, if the reversing symmetry acts symplectically there is typically a doubly degenerate pair of purely imaginary eigenvalues, and the structure of the set of periodic solutions is more complicated. The persistent double degeneracy of purely imaginary eigenvalues in this setting was (to our knowledge) first observed by Buzzi and Teixeira [5]. In fact, the double degeneracy is of the type of "semisimple $1 : -1$ resonance", which in the present setting thus has codimension zero. In the absence of symmetry, the codimension-two Hamiltonian semisimple $1 : -1$ resonance was studied in [6], and the codimension-one non-semisimple Hamiltonian $1 : -1$ resonance in [20]. For related studies of codimension-one reversible equivariant and reversible equivariant Hamiltonian Hopf bifurcation, see [3, 4].

The overall aim is to extend our results to the reversible equivariant Hamiltonian category in the future, extending previous results in the reversible equivariant and reversible equivariant Hamiltonian categories obtained by Golubitsky, Krupa and Lim [8] and Montaldi, Roberts and Stewart [15, 16], respectively. Although these works work with general types of equivariance, they assume certain conditions on the representation of the reversing symmetry. Also, ideally, the aim is to develop a full description of all robust families of periodic solutions near generic (partially) elliptic equilibria for reversible-equivariant and reversible-equivariant Hamiltonian vector fields. In [8, 15, 16], the existence of many families of periodic solutions is established, but the existence of additional periodic solutions is not ruled out.

Similar techniques may be used to study periodic solutions in the neighbourhood of degenerate (codimension one) elliptic equilibria, so-called reversible and/or Hamiltonian Hopf bifurcation [9]. Recently, in [3] we studied such Hopf bifurcation in reversible equivariant systems, where apart from *splitting* and *passing* eigenvalue scenarios, and alternative *crossing* scenario exists [14]. We intend to report on the corresponding bifurcations in Hamiltonian systems in the near future.

This paper is organized as follows. In Section 2, we introduce our setting of reversible Hamiltonian systems. In Section 3, we use results from linear normal form theory to study linear reversible Hamiltonian vector fields and establish the two different types of centers, depending on whether the reversing symmetry acts symplectically or anti-symplectically. In Section 4 we discuss the main tool by which we obtain our results: viewing the ODE as a functional differential operator on a loop space of periodic functions and then obtaining a finite-dimensional bifurcation problem by Liapunov-Schmidt reduction. We discuss the Liapunov Center theorems in the cases of anti-symplectic and symplectic reversing symmetries in Sections 5 and 6. Our main results are stated in Theorem 5.1 and Theorem 6.1.

2. **Reversible Hamiltonian systems.** In this section we define our setting of reversible Hamiltonian systems.

We consider a vector field $f : \mathbb{R}^N \to \mathbb{R}^N$, generating a dynamical systems, i.e. the flow of

$$\frac{d}{dt}x = f(x). \tag{1}$$

Let R be a diffeomorphism of $\mathbb{R}^N$ to itself. Then f is called *R-reversible* if $R_* f = -f$, so that (1) has a solution $x(t)$ if and only if $Rx(-t)$ is too. Analogously, f is called S-equivariant if $S_* f = f$ in which case any solution $x(t)$ implies the existence of the solution $Sx(t)$. If the vector field is assumed not to have any nontrivial equivariance, it follows that $R^2 = I$ and we say that f is *purely* reversible, to contrast the case in which the vector field is reversible-equivariant.

In order to discuss Hamiltonian vector fields, we focus on the case that the phase space has even dimension $N = 2n$. Let ω be a symplectic form, i.e. a nondegenerate skew symmetric bilinear form on $\mathbb{R}^{2n}$. Let $< \cdot, \cdot >$ denote the standard inner product on $\mathbb{R}^{2n}$. Then there is a (linear) *structure map* J satisfying $J^* = -J$ (with * denoting transpose) and $J^2 = -I$ such that $\omega(x, y) =< x, Jy >$ for all $x, y \in \mathbb{R}^{2n}$. f is a Hamiltonian vector field if its flow is *symplectic*, i.e. preserves the symplectic form ω. This coincides with the condition that $f = J\nabla H$, where $H : \mathbb{R}^{2n} \to \mathbb{R}$ is the so-called *Hamiltonian*.

By the Darboux Theorem [1], we may assume without loss of generality that J assumes the standard form

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{2}$$

On the other hand, by Bochner's Theorem [2] in the case of reversible vector fields one may assume without loss of generality that R acts linearly. It remains the question whether R and J can be simultaneously brought into these forms. In general the answer to this question is negative.

However, when we assume that the reversing symmetry R acts (anti-)symplectically, it follows that one can in fact linearize R using a symplectic coordinate transformation. We recall that R is symplectic if $\omega(Rx, Ry) = \omega(x, y)$ for all $x, y \in \mathbb{R}^{2n}$ and anti-symplectic if $\omega(Rx, Ry) = -\omega(x, y)$ for all $x, y \in \mathbb{R}^{2n}$. In other words, R is (anti-)symplectic if $RJ = \pm JR^*$. Hence, whenever R is (anti-)symplectic it preserves the absolute value of the symplectic form, which can be viewed as some kind of *symplectic volume*.

Although examples of Hamiltonian vector fields with non-(anti-)symplectic symmetries and reversing symmetries can be constructed, from a theoretical point of view - in the context of dynamical systems with symplectic structure - it appears natural to focus on (reversing) symmetries that act (anti-)symplectically. In fact, in the literature - in particular in mechanics - it is often assumed that reversing symmetries act anti-symplectically, which coincides with the property that the Hamilton (energy) function H is invariant under all symmetry transformations. However, in general, it may happen that some symmetries act anti-symplectically and some reversing symmetries symplectically, in which case such (reversing) symmetries only fix the absolute value of H, but change its sign (see for instance [11] for an example where symplectic time-reversal symmetries arise).

Thus by assumption of the (anti-)symplectic action of R, R and J can be chosen to be simultaneously linear. Moreover, by [12][appendix B], R and J can be chosen to be simultaneously orthogonal (with respect to the natural inner product).

3. **Linear reversible Hamiltonian vector fields with purely imaginary eigenvalues.** We now consider in more detail the persistent occurrence of purely imaginary eigenvalues in reversible Hamiltonian vector fields.

Let $L \in sp_J(2n, \mathbb{R})$ represent a linear Hamiltonian vector field so that

$$LJ = -JL^*.$$

Using the Darboux Theorem one may assume without loss of generality that J is in the standard form (2). By the discussion in the previous section, the reversing symmetry R can at the same time be chosen to be linear and orthogonal. R-reversibility of L then implies that

$$LR = -RL.$$

We now use the important assumption that R is (anti-)symplectic

$$RJ = \pm JR.$$

The linear normal form theory for reversible Hamiltonian systems can be studied along the lines set out by Hoveijn, Lamb and Roberts [12], as the structure is characterized by two commuting Lie-algebra (anti-) automorphisms

$$\psi_J(L) = J^{-1}L^*J \text{ and } \phi_R(L) = R^{-1}LR.$$

In fact, $L \in gl(2n, \mathbb{R})$ is precisely reversible Hamiltonian if it is in the -1 eigenspaces of both ψ_J and ϕ_R. The action of $\mathbb{Z}_2(\psi_J)\times\mathbb{Z}_2(\phi_R)$ on $gl(2n, \mathbb{R})$ induces the following isotypic decomposition of $gl(2n, \mathbb{R})$:

$$gl(2n, \mathbb{R}) = gl_{J+,R+}(2n, \mathbb{R}) \oplus gl_{J+,R-}(2n, \mathbb{R}) \oplus gl_{J-,R+}(2n, \mathbb{R}) \oplus gl_{J-,R-}(2n, \mathbb{R}),$$

where the righthand components are exactly the combined ± 1 eigenspaces of ψ_J and ϕ_R. The component corresponding to linear reversible Hamiltonian vector fields is thus $gl_{J-,R-}(2n, R)$.

Hoveijn *et al.* [12] treat the linear normal form theory for such components. Normal forms are based on the construction of minimal $< J, R >$-invariant subspaces. By [12], as we are only interested in codimension zero cases, we can restrict our attention to such minimal invariant subspaces on which L is semi-simple (i.e. is diagonalizable over $\mathbb{C}$).

As we are interested in (partially) elliptic equilibria, we furthermore assume that L has at least one pair of purely imaginary eigenvalues $\lambda, -\lambda$ with corresponding (real) invariant subspace spanned by $e_1, \bar{e_1}$. Note that if e_1 has eigenvalue λ then $\bar{e_1}$ has eigenvalue $-\lambda = \overline{\lambda}$.

Now we consider the situation that L is not only Hamiltonian, but also (purely) R-reversible. This implies that if e is an eigenvector for eigenvalue λ, then Re is an eigenvector for eigenvalue $-\lambda$. The type of minimal invariant subspace now depends on whether R acts symplectically or anti-symplectically.

Lemma 3.1. *Consider a linear (purely) R-reversible Hamiltonian vector field L, with R acting (anti-)symplectically. Let V be a minimal (L, J, R)-invariant subspace on which L has purely imaginary eigenvalues. Then, $L|_V$, $J|_V$ and $R|_V$ can be brought in the following normal forms:*

- *R acting anti-symplectically:*
  $\dim V = 2$ *and*

$$R|_V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; J|_V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; L|_V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- <u>*R acting symplectically:*</u>
  $\dim V = 4$ *and*

$$R|_V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$J|_V = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$L|_V = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The above situations arise generically in (purely) reversible Hamiltonian vector fields when R acts anti-symplectically or symplectically, respectively.

Proof. Let W be a 2-dimensional symplectic subspace so that L has purely imaginary eigenvalues. Then, by standard Hamiltonian theory, L and J can be normalized to take the same form on W (taking into account multiplication of time by a scalar). If the reversing symmetry R acts anti-symplectically we have $RL = -LR$ and $RJ = -JR$, and R maps eigenvectors of L and J with eigenvalue λ into eigenvectors with eigenvalue $-\lambda$. Because L and J commute on W, they share their eigenvectors. Let e_1 and $\overline{e_1}$ be the (complexified) eigenvectors of L, then - taking into account the anti-commutation between R and L, one obtains a minimal invariant subspace by choosing $Re_1 = \overline{e_1}$. As $J = L$ on W, the anti-commutation relation with J is automatically satisfied.

If R acts symplectically the dimension of the minimal invariant subspace can no longer be taken to be equal to two. Namely, starting with a 2-dimensional W as above, it follows that if R leaves W invariant and anti-commutes with L it also must anti-commute with J. Hence, $R(W) = W' \neq W$ and a minimal invariant subspace is given by $V = W \oplus W'$, so that $\dim V = 4$. Because of the commutation relations, when we now choose $J|_W = L|_W$ and have J acting identically on W and W', then $J_{W'} = -L_{W'}$.

The generic (codimension zero) occurrence of invariant subspaces with purely imaginary eigenvalues of the above kind, can be easily verified by calculation of the centralizer unfolding, the versality of which has been proven in [12]. Such unfolding of L moves the eigenvalues along the imaginary axis. □

In this paper we focus on the situation in which L has purely imaginary eigenvalues $\pm i$, either a single or double pair depending on whether R acts anti-symplectically or symplectically. Furthermore, we assume that these imaginary eigenvalues of L are nonresonant in the sense that L has no other eigenvalues of the form $\pm ki$ with $k \in \mathbb{Z}$. This condition is clearly generic (codimension zero). The main aim is to describe the (families of) periodic solutions in the neighbourhood of the equilibrium point. The main technique we employ is a Liapunov-Schmidt reduction, which we discuss in the next section.

4. **Liapunov-Schmidt reduction.** In this section we discuss the Liapunov-Schmidt reduction procedure for periodic solution near equilibria for reversible equivariant Hamiltonian systems. We assume our vector field $f : \mathbb{R}^N \to \mathbb{R}^N$ has a reversing symmetry group G, implying the existence of representations $\rho : G \to O(N)$ and $\sigma : G \to \{\pm 1\}$ (reversible sign) such that

$$f\rho(g) = \sigma(g)\rho(g)f, \ \forall\ g \in G.$$

We closely follow the exposition of Liapunov-Schmidt reduction in [10, 9].

Define $\Phi : \mathcal{C}^1_{2\pi} \times \mathbb{R} \to \mathcal{C}_{2\pi}$ by

$$\Phi(u, \tau) = (1 + \tau)\frac{du}{dt} - f(u) \tag{3}$$

where $\mathcal{C}_{2\pi}$ is the Banach Space of $\mathbb{R}^N$-valued, 2π-periodic mappings and $\mathcal{C}^1_{2\pi}$ consists of those $u(t) \in \mathcal{C}_{2\pi}$ that are continuously differentiable. Observe that solutions to $\Phi(u, \tau) = 0$ corresponds to $\frac{2\pi}{1+\tau}$-periodic solutions of (1).

We can define an action $T : \tilde{G} \times \mathcal{C}_{2\pi} \to \mathcal{C}_{2\pi}$, or in $\mathcal{C}^1_{2\pi}$, by

$$(T_g u)(t) = \rho(\gamma)(u(\sigma(\gamma)t + \theta)) \tag{4}$$

where $g = \gamma\theta$ is an element of

$$\tilde{G} = G \ltimes S^1.$$

From the G-reversible equivariance of f, it is readily verified that Φ is $\tilde{G}$-reversible equivariant:

$$\Phi(T_g u, \tau) = \sigma(\gamma)T_g\Phi(u, \tau)\ \forall g = \gamma\theta \in \tilde{G} = G \ltimes S^1 \tag{5}$$

In the first step of the Liapunov-Schmidt reduction we consider the linear part of Φ

$$\mathcal{L} = (d\Phi)_{0,0},$$

where

$$\mathcal{L}u = \frac{du}{dt} - Lu,$$

and $L = (df)_0$. The elements of Ker $\mathcal{L}$ correspond to solutions of the linear system $\frac{du}{dt} = Lu$ satisfying $u(t) = u(t + 2\pi)$. Hence, Ker $\mathcal{L}$ is the union of the (generalized) eigenspaces of L with eigenvalues of the form ki, with $k \in \mathbb{Z}$.

Naturally, $\mathcal{L}$ is also $\tilde{G}$-reversible equivariant

$$\mathcal{L}T_g = \sigma(\gamma)T_g\mathcal{L},$$

and it follows that T_g preserves Ker $\mathcal{L}$ and Range $\mathcal{L}$.

The second step in the Liapunov-Schmidt reduction is to consider the splittings

$$\mathcal{C}^1_{2\pi} = \text{Ker } \mathcal{L} \oplus (\text{Ker } \mathcal{L})^\perp \text{ and } \mathcal{C}_{2\pi} = (\text{Range } \mathcal{L})^\perp \oplus \text{Range } \mathcal{L}. \tag{6}$$

Here, we choose the orthogonal complement to be taken with respect to the natural inner product in $\mathcal{C}_{2\pi}$ and $\mathcal{C}^1_{2\pi}$

$$[u, v] = \int_{\tilde{G}} < T_g u, T_g v > d\mu \tag{7}$$

where $< u, v >= \int_0^{2\pi}[u(t)]^t v(t)dt$ and μ is a normalized Haar measure for $\tilde{G}$. It is readily verified that $[T_g u, T_g v] = [u, v]$ for all $g \in \tilde{G}$ so that the complements (Ker $\mathcal{L})^\perp$ and (Range $\mathcal{L})^\perp$, and hence the splittings (6), are T_g-invariant.

Using the fact that (Range $\mathcal{L}$) and (Range $\mathcal{L})^{\perp}$ are T_g-invariant, it follows that the projections

$$\begin{array}{c} E : \mathcal{C}_{2\pi} \to \text{Range } \mathcal{L} \\ (I - E) : \mathcal{C}_{2\pi} \to (\text{Range } \mathcal{L})^{\perp} \end{array} \tag{8}$$

commute with T_g. Namely, if $u \in \mathcal{C}_{2\pi}$, $u = v + w$, $v \in$ (Range $\mathcal{L})^{\perp}$ and $w \in$ Range $\mathcal{L}$. In this case, $T_g(E(u)) = T_g w = E(T_g w) = E(T_g v + T_g w) = E(T_g u)$ and $T_g[(I - E)(u)] = T_g v = (I - E)(T_g v) = (I - E)(T_g v + T_g w) = (I - E)(T_g u)$.

The third step in the Liapunov-Schmidt reduction procedure is to solve the equation

$$E\Phi(v + w, \tau) = 0 \tag{9}$$

by using the implicit function theorem to find a function W : Ker $\mathcal{L} \times \mathbb{R} \to$ (Ker $\mathcal{L})^{\perp}$ such that $E\Phi(v + W(v, \tau), \tau) = 0$ for all $v \in$ Ker $\mathcal{L}$.

It is easy to verify that W commutes with T_g. Namely, define W_g : Ker $\mathcal{L} \times \mathbb{R} \to$ (Ker $\mathcal{L})^{\perp}$ by $W_g(v, \tau) = T_{g^{-1}} W(T_g v, \tau)$. Observe that $E\Phi(v + W_g(v, \tau), \tau) = E\Phi(T_{g^{-1}}(T_g v + W(T_g v, \tau), \tau)) = \sigma(\gamma) T_{g^{-1}} E\Phi(T_g v + W(T_g v, \tau), \tau) = 0$ since $T_g v \in$ Ker $\mathcal{L}$. Then W_g also solves the equation (9) and $W_g(0, 0) = W(0, 0) = 0$, so we have by uniqueness $W_g(v, \tau) = W(v, \tau)$, which implies that W commutes with T_g.

The above Liapunov-Schmidt reduction thus reduces our problem to the problem of finding zeros of the map φ : Ker $\mathcal{L} \times \mathbb{R} \to$ (Range $\mathcal{L})^{\perp}$ that is defined by

$$\varphi(u, \tau) = (I - E)\Phi(v + W(v, \tau), \tau). \tag{10}$$

We refer to φ as the *bifurcation map*. Importantly, the bifurcation map is $\tilde{G}$-reversible equivariant:

Proposition 4.1. *If f is G reversible equivariant then the bifurcation map φ is also $\tilde{G}$ reversible equivariant:*

$$\varphi(T_g u, \tau) = \sigma(\gamma) T_g \varphi(u, \tau) \ \forall g \in G \ltimes S^1 \tag{11}$$

Proof. This is a direct consequence of the $\tilde{G}$-equivariance of W and the projection $(I - E)$. □

Having seen above how symmetry enters the bifurcation equation φ, we now consider the consequences of the Hamiltonian structure, following the treatment of constrained Liapunov-Schmidt reduction in [9].

Since the vector field f is Hamiltonian, i.e. $f = X_H$, the identity

$$\omega(X_H(u), v) = < dH(u), v >$$

holds for all $v \in \mathbb{R}^{2n}$. One may define the map

$$F : \mathcal{C}^1_{2\pi} \times \mathbb{R} \times \mathcal{C}^0_{2\pi} \to [\mathcal{C}^1_{2\pi}]^*$$

by

$$F(u, \tau, v).U = \int_0^{2\pi} \left\{ \omega \left(v - (1 + \tau) \frac{du}{ds}, U \right) + dH(u).U \right\} ds.$$

Then, the implicit constraint

$$F(u, \tau, \Phi(u, \tau)) = 0$$

is a restatement of the condition that the vector field f is Hamiltonian with Hamiltonian function H. Alternatively, Φ may be regarded as a (parameter-dependent) vector field on $\mathcal{C}^1_{2\pi}$ that is Hamiltonian with respect to the weak symplectic form

$$\Omega(u, v) = \frac{1}{2\pi} \int_0^{2\pi} \omega(u(s), v(s)) ds,$$

and with the Hamiltonian function

$$\mathcal{H}(u,\tau) = \frac{1}{2\pi}\int_0^{2\pi} \left\{ \frac{1}{2}\omega\left((1+\tau)\frac{du}{ds}, u\right) - H(u) \right\} ds. \tag{12}$$

In the Hamiltonian context, we now assume that the actions of G are (anti-)symplectic. Hence we introduce a *symplectic sign* $\chi :\to \{\pm 1\}$, so that

$$\omega(gx, gy) = \chi(g)\omega(x,y)$$

Consequently, Ω is $\tilde{G} = G \ltimes S^1$ (anti-)invariant,

$$\Omega(\gamma u, \gamma v) = \chi(g)\Omega(u,v),\ \ \gamma = g\theta,\ \ \theta \in S^1,$$

and the Hamiltonian $\mathcal{H}$ satisfies

$$\mathcal{H}(\gamma u, \gamma v) = \sigma(g)\chi(g)\mathcal{H}(u,v). \tag{13}$$

By [9, Theorem 6.2], it follows that as long as the kernel of $\mathcal{L}$ is symplectic and invariant under the complex structure J, i.e.

$$\mathrm{Ker}\ \mathcal{L} = \mathrm{Ker}\ \mathcal{L}^*, \tag{14}$$

then the bifurcation equation is a Hamiltonian vector field.

To apply the above theory, one thus needs to work out the linearized equations and calculate the induced symplectic structure on the kernel of the linearization. Thereafter the induced Hamiltonian can be derived and the resulting bifurcation equation studied.

In the situations we consider in this paper, Ker $\mathcal{L}$ is finite dimensional and (14) is satisfied. In that case, the bifurcation equation is thus a Hamiltonian vector field. The corresponding symplectic form is the restriction of Ω to Ker $\mathcal{L}$ and the corresponding Hamiltonian satisfies the same (anti-)invariance relations (13), of course with respect to the actions of $\tilde{G}$ restricted to Ker $\mathcal{L}$. Also, in this paper we always have $G = \mathbb{Z}_2$ and $\tilde{G} = \mathbb{Z}_2 \ltimes S^1 \cong O(2)$.

5. **Anti-symplectic time-reversal symmetry.**

We first discuss the case that the reversing symmetry R is anti-symplectic. Anti-symplectic reversing symmetries are well known to arise in classical mechanics, corresponding to the "natural" time-reversal transformation $(x,p,t) \to (x,-p,-t)$.

From the linear analysis in Section 3 above, we know that purely imaginary eigenvalue pairs typically arise isolated. We are interested in the existence of periodic solutions close to an equilibrium point with such an isolated pair of purely imaginary eigenvalues. We are also interested in the symmetry properties of such periodic solutions. Recall that an orbit is called "symmetric" if it is (setwise) R-invariant.

Theorem 5.1. *Consider a symmetric equilibrium* 0 *of a purely reversible Hamiltonian vector field f, with the reversing symmetry acting anti-symplectically. Suppose that $Df(0)$ has one purely imaginary pair of eigenvalues $\pm i$, and that the spectrum contains no other eigenvalues of the form $\pm ki$, $k \in \mathbb{Z}$. Then, typically, the equilibrium is contained in a smooth two-dimensional flow-invariant manifold consisting of a one-parameter family of symmetric periodic solutions whose period tends to 2π as they approach the equilibrium. Furthermore, there are no other periodic solutions with period close to 2π in the neighbourhood of* 0.

It is not surprising that the Liapunov Center Theorem takes the above form. In this case there are many ways in which the existence of a Liapunov Center family of periodic solutions can be proven, using the fact that the flow has an integral [9], using the fact that the flow is reversible [19, 18, 9], and using the fact that the flow is Hamiltonian [1, 9]. Moreover, the linear analysis shows that in case R acts anti-symplectically, one would not expect more periodic solutions than a one-parameter family, in which case by uniqueness it also follows that all orbits in this family then should be symmetric.

For completeness, we give a proof using the Liapunov-Schmidt strategy set out above, following closely the treatment in [9].

Proof (of Theorem 5.1) The linearization of Φ at the equilibrium 0 is

$$\mathcal{L} : \mathcal{C}^1_{2\pi} \to \mathcal{C}_{2\pi}, \ \mathcal{L}v = \frac{dv}{ds} - Lv,$$

where $L = Df(0)$. Since L is Fredholm of index zero, we may use the orthogonal splittings

$$\mathcal{C}^1_{2\pi} = \text{Ker } \mathcal{L} \oplus \text{Range } \mathcal{L}^*, \ \mathcal{C}_{2\pi} = \text{Range } \mathcal{L} \oplus \text{Ker } \mathcal{L}^*.$$

As L has a single pair of eigenvalues $\pm i$, $\dim \text{Ker } \mathcal{L} = 2$, with Ker $\mathcal{L} \subset \mathcal{C}_{2\pi}(V)$ being generated by

$$v_1 = \begin{pmatrix} \sin(s) \\ -\cos(s) \end{pmatrix}, \ v_2 = \begin{pmatrix} \cos(s) \\ \sin(s) \end{pmatrix},$$

where $V \cong \mathbb{R}^2$ is the eigenspace for $\pm i$ of L, assuming without loss of generality that $L|_V$ takes the (normal) form of Lemma 3.1.

Consequently, the symplectic form Ω for the reduced bifurcation equation satisfies

$$\Omega(v_n, v_m) = \int_0^{2\pi} < v_n, Jv_m > ds = J_{n,m}$$

where J is the symplectic structure map of Lemma 3.1, and $J_{n,m}$ with $n, m \in \{1, 2\}$ denote its components as usual. Hence with basis $\{v_1, v_2\}$ for Ker $\mathcal{L} \cong \mathbb{R}^2$ we obtain

$$\Omega(x, y) =< x, Jy >,$$

for all $x, y \in$ Ker $\mathcal{L}$, so that our reduced bifurcation equation has the same symplectic form as the original Hamiltonian vector field.

By similar calculation, the reduced Hamiltonian can be computed from $\mathcal{H}$ to take the form

$$h(x, y, \tau) = \frac{\tau}{2}(x^2 + y^2) + (x^2 + y^2)\tilde{h}(x^2 + y^2, \tau),$$

with $\tilde{h}$ smooth and $\tilde{h}(0, 0) = 0$.

We note that the eigenvalue conditions and Liapunov-Schmidt reduction coordinate τ arise prominently in the lowest order (quadratic) part of the reduced Hamiltonian. The higher order terms of h can be characterized by its $\tilde{G}$ (anti-)invariance properties that are induced from $\mathcal{H}$. Here,

$$\tilde{G} = \mathbb{Z}_2(R) \ltimes S^1 \cong O(2),$$

acts on Ker $\mathcal{L}$ as the matrix in Lemma 3.1 and $\theta \in S^1$ acts as the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Because of the S^1-invariance of h we have $h(x, y, \tau) = \hat{h}(x^2 + y^2, \tau)$. But then h is automatically also R-invariant, with $R(x, y) = (x, -y)$. (Hence the presence of reversibility has few consequences.)

Writing $x^2 + y^2 = r^2$, we may write

$$h(r^2, \tau) = r^2(\frac{\tau}{2} + \tilde{h}(r^2, \tau)).$$

The equilibria for the corresponding vector field are given by:

$$r = 0 \text{ (trivial equilibrium) and } \frac{\tau}{2} + \tilde{h}(r^2, \tau) + r^2\tilde{h}'(r^2, \tau) = 0.$$

By application of the implicit function theorem, there is a function $\tau(r^2)$ so that for all r^2 sufficiently small

$$\frac{\tau(r^2)}{2} + \tilde{h}(r^2, \tau(r^2)) + r^2\tilde{h}'(r^2, \tau(r^2)) = 0.$$

Thus we obtain S^1-orbits of solutions, i.e. circles if $r^2 > 0$, each circle representing a periodic orbit. They are symmetric because they are setwise R-invariant (and hence intersect Fix$R$ twice). Alternatively, it is readily verified that every point in Ker $\mathcal{L}$ fixed by θR for some $\theta \in S^1$, so that all solutions must be symmetric.

Finally, we note that $\tau \to 0$ as $r^2 \to 0$. By the definition of τ in Section 4, the corresponding family of periodic orbits thus have their period tending to 2π as $r \to 0$. □

6. **Symplectic time-reversal symmetry.** We now consider the case that the time-reversal symmetry acts symplectically. In this case, (partially) elliptic equilibria typically have all their purely imaginary eigenvalues doubly degenerate. The structure of nearby periodic solutions is also more complicated.

Theorem 6.1. *Consider a symmetric equilibrium* 0 *of a purely reversible Hamiltonian vector field f, with the reversing symmetry acting symplectically. Suppose that $Df(0)$ has two purely imaginary pairs of eigenvalues $\pm i$, and that the spectrum contains no other eigenvalues of the form $\pm ki$, $k \in \mathbb{Z}$. Then, typically, the equilibrium is contained in a three-dimensional flow-invariant surface consisting of a two-parameter family of symmetric periodic solutions whose period tends to 2π as they approach the equilibrium. Moreover, typically, the equilibrium is contained in two smooth two-dimensional flow-invariant manifolds each containing a one-parameter family of non-symmetric periodic solutions whose period tends to 2π as they approach the equilibrium. Furthermore, there are no other periodic solutions with period close to 2π in the neighbourhood of* 0.

The remainder of this section is devoted to the proof of this theorem, which consists of Lemma 6.2 and Lemma 6.4. Moreover, more detail on the structure of two-parameter family of symmetric periodic solutions is described in Lemma 6.3.

We assume without loss of generality that the actions of R, J and L on the four-dimensional eigenspace V of $\pm i$ for L are as in Lemma 3.1.

We proceed to apply the Liapunov-Schmidt reduction of Section 4. First of all, we find that $\dim \text{Ker } \mathcal{L} = 4$, and we can find a basis for Ker $\mathcal{L}$ such the symplectic structure map J and reversing symmetry R act in the same way as they act on V in Lemma 3.1. Clearly, again Ker $\mathcal{L}$ satisfies the conditions of [9, Theorem 6.2], and consequently the reduced bifurcation equation φ is a Hamiltonian vector field.

To facilitate the notation, we furthermore identify Ker $\mathcal{L} \cong \mathbb{R}^4$ with $\mathbb{C}^2$ in a natural way

$$\mathbb{R}^4 \ni (x_1, y_1, x_2, y_2) \simeq (x_1 + iy_1, x_2 + iy_2) = (z_1, z_2) \in \mathbb{C}^2,$$

so that our bifurcation map takes the form

$$\varphi : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2,$$

φ is Hamiltonian

$$\varphi = 2J\nabla_{\bar{z}} h \tag{15}$$

with Hamilton function

$$h : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{R}.$$

Moreover, h is S^1-invariant, $h \circ \theta = h$ with $\theta \in S^1$ acting on $\mathbb{C}^2$ as

$$\theta(z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_1),$$

and h is R-anti-invariant, $h \circ R = -h$, with

$$R(z_1, z_2) = (z_2, z_1).$$

Finally, J acts on $\mathbb{C}^2$ simply as multiplication by i.

By S^1-invariance we have

$$h = h(|z_1|^2, |z_2|^2, z_1 z_2, \overline{z_1 z_2}, \tau).$$

Using moreover the R-anti-invariance and reality of h, we have

$$\begin{aligned} h &= (|z_1|^2 - |z_2|^2) g(|z_1|^2 + |z_2|^2, |z_1 z_2|^2, z_1 z_2, \overline{z_1 z_2}, \tau), \\ &= (|z_1|^2 - |z_2|^2) \left[\, g^1(|z_1|^2 + |z_2|^2, |z_1 z_2|^2, \tau) \right. \\ &\quad \left. + 2\mathrm{Re}[z_1 z_2 g^2(|z_1|^2 + |z_2|^2, |z_1 z_2|^2, z_1 z_2, \tau)] \right], \end{aligned} \tag{16}$$

where g is decomposed as $g = g^1 + 2Re(z_1 z_2 g^2)$ so as to remove the ambiguity of coefficients for the Taylor expansion of g.

Finally, from (12) the τ-dependence of the lowest (quadratic) order contribution to h can be calculated to yield

$$h = (|z_1|^2 - |z_2|^2)\frac{\tau}{2} + hot,$$

or equivalently that $g^1(0, 0, \tau) = \frac{\tau}{2}$.

6.1. **Symmetric periodic solutions.** As $h \circ R = -h$ it follows that all symmetric (R-invariant) periodic solutions must be presented by (S^1-orbits of) zeroes of the bifurcation map φ that lie in the level set $h = 0$. In fact, it turns out that all points in This level set are zeroes of the bifurcation map, corresponding to symmetric periodic solutions:

Lemma 6.2. *In the neighbourhood of the equilibrium* $(z_1, z_2, \tau) = (0, 0, 0)$*, all points in the level set* $h = 0$ *are solutions of the bifurcation map* φ*.*

Proof. The bifurcation map (15), evaluated in $\mathrm{Fix}(R) = \{(z, z) \in \mathbb{C}^2\}$, yields

$$z = 0, \text{ or } g(z, \tau) = 0.$$

The orbit of (z, z) is an equilibrium if and only if $z = 0$. By the form of g we know that $\partial g/\partial\tau(0, 0) = \frac{1}{2} \neq 0$. Subsequent application of the Implicit Function Theorem yields that for each nonzero small z there exists a τ such that (z, z) lies on a periodic solution with period $2\pi/(\tau + 1)$. $\square$

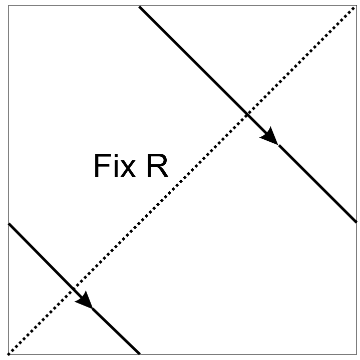

FIGURE 1. Sketch of typical orbit of the linear part of the vector field on the two-tori characterized by $r_1 = r_2 = r$, and parameterized by ϕ_1, ϕ_2. Each orbit intersects $\mathrm{Fix}R = \{\phi_1 = \phi_2\}$ exactly twice.

For a more geometrical approach to the proof of Lemma 6.2, we consider the simplest situation, that is a two-degree of freedom Hamiltonian system with symplectic reversing symmetry.

In that case, in complex coordinates, the Hamiltonian has the form

$$H = (|z_1|^2 - |z_2|^2)g(|z_1|^2 + |z_2|^2, |z_1 z_2|^2, z_1 z_2, \overline{z_1 z_2}),$$

with $g(0,0,0,0) = 1$. Clearly, the origin lies in $\mathrm{Fix}R = \{z_1 = z_2\}$ and in the level set $H = 0$. As $g(0,0,0,0) \neq 0$, locally near the origin, this level set is given by $|z_1|^2 - |z_2|^2 = 0$, or in polar coordinates $r_1 = r_2$. Considering a small sphere around the origin, the level set $H = 0$ is hence a three-dimensional surface.

By restricting this surface to some nonzero r, one obtains a two-torus, parametrized by the two angles ϕ_1 and ϕ_2. The surface thus looks like a one-parameter (r) family of two-tori, that is pinched at the origin, yielding a singular point at the origin with conical structure.

Now we consider $\mathrm{Fix}R = \{z_1 = z_2\}$. $\mathrm{Fix}R$ intersects each of the above two-tori in exactly one line $\phi_1 = \phi_2$. The flow induced on the two-tori by the linear part of the vector field

$$\dot{\phi}_1 = 1, \ \dot{\phi}_2 = -1,$$

traces orbits on the torus, intersecting $\mathrm{Fix}R$ exactly twice, see the illustration in Fig. 1.

Taking r again into account, we observe that all orbits of the linear vector field in the level set $H = 0$ intersect $\mathrm{Fix}R$ transversally. We note that $\mathrm{Fix}R$ is a two-dimensional submanifold of the three-dimensional level set $H = 0$, and thus has codimension one. Hence, it is not at all surprising that when taking higher order terms of the nonlinear vector field into account, all orbits continue to intersect $\mathrm{Fix}R$ transversally in two points, and hence are symmetric periodic orbits. The latter follows from the well-known fact that an orbit of a reversible flow is a symmetric periodic orbit if and only if it intersects $\mathrm{Fix}R$ (exactly) twice, see [13].

There is one remaining feature that we would like to highlight. Namely, examination of the period distribution within the set of periodic solutions reveals an

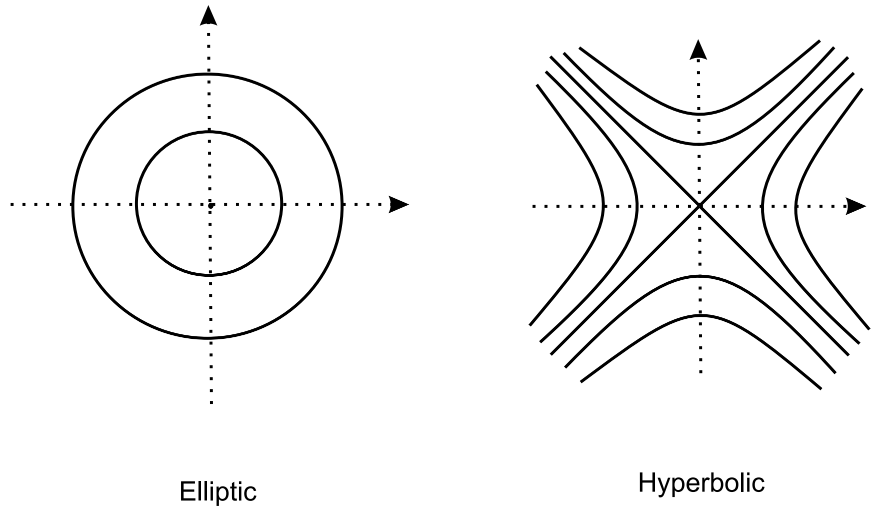

FIGURE 2. Structure of period-distribution for symmetric periodic solutions: sketch of the level sets of $\tau = \varepsilon_1 x^2 + \varepsilon_2 y^2$.

interesting feature. Let $z = x + iy$, then the level sets of $\tau(x, y)$ can be typically shown to be one of two types: in suitable coordinates $(\tilde{x}, \tilde{y})$, near $(0, 0)$ τ is equivalent to $\tau = \varepsilon_1 \tilde{x}^2 + \varepsilon_2 \tilde{y}^2$ with $\varepsilon_j = \pm 1$, where the signs depend on details of h (and H). If the singularity is elliptic ($\varepsilon_1 \varepsilon_2 = 1$) the level sets form circles and τ typically increases or decreases monotonically with increasing radius. If it is hyperbolic ($\varepsilon_1 \varepsilon_2 = -1$), the level sets form two families of hyperbolae, one family with positive increasing τ and one with negative decreasing τ. The elliptic and hyperbolic situations are sketched in Fig. 2. Recall that the periodic solutions are in a one-to-two relationship with the zeroes of τ, as the points (x, y) and $(-x, -y)$ refer to two points on the same orbit, half-a-period apart.

It remains to give an interpretation of this feature. In the elliptic case, the equilibrium is surrounded by periodic solutions with period close but not equal to 2π, and the periods converge to 2π, from one side, as they approach the equilibrium. In the hyperbolic case, however, there are two smooth local two-dimensional invariant manifolds, each containing a one-parameter family of symmetric periodic solutions with period exactly equal to 2π. In this case, the remainder of the symmetric periodic solutions have period slightly higher or lower: on one side the period is higher ($\tau < 0$) and on the other lower ($\tau > 0$), cf. Fig. 2. Recall in this respect that the points (x, y) and $(-x, -y)$ correspond to the same periodic solution.

We summarize this conclusion in the following lemma:

Lemma 6.3. *Depending on quartic terms of the Hamiltonian (and quadratic terms of the function g), among the three-dimensional surface of symmetric periodic solutions near the equilibrium point, typically either there exist two local two-dimensional invariant manifolds containing periodic solutions whose period is exactly equal to 2π (hyperbolic case), or no such periodic solutions exist (elliptic case).*

Proof. We already discussed the main arguments of the proof, above. We here give some details on the computation of $\tau(x, y)$, and explicit conditions on the function g for the singularity to be elliptic or hyperbolic.

From the proof of Lemma 6.2, we see that the profile of $\tau(z)$ is determined by the equation $g(z,\tau)=0$. The lowest (quadratic) order terms of this equation read:

$$
\begin{aligned}
-\frac{\tau}{2} &= 2|z|^2 g_1^1(0,0,0) + 2\mathrm{Re}[z^2 g^2(0,0,0,0)],\\
&= 2g_1^1(0,0,0)(x^2+y^2) + 2\mathrm{Re}[g^2(0,0,0,0)](x^2-y^2)\\
&\quad -4\mathrm{Im}[g^2(0,0,0,0)]xy,
\end{aligned}
\tag{17}
$$

where $z = x+iy$. The shape of $\tau(x,y)$ near 0 is typically determined by the quadratic terms by Morse's Lemma. The nature of the singularity is determined by the determinant

$$D = g_1^1(0,0,0)^2 - (\mathrm{Re}[g^2(0,0,0,0)])^2 - 4(\mathrm{Im}[g^2(0,0,0,0)])^2.$$

It is elliptic if $D>0$ and hyperbolic if $D<0$.

When we write $\tau = \varepsilon_1 x^2 + \varepsilon_2 y^2$, we have $\mathrm{sign}(D) = \varepsilon_1\varepsilon_2$ and, in the elliptic case (where $\mathrm{sign}(\varepsilon_1)$ has a meaining), $\varepsilon_1 = -\mathrm{sign}(g_1^1(0,0,0))$. □

6.2. **Non-symmetric periodic solutions.** It now remains to be determined if there are also non-symmetric periodic solutions close to the equilibrium point.

Lemma 6.4. *In addition to the symmetric Liapunov Center families described in Lemma 6.2, there are typically two nonsymmetric Liapunov Center families of periodic solutions, each filling out a local two-dimensional smooth manifold with the period of the periodic solutions converging to 2π as the solutions tend to the equilibrium point.*

Proof. We divide the proof into the discussion of three cases: $z_1 \neq 0$ and $z_2 \neq 0$, $z_1 = 0$ and $z_2 \neq 0$, and $z_1 \neq 0$ and $z_2 = 0$.

$z_1 \neq 0$ and $z_2 \neq 0$

As before, we write $h = I_1 g$, with $I_1 = (|z_1|^2 - |z_2|^2)$. As J is invertible, we have

$$\varphi = 0 \;\Leftrightarrow\; \nabla_{\overline{z}} h = 0.$$

The latter equality can be rewritten as

$$\begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} g + I_1 \nabla_{\overline{z}} g = 0.$$

After writing out these equations, using (16), and multiplying the first one by $\overline{z_1}$ and the second one by $\overline{z_2}$ (looking for solutions with $z_1 \neq 0$ and $z_2 \neq 0$), we obtain

$$
\left\{
\begin{aligned}
0 &= |z_1|^2\left(g + I_1(g_1^1 + |z_2|^2 g_2^1 + 2\mathrm{Re}[z_1 z_2(g_1^2 + |z_2|^2 g_2^2)]\right) + \overline{z_1 z_2(g^2 + z_1 z_2 g_2^2)},\\
0 &= |z_2|^2\left(g + I_1(g_1^1 + |z_1|^2 g_2^1 + 2\mathrm{Re}[z_1 z_2(g_1^2 + |z_1|^2 g_2^2)]\right) + \overline{z_1 z_2(g^2 + z_1 z_2 g_3^2)},
\end{aligned}
\right.
\tag{18}
$$

where g_i^j the partial derivative of g^j with respect to X_i, where $X_1 = |z_1|^2 + |z_2|^2$, $X_2 = |z_1 z_2|^2$, and $X_3 = z_1 z_2$.

We now note that all but the last identical term of (18) are real. Hence the imaginary part of this last term must be equal to zero:

$$\mathrm{Im}[z_1 z_2(g^2 + z_1 z_2 g_3^2)] = 0. \tag{19}$$

We now write $z_j = r_j e^{i\phi_j}$. We now note that on each S^1-orbit there lies a point (z_1, z_2) with $\phi_1 = \phi_2 = \phi$ for some ϕ. Since periodic orbits are represented by S^1-orbits of solutions of the bifurcation map, without loss of generality we may confine our search for solutions to points $(z_1, z_2) = (r_1 e^{i\phi}, r_2 e^{i\phi})$.

In these coordinates, (19) reads

$$\mathrm{Im}[r_1r_2e^{2i\phi}(g^2 + r_1r_2e^{2i\phi}g_3^2)] = 0.$$

We now perform a "blow-up" by dividing this equation by r_1r_2, and setting $r_1 = r_2 = \tau = 0$, yielding

$$\mathrm{Im}[e^{2i\phi}g^2(0,0,0,0)] = 0.$$

Writing $g^2(0,0,0) = a + ib$, with $a, b \in \mathbb{R}$, we find that

$$a\sin(2\phi) + b\cos(2\phi) = 0. \tag{20}$$

Assuming that $|g^2(0,0,0,0)| \neq 0$, this yields four solutions for ϕ, each $\frac{\pi}{2}$ apart from each other. In order to extend this result to nonzero r_1 and r_2, we would like to invoke the Implicit Function Theorem with respect to ϕ. We have

$$\frac{\partial}{\partial\phi}\mathrm{Im}\left[e^{2i\phi}(g^2 + r_1r_2e^{2i\phi}g_3^2)\right]_{r_1=r_2=\tau=0} = 2a\cos(2\phi) - 2b\sin(2\phi),$$

which, evaluated at the value of ϕ satisfying (20), amounts to

$$2a\cos(2\phi) - 2b\sin(2\phi) = 2(a^2+b^2)\frac{\cos(2\phi)}{a} = -2(a^2+b^2)\frac{\cos(2\phi)}{b}.$$

This expression is nonzero whenever $|g^2(0,0,0,0)| \neq 0$. Clearly, this condition holds generically. And when this derivative is nonzero, by application of the Implicit Function Theorem, we have four solutions for ϕ for all sufficiently small r_1, r_2 and τ.

We thus assume now that ϕ has been fixed by the above, and we are left with the variables r_1, r_2 and τ to solve the pair of real equations

$$\begin{cases} 0 = |z_1|^2\left(g + I_1(g_1^1 + |z_2|^2g_2^1 + 2\mathrm{Re}[z_1z_2(g_1^2 + |z_2|^2g_2^2)]\right) + \mathrm{Re}[z_1z_2(g^2 + z_1z_2g_2^2)], \\ 0 = |z_2|^2\left(g + I_1(g_1^1 + |z_1|^2g_2^1 + 2\mathrm{Re}[z_1z_2(g_1^2 + |z_1|^2g_2^2)]\right) + \mathrm{Re}[z_1z_2(g^2 + z_1z_2g_2^2)], \end{cases} \tag{21}$$

Subtracting the second equation of (21) from the first we obtain

$$-I_1^3(g_2^1 + 2\mathrm{Re}[z_1z_2g_1^2]) = 0.$$

As we are looking for nonsymmetric solutions, we have $I_1 \neq 0$. However, the remaining expression examined at $r_1 = r_2 = \tau = 0$ yields $g_2^1(0,0,0)$ which is typically nonzero. Hence, we find no small solutions to the bifurcation map with $z_1 \neq 0$ and $z_2 \neq 0$.

<u>$z_1 = 0$ and $z_2 \neq 0$</u>

The same procedure is followed as in the previously discussed case. However, due to the fact that $z_1 = 0$, the equations (18) simplify significantly to the single equation

$$|z_2|^2(g^1(|z_2^2|,\tau) + |z_2|^2g_1^1(|z_2|^2,\tau)) = 0.$$

Now we use the fact that $g^1(0,\tau) = \frac{\tau}{2}$. Hence, since $g^1(0,0) = 0$, we can use the Implicit Function Theorem with respect to τ, to find a smooth function $\tau(|z_2|)$ with $\tau(0) = 0$ so that for sufficiently small $|z_2|$ we have $g^1(|z_2^2|, \tau(|z_2|^2) + |z_2|^2g_1^1(|z_2|^2, \tau(|z_2|^2)) = 0$. Correspondingly, we have a one-parameter family of nonsymmetric periodic solutions filling out a local smooth two-dimensional invariant manifold.

<u>$z_1 \neq 0$ and $z_2 = 0$</u>

The analysis in this case is identical to the previous one, yielding another one-parameter family which is in fact the R-image of the family with $z_1 = 0$. □

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