Stochastic Processes, 02407
Final Exam

fall 21

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Contents

Preliminary remarks

The following is my exam paper in the course Stochastic Processes, 02407, taken in the fall semester, 2021. The reader should be made aware of the following when reading the paper.

1. Every question will start with the question formulation, followed by the answer. The first word in the answer will be in **bold**.
2. If any special assumptions are made regarding the question formulation it will be given in a footnote related to the question formulation as this.[1]
3. All matrices in this exercise will be indexed from 0.

1 Part 1

In the first, rather primitive, model we assume that the cod has a daily meal opportunity at dawn and that the size of this meal at dawn expressed in number of capelin can be described by a geometrically distributed random variable with probability mass function $p(1-p)^i$, where $i = 0, 1, \ldots$ Each food item (capelin) present in the stomach at the beginning of a day has a probability q of being digested (leave the stomach) before the meal opportunity at dawn the following day. The cod stomach has finite capacity of K capelin. If the combined meal size and stomach content exceeds this value the stomach content will be K.

1.1 Question 1

We will now formulate a model describing the daily content of capelin in a cod stomach at dawn, immediately before the meal opportunity.

We first observe that time is discrete and 'space' is both discrete and finite. Hence we can model the situation using a discrete-time finite-state Markov chain with $K+1$ states.

Instead of trying to model everything at once we will start modelling the feeding process. We are given that the cod-capelin interaction can be described by a geometric distribution

$$\Pr\{\text{Seeing } i \text{ capelins}\} = p(1-p)^i \tag{1}$$

Hence we can model the feeding process as

$$F_{ij} = \begin{cases} p(1-p)^{j-i} & \text{if } j \geq i \ \wedge \ j \neq K \ \wedge \ i \neq K \\ 1 - \sum_{l=0}^{K-(i+1)} p(1-p)^l & \text{if } j = K \ \wedge \ i \neq K \\ 1 & \text{if } j = k \ \wedge \ i = K \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

[1]Example

Where

$$\Pr\{\text{Stomach contains } j \text{ at time } t \big| \text{Stomach contains } i \text{ at time } t-1\} = F_{ij} \quad (3)$$

Next we will describe the digestion of capelins. We are given that each capelin in the stomach has a q probability of being digested and the digestion of each fish happens independently. Hence we can describe the digestion process using the binomial distribution.

$$\begin{aligned}\Pr\{\text{Digesting } i \text{ capelins of a total } N\} &= \text{Bin}(i, N, q) \\ &= \binom{N}{i} q^i (1-q)^{i-N}\end{aligned} \quad (4)$$

Hence we can model the digestion process as

$$D_{ij} = \begin{cases} 1 & \text{if } i = 0 \ \wedge \ j = 0 \\ 1 - \sum_{l=1}^{j} \text{Bin}(l, K, q) & \text{if } i \neq 0 \ \wedge \ j = i \\ \text{Bin}(i, j, q) & \text{if } j > i \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

The total process we are to model consists of first eating and then digesting. Both the feeding process and the digestion process describes stomach content and are discrete-time finite-state Markov chains with $K+1$ states. Hence we can just multiply them together to obtain the total process.

$$P = FD \quad (6)$$

We denote the daily process, which P describes, by X_t.

1.2 Question 2

We now consider specific values of the model parameters $K = 6$, $p = \frac{1}{3}$, and $q = \frac{1}{2}$.

We will now determine the mean number of capelins one would expect to find in a cod stomach.

We will first determine the limiting distribution of P. To do this we will use Threorem 4.1 in [1] which, in matrix form, states.

$$\pi = \pi P \quad (7)$$

Where π denotes the limiting distribution. We see π is the first left eigenvector of P. It can easily be found by using a standard eigenvector function on P^T. The eigenvector obtained for the eigenvalue 1, is then the limiting distribution after being normalized. We find the limiting distribution to be.

$$\pi \approx \begin{bmatrix} 0.232 \\ 0.293 \\ 0.234 \\ 0.146 \\ 0.070 \\ 0.022 \\ 0.003 \end{bmatrix} \tag{8}$$

To obtain the mean number of capelins in the stomach we calculate

$$\mathbb{E}[X_t] = \sum_{i=0}^{6} i\pi \approx 1.606 \tag{9}$$

1.3 Question 3

We now want to find the expected amount of capelins being part of a potential meal, which is actually not ingested due to lack of stomach capacity.

We will denote the probability of not being eaten as $Pr\{\text{Capelin not eaten}\} = \Pr\{CNE\}$. We can hence calculate

$$\begin{aligned} \Pr\{CNE\} &= \sum_{i=0}^{6} \Pr\{X_t = i\} \Pr\{CNE \mid X_t = i\} \\ &= \sum_{i=0}^{6} \pi_i \sum_{j=7-i}^{\infty} (j-6+i)p(1-p)^j \\ &\approx 0.394 \end{aligned} \tag{10}$$

1.4 Question 4

We say that a cod had a meal of size j if the actual number of capelin ingested was j irrespective of the size of the original or potential meal.

We now want to find the probability that a capelin was part of a meal of size j?

First we find the probability of different meal sizes.

$$\begin{aligned}\Pr\{\text{Meal size} = j\} &= \sum_{i=0}^{K} Pr\{X_t = i\} \Pr\{\text{Meal size} = j \mid X_t = i\} \\ &= \sum_{i=0}^{K} \pi_i F_{i,((j+i) \bmod 6)} \\ &\approx \begin{bmatrix} 0.335 \\ 0.231 \\ 0.165 \\ 0.118 \\ 0.081 \\ 0.049 \\ 0.020 \end{bmatrix} \end{aligned} \tag{11}$$

To get the probability of a capelin being part of a meal size j instead of the probability of a meal size j, we weight (11) by the number of capelins in each meal.

$$\Pr\{\text{A random capelin being part of a meal size } j\} \propto j \cdot \Pr\{\text{Meal size} = j\} \tag{12}$$

For (12) to be a true probability distribution it needs to sum to 1. We normalize

$$\begin{aligned}\Pr\{\text{A random capelin being part of a meal size } j\} &= \frac{j \cdot \Pr\{\text{Meal size} = j\}}{\sum_{i=0}^{K} i \cdot \Pr\{\text{Meal size} = i\}} \\ &\approx \begin{bmatrix} 0 \\ 0.144 \\ 0.206 \\ 0.221 \\ 0.201 \\ 0.152 \\ 0.076 \end{bmatrix} \end{aligned} \tag{13}$$

1.5 Question 5

Fisheries scientists on R/V DANA are studying cod in the Barents Sea. During the night they catch cod, empty their stomachs (flush the stomach with water) and equip cod with electronic devices that are able to register their feeding activities. Since the devices are very expensive a trade off has to be made between the amount of data and the probability of recuperating the device. Five days is the maximum time that the scientists are willing to wait for the results, considering the severe weather conditions that can occur in the Arctic. Now suppose a cod is caught, examined and released at sea with an empty stomach.

We now want to find the probability that there will be 3 capelin in the stomach after 5 days.

We want to find the probability

$$\Pr\{X_5 = 3 \big| X_0 = 0\} \tag{14}$$

From eq. (3.12) in [1] we know (14) can be found using n-step transitions.

$$\Pr\{X_n = j \big| X_0 = i\} = P_{ij}^n \tag{15}$$

Hence

$$\begin{aligned}\Pr\{X_5 = 3 \big| X_0 = 0\} &= P_{03}^5 \\ &\approx 0.144\end{aligned}$$

1.6 Question 6

We now want to find the probability that the cod has had at least 3 capelin in the stomach during the first 5 days.

We want to find the probability

$$\Pr\{X_5 \geq 3 \big| X_0 = 0\} \tag{16}$$

We can find this probability by making state 3-6 in P absorbing states, giving the matrix

$$P^{\text{absorb}} = \begin{bmatrix} Q^{3\times 3} & R^{3\times 4} \\ 0^{4\times 3} & I^{4\times 4} \end{bmatrix} \tag{17}$$

We can now find the probability of not being absorbed starting in state 0 by

$$\Pr\{X_5 < 3 \big| X_0 = 0\} = \sum_{i=0}^{2} Q_{0,i}^5 \tag{18}$$

And hence (16) is

$$\Pr\{X_5 \geq 3 \big| X_0 = 0\} = 1 - \Pr\{X_5 < 3 \big| X_0 = 0\} \tag{19}$$
$$\approx 0.560 \tag{20}$$

1.7 Question 7

We now assume that each capelin stays exactly two days in the cod stomach. We will denote the process of the number of capelin in the cod stomach right after dawn, immediately before the meal opportunity, at day t by Y_t.

Is the process Y_t Markovian? Depending on your answer: Specify the model or formulate a Markovian model that describes Y_t.

In [1] the Markov property is given in eq. (3.1) as

$$\begin{aligned} &\Pr\left\{X_{n+1} = j \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i\right\} \\ &\qquad = \Pr\left\{X_{n+1} = j \mid X_n = i\right\} \end{aligned} \tag{21}$$

Hence for the model to be markovian only the current state of the process should be necessary to predict the future. This is obviously violated illustrated in following example

$$\begin{aligned} &\text{Scenario 1:} \quad Y_0^{(1)} = 0 \quad Y_1^{(1)} = 2 \quad Y_2^{(1)} = 4 \\ &\text{Scenario 2:} \quad Y_0^{(2)} = 0 \quad Y_1^{(2)} = 4 \quad Y_2^{(2)} = 4 \end{aligned}$$

We see that obviously $\Pr\{Y_3^{(1)} \big| Y_2^{(1)}\} \neq \Pr\{Y_3^{(2)} \big| Y_2^{(2)}\}$.
A question now is if we can model Y_t as a higher order Markov chain and then transform it to a first order chain. It turns out this is nor a possibility because to remove exactly what has been added to the stomach is not always possible. Capelins are just filled into the stomach and we cannot distinguish between a 1 day old capelin and a 2 day old capelin only having information about the state of the stomach, Y_t. This is given as an example in figure 1.

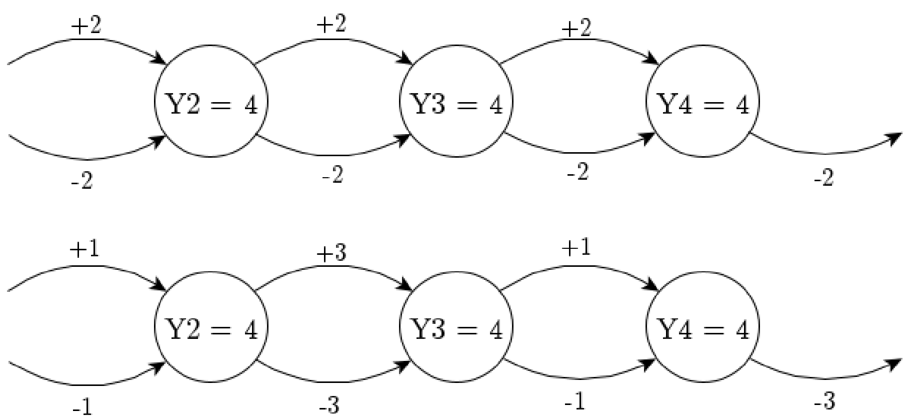

Figure 1 – Two possible realizations of the process Y_t

To model Y_t we need information about the feeding process, F_t, separately. If we have this we can model is as follows.

$$Y_t = F_t - F_{t-2} \tag{22}$$

1.8 Question 8

We now find the mean time (in days) between periods, where the stomach is empty.[2]

In [1], Theorem 4.3 tells us that if P is a recurrent irreducible aperiodic Markov chain, then the mean time between visits is given by

$$\lim_{n \leftarrow \infty} P_{ii}^{(n)} = \frac{1}{m_i} \tag{23}$$

We will not prove that our chain is irreducible and aperiodic but we can use Theorem 4.2 in [1] to show it is recurrent. It says that if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \tag{24}$$

Then P is recurrent and this is indeed the case. We will hence proceed and calculate m_0.

$$\begin{aligned} \pi_0 &= \frac{1}{m_0} \Leftarrow \\ m_0 &= \frac{1}{\pi_0} \\ &\approx 4.316 \end{aligned}$$

[2]It is not clear from the problem formulation of we are to work with the model from Question 7 or Question 2-6. I have hence decided to use the model from Question 2-6.

2 Part 2

During the polar summer it is perhaps more suitable to assume that food is accessible throughout the day and the capelins are caught individually on a continuous basis. The mean number of encountered capelin is 1 per day, and we assume that the probability of encountering one is constant over the day and proportional to the length of a small time interval-this is a standard assumption in population dynamics. The digestion time can be described by a scaled $\chi^2(1)$-distribution with mean 18 hours. In order to simplify the analysis we assume that there are no size constraints on the stomach. The capelins are assumed to be digested sequentially.

2.1 Question 9

We want to determine the mean number of capelins in a cod stomach but before doing so we need to set a modeling framework up which can describe the given setting.

We know that the capelins are caught continuously with a rate of 1/day. The digestion is continuously and following a $\frac{24}{18}\chi^2(1)$. Hence the system can be described by a birth and death process with $\lambda = 1$ and $\mu = \mathbb{E}[\frac{24}{18}\chi^2(1)] = \frac{24}{18}$.
This gives the following intensity matrix

$$A = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{25}$$

The process, which A describes, we will refer to as X_t.

To calculate the mean number of capelins in the cod's stomach we will need the stationary distribution of (25). To find it we will use eq. (6.37) in [1].

$$\pi_0 = \left(\sum_{k=0}^{\infty} \theta_k\right)^{-1} \tag{26}$$

$$\pi_j = \theta_j \pi_0 \tag{27}$$

To use (26) and (27) we need the θ's. These are given in eq. (6.36) in [1].

$$\theta_j = \prod_{k=0}^{j-1} \frac{\lambda_k}{\mu_{k+1}} \tag{28}$$

Because $\lambda_k = 1,\ \forall k$ and $\mu_k = \frac{24}{18},\ \forall k$ then

$$\theta_j = \left(\frac{3}{4}\right)^j \tag{29}$$

We can now calculate π_0 and π_j

$$\begin{aligned}
\pi_0 &= \left(\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^j \right)^{-1} \\
&= \frac{1}{4} \qquad (30) \\
\pi_j &= \pi_0 \; \theta_j \\
&= \frac{1}{4} \left(\frac{3}{4}\right)^j \qquad (31)
\end{aligned}$$

We can now calculate the mean number of capelins in the stomach.

$$\begin{aligned}
\mathbb{E}[X_t] &= \sum_{k=0}^{\infty} k\pi_k \\
&= 3 \qquad (32)
\end{aligned}$$

2.2 Question 10

We will now also calculate the variance of the number of capelins in a cod stomach.

To do this we will use

$$\begin{aligned}
\mathbb{V}[X_t] &= \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 \\
&= \left(\sum_{k=0}^{\infty} k^2 \pi_k \right) - \left(\sum_{k=0}^{\infty} k\pi_k \right)^2 \\
&= 21 - 9 \\
&= 12
\end{aligned}$$

Hence the variance of the number of capelins in a cod stomach is 12.

2.3 Question 11

Large migrating cod (skrei in Norwegian) is the most important fish in the northern part of Norway. The local communities are severely hit by recent collapses of the cod population due to over fishing of the stock of capelin. We are asked by fellow scientists at R/V G. O. Sars to make a more accurate model of the dynamics of the stomach of a large cod in order to refine the estimates of the daily ration. Recently it has been shown that the digestion rate depends in a nonlinear way on the number of capelin in the stomach. However, we will make the approximation that the digestion rate increases linearly with number of capelin in the stomach until 6 capelin are found in the stomach. Whenever 6 or more capelin are present in the stomach the digestion rate is at its maximum (6 times the digestion rate of one). Since a large *skrei* is about 1.20 m and capelin are very small in relation to this we assume that the stomach capacity is unlimited. The mean digestion time is 4 days, for one capelin. Knowing that it has been in the stomach for x time units does not provide us with further information of its total time in the stomach. The encounter process is as above.

We will now determine a upper bound for the feeding rate λ such that the stomach content will not grow without bounds.

We now have an varying digestion rate given by

$$\mu_k = \begin{cases} k\frac{1}{4} & \text{for } k \leq 6 \\ \frac{6}{4} & \text{for } k > 6 \end{cases} \tag{33}$$

We will refer to the base rate μ_1 as μ in the rest of part 2.

Mere intuitively λ should probably be under the maximum digestion rate, hence $\lambda < \frac{6}{4}$. To check if this is indeed the case we will use the fact, stated on page 306 in [1], that

$$\sum_{k=0}^{\infty} \theta_k < \infty \tag{34}$$

for there to be a limiting distribution and hence a bound on the stomach content.

We set $\lambda = \frac{x}{4}$ to have the same denominater as μ_k and get the following sum.

$$\sum_{k=0}^{\infty} \theta_k = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \sum_{k=6}^{\infty} \frac{\left(\frac{x}{6}\right)^k x^6}{6!} \tag{35}$$

For (35) to converge, it must hold that $\left(\frac{x}{6}\right)^k < 1$. This means $x < 6$ and hence $\lambda < \frac{6}{4}$. We see the intuitive bound appears also to be the correct bound.

2.4 Question 12

We return to the estimate of a stock assessment group aboard R/V DANA who estimated that the encounter rate is 1 capelin per day.

We will now also determine the mean number of capelins in the stomach of a large cod.

To calculate the mean number of capelins in the stomach of a large cod we will follow the exact same procedure as in Question 9 except now the first 6 μ's changes. We will hence not restate the theory here. We find the mean number of capelins in a large cods stomach to be.

$$\begin{aligned} \mathbb{E}[X_t^{\text{large}}] &= \sum_{k=0}^{\infty} k\pi_k \\ &\approx 3.647 \end{aligned}$$

2.5 Question 13

Suppose that at some point in time a cod has an empty stomach. We will now find an expression for the probability that the stomach has *not* reached a level (for the first time) with at least 5 capelin present.[3]

To calculate this we will use the theory given in section 1.2 of [2]. We will only use from state 0 to 5 and change state 5 to an absorbing state. This gives a new finite intensity matrix.

$$A^{\text{absorb}} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 & 0 \\ 0 & 0 & 3\mu & -(\lambda+3\mu) & \lambda & 0 \\ 0 & 0 & 0 & 4\mu & -(\lambda+4\mu) & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

We now find the matrix exponential of A^{absorb} and take out the $(0,5)$-element.

$$\left[e^{A^{\text{absorb}}t}\right]_{05} \approx 1.00 - 1.66\mathrm{e}^{-0.16t} + 1.15\mathrm{e}^{-0.70t} - 0.67\mathrm{e}^{-1.35t} + 0.20\mathrm{e}^{-2.14t} - 0.02\mathrm{e}^{-3.16t} \tag{37}$$

This is the probability of absorption from state 0 to 5. We want the probability of not being absorb and hence

$$\Pr\{X_t^{\text{large}} < 5\} = 1 - \left[e^{A^{\text{absorb}}t}\right]_{05} \tag{38}$$

The probability (38) is illustrated in figure 2.

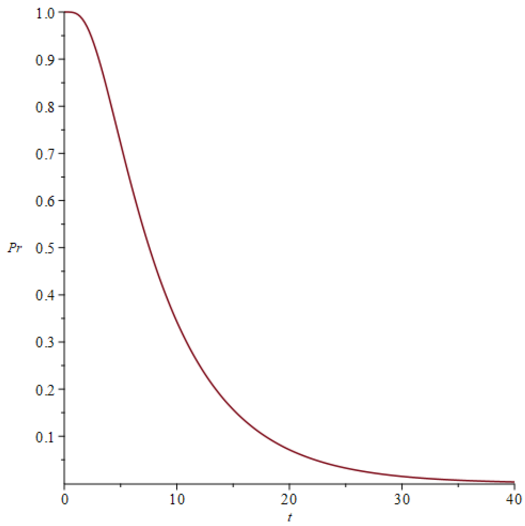

Figure 2 – The probability (38) is illustrated in the figure

[3]It has not been specified in the exam paper if we are continuing to work with the large cod or going back to the small cod. Hence I have decided to continue with the large cod.

2.6 Question 14

Suppose, as in the previous question, that at some point in time a cod has an empty stomach. We will now calculate the expectation and the variance of the time it takes for the stomach to reach a level (for the first time) with at least 5 capelin present.

In Question 13 we found the cumulative distribution function for the probability of being absorbed in state 5 from state 0, i.e.

$$F_{05}^{\text{absorb}}(t) = \left[e^{A^{\text{absorb}}t}\right]_{05} \tag{39}$$

Hence we can obtain the probability density function by differentiating $F_{05}^{\text{absorb}}(x)$.

$$f_{05}^{\text{absorb}}(t) = \frac{d}{dt}\left[e^{A^{\text{absorb}}t}\right]_{05} \tag{40}$$

We can now calculate the expectation and variance.

$$\begin{aligned}\mathbb{E}\left[T_{05}^{\text{absorb}}\right] &= \int_0^\infty t f_{05}^{\text{absorb}}(t)dt \\ &= 9.3125 \end{aligned} \tag{41}$$

$$\begin{aligned}\mathbb{V}\left[T_{05}^{\text{absorb}}\right] &= \left(\int_0^\infty t^2 f_{05}^{\text{absorb}}(t)dt\right) - \left(\int_0^\infty t f_{05}^{\text{absorb}}(t)dt\right)^2 \\ &= 43.285 \end{aligned} \tag{42}$$

Where T_{05}^{absorb} is the time to absorption in state 5 from state 0.

2.7 Question 15

It has been known for some time that even fish can become seasick. This happens when there are large waves (as with many people). We are asked to refine our model to incorporate this knowledge. The consequences to the cod is that sometimes it will vomit thus empty the stomach instantly and completely. With the harsh weather conditions in the Barents Sea it has been estimated that such bad weather may occur randomly with a rate of 1 every 2 weeks. For the remaining questions in this part you can assume that the stomach capacity is finite at 20 capelin. The digestion process is considered to be the same as the one used for questions 12-14.

We will now formulate a model to describe this scenario of the stomach content.

We are given that fish can be seasick in rough sea and when this happens they throw up, i.e. return to state 0 immediately. This happens every 14th day hence we can add a rate, $\rho = \frac{1}{14}$, in every $(0, x)$-position of the new intensity matrix except $(0, 0)$. Furthermore we are given that we limit the size of the cod's stomach to 20 capelins. Hence we will now have a 21 by 21 intensity matrix for which the first 5 states are shown here below.

$$A^{\text{seasick}}_{0:4\times 0:4} =$$
$$\begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ \mu+\rho & -(\lambda+\mu+\rho) & \lambda & 0 & 0 \\ \rho & 2\mu & -(\lambda+2\mu+\rho) & \lambda & 0 \\ \rho & 0 & 3\mu & -(\lambda+3\mu+\rho) & \lambda \\ \rho & 0 & 0 & 4\mu & -(\lambda+4\mu+\rho) \end{bmatrix}$$

One should remember that the digestion rate in the intensity matrix A^{seasick} only increase until state 7 and then afterwards stays constant at $6\mu = \frac{6}{4}$.

2.8 Question 16

We will now determine the mean number of capelin in the stomach and the long run fraction of time with j capelin in the stomach, i.e. the stationary distribution.

We will first determine the stationary distribution of A^{seasick}. To do this we will use eq. (6.68) in [1] which states

$$A^{\text{seasick}}\pi^{\text{seasick}} = 0 \tag{43}$$

When we try to solve π^{seasick} in maple, the calculation fail due to A^{seasick} being near singular. Hence we will resort to approximate π^{seasick}. We know from eq. (6.67) in [1] that the probability distribution is given by

$$P^{\text{seasick}}(t) = e^{A^{\text{seasick}}t} \tag{44}$$

and the stationary distribution is defined by

$$\pi^{\text{seasick}} = \lim_{t\to\infty} P^{\text{seasick}}_{0\times 0:20}(t) \tag{45}$$

Hence we define a function

$$g(t) = \left[\|\left| P^{\text{seasick}}(t) - P^{\text{seasick}}(t+1) \right|\|_1 \right]_{0\times 0:20} \tag{46}$$

We find that for $g(10^{10}) < \varepsilon_{machine}$ where $\varepsilon_{machine}$ is the machine precision in maple. Hence we will use $P^{\text{seasick}}_{0\times 0:20}(10^{10})$ as an approximation for π^{seasick}.

$$\pi^{\text{seasick}} \approx P^{\text{seasick}}_{0\times 0:20}(10^{10})$$

We will not show the vector π^{seasick} of size 21 here but it can be found in Appendix under Question 16. Here we have illustrate it in figure 3 using a point plot.

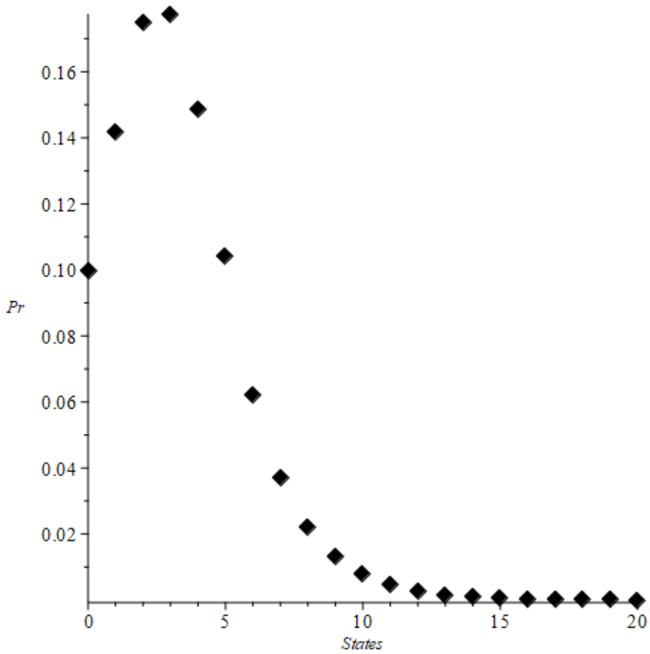

Figure 3 – Illustration of π^{seasick}.

Next we determine the mean number of capelins in the stomach. This is simple when we have the stationary distribution.

$$\mathbb{E}\left[X_t^{\text{seasick}}\right] = \sum_{k=0}^{20} k\pi_k^{\text{seasick}} \tag{47}$$

$$\approx 3.287 \tag{48}$$

2.9 Question 17

We will now calculate the mean time a capelin will be in the digestion system of a cod.

We can probabilistically formulate the mean time for a capelin to stay in the system as

$$\mathbb{E}\left[T_{i0}^{seasick}\right] = \sum_{k=1}^{20} \mathbb{E}\left[T_{i0}^{seasick} \middle| X_t = k\right] \Pr\{X_t = k\} \tag{49}$$

$\mathbb{E}\left[T_{i0}^{seasick} \middle| X_t = k\right]$ is the mean time for a capelin to be stay in the system when found in state k. In state k we have a constant risk of puking, ρ, where all capelins will leave the system at once and the individual rate of being digested given by $\mu_k = \min(k\mu, 6\mu)$. The mean time to leave the system in state k is hence.

$$\mathbb{E}\left[T_{i0}^{seasick} \middle| X_t = k\right] = k\frac{1}{k\rho + \mu_k} \tag{50}$$

We have plotted (50) in figure 4 and see that the mean time to stay in the system goes towards 14 days which makes sense due to the puking affecting all of the fish. For the first 6 states it stays constant due to the digestion rate μ_k increases with the number of capelins.

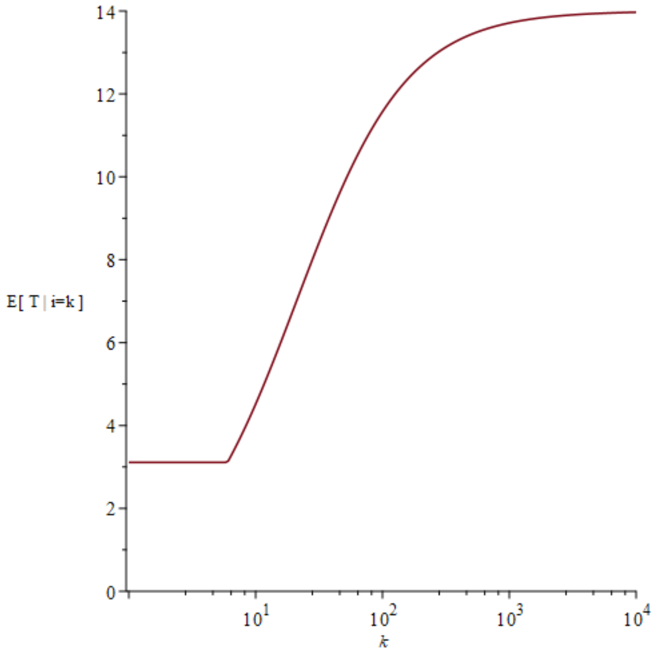

Figure 4 – $\mathbb{E}\left[T_{i0}^{seasick} \middle| X_t = k\right]$ plotted where the x-axis is a logarithmic scale

Next we have the probability of being in state k. One could think this is just $\pi^{seasick}$ but as in Question 4 we are calculating a probability concerning the capelins and not the stomach content. Hence we should only consider state 1 to 20 and normalize accordingly.
Hence

$$\begin{aligned}\mathbb{E}\left[T_{i0}^{seasick}\right] &= \sum_{k=1}^{20} \mathbb{E}\left[T_{i0}^{seasick} \middle| X_t = k\right] \Pr\{X_t = k\} \\ &= \sum_{k=1}^{20} \frac{k}{k\rho + \mu_k} \frac{\pi_k^{seasick}}{\sum_{k=1}^{20} \pi_k^{seasick}} \\ &= 3.20 \end{aligned} \tag{51}$$

3 Part 3

We are asked to consider yet another scenario. Cod encounters individual capelin with constant intensity of $\frac{3}{2}$ per day, but immediately after an encounter the cod has to handle the caught capelin for some exponentially distributed random time. During this time the cod is prevented from catching a new capelin during this process and the mean handling time is $\frac{1}{3}$ day.

3.1 Question 18

We will now calculate what the mean number of capelin ingested during a 10 day period is.

To model the situation we will use a continuous time Markov chain. State 0 is the hunting state and state 1 is the eating state. In state 0 we have a catching-rate, λ, of $\frac{3}{2}$ per day and in state 1 we have a eating-rate, μ, of 3 per day. Hence our intensity matrix becomes

$$A = \begin{bmatrix} -3/2 & 3/2 \\ 3 & -3 \end{bmatrix} \tag{52}$$

We calculate the stationary probability of A by using equation (6.8) in [1].

$$\begin{aligned} A\pi &= 0 \Rightarrow \\ \pi &= \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \end{aligned} \tag{53}$$

We can now calculate the expected number of capelins catched over 10 days which we will denote F_{10}

$$\begin{aligned} \mathbb{E}[F_{10}] &= t\lambda\pi_{\text{Hunting}} \\ &= 10\,\frac{3}{2}\,\frac{2}{3} \\ &= 10 \end{aligned} \tag{54}$$

3.2 Question 19

A cod has been caught. We now want to find out if the cod hadn't been disturbed (caught), how long would it have had to wait until the next capture of a capelin?

The waiting times are exponentially distributed and hence memoryless in the following sense

$$\Pr(T > s + t \mid T > s) = \Pr(T > t), \quad \forall s, t \geq 0 \tag{55}$$

Hence we can just calculate the expected time to absorbtion in the eating state from the hunting state.

$$A^{\text{absorb}} = \begin{bmatrix} -3/2 & 3/2 \\ 0 & 0 \end{bmatrix} \tag{56}$$

$$F_{01}^{\text{absorb}}(t) = \left[e^{A^{\text{absorb}}t}\right]_{01} \tag{57}$$

$$f_{01}^{\text{absorb}}(t) = \frac{d}{dt}F_{01}^{\text{absorb}}(t) \tag{58}$$

We can now calculate the mean time to absorbtion

$$\begin{aligned}\mathbb{E}\left[T_{01}^{\text{absorb}}\right] &= \int_0^\infty t f_{01}^{\text{absorb}}(t)dt \\ &= \frac{2}{3}\end{aligned}$$

3.3 Question 20

The stock assessment group asks for the biomass consumed by a cod in a 10 day period. We are given the information that there is variation in the size of capelin. This variation can be described by a log-normal distribution with mean $\mu = 10$ cm and variance $\sigma^2 = 5$ cm^2. The variation in size does not alter our previous assumptions on handling times and digestion times. We are told by a computer wizard on board R/V Akademik Feodorov that the variation in the arrival process after 10 days is 5.6543, we trust her.

We now want to find the mean and the variance of the biomass of capelin ingested by a cod during a 10 day period, starting immediately after a cod was caught.

We realize that the capelins are given as a stochastic variable which we will refer to as C. We now have to find the mean and variance of the product of two stochastic variables.

We assume the two stochastic variables are independent and hence we know from [3] that the expected values are multiplicative.

$$\begin{aligned}\mathbb{E}[C\ F_{10}] &= \mathbb{E}[C]\mathbb{E}[F_{10}] \\ &= 10 \cdot 10 \\ &= 100\end{aligned}$$

We know from [4] that the variance of the product of independent variables can be found by

$$\mathbb{V}[XY] = \mathbb{E}[X^2]\mathbb{E}[Y^2] - \mathbb{E}[X]^2\mathbb{E}[Y]^2 \tag{59}$$

We hence calculate first calculate $\mathbb{E}[C^2]$ and $\mathbb{E}[F_{10}^2]$

$$\begin{aligned}\mathbb{V}[C] &= \mathbb{E}[C^2] - \mathbb{E}[C]^2 \Rightarrow \\ 5 &= \mathbb{E}[C^2] - 100 \Rightarrow \\ \mathbb{E}[C^2] &= 105\end{aligned} \tag{60}$$

$$\begin{aligned}\mathbb{V}[F_{10}] &= \mathbb{E}[F_{10}^2] - \mathbb{E}[F_{10}]^2 \Rightarrow \\ 5.6543 &= \mathbb{E}[F_{10}^2] - 100 \Rightarrow \\ \mathbb{E}[F_{10}^2] &= 105.6543\end{aligned} \tag{61}$$

We can now calculate the variance of their product.

$$\begin{aligned}\mathbb{V}[C\ F_{10}] &= \mathbb{E}[C^2]\mathbb{E}[F_{10}^2] - \mathbb{E}[C]^2\mathbb{E}[F_{10}]^2 \\ &= 1093.7\end{aligned}$$

4 Part 4

Consider again the case, where cods are caught, equipped with electronic devices and released. Initially we assume that a cod will roam horizontally and directionless. The distance moved in any two or more time intervals will be independent from the movement in any other interval. If we consider space described by a coordinate system then the standard deviation of the daily movement in each of the x and y directions is 2 km.

4.1 Question 21

We now want to find the probability that the cod is within a distance of 5km of the place where it was dropped three days earlier.

To do this we can describe movement in the x-axis and y-axis with two standard Brownian motons.

$$X_t = \sigma B(t)$$
$$Y_t = \sigma B(t)$$

Where $\sigma = 2$. Both the x- and y-axis' position follows a normal distribution with variance $2^2 t$ and mean zero and hence this is a special case described by the Reighley distribution. The CDF for the Reighley distribution is found here [5] and is given by

$$F(x, \sigma) = 1 - \exp^{-\frac{1}{2}(\frac{x}{\sigma})^2} \tag{62}$$

We hence calculate desired probability.

$$\begin{aligned} F(5, 2\sqrt{3}) &= 1 - \exp^{-\frac{1}{2}(\frac{5}{2\sqrt{3}})^2} \\ &\approx 0.6472 \end{aligned} \tag{63}$$

4.2 Question 22

A cod is now catched three days later at the same x-coordinate at which it was released. We now want to find the probability that the cod was more than 2 kilometres away in the same x-direction at both one and two days after its release.

Here we can describe the scenario with a Brownian bridge. We are given a start and end position, and the dynamics follow a standard Brownian motion.
The probability we want to find is

$$\Pr\{\, |X_1| \geq 2 \ \wedge \ |X_2| \geq 2 \mid X_0 = 0 \wedge X_3 = 0\} \tag{64}$$

By using the Brownian bridge we can handle the condition $X_0 = 0 \wedge X_3 = 0$ in (64).

We hence continue with

$$\begin{aligned}\Pr\{\,|X_1| \geq 2 \ \wedge\ |X_2| \geq 2\} =\\ \Pr\{\,|X_1| \geq 2\}\Pr\{\,|X_2| \geq 2 \,\big|\, |X_1| \geq 2\} =\\ \Pr\{\ X_1 \geq 2\}\Pr\{\ X_2 \geq 2 \,\big|\, X_1 \geq 2\}+\\ \Pr\{\ X_1 \geq 2\}\Pr\{\ X_2 \leq -2 \,\big|\, X_1 \geq 2\}+\\ \Pr\{\ X_1 \leq -2\}\Pr\{\ X_2 \geq 2 \,\big|\, X_1 \leq -2\}+\\ \Pr\{\ X_1 \leq -2\}\Pr\{\ X_2 \leq -2 \,\big|\, X_1 \leq -2\}\end{aligned} \tag{65}$$

We will start breaking down the $X_1 \geq 2 \wedge X_2 \geq 2$ case.

$$\begin{aligned}\Pr\{\ X_1 \geq 2\}\Pr\{\ X_2 \geq 2 \,\big|\, X_1 \geq 2\} =\\ \int_2^\infty \phi(z, \mathbb{E}\left[X_1 \middle| X_0 = 0 \wedge X_3 = 0\right], \sqrt{\mathbb{V}\left[X_1 \middle| X_0 = 0 \wedge X_3 = 0\right]})\\ \int_2^\infty \phi(z, \mathbb{E}\left[X_2 \middle| X_1 = z \wedge X_3 = 0\right], \sqrt{\mathbb{V}\left[X_2 \middle| X_1 = z \wedge X_3 = 0\right]})\mathrm{d}x\mathrm{d}z\end{aligned} \tag{66}$$

Where $\phi(x, \mu, \sigma)$ is the density function for the normal distribution. From [6] the expectation and variance for the Brownian bridge are given as

$$\mathbb{E}\left[X_t \middle| X_{t_1} = a \wedge X_{t_2} = b\right] = a + \frac{t - t_1}{t_2 - t_1}(b - a) \tag{67}$$

$$\mathbb{V}\left[X_t \middle| X_{t_1} = a \wedge X_{t_2} = b\right] = \sigma^2 \frac{(t_2 - t)(t - t_1)}{t_2 - t_1} \tag{68}$$

Hence we calculate

$$\mathbb{E}\left[X_1 \middle| X_0 = 0 \wedge X_3 = 0\right] = 0 + \frac{1 - 0}{3 - 0}(0 - 0) = 0 \tag{69}$$

$$\mathbb{E}\left[X_2 \middle| X_1 = z \wedge X_3 = 0\right] = z + \frac{2 - 1}{3 - 1}(0 - z) = \frac{1}{2}z \tag{70}$$

$$\mathbb{V}\left[X_1 \middle| X_0 = 0 \wedge X_3 = 0\right] = 2^2 \frac{(3 - 1)(1 - 0)}{3 - 0} = \frac{8}{3} \tag{71}$$

$$\mathbb{V}\left[X_2 \middle| X_1 = z \wedge X_3 = 0\right] = 2^2 \frac{(3 - 2)(2 - 1)}{3 - 1} = 2 \tag{72}$$

We now have all parts to be able to evaluate (66).

$$\begin{aligned}\Pr\{\ X_1 \geq 2\}\Pr\{\ X_2 \geq 2 \,\big|\, X_1 \geq 2\} =\\ \int_2^\infty \phi(z, 0, \sqrt{\frac{8}{3}}) \int_2^\infty \phi(z, \frac{1}{2}z, \sqrt{2})\mathrm{d}x\mathrm{d}z \approx 0.0373\end{aligned}$$

The rest of the cases in (65) only differ from (66) by having different limits on the integrals so we will skip these derivations. Hence we get

$$\begin{aligned}
&\Pr\{\, X_1 \geq 2\} \Pr\{\, X_2 \geq 2 \mid X_1 \geq 2\} + \\
&\Pr\{\, X_1 \geq 2\} \Pr\{\, X_2 \leq -2 \mid X_1 \geq 2\} + \\
&\Pr\{\, X_1 \leq -2\} \Pr\{\, X_2 \geq 2 \mid X_1 \leq -2\} + \\
&\Pr\{\, X_1 \leq -2\} \Pr\{\, X_2 \leq -2 \mid X_1 \leq -2\} \approx \\
&0.0373 + 0.0010 + 0.0010 + 0.0373 = 0.0766
\end{aligned}$$

4.3 Question 23

The biologists are also interested in the behaviour of the cods with respect to vertical movements immediately upon release. Suppose that these movements can be described in a similar way as the horizontal movements with a downward drift of 0.5 m per half hour with standard deviation of 1.5 m per half hour.

We now want to find the probability that a cod released at a depth of 4 m will reach the surface before the sea floor, when the water depth is 20 m.

First we realize that the situation can be described by a Brownian motion with drift.

$$\begin{aligned}
X_t &= \mu t + \sigma B(t) \\
&= -\frac{1}{2}t + \frac{3}{2}B(t) \qquad (73)
\end{aligned}$$

Next we notice that the probability we want to find can be described by the phenomena called gamblers ruin. Theorem 8.1 in [1] is as follows

Theorem 8.1. *For a Brownian motion with drift parameter μ and variance parameter σ^2, and $a < x < b$,*

$$u(x) = \Pr\left\{X(T_{ab}) = b \mid X(0) = x\right\} = \frac{\mathrm{e}^{-2\mu x/\sigma^2} - \mathrm{e}^{-2\mu a/\sigma^2}}{\mathrm{e}^{-2\mu b/\sigma^2} - \mathrm{e}^{-2\mu a/\sigma^2}}.$$

We see that in our case $b = 0$, $a = -20$, $\sigma^2 = \frac{9}{4}$ amd $\mu = -\frac{1}{2}$. Hence

$$u(-4) \approx 0.1689$$

4.4 Question 24

We will consider an equation related to an enzymatic process in the cods digestion system. This process can be written as $Z(t) = X(t)e^{X(t)}$, where $X(t)$ is a Brownian motion with drift -0.5 and variance parameter 1.

We now want to calculate $\mathbb{E}[Z(t)]$.

We calculate

$$\begin{aligned}\mathbb{E}[Z(t)] &= \mathbb{E}\left[X(t)e^{X(t)}\right] \\ &= \mathbb{E}\left[\left(\mu t+\sigma B(t)\right)e^{\mu t+\sigma B(t)}\right] \\ &= \mathbb{E}\left[\mu t e^{\mu t+\sigma B(t)}\right]+\mathbb{E}\left[\sigma B(t)e^{\mu t+\sigma B(t)}\right]\end{aligned}$$

From the teaching notes for lesson 11 we recognize $\mathbb{E}\left[\mu t e^{\mu t+\sigma B(t)}\right]$ as the expectation of a geometric Brownian motion.

$$\begin{aligned}\mathbb{E}\left[\mu t e^{\mu t+\sigma B(t)}\right]+\mathbb{E}\left[\sigma B(t)e^{\mu t+\sigma B(t)}\right] &= \mu t e^{(\mu+\frac{1}{2}\sigma^2)t}+\mathbb{E}\left[\sigma B(t)e^{\mu t+\sigma B(t)}\right] \\ &= \mu t e^{(\mu+\frac{1}{2}\sigma^2)t}+\sigma e^{\mu t}\mathbb{E}\left[B(t)e^{\sigma B(t)}\right] \\ &= \mu t e^{(\mu+\frac{1}{2}\sigma^2)t}+\sigma e^{\mu t}\int_{-\infty}^{\infty} x e^{x\sigma}\phi(x,0,\sqrt{t})dx \\ &= \left(\mu+\sigma^2\right)te^{\left(\mu+\frac{1}{2}\sigma^2\right)t} \qquad (74)\end{aligned}$$

We insert $\mu=-\frac{1}{2}$ and $\sigma^2=1$ in (74) and obtain

$$\mathbb{E}[Z(t)]=\frac{1}{2}t$$

4.5 Question 25

We now observe the horizontal movements of the cod according to a Poisson process with rate λ, i.e. at discrete but random time epochs. We are interested in the position of the fish in the x direction only, and suppose that the parameters are as in Question 23[4].

We will now show that the value of the position process at the time of the first observation is given by a modified Laplace distribution. Furthermore we will also determine the parameters of the distribution.

We know the position process follows a standard Brownian motion with standard deviation σ.

$$X_t=\sigma B(t)\sim\mathcal{N}(0,\sigma^2 t) \qquad (75)$$

From Theorem 5.4 in [1] we know the time of the first observation follows a exponential distribution with parameter λ.

$$T_1\sim\text{Exp}(\lambda) \qquad (76)$$

[4] I am not quite sure which parameters are referenced here. In Question 23 we were working with vertical movement and here we are moving in the horizontal direction. Hence I have decided to model the movement as a standard Brownian motion as in Question 21 and 22, and do not assume specific values for the parameters

The probability distribution we want to find is

$$\begin{aligned}\Pr\{X_{T_1}\} &= \Pr\{X_t \big| \ t = T_1\} \Pr\{T_1\} \\ &= \int_0^\infty f_{X_t|\ T_1=t}(x) f_{T_1}(t) dt \qquad (77)\end{aligned}$$

The density functions in (77) are given as

$$f_{X_t|\ T_1=t}(x) = \phi(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \qquad (78)$$

$$f_{T_1}(t) = f_{\mathrm{Exp}}(t,\lambda) = \lambda e^{-\lambda t} \qquad (79)$$

We insert the density functions in (77).

$$\begin{aligned}f_{X_t \big|\ T_1=t}(x) &= \int_0^\infty f_{X_t|\ T_1=t}(x) f_{T_1}(t) dt \\ &= \int_0^\infty \phi(x,0,\sigma\sqrt{t}) f_{\mathrm{Exp}}(t,\lambda) dt \\ &= \frac{\sqrt{2}\sqrt{\lambda}}{2} e^{-\sqrt{2}\sqrt{\lambda}\ |x|} \qquad (80)\end{aligned}$$

We have the Laplace distribution given by

$$f_{Laplace}(x \mid \mu, b) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}} \qquad (81)$$

We see we can rewrite (80) as

$$f_{X_t|\ T_1=t}(x) = \left(2\frac{\sqrt{2}}{2\sqrt{\lambda}}\right)^{-1} e^{-\ |x-0|\ \left(\frac{\sqrt{2}}{2\sqrt{\lambda}}\right)^{-1}} \qquad (82)$$

Hence we see that $b = \frac{\sqrt{2}}{2\sqrt{\lambda}}$ and $\mu = 0$ in our case.

4.6 Question 26

Although most cod migrate, particularly during the spawning season some cod tend to revert around the same position, lets call it the place of return. For simplicity we will (again) only consider movements in one (e.g. the x) direction. The movements occur according to a Gaussian process such that the mean of the change during a small time interval is approximately equal to the product of the value of the current position, the length of the time interval and a certain constant γ. The variance of the change during a small interval is approximately equal to the length of the time interval multiplied with the value ϕ.

We will now formulate a model for the movement of the cod in the x-direction.

We can write up the mean and variance of the process as

$$\mathbb{E}\left[X_{t+} \mid X_t = x\right] = \gamma x \Delta t \qquad (83)$$

$$\mathbb{V}\left[X_{t+} \mid X_t = x\right] = \phi \Delta t \qquad (84)$$

We see this matches the mean and variance given for the Ohrnstein-Uhlenbeck process on page 435 in [1]. Hence we can model the movement of the cod in the x-direction using the Ohrnstein-Uhlenbeck process.
In the book they use other greek letters as constants and because we use ϕ as the density function for the normal distribution in this assignment we will change the constants as follows.

$$\phi = \sigma^2 \tag{85}$$
$$\gamma = -\beta \tag{86}$$

4.7 Question 27

Suppose the standard deviation of the change in the location of the cod in the x direction during one day is 2 km and the correlation between the location of the cod in the x-direction from any day to another is $\frac{1}{2}$.

We will now calculate the probability that the cod is less than 2 km away from its place of return at an arbitrary point in time.

Before we are able to do any calculation regarding the asked question, we must find the constants β and σ^2. We are told that the correlation at any point in time is $\frac{1}{2}$. We know that

$$Corr(X_s, X_u) = \frac{Cov(X_s, X_u)}{Cov(X_s, X_s)} \tag{87}$$

Because the wording 'in any point in time', we will use the stationary covariance given in eq. (8.69) in [1].

$$Cov(X_s, X_u) = \frac{\sigma^2}{2\beta} e^{-\beta\,|s-u|} \tag{88}$$

Hence the stationary correlation is given by

$$\begin{aligned} Corr(X_s, X_u) &= \frac{\frac{\sigma^2}{2\beta} e^{-\beta\,|s-u|}}{\frac{\sigma^2}{2\beta} e^{-\beta\,|s-s|}} \\ &= e^{-\beta\,|s-u|} \end{aligned} \tag{89}$$

We can now solve for β.

$$\begin{aligned} Corr(X_s, X_{s+1}) &= e^{-\beta\,\left|s-(s+1)\right|} \;\Rightarrow \\ \frac{1}{2} &= e^{-\beta} \;\Rightarrow \\ \beta &= \ln(2) \end{aligned} \tag{90}$$

We can now again use eq (8.69) given in (88) to solve for σ^2.

$$\begin{aligned}\mathbb{V}[X_s] &= Cov(X_s, X_s) \\ &= \frac{\sigma^2}{2\beta} \Rightarrow \\ 2^2 &= \frac{\sigma^2}{2\ln(2)} \Rightarrow \\ \sigma^2 &= 8\ln(2)\end{aligned} \tag{91}$$

The probability we want to find is

$$\Pr\{-2 \geq X_t \geq 2 \mid X_0 = 0\} = \int_{-2}^{2} \phi(x, \mathbb{E}\left[X_t \mid X_0 = 0\right], \sqrt{\mathbb{V}\left[X_t \mid X_0 = 0\right]})dx \tag{92}$$

We hence need to find the conditioned expectation and variance. The conditioned expectation and conditioned variance are given in [1] as eq. (8.58) and (8.59) respectively.

$$\mathbb{E}\left[X_t \mid X_0 = x\right] = xe^{-\beta t} \tag{93}$$

$$\mathbb{V}\left[X_t \mid X_0 = x\right] = \sigma^2 \left(\frac{1 - e^{-2\beta t}}{2\beta}\right) \tag{94}$$

Hence we have

$$\mathbb{E}\left[X_t \mid X_0 = 0\right] = 0 \tag{95}$$

$$\mathbb{V}\left[X_t \mid X_0 = 0\right] = 4 - 4e^{-2\ln(2)t} \tag{96}$$

We calculate

$$\Pr\{-2 \geq X_t \geq 2 \mid X_0 = 0\} = \int_{-2}^{2} \phi(x, 0, \sqrt{4 - 4e^{-2\ln(2)t}})dx \tag{97}$$

The evaluation of (97) is really nasty and hence we will not include it. Instead the probability is plotted versus time in figure 5 and the stationary probability given in (98)

$$\lim_{t\to\infty} \Pr\{-2 \geq X_t \geq 2\} \approx 0.6827 \tag{98}$$

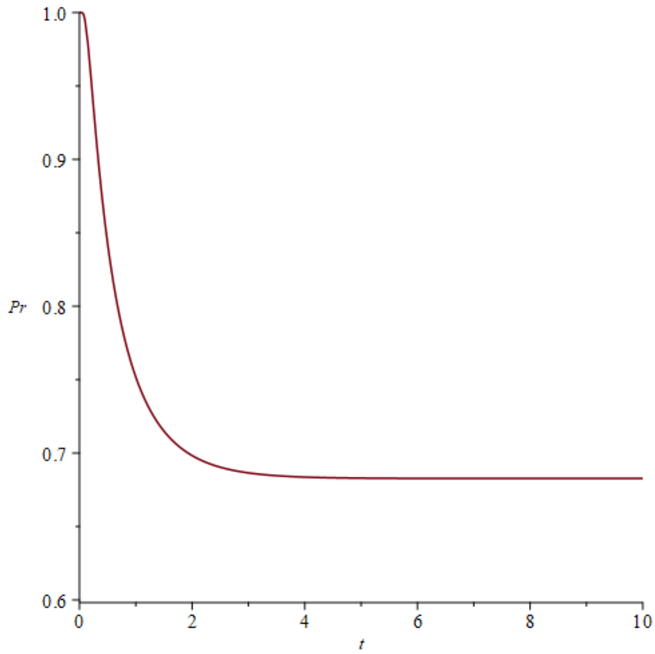

Figure 5 – The probability (97) is illustrated in the figure

4.8 Question 28

We will now calculate the probability that the cod is less than 2 km away from its place of return at an arbitrary point in time and also exactly one day after.

Hence we want to find

$$\begin{aligned}\Pr\{-2 \geq X_t \geq 2 \ \wedge \ -2 \geq X_{t+1} \geq 2\} = \\ \Pr\{-2 \geq X_t \geq 2\} \Pr\{-2 \geq X_{t+1} \geq 2 \,\Big|\, -2 \geq X_t \geq 2\} = \\ \int_{-2}^{2} \phi(z, \mathbb{E}\left[X_t \mid X_0 = 0\right], \sqrt{\mathbb{V}\left[X_t \mid X_0 = 0\right]}) \\ \int_{-2}^{2} \phi(x, \mathbb{E}\left[X_1 \mid X_0 = z\right], \sqrt{\mathbb{V}\left[X_1 \mid X_0 = z\right]}) dx dz\end{aligned} \tag{99}$$

We hence find the expectations and variances using 93 and 94.

$$\mathbb{E}\left[X_t \mid X_0 = 0\right] = 0e^{-\ln(2)t} = 0 \tag{100}$$

$$\mathbb{E}\left[X_1 \mid X_0 = z\right] = ze^{-\ln(2)1} = \frac{1}{2}z \tag{101}$$

$$\mathbb{V}\left[X_t \mid X_0 = 0\right] = 8\ln(2)\left(\frac{1 - e^{-2\ln(2)t}}{2\ln(2)}\right) = 4 - 4e^{-2\ln(2)t} \tag{102}$$

$$\mathbb{V}\left[X_1 \mid X_0 = z\right] = 8\ln(2)\left(\frac{1 - e^{-2\ln(2)1}}{2\ln(2)}\right) = 3 \tag{103}$$

We can hence evaluate (99).

$$\Pr\{-2 \geq X_t \geq 2 \ \wedge \ -2 \geq X_{t+1} \geq 2\} = \int_{-2}^{2} \phi(z, 0, \sqrt{4 - 4e^{-2\ln(2)t}}) \int_{-2}^{2} \phi(x, \frac{1}{2}z, \sqrt{3}) dx dz \tag{104}$$

Again the evaluation of (104) results is a really nasty integral and hence we will again plot the probability versus time in figure 6 and the stationary probability is given in (105).

$$\lim_{t \to \infty} \Pr\{-2 \geq X_t \geq 2 \ \wedge \ -2 \geq X_{t+1} \geq 2\} \approx 0.49797 \tag{105}$$

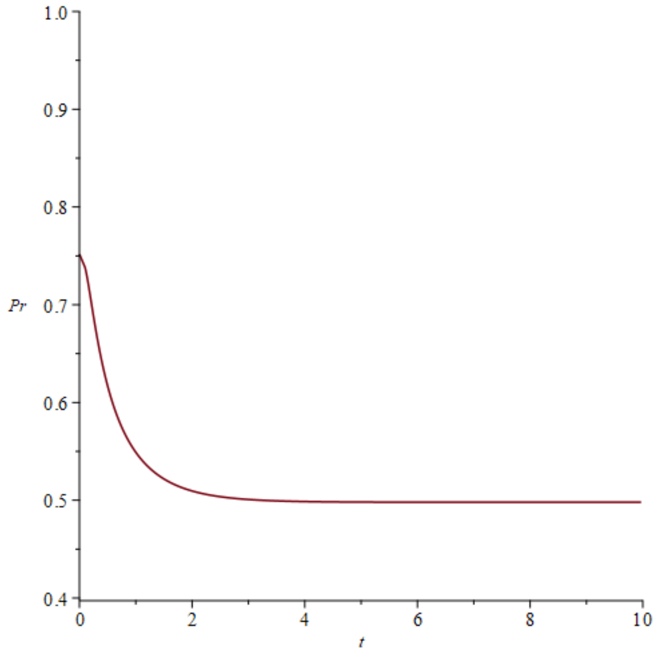

Figure 6 – The probability (104) is illustrated in the figure

References

[1] M. A. Pinsky and S. Karlin, *An Introduction to Stochastic Modeling.* Academic Press, 2010.

[2] A. Mitrofanova, "lecture notes, computational systems biology, new york university."

[3] "Expected value," https://en.wikipedia.org/wiki/Expected_value#Uses_and_applications, accessed: 2021-12-22.

[4] "Variance," https://en.wikipedia.org/wiki/Variance#Properties, accessed: 2021-12-22.

[5] “Rayleigh distribution,” https://en.wikipedia.org/wiki/Rayleigh_distribution, accessed: 2021-12-20.

[6] “Brownian bridge,” https://en.wikipedia.org/wiki/Brownian_bridge, accessed: 2021-12-20.

5 Appendix

5.1 Question 16

$$\pi^{\text{seasick}} = \begin{bmatrix} 0.10 \\ 0.14 \\ 0.17 \\ 0.18 \\ 0.15 \\ 0.10 \\ 0.06 \\ 0.04 \\ 0.02 \\ 0.01 \\ 7.85 \times 10^{-3} \\ 4.68 \times 10^{-3} \\ 2.79 \times 10^{-3} \\ 1.67 \times 10^{-3} \\ 9.96 \times 10^{-4} \\ 5.96 \times 10^{-4} \\ 3.57 \times 10^{-4} \\ 2.15 \times 10^{-4} \\ 1.31 \times 10^{-4} \\ 8.11 \times 10^{-5} \\ 5.16 \times 10^{-5} \end{bmatrix} \tag{106}$$