SUPERFLAT GRAPHS ARE DP-MINIMAL

ANTON BOBKOV

Abstract. We show that the theory of superflat graphs is dp-minimal.

1. Preliminaries

Superflat graphs in model theoretic context were first introduced in [1] as a natural class of stable graphs. In this paper we further explore model theoretic structure of such graphs by directly showing that they are dp-minimal.

First, we introduce some basic definition regarding connectivity in graphs.

Definition 1.1. Work in an infinite graph G. Let $A, V \subset V(G)$

(1) Then for $a, b \in V(G)$ define $d_A(a, b)$ to be the *distance* (length of the smallest path) in an induced subgraph of G after removing vertices A.
(2) We say that A *separates* V if for all $a, b \in V$, $d_A(a, b) = \infty$.
(3) We say that V has *connectivity* n if there are no sets of size $n-1$ in $V(G)$ that separates V.
(4) Suppose V has finite connectivity n. *Connectivity hull* of V is defined to be the union of all sets separating V of size $n-1$.

Connectivity of graphs is well described by Megner's Theorem. Here is a simple modification of the result in [2] concerning generalization of Megner's Theorem to infinite graphs.

Theorem 1.2. *Let V be a subset of a graph G with connectivity n. Then there exists a set of n disjoint paths from V into itself.*

Corollary 1.3. *With assumptions as above, connectivity hull of V is finite.*

Proof. All the separating sets have to have exactly one vertex in each of those paths. □

2. Indiscernible sequences

In this section we work in a superflat graph. It is stable so all the indiscernible sequences are totally indiscernible. Also note that by indiscernibility all pairwise distances between points are the same.

Denote by K_n^m an m-subdivision of the complete graph on n vertices. Graph is called superflat if for every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that the graph avoids K_n^m as a subgraph. We have the following useful equivalent characterization as given in [1], Theorem 2

Theorem 2.1. *The following are equivalent*

(1) *G is superflat*
(2) *For every $n \in \mathbb{N}$ and an infinite set $A \subset V(G)$, there exists a finite $B \subset V(G)$ and infinite $A' \subseteq A$ such that for all $x, y \in A'$ we have $d_B(x, y) > n$.*

If the graph is superflat then, roughly, the intuition is that from every infinite set we can extract a sparse infinite subset (after throwing away finitely many nodes).

Let $V \subset V(G)$. Denote $P_n(V)$ a union of all paths of length $\leq n$ between points of V. It is a subgraph of G.

Lemma 2.2. *Let $(a_i)_{i\in I}$ be a countable indiscernible sequence over A. Fix $n \in \mathbb{N}$. There exists a finite set B such that*

$$\forall i \neq j \ d_B(a_i, a_j) > n$$

Proof. By a flatness result we can find an infinite $J \subset I$ and a finite set B' such that each pair from $(a_j)_{j\in J}$ have distance $> n$ over B'. Using total indiscernibility we have an automorphism sending $(a_j)_{j\in J}$ to $(a_i)_{i\in I}$. Image of B' under this automorphism is the required set B. $\square$

In other words, B disconnects $P_n(\{a_i\})$. This shows that $\{a_i\}$ has finite connectivity in $P(\{a_i\})$. Applying lemma from last section we obtain that connectivity hull of $\{a_i\}$ in $P_n(\{a_i\})$ is finite.

Lemma 2.3. *Connectivity hull of $\{a_i\}$ in $P_n(\{a_i\})$ is $\{a_i\}$-definable as a subset of G.*

Definition 2.4. Given a graph G and $V \subset V(G)$ define $H(G, V)$ to be connectivity hull of V in G.

Note 2.5. Given a finite V we have $H(P_n(V), V)$ is V-definable.

Proof. Consider finite parts of the sequence $I_i = \{a_1, a_2, \ldots, a_i\}$. We study $H_i = H(P_n(I_i), I_i)$ as a function of i as approximations of the hull in question. We have the following properties

$$\forall i \ H(P_n(I_i), I_i) \subseteq H(P_n(I), I_i)$$
$$\forall i \leq j \ H(P_n(I), I_i) \supseteq H(P_n(I), I_j)$$

Eventually $H(P_n(I_i))$ contains n disjoint paths for the whole graph, thus stabilizes at $H(P_n(I), I)$. This shows that for large enough $i > N$ we have $H_i = H_{i+m}$. By symmetry of indiscernible sequence we have that any subset of size N defines the connectivity hull. $\square$

Lemma 2.6. *I is indiscernible over the $A \cup H(P_n(I), I)$.*

Proof. Denote the hull by H. Fix an A-formula $\phi(x, y)$. Consider a collection of traces $\phi(\vec{a}, H^{\{}|y|\})$ for $\vec{a} \in I^{|x|}$. As H is I definable those are either all distinct or all the same. Finiteness of H forces latter. This shows indiscernability. $\square$

Corollary 2.7. *Let $(a_i)_{i\in I}$ be a countable indiscernible sequence over A. Then there is a countable B such that (a_i) is indiscernible over $A \cup B$ and*

$$\forall i \neq j \ d_B(a_i, a_j) = \infty$$

Proof. Let $B_i = H(P_i(I), I)$. Successive applications of previous lemma yield the appropriate set $B = \bigcup B_i$. $\square$

That is every indiscernible sequence can be upgraded to have infinite distance over its parameter set.

3. Superflat graphs are dp-minimal

Lemma 3.1. *Suppose $x \equiv_A y$ and $d_A(x,c) = d_A(y,c) = \infty$. Then $x \equiv_{Ac} y$*

Proof. Define an equivalence relation $G - A$. Two points p, q are equivalent if $d_A(p,q)$ is finite. There is an automorphism f of G fixing A sending x to y. Denote by X and Y equivalence classes of x and y respectively. It's easy to see that $f(X) = Y$. Define the following function

$$g = f \text{ on } X$$
$$g = f^{-1} \text{ on } Y$$
$$\text{identity otherwise}$$

It is easy to see that g is an automorphism fixing Ac that sends x to y. □

Theorem 3.2. *Let G be a flat graph with $(a_i)_{i\in\mathbb{Q}}$ indiscernible over A and $b \in G$. There exists $c \in \mathbb{Q}$ such that all $(a_i)_{i\in\{\mathbb{Q}-c\}}$ have the same type over Ab.*

Proof. Find $B \supseteq A$ such that (a_i) is indiscernible over B and has infinite distance over B. All the elements of the indiscernible sequence fall into distinct equivalence classes. b can be in at most one of them. Exclude that element from the sequence. Remaining sequence elements are all infinitely far away from b. By previous lemma we have that elements of indiscernible sequence all have the same type over Bb. □

But this is exactly what it means to be dp-minimal, as given, say, in [3] Lemma 1.4.4

Corollary 3.3. *Flat graphs are dp-minimal.*

References

[1] Klaus-Peter Podewski and Martin Ziegler. Stable graphs. *Fund. Math.*, 100:101-107, 1978.
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[3] P. Simon, *On dp-minimal ordered structures*, J. Symbolic Logic 76 (2011), no. 2, 448460.
[4] Reinhard Diestel. *Graph Theory*, volume 173 of *Grad. Texts in Math.* Springer, 2005.

E-mail address: bobkov@math.ucla.edu