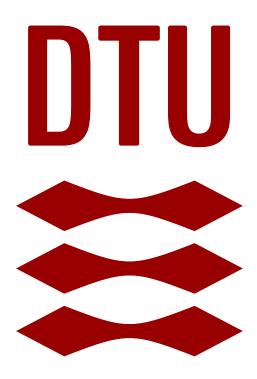
Week 2: 01125 Fundamental Topological Concepts and Metric Spaces

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3.17

Proof by contradiction. We assume $\bigcap_{n=1}^{\infty} K_n = \emptyset$. We then know via De Morgan's Laws:

$$M \setminus (\cap_{n=1}^{\infty} K_n) = \cup_{n=1}^{\infty} (M \setminus K_n)
\Rightarrow
M \setminus \emptyset = M
\Leftrightarrow
\cup_{n=1}^{\infty} (M \setminus K_n) = M$$

 $\{M\backslash K_n|n\in \mathbf{N}\}$ are all open subsets covering M, in particular covering K_1 . K_1 is compact and therefore it has a finite sub-covering:

$$K_1 \subseteq \bigcup_{n=1}^S (M \backslash K_n)$$

Since

$$\begin{split} \mathbf{K}_1 \supseteq \mathbf{K}_2 \supseteq \mathbf{K}_3 \supseteq ... \supseteq \mathbf{K}_S &\Leftrightarrow \mathbf{M} \backslash \mathbf{K}_1 \subseteq \mathbf{M} \backslash \mathbf{K}_2 \subseteq ... \subseteq \mathbf{M} \backslash \mathbf{K}_S \\ &\Rightarrow \\ \mathbf{M} \backslash \mathbf{K}_S \supseteq \mathbf{K}_1 \supseteq \mathbf{K}_S \\ &\Rightarrow \\ \mathbf{M} \backslash \mathbf{K}_S \supseteq \mathbf{K}_S \end{split}$$

Hereby we have a contradiction because for $M \setminus K_S \supseteq K_S$ to be true $M = \emptyset$, and in the exercise we are told the space is non-empty.

3.20(2)

Direct proof. Assume f is uniformly continuous in K, then it holds:

$$\forall \gamma > 0 \ \exists \delta : \forall x, y \ d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \gamma$$

We further know that all points, $x \in K$, will lie in some open ball with its center at some x_0 and with radius ϵ , such that:

$$d(x_n, x) < \epsilon$$

We can now choose $\delta=\epsilon$ and from here it follows via the definition of uniform continuity that:

$$d(x_n, x) < \delta = \epsilon \implies d(f(x_n), f(x)) < \gamma$$

We now have a finite covering of the image set $f(K) \in Y$ of open balls with centers in $f(x_1), ..., f(x_p)$ and all radius γ . Therefore f(K) must by definition be precompact.

3.29

1)

Proof by contradiction. Assume $\exists x_0, y_0 \in M$ for which $x_0 \neq y_0$ and $Tx_0 = x_0$ and $Ty_0 = y_0$. Hence it follows

$$d(x_0, y_0) = d(Tx_0, Ty_0)$$

From the definition of a weak contraction it follows that

$$d(Tx_0, Ty_0) < d(x_0, y_0)$$

From the assumptions this leads to the contradiction

$$d(x_0, y_0) = d(Tx_0, Ty_0) < d(x_0, y_0)$$

Hence, the assumptions were wrong. It then follows that T does not have two or more fixed points, which implies it has at most one fixed point.

2)

Proof by example. Define $T: M \to M$ to be $Tx = x + \frac{1}{x}$ for $x \ge 1$ where $x \in \mathbb{R}$. Let y be another point in the same interval. By the usual metric in R we have that d(x,y) = |x-y|. At first we want to show that $Tx = x + \frac{1}{x}$ is a weak contraction.

$$d(Tx,Ty) = |Tx - Ty| = |(x + \frac{1}{x}) - (y + \frac{1}{y})| = |x - y| \cdot |1 - \frac{1}{x \cdot y}|$$

Since $x,y\geq 1$ we know that $0<\frac{1}{x\cdot y}\leq 1$. Then it follows that $0\leq |1-\frac{1}{x\cdot y}|<1$. Then we have that

$$d(Tx, Ty) < d(x, y) \ \forall x, y \in M \ \text{for} \ x \neq y$$

Hence, $Tx = x + \frac{1}{x}$ for $x \ge 1$ for \mathbb{R} is a weak contraction. However, since Tx = x would imply that x = 0, the weak contraction $Tx = x - \frac{1}{x}$ does not have a fixed point.