# VC-DENSITY IN AN ADDITIVE REDUCT OF $p$-ADIC NUMBERS

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Abstract. Aschenbrenner et. al. computed a bound $\mathrm{vc}(n) \leq 2n - 1$ for the VC density function in the field of $p$-adic numbers, but it is not known to be optimal. I investigate a certain $P$-minimal additive reduct of the field of $p$-adic numbers and use a cell decomposition result of Leenknegt to compute an optimal bound $\mathrm{vc}(n) = n$ for that structure.

VC density was introduced into model theory in [1] by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko as a natural notion of dimension for definable families of sets in NIP theories. In a NIP theory $T$ we can define the vc-function

$$\mathrm{vc}_T = \mathrm{vc} : \mathbb{N} \longrightarrow \mathbb{N}$$

where $\mathrm{vc}(n)$ measures the worst-case complexity of families of definable sets in an $n$-dimensional space. The simplest possible behavior is $\mathrm{vc}(n) = n$ for all $n$. For $T = \mathrm{Th}(\mathbb{Q}_p)$, the paper [1] computes an upper bound for this function to be $2n - 1$, and it is not known whether it is optimal. This same bound would hold in any reduct of the field of $p$-adic numbers, so one may expect that the simplified structure of the reduct would allow a better bound. In [2], Leenknegt provides a cell decomposition result for a certain P-minimal additive reduct of the field $p$-adic numbers. Using this result, in this paper we improve the bound for the VC function, showing that in Leenknegt's structure $\mathrm{vc}(n) = n$.

*Explain organization of this paper, notation*

## 1. VC-dimension and VC-density

**Definition 1.1.** Throughout this section we work with a collection $\mathcal{F}$ of subsets of a set $X$. We call the pair $(X, \mathcal{F})$ a set system.

- Given a subset $A$ of $X$, we define the set system $(A, A \cap \mathcal{F})$ where $A \cap \mathcal{F} = \{A \cap F\}_{F \in \mathcal{F}}$.
- For $A \subset X$ we say that $\mathcal{F}$ shatters $A$ if $A \cap \mathcal{F} = \mathcal{P}(A)$.

**Definition 1.2.** We say $(X, \mathcal{F})$ has VC-dimension $n$ if the largest subset of $X$ shattered by $\mathcal{F}$ is of size $n$. If $\mathcal{F}$ shatters arbitrarily large subsets of $X$, we say that $(x, \mathcal{F})$ has infinite VC-dimension. We denote the VC-dimension of $(X, \mathcal{F})$ by $\mathrm{VC}(\mathcal{F})$.

**Note 1.3.** We may drop $X$ from the previous definition, as it VC-dimension doesn't depend on the base set and is determined by $(\bigcup \mathcal{F}, \mathcal{F})$.

This allows us to distinguish between well behaved set systems of finite VC-dimension which tend to have good combinatorial properties and poorly behaved set systems with infinite VC dimension.

Another natural combinatorial notion is that of a dual system:

**Definition 1.4.** For $a \in X$ define $X_a = \{F \in \mathcal{F} \mid a \in F\}$. Let $\mathcal{F}^* = \{X_a\}_{a \in X}$. We define $(\mathcal{F}, \mathcal{F}^*)$ as the dual system of $(X, \mathcal{F})$. The VC-dimension of the dual system of $(X, \mathcal{F})$ is referred to as the dual VC-dimension of $(X, \mathcal{F})$ and denoted by $\mathrm{VC}^*(\mathcal{F})$. (As before, this notion doesn't depend on $X$.)

**Lemma 1.5.** *A set system has finite VC-dimension if and only if its dual system has finite VC-dimension. More precisely*

$$\mathrm{VC}^*(\mathcal{F}) \leq 2^{1+\mathrm{VC}(\mathcal{F})}.$$

For a more refined notion we look at the traces of our family on finite sets:

**Definition 1.6.** Define the shatter function $\pi_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ and the dual shatter function $\pi^*_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ by

$\pi_{\mathcal{F}}(n) = \max\{|A \cap \mathcal{F}| \mid A \subset X \text{ and } |A| = n\}$

$\pi^*_{\mathcal{F}}(n) = \max\{\text{number of atoms in Boolean algebra generated by } B \mid B \subset \mathcal{F}, |B| = n\}$

Note that the dual shatter function is precisely the shatter function of the dual system: $\pi^*_{\mathcal{F}} = \pi_{\mathcal{F}^*}$

A simple upper bound is $\pi_{\mathcal{F}}(n) \leq 2^n$ (same for the dual). If VC-dimension is infinite then clearly $\pi_{\mathcal{F}}(n) = 2^n$ for all $n$. Conversely we have the following remarkable fact:

**Theorem 1.7** (Sauer-Shelah '72)**.** *If the set system $(X, \mathcal{F})$ has finite VC-dimension $d$ then $\pi_{\mathcal{F}}(n) \leq \binom{n}{\leq d}$ where $\binom{n}{\leq d} = \binom{n}{d} + \binom{n}{d-1} + \ldots + \binom{n}{1}$.*

Thus the systems with a finite VC-dimension are precisely the systems where the shatter function grows polynomially. Define VC-density to be the degree of that polynomial:

**Definition 1.8.** Define vc-density and dual vc-density of $\mathcal{F}$ as

$$\mathrm{vc}(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}$$
$$\mathrm{vc}^*(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}$$

Generally speaking a shatter function that is bounded by a polynomial doesn't itself have to be a polynomial. Proposition 4.12 in [1] gives an example of a shatter function that grows like $n \log n$ (so it has VC-density 1).

So far the notions that we have defined are purely combinatorial. We now adapt VC-dimension and VC-density to the model theoretic context.

**Definition 1.9.** Work in a structure $M$. Fix a finite collection of formulas $\Phi(x, y) = \{\phi_i(x, y)\}$.

- For $\phi(x,y) \in \mathcal{L}(M)$ and $b \in M^{|y|}$ let $\phi(M^{|x|}, b) = \{a \in M^{|x|} \mid \phi(a,b)\} \subseteq M^{|x|}$.
- Let $\Phi(M^{|x|}, M^{|y|}) = \{\phi_i(M^{|x|}, b) \mid \phi_i \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|})$.
- Let $\mathcal{F}_\Phi = \Phi(M^{|x|}, M^{|y|})$ giving a set system $(M^{|x|}, \mathcal{F}_\Phi)$.
- Define VC-dimension of $\Phi$, $\mathrm{VC}(\Phi)$ to be the dual VC-dimension of $(M^{|x|}, \mathcal{F}_\Phi)$.
- Define VC-density of $\Phi$, $\mathrm{vc}(\Phi)$ to be the dual VC-density of $(M^{|x|}, \mathcal{F}_\Phi)$.

We will also refer to the VC-density and VC-dimension of a single formula $\phi$ viewing it as a one element collection $\{\phi\}$.

Counting atoms of a Boolean algebra in a model theoretic setting corresponds to counting types, so it is instructive to rewrite the shatter function in terms of types.

**Definition 1.10.**

$$\pi_\Phi(n) = \max\{\text{number of } \Phi\text{-types over } B \mid B \subset M, |B| = n\}$$

**Lemma 1.11.**

$$\mathrm{vc}(\Phi) = \textit{degree of polynomial growth of } \pi_\Phi(n) = \limsup_{n\to\infty} \frac{\log \pi_\Phi(n)}{\log n}$$

One can check that the shatter function and hence VC-dimension and VC-density of a formula are elementary notions, so they only depend on the first-order theory of the structure.

NIP theories are a natural context for studying VC-density. In fact we can take the following as the definition of NIP:

**Definition 1.12.** Define $\phi$ to be NIP if it has finite VC-dimension.

[**?**] shows that in a general combinatorial context, VC-density can be any real number in $0 \cup [1, \infty)$. Less is known if we restrict our attention to NIP theories. Proposition 4.6 in [1] gives examples of formulas that have non-integer rational VC-density in an NIP theory, however it is open whether one can get an irrational VC-density in this context.

In general, instead of working with a theory formula by formula, we can look for a uniform bound for all formulas:

**Definition 1.13.** For a given NIP structure $M$, define the vc-function

$$\mathrm{vc}^M(n) = \sup\{\mathrm{vc}(\phi(x,y)) \mid \phi \in \mathcal{L}(M), |x| = n\}$$

As before this definition is elementary, so it only depends on the theory of $M$. We omit the superscript $M$ if it is understood from the context. One can easily check the following bounds:

**Lemma 1.14** (Lemma 3.22 in [1])**.**

$$\mathrm{vc}(1) \geq 1$$
$$\mathrm{vc}(n) \geq n\,\mathrm{vc}(1)$$

However, it is not known whether the second inequality can be strict or even whether $\mathrm{vc}(1) < \infty$ implies $\mathrm{vc}(n) < \infty$.

## 2. $P$-adic numbers

The field of $p$-adic numbers is often studied in the language of Macintyre $\mathcal{L}_{Mac} = \{0, 1, +, -, \cdot, |, P_n\}$. which is a language of fields together with unary predicates $\{P_n\}_{n\in\mathbb{N}}$ interpreted in $\mathbb{Q}_p$ by

$$P_n x \leftrightarrow \exists y\ y^n = x$$

and a divisibility relation where $a|b$ holds when $\mathrm{val}\, a \leq \mathrm{val}\, b$.

Note that $P_n \setminus \{0\}$ is a multiplicative subgroup of $\mathbb{Q}_p$ with finitely many cosets.

**Theorem 2.1** (Macintyre '76)**.** *The $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$ has quantifier elimination.*

There is also a cell decomposition result.

**Definition 2.2.** Define <u>$n$-cell</u> recursively. 0-cells are points in $\mathbb{Q}_p$. An $n+1$-cell is a subset of $\mathbb{Q}_p^{n+1}$ of the following form:

$$\{(x,t) \in \mathbb{Q}_p \times D \mid \operatorname{val} a_1(x) \,\square_1\, \operatorname{val}(t-c(x)) \,\square_2\, \operatorname{val} a_2(x), t-c(x) \in \lambda P_n\}$$

where $D$ is an $n$-cell, $a_1(x), a_2(x), c(x)$ are $\emptyset$-definable, $\square$ is $<, \leq$ or no condition, and $\lambda \in \mathbb{Q}_p$.

**Theorem 2.3** (Denef '84). *Any subset of $\mathbb{Q}_p$ defined by a $\mathcal{L}_{Mac}$-formula $\phi(x,t)$ with $|t| = 1$ and $|x| = n$ decomposes into a finite union of $n+1$-cells.*

In [1], Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko show that this structure has $\operatorname{vc}(n) \leq 2n-1$, however it is not known whether this bound is optimal.

In [2], Leenknegt analyzes the reduct of $p$-adic numbers to the language

$$\mathcal{L}_{aff} = \left\{0, 1, +, -, \{\bar{c}\}_{c \in \mathbb{Q}_p}, |, \{Q_{m,n}\}_{m,n \in \mathbb{N}}\right\}$$

where $\bar{c}$ is a scalar multiplication by $c$, $a|b$ stands for $\operatorname{val} a \leq \operatorname{val} b$, and $Q_{m,n}$ is a unary predicate

$$Q_{m,n} = \bigcup_{k \in \mathbb{Z}} p^{km}(1 + p^n \mathbb{Z}_p).$$

Note that $Q_{m,n}$ is a subgroup of the multiplicative group of $\mathbb{Q}_p$ with finitely many cosets. One can check that the extra relation symbols are definable in the $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$. The paper [2] provides a cell decomposition result with the following cells:

**Definition 2.4.** A 0-cell is a point in $\mathbb{Q}_p$. An $n+1$-cell is a subset of $\mathbb{Q}_p^{n+1}$ of the following form:

$$\{(x,t) \in K \times D \mid \operatorname{val} a_1(x) \,\square_1\, \operatorname{val}(t-c(x)) \,\square_2\, \operatorname{val} a_2(x), t-c(x) \in \lambda Q_{m,n}\}$$

where $D$ is an $n$-cell called the <u>base</u> of the cell, $a_1(x), a_2(x), c(x)$ are linear polynomials, $\square$ is $<$ or no condition, and $\lambda \in \mathbb{Q}_p$.

**Theorem 2.5** (Leenknegt '12). *Any formula $\phi(x,t)$ in $(\mathbb{Q}_p, \mathcal{L}_{aff})$ with $|t| = 1$ and $|x| = n$ decomposes into a union of $n+1$-cells.*

Moreover, [2] shows that $(\mathbb{Q}_p, \mathcal{L}_{aff})$ is a $P$-minimal reduct, that is the one-dimensional definable sets of $(\mathbb{Q}_p, \mathcal{L}_{aff})$ coincide with the one-dimensional definable sets in the full structure $(\mathbb{Q}_p, \mathcal{L}_{Mac})$.

I am able to compute the vc-function for this structure

**Theorem 2.6.** *Theorem (B.) $(\mathbb{Q}_p, \mathcal{L}_{aff})$ has* $\mathrm{vc}(n) = n$.

## 3. Key Lemmas and Definitions

To show that $\mathrm{vc}(n) = n$ it suffices to bound $\mathrm{vc}(\phi) \leq |x|$ for every formula $\phi(x;y)$. Fix such a formula $\phi(x;y)$. Instead of working with it directly, we simplify it using quantifier elimination. Quantifier elimination result can be easily obtained from cell decomposition:

**Lemma 3.1.** *Any formula $\phi(x;y)$ in $(\mathbb{Q}_p, \mathcal{L}_{aff})$ can be written as a boolean combination of formulas from the following collection*

$$\Psi(x;y) = \{\mathrm{val}(p_i(x) - c_i(y)) < \mathrm{val}(p_j(x) - c_j(y))\}_{i,j \in I} \cup$$
$$\{p_i(x) - c_i(y) \in \lambda_k Q_{m,n}\}_{i \in I, k \in K}$$

*where $I, K$ are finite index sets, each $p_i$ is a linear polynomial in $x$ without a constant term, each $c_i$ is a linear polynomial in $y$, and $\lambda_k \in \mathbb{Q}_p$.*

*Proof.* Let $l = |x| + |y|$. Apply cell decomposition theorem to $\phi(x;y)$ to obtain $\mathscr{D}^l$, a collection of $l$-cells. Let $\mathscr{D}^{l-1}$ be a collection $l-1$ of bases of cells in $\mathscr{D}^l$. Similarly, construct by induction $\mathscr{D}^i$ for each $0 \leq j < l$, where $\mathscr{D}_j$ is a collection of $j$-cells which are the bases of cells in $\mathscr{D}_{j+1}$. Let $\mathscr{D} = \bigcup \mathscr{D}_j$. Choose $n, m$ large enough to cover all $n', m'$ that come up in the cells for $Q_{n',m'}$. Choose $\lambda_k$ to go over all the cosets of $Q_{n,m}$. Let $q_i(x,y)$ enumerate all of the polynomials $a_1(\bar{x}), a_2(\bar{x}), t - c(\bar{x})$ that show up in the cells of $\mathscr{D}$. Those are all polynomials of degree $\leq 1$ in variables

$x, y$. We can split each of them as $q_i(x, y) = p_i(x) - c_j(y)$ where the constant term goes into $c_j$. This gives us the appropriate finite collection of formulas $\Psi$. From cell decomposition it is easy to see that when $a, a'$ have the same $\Psi$-type, then they would have they have the same $\phi$-type. Thus $\phi$ can be written as a boolean combination of formulas from $\Psi$. $\square$

**Lemma 3.2.** *If $\phi$ can be written as a Boolean combination of formulas from $\Psi$ then*

$$\mathrm{vc}(\Psi) \leq n \implies \mathrm{vc}(\phi) \leq n$$

*Proof.* If $a, a'$ have the same $\Psi$-type over $B$, then they have the same $\phi$-type over $B$, where $B$ is some parameter set. Therefore the number of $\phi$-types is bounded by the number of $\Psi$-types. The bound follows from Lemma 1.11. $\square$

Therefore to show that $\mathrm{vc}(\phi) \leq |x|$, it suffices to bound $\mathrm{vc}(\Psi) \leq |x|$. More precisely, it is sufficient to show that if there is a parameter set $B$ of size $N$ then the number of $\Psi$-types over $B$ is $O(N^{|x|})$. Fix such a parameter set $B$ and work with it from now on. We will compute a bound for the number of $\Psi$-types over $B$.

Consider a set $T = \{c_i(b) \mid b \in B, i \in I\} \subset \mathbb{Q}_p$. In this definition $B$ is the parameter set that we fixed and $c_i(b)$ come from collection of formulas $\Psi$ from the quantifier elimination above. View $T$ as a tree as follows:

**Definition 3.3.**

- For $c \in \mathbb{Q}_p, \alpha \in \mathbb{Z}$ define a <u>ball</u>

$$B(c, \alpha) = \{c' \in \mathbb{Q}_p \mid \mathrm{val}\,(c' - c) \leq \alpha\}.$$

- Define a collection of balls $\mathscr{B} = \{B(t_1, \mathrm{val}(t_1 - t_2))\}_{t_1, t_2 \in T}$. An <u>interval</u> $(B_1, B_2)$ is a set $B_1 \backslash B_2$ for $B_1, B_2 \in \mathscr{B}$ with $B_1 \supset B_2$ and no balls from $\mathscr{B}$ in between. We also define an interval $(-\infty, B)$ as a set $\mathbb{Q}_p \backslash B$ for a ball $B(c, v) \in \mathscr{B}$ with the smallest valuation $v$ of all the balls in $\mathscr{B}$. Note that there are at most $2|T| = 2N|I| = O(N)$ different intervals and they partition $\mathbb{Q}_p$.

- Define a collection of balls $\mathscr{B}' = \mathscr{B} \cup \{B(c_{i_1}(b), \mathrm{val}(c_{i_2}(b) - c_{i_3}(b)))\}_{i_1,i_2,i_3 \in I, b \in B}$. An sub-interval is defined the same as an interval except using collection $\mathscr{B}'$ instead of $\mathscr{B}$. Sub-intervals refine intervals, and there are at most $2|T| + |B| \cdot |I|^3 = O(N)$ many of them.

**Definition 3.4.** Suppose $a \in \mathbb{Q}_p$ lies in an interval $B(t_L, \alpha_L) \backslash B(t_U, \alpha_U)$.

- Define T-valuation of $a$ to be $\text{T-val}(a) = \mathrm{val}(a - t_U)$.
- Define floor of $a$ to be $F(a) = \alpha_L$.

**Definition 3.5.** Suppose $a_1, a_2 \in \mathbb{Q}_p$ lie in our tree in the same interval $B(t_L, \alpha_L) \backslash B(t_U, \alpha_U)$. We say that $a_i$ is close to boundary if $|\,\text{T-val}(a_i) - \alpha_L| \leq m$ or $|\,\text{T-val}(a_i) - \alpha_U| \leq m$. Otherwise we say that it is far from boundary.

**Definition 3.6.** We say $a_1, a_2$ have the same interval type if one of the following holds:

- Both $a_1, a_2$ are far from boundary and $a_1 - t_U, a_2 - t_U$ are in the same $Q_{m,n}$ coset.
- Both $a_1, a_2$ are close to boundary and $\mathrm{val}(a_1 - a_2) > \text{T-val}(a_1) + n = \text{T-val}(a_2) + n$.

The following lemma is an adaptation of lemma 7.4 in [1].

**Lemma 3.7.** *For $n, m$ there exists $D = D(n, m) \in \mathbb{Z}$ such that for any $x, y, a \in \mathbb{Q}_p$ if*

$$\mathrm{val}(x - c) = \mathrm{val}(y - c) < \mathrm{val}(x - y) - D$$

*then $x - c, y - c$ are in the same coset of $Q_{n,m}$.*

*Proof.* Define that $a, b \in \mathbb{Q}_p$ are similar if $\mathrm{val}\, a = \mathrm{val}\, b$ and

$$a \restriction [\mathrm{val}\, a, \mathrm{val}\, a + (m + n)] = b \restriction [\mathrm{val}\, b, \mathrm{val}\, b + (m + n)]$$

If $a, b$ are similar then

$$a \in Q_{n,m} \leftrightarrow b \in Q_{n,m}$$

Moreover for any $\lambda \in \mathbb{Q}_p$, if $a, b$ are similar we would also have $a/\lambda, b/\lambda$ are similar. Thus if $a, b$ are similar, then they belong in the same coset of $Q_{n,m}$. If we pick $D = n + m$ then conditions of the lemma force $x - c, y - c$ to be similar. □

The following construction is along the lines of lemmas 7.3, 7.5 of [1].

**Definition 3.8.** For two balls $B(a, \alpha), B(b, \beta)$ let $\gamma = \min(\alpha, \beta, \mathrm{val}(a - b))$. Define the distance between those two balls to be $|\alpha - \gamma| + |\beta - \gamma|$. In $\mathbb{Q}_p$ value group is discrete and residue field is finite, so there are finitely many balls at a fixed distance from a given ball. Near balls of $B(a, \alpha)$ are defined to be balls with distance $\mathcal{D}$ from $B(a, \alpha)$. Enumerate those as:

$$B_1(a, \alpha), B_2(c, \alpha), \ldots B_{N_D}(a, \alpha)$$

Near balls partition the space

$$\{b \in \mathbb{Q}_p \mid |\,\mathrm{val}(a - b) - \alpha| \leq D\}$$

**Lemma 3.9.** *Suppose $c_1, c_2 \in \mathbb{Q}_p^{|x|}$ satisfy the follwing three conditions*

- *For all $i \in I$ $p_i(c_1)$ and $p_i(c_2)$ are in the same sub-interval.*
- *For all $i \in I$ $p_i(c_1)$ and $p_i(c_2)$ have the same interval type.*
- *For all $i, j \in I$,* $\text{T-val}(p_i(c_1)) > \text{T-val}(p_j(c_1))$ *iff* $\text{T-val}(p_i(c_2)) > \text{T-val}(p_j(c_2))$.

*Then $c_1, c_2$ have the same $\Psi$-type over $B$.*

*Proof.* There are two kinds of formulas in $\Psi$ (see Lemma 3.1). First we show that $d_1, d_2$ agree on formulas of the form $p_i(x) - c_i(y) \in \lambda_k Q_{m,n}$. It is enough to show that for every $i \in I, b \in B$ we have $p_i(d_1) - c_i(b), p_i(d_2) - c_i(b)$ are in the same $Q_{m,n}$-coset. Fix such $i, b$. For brievety let $a = p_i(d_1), a' = p_i(d_2)$ and $Q = Q_{m,n}$. We want to show that $a - c_i(b), a' - c_i(b)$ are in the same $Q$-coset.

Suppose $a$ is in one of the near balls. As $a'$ has the same interval type, it has to be in the same near ball. By definition of the near ball we then have $\mathrm{val}(a - c_i(b)) = \mathrm{val}(a' - c_i(b)) < \mathrm{val}(a - a') - D$. Thus by Lemma 3.7 we have $a - c_i(b), a' - c_i(b)$ in the same $Q$-coset.

Now, suppose both $a, a'$ aren't in any near balls. Label their interval as $B(c_L, \alpha_L) \backslash B(c_U, \alpha_U)$. Then we have

$$\alpha_L + D < \mathrm{val}(a - c_U) < \alpha_U - D$$

$$\alpha_L + D < \mathrm{val}(a' - c_U) < \alpha_U - D$$

as otherwise one (both) of them would be in one of the near balls. We have either $\mathrm{val}(c_U - c_i(b)) \geq \alpha_U$ or $\mathrm{val}(c_U - c_i(b)) \leq \alpha_L$ as otherwise it would contradict the definition of an interval.

Suppose it is the first case $\mathrm{val}(c_U - c_i(b)) \geq \alpha_U$. Then

$$\mathrm{val}(a - c_i(b)) = \mathrm{val}(a - c_U) < \alpha_U - D \leq \mathrm{val}(c_U - c_i(b)) - D$$

so by Lemma 3.7 we have $a - c_i(b), a - c_U$ are in the same $Q$-coset. By a parallel argument we have $a' - c_i(b), a' - c_U$ are in the same $Q$-coset. As we are assuming $a, a'$ have the same tree type it implies that $a - c_U, a' - c_U$ are in the same $Q$-coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same $Q$-coset.

For the second case, suppose $\mathrm{val}(c_U - c_i(b)) \leq \alpha_L$. Then

$$\mathrm{val}(a - c_i(b)) = \mathrm{val}(c_U - c_i(b)) \leq \alpha_L < \mathrm{val}(a - c_U) - D$$

so by Lemma 3.7 we have $a - c_i(b), c_U - c_i(b)$ are in the same $Q$-coset. By a parallel argument we have $a' - c_i(b), c_U - c_i(b)$ are in the same $Q$-coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same $Q$-coset.

Next, we need to show that $d_1, d_2$ agree on formulas of the form $\mathrm{val}(p_i(x) - c_i(y)) < \mathrm{val}(p_j(x) - c_j(y))$ (see Lemma 3.1). $\square$

This gives us an upper bound on the number of types - there are at most $|I|!$ many choices for the order of T-val, $O(N)$ many choices for the interval for each $p_i$, and $K$ many choices for the interval type for each $p_i$, giving a total of $O(N^{|I|}) \cdot K^{|I|} \cdot |I|! = O(N^{|I|})$ many types. This implies $\mathrm{vc}(\Psi) \leq |I|$. The biggest contribution to this bound are the choices among the $O(N)$ many intervals for each $p_i$ with $i \in I$. Are all of those choices realized? Intuitively there are $|x|$ many variables and $|I|$ many equations, so once we choose an interval for $|x|$ many $p_i$'s, the interval for the rest should be determined. This would give the required $\mathrm{vc}(\Psi) \leq |x|$ bound. The next section outlines this proof formally.

## 4. Main Proof

**Lemma 4.1.** *Suppose we have a finite collection of vectors $\{\vec{p}_i\}_{i\in I}$ with each $\vec{p}_i \in \mathbb{Q}_p^{|x|}$. Suppose $J \subset I$ and $i \in I$ satisfy*

$$\vec{p}_i \in \operatorname{span}\{\vec{p}_j\}_{j\in J},$$

*and we have $\vec{x} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$ with*

$$\mathrm{val}(\vec{p}_j \cdot \vec{x}) > \alpha \text{ for all } j \in J$$

*Then*

$$\mathrm{val}(\vec{p}_i \cdot \vec{x}) > \alpha - \gamma$$

*for some $\gamma \in \mathbb{N}$. Moreover $\gamma$ can be chosen independently from $J, j, \vec{x}, \alpha$ depending only on $\{\vec{p}_i\}_{i\in I}$.*

*Proof.* Fix $i, J$ satisfying the conditions of the lemma. For some $c_j \in \mathbb{Q}_p$ for $j \in J$ we have

$$\vec{p}_i = \sum_{j\in J} c_j \vec{p}_j,$$

hence

$$\vec{p}_i \cdot \vec{x} = \sum_{j \in J} c_j \vec{p}_j \cdot \vec{x}.$$

We have

$$\operatorname{val}(c_j \vec{p}_j \cdot \vec{x}) = \operatorname{val}(c_j) + \operatorname{val}(\vec{p}_j \cdot \vec{x}) > \operatorname{val}(c_j) + \alpha.$$

Let $\gamma = \max(0, \min - \operatorname{val}(c_j))$. Then we have

$$\begin{aligned} &\operatorname{val}(c_j \vec{p}_j \cdot \vec{x}) > \alpha - \gamma && \text{for all } j \in J \\ &\sum_{j \in J} c_j \vec{p}_j \cdot \vec{x} > \alpha - \gamma \end{aligned}$$

This shows that we can pick such $\gamma$ for a given choice of $i, J$, but independent from $\alpha, \vec{x}$. To get a choice independent from $i, J$, go over all such eligible choices ($i$ ranges over $I$ and $J$ ranges over subsets of $I$), pick $\gamma$ for each, and then take the maximum of those values. □

Alternative way to write $p_i(x)$ is $\vec{p}_i \cdot \vec{x}$, where $\vec{p}_i$ and $\vec{x}$ are vectors in $\mathbb{Q}_p^{|x|}$. The lemma above is a general result, but we only use it applied to the vectors $\vec{p}_i$ given by our collection of formulas $\Psi$.

**Definition 4.2.** For $c \in \mathbb{Q}_p$ and $\alpha, \beta \in \mathbb{Z}$ define $c \restriction [\alpha, \beta] \in (\mathbb{Z}/p\mathbb{Z})^{\beta - \alpha + 1}$ to be the record of the coefficients of $c$ for the valuations between $\alpha, \beta$. More precisely write $c$ in its power series form

$$c = \sum_{\gamma \in \mathbb{Z}} c_\gamma p^\gamma \text{ with } c_\gamma \in \mathbb{Z}/p\mathbb{Z}$$

Then $c \restriction [\alpha, \beta]$ is just $(c_\alpha, c_{\alpha+1}, \ldots c_\beta)$.

Fix $\gamma$ corresponding to $\{\vec{p}_i\}_{i \in I}$ according to Lemma 4.1.

**Definition 4.3.** Denote $\mathbb{Z}/p\mathbb{Z}^\gamma$ as Ct.

**Definition 4.4.** Let $f : \mathbb{Q}_p^{|x|} \longrightarrow \mathbb{Q}_p^I$ with $f(\bar{c}) = (p_i(\bar{c}))_{i \in I}$. Define the segment space Sg to be the image of $f$.

Given a tuple $(a_i)_{i \in I}$in the segment space look at the corresponding floors $\{F(a_i)\}_{i \in I}$. Those are ordered as elements of $\mathbb{Z}$. Partition the segment space by order type of $\{F(a_i)\}$. Work in a fixed partition $\mathrm{Sg}'$. After relabeling we may assume that

$$F(a_1) \geq F(a_2) \geq \dots$$

Consider the (relabeled) sequence of vectors $\vec{p}_1, \vec{p}_2, \dots, \vec{p}_I$. There is a unique subset $J \subset I$ such that all vectors with indices in $J$ are linearly independent, and all vectors with indices outside of $J$ are a linear combination of preceding vectors. For any index $i \in I$ we call it independent if $i \in J$ and we call it dependent otherwise.

Now, we define the following function

$$g : \mathrm{Sg}' \longrightarrow \mathrm{Bt}^I \times \mathrm{Pt}^J \times \mathrm{Ct}^{I-J}$$

Let $\bar{a} = (a_i)_{i \in I} \in \mathrm{Sg}'$. To define $g(\bar{a})$ we need to specify where it maps $\bar{a}$ in each individual component of the product.

For all $a_i$ record its interval type $\in$ Bt, giving the first component.

For $a_j$ with $j \in J$, record the interval of $a_j$, giving the second component.

For the third component do the following computation. Pick $a_i$ with $i$ dependent. Let $j$ be the largest independent index with $j < i$. Record $a_i \restriction [F(a_j) - \gamma, F(a_j)]$.

**Lemma 4.5.** *For $\bar{a}, \bar{a}' \in \mathrm{Sg}'$ if $g(\bar{a}) = g(\bar{a}')$ then $a_i, a_i'$ have the same tree type for all $i \in I$.*

*Proof.* For each $i$ we show that $a_i, a_i'$ are in the same interval and have the same interval type, so the conclusion follows by Lemma **??**. Bt records the interval type of each element, so if $g(\bar{a}) = g(\bar{a}')$ then $a_i, a_i'$ have the same interval type for all $i \in I$. Thus it remains to show that $a_i, a_i'$ lie in the same interval for all $i \in I$.

Suppose $i$ is an independent index. Then by construction, Pt records the interval for $a_i, a'_i$, so those have to belong to the same interval. Now suppose $i$ is dependent. Pick the largest $j < i$ such that $j$ is independent. We have $F(a_i) \leq F(a_j)$ and $F(a'_i) \leq F(a'_j)$. Moreover $F(a_j) = F(a'_j)$ as they are mapped to the same interval (using the earlier part of the argument as $j$ is independent).

**Claim 4.6.** $\mathrm{val}(a_i - a'_i) > F(a_j) - \gamma$

*Proof.* Let $\vec{x}, \vec{x}' \in \mathbb{Q}_p^{|x|}$ be some elements with

$$\vec{p}_k \cdot \vec{x} = a_k$$

$$\vec{p}_k \cdot \vec{x}' = a'_k \text{ for all } k \in I$$

It is always possible to do that as $\bar{a}, \bar{a}' \in \mathrm{Sg}'$. Let $J'$ be the set of the independent indices less than $i$. We have

$$\mathrm{val}(a_k - a'_k) > F(a_k) \text{ for all } k \in J'$$

as for the independent indices $a_k, a'_k$ lie in the same interval.

$$\mathrm{val}(a_k - a'_k) > F(a_j) \text{ for all } k \in J' \text{ by monotonicity of } F(a_k)$$

$$\mathrm{val}(\vec{p}_k \cdot \vec{x} - \vec{p}_k \cdot \vec{x}') > F(a_j) \text{ for all } k \in J'$$

$$\mathrm{val}(\vec{p}_k \cdot (\vec{x} - \vec{x}')) > F(a_j) \text{ for all } k \in J'$$

$J'$ and $i$ match the requirements of Lemma 4.1 so we conclude

$$\mathrm{val}(\vec{p}_i \cdot (\vec{x} - \vec{x}')) > F(a_j) - \gamma$$

$$\mathrm{val}(\vec{p}_i \cdot \vec{x} - \vec{p}_i \cdot \vec{x}') > F(a_j) - \gamma$$

$$\mathrm{val}(a_i - a'_i)) > F(a_j) - \gamma$$

as needed, finishing the proof of the claim. □

Additionally $a_i, a_i'$ have the same image in Ct component, so we have

$$\mathrm{val}(a_i - a_i') > F(a_j)$$

As $F(a_i) \leq F(a_j)$, $a_i, a_i'$ have to lie in the same interval. □

**Corollary 4.7.** *$\Psi(x, y)$ has VC-density $\leq |x|$*

*Proof.* Suppose we have $c, c' \in \mathbb{Q}_p^{|x|}$ such that $f(c), f(c')$ are in the same partition and $g(f(c)) = g(f(c'))$. Then by the previous lemma $p_i(c)$ has the same tree type as $p_i(c')$ for all $i \in I$. Then by Lemma **??** $c, c'$ have the same $\Psi$-type. Thus the number of possible $\Psi$-types is bounded by the size of the range of $g$ times the number of possible partitions

$$(\text{number of partitions}) \cdot |Bt|^{|I|} \cdot |Pt|^{|J|} \cdot |Ct|^{|I-J|}$$

We have

$$|\,\mathrm{Bt}\,| = N_D + \text{number of cosets of } Q |\,\mathrm{Pt}\,| \leq N \cdot I^2 \text{ (the only component dependent on } N)$$

$$|\,\mathrm{Ct}\,| = p^{\gamma}$$

and there are at most $|I|!$ many partitions of Sg. This gives us a bound

$$|I|! \cdot |Bt|^{|I|} \cdot (N \cdot |I|^2)^{|J|} \cdot p^{\gamma|I-J|} = O(N^{|J|})$$

Every $p_i$ is an element of a $|x|$-dimensional vector space, so there can be at most $|x|$ many independent vectors. Thus we have $|J| \leq |x|$ and the bound follows. □

**Corollary 4.8.** *In the language $\mathcal{L}_{aff}$ we have* $\mathrm{vc}(n) = n$.

*Proof.* Previous lemma implies that $\mathrm{vc}(\phi) \leq \mathrm{vc}(\Psi) \leq |x|$. As choice of $\phi$ was arbitrary, this implies that VC-density of any formula is bounded by the arity of $x$. $\square$

## References

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