

# A NOTE ON QUANTIFIER ELIMINATION IN SHELAH-SPENCER GRAPHS

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ABSTRACT. We simplify [1]’s proof of quantifier elimination in Shelah-Spencer graphs.

## 1. INTRODUCTION

Laskowski’s paper [1] provides a combinatorial proof of quantifier elimination in Shelah-Spencer graphs. Here we provide a simplification of the proof using only maximal chains and avoiding the use of proposition 3.1 and technical lemmas of section 4.

We will use notation of [1], in particular things like  $K_\alpha$ ,  $\delta(\mathcal{A}/\mathcal{B})$ ,  $X_m(\mathcal{A})$ ,  $S_\alpha$ ,  $\mathcal{B}^* \sqsubseteq \mathcal{B}'$ , maximal embedding,  $\Delta_{\mathcal{A}}(x)$ ,  $\Psi_{\mathcal{A},\mathcal{B}}(x)$  etc. However we will give a different definition of  $Y(\dots)$ . When we write formulas  $\theta(x, y)$  we may have  $x, y$  to be tuples.

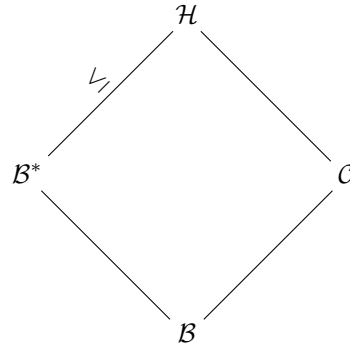
## 2. OMITTING LEMMA

**Definition 2.1.** Let  $\mathcal{M} \models S_\alpha$ ,  $\mathcal{B} \in K_\alpha$ , embedding  $f: \mathcal{B} \rightarrow \mathcal{M}$ ,  $\Phi$  finite subset of  $K_\alpha$

- (1) Say that  $f$  *omits*  $\Phi$  if there are no  $\mathcal{C} \in \Phi$  and  $g: \mathcal{C} \rightarrow \mathcal{M}$  extending  $f$ .
- (2) Say that  $f$  *admits*  $\Phi$  if for every  $\mathcal{C} \in \Phi$  there is  $g: \mathcal{C} \rightarrow \mathcal{M}$  extending  $f$ .

**Note 2.2.** Take notation as above and a structure  $\mathcal{C} \in K_\alpha$  extending  $\mathcal{B}$ . Then  $f$  doesn’t omit  $\{\mathcal{C}\}$  iff  $f$  admits  $\{\mathcal{C}\}$ .

**Definition 2.3.** Fix  $\mathcal{B}, \mathcal{C} \in K_\alpha$ , and  $m \in \omega$  such that  $|C \setminus B| < m$ . Define  $Z(\mathcal{B}, \mathcal{C}, m)$  to be all  $\mathcal{B}^* \in X_m(\mathcal{B})$  such that there are no  $\mathcal{H}$  with  $|H \setminus B^*| < m$  satisfying



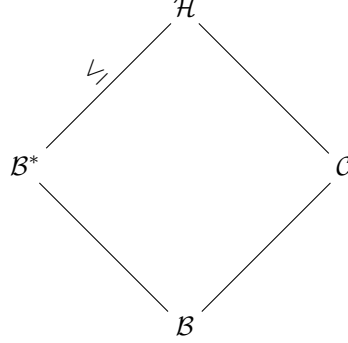
**Lemma 2.4.** Let  $\mathcal{B}, \mathcal{C} \in K_\alpha$ , and  $m \in \omega$  such that  $|C \setminus B| < m$ . Also let  $\mathcal{M} \models S_\alpha$  and  $f: \mathcal{B} \rightarrow \mathcal{M}$  an embedding. The following are equivalent:

- (1)  $f$  *omits*  $\{\mathcal{C}\}$ .

(2) *There exists  $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$  maximally embeddable into  $\mathcal{M}$  over  $f$ .*

*Proof.* For the proof we identify  $\mathcal{B}$  with  $f(\mathcal{B})$ , i.e. for ease of notation assume that  $\mathcal{B} \subset \mathcal{M}$ .

(1)  $\Rightarrow$  (2) By remark 5.3 of [1] there is some  $B^* \in X_m(\mathcal{B})$  maximally embeddable in  $\mathcal{M}$  over  $f$ . Such embedding is unique by Lemma 3.8 of [1]. Again, we identify  $B^*$  with its maximal embedding into  $\mathcal{M}$ . To show (2) we need to verify that  $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$ . Suppose not. Then there is  $\mathcal{H}$  with  $|H \setminus B^*| < m$  satisfying



As  $\mathcal{B}^* \leq \mathcal{H}$  and  $\mathcal{B} \subset \mathcal{M}$  we can embed  $\mathcal{H}$  into  $\mathcal{M}$  (as  $\mathcal{M} \models S_\alpha$ ). But this would witness  $\mathcal{C}$  extending  $\mathcal{B}$  in  $\mathcal{M}$  which is impossible as we assumed that  $f$  omits  $\{\Phi\}$ .

(2)  $\Rightarrow$  (1) Suppose  $f$  doesn't omit  $\{\mathcal{C}\}$ . Then by the note 2.2  $f$  admits  $\{\mathcal{C}\}$ , i.e. there is an embedding of  $\mathcal{C}$  into  $\mathcal{M}$  over  $f$ . We identify  $\mathcal{C}$  with the image of that embedding. Similarly we identify  $\mathcal{B}^*$  with the image of its maximal embedding over  $f$ . That is we may assume  $\mathcal{C}, \mathcal{B}^* \subset \mathcal{M}$ . Let  $H$  be the substructure of  $\mathcal{M}$  induced by vertices  $\mathcal{C} \cup \mathcal{B}^*$ . As  $|\mathcal{C} \setminus \mathcal{B}| < m$  we have  $|H \setminus \mathcal{B}^*| < m$ .  $\mathcal{B}^*$  is  $m$ -strong by remark 5.3 of [1]. This forces  $\mathcal{B}^* \leq H$ . But this contradicts the fact that  $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$ .  $\square$

**Corollary 2.5.** *With the setup of the previous lemma, the following are equivalent:*

- (1)  $f$  admits  $\{\mathcal{C}\}$ .
- (2) *There exists  $\mathcal{B}^* \in X_m(\mathcal{B}) \setminus Z(\mathcal{B}, \mathcal{C}, m)$  maximally embeddable into  $\mathcal{M}$  over  $f$ .*

For quantifier elimination we need to track multiple structures being admitted and omitted, hence the following definition.

**Definition 2.6.** Let  $\mathcal{B} \in \mathbf{K}_\alpha$ ,  $\Phi, \Gamma$  finite subsets of  $\mathbf{K}_\alpha$ , and  $m \in \omega$  such that for each  $\mathcal{C} \in \Phi$  or  $\mathcal{C} \in \Gamma$  we have  $\mathcal{B} \subseteq \mathcal{C}$  and  $|\mathcal{C} \setminus \mathcal{B}| < m$ . Define

$$Y(\mathcal{B}, \Phi, \Gamma, m) = \{B^* \in X_m(\mathcal{B}) \mid \forall \mathcal{C} \in \Phi \ B^* \in Z(\mathcal{B}, \mathcal{C}, m) \text{ and } \forall \mathcal{D} \in \Gamma \ B^* \notin Z(\mathcal{B}, \mathcal{D}, m)\}$$

**Lemma 2.7.** *Let  $\mathcal{B} \in \mathbf{K}_\alpha$ ,  $\Phi, \Gamma$  finite subsets of  $\mathbf{K}_\alpha$ , and  $m \in \omega$  such that for each  $\mathcal{C} \in \Phi$  or  $\mathcal{C} \in \Gamma$  we have  $\mathcal{B} \subseteq \mathcal{C}$  and  $|\mathcal{C} \setminus \mathcal{B}| < m$ . The following are equivalent:*

- (1)  $f$  omits  $\Phi$  and admits  $\Gamma$ .
- (2) *There exists  $\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$  maximally embeddable into  $\mathcal{M}$  over  $f$ .*

*Proof.* Easy corollary of 2.4 and 2.5.  $\square$

## 3. QUANTIFIER ELIMINATION

Following proof of 5.6 in [1], we have a formula  $\theta(x, y)$ , some  $\mathcal{A} \subseteq \mathcal{B} \in \mathbf{K}_\alpha$  with  $\theta(x, y) \vdash \Delta_{\mathcal{A}}(x) \wedge \Delta_{\mathcal{B}}(x, y)$ . We may also assume that  $\theta(x, y)$  is a conjunction of formulas of the type  $\Psi_{\mathcal{B}, \mathcal{C}}(x, y)$  and their negations. More precisely

$$\theta(x, y) \Leftrightarrow \bigwedge_{\mathcal{C} \in \Phi} \Psi_{\mathcal{B}, \mathcal{C}}(x, y) \wedge \bigwedge_{\mathcal{D} \in \Gamma} \neg \Psi_{\mathcal{B}, \mathcal{D}}(x, y)$$

for some finite subsets  $\Phi, \Gamma$  of  $\mathbf{K}_\alpha$ . Let  $m = \max\{|C \setminus B| \mid \mathcal{C} \in \Phi \text{ or } \Gamma\}$ . We claim that in  $\mathcal{M} \models S_\alpha$

$$\begin{aligned} \exists y \theta(x, y) &\Leftrightarrow \bigvee_{\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)} (\mathcal{B}^* \text{ maximally embeds into } \mathcal{M} \text{ over } \mathcal{A}) \\ &\Leftrightarrow \bigvee_{\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)} \left( \Psi_{\mathcal{A}, \mathcal{B}^*}(x) \wedge \bigwedge_{\mathcal{B}^* \sqsubseteq \mathcal{B}', \mathcal{B}' \in X_m(\mathcal{B})} \neg \Psi_{\mathcal{A}, \mathcal{B}'}(x) \right) \end{aligned}$$

*Proof.* ( $\Rightarrow$ ) Fix  $\mathcal{B} \subset \mathcal{M}$  witnessing existential statement. By remark 5.3 and lemma 3.8 There is a unique  $\mathcal{B}^* \in X_m$  (uniquely embedded) maximally embeddable into  $\mathcal{M}$  over  $\mathcal{B}$ . By lemma 2.7  $\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$ .

( $\Leftarrow$ ) Take the embedding  $g: \mathcal{B}^* \rightarrow \mathcal{M}$  and restrict it to  $\mathcal{B} \subseteq \mathcal{B}^*$  i.e.  $f = g \upharpoonright \mathcal{B}$ . As  $\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$  by lemma 2.7  $f$  omits  $\Phi$  and admits  $\Gamma$ . Thus is is a witness to  $\exists y \theta(x, y)$ .  $\square$

## REFERENCES

- [1] Michael C. Laskowski, *A simpler axiomatization of the Shelah-Spencer almost sure theories*, Israel J. Math. **161** (2007), 157-186. MR MR2350161  
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