

A NOTE ON QUANTIFIER ELIMINATION IN SHELAH-SPENCER GRAPHS

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ABSTRACT. We simplify [1]’s proof of quantifier elimination in Shelah-Spencer graphs.

1. INTRODUCTION

Laskowski’s paper [1] provides a combinatorial proof of quantifier elimination in Shelah-Spencer graphs. Here we provide a simplification of the proof using only maximal chains and avoiding the use of proposition 3.1 and technical lemmas of section 4.

We will use notation of [1], in particular things like \mathbf{K}_α , $\delta(\mathcal{A}/\mathcal{B})$, $X_m(\mathcal{A})$, S_α , $\mathcal{B}^* \sqsubseteq \mathcal{B}'$, maximal embedding, $\Delta_{\mathcal{A}}(x)$, $\Psi_{\mathcal{A},\mathcal{B}}(x)$ etc. However we will give a different definition of $Y(\dots)$. When we write formulas $\theta(x, y)$ we may have x, y to be tuples.

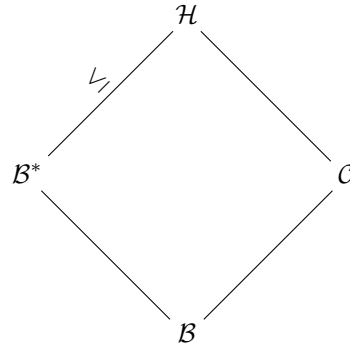
2. OMITTING LEMMA

Definition 2.1. Let $\mathcal{M} \models S_\alpha$, $\mathcal{B} \in \mathbf{K}_\alpha$, embedding $f: \mathcal{B} \rightarrow \mathcal{M}$, Φ finite subset of \mathbf{K}_α

- (1) Say that f *omits* Φ if there are no $\mathcal{C} \in \Phi$ and $g: \mathcal{C} \rightarrow \mathcal{M}$ extending f .
- (2) Say that f *admits* Φ if for every $\mathcal{C} \in \Phi$ there is $g: \mathcal{C} \rightarrow \mathcal{M}$ extending f .

Note 2.2. Take notation as above and a structure $\mathcal{C} \in \mathbf{K}_\alpha$ extending \mathcal{B} . Then f doesn’t omit $\{\mathcal{C}\}$ iff f admits $\{\mathcal{C}\}$.

Definition 2.3. Fix $\mathcal{B}, \mathcal{C} \in \mathbf{K}_\alpha$, and $m \in \omega$ such that $|C \setminus B| < m$. Define $Z(\mathcal{B}, \mathcal{C}, m)$ to be all $\mathcal{B}^* \in X_m(\mathcal{B})$ such that there are no \mathcal{H} with $|H \setminus B^*| < m$ satisfying



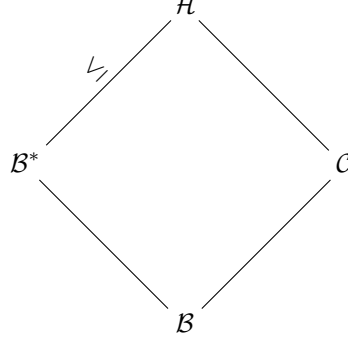
Lemma 2.4. Let $\mathcal{B}, \mathcal{C} \in \mathbf{K}_\alpha$, and $m \in \omega$ such that $|C \setminus B| < m$. Also let $\mathcal{M} \models S_\alpha$ and $f: \mathcal{B} \rightarrow \mathcal{M}$ an embedding. The following are equivalent:

- (1) f *omits* $\{\mathcal{C}\}$.

(2) *There exists $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into \mathcal{M} over f .*

Proof. For the proof we identify \mathcal{B} with $f(\mathcal{B})$, i.e. for ease of notation assume that $\mathcal{B} \subset \mathcal{M}$.

(1) \Rightarrow (2) By remark 5.3 of [1] there is some $B^* \in X_m(\mathcal{B})$ maximally embeddable in \mathcal{M} over f . Such embedding is unique by Lemma 3.8 of [1]. Again, we identify B^* with its maximal embedding into \mathcal{M} . To show (2) we need to verify that $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$. Suppose not. Then there is \mathcal{H} with $|H \setminus B^*| < m$ satisfying



As $\mathcal{B}^* \leq \mathcal{H}$ and $\mathcal{B} \subset \mathcal{M}$ we can embed \mathcal{H} into \mathcal{M} (as $\mathcal{M} \models S_\alpha$). But this would witness \mathcal{C} extending \mathcal{B} in \mathcal{M} which is impossible as we assumed that f omits $\{\Phi\}$.

(2) \Rightarrow (1) Suppose f doesn't omit $\{\mathcal{C}\}$. Then by the note 2.2 f admits $\{\mathcal{C}\}$, i.e. there is an embedding of \mathcal{C} into \mathcal{M} over f . We identify \mathcal{C} with the image of that embedding. Similarly we identify \mathcal{B}^* with the image of its maximal embedding over f . That is we may assume $\mathcal{C}, \mathcal{B}^* \subset \mathcal{M}$. Let H be the substructure of \mathcal{M} induced by vertices $\mathcal{C} \cup \mathcal{B}^*$. As $|\mathcal{C} \setminus \mathcal{B}| < m$ we have $|H \setminus \mathcal{B}^*| < m$. \mathcal{B}^* is m -strong by remark 5.3 of [1]. This forces $\mathcal{B}^* \leq H$. But this contradicts the fact that $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$. \square

Corollary 2.5. *With the setup of the previous lemma, the following are equivalent:*

- (1) f admits $\{\mathcal{C}\}$.
- (2) *There exists $\mathcal{B}^* \in X_m(\mathcal{B}) \setminus Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into \mathcal{M} over f .*

For quantifier elimination we need to track multiple structures being admitted and omitted, hence the following definition.

Definition 2.6. Let $\mathcal{B} \in \mathbf{K}_\alpha$, Φ, Γ finite subsets of \mathbf{K}_α , and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|\mathcal{C} \setminus \mathcal{B}| < m$. Define

$$Y(\mathcal{B}, \Phi, \Gamma, m) = \{B^* \in X_m(\mathcal{B}) \mid \forall \mathcal{C} \in \Phi \ B^* \in Z(\mathcal{B}, \mathcal{C}, m) \text{ and } \forall \mathcal{D} \in \Gamma \ B^* \notin Z(\mathcal{B}, \mathcal{D}, m)\}$$

Lemma 2.7. *Let $\mathcal{B} \in \mathbf{K}_\alpha$, Φ, Γ finite subsets of \mathbf{K}_α , and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|\mathcal{C} \setminus \mathcal{B}| < m$. The following are equivalent:*

- (1) f omits Φ and admits Γ .
- (2) *There exists $\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$ maximally embeddable into \mathcal{M} over f .*

Proof. Easy corollary of 2.4 and 2.5. \square

3. QUANTIFIER ELIMINATION

Following proof of 5.6 in [1], we have a formula $\theta(x, y)$, some $\mathcal{A} \subseteq \mathcal{B} \in \mathbf{K}_\alpha$ with $\theta(x, y) \vdash \Delta_{\mathcal{A}}(x) \wedge \Delta_{\mathcal{B}}(x, y)$. We may also assume that $\theta(x, y)$ is a conjunction of formulas of the type $\Psi_{\mathcal{B}, \mathcal{C}}(x, y)$ and their negations. More precisely

$$\theta(x, y) \Leftrightarrow \bigwedge_{\mathcal{C} \in \Phi} \Psi_{\mathcal{B}, \mathcal{C}}(x, y) \wedge \bigwedge_{\mathcal{D} \in \Gamma} \neg \Psi_{\mathcal{B}, \mathcal{D}}(x, y)$$

for some finite subsets Φ, Γ of \mathbf{K}_α . Let $m = \max\{|C \setminus B| : \mathcal{C} \in \Phi \text{ or } \mathcal{C} \in \Gamma\}$. We claim that in $\mathcal{M} \models S_\alpha$

$$\begin{aligned} \exists y \theta(x, y) &\Leftrightarrow \bigvee_{\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)} (\mathcal{B}^* \text{ maximally embeds into } \mathcal{M} \text{ over } \mathcal{A}) \\ &\Leftrightarrow \bigvee_{\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)} \left(\Psi_{\mathcal{A}, \mathcal{B}^*}(x) \wedge \bigwedge_{\mathcal{B}^* \sqsubseteq \mathcal{B}', \mathcal{B}' \in X_m(\mathcal{B})} \neg \Psi_{\mathcal{A}, \mathcal{B}'}(x) \right) \end{aligned}$$

Proof. (\Rightarrow) Fix $\mathcal{B} \subset \mathcal{M}$ witnessing existential statement. By remark 5.3 and lemma 3.8 There is a unique $\mathcal{B}^* \in X_m$ (uniquely embedded) maximally embeddable into \mathcal{M} over \mathcal{B} . By lemma 2.7 $\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$.

(\Leftarrow) Take the embedding $g: \mathcal{B}^* \rightarrow \mathcal{M}$ and restrict it to $\mathcal{B} \subseteq \mathcal{B}^*$ i.e. $f = g \upharpoonright \mathcal{B}$. As $\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$ by lemma 2.7 f omits Φ and admits Γ . Thus is is a witness to $\exists y \theta(x, y)$. \square

REFERENCES

- [1] Michael C. Laskowski, *A simpler axiomatization of the Shelah-Spencer almost sure theories*, Israel J. Math. **161** (2007), 157-186. MR MR2350161
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