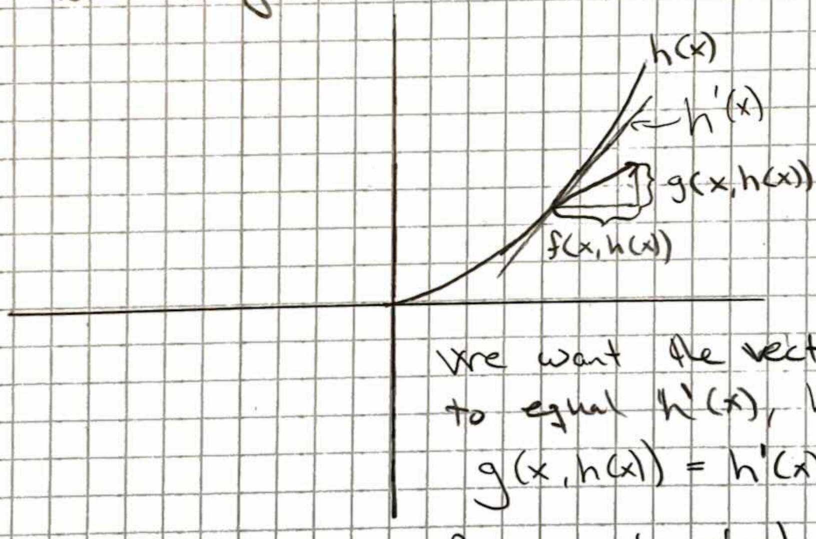
Test exercise 1 - Consider the system

$$\dot{x} = -x \quad = f(x,y)$$
$$\dot{y} = y + x^3 = g(x,y)$$

The origin is a saddel, and the y-axis is the unstable manifold. In addition, there exists a real number k so that the stable manifold is the graph of the function $y = h(x)$, with $h(x) = kx^3$. Find k.

We will use the invariance equation from week 4, slides. We will shortly derive it here. We know that for an invariant manifold all vectors in the phase field must be tangent.



We want the vector slope $\frac{g(x,h(x))}{f(x,h(x))}$ to equal $h'(x)$, hence

$$g(x,h(x)) = h'(x)\, f(x,h(x))$$

for an invariant manifold.

We plug-in.

$h'(x) = 3kx^2$ $\quad f(x,h(x)) = -x \quad , \quad g(x,h(x)) = kx^3 + x^3$

So

$$kx^3 + x^3 = 3kx^2(-x) \quad \Rightarrow \quad \underline{\underline{k = -\tfrac{1}{4}}}$$

Test exercise 2: Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field, and that x^* is the particular kind of hyperbolic equilirium point know as a sink. The stable manifold then gaurentes what?

We know that a sink is a stable node and because such a fixed point is hyperbolic we know f follows the same dynamics as the linerization in a small neighboorhood U. If x^* is the only fixed point it would hold for all of $\mathbb{R}^n$ but this we are not gurenteed. Hence

option 2

Exercise 1: Hartman-Grobman establish local topological equivalence between the non-linear system and its linerization around a hyperbolic fixed point. Local here refers to the fact that the set U, when the homeomorphism in the HG thm is defined is a small set.

Determine if each of the following statements is true or false.

a) 2D system

$$f: \begin{aligned} \dot{x} &= x + y + xy \\ \dot{y} &= -y + x^2 y \end{aligned}$$

is topological equivalent in a neighborhood around the origin with

$$g: \begin{aligned} \dot{x} &= x + y \\ \dot{y} &= -y \end{aligned}$$

— We linerize

$$Df(0,0) = \begin{bmatrix} 1+y & 1+x \\ 2xy & -1+x^2 \end{bmatrix}(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$Dg(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

We see they have the same linerization and hence $H(x) = Ix$ in the HG thm

Ex 1 con't:

(b)

$$f: \begin{aligned} \dot{x} &= -x - 3y + x^6 \\ \dot{y} &= -3x - y + x^2 y^{1964} \end{aligned}$$

$$g: \begin{aligned} \dot{x} &= -x - 3y \\ \dot{y} &= -3x - y \end{aligned}$$

- we can see the linerized system is the same hence top. equal. **TRUE**

(c)

$$f: \begin{aligned} \dot{x} &= -x - 3y + xy^{666} \\ \dot{y} &= -3x - y + x^2 y^{1992} \end{aligned}$$

$$g: \begin{aligned} \dot{x} &= 2x \\ \dot{y} &= -4y \end{aligned}$$

We linerize

$$Df(0,0) = \begin{bmatrix} -1 & -3 \\ -3 & -1 \end{bmatrix}$$

$$Dg(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$$

We find the eigenvalues of $Df(0,0)$ and see they are 2 and -4. Hence the orthonormal basis for $Df(0,0)$ is the matrix in $H(x) = \underline{\underline{A}}\,x$

Exercise 1 con't:

f: $\dot{x} = x + xy$

$\dot{y} = -y + x^2 y$

g: $\dot{x} = -x$

$\dot{y} = y$

$$Df(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Df(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

So the matrix in $H(x) = \underline{\underline{A}}x$ is $-I$. So true

Exersise 2: Consider

$$\dot{x} = Ax \qquad (1)$$

and

$$\dot{y} = By \qquad (2)$$

with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$. Suppose that A and B are similar: There exists an invertible $V \in \mathbb{R}^{n \times n}$ such that

$$A = V^{-1} B V$$

Show that (1) and (2) are topological conjugate.

- It follows from example 1 is sec. 2.8 in the book. $\square$