

A Tiny Example

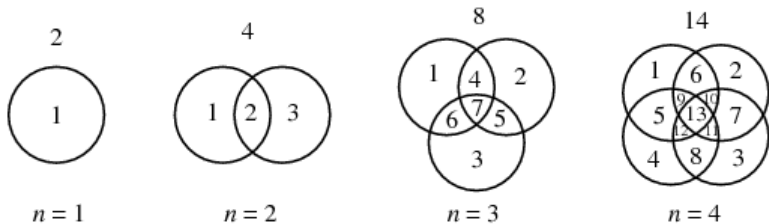
Andrew Mertz and William Slough

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Suppose we have an (infinite) collection of sets \mathcal{F} .
We define a shatter function $\pi_{\mathcal{F}}(n)$

$$\pi_{\mathcal{F}}(n) = \max\{\# \text{ of atoms in boolean algebra generated by } S \\ | S \subset \mathcal{F} \text{ with } |S| = n\}$$

Example: Let \mathcal{F} consist of all discs on a plane.



$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 8 \quad \pi_{\mathcal{F}}(4) = 14$$

$$\pi_{\mathcal{F}}(n) = n^2 - n + 2$$

More examples:

1. Lines on a plane $\pi_{\mathcal{F}}(n) = n^2/2 + n/2 + 1$
2. Disks on a plane $\pi_{\mathcal{F}}(n) = n^2 - n + 2$
3. Balls in \mathbb{R}^3 $\pi_{\mathcal{F}}(n) = n^3/3 - n^2 + 8n/3$
4. Intervals on a line $\pi_{\mathcal{F}}(n) = 2n$
5. Half-planes on a plane $\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$
6. Finite subsets of \mathbb{N} $\pi_{\mathcal{F}}(n) = 2^n$
7. Polygons in a plane $\pi_{\mathcal{F}}(n) = 2^n$

Theorem (Sauer-Shelah)

Shatter function is either 2^n or bounded by a polynomial.

Definition

Suppose growth of shatter function for \mathcal{F} is polynomial. Let r be the smallest real such that

$$\pi_{\mathcal{F}}(n) = O(n^r)$$

We define such r to be the vc-density of \mathcal{F} , $\text{vc}(\mathcal{F})$. If shatter function grows exponentially, we let the vc-density to be infinite.

Applications

- ▶ VapnikChervonenkis Theorem in probability
- ▶ Computability
- ▶ NIP theories

Model Theory

Model Theory studies definable sets in first-order structures.

$$(\mathbb{Q}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) = \exists y \ y \cdot y = x$$

In the structure above $\phi(x)$ defines a set of numbers that are a square.

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) = \exists y \ y \cdot y = x$$

In the structure above $\phi(x)$ defines the set $[0, \infty)$.

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\psi(x_1, x_2) = (x_1 \cdot x_1 + x_2 \cdot x_2 \leq 1.5) \wedge (x_1^2 \leq x_2)$$