

# QUANTIFIER ELIMINATION IN SHELAH-SPENCER GRAPHS

ANTON BOBKOV

ABSTRACT. We simplify [?] proof of quantifier elimination in Shelah-Spencer graphs.

## 1. INTRODUCTION

Laskowski's paper [?] provides a combinatorial proof of quantifier elimination in Shelah-Spencer graphs. Here we provide a simplification of the proof using only maximal chains and avoiding the use of proposition 3.1 and technical lemmas of section 4.

We will use notation of [?], in particular things like  $\mathbf{K}_\alpha$ ,  $\delta(\mathcal{A}/\mathcal{B})$ ,  $X_m(\mathcal{A})$ ,  $S_\alpha$ , maximal embedding, etc.

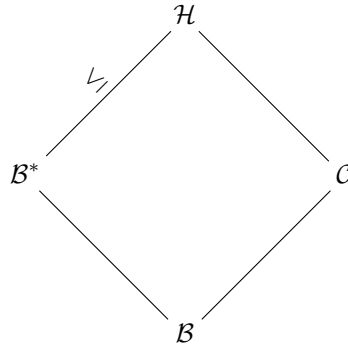
## 2. OMITTING LEMMA

**Definition 2.1.** Let  $\mathcal{M} \models S_\alpha$ ,  $\mathcal{B} \in \mathbf{K}_\alpha$ , embedding  $f: \mathcal{B} \rightarrow \mathcal{M}$ ,  $\Phi$  finite subset of  $\mathbf{K}_\alpha$

- (1) Say that  $f$  *omits*  $\Phi$  if there are no  $\mathcal{C} \in \Phi$  and  $g: \mathcal{C} \rightarrow \mathcal{M}$  extending  $f$ .
- (2) Say that  $f$  *admits*  $\Phi$  if for every  $\mathcal{C} \in \Phi$  there is  $g: \mathcal{C} \rightarrow \mathcal{M}$  extending  $f$ .

**Note 2.2.** Take notation as above and a structure  $\mathcal{C} \in \mathbf{K}_\alpha$  extending  $\mathcal{B}$ . Then  $f$  doesn't omit  $\{\mathcal{C}\}$  iff  $f$  admits  $\{\mathcal{C}\}$ .

**Definition 2.3.** Fix  $\mathcal{B}, \mathcal{C} \in \mathbf{K}_\alpha$ , and  $m \in \omega$  such that  $|C \setminus B| < m$ . Define  $Z(\mathcal{B}, \mathcal{C}, m)$  to be all  $\mathcal{B}^* \in X_m(\mathcal{B})$  such that there are no  $\mathcal{H}$  with  $|H \setminus B^*| < m$  satisfying

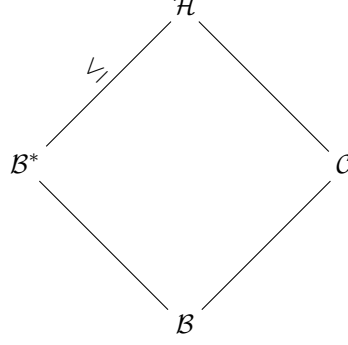


**Lemma 2.4.** Let  $\mathcal{B}, \mathcal{C} \in \mathbf{K}_\alpha$ , and  $m \in \omega$  such that  $|C \setminus B| < m$ . Also let  $\mathcal{M} \models S_\alpha$  and  $f: \mathcal{B} \rightarrow \mathcal{M}$  an embedding. The following are equivalent:

- (1)  $f$  *omits*  $\{\mathcal{C}\}$ .
- (2) There exists  $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$  maximally embeddable into  $\mathcal{M}$  over  $f$ .

*Proof.* For the proof we identify  $\mathcal{B}$  with  $f(\mathcal{B})$ , i.e. for ease of notation assume that  $\mathcal{B} \subset \mathcal{M}$ .

(1)  $\Rightarrow$  (2) By remark 5.3 of [?] there is some  $B^* \in X_m(\mathcal{B})$  maximally embeddable in  $\mathcal{M}$  over  $f$ . Such embedding is unique by Lemma 3.8 of [?]. Again, we identify  $B^*$  with its maximal embedding into  $\mathcal{M}$ . To show (2) we need to verify that  $B^* \in Z(\mathcal{B}, \mathcal{C}, m)$ . Suppose not. Then there is  $\mathcal{H}$  with  $|H \setminus B^*| < m$  satisfying



As  $B^* \leq \mathcal{H}$  and  $\mathcal{B} \subset \mathcal{M}$  we can embed  $\mathcal{H}$  into  $\mathcal{M}$  (as  $\mathcal{M} \models S_\alpha$ ). But this would witness  $\mathcal{C}$  extending  $\mathcal{B}$  in  $\mathcal{M}$  which is impossible as we assumed that  $f$  omits  $\{\Phi\}$ .

(2)  $\Rightarrow$  (1) Suppose  $f$  doesn't omit  $\{C\}$ . Then by the note above  $f$  admits  $\{C\}$ , i.e. there is an embedding of  $\mathcal{C}$  into  $\mathcal{M}$  over  $f$ . We identify  $\mathcal{C}$  with the image of that embedding. Similarly we identify  $B^*$  with the image of its maximal embedding over  $f$ . That is we may assume  $\mathcal{C}, B^* \subset \mathcal{M}$ . Let  $H$  be the substructure of  $\mathcal{M}$  induced by vertices  $C \cup B^*$ . As  $|C \setminus B| < m$  we have  $|H \setminus B^*| < m$ .  $B^*$  is  $m$ -strong by remark 5.3 of [?]. This forces  $B^* \leq H$ . But this contradicts the fact that  $B^* \in Z(\mathcal{B}, \mathcal{C}, m)$ .  $\square$

**Corollary 2.5.** *With the setup of the previous lemma, the following are equivalent:*

- (1)  $f$  admits  $\{C\}$ .
- (2) There exists  $B^* \in X_m(\mathcal{B}) \setminus Z(\mathcal{B}, \mathcal{C}, m)$  maximally embeddable into  $\mathcal{M}$  over  $f$ .

For quantifier elimination we need to track multiple structures being admitted and omitted, hence the following definition.

**Definition 2.6.** Let  $\mathcal{B} \in \mathbf{K}_\alpha$ ,  $\Phi, \Gamma$  finite subsets of  $\mathbf{K}_\alpha$ , and  $m \in \omega$  such that for each  $\mathcal{C} \in \Phi$  or  $\mathcal{C} \in \Gamma$  we have  $\mathcal{B} \subseteq \mathcal{C}$  and  $|C \setminus B| < m$ . Define

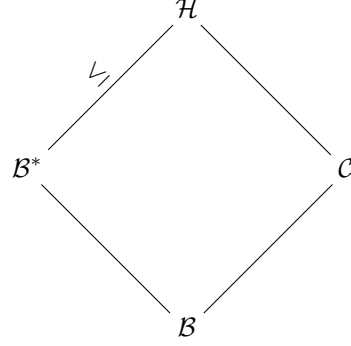
$$Y(\mathcal{B}, \Phi, \Gamma, m) = \{B^* \in X_m(\mathcal{B}) \mid \forall \mathcal{C} \in \Phi \ B^* \in Z(\mathcal{B}, \mathcal{C}, m) \text{ and } \forall \mathcal{D} \in \Gamma \ B^* \notin Z(\mathcal{B}, \mathcal{D}, m)\}$$

**Lemma 2.7.** *Let  $\mathcal{B} \in \mathbf{K}_\alpha$ ,  $\Phi, \Gamma$  finite subsets of  $\mathbf{K}_\alpha$ , and  $m \in \omega$  such that for each  $\mathcal{C} \in \Phi$  or  $\mathcal{C} \in \Gamma$  we have  $\mathcal{B} \subseteq \mathcal{C}$  and  $|C \setminus B| < m$ . The following are equivalent:*

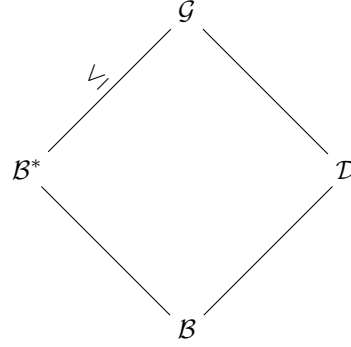
- (1)  $f$  omits  $\Phi$  and admits  $\Gamma$ .
- (2) There exists  $B^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$  maximally embeddable into  $\mathcal{M}$  over  $f$ .

*Proof.* (1)  $\Rightarrow$  (2) Identify  $\mathcal{B}$  with  $f(\mathcal{B})$ , i.e. for ease of notation assume that  $\mathcal{B} \subset \mathcal{M}$ . By remark 5.3 of [?] there is some  $B^* \in X_m(\mathcal{B})$  maximally embeddable in  $\mathcal{M}$  over  $f$ . Such embedding is unique by Lemma 3.8 of [?]. Again, we identify  $B^*$  with its maximal embedding into  $\mathcal{M}$ . To show (2) we need to verify that  $B^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$ .

Suppose not. Two things can go wrong. First, there can be  $\mathcal{H}$  with  $|H \setminus B^*| < m$  and  $\mathcal{C} \in \Phi$  satisfying



As  $\mathcal{B}^* \leq \mathcal{H}$  and  $\mathcal{B} \subset \mathcal{M}$  we can embed  $\mathcal{H}$  into  $\mathcal{M}$  (as  $\mathcal{M} \models S_\alpha$ ). But this would witness  $\mathcal{C}$  extending  $\mathcal{B}$  in  $\mathcal{M}$  which is impossible as we assumed that  $f$  omits  $\Phi$ . Another thing that could go wrong is that there could be  $\mathcal{D} \in \Gamma$  and no  $\mathcal{G}$  with  $|G \setminus B^*| < m$  satisfying



As  $f$  admits

□

#### REFERENCES

- [1] Klaus-Peter Podewski and Martin Ziegler. Stable graphs. *Fund. Math.*, 100:101-107, 1978.
- [2] Aharoni, Ron and Berger, Eli (2009). "Menger's Theorem for infinite graphs". *Inventiones Mathematicae* 176: 162
- [3] P. Simon, *On dp-minimal ordered structures*, J. Symbolic Logic 76 (2011), no. 2, 448460.  
E-mail address: bobkov@math.ucla.edu