VC-DENSITY FOR TREES

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ABSTRACT. We show that for the theory of infinite trees we have $\mathrm{vc}(n) = n$ for all n.

VC-density was studied in [1] by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko as a natural notion of dimension for NIP theories. In an NIP theory we can define a vc-function

$$\mathrm{vc} : \mathbb{N} \longrightarrow \mathbb{N}$$

Where $\mathrm{vc}(n)$ measures the worst-case complexity of definable sets in an n-dimensional space. Simplest possible behavior is $\mathrm{vc}(n) = n$ for all n. Theories with the property that $\mathrm{vc}(1) = 1$ are known to be dp-minimal, i.e. having the smallest possible dp-rank. In general, it is not known whether there can be a dp-minimal theory which doesn't satisfy $\mathrm{vc}(n) = n$.

In this paper we work with trees viewed as posets. Parigot in [3] showed that such structures have NIP. This result was strengthened by Simon in [2] showing that trees are dp-minimal. The paper [1] poses the following problem:

Problem 0.1. ([1] p. 47) Determine the VC density function of each (infinite) tree.

Here we settle this question by showing any model of trees has $\mathrm{vc}(n) = n$.

Section 1 of the paper consists of a basic introduction to the concepts of VC-dimension and vc-density. In Section 2 we introduce proper subdividions – the main tool that we use to analyze trees. We also prove some basic properties of proper subdivisions. Section 3 introduces the key constructions of proper subdivisions in tree which will be used in the proof. Section 4 presents the proof of $\mathrm{vc}(n) = n$ via the subdivisions.

We use notation $a \in T^n$ for the tuples of size n. For a variable x or a tuple a we denote their arity by $|x|$ and $|a|$ respectively.

The language for the trees consists of a single binary predicate $\{\leq\}$. The theory of trees states that $\leq$ defines a partial order and for every element a we have that $\{x \mid x < a\}$ is a linear order. For visualization purposes we assume that trees grow upwards, with the smaller elements on the bottom and the larger elements on the top. If $a \leq b$ we will say that a is below b and b is above a.

Definition 0.2. Work in a tree $\boldsymbol{T} = (T, \leq)$. For $x \in T$ let $I(x) = \{t \in T \mid t \leq x\}$ denote all the elements below x. The *meet* of two tree elements a, b is the greatest element of $I(a) \cap I(b)$ (if one exists) and is denoted by $a \wedge b$.

The theory of meet trees requires that any two elements in the same connected component have a meet. Colored trees are trees with a finite number of colors added via unary predicates.

From now on assume that all trees are colored. We allow our trees to be disconnected (so really, we work with forests) or finite unless otherwise stated.

1. VC-dimension and vc-density

Definition 1.1. Throughout this section we work with a collection $\mathcal{F}$ of subsets of a set X. We call the pair $(X, \mathcal{F})$ a set system.

- Given a subset A of X, we define the set system $(A, A \cap \mathcal{F})$ where $A \cap \mathcal{F} = \{A \cap F\}_{F \in \mathcal{F}}$.
- For $A \subset X$ we say that $\mathcal{F}$ shatters A if $A \cap \mathcal{F} = \mathcal{P}(A)$.

Definition 1.2. We say $(X, \mathcal{F})$ has VC-dimension n if the largest subset of X shattered by $\mathcal{F}$ is of size n. If $\mathcal{F}$ shatters arbitrarily large subsets of X, we say that $(X, \mathcal{F})$ has infinite VC-dimension. We denote the VC-dimension of $(X, \mathcal{F})$ by $\mathrm{VC}(\mathcal{F})$.

Note 1.3. We may drop X from the previous definition, as the VC-dimension doesn't depend on the base set and is determined by $(\bigcup \mathcal{F}, \mathcal{F})$.

This allows us to distinguish between well-behaved set systems of finite VC-dimension which tend to have good combinatorial properties and poorly behaved set systems with infinite VC-dimension.

Another natural combinatorial notion is that of a dual system:

Definition 1.4. For $a \in X$ define $X_a = \{F \in \mathcal{F} \mid a \in F\}$. Let $\mathcal{F}^* = \{X_a\}_{a \in X}$. We define $(\mathcal{F}, \mathcal{F}^*)$ as the dual system of $(X, \mathcal{F})$. The VC-dimension of the dual system of $(X, \mathcal{F})$ is referred to as the dual VC-dimension of $(X, \mathcal{F})$ and denoted by $\mathrm{VC}^*(\mathcal{F})$. (As before, this notion doesn't depend on X.)

Lemma 1.5. *A set system has finite VC-dimension if and only if its dual system has finite VC-dimension. More precisely*

$$\mathrm{VC}^*(\mathcal{F}) \leq 2^{1+\mathrm{VC}(\mathcal{F})}.$$

For a more refined notion we look at the traces of our family on finite sets:

Definition 1.6. Define the shatter function $\pi_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ and the dual shatter function $\pi^*_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ by

$$\pi_{\mathcal{F}}(n) = \max\{|A \cap \mathcal{F}| \mid A \subset X \text{ and } |A| = n\}$$

$$\pi^*_{\mathcal{F}}(n) = \max\{\mathrm{atoms}(B) \mid B \subset \mathcal{F}, |B| = n\}$$

where $\mathrm{atoms}(B)$ = number of atoms in the Boolean algebra generated by B. Note that the dual shatter function is precisely the shatter function of the dual system: $\pi^*_{\mathcal{F}} = \pi_{\mathcal{F}^*}$.

A simple upper bound is $\pi_{\mathcal{F}}(n) \leq 2^n$ (same for the dual). If VC-dimension is infinite then clearly $\pi_{\mathcal{F}}(n) = 2^n$ for all n. Conversely we have the following remarkable fact:

Theorem 1.7 (Sauer-Shelah '72)**.** *If the set system $(X, \mathcal{F})$ has finite VC-dimension d then $\pi_{\mathcal{F}}(n) \leq \binom{n}{\leq d}$ where $\binom{n}{\leq d} = \binom{n}{d} + \binom{n}{d-1} + \ldots + \binom{n}{1}$.*

Thus the systems with a finite VC-dimension are precisely the systems where the shatter function grows polynomially. Define vc-density to be the degree of that polynomial:

Definition 1.8. Define vc-density and dual vc-density of $\mathcal{F}$ as

$$\mathrm{vc}(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}$$
$$\mathrm{vc}^*(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}$$

Generally speaking a shatter function that is bounded by a polynomial doesn't itself have to be a polynomial. Proposition 4.12 in [1] gives an example of a shatter function that grows like $n \log n$ (so it has vc-density 1).

So far the notions that we have defined are purely combinatorial. We now adapt VC-dimension and vc-density to the model theoretic context.

Definition 1.9. Work in a structure M. Fix a finite collection of formulas $\Phi(x, y) = \{\phi_i(x, y)\}$.

- For $\phi(x, y) \in \mathcal{L}(M)$ and $b \in M^{|y|}$ let
$$\phi(M^{|x|}, b) = \{a \in M^{|x|} \mid \phi(a, b)\} \subseteq M^{|x|}.$$
- Let $\Phi(M^{|x|}, M^{|y|}) = \{\phi_i(M^{|x|}, b) \mid \phi_i \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|})$.
- Let $\mathcal{F}_\Phi = \Phi(M^{|x|}, M^{|y|})$ giving a set system $(M^{|x|}, \mathcal{F}_\Phi)$.
- Define VC-dimension of Φ, $\mathrm{VC}(\Phi)$ to be the VC-dimension of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.
- Define vc-density of Φ, $\mathrm{vc}(\Phi)$ to be the vc-density of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.

We will also refer to the vc-density and VC-dimension of a single formula ϕ viewing it as a one element collection $\{\phi\}$.

Counting atoms of a Boolean algebra in a model theoretic setting corresponds to counting types, so it is instructive to rewrite the shatter function in terms of types.

Definition 1.10.

$$\pi^*_\Phi(n) = \max\{\text{number of } \Phi\text{-types over } B \mid B \subset M, |B| = n\}$$

Lemma 1.11.

$$\mathrm{vc}^*(\Phi) = \textit{degree of polynomial growth of } \pi^*_\Phi(n) = \limsup_{n\to\infty} \frac{\log \pi^*_\Phi(n)}{\log n}$$

One can check that the shatter function and hence VC-dimension and vc-density of a formula are elementary notions, so they only depend on the first-order theory of the structure.

NIP theories are a natural context for studying vc-density. In fact we can take the following as the definition of NIP:

Definition 1.12. Define ϕ to be NIP if it has finite VC-dimension.

In a general combinatorial context, vc-density can be any real number in $0 \cup [1, \infty)$. Less is known if we restrict our attention to NIP theories. Proposition 4.6 in [1] gives examples of formulas that have non-integer rational vc-density in an NIP theory, however it is open whether one can get an irrational vc-density in this context.

In general, instead of working with a theory formula by formula, we can look for a uniform bound for all formulas:

Definition 1.13. For a given NIP structure M, define the <u>vc-function</u>

$$\begin{aligned}\mathrm{vc}^M(n) &= \sup\{\mathrm{vc}^*(\phi(x,y)) \mid \phi \in \mathcal{L}(M), |x| = n\}\\ &= \sup\{\mathrm{vc}(\phi(x,y)) \mid \phi \in \mathcal{L}(M), |y| = n\}\end{aligned}$$

As before this definition is elementary, so it only depends on the theory of M. We omit the superscript M if it is understood from the context. One can easily check the following bounds:

Lemma 1.14 (Lemma 3.22 in [1])**.**

$$\mathrm{vc}(1) \geq 1$$

$$\mathrm{vc}(n) \geq n\,\mathrm{vc}(1)$$

However, it is not known whether the second inequality can be strict or even whether $\mathrm{vc}(1) < \infty$ implies $\mathrm{vc}(n) < \infty$.

2. Proper Subdivisions: Definition and Properties

We work with finite relational languages. Given a formula we define its complexity as the depth of quantifiers used to build up the formula. More precisely:

Definition 2.1. Define *complexity* of a formula by induction:

$$\mathrm{Complexity}(\text{q.f. formula}) = 0$$

$$\mathrm{Complexity}(\exists x \phi(x)) = \mathrm{Complexity}(\phi(x)) + 1$$

$$\mathrm{Complexity}(\phi \wedge \psi) = \max(\mathrm{Complexity}(\phi), \mathrm{Complexity}(\psi))$$

$$\mathrm{Complexity}(\neg\phi) = \mathrm{Complexity}(\phi)$$

A simple inductive argument verifies that there are (up to equivalence) only finitely many formulas when the complexity and the number of free variables are fixed. We will use the following notation for types:

Definition 2.2. Let $\boldsymbol{B}$ be a structure, $A \subset B$ be a finite parameter set, and a, b be tuples in $\boldsymbol{B}$, and m, n be natural numbers.

- $\mathrm{tp}^n_{\boldsymbol{B}}(a/A)$ will stand for the set of all A-formulas of complexity $\leq n$ that are true of a in $\boldsymbol{B}$. If $A = \emptyset$ we may also write this as $\mathrm{tp}^n_{\boldsymbol{B}}(a)$. The subscript $\boldsymbol{B}$ will be omitted as well if it is clear from context. Note that if A is finite, there are finitely many such formulas (up to equivalence). The conjunction of those formulas still has complexity $\leq n$ and so we can just associate a single formula to every type $\mathrm{tp}^n_{\boldsymbol{B}}(a/A)$.

- $\boldsymbol{B} \models a \equiv^n_A b$ means that a, b have the same type with complexity $\leq n$ over A in $\boldsymbol{B}$, i.e., $\mathrm{tp}^n_{\boldsymbol{B}}(a/A) = \mathrm{tp}^n_{\boldsymbol{B}}(b/A)$.
- $S^n_{\boldsymbol{B},m}(A)$ will stand for the set of all m-types of complexity $\leq n$ over A:

$$S^n_{\boldsymbol{B},m}(A) = \{\mathrm{tp}^n_{\boldsymbol{B}}(a/A) \mid a \in B^m\}.$$

Definition 2.3.

- Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{T}$ be structures in some (possibly different) finite relational languages. If the underlying sets A, B of $\boldsymbol{A}, \boldsymbol{B}$ partition the underlying set T of $\boldsymbol{T}$ (i.e. $T = A \sqcup B$), then we say that $(\boldsymbol{A}, \boldsymbol{B})$ is a *subdivision* of $\boldsymbol{T}$.
- A subdivision $(\boldsymbol{A}, \boldsymbol{B})$ of $\boldsymbol{T}$ is called *n-proper* if given $p, q \in \mathbb{N}$, $a_1, a_2 \in A^p$ and $b_1, b_2 \in B^q$ with

$$\boldsymbol{A} \models a_1 \equiv_n a_2$$

$$\boldsymbol{B} \models b_1 \equiv_n b_2$$

  we have

$$\boldsymbol{T} \models a_1 b_1 \equiv_n a_2 b_2.$$

- A subdivision $(\boldsymbol{A}, \boldsymbol{B})$ of $\boldsymbol{T}$ is called *proper* if it is n-proper for all $n \in \mathbb{N}$.

Lemma 2.4. *Consider a subdivision $(\boldsymbol{A}, \boldsymbol{B})$ of $\boldsymbol{T}$. If $(\boldsymbol{A}, \boldsymbol{B})$ is 0-proper then it is proper.*

Proof. We prove that the subdivision is k-proper for all k by induction. The case $k = 0$ is given by the assumption. Suppose we have $\boldsymbol{T} \models \exists x\, \phi^k(x, a_1, b_1)$ where ϕ^k is some formula of complexity k. Let $a \in T$ witness the existential claim, i.e., $\boldsymbol{T} \models \phi^k(a, a_1, b_1)$. We can have $a \in A$ or $a \in B$. Without loss of generality assume

$a \in A$. Let $\boldsymbol{p} = \mathrm{tp}^k_{\boldsymbol{A}}(a, a_1)$. Then we have

$$\boldsymbol{A} \models \exists x \ \mathrm{tp}^k_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$$

(with $\mathrm{tp}^k_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$ a shorthand for $\phi_{\boldsymbol{p}}(x)$ where $\phi_{\boldsymbol{p}}$ is a formula that determines the type $\boldsymbol{p}$). The formula $\mathrm{tp}^k_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$ is of complexity $\leq k$ so $\exists x \ \mathrm{tp}^k_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$ is of complexity $\leq k + 1$. By the inductive hypothesis we have

$$\boldsymbol{A} \models \exists x \ \mathrm{tp}^k_{\boldsymbol{A}}(x, a_2) = \boldsymbol{p}\,.$$

Let a' witness this existential claim, so that $\mathrm{tp}^k_{\boldsymbol{A}}(a', a_2) = \boldsymbol{p}$, hence $\mathrm{tp}^k_{\boldsymbol{A}}(a', a_2) = \mathrm{tp}^k_{\boldsymbol{A}}(a, a_1)$, that is, $\boldsymbol{A} \models a'a_2 \equiv_k aa_1$. By the inductive hypothesis we therefore have $\boldsymbol{T} \models aa_1b_1 \equiv_k a'a_2b_2$; in particular $\boldsymbol{T} \models \phi^k(a', a_2, b_2)$as $\boldsymbol{T} \models \phi^k(a, a_1, b_1)$, and $\boldsymbol{T} \models \exists x \phi^k(x, a_2, b_2)$. $\square$

This lemma is general, but we will use it specifically applied to (colored) trees. Suppose $\boldsymbol{T}$ is a (colored) tree in some language $\mathcal{L} = \{\leq, \ldots\}$. Suppose $\boldsymbol{A}, \boldsymbol{B}$ are some structures in languages $\mathcal{L}_A, \mathcal{L}_B$ which expand $\mathcal{L}$, with the $\mathcal{L}$-reducts of $\boldsymbol{A}, \boldsymbol{B}$ substructures of $\boldsymbol{T}$. Furthermore suppose that $(\boldsymbol{A}, \boldsymbol{B})$ is 0-proper. Then by the previous lemma $(\boldsymbol{A}, \boldsymbol{B})$ is a proper subdivision of $\boldsymbol{T}$. From now on all the subdivisions we work with will be of this form.

Example 2.5. Suppose a tree consists of two connected components C_1, C_2. Then those components $(C_1, \leq)$, $(C_2, \leq)$ interpreted as substructures form a proper subdivision. To see that we only need to show that this subdivision is 0-proper. But that is immediate as any $c_1 \in C_1$ and $c_2 \in C_2$ are incomprable.

Example 2.6. Fix a tree $\boldsymbol{T}$ in the language $\{\leq\}$ and $a \in T$. Let $B = \{t \in T \mid a < t\}$, $S = \{t \in T \mid t \leq a\}$, $A = T - B$. Then $(A, \leq, S)$ and $(B, \leq)$ form a proper subdivision, where $\mathcal{L}_A$ has a unary predicate interpreted by S. To see this, again, we show that the subdivision is 0-proper. The only time $a \in A$ and $b \in B$ are comparable is when $a \in S$, so that is captured by the language. (See proof of Lemma 3.7 for more details.)

Definition 2.7. For $\phi(x, y)$, $A \subseteq T^{|x|}$ and $B \subseteq T^{|y|}$

- Let $\phi(A, b) = \{a \in A \mid \phi(a, b)\} \subseteq A$
- Let $\phi(A, B) = \{\phi(A, b) \mid b \in B\} \subseteq \mathcal{P}(A)$

$\phi(A, B)$ is a collection of subsets of A definable by ϕ with parameters from B. We notice the following bound when A, B are parts of a proper subdivision.

Corollary 2.8. *Let $\boldsymbol{A}, \boldsymbol{B}$ be a proper subdivision of $\boldsymbol{T}$ and $\phi(x, y)$ a formula of complexity n. Then $|\phi(A^{|x|}, B^{|y|})|$ is bounded by $|S^n_{\boldsymbol{B},|y|}|$.*

Proof. Take some $a \in A^{|x|}$ and $b_1, b_2 \in B^{|y|}$ with $\mathrm{tp}^n_{\boldsymbol{B}}(b_1) = \mathrm{tp}^n_{\boldsymbol{B}}(b_2)$. We have $\boldsymbol{B} \models b_1 \equiv_n b_2$ and (trivially) $\boldsymbol{A} \models a \equiv_n a$. Thus by the Lemma 2.4 we have $\boldsymbol{T} \models ab_1 \equiv_n ab_2$ so $\phi(a, b_1) \leftrightarrow \phi(a, b_2)$. Since a was arbitrary we have $\phi(A^{|x|}, b_1) = \phi(A^{|x|}, b_2)$ as different traces can only come from parameters of different types. Thus $|\phi(A^{|x|}, B^{|y|}) \leq |S^n_{\boldsymbol{B},|y|}|$. □

We note that the size of type space $|S^n_{\boldsymbol{B},|y|}|$ can be bounded uniformly:

Definition 2.9. Fix a (finite relational) language $\mathcal{L}_B$, and n, $|y|$. Let $N = N(n, |y|, \mathcal{L}_B)$ be smallest number such that for any structure $\boldsymbol{B}$ in $\mathcal{L}_B$ we have $|S^n_{\boldsymbol{B},|y|}| \leq N$. This number is well-defined as there is a finite number (up to equivalence) of possible formulas of complexity $\leq n$ with $|y|$ free variables. Note the following easy inequalities

$$n \leq m \Rightarrow N(n, |y|, \mathcal{L}_B) \leq N(m, |y|, \mathcal{L}_B)$$

$$|y| \leq |z| \Rightarrow N(n, |y|, \mathcal{L}_B) \leq N(n, |z|, \mathcal{L}_B)$$

$$\mathcal{L}_A \subseteq \mathcal{L}_B \Rightarrow N(n, |y|, \mathcal{L}_A) \leq N(n, |y|, \mathcal{L}_B)$$

3. Proper Subdivisions: Constructions

From now on work in meet trees unless mentioned otherwise. First, we describe several constructions of proper subdivisions that are needed for the proof.

Definition 3.1. If b and c are in the same connected component we denote it as $E(b,c)$

$$E(b,c) \Leftrightarrow \exists x\,(b \geq x) \wedge (c \geq x)$$

Definition 3.2. Given a tree element a we can look at all the elements above a, i.e. $\{x \mid x \geq a\}$. We can think about it as a *closed cone* above a. Connected components of that cone can be thought of as *open cones* over a. With that interpretation in mind, notation $E_a(b,c)$ means that b and c are in the same open cone over a. More formally:

$$E_a(b,c) \Leftrightarrow E(b,c) \text{ and } (b \wedge c) > a$$

Fix a language $\mathcal{L}$ for a colored tree $\mathcal{L} = \{\leq, C_1, \dots C_n\} = \{\leq, \vec{C}\}$. In the following four definitions $\boldsymbol{B}$-structures are going to be in the same language $\mathcal{L}_B = \mathcal{L} \cup \{U\}$ with U a unary predicate. It is not always necessary to have this predicate but we keep it for the sake of uniformity. $\boldsymbol{A}$-structures will have different $\mathcal{L}_A$ languages (those are not as important in later applications).

Definition 3.3. Fix $c_1 < c_2$ in T. Let

$$B = \{b \in T \mid E_{c_1}(c_2, b) \wedge \neg(b \geq c_2)\}$$

$$A = T - B$$

$$S_1 = \{t \in T \mid t < c_1\}$$

$$S_2 = \{t \in T \mid t < c_2\}$$

$$S_B = S_2 - S_1$$

$$T_A = \{t \in T \mid c_2 \leq t\}$$

Define structures $\boldsymbol{A}^{c_1}_{c_2} = (A, \leq, \vec{C}, S_1, T_A)$ and $\boldsymbol{B}^{c_1}_{c_2} = (B, \leq, \vec{C}, S_B)$ where $\mathcal{L}_A$ is an expansion of $\mathcal{L}$ by two unary predicates (and $\mathcal{L}_B$ as defined above). Note that $c_1, c_2 \notin B$.

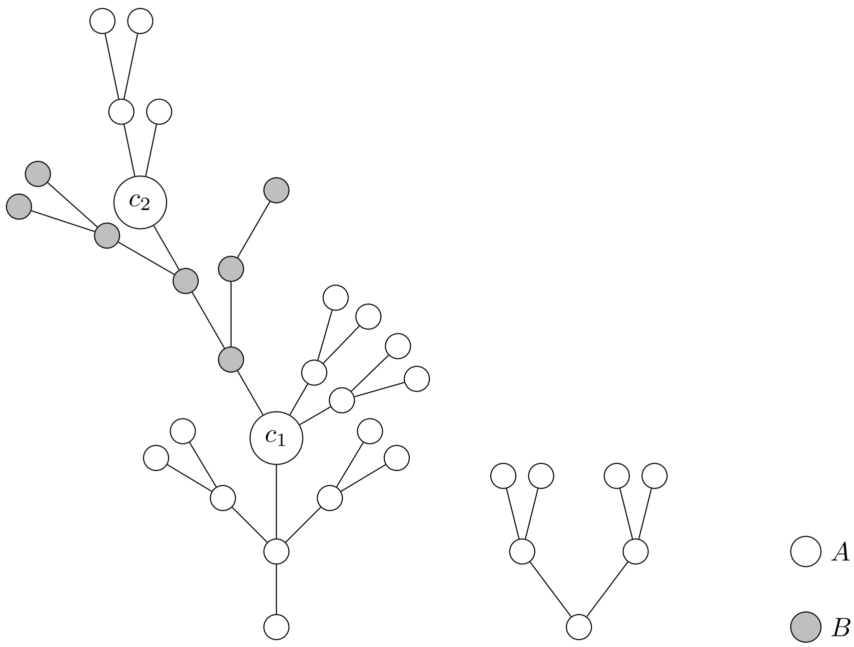

FIGURE 1. Proper subdivision for $(A, B) = (A_{c_2}^{c_1}, B_{c_2}^{c_1})$

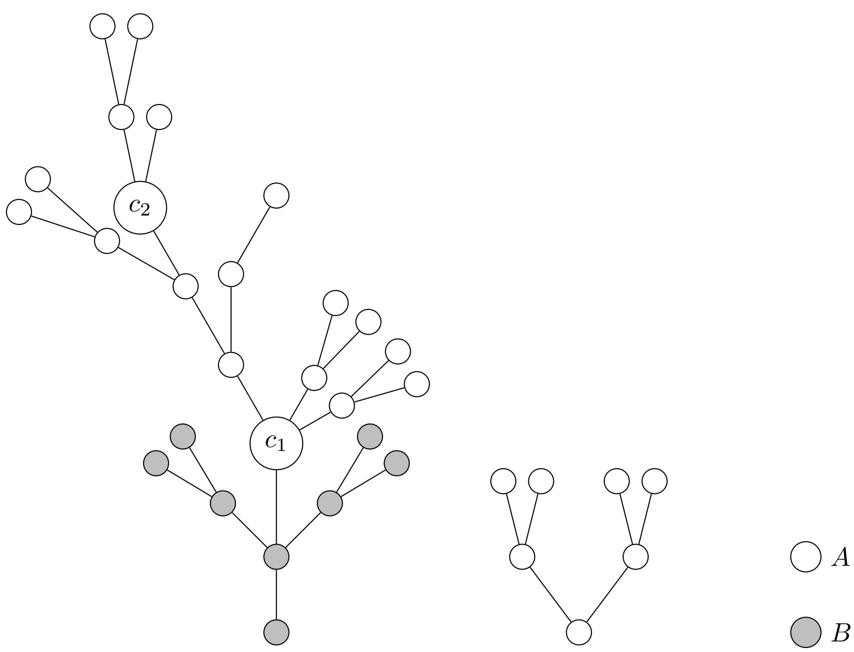

FIGURE 2. Proper subdivision for $(A, B) = (A_{c_1}, B_{c_1})$

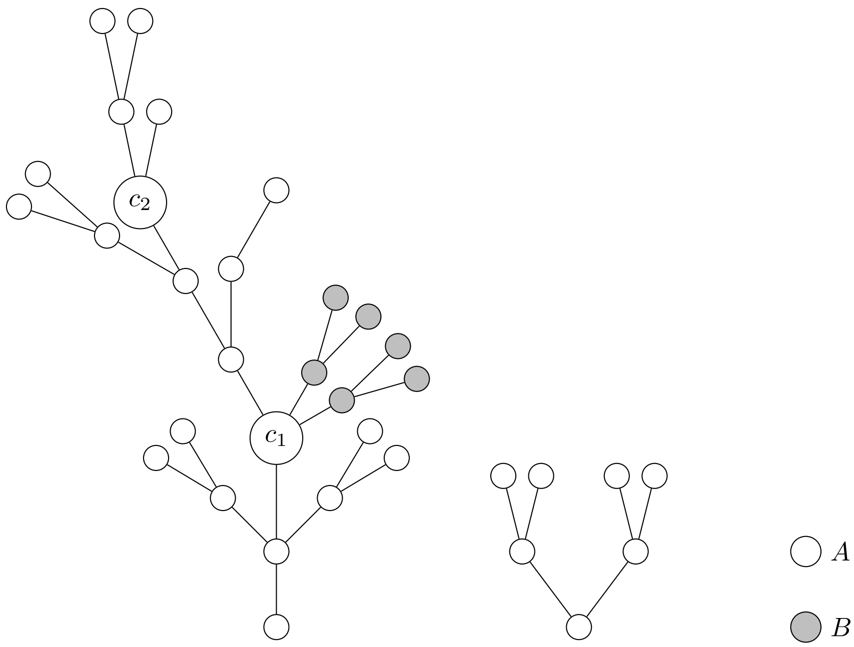

FIGURE 3. Proper subdivision for $(A, B) = (A_G^{c_1}, B_G^{c_1})$ for $G = \{c_2\}$

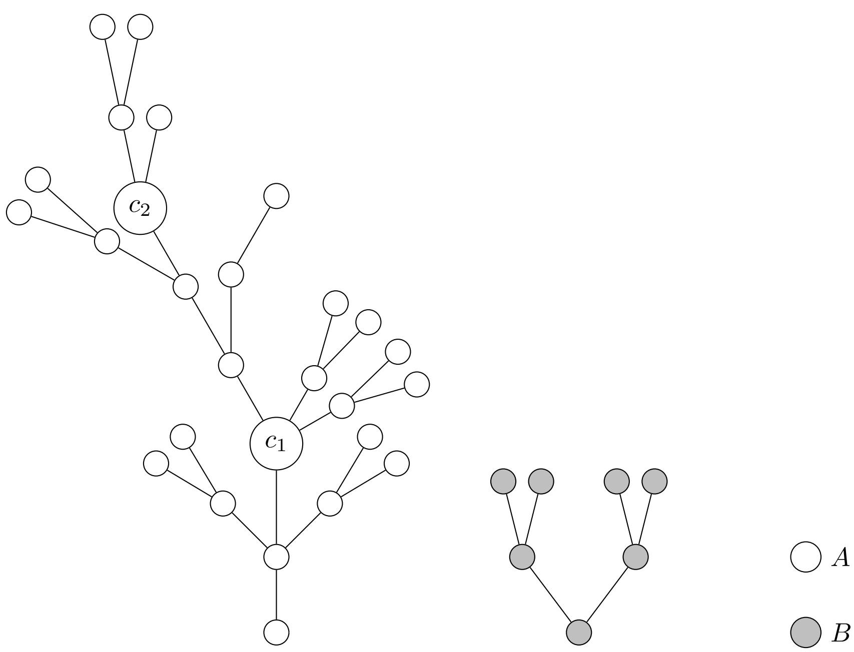

FIGURE 4. Proper subdivision for $(A, B) = (A_G, B_G)$ for $G = \{c_1, c_2\}$

Definition 3.4. Fix c in T. Let

$$B = \{b \in T \mid \neg(b \geq c) \wedge E(b,c)\}$$

$$A = T - B$$

$$S_1 = \{t \in T \mid t < c\}$$

Define structures $\boldsymbol{A}_c = (A, \leq, \vec{C})$ and $\boldsymbol{B}_c = (B, \leq, \vec{C}, S_1)$ where $\mathcal{L}_A = \mathcal{L}$ (and $\mathcal{L}_B$ as defined above). Note that $c \notin B$. (cf example 2.6).

Definition 3.5. Fix c in T and $S \subseteq T$ a finite subset. Let

$$B = \{b \in T \mid (b > c) \text{ and for all } s \in S \text{ we have } \neg E_c(s,b)\}$$

$$A = T - B$$

$$S_1 = \{t \in T \mid t \leq c\}$$

Define structures $\boldsymbol{A}^c_S = (A, \leq, \vec{C}, S_1)$ and $\boldsymbol{B}^c_S = (B, \leq, \vec{C}, B)$ where L_A is an expansion of $\mathcal{L}$ by a single unary predicate (and $U \in \mathcal{L}_B$ vacuously interpreted by B). Note that $c \notin B$ and $S \cap B = \emptyset$.

Definition 3.6. Fix $S \subseteq T$ a finite subset. Let

$$B = \{b \in T \mid \text{ for all } s \in S \text{ we have } \neg E(s,b)\}$$

$$A = T - B$$

Define structures $\boldsymbol{A}_S = (A, \leq)$ and $\boldsymbol{B}_S = (B, \leq, \vec{C}, B)$ where $\mathcal{L}_A = \mathcal{L}$ (and $U \in \mathcal{L}_B$ vacuously interpreted by B). Note that $S \cap B = \emptyset$. (cf example 2.5)

Lemma 3.7. *The pairs of structures defined above are all proper subdivisions.*

Proof. We only show this holds for the first definition $\boldsymbol{A} = \boldsymbol{A}^{c_1}_{c_2}$ and $\boldsymbol{B} = \boldsymbol{B}^{c_1}_{c_2}$. Other cases follow by a similar argument. A, B partition T by definition, so it is a subdivision. To show that it is proper by Lemma 2.4 we only need to check that it is 0-proper. Suppose we have

$$a = (a_1, a_2, \dots, a_p) \in A^p$$

$$a' = (a'_1, a'_2, \dots, a'_p) \in A^p$$

$$b = (b_1, b_2, \dots, b_q) \in B^q$$

$$b' = (b'_1, b'_2, \dots, b'_q) \in B^q$$

with $(\boldsymbol{A}, a) \equiv_0 (\boldsymbol{A}, a')$ and $(\boldsymbol{B}, b) \equiv_0 (\boldsymbol{B}, b')$. We need to make sure that ab has the same quantifier free type as $a'b'$. Any two elements in T can be related in the four following ways

$$x = y$$

$$x < y$$

$$x > y$$

$$x, y \text{ are incomparable}$$

We need to check that the same relations hold for the pairs of $(a_i, b_j), (a'_i, b'_j)$ for all i, j.

- It is impossible that $a_i = b_j$ as they come from disjoint sets.
- Suppose $a_i < b_j$. This forces $a_i \in S_1$ thus $a'_i \in S_1$ and $a'_i < b'_j$.
- Suppose $a_i > b_j$ This forces $b_j \in S_B$ and $a \in T_A$, thus $b'_j \in S_B$ and $a'_i \in T_A$ so $a'_i > b'_j$.
- Suppose a_i and b_j are incomparable. Two cases are possible:
  - $b_j \notin S_B$ and $a_i \in T_A$. Then $b'_j \notin S_B$ and $a'_i \in T_A$ making a'_i, b'_j incomparable.
  - $b_j \in S_B$, $a_i \notin T_A$, $a_i \notin S_1$. Similarly this forces a'_i, b'_j incomparable.

Also we need to check that ab has the same colors as $a'b'$. But that is immediate as having the same color in the substructure means having the same color in the whole tree. $\square$

4. Main proof

Basic idea for the proof is as follows. Suppose we have a formula with q parameters. We are able to split our parameter space into $O(n)$ many partitions. Each of q parameters can come from any of those $O(n)$ partitions giving us $O(n)^q$ many choices for parameter configuration. When every parameter is coming from a fixed partition the number of definable sets is constant and in fact is uniformly bounded above by some N. This gives us at most $N \cdot O(n)^q$ possibilities for different definable sets.

First, we generalize Corollary 2.8. (This is required for computing vc-density for formulas $\phi(x, y)$ with $|y| > 1$).

Lemma 4.1. *Consider a finite collection $(\boldsymbol{A}_i, \boldsymbol{B}_i)_{i \leq n}$ where each $(\boldsymbol{A}_i, \boldsymbol{B}_i)$ is either a proper subdivision or a singleton: $B_i = \{b_i\}$ with $A_i = T$. Also assume that all $\boldsymbol{B}_i$ have the same language $\mathcal{L}_B$. Let $A = \bigcap_{i \in I} A_i$. Fix a formula $\phi(x, y)$ of complexity m . Let $N = N(m, |y|, \mathcal{L}_B)$ as in Definition 2.9. Consider any $B \subseteq T^{|y|}$ of the form*

$$B = B_1^{i_1} \times B_2^{i_2} \times \ldots \times B_n^{i_n} \text{ with } i_1 + i_2 + \ldots + i_n = |y|$$

(some of the indexes can be zero). Then we have the following bound

$$\phi(A^{|x|}, B) \leq N^{|y|}$$

Proof. We show this result by counting types. Suppose we have

$$b_1, b_1' \in B_1^{i_1} \text{ with } b_1 \equiv_m b_1' \text{ in } \boldsymbol{B}_1$$

$$b_2, b_2' \in B_2^{i_2} \text{ with } b_2 \equiv_m b_2' \text{ in } \boldsymbol{B}_2$$

$$\ldots$$

$$b_n, b_n' \in B_n^{i_n} \text{ with } b_n \equiv_m b_n' \text{ in } \boldsymbol{B}_n$$

Then we have

$$\phi(A^{|x|}, b_1, b_2, \dots b_n) \Leftrightarrow \phi(A^{|x|}, b_1', b_2', \dots b_n')$$

This is easy to see by applying Corollary 2.8 one by one for each tuple. This works if $\boldsymbol{B}_i$ is a part of a proper subdivision; if it is a singleton then the implication is trivial as $b_i = b_i'$. Thus $\phi(A^{|x|}, B)$ only depends on the choice of the types for the tuples

$$|\phi(A^{|x|}, B)| \le |S^m_{\boldsymbol{B}_1, i_1}| \cdot |S^m_{\boldsymbol{B}_2, i_2}| \cdot \dots \cdot |S^m_{\boldsymbol{B}_n, i_n}|$$

Now for each type space we have an inequality

$$|S^m_{\boldsymbol{B}_j, i_j}| \le N(m, i_j, \mathcal{L}_B) \le N(m, |y|, \mathcal{L}_B) \le N$$

(For singletons $|S^m_{\boldsymbol{B}_j, i_j}| = 1 \le N$). Only non-zero indexes contribute to the product and there are at most $|y|$ of those (by equality $i_1 + i_2 + \dots + i_n = |y|$). Thus we have

$$|\phi(A^{|x|}, B)| \le N^{|y|}$$

as needed. □

For subdivisions to work out properly, we will need to work with subsets closed under meets. We observe that the closure under meets doesn't add too many new elements.

Lemma 4.2. *Suppose $S \subseteq T$ is a finite subset of size n in a meet tree and S' is its closure under meets. Then $|S'| \le 2n$.*

Proof. We can partition S into connected components and prove the result separately for each component. Thus we may assume elements of S lie in the same connected component. We prove the claim by induction on n. Base case $n = 1$ is clear. Suppose we have S of size k with closure of size at most $2k - 1$. Take a new point s, and look at its meets with all the elements of S. Pick the smallest one, s'. Then $S \cup \{s, s'\}$ is closed under meets. $\square$

Putting all of those results together we are able to compute the vc-density of formulas in meet trees.

Theorem 4.3. *Let $\boldsymbol{T}$ be an infinite (colored) meet tree and $\phi(x, y)$ a formula with $|x| = p$ and $|y| = q$. Then* $\mathrm{vc}(\phi) \leq q$.

Proof. Pick a finite subset of $S_0 \subset T^p$ of size n. Let $S_1 \subset T$ consist of coordinates of S_0. Let $S \subset T$ be a closure of S_1 under meets. Using Lemma 4.2 we have $|S| \leq 2|S_1| \leq 2p|S_0| = 2pn = O(n)$. We have $S_0 \subseteq S^p$, so $|\phi(S_0, T^q)| \leq |\phi(S^p, T^q)|$. Thus it is enough to show $|\phi(S^p, T^q)| = O(n^q)$.

Label $S = \{c_i\}_{i \in I}$ with $|I| \leq 2pn$. For every c_i we construct two partitions in the following way. We have that c_i is either minimal in S or it has a predecessor in S (greatest element less than c). If it is minimal, construct $(\boldsymbol{A}_{c_i}, \boldsymbol{B}_{c_i})$. If there is a predecessor p, construct $(\boldsymbol{A}^p_{c_i}, \boldsymbol{B}^p_{c_i})$. For the second subdivision let G be all the elements in S greater than c_i and construct $(\boldsymbol{A}^c_G, \boldsymbol{B}^c_G)$. So far we have constructed two subdivisions for every $i \in I$. Additionally construct $(\boldsymbol{A}_S, \boldsymbol{B}_S)$. We end up with a finite collection of proper subdivisons $(\boldsymbol{A}_j, \boldsymbol{B}_j)_{j \in J}$ with $|J| = 2|I| + 1$. Before we proceed, we note the following two lemmas describing our partitions.

Lemma 4.4. *For all $j \in J$ we have $S \subseteq A_j$. Thus $S \subseteq \bigcap_{j \in J} A_j$ and $S^p \subseteq \bigcap_{j \in J} (A_j)^p$.*

Proof. Check this for each possible choice of partition. Cases for partitions of the type $\boldsymbol{A}_S, \boldsymbol{A}^c_G, \boldsymbol{A}_c$ are easy. Suppose we have a partition $(\boldsymbol{A}, \boldsymbol{B}) = (\boldsymbol{A}^{c_1}_{c_2}, \boldsymbol{B}^{c_1}_{c_2})$. We need to show that $B \cap S = \emptyset$. By construction we have $c_1, c_2 \notin B$. Suppose we have some other $c \in S$ with $c \in B$. We have $E_{c_1}(c_2, c)$ i.e. there is some b such that $(b > c_1)$, $(b \leq c_2)$ and $(b \leq c)$. Consider the meet $(c \wedge c_2)$. We have $(c \wedge c_2) \geq b > c_1$. Also as $\neg(c \geq c_2)$ we have $(c \wedge c_2) < c_2$. To summarize: $c_2 > (c \wedge c_2) > c_1$. But this contradicts our construction as S is closed under meets, so $(c \wedge c_2) \in S$ and c_1 is supposed to be a predecessor of c_2 in S. $\square$

Lemma 4.5. *$\{B_j\}_{j\in J}$ is a disjoint partition of $T-S$ i.e. $T=\bigsqcup_{j\in J} B_j \sqcup S$*

Proof. This more or less follows from the choice of partitions. Pick any $b\in S-T$. Take all the elements in S greater than b and take the minimal one a. Take all the elements in S less than b and take the maximal one c (possible as S is closed under meets). Also take all the elements in S incomparable to b and denote them G. If both a and c exist we have $b\in \boldsymbol{B}_c^a$. If only the upper bound exists we have $b\in \boldsymbol{B}_G^a$. If only the lower bound exists we have $b\in \boldsymbol{B}_c$. If neither exists we have $b\in \boldsymbol{B}_G$. □

Note 4.6. Those two lemmas imply $S=\bigcap_{j\in J} A_j$.

Note 4.7. For one-dimensional case $q=1$ we don't need to do any more work. We have partitioned the parameter space into $|J|=O(n)$ many pieces and over each piece the number of definable sets is uniformly bounded. By Corollary 2.8 we have that $|\phi((A_j)^p,B_j)|\le N$ for any $j\in J$ (letting $N=N(n_\phi,q,\mathcal{L}\cup\{S\})$ where n_ϕ is the complexity of ϕ and S is a unary predicate). Compute

$$
\begin{aligned}
|\phi(S^p,T)| &= \left|\bigcup_{j\in J}\phi(S^p,B_j)\cup\phi(S^p,S)\right| \le \\
&\le \sum_{j\in J}|\phi(S^p,B_j)|+|\phi(S^p,S)| \le \\
&\le \sum_{j\in J}|\phi((A_j)^p,B_j)|+|S| \le \\
&\le \sum_{j\in J} N+|I| \le \\
&\le (4pn+1)N+2pn=(4pN+2p)n+N=O(n)
\end{aligned}
$$

Basic idea for the general case $q\ge 1$ is that we have q parameters and $|J|=O(n)$ many partitions to pick each parameter from giving us $|J|^q=O(n^q)$ choices for the parameter configuration, each giving a uniformly constant number of definable subsets of S. (If every parameter is picked from a fixed partition, Lemma 4.1

provides a uniform bound). This yields $\mathrm{vc}(\phi) \leq q$ as needed. The rest of the proof is stating this idea formally.

First, we extend our collection of subdivisions $(\boldsymbol{A}_j, \boldsymbol{B}_j)_{j\in J}$ by the following singleton sets. For each $c_i \in S$ let $B_i = \{c_i\}$ and $A_i = T$ and add $(\boldsymbol{A}_i, \boldsymbol{B}_i)$ to our collection with $\mathcal{L}_B$ the language of B_i interpreted arbitrarily. We end up with a new collection $(\boldsymbol{A}_k, \boldsymbol{B}_k)_{k\in K}$ indexed by some K with $|K| = |J| + |I|$ (we added $|S|$ new pairs). Now $B_{k k\in K}$ partitions T, so $T = \bigsqcup_{k\in K} B_k$ and $S = \bigcap_{j\in J} A_j = \bigcap_{k\in K} A_k$. For $(k_1, k_2, \ldots k_q) = \vec{k} \in K^q$ denote

$$B_{\vec{k}} = B_{k_1} \times B_{k_2} \times \ldots \times B_{k_q}$$

Then we have the following identity

$$T^q = (\bigsqcup_{k\in K} B_k)^q = \bigsqcup_{\vec{k}\in K^q} B_{\vec{k}}$$

Thus we have that $\{B_{\vec{k}}\}_{\vec{k}\in K^q}$ partition T^q. Compute

$$\begin{aligned} |\phi(S^p, T^q)| = \left| \bigcup_{\vec{k}\in K^q} \phi(S^p, B_{\vec{k}}) \right| \leq \\ \leq \sum_{\vec{k}\in K^q} |\phi(S^p, B_{\vec{k}})| \end{aligned}$$

We can bound $|\phi(S^p, B_{\vec{k}})|$ uniformly using Lemma 4.1. $(\boldsymbol{A}_k, \boldsymbol{B}_k)_{k\in K}$ satisfies the requirements of the lemma and $B_{\vec{k}}$ looks like B in the lemma after possibly permuting some variables in ϕ. Applying the lemma we get

$$|\phi(S^p, B_{\vec{k}})| \leq N^q$$

with N only depending on q and complexity of ϕ. We complete our computation

$$\begin{aligned}|\phi(S^p, T^q)| &\leq \sum_{\vec{k} \in K^q} |\phi(S^p, B_{\vec{k}})| \leq \\ &\leq \sum_{\vec{k} \in K^q} N^q \leq \\ &\leq |K^q| N^q \leq \\ &\leq (|J| + |I|)^q N^q \leq \\ &\leq (4pn + 1 + 2pn)^q N^q = N^q (6p + 1/n)^q n^q = O(n^q)\end{aligned}$$

□

Corollary 4.8. *In the theory of infinite (colored) meet trees we have $vc(n) = n$ for all n.*

We get the general result for the trees that aren't necessarily meet trees via an easy application of interpretability.

Corollary 4.9. *In the theory of infinite (colored) trees we have $vc(n) = n$ for all n.*

Proof. Let $\boldsymbol{T}'$ be a tree. We can embed it in a larger tree that is closed under meets $\boldsymbol{T}' \subset \boldsymbol{T}$. Expand $\boldsymbol{T}$ by an extra color and interpret it by coloring the subset $\boldsymbol{T}'$. Thus we can interpret $\boldsymbol{T}'$ in T^1. By Corollary 3.17 in [1] we get that $\mathrm{vc}^{\boldsymbol{T}'}(n) \leq \mathrm{vc}^T(1 \cdot n) = n$ thus $\mathrm{vc}^{\boldsymbol{T}'}(n) = n$ as well. □

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