1. VC-dimension

Suppose we have an infinite collection of sets $\mathcal{F}$. Take n many of those sets. They generate a boolean algebra. Count the number of atoms in it. There can be at most 2^n atoms, though depending on the collection there may be much less. For a given n, out of all choices of n sets, record the highest possible number of atoms generated. We define that to be a shatter function.

Definition 1.1.

$$\pi_{\mathcal{F}}(n) = \max\{\# \text{ of atoms in boolean algebra generated by } S \mid S \subset \mathcal{F} \text{ and } |S| = n\}$$

Example 1.2.

(1) Let $\mathcal{F}$ be a set of lines on a plane. Then
$$\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$$
(2) Let $\mathcal{F}$ be a set of disks on a plane. Then
$$\pi_{\mathcal{F}}(n) = n^2 - n + 2$$
(3) Let $\mathcal{F}$ be a set of balls in $\mathbb{R}^3$. Then
$$\pi_{\mathcal{F}}(n) = n(n^2 - 3n + 8)/3$$
(4) Let $\mathcal{F}$ be a set of intervals on a line. Then
$$\pi_{\mathcal{F}}(n) = 2n$$
(5) Let $\mathcal{F}$ be a set of half-planes. Then
$$\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$$
(6) Let $\mathcal{F}$ be a collection of finite subsets of $\mathbb{N}$. Then
$$\pi_{\mathcal{F}}(n) = 2^n$$
(7) Let $\mathcal{F}$ be a collection of polygons in a plane. Then
$$\pi_{\mathcal{F}}(n) = 2^n$$

Theorem 1.3 (Sauer-Shelah)**.** *Shatter function is either 2^n or bounded by a polynomial.*

Definition 1.4. Families of sets with polynomially bounded shatter functions are said to have a finite VC-dimension.

Definition 1.5. Suppose $\mathcal{F}$ has a finite VC-dimension. Let k be the smallest real such that
$$\pi_{\mathcal{F}}(n) = O(n^k)$$
We define such k to be the vc-density of $\mathcal{F}$.

2. Model Theory

Consider a structure with a language
$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

We work with subsets of the underlying set definable by first-order formulas. Those are called definable sets.

$$\phi(x) = 5 \leq x \leq 7.7 \vee x \leq 0$$
$$\psi(x) = \exists y \; y \cdot y = x$$
$$\gamma(x) = x \cdot x \cdot x \cdot x = 2$$

$\phi(x)$ defines the set $[5, 7.7] \cup (-\infty, 0]$ in the structure above. $\psi(x)$ defines the set $[0, \infty)$ in the structure above.

(1) in $(\mathbb{Q}, \cdot)$ $\gamma(x)$ defines an empty subset
(2) in $(\mathbb{R}, \cdot)$ $\gamma(x)$ defines a subset with two elements
(3) in $(\mathbb{C}, \cdot)$ $\gamma(x)$ defines a subset with four elements
(4) in $(\mathbb{H}, \cdot)$ $\gamma(x)$ defines an infinite subset

$$\theta(x) = \forall y \exists z \; x \leq z \leq y$$

(1) in $(\mathbb{Q}, \leq)$ $\theta(x)$ defines an empty subset
(2) in $(\mathbb{N}, \leq)$ $\theta(x)$ defines an empty subset
(3) in $(\mathbb{Q}^{\geq 0}, \leq)$ $\theta(x)$ defines the set $\{0\}$

Definition 2.1. for a formula $\phi(x_1 \ldots x_n, y_1, \ldots y_n)$ we can plug in elements of our structure as parameters in places of y variables. This gives us a collection of definable sets.

Example 2.2.

$$\phi(x_1, x_2, y_1, y_2, y_3) = (x_1 - y_1) \cdot (x_1 - y_1) + (x_2 - y_2) \cdot (x_2 - y_2) \leq y_3 \cdot y_3$$