A NOTE ON QUANTIFIER ELIMINATION IN SHELAH-SPENCER GRAPHS

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Abstract. We simplify [1]'s proof of quantifier elimination in Shelah-Spencer graphs.

1. Introduction

Laskowski's paper [1] provides a combinatorial proof of quantifier elimination in Shelah-Spencer graphs. Here we provide a simplification of the proof using only maximal chains and avoiding the use of proposition 3.1 and technical lemmas of section 4.

We will use notation of [1], in particular things like K_{α} , $\delta(\mathcal{A}/\mathcal{B})$, $X_m(\mathcal{A})$, S_{α} , $\mathcal{B}^* \sqsubseteq \mathcal{B}'$,maximal embedding, $\Delta_{\mathcal{A}}(x)$, $\Psi_{\mathcal{A},\mathcal{B}}(x)$ etc. However we will give a different definition of $Y(\ldots)$. When we write formulas $\theta(x,y)$ we may have x,y to be tuples.

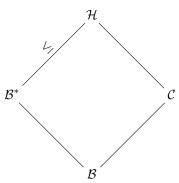
2. Omitting Lemma

Definition 2.1. Let $\mathcal{M} \models S_{\alpha}$, $\mathcal{B} \in \mathbf{K}_{\alpha}$, embedding $f : \mathcal{B} \to \mathcal{M}$, Φ finite subset of \mathbf{K}_{α}

- (1) Say that f omits Φ if there are no $\mathcal{C} \in \Phi$ and $g: \mathcal{C} \to \mathcal{M}$ extending f.
- (2) Say that f admits Φ if for every $\mathcal{C} \in \Phi$ there is $g: \mathcal{C} \to \mathcal{M}$ extending f.

Note 2.2. Take notation as above and a structure $C \in K_{\alpha}$ extending \mathcal{B} . Then f doesn't omit $\{C\}$ iff f admits $\{C\}$.

Definition 2.3. Fix $\mathcal{B}, \mathcal{C} \in K_{\alpha}$, and $m \in \omega$ such that $|C \setminus B| < m$. Define $Z(\mathcal{B}, \mathcal{C}, m)$ to be all $\mathcal{B}^* \in X_m(\mathcal{B})$ such that there are no \mathcal{H} with $|H \setminus B^*| < m$ satisfying



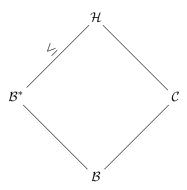
Lemma 2.4. Let $\mathcal{B}, \mathcal{C} \in K_{\alpha}$, and $m \in \omega$ such that $|C \setminus B| < m$. Also let $\mathcal{M} \models S_{\alpha}$ and $f : \mathcal{B} \to \mathcal{M}$ an embedding. The following are equivalent:

(1) f omits $\{C\}$.

(2) There exists $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into \mathcal{M} over f.

Proof. For the proof we identify \mathcal{B} with $f(\mathcal{B})$, i.e. for ease of notation assume that $\mathcal{B} \subset \mathcal{M}$.

 $(1) \Rightarrow (2)$ By remark 5.3 of [1] there is some $B^* \in X_m(\mathcal{B})$ maximally embeddable in \mathcal{M} over f. Such embedding is unique by Lemma 3.8 of [1]. Again, we identify B^* with its maximal embedding into \mathcal{M} . To show (2) we need to verify that $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$. Suppose not. Then there is \mathcal{H} with $|H \setminus B^*| < m$ satisfying



As $\mathcal{B}^* \leq \mathcal{H}$ and $\mathcal{B} \subset \mathcal{M}$ we can embed \mathcal{H} into \mathcal{M} (as $\mathcal{M} \models S_{\alpha}$). But this would witness \mathcal{C} extending \mathcal{B} in \mathcal{M} which is impossible as we assumed that f omits $\{\Phi\}$.

 $(2)\Rightarrow (1)$ Suppose f doesn't omit $\{C\}$. Then by the note 2.2 f admits $\{C\}$, i.e. there is an embedding of $\mathcal C$ into M over f. We identify $\mathcal C$ with the image of that embedding. Similarly we identify $\mathcal B^*$ with the image of its maximal embedding over f. That is we may assume $\mathcal C, \mathcal B^* \subset \mathcal M$. Let H be the substructure of M induced by vertices $C \cup B^*$. As $|C \setminus B| < m$ we have $|H \setminus B^*| < m$. $\mathcal B^*$ is m-strong by remark 5.3 of [1]. This forces $\mathcal B^* \leq H$. But this contradicts the fact that $\mathcal B^* \in Z(\mathcal B, \mathcal C, m)$. \square

Corollary 2.5. With the setup of the previous lemma, the following are equivalent:

- (1) f admits $\{C\}$.
- (2) There exists $\mathcal{B}^* \in X_m(\mathcal{B}) \setminus Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into \mathcal{M} over f.

For quantifier elimination we need to track multiple structures being admitted and omitted, hence the following definition.

Definition 2.6. Let $\mathcal{B} \in \mathbf{K}_{\alpha}$, Φ, Γ finite subsets of \mathbf{K}_{α} , and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|C \setminus B| < m$. Define

$$Y(\mathcal{B}, \Phi, \Gamma, m) = \{ B^* \in X_m(\mathcal{B}) \mid \forall \mathcal{C} \in \Phi \ B^* \in Z(\mathcal{B}, \mathcal{C}, m) \text{ and } \forall \mathcal{D} \in \Gamma \ B^* \notin Z(\mathcal{B}, \mathcal{D}, m) \}$$

Lemma 2.7. Let $\mathcal{B} \in K_{\alpha}$, Φ, Γ finite subsets of K_{α} , and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|\mathcal{C} \setminus \mathcal{B}| < m$. The following are equivalent:

- (1) f omits Φ and admits Γ .
- (2) There exists $\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$ maximally embeddable into \mathcal{M} over f.

Proof. Easy corollary of 2.4 and 2.5.

3. Quantifier Elimination

Following proof of 5.6 in [1], we have a formula $\theta(x,y)$, some $\mathcal{A} \subseteq \mathcal{B} \in \mathbf{K}_{\alpha}$ with $\theta(x,y) \vdash \Delta_A(x) \land \Delta_{\mathcal{B}}(x,y)$. We may also assume that $\theta(x,y)$ is a conjunction of formulas of the type $\Psi_{\mathcal{B},\mathcal{C}}(x,y)$ and their negations. More precisely

$$\theta(x,y) \Leftrightarrow \bigwedge_{\mathcal{C} \in \Phi} \Psi_{\mathcal{B},\mathcal{C}}(x,y) \wedge \\ \bigwedge_{\mathcal{D} \in \Gamma} \neg \Psi_{\mathcal{B},\mathcal{D}}(x,y)$$

for some finite subsets Φ, Γ of \mathbf{K}_{α} . Let $m = \max\{|C \setminus B| \mid C \in \Phi \text{ or } \Gamma\}$. We claim that in S_{α}

$$\exists y \theta(x,y) \Leftrightarrow \bigcup_{B^* \in Y(\mathcal{B},\Phi,\Gamma,m)} (\mathcal{B}^* \text{ maximally embeds over } \mathcal{B})$$

References

[1] Michael C. Laskowski, A simpler axiomatization of the Shelah-Spencer almost sure theories, Israel J. Math. **161** (2007), 157-186. MR MR2350161

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