ex 1

$$\dot{x} = xy - x^3 + xy^2 \qquad (*)$$

$$\dot{y} = -y + x^2 + x^2 y$$

x is the center subspace | $F(x,y) = xy - x^3 + xy^2$

y is the stable subspace | $G(x,y) = x^2 + x^2 y$

Hence $A = 0$, $B = -1$

We have to determine $y = h(x)$ to order $O(x^4)$ hence

$$y = h(x) = ax^2 + bx^3 + cx^4 + dx^5 + O(x^6)$$

and

$$Dh(x) = 2ax + 3bx^2 + 4cx^3 + 5dx^4 + O(x^5)$$

We now use the equation

$$Dh(x)\dot{x} - \dot{y} = 0 \quad \Rightarrow$$

$$Dh(x)[Ax + F(x,h(x))] - Bh(x) - G(x,h(x)) = 0$$

We insert

$$[a + 2bx + 3cx^2 + 4dx^3 + 5ex^4 + O(x^5)] \cdot [x \cdot h(x) - x^3 + x(h(x))^2]$$

$$+ h(x) - x^2 - x^2 h(x) = 0$$

If we expand this and let $O(x^6)$ absorb all higher terms we get

$$(a-1)x^2 + bx^3 + (c - 3a + 2a^2)x^4 + (5ab - 3b + d)x^5 = 0$$

hence we get $a = 1$, $b = 0$, $c = 1$ and $d = 0$. We can then conclude that our 5'th ~~orde~~ order approx to W^c is

$$y = h(x) = x^2 + x^4$$

We can now approx the vector field on the manifold by $(*)$

$$\dot{x} = xh(x) - x^3 + xh(x)^2 = 2x^5 + 2x^7 + x^9 \approx 2x^5$$

ex2

Compute the center manifold to order $O(x^3)$ of

$$\dot{x} = 10(y-x) + x^2$$

$$\dot{y} = -y + x - \left(\tfrac{1}{11}x + \tfrac{10}{11}y\right)^2$$

We see that the system is coupled and to use the center manifold thm. we must decouple the system

$$Df(0,0) = \begin{bmatrix} -10 & 10 \\ 0 & -1 \end{bmatrix}$$

$$\Lambda = P^{-1} Df\, P$$

$$= \begin{bmatrix} 1 & -10 \\ 1 & 1 \end{bmatrix}^{-1} Df \begin{bmatrix} 1 & -10 \\ 1 & 1 \end{bmatrix}$$

(where the first eigenvector has $\lambda = 0$ and hence x will be E^c)

Our total system will now follow dynamics by the transform.

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} u \\ v \end{bmatrix} \iff \begin{bmatrix} u \\ v \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = P^{-1} Df\, P \begin{bmatrix} u \\ v \end{bmatrix} + P^{-1} \begin{bmatrix} f(x(u,v), y(u,v)) \\ g(x(u,v), y(u,v)) \end{bmatrix}$$

[See maple sheet for details], The new system becomes

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -11 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1/11(u+10v)^2 - 10/11\, u^2 \\ -1/11(u-10v)^2 - 1/11\, u^2 \end{bmatrix}$$

We can now proceed with the form

$$\dot{u} = Au + f(u,v) \qquad (E^c)$$

$$\dot{v} = Bv + g(u,v) \qquad (E^s)$$

$v = h(u) = au^2 + O(u^3)$ and we have the tangency condition

$$\dot{v} = Dh(u)\dot{u}$$

We insert

$$Bh(u) + g(u,h(u)) = Dh(u)\left[Au + f(u,h(u))\right]$$

We simplify and let $O(u^3)$ absorb all higher order terms

[The rest is done in maple]