A Tiny Example

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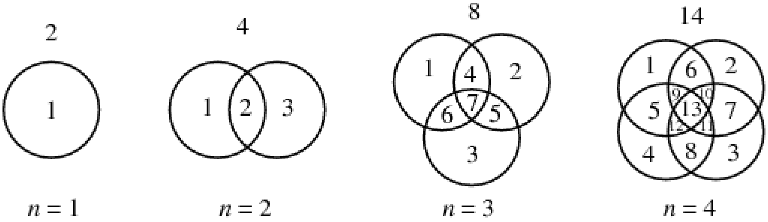
Suppose we have an (infinite) collection of sets $\mathcal{F}$.
We define a shatter function $\pi_{\mathcal{F}}(n)$

$$\pi_{\mathcal{F}}(n) = \max\{\#\text{ of atoms in boolean algebra generated by } S \mid S \subset \mathcal{F} \text{ with } |S| = n\}$$

Example: Let $\mathcal{F}$ consist of all discs on a plane.



$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 8 \quad \pi_{\mathcal{F}}(4) = 14$$

$$\pi_{\mathcal{F}}(n) = n^2 - n + 2$$

More examples:

1. Lines on a plane $\pi_{\mathcal{F}}(n) = n^2/2 + n/2 + 1$
2. Disks on a plane $\pi_{\mathcal{F}}(n) = n^2 - n + 2$
3. Balls in $\mathbb{R}^3$ $\pi_{\mathcal{F}}(n) = n^3/3 - n^2 + 8n/3$
4. Intervals on a line $\pi_{\mathcal{F}}(n) = 2n$
5. Half-planes on a plane $\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$
6. Finite subsets of $\mathbb{N}$ $\pi_{\mathcal{F}}(n) = 2^n$
7. Polygons in a plane $\pi_{\mathcal{F}}(n) = 2^n$

Theorem (Sauer-Shelah)

Shatter function is either 2^n or bounded by a polynomial.

Definition

Suppose growth of shatter function for $\mathcal{F}$ is polynomial. Let r be the smallest real such that

$$\pi_{\mathcal{F}}(n) = O(n^r)$$

We define such r to be the vc-density of $\mathcal{F}$. If shatter function grows exponentially, we let vc-density to be infinite.

Applications

- VapnikChervonenkis Theorem in probability
- Computability
- NIP theories