Week 1: 01125 Fundamental Topological Concepts and Metric Spaces

Andreas Heidelbach Engly [s170303], Anton Ruby Larsen [s174356], and Mads Esben Hansen [s174434]

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1.21

We proceed with a proof by contradiction.

Assume S is a countable set. Then $\exists f : \mathbb{N} \to S$. That means all natural number are mapped to distinct binary sequence. In other words, we create a bijection from $n \in \mathbb{N}$ to $s \in S$.

Listing out $f(n)\ \forall n \in \mathbb{N}$ gives us infinite many rows.

$f(1) = s_1$
$f(2) = s_2$
...
$f(n) = s_n$
...

By the definition of countability we should have listed all sequences $s \in S$. Let $s_n[n]$ denote the n'th element of the n'th sequence in the listing. We now create a new sequence $b =< x_n >_{n\in\mathbb{N}}$ in the following way.

1. If $s_n[n] = 1$, then $x_n = 0$.
2. If $s_n[n] \neq 1$, then $x_n = 1$.

Since each element in the sequence differs from at least 1 number in the diagonal it follows that $b \notin S$. This procedure is known as Cantor's diagonal argument and leads to the contradiction of the statement that all possible binary sequences were listed. Hence, S is uncountable.

2.16

I)

To show that D is a metric on $F(K, M)$ we start by noticing that D is defined by $d(x, y)$. Since $d(x, y)$ is a metric of the metric space (M, d), d satisfies the three conditions presented on page 20. Hence it follows:

1. $d(x, y) \geq 0\ \forall x, y \in M$ and $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)\ \forall x, y \in M$.
3. $d(x, y) \leq d(x, z) + d(z, y)\ \forall x, y, z \in M$.

Furthermore it is stated that $0 \leq d(x, y) \leq 1\ \forall x, y \in M$. Since the supremum is the least element of the upper bounds, it follows

$$0 \leq sup_{t\in K}(d(f(t), g(t))) = D(f, g) \leq 1 \quad \forall f, g \in F(K, M)$$

Utilizing the above properties it can be shown that $D(f,g)$ satisfies the properties of a metric.

1. If $D(f,g) = 0$, then it must hold that $d(x,y) \leq 0 \ \forall t \in K$. Since $0 \leq d(x,y) \leq 1$, then $0 \leq d(x,y) \leq 0$ implies that $d(x,y) = 0$. Since $d(x,y)$ is a metric and $D(f,g) = 0$, the values $f(t) = g(t) = 0 \ \forall t \in K$. Functions that produce identical values from identical arguments are identical. Hence it can be concluded that whenever $D(f,g) = 0$, then $f = g$.
2. Since $d(x,y) = d(y,x) \ \forall x,y \in M$, it follows that $D(f,g) = sup_{t\in K}(d(f(t),g(t))) = sup_{t\in K}(d(g(t),t(t))) = D(g,f)$.
3. We know that the sup of a sum cannot be larger than the sum of the sups of each element in the sum, i.e. $sup\{a,b\} \leq sup\{a\} + sup\{b\} \ \forall a,b$. Combined with the triangular inequality of d, it follows directly that $D(f,g) = sup_{t\in K}(d(f(t),g(t))) \leq sup_{t\in K}(d(f(t),h(t))) + sup_{t\in K}(d(h(t),g(t))) = D(f,h) + D(h,g)$.

Thereby we have verified that D is a metric on $F(K,M)$.

II)

For any given $\epsilon > 0$ we must be able to choose $\delta > 0$ s.t. $D(f,g) < \delta \Rightarrow \mathrm{d}(\mathrm{E}_{\nu_{t_0}}(f), E_{\nu_{t_0}}(g)) < \epsilon$.

Since $d(E_{\nu_{t_0}}(f), E_{\nu_{t_0}}(g)) = d(f(t_0),g(t_0)) \leq sup_{t\in K}(d(f(t_0),g(t_0))) = D(f,g)$, then letting $\delta = \epsilon$ will guarantee that the above is satisfied.

Let the enemy demand an error-margin $d(E_{\nu_{t_0}}(f), E_{\nu_{t_0}}(g)) < \epsilon$. Then $D(f,g) < \delta = \epsilon$ because $d(f(t_0),g(t_0)) \leq D(f,g)$.

Hence, the evaluation map $E_{\nu_{t_0}}$ is continuous.

2.27 (I)

To show that $\mathcal{T}$ is a topology on $\mathbb{R}$, $\mathcal{T}$ needs to satisfy the three requirements TOP1, TOP2, and TOP3.

We start by showing $\mathcal{T}$ satisfy TOP3.
TOP3 states that both the empty set Ø and $\mathbb{R}$ itself should belong to the topology.
We start with Ø and observe that it cannot be of type 2 since it demands 0 to be part of the set. However, type 1 only demands that 0 is not in the set. Therefore the empty set, Ø, is part of type 1.
$\mathbb{R}$ cannot be of type 1 since $0 \in \mathbb{R}$ 0, but $\mathbb{R} \backslash \mathbb{R}$=Ø which is finite so it complies

with all the requirements of type 2.
Therefore $\mathcal{T}$ complies with TOP3.

We then want to show TOP2.

TOP2 states that the intersection of an arbitrary finite set of subsets should still belong to the topology.
We need to consider three cases:

1. None of the U_i is of type 2.

   We therefore have n subsets of type 1.

   $$\cap_{i=1}^{n} U_i = U_{type1}$$

   This new set will be of type 1 due to the absence of 0 in all U_i and therefore the intersection cannot include 0 either.

2. Some but not all of the U_i is of type 2.

   Because we still have sets of type 1 and they do not include 0 the set we get after a intersection of a finite number of sets will still be of type 1.

3. All of the U_i is of type 2.

   If we took the intersection over infinitely many subsets of type 2 the compliment of the intersection would be infinite, but because we only take the intersection over finitely many subsets the compliment of the intersection will also only contain finitely many elements. Therefore it will still belong to type 2. This is a result of De Morgans Laws that states that the compliment of an intersection is the union of complements. The complement of our intersection will then be the finite union of finite sets, thus finite itself.

We see that in all three cases the intersection still belongs to the topology $\mathcal{T}$ and therefore $\mathcal{T}$ complies with TOP2.

Lastly we need to show TOP1

TOP1 says that any union of subsets should still belong to the topology.
Here we need to show two cases:

1. None of the U_i is of type 2.
   This implies that no matter how you combine type 1 sets you will always get a type 1 because 0 cannot be part of a type 1 set and therefore the union of them cannot either.

2. Some of the U_i is of type 2.

   Because some of the sets in the union now are of type 2 the union cannot be of type 1 due to the presence of 0. We then need to investigate if the union is part of type 2.
   Because the type 2 subsets all have a finite compliment their union will also have a finite compliment. If we let the number of subsets of type 2 go to infinity we will end up with the real number line whos compliment is the empty set which is finite.
   Therefore the union with some subsets of type 2 will always belong to type 2.

We see here that the topology $\mathcal{T}$ also complies with TOP1 and therefore complies with all 3 requirements and is therefore a topology on $\mathbb{R}$.