QUANTIFIER ELIMINATION IN SHELAH-SPENCER GRAPHS

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Abstract. We simplify [?] proof of quantifier elimination in Shelah-Spencer graphs.

1. Introduction

Laskowski's paper [?] provides a combinatorial proof of quantifier elimination in Shelah-Spencer graphs. Here we provide a simplification of the proof using only maximal chains and avoiding the use of proposition 3.1 and technical lemmas of section 4.

We will use notation of [?], in particular things like $\boldsymbol{K}_\alpha$, $\delta(\mathcal{A}/\mathcal{B})$, $X_m(\mathcal{A})$, S_α, maximal embedding, etc.

2. Omitting lemma

Definition 2.1. Let $\mathcal{M} \models S_\alpha$, $\mathcal{B} \in \boldsymbol{K}_\alpha$, embedding $f\colon \mathcal{B} \to \mathcal{M}$, Φ finite subset of $\boldsymbol{K}_\alpha$

(1) Say that f *omits* Φ if there are no $\mathcal{C} \in \Phi$ and $g\colon \mathcal{C} \to \mathcal{M}$ extending f.
(2) Say that f *admits* Φ if for every $\mathcal{C} \in \Phi$ there is $g\colon \mathcal{C} \to \mathcal{M}$ extending f.

Note 2.2. Take notation as above and a structure $\mathcal{C} \in \boldsymbol{K}_\alpha$ extending $\mathcal{B}$. Then f doesn't omit $\{\mathcal{C}\}$ iff f admits $\{\mathcal{C}\}$.

Definition 2.3. Fix $\mathcal{B}, \mathcal{C} \in \boldsymbol{K}_\alpha$, and $m \in \omega$ such that $|C \backslash B| < m$. Define $Z(\mathcal{B}, \mathcal{C}, m)$ to be all $\mathcal{B}^* \in X_m(\mathcal{B})$ such that there are no $\mathcal{H}$ with $|H \backslash B^*| < m$ satisfying

$$\begin{tikzcd}
 & \mathcal{H} \arrow[dr, dash] & \\
\mathcal{B}^* \arrow[ur, dash, "\leq"] & & \mathcal{C} \\
 & \mathcal{B} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Lemma 2.4. *Let $\mathcal{B}, \mathcal{C} \in \boldsymbol{K}_\alpha$, and $m \in \omega$ such that $|C \backslash B| < m$. Also let $\mathcal{M} \models S_\alpha$ and $f\colon \mathcal{B} \to \mathcal{M}$ an embedding. The following are equivalent:*

(1) *f omits $\{\mathcal{C}\}$.*
(2) *There exists $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into $\mathcal{M}$ over f.*

Proof. For the proof we identify $\mathcal{B}$ with $f(\mathcal{B})$, i.e. for ease of notation assume that $\mathcal{B} \subset \mathcal{M}$.

(1) $\Rightarrow$ (2) By remark 5.3 of [**?**] there is some $B^* \in X_m(\mathcal{B})$ maximally embeddable in $\mathcal{M}$ over f. Such embedding is unique by Lemma 3.8 of [**?**]. Again, we identify B^* with its maximal embedding into $\mathcal{M}$. To show (2) we need to verify that $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$. Suppose not. Then there is $\mathcal{H}$ with $|H \backslash B^*| < m$ satisfying

$$\begin{tikzcd} & \mathcal{H} & \\ \mathcal{B}^* \arrow[ur, dash, "\not\leq"] & & \mathcal{C} \arrow[ul, dash] \\ & \mathcal{B} \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

As $\mathcal{B}^* \leq \mathcal{H}$ and $\mathcal{B} \subset \mathcal{M}$ we can embed $\mathcal{H}$ into $\mathcal{M}$ (as $\mathcal{M} \models S_\alpha$). But this would witness $\mathcal{C}$ extending $\mathcal{B}$ in $\mathcal{M}$ which is impossible as we assumed that f omits $\{\Phi\}$.

(2) $\Rightarrow$ (1) Suppose f doesn't omit $\{C\}$. Then by the note above f admits $\{C\}$, i.e. there is an embedding of $\mathcal{C}$ into M over f. We identify $\mathcal{C}$ with the image of that embedding. Similarly we identify $\mathcal{B}^*$ with the image of its maximal embedding over f. That is we may assume $\mathcal{C}, \mathcal{B}^* \subset \mathcal{M}$. Let H be the substructure of M induced by vertices $C \cup B^*$. As $|C \backslash B| < m$ we have $|H \backslash B^*| < m$. $\mathcal{B}^*$ is m-strong by remark 5.3 of [**?**]. This forces $\mathcal{B}^* \leq H$. But this contradicts the fact that $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$. $\square$

Corollary 2.5. *With the setup of the previous lemma, the following are equivalent:*

(1) *f admits $\{\mathcal{C}\}$.*
(2) *There exists $\mathcal{B}^* \in X_m(\mathcal{B}) \backslash Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into $\mathcal{M}$ over f.*

For quantifier elimination we need to track multiple structures being admitted and omitted, hence the following definition.

Definition 2.6. Let $\mathcal{B} \in \boldsymbol{K}_\alpha$, Φ, Γ finite subsets of $\boldsymbol{K}_\alpha$, and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|C \backslash B| < m$. Define

$$\begin{aligned} Y(\mathcal{B}, \Phi, \Gamma, m) = \{B^* \in X_m(\mathcal{B}) \; | \forall \mathcal{C} \in \Phi \; B^* \in Z(\mathcal{B}, \mathcal{C}, m)\text{'' and''} \\ \forall \mathcal{D} \in \Gamma \; B^* \notin Z(\mathcal{B}, \mathcal{C}, m)\} \end{aligned}$$

Lemma 2.7. *Let $\mathcal{B} \in \boldsymbol{K}_\alpha$, Φ, Γ finite subsets of $\boldsymbol{K}_\alpha$, and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|C \backslash B| < m$. The following are equivalent:*

(1) *f omits Φ and admits Γ.*
(2) *There exists $\mathcal{B}^* \in Z(\mathcal{B}, \Phi, \Gamma, m)$ maximally embeddable into $\mathcal{M}$ over f.*

Proof. (1) $\Rightarrow$ (2) Identify $\mathcal{B}$ with $f(\mathcal{B})$, i.e. for ease of notation assume that $\mathcal{B} \subset \mathcal{M}$. By remark 5.3 of [**?**] there is some $B^* \in X_m(\mathcal{B})$ maximally embeddable in $\mathcal{M}$ over f. Such embedding is unique by Lemma 3.8 of [**?**]. Again, we identify B^* with its maximal embedding into $\mathcal{M}$. To show (2) we need to verify that $\mathcal{B}^* \in Z(\mathcal{B}, \Phi, \Gamma, m)$.

Suppose not. Two things can go wrong. First, there can be $\mathcal{H}$ with $|H \backslash B^*| < m$ and $\mathcal{C} \in \Phi$ satisfying

$$\begin{tikzcd}
& \mathcal{H} & \\
\mathcal{B}^* \arrow[ur, dash, "\leq"] & & \mathcal{C} \arrow[ul, dash] \\
& \mathcal{B} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

As $\mathcal{B}^* \leq \mathcal{H}$ and $\mathcal{B} \subset \mathcal{M}$ we can embed $\mathcal{H}$ into $\mathcal{M}$ (as $\mathcal{M} \models S_\alpha$). But this would witness $\mathcal{C}$ extending $\mathcal{B}$ in $\mathcal{M}$ which is impossible as we assumed that f omits Φ. Another thing that could go wrong is that there could be $\mathcal{D} \in \Gamma$ and no $\mathcal{G}$ with $|G \backslash B^*| < m$ satisfying

$$\begin{tikzcd}
& \mathcal{G} & \\
\mathcal{B}^* \arrow[ur, dash, "\leq"] & & \mathcal{D} \arrow[ul, dash] \\
& \mathcal{B} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

As f admits $\square$

References

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