Math 285D Notes: 11/17, 11/19, 11/21

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0.1 Lemma (Associativity)**.** Let $\alpha, \beta \in \mathbf{On}$, $(a_i)_{i<\alpha+\beta}$ be a strictly decreasing sequence in **No** and $f_i \in \mathbb{R}$ for $i < \alpha + \beta$. Then,

$$\sum_{i<\alpha+\beta} f_i \omega^{a_i} = \sum_{i<\alpha} f_i \omega^{a_i} + \sum_{j<\beta} f_{\alpha+j} \omega^{a_{\alpha+j}}.$$

Proof. We proceed by induction on β. In the case that $\beta = \gamma + 1$ is a successor ordinal, we have

$$\begin{aligned}
\sum_{i<\alpha+(\gamma+1)} f_i \omega^{a_i} &= \sum_{i<\alpha+\gamma} f_i \omega^{a_i} + f_{\alpha+\gamma}\omega^{a_{\alpha+\gamma}} \\
&= \sum_{i<\alpha} f_i \omega^{a_i} + \sum_{j<\gamma} f_{\alpha+j}\omega^{a_{\alpha+j}} + f_{\alpha+\gamma}\omega^{a_{\alpha+\gamma}} \\
&= \sum_{i<\alpha} f_i \omega^{a_i} + \sum_{j<\gamma+1} f_{\alpha+j}\omega^{a_{\alpha+j}},
\end{aligned}$$

where the first and third equality use the definition of $\sum$ and the second equality uses the induction hypothesis.

In the case where β is a limit ordinal, we let

$$\{L|R\} = \sum_{j<\beta} f_{\alpha+j}\omega^{a_{\alpha+j}}.$$

Using the definition of addition between surreal numbers and a simple cofinality argument, we obtain

$$\sum_{i<\alpha} f_i \omega^{a_i} + \sum_{j<\beta} f_{\alpha+j}\omega^{a_{\alpha+j}} = \left\{ \sum_{i<\alpha} f_i \omega^{a_i} + L \middle| \sum_{i<\alpha} f_i \omega^{a_i} + R \right\}.$$

A typical element of this cut is

$$\sum_{i<\alpha} f_i \omega^{a_i} + \sum_{j\leq\gamma} f_{\alpha+j}\omega^{a_{\alpha+j}} - \varepsilon\omega^{a_{\alpha+\gamma}} \qquad (\gamma < \beta, \varepsilon \in \mathbb{R}^{>0}).$$

By inductive hypothesis, this equals

$$\sum_{i<\alpha+\gamma} f_i \omega^{a_i} - \varepsilon\omega^{a_{\alpha+\gamma}}.$$

But these elements are cofinal in the cut defining $\sum_{i<\alpha+\beta} f_i\omega^{a_i}$; hence, the claim follows by cofinality. $\square$

0.2 Proposition. Let $\alpha \in \mathbf{On}$, $(a_i)_{i<\alpha}$ be a strictly decreasing sequence in **No** and $f_i, g_i \in \mathbb{R}$ for $i < \alpha$. Then,

$$\sum_{i<\alpha} f_i \omega^{a_i} + \sum_{i<\alpha} g_i \omega^{a_i} = \sum_{i<\alpha} (f_i + g_i)\omega^{a_i}.$$

Proof. We proceed by induction on α. If $\alpha = \beta + 1$ is a successor, then

$$\begin{aligned}
\sum_{i<\beta+1} f_i \omega^{a_i} + \sum_{i<\beta+1} g_i \omega^{a_i} &= \left(\sum_{i<\beta} f_i \omega^{a_i} + f_\beta \omega^{a_\beta}\right) + \left(\sum_{i<\beta} g_i \omega^{a_i} + g_\beta \omega^{a_\beta}\right) \\
&= \left(\sum_{i<\beta} f_i \omega^{a_i} + \sum_{i<\beta} g_i \omega^{a_i}\right) + (f_\beta \omega^{a_\beta} + g_\beta \omega^{a_\beta}) \\
&= \sum_{i<\beta} (f_i + g_i)\omega^{a_i} + (f_\beta + g_\beta)\omega^{a_\beta} \\
&= \sum_{i<\beta+1} (f_i + g_i)\omega^{a_i},
\end{aligned}$$

where the third equality uses the induction hypothesis.

Now suppose α is a limit. One type of element from the lef-hand-side of the cut defining $\sum_{i<\alpha} f_i \omega^{a_i} + \sum_{i<\alpha} g_i \omega^{a_i}$ is of the form

$$\sum_{i\le\beta} f_i \omega^{a_i} - \varepsilon \omega^{a_\beta} + \sum_{i<\alpha} g_i \omega^{a_i}$$

or of the form

$$\sum_{i<\alpha} f_i \omega^{a_i} + \sum_{i\le\beta} g_i \omega^{a_i} - \varepsilon \omega^{a_\beta}.$$

We have

$$\begin{aligned}
\sum_{i\le\beta} f_i \omega^{a_i} - \varepsilon \omega^{a_\beta} + \sum_{i<\alpha} g_i \omega^{a_i} &= \sum_{i\le\beta} f_i \omega^{a_i} + \sum_{i\le\beta} g_i \omega^{a_i} + \sum_{\beta<i<\alpha} g_i \omega^{a_i} - \varepsilon \omega^{a_\beta} \\
&= \sum_{i\le\beta} (f_i + g_i)\omega^{a_i} + \sum_{\beta<i<\alpha} g_i \omega^{a_i} - \varepsilon \omega^{a_\beta},
\end{aligned}$$

where the first equality follows from (0.1) and the second equality uses the inductive hypothesis. But this is mutually cofinal with

$$\sum_{i\le\beta} (f_i + g_i)\omega^{a_i} - \varepsilon \omega^{a_\beta}.$$

Similarly if we star with $\sum_{i<\alpha} f_i \omega^{a_i} + \sum_{i\le\beta} g_i \omega^{a_i} - \varepsilon \omega^{a_\beta}$. $\square$

0.3 Lemma. Let $\alpha \in \mathbf{On}$, $(a_i)_{i<\alpha}$ be a strictly decreasing sequence in $\mathbf{No}$, $b \in \mathbf{No}$, and $f_i \in \mathbb{R}$ for $i < \alpha$. Then,

$$\left(\sum_{i<\alpha} f_i \omega^{a_i}\right)\omega^b = \sum_{i<\alpha} f_i \omega^{a_i+b}.$$

Note that the sequence $(a_i + b)_i$ is also strictly decreasing.

Proof. We proceed by induction on α. If $\alpha = \beta + 1$, then

$$\begin{aligned}\left(\sum_{i<\beta+1} f_i \omega^{a_i}\right)\omega^b &= \left(\sum_{i<\beta} f_i \omega^{a_i} + f_\beta \omega^{a_\beta}\right)\omega^b \\ &= \left(\sum_{i<\beta} f_i \omega^{a_i}\right)\omega^b + f_\beta \omega^{a_\beta}\cdot\omega^b \\ &= \sum_{i<\beta} f_i \omega^{a_i+b} + f_\beta \omega^{a_\beta+b} \\ &= \sum_{i<\beta+1} f_i \omega^{a_i+b},\end{aligned}$$

where the third equality uses the inductive hypothesis.

Now suppose α is a limit. Recall that, by their respective definitions,

$$\omega^b = \{0, s\omega^{b'} \mid t\omega^{b''}\}$$

and

$$\sum_{i<\alpha} f_i \omega^{a_i} = \left\{ \sum_{i\le\beta} f_i \omega^{a_i} - \varepsilon\omega^{a_\beta} \ :\ \beta < \alpha, \varepsilon \in \mathbb{R}^{>0} \ \middle|\ \sum_{i\le\beta} f_i \omega^{a_i} + \varepsilon\omega^{a_\beta} \ :\ \beta < \alpha, \varepsilon \in \mathbb{R}^{>0} \right\}.$$

Set $d := \sum_{i<\alpha} f_i \omega^{a_i}$ and let d', d'' be elements from the left and right-hand sides, respectively, of the defining cut determined by the same choice of ε. Note that

$$d - d' = \varepsilon\omega^{a_\beta} + c', \quad \text{where } c' \ll \omega^{a_\beta},$$

and

$$d'' - d = \varepsilon\omega^{a_\beta} + c'', \quad \text{where } c'' \ll \omega^{a_\beta}.$$

It follows that

$$\varepsilon_1\omega^{a_\beta} < d - d', d'' - d < \varepsilon_2\omega^{a_\beta}, \quad \text{for all } \varepsilon_1 < \varepsilon < \varepsilon_2 \text{ in } \mathbb{R}, \tag{$*$}$$

where ε is given by the choice of d', d''. Now,

$$\begin{aligned} d\omega^b &= \{d' \mid d''\} \cdot \{0, s\omega^{b'} \mid t\omega^{b''}\} \\ &= \{d'\omega^b, d'\omega^b + (d-d')s\omega^{b'}, \underline{d''\omega^b - (d''-d)\,t\omega^{b''}} \mid \\ &\qquad d''\omega^b, \underline{d'\omega^b + (d-d')\,t\omega^{b''}}, d''\omega^b - (d''-d)s\omega^{b'}\}, \end{aligned}$$

and we claim that the underlined terms are superfluous; in particular,

(1) $d''\omega^\beta - (d''-d)\,t\omega^{b''} \leq d'\omega^b + (d-d')s\omega^{b'}$;

(2) $d''\omega^b - (d''-d)s\omega^{b'} \leq d'\omega^b + (d-d')\,t\omega^{b''}$.

To show (1), note that $\omega^{b''} \gg \omega^b \gg \omega^{b'}$ implies

$$(d''-d)\,t\omega^{b''} + (d-d')s\omega^{b'} \geq \varepsilon_1\omega^{a_\beta}t\omega^{b''} > 2\varepsilon_2\omega^{a_\beta}\omega^b \geq (d''-d)\omega^b.$$

The verification for (2) is similar. So, by (1), (2) and confinality,

$$d\omega^b = \{d'\omega^b, d'\omega^b + (d-d')s\omega^{b'} \mid d''\omega^b, d''\omega^b - (d''-d)s\omega^{b'}\}.$$

We claim that we can further simplify this to

$$d\omega^b = \{d'\omega^b \mid d''\omega^b\},$$

then we are done by inductive hypothesis. Let now $\varepsilon_{1,2} \in \mathbb{R}^{>0}$ with $\varepsilon_1 < \varepsilon < \varepsilon_2$ and

$$d_1' = \sum_{i\geq\beta} f_i\omega^{a_i} - \varepsilon_1\omega^{a_\beta}, \quad d_1'' = \sum_{i\geq\beta} f_i\omega^{a_i} + \varepsilon_1\omega^{a_\beta}.$$

We claim that

$$d_1'\omega^b > d'\omega^b + (d-d')s\omega^{b'}, \quad d_1''\omega^b < d''\omega^b - (d''-d)s\omega^{b'}.$$

Notice that the first claim holds if and only if $(d_1'-d')\omega^b > (d-d')s\omega^{b'}$. But this inequality holds since

$$(d_1'-d)\omega^b = (\varepsilon-\varepsilon_2)\omega^{a_\beta}\omega^b > \varepsilon_2 s\omega^{a_\beta}\omega^{b'} \geq (d-d')s\omega^{b'},$$

where the first inequality holds since $\omega^b \gg \omega^{b'}$ and the second inequality holds by $(*)$. The second part of the claim is proved similarly. □

0.4 Proposition. Let $\alpha, \beta \in \mathbf{On}$, $(a_i)_{i<\alpha}$, $(b_j)_{j<\beta}$ be strictly decreasing sequences in **No**, and $f_i, g_i \in \mathbb{R}$ for $i < \alpha$. Then,

$$\left(\sum_{i<\alpha} f_i\omega^{a_i}\right)\left(\sum_{j<\beta} g_j\omega^{b_j}\right) = \sum_{i<\alpha, j<\beta} f_i g_j \omega^{a_i+b_j}.$$

Proof. If either α or β are successor ordinals, we verify the proposition by using the inductive hypothesis and lemma (0.3). Thus, we only need to consider the case where α and β are both limits. Put

$$f = \sum_{i<\alpha} f_i X^{a_i}, \quad g = \sum_{j<\beta} g_j X^{a_j} \in K.$$

Recall that the typical element in the cut of $f(\omega)\cdot g(\omega)$ is

$$f(\omega)g(\omega)_{**} + f(\omega)_* g(\omega) - f(\omega)_* g(\omega)_{**}, \tag{†}$$

where $*, **$ are either L or R. Moreover, this element is $< f(\omega)g(\omega)$ if and only if $(*, **) = (L, L)$ or (R, R). Take $f_*, g_{**} \in K$ such that $f_*(\omega) = f(\omega)_*$ and $g_{**}(\omega) = g(\omega)_{**}$. Then, by inductive hypothesis, † equals

$$(f\cdot g)(\omega) - ((f - f_*)(g - g_{**}))(\omega).$$

For example,

$$f_* = \sum_{i<\gamma} f_i X^{a_i} + (f_\gamma \pm \varepsilon_1)X^{a_\gamma}, \quad \gamma < \alpha$$

implies $f - f_* = \pm\varepsilon_1 X^{a_\gamma} + h_1$, where all the terms in h_1 have degree $> \gamma$. Similarly, $g - g_{**} = \pm\varepsilon_2 X^{b_\delta} + h_2$, where $\delta < \beta$ and all the terms in h_2 have degree $> \delta$. Thus,

$$(f - f_*)(g - g_{**}) = \pm\varepsilon_1\varepsilon_2 X^{a_\gamma + b_\delta} + \text{higher order terms},$$

and

$$[(f - f_*)(g - g_{**})](\omega) = \pm\varepsilon_1\varepsilon_2\omega^{\alpha_\gamma + b_\delta} + h_3(\omega),$$

where $h_3(\omega) \ll \omega^{a_\gamma + b_\delta}$. So by cofinality,

$$\begin{aligned} f(\omega)g(\omega) = \{(f\cdot g)(\omega) - \varepsilon\omega^{a_\gamma+b_\delta} \;:\; \gamma<\alpha, \delta<\beta, \varepsilon\in\mathbb{R}^{>0} \mid \\ (f\cdot g)(\omega) + \varepsilon\omega^{a_\gamma+b_\delta} \;:\; \gamma<\alpha, \delta<\beta, \varepsilon\in\mathbb{R}^{>0}\}. \end{aligned}$$

Now,

$$\begin{aligned} (f\cdot g)(\omega) = \{(f\cdot g)(\omega) - \varepsilon\omega^{a_\gamma+b_\delta} \;:\; \gamma<\alpha, \delta<\beta \text{ s.t } a_\alpha + b_\delta \in \operatorname{supp}(f\cdot g), \varepsilon\in\mathbb{R}^{>0} \mid \\ f\cdot g(\omega) + \varepsilon\omega^{a_\gamma+b_\delta} \;:\; \gamma<\alpha, \delta<\beta \text{ s.t } a_\alpha + b_\delta \in \operatorname{supp}(f\cdot g), \varepsilon\in\mathbb{R}^{>0}\}. \end{aligned}$$

Thus, $(f\cdot g)(\omega)$ satisfies the cut for $f(\omega)\cdot g(\omega)$ and the claim follows by cofinality. □

All together, this completes the proof of the following theorem.

0.5 Theorem. *The map*

$$\mathbb{R}((t^{\mathbf{No}})) \xrightarrow{\sim} \mathbf{No}, \quad \sum_{i<\alpha} f_i X^{a_i} \mapsto \sum_{i<\alpha} f_i \omega^{a_i},$$

is an ordered field isomorphism.

1 The Surreals as a Real Closed Field

Let K be a field. We call K *orderable* if some ordering on K makes it an ordered field. If K is orderable, then $\mathrm{char}(K) = 0$ and K is not algebraically closed. [1] We call K *euclidean* if $x^2 + y^2 \neq -1$ for all $x, y \in K$ and $K = \{\pm x^2 \ : \ x \in K\}$. If K is euclidean, then K isa an ordered field for a unique ordering—namely, $a \geq 0 \iff \exists x \in K. x^2 = a$.

1.1 Theorem (Artin & Schreier, 1927)**.** *For a field K, the following are equivalent.*

(1) K is orderable, but has no proper orderable algebraic field extension.

(2) K is euclidean and every polynomial $p \in K[X]$ of odd degree has a zero in K.

(3) K is not algebraically closed, but $K(i)$, $i^2 = -1$, is algebraically closed.

(4) K is not algebraically closed, but has an algebraically closed field extension L with $[L:K] < \infty$.

We call K real closed if it satisfies one of these equivalent conditions.[2]

1.2 Corollary. Let K' be a subfield of a real closed field K. Then K' is real closed if and only if K' is algebraically closed in K.

Proof. Suppose K' is not algebraically closed in K. Fix $a \in K \setminus K'$ that is algebraic over K'. Then, $K'(a)$ is an proper orderable algebraic field extension of K'. Thus, K' is not real closed by (1) of theorem (1.1).

Conversely, suppose K' is algebraically closed in K. We verify that condition (2) of theorem (1.1) holds for K'. Since K' is algebraically closed in K, any zero of a polynomial of the form $X^2 - a$ or $-X^2 - a$, where $a \in K'$, must be in K'. This along with the fact that K is euclidean implies that K' is euclidean. Moreover, if $p \in K'[X]$ has odd degree, then since K satisfies (2), p has a zero $a \in K$. But, $a \in K'$ since K' is algebraically closed in K. Thus, K is real closed. $\square$

The archetypical example of a real closed field is $\mathbb{R}$. By corollary (1.2), the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$ is also real closed. In fact, the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$ can be embedded into any real closed field.

1.3 Proposition. Suppose K is real closed and $p \in K[X]$. Then,

[1] To prove that K is not algebraically closed: suppose K is an algebraically closed ordered field and derive a contradiction using i, the square root of -1.

[2] See Lange's *Algebra* for partial proof.

(1) p is monic and irreducible if and only if $p = X - a$ for some $a \in K$ or $p = (X-a)^2 + b^2$ for some $a, b \in K$, $b \neq 0$.

(2) The map $x \mapsto p(x) : K \to K$ has the intermediate value theorem.

1.4 Theorem (Tarksi)**.** *The theory of real closed ordered fields in the language* $\mathscr{L} = \{0, 1, +, -, \cdot, \leq\}$ *of ordered rings admits quantifier elimination. Hence, for any real closed field* K*,* $\mathbb{R} \equiv K$ *and, if* $\mathbb{R}$ *is a subfield of* K*, then* $\mathbb{R} \preceq K$.

1.5 Theorem. *Let* Γ *be a divisible ordered abelian group and let* k *be a real closed field. Then,* $K = k((t^\Gamma))$ *is real closed.*

We have $K[i] \cong k[i]((t^\Gamma))$, so it's enough to show the following theorem.

1.6 Theorem. *Let* Γ *be a divisible ordered abelian group and let* k *be an algebraically closed field of characteristic* 0*. Then,* $K = k((t^\Gamma))$ *is algebraically closed.*

Remark. This theorem is still true if we drop the characteristic 0 assumption, but it would require a different proof than the one given below.

Proof. Let $P \in K[X]$ be monic and irreducible, and write

$$P = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \quad (a_i \in K, n \geq n).$$

By replacing $P(X)$ by $P(X - a_{n-1})$, we get

$$P\left(X - \frac{a_{n-1}}{n}\right) = X^n + \text{terms of degree} < n-1.$$

Thus, we may assume $a_{n-1} = 0$. Put $\gamma_i := va_i \in \Gamma \cup \{\infty\}$ (recall that $vf := \min \operatorname{supp} f$ for $f \in K$) and put

$$\gamma := \min\left\{\frac{1}{n-i}\gamma_i \ : \ i = 0, \ldots, n-2\right\} \in \Gamma.$$

Then,

$$t^{-n\gamma}P(t^\gamma X) = X^n + \sum_{i=0}^{n-2} a_i t^{(i-n)\gamma} X^i,$$

where $v(a_i t^{(i-n)\gamma}) = \gamma_i + (i-n)\gamma \geq 0$, with equality holding for some i. Thus, we may assume $va_i \geq 0$ for all i, and $va_i = 0$ for some i.

Let $\mathcal{O} := \{f \in K \ : \ vf \geq 0\}$. It is readily verified that this is a subring of K which contains k. We have a ring morphism $\mathcal{O} \to k$ define by

$$f = \sum_{\gamma \geq 0} f_\gamma t^\gamma \mapsto f_0 =: \overline{f}.$$

1.7 Lemma. Let $P \in \mathscr{O}[X]$ be monic and $\overline{P} = Q_0 R_0$, where $Q_0, R_0 \in k[X]$ are monic and relatively prime. Then there are monic $Q, R \in \mathscr{O}[X]$ with $P = QR$ and $\overline{Q} = Q_0$, $\overline{R} = R_0$.

The lemma applies to our P. Since P is assumed irreducible, the lemma implies $\overline{P} = (X-a)^n$ for some $a \in k$, i.e.,

$$\overline{P} = X^n - naX^{n-1} + \text{lower degree terms.}$$

Since $a_{n-1} = 0$, we have $na = 0$; hence, $a = 0$ since k has characteristic 0. Thus, $\overline{P} = X^n$. But, $va_i = 0$ for some i, so we have a contradiction.

We now prove the lemma. Write $P = \sum_{i<\alpha} P_i(X) t^{a_i} \in k[X]((t^\Gamma))$, where a_i is strictly increasing in Γ, $a_0 = 0$, $P_i(X) \in k[X]$ are of degree $< n$ for $i > 0$, and $P_0 = \overline{P}$. Suppose we have a strictly increasing sequence $(b_i)_{i<\beta}$ in Γ and sequences $(Q_i)_{i<\beta}$, $(R_i)_{i<\beta}$ of polynomials in $k[X]$ of degree $< \deg Q_0$ and $< \deg R_0$, respectively, such that for

$$Q_{<\beta} := \sum_{i<\beta} Q_i t^{b_i}, \quad R_{<\beta} := \sum_{i<\beta} R_i t^{b_i}$$

we have

$$P \equiv Q_{<\beta} R_{<\beta} \mod (t^b \mathscr{O})$$

for all $b \in \Gamma$ with $b \le b_i$ for some i. Suppose $P \ne Q_{<\beta} R_{<\beta}$; we are going to find $b_\beta \in \Gamma$ and $Q_\beta, R_\beta \in k[X]$ of degrees $< \deg Q_0$ and $< \deg R_0$, respectively, such that

- $b_\beta > b_i$ for all $i < \beta$.
- $P \equiv (Q_{<\beta} + Q_\beta t^{b_\beta})(R_{<\beta} + R_\beta t^{b_\beta}) \mod (t^b \mathscr{O})$ for all $b \le b_\beta$.

To this end, let $\gamma := v(P - R_{<\beta} Q_{<\beta}) \in \Gamma$. Then, $b_\beta := \gamma > b_i$ for all $i < \beta$. Consider any $G, H \in k[X]$; then

$$P \equiv (Q_{<\beta} + Q_\beta t^{b_\beta})(R_{<\beta} + R_\beta t^{b_\beta}) \mod (t^b \mathscr{O})$$

for all $b \le b_\beta$. To get this congruence to hold also for $b = b_\beta$, we need G, H to satisfy an equation

$$S = Q_0 H + R_0 G,$$

where $S \in k[X]$ has degree < 0. But we can find such G, H since Q_0, R_0 are relatively prime. Then, take $Q_\beta = G$ and $R_\beta = G$ for such G, H.

$\square$