# VC-DENSITY IN AN ADDITIVE REDUCT OF THE $P$-ADIC NUMBERS

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Abstract. Aschenbrenner et. al. computed a linear bound for the vc-density function in the field of $p$-adic numbers, but this bound is not known to be optimal. In this paper we investigate a certain $P$-minimal additive reduct of the field of $p$-adic numbers and use a cell decomposition result of Leenknegt to compute an optimal bound for that structure.

VC-density was studied in model theory in [1] by Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko as a natural notion of dimension for definable families of sets in NIP theories. In a complete NIP theory $T$ we can define the vc-function

$$\mathrm{vc}^T = \mathrm{vc} : \mathbb{N} \longrightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$$

where $\mathrm{vc}(n)$ measures the worst-case complexity of families of definable sets in an $n$-fold cartesian power of the underlying set of a model of $T$ (see 1.13 below for a precise definition of $\mathrm{vc}^T$). The simplest possible behavior is $\mathrm{vc}(n) = n$ for all $n$, satisfied, for example, if $T$ is o-minimal. For $T = \mathrm{Th}(\mathbb{Q}_p)$, the paper [1] computes an upper bound for this function to be $2n - 1$, and it is not known whether this is optimal. This same bound holds in any reduct of the field of $p$-adic numbers, but one may expect that the simplified structure of suitable reducts would allow a better bound. In [2], Leenknegt provides a cell decomposition result for a certain $P$-minimal additive reduct of the field of $p$-adic numbers. Using this result, in this paper we improve the bound for the vc-function, showing that in Leenknegt's structure $\mathrm{vc}(n) = n$.

Section 1 defines vc-density and states some basic lemmas about it. A more in depth exposition of vc-density can be found in [1]. Section 2 defines and states some basic facts about the theory of $p$-adic numbers. Here we also introduce the reduct which we will be working with. Section 3 sets up basic definitions and lemmas that will be needed for the proof. We define trees and intervals and show how they help with vc-density calculations. Section 4 concludes the proof.

Throughout the paper, variables and tuples of elements will be simply denoted as $x, y, a, b, \ldots$. We will occasionally write $\vec{a}$ instead of $a$ for a tuple in $\mathbb{Q}_p^n$ to emphasize it as an element of the $\mathbb{Q}_p$-vector space $\mathbb{Q}_p^n$. We denote the arity of a tuple $x$ of variables by $|x|$. The set of natural numbers is denoted by $\mathbb{N} = \{0, 1, \ldots\}$.

## 1. VC-dimension and vc-density

Throughout this section we work with a collection $\mathcal{F}$ of subsets of an infinite set $X$. We call the pair $(X, \mathcal{F})$ a set system.

**Definition 1.1.**

- Given a subset $A$ of $X$, we define the set system $(A, A \cap \mathcal{F})$ where $A \cap \mathcal{F} = \{A \cap F \mid F \in \mathcal{F}\}$.
- For $A \subseteq X$ we say that $\mathcal{F}$ shatters $A$ if $A \cap \mathcal{F} = \mathcal{P}(A)$ (the power set of $A$).

**Definition 1.2.** We say $(X, \mathcal{F})$ has VC-dimension $n$ if the largest subset of $X$ shattered by $\mathcal{F}$ is of size $n$. If $\mathcal{F}$ shatters arbitrarily large subsets of $X$, we say that $(X, \mathcal{F})$ has infinite VC-dimension. We denote the VC-dimension of $(X, \mathcal{F})$ by $\mathrm{VC}(X, \mathcal{F})$.

**Note 1.3.** We may drop $X$ from the notation $\mathrm{VC}(X, \mathcal{F})$, as the VC-dimension doesn't depend on the base set and is determined by $(\bigcup \mathcal{F}, \mathcal{F})$.

Set systems of finite VC-dimension tend to have good combinatorial properties, and we consider set systems with infinite VC-dimension to be poorly behaved.

Another natural combinatorial notion is that of the dual system of a set system:

**Definition 1.4.** For $a \in X$ define $X_a = \{F \in \mathcal{F} \mid a \in F\}$. Let $\mathcal{F}^* = \{X_a \mid a \in X\}$. We call $(\mathcal{F}, \mathcal{F}^*)$ the dual system of $(X, \mathcal{F})$. The VC-dimension of the dual system of $(X, \mathcal{F})$ is referred to as the dual VC-dimension of $(X, \mathcal{F})$ and denoted by $\mathrm{VC}^*(\mathcal{F})$. (As before, this notion doesn't depend on $X$.)

**Lemma 1.5** (see 2.13b in [3]). *A set system $(X, \mathcal{F})$ has finite VC-dimension if and only if its dual system has finite VC-dimension. More precisely*

$$\mathrm{VC}^*(\mathcal{F}) \leq 2^{1+\mathrm{VC}(\mathcal{F})}.$$

For a more refined notion of complexity of $(X, \mathcal{F})$ we look at the traces of our family on finite sets:

**Definition 1.6.** Define the shatter function $\pi_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ and the dual shatter function $\pi^*_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ by

$$\pi_{\mathcal{F}}(n) = \max \{ |A \cap \mathcal{F}| \mid A \subseteq X \text{ and } |A| = n \}$$

$$\pi^*_{\mathcal{F}}(n) = \max \{ \mathrm{atoms}(B) \mid B \subseteq \mathcal{F}, |B| = n \}$$

where $\mathrm{atoms}(B)$ = number of atoms in the boolean algebra of sets generated by $B$. Note that the dual shatter function is precisely the shatter function of the dual system: $\pi^*_{\mathcal{F}} = \pi_{\mathcal{F}^*}$.

A simple upper bound is $\pi_{\mathcal{F}}(n) \leq 2^n$ (same for the dual). If the VC-dimension of $\mathcal{F}$ is infinite then clearly $\pi_{\mathcal{F}}(n) = 2^n$ for all $n$. Conversely we have the following remarkable fact:

**Theorem 1.7** (Sauer-Shelah '72, see [5], [6]). *If the set system $(X, \mathcal{F})$ has finite VC-dimension $d$ then $\pi_{\mathcal{F}}(n) \leq \binom{n}{\leq d}$ for all $n$, where $\binom{n}{\leq d} = \binom{n}{d} + \binom{n}{d-1} + \ldots + \binom{n}{1}$.*

Thus the systems with a finite VC-dimension are precisely the systems where the shatter function grows polynomially. The vc-density of $\mathcal{F}$ quantifies the growth of the shatter function of $\mathcal{F}$:

**Definition 1.8.** Define the vc-density and dual vc-density of $\mathcal{F}$ as

$$\mathrm{vc}(\mathcal{F}) = \limsup_{n \to \infty} \frac{\log \pi_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\},$$

$$\mathrm{vc}^*(\mathcal{F}) = \limsup_{n \to \infty} \frac{\log \pi^*_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}.$$

Generally speaking a shatter function that is bounded by a polynomial doesn't itself have to be a polynomial. Proposition 4.12 in [1] gives an example of a shatter function that grows like $n \log n$ (so it has vc-density 1).

So far the notions that we have defined are purely combinatorial. We now adapt VC-dimension and vc-density to the model theoretic context.

**Definition 1.9.** Work in a first-order structure $M$. Fix a finite collection of formulas $\Phi(x, y)$ in the language $\mathcal{L}(M)$ of $M$.

- For $\phi(x, y) \in \mathcal{L}(M)$ and $b \in M^{|y|}$ let

$$\phi(M^{|x|}, b) = \{a \in M^{|x|} \mid \phi(a, b)\} \subseteq M^{|x|}.$$

- Let $\Phi(M^{|x|}, M^{|y|}) = \{\phi(M^{|x|}, b) \mid \phi_i \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|})$.
- Let $\mathcal{F}_\Phi = \Phi(M^{|x|}, M^{|y|})$, giving rise to a set system $(M^{|x|}, \mathcal{F}_\Phi)$.
- Define the VC-dimension $\mathrm{VC}(\Phi)$ of $\Phi$ to be the VC-dimension of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.
- Define the vc-density $\mathrm{vc}(\Phi)$ of $\Phi$ to be the vc-density of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.

We will also refer to the vc-density and VC-dimension of a single formula $\phi$ viewing it as a one element collection $\Phi = \{\phi\}$.

Counting atoms of a boolean algebra in a model theoretic setting corresponds to counting types, so it is instructive to rewrite the shatter function in terms of types.

**Definition 1.10.**

$$\pi^*_\Phi(n) = \max \{\text{number of } \Phi\text{-types over } B \mid B \subseteq M, |B| = n\}.$$

Here a $\Phi$-type over $B$ is a maximal consistent collection of formulas of the form $\phi(x, b)$ or $\neg\phi(x, b)$ where $\phi \in \Phi$ and $b \in B$.

The functions $\pi^*_\Phi$ and $\pi^*_{\mathcal{F}_\Phi}$ do not have to agree, as one fixes the number of generators of a boolean algebra of sets and the other fixes the size of the parameter set.

However, as the following lemma demonstrates, they both give the same asymptotic definition of dual vc-density.

**Lemma 1.11.**

$$\mathrm{vc}^*(\Phi) = \textit{degree of polynomial growth of } \pi^*_\Phi(n) = \limsup_{n\to\infty} \frac{\log \pi^*_\Phi(n)}{\log n}.$$

*Proof.* With a parameter set $B$ of size $n$, we get at most $|\Phi|n$ sets $\phi(M^{|x|}, b)$ with $\phi \in \Phi, b \in B$. We check that asymptotically it doesn't matter whether we look at growth of boolean algebra of sets generated by $n$ or by $|\Phi|n$ many sets. We have:

$$\pi^*_{\mathcal{F}_\Phi}(n) \leq \pi^*_\Phi(n) \leq \pi^*_{\mathcal{F}_\Phi}(|\Phi|n).$$

Hence:

$$\begin{aligned}
&\mathrm{vc}^*(\Phi) \leq \limsup_{n\to\infty} \frac{\log \pi^*_\Phi(n)}{\log n} \leq \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log n} = \\
&= \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log |\Phi|n} \frac{\log |\Phi|n}{\log n} = \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log |\Phi|n} \leq \\
&\leq \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(n)}{\log n} = \mathrm{vc}^*(\Phi).
\end{aligned}$$

□

One can check that the shatter function and hence VC-dimension and vc-density of a formula are elementary notions, so they only depend on the first-order theory of the structure $M$.

NIP theories are a natural context for studying vc-density. In fact we can take the following as the definition of NIP:

**Definition 1.12.** Define $\phi$ to be NIP if it has finite VC-dimension in a theory $T$. A theory $T$ is NIP if all the formulas in $T$ are NIP.

In a general combinatorial context (for arbitrary set systems), vc-density can be any real number in $0 \cup [1, \infty)$ (see [4]). Less is known if we restrict our attention to NIP theories. Proposition 4.6 in [1] gives examples of formulas that have non-integer rational vc-density in an NIP theory, however it is open whether one can get an irrational vc-density in this model-theoretic setting.

Instead of working with a theory formula by formula, we can look for a uniform bound for all formulas:

**Definition 1.13.** For a given NIP structure $M$, define the <u>vc-function</u>

$$\begin{aligned} \mathrm{vc}^M(n) &= \sup\{\mathrm{vc}^*(\phi(x,y)) \mid \phi \in \mathcal{L}(M), |x| = n\} \\ &= \sup\{\mathrm{vc}(\phi(x,y)) \mid \phi \in \mathcal{L}(M), |y| = n\} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}\,. \end{aligned}$$

As before this definition is elementary, so it only depends on the theory of $M$. We omit the superscript $M$ if it is understood from the context. One can easily check the following bounds:

**Lemma 1.14** (Lemma 3.22 in [1])**.** *We have* $\mathrm{vc}(1) \geq 1$ *and* $\mathrm{vc}(n) \geq n\,\mathrm{vc}(1)$.

However, it is not known whether the second inequality can be strict or even whether $\mathrm{vc}(1) < \infty$ implies $\mathrm{vc}(n) < \infty$.

## 2. $P$-adic numbers

The field $\mathbb{Q}_p$ of $p$-adic numbers is often studied in the language of Macintyre

$$\mathcal{L}_{Mac} = \{0, 1, +, -, \cdot, |, \{P_n\}_{n\in\mathbb{N}}\}$$

which is a language $\{0, 1, +, -, \cdot\}$ of rings together with unary predicates $P_n$ interpreted in $\mathbb{Q}_p$ so as to satisfy

$$P_n x \leftrightarrow \exists y\; y^n = x$$

and a divisibility relation where $a|b$ holds in $\mathbb{Q}_p$ when $\operatorname{val} a \leq \operatorname{val} b$.

Note that $P_n \setminus \{0\}$ is a multiplicative subgroup of $\mathbb{Q}_p$ with finitely many cosets.

**Theorem 2.1** (Macintyre '76, [7])**.** *The* $\mathcal{L}_{Mac}$*-structure* $\mathbb{Q}_p$ *has quantifier elimination.*

There is also a cell decomposition result:

**Definition 2.2.** Define $\underline{k\text{-cells}}$ recursively as follows. A $\underline{0\text{-cell}}$ is the singleton $\mathbb{Q}_p^0$. A $\underline{(k+1)\text{-cell}}$ is a subset of $\mathbb{Q}_p^{k+1}$ of the following form:

$$\{(x,t) \in D \times \mathbb{Q}_p \mid \operatorname{val} a_1(x) \,\Box_1\, \operatorname{val}(t - c(x)) \,\Box_2\, \operatorname{val} a_2(x), t - c(x) \in \lambda P_n\}$$

where $D$ is a $k$-cell, $a_1(x), a_2(x), c(x)$ are definable functions $D \longrightarrow \mathbb{Q}_p$, each of $\Box_i$ is $<, \leq$ or no condition, $n \in \mathbb{N}$, and $\lambda \in \mathbb{Q}_p$.

**Theorem 2.3** (Denef '84, [8])**.** *Any definable subset of $\mathbb{Q}_p^k$ defined by an $\mathcal{L}_{Mac}$-formula decomposes into a finite disjoint union of $k$-cells.*

In [1], Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko show that $\mathbb{Q}_p$ as $\mathcal{L}_{Mac}$-structure satisfies $\mathrm{vc}(n) \leq 2n - 1$, however it is not known whether this bound is optimal.

In [2], Leenknegt analyzes the reduct of $\mathbb{Q}_p$ to the language

$$\mathcal{L}_{aff} = \left\{0, 1, +, -, \{\bar{c}\}_{c \in \mathbb{Q}_p}, |, \{Q_{m,n}\}_{m,n \in \mathbb{N}}\right\}$$

where $\bar{c}$ denotes the scalar multiplication by $c$, $a|b$ as above stands for $\operatorname{val} a \leq \operatorname{val} b$, and $Q_{m,n}$ is a unary predicate interpreted as

$$Q_{m,n} = \bigcup_{k \in \mathbb{Z}} p^{km}(1 + p^n \mathbb{Z}_p).$$

Note that $Q_{m,n} \setminus \{0\}$ is a subgroup of the multiplicative group of $\mathbb{Q}_p$ with finitely many cosets. One can check that these extra relation symbols are definable in the $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$. The paper [2] provides a cell decomposition result with the following cells:

**Definition 2.4.** A $\underline{0\text{-cell}}$ is the singleton $\mathbb{Q}_p^0$. A $\underline{(k+1)\text{-cell}}$ is a subset of $\mathbb{Q}_p^{k+1}$ of the following form:

$$\{(x,t) \in D \times \mathbb{Q}_p \mid \operatorname{val} a_1(x) \,\Box_1\, \operatorname{val}(t - c(x)) \,\Box_2\, \operatorname{val} a_2(x), t - c(x) \in \lambda Q_{m,n}\}$$

where $D$ is a $k$-cell, called the base of the cell, $a_1(x), a_2(x), c(x)$ are polynomials of degree $\leq 1$, called the defining polynomials, each of $\square_1, \square_2$ is $<$ or no condition, $m, n \in \mathbb{N}$, and $\lambda \in \mathbb{Q}_p$. We call $Q_{m,n}$ the defining predicate of our cell.

**Theorem 2.5** (Leenknegt '12)**.** *Any definable subset of $\mathbb{Q}_p^k$ defined by an $\mathcal{L}_{aff}$-formula decomposes into a finite disjoint union of $k$-cells.*

Moreover, [2] shows that $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ is a $P$-minimal reduct, that is, the one-variable definable sets of $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ coincide with the one-variable definable sets in the full structure $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$.

The main result of this paper is the computation of the vc-function for this structure:

**Theorem 2.6.** *The $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ satisfies $\mathrm{vc}(n) = n$ for all $n$.*

## 3. Key Lemmas and Definitions

To show that $\mathrm{vc}(n) = n$ it suffices to bound $\mathrm{vc}^*(\phi) \leq |x|$ for every $\mathcal{L}_{aff}$-formula $\phi(x; y)$. Fix such a formula $\phi(x; y)$. Instead of working with this formula directly, we first simplify it using quantifier elimination. The required quantifier elimination result can be easily obtained from cell decomposition:

**Lemma 3.1.** *Any $\mathcal{L}_{aff}$-formula $\phi(x; y)$ is equivalent in the $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ to a boolean combination of formulas from a collection*

$$\Phi(x; y) = \{\mathrm{val}(p_i(x) - c_i(y)) < \mathrm{val}(p_j(x) - c_j(y))\}_{i,j\in I} \cup$$
$$\{p_i(x) - c_i(y) \in \lambda_k Q_{m,n}\}_{i\in I, k\in K}$$

*of $\mathcal{L}_{aff}$-formulas where $I, K$ are finite index sets, each $p_i$ is a degree $\leq 1$ polynomial in $x$ without a constant term, each $c_i$ is a degree $\leq 1$ polynomial in $y$, $m, n \in \mathbb{N}$, and $\lambda_k \in \mathbb{Q}_p$.*

*Proof.* Let $l = |x| + |y|$. Using Theorem 2.5 partition the subset of $\mathbb{Q}_p^l$ defined by $\phi$ to obtain $\mathscr{D}^l$, a collection of $l$-cells. Let $\mathscr{D}^{l-1}$ be the collection of the bases of the cells in $\mathscr{D}^l$. Similarly, construct by induction $\mathscr{D}^i$ for each $0 \leq j < l$, where $\mathscr{D}^j$ is the collection of $j$-cells which are the bases of cells in $\mathscr{D}^{j+1}$. Set

$$m = \prod \left\{ m' \mid Q_{m',n'} \text{ is the defining predicate of a cell in } \mathscr{D}^j \text{ for } 0 \leq j \leq l \right\}$$

$$n = \max \left\{ n' \mid Q_{m',n'} \text{ is the defining predicate of a cell in } \mathscr{D}^j \text{ for } 0 \leq j \leq l \right\}.$$

This way, if $a, a'$ are in the same coset of the definable predicate $Q_{m',n'}$ of a cell in $\mathscr{D}^j$ ($0 \leq j \leq l$), then they are in the same coset of $Q_{m,n}$. Choose $\{\lambda_k\}_{k \in K}$ to range over all representations of cosets of $Q_{m,n}$. Let $q_i(x, y)$ enumerate all of the defining polynomials $a_1(x), a_2(x), t - c(x)$ that show up in the cells of $\mathscr{D}^j$ for any $j$. All of those are polynomials of degree $\leq 1$ in the variables $x, y$. We can split each of them as $q_i(x, y) = p_i(x) - c_i(y)$ where the constant term of $q_i$ is substituted by $c_i$. This gives us the appropriate finite collection $\Phi$ of formulas. From the cell decomposition it is easy to see that when $a, a'$ have the same $\Phi$-type, then they have the same $\phi$-type. Thus $\phi$ can be written as a boolean combination of formulas from $\Phi$. □

**Lemma 3.2.** *Let $\Phi(x; y)$ be a finite collection of formulas. If $\phi$ can be written as a boolean combination of formulas from $\Phi$ then* $\mathrm{vc}^*(\phi) \leq \mathrm{vc}^*(\Phi)$.

*Proof.* If $a, a'$ have the same $\Phi$-type over $B$, then they have the same $\phi$-type over $B$, where $B$ is some parameter set. Therefore the number of $\phi$-types is bounded by the number of $\Phi$-types. The bound follows from Lemma 1.11. □

For the remainder of the paper fix $\Phi(x; y)$ to be a collection of formulas as in Lemma 3.1. By the previous lemma, to show that $\mathrm{vc}^*(\phi) \leq |x|$, it suffices to bound $\mathrm{vc}^*(\Phi) \leq |x|$. More precisely, it is sufficient to show that given a parameter set $B$ of size $N$, the number of $\Phi$-types over $B$ is $O(N^{|x|})$. Fix such a parameter set $B$ and work with it from now on. We will compute a bound for the number of $\Phi$-types over $B$.

Consider the finite set $T = T(\Phi, B) = \{c_i(b) \mid b \in B, i \in I\} \subseteq \mathbb{Q}_p$. In this definition $B$ is the parameter set that we have fixed and $c_i(b)$ come from the collection of formulas $\Phi$ from the quantifier elimination above. View $T$ as a tree as follows:

**Definition 3.3.**

- For $c \in \mathbb{Q}_p, \alpha \in \mathbb{Z}$ define the (open) ball

$$B(c, \alpha) = \{c' \in \mathbb{Q}_p \mid \operatorname{val}(c' - c) > \alpha\}$$

  of radius $\alpha$ and center $c$. We also let $B(c, -\infty) = \mathbb{Q}_p$ and $B(c, +\infty) = \emptyset$.
- Define the collection of balls $\mathscr{B} = \{B(t_1, \operatorname{val}(t_1 - t_2))\}_{t_1, t_2 \in T}$. Note that $\mathscr{B}$ is a (directed) boolean algebra of sets in $\mathbb{Q}_p$. We refer to the atoms in that algebra as intervals. Note that the intervals partition $\mathbb{Q}_p$ so any element $a \in \mathbb{Q}_p$ belongs to a unique interval.
- Let's introduce some notation for the intervals. For $t \in T$ and $\alpha_L, \alpha_U \in \mathbb{Z} \cup \{-\infty, +\infty\}$ define

$$\mathrm{I}(t, \alpha_L, \alpha_U) = B(t, \alpha_L) \backslash \bigcup \{B(t', \alpha_U) \mid t' \in T, \operatorname{val}(t' - t) \geq \alpha_U\}$$

  (this is sometimes referred to as the swiss cheese construction). One can check that every interval is of the form $\mathrm{I}(t, \alpha_L, \alpha_U)$ for some values of $t, \alpha_L, \alpha_U$. The quantities $\alpha_L, \alpha_U$ are uniquely determined by the interval $\mathrm{I}(t, \alpha_L, \alpha_U)$, while $t$ might not be.
- Intervals are a natural construction for trees, however we will require a more refined notion to make Lemma 3.12 below work. Define a larger collection of balls

$$\mathscr{B}' = \mathscr{B} \cup \{B(c_i(b), \operatorname{val}(c_j(b) - c_k(b)))\}_{i,j,k \in I, b \in B}.$$

  Similarly to the previous definition, we define a subinterval to be an atom of the boolean algebra generated by $\mathscr{B}'$. Subintervals refine intervals. Moreover, as before, each subinterval can be written as $\mathrm{I}(t, \alpha_L, \alpha_U)$ for some values of $t, \alpha_L, \alpha_U$. As before, $\alpha_L, \alpha_U$ are uniquely determined by the subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$, while $t$ might not be.

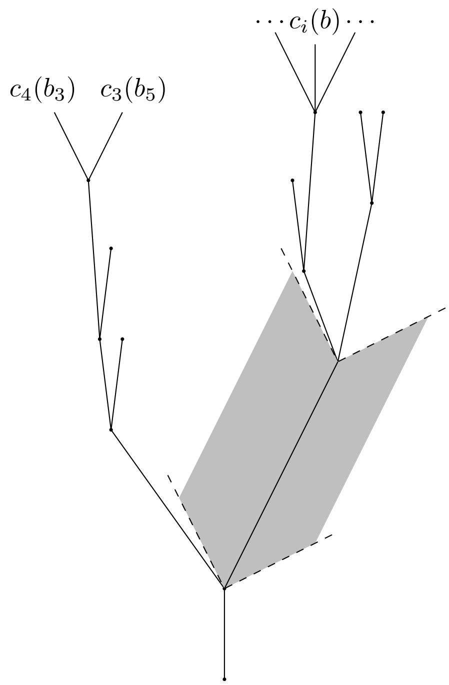


FIGURE 1. A typical interval (left) and subinterval (right) on a tree $\{c_i(b) \mid i \in I, b \in B\}$.

Subintervals are fine enough to make Lemma 3.12 below work while coarse enough to be $O(N)$ small:

**Lemma 3.4.**

- *There are at most* $2|T| = 2N|I| = O(N)$ *different intervals.*
- *There are at most* $2|T| + |B| \cdot |I|^3 = O(N)$ *different subintervals.*

*Proof.* Each new element in the tree $T$ adds at most two intervals to the total count, so by induction there can be at most $2|T|$ many intervals. Each new ball in $\mathscr{B}' \backslash \mathscr{B}$ adds at most one subinterval to the total count, so by induction there are at most $|\mathscr{B}' \backslash \mathscr{B}|$ more subintervals than there are intervals. $\square$

**Definition 3.5.** Suppose $a \in \mathbb{Q}_p$ lies in the interval $\mathrm{I}(t, \alpha_L, \alpha_U)$. Define the T-valuation of $a$ to be $\text{T-val}(a) = \mathrm{val}(a - t)$.

This is a natural notion having the following properties:

**Lemma 3.6.**

*(a)* $\text{T-val}(a)$ *is well-defined, independent of choice of* $t$ *to represent the interval.*
*(b) If* $a \in \mathbb{Q}_p$ *lies in the subinterval* $\mathrm{I}(t, \alpha_L, \alpha_U)$*, then* $\text{T-val}(a) = \mathrm{val}(a - t)$.
*(c) If* $a \in \mathbb{Q}_p$ *lies in the (sub)interval* $\mathrm{I}(t, \alpha_L, \alpha_U)$ *then* $\alpha_L < \text{T-val}(a) \le \alpha_U$.
*(d) For any* $a \in \mathbb{Q}_p$ *lying in the (sub)interval* $\mathrm{I}(t, \alpha_L, \alpha_U)$ *and* $t' \in T$*:*
- *If* $\mathrm{val}(t - t') \ge \alpha_U$*, then* $\mathrm{val}(a - t') = \text{T-val}(a)$.
- *If* $\mathrm{val}(t - t') \le \alpha_L$*, then* $\mathrm{val}(a - t') = \mathrm{val}(t - t')\,(\le \alpha_L < \text{T-val}(a))$.

*Proof.* (a)-(c) are clear. For (d) fix $t' \in T$ and suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\mathrm{I}(t, \alpha'_L, \alpha'_U)$. This subinterval lies inside of a unique interval $\mathrm{I}(t, \alpha_L, \alpha_U)$ for some choice of $\alpha_L, \alpha_U$ and by the definition of intervals (or more specifically $\mathscr{B}$):

$$\mathrm{val}(t - t') \ge \alpha_U \iff \mathrm{val}(t - t') \ge \alpha'_U,$$

$$\mathrm{val}(t - t') \ge \alpha_L \iff \mathrm{val}(t - t') \ge \alpha'_L.$$

Therefore without loss of generality we may assume that $a \in \mathbb{Q}_p$ lies in an interval $\mathrm{I}(t, \alpha_L, \alpha_U)$. By (c) and the definition of intervals one of the three following cases has to hold.

Case 1: $\mathrm{val}(t - t') \ge \alpha_U$ and $\text{T-val}(a) < \alpha_U$. Then

$$\mathrm{val}(t - t') \ge \alpha_U > \text{T-val}(a) = \mathrm{val}(a - t),$$

thus $\mathrm{val}(a - t') = \mathrm{val}(a - t) = \text{T-val}(a)$ as needed.

Case 2: $\mathrm{val}(t - t') \ge \alpha_U$ and $\text{T-val}(a) = \alpha_U$. Then

$$\text{T-val}(a) = \mathrm{val}(a - t) = \mathrm{val}(t - t') \ge \alpha_U,$$

thus $\mathrm{val}(a-t') \geq \alpha_U$. The interval $\mathrm{I}(t, \alpha_L, \alpha_U)$ is disjoint from the ball $B(t', \alpha_U)$, so $a \notin B(t', \alpha_U)$, that is, $\mathrm{val}(a-t') \leq \alpha_U$. Combining this with the previous inequality we get that $\mathrm{val}(a-t') = \alpha_U = \text{T-val}(a)$ as needed.

Case 3: $\mathrm{val}(t-t') \leq \alpha_L$. Then

$$\mathrm{val}(t-t') \leq \alpha_L < \text{T-val}(a) = \mathrm{val}(a-t),$$

thus $\mathrm{val}(a-t') = \mathrm{val}(t-t')$ as needed. □

**Definition 3.7.** Suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$. We say that $a$ is <u>far from the boundary</u> (tacitly: of $\mathrm{I}(t, \alpha_L, \alpha_U)$) if

$$\alpha_L + n \leq \text{T-val}(a) \leq \alpha_U - n.$$

Here $n$ is as in Lemma 3.1. Otherwise we say that it is <u>close to the boundary</u> (of $\mathrm{I}(t, \alpha_L, \alpha_U)$).

**Definition 3.8.** Suppose $a_1, a_2 \in \mathbb{Q}_p$ lie in the same subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$. We say $a_1, a_2$ have the same <u>subinterval type</u> if one of the following holds:

- Both $a_1, a_2$ are far from the boundary and $a_1 - t, a_2 - t$ are in the same $Q_{m,n}$-coset. (Here $Q_{m,n}$ is an in Lemma 3.1.)
- Both $a_1, a_2$ are close to the boundary and

$$\text{T-val}(a_1) = \text{T-val}(a_2) \leq \mathrm{val}(a_1 - a_2) - n.$$

**Definition 3.9.** For $c \in \mathbb{Q}_p$ and $\alpha, \beta \in \mathbb{Z}, \alpha < \beta$ define $c \restriction [\alpha, \beta)$ to be the record of the coefficients of $c$ for the valuations between $[\alpha, \beta)$. More precisely write $c$ in its power series form

$$c = \sum_{\gamma \in \mathbb{Z}} c_\gamma p^\gamma \text{ with } c_\gamma \in \{0, 1, \ldots, p-1\}.$$

Then $c \restriction [\alpha, \beta)$ is just $(c_\alpha, c_{\alpha+1}, \ldots c_{\beta-1}) \in \{0, 1, \ldots, p-1\}^{\beta-\alpha}$.

The following lemma is an adaptation of Lemma 7.4 in [1].

**Lemma 3.10.** *Fix $m, n \in \mathbb{N}$. For any $x, y, c \in \mathbb{Q}_p$, if*

$$\operatorname{val}(x-c) = \operatorname{val}(y-c) \leq \operatorname{val}(x-y) - n,$$

*then $x-c, y-c$ are in the same coset of $Q_{m,n}$.*

*Proof.* Call $a, b \in \mathbb{Q}_p$ similar if $\operatorname{val} a = \operatorname{val} b$ and

$$a \restriction [\operatorname{val} a, \operatorname{val} a + n) = b \restriction [\operatorname{val} b, \operatorname{val} b + n).$$

If $a, b$ are similar then

$$a \in Q_{m,n} \iff b \in Q_{m,n}.$$

Moreover for any $\lambda \in \mathbb{Q}_p^\times$, if $a, b$ are similar then so are $\lambda a, \lambda b$. Thus if $a, b$ are similar, then they belong to the same coset of $Q_{m,n}$. The hypothesis of the lemma force $x-c, y-c$ to be similar, thus belonging to the same coset. $\square$

**Lemma 3.11.** *For each subinterval there are at most $K = K(Q_{m,n})$ many subinterval types (with $K$ not depending on $B$ or on the subinterval).*

*Proof.* Let $a, a' \in \mathbb{Q}_p$ lie in the same subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$.

Suppose $a, a'$ are far from the boundary. Then they have the same subinterval type if $a-t, a'-t$ are in the same $Q_{m,n}$-coset. So the number of such subinterval types is bounded by the number of $Q_{m,n}$-cosets.

Suppose $a, a'$ are close to the boundary and

$$\begin{gathered}\text{T-val}(a) - \alpha_L = \text{T-val}(a') - \alpha_L < n \text{ and} \\ a \restriction [\text{T-val}(a), \text{T-val}(a) + n) = a' \restriction [\text{T-val}(a'), \text{T-val}(a') + n).\end{gathered}$$

Then $a, a'$ have the same subinterval type. Such a subinterval type is thus determined by $\text{T-val}(a) - \alpha_L$ and the tuple $a \restriction [\text{T-val}(a), \text{T-val}(a) + n)$, therefore there are at most $np^n$ many such types.

A similar argument works for $a$ with $\alpha_U - \text{T-val}(a) \leq n$.

Adding all this up we get that there are at most

$$K = (\text{number of } Q_{m,n} \text{ cosets}) + 2np^n$$

many subinterval types. $\square$

The following critical lemma relates tree notions to $\Phi$-types.

**Lemma 3.12.** *Suppose $d, d' \in \mathbb{Q}_p^{|x|}$ satisfy the follwing three conditions:*

- *For all $i \in I$ $p_i(d)$ and $p_i(d')$ are in the same subinterval.*
- *For all $i \in I$ $p_i(d)$ and $p_i(d')$ have the same subinterval type.*
- *For all $i, j \in I$, $\text{T-val}(p_i(d)) > \text{T-val}(p_j(d))$ iff $\text{T-val}(p_i(d')) > \text{T-val}(p_j(d'))$.*

*Then $d, d'$ have the same $\Phi$-type over $B$.*

*Proof.* There are two kinds of formulas in $\Phi$ (see Lemma 3.1). First we show that $d, d'$ agree on formulas of the form $p_i(x) - c_i(y) \in \lambda_k Q_{m,n}$. It is enough to show that for every $i \in I, b \in B$, $p_i(d) - c_i(b), p_i(d') - c_i(b)$ are in the same $Q_{m,n}$-coset. Fix such $i, b$. For brevity let $a = p_i(d), a' = p_i(d')$ and $Q = Q_{m,n}$. We want to show that $a - c_i(b), a' - c_i(b)$ are in the same $Q$-coset.

Suppose $a, a'$ are close to the boundary. Then $\text{T-val}(a) = \text{T-val}(a') \leq \text{val}(a - a') - n$. Using Lemma 3.6d, we have

$$\text{val}(a - c_i(b)) = \text{val}(a' - c_i(b)) \leq \text{T-val}(a) \leq \text{val}(a - a') - n.$$

Lemma 3.10 shows that $a - c_i(b), a' - c_i(b)$ are in the same $Q$-coset.

Now, suppose both $a, a'$ are far from the boundary. Let $\mathrm{I}(t, \alpha_L, \alpha_U)$ be the interval containing $a, a'$. Then we have

$$\alpha_L + n \leq \text{val}(a - t) \leq \alpha_U - n,$$

$$\alpha_L + n \leq \text{val}(a' - t) \leq \alpha_U - n$$

(as being far from the subinterval's boundary also makes $a, a'$ far from interval's boundary). We have either $\mathrm{val}(t - c_i(b)) \geq \alpha_U$ or $\mathrm{val}(t - c_i(b)) \leq \alpha_L$ (as otherwise it would contradict the definition of intervals, or more specifically $\mathscr{B}$).

Suppose it is the first case $\mathrm{val}(t - c_i(b)) \geq \alpha_U$. Then using Lemma 3.6d

$$\mathrm{val}(a - c_i(b)) = \mathrm{val}(a - t) \leq \alpha_U - n \leq \mathrm{val}(t - c_i(b)) - n.$$

So by Lemma 3.10 $a - c_i(b), a - t$ are in the same $Q$-coset. By an analogous argument, $a' - c_i(b), a' - t$ are in the same $Q$-coset. As $a, a'$ have the same subinterval type, $a - t, a' - t$ are in the same $Q$-coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same $Q$-coset.

For the second case, suppose $\mathrm{val}(t - c_i(b)) \leq \alpha_L$. Then using Lemma 3.6d

$$\mathrm{val}(a - c_i(b)) = \mathrm{val}(t - c_i(b)) \leq \alpha_L \leq \mathrm{val}(a - t) - n,$$

so by Lemma 3.10, $a - c_i(b), t - c_i(b)$ are in the same $Q$-coset. Similarly $a' - c_i(b), t - c_i(b)$ are in the same $Q$-coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same $Q$-coset.

Next, we need to show that $d, d'$ agree on formulas of the form $\mathrm{val}(p_i(x) - c_i(y)) < \mathrm{val}(p_j(x) - c_j(y))$ (again, referring to the presentation in Lemma 3.1). Fix $i, j \in I, b \in B$. We would like to show the following equivalence:

$$\begin{aligned} \text{(3.1)} \qquad & \mathrm{val}(p_i(d) - c_i(b)) < \mathrm{val}(p_j(d) - c_j(b)) \iff \\ & \iff \mathrm{val}(p_i(d') - c_i(b)) < \mathrm{val}(p_j(d') - c_j(b)) \end{aligned}$$

Suppose $p_i(d), p_i(d')$ are in the subinterval $\mathrm{I}(t_i, \alpha_i, \beta_i)$ and $p_j(d), p_j(d')$ are in the subinterval $\mathrm{I}(t_j, \alpha_j, \beta_j)$. Lemma 3.6d yields the following four cases.

Case 1:

$$\mathrm{val}(p_i(d) - c_i(b)) = \mathrm{val}(p_i(d') - c_i(b)) = \mathrm{val}(t_i - c_i(b))$$

$$\mathrm{val}(p_j(d) - c_j(b)) = \mathrm{val}(p_j(d') - c_j(b)) = \mathrm{val}(t_j - c_j(b))$$

Then it is clear that the equivalence (3.1) holds.

Case 2:

$$\text{val}(p_i(d) - c_i(b)) = \text{T-val}(p_i(d)) \text{ and } \text{val}(p_i(d') - c_i(b)) = \text{T-val}(p_i(d'))$$

$$\text{val}(p_j(d) - c_j(b)) = \text{T-val}(p_j(d)) \text{ and } \text{val}(p_j(d') - c_j(b)) = \text{T-val}(p_j(d'))$$

Then the equivalence (3.1) holds by the third hypothesis of the lemma (that order of T-valuations is preserved).

Case 3:

$$\text{val}(p_i(d) - c_i(b)) = \text{val}(p_i(d') - c_i(b)) = \text{val}(t_i - c_i(b))$$

$$\text{val}(p_j(d) - c_j(b)) = \text{T-val}(p_j(d)) \text{ and } \text{val}(p_j(d') - c_j(b)) = \text{T-val}(p_j(d'))$$

If $p_j(d), p_j(d')$ are close to the boundary, then $\text{T-val}(p_j(d)) = \text{T-val}(p_j(d'))$ and the equivalence (3.1) clearly holds. Suppose then that $p_j(d), p_j(d')$ are far from the boundary.

$$\alpha_j + n \leq \text{T-val}(p_j(d)), \text{T-val}(p_j(d')) \leq \beta_j - n$$

$$\alpha_j < \text{T-val}(p_j(d)), \text{T-val}(p_j(d')) < \beta_j$$

and $\text{val}(t_i - c_i(b))$ lies outside of the $(\alpha_j, \beta_j)$ by the definition of subinterval (more specifically definition of $\mathscr{B}'$). Therefore (3.1) has to hold. (Note that we always have $\text{T-val}(p_j(d)), \text{T-val}(p_j(d')) \in (\alpha_j, \beta_j]$ by Lemma 3.6c, so we only need the condition on being far from the boundary to avoid the edge case of equality to $\beta_j$.)

Case 4:

$$\text{val}(p_i(d) - c_i(b)) = \text{T-val}(p_i(d)) \text{ and } \text{val}(p_i(d') - c_i(b)) = \text{T-val}(p_i(d'))$$

$$\text{val}(p_j(d) - c_j(b)) = \text{val}(p_j(d') - c_j(b)) = \text{val}(t_j - c_j(b)).$$

Similar to case 3 (switching $i, j$). $\square$

The previous lemma gives us an upper bound on the number of types - there are at most $|2I|!$ many choices for the order of T-val, $O(N)$ many choices for the subinterval for each $p_i$, and $K$ many choices for the subinterval type for each $p_i$ (where $K$ is as in Lemma 3.11), giving a total of $O(N^{|I|}) \cdot K^{|I|} \cdot |I|! = O(N^{|I|})$ many types. This implies $\mathrm{vc}^*(\Phi) \leq |I|$. The biggest contribution to this bound are the choices among the $O(N)$ many subintervals for each $p_i$ with $i \in I$. Are all of those choices realized? Intuitively there are $|x|$ many variables and $|I|$ many equations, so once we choose a subinterval for $|x|$ many $p_i$'s, the subintervals for the rest should be determined. This would give the required bound $\mathrm{vc}^*(\Phi) \leq |x|$. The next section outlines this idea formally.

## 4. Main Proof

An alternative way to write $p_i(c)$ is as a scalar product $\vec{p_i} \cdot \vec{c}$, where $\vec{p_i}$ and $\vec{c}$ are vectors in $\mathbb{Q}_p^{|x|}$ (as $p_i(x)$ is homogeneous linear).

**Lemma 4.1.** *Suppose we have a finite collection of vectors $\{\vec{p_j}\}_{j \in J}$ with each $\vec{p_j} \in \mathbb{Q}_p^{|x|}$. Suppose $\vec{p} \in \mathbb{Q}_p^{|x|}$ satisfies $\vec{p} \in \operatorname{span}\{\vec{p_j}\}_{j \in J}$, and we have $\vec{c} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$ with $\mathrm{val}(\vec{p_j} \cdot \vec{c}) > \alpha$ for all $j \in J$. Then $\mathrm{val}(\vec{p} \cdot \vec{c}) > \alpha - \gamma$ for some $\gamma \in \mathbb{N}$. Moreover $\gamma$ can be chosen independently from $\vec{c}, \alpha$ depending only on $\{\vec{p_j}\}_{j \in J}$.*

*Proof.* For some $c_j \in \mathbb{Q}_p$ for $j \in J$ we have $\vec{p} = \sum_{j \in J} c_j \vec{p_j}$, hence $\vec{p} \cdot \vec{c} = \sum_{j \in J} c_j \vec{p_j} \cdot \vec{c}$. Thus

$$\mathrm{val}\left(c_j \vec{p_j} \cdot \vec{c}\right) = \mathrm{val}\left(c_j\right) + \mathrm{val}\left(\vec{p_j} \cdot \vec{c}\right) > \mathrm{val}\left(c_j\right) + \alpha.$$

Let $\gamma = \max(0, -\max_{j \in J} \mathrm{val}\left(c_j\right))$. Then we have

$$\mathrm{val}(\vec{p} \cdot \vec{c}) = \mathrm{val}\left(\sum_{j \in J} c_j \vec{p_j} \cdot \vec{c}\right) \geq$$

$$\geq \min_{j \in J} \mathrm{val}\left(\sum_{j \in J} c_j \vec{p_j} \cdot \vec{c}\right) > \min_{j \in J} \mathrm{val}(c_j) + \alpha \geq \alpha - \gamma$$

as required. □

**Corollary 4.2.** *Suppose we have a finite collection of vectors* $\{\vec{p}_i\}_{i\in I}$ *with each* $\vec{p}_i \in \mathbb{Q}_p^{|x|}$. *Suppose* $J \subseteq I$ *and* $i \in I$ *satisfy* $\vec{p}_i \in \operatorname{span}\{\vec{p}_j\}_{j\in J}$, *and we have* $\vec{c} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$ *with* $\operatorname{val}(\vec{p}_j \cdot \vec{c}) > \alpha$ *for all* $j \in J$. *Then* $\operatorname{val}(\vec{p}_i \cdot \vec{c}) > \alpha - \gamma$ *for some* $\gamma \in \mathbb{N}$. *Moreover* $\gamma$ *can be chosen independently from* $J, j, \vec{c}, \alpha$ *depending only on* $\{\vec{p}_i\}_{i\in I}$.

*Proof.* The previous lemma shows that we can pick such $\gamma$ for a given choice of $i, J$, but independent from $\alpha, \vec{c}$. To get a choice independent from $i, J$, go over all such eligible choices ($i$ ranges over $I$ and $J$ ranges over subsets of $I$), pick $\gamma$ for each, and then take the maximum of those values. □

Fix $\gamma$ according to Corollary 4.2 corresponding to $\{\vec{p}_i\}_{i\in I}$ given by our collection of formulas $\Phi$. (The lemma above is a general result, but we only use it applied to the vectors given by $\Phi$.)

**Definition 4.3.** Suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$. Define the $\underline{T\text{-floor}}$ of $a$ to be $\text{T-fl}(a) = \alpha_L$.

**Definition 4.4.** Let $f : \mathbb{Q}_p^{|x|} \longrightarrow \mathbb{Q}_p^I$ with $f(c) = (p_i(c))_{i\in I}$. Define the segment space Sg to be the image of $f$. Equivalently:

$$\mathrm{Sg} = \left\{ (p_i(c))_{i\in I} \mid c \in \mathbb{Q}_p^{|x|} \right\} \subseteq \mathbb{Q}_p^I.$$

Without loss of generality, we may assume that $I = \{1, 2, \ldots, k\}$ (that is the formulas are labeled by consecutive natural numbers). Given a tuple $(a_i)_{i\in I}$ in the segment space, look at the corresponding $T$-floors $\{\text{T-fl}(a_i)\}_{i\in I}$ and T-valuations $\{\text{T-val}(a_i)\}_{i\in I}$. Partition the segment space by the order types of $\{\text{T-fl}(a_i)\}_{i\in I}$ and $\{\text{T-val}(a_i)\}_{i\in I}$ (as subsets of $\mathbb{Z}$).

Work in a fixed set $\mathrm{Sg}'$ of the partition. After relabeling the $p_i$ we may assume that

$$\text{T-fl}(a_1) \geq \text{T-fl}(a_2) \geq \ldots \text{ for all } a_i \in \mathrm{Sg}'.$$

Consider the (relabeled) sequence of vectors $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_I$. There is a unique subset $J \subseteq I$ such that the set of all vectors with indices in $J$ is linearly independent, and all vectors with indices outside of $J$ are a linear combination of preceding vectors. (We can pick those using a greedy algorithm for finding a linearly independent subset of vectors.) We call indices in $I$ independent and we call the indices in $I \setminus J$ dependent.

**Definition 4.5.**

- Denote $\{0, 1, \ldots, p-1\}$ as Ct.
- Let Tp be the space of all subinterval types. By Lemma 3.11 we have $|\operatorname{Tp}| \le K$.
- Let Sub be the space of all subintervals. By Lemma 3.4 we have $|\operatorname{Sub}| \le 3|I|^2 \cdot N = O(N)$.

**Definition 4.6.** Now, we define a function

$$g_{\mathrm{Sg}'} : \mathrm{Sg}' \longrightarrow \mathrm{Tp}^I \times \mathrm{Sub}^J \times \mathrm{Ct}^{I \setminus J}$$

as follows:

Let $a = (a_i)_{i \in I} \in \mathrm{Sg}'$. To define $g_{\mathrm{Sg}'}(a)$ we need to specify where it maps $a$ in each individual component of the product.

For each $a_i$ record its subinterval type, giving the first component in $\mathrm{Tp}^I$.

For $a_j$ with $j \in J$, record the subinterval of $a_j$, giving the second component in $\mathrm{Sub}^J$.

For the third component (an element of $\mathrm{Ct}^{I \setminus J}$) do the following computation. Pick $a_i$ with $i$ dependent. Let $j$ be the largest independent index with $j < i$. Record $a_i \restriction [\text{T-fl}(a_j) - \gamma, \text{T-fl}(a_j))$.

Combine $g_{\mathrm{Sg}'}$ for all sets $\mathrm{Sg}'$ in our partition of Sg to get a function

$$g : \mathrm{Sg} \longrightarrow \mathrm{Tp}^I \times \mathrm{Sub}^J \times \mathrm{Ct}^{I \setminus J}.$$

**Lemma 4.7.** *Suppose we have $c, c' \in \mathbb{Q}_p^{|x|}$ such that $f(c), f(c')$ are in the same set $\mathrm{Sg}'$ of the partition of $\mathrm{Sg}$ and $g(f(c)) = g(f(c'))$. Then $c, c'$ have the same $\Phi$-type over $B$.*

*Proof.* Let $a_i = \vec{p}_i \cdot \vec{c}$ and $a'_i = \vec{p}_i \cdot \vec{c}'$ so that

$$f(c) = (p_i(c))_{i\in I} = (\vec{p}_i \cdot \vec{c})_{i\in I} = (a_i)_{i\in I}$$

$$f(c') = (p_i(c'))_{i\in I} = (\vec{p}_i \cdot \vec{c}')_{i\in I} = (a'_i)_{i\in I}$$

For each $i$ we show that $a_i, a'_i$ are in the same subinterval and have the same subinterval type, so the conclusion follows by Lemma 3.12 (the tuples $f(c), f(c')$ are in the same partition ensuring the proper order of T-valuations for the 3rd condition of the lemma). Tp records the subinterval type of each element, so if $g(\bar{a}) = g(\bar{a}')$ then $a_i, a'_i$ have the same subinterval type for all $i \in I$. Thus it remains to show that $a_i, a'_i$ lie in the same subinterval for all $i \in I$. Suppose $i$ is an independent index. Then by construction, Sub records the subinterval for $a_i, a'_i$, so those have to belong to the same subinterval. Now suppose $i$ is dependent. Pick the largest $j < i$ such that $j$ is independent. We have $\text{T-fl}(a_i) \leq \text{T-fl}(a_j)$ and $\text{T-fl}(a'_i) \leq \text{T-fl}(a'_j)$. Moreover $\text{T-fl}(a_j) = \text{T-fl}(a'_j)$ as $a_j, a'_j$ lie in the same subinterval (using the earlier part of the argument as $j$ is independent).

**Claim 4.8.** $\mathrm{val}(a_i - a'_i) > \text{T-fl}(a_j) - \gamma$

*Proof.* Let $K$ be the set of the independent indices less than $i$. Note that by the definition for dependent indices we have $\vec{p}_i \in \mathrm{span}\,\{\vec{p}_k\}_{k\in K}$. We also have

$$\mathrm{val}(a_k - a'_k) > \text{T-fl}(a_k) \text{ for all } k \in K$$

as $a_k, a'_k$ lie in the same subinterval (using the earlier part of the argument as $k$ is independent). Now $\mathrm{val}(a_k - a'_k) > \text{T-fl}(a_j)$ for all $k \in K$ by monotonicity of $\text{T-fl}(a_k)$. Moreover $a_k - a'_k = \vec{p}_k \cdot \vec{c} - \vec{p}_k \cdot \vec{c}' = \vec{p}_k \cdot (\vec{c} - \vec{c}')$. Combining the two, we get that $\mathrm{val}(\vec{p}_k \cdot (\vec{c} - \vec{c}')) > \text{T-fl}(a_j)$ for all $k \in K$. Now observe that $K \subseteq I, i \in I, \vec{c} - \vec{c}' \in \mathbb{Q}_p^{|x|}, \text{T-fl}(a_j) \in \mathbb{Z}$ satisfy the requirements of Lemma 4.2, so we apply it to obtain $\mathrm{val}(\vec{p}_i \cdot (\vec{c} - \vec{c}\,')) > \text{T-fl}(a_j) - \gamma$. Similarly to before, we have $\vec{p}_i \cdot (\vec{c} - \vec{c}\,') = \vec{p}_i \cdot \vec{c} - \vec{p}_i \cdot \vec{c}\,' = a_i - a_i'$. Therefore we can conclude that $\mathrm{val}(a_i - a_i') > \text{T-fl}(a_j) - \gamma$ as needed, finishing the proof of the claim. $\square$

Additionally $a_i, a_i'$ have the same image in the Ct component, so we have $\mathrm{val}(a_i - a_i') > \text{T-fl}(a_j)$. We now would like to show that $a_i, a_i'$ lie in the same subinterval. As $\text{T-fl}(a_i) \leq \text{T-fl}(a_j)$, $\text{T-fl}(a_i') \leq \text{T-fl}(a_j')$ and $\text{T-fl}(a_j) = \text{T-fl}(a_j')$ we have that $\mathrm{val}(a_i - a_i') > \text{T-fl}(a_i)$ and $\mathrm{val}(a_i - a_i') > \text{T-fl}(a_i')$. Suppose that $a_i$ lies in the subinterval $\mathrm{I}(t, \text{T-fl}(a_i), \alpha_U)$ and that $a_i'$ lies in the subinterval $\mathrm{I}(t', \text{T-fl}(a_i'), \alpha_U')$. Without loss of generality assume that $\text{T-fl}(a_i) \leq \text{T-fl}(a_i')$. As $\mathrm{val}(a_i - a_i') > \text{T-fl}(a_i')$, this implies that $a_i \in B(a_i', \text{T-fl}(a_i'))$. Then $a_i \in B(t', \text{T-fl}(a_i'))$ as $\mathrm{val}(a_i - t') > \text{T-fl}(a_i')$. This implies that $B(t, \text{T-fl}(a_i)) \cap B(t', \text{T-fl}(a_i')) \neq \emptyset$ as they both contain $a_i$. As balls are directed, the non-zero intersection means that one ball has to be contained in another. Given our assumption that $\text{T-fl}(a_i) \leq \text{T-fl}(a_i')$, we have $B(t, \text{T-fl}(a_i)) \subseteq B(t', \text{T-fl}(a_i'))$. For the subintervals to be disjoint we need $\mathrm{I}(t, \text{T-fl}(a_i), \alpha_U) \cap B(t', \text{T-fl}(a_i')) = \emptyset$. But $\mathrm{val}(t' - a_i) > \text{T-fl}(a_i')$ implying that $a_i \in \mathrm{I}(t, \text{T-fl}(a_i), \alpha_U) \cap B(t', \text{T-fl}(a_i'))$ giving a contradiction. Therefore the subintervals coincide finishing the proof. $\square$

**Corollary 4.9.** *The dual* vc*-density of* $\Phi(x, y)$ *is* $\leq |x|$.

*Proof.* Suppose we have $c, c' \in \mathbb{Q}_p^{|x|}$ such that $f(c), f(c')$ are in the same partition and $g(f(c)) = g(f(c'))$. Then by the previous lemma $c, c'$ have the same $\Phi$-type. Thus the number of possible $\Phi$-types is bounded by the size of the range of $g$ times the number of possible partitions

$$(\text{number of partitions}) \cdot |\,\mathrm{Tp}\,|^{|I|} \cdot |\,\mathrm{Sub}\,|^{|J|} \cdot |\,\mathrm{Ct}\,|^{|I-J|}.$$

There are at most $(|2I|!)^2$ many partitions of Sg, so in the product above, the only component dependent on $B$ is

$$|\operatorname{Sub}|^{|J|} \leq (N \cdot 3|I|^2)^{|J|} = O(N^{|J|}).$$

Every $p_i$ is an element of a $|x|$-dimensional vector space, so there can be at most $|x|$ many independent vectors. Thus we have $|J| \leq |x|$ and the bound follows. $\square$

**Corollary 4.10** (Theorem 2.6)**.** *The $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ satisfies* $\mathrm{vc}(n) = n$.

*Proof.* The previous lemma implies that $\mathrm{vc}^*(\phi) \leq \mathrm{vc}^*(\Phi) \leq |x|$. As choice of $\phi$ was arbitrary, this implies that the vc-density of any formula is bounded by the arity of $x$. $\square$

This proof relies heavily on the linearity of the defining polynomials $a_1, a_2, c$ in the cell decomposition result (see Definition 2.4). Linearity is used to separate the $x$ and $y$ variables as well as for Corollary 4.2 to reduce the number of independent factors from $|I|$ to $|x|$. The paper [2] has cell decomposition results for more expressive reducts of $\mathbb{Q}_p$, including, for example, restricted multiplication. While our results don't apply to them directly, it is this author's hope that similar techniques can be used to also compute the vc-function for those structures.

Another interesting question is whether the reduct studied in this paper has the VC 1 property (see [1], 5.2 for the definition). If so, this would imply the linear vc-density bound directly. The techniques used in paper [1] make it seem likely that the reduct has VC 2 property. While there are techniques for showing that a structure has a given VC property, less is known about showing that a structure doesn't have a given VC property. Perhaps the simple structure of the $\mathcal{L}_{aff}$-reduct can help understand this property better.

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