

# VC-DENSITY IN AN ADDITIVE REDUCT OF THE $P$ -ADIC NUMBERS

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ABSTRACT. Aschenbrenner et. al. computed a linear bound for the vc-density function in the field of  $p$ -adic numbers, but it is not known to be optimal. In this paper we investigate a certain  $P$ -minimal additive reduct of the field of  $p$ -adic numbers and use a cell decomposition result of Leenknegt to compute an optimal bound for that structure.

VC-density was studied in model theory in [1] by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko as a natural notion of dimension for definable families of sets in NIP theories. In a complete NIP theory  $T$  we can define the vc-function

$$\text{vc}^T = \text{vc} : \mathbb{N} \longrightarrow \mathbb{R} \cup \{\infty\}$$

where  $\text{vc}(n)$  measures the worst-case complexity of families of definable sets in an  $n$ -fold cartesian power of the underlying set of a model of  $T$  (see 1.13 below for a precise definition of  $\text{vc}^T$ ). The simplest possible behavior is  $\text{vc}(n) = n$  for all  $n$ , satisfied, for example, if  $T$  is o-minimal. For  $T = \text{Th}(\mathbb{Q}_p)$ , the paper [1] computes an upper bound for this function to be  $2n - 1$ , and it is not known whether this is optimal. This same bound holds in any reduct of the field of  $p$ -adic numbers, but one may expect that the simplified structure of the reduct would allow a better bound. In [2], Leenknegt provides a cell decomposition result for a certain  $P$ -minimal additive reduct of the field of  $p$ -adic numbers. Using this result, in this paper we improve the bound for the vc-function, showing that in Leenknegt's structure  $\text{vc}(n) = n$ .

Section 1 defines vc-density and states some basic lemmas about it. A more in depth exposition of vc-density can be found in [1]. Section 2 defines and states some

basic facts about the theory of  $p$ -adic numbers. Here we also introduce the reduct which we will be working with. Section 3 sets up basic definitions and lemmas that will be needed for the proof. We define trees and intervals and show how they help with vc-density calculations. Section 4 concludes the proof.

Throughout the paper, variables and tuples of elements will be simply denoted as  $x, y, a, b, \dots$ . We will occasionally write  $\vec{a}$  instead of  $a$  for a tuple in  $\mathbb{Q}_p^n$  to emphasize it as an element of the  $\mathbb{Q}_p$ -vector space  $\mathbb{Q}_p^n$ . We denote the arity of a tuple  $x$  of variables by  $|x|$ . Natural numbers are  $\mathbb{N} = \{0, 1, \dots\}$ .

### 1. VC-DIMENSION AND VC-DENSITY

Throughout this section we work with a collection  $\mathcal{F}$  of subsets of a set  $X$ . We call the pair  $(X, \mathcal{F})$  a set system.

**Definition 1.1.**

- Given a subset  $A$  of  $X$ , we define the set system  $(A, A \cap \mathcal{F})$  where  $A \cap \mathcal{F} = \{A \cap F \mid F \in \mathcal{F}\}$ .
- For  $A \subset X$  we say that  $\mathcal{F}$  shatters  $A$  if  $A \cap \mathcal{F} = \mathcal{P}(A)$  (the power set of  $A$ ).

**Definition 1.2.** We say  $(X, \mathcal{F})$  has VC-dimension  $n$  if the largest subset of  $X$  shattered by  $\mathcal{F}$  is of size  $n$ . If  $\mathcal{F}$  shatters arbitrarily large subsets of  $X$ , we say that  $(X, \mathcal{F})$  has infinite VC-dimension. We denote the VC-dimension of  $(X, \mathcal{F})$  by  $\text{VC}(X, \mathcal{F})$ .

**Note 1.3.** We may drop  $X$  from the notation  $\text{VC}(X, \mathcal{F})$ , as the VC-dimension doesn't depend on the base set and is determined by  $(\bigcup \mathcal{F}, \mathcal{F})$ .

Set systems of finite VC-dimension tend to have good combinatorial properties, and we consider set systems with infinite VC-dimension to be poorly behaved.

Another natural combinatorial notion is that of a dual system:

**Definition 1.4.** For  $a \in X$  define  $X_a = \{F \in \mathcal{F} \mid a \in F\}$ . Let  $\mathcal{F}^* = \{X_a \mid a \in X\}$ . We call  $(\mathcal{F}, \mathcal{F}^*)$  the dual system of  $(X, \mathcal{F})$ . The VC-dimension of the dual system of

$(X, \mathcal{F})$  is referred to as the dual VC-dimension of  $(X, \mathcal{F})$  and denoted by  $\text{VC}^*(\mathcal{F})$ .  
(As before, this notion doesn't depend on  $X$ .)

**Lemma 1.5** (see 2.13b in [3]). *A set system  $(X, \mathcal{F})$  has finite VC-dimension if and only if its dual system has finite VC-dimension. More precisely*

$$\text{VC}^*(\mathcal{F}) \leq 2^{1+\text{VC}(\mathcal{F})}.$$

For a more refined notion of complexity of  $(X, \mathcal{F})$  we look at the traces of our family on finite sets:

**Definition 1.6.** Define the shatter function  $\pi_{\mathcal{F}}: \mathbb{N} \longrightarrow \mathbb{N}$  of  $\mathcal{F}$  and the dual shatter function  $\pi_{\mathcal{F}}^*: \mathbb{N} \longrightarrow \mathbb{N}$  of  $\mathcal{F}$  by

$$\begin{aligned} \pi_{\mathcal{F}}(n) &= \max \{ |A \cap \mathcal{F}| \mid A \subset X \text{ and } |A| = n \} \\ \pi_{\mathcal{F}}^*(n) &= \max \{ \text{atoms}(B) \mid B \subset \mathcal{F}, |B| = n \} \end{aligned}$$

where  $\text{atoms}(B)$  = number of atoms in the boolean algebra of sets generated by  $B$ . Note that the dual shatter function is precisely the shatter function of the dual system:  $\pi_{\mathcal{F}}^* = \pi_{\mathcal{F}^*}$ .

A simple upper bound is  $\pi_{\mathcal{F}}(n) \leq 2^n$  (same for the dual). If the VC-dimension of  $\mathcal{F}$  is infinite then clearly  $\pi_{\mathcal{F}}(n) = 2^n$  for all  $n$ . Conversely we have the following remarkable fact:

**Theorem 1.7** (Sauer-Shelah '72, see [5], [6]). *If the set system  $(X, \mathcal{F})$  has finite VC-dimension  $d$  then  $\pi_{\mathcal{F}}(n) \leq \binom{n}{\leq d}$  for all  $n$ , where  $\binom{n}{\leq d} = \binom{n}{d} + \binom{n}{d-1} + \dots + \binom{n}{1}$ .*

Thus the systems with a finite VC-dimension are precisely the systems where the shatter function grows polynomially. Define the *vc-density* of  $\mathcal{F}$  to quantify the growth of the shatter function of  $\mathcal{F}$ :

**Definition 1.8.** Define the vc-density and dual vc-density of  $\mathcal{F}$  as

$$\begin{aligned} \text{vc}(\mathcal{F}) &= \limsup_{n \rightarrow \infty} \frac{\log \pi_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}, \\ \text{vc}^*(\mathcal{F}) &= \limsup_{n \rightarrow \infty} \frac{\log \pi_{\mathcal{F}}^*(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}. \end{aligned}$$

Generally speaking a shatter function that is bounded by a polynomial doesn't itself have to be a polynomial. Proposition 4.12 in [1] gives an example of a shatter function that grows like  $n \log n$  (so it has vc-density 1).

So far the notions that we have defined are purely combinatorial. We now adapt VC-dimension and vc-density to the model theoretic context.

**Definition 1.9.** Work in a first-order structure  $M$ . Fix a finite collection of formulas  $\Phi(x, y)$ .

- For  $\phi(x, y) \in \mathcal{L}(M)$  and  $b \in M^{|y|}$  let

$$\phi(M^{|x|}, b) = \{a \in M^{|x|} \mid \phi(a, b)\} \subseteq M^{|x|}.$$

- Let  $\Phi(M^{|x|}, M^{|y|}) = \{\phi(M^{|x|}, b) \mid \phi \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|})$ .
- Let  $\mathcal{F}_{\Phi} = \Phi(M^{|x|}, M^{|y|})$ , giving rise to a set system  $(M^{|x|}, \mathcal{F}_{\Phi})$ .
- Define the VC-dimension  $\text{VC}(\Phi)$  of  $\Phi$ , to be the VC-dimension of  $(M^{|x|}, \mathcal{F}_{\Phi})$ , similarly for the dual.
- Define the vc-density  $\text{vc}(\Phi)$  of  $\Phi$ , to be the vc-density of  $(M^{|x|}, \mathcal{F}_{\Phi})$ , similarly for the dual.

We will also refer to the vc-density and VC-dimension of a single formula  $\phi$  viewing it as a one element collection  $\Phi = \{\phi\}$ .

Counting atoms of a boolean algebra in a model theoretic setting corresponds to counting types, so it is instructive to rewrite the shatter function in terms of types.

**Definition 1.10.**

$$\pi_{\Phi}^*(n) = \max \{\text{number of } \Phi\text{-types over } B \mid B \subset M, |B| = n\}$$

Here a  $\Phi$ -type over  $B$  is a maximal consistent collection of formulas of the form  $\phi(x, b)$  or  $\neg\phi(x, b)$  where  $\phi \in \Phi$  and  $b \in B$ .

Functions  $\pi_\Phi^*$  and  $\pi_{\mathcal{F}_\Phi}^*$  are not equal, as one fixes the size of boolean algebra and another fixes the size of the parameter set. However, as the following lemma demonstrates, they both give the same asymptotic definition of dual vc-density.

**Lemma 1.11.**

$$\text{vc}^*(\Phi) = \text{degree of polynomial growth of } \pi_\Phi^*(n) = \limsup_{n \rightarrow \infty} \frac{\log \pi_\Phi^*(n)}{\log n}$$

*Proof.* With parameter set of size  $n$ , we get  $|\Phi|n$  elements in the boolean algebra. We check that asymptotically it doesn't matter whether we look at growth of boolean algebra of size  $n$  or size  $|\Phi|n$ .

$$\begin{aligned} \pi_{\mathcal{F}_\Phi}^*(n) &\leq \pi_\Phi^*(n) \leq \pi_{\mathcal{F}_\Phi}^*(|\Phi|n) \\ \text{vc}^*(\Phi) &\leq \limsup_{n \rightarrow \infty} \frac{\log \pi_\Phi^*(n)}{\log n} \leq \limsup_{n \rightarrow \infty} \frac{\log \pi_{\mathcal{F}_\Phi}^*(|\Phi|n)}{\log n} = \\ &= \limsup_{n \rightarrow \infty} \frac{\log \pi_{\mathcal{F}_\Phi}^*(|\Phi|n)}{\log |\Phi|n} \frac{\log |\Phi|n}{\log n} = \limsup_{n \rightarrow \infty} \frac{\log \pi_{\mathcal{F}_\Phi}^*(|\Phi|n)}{\log |\Phi|n} \leq \\ &\leq \limsup_{n \rightarrow \infty} \frac{\log \pi_{\mathcal{F}_\Phi}^*(n)}{\log n} = \text{vc}^*(\Phi) \end{aligned}$$

□

One can check that the shatter function and hence VC-dimension and vc-density of a formula are elementary notions, so they only depend on the first-order theory of the structure  $M$ .

NIP theories are a natural context for studying vc-density. In fact we can take the following as the definition of NIP:

**Definition 1.12.** Define  $\phi$  to be NIP if it has finite VC-dimension in a theory  $T$ . A theory  $T$  is NIP if all the formulas in  $T$  are NIP.

In a general combinatorial context for arbitrary set systems, vc-density can be any real number in  $0 \cup [1, \infty)$  (see [4]). Less is known if we restrict our attention

to NIP theories. Proposition 4.6 in [1] gives examples of formulas that have non-integer rational vc-density in an NIP theory, however it is open whether one can get an irrational vc-density in this model-theoretic setting.

Instead of working with a theory formula by formula, we can look for a uniform bound for all formulas:

**Definition 1.13.** For a given NIP structure  $M$ , define the vc-function

$$\begin{aligned} \text{vc}^M(n) &= \sup\{\text{vc}^*(\phi(x, y)) \mid \phi \in \mathcal{L}(M), |x| = n\} \\ &= \sup\{\text{vc}(\phi(x, y)) \mid \phi \in \mathcal{L}(M), |y| = n\} \in \mathbb{R}^{\geq 0} \cup \{+\infty\} \end{aligned}$$

As before this definition is elementary, so it only depends on the theory of  $M$ . We omit the superscript  $M$  if it is understood from the context. One can easily check the following bounds:

**Lemma 1.14** (Lemma 3.22 in [1]). *We have  $\text{vc}(1) \geq 1$  and  $\text{vc}(n) \geq n \text{vc}(1)$ .*

However, it is not known whether the second inequality can be strict or even whether  $\text{vc}(1) < \infty$  implies  $\text{vc}(n) < \infty$ .

## 2. $P$ -ADIC NUMBERS

The field  $\mathbb{Q}_p$  of  $p$ -adic numbers is often studied in the language of Macintyre

$$\mathcal{L}_{Mac} = \{0, 1, +, -, \cdot, |, \{P_n\}_{n \in \mathbb{N}}\}$$

which is a language  $\{0, 1, +, -, \cdot\}$  of rings together with unary predicates  $P_n$  interpreted in  $\mathbb{Q}_p$  so as to satisfy

$$P_n x \leftrightarrow \exists y \ y^n = x$$

and a divisibility relation where  $a|b$  holds in  $\mathbb{Q}_p$  when  $\text{val } a \leq \text{val } b$ .

Note that  $P_n \setminus \{0\}$  is a multiplicative subgroup of  $\mathbb{Q}_p$  with finitely many cosets.

**Theorem 2.1** (Macintyre '76). *The  $\mathcal{L}_{Mac}$ -structure  $\mathbb{Q}_p$  has quantifier elimination.*

There is also a cell decomposition result:

**Definition 2.2.** Define  $k$ -cells recursively as follows. A 0-cell is a singleton subset of  $\mathbb{Q}_p$ . A  $(k+1)$ -cell is a subset of  $\mathbb{Q}_p^{k+1}$  of the following form:

$$\{(x, t) \in D \times \mathbb{Q}_p \mid \text{val } a_1(x) \square_1 \text{val}(t - c(x)) \square_2 \text{val } a_2(x), t - c(x) \in \lambda P_n\}$$

where  $D$  is a  $k$ -cell,  $a_1(x), a_2(x), c(x)$  are definable functions  $D \rightarrow \mathbb{Q}_p$ , each of  $\square_i$  is  $<, \leq$  or no condition,  $n \in \mathbb{N}$ , and  $\lambda \in \mathbb{Q}_p$ .

**Theorem 2.3** (Denef '84). *Any definable subset of  $\mathbb{Q}_p^n$  defined by an  $\mathcal{L}_{Mac}$ -formula decomposes into a finite disjoint union of  $n$ -cells.*

In [1], Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko show that  $\mathbb{Q}_p$  as  $\mathcal{L}_{Mac}$ -structure satisfies  $\text{vc}(n) \leq 2n - 1$ , however it is not known whether this bound is optimal.

In [2], Leenknegt analyzes the reduct of  $\mathbb{Q}_p$  to the language

$$\mathcal{L}_{aff} = \{0, 1, +, -, \{\bar{c}\}_{c \in \mathbb{Q}_p}, |, \{Q_{m,n}\}_{m,n \in \mathbb{N}}\}$$

where  $\bar{c}$  denotes a scalar multiplication by  $c$ ,  $a|b$  as above stands for  $\text{val } a \leq \text{val } b$ , and  $Q_{m,n}$  is a unary predicate interpreted as

$$Q_{m,n} = \bigcup_{k \in \mathbb{Z}} p^{km} (1 + p^n \mathbb{Z}_p).$$

Note that  $Q_{m,n} \setminus \{0\}$  is a subgroup of the multiplicative group of  $\mathbb{Q}_p$  with finitely many cosets. One can check that these extra relation symbols are definable in the  $\mathcal{L}_{Mac}$ -structure  $\mathbb{Q}_p$ . The paper [2] provides a cell decomposition result with the following cells:

**Definition 2.4.** A 0-cell is a singleton subset of  $\mathbb{Q}_p$ . A  $(k+1)$ -cell is a subset of  $\mathbb{Q}_p^{k+1}$  of the following form:

$$\{(x, t) \in D \times \mathbb{Q}_p \mid \text{val } a_1(x) \square_1 \text{val}(t - c(x)) \square_2 \text{val } a_2(x), t - c(x) \in \lambda Q_{m,n}\}$$

where  $D$  is a  $k$ -cell, called the base of the cell,  $a_1(x), a_2(x), c(x)$  are polynomials of degree  $\leq 1$ , called the defining polynomials each of  $\square_1, \square_2$  is  $<$  or no condition,  $m, n \in \mathbb{N}$ , and  $\lambda \in \mathbb{Q}_p$ . We call  $Q_{m,n}$  the defining predicate.

**Theorem 2.5** (Leenknecht '12). *Any definable subset of  $\mathbb{Q}_p^n$  defined by an  $\mathcal{L}_{aff}$ -formula decomposes into a finite disjoint union of  $n$ -cells.*

Moreover, [2] shows that  $\mathcal{L}_{aff}$ -structure  $\mathbb{Q}_p$  is a  $P$ -minimal reduct, that is, the one-dimensional definable sets of  $\mathcal{L}_{aff}$ -structure  $\mathbb{Q}_p$  coincide with the one-dimensional definable sets in the full structure  $\mathcal{L}_{Mac}$ -structure  $\mathbb{Q}_p$ .

The main result of this paper is the computation of the vc-function for this structure:

**Theorem 2.6.**  *$\mathcal{L}_{aff}$ -structure  $\mathbb{Q}_p$  has  $\text{vc}(n) = n$ .*

### 3. KEY LEMMAS AND DEFINITIONS

To show that  $\text{vc}(n) = n$  it suffices to bound  $\text{vc}^*(\phi) \leq |x|$  for every  $\mathcal{L}_{aff}$ -formula  $\phi(x; y)$ . Fix such a formula  $\phi(x; y)$ . Instead of working with it directly, we first simplify it using quantifier elimination. The required quantifier elimination result can be easily obtained from cell decomposition:

**Lemma 3.1.** *Any formula  $\phi(x; y)$  in  $\mathcal{L}_{aff}$ -structure  $\mathbb{Q}_p$  can be written as a boolean combination of formulas from a collection*

$$\begin{aligned} \Phi(x; y) = & \{\text{val}(p_i(x) - c_i(y)) < \text{val}(p_j(x) - c_j(y))\}_{i,j \in I} \cup \\ & \{p_i(x) - c_i(y) \in \lambda_k Q_{m,n}\}_{i \in I, k \in K} \end{aligned}$$



of  $\mathcal{L}_{aff}$ -formulas where  $I, K$  are finite index sets, each  $p_i$  is a degree  $\leq 1$  polynomial in  $x$  without a constant term, each  $c_i$  is a degree  $\leq 1$  polynomial in  $y$ ,  $m, n \in \mathbb{N}$ , and  $\lambda_k \in \mathbb{Q}_p$ .

*Proof.* Let  $l = |x| + |y|$ . Partition the subset of  $\mathbb{Q}_p^l$  defined by  $\phi$  to obtain  $\mathcal{D}^l$ , a collection of  $l$ -cells. Let  $\mathcal{D}^{l-1}$  be the collection of the bases of the cells in  $\mathcal{D}^l$ . Similarly, construct by induction  $\mathcal{D}^j$  for each  $0 \leq j < l$ , where  $\mathcal{D}^j$  is the collection of  $j$ -cells which are the bases of cells in  $\mathcal{D}^{j+1}$ . Set

$$m = \prod \{m' \mid Q_{m',n'} \text{ is the defining predicate of a cell in } \mathcal{D}^j \text{ for } 0 \leq j \leq l\}$$

$$n = \max \{n' \mid Q_{m',n'} \text{ is the defining predicate of a cell in } \mathcal{D}^j \text{ for } 0 \leq j \leq l\}$$

This way if  $a, a'$  are in the same coset of  $Q_{m',n'}$  then they are in the same coset of  $Q_{m,n}$ . Choose  $\{\lambda_k\}_{k \in K}$  to range over all the cosets of  $Q_{m,n}$ . Let  $q_i(x, y)$  enumerate all of the defining polynomials  $a_1(x), a_2(x), t - c(x)$  that show up in the cells of  $\mathcal{D}^j$  for any  $j$ . All if those are all polynomials of degree  $\leq 1$  in variables  $x, y$ . We can split each of them as  $q_i(x, y) = p_i(x) - c_i(y)$  where the constant term of  $q_i$  goes into  $c_i$ . This gives us the appropriate finite collection of formulas  $\Phi$ . From the cell decomposition it is easy to see that when  $a, a'$  have the same  $\Phi$ -type, then they have the same  $\phi$ -type. Thus  $\phi$  can be written as a boolean combination of formulas from  $\Phi$ .  $\square$

**Lemma 3.2.** *Let  $\Phi(x; y)$  be a finite collection of formulas. If  $\phi$  can be written as a boolean combination of formulas from  $\Phi$  then*

$$\text{vc}^*(\Phi) \leq r \implies \text{vc}^*(\phi) \leq r \text{ for all } r \in \mathbb{R}.$$

*Proof.* If  $a, a'$  have the same  $\Phi$ -type over  $B$ , then they have the same  $\phi$ -type over  $B$ , where  $B$  is some parameter set. Therefore the number of  $\phi$ -types is bounded by the number of  $\Phi$ -types. The bound follows from Lemma 1.11.  $\square$

For the remainder of the paper fix  $\Phi(x; y)$  to be the collection of formulas defined by Lemma 3.1. By the previous lemma, to show that  $\text{vc}^*(\phi) \leq |x|$ , it suffices to

bound  $\text{vc}^*(\Phi) \leq |x|$ . More precisely, it is sufficient to show that if there is a parameter set  $B$  of size  $N$  then the number of  $\Phi$ -types over  $B$  is  $O(N^{|x|})$ . Fix such a parameter set  $B$  and work with it from now on. We will compute a bound for the number of  $\Phi$ -types over  $B$ .

Consider the set  $T = T(\Phi, B) = \{c_i(b) \mid b \in B, i \in I\} \subset \mathbb{Q}_p$ . In this definition  $B$  is the parameter set that we have fixed and  $c_i(b)$  come from the collection of formulas  $\Phi$  from the quantifier elimination above. View  $T$  as a tree as follows:

**Definition 3.3.**

- For  $c \in \mathbb{Q}_p, \alpha \in \mathbb{Z}$  define a ball

$$B(c, \alpha) = \{c' \in \mathbb{Q}_p \mid \text{val}(c' - c) > \alpha\}.$$

We also let  $B(c, -\infty) = \mathbb{Q}_p$  and  $B(c, +\infty) = \emptyset$ .

- Define a collection of balls  $\mathcal{B} = \{B(t_1, \text{val}(t_1 - t_2))\}_{t_1, t_2 \in T}$ . Note that  $\mathcal{B}$  is a (directed) boolean algebra of sets in  $\mathbb{Q}_p$ . We refer to the atoms in that algebra as intervals. Note that the intervals partition  $\mathbb{Q}_p$  so any element  $a \in \mathbb{Q}_p$  belongs to a unique interval.
- Let's introduce some notation for the intervals. For  $t \in T$  and  $\alpha_L, \alpha_U \in \mathbb{Z} \cup \{-\infty, +\infty\}$  define

$$I(t, \alpha_L, \alpha_U) = B(t, \alpha_L) \setminus \bigcup \{B(t', \alpha_U) \mid t' \in T, \text{val}(t' - t) \geq \alpha_U\}$$

(this is sometimes referred to as the swiss cheese construction). One can check that every interval is of the form  $I(t, \alpha_L, \alpha_U)$  for some values of  $t, \alpha_L, \alpha_U$ . The quantities  $\alpha_L, \alpha_U$  are uniquely determined by the interval  $I(t, \alpha_L, \alpha_U)$ , while  $t$  might not be.

- Intervals are a natural construction for trees, however we will require a more refined notion to make Lemma 3.12 below work. Define a larger collection of balls

$$\mathcal{B}' = \mathcal{B} \cup \{B(c_i(b), \text{val}(c_j(b) - c_k(b)))\}_{i,j,k \in I, b \in B}.$$

Similar to the previous definition, we define a subinterval to be an atom of the boolean algebra generated by  $\mathcal{B}'$ . Subintervals refine intervals. Moreover, as before, each subinterval can be written as  $I(t, \alpha_L, \alpha_U)$  for some values of  $t, \alpha_L, \alpha_U$ . As before,  $\alpha_L, \alpha_U$  are uniquely determined by the subinterval  $I(t, \alpha_L, \alpha_U)$ , while  $t$  might not be.

Subintervals are fine enough to make Lemma 3.12 below work while coarse enough to be  $O(N)$  small:

**Lemma 3.4.**

- *There are at most  $2|T| = 2N|I| = O(N)$  different intervals.*
- *There are at most  $2|T| + |B| \cdot |I|^3 = O(N)$  different subintervals.*

*Proof.* Each new element in the tree  $T$  adds at most two intervals to the total count, so by induction there can be at most  $2|T|$  many intervals. Each new ball in  $\mathcal{B}' \setminus \mathcal{B}$  adds at most one subinterval to the total count, so by induction there are at most  $|\mathcal{B}' \setminus \mathcal{B}|$  more subintervals than there are intervals.  $\square$

**Definition 3.5.** Suppose  $a \in \mathbb{Q}_p$  lies in the interval  $I(t, \alpha_L, \alpha_U)$ . Define the T-valuation of  $a$  to be  $\text{T-val}(a) = \text{val}(a - t)$ .

This is a natural notion having the following properties:

**Lemma 3.6.**

- (a) *T-val( $a$ ) is well-defined, independent of choice of  $t$  to represent the interval.*
- (b) *If  $a \in \mathbb{Q}_p$  lies in the subinterval  $I(t, \alpha_L, \alpha_U)$ , then  $\text{T-val}(a) = \text{val}(a - t)$ .*
- (c) *If  $a \in \mathbb{Q}_p$  lies in the (sub)interval  $I(t, \alpha_L, \alpha_U)$  then  $\alpha_L < \text{T-val}(a) \leq \alpha_U$ .*
- (d) *For any  $a \in \mathbb{Q}_p$  lying in the (sub)interval  $I(t, \alpha_L, \alpha_U)$  and  $t' \in T$ :*
  - *If  $\text{val}(t - t') \geq \alpha_U$ , then  $\text{val}(a - t') = \text{T-val}(a)$ .*
  - *If  $\text{val}(t - t') \leq \alpha_L$ , then  $\text{val}(a - t') = \text{val}(t - t') (\leq \alpha_L < \text{T-val}(a))$ .*

*Proof.* (a)-(c) are clear. For (d) fix  $t' \in T$  and suppose  $a \in \mathbb{Q}_p$  lies in the subinterval  $I(t, \alpha'_L, \alpha'_U)$ . This subinterval lies inside of a unique interval  $I(t, \alpha_L, \alpha_U)$  for some

choice of  $\alpha_L, \alpha_U$  and by the definition of intervals (or more specifically  $\mathcal{B}$ ):

$$\text{val}(t - t') \geq \alpha_U \iff \text{val}(t - t') \geq \alpha'_U,$$

$$\text{val}(t - t') \geq \alpha_L \iff \text{val}(t - t') \geq \alpha'_L.$$

Therefore without loss of generality we may assume that  $a \in \mathbb{Q}_p$  lies in an interval  $I(t, \alpha_L, \alpha_U)$ . By (c) and the definition of intervals one of the three following cases has to hold.

Case 1:  $\text{val}(t - t') \geq \alpha_U$  and  $\text{T-val}(a) < \alpha_U$ . Then

$$\text{val}(t - t') \geq \alpha_U > \text{T-val}(a) = \text{val}(a - t),$$

thus  $\text{val}(a - t') = \text{val}(a - t) = \text{T-val}(a)$  as needed.

Case 2:  $\text{val}(t - t') \geq \alpha_U$  and  $\text{T-val}(a) = \alpha_U$ . Then

$$\text{T-val}(a) = \text{val}(a - t) = \text{val}(t - t') \geq \alpha_U,$$

thus  $\text{val}(a - t') \geq \alpha_U$ . The interval  $I(t, \alpha_L, \alpha_U)$  is disjoint from the ball  $B(t', \alpha_U)$ , so  $a \notin B(t', \alpha_U)$ , that is,  $\text{val}(a - t') \leq \alpha_U$ . Combining this with the previous inequality we get that  $\text{val}(a - t') = \alpha_U = \text{T-val}(a)$  as needed.

Case 3:  $\text{val}(t - t') \leq \alpha_L$ . Then

$$\text{val}(t - t') \leq \alpha_L < \text{T-val}(a) = \text{val}(a - t),$$

thus  $\text{val}(a - t') = \text{val}(t - t')$  as needed. □

**Definition 3.7.** Suppose  $a \in \mathbb{Q}_p$  lies in the subinterval  $I(t, \alpha_L, \alpha_U)$ . We say that  $a$  is far from the boundary tacitly (of  $I(t, \alpha_L, \alpha_U)$ ) if

$$\alpha_L + n \leq \text{T-val}(a) \leq \alpha_U - n.$$

Here  $n$  is as in Lemma 3.1. Otherwise we say that it is close to the boundary (of  $I(t, \alpha_L, \alpha_U)$ ).

**Definition 3.8.** Suppose  $a_1, a_2 \in \mathbb{Q}_p$  lie in the same subinterval  $I(t, \alpha_L, \alpha_U)$ . We say  $a_1, a_2$  have the same subinterval type if one of the following holds:

- Both  $a_1, a_2$  are far from the boundary and  $a_1 - t, a_2 - t$  are in the same  $Q_{m,n}$ -coset. (Here  $Q_{m,n}$  is as in Lemma 3.1.)
- Both  $a_1, a_2$  are close to the boundary and

$$\text{T-val}(a_1) = \text{T-val}(a_2) \leq \text{val}(a_1 - a_2) - n.$$

**Definition 3.9.** For  $c \in \mathbb{Q}_p$  and  $\alpha, \beta \in \mathbb{Z}, \alpha < \beta$  define  $c \upharpoonright [\alpha, \beta)$  to be the record of the coefficients of  $c$  for the valuations between  $[\alpha, \beta)$ . More precisely write  $c$  in its power series form

$$c = \sum_{\gamma \in \mathbb{Z}} c_\gamma p^\gamma \text{ with } c_\gamma \in \{0, 1, \dots, p-1\}$$

Then  $c \upharpoonright [\alpha, \beta)$  is just  $(c_\alpha, c_{\alpha+1}, \dots, c_{\beta-1}) \in \{0, 1, \dots, p-1\}^{\beta-\alpha}$ .

The following lemma is an adaptation of Lemma 7.4 in [1].

**Lemma 3.10.** Fix  $m, n \in \mathbb{N}$ . For any  $x, y, c \in \mathbb{Q}_p$ , if

$$\text{val}(x - c) = \text{val}(y - c) \leq \text{val}(x - y) - n,$$

then  $x - c, y - c$  are in the same coset of  $Q_{m,n}$ .

*Proof.* Call  $a, b \in \mathbb{Q}_p$  similar if  $\text{val } a = \text{val } b$  and

$$a \upharpoonright [\text{val } a, \text{val } a + n) = b \upharpoonright [\text{val } b, \text{val } b + n).$$

If  $a, b$  are similar then

$$a \in Q_{m,n} \iff b \in Q_{m,n}.$$

Moreover for any  $\lambda \in \mathbb{Q}_p^\times$ , if  $a, b$  are similar then so are  $\lambda a, \lambda b$ . Thus if  $a, b$  are similar, then they belong to the same coset of  $Q_{m,n}$ . The hypothesis of the lemma force  $x - c, y - c$  to be similar, thus belonging to the same coset.  $\square$

**Lemma 3.11.** *For each subinterval there are at most  $K = K(Q_{m,n})$  many subinterval types (with  $K$  not depending on  $B$  or on the subinterval).*

*Proof.* Let  $a, a' \in \mathbb{Q}_p$  lie in the same subinterval  $I(t, \alpha_L, \alpha_U)$ .

Suppose  $a, a'$  are far from the boundary. Then they have the same subinterval type if  $a - t, a' - t$  are in the same  $Q_{m,n}$ -coset. So the number of such subinterval types is bounded by the number of  $Q_{m,n}$ -cosets.

Suppose  $a, a'$  are close to the boundary and

$$\text{T-val}(a) - \alpha_L = \text{T-val}(a') - \alpha_L < n \text{ and}$$

$$a \upharpoonright [\text{T-val}(a), \text{T-val}(a) + n) = a' \upharpoonright [\text{T-val}(a'), \text{T-val}(a') + n).$$

Then  $a, a'$  have the same subinterval type. Such a subinterval type is thus determined by  $\text{T-val}(a) - \alpha_L$  and the tuple  $a \upharpoonright [\text{T-val}(a), \text{T-val}(a) + n)$ , therefore there are at most  $np^n$  many such types.

A similar argument works for  $a$  with  $\alpha_U - \text{T-val}(a) \leq n$ .

Adding all this up we get that there are at most

$$K = (\text{number of } Q_{m,n} \text{ cosets}) + 2np^n$$

many subinterval types. □

The following critical lemma relates tree notions to  $\Phi$ -types.

**Lemma 3.12.** *Suppose  $d, d' \in \mathbb{Q}_p^{|x|}$  satisfy the following three conditions:*

- *For all  $i \in I$   $p_i(d)$  and  $p_i(d')$  are in the same subinterval.*
- *For all  $i \in I$   $p_i(d)$  and  $p_i(d')$  have the same subinterval type.*
- *For all  $i, j \in I$ ,  $\text{T-val}(p_i(d)) > \text{T-val}(p_j(d))$  iff  $\text{T-val}(p_i(d')) > \text{T-val}(p_j(d'))$ .*

*Then  $d, d'$  have the same  $\Phi$ -type over  $B$ .*

*Proof.* There are two kinds of formulas in  $\Phi$  (see Lemma 3.1). First we show that  $d, d'$  agree on formulas of the form  $p_i(x) - c_i(y) \in \lambda_k Q_{m,n}$ . It is enough to show that for every  $i \in I, b \in B$ ,  $p_i(d) - c_i(b), p_i(d') - c_i(b)$  are in the same  $Q_{m,n}$ -coset.

Fix such  $i, b$ . For brevity let  $a = p_i(d), a' = p_i(d')$  and  $Q = Q_{m,n}$ . We want to show that  $a - c_i(b), a' - c_i(b)$  are in the same  $Q$ -coset.

Suppose  $a, a'$  are close to the boundary. Then  $\text{T-val}(a) = \text{T-val}(a') \leq \text{val}(a - a') - n$ . Using Lemma 3.6d, we have

$$\text{val}(a - c_i(b)) = \text{val}(a' - c_i(b)) \leq \text{T-val}(a) \leq \text{val}(a - a') - n.$$

Lemma 3.10 shows that  $a - c_i(b), a' - c_i(b)$  are in the same  $Q$ -coset.

Now, suppose both  $a, a'$  are far from the boundary. Let  $I(t, \alpha_L, \alpha_U)$  be the interval containing  $a, a'$ . Then we have

$$\alpha_L + n \leq \text{val}(a - t) \leq \alpha_U - n,$$

$$\alpha_L + n \leq \text{val}(a' - t) \leq \alpha_U - n$$

(as being far from the subinterval's boundary also makes  $a, a'$  far from interval's boundary). We have either  $\text{val}(t - c_i(b)) \geq \alpha_U$  or  $\text{val}(t - c_i(b)) \leq \alpha_L$  (as otherwise it would contradict the definition of intervals, or more specifically  $\mathcal{B}$ ).

Suppose it is the first case  $\text{val}(t - c_i(b)) \geq \alpha_U$ . Then using Lemma 3.6d

$$\text{val}(a - c_i(b)) = \text{val}(a - t) \leq \alpha_U - n \leq \text{val}(t - c_i(b)) - n.$$

So by Lemma 3.10  $a - c_i(b), a - t$  are in the same  $Q$ -coset. By an analogous argument,  $a' - c_i(b), a' - t$  are in the same  $Q$ -coset. As  $a, a'$  have the same subinterval type,  $a - t, a' - t$  are in the same  $Q$ -coset. Thus by transitivity we get that  $a - c_i(b), a' - c_i(b)$  are in the same  $Q$ -coset.

For the second case, suppose  $\text{val}(t - c_i(b)) \leq \alpha_L$ . Then using Lemma 3.6d

$$\text{val}(a - c_i(b)) = \text{val}(t - c_i(b)) \leq \alpha_L \leq \text{val}(a - t) - n,$$

so by Lemma 3.10,  $a - c_i(b), t - c_i(b)$  are in the same  $Q$ -coset. Similarly  $a' - c_i(b), t - c_i(b)$  are in the same  $Q$ -coset. Thus by transitivity we get that  $a - c_i(b), a' - c_i(b)$  are in the same  $Q$ -coset.

Next, we need to show that  $d, d'$  agree on formulas of the form  $\text{val}(p_i(x) - c_i(y)) < \text{val}(p_j(x) - c_j(y))$  (again, referring to the presentation in Lemma 3.1). Fix  $i, j \in I, b \in B$ . We would like to show the following equivalence:

$$(3.1) \quad \text{val}(p_i(d) - c_i(b)) < \text{val}(p_j(d) - c_j(b)) \iff \\ \iff \text{val}(p_i(d') - c_i(b)) < \text{val}(p_j(d') - c_j(b))$$

Suppose  $p_i(d), p_i(d')$  are in the subinterval  $I(t_i, \alpha_i, \beta_i)$  and  $p_j(d), p_j(d')$  are in the subinterval  $I(t_j, \alpha_j, \beta_j)$ . Lemma 3.6d yields the following four cases.

Case 1:

$$\begin{aligned} \text{val}(p_i(d) - c_i(b)) &= \text{val}(p_i(d') - c_i(b)) = \text{val}(t_i - c_i(b)) \\ \text{val}(p_j(d) - c_j(b)) &= \text{val}(p_j(d') - c_j(b)) = \text{val}(t_j - c_j(b)) \end{aligned}$$

Then it is clear that the equivalence (3.1) holds.

Case 2:

$$\begin{aligned} \text{val}(p_i(d) - c_i(b)) &= \text{T-val}(p_i(d)) \text{ and } \text{val}(p_i(d') - c_i(b)) = \text{T-val}(p_i(d')) \\ \text{val}(p_j(d) - c_j(b)) &= \text{T-val}(p_j(d)) \text{ and } \text{val}(p_j(d') - c_j(b)) = \text{T-val}(p_j(d')) \end{aligned}$$

Then the equivalence (3.1) holds by the third hypothesis of the lemma (that order of T-valuations is preserved).

Case 3:

$$\begin{aligned} \text{val}(p_i(d) - c_i(b)) &= \text{val}(p_i(d') - c_i(b)) = \text{val}(t_i - c_i(b)) \\ \text{val}(p_j(d) - c_j(b)) &= \text{T-val}(p_j(d)) \text{ and } \text{val}(p_j(d') - c_j(b)) = \text{T-val}(p_j(d')) \end{aligned}$$

If  $p_j(d), p_j(d')$  are close to the boundary, then  $\text{T-val}(p_j(d)) = \text{T-val}(p_j(d'))$  and the equivalence (3.1) clearly holds. Suppose then that  $p_j(d), p_j(d')$  are far from the



boundary.

$$\alpha_j + n \leq \text{T-val}(p_j(d)), \text{T-val}(p_j(d')) \leq \beta_j - n$$

$$\alpha_j < \text{T-val}(p_j(d)), \text{T-val}(p_j(d')) < \beta_j$$

and  $\text{val}(t_i - c_i(b))$  lies outside of the  $(\alpha_j, \beta_j)$  by the definition of subinterval (more specifically definition of  $\mathcal{B}'$ ). Therefore (3.1) has to hold. (Note that we always have  $\text{T-val}(p_j(d)), \text{T-val}(p_j(d')) \in (\alpha_j, \beta_j]$  by Lemma 3.6c, so we only need the condition on being far from the boundary to avoid the edge case of equality to  $\beta_j$ .)

Case 4:

$$\text{val}(p_i(d) - c_i(b)) = \text{T-val}(p_i(d)) \text{ and } \text{val}(p_i(d') - c_i(b)) = \text{T-val}(p_i(d'))$$

$$\text{val}(p_j(d) - c_j(b)) = \text{val}(p_j(d') - c_j(b)) = \text{val}(t_j - c_j(b))$$

Similar to case 3 (switching  $i, j$ ). □

The previous lemma gives us an upper bound on the number of types - there are at most  $|2I|!$  many choices for the order of T-val,  $O(N)$  many choices for the subinterval for each  $p_i$ , and  $K$  many choices for the subinterval type for each  $p_i$  (where  $K$  is as in Lemma 3.11), giving a total of  $O(N^{|I|}) \cdot K^{|I|} \cdot |I|! = O(N^{|I|})$  many types. This implies  $\text{vc}^*(\Phi) \leq |I|$ . The biggest contribution to this bound are the choices among the  $O(N)$  many subintervals for each  $p_i$  with  $i \in I$ . Are all of those choices realized? Intuitively there are  $|x|$  many variables and  $|I|$  many equations, so once we choose a subinterval for  $|x|$  many  $p_i$ 's, the subintervals for the rest should be determined. This would give the required bound  $\text{vc}^*(\Phi) \leq |x|$ . The next section outlines this idea formally.

#### 4. MAIN PROOF

An alternative way to write  $p_i(c)$  is as a scalar product  $\vec{p}_i \cdot \vec{c}$ , where  $\vec{p}_i$  and  $\vec{c}$  are vectors in  $\mathbb{Q}_p^{|x|}$  (as  $p_i(x)$  is homogeneous linear).

**Lemma 4.1.** *Suppose we have a finite collection of vectors  $\{\vec{p}_j\}_{j \in J}$  with each  $\vec{p}_j \in \mathbb{Q}_p^{|x|}$ . Suppose  $\vec{p} \in \mathbb{Q}_p^{|x|}$  satisfies  $\vec{p} \in \text{span}\{\vec{p}_j\}_{j \in J}$ , and we have  $\vec{c} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$  with  $\text{val}(\vec{p}_j \cdot \vec{c}) > \alpha$  for all  $j \in J$ . Then  $\text{val}(\vec{p} \cdot \vec{c}) > \alpha - \gamma$  for some  $\gamma \in \mathbb{N}$ . Moreover  $\gamma$  can be chosen independently from  $\vec{c}, \alpha$  depending only on  $\{\vec{p}_j\}_{j \in J}$ .*

*Proof.* For some  $c_j \in \mathbb{Q}_p$  for  $j \in J$  we have  $\vec{p} = \sum_{j \in J} c_j \vec{p}_j$ , hence  $\vec{p} \cdot \vec{c} = \sum_{j \in J} c_j \vec{p}_j \cdot \vec{c}$ .

Thus

$$\text{val}(c_j \vec{p}_j \cdot \vec{c}) = \text{val}(c_j) + \text{val}(\vec{p}_j \cdot \vec{c}) > \text{val}(c_j) + \alpha.$$

Let  $\gamma = \max(0, -\max_{j \in J} \text{val}(c_j))$ . Then we have

$$\begin{aligned} \text{val}(\vec{p} \cdot \vec{c}) &= \text{val}\left(\sum_{j \in J} c_j \vec{p}_j \cdot \vec{c}\right) \geq \\ &\geq \min_{j \in J} \text{val}\left(\sum_{j \in J} c_j \vec{p}_j \cdot \vec{c}\right) > \min_{j \in J} \text{val}(c_j) + \alpha \geq \alpha - \gamma \end{aligned}$$

as required.  $\square$

**Corollary 4.2.** *Suppose we have a finite collection of vectors  $\{\vec{p}_i\}_{i \in I}$  with each  $\vec{p}_i \in \mathbb{Q}_p^{|x|}$ . Suppose  $J \subseteq I$  and  $i \in I$  satisfy  $\vec{p}_i \in \text{span}\{\vec{p}_j\}_{j \in J}$ , and we have  $\vec{c} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$  with  $\text{val}(\vec{p}_j \cdot \vec{c}) > \alpha$  for all  $j \in J$ . Then  $\text{val}(\vec{p}_i \cdot \vec{c}) > \alpha - \gamma$  for some  $\gamma \in \mathbb{N}$ . Moreover  $\gamma$  can be chosen independently from  $J, j, \vec{c}, \alpha$  depending only on  $\{\vec{p}_i\}_{i \in I}$ .*

*Proof.* The previous lemma shows that we can pick such  $\gamma$  for a given choice of  $i, J$ , but independent from  $\alpha, \vec{c}$ . To get a choice independent from  $i, J$ , go over all such eligible choices ( $i$  ranges over  $I$  and  $J$  ranges over subsets of  $I$ ), pick  $\gamma$  for each, and then take the maximum of those values.  $\square$

Fix  $\gamma$  according to Corollary 4.2 corresponding to  $\{\vec{p}_i\}_{i \in I}$  given by our collection of formulas  $\Phi$ . (The lemma above is a general result, but we only use it applied to the vectors given by  $\Phi$ .)

**Definition 4.3.** Suppose  $a \in \mathbb{Q}_p$  lies in the subinterval  $I(t, \alpha_L, \alpha_U)$ . Define the  $T$ -floor of  $a$  to be  $\text{T-fl}(a) = \alpha_L$ .

**Definition 4.4.** Let  $f : \mathbb{Q}_p^{|x|} \rightarrow \mathbb{Q}_p^I$  with  $f(c) = (p_i(c))_{i \in I}$ . Define the segment space  $\text{Sg}$  to be the image of  $f$ . Equivalently:

$$\text{Sg} = \left\{ (p_i(c))_{i \in I} \mid c \in \mathbb{Q}_p^{|x|} \right\} \subseteq \mathbb{Q}_p^I$$

Without loss of generality, we may assume that  $I = \{1, 2, \dots, k\}$  (that is the formulas are labeled by consecutive natural numbers). Given a tuple  $(a_i)_{i \in I}$  in the segment space, look at the corresponding  $T$ -floors  $\{\text{T-fl}(a_i)\}_{i \in I}$  and  $T$ -valuations  $\{\text{T-val}(a_i)\}_{i \in I}$ . Partition the segment space by the order types of  $\{\text{T-fl}(a_i)\}_{i \in I}$  and  $\{\text{T-val}(a_i)\}_{i \in I}$  (as subsets of  $\mathbb{Z}$ ).

Work in a fixed set  $\text{Sg}'$  of the partition. After relabeling the  $p_i$  we may assume that

$$\text{T-fl}(a_1) \geq \text{T-fl}(a_2) \geq \dots \text{ for all } a_i \in \text{Sg}'$$

Consider the (relabelled) sequence of vectors  $\vec{p}_1, \vec{p}_2, \dots, \vec{p}_I$ . There is a unique subset  $J \subset I$  such that the set of all vectors with indices in  $J$  is linearly independent, and all vectors with indices outside of  $J$  are a linear combination of preceding vectors. (We can pick those using a greedy algorithm for finding a linearly independent subset of vectors.) We call indices in  $I$  independent and we call the indices in  $I \setminus J$  dependent.

**Definition 4.5.**

- Denote  $\{0, 1, \dots, p-1\}$  as Ct.
- Let Tp be the space of all subinterval types. By Lemma 3.11 we have  $|\text{Tp}| \leq K$ .
- Let Sub be the space of all subintervals. By Lemma 3.4 we have  $|\text{Sub}| \leq 3|I|^2 \cdot N = O(N)$ .

**Definition 4.6.** Now, we define a function

$$g_{\text{Sg}'} : \text{Sg}' \longrightarrow \text{Tp}^I \times \text{Sub}^J \times \text{Ct}^{I \setminus J}$$

as follows:

Let  $a = (a_i)_{i \in I} \in \text{Sg}'$ . To define  $g_{\text{Sg}'}(a)$  we need to specify where it maps  $a$  in each individual component of the product.

For each  $a_i$  record its subinterval type, giving the first component in  $\text{Tp}^I$ .

For  $a_j$  with  $j \in J$ , record the subinterval of  $a_j$ , giving the second component in  $\text{Sub}^J$ .

For the third component (an element of  $\text{Ct}^{I \setminus J}$ ) do the following computation. Pick  $a_i$  with  $i$  dependent. Let  $j$  be the largest independent index with  $j < i$ . Record  $a_i \upharpoonright [\text{T-fl}(a_j) - \gamma, \text{T-fl}(a_j))$ .

Combine  $g_{\text{Sg}'}$  for all the partitions to get a function

$$g : \text{Sg} \longrightarrow \text{Tp}^I \times \text{Sub}^J \times \text{Ct}^{I \setminus J}.$$

**Lemma 4.7.** *Suppose we have  $c, c' \in \mathbb{Q}_p^{|x|}$  such that  $f(c), f(c')$  are in the same set  $\text{Sg}'$  of the partition of  $\text{Sg}$  and  $g(f(c)) = g(f(c'))$ . Then  $c, c'$  have the same  $\Phi$ -type over  $B$ .*

*Proof.* Let  $a_i = \vec{p}_i \cdot \vec{c}$  and  $a'_i = \vec{p}_i \cdot \vec{c}'$  so that

$$f(c) = (p_i(c))_{i \in I} = (\vec{p}_i \cdot \vec{c})_{i \in I} = (a_i)_{i \in I}$$

$$f(c') = (p_i(c'))_{i \in I} = (\vec{p}_i \cdot \vec{c}')_{i \in I} = (a'_i)_{i \in I}$$

For each  $i$  we show that  $a_i, a'_i$  are in the same subinterval and have the same subinterval type, so the conclusion follows by Lemma 3.12 ( $f(c), f(c')$  are in the same partition ensuring the proper order of T-valuations for the 3rd condition of the lemma).  $\text{Tp}$  records the subinterval type of each element, so if  $g(\vec{a}) = g(\vec{a}')$  then  $a_i, a'_i$  have the same subinterval type for all  $i \in I$ . Thus it remains to show

that  $a_i, a'_i$  lie in the same subinterval for all  $i \in I$ . Suppose  $i$  is an independent index. Then by construction, Sub records the subinterval for  $a_i, a'_i$ , so those have to belong to the same subinterval. Now suppose  $i$  is dependent. Pick the largest  $j < i$  such that  $j$  is independent. We have  $\text{T-fl}(a_i) \leq \text{T-fl}(a_j)$  and  $\text{T-fl}(a'_i) \leq \text{T-fl}(a'_j)$ . Moreover  $\text{T-fl}(a_j) = \text{T-fl}(a'_j)$  as  $a_j, a'_j$  lie in the same subinterval (using the earlier part of the argument as  $j$  is independent).

**Claim 4.8.**  $\text{val}(a_i - a'_i) > \text{T-fl}(a_j) - \gamma$

*Proof.* Let  $K$  be the set of the independent indices less than  $i$ . Note that by the definition for dependent indices we have  $\vec{p}_i \in \text{span}\{\vec{p}_k\}_{k \in K}$ . We also have

$$\text{val}(a_k - a'_k) > \text{T-fl}(a_k) \text{ for all } k \in K$$

as  $a_k, a'_k$  lie in the same subinterval (using the earlier part of the argument as  $k$  is independent). Now  $\text{val}(a_k - a'_k) > \text{T-fl}(a_j)$  for all  $k \in K$  by monotonicity of  $\text{T-fl}(a_k)$ . Moreover  $a_k - a'_k = \vec{p}_k \cdot \vec{c} - \vec{p}_k \cdot \vec{c}' = \vec{p}_k \cdot (\vec{c} - \vec{c}')$ . Combining the two, we get that  $\text{val}(\vec{p}_k \cdot (\vec{c} - \vec{c}')) > \text{T-fl}(a_j)$  for all  $k \in K$ . Now observe that  $K \subset I, i \in I, \vec{c} - \vec{c}' \in \mathbb{Q}_p^{|x|}, \text{T-fl}(a_j) \in \mathbb{Z}$  satisfy the requirements of Lemma 4.2, so we apply it to obtain  $\text{val}(\vec{p}_i \cdot (\vec{c} - \vec{c}')) > \text{T-fl}(a_j) - \gamma$ . Similarly to before, we have  $\vec{p}_i \cdot (\vec{c} - \vec{c}') = \vec{p}_i \cdot \vec{c} - \vec{p}_i \cdot \vec{c}' = a_i - a'_i$ . Therefore we can conclude that  $\text{val}(a_i - a'_i) > \text{T-fl}(a_j) - \gamma$  as needed, finishing the proof of the claim.  $\square$

Additionally  $a_i, a'_i$  have the same image in the Ct component, so we have

$$\text{val}(a_i - a'_i) > \text{T-fl}(a_j).$$

We now would like to show that  $a_i, a'_i$  lie in the same subinterval. As  $\text{T-fl}(a_i) \leq \text{T-fl}(a_j)$ ,  $\text{T-fl}(a'_i) \leq \text{T-fl}(a'_j)$  and  $\text{T-fl}(a_j) = \text{T-fl}(a'_j)$  we have that  $\text{val}(a_i - a'_i) > \text{T-fl}(a_i)$  and  $\text{val}(a_i - a'_i) > \text{T-fl}(a'_i)$ . Suppose that  $a_i$  lies in the subinterval  $I(t, \text{T-fl}(a_i), \alpha_U)$  and that  $a'_i$  lies in the subinterval  $I(t', \text{T-fl}(a'_i), \alpha'_U)$ . Without loss of generality assume that  $\text{T-fl}(a_i) \leq \text{T-fl}(a'_i)$ . As  $\text{val}(a_i - a'_i) > \text{T-fl}(a'_i)$ , this implies that  $a_i \in B(a'_i, \text{T-fl}(a'_i))$ . Then  $a_i \in B(t', \text{T-fl}(a'_i))$  as  $\text{val}(a_i - t') > \text{T-fl}(a'_i)$ . This implies

that  $B(t, \text{T-fl}(a_i)) \cap B(t', \text{T-fl}(a'_i)) \neq \emptyset$  as they both contain  $a_i$ . As balls are directed, the non-zero intersection means that one ball has to be contained in another. Given our assumption that  $\text{T-fl}(a_i) \leq \text{T-fl}(a'_i)$ , we have  $B(t, \text{T-fl}(a_i)) \subset B(t', \text{T-fl}(a'_i))$ . For the subintervals to be disjoint we need  $I(t, \text{T-fl}(a_i), \alpha_U) \cap B(t', \text{T-fl}(a'_i)) = \emptyset$ . But  $\text{val}(t' - a_i) > \text{T-fl}(a'_i)$  implying that  $a_i \in I(t, \text{T-fl}(a_i), \alpha_U) \cap B(t', \text{T-fl}(a'_i))$  giving a contradiction. Therefore the subintervals coincide finishing the proof.  $\square$

**Corollary 4.9.** *The dual vc-density of  $\Phi(x, y)$  is  $\leq |x|$ .*

*Proof.* Suppose we have  $c, c' \in \mathbb{Q}_p^{|x|}$  such that  $f(c), f(c')$  are in the same partition and  $g(f(c)) = g(f(c'))$ . Then by the previous lemma  $c, c'$  have the same  $\Phi$ -type. Thus the number of possible  $\Phi$ -types is bounded by the size of the range of  $g$  times the number of possible partitions

$$(\text{number of partitions}) \cdot |\text{Tp}|^{|I|} \cdot |\text{Sub}|^{|J|} \cdot |\text{Ct}|^{|I-J|}.$$

There are at most  $(|2I|!)^2$  many partitions of  $\text{Sg}$ , so in the product above, the only component dependent on  $B$  is

$$|\text{Sub}|^{|J|} \leq (N \cdot 3|I|^2)^{|J|} = O(N^{|J|}).$$

Every  $p_i$  is an element of a  $|x|$ -dimensional vector space, so there can be at most  $|x|$  many independent vectors. Thus we have  $|J| \leq |x|$  and the bound follows.  $\square$

**Corollary 4.10** (Theorem 2.6). *The  $\mathcal{L}_{\text{aff}}$ -structure  $\mathbb{Q}_p$  satisfies  $\text{vc}(n) = n$ .*

*Proof.* The previous lemma implies that  $\text{vc}^*(\phi) \leq \text{vc}^*(\Phi) \leq |x|$ . As choice of  $\phi$  was arbitrary, this implies that the vc-density of any formula is bounded by the arity of  $x$ .  $\square$

This proof relies heavily on the linearity of the defining polynomials  $a_1, a_2, c$  in the cell decomposition result (see Definition 2.4). Linearity is used to separate the

$x$  and  $y$  variables as well as for Corollary 4.2 to reduce the number of independent factors from  $|I|$  to  $|x|$ . The paper [2] has cell decomposition results for more expressive reducts of  $\mathbb{Q}_p$ , including, for example, restricted multiplication. While our results don't apply to it directly, it is this author's hope that similar techniques can be used to also compute the vc-function for those structures.

Another interesting question whether the reduct studied in this paper has VC 1 property (see [1] 5.2 for the definition). If so, this would imply the linear vc-density bound directly. The paper [1] implies that the reduct has VC 2 property. While there are techniques for showing that a structure has a given VC property, less is known about showing that a structure doesn't have a given VC property. Perhaps the simple structure of the  $\mathcal{L}_{aff}$ -reduct can help understand this property better.

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