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VC-density Computations in Various Model-Theoritic Structures

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ABSTRACT OF THE DISSERTATION

VC-density Computations
in Various Model-Theoritic Structures

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Aschenbrenner et. al. have studied vc-density in the model-theoretic context. We investigate it further by computing it in some common structures: trees, Shelah-Spencer graphs, and an additive reduct of p-adic numbers. We show that in the theory of infinite trees the vc-function is optimal. This generalizes a result of Simon showing that the trees are dp-minimal. In Shelah-Spencer graphs we provide an upper bound on a formula-by-formula basis and show that there isn't a uniform lower bound, forcing the vc-function to be infinite. In addition we show that Shelah-Spencer graphs do not have a finite dp-rank, so they are not dp-minimal. There is a linear bound for the vc-density function in the field of p-adic numbers, but this bound is not known to be optimal. We investigate a certain P-minimal additive reduct of the field of p-adic numbers and use a cell decomposition result of Leenknegt to compute an optimal bound for that structure. Following results of Podewski and Ziegler we show that flat graphs are dp-minimal.

The dissertation of Anton Bobkov is approved.

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2017

To my family and friends

who have been unerringly supportive

throughout my career path

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CHAPTER 1

Inroduction and Preliminaries

1.1 Inroduction

My research concentrates on the concept of vc-density, a recent notion of rank in NIP theories. The study of a structure in model theory usually starts with quantifier elimination, followed by a finer analysis of definable functions and interpretability. The study of vc-density goes one step further, looking at a structure of the asymptotic growth of finite definable families. In the most geometric examples, vc-density coincides with the natural notion of dimension. However, no geometric structure is required for the definition of vc-density, thus we can get some notion of geometric dimension for families of sets given without any geometric context.

In 2013, Aschenbrenner et al. investigated and developed a notion of vc-density for NIP structures, an analog of geometric dimension in an abstract setting [ADH16]. Their applications included a bound for p-adic numbers, an object of great interest and a very active area of research in mathematics. My research concentrates on improving and expanding techniques of that paper to improve the known bounds as well as computing vc-density for other NIP structures of interest. I am able to obtain new bounds for the additive reduct of p-adic numbers, trees, and certain families of graphs. Recent research by Chernikov and Starchenko in 2015 [CS15] suggests that having good bounds on vc-density in p-adic numbers opens a path for applications to incidence combinatorics (e.g. Szemeredi-Trotter theorem).

The concept of VC-dimension was first introduced in 1971 by Vapnik and Chervonenkis for set systems in a probabilistic setting [VC71]. The theory grew rapidly and found wide use in geometric combinatorics, computational learning theory, and machine learning. Around the same time Shelah was developing the notion of NIP ("not having the independence property"), a natural tameness property of (complete theories of) structures in model theory [She71]. In 1992 Laskowski noticed the connection between the two: theories where all uniformly definable families of sets have finite VC-dimension are exactly NIP theories [Las92]. It is a wide class of theories including algebraically closed fields, differentially closed fields, modules, free groups, o-minimal structures, and ordered abelian groups. A variety of valued fields fall into this category as well, including the p-adic numbers.

The p-adic numbers were first introduced by Hensel in 1897 in [Hen97], and over the following century a powerful theory was developed around them with numerous applications across a variety of disciplines, primarily in number theory, but also in physics and computer science. In 1965 Ax, Kochen [AK65] and Ershov [Ers65] axiomatized the theory of p-adic numbers and proved a quantifier elimination result. A key insight was to connect properties of the value group and residue field to the properties of the valued field itself. In 1984 Denef proved a cell decomposition result for more general valued fields [Den84]. This result was soon generalized to p-adic subanalytic and rigid analytic extensions, allowing for the later development of a more powerful technique of motivic integration. The conjunction of those model theoretic results allowed to solve a number of outstanding open problems in number theory (e.g., Artin's Conjecture on p-adic homogeneous forms).

In 1997, Karpinski and Macintyre computed vc-density bounds for o-minimal structures and asked about similar bounds for p-adic numbers [KM97]. vc-density is a concept closely related to VC-dimension. It comes up naturally in combinatorics with relation to packings, Hamming metric, entropic dimension and discrepancy. vc-density is also the decisive parameter in the Epsilon-Approximation Theorem, which is one of the crucial tools for applying VC theory in computational geometry. In a model theoretic setting it is computed for families of uniformly definable sets. In 2013, Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko computed a bound for vc-density in p-adic numbers and a number of other NIP structures [ADH16]. They observed connections to dp-rank and dp-minimality, notions first introduced by Shelah. In well behaved NIP structures families of uniformly definable sets tend to have vc-density bounded by a multiple of their dimension, a simple linear behavior. In a lot of cases including p-adic numbers this bound is not known to be optimal. My research concentrates on improving those bounds and adapting those techniques to compute vc-density in other common NIP structures of interest to mathematicians.

Some of the other well behaved NIP structures are Shelah-Spencer graphs and flat graphs. Shelah-Spencer graphs are limit structures for random graphs arising naturally in a combinatorial context. Their model theory was studied by Baldwin, Shi, and Shelah in 1997 [BS96], [BS97]. Later work of Laskowski in 2006 [Las07] has provided a quantifier simplification result. Flat graphs were first studied by Podewski-Ziegler in 1978, showing that those are stable [PZ78], and later results gave a criterion for super stability. Flat graphs also come up naturally in combinatorics in work of Nesetril and Ossona de Mendez [NM11].

The first chapter of my dissertation introduces some basics of model theory and defines vc-density and VC-dimension.

The second chapter concentrates on trees. I answer an open question from [ADH16], computing vc-density for trees viewed as a partial order. The main idea is to adapt a technique of Parigot [Par82] to partition trees into weakly interacting parts, with simple bounds of vc-density on each.

In the third chapter of my dissertation I work with Shelah-Spencer graphs. I have shown that they have infinite dp-rank, so they are poorly behaved as NIP structures. I have also shown that one can obtain arbitrarily high vc-density when looking at uniformly definable families in a fixed dimension. However I'm able to bound vc-density of individual formulas in terms of edge density of the graphs they define.

The fourth chapter deals with p-adic numbers. I have shown that vc-density is linear for an additive reduct of p-adic numbers using a cell decomposition result from the work of Leenknegt in 2013 [Lee14].

In chapter five I investigate flat graphs using the work of Podewski-Ziegler [PZ78]. I am able to show that flat graphs are dp-minimal, an important first step before establishing bounds on vc-density.

1.2 Basic Model Theory

This section goes through the basics of the model theory used throughout this text. It is meant to be used mostly as a reference on the notation as opposed to a comprehensive summary. For a complete and more thorough introduction to the material, we refer the reader to Chapters 1 and 2 of [TZ12]. We begin with a short summary of languages, formulas, and structures:

Definition 1.2.1.

- A language is a collection of predicate, function, and constant symbols.
- Fix a language $\mathcal{L}$ and a collection of variables. A term is an expression constructed out of constants, variables, and functions.
- An atomic formula is an expression constructed out of the equality symbol or a predicate applied to terms.
- A (first-order) formula is an expression constructed out of atomic formulas using boolean connectives $\wedge, \vee, \neg$ and quantifiers $\exists, \forall$. We denote such a formula as $\phi(x)$ where x is a tuple of free variables, that is the variables used in ϕ that are not bound by quantifiers. Abusing notation, we denote $\mathcal{L}$ to be the set of all such formulas (so we have $\phi \in \mathcal{L}$).
- A formula without free variables is called a sentence.
- A quantifier-free formula is a formula that doesn't contain any quantifiers.
- A structure $\mathbb{M}$ consists of an infinite universe M and functions, predicates, and constants matching those of $\mathcal{L}$.
- For a variable tuple x, let $|x|$ be the arity of the tuple. Similarly, for a tuple $a \in M^n$ let $|a| = n$.

- Suppose we have a formula $\phi(x)$, structure $\mathbb{M}$, and $a \in M^{|x|}$. Then we say that $\mathbb{M}$ models $\phi(a)$, denoted as $\mathbb{M} \models \phi(a)$, if the formula ϕ holds $\mathbb{M}$ when we plug in a into x.

- Suppose we have a structure $\mathbb{M}$ and $A \subseteq M$. Then $\mathcal{L}(A)$ denotes an expansion of $\mathcal{L}$ by constant symbols correspoding to elements in A. The structure $\mathbb{M}$ then can be viewed as a $\mathcal{L}(A)$-structure with the appropriate interpretations. Formulas $\phi \in \mathcal{L}(A)$ will be referred to as formulas with parameters from A or simply as A-formulas. In this context A is usually referred to as a parameter set.

- A theory is a collection of sentences.

- For a theory T and a structure $\mathbb{M}$, we say that $\mathbb{M}$ models T, or that $\mathbb{M}$ is a model of T, if $\mathbb{M}$ models every sentence in T.

- For a structure $\mathbb{M}$, a theory of $\mathbb{M}$ is a collection of all sentences that are modelled by $\mathbb{M}$.

- A theory is called complete if it is a theory of some structure $\mathbb{M}$.

Throughout this text we often confuse complete theories with their models. This is justfied for properties that can be described by a collection of first-order sentences. Then a theory has this property if and only any (all) models have this property. An example of that is a notion of stability.

Stability is a deep subject, with a lot of theory developed around it. We won't work with it directly, but it is a property of some of the structures we study. We present a definitnon for completeness and refer the reader to Chapter 8 of [TZ12] or to [Pil13] for a more complete introduction.

Definition 1.2.2.

- Suppose we have a structure $\mathbb{M}$. The formula $\phi(x, y)$ is called unstable if for all natural n there exist $a_i \in M^{|x|}, b_i \in M^{|y|}$ for $0 \leq i \leq n$ such that

$$\mathbb{M} \models \phi(a_i, b_j) \iff i \leq j.$$

- A formula is stable if it is not unstable.
- A structure $\mathbb{M}$ is stable if all of its formulas are stable.
- A complete theory T is stable if any (all) of its models are stable.

Definable sets are subsets of our structure given by formulas. More precisely:

Definition 1.2.3. Suppose we have a structure $\mathbb{M}$, a paramter set $A \subseteq M$ and an A-formula $\phi(x)$. Then

$$\phi(M^{|x|}) = \left\{m \in M^{|x|} \mid \mathbb{M} \models \phi(m)\right\}$$

is referred to as an A-definable subset of $M^{|x|}$ defined by ϕ.

More generally, we will need a slightly more refined notion of a trace:

Definition 1.2.4. Suppose we have a structure $\mathbb{M}$, a formula $\phi(x, y)$, tuples $a \in M^{|x|}, b \in M^{|y|}$, and sets $A \subseteq M^{|x|}, B \subseteq M^{|y|}$. Define

$$\phi(A, b) = \{a \in A \mid \mathbb{M} \models \phi(a, b)\},$$
$$\phi(a, B) = \{b \in B \mid \mathbb{M} \models \phi(a, b)\}.$$

These sets will be informally referred to as traces. Similarly, let

$$\phi(A, B) = \{\phi(A, b) \mid b \in B\} \in \mathcal{P}(A)$$

denote a collection of traces.

Types is one of the main tools of study in model theory.

Definition 1.2.5. Suppose $\mathbb{M}$ is a structure, $B \subseteq M$. Also fix a variable tuple x.

- A partial type over B is a collection of formulas in variable x with parameters from B.
- A partial type $p(x)$ has a realization in $\mathbb{M}$ if there exists $a \in M^{|x|}$ such that $\mathbb{M} \models \phi(a)$ for all $\phi(x) \in p(x)$.

- A partial type is consistent if its every finite subset of formulas has a realization.
- Suppose $a \in M^{|x|}$ and $\Delta \subseteq \mathcal{L}(B)$ a collection of formulas in x. Define the Δ-type of a over B to be a collection of formulas $\phi(x) \in \Delta$ such that $\mathbb{M} \models \phi(a)$. Denote it as $\mathrm{tp}_\Delta(a/B)$.
- Suppose $a \in M^{|x|}$. Define the type of a over B as the Δ-type of a over B for $\Delta = \mathcal{L}(B)$. Denote it as $\mathrm{tp}(a/B)$.

A lot of model theoretic computations are simplified when done inside of saturated structures. This is the next important construction that we turn our attention to:

Definition 1.2.6. Let κ be a cardinal. A structure $\mathbb{M}$ is called κ-saturated if for all $B \subseteq M$ with $|B| < \kappa$ we have that all consistent partial types over B are realized in $\mathbb{M}$.

Indiscernible sequences will be useful to us to describe dp-rank and dp-minimality. They come up often in model theory as a way to leverage symmetry present in sequences and sets.

Definition 1.2.7.

- Suppose we have a sequence $(a_i)_{i\in\mathcal{I}}$ where $\mathcal{I}$ is an ordered index set. For a subsequence $\mathcal{J} \subseteq \mathcal{I}$ let $a_\mathcal{J}$ denote the tuple obtained by the concatenation of the sequence $(a_j)_{j\in\mathcal{J}}$ (where the sequence is ordered using the order of $\mathcal{I}$).
- Suppose $\mathbb{M}$ is a structure, $B \subseteq M$, and $\mathcal{I}$ is an oredred index set. A sequence $(a_i)_{i\in\mathcal{I}}$ is called indiscernible over B if for any two subsets $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{I}$ of equal length we have

$$\mathrm{tp}(a_{J_1}/B) = \mathrm{tp}(a_{J_2}/B).$$

- If we use the same definition, but allow tuples a_{J_1}, a_{J_2} to be concatenated in arbitrary order, then we obtain the definition a sequence that is totally indiscernible over B (alternatively an indiscernible set).

Here is an important property of indiscernible sequences in stable thoeries:

Lemma 1.2.8 (see Lemma 9.1.1 in [TZ12])**.** *If a structure is stable then every indiscernible sequence is totally indiscernible.*

Sometimes instead of starting with an indiscernible sequence, we wish to construct one from a sequence with some degree of symmetry:

Lemma 1.2.9 (see Lemma 5.1.3 in [TZ12])**.** *Work in a $\aleph_1$-saturated structure $\mathbb{M}$. Suppose $B \subseteq M$. Fix a variable tuple x and a collection of formulas $\Delta(x_1, \ldots, x_n)$ with $|x_i| = |x|$. Suppose we can find an arbitrarily long sequence $(a_i)_{i \in \mathcal{I}}$ with $a_i \in M^{|x|}$ such that for any subset $\mathcal{J} \subseteq \mathcal{I}$ of length n we have*

$$\mathbb{M} \models \Delta(a_{\mathcal{J}}).$$

Then there exists an infinite indiscernible sequence $(a'_i)_{i \in \mathbb{N}}$ with

$$\mathbb{M} \models \Delta(a'_1, a'_2, \ldots, a'_n).$$

Instead of working with types directly, it is often more convenient to work with automorphisms:

Definition 1.2.10. Suppose $\mathbb{M}$ is a structure and $A \subseteq M$. An automorphism of $\mathbb{M}$ over A is a bijection of $f : M \longrightarrow M$ that fixes A and preserves constants, relations, and functions of $\mathbb{M}$. We use notation $f \in \operatorname{Aut}(\mathbb{M}/A)$. For a tuple $a = (a_1, \ldots, a_m)$ let $f(a) = (f(a_1), \ldots, f(a_m))$.

The following lemma is easy to show directly from the definition of an automorphism:

Lemma 1.2.11. *Suppose $\mathbb{M}$ is a structure, $A \subseteq M$, and $f \in \operatorname{Aut}(\mathbb{M}/A)$. Suppose also that we have $a, b \in M^n$ such that $f(a) = b$. Then $tp(a/A) = tp(b/A)$.*

The converse of this result holds in a special type of structures:

Definition 1.2.12. Let $\mathbb{M}$ be a structure and κ a cardinal. Then $\mathbb{M}$ is called strongly κ-homogeneous if for all $A \subseteq M$ with $|A| < \kappa$ we have that for all $a, b \in M^n$ if $tp(a/A) = tp(b/A)$ then there exists $f \in \operatorname{Aut}(\mathbb{M}/A)$ such that $f(a) = b$.

Luckily, for a given theory one can always find a model sufficiently saturated and homogeneous:

Lemma 1.2.13 (see Theorem 6.1.7 in [TZ12])**.** *Let T be a complete theory and κ a cardinal. There exists a model of T that is κ-saturated and strongly κ-homogeneous.*

1.3 VC-dimension and vc-density

Throughout this section we work with a collection $\mathcal{F}$ of subsets of an infinite set X. We call the pair $(X, \mathcal{F})$ a set system.

Definition 1.3.1.

- Given a subset A of X, we define the set system $(A, A \cap \mathcal{F})$ where $A \cap \mathcal{F} = \{A \cap F \mid F \in \mathcal{F}\}$.
- For $A \subseteq X$ we say that $\mathcal{F}$ shatters A if $A \cap \mathcal{F} = \mathcal{P}(A)$ (the power set of A).

Definition 1.3.2. We say $(X, \mathcal{F})$ has VC-dimension n if the largest subset of X shattered by $\mathcal{F}$ is of size n. If $\mathcal{F}$ shatters arbitrarily large subsets of X, we say that $(X, \mathcal{F})$ has infinite VC-dimension. We denote the VC-dimension of $(X, \mathcal{F})$ by $\mathrm{VC}(X, \mathcal{F})$.

Note 1.3.3. We may drop X from the notation $\mathrm{VC}(X, \mathcal{F})$, as the VC-dimension doesn't depend on the base set and is determined by $(\bigcup \mathcal{F}, \mathcal{F})$.

Set systems of finite VC-dimension tend to have good combinatorial properties, and we consider set systems with infinite VC-dimension to be poorly behaved.

Another natural combinatorial notion is that of the dual system of a set system:

Definition 1.3.4. For $a \in X$ define $X_a = \{F \in \mathcal{F} \mid a \in F\}$. Let $\mathcal{F}^* = \{X_a \mid a \in X\}$. We call $(\mathcal{F}, \mathcal{F}^*)$ the dual system of $(X, \mathcal{F})$. The VC-dimension of the dual system of $(X, \mathcal{F})$ is referred to as the dual VC-dimension of $(X, \mathcal{F})$ and denoted by $\mathrm{VC}^*(\mathcal{F})$. (As before, this notion doesn't depend on X.)

Lemma 1.3.5 (see 2.13b in [Ass83]). *A set system $(X, \mathcal{F})$ has finite VC-dimension if and only if its dual system has finite VC-dimension. More precisely*

$$\mathrm{VC}^*(\mathcal{F}) \leq 2^{1+\mathrm{VC}(\mathcal{F})}.$$

For a more refined notion of complexity of $(X, \mathcal{F})$ we look at the traces of our family on finite sets:

Definition 1.3.6. Define the shatter function $\pi_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ and the dual shatter function $\pi^*_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ by

$$\pi_{\mathcal{F}}(n) = \max \{|A \cap \mathcal{F}| \mid A \subseteq X \text{ and } |A| = n\}$$
$$\pi^*_{\mathcal{F}}(n) = \max \{\mathrm{atoms}(B) \mid B \subseteq \mathcal{F}, |B| = n\}$$

where $\mathrm{atoms}(B)$ = number of atoms in the boolean algebra of sets generated by B. Note that the dual shatter function is precisely the shatter function of the dual system: $\pi^*_{\mathcal{F}} = \pi_{\mathcal{F}^*}$.

A simple upper bound is $\pi_{\mathcal{F}}(n) \leq 2^n$ (same for the dual). If the VC-dimension of $\mathcal{F}$ is infinite then clearly $\pi_{\mathcal{F}}(n) = 2^n$ for all n. Conversely we have the following remarkable fact:

Theorem 1.3.7 (Sauer-Shelah '72, see [Sau72], [She72]). *If the set system $(X, \mathcal{F})$ has finite VC-dimension d then $\pi_{\mathcal{F}}(n) \leq \binom{n}{\leq d}$ for all n, where $\binom{n}{\leq d} = \binom{n}{d} + \binom{n}{d-1} + \ldots + \binom{n}{1}$.*

Thus the systems with a finite VC-dimension are precisely the systems where the shatter function grows polynomially. The vc-density of $\mathcal{F}$ quantifies the growth of the shatter function of $\mathcal{F}$:

Definition 1.3.8. Define the vc-density and dual vc-density of $\mathcal{F}$ as

$$\mathrm{vc}(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\},$$
$$\mathrm{vc}^*(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}.$$

Generally speaking a shatter function that is bounded by a polynomial doesn't itself have to be a polynomial. Proposition 4.12 in [ADH16] gives an example of a shatter function that grows like $n \log n$ (so it has vc-density 1).

So far the notions that we have defined are purely combinatorial. We now adapt VC-dimension and vc-density to the model theoretic context.

Definition 1.3.9. Work in a first-order structure M. Fix a finite collection of formulas $\Phi(x, y)$ in the language $\mathcal{L}(M)$ of M.

- For $\phi(x, y) \in \mathcal{L}(M)$ and $b \in M^{|y|}$ let

$$\phi(M^{|x|}, b) = \{a \in M^{|x|} \mid \phi(a, b)\} \subseteq M^{|x|}.$$

- Let $\Phi(M^{|x|}, M^{|y|}) = \{\phi(M^{|x|}, b) \mid \phi_i \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|})$.

- Let $\mathcal{F}_\Phi = \Phi(M^{|x|}, M^{|y|})$, giving rise to a set system $(M^{|x|}, \mathcal{F}_\Phi)$.

- Define the VC-dimension $\mathrm{VC}(\Phi)$ of Φ to be the VC-dimension of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.

- Define the vc-density $\mathrm{vc}(\Phi)$ of Φ to be the vc-density of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.

We will also refer to the vc-density and VC-dimension of a single formula ϕ viewing it as a one element collection $\Phi = \{\phi\}$.

Counting atoms of a boolean algebra in a model theoretic setting corresponds to counting types, so it is instructive to rewrite the shatter function in terms of types.

Definition 1.3.10.

$$\pi^*_\Phi(n) = \max \{\text{number of } \Phi\text{-types over } B \mid B \subseteq M, |B| = n\}.$$

Here a Φ-type over B is a maximal consistent collection of formulas of the form $\phi(x, b)$ or $\neg\phi(x, b)$ where $\phi \in \Phi$ and $b \in B$.

The functions π^*_Φ and $\pi^*_{\mathcal{F}_\Phi}$ do not have to agree, as one fixes the number of generators of a boolean algebra of sets and the other fixes the size of the parameter set. However, as the following lemma demonstrates, they both give the same asymptotic definition of dual vc-density.

Lemma 1.3.11.

$$\mathrm{vc}^*(\Phi) = \textit{degree of polynomial growth of } \pi^*_\Phi(n) = \limsup_{n\to\infty} \frac{\log \pi^*_\Phi(n)}{\log n}.$$

Proof. With a parameter set B of size n, we get at most $|\Phi|n$ sets $\phi(M^{|x|}, b)$ with $\phi \in \Phi, b \in B$. We check that asymptotically it doesn't matter whether we look at growth of boolean algebra of sets generated by n or by $|\Phi|n$ many sets. We have:

$$\pi^*_{\mathcal{F}_\Phi}(n) \le \pi^*_\Phi(n) \le \pi^*_{\mathcal{F}_\Phi}(|\Phi|n).$$

Hence:

$$\begin{aligned}
&\mathrm{vc}^*(\Phi) \le \limsup_{n\to\infty} \frac{\log \pi^*_\Phi(n)}{\log n} \le \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log n} = \\
&= \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log |\Phi|n} \frac{\log |\Phi|n}{\log n} = \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log |\Phi|n} \le \\
&\le \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(n)}{\log n} = \mathrm{vc}^*(\Phi).
\end{aligned}$$

□

One can check that the shatter function and hence VC-dimension and vc-density of a formula are elementary notions, so they only depend on the first-order theory of the structure M.

NIP theories are a natural context for studying vc-density. In fact we can take the following as the definition of NIP:

Definition 1.3.12. Define ϕ to be NIP if it has finite VC-dimension in a theory T. A theory T is NIP if all the formulas in T are NIP.

In a general combinatorial context (for arbitrary set systems), vc-density can be any real number in $0 \cup [1, \infty)$ (see [Ass85]). Less is known if we restrict our attention to NIP theories. Proposition 4.6 in [ADH16] gives examples of formulas that have non-integer rational vc-density in an NIP theory, however it is open whether one can get an irrational vc-density in this model-theoretic setting.

Instead of working with a theory formula by formula, we can look for a uniform bound for all formulas:

Definition 1.3.13. For a given NIP structure M, define the vc-function

$$\begin{aligned}\mathrm{vc}^M(n) &= \sup\{\mathrm{vc}^*(\phi(x,y)) \mid \phi \in \mathcal{L}(M), |x| = n\} \\ &= \sup\{\mathrm{vc}(\phi(x,y)) \mid \phi \in \mathcal{L}(M), |y| = n\} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}\,.\end{aligned}$$

As before this definition is elementary, so it only depends on the theory of M. We omit the superscript M if it is understood from the context. One can easily check the following bounds:

Lemma 1.3.14 (Lemma 3.22 in [ADH16])**.** *We have* $\mathrm{vc}(1) \geq 1$ *and* $\mathrm{vc}(n) \geq n\,\mathrm{vc}(1)$.

However, it is not known whether the second inequality can be strict or even just whether $\mathrm{vc}(1) < \infty$ implies $\mathrm{vc}(n) < \infty$.

1.4 Dp-rank and dp-minimality

Dp-rank is a popular dimension notion used in the study of NIP theories, and is used to define dp-minimality. Those notions originated in [She14], and were further studied in [KOU13], where it was shown, for example, that dp-rank is additive. Here it is easiest for us to introduce dp-rank in terms of vc-density over indiscernible sequences.

Definition 1.4.1.

- Work in a $\aleph_1$-saturated structure M. Fix a finite collection of formulas $\Phi(x,y)$ in the language of M. Suppose $A = (a_i)_{i\in\mathbb{N}}$ is an indiscernible sequence with each $a_i \in M^{|x|}$. Let

$$\mathscr{I}(A,\Phi) = \{\phi(A,b) \mid \phi \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|}).$$

  This gives rise to a set system $(M^{|x|}, \mathscr{I}(A,\Phi))$.

- Define

$$\mathrm{vc}_{\mathrm{ind}}(\Phi) = \sup\left\{\mathrm{vc}(\mathscr{I}(A,\Phi)) \mid A = (a_i)_{i\in\mathbb{N}} \text{ is indiscernible}\right\}.$$

- The dp-rank of an $\aleph_1$-saturated structure M is $\leq n$ if $\mathrm{vc}_{\mathrm{ind}}(\phi) \leq n$ for all formulas ϕ.

- The dp-rank of a theory T is $\leq n$ if dp-rank is $\leq n$ for any (all) $\aleph_1$-saturated model of T.

- A theory T is said to have finite dp-rank if its dp-rank is $\leq n$ for some n.

- A theory T is dp-minimal if its dp-rank ≤ 1.

Refer to [GH14] for the connection between the classical definition of dp-rank and the definition given here.

There is a useful characterization of dp-minimality in terms of indiscernible sequences that will be useful for what we do:

Lemma 1.4.2 (see Lemma 1.4 in [Sim11])**.** *Suppose* $\mathbb{M}$ *is an* $\aleph_1$*-saturated structure. Then the following are equivalent:*

- $\mathbb{M}$ *is dp-minimal.*

- *For any countable indiscernible sequence* $(a_i)_{i \in \mathcal{I}}$ *indexed by a dense linear order* $\mathcal{I}$*, and any* $c \in M$*, there is* i_0 *in the completion of* I *such that the two sequences* $(\mathrm{tp}(a_i/c) \mid i < i_0)$ *and* $(\mathrm{tp}(a_i/c) \mid i > i_0)$ *are constant.*

CHAPTER 2

Trees

(work in progress)

CHAPTER 3

Shelah-Spencer Graphs

(work in progress)

CHAPTER 4

An Additive Reduct of the P-adic Numbers

(work in progress)

CHAPTER 5

Dp-minimality in Flat Graphs

In this chapter we show that the theory of superflat graphs is dp-minimal.

5.1 Preliminaries

Superflat graphs were introduced in [PZ78] as a natural class of stable graphs. Here we present a direct proof showing dp-minimality.

First, we introduce some basic graph-theoretic definitions.

Definition 5.1.1. Work in a possibly infinite graph $\mathbb{G}$. Let $A, B, S, V \subseteq G$ where G is the set of vertices of G.

1. A path is a subgraph of $\mathbb{G}$ with distinct vertices $v_0, v_1, \ldots, v_n$ and an edge between v_{i-1}, v_i for all $i = 1, \ldots n$. It is called a path from A to B if $v_0 \in A$ and $v_n \in B$. A length of such a path is n.

2. Two paths are disjoint if, excluding endpoints, they have no vertices in common.

3. For $a, b \in G$ define the distance $d(a, b)$ to be the length of the shortest path from a to b in G. If no such path exists then the distance is infinite.

4. For $a, b \in G - A$ define $d_A(a, b)$ to be the distance between a and b in the subgraph of $\mathbb{G}$ induced on the vertices $G - A$. Equivalently it is the shortest path between a and b that avoids vertices in A.

5. We say that S separates A from B if there exists $a \in A - S$, $b \in B - S$, with $d_S(a, b) = \infty$.

6. We say that A separates V if it separates V from itself.

7. We say that V has connectivity n if there is a set of size n that separates V, but there are no sets of size $n-1$ that separate V.

8. Suppose V has connectivity n. A connectivity hull of V is defined to be the union of all sets of size n separating V.

In [AB09] we find a generalization of Megner's Theorem for infinite graphs:

Theorem 5.1.2. *Let A and B be two sets of vertices in a possibly infinite graph. Then there exists a set P of disjoint paths from A to B, and a set S of vertices separating A from B, such that S consists of a choice of precisely one vertex from each path in P.*

We use the following easy consequences:

Corollary 5.1.3. *Let V be a subset of vertices of a graph $\mathbb{G}$ with connectivity n. Then there exists a set of n disjoint paths from V into itself.*

Corollary 5.1.4. *With assumptions as above, the connectivity hull of V is finite.*

Proof. All the separating sets have to have exactly one vertex in each of those paths. □

Definition 5.1.5.

- A graph K_n^m denotes a graph obtained from a complete graph on n vertices by adding m vertices to every edge.
- A graph is called superflat if for every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that the graph avoids K_n^m as a subgraph.

Theorem 2 in [PZ78] gives a useful characterization of the superflat graphs.

Theorem 5.1.6. *The following are equivalent:*

1. *$\mathbb{G}$ is superflat.*

2. *For every $n \in \mathbb{N}$ and an infinite set $A \subseteq G$, there exists a finite $B \subseteq G$ and infinite $A' \subseteq A$ such that for all $x, y \in A'$ we have $d_B(x, y) > n$.*

Roughly, in superflat graphs every infinite set contains a sparse infinite subset (possibly after throwing away finitely many nodes).

5.2 Indiscernible sequences

Fix an uncoutable cardinal κ. Work in a superflat graph $\mathbb{S}$ that is κ-saturated and strongly κ-homogeneous. Fix a parameter set $A \subseteq S$ with $|A| < \kappa$. Let $I = (a_i)_{i \in \mathcal{I}}$ be a countable A-indiscernible sequence.Stability implies that I is totally indiscernible (see Lemma 1.2.8).

Definition 5.2.1.

- For a subsequence $\mathcal{J} \subseteq \mathcal{I}$ let $a(\mathcal{J})$ denote the tuple obtained by concatenating $(a_j)_{j \in \mathcal{J}}$.
- Let m be the arity of elements of I, that is $a_i \in S^m$. We call a set $H \subseteq S$ uniformly definable from I if there is a formula $\phi(x, y_1, \ldots, y_k)$ with $|y_i| = m$ such that for every $\mathcal{J} \subseteq \mathcal{I}$ of size k we have $H = \phi(G, a(\mathcal{J}))$.

First suppose that I consists of singletons, that is $a_i \in S$.

Definition 5.2.2. Let $V \subseteq S$. Define $P_n(V)$, a subgraph of $\mathbb{S}$, to be a union of all paths of length $\leq n$ between the vertices of V.

Lemma 5.2.3. *Let $n \in \mathbb{N}$. There exists a finite set $B \subseteq S$ such that*

$$\forall i \neq j \; d_B(a_i, a_j) > n.$$

Proof. By a 5.1.6 we can find an infinite $\mathcal{J} \subseteq \mathcal{I}$ and a finite set B' such that each pair from $J = (a_j)_{j \in \mathcal{J}}$ has distance $> n$ over B'. By total indiscernibility there exists an automorphism mapping J to I and fixing A. Image of B' under this automorphism is the required set B. $\square$

In other words, B separates I when viewed inside the subgraph $P_n(I)$. This shows that I has finite connectivity in $P_n(I)$. Applying Corollary 5.1.4 we obtain that the connectivity hull of I in $P_n(I)$ is finite.

Definition 5.2.4. Given a graph $\mathbb{G}$ and $V \subseteq G$ define $H(\mathbb{G}, V) \subseteq G$ to be the connectivity hull of V in $\mathbb{G}$. Note that if V is finite, we have that $H(P_n(V), V)$ is V-definable.

Lemma 5.2.5. *Let H be the connectivity hull of I inside of graph $P_n(I)$, that is $H = H(P_n(I), I)$. Then H is uniformly definable from I in $\mathbb{S}$.*

Proof. Using total indiscernability we may assume without the loss of generality that I is indexed by $\mathbb{N}$. Let $I_i = \{a_0, a_1, \ldots, a_{i-1}\}$ a finite segment of the sequence. Let N be the connectivity of I inside of $P_n(I)$.

First note that any finite set $H \subseteq P_n(I)$ will be contained in $P_n(I_i)$ for large enough i. Every element of H is inside a path of length $\leq n$ and endpoints of that path are eventually going to be inside I_i. (Here the assumption that I is enumerated by $\mathbb{N}$ is important.)

Vertices a_0, a_1 cannot be separated by less than N elements inside of $P_n(I)$ (as this would contradict connectivity being N). Thus by Theorem 5.1.2 there are N disjoint paths inside of $P_n(I)$ connecting a_0 to a_1. For large enough i, say $i \geq M_1$, all these paths are contained inside of $P_n(I_i)$. Those paths witness that vertices a_0, a_1 cannot be separated by less than N elements inside of $P_n(I_i)$. As the set $P_n(I_i)$ is I_i-definable and I is indiscernible, we have that no two vertices can be separated by less than N elements inside of $P_n(I_i)$ Thus I_i has connectivity $\geq N$ inside of $P_n(I_i)$ for $i \geq M_1$.

Consider a set S of size N that separates I inside of $P_n(I)$. This is witnessed by two elements of I that are separated. There are finitely many such sets S as connectivity hull is finite. Thus for large enough i, say $i \geq M_2$, for each such S the segment I_i contains a pair of vertices witnessing that S is a separating set.

Corollary 5.1.3 tells us that there are finitely many paths between elements of V such that $H(P_n(I), I)$ is inside the union of those paths. For large enough i, say $i \geq M_3$, $P_n(I_i)$ will contain all of those paths, and thus $H(P_n(I), I) \subseteq P_n(I_i)$.

Combine those three observations. Let $M = \max(M_1, M_2, M_3)$. Then for $i \geq M$ the set $P_n(I_i)$ contains all the N-element sets separating I in $P_n(I)$, those sets separate I_i in $P_n(I_i)$, and the connectity of I_i in $P_n(I_i)$ is at most N. But this means that the connectity of I_i in $P_n(I_i)$ has to be exactly N, and $H(P_n(I), I) \subseteq H(P_n(I_i), I_i)$.

For $i \geq M$ define

$$E_i = \bigcap_{j=M}^{i} H(P_n(I_j), I_j).$$

We have $H(P_n(I), I) \subseteq E_i$ and E_i is a decreasing chain. Suppose $H(P_n(I), I) \subsetneq H(P_n(I_M), I_M)$, that is $H(P_n(I_M), I_M) - H(P_n(I), I) \neq \emptyset$. Then there exists a set S of size N that separates I_M in $P_n(I_M)$ but does not separate I in $P_n(I)$. Thus there has to be a finite subgraph of $P_n(I)$ disjoint from S that connects all the elements of I_M (witness of failure of separation). For large enough i, say $i \geq M_S$, this subgraph is contained in $P_n(I_i)$. There are finitely many possibilities for S (as connectivity hull of I_M in $P_n(I_M)$ is finite). Let $M_4 = \max_S(M_S)$. Then for $i \geq \max(M_4, M)$ we have

$$H(P_n(I_i), I_i) \cap (H(P_n(I_M), I_M) - H(P_n(I), I)) = \emptyset,$$

and thus $E_i = H(P_n(I), I)$. As E_i is I_i-definable, this shows that $H(P_n(I), I)$ is I_i-definable. Now we need to show uniform definability. Suppose I' is a subsequence of I of length i. There is an automorphism mapping I_i to I' that fixes I setwise. But then it has to fix $H(P_n(I), I)$ setwise. Then it maps I_i-definition of $H(P_n(I), I)$ to a I'-definition of $H(P_n(I), I)$. As I' was arbitrary this shows uniformity. □

Corollary 5.2.6. *Let $H_n = H(P_n(I), I)$. Then*

$$\forall i \neq j \ d_{H_n}(a_i, a_j) > n.$$

Proof. The set H_n separates I inside of $P_n(I)$. In particular there exist $i \neq j$ such that $d_{H_n}(a_i, a_j) = \infty$ inside $P_n(I)$. This means that $d_{H_n}(a_i, a_j) > n$ inside of $\mathbb{S}$. But then by total indiscernibility and using the fact that H_n is uniformly I-definable, this holds for all $i \neq j$. □

We would like to start working with tuples now instead of singletons. We need some notation to extract individual elements of a tuple:

Definition 5.2.7. Suppose $a = (a_1, \ldots, a_m)$ is a tuple of arity m. Let $a^{(j)}$ denote the i'th component, that is $a^{(j)} = a_j$.

More generally, now suppose that I consists of tuples of arity m, that is $a_i \in S^m$.

Definition 5.2.8.

- We would like extract j'th components out of elements of I. Let $I^{(j)} = ({a_i}^{(j)})_{i \in \mathcal{I}}$, an A-indiscernible sequence of singletons.
- Let ${H_n}^{(j)} = H(P_n(I^{(j)}), I^{(j)})$.
- Let
$$B_n = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} {H_n}^{(j)}.$$
Note that B_n is finite as each ${H_n}^{(j)}$ is finite by Corollary 5.1.4.

Lemma 5.2.9. *The sequence I is indiscernible over the $A \cup B_n$.*

Proof. By Lemma 5.2.5 the set ${H_n}^{(j)}$ is uniformly $I^{(j)}$-definable. Thus it is uniformly I-definable. Then B_n is a finite union of uniformly I-definable sets, thus also uniformly I-definable.

By uniform definability there is a formula $\phi(z, w_1, \dots, w_k)$ with $|z| = 1$ and $|w_i| = m$ such that for any subsequence $\mathcal{J} \subseteq \mathcal{I}$ of length k we have $\phi(G, a(\mathcal{J})) = B_n$. Fix such a subsequence $\mathcal{J}$.

Let $\psi(x_1, \dots, x_l, y)$ be an arbitrary A-formula with $|x_i| = m$. Consider a collection of traces (i.e a collection of subsets of $B_n^{|y|}$)

$$\left\{ \psi(a(\mathcal{J}'), B_n^{|y|}) \mid \ \mathcal{J}' \text{ a subsequence of } \mathcal{I} \text{ of length } l \text{ disjoint from } \mathcal{J} \right\}.$$

If two of the traces are distinct, then by indiscernibility all of them are (using the fact that B_n is uniformly definable). But that is impossible as B_n is finite and thus has finitely many subsets. Thus all such traces are identical. As the choice of $\mathcal{J}$ was arbitrary, we can drop the condition that $\mathcal{J}'$ is disjoint from $\mathcal{J}$. This shows that for any $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{I}$ of length l and $h \in B_n^{|y|}$ we have

$$\mathbb{S} \models \psi(a(\mathcal{J}_1), h) \iff \mathbb{S} \models \psi(a(\mathcal{J}_2), h).$$

As the choice of ψ was arbitrary, this shows that I is indiscernible over $A \cup B_n$ as needed. □

Definition 5.2.10. For tuples a, b of the same arity m and $B \subseteq S$ define

$$d_B(a,b) = \min_{1 \le i,j \le m} d_B(a^{(i)}, b^{(j)}).$$

Lemma 5.2.11.

$$\forall i \neq j \ d_{B_n}(a_i, a_j) > n/2.$$

Proof. Towards a contradiction suppose we have some $i \neq j$ and k, l such that

$$d_{B_n}({a_i}^{(k)}, {a_j}^{(l)}) \le n/2.$$

As B_n is uniformly I-definable, by total indiscernability we have that this inequality holds for all $i \neq j$. Assuming for convenience that I is enumerated by naturals, let $b_1 = {a_1}^{(k)}$, $b_2 = {a_2}^{(l)}$, $b_3 = {a_3}^{(k)}$ (note the superscripts). Then we have

$$d_{B_n}(b_1, b_2) \le n/2,$$
$$d_{B_n}(b_3, b_2) \le n/2.$$

By the triangle inequality

$$d_{B_n}(b_1, b_3) \le n,$$
$$d_{B_n}({a_1}^{(k)}, {a_3}^{(k)}) \le n.$$

But this is a contradiction, as Lemma 5.2.12 gives us

$$\forall i \neq j \ d_{{H_n}^{(k)}}({a_i}^{(k)}, {a_j}^{(k)}) > n$$

and we have ${H_n}^{(k)} \subseteq B_n$. □

Corollary 5.2.12. *There is a countable B such that I is indiscernible over $A \cup B$ and*

$$\forall i \neq j \ d_B(a_i, a_j) = \infty.$$

Proof. Let B_n as above. By Lemma 5.2.11 we have

$$\forall i \neq j \ d_{B_n}(a_i, a_j) > n,$$

and I is indiscernible over $A \cup B_n$ by Lemma 5.2.9. Let $B = \bigcup_{n \in \mathbb{N}} B_n$. Then

$$\forall i \neq j \; d_B(a_i, a_j) = \infty.$$

As $B_n \subseteq B_{n+1}$, the sequence I is indiscernible over $A \cup B$ as needed. $\square$

Thus I can be upgraded to have infinite distance over its parameter set.

5.3 Superflat graphs are dp-minimal

Definition 5.3.1. For $B \subseteq S$ define an equivalence relation $\sim_B$ on $S - B$. Two vertices b, c are $\sim_B$-equivalent if $d_B(b, c)$ is finite.

Lemma 5.3.2. *Fix tuples a, b, c in S, with a, b having the same arity. Also let $B \subseteq S$. Suppose* $\mathrm{tp}(a/B) = \mathrm{tp}(b/B)$ *and* $d_B(a, c) = d_B(b, c) = \infty$. *Then* $\mathrm{tp}(a/Bc) = \mathrm{tp}(b/Bc)$.

Proof. Suppose $a = (a_1, a_2, \ldots, a_m)$ and $b = (b_1, b_2, \ldots, b_m)$. Define X_j to be the $\sim_B$-equivalence class of a_j or $X_j = \emptyset$ if $a_j \in B$. Similarly define Y_j for b_j. There is an automorphism f of $\mathbb{S}$ fixing B with $f(a) = b$. It's easy to see that $f(X_j) = Y_j$ setwise. We would like to define a function $g \colon S \longrightarrow S$. For each j let $g = f$ on X_j. Additionally if $X_j \neq Y_j$ then also let $g = f^{-1}$ on Y_j. Define g to be identity on the rest of S. It is easy to check that g is a well-defined automorphism fixing Bc that maps a to b. This shows that $\mathrm{tp}(a/Bc) = \mathrm{tp}(b/Bc)$. $\square$

Lemma 5.3.3. *Let $b \in G$. There exists $c \in \mathcal{I}$ such that all $(a_i)_{i \in \{\mathcal{I} - c\}}$ have the same type over Ab.*

Proof. Use Corollary 5.2.12 to find $B \supseteq A$ such that I is indiscernible over B and has infinite distance over B. All the tuples of the indiscernible sequence fall into distinct $\sim_B$-equivalence classes. If $b \in B$ we are done. Otherwise, there can be at most one element of the sequence that is in the same $\sim_B$-equivalence class as b. Exclude that element from the sequence. Remaining sequence elements are all infinitely far away from b over B. By the previous lemma we have that elements of indiscernible sequence all have the same type over Bb as needed. $\square$

Corollary 5.3.4. *Flat graphs are dp-minimal.*

Proof. It suffices to show that $\mathbb{S}$ is dp-minimal. Using Lemma 1.4.2, by total indiscernibility it is enough to show that if $b \in S$ and I is a countable sequence indiscernible over $\emptyset$, then one element can be excluded from I, so that the remaining elements have the same type over b. But this is precisely Lemma 5.3.3. □

5.4 Conclusion

The determination of dp-minimality is the first step towards establishing bounds on vc-density. It is this author's hope that the simple structure of flat graphs yields nicely behaved vc-density. We pose the following question for the future work:

Open Question: What are bounds on vc-function $\mathrm{vc}(n)$ in flat graphs? In particular do we have $\mathrm{vc}(1) = 1$ or $\mathrm{vc}(n) = n\,\mathrm{vc}(1)$? Are the bounds better in specific classes of flat graphs, such as planar graphs, graphs with bounded tree-width, or graphs excluding certain classes of subgraphs?

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