DTU Compute

02425 Diffusions and SDEs
September 12, 2022
UHT/uht

Exercise for session 3: Stochastic processes

Question 1 Simulation of Brownian motion: (Compare exercise in the notes; p. 61). Implement a function which simulates Brownian motion on an interval $[0, T]$; for example as given in the notes: The function should take as input a partition $0 \le t_1 < \cdots < t_n = T$, and should compute and return $B_{t_1}, B_{t_2}, \ldots, B_{t_{n-1}}, B_T$. Test the function by simulating sufficiently many replicates of $(B_0, B_{1/2}, B_{3/2}, B_2)$ to verify the covariance of this vector, and the distribution of B_2. Save the function for future use.

Solution:

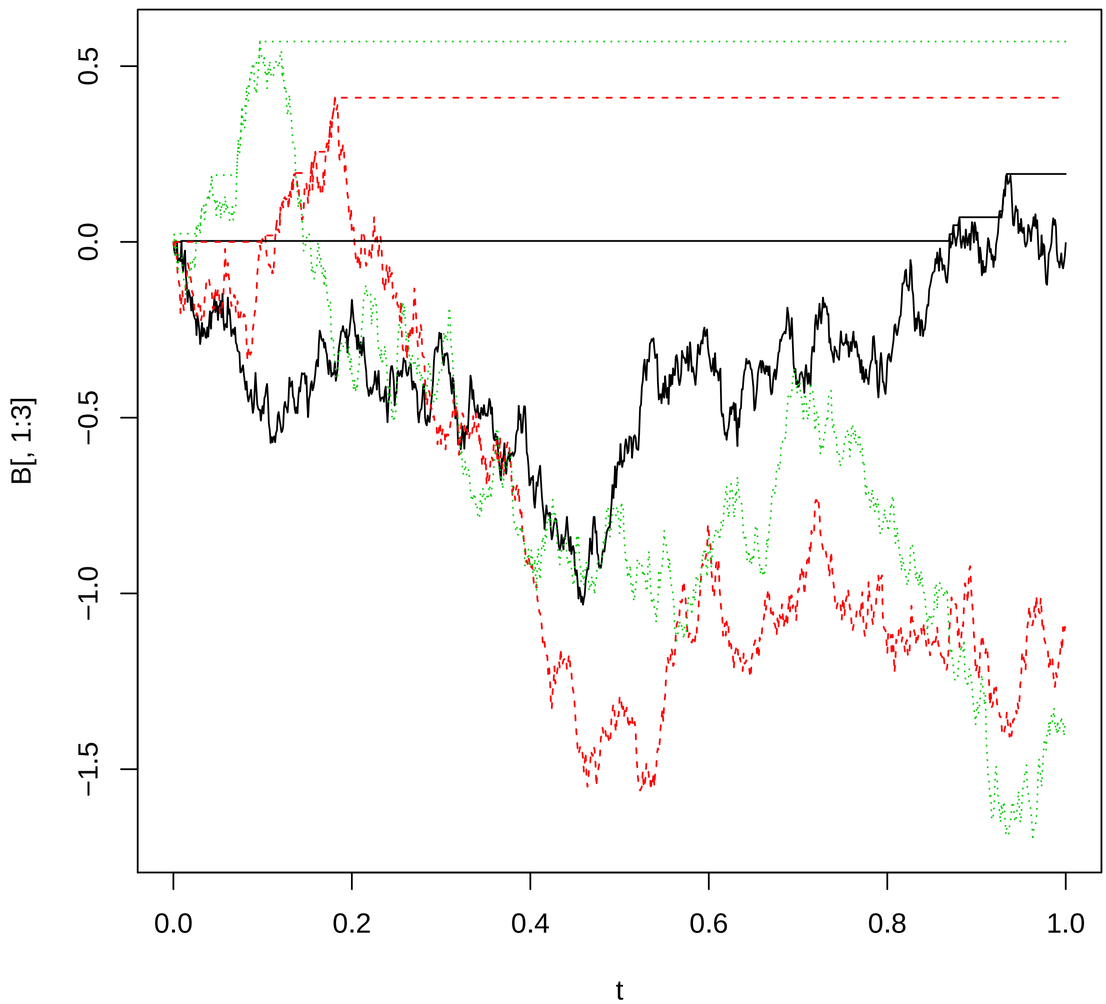
```
  rBM <- function(t) cumsum(rnorm(length(t),mean=0,sd=sqrt(diff(c(0,t)))))

  t <- c(0,0.5,1.5,2)
  N <- 1000

  B <- sapply(1:N,function(i)rBM(t))
  print(apply(B,1,mean))


## [1]  0.000000000  0.032113092  0.003370709 -0.024056772


  print(var(t(B)))


##      [,1]      [,2]      [,3]      [,4]
## [1,]    0 0.0000000 0.0000000 0.0000000
## [2,]    0 0.5032123 0.4668747 0.4736281
## [3,]    0 0.4668747 1.3574657 1.3923866
## [4,]    0 0.4736281 1.3923866 1.9439795
```

Question 2 Extrema of Brownian motion and hitting times:

1. Generate $N = 1000$ sample paths of Brownian motion on the time interval [0,1] using a time step of 0.001.
2. For each sample path $\{B_t : 0 \le t \le 1\}$, compute the maximum $S_1 = \max\{B_t : 0 \le t \le 1\}$. Plot the histogram of S_1 (or the empirical c.d.f.) and compare with the theoretical distribution of S_1.
3. For each sample path $\{B_t : 0 \le t \le 1\}$, compute the hitting time $\tau = \min\{t : B_t \ge b\}$ with $b = 0.5$. *Note:* If the sample path does not hit b in the time interval $[0, 1]$, then define $\tau = 1$. Plot the histogram of τ (or the emprirical c.d.f.) and compare with the theoretical distribution.

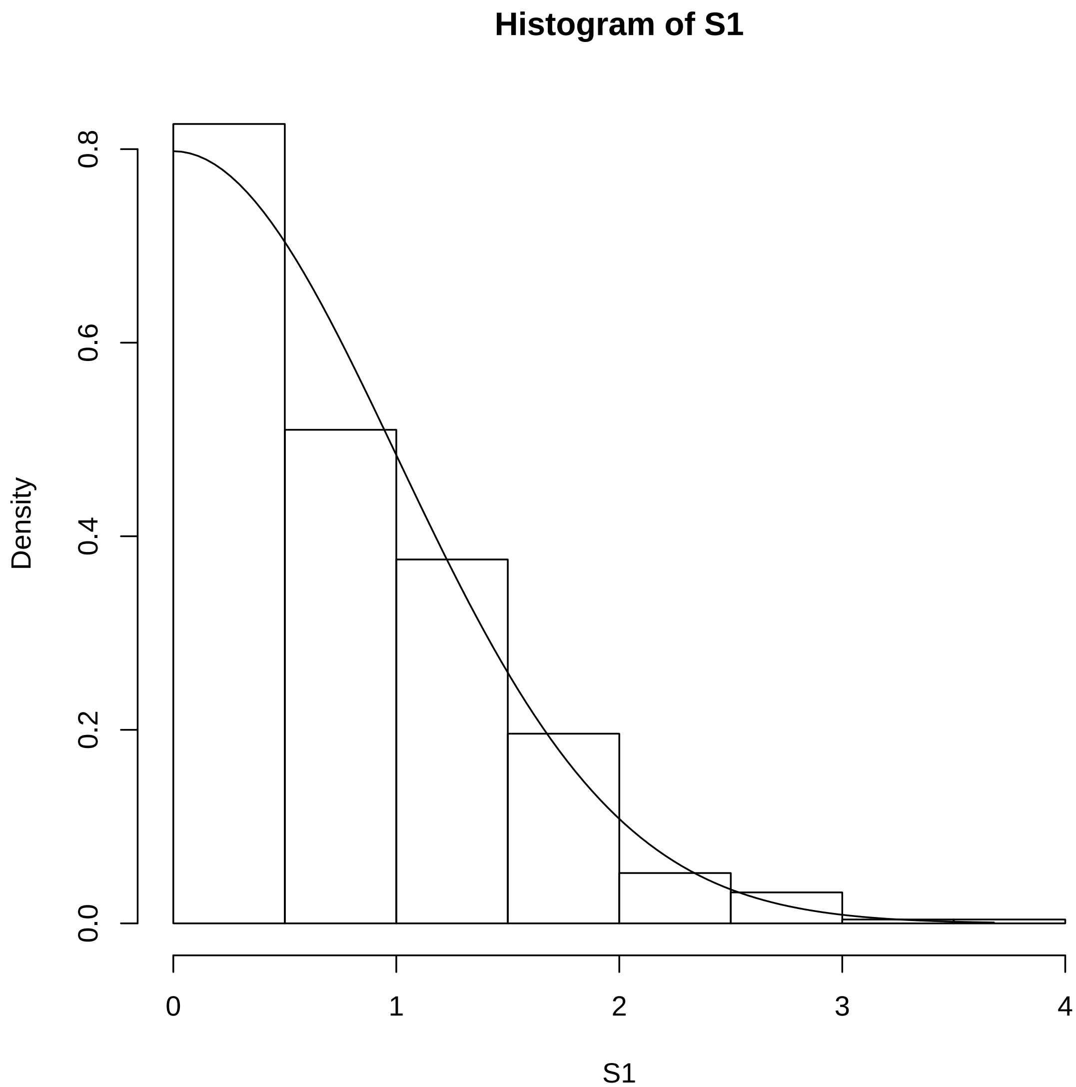
Solution: We first generate the sample paths and show the running max (even if it is not asked for).

```
N <- 1000
t <- seq(0,1,0.001)
B <- sapply(1:N,function(i)rBM(t))
S <- apply(B,2,cummax)
S1 <- apply(B,2,max)
matplot(t,B[,1:3],type="l")
matplot(t,S[,1:3],type="l",add=TRUE)
```



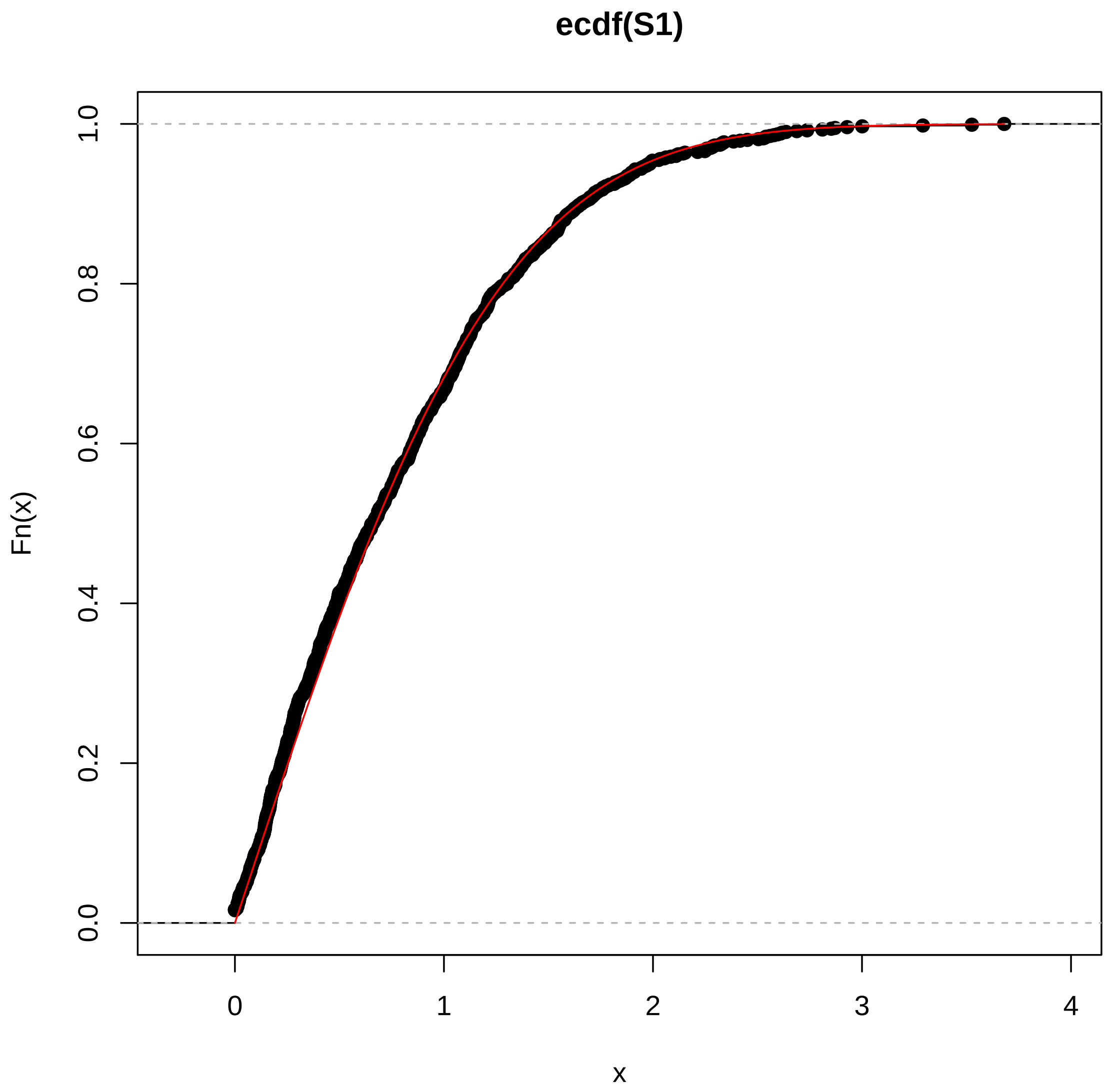
We plot the histogram and compare with the pdf:

```
hist(S1,freq=FALSE)
Spdf <- function(x) 2*dnorm(x)
plot(Spdf,add=TRUE,from=0,to=max(S1))
```



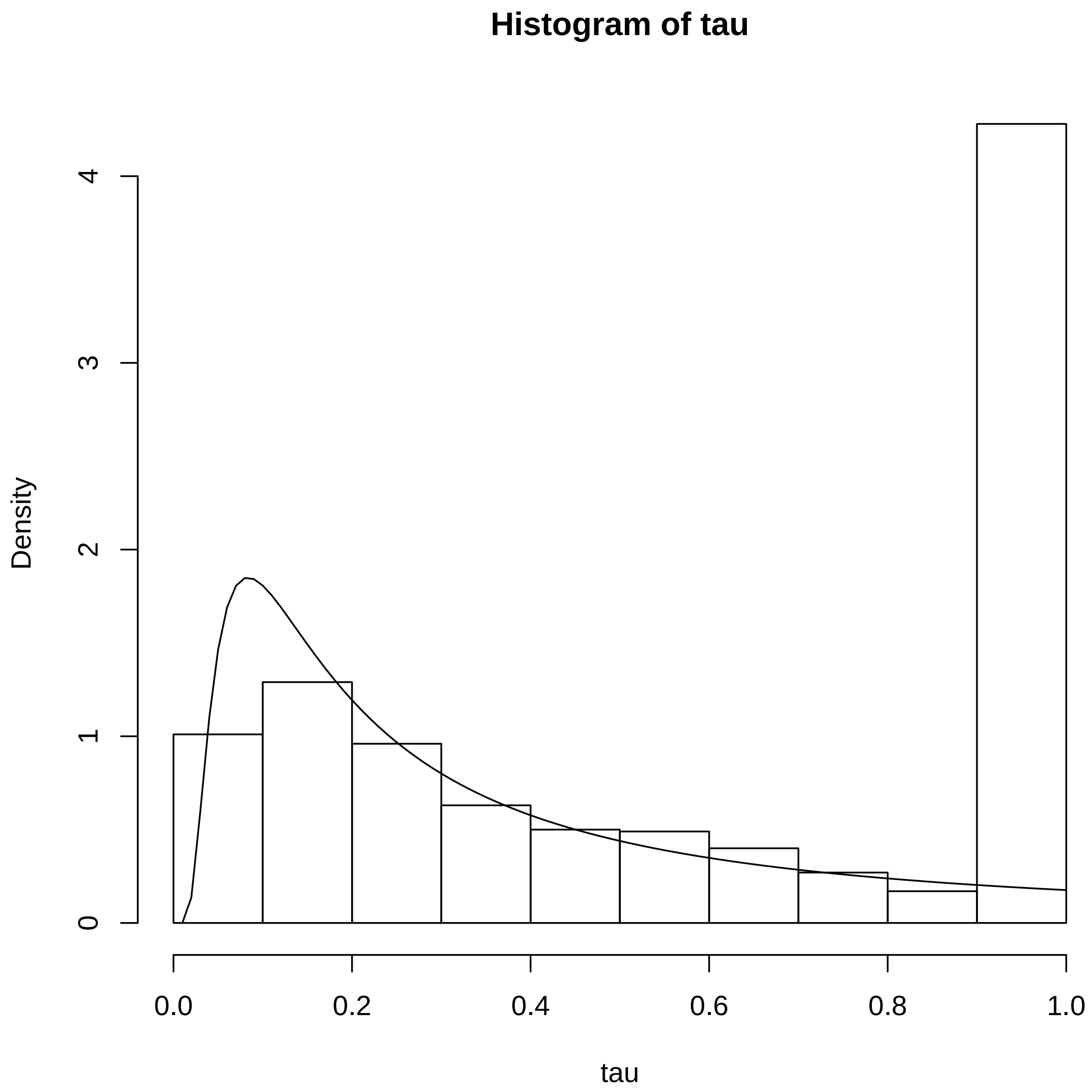
We also do this for the empirical and theoretical cdf:

```
plot(ecdf(S1))
Scdf <- function(x) 2*pnorm(x)-1
plot(Scdf,add=TRUE,from=0,to=max(S1),col="red")
```



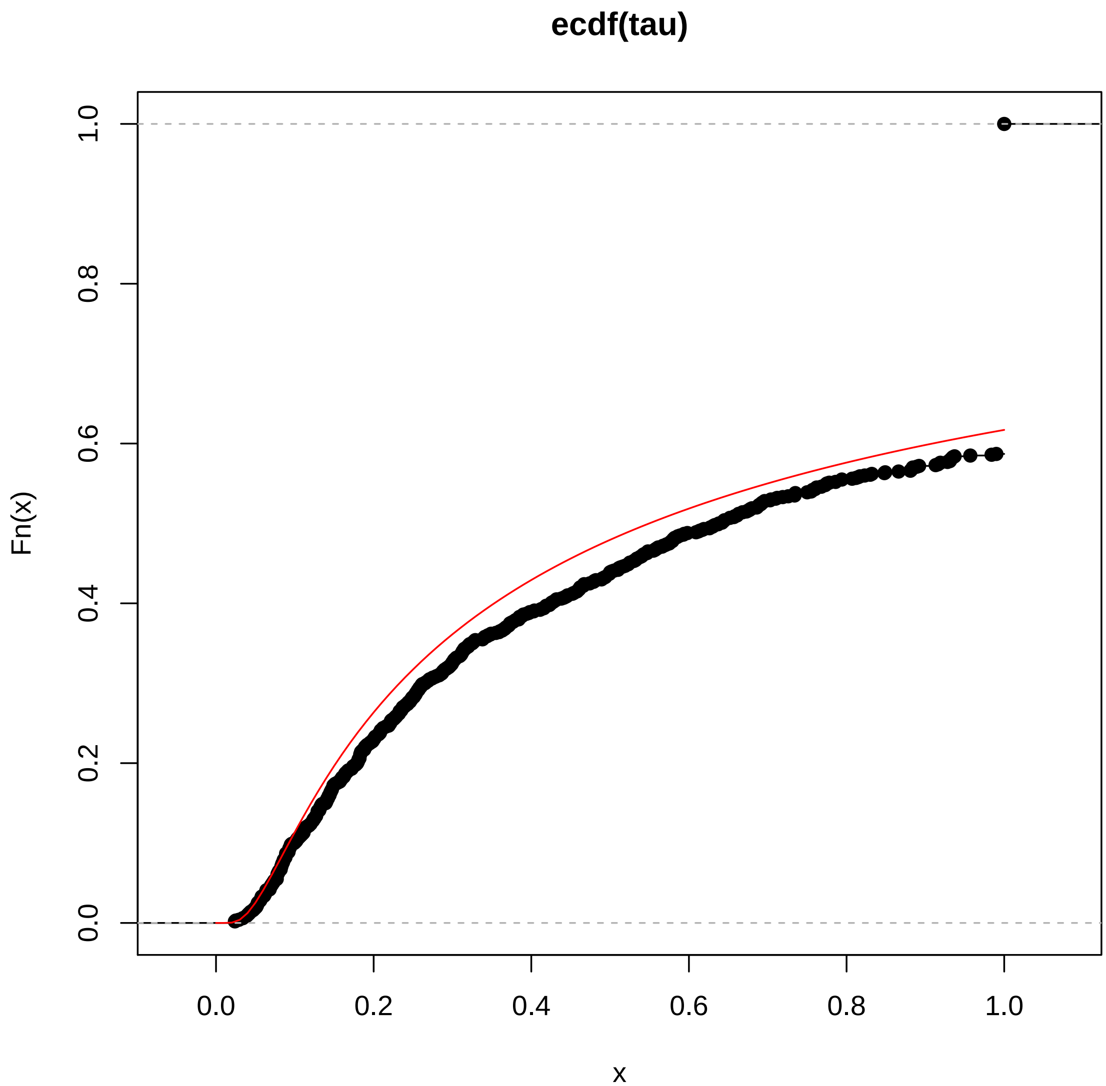
We compute *tau* for each sample path (a little coarsely) and plot the empirical and theoretical p.d.f.:

```
s <- 0.5
tau <- apply(S,2,function(x)t[sum(x<s)])
hist(tau,freq=FALSE)
taucdf <- function(t)2-2*pnorm(s/sqrt(t))
taupdf <- function(t)dnorm(s/sqrt(t))*s*t^(-3/2)
curve(taupdf,add=TRUE,from=0,to=max(tau))
```



... repeat for empirical and theoretical c.d.f.:

```
plot(ecdf(tau))
curve(taucdf,from=0,to=max(tau),add=TRUE,col="red")
```



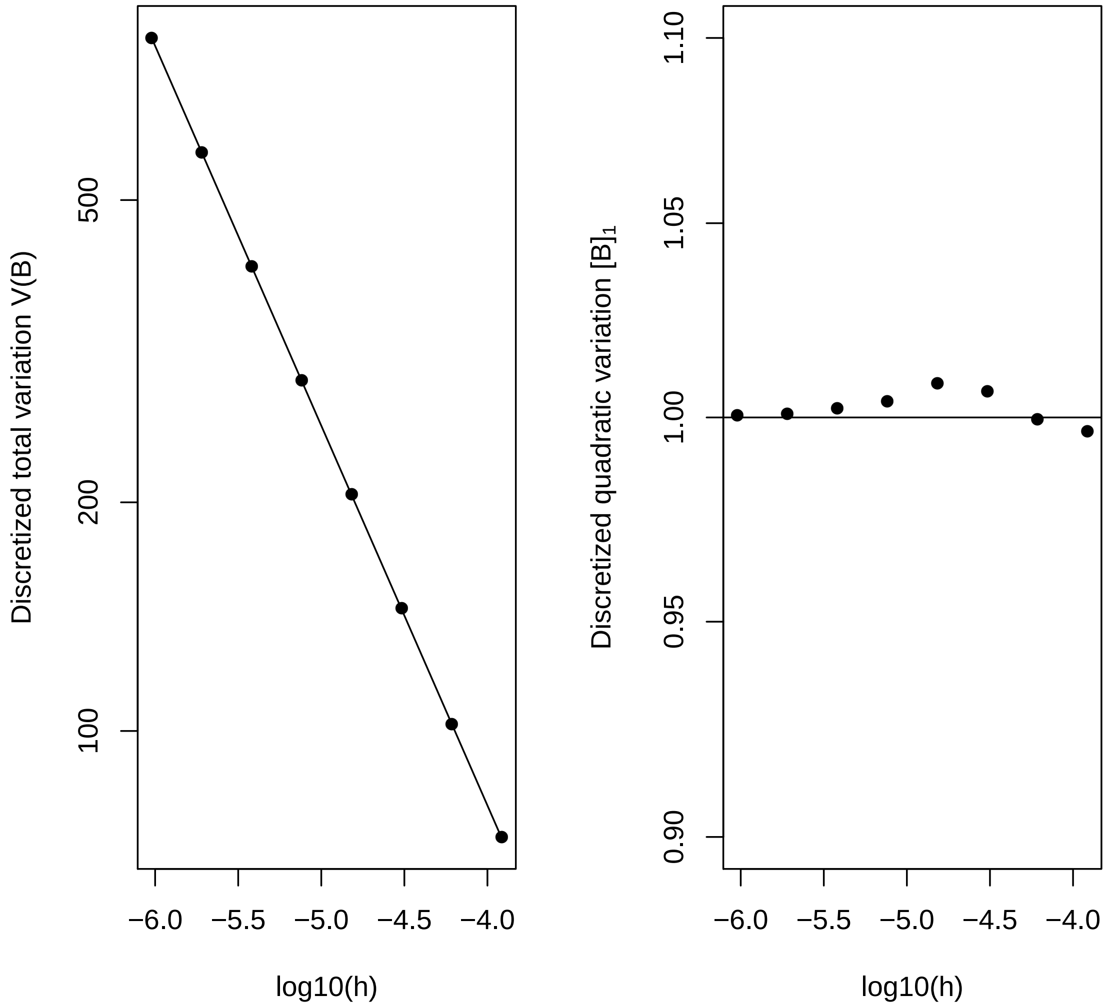
Question 3 **Total and quadratic variation of Brownian motion:** Reproduce figure 4.3 (page 74) in the notes:

1. Generate one sample path of standard Brownian motion on the time interval $[0, 1]$ using a time step of 2^{-20}.
2. Compute the discretized total variation $V_\Delta = \sum |\Delta B|$ and quadratic variation $[B]_1 = \sum |\Delta B|^2$.
3. Subsample the Brownian motion at every other time step.
4. Repeat the previous steps until you have reached a time step of 2^{-10}.
5. Plot the discretized total variation and quadratic variation as function of the time step in double logarithmic plot.

Include the analytical predictions in the graphs, using exercise 4.34 (page 63) for the total variation. *Optional:* Solve exercise 4.34 so that you know where the analytical prediction comes from.

Solution:

```
T <- 1

N <- 2^20
h <- T/N

Ndouble <- 8

B <- rBM(seq(0,T,h))
dB <- diff(B)

var <- array(0,c(Ndouble,2))

for(i in 1:Ndouble)
    {
        var[i,1] <- sum(abs(dB))
        var[i,2] <- sum(dB^2)
        dB <- apply(array(dB,c(2,length(dB)/2)),2,sum)
    }

hs <- h*2^(1:Ndouble)/2

par(mfrow=c(1,2))
plot(log10(hs),var[,1],xlab='log10(h)',ylab="Discretized total variation V(B)",pch=16,log="y")
lines(log10(hs),sqrt(2/pi/hs))
plot(log10(hs),var[,2],xlab='log10(h)',ylab=expression("Discretized quadratic variation [B]"[1]),ylim=c
abline(h=1)
```



Basic Martingales

Question 4: Show that $\{B_t^2 - t\}$ is a martingale (exercise 4.9 in the notes; p. 83).

Solution: See the solution in the notes.

Question 5: Solve exercise 4.20 in the notes (p. 91) concerning *Doob's martingale.*

Solution: See the solution in the notes.

Question 6: Solve exercise 4.11 (p. 86) concerning the increasing variance of a martingale. *Extra:* When we apply this result to Doob's martingale (exercise 4.20), we conclude that the variance of the estimator increases as we accumulate information. Does this sound obvious or counter-intuitive to you? In the latter case, think carefully about how it should be understood; for example by decomposing the variance of X according to the information $\mathcal{F}_t$.

Solution: See the solution in the notes for the first part. For the "extra" question regarding Doob's martingale: We see that the variance of the estimator increases with time. To many, this sounds counter-intuitive; they would expect that the estimator becomes more precise with time and therefore that the variance decreases. This seeming paradox stems from a confusion about what is the variance of an estimator.

We can write $X = M_t + \tilde{X}_t$, where $\tilde{X}_t$ is the estimation error. An observer at time t knows M_t but does not know $\tilde{X}_t$. Since M_t and X_t are uncorrelated, we have

$$\mathbf{V}X = \mathbf{V}M_t + \mathbf{V}\tilde{X}_t$$

and since the variance $\mathbf{V}M_t$ is increasing, the variance of the estimation error $\mathbf{V}\tilde{X}_t$ is decreasing. This agrees with intuition. As for the variance of the estimator, $\mathbf{V}M_t$, note that this is the *prior* variance in M_t.

To expand on this distinction, note that $\mathbf{V}\tilde{X}_t = \mathbf{E}\mathbf{V}\{\tilde{X}_t|\mathcal{F}_t\} = \mathbf{E}\mathbf{V}\{X|\mathcal{F}_t\}$ so that the observer expects to know X with less and less variance as time progresses. Note that this is only an expectation; there may be sample paths where $\mathbf{V}\{X|\mathcal{F}_t\}$ is increasing (compare **??** to construct such examples).