02424 - week2, Likelihood theory

ex 1

a) Cf. slide 21/60. $f_{\underline{Y}}(y_i;\mu) = \frac{1}{(\sqrt{2\pi})^{12}} \exp\left(-\sum_{i=1}^{12} \frac{(y_i-\mu)^2}{2}\right)$

b) Cf. slide 22 ^ 24 $\bar{y}$ is a sufficient statistic for the mean and hence

$$f_{\underline{Y}}(y_i;\mu) = \frac{1}{(\sqrt{2\pi})^{12}} \exp\left(\frac{12\cdot(\bar{y}-\mu)^2}{2}\right) \cdot \text{const.}$$

c) solved in R

NB! Der er en fejl i a) og b) da σ^2 er antaget lig 1 men man skal antage observed var i y.

02424 - week 2

ex2)

a) - likelihood function

$$L(y_1,\dots,y_n;\lambda) = \prod_{i=1}^{n} \frac{\lambda^{y_i}\exp(-\lambda)}{y_i!}$$

- log-likelihood function

$$\log\text{-}L(y_1,\dots,y_n;\lambda) = \sum_{i=1}^{n} \log\left(\frac{\lambda^{y_i}\exp(-\lambda)}{y_i!}\right)$$

$$= -n\lambda + \log(\lambda)\sum_{i=1}^{n} y_i + \log\left(\prod_{i=1}^{n} y_i!\right)$$

$$= -n\lambda + \log(\lambda)\sum_{i=1}^{n} y_i + \text{const.}$$

- score-function

$$\frac{d}{d\lambda}\log\text{-}L(y_1,\dots,y_n;\lambda) = -n + \frac{\sum_{i=1}^{n} y_i}{\lambda}$$

- observed information

$$j(y;\lambda) = \frac{d^2}{d\lambda^2}\log\text{-}L(y_1,\dots,y_n;\lambda) = -\frac{\sum_{i=1}^{n} y_i}{\lambda^2}$$

b) We derive the MLE

$$\frac{d}{d\lambda}\log\text{-}L(y_1,\dots,y_n;\lambda) = 0 \Rightarrow \hat{\lambda} = \frac{\sum_{i=1}^{n} y_i}{n}$$

We calculate the observed information using $\hat{\lambda}$

$$j(y;\hat{\lambda}) = -\frac{n^2}{\sum_{i=1}^{n} y_i}$$

02424 - week2

ex4

samples	vol	concentration
1	1	λ bact/ml
2	10	

bactiria count of sample 1: $Y_1 \sim Pois(\lambda)$

bacterria count of sample 2: $Y_2 \sim Pois(\lambda/10)$

a) Consider the estimator

$$T_1 = \frac{Y_1 + 10Y_2}{2}$$

- Varify that T_1 is unbiased, i.e. that $\mathbb{E}[T_1] = \lambda$.

$$\mathbb{E}[T_1] = \mathbb{E}\left[\frac{Y_1 + 10Y_2}{2}\right] = \frac{\mathbb{E}[Y_1] + 10\,\mathbb{E}[Y_2]}{2}$$

$$= \frac{\lambda + 10\,\lambda/10}{2} = \lambda$$

- Estimate the variance

$$Var[T_1] = Var\left[\frac{Y_1 + 10Y_2}{2}\right] \overset{\text{(indpendent)}}{=}$$

$$Var\left[\tfrac{1}{2}Y_1\right] + Var\left[5Y_2\right] =$$

$$\tfrac{1}{4}Var[Y_1] + 25\,Var[Y_2] =$$

$$\tfrac{1}{4}\cdot\lambda + 25\cdot\frac{\lambda}{10} = {}^{11}/_{4}\,\lambda$$

- compared to

$$Var[Y_1] = \lambda$$

02424 - week2

ex4 con't

b) we now consider an unbiased estimator

$$T_\omega = \omega Y_1 + (1-\omega) 10 Y_2 \quad , \text{ with } 0 \le \omega \le 1$$

Derive $Var[T_\omega]$

$$Var[T_\omega] = Var[\omega Y_1 + (1-\omega) 10 Y_2]$$

$$\stackrel{(\text{independence})}{=} Var[\omega Y_1] + Var[(1-\omega) 10 Y_2]$$

$$= \omega^2 Var[Y_1] + 100 (1-\omega)^2 Var[Y_2]$$

$$= \omega^2 \lambda + (1-\omega)^2 10 \lambda$$

we now minimize w.r.t. ω

$$\frac{d}{\partial \omega} Var[T_\omega] = 2\omega\lambda - 20(1-\omega)\lambda$$

$$\frac{d}{d\omega} Var[T_\omega] = 0 \Rightarrow \omega^* = 10/11$$

We now find the variance of ω^*

$$Var[T_{\omega^*}] = {}^{10}/_{11} \lambda$$