

# VC-density in model theoretic structures

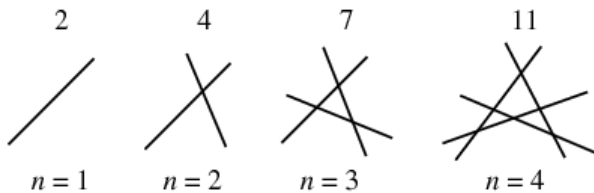
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Suppose we have an (infinite) collection of sets  $\mathcal{F}$ .  
We define the *shatter function*  $\pi_{\mathcal{F}}: \mathbb{N} \longrightarrow \mathbb{N}$  of  $\mathcal{F}$

$$\pi_{\mathcal{F}}(n) = \max\{\# \text{ of atoms in the boolean algebra generated by } \mathcal{S} \\ | \mathcal{S} \subset \mathcal{F} \text{ with } |\mathcal{S}| = n\}$$

Example: Let  $\mathcal{F}$  consist of all half-planes in the plane.



$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 7 \quad \pi_{\mathcal{F}}(4) = 11$$

$$\pi_{\mathcal{F}}(n) = n^2/2 + n/2 + 1$$

More examples:

1. Disks in the plane:  $\pi_{\mathcal{F}}(n) = n^2 - n + 2$
2. Balls in  $\mathbb{R}^3$ :  $\pi_{\mathcal{F}}(n) = n^3/3 - n^2 + 8n/3$
3. Intervals in the line:  $\pi_{\mathcal{F}}(n) = 2n$
4. Finite subsets of  $\mathbb{N}$ :  $\pi_{\mathcal{F}}(n) = 2^n$
5. Convex polygons in the plane:  $\pi_{\mathcal{F}}(n) = 2^n$

## Theorem (Sauer-Shelah '72)

*The shatter function is either  $2^n$  or bounded by a polynomial.*

## Definition

Suppose the growth of the shatter function of  $\mathcal{F}$  is polynomial. Let  $\text{vc}(\mathcal{F})$  be the infimum of all positive reals  $r$  such that

$$\pi_{\mathcal{F}}(n) = O(n^r)$$

Call  $\text{vc}(\mathcal{F})$  the  $\text{vc}$ -density of  $\mathcal{F}$ . If the shatter function grows exponentially, we let  $\text{vc}(\mathcal{F}) := \infty$ .

# Applications

- ▶ Model Theory (NIP theories)
- ▶ VC-Theorem in probability (Vapnik-Chervonenkis '71)
- ▶ Computational learning theory (PAC learning, Warmuth conjecture)
- ▶ Computational geometry
- ▶ Functional analysis (Bourgain-Fremlin-Talagrand theory)
- ▶ Abstract topological dynamics (tame dynamical systems)

# History

- ▶ VC-dimension defined by Vapnik-Chervonenkis '71
- ▶ NIP theories studied by Shelah '71
- ▶ vc-density in model theoretic context introduced by Aschenbrenner, Dolich, Haskell, Macpherson, Starchenko '13

# Model Theory

Model Theory studies definable sets in first-order structures.

$$(\mathbb{Q}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) := \exists y \ y \cdot y = x$$

$\phi(\mathbb{Q})$  defines the set of rationals that are a square.



$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) := \exists y \ y \cdot y = x$$

$\phi(\mathbb{R})$  defines the set  $[0, \infty)$ .

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\psi(x_1, x_2) := (x_1 \cdot x_1 + x_2 \cdot x_2 \leq 1.5) \wedge (x_1^2 \leq x_2)$$

$\phi(\mathbb{R}^2)$  defines the set in  $\mathbb{R}^2$  that is an intersection of a disc with an inside of a parabola.

We work with families of uniformly definable sets. Fix a formula  $\phi(x_1 \dots x_m, y_1, \dots y_n) = \phi(\vec{x}, \vec{y})$ . Plug in elements from  $M$  for  $y$  variables to get a family of definable sets in  $M^m$ .

$$\mathcal{F}_\phi^M = \{\phi(M^m, a_1, \dots a_n) \mid a_1, \dots a_n \in M\}$$

Define  $\text{vc}^M(\phi)$  to be the vc-density of the family  $\mathcal{F}_\phi^M$

$$\phi(x_1, x_2, y_1, y_2, y_3) := (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq y_3^2$$

In structure  $(\mathbb{R}, 0, 1, +, \cdot, \leq)$  given  $a, b, r \in \mathbb{R}$  the formula  $\phi(x_1, x_2, a, b, r)$  defines a disk in  $\mathbb{R}^2$  with radius  $r$  with center  $(a, b)$ . Thus  $\mathcal{F}_\phi^\mathbb{R}$  is a collection of all disks in  $\mathbb{R}^2$ .

Shelah ('78) classified number of isomorphism classes for structures elementarily equivalent to structure  $M$ . One of the important classes is NIP structures. Structure  $M$  is said to be NIP if all uniformly definable families in it have finite vc-density.

- ▶  $(\mathbb{C}, 0, 1, +, \cdot)$  is NIP
- ▶  $(\mathbb{R}, 0, 1, +, \cdot, \leq)$  is NIP
- ▶  $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$  is NIP
- ▶ Random graph  $(V, R)$  is not NIP
- ▶  $(\mathbb{Q}, 0, 1, +, \cdot)$  is not NIP.

Given an NIP structure  $M$  we define a vc-function of  $n$  to be the largest vc-density achieved by families of uniformly definable sets in  $M^n$ .

$$\text{vc}^M(n) = \max \left\{ \text{vc}^M(\phi) \mid \phi(\vec{x}, \vec{y}) \text{ with } |\vec{x}| = n \right\}$$

Easy to show  $\text{vc}^M(n) \geq n \text{vc}^M(1) \geq n$

Open Question: If  $M$  is NIP, is it possible for  $\text{vc}^M(\phi)$  to be irrational? Open Question: Is  $\text{vc}^M(n) = n \text{vc}^M(1)$ ? If not, is there a linear relationship? If  $\text{vc}(1) < \infty$  do we have  $\text{vc}(2) < \infty$ ?

## Examples

- ▶  $(\mathbb{R}, 0, 1, +, \cdot, \leq)$  has  $\text{vc}(n) = n$  (true for o-minimal structures)
- ▶  $(\mathbb{C}, 0, 1, +, \cdot)$  has  $\text{vc}(n) = n$
- ▶  $(\mathbb{Q}_p, 0, 1, +, \cdot)$  has  $\text{vc}(n) \leq 2n - 1$

## vc-density in trees

Consider structure  $(T, \leq)$  where elements of  $T$  are vertices of a rooted tree and we say that  $a \leq b$  if  $a$  is below  $b$  in the tree.

- ▶ Trees are NIP (Parigot '82)
- ▶ Trees are dp-minimal (Simon '11)
- ▶ Trees have  $vc(n) = n$  (B. '13)



## proof background

$\text{tp}(a)$ , a type of an element  $a$  is a set of all the formulas that are true about  $a$ .

Parigot's observation: there is a natural way to split a tree into parts  $A, B$  such that for  $a \in A$  and  $b \in B$  we have

$$\text{tp}(a), \text{tp}(b) \vdash \text{tp}(ab)$$

This allows us to decompose complex types into simple parts, which we can use to compute vc-density.

# Shelah-Spencer graphs

Let  $\alpha$  irrational  $\in (0, 1)$ . Consider a random graph on  $n$  vertices where the probability of any given two vertices having an edge is  $n^{-\alpha}$ . Shelah-Spencer ('88) showed that 0-1 law holds for first-order formulas. A structure satisfying those axioms is called a Shelh-Spencer graph.

- ▶ Shelah-Spencer graphs are stable (Baldwin-Shi '96, Baldwin-Shelah '97)

# Background

## Definition

- ▶ To a finite graph  $A$  assign a dimension  $\delta(A) = |V| - \alpha|E|$ .
- ▶  $B/A$  is called a positive extension if quantity  $\delta(B/A) = |V_B/V_A| - \alpha|E_B/E_A|$  is positive.
- ▶  $B/A$  is called minimal if its dimension is negative, but every subextension is positive.
- ▶  $(A_0, \dots, A_n)$  is a minimal chain if each  $A_{i+1}/A_i$  is minimal.

For  $B/A$  chain-minimal define

$$\phi_{A,B}(\vec{x}) = \exists \vec{x}^* \text{ such that } \vec{x}^* / \vec{x} \text{ is isomorphic to } B/A$$

## Theorem (quantifier elimination, Laskowski '06)

*In Shelah-Spencer graph every definable set can be defined by a boolean combination of formulas  $\phi_{A_i, B_i}(\vec{x})$ .*

# vc-density in Shelah-Spencer graphs

## Theorem (B., '15)

*For a formula  $\phi(\vec{x}, \vec{y})$  we can define  $\epsilon_L, \epsilon_U$  explicitly computable from  $\delta(B_i/A_i)$  such that*

$$\epsilon_L |\vec{x}| \leq \text{vc}(\phi) \leq \epsilon_U |\vec{x}|$$

## Corollary

$\text{vc}(1) = \infty$ , so *vc-function is not well-behaved for this structure.*

# Future work

- ▶  $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$
- ▶ Other partial orderings, lattices
- ▶ Other graph structures, in particular flat graphs