

# QUANTIFIER ELIMINATION IN SHELAH-SPENCER GRAPHS

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ABSTRACT. We simplify [?] proof of quantifier elimination in Shelah-Spencer graphs.

## 1. INTRODUCTION

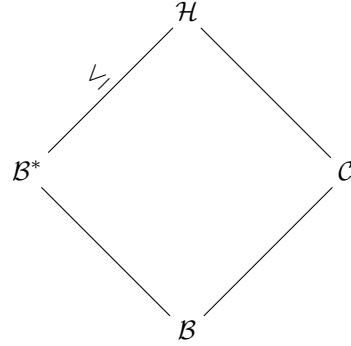
Laskowski's paper [?] provides a combinatorial proof of quantifier elimination in Shelah-Spencer graphs. Here we provide a simplification of the proof using only maximal chains and avoiding technical lemmas of sections 3 and 4.

We will use notation of [?], in particular things like  $\mathbf{K}_\alpha$ ,  $\delta(\mathcal{A}/\mathcal{B})$ ,  $X_m(\mathcal{A})$ ,  $S_\alpha$ , maximal embedding, etc.

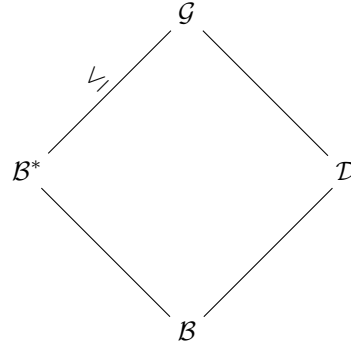
## 2. OMITTING LEMMA

**Definition 2.1.** Fix  $\mathcal{B} \in \mathbf{K}_\alpha$ ,  $\Phi, \Gamma$  finite subsets of  $\mathbf{K}_\alpha$ , and  $m \in \omega$  such that for each  $\mathcal{C} \in \Phi$  or  $\mathcal{C} \in \Gamma$  we have  $\mathcal{B} \subseteq \mathcal{C}$  and  $|C \setminus B| < m$ . Define  $Z(\mathcal{B}, \Phi, \Gamma, m)$  to be all  $\mathcal{B}^* \in X_m(\mathcal{B})$  such that

- (1) For every  $\mathcal{C} \in \Phi$  there are no  $\mathcal{H}$  with  $|H \setminus B^*| < m$  satisfying



- (2) For every  $\mathcal{D} \in \Gamma$  there is some  $\mathcal{G}$  with  $|G \setminus B^*| < m$  satisfying



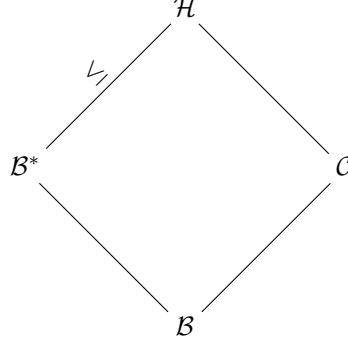
**Definition 2.2.** Let  $\mathcal{M} \models S_\alpha$ ,  $\mathcal{B} \in \mathbf{K}_\alpha$ , embedding  $f: \mathcal{B} \rightarrow \mathcal{M}$ ,  $\Phi$  finite subset of  $\mathbf{K}_\alpha$

- (1) Say that  $f$  *omits*  $\Phi$  if there are no  $\mathcal{C} \in \Phi$  and  $g: \mathcal{C} \rightarrow \mathcal{M}$  extending  $f$ .
- (2) Say that  $f$  *admits*  $\Phi$  if for every  $\mathcal{C} \in \Phi$  there is  $g: \mathcal{C} \rightarrow \mathcal{M}$  extending  $f$ .

**Lemma 2.3.** Let  $\mathcal{B} \in \mathbf{K}_\alpha$ ,  $\Phi, \Gamma$  finite subsets of  $\mathbf{K}_\alpha$ , and  $m \in \omega$  such that for each  $\mathcal{C} \in \Phi$  or  $\mathcal{C} \in \Gamma$  we have  $\mathcal{B} \subseteq \mathcal{C}$  and  $|C \setminus B| < m$ . The following are equivalent:

- (1)  $f$  omits  $\Phi$  and admits  $\Gamma$ .
- (2) There exists  $\mathcal{B}^* \in Z(\mathcal{B}, \Phi, \Gamma, m)$  maximally embeddable into  $\mathcal{M}$  over  $f$ .

*Proof.* (1)  $\Rightarrow$  (2) Identify  $\mathcal{B}$  with  $f(\mathcal{B})$ , i.e. for ease of notation assume that  $\mathcal{B} \subset \mathcal{M}$ . By remark 5.3 of [?] there is some  $B^* \in X_m(\mathcal{B})$  maximally embeddable in  $\mathcal{M}$  over  $f$ . Such embedding is unique by Lemma 3.8 of [?]. Again, we identify  $B^*$  with its maximal embedding into  $\mathcal{M}$ . To show (2) we need to verify that  $\mathcal{B}^* \in Z(\mathcal{B}, \Phi, \Gamma, m)$ . Suppose not. Two things can go wrong. First, there can be  $\mathcal{H}$  with  $|H \setminus B^*| < m$  and  $\mathcal{C} \in \Phi$  satisfying



□

#### REFERENCES

- [1] Klaus-Peter Podewski and Martin Ziegler. Stable graphs. *Fund. Math.*, 100:101-107, 1978.
- [2] Aharoni, Ron and Berger, Eli (2009). "Menger's Theorem for infinite graphs". *Inventiones Mathematicae* 176: 162
- [3] P. Simon, *On dp-minimal ordered structures*, J. Symbolic Logic 76 (2011), no. 2, 448460.  
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