SOME VC-DENSITY COMPUTATIONS IN SHELAH-SPENCER GRAPHS

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Abstract. We compute vc-densities of minimal extension formulas in Shelah-Spencer random graphs.

We fix the density of the graph α.

Lemma 0.1. *For any $\mathcal{A} \in \boldsymbol{K}_\alpha$ and $\epsilon > 0$ there exists an $\mathcal{B}$ such that $(\mathcal{A}, \mathcal{B})$ is minimal and $\delta(\mathcal{B}/\mathcal{A}) < \epsilon$.*

Proof. Let m be an integer such that $m\alpha < 1 < (m+1)\alpha$. Suppose $\mathcal{A}$ has less than $m+1$ vertices. Make a construction $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_{i+1}$ is $\mathcal{A}_i$ with one extra vertex connected to every single vertex of A_i. Stop when the total number of vertices is $m+1$. Proceed as in [1] 4.1. Resulting construction is still minimal. $\square$

Lemma 0.2. *Let $\mathcal{A}_1 \subset \mathcal{B}_1$ and $\mathcal{A}_2 \subset \mathcal{B}_2$ be $\boldsymbol{K}_\alpha$ structures with $(\mathcal{A}_2, \mathcal{B}_2)$ a minimal pair with $\epsilon = \delta(\mathcal{B}_2/\mathcal{A}_2)$. Let M be some ambient structure. Fix embeddings of $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2$ into M. Assume that it is not that case that $\mathcal{A}_2 \subset \mathcal{B}_2$ and $\mathcal{A}_1$ is disjoint from $\mathcal{A}_2$. Now consider all possible embeddings $f \colon \mathcal{B}_2 \to M$ over $\mathcal{A}_1$. Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{B}_f = \mathcal{B}_1 \cup f(\mathcal{B}_2)$ with $\delta_f = \delta(\mathcal{B}_f/\mathcal{A})$. Then δ_f is at most $\delta(\mathcal{B}_1 \cup \mathcal{A}/\mathcal{A}) + \epsilon$*

Fix an embedding f. It induces the following substructure diagram in M. Denote

$$
\begin{aligned}
\mathcal{A} &= \mathcal{A}_1 \cup \mathcal{A}_2 \\
\mathcal{B}_f^* &= \mathcal{B}_1 \cup f(\mathcal{B}_2) \\
\mathcal{B}_1^* &= \mathcal{B}_1 \cup \mathcal{A} \\
\mathcal{B}_2^* &= f(\mathcal{B}_2) \cup \mathcal{A} \\
\mathcal{B}^* &= (\mathcal{B}_1 \cap f(\mathcal{B}_2)) \cup \mathcal{A}
\end{aligned}
$$

$$\begin{tikzcd}
& \mathcal{B}_f \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{B}_1^* \arrow[dr, dash] & & \mathcal{B}_2^* \arrow[dl, dash] \\
& \mathcal{B}^* \arrow[d, dash] & \\
& \mathcal{A} &
\end{tikzcd}$$

From the diagram we see that

$$\delta(\mathcal{B}_f/\mathcal{A}) \leq \delta(\mathcal{B}_1 \cup \mathcal{A}/\mathcal{A}) + \delta\left((f(\mathcal{B}_2) \cup \mathcal{A})/((\mathcal{B}_1 \cap f(\mathcal{B}_2)) \cup \mathcal{A})\right)$$

Thus all we need to do is to verify that

$$\delta\left((f(\mathcal{B}_2) \cup \mathcal{A})/((\mathcal{B}_1 \cap f(\mathcal{B}_2)) \cup \mathcal{A})\right) \leq \delta(\mathcal{B}_2/\mathcal{A}_2)$$

Let $\mathcal{B}^*$ denote all the vertices in $f(\mathcal{B}_2)$ that are not in $\mathcal{B}_1 - \mathcal{A}_2$. Then $\delta(\mathcal{B}^*/A_2)$ has to be less than $\delta(\mathcal{B}_2/\mathcal{A}_2)$ by minimality of $(\mathcal{B}_2, \mathcal{A}_2)$. Relative dimension of the whole construction has to be even smaller. It is easy to show that this construction induces a proper subpair in $(\mathcal{A}_2, \mathcal{B}_2)$ which has to have smaller dimension.

Let $\phi(x, y)$ be a formula in a random graph with $|x| = |y| = 1$ saying that there exists a minimal extension M over $\{x, y\}$ of relative dimension ϵ. Let n be such that $n\epsilon < 1 < (n + 1)\epsilon$. Then we argue that $vc(\phi) = n$.

Fix a m-strong (for any $m > |M|$) set of non-connected vertices B. Fix some a. We invesitgate the trace of $\phi(x, a)$ on B. Suppose we have $b_1, \ldots, b_k$ satisfying $\phi(b_i, a)$ as witnessed by M_j. Relative dimension of $M_1 \cup M_2 \cup \ldots \cup M_j \cup a$ is minimized when all M_j are disjoint (by minimality). Thus for that dimension to be positive we can have at most n extensions.

References

[1] Michael C. Laskowski, *A simpler axiomatization of the Shelah-Spencer almost sure theories,* Israel J. Math. **161** (2007), 157-186. MR MR2350161

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