

SOME VC-DENSITY COMPUTATIONS IN SHELAH-SPENCER GRAPHS

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ABSTRACT. We compute vc-densities of minimal extension formulas in Shelah-Spencer random graphs.

We fix the density of the graph α .

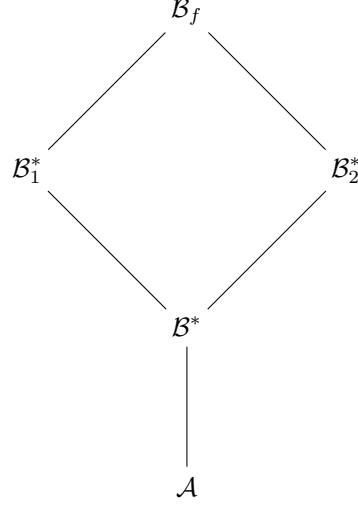
Lemma 0.1. *For any $\mathcal{A} \in K_\alpha$ and $\epsilon > 0$ there exists an \mathcal{B} such that $(\mathcal{A}, \mathcal{B})$ is minimal and $\delta(\mathcal{B}/\mathcal{A}) < \epsilon$.*

Proof. Let m be an integer such that $m\alpha < 1 < (m+1)\alpha$. Suppose \mathcal{A} has less than $m+1$ vertices. Make a construction $\mathcal{A}_0 = \mathcal{A}$ and \mathcal{A}_{i+1} is \mathcal{A}_i with one extra vertex connected to every single vertex of \mathcal{A}_i . Stop when the total number of vertices is $m+1$. Proceed as in [1] 4.1. Resulting construction is still minimal. \square

Lemma 0.2. *Let $\mathcal{A}_1 \subset \mathcal{B}_1$ and $\mathcal{A}_2 \subset \mathcal{B}_2$ be K_α structures with $(\mathcal{A}_2, \mathcal{B}_2)$ a minimal pair with $\epsilon = \delta(\mathcal{B}_2/\mathcal{A}_2)$. Let M be some ambient structure. Fix embeddings of $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2$ into M . Assume that it is not the case that $\mathcal{A}_2 \subset \mathcal{B}_2$ and \mathcal{A}_1 is disjoint from \mathcal{A}_2 . Now consider all possible embeddings $f: \mathcal{B}_2 \rightarrow M$ over \mathcal{A}_1 . Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{B}_f = \mathcal{B}_1 \cup f(\mathcal{B}_2)$ with $\delta_f = \delta(\mathcal{B}_f/\mathcal{A})$. Then δ_f is at most $\delta(\mathcal{B}_1 \cup \mathcal{A}/\mathcal{A}) + \epsilon$.*

Fix an embedding f . It induces the following substructure diagram in M . Denote

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_1 \cup \mathcal{A}_2 \\ \mathcal{B}_f^* &= \mathcal{B}_1 \cup f(\mathcal{B}_2) \\ \mathcal{B}_1^* &= \mathcal{B}_1 \cup \mathcal{A} \\ \mathcal{B}_2^* &= f(\mathcal{B}_2) \cup \mathcal{A} \\ \mathcal{B}^* &= (\mathcal{B}_1 \cap f(\mathcal{B}_2)) \cup \mathcal{A}\end{aligned}$$



From the diagram we see that

$$\delta(\mathcal{B}_f/\mathcal{A}) \leq \delta(\mathcal{B}_1^*/\mathcal{A}) + \delta(\mathcal{B}_2^*/\mathcal{B}^*)$$

Thus all we need to do is to verify that

$$\delta(\mathcal{B}_2^*/\mathcal{B}^*) \leq \epsilon$$

Let \mathcal{B}' denote graph induced on all the vertices in $(f(B_2)/B_1) \cup A_2$. Then \mathcal{B}' is a substructure of \mathcal{B}_2 over \mathcal{A}_2 . By minimality we get that $\delta(\mathcal{B}'/\mathcal{A}_2) \leq \epsilon$. Moreover $\delta(\mathcal{B}_2^*/\mathcal{B}^*) \leq \delta(\mathcal{B}'/\mathcal{A}_2)$ thus proving the claimed statement. Then $\delta(\mathcal{B}^*/\mathcal{A}_2)$ has to be less than $\delta(\mathcal{B}_2/\mathcal{A}_2)$ by minimality of $(\mathcal{B}_2, \mathcal{A}_2)$. Relative dimension of the whole construction has to be even smaller. It is easy to show that this construction induces a proper subpair in $(\mathcal{A}_2, \mathcal{B}_2)$ which has to have smaller dimension.

Let $\phi(x, y)$ be a formula in a random graph with $|x| = |y| = 1$ saying that there exists a minimal extension M over $\{x, y\}$ of relative dimension ϵ . Let n be such that $n\epsilon < 1 < (n+1)\epsilon$. Then we argue that $vc(\phi) = n$.

Fix a m -strong (for any $m > |M|$) set of non-connected vertices B . Fix some a . We investigate the trace of $\phi(x, a)$ on B . Suppose we have b_1, \dots, b_k satisfying $\phi(b_i, a)$ as witnessed by M_j . Relative dimension of $M_1 \cup M_2 \cup \dots \cup M_j \cup a$ is minimized when all M_j are disjoint (by minimality). Thus for that dimension to be positive we can have at most n extensions.

REFERENCES

- [1] Michael C. Laskowski, *A simpler axiomatization of the Shelah-Spencer almost sure theories*, Israel J. Math. **161** (2007), 157-186. MR MR2350161
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