# VC-density in model theoretic structures

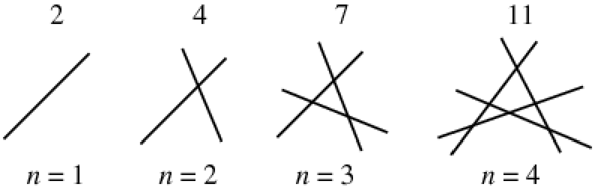
Anton Bobkov

June 3, 2015

Suppose we have an (infinite) collection of sets $\mathcal{F}$.
We define the <u>shatter function</u> $\pi_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$

$$\pi_{\mathcal{F}}(n) = \max\{\#\text{ of atoms in the boolean algebra generated by } \mathcal{S} \mid \mathcal{S} \subset \mathcal{F} \text{ with } |\mathcal{S}| = n\}$$

Example: Let $\mathcal{F}$ consist of all half-planes in the plane.




$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 7 \quad \pi_{\mathcal{F}}(4) = 11$$

$$\pi_{\mathcal{F}}(n) = n^2/2 + n/2 + 1$$

More examples:

1. Disks in the plane: $\pi_{\mathcal{F}}(n) = n^2 - n + 2$
2. Balls in $\mathbb{R}^3$: $\pi_{\mathcal{F}}(n) = n^3/3 - n^2 + 8n/3$
3. Intervals in the line: $\pi_{\mathcal{F}}(n) = 2n$
4. Finite subsets of $\mathbb{N}$: $\pi_{\mathcal{F}}(n) = 2^n$
5. Convex polygons in the plane: $\pi_{\mathcal{F}}(n) = 2^n$

### Theorem (Sauer-Shelah '72)

*The shatter function is either $2^n$ or bounded by a polynomial.*

### Definition

Suppose the growth of the shatter function of $\mathcal{F}$ is polynomial. Let $\mathrm{vc}(\mathcal{F})$ be the infimum of all positive reals $r$ such that

$$\pi_{\mathcal{F}}(n) = O(n^r)$$

Call $\mathrm{vc}(\mathcal{F})$ the vc-density of $\mathcal{F}$. If the shatter function grows exponentially, we let $\mathrm{vc}(\mathcal{F}) := \infty$.

# Applications

- Model Theory (NIP theories)
- VC-Theorem in probability (Vapnik-Chervonenkis '71)
- Computational learning theory (PAC learning, Warmuth conjecture)
- Computational geometry
- Functional analysis (Bourgain-Fremlin-Talagrand theory)
- Abstract topological dynamics (tame dynamical systems)

# History

- VC-dimension defined by Vapnik-Chervonenkis '71
- NIP theories studied by Shelah '71
- vc-density in model theoretic context introduced by Aschenbrenner, Dolich, Haskell, Macpherson, Starchenko '13

# Model Theory

Model Theory studies definable sets in first-order structures.

$$(\mathbb{Q}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) := (\exists y \;\; y \cdot y = x)$$

$\phi(\mathbb{Q})$ defines the set of rationals that are a square.

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) := (\exists y \ y \cdot y = x)$$

$\phi(\mathbb{R})$ defines the set $[0, \infty)$.

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\psi(x_1, x_2) := (x_1 \cdot x_1 + x_2 \cdot x_2 \leq 1.5) \wedge (x_1^2 \leq x_2)$$

$\phi(\mathbb{R}^2)$ defines the set in $\mathbb{R}^2$ that is an intersection of a disc with an inside of a parabola.

We work with families of uniformly definable sets. Fix a formula $\phi(x_1 \dots x_m, y_1, \dots y_n) = \phi(\vec{x}, \vec{y})$. Plug in elements from $M$ for $y$ variables to get a family of definable sets in $M^m$.

$$\mathcal{F}_\phi^M = \{\phi(M^m, a_1, \dots a_n) \mid a_1, \dots a_n \in M\}$$

Define $\mathrm{vc}^M(\phi)$ to be the vc-density of the family $\mathcal{F}_\phi^M$

$$\phi(x_1, x_2, y_1, y_2, y_3) := (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq y_3^2$$

In structure $(\mathbb{R}, 0, 1+, \cdot, \leq)$ given $a, b, r \in \mathbb{R}$ the formula $\phi(x_1, x_2, a, b, r)$ defines a disk in $\mathbb{R}^2$ with radius $r$ with center $(a, b)$. Thus $\mathcal{F}_\phi^\mathbb{R}$ is a collection of all disks in $\mathbb{R}^2$.

Shelah ('78) classified number of isomorphism classes for structures elementarily equivalent to structure $M$. One of the important classes is NIP structures. Structure $M$ is said to be NIP if all uniformly definable families in it have finite vc-density.

- $(\mathbb{C}, 0, 1, +, \cdot)$ is NIP
- $(\mathbb{R}, 0, 1, +, \cdot, \leq)$ is NIP
- $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$ is NIP
- Random graph $(V, R)$ is not NIP
- $(\mathbb{Q}, 0, 1, +, \cdot)$ is not NIP.

Given an NIP structure $M$ we define a vc-function of $n$ to be the largest vc-density achieved by families of uniformly definable sets in $M^n$.

$$\mathrm{vc}^M(n) = max\left\{\mathrm{vc}^M(\phi) \mid \phi(\vec{x},\vec{y}) \text{ with } |\vec{x}| = n\right\}$$

Easy to show $\mathrm{vc}^M(n) \geq n\,\mathrm{vc}^M(1) \geq n$
Open Question: If $M$ is NIP, is it possible for $\mathrm{vc}^M(\phi)$ to be irrational? Open Question: Is $\mathrm{vc}^M(n) = n\,\mathrm{vc}^M(1)$? If not, is there a linear relationship? If $\mathrm{vc}(1) < \infty$ do we have $\mathrm{vc}(2) < \infty$?

Examples

- $(\mathbb{R}, 0, 1, +, \cdot, \leq)$ has $\mathrm{vc}(n) = n$ (true for o-minimal structures)
- $(\mathbb{C}, 0, 1, +, \cdot)$ has $\mathrm{vc}(n) = n$
- $(\mathbb{Q}_p, 0, 1, +, \cdot)$ has $\mathrm{vc}(n) \leq 2n - 1$

# vc-density in trees

Consider structure $(T, \leq)$ where elements of $T$ are vertices of a rooted tree and we say that $a \leq b$ if $a$ is below $b$ in the tree.

- Trees are NIP (Parigot '82)
- Trees are dp-minimal (Simon '11)
- Trees have $vc(n) = n$ (B. '13)

# proof background

tp($a$), a type of an element $a$ is a set of all the formulas that that are true about $a$.
Parigot's observation: there is a natural way to split a tree into parts $A, B$ such that for $a \in A$ and $b \in B$ we have

$$\mathrm{tp}(a), \mathrm{tp}(b) \vdash \mathrm{tp}(ab)$$

This allows us to decompose complex types into simple parts, which we can use to compute vc-density.

# Shelah-Spencer graphs

Let $\alpha$ irrational $\in (0,1)$. Consider a random graph on $n$ vertices where the probability of any given two vertices having an edge is $n^{-\alpha}$. Shelah-Spencer ('88) showed that 0-1 law holds for first-order formulas. A structure satisfying those axioms is called a Shelh-Spencer graph.

- Shelah-Spencer graphs are stable (Baldwin-Shi '96, Baldwin-Shelah '97)

# Background

## Definition

- To a finite graph $A$ assign a dimension $\delta(A) = |V| - \alpha|E|$.
- $B/A$ is called a positive extension if quantity $\delta(B/A) = |V_B/V_A| - \alpha|E_B/E_A|$ is positive.
- $B/A$ is called minimal if its dimension is negative, but every subextension is positive.
- $(A_0, \dots A_n)$ is a minimal chain if each $A_{i+1}/A_i$ is minimal.

For $B/A$ chain-minimal define

$$\phi_{A,B}(\vec{x}) = \exists \vec{x}* \text{ such that } \vec{x}*/\vec{x} \text{ is isomorphic to } B/A$$

## Theorem (quantifier elimination, Laskowski '06)

*In Shelah-Spencer graph every definable set can be defined by a boolean combination of formulas* $\phi_{A_i,B_i(\vec{x})}$.

# vc-density in Shelah-Spencer graphs

### Theorem (B., '15)

*For a formula* $\phi(\vec{x}, \vec{y})$ *we can define* $\epsilon_L, \epsilon_U$ *explicitly computable from* $\delta(B_i/A_i)$ *such that*

$$\epsilon_L|\vec{x}| \leq \mathrm{vc}(\phi) \leq \epsilon_U|\vec{x}|$$

### Corollary

$\mathrm{vc}(1) = \infty$, *so vc-function is not well-behaved for this structure.*

# Future work

- $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$
- Other partial orderings, lattices
- Other graph structures, in particular flat graphs