1. Graph Combinatorics

Throughout this paper A, B, C, M will denote finite graphs, and $\mathbb{D}$ will be used to denote potentially infinite graphs. For a graph A the set of its vertices is denoted by $v(A)$, and the set of its edges by $e(A)$. Number of vertices of A will be denoted as $|A|$. Subgraph always means induced subgraph and $A \subseteq B$ means that A is a subgraph of B. For two subgraphs A, B of a larger graph, the union $A \cup B$ denotes the graph induced by $v(A) \cup v(B)$. Similarly, $A - B$ means a subgraph of A induced by the vertices of $v(A) - v(B)$. For $A \subseteq B \subseteq D$ and $A \subseteq C \subseteq D$, graphs B, C are said to be disjoint over A if $v(B) - v(A)$ is disjoint from $v(C) - v(A)$ and there are no edges from $v(B) - v(A)$ to $v(C) - v(A)$ in D.

For the remainder of the paper fix $\alpha \in (0, 1)$, irrational.

Definition 1.1.

- For a graph A let $\dim(A) = |A| - \alpha|e(A)|$.
- For A, B with $A \subseteq B$ define $\dim(B/A) = \dim(B) - \dim(A)$.
- We say that $A \leq B$ if $A \subseteq B$ and $\dim(A'/A) > 0$ for all $A \subsetneq A' \subseteq B$.
- Define A to be positive if for all $A' \subseteq A$ we have $\dim(A') \geq 0$.
- We work in theory S_α in the language of graphs axiomatized by:
  - Every finite substructure is positive.
  - Given a model $\mathbb{G}$ and graphs $A \leq B$, every embedding $f : A \longrightarrow \mathbb{G}$ extends to an embedding $g : B \longrightarrow \mathbb{G}$.

  (Here an embedding maps edges to edges and nonedges to nonedges.) This theory is complete and stable (see 5.7 and 7.1 in [2]). From now on fix an ambient model $\mathbb{G} \models S_\alpha$. This will be the only infinite graph we work with.
- For A, B positive, (A, B) is called a minimal pair if $A \subseteq B$, $\dim(B/A) < 0$ but $\dim(A'/A) \geq 0$ for all proper $A \subseteq A' \subsetneq B$. We call B a minimal extension of A. The dimension of a minimal pair is defined as $|\dim(B/A)|$.
- A sequence $\langle M_i \rangle_{0 \leq i \leq n}$ is called a minimal chain if (M_i, M_{i+1}) is a minimal pair for all $0 \leq i < n$.

- For a graph A with the tuple of vertices x let $\mathrm{diag}_A(x)$ be the atomic diagram of A, i.e. the first-order formula recording whether there is an edge between every pair of vertices.
- Given $A \subseteq B$ let

$$\phi_{A,B}(x) = \mathrm{diag}_A(x) \wedge \exists z \ \mathrm{diag}_B(x, z).$$

  Any graph isomorphic to B is called a witness of $\phi_{A,B}$.
- A formula $\phi_{A,B}$ is called a basic formula if there is a minimal chain $\langle M_i \rangle_{0 \leq i \leq n}$ such that $A = M_0$ and $B = M_n$.

Theorem 1.2 (Quantifier elimination, 5.6 in [2])**.** *In theory S_α every formula is equivalent to a boolean combination of basic formulas.*

Definition 1.3. A graph $S \subseteq \mathbb{D}$ is called N-strong if for any $S \subseteq T \subseteq D$ with $|T| - |S| \leq N$ we have $S \leq T$.

2. Basic Definitions and Lemmas

Definition 2.1. Suppose $\phi(x, y)$ is a basic formula. Define X to be the graph on vertices x with edges defined by ϕ. Similarly define Y. Note that X, Y are positive. Additionally, let Y' be a subgraph of Y induced by vertices of Y that are connected to $W - (X \cup Y)$, where W is a witness of ϕ.

We will require the following lemmas from [2]:

Lemma 2.2. *[See 2.3 in [2]] Let $A, B \subseteq \mathbb{D}$. Then*

$$\dim(A \cup B/A) \leq \dim(B/A \cap B).$$

Moreover,

$$\dim(A \cup B/A) = \dim(B/A \cap B) - \alpha E,$$

where E is the number of edges connecting the vertices of $A \cup B - A$ to the vertices of $A - A \cap B$.

Lemma 2.3. *[See 4.1 in [2]] Suppose A is a positive graph, with at least $1/\alpha + 2$ vertices. Then for any $\epsilon > 0$ there exists a graph B such that (A, B) is a minimal pair with dimension $\leq \epsilon$. Moreover, every vertex in A is connected to a vertex in $B - A$.*

Lemma 2.4. *[See 4.4 in [2]] Suppose A is a positive graph, and $\mathcal{G}$ a model of S_α. Then for any integer S there exists an embedding $f\colon A \longrightarrow \mathcal{G}$ such that $f(A)$ is S-strong in $\mathcal{G}$.*

We conclude this section by stating a couple of technical lemmas that will be useful in our proofs later.

Lemma 2.5. *Work in an ambient graph $\mathbb{D}$. Suppose we have a set B and a minimal pair (A, M) with $A \subseteq B$ and $\dim(M/A) = -\epsilon$. Then either $M \subseteq B$ or $\dim(M \cup B/B) < -\epsilon$.*

Proof. By Lemma 2.2

$$\dim(M \cup B/B) \leq \dim(M/M \cap B),$$

and as $A \subseteq M \cap B \subseteq M$

$$\dim(M/A) = \dim(M/M \cap B) + \dim(M \cap B/A).$$

In addition we are given $\dim(M/A) = -\epsilon$. If $M \not\subseteq B$ then $A \subseteq M \cap B \subsetneq M$ and by minimality $\dim(M \cap B/A) > 0$. Combining the inequalities above we obtain the desired result:

$$\dim(M \cup B/B) \leq \dim(M/M \cap B) = \dim(M/A) - \dim(M \cap B/A) < -\epsilon.$$

$\square$

Lemma 2.6. *Work in an ambient graph $\mathbb{D}$. Suppose we have a set B and a minimal chain $\langle M_i \rangle_{0 \leq i \leq n}$ with dimensions*

$$\dim(M_{i+1}/M_i) = -\epsilon_i.$$

Let $\epsilon = \min_{0 \leq i \leq n} \epsilon_i$. Then either $M_n \subseteq B$ or $\dim((M_n \cup B)/B) < -\epsilon$.

Proof. Let $\bar{M}_i = M_i \cup B$. Then:

$$\dim(\bar{M}_n/B) = \dim(\bar{M}_n/\bar{M}_{n-1}) + \ldots + \dim(\bar{M}_2/\bar{M}_1) + \dim(\bar{M}_1/B).$$

Either $M_n \subseteq B$ or at least one of the summands above is nonzero. Apply previous lemma. $\square$

Lemma 2.7. *Suppose we have a minimal pair (A, M) with dimension ϵ. Suppose we have some $B \subseteq M$. Then $\dim B/(A \cap B) \geq -\epsilon$. Moreover if $B \cup A \neq M$ then $\dim B/(A \cap B) \geq 0$.*

Proof. We have $\dim(B \cup A/A) \leq \dim B/(A \cap B)$ by Lemma 2.2. As $A \subseteq B \cup A \subseteq M$ we have $\dim(B \cup A/A) \geq -\epsilon$ by minimality. Moreover, minimality implies that it is positive if $B \cup A \neq M$. $\square$

Lemma 2.8. *Suppose we have a minimal chain $\langle M_i \rangle_{0 \leq i \leq n}$ with dimensions*

$$\dim(M_{i+1}/M_i) = -\epsilon_i.$$

Let ϵ be the sum of all ϵ_i. Suppose we have a graph B with $B \subseteq M_n$. Then $\dim B/(M_0 \cap B) \geq -\epsilon$.

Proof. Let $B_i = B \cap M_i$. We have $\dim B_{i+1}/B_i \geq \dim M_{i+1}/M_i$ by the previous lemma. Thus

$$\dim B/(M_0 \cap B) = \dim B_n/B_0 = \sum \dim B_{i+1}/B_i \geq -\epsilon.$$

$\square$

3. Lower bound

In this section we restrict our attention to the following family of basic formulas $\phi(x,y)$:

- All formulas have $Y' = Y$ (see Definition 2.1).
- All formulas define no edges between X and Y.
- Minimal chain of $\phi(x,y)$ consists of one step, that is we only have one minimal extension as opposed to a chain of minimal extensions.
- The dimension of that minimal extension is smaller than α.

We obtain a lower bound for the formulas that are boolean combinations of basic formulas written in the disjunctive-conjunctive form. First, define $\epsilon_L(\phi)$.

Definition 3.1. For a basic formula $\phi = \phi_{\langle M_i \rangle_{0 \le i \le n}}(x,y)$ let

- $\epsilon_i(\phi) = -\dim(M_i/M_{i-1})$.
- $\epsilon_L(\phi) = \sum_1^n \epsilon_i(\phi)$.

Definition 3.2 (Negation)**.** If ϕ is a basic formula, then define

$$\epsilon_L(\neg\phi) = \epsilon_L(\phi).$$

Definition 3.3 (Conjunction)**.** Take a collection of formulas $\phi_i(x,y)$ where each ϕ_i is a positive or a negative basic formula. If both positive and negative formulas are present then $\epsilon_L(\phi) = \infty$. We don't have a lower bound for that case. If different formulas define X or Y differently then $\epsilon_L(\phi) = \infty$. In the case of conflicting definitions the formula would have no realizations. Otherwise let

$$\epsilon_L\left(\bigwedge \phi_i\right) = \sum \epsilon_L(\phi_i).$$

Definition 3.4 (Disjunction)**.** Take a collection of formulas ψ_i where each instance is a conjunction as above all agreing on X and Y. Then

$$\epsilon_L\left(\bigvee \psi_i\right) = \min \epsilon_L(\psi_i).$$

Theorem 3.5. *For a formula ψ as above we have*

$$\operatorname{vc}\psi \geq \left\lfloor \frac{Y(\psi)}{\epsilon_L(\psi)} \right\rfloor,$$

where $Y(\psi)$ is $\dim(Y)$ (as all basic componenets agree on Y).

Proof. First, work with a formula that is a conjunction of positive basic formulas $\psi = \bigwedge_{i\in I}\phi_i$. Then as we have defined above

$$\epsilon_L(\psi) = \sum_{i\in I}\epsilon_L(\phi_i).$$

If W_i is a witness of ϕ_i, let $S_i = |W_i|$. Let n_1 be the largest natural number such that

$$n_1\epsilon_L(\psi) < Y(\psi).$$

Let ϵ' be the smallest value among $\epsilon_L(\phi_i)$. Suppose it corresponds to the formula ϕ'. Let n_2 be the largest natural number such that

$$n_1\epsilon_L(\psi) + n_2\epsilon' < Y(\psi).$$

Fix some $N > n_1 + n_2$. Let

$$J = \{0 \leq j < N\} \subseteq \mathbb{N}.$$

Let a_j be a graph isomorphic to X for each $j \in J$, pairwise disjoint. Let $A = \bigcup_{1\leq j\leq N} a_j$. Let

$$S = |Y| + (n_1 + n_2 + 1)\sum_{i\in I} S_i.$$

By Lemma 2.4 the graph A can be embedded into $\mathbb{G}$ as an S-strong graph. Abusing notation, we identify A with this embedding. Thus we have $A \subseteq \mathbb{G}$, S-strong.

Let J_1, J_2 be disjoint subsets of J, of sizes n_1, n_2 respectively. Let b be a graph isomorphic to Y. For each $i \in I, j \in J_1$ let W_{ij} be a witness of $\phi_i(a_j, b)$. (Note that then $(a_j \cup b, W_{ij})$ is a minimal pair.) For each $j \in J_1$ let W_j be a union of $\{W_{ij}\}_{i\in I}$ disjoint over $a_j \cup b$. For each $j \in J_2$ let W_j be a witness of $\phi'(a_j, b)$. Let W' be a union of $\{W_j\}_{j\in J_1\cup J_2}$ disjoint over b. Let W be a union of W' and A disjoint over $\{a_j\}_{j\in J_1\cup J_2}$.

Claim 3.6. *We have $A \leq W$.*

Proof. Consider some $A \subsetneq B \subseteq W$. We need to show $\dim(B/A) > 0$. Let $\bar{A} = A \cup b$. We have

$$\dim(B/A) = \dim(B/B \cap \bar{A}) + \dim(B \cap \bar{A}/A).$$

Let $B_{ij} = B \cap W_{ij}$. Let $B_j = B \cap W_j$. To unify indices, relabel all the graphs above as $\{B_k\}_{k\in K}$ for some index set K. By the construction of W we have

$$\dim(B/B \cap \bar{A}) = \sum_{k\in K} \dim(B_k/B_k \cap \bar{A}).$$

Fix k. We have $B_k \subseteq W_k$, where W_k is a minimal extension of $M_0^k = a \cup b$ for some $a \in A$. Let ϵ_k be the dimension of this minimal extension. We have $\dim(B_k/B_k \cap \bar{A}) = \dim(B_k/a \cup (B \cap b))$.

Case 1: $B \cap b = b$. Then $M_0^k \subseteq B_k \subseteq W_k$ and

$$\dim(B_k/a \cup (B \cap b)) = \dim(B_k/M_0^k).$$

By minimality of (M_0^k, B_k) we have $\dim(B_k/M_0^k) \geq -\epsilon_k$. Thus

$$\dim(B/B \cap \bar{A}) \geq -\sum_{k\in K} \epsilon_k = -\left(n_1\epsilon_L(\psi) + n_2\epsilon'\right).$$

In addition

$$\dim(B \cap \bar{A}/A) = \dim(b) = Y(\psi).$$

Combining the two, we get

$$\dim(B/A) \geq Y(\psi) - \left(n_1\epsilon_L(\psi) + n_2\epsilon'\right),$$

which is positive by the construction of n_1, n_2 as needed.

Case 2: $B \cap b \subsetneq b$.

Claim 3.7. *We have* $\dim(B_k/B_k \cap \bar{A}) > 0$.

Proof. Recall that $\dim(B_k/B_k \cap \bar{A}) = \dim(B_k/a \cup (B \cap b))$. First, suppose that $B_k \cup M_0^k \neq W_k$. Then by Lemma 2.7 we get the required inequality. Thus we may assume that $B_k \cup M_0^k = W_k$. By Lemma 2.2 we have

$$\dim(B_k \cup M_0^k/M_0^k) = \dim(B_k/B_k \cap M_0^k) - \alpha E,$$

where E is the number of edges connecting the vertices of $B_k \cup M_0^k - M_0^k$ to the vertices of $M_0^k - B_k \cap M_0^k$. Noting that $B_k \cup M_0^k = W_k$, $\dim W_k/M_0^k = -\epsilon_k$, and $B_k \cap M_0^k = a \cup (B \cap b)$ we may rewrite the equality above as

$$\dim(B_k/a \cup (B \cap b)) = \alpha E - \epsilon,$$

and E is the number of edges connecting the vertices of $W_k - M_0^k$ to the vertices of $M_0^k - a \cup (B \cap b)$. As $Y = Y'$ and $B \cap b \subsetneq b$ we must have $E \geq 1$. But then as $\alpha > \epsilon$ we have $\dim(B_k/a \cup (B \cap b)) > 0$ as needed. □

Now, recall that

$$\dim(B/A) = \dim(B \cap \bar{A}/A) + \sum_{k \in K} \dim(B_k/B_k \cap \bar{A}).$$

By the claim above each of $\dim(B_k/B_k \cap \bar{A}) > 0$, thus

$$\dim(B/A) > \dim(B \cap \bar{A}/A).$$

In addition

$$\dim(B \cap \bar{A}/A) = \dim(B \cap b) \geq 0,$$

as b is postive. Thus $\dim(B/A) > 0$ as needed. □

As $A \leq W$ and $A \subseteq \mathbb{G}$, we can embed W into $\mathbb{G}$ over A. Abusing notation again, we identify W with its embedding $A \leq W \subseteq \mathbb{G}$. In particular, now we have $b \in \mathbb{G}$. Also note that

$$\dim(W/A) = Y(\psi) - (n_1\epsilon_L(\psi) + n_2\epsilon')\,,$$
$$|W| - |A| \leq |b| + (n_1 + n_2)\sum_{i \in I} S_i.$$

Lemma 3.8. *We have*

$$\{a_j\}_{j \in J_1} \subseteq \psi(A, b) \subseteq \{a_j\}_{j \in J_1 \cup J_2}\,.$$

Proof. First inclusion $\{a_j\}_{j\in J_1} \subseteq \psi(A,b)$ is immediate from the construction of W, as W_{ij} witnesses that $\phi_i(a_j, b)$ holds. For the second inclusion, suppose that there is $a \in A - \{a_j\}_{j \in J_1 \cup J_2}$ such that $\psi(a, b)$ holds. Let $W' \subseteq \mathbb{G}$ be a witness of $\phi_1(a, b)$. First, note that the case $W' \subseteq W$ is impossible as there are no edges between a and $W - a$, but there are edges between a and $W' - a$. Thus assume $W' \not\subseteq W$. As $(a \cup b, W')$ is minimal, by Lemma 2.5 we have $\dim(W' \cup W/W) < -\epsilon_1$. Therefore

$$\dim(W' \cup W/A) = \dim(W' \cup W/W) + \dim(W/A) < Y(\psi) - (n_1\epsilon_L(\psi) + n_2\epsilon') - \epsilon_1,$$

which is negative by the construction of n_1, n_2. Thus $A \not\leq W \cup W'$, as then it would have a positive dimension. Additionally,

$$|W' \cup W| - |A| \leq |W' - W| + |W| - |A| \leq S_1 + |b| + (n_1 + n_2)\sum_{i \in I} S_i \leq S,$$

but then this contradicts that A is S-strong, as then we would have $A \leq W \cup W'$. □

In the construction of W we have chosen indices J_1, J_2 arbitrarily. In particular, suppose we let J_2 to be the last n_2 indices of J and J_1 an arbitrary n_1-element subset of the first $N - n_2$ elements of J. Each of those choices would then yield a different trace $\psi(A, b)$ by the lemma above. Thus $\psi(A, M^{|y|}) \geq \binom{N-n_2}{n_1}$ and therefore $\mathrm{vc}(\psi) \geq n_1$. By the definition of n_1 we have $n_1 = \left\lfloor \frac{Y(\psi)}{\epsilon_L(\psi)} \right\rfloor$, so this proves the theorem for ψ.

Now consider a formula which is a conjunction consisting of negative basic formulas $\psi = \bigwedge_{i\in I} \neg\phi_i$. Let $\bar{\psi} = \bigwedge_{i\in I} \phi_i$. Do the construction above for $\bar{\psi}$ and suppose its trace is $X \subseteq A$ for some b. Then over b the same construction gives trace $(A - X)$ for ψ. Thus we get as many traces as above, and the same bound.

Finally consider a formula which is a disjunction of formulas considered above $\theta = \bigvee_{k\in K} \psi_k$. Choose the one with the smallest ϵ_L, say ψ_k, and repeat the construction above for ψ_k. Any trace we obtain is automatically a trace for θ, and thus we get as many traces as above, and the same bound. □

Corollary 3.9. *VC-function is infinite in Shelah-Spencer random graphs:*

$$\mathrm{vc}(n) = \infty.$$

Proof. Let A be a graph consisting of $1/\alpha + 2 + n$ disconnected vertices. Fix $\epsilon > 0$. By Lemma 2.3, there exists B such that (A, B) is minimal with dimension $\leq \epsilon$. Consider a basic formula $\psi_{A,B}(x, y)$ where $|x| = 1/\alpha + 2$ and $|y| = n$. Then by the theorem above $\mathrm{vc}(n) \geq \mathrm{vc}(\psi_{A,B}) \geq \frac{n}{\epsilon}$. As ϵ was arbitrary, this number can be made arbitrarily large, giving $vc(n) = \infty$ as needed. □

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