VC-density in model theoretic structures

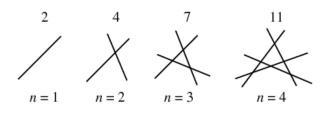
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June 3, 2015

Suppose we have an (infinite) collection of sets \mathcal{F} . We define the shatter function $\pi_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of \mathcal{F}

$$\pi_{\mathcal{F}}(n)=\max\{\# \text{ of atoms in the boolean algebra generated by } \mathcal{S} \ | \ \mathcal{S} \subset \mathcal{F} \text{ with } |\mathcal{S}|=n\}$$

Example: Let \mathcal{F} consist of all half-planes in the plane.



$$\pi_{\mathcal{F}}(1) = 2$$
 $\pi_{\mathcal{F}}(2) = 4$ $\pi_{\mathcal{F}}(3) = 7$ $\pi_{\mathcal{F}}(4) = 11$ $\pi_{\mathcal{F}}(n) = n^2/2 + n/2 + 1$

More examples:

- 1. Disks in the plane: $\pi_{\mathcal{F}}(n) = n^2 n + 2$
- 2. Balls in \mathbb{R}^3 : $\pi_{\mathcal{F}}(n) = n^3/3 n^2 + 8n/3$
- 3. Intervals in the line: $\pi_{\mathcal{F}}(n) = 2n$
- 4. Finite subsets of \mathbb{N} : $\pi_{\mathcal{F}}(n) = 2^n$
- 5. Convex polygons in the plane: $\pi_{\mathcal{F}}(n) = 2^n$

Theorem (Sauer-Shelah '72)

The shatter function is either 2^n or bounded by a polynomial.

Definition

Suppose the growth of the shatter function of \mathcal{F} is polynomial. Let $vc(\mathcal{F})$ be the infimum of all positive reals r such that

$$\pi_{\mathcal{F}}(n) = O(n^r)$$

Call $vc(\mathcal{F})$ the vc-density of \mathcal{F} . If the shatter function grows exponentially, we let $vc(\mathcal{F}) := \infty$.

Applications

- Model Theory (NIP theories)
- VC-Theorem in probability (Vapnik-Chervonenkis '71)
- Computational learning theory (PAC learning, Warmuth conjecture)
- Computational geometry
- Functional analysis (Bourgain-Fremlin-Talagrand theory)
- Abstract topological dynamics (tame dynamical systems)

History

- VC-dimension defined by Vapnik-Chervonenkis '71
- ▶ NIP theories studied by Shelah '71
- vc-density in model theoretic context introduced by Aschenbrenner, Dolich, Haskell, Macpherson, Starchenko '13

Model Theory

Model Theory studies definable sets in first-order structures.

$$(\mathbb{Q},0,1,+,\cdot,\leq)$$

$$\phi(x) := \exists y \ y \cdot y = x$$

 $\phi(\mathbb{Q})$ defines the set of rationals that are a square.

$$(\mathbb{R},0,1,+,\cdot,\leq)$$

$$\phi(x) := \exists y \ y \cdot y = x$$

 $\phi(\mathbb{R})$ defines the set $[0,\infty)$.

$$\big(\mathbb{R},0,1,+,\cdot,\leq\big)$$

$$\psi(x_1,x_2) := (x_1 \cdot x_1 + x_2 \cdot x_2 \le 1.5) \land (x_1^2 \le x_2)$$

 $\phi(\mathbb{R}^2)$ defines the set in \mathbb{R}^2 that is an intersection of a disc with an inside of a parabola.

We work with families of uniformly definable sets. Fix a formula $\phi(x_1 \dots x_m, y_1, \dots y_n) = \phi(\vec{x}, \vec{y})$. Plug in elements from M for y variables to get a family of definable sets in M^m .

$$\mathcal{F}_{\phi}^{M} = \{\phi(M^{m}, a_{1}, \dots a_{n}) \mid a_{1}, \dots a_{n} \in M\}$$

Define $\mathrm{vc}^M(\phi)$ to be the vc-density of the family \mathcal{F}_ϕ^M



$$\phi(x_1, x_2, y_1, y_2, y_3) := (x_1 - y_1)^2 + (x_2 - y_2)^2 \le y_3^2$$

In structure $(\mathbb{R},0,1+,\cdot,\leq)$ given $a,b,r\in\mathbb{R}$ the formula $\phi(x_1,x_2,a,b,r)$ defines a disk in \mathbb{R}^2 with radius r with center (a,b). Thus $\mathcal{F}_\phi^\mathbb{R}$ is a collection of all disks in \mathbb{R}^2 .

Shelah ('78) classified number of isomorphism classes for structures elementarily equivalent to structure M. One of the important classes is NIP structures. Structure M is said to be NIP if all uniformly definable families in it have finite vc-density.

- $ightharpoonup (\mathbb{C},0,1,+,\cdot)$ is NIP
- $ightharpoonup (\mathbb{R},0,1,+,\cdot,\leq)$ is NIP
- $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$ is NIP
- ▶ Random graph (V, R) is not NIP
- $(\mathbb{Q}, 0, 1, +, \cdot)$ is not NIP.

Given an NIP structure M we define a vc-function of n to be the largest vc-density achieved by families of uniformly definable sets in M^n .

$$\operatorname{vc}^{M}(n) = \max \left\{ \operatorname{vc}^{M}(\phi) \mid \phi(\vec{x}, \vec{y}) \text{ with } |\vec{x}| = n \right\}$$

Easy to show $\operatorname{vc}^M(n) \geq n \operatorname{vc}^M(1) \geq n$ Open Question: If M is NIP, is it possible for $\operatorname{vc}^M(\phi)$ to be irrational? Open Question: Is $\operatorname{vc}^M(n) = n \operatorname{vc}^M(1)$? If not, is there a linear relationship? If $\operatorname{vc}(1) < \infty$ do we have $\operatorname{vc}(2) < \infty$?

Examples

- ▶ $(\mathbb{R}, 0, 1, +, \cdot, \leq)$ has vc(n) = n (true for o-minimal structures)
- $(\mathbb{C}, 0, 1, +, \cdot)$ has vc(n) = n
- $(\mathbb{Q}_p, 0, 1, +, \cdot)$ has $vc(n) \leq 2n 1$

vc-density in trees

Consider structure (T, \leq) where elements of T are vertices of a rooted tree and we say that $a \leq b$ if a is below b in the tree.

- ► Trees are NIP (Parigot '82)
- ► Trees are dp-minimal (Simon '11)
- ▶ Trees have vc(n) = n (B. '13)

proof background

tp(a), a type of an element a is a set of all the formulas that that are true about a.

Parigot's observation: there is a natural way to split a tree into parts A, B such that for $a \in A$ and $b \in B$ we have

$$tp(a), tp(b) \vdash tp(ab)$$

This allows us to decompose complex types into simple parts, which we can use to compute vc-density.

Shelah-Spencer graphs

Let α irrational \in (0,1). Consider a random graph on n vertices where the probability of any given two vertices having an edge is $n^{-\alpha}$. Shelah-Spencer ('88) showed that 0-1 law holds for first-order formulas. A structure satisfying those axioms is called a Shelh-Spencer graph.

 Shelah-Spencer graphs are stable (Baldwin-Shi '96, Baldwin-Shelah '97)

Background

Definition

- ▶ To a finite graph A assign a dimension $\delta(A) = |V| \alpha |E|$.
- ▶ B/A is called a positive extension if quantity $\delta(B/A) = |V_B/V_A| \alpha |E_B/E_A|$ is positive.
- ▶ *B/A* is called minimal if its dimension is negative, but every subextension is positive.
- $(A_0, ... A_n)$ is a minimal chain if each A_{i+1}/A_i is minimal.

For B/A chain-minimal define

$$\phi_{A,B}(\vec{x}) = \exists \vec{x} * \text{ such that } \vec{x} * / \vec{x} \text{ is isomorphic to } B/A$$

Theorem (quantifier elimination, Laskowski '06)

In Shelah-Spencer graph every definable set can be defined by a boolean combination of formulas $\phi_{A_i,B_i(\vec{x})}$.



vc-density in Shelah-Spencer graphs

Theorem (B., '15)

For a formula $\phi(\vec{x}, \vec{y})$ we can define ϵ_L, ϵ_U explicitly computable from $\delta(B_i/A_i)$ such that

$$\epsilon_L |\vec{x}| \le \mathsf{vc}(\phi) \le \epsilon_U |\vec{x}|$$

Corollary

 $vc(1) = \infty$, so vc-function is not well-behaved for this structure.

Future work

- $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$
- Other partial orderings, lattices
- ▶ Other graph structures, in particular flat graphs