SOME VC-DENSITY COMPUTATIONS IN SHELAH-SPENCER GRAPHS

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1. Preliminaries

VC density was introduced in [1] by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko as a natural notion of dimension for NIP theories. In a NIP theory we can define a VC function

$$\mathrm{vc} : \mathbb{N} \longrightarrow \mathbb{N}$$

Where $vc(n)$ measures complexity of definable sets in an n-dimensional space. Simplest possible behavior is $\mathrm{vc}(n) = n$ for all n. Theories with that property are known to be dp-minimal, i.e. having the smallest possible dp-rank. In general, it is not known whether there can be a dp-minimal theory which doesn't satisfy $\mathrm{vc}(n) = n$.

In this paper, we investigate vc-density of definable sets in Shelah-Spencer structures. We follow notations in [2]. In this paper we work with limit of random structure $G(n, n^{-\alpha})$ for $\alpha \in (0, 1)$, irrational. This structure is axiomatized by S_α. Our ambient model is $\mathscr{M}$. Notations we use are $\delta(\mathcal{A}), \delta(\mathcal{A}/\mathcal{B}), \mathcal{A} \leq \mathcal{B}$ as well as notions of N-strong substructure, minimal extension, chain minimal extension, minimal pair, and N-strong closure.

2. Graph Combinatorics

We denote graph by $\mathcal{A}$, set of its vertices by A. When we say $\mathcal{A} \subseteq \mathcal{B}$ we mean that $A \subseteq B$ and edges of $\mathcal{A}$ are also edges of $\mathcal{B}$. However $\mathcal{B}$ may add new edges between vertices of $\mathcal{A}$.

Fix $\alpha \in (0, 1)$, irrational. For a finite graph $\mathcal{A}$ let

$$\delta(\mathcal{A}) = |A| - \alpha e(\mathcal{A})$$

where $e(\mathcal{A})$ is the number of edges in $\mathcal{A}$.

For finite $\mathcal{A}, \mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B}$ define $\delta(\mathcal{B}/\mathcal{A}) = \delta(\mathcal{B}) - \delta(\mathcal{A})$. We say that $\mathcal{A} \leq \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and $\delta(\mathcal{A}'/\mathcal{B}) > 0$ for all $\mathcal{A} \subseteq \mathcal{A}' \subsetneq \mathcal{B}$.

We say that finite $\mathcal{A}$ is positive if for all $\mathcal{A}' \subseteq \mathcal{A}$ we have $\delta(\mathcal{A}') \geq 0$.

Definition 2.1. We work in theory S_α axiomatized by

- Every finite substructure is positive
- For a model $\mathscr{M}$ given $\mathcal{A} \leq \mathcal{B}$ every embedding $f : \mathcal{A} \longrightarrow \mathscr{M}$ extends to $g : \mathcal{B} \longrightarrow \mathscr{M}$.

For $\mathcal{A}, \mathcal{B}$ positive, $(\mathcal{A}, \mathcal{B})$ is called a minimal pair if $\mathcal{A} \subseteq \mathcal{B}$, $\delta(\mathcal{B}/\mathcal{A}) < 0$ but $\delta(\mathcal{A}'/\mathcal{A}) \geq 0$ for all proper $\mathcal{A} \subseteq \mathcal{A}' \subsetneq \mathcal{B}$.

$\langle \mathcal{A}_i \rangle_{i \leq m}$ is called a minimal chain if $(\mathcal{A}_i, \mathcal{A}_i + 1)$ is a minimal pair (for all $i < m$).

For a positive $\mathcal{A}$ let $\delta_{\mathcal{A}}(\bar{x})$ be the atomic diagram of $\mathcal{A}$. For positive $\mathcal{A} \subset \mathcal{B}$ let

$$\Psi_{\mathcal{A},\mathcal{B}}(\bar{x}) = \delta_{\mathcal{A}}(\bar{x}) \wedge \exists \bar{y}\ \delta_{\mathcal{B}}(\bar{x}, \bar{y})$$

Such formula is called chain-minimal extension formula if in addition we have that there is a minimal chain starting at $\mathcal{A}$ and ending in $\mathcal{B}$. Denote such formulas as $\Psi_{\langle \mathcal{M}_i \rangle}$

Theorem 2.2 (5.6 in [2])**.** *S_α admits quantifier elimination down to boolean combination of chain-minimal extension formulas.*

3. Definitions

Fix tuples $x = (x_1, \dots x_n), y = (y_1, \dots, y_m)$. We refer to chain-minimal extension formulas as basic formulas. Let $\phi_{\langle \mathcal{M}_i \rangle}(x, y)$ be a basic formula.

Definition 3.1. Define $\mathcal{X}$ to be the graph on vertices $\{x_i\}$ with edges as defined by $\phi_{\langle \mathcal{M}_i \rangle}$. Similarly define $\mathcal{Y}$. We define those abstractly, i.e. on a new set of vertices disjoint from $\mathscr{M}$.

Note that $\mathcal{X}$, $\mathcal{Y}$ are positive as they are subgraphs of $\mathcal{M}_0$. As usual X, Y will refer to vertices of those graphs.

We restrict our attention to formulas that define no edges between X and Y.

Note 3.2. We can handle edges between x and y as separate elements of the minimal chain extension.

Definition 3.3. For a basic formula $\phi = \phi_{\langle \mathcal{M}_i \rangle_{i \leq k}}(x, y)$ let

- $\epsilon_i(\phi) = -\dim(M_i/M_{i-1})$.
- $\epsilon_L(\phi) = \sum_{[1..k]} \epsilon_i(\phi)$.
- $\epsilon_U(\phi) = \min_{[1..k]} \epsilon_i(\phi)$.
- Let $\mathcal{Y}'$ be a subgraph of $\mathcal{Y}$ induced by vertices of $\mathcal{Y}$ that are connected to $M_k - (X \cup Y)$.
- Let $Y(\phi) = \dim(\mathcal{Y}')$. In particular if $\mathcal{Y} = \mathcal{Y}'$ and $\mathcal{Y}$ is disconnected then $Y(\phi)$ is just the arity of the tuple y.

4. Lower bound

As a simplification for our lower bound computation we assume that all the basic formulas involved we have $\mathcal{Y}' = \mathcal{Y}$ (see Definition 3.3).

We work with formulas that are boolean combinations of basic formulas written in disjunctive-conjunctive form. First, we extend our definition of ϵ.

Definition 4.1 (Negation)**.** If ϕ is a basic formula, then define

$$\epsilon_L(\neg\phi) = \epsilon_L(\phi)$$

Definition 4.2 (Conjunction)**.** Take a collection of formulas $\phi_i(x, y)$ where each ϕ_i is positive or negative basic formula. If both positive and negative formulas are present then $\epsilon_L(\phi) = \infty$. We don't have a lower bound for that case. If different formulas define $\mathcal{X}$ or $\mathcal{Y}$ differently then $\epsilon_L(\phi) = \infty$. In that case of the conflicting definitions would make the formula have no realizations. Otherwise

$$\epsilon_L(\bigwedge \phi_i) = \sum \epsilon_L(\phi_i)$$

Definition 4.3 (Disjunction)**.** Take a collection of formulas ψ_i where each instance is a conjunction of positive and negative instances of basic formulas that agree on $\mathcal{X}$ and $\mathcal{Y}$.

$$\epsilon_L(\bigvee \psi_i) = \min \epsilon_L(\psi_i)$$

Theorem 4.4. *For a formula ϕ as above*

$$\mathrm{vc}\,\phi \geq \left\lfloor \frac{Y(\phi)}{\epsilon_L(\phi)} \right\rfloor$$

where $Y(\phi)$ is $Y(\psi)$ for ψ one the basic components of ϕ (all basic componenets agree on $\mathcal{Y}$).

Proof. First work with a formula that is a conjunction of positive basic formulas.

$$\psi = \bigwedge_{j \leq J} \phi_j$$

Then as we defined above

$$\epsilon_L(\psi) = \sum \epsilon_L(\phi_j)$$

Let ϕ be one of the basic formulas in ψ with a chain $\langle M_i \rangle_{i \leq k}$. Let $K_\phi = |M_k|$ i.e. the size of the extension. Let K be the largest such size among all ϕ_i.

Let n be the integer such that $n\epsilon_L(\psi) < Y$ and $(n+1)\epsilon_L(\psi) > Y$.

Label $\mathcal{Y}$ by an tuple b.

Pick parameter set $A \subset \mathscr{M}$ such that

$$A = \bigcup_{i<N} b_i$$

a disjoint union where each b_i is an ordered tuple of size $|x|$ connected according to ψ. We also require A to be $N \cdot I \cdot K$-strong.

Fix n arbitrary elements out of A, label them a_j.

For each ϕ_i, a_j pick an abstract realization M_{ij} of ϕ_i over (a_j, b) (abstract meaning disjoint from $\mathscr{M}$).

Let $\bar{M}$ be an abstract disjoint union of all those realizations.

Claim 4.5. $(A \cap \bar{M}) \leq \bar{M}$.

Proof. Consider some $(A \cap \bar{M}) \subseteq B \subseteq \bar{M}$. Let $B_{ij} = B \cap M_{ij} \subseteq M_{ij}$. Then B_{ij}'s are disjoint over $\bar{A} = A \cup b$. In particular $\dim B/(\bar{A} \cap B) = \sum \dim B_{ij}/(\bar{A} \cap B_{ij})$. $\dim B_{ij}/\bar{A} \geq -\epsilon_L(\phi_i)$ by Lemma 6.3. Thus $\dim B/(\bar{A} \cap B) \geq -n\epsilon(\psi)$. Thus $\dim B/(A \cap B) \geq \dim(B) - n\epsilon(\psi)$. By construction we have $Y - n\epsilon_L(\psi) > 0$ as needed. □

$|\bar{M}| \leq N \cdot I \cdot K$ and A is $\leq N \cdot I \cdot K$-strong. Thus a copy of $\bar{M}$ can be embedded over A into our ambient model $\mathscr{M}$. Our choice of b_i's was arbitrary, so we get $\binom{N}{n}$ choices out of $N|x|$ many elements. Thus we have $O(|A|^n)$ many traces.

Lemma 4.6. *There are arbitrarily large sets with properties of A.*

Proof. proof goes here. use lemma in laskoswki paper □

This shows

$$\operatorname{vc}\psi \geq n = \left\lfloor \frac{Y}{\epsilon_L} \right\rfloor$$

Now consider the formula which is a conjunction consists of negative basic formulas

$$\psi = \bigwedge \neg\phi_i$$

Let

$$\bar{\psi} = \bigwedge \phi_i$$

Do the construction above for $\bar{\psi}$ and suppose its trace is $X \subset A$ for some b. Then over b the same construction gives trace $(A - X)$ for ψ. Thus we get as many traces.

Finally consider a formula which is a disjunction of formulas considered above. Choose the one with the smallest ϵ_L, this yields the lower bound for the entire formula. □

Claim 4.7. *We can find a minimal extension $M/\{x,y\}$ with arbitrarily small dimension.*

This shows that vc function is infinite in Shelah-Spencer random graphs.

$$\operatorname{vc}(n) = \infty$$

5. Upper bound

Consider a case of a single basic formula $\phi(\vec{x}, \vec{y})$.

Suppose it defines a minimal chain extension over $\{x, y\}$. Record the size of that extension as $K(\phi)$ and its total dimension $\epsilon(\phi) = \epsilon_U(\phi)$.

In general we have parameter set $A \subset \mathscr{M}^{|x|}$, however without loss of generality we may work with a parameter set $A^{|x|}$, with $A \subset \mathscr{M}$.

Let $S = \left\lfloor \frac{K(\phi)Y(\phi)}{\epsilon(\phi)} \right\rfloor$ (dependent only on ϕ).

For our proof to work we also need A to be S-strong. We can achieve this by taking (the unique) S-strong closure of A. If size of A is N then the size of its closure is $O(N)$. So without loss of generality we can assume that A is S-strong.

Definition 5.1. Define a b-trace of ϕ on A

$$A_b = \phi(A, b) = \left\{ a \in A^{|x|} \mid \phi(a, b) \right\}$$

Let $\bar{A} = A \cup b$.

Definition 5.2. For a set C define the boundary of C over $\bar{A}$

$$\partial(C, \bar{A}) = \left\{ a \in \bar{A} \mid \text{there is an edge between } a \text{ and element of } C - \bar{A} \right\}$$

Definition 5.3. A *witness* of $\phi(a, b)$ is a realization of the existential formula together with a_i, b.

Definition 5.4. For a trace $A_b = \{a_1, \ldots, a_I\}$ for each $\phi(a_i, b)$ pick a witness and then take a union of all those witnesses. Call this a witness of the trace A_b.

Definition 5.5. For each b pick some $\bar{M}_b$ to be a witness of A_b. Define two quantities

- ∂_b is the boundary $\partial(\bar{M}_b, \bar{A})$
- Call $f\colon G_1 \longrightarrow G_2$ a ∂-isomorphism if it is a graph isomorphism and f and f^{-1} are constant on $\bar{A}$. Define $\mathscr{I}_b$ as the ∂-isomorphism class of the graph induced on vertices $(\bar{M}_b - \bar{A}) \cup \partial_b$.

Lemma 5.6. *If* $\mathscr{I}_{b_1} = \mathscr{I}_{b_2}$ *then* $A_{b_1} = A_{b_2}$.

Proof. Fix witnesses $\bar{M}_{b_1}, \bar{M}_{b_2}$. Suppose we have $\phi(a, b_1)$ for some a. Pick its witness $M_1 \subset \bar{M}_{b_1}$. This gives us a witness M_2 via the ∂-isomorphism. □

Thus to bound the number of traces it is sufficient to bound the number of possibilities for $\mathscr{I}_b$.

Theorem 5.7.

$$|\partial_b| \leq K(\phi)\frac{Y(\phi)}{\epsilon(\phi)}$$

$$|\bar{M}_b - \bar{A}| \leq K(\phi)\frac{Y(\phi)}{\epsilon(\phi)}$$

Corollary 5.8.

$$\mathrm{vc}(\phi) \leq K(\phi)\frac{Y(\phi)}{\epsilon(\phi)}$$

Proof. We count possible ∂-isomorphism classes $\mathscr{I}_b$. Let $W = K(\phi)\frac{Y(\phi)}{\epsilon(\phi)}$. If the parameter set A is of size N then there are $\binom{N}{W}$ choices for boundary ∂_b. On top of the boundary there are at most W extra vertices and $(2W)^2$ extra edges. Thus there are at most

$$W \cdot 2^{(2W)^2}$$

configurations up to a graph isomorphism. In total this gives us

$$\binom{N}{W} \cdot W \cdot 2^{(2W)^2} = O(N^W)$$

options for ∂-isomorphism classes. By Lemma 5.6 there are at most $O(N^W)$ many traces, giving the required bound. □

Proof. (of Theorem 5.7) Fix some b-trace A_b. Enumerate $A_b = \{a_1, \dots, a_I\}$.

Let $M_i/\{a_i, b\}$ be a witness of $\phi(a_i, b)$ for each $i \leq I$. Let $\bar{M}_i = \bigcup_{j<i} M_j$. Let $\bar{M} = \bigcup M_i$, a witness of A_b

Claim 5.9.

$$\left|\partial(M_i M, \bar{A}) - \partial(M, \bar{A})\right| \leq |M_i| = K(\phi)$$

$$\dim(M_i M\bar{A}/M\bar{A}) > -\epsilon(\phi)$$

Definition 5.10. $(j-1, j)$ is called a *jump* if some of the following conditions happen

- New vertices are added outside of $\bar{A}$ i.e.

$$\bar{M}_j - \bar{A} \neq \bar{M}_{j-1} - \bar{A}$$

- New vertices are added to the boundary, i.e.

$$\partial(\bar{M}_j, \bar{A}) \neq \partial(\bar{M}_{j-1}, \bar{A})$$

Definition 5.11. We now let m_i count all jumps below i

$$m_i = |\{j < i \mid (j-1, j) \text{ is a jump}\}|$$

Lemma 5.12.

$$\dim(\bar{M}_i/\bar{A}) \leq -m_i \cdot \epsilon(\phi)$$
$$|\partial(\bar{M}_i, \bar{A})| \leq m_i \cdot K(\phi)$$
$$|\bar{M}_j - \bar{A}| \leq m_i \cdot K(\phi)$$

Proof. (of Lemma 5.12) Proceed by induction. Second and third propositions are clear. For the first proposition base case is clear.

Induction step. Suppose $\bar{M}_j \cap (A \cup b) = \bar{M}_{j+1}$ and $\partial(\bar{M}_j, A) = \partial(\bar{M}_{j+1}, A)$. Then $m_i = m_{i+1}$ and the quantities don't change. Thus assume at least one of these equalities fails.

Apply Lemma 6.2 to $\bar{M}_j \cup (A \cup b)$ and $(M_{j+1}, a_{j+1}b)$. There are two options

- $\dim(\bar{M}_{j+1} \cup (A \cup b)/\bar{M}_i \cup (A \cup b)) \leq -\epsilon_U$. This implies the proposition.
- $M_{j+1} \subset \bar{M}_j \cup (A \cup b)$. Then by our assumption it has to be $\partial(\bar{M}_j, A) \neq \partial(\bar{M}_{j+1}, A)$. There are edges between $M_{j+1} \cap (\partial(\bar{M}_{j+1}, A) - \partial(\bar{M}_j, A))$ so they contribute some negative dimension $\leq \epsilon_U$.

This ends the proof for Lemma 5.12. □

(Proof of Theorem 5.7 continued) First part of lemma 5.12 implies that we have $\dim(\bar{M}/\bar{A}) \leq -m_I \cdot \epsilon(\phi)$. The requirement of A to be S-strong forces

$$m_I \cdot \epsilon(\phi) < Y(\phi)$$
$$m_I < \frac{Y(\phi)}{\epsilon(\phi)}$$

Applying the rest of 5.12 gives us

$$|\partial(\bar{M}, A)| \leq m_I \cdot K(\phi) \leq \frac{K(\phi)Y(\phi)}{\epsilon(\phi)}$$
$$|\bar{M} \cap A| \leq m_I \cdot K(\phi) \leq \frac{K(\phi)Y(\phi)}{\epsilon(\phi)}$$

as needed. This ends the proof for Theorem 5.7. □

So far we have computed an upper bound for a single basic formula ϕ.

To bound an arbitrary formula, write it as a boolean combination of basic formulas ϕ_i (via quantifier elimination) It suffices to bound vc-density for collection of formulas $\{\phi_i\}$ to obtain a bound for the original formula.

In general work with a collection of basic formulas $\{\phi_i\}_{i \in I}$. The proof generalizes in a straightforward manner. Instead of $A^{|x|}$ we now work with $A^{|x|} \times I$ separating traces of different formulas. Formula with the largest quantity $Y(\phi)\frac{K(\phi)}{\epsilon(\phi)}$

contributes the most to the vc-density. Thus we have

$$\Phi = \{\phi_i\}_{i\in I}$$
$$\mathrm{vc}(\Phi) = \max_{i\in I} Y(\phi_i)\frac{K(\phi_i)}{\epsilon_{\phi_i}}$$

6. Technical Lemmas

Lemma 6.1. *Suppose we have a set B and a minimal pair (M, A) with $A \subset B$ and $\dim(M/A) = -\epsilon$. Then either $M \subseteq B$ or $\dim((M \cup B)/B) < -\epsilon$.*

Proof. By diamond construction

$$\dim((M \cup B)/B) \leq \dim(M/(M \cap B))$$

and

$$\begin{aligned} \dim(M/(M \cap B)) &= \dim(M/A) - \dim(M/(M \cap B)) \\ \dim(M/A) &= -\epsilon \\ \dim(M/(M \cap B)) &> 0 \end{aligned}$$

□

Lemma 6.2. *Suppose we have a set B and a minimal chain M_n with $M_0 \subset B$ and dimensions $-\epsilon_i$. Let ϵ be the minimal of ϵ_i. Then either $M_n \subseteq B$ or $\dim((M_n \cup B)/B) < -\epsilon$.*

Proof. Let $\bar{M}_i = M_i \cup B$

$$\dim(\bar{M}_n/B) = \dim(\bar{M}_n/\bar{M}_{n-1}) + \ldots + \dim(\bar{M}_2/\bar{M}_1) + \dim(\bar{M}_1/B)$$

Either $M_n \subseteq B$ or one of the summands above is nonzero. Apply previous lemma. □

Lemma 6.3. *Suppose we have a minimal chain M_n with dimensions $-\epsilon_i$. Let ϵ be the sum of all ϵ_i. Suppose we have some B with $B \subseteq M_n$. Then $\dim B/(M_0 \cap B) \geq -\epsilon$.*

Proof. Let $B_i = B \cap M_i$. We have $\dim B_{i+1}/B_i \geq \dim M_{i+1}/M_i$ by minimality. $\dim B/(M_0 \cap B) = \dim B_n/B_0 = \sum \dim B_{i+1}/B_i \geq -\epsilon$. □

7. Counterexamples

8. Upper bound on $\mathscr{A}$

Definition 8.1.

$$\mathscr{A} = \{A \subset \mathcal{U}^y \mid \text{finite, disconnected, strongly embedded}\}$$

Let n be the integer such that $n\epsilon_U < Y$ and $(n+1)\epsilon_U > Y$.

Pick a trace of $\phi(x, y)$ on $A^{|x|}$ by a parameter b.

$$B = \left\{a \in A^{|x|} \mid \phi(a, b)\right\}$$

Pick $B' \subset B$, ordered $B' = \{a_i\}_{i \in I}$ such that

$$a_i \cap \bigcup_{j<i} a_j \neq \emptyset$$

This is always possible by starting with B and taking away elements one by one. Call such a set a *generating set* of B.

Let $M_i/\{a_i, b\}$ be a witness of $\phi(a_i, b)$ for each $i \in I$. Let $\bar{M} = \bigcup M_i$. Consider $\bar{M}/A$.

Pick $\bar{M}$ such that $\dim(\bar{M}/A)$ is maximized.

$\bar{M} \cap A \leq \bar{M}$ as A is strong. (Make sure M is not too big!) Let $\bar{A} = A - \{a_i\}_{i \in I}$. Suppose $\bar{A} \cap \bar{M} \neq \emptyset$. Then we can abstractly reembed $\mathcal{M}$ over A such that $\bar{A} \cap \bar{M} = \emptyset$. This would increase the dimension, contradicting maximality. Thus we can assume $A \cap \bar{M} = \{a_i\}_{i \in I}$

Let $\bar{M}_j = \bigcup_{i<j} M_i$.

Lemma 8.2. $\dim(\bar{M}_j/A) \leq j \cdot \epsilon_U$

Proof. Proceed by induction. Base case is clear.

For induction case apply lemma to $\bar{M}_j \cup \{a_j\}$ and $M_j/\{a_j, b\}$. There are two cases

(1) $M_j \subset \bar{M}' \cup \{a_j\}$. In this case there are edges between $\{a_j\}$ and M_j that contribute to dimension less than $-\epsilon_U$.
(2) Otherwise M_j adds extra dimension less than $-\epsilon_U$

$\square$

Thus we have $\dim(\bar{M}/A) = \dim(\bar{M}_n/A) \leq -\epsilon_U n$.

Thus as A is strong we need $|B'|\epsilon_U < Y$. This gives us $|B'| \leq n$. Finally we need to relate $|B'|$ to $|B|$.

Suppose we have $C \subset A^{|x|}$, finite with $|C| = N$. A generating set for a trace has to have size $\leq n$. Thus there are $\binom{N}{n} \leq N^n$ choices for a generating set. A set generated from set of size n can have at most $(x|n|)^{|x|}$ elements. Thus a given set of size n can generate at most

$$2^{(x|n|)^{|x|}}$$

sets. Thus the number of possible traces on C is bounded above by

$$2^{(x|n|)^{|x|}} \cdot N^n = O(N^n)$$

This bounds the vc-density by n.

$$\mathrm{vc}_{\mathscr{A}}(\phi) \geq \left\lfloor \frac{Y}{\epsilon_U} \right\rfloor$$

References

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