Test exercise 1: Consider the linear system

$\dot{x} = Ax$, where

$$A = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}$$

Do there exist a solution x with $|x(t)| \to \infty$

as $t \to \infty$.

Because A is a upper diagonal, the diagonal

are the eigenvalues, i.e. $\lambda_1 = 2 \wedge \lambda_2 = -4$.

We hence have an saddel point as fixed point

where

$|x(t)| \to 0$ on the x-axis

$|x(t)| \to \infty$ on the y-axis.

So $\underline{\underline{\text{yes}}}$.

Test exercise 2: Consider again $\dot{x} = Ax$, from before. If we take an initial value, x_0, which is not an element of neither E_s or E_u, what can we say about the limit behaviour of the corresponding solution $|x(t)|$ when $t \to \infty$.

E_s is the x-axis and E_u is the y-axis. Hence we start some where else than the axis and in a "saddel-space" we will diverge to ∞.

the solution is $\underline{\underline{|x(t)| \to \infty}}$.

Exercise 1: Consider the following model of the water height of a leaky cylindrical bucket

$$\dot{x} = -2\sqrt{x}, \quad x \geq 0 \qquad (2)$$

(a) Show that

$$x = \begin{cases} (\sqrt{x_0} - t)^2, & t \leq \sqrt{x_0} \\ 0, & t \geq \sqrt{x_0} \end{cases}$$

is a solution

$$\frac{dx}{dt} = -2\sqrt{x(t)} \Rightarrow dx = -2\sqrt{x(t)}\,dt \Rightarrow \frac{1}{-2\sqrt{x(t)}}\,dx = dt$$

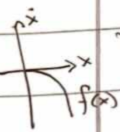
$$\Rightarrow \int_{x_0}^{x} (-2\sqrt{x})^{-1}\,dx = \int_0^t dt \Rightarrow \sqrt{x_0} - \sqrt{x} = t \Rightarrow$$

$$-\sqrt{x} = t - \sqrt{x_0} \Rightarrow \sqrt{x} = -t + \sqrt{x_0} \Rightarrow \underline{\underline{x = (\sqrt{x_0} - t)^2}}$$

So $x = (\sqrt{x_0} - t)^2$ when $t \leq \sqrt{x_0}$ hence the water is pouring out at some rate but when $t \geq \sqrt{x_0}$ the model would suggest the water would flow in reverse which makes no sense, hence we set it to zero.

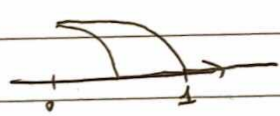
Exercise 1, con't.

(b) Are initial value solutions of (2) unique?

Because $f(x) = -2\sqrt{x} \notin C^1$ (see drawing )
the thm does not hold and we can
have non-unique solutions.

(c) Is it true that any solution of (2) is strictly
increasing/decreasing or constant as we
have learned? can you explain the apparent
paradox?

We have solutions to $x(1) = 0$
which is non-zero when $0 \le t < 1$ and the
one when $t << 1$



violates but it can ~~be~~ happen due
to $f(x) \notin C^1$.

Exercise 2: As in example 1, sketch the region Ω in the (t, x_0) plane for the initial value problem

$$\dot{x} = x^2, \quad x(0) = x_0 \qquad (1)$$

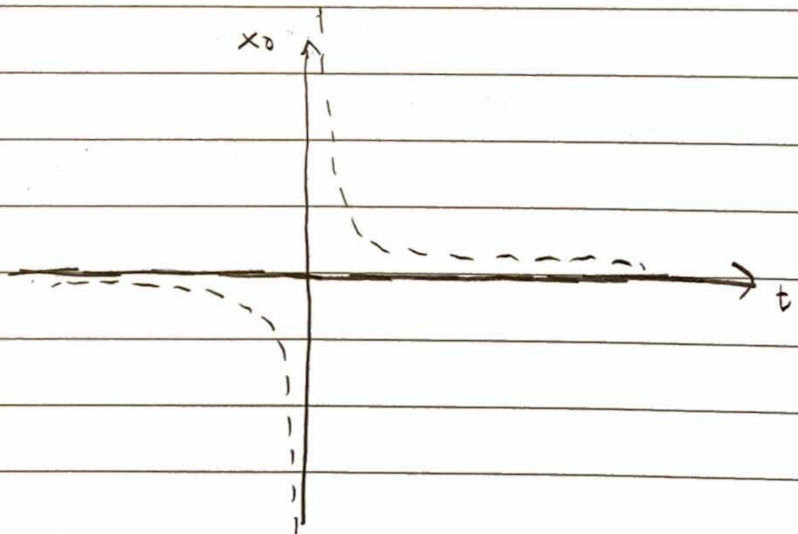
we solve (1)

$$\frac{dx}{dt} = x^2 \Rightarrow dx = x^2 dt \Rightarrow \frac{1}{x^2} dx = dt$$

$$\Rightarrow \int_{x_0}^{x} \frac{1}{x^2} dx = \int_0^t dt \Rightarrow -\frac{1}{x} + \frac{1}{x_0} = t \Rightarrow$$

$$-\frac{1}{x} = t - \frac{1}{x_0} \Rightarrow x = -\frac{x_0}{tx_0 - 1}$$

So we can't have that $tx_0 = 1 \Rightarrow t = \frac{1}{x_0}$
Hence we can be everywhere in figure 1 except on the lines



Because our initial value is defined for $t=0$ we have two sets

for $x_0 \geq 0 \rightarrow -\infty \leq t < \frac{1}{x_0}$

for $x_0 \leq 0 \rightarrow \frac{1}{x_0} < t \leq \infty$

Exercise 3: Determine the folw $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ for the non-linear system $\dot{x} = f(x)$ when

$$f(x) = \begin{bmatrix} -x_1 \\ 2x_2 + x_1^2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \quad (1)$$

and show that the set $S = \{ x \in \mathbb{R}^2 \mid x_2 = -x_1^2/4 \}$ is invariant w.r.t. the flow ϕ_t.

We solve (1)

$$\dot{x}_1 = -x_1 \Rightarrow x_1 = x_{01} e^{-t}$$

$$\dot{x}_2 = 2x_2 + (x_{01} e^{-t})^2 = 2x_2 + x_{01}^2 e^{-2t} \quad (2)$$

This is an inhomogenous linear differential equation.
A solution to such we can find by

$$x_2(t) = x_{2\,hom}(t) + x_{2\,inhom}(t)$$

We first solve the homogenous

$$\dot{x}_2 = 2x_2 \Rightarrow x_2 = ke^{2t}$$

We now proceed to the inhomogenous. Because we know e^{-2t} we guess ~~on~~ on such a solution.

$$x_{2\,inhom}(t) = Ae^{-2t} \quad (3)$$

we insert (3) in (2)

$$-2Ae^{-2t} = 2Ae^{-2t} + x_{01}^2 e^{-2t} \Rightarrow$$

$$-2A = 2A + x_{01}^2 \Rightarrow$$

$$A = -x_{01}^2/4$$

We hence know

$$x_{2\,inhom}(t) = -\frac{x_{01}^2}{4} e^{-2t}$$

Exercise 3: con't

We can now stitch the solution togeth

$$X_2(t) = ke^{2t} - \frac{x_{01}^2}{4} e^{-2t}$$

We know $X_2(0) = X_{02}$ so we solve for k.

$$X_2(0) = k - \frac{x_{01}^2}{4} = X_{02} \Rightarrow k = X_{02} + \frac{x_{01}^2}{4}$$

So we have

$$X_2(t) = \left(X_{02} + \frac{x_{01}^2}{4}\right) e^{2t} - \frac{x_{01}^2}{4} e^{-2t}$$

giving the total solution

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} x_{01} e^{-t} \\ \left(x_{02} + \frac{1}{4} x_{01}^2\right) e^{2t} - \frac{1}{4} x_{01}^2 e^{-2t} \end{bmatrix} \quad (4)$$

We now want to show that the set $S = \{ x \in \mathbb{R}^2 \mid X_2 = -X_1^2/4 \}$ is invariant. We insert (4) in the set

$$X_2(t) = -(X_1(t))^2/4 \Rightarrow$$

$$\left(X_{02} + \frac{1}{4} X_{01}^2\right) e^{2t} - \frac{1}{4} X_{01}^2 e^{-2t} = -(X_{01} e^{-t})^2/4 \Rightarrow$$

$$\left(X_{02} + \frac{1}{4} X_{01}^2\right) e^{2t} - \frac{1}{4} X_{01}^2 e^{-2t} = -X_{01}^2 e^{-2t}/4 \Rightarrow$$

$$X_{02} e^{2t} + \frac{1}{4} X_{01}^2 e^{2t} = 0 \Rightarrow$$

$$X_{02} e^{2t} = -\frac{1}{4} X_{01}^2 e^{2t} \Rightarrow \text{(multiply by } e^{-2t} \text{ on both sides)}$$

$$X_{02} = -\frac{1}{4} X_{01}^2$$

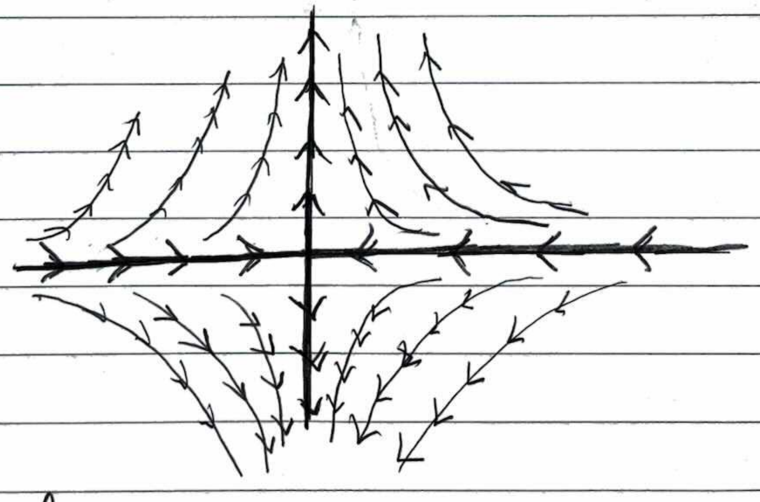
We see the relation holds and hence the ratio in S will be constant and the set is hence invariant.

Exercise 4: Sketch the flow for the linear system $\dot{x} = Ax$ with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \qquad (1)$$

and in particular describe (with a rough sketch) what happens to a small neighborhood $N_\varepsilon(x_0)$ of a point x_0 on the negative x_1-axis, say $N_\varepsilon(-3,0)$, with $\varepsilon = 0.2$.

- From A it is apperent that $\lambda_1 = -1$ and $\lambda_2 = 2$. We hence have a saddel which contracts in the x-axis and expands at twice the pace in the y-axis as sketched below



the flow of (1) is given by

$$\phi_t(x_1, x_2) = \begin{pmatrix} e^{-t} & \\ & e^{2t} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the small neighbourhood around $x_0 = (-3,0)$ will be alongated in the y direction twice as fast as it will shrink in the x diretion