Math 285D Notes: 12/5

Tyler Arant

Correction for proof of Weierstrass Preparation. We had $f, g \in \mathbb{C}\{X, T\}$, f regular of order d, and

$$F = u^{-1} \sum_{i<d} f_i T^i, \quad u = f_d + f_{d+1} T + \cdots \in \mathbb{C}\{X, T\}^{\times}.$$

Choose $(r', r_{m+1}) \in (\mathbb{R}^{>0})^{m+1}$ such that

$$\|g\|_r, \|u^{-1}\|_r, \|f_0\|_{r'}, \ldots, \|f_{d+1}\|_{r'} < \infty.$$

Then, we can achieve

$$\|F\|_r \leq \|u^{-1}\| \cdot \sum_{i<d} \|f_i\|_{r'} r_{m+1}^i < r_{m+1}^d,$$

since the f_i vanish at 0 we can make the norms as small as we want by choosing r' small enough.

Let $R \subset S$ be an extension of commutative rings.

0.1 Definition. S is *flat over R* if each solution in S to an equation

$$r_1 x_1 + \cdots + r_n x_n = 0 \qquad (r_i \in R) \tag{$*$}$$

is an S-linear combination of solutions in R.

0.2 Lemma. If S free as an R-module, then S is flat over R.

Proof. Let $s = (s_1, \ldots, s_n) \in S^n$ be a solution to $(*)$. Take R-linearly independent $e_1, \ldots, e_k \in S$ such that

$$s_i = \sum_j w_{ij} e_j \qquad (w_{ij} \in R).$$

Put $w_j = (w_{1j}, \ldots, w_{nj})$. Then w_j is a solution to $(*)$ and $s = \sum_j e_j w_j$. $\square$

We give some examples

- If R is a field, then each S is flat over R.
- $S = R[X_1, \ldots, X_n]$ flat over R.

0.3 Lemma. Suppose S is flat over R. Then each solution in S to a system

$$r_{i1}x_1 + \cdots + r_{in}x_n = 0 \qquad (i = 1, \ldots, m; r_{ij} \in R)$$

is an S-linear combination of solutions in R.

Proof. By induction on m. □

0.4 Definition. We say that S is *faithfully flat* over R if

- S is flat over R.
- Each equation
$$r_1x_1 + \cdots + r_nx_n = 1 \qquad (r_i \in R)$$
that has a solution in S has a solution in R.

0.5 Lemma. Suppose S is flat over R. The following are equivalent.

(1) S is faithfully flat over R.

(2) For each maximal ideal $\mathfrak{m}$ of R, we have $\mathfrak{m}S \neq S$.

(3) Each system
$$\sum_{j=1}^{n} r_{ij}x_j = t_i \qquad (i = 1, \ldots, m; r_{ij}, t_i \in R) \tag{$*$}$$
that has a solution in S has a solution in R.

(1) $\implies$ (2): Suppose, by means of contradiction, that $\mathfrak{m}S = S$. Then, there are $r_i \in \mathfrak{m}$ and $s_i \in S$ such that
$$r_1s_1 + \cdots + r_ns_n = 1,$$
i.e., $s = (s_1, \ldots, s_n)$ is a solution to the equation

$$r_1x_1 + \cdots + r_nx_n = 1.$$

Since S is faithfully flat over R, there is a solution $w = (w_1, \ldots, w_n)$ so that

$$1 = r_1w_1 + \cdots + r_nw_n \in \mathfrak{m},$$

which is a contradiction.

(2) $\implies$ (1): Suppose S is not faithfully flat over R; then, there are $r_i \in R$ such that
$$r_1x_1 + \cdots + r_nx_n = 1$$

has a solution s in S but not a solution in R. Then the ideal $\mathfrak{a} = (r_1, \dots, r_n)$ in R is proper. Let $\mathfrak{m}$ be an ideal in R that contains $\mathfrak{a}$. Then, $\mathfrak{m}S = S$ since

$$1 = r_1 s_1 + \cdots r_n s_n \in \mathfrak{m}S.$$

(3) $\implies$ (1) trivially.

(1) $\implies$ (3): Suppose $(*)$ has a solution $s = (s_1, \dots, s_n) \in S^n$. Then, $(1, s) = (1, s_1, \dots, s_n)$ is a solution to the homogenous system

$$\begin{aligned} -t_1 x_0 + \sum_{j=1}^{n} r_{1j} x_j &= 0 \\ &\vdots \\ -t_m x_0 + \sum_{j=1}^{n} r_{mj} x_j &= 0 \end{aligned} \tag{$**$}$$

Since S is flat over R, $(1, s)$ is an S-linear combination of solutions $(u_1, v_1), \dots, (u_k, v_k)$ in R^{1+n}. So,

$$(1, s) = w_1(u_1, v_1) + \cdots + w_k(u_k, v_k) \qquad (w_i \in S),$$

hence

$$1 = w_1 u_1 + \cdots + w_k u_k.$$

By (1), there exists $\omega_1, \dots, \omega_k \in R$ such that

$$1 = \omega_1 u_1 + \cdots + \omega_k u_k.$$

Then, $\omega_1 v_1 + \cdots + \omega_k v_k \in R^n$ solves $(*)$.

0.6 Theorem. *$\mathbb{C}[[X]]$ is faithfully flat over $\mathbb{C}\{X\}$.*

Proof sketch. We proceed by induction on the number of variable. Consider

$$f_1 y_1 + \cdots + f_n y_n = 0 \qquad (f_i \in \mathbb{C}\{X, T\}. \tag{$*$}$$

We may assume that all f_i, if nonzero, are regular in T. Then, apply Weierstass Preparation in $\mathbb{C}\{X, T\}$ to the $f_i \neq 0$. Then, we can assume that the $f_i \neq 0$ are Weierstrass polynomials: $f_i \in \mathbb{C}\{X\}[T]$ monic of some degree d_i. Set

$$z_2 = \begin{bmatrix} f_2 \\ -f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, z_3 = \begin{bmatrix} f_3 \\ 0 \\ -f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, z_n = \begin{bmatrix} f_n \\ 0 \\ \vdots \\ 0 \\ -f_1 \end{bmatrix},$$

which together is a solution of $(*)$. We have $y_i = q_i f_1 + r_i$, where $q_i \in \mathbb{C}[[X,T]]$, and $r_i \in \mathbb{C}[[X]][T]$ is of degree $< d_1$. Then consider

$$y + q_2 z_2 = \begin{bmatrix} * \\ r_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}, \dots, y + q_2 z_2 + \cdots f_n y_n = \begin{bmatrix} * \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix},$$

and conclude that we can assume $y_2, \dots, y_n \in \mathbb{C}[[X]][T]$. We have

$$g := f_1 y_1 = -(f_2 y_2 + \cdots + f_n y_n) \in \mathbb{C}\{X\}[T].$$

We can find h, r with $g = f_1 h + r$ with $h, r \in \mathbb{C}[[X]][T]$, $\deg r < d_1$. So, $g = f_1 y_1 + 0$ in $\mathbb{C}[[X,T]]$, hence $r = 0$ and $y_1 = h \in \mathbb{C}[[X]][T]$. This reduces the proof to showing: $R \subset S$ flat $\implies R[T] \subset S[T]$ flat. $\square$

1 Restricted Analytic Functions

1.1 Lemma (Taylor expansion)**.** Suppose $f \in \mathbb{C}\{X\}_s$ and $b \in D_s(0)$, $j \in \mathbb{N}^m$. Then,

(1) $\partial^j f := \left(\frac{\partial}{\partial X_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial X_m}\right)^{j_m} f \in \mathbb{C}\{X\}_r$ for all $r < s$.

(2) $(\partial^j f)(b) := \sum_{i \geq j} f_i \frac{i!}{(i-j)!} b^{i-j}$ converges absolutely.

(3) $\sum_j \frac{1}{j!} (\partial^j f)(b) X^j \in \mathbb{C}\{X\}_{s(b)}$, where

$$s(b) = (s_1 - |b_1|, \dots, s_m - |b_m|),$$

and

$$f(x+b) = \sum_j \frac{1}{j!} (\partial^j f)(b) x^j \qquad (x \in \overline{D_{s(b)}}(0)).$$

Proof. (1) By induction on $|j|$. The case $\frac{\partial}{\partial x + k}$ follows from Abel's Lemma: a finite bound on $|f_i| s^i$ gives a finite bound on $i_k |f_i| r^{i - e_k}$ (for given $r < s$, with e_k is the $k^t h$ standard basis vector).

(2) Follows easily from (1).

(3) Left as an exercise using (2) and multivariate binomial theorem.

$\square$