Introduction to Dynamical Systems: Assignment 2

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Problem 1

We shall analyse the system of differential equations

$$\dot{x}_1 = f_1(x_1, x_2) = x_2 \tag{1a}$$
$$\dot{x}_2 = f_2(x_1, x_2) = \mu x_1 - \frac{1}{2}x_1^2 \tag{1b}$$

Here the two variables x_1 and x_2 depend on time t. A dot above a time dependent variable denotes differentiation with respect to t. The parameter μ is a real number.

1.A

Find all critical points of the two dimensional vectorfield $f = (f_1, f_2)^T$ in the system (1).

The critical points are determined by:

$$\dot{x}_1 = x_2 = 0 \tag{2}$$
$$\dot{x}_2 = \mu x_1 - \frac{1}{2}x_1^2 = 0 \tag{3}$$

We see that x_2 must equal 0 for the system to have a critical point. We calculate the values for x_1.

$$\mu x_1 - \frac{1}{2}x_1^2 = 0 \quad \Leftrightarrow \quad x_1 = 2\mu \ \vee \ x_1 = 0 \tag{4}$$

We can hence conclude that the system (1) have critical points in

$$(x_1, x_2) = (0, 0) \ \wedge \ (x_1, x_2) = (2\mu, 0) \tag{5}$$

1.B

Determine the stability of all critical points for the cases of $\mu > 0$ and $\mu < 0$. Classify these critical points according to the list: saddles, centers, cusps, nodes and foci.

Before we classify any critical points of (1) we see that the system is Hamiltonian which is shown in 1.D. Besides being Hamiltonian the system possesses even more structure. C.f. equation (3) on page 173 in [1] the system is also Newtonian. This allows us to use theorem 3 from section 2.14 in [1] to classify the critical points. We start by finding the kinetic energy of (1) given by $U(x_1)$.

$$\begin{aligned} U(x_1) &= -\int_{x_1^*}^{x_1} f(s)ds \\ &= -\int_{x_1^*}^{x_1} \mu s - \frac{1}{2}s^2 ds \\ &= -\frac{1}{2}\mu x_1^2 + \frac{1}{2}\mu(x_1^*)^2 + \frac{1}{6}x_1^3 - \frac{1}{6}(x_1^*)^3 \end{aligned} \tag{6}$$

Where x_1^* is the x_1-value at a critical point. We differentiate (6) and solve for roots.

$$\frac{\partial U(x_1)}{\partial x_1} = -\mu x_1 + \frac{1}{2}x_1^2 = 0 \quad \Leftrightarrow \quad x_1 = 0 \ \vee \ x_1 = 2\mu \tag{7}$$

We see the roots are independent of x_1^* so we can continue without considering the critical points one by one. We calculate the second derivative of (6) to investigate the extrema.

$$\frac{\partial^2 U(x_1)}{\partial x_1^2} = x_1 - \mu \tag{8}$$

We can now analyse the critical points for positive and negative values of μ.

1. $\boldsymbol{\mu < 0}$**:** In this case we see that (8) is positive for $x_1 = 0$ giving a local minimum and negative for $x_1 = 2\mu$ giving a local maximum. This means that c.f. theorem 3 from section 2.14 in [1] that $(x_1, x_2) = (0, 0)$ is a center which is stable but not asymptotically stable and $(x_1, x_2) = (2\mu, 0)$ is a saddle which is unstable.

2. $\boldsymbol{\mu > 0}$**:** In this case we see that (8) is positive for $x_1 = 2\mu$ giving a local minimum and negative for $x_1 = 0$ giving a local maximum. This means that c.f. theorem 3 from section 2.14 in [1] that $(x_1, x_2) = (0, 0)$ is a saddle which is unstable and $(x_1, x_2) = (2\mu, 0)$ is a center which is stable but not asymptotically stable.

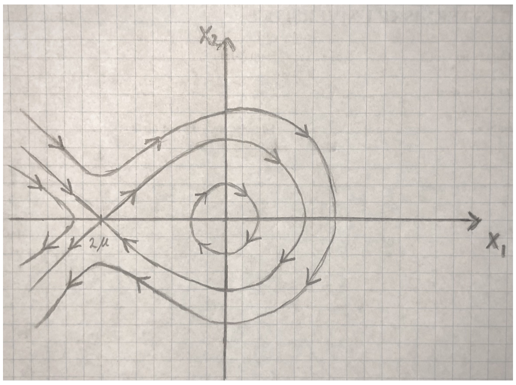
We summarize in a table

For	**Critical point**	**Stability**	**Type**
$\mu < 0$	$x = (0, 0)$	Stable	Center
	$x = (2\mu, 0)$	Unstable	Saddle
$\mu > 0$	$x = (0, 0)$	Unstable	Saddle
	$x = (2\mu, 0)$	Stable	Center

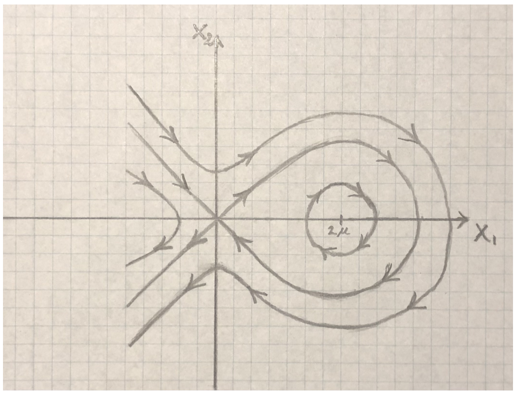
1.C

Sketch the flow in a phase plane plot for $\mu > 0$ and $\mu < 0$. Use pencil and paper and support your findings using pplane or Maple.

First we make a sketch using pen and paper.



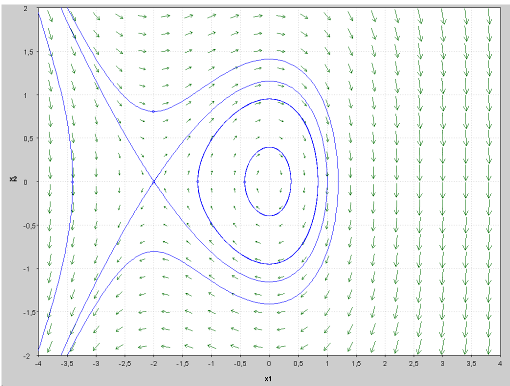
(a) Sketch of flow for $\mu < 0$.



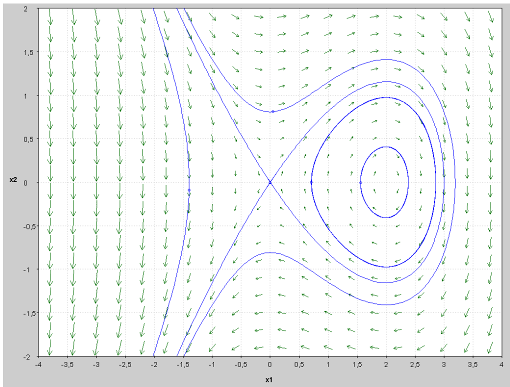
(b) Sketch of flow for $\mu > 0$.

Figure 1

We can now make equivalent phase plots using pplane.



(a) Pplane of flow for $\mu < 0$.



(b) Pplane of flow for $\mu > 0$.

Figure 2

We see that the hand made sketches and the ones made in pplane are qualitatively equivalent.

1.D

For $\mu \neq 0$ there are homoclinic orbits. Similar to finding center manifolds we can determine the homoclinic orbits by introducing a function $h \in \mathcal{C}^\infty$ mapping $\mathcal{R} \to \mathcal{R}$ according to

$$x_2 = h(x_1) \tag{9}$$

Which condition must h satisfy? Now show that h is given by

$$h(x_1)^2 = a_1 x_1^2 + b x_1^3 + c \tag{10}$$

and determine the parameters a, b and c for $\mu > 0$ and $\mu < 0$.

If h defines a homoclinic orbit it must, according to page 161 in [2], satisfy:

- h must define trajectories of (1).
- h must connect the saddlepoint of (1) to itself.

As we mentioned in 1.B our system (1) is Hamiltonian given in definition 1 page 171 in [1]. Therefore h defines a trajectory, for which the energy-level is constant via theorem 1 page 172 in [1].
Let us verify that (1) is Hamiltonian by use of the given structure of h.

$$H(x_1, x_2) = a_1 x_1^2 + a_2 x_1^3 + b_1 x_2^2$$
$$\rightarrow$$
$$\frac{\partial H}{\partial x_2} = 2b_1 x_2 \tag{11a}$$
$$-\frac{\partial H}{\partial x_1} = -2a_1 x_1 - 3a_2 x_1^2 \tag{11b}$$

We see that if we choose the Hamiltonian, $H(x_1, x_2)$, as given in (12), (11) is equivalent to (1); therefore, our system must be Hamiltonian.

$$H(x_1, x_2) = -\frac{1}{2}\mu x_1^2 + \frac{1}{6}x_1^3 + \frac{1}{2}x_2^2 \tag{12}$$

We now isolate $h(x_1)^2 = x_2^2$, and determine the parameters a and b, and determine c based on the energy level at the saddle point.

$$H(x_1, x_2) = -\frac{1}{2}\mu x_1^2 + \frac{1}{6}x_1^3 + \frac{1}{2}x_2^2$$
$$\rightarrow$$
$$h(x_1)^2 = x_2^2 = \mu x_1^2 - \frac{1}{3}x_1^3 + 2H(x_1, x_2)$$

We can now see that $a = \mu$ and $b = -\frac{1}{3}$.

We now need to determine c. As mentioned, this is determined based on the energy level at the saddle points. The reasoning behind this is, that we know the homoclinic orbit must connect the saddle point to itself. Since the saddle point is different for $\mu < 0$ and $\mu > 0$ we must find c for each of these possibilities.

For $\mu < 0$ the saddle point is located at $(2\mu, 0)$

$$\begin{aligned} 0 &= \mu(2\mu)^2 - \frac{1}{3}(2\mu)^3 + c \\ \Rightarrow& \\ c &= \frac{8}{3}\mu^3 - 4\mu^3 \\ &= -\frac{4}{3}\mu^3 \end{aligned}$$

For $\mu > 0$ the saddle point is located at $(0, 0)$

$$\begin{aligned} &0 = 0 - 0 + c \Rightarrow \\ &c = 0 \end{aligned}$$

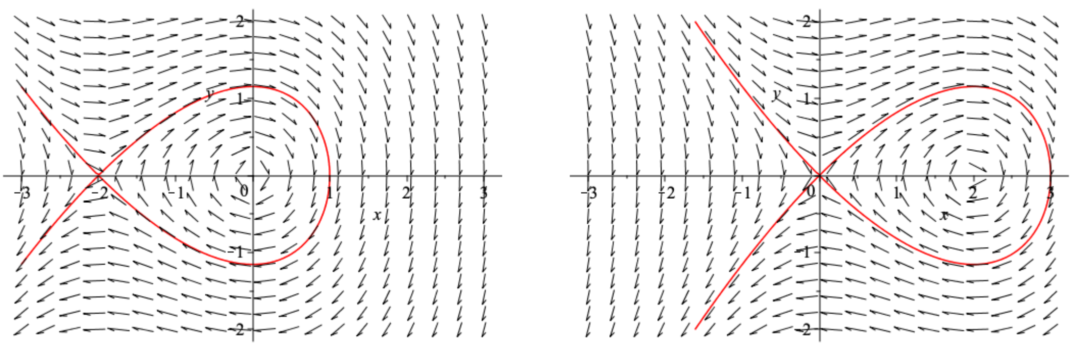
To summarize we have:

1. $\boldsymbol{\mu > 0}:\quad h(x_1)^2 = \mu x_1^2 - \frac{1}{3}x_1^3$
2. $\boldsymbol{\mu < 0}:\quad h(x_1)^2 = \mu x_1^2 - \frac{1}{3}x_1^3 - \frac{4}{3}\mu^3$

1.E

Plot your analytical expression for $h(x_1)$ together with the vectorfield. Does the figure show what you expect?

We plot the analytical solutions together with the vectorfield using maple.



(a) Solution and vectorfield for $\mu < 0$.

(b) Solution and vectorfield for $\mu > 0$.

Figure 3

In figure 3 we see the analytical expressions for h together with the vector field. We expected a homoclinical orbit similar to the one shown in figure 2 on page 206 in [1].

Comparing this with the hand sketched solutions from 1.C, this plot is very much as expected.
Specifically, the curve right of the saddle point define the homoclinic orbit, which is an example of an elliptic sector. The curves left of the saddle point define the stable and unstable hyperbolic sectors. The homoclic orbit together with the saddle point form the *sepratrix cycle*, if we further include the hyperbolic sectors it forms the entire *seperatrix* of (1) c.f. page 207 and definition 1 section 3.11 in [1].

Problem 2

The results from problem 1 indicate that a bifurcation occurs at $\mu = 0$ and that μ is a bifurcation parameter. Secondly, for $\mu = 0$ the critical point $(x_1, x_2) = (0, 0)$ is different compared to the cases where $\mu \neq 0$.

2.A

Construct a bifurcation diagram for the system (1). In doing so sketch in a figure x_1 as function of μ, for each critical point (x_1, x_2).

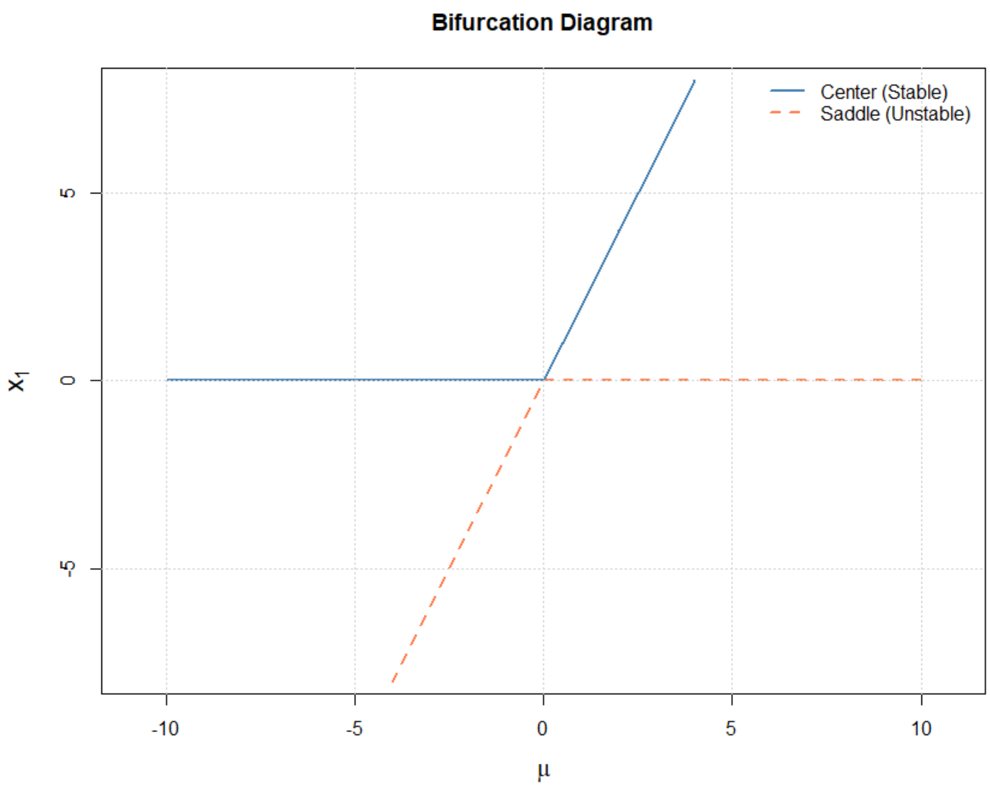


Figure 4 – The transcritical bifurcation diagram for the given system.

2.B

Determine the type of bifurcation you observe.

Figure 4 clearly shows it is an example of a *transcritical bifurcation*, as given in example 2 on page 335 in [1].

2.C

Is the critical point $(x_1, x_2) = (0, 0)$ hyperbolic or nonhyperbolic for $\mu = 0$?

For $\mu = 0$ the eigenvalues becomes

$$\begin{aligned} eig(Df(0,0)) &= eig\left(\begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix}\right) \\ &\xrightarrow{\mu=0} \\ \lambda =& 0 \ \vee \ \lambda = 0 \end{aligned}$$

So the equilibrium point is nonhyperbolic c.f. definition 1 page 102 in [1].

2.D

For $\mu = 0$ the system (1) becomes

$$\dot{x}_1 = x_2 \tag{13a}$$
$$\dot{x}_2 = -\frac{1}{2}x_1^2 \tag{13b}$$

At the critical point $(0, 0)$ the system possesses locally a center manifold $W^c(0, 0)$. Determine this center manifold by introducing the mapping

$$x_2 = h(x_1) \tag{14}$$

and find an analytical expression for h. If we strictly follow Theorem 1, p155, in the textbook by Perko, we would choose a function h_c, where $x_1 = h_c(x_2)$ instead of the expression in (14). However, the calculations using the mapping h is more straight forward than using the mapping defined by h_c. By the way they are each others inverse.

We first notice from 2.C that the linearized system only contains a center subspace and hence theorem 1 on page 155 [1] will fail because it needs a non-zero dimensional stable or unstable subspace. We will therefore attack the problem in an alternative fashion.
First we analyze the single critical point for $\mu = 0$, the same way as in 1.B. We first find the roots of $\left.\frac{\partial U(x_1)}{\partial x_1}\right|_{\mu=0}$.

$$\left.\frac{\partial U(x_1)}{\partial x_1}\right|_{\mu=0} = \frac{1}{2}x_1^2 = 0 \quad \Leftrightarrow \quad x_1 = 0 \tag{15}$$

We analyze the extremum by consulting the second derivative.

$$\left.\frac{\partial^2 U(x_1)}{\partial x_1^2}\right|_{\mu=0} = x_1 \tag{16}$$

We see that the second derivative is zero at the critical point, $(x_1, x_2) = (0, 0)$, meaning we have a horizontal inflection point. C.f. theorem 3 in section 2.14 of [1], we can conclude that the critical point must be a cusp.
From page 149-150 in [1] we know that a cusp only contains two hyperbolic sectors, which are defined in definition 1, section 2.11 of [1]. The sectors are the only part of a system which is invariant, c.f. page 147 in [1], and hence $W^c(0, 0)$ must consist of these two sectors. Because the system is Hamiltonian we can find the expression for these two sectors by equating $H(x_1, x_2)|_{\mu=0}$ with the energy level at the critical point.

$$c_{\text{cusp}} = H(0, 0)|_{\mu=0} = 0$$

giving

$$\begin{aligned} H(x_1, x_2)|_{\mu=0} &= c_{\text{cusp}} && \Rightarrow \\ \frac{x_2^2}{2} + \frac{x_1^3}{6} &= 0 && \Rightarrow \\ h_c(x_1)^2 &= -\frac{x_1^3}{6} \end{aligned}$$

Hence the center-manifold, $W^c(0, 0)$, is given by

$$h_c(x_1)^2 = -\frac{x_1^3}{6} \tag{17}$$

The method is verified in chapter 5 of the PhD, [3], and the center-manifold for the normal form of (1) is given in remark 5.2.1.

2.E

Plot your analytical expression for $h(x_1)$ together with the vectorfield f for $\mu = 0$ by using Maple or pplane. Determine the type of the critical point $(0, 0)$ at $\mu = 0$ and compare to Theorem 3 p151 in section 2.11 of the textbook by Perko.

We plot (1) for $\mu = 0$ together with the found center-manifold in figure 5. The type of the critical point was determined to be a cusp in 2.D so here we will only compare our finding with theorem 3 on page 151 in [1]. To use theorem 3 we must formulate (13) according to equation (3) on page 151 in [1] here given by (18).

$$\dot{x} = y \tag{18a}$$
$$\dot{y} = a_k x^k[1 + h(x)] + b_n x^n y[1 + g(x)] + y^2 R(x, y) \tag{18b}$$

We see that for (13) to match (18), $a_k = -\frac{1}{2}$, $k = 2$, $h(x) = 0$, $b_n = 0$, and $R(x, y) = 0$. Then according to theorem 3 the critical point must be a cusp which match our finding.

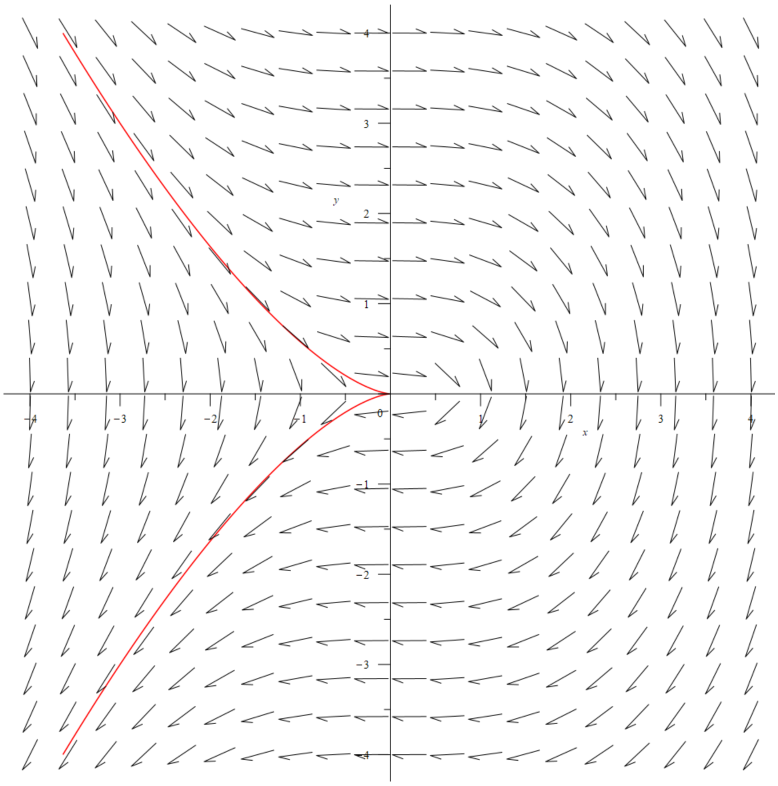


Figure 5 – The figure illustrates the system (1) for $\mu = 0$. The center-manifold of the system is given by the red curve.

References

[1] L. Perko, *Differential Equations and Dynamical Systems*. Springer, 2001.

[2] S. H. Strogatz, *Nonlinear dynamics and Chaos*, 1994.

[3] W. Giles, "On homoclinic orbits to center manifolds in hamiltonian systems," 2015.