Test exercise 1: Consider the linear system $\dot{x} = Ax$, where

$$A = \begin{pmatrix} -2 & -1 \\ 5 & 0 \end{pmatrix}$$

what kind of fixed point is the origin?

We solve the charactaristic polynomial

$$P(\lambda) = \operatorname{Det}\begin{pmatrix} -2-\lambda & -1 \\ 5 & -\lambda \end{pmatrix} = (-2-\lambda)(-\lambda) - (5(-1))$$

$$= +2\lambda + \lambda^2 + 5$$

$$= \lambda^2 + 2\lambda + 5$$

We solve $P(\lambda)$ by the 2nd degree poly. formula

$$d = b^2 - 4ac$$

$$= 2^2 - 4 \cdot 1 \cdot 5$$

$$= -16$$

$$x = \frac{-b \pm \sqrt{d}}{2a}$$

$$= \frac{-2 \pm \sqrt{-16}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{16}\sqrt{-1}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

Hence we see that we have two complex eigenvalues with negative real part which means we have

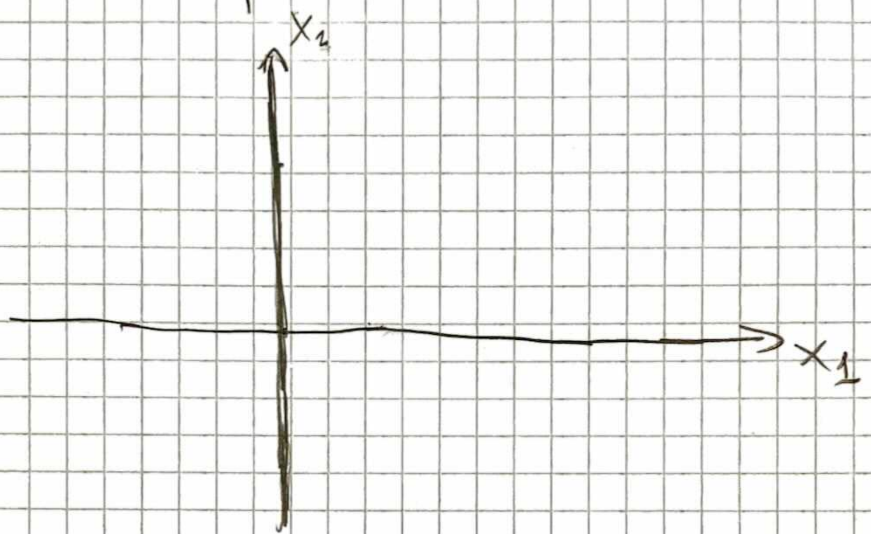
<u>Stable Spiral</u>

Test exercise 2

Consider the system

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = x_1 x_2$$

Analyze invariance of the axes.



- We first analyze the x_2-axis. If $x_1 = 0$ then

$$\dot{x}_1 = 0$$
$$\dot{x}_2 = 0$$

Hence we conclude that the x_2-axis is invariant because if we first are there we can't escape.

- Next we analyze the x_1-axis. We set $x_2 = 0$ and then

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = 0$$

We hence can conclude that also the x_1-axis is invariant because we can't escape.

Exercise 1:

Consider

$$\dot{x}_1 = -x_1(1-x_1)$$

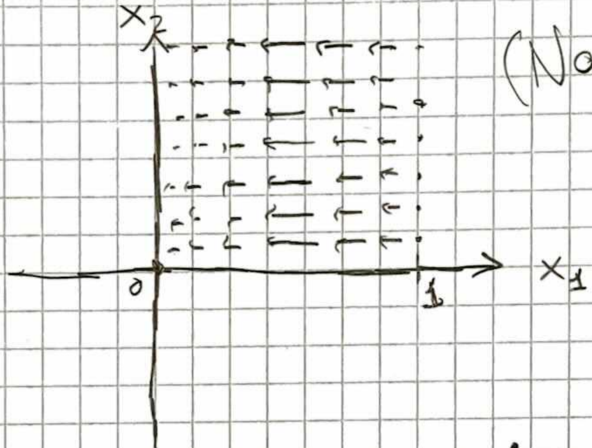
$$\dot{x}_2 = -x_2$$

Show the following statements geometrically:

A) The set

$$S_1 = \{(x_1, x_2); x_1 \in (0,1)\}$$

is invariant?



(No $\dot{x}_2$ component)

We see that when $x_1 = 0$ then $\dot{x}_1 = 0$ and for $x_1 = 1$ then $\dot{x}_1 = 0$. Hence we can't cross these points in finite time so the set is invariant.

Exercise 1, con't

B) $S_2 = \{(x_1, x_2) : x_2 = 0\}$

Hence is the x_1-axis invariant?

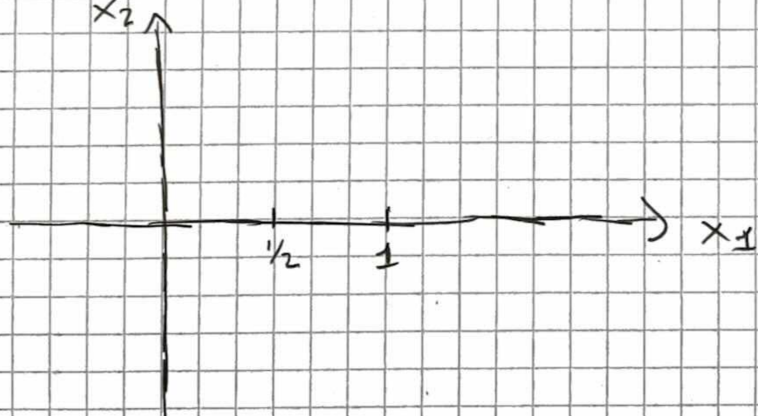
We analyze for $x_2 = 0$

$$\dot{x}_1 = -x_1(1-x_1)$$
$$\dot{x}_2 = 0$$

We dont have a change in the x_2 direction and hence is S_2 invariant.

C) $S_3 = \{(x_1, x_2) : x_1 < \frac{1}{2}\}$

is the set forward invariant?



Because there is no x_1 component in $\dot{x}_2$ so we need only to analyze $\dot{x}_1$.
The fixed points of $\dot{x}_1$ are 0 and 1 so we need to analyze the behaviour for $x_1 < 0$ and $\frac{1}{2} > x_1 > 0$
$\frac{1}{2} > x_1 > 0$ we know it will move to 0 in forward time from A).

Exercise 1 con't:

C) $x_1 < 0$ we see

$$\dot{x}_1 = -x_1(1-x_1) > 0 \quad \forall x_1 < 0.$$

We hence know all points will move to the x_2-axis ($x_1 = 0$) for forward time.

Hence the set is forward-invariant.

(Not backward invariant because $x_1 \in [0, \frac{1}{2})$ will move to $x_1 = 1$ in backward time)

D) are other sets invariant?

$$S_4 = \{(x_1, x_2) : x_1 = 0\} \quad , \quad S_5 = \{(x_1, x_2) : x_2 = 1\}$$

These are the two "x_2-axis" going through the fixed points

Exercise 2.: Consider

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

with f and g both C^1-functions. The x-nullcline is the subset of the xy-plane defined by $\dot{x}=0$ or simply

$$f(x,y)=0$$

Similarly, the y-nullcline is $\dot{y}=0$ or

$$g(x,y)=0$$

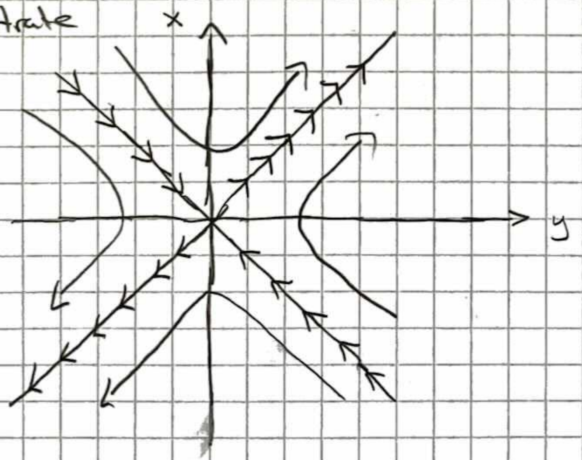
Decide the boolean value of:

a) Nullclines are always invariant sets?

No, because if we have some coupled behaviour it will not hold.

Ex $\quad \dot{x} = y \qquad \dot{y} = x \qquad$ Nullclines: $\dot{x}=0 \rightarrow y=0$, $\dot{y}=0 \rightarrow x=0$ [So we have that the axes are the nullclients]

We illustrate



So here we the nullclines are not forward or backward invariant.

FALSE

Exercise 2 con't:

(b) Nullclines are never invariant sets?

No if we have no coupled behaviour as in ex1 then the nullclines are invariant sets.

False

(c) Orbits intersect the x-nullcline vertically?

In the moment a flow/orbit intersects the x-nullcline, $\dot{x}=0$ so we do not have any horizontal movemement. Hence only vertical movement be left. So True

(d) Orbits intersects the y-nullcline vertically?

At these points $\dot{y}=0$ so only horizontal behaviour will be left.

FALSE

(e) If an x-nullcline and a y-nullcline intersects at $\mathbf{x}$ then $\mathbf{x}$ is a fixed point?

Yes beause $\dot{x}=0$, $\dot{y}=0$

TRUE

Exercise 3: Find $S = W^s$ and $U = W^u$ for

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = x_2 + x_1^2 \qquad (1)$$

Show that S and U are tangent to E^s and E^u respectively, for the linerized system.

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = x_2$$

- First we calculate the stable and unstable subspace E^s and E^u.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = -1 \quad \lambda_2 = 1$$

Where we have the eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence the x-axis makes up the stable subspace and the y-axis makes up the unstable subspace.

Next we proceed to the stable and unstable manifold W^s and W^u. We start by defining

$$x_1(0) = c_1 \quad \wedge \quad x_2(0) = c_2$$

We solve the first equation of (1)

$$x_1(t) = c_1 e^{-t}$$

Exercise 3 con't:

Next we solve equation 2 of (1)

$$\dot{x}_2 = x_2 + x_1^2$$
$$= x_2 + c_1^2 e^{-2t}$$

This is a inhomogenous linear differential equation and if one wants to see how to solve it step by step look in the Sloution for week3 (own solution). We will just state the answer here

$$x_2(t) = \left(c_2 + \frac{c_1^2}{3}\right)e^{t} - \frac{c_1^2}{3} e^{-2t}.$$

we hence have

$$\phi_t(c_1,c_2) = \begin{bmatrix} c_1 e^{-t} \\ \left(c_2 + \frac{c_1^2}{3}\right)e^{t} - \frac{c_1^2}{3} e^{-2t} \end{bmatrix}$$

To find W^s and W^u we consider the equation below where p is the fixed point $(0,0)$

$$\lim_{t \to \infty} \phi_t(c_1,c_2) = p$$

W^s goes to p in forward time and W^u goes to p in backward time. Hence it must hold for W^s that

$$c_2 + \frac{c_1^2}{3} = 0$$

meaning that the stable manifold follows the line

$$W^s = \left\{(c_1, c_2) : c_2 = \frac{-c_1^2}{3}\right\} \equiv \left\{(x(t), y(t)) : y(t) = \frac{-x(t)^2}{3}\right\}$$

Exercise 3 con't:

Next we find W^u for which it must hold that

$$C_1 = 0$$

meaning

$$W^u = \{(C_1, C_2) : C_1 = 0\} \equiv \{(x(t), y(t)) : x_1(t) = 0\}$$

We have now found the manifolds and now find the nullclines

x-nullcline: $\dot{x} = 0 \Rightarrow$

$0 = -x_1 \Rightarrow$

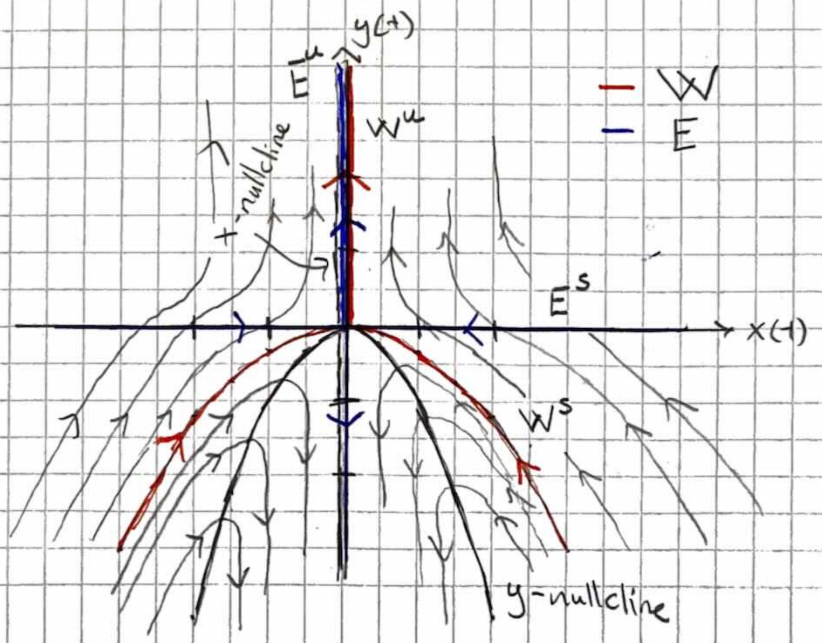
$\underline{\underline{x_1 = 0}}$

y-nullcline: $\dot{y} = 0 \Rightarrow$

$0 = x_2 + x_1^2 \Rightarrow$

$x_2 = -x_1^2 \Rightarrow$

We can now plot everything



Exercise 3 con't:

Lastly we must show that W^s is tangent to E^s in the fixed point $p=(0,0)$. (and for W^u and E^u).

Two curves $y=f(x)$ and $y=g(x)$ are tangent at $x=a$ if and only if

- They intersect at $f(a)=g(a)$
- Their tangent lines have equal slope: $f'(a)=g'(a)$

We start with the stable case

E^s is the x-axis so $f(x)=0$

W^s is the line $y(t)=-x(t)^2/3$ so $g(x)=-x^2/3$

and $f'(x)=0$ and $g'(x)=-2x/3$.

We test $f(0)=0 \wedge g(0)=0$ ✓

$f'(0)=0 \wedge g'(0)=0$ ✓

Unstable case:

Here E^u is the y-axis which means $f(x)=\infty$ but we can talk us out of this one. W^u is namely also the y-axis but only $y\in[0,\infty)$ but because they both exists at zero they are tangent there.

Exercise 4:

Classify the fixed points of

$$\dot{x} = \begin{bmatrix} x_1 - x_1 x_2 \\ x_2 - x_1^2 \end{bmatrix} = f(x)$$

We factorize

$$\dot{x} = \begin{bmatrix} x_1(1 - x_2) \\ x_2 - x_1^2 \end{bmatrix}$$

We see we have fixed points at $P_1 = (0,0)$, $P_2 = (1,1)$, $P_3 = (-1,1)$

We find the Jacobian of $f(x)$

$$Df(x) = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix} = \begin{bmatrix} 1 - x_2 & -x_1 \\ -2x_1 & 1 \end{bmatrix}$$

We test $P_1 = (0,0)$

$$Df(P_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$, hence a unstable node

We test $P_2 = (1,1)$

$$Df(P_2) = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$$, we calculate the char. poly.

$$\det \begin{bmatrix} -\lambda & -1 \\ -2 & 1-\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - (-1)(-2)$$

$$= \lambda^2 - \lambda - 2$$

We solve for roots

$d = b^2 - 4ac$ $\qquad x = \frac{-b \pm \sqrt{d}}{2a}$

Exercise 4, con't:

$$d = (-1)^2 - 4 \cdot 1 \cdot (-2) = 1 + 8 = 9$$

$$\lambda = \frac{1 \pm 3}{2} \Rightarrow \lambda_1 = 2 \wedge \lambda_2 = -1$$

So a saddel.

We test $P_3 = (-1, 1)$

$$Df(P_3) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix},$$ we find the char. poly.

$$\text{Det} \begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - (2 \cdot 1)$$

$$= \lambda^2 - \lambda - 2$$

This is the same as for P_2 so also a saddel.

We draw our found approx.