Week 1: 02953 Convex Optimization

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May 13, 2023

A1.8

a) Determine if $\{(x, y) \in \mathbf{R}^2_{++} \mid x/y \leq 1\}$ is convex

Solution We start by rewriting the set.

$$\{(x, y) \in \mathbf{R}^2_{++} \mid x/y \leq 1\} \Rightarrow \{(x, y) \in \mathbf{R}^2_{++} \mid x \leq y\}$$

We can now plot the set.

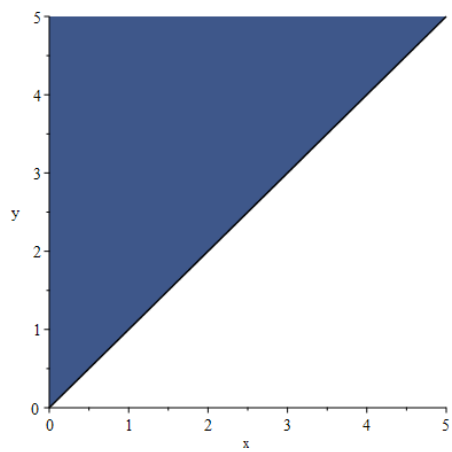

Figure 1: The set $\{(x, y) \in \mathbf{R}^2_{++} \mid x/y \leq 1\}$ is plotted in blue

We see from figure 1 that the set is convex.

b) Determine if $\{(x, y) \in \mathbf{R}^2_{++} \mid x/y \geq 1\}$ is convex

Solution We start by rewriting the set

$$\{(x, y) \in \mathbf{R}^2_{++} \mid x/y \geq 1\} \Rightarrow \{(x, y) \in \mathbf{R}^2_{++} \mid x \geq y\}$$

We can now plot the set.

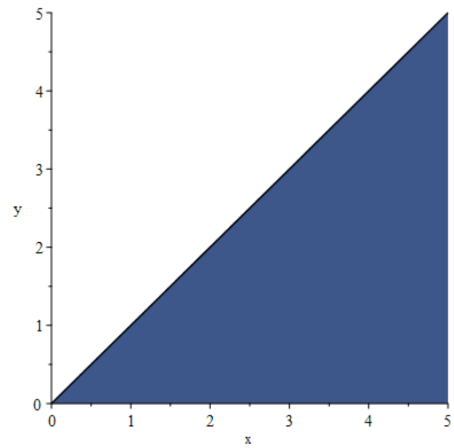

Figure 2: The set $\left\{(x, y) \in \mathbf{R}^2_{++} \mid x/y \geq 1\right\}$ is plotted in blue

We see from figure 2 that the set is convex.

c) Determine if $\left\{(x, y) \in \mathbf{R}^2_{++} \mid xy \leq 1\right\}$ is convex

Solution We start by rewriting the set

$$\left\{(x, y) \in \mathbf{R}^2_{++} \mid xy \leq 1\right\} \Rightarrow \left\{(x, y) \in \mathbf{R}^2_{++} \mid x \leq \frac{1}{y}\right\}$$

We can now plot the set.

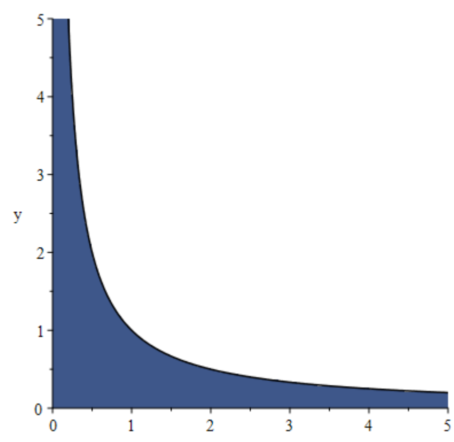

Figure 3: The set $\left\{(x, y) \in \mathbf{R}^2_{++} \mid xy \leq 1\right\}$ is plotted in blue

We see from figure 3 that the set is not convex.

d) Determine if $\left\{(x,y) \in \mathbf{R}^2_{++} \mid xy \geq 1\right\}$ is convex

Solution We start by rewriting the set

$$\left\{(x,y) \in \mathbf{R}^2_{++} \mid xy \geq 1\right\} \Rightarrow \left\{(x,y) \in \mathbf{R}^2_{++} \mid x \geq \frac{1}{y}\right\}$$

We can now plot the set.

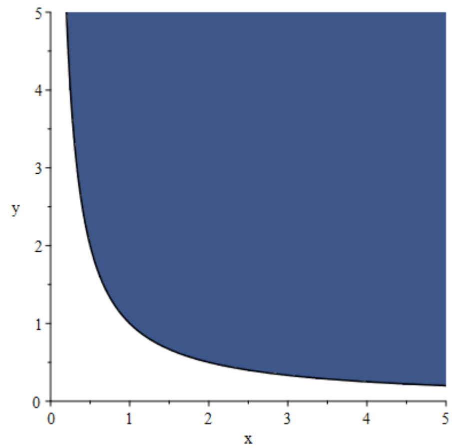

Figure 4: The set $\left\{(x,y) \in \mathbf{R}^2_{++} \mid xy \geq 1\right\}$ is plotted in blue

We see from figure 4 that the set is convex.

A2.3

Logarithmic barrier for the second-order cone. The function $f(x,t) = -log(t^2 - x^T x)$, with $dom(f) = \{(x,t) \in R_n \times R \quad | \quad t > |x|_2\}$ is convex. We are to show that f is convex.

a) Explain why $t - (1/t)u^T u$ is a concave function on $dom(f)$.

Solution Cf. the textbook p. 72, $f(x,y) = \frac{x^2}{y}, \quad y > 0$ is convex. Hence, $-\frac{u^T u}{t}$ is concave. $f(x,y) = t$ is concave (and convex), hence $f(x,t) = t + (-\frac{u^T u}{t})$ is the sum of concave functions, and is therefore also concave.

b) Show that $-log(t - (1/t)u^T u)$ is a convex function on $dom(f)$.

Solution We know that $t > ||u||_2$, hence $\frac{u^T u}{t} < t$ which implies that $t - \frac{u^T u}{t} > 0$. Cf. the textbook p. 86, if g is concave and positive, then $log(g)$

is concave. Therefore $log(t - \frac{u^T u}{t})$ is concave, which means that $-log(t - \frac{u^T u}{t})$ is convex.

c) Show that f is convex.

Solution We re-write $log(t^2 - x^T x)$ as $log(t - \frac{x^T x}{t}) + log(t)$. We have just shown that $log(t - \frac{x^T x}{t})$ and $log(t)$ is concave. Hence $log(t^2 - x^T x)$ is the sum of 2 concave function and is therefore also concave. This means that $-log(t^2 - x^T x)$ must be convex.

A3.3

a) Rewrite `norm([x + 2*y, x - y]) == 0`

Solution We can rewrite it as

i. `x + 2*y == 0`
ii. `x - y == 0`

because the $l_2 - norm$ of a vector is only zero if the vector element-wise is zero.

b) Rewrite `square(square(x + y)) <= x - y`

Solution CVX can not recognize that `square(square())` will always be positive. We therefore rewrite

i. `(x + y)^4 <= x - y`

CVX recognizes `()^4` so now it works.

c) Rewrite `1/x + 1/y <= 1; x >= 0; y >= 0`

Solution CVX can not recognize that x in `1/x` will never get negative and the same for y. We therefore use the CVX function `inv_pos()` which encodes this.

i. `inv_pos(x) + inv_pos(y) <= 1;`
ii. `x >= 0;`
iii. `y >= 0;`

d) Rewrite `norm([max(x,1), max(y,2)]) <= 3*x + y`

Solution CVX cannot eat `norm(max(),max())` so we introduce two dummy variables `t1,t2` and add two constraints.

i. `max(x,1) >= t1`
ii. `max(y,2) >= t2`
iii. `norm([x,y]) <= 3*x + y`

e) Rewrite `x*y >= 1; x >= 0; y >= 0`

Solution CVX cannot see that `xy` will always be positive so rewrite.

i. `x >= inv_pos(y);`
ii. `x >= 0;`
iii. `y >= 0`

f) Rewrite `(x + y)^2/sqrt(y) <= x - y + 5`

Solution We use CVX's own `quad_over_lin()` function.

i. `quad_over_lin(x+y,sqrt(y))<=x-y+5`

g) Rewrite `x^3 + y^3 <= 1; x >= 0; y >= 0`

Solution An uneven power can be negative but we only have positive variables. We therefore encode this information.

i. `pow_pos(x,3) + pow_pos(y,3) <= 1;`
ii. `x >= 0;`
iii. `y >= 0`

h) Rewrite `x + z <= 1 + sqrt(x*y - z^2); x >= 0; y >= 0`

Solution We can rewrite by

i. `x + z <= 1 + geo_mean([x - quad_over_lin(z,y), y]);`
ii. `x >= 0;`
iii. `y >= 0`

Code that tests the constraints is attached.

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We are given the linear program

$$\begin{array}{rl} \text{minimize}_x & x_1 + 2x_2 \\ \text{subject to} & x_1 - x_2 \leq 0 \\ & -x_1 + \alpha x_2 \leq -1 \end{array} \tag{1}$$

We want the optimal value as a function of α and for which values of α the problem is unbounded?

We first derive the dual of 1.

$$\begin{aligned} g(\lambda) &= \inf_{x\in\mathcal{D}} \mathcal{L}(x,\lambda) \\ &= \inf_{x\in\mathcal{D}} x_1 + 2x_2 + \lambda_1(x_1 - x_2) + \lambda_2(\alpha x_2 - x_1 + 1) \\ &= \inf_{x\in\mathcal{D}} x_1(1 + \lambda_1 - \lambda_2) + x_2(2 - \lambda_1 + \alpha\lambda_2) + \lambda_2 \\ &= \begin{cases} \lambda_2 & 1 + \lambda_1 - \lambda_2 = 0 \wedge 2 - \lambda_1 + \alpha\lambda_2 = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

We can now write the dual

$$
\begin{array}{rl}
\text{minimize}_\lambda & \lambda_2 \\
\text{subject to} & 1 + \lambda_1 - \lambda_2 = 0 \\
& 2 - \lambda_1 + \alpha\lambda_2 = 0 \\
& \lambda_1, \lambda_2 \geq 0
\end{array} \tag{2}
$$

We observe that λ_1 and λ_2 can be fully determined from from the constraints. We get

$$\lambda_1 = -\tfrac{\alpha+2}{\alpha-1} \quad \lambda_2 = -\tfrac{3}{\alpha-1}$$

We hence have that the optimal value for dual is determined by

$$d^* = \begin{cases} -\tfrac{3}{\alpha-1} & \alpha \in [-2, 1) \\ \text{Infeasible} & \text{otherwise} \end{cases}$$

We know that for a LP $p^* = d^*$ except when the primal and the dual are infeasible. When the dual is infeasible the primal is unbounded. When $\alpha = 1$ we see that the two constraints of 1 are parallel but in opposite directions. Hence their normal vectors will point in opposite direction and because the two lines do not lie on top of each other the program is infeasible. We therefore have

$$p^* = \begin{cases} -\tfrac{3}{\alpha-1} & \alpha \in [-2, 1) \\ \text{Infeasible} & \alpha = 1 \\ -\infty & \text{otherwise} \end{cases}$$

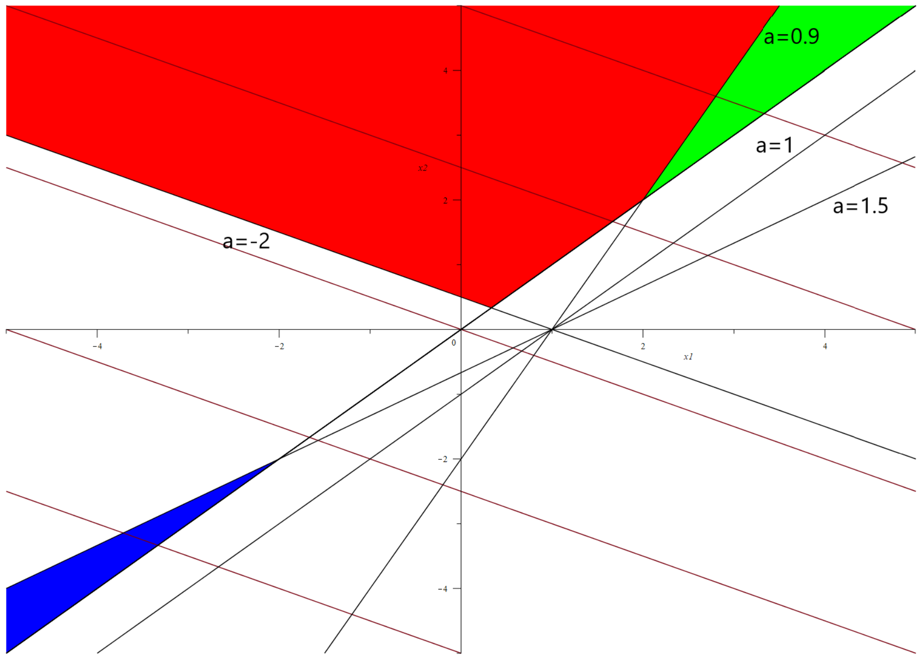

Figure 5: Illustration of 1 with different values of α. We see that the constraint rotates around (1,0)

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We are given the semi-definite program

$$\begin{array}{ll} \text{minimize}_{x,Z} & \operatorname{tr}(A_0 Z) + b_0^T x \\ \text{subject to} & \operatorname{tr}(A_i Z) + b_i^T x + c_i = 0, \quad i = 1, \ldots, m \\ & x \in \mathbf{R}_+^n, Z \in \mathbf{S}_+^p \end{array} \tag{3}$$

We want the dual of 3 but before deriving that we will rewrite 3

$$\begin{array}{ll} \text{minimize}_{x,Z} & \operatorname{tr}(A_0 Z) + b_0^T x \\ \text{subject to} & \operatorname{tr}(A_i Z) + b_i^T x + c_i = 0, \quad i = 1, \ldots, m \\ & -x \leq 0 \\ & -Z \preceq 0 \end{array} \tag{4}$$

We now derive the dual

$$
\begin{aligned}
g(\lambda, \nu, W) &= \inf_{x, Z \in \mathcal{D}} \mathcal{L}(x, Z, \lambda, \nu, W) \\
&= \inf_{x, Z \in \mathcal{D}} \operatorname{tr}(A_0 Z) + b_0^T x + \left(\sum_{i=1}^{m} \lambda_i \left(\operatorname{tr}(A_i Z) \right) + b_i^T x + c_i \right) - \nu^T x - \operatorname{tr}(WZ) \\
&= \inf_{x, Z \in \mathcal{D}} \left(-\nu + \left(\sum_{i=1}^{m} \lambda_i b_i \right) + b_0 \right)^T x + \operatorname{tr}\left(\left(\left(\sum_{i=1}^{m} \lambda_i A_i \right) - W + A_0 \right) Z + \left(\sum_{i=1}^{m} \lambda_i c_i \right) \right. \\
&= \begin{cases} \lambda^T c & \left(\sum_{i=1}^{m} \lambda_i A_i\right) - W + A_0 = 0 \wedge \left(-\nu + \left(\sum_{i=1}^{m} \lambda_i b_i\right) + b_0\right)^T = 0 \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

We can now write the dual program

$$
\begin{array}{ll}
\text{minimize}_{\lambda, \nu, W} & \lambda^T c \\
\text{subject to} & \left(\sum_{i=1}^{m} \lambda_i A_i\right) - W + A_0 = 0 \\
& \left(-\nu + \left(\sum_{i=1}^{m} \lambda_i b_i\right) + b_0\right)^T = 0 \\
& \lambda \geq 0 \\
& \nu \geq 0 \\
& W \succeq 0
\end{array}
\tag{5}
$$