VC-DENSITY IN AN ADDITIVE REDUCT OF THE P-ADIC NUMBERS

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Abstract. Aschenbrenner et. al. computed a linear bound for the vc-density function in the field of p-adic numbers, but this bound is not known to be optimal. In this paper we investigate a certain P-minimal additive reduct of the field of p-adic numbers and use a cell decomposition result of Leenknegt to compute an optimal bound for that structure.

VC-density was studied in model theory in [1] by Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko as a natural notion of dimension for definable families of sets in NIP theories. In a complete NIP theory T we can define the vc-function

$$\mathrm{vc}^T = \mathrm{vc} : \mathbb{N} \longrightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$$

where $\mathrm{vc}(n)$ measures the worst-case complexity of families of definable sets in an n-fold cartesian power of the underlying set of a model of T (see 1.13 below for a precise definition of vc^T). The simplest possible behavior is $\mathrm{vc}(n) = n$ for all n, satisfied, for example, if T is o-minimal. For $T = \mathrm{Th}(\mathbb{Q}_p)$, the paper [1] computes an upper bound for this function to be $2n - 1$, and it is not known whether this is optimal. This same bound holds in any reduct of the field of p-adic numbers, but one may expect that the simplified structure of suitable reducts would allow a better bound. In [5], Leenknegt provides a cell decomposition result for a certain P-minimal additive reduct of the field of p-adic numbers. Using this result, in this paper we improve the bound for the vc-function, showing that in Leenknegt's structure $\mathrm{vc}(n) = n$ for each n. By Corollary 5.13 in [1] this also gives a proof that this structure is dp-minimal which is more direct than using that the field of p-adics is dp-minimal.

Section 1 defines vc-density and states some basic lemmas about this notion. A more in depth exposition of vc-density can be found in [1]. In Section 2 we recall some basic facts about the theory of p-adic numbers. Here we also introduce the reduct which we will be working with. Section 3 sets up basic definitions and lemmas that will be needed for the proof. We define trees and intervals and show how they help with vc-density calculations. Section 4 concludes the proof. In concluding section we discuss some possible future work.

Throughout the paper, variables and tuples of elements will be simply denoted as $x, y, a, b, \ldots$. We will occasionally write $\vec{a}$ instead of a for a tuple in $\mathbb{Q}_p^n$ to emphasize it as an element of the $\mathbb{Q}_p$-vector space $\mathbb{Q}_p^n$. We denote the arity of a tuple x of variables by $|x|$. The set of natural numbers is denoted by $\mathbb{N} = \{0, 1, \ldots\}$.

1. VC-dimension and vc-density

Throughout this section we work with a collection $\mathcal{F}$ of subsets of an infinite set X. We call the pair $(X, \mathcal{F})$ a set system.

Definition 1.1.

- Given a subset A of X, we define the set system $(A, A \cap \mathcal{F})$ where $A \cap \mathcal{F} = \{A \cap F \mid F \in \mathcal{F}\}$.
- For $A \subseteq X$ we say that $\mathcal{F}$ shatters A if $A \cap \mathcal{F} = \mathcal{P}(A)$ (the power set of A).

Definition 1.2. We say $(X, \mathcal{F})$ has VC-dimension n if the largest subset of X shattered by $\mathcal{F}$ is of size n. If $\mathcal{F}$ shatters arbitrarily large subsets of X, we say that $(X, \mathcal{F})$ has infinite VC-dimension. We denote the VC-dimension of $(X, \mathcal{F})$ by $\mathrm{VC}(X, \mathcal{F})$.

Note 1.3. We may drop X from the notation $\mathrm{VC}(X, \mathcal{F})$, as the VC-dimension doesn't depend on the base set and is determined by $(\bigcup \mathcal{F}, \mathcal{F})$.

Set systems of finite VC-dimension tend to have good combinatorial properties, and we consider set systems with infinite VC-dimension to be poorly behaved.

Another natural combinatorial notion is that of the dual system of a set system:

Definition 1.4. For $a \in X$ define $X_a = \{F \in \mathcal{F} \mid a \in F\}$. Let $\mathcal{F}^* = \{X_a \mid a \in X\}$. We call $(\mathcal{F}, \mathcal{F}^*)$ the dual system of $(X, \mathcal{F})$. The VC-dimension of the dual system of $(X, \mathcal{F})$ is referred to as the dual VC-dimension of $(X, \mathcal{F})$ and denoted by $\mathrm{VC}^*(\mathcal{F})$. (As before, this notion doesn't depend on X.)

Lemma 1.5 (see 2.13b in [2])**.** *A set system $(X, \mathcal{F})$ has finite VC-dimension if and only if its dual system has finite VC-dimension. More precisely*

$$\mathrm{VC}^*(\mathcal{F}) \leq 2^{1+\mathrm{VC}(\mathcal{F})}.$$

For a more refined notion of complexity of $(X, \mathcal{F})$ we look at the traces of our family on finite sets:

Definition 1.6. Define the shatter function $\pi_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ and the dual shatter function $\pi^*_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ by

$$\pi_{\mathcal{F}}(n) = \max\{|A \cap \mathcal{F}| \mid A \subseteq X \text{ and } |A| = n\}$$

$$\pi^*_{\mathcal{F}}(n) = \max\{\mathrm{atoms}(B) \mid B \subseteq \mathcal{F}, |B| = n\}$$

where $\mathrm{atoms}(B)$ = number of atoms in the boolean algebra of sets generated by B. Note that the dual shatter function is precisely the shatter function of the dual system: $\pi^*_{\mathcal{F}} = \pi_{\mathcal{F}^*}$.

A simple upper bound is $\pi_{\mathcal{F}}(n) \leq 2^n$ (same for the dual). If the VC-dimension of $\mathcal{F}$ is infinite then clearly $\pi_{\mathcal{F}}(n) = 2^n$ for all n. Conversely we have the following remarkable fact:

Theorem 1.7 (Sauer-Shelah '72, see [7], [8])**.** *If the set system $(X, \mathcal{F})$ has finite VC-dimension d then $\pi_{\mathcal{F}}(n) \leq \binom{n}{\leq d}$ for all n, where $\binom{n}{\leq d} = \binom{n}{d} + \binom{n}{d-1} + \ldots + \binom{n}{1}$.*

Thus the systems with a finite VC-dimension are precisely the systems where the shatter function grows polynomially. The vc-density of $\mathcal{F}$ quantifies the growth of the shatter function of $\mathcal{F}$:

Definition 1.8. Define the vc-density and dual vc-density of $\mathcal{F}$ as

$$\mathrm{vc}(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\},$$
$$\mathrm{vc}^*(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}.$$

Generally speaking a shatter function that is bounded by a polynomial doesn't itself have to be a polynomial. Proposition 4.12 in [1] gives an example of a shatter function that grows like $n \log n$ (so it has vc-density 1).

So far the notions that we have defined are purely combinatorial. We now adapt VC-dimension and vc-density to the model theoretic context.

Definition 1.9. Work in a first-order structure M. Fix a finite collection of formulas $\Phi(x, y)$ in the language $\mathcal{L}(M)$ of M.

- For $\phi(x, y) \in \mathcal{L}(M)$ and $b \in M^{|y|}$ let

$$\phi(M^{|x|}, b) = \{a \in M^{|x|} \mid \phi(a, b)\} \subseteq M^{|x|}.$$

- Let $\Phi(M^{|x|}, M^{|y|}) = \{\phi(M^{|x|}, b) \mid \phi_i \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|})$.
- Let $\mathcal{F}_\Phi = \Phi(M^{|x|}, M^{|y|})$, giving rise to a set system $(M^{|x|}, \mathcal{F}_\Phi)$.
- Define the VC-dimension $\mathrm{VC}(\Phi)$ of Φ to be the VC-dimension of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.
- Define the vc-density $\mathrm{vc}(\Phi)$ of Φ to be the vc-density of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.

We will also refer to the vc-density and VC-dimension of a single formula ϕ viewing it as a one element collection $\Phi = \{\phi\}$.

Counting atoms of a boolean algebra in a model theoretic setting corresponds to counting types, so it is instructive to rewrite the shatter function in terms of types.

Definition 1.10.

$$\pi^*_\Phi(n) = \max\{\text{number of } \Phi\text{-types over } B \mid B \subseteq M, |B| = n\}.$$

Here a Φ-type over B is a maximal consistent collection of formulas of the form $\phi(x,b)$ or $\neg\phi(x,b)$ where $\phi \in \Phi$ and $b \in B$.

The functions π^*_Φ and $\pi^*_{\mathcal{F}_\Phi}$ do not have to agree, as one fixes the number of generators of a boolean algebra of sets and the other fixes the size of the parameter set. However, as the following lemma demonstrates, they both give the same asymptotic definition of dual vc-density.

Lemma 1.11.

$$\mathrm{vc}^*(\Phi) = \textit{degree of polynomial growth of } \pi^*_\Phi(n) = \limsup_{n\to\infty} \frac{\log \pi^*_\Phi(n)}{\log n}.$$

Proof. With a parameter set B of size n, we get at most $|\Phi|n$ sets $\phi(M^{|x|}, b)$ with $\phi \in \Phi, b \in B$. We check that asymptotically it doesn't matter whether we look at growth of boolean algebra of sets generated by n or by $|\Phi|n$ many sets. We have:

$$\pi^*_{\mathcal{F}_\Phi}(n) \leq \pi^*_\Phi(n) \leq \pi^*_{\mathcal{F}_\Phi}(|\Phi|n).$$

Hence:

$$\begin{aligned}
\mathrm{vc}^*(\Phi) &\leq \limsup_{n\to\infty} \frac{\log \pi^*_\Phi(n)}{\log n} \leq \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log n} = \\
&= \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log |\Phi|n} \frac{\log |\Phi|n}{\log n} = \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log |\Phi|n} \leq \\
&\leq \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(n)}{\log n} = \mathrm{vc}^*(\Phi).
\end{aligned}$$

□

One can check that the shatter function and hence VC-dimension and vc-density of a formula are elementary notions, so they only depend on the first-order theory of the structure M.

NIP theories are a natural context for studying vc-density. In fact we can take the following as the definition of NIP:

Definition 1.12. Define ϕ to be NIP if it has finite VC-dimension in a theory T. A theory T is NIP if all the formulas in T are NIP.

In a general combinatorial context (for arbitrary set systems), vc-density can be any real number in $0 \cup [1, \infty)$ (see [3]). Less is known if we restrict our attention to NIP theories. Proposition 4.6 in [1] gives examples of formulas that have non-integer rational vc-density in an NIP theory, however it is open whether one can get an irrational vc-density in this model-theoretic setting.

Instead of working with a theory formula by formula, we can look for a uniform bound for all formulas:

Definition 1.13. For a given NIP structure M, define the vc-function

$$\begin{aligned} \mathrm{vc}^M(n) &= \sup\{\mathrm{vc}^*(\phi(x, y)) \mid \phi \in \mathcal{L}(M), |x| = n\} \\ &= \sup\{\mathrm{vc}(\phi(x, y)) \mid \phi \in \mathcal{L}(M), |y| = n\} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}. \end{aligned}$$

As before this definition is elementary, so it only depends on the theory of M. We omit the superscript M if it is understood from the context. One can easily check the following bounds:

Lemma 1.14 (Lemma 3.22 in [1])**.** *We have* $\mathrm{vc}(1) \geq 1$ *and* $\mathrm{vc}(n) \geq n\,\mathrm{vc}(1)$.

However, it is not known whether the second inequality can be strict or even just whether $\mathrm{vc}(1) < \infty$ implies $\mathrm{vc}(n) < \infty$.

2. P-adic numbers

The field $\mathbb{Q}_p$ of p-adic numbers is often studied in the language of Macintyre

$$\mathcal{L}_{Mac} = \{0, 1, +, -, \cdot, |, \{P_n\}_{n \in \mathbb{N}}\}$$

which is a language $\{0, 1, +, -, \cdot\}$ of rings together with unary predicates P_n interpreted in $\mathbb{Q}_p$ so as to satisfy

$$P_n x \leftrightarrow \exists y \; y^n = x$$

and a divisibility relation where $a|b$ holds in $\mathbb{Q}_p$ when $\operatorname{val} a \leq \operatorname{val} b$.

Note that $P_n \setminus \{0\}$ is a multiplicative subgroup of $\mathbb{Q}_p$ with finitely many cosets.

Theorem 2.1 (Macintyre '76, [6]). *The $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$ has quantifier elimination.*

There is also a cell decomposition result for definable sets in this structure:

Definition 2.2. Define $\underline{k\text{-cells}}$ recursively as follows. A $\underline{0\text{-cell}}$ is the singleton $\mathbb{Q}_p^0$. A $\underline{(k+1)\text{-cell}}$ is a subset of $\mathbb{Q}_p^{k+1}$ of the following form:

$$\{(x,t) \in D \times \mathbb{Q}_p \mid \operatorname{val} a_1(x) \,\square_1\, \operatorname{val}(t - c(x)) \,\square_2\, \operatorname{val} a_2(x), t - c(x) \in \lambda P_n\}$$

where D is a k-cell, $a_1(x), a_2(x), c(x)$ are definable functions $D \longrightarrow \mathbb{Q}_p$, each of $\square_i$ is $<, \leq$ or no condition, $n \in \mathbb{N}$, and $\lambda \in \mathbb{Q}_p$.

Theorem 2.3 (Denef '84, [4]). *Any definable subset of $\mathbb{Q}_p^k$ defined by an $\mathcal{L}_{Mac}$-formula decomposes into a finite disjoint union of k-cells.*

In [1], Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko show that $\mathbb{Q}_p$ as $\mathcal{L}_{Mac}$-structure satisfies $\operatorname{vc}(n) \leq 2n - 1$ for each $n \geq 1$, however it is not known whether this bound is optimal.

In [5], Leenknegt analyzes the reduct of $\mathbb{Q}_p$ to the language

$$\mathcal{L}_{aff} = \left\{0, 1, +, -, \{\bar{c}\}_{c \in \mathbb{Q}_p}, |, \{Q_{m,n}\}_{m,n \in \mathbb{N}}\right\}$$

where $\bar{c}$ denotes the scalar multiplication by c, $a|b$ as above stands for $\operatorname{val} a \leq \operatorname{val} b$, and $Q_{m,n}$ is a unary predicate interpreted as

$$Q_{m,n} = \bigcup_{k \in \mathbb{Z}} p^{km}(1 + p^n \mathbb{Z}_p).$$

Note that $Q_{m,n} \setminus \{0\}$ is a subgroup of the multiplicative group of $\mathbb{Q}_p$ with finitely many cosets. One can check that these extra relation symbols are definable in the $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$. The paper [5] provides a cell decomposition result based on the following notion of cell:

Definition 2.4. A 0-cell is the singleton $\mathbb{Q}_p^0$. A $(k+1)$-cell is a subset of $\mathbb{Q}_p^{k+1}$ of the following form:

$$\{(x,t) \in D \times \mathbb{Q}_p \mid \operatorname{val} a_1(x) \,\square_1\, \operatorname{val}(t - c(x)) \,\square_2\, \operatorname{val} a_2(x), t - c(x) \in \lambda Q_{m,n}\}$$

where D is a k-cell, called the base of the cell, $a_1(x), a_2(x), c(x)$ are polynomials of degree ≤ 1, called the defining polynomials, each of $\square_1, \square_2$ is $<$ or no condition, $m, n \in \mathbb{N}$, and $\lambda \in \mathbb{Q}_p$. We call $Q_{m,n}$ the defining predicate of our cell.

Theorem 2.5 (Leenknegt '12)**.** *Any definable subset of $\mathbb{Q}_p^k$ defined by an $\mathcal{L}_{aff}$-formula decomposes into a finite disjoint union of k-cells.*

Moreover, [5] shows that $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ is a P-minimal reduct, that is, the one-variable definable sets of $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ coincide with the one-variable definable sets in the full structure $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$.

The main result of this paper is the computation of the vc-function for this structure:

Theorem 2.6. *The $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ satisfies $\operatorname{vc}(n) = n$ for all n.*

3. Key Lemmas and Definitions

To show that $\operatorname{vc}(n) = n$ it suffices to bound $\operatorname{vc}^*(\phi) \leq |x|$ for every $\mathcal{L}_{aff}$-formula $\phi(x; y)$. Fix such a formula $\phi(x; y)$. Instead of working with this formula directly, we first simplify it using quantifier elimination. The required quantifier elimination result can be easily obtained from cell decomposition:

Lemma 3.1. *Any $\mathcal{L}_{aff}$-formula $\phi(x; y)$ is equivalent in the $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ to a boolean combination of formulas from a collection*

$$\Phi(x; y) = \{\operatorname{val}(p_i(x) - c_i(y)) < \operatorname{val}(p_j(x) - c_j(y))\}_{i,j \in I} \cup$$
$$\{p_i(x) - c_i(y) \in \lambda_k Q_{m,n}\}_{i \in I, k \in K}$$

of $\mathcal{L}_{aff}$-formulas where I, K are finite index sets, each p_i is a degree ≤ 1 polynomial in x without a constant term, each c_i is a degree ≤ 1 polynomial in y, $m, n \in \mathbb{N}$, and $\lambda_k \in \mathbb{Q}_p$.

Proof. Let $l = |x| + |y|$. Using Theorem 2.5 partition the subset of $\mathbb{Q}_p^l$ defined by ϕ to obtain $\mathscr{D}^l$, a collection of l-cells. Let $\mathscr{D}^{l-1}$ be the collection of the bases of the cells in $\mathscr{D}^l$. Similarly, construct by induction $\mathscr{D}^i$ for each $0 \leq j < l$, where $\mathscr{D}^j$ is the collection of j-cells which are the bases of cells in $\mathscr{D}^{j+1}$. Set

$$m = \prod \left\{ m' \mid Q_{m',n'} \text{ is the defining predicate of a cell in } \mathscr{D}^j \text{ for } 0 \leq j \leq l \right\}$$
$$n = \max \left\{ n' \mid Q_{m',n'} \text{ is the defining predicate of a cell in } \mathscr{D}^j \text{ for } 0 \leq j \leq l \right\}.$$

This way, if a, a' are in the same coset of the definable predicate $Q_{m',n'}$ of a cell in $\mathscr{D}^j$ $(0 \leq j \leq l)$, then they are in the same coset of $Q_{m,n}$. Choose $\{\lambda_k\}_{k \in K}$ to range over all representations of cosets of $Q_{m,n}$. Let $q_i(x, y)$ enumerate all of the defining polynomials $a_1(x), a_2(x), t - c(x)$ that show up in the cells of $\mathscr{D}^j$ for any j. All of those are polynomials of degree ≤ 1 in the variables x, y. We can split each of them as $q_i(x, y) = p_i(x) - c_i(y)$ where the constant term of q_i is substituted by c_i. This gives us the appropriate finite collection Φ of formulas. From the cell decomposition it is easy to see that when a, a' have the same Φ-type, then they have the same ϕ-type. Thus ϕ can be written as a boolean combination of formulas from Φ. □

Lemma 3.2. *Let $\Phi(x; y)$ be a finite collection of formulas. If ϕ can be written as a boolean combination of formulas from Φ then* $\mathrm{vc}^*(\phi) \leq \mathrm{vc}^*(\Phi)$.

Proof. If a, a' have the same Φ-type over B, then they have the same ϕ-type over B, where B is some parameter set. Therefore the number of ϕ-types is bounded by the number of Φ-types. The bound follows from Lemma 1.11. □

For the remainder of the paper fix $\Phi(x; y)$ to be a collection of formulas as in Lemma 3.1. By the previous lemma, to show that $\mathrm{vc}^*(\phi) \leq |x|$, it suffices to bound

$\mathrm{vc}^*(\Phi) \leq |x|$. More precisely, it is sufficient to show that given a parameter set B of size N, the number of Φ-types over B is $O(N^{|x|})$. Fix such a parameter set B and work with it from now on. We will compute a bound for the number of Φ-types over B.

Consider the finite set $T = T(\Phi, B) = \{c_i(b) \mid b \in B, i \in I\} \subseteq \mathbb{Q}_p$. In this definition B is the parameter set that we have fixed and $c_i(b)$ come from the collection of formulas Φ from the quantifier elimination above. View T as a tree as follows:

Definition 3.3.

- For $c \in \mathbb{Q}_p, \alpha \in \mathbb{Z}$ define the (open) ball

$$B(c, \alpha) = \{c' \in \mathbb{Q}_p \mid \mathrm{val}\,(c' - c) > \alpha\}$$

  of radius α and center c. We also let $B(c, -\infty) = \mathbb{Q}_p$ and $B(c, +\infty) = \emptyset$.
- Define the collection of balls $\mathscr{B} = \{B(t_1, \mathrm{val}(t_1 - t_2))\}_{t_1, t_2 \in T}$. Note that $\mathscr{B}$ is a (directed) boolean algebra of sets in $\mathbb{Q}_p$. We refer to the atoms in that algebra as intervals. Note that the intervals partition $\mathbb{Q}_p$ so any element $a \in \mathbb{Q}_p$ belongs to a unique interval.
- Let's introduce some notation for the intervals. For $t \in T$ and $\alpha_L, \alpha_U \in \mathbb{Z} \cup \{-\infty, +\infty\}$ define

$$\mathrm{I}(t, \alpha_L, \alpha_U) = B(t, \alpha_L) \setminus \bigcup \{B(t', \alpha_U) \mid t' \in T, \mathrm{val}(t' - t) \geq \alpha_U\}$$

  (this is sometimes referred to as the swiss cheese construction). One can check that every interval is of the form $\mathrm{I}(t, \alpha_L, \alpha_U)$ for some values of t, α_L, α_U. The quantities α_L, α_U are uniquely determined by the interval $\mathrm{I}(t, \alpha_L, \alpha_U)$, while t might not be.
- Intervals are a natural construction for trees, however we will require a more refined notion to make Lemma 3.12 below work. Define a larger collection of balls

$$\mathscr{B}' = \mathscr{B} \cup \{B(c_i(b), \mathrm{val}(c_j(b) - c_k(b)))\}_{i,j,k \in I, b \in B}.$$

Similarly to the previous definition, we define a subinterval to be an atom of the boolean algebra generated by $\mathscr{B}'$. Subintervals refine intervals. Moreover, as before, each subinterval can be written as $\mathrm{I}(t, \alpha_L, \alpha_U)$ for some values of t, α_L, α_U. As before, α_L, α_U are uniquely determined by the subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$, while t might not be.

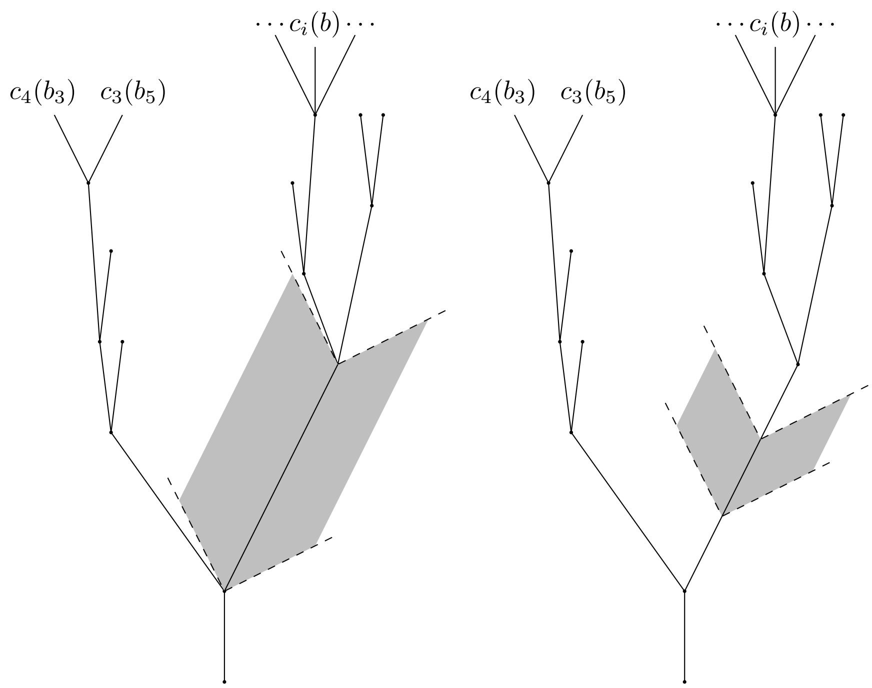

FIGURE 1. A typical interval (left) and subinterval (right) on a tree $\{c_i(b) \mid i \in I, b \in B\}$.

Subintervals are fine enough to make Lemma 3.12 below work while coarse enough to be $O(N)$ small:

Lemma 3.4.

- *There are at most* $2|T| = 2N|I| = O(N)$ *different intervals.*
- *There are at most* $2|T| + |B| \cdot |I|^3 = O(N)$ *different subintervals.*

Proof. Each new element in the tree T adds at most two intervals to the total count, so by induction there can be at most $2|T|$ many intervals. Each new ball in $\mathscr{B}' \backslash \mathscr{B}$

adds at most one subinterval to the total count, so by induction there are at most $|\mathscr{B}'\backslash\mathscr{B}|$ more subintervals than there are intervals. □

Definition 3.5. Suppose $a \in \mathbb{Q}_p$ lies in the interval $\mathrm{I}(t, \alpha_L, \alpha_U)$. Define the T-valuation of a to be $\text{T-val}(a) = \mathrm{val}(a - t)$.

This is a natural notion having the following properties:

Lemma 3.6.

(a) $\text{T-val}(a)$ *is well-defined, independent of choice of* t *to represent the interval.*
(b) If $a \in \mathbb{Q}_p$ *lies in the subinterval* $\mathrm{I}(t, \alpha_L, \alpha_U)$*, then* $\text{T-val}(a) = \mathrm{val}(a - t)$*.*
(c) If $a \in \mathbb{Q}_p$ *lies in the (sub)interval* $\mathrm{I}(t, \alpha_L, \alpha_U)$ *then* $\alpha_L < \text{T-val}(a) \leq \alpha_U$*.*
(d) For any $a \in \mathbb{Q}_p$ *lying in the (sub)interval* $\mathrm{I}(t, \alpha_L, \alpha_U)$ *and* $t' \in T$*:*
- *If* $\mathrm{val}(t - t') \geq \alpha_U$*, then* $\mathrm{val}(a - t') = \text{T-val}(a)$*.*
- *If* $\mathrm{val}(t - t') \leq \alpha_L$*, then* $\mathrm{val}(a - t') = \mathrm{val}(t - t')\,(\leq \alpha_L < \text{T-val}(a))$*.*

Proof. (a)-(c) are clear. For (d) fix $t' \in T$ and suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\mathrm{I}(t, \alpha'_L, \alpha'_U)$. This subinterval lies inside of a unique interval $\mathrm{I}(t, \alpha_L, \alpha_U)$ for some choice of α_L, α_U and by the definition of intervals (or more specifically $\mathscr{B}$):

$$\mathrm{val}(t - t') \geq \alpha_U \iff \mathrm{val}(t - t') \geq \alpha'_U,$$

$$\mathrm{val}(t - t') \geq \alpha_L \iff \mathrm{val}(t - t') \geq \alpha'_L.$$

Therefore without loss of generality we may assume that $a \in \mathbb{Q}_p$ lies in an interval $\mathrm{I}(t, \alpha_L, \alpha_U)$. By (c) and the definition of intervals one of the three following cases has to hold.

Case 1: $\mathrm{val}(t - t') \geq \alpha_U$ and $\text{T-val}(a) < \alpha_U$. Then

$$\mathrm{val}(t - t') \geq \alpha_U > \text{T-val}(a) = \mathrm{val}(a - t),$$

thus $\mathrm{val}(a - t') = \mathrm{val}(a - t) = \text{T-val}(a)$ as needed.

Case 2: $\mathrm{val}(t-t') \geq \alpha_U$ and $\text{T-val}(a) = \alpha_U$. Then

$$\text{T-val}(a) = \mathrm{val}(a-t) = \mathrm{val}(t-t') \geq \alpha_U,$$

thus $\mathrm{val}(a-t') \geq \alpha_U$. The interval $\mathrm{I}(t, \alpha_L, \alpha_U)$ is disjoint from the ball $B(t', \alpha_U)$, so $a \notin B(t', \alpha_U)$, that is, $\mathrm{val}(a-t') \leq \alpha_U$. Combining this with the previous inequality we get that $\mathrm{val}(a-t') = \alpha_U = \text{T-val}(a)$ as needed.

Case 3: $\mathrm{val}(t-t') \leq \alpha_L$. Then

$$\mathrm{val}(t-t') \leq \alpha_L < \text{T-val}(a) = \mathrm{val}(a-t),$$

thus $\mathrm{val}(a-t') = \mathrm{val}(t-t')$ as needed. □

Definition 3.7. Suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$. We say that a is far from the boundary (tacitly: of $\mathrm{I}(t, \alpha_L, \alpha_U)$) if

$$\alpha_L + n \leq \text{T-val}(a) \leq \alpha_U - n.$$

Here n is as in Lemma 3.1. Otherwise we say that it is close to the boundary (of $\mathrm{I}(t, \alpha_L, \alpha_U)$).

Definition 3.8. Suppose $a_1, a_2 \in \mathbb{Q}_p$ lie in the same subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$. We say a_1, a_2 have the same subinterval type if one of the following holds:

- Both a_1, a_2 are far from the boundary and $a_1 - t, a_2 - t$ are in the same $Q_{m,n}$-coset. (Here $Q_{m,n}$ is an in Lemma 3.1.)
- Both a_1, a_2 are close to the boundary and

$$\text{T-val}(a_1) = \text{T-val}(a_2) \leq \mathrm{val}(a_1 - a_2) - n.$$

Definition 3.9. For $c \in \mathbb{Q}_p$ and $\alpha, \beta \in \mathbb{Z}, \alpha < \beta$ define $c \upharpoonright [\alpha, \beta)$ to be the record of the coefficients of c for the valuations between $[\alpha, \beta)$. More precisely write c in its power series form

$$c = \sum_{\gamma \in \mathbb{Z}} c_\gamma p^\gamma \text{ with } c_\gamma \in \{0, 1, \ldots, p-1\}.$$

Then $c \upharpoonright [\alpha, \beta)$ is just $(c_\alpha, c_{\alpha+1}, \dots c_{\beta-1}) \in \{0, 1, \dots, p-1\}^{\beta-\alpha}$.

The following lemma is an adaptation of Lemma 7.4 in [1].

Lemma 3.10. *Fix $m, n \in \mathbb{N}$. For any $x, y, c \in \mathbb{Q}_p$, if*

$$\operatorname{val}(x-c) = \operatorname{val}(y-c) \leq \operatorname{val}(x-y) - n,$$

then $x-c, y-c$ are in the same coset of $Q_{m,n}$.

Proof. Call $a, b \in \mathbb{Q}_p$ similar if $\operatorname{val} a = \operatorname{val} b$ and

$$a \upharpoonright [\operatorname{val} a, \operatorname{val} a + n) = b \upharpoonright [\operatorname{val} b, \operatorname{val} b + n).$$

If a, b are similar then

$$a \in Q_{m,n} \iff b \in Q_{m,n}.$$

Moreover for any $\lambda \in \mathbb{Q}_p^\times$, if a, b are similar then so are $\lambda a, \lambda b$. Thus if a, b are similar, then they belong to the same coset of $Q_{m,n}$. The hypothesis of the lemma force $x-c, y-c$ to be similar, thus belonging to the same coset. $\square$

Lemma 3.11. *For each subinterval there are at most $K = K(Q_{m,n})$ many subinterval types (with K not depending on B or on the subinterval).*

Proof. Let $a, a' \in \mathbb{Q}_p$ lie in the same subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$.

Suppose a, a' are far from the boundary. Then they have the same subinterval type if $a-t, a'-t$ are in the same $Q_{m,n}$-coset. So the number of such subinterval types is bounded by the number of $Q_{m,n}$-cosets.

Suppose a, a' are close to the boundary and

$$\begin{aligned}
&\text{T-val}(a) - \alpha_L = \text{T-val}(a') - \alpha_L < n \text{ and} \\
&a \upharpoonright [\text{T-val}(a), \text{T-val}(a) + n) = a' \upharpoonright [\text{T-val}(a'), \text{T-val}(a') + n).
\end{aligned}$$

Then a, a' have the same subinterval type. Such a subinterval type is thus determined by $\text{T-val}(a) - \alpha_L$ and the tuple $a \restriction [\text{T-val}(a), \text{T-val}(a) + n)$, therefore there are at most np^n many such types.

A similar argument works for a with $\alpha_U - \text{T-val}(a) \leq n$.

Adding all this up we get that there are at most

$$K = (\text{number of } Q_{m,n} \text{ cosets}) + 2np^n$$

many subinterval types. □

The following critical lemma relates tree notions to Φ-types.

Lemma 3.12. *Suppose $d, d' \in \mathbb{Q}_p^{|x|}$ satisfy the follwing three conditions:*

- *For all $i \in I$ $p_i(d)$ and $p_i(d')$ are in the same subinterval.*
- *For all $i \in I$ $p_i(d)$ and $p_i(d')$ have the same subinterval type.*
- *For all $i, j \in I$, $\text{T-val}(p_i(d)) > \text{T-val}(p_j(d))$ iff $\text{T-val}(p_i(d')) > \text{T-val}(p_j(d'))$.*

Then d, d' have the same Φ-type over B.

Proof. There are two kinds of formulas in Φ (see Lemma 3.1). First we show that d, d' agree on formulas of the form $p_i(x) - c_i(y) \in \lambda_k Q_{m,n}$. It is enough to show that for every $i \in I, b \in B$, $p_i(d) - c_i(b), p_i(d') - c_i(b)$ are in the same $Q_{m,n}$-coset. Fix such i, b. For brevity let $a = p_i(d), a' = p_i(d')$ and $Q = Q_{m,n}$. We want to show that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

Suppose a, a' are close to the boundary. Then $\text{T-val}(a) = \text{T-val}(a') \leq \text{val}(a - a') - n$. Using Lemma 3.6d, we have

$$\text{val}(a - c_i(b)) = \text{val}(a' - c_i(b)) \leq \text{T-val}(a) \leq \text{val}(a - a') - n.$$

Lemma 3.10 shows that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

Now, suppose both a, a' are far from the boundary. Let $\mathrm{I}(t, \alpha_L, \alpha_U)$ be the interval containing a, a'. Then we have

$$\alpha_L + n \leq \mathrm{val}(a - t) \leq \alpha_U - n,$$
$$\alpha_L + n \leq \mathrm{val}(a' - t) \leq \alpha_U - n$$

(as being far from the subinterval's boundary also makes a, a' far from interval's boundary). We have either $\mathrm{val}(t - c_i(b)) \geq \alpha_U$ or $\mathrm{val}(t - c_i(b)) \leq \alpha_L$ (as otherwise it would contradict the definition of intervals, or more specifically $\mathscr{B}$).

Suppose it is the first case $\mathrm{val}(t - c_i(b)) \geq \alpha_U$. Then using Lemma 3.6d

$$\mathrm{val}(a - c_i(b)) = \mathrm{val}(a - t) \leq \alpha_U - n \leq \mathrm{val}(t - c_i(b)) - n.$$

So by Lemma 3.10 $a - c_i(b), a - t$ are in the same Q-coset. By an analogous argument, $a' - c_i(b), a' - t$ are in the same Q-coset. As a, a' have the same subinterval type, $a - t, a' - t$ are in the same Q-coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

For the second case, suppose $\mathrm{val}(t - c_i(b)) \leq \alpha_L$. Then using Lemma 3.6d

$$\mathrm{val}(a - c_i(b)) = \mathrm{val}(t - c_i(b)) \leq \alpha_L \leq \mathrm{val}(a - t) - n,$$

so by Lemma 3.10, $a - c_i(b), t - c_i(b)$ are in the same Q-coset. Similarly $a' - c_i(b), t - c_i(b)$ are in the same Q-coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

Next, we need to show that d, d' agree on formulas of the form $\mathrm{val}(p_i(x) - c_i(y)) < \mathrm{val}(p_j(x) - c_j(y))$ (again, referring to the presentation in Lemma 3.1). Fix $i, j \in I, b \in B$. We would like to show the following equivalence:

$$\mathrm{val}(p_i(d) - c_i(b)) < \mathrm{val}(p_j(d) - c_j(b)) \iff \tag{3.1}$$
$$\iff \mathrm{val}(p_i(d') - c_i(b)) < \mathrm{val}(p_j(d') - c_j(b))$$

Suppose $p_i(d), p_i(d')$ are in the subinterval $\mathrm{I}(t_i, \alpha_i, \beta_i)$ and $p_j(d), p_j(d')$ are in the subinterval $\mathrm{I}(t_j, \alpha_j, \beta_j)$. Lemma 3.6d yields the following four cases.

Case 1:

$$\mathrm{val}(p_i(d) - c_i(b)) = \mathrm{val}(p_i(d') - c_i(b)) = \mathrm{val}(t_i - c_i(b))$$

$$\mathrm{val}(p_j(d) - c_j(b)) = \mathrm{val}(p_j(d') - c_j(b)) = \mathrm{val}(t_j - c_j(b))$$

Then it is clear that the equivalence (3.1) holds.

Case 2:

$$\mathrm{val}(p_i(d) - c_i(b)) = \text{T-val}(p_i(d)) \text{ and } \mathrm{val}(p_i(d') - c_i(b)) = \text{T-val}(p_i(d'))$$

$$\mathrm{val}(p_j(d) - c_j(b)) = \text{T-val}(p_j(d)) \text{ and } \mathrm{val}(p_j(d') - c_j(b)) = \text{T-val}(p_j(d'))$$

Then the equivalence (3.1) holds by the third hypothesis of the lemma (that order of T-valuations is preserved).

Case 3:

$$\mathrm{val}(p_i(d) - c_i(b)) = \mathrm{val}(p_i(d') - c_i(b)) = \mathrm{val}(t_i - c_i(b))$$

$$\mathrm{val}(p_j(d) - c_j(b)) = \text{T-val}(p_j(d)) \text{ and } \mathrm{val}(p_j(d') - c_j(b)) = \text{T-val}(p_j(d'))$$

If $p_j(d), p_j(d')$ are close to the boundary, then $\text{T-val}(p_j(d)) = \text{T-val}(p_j(d'))$ and the equivalence (3.1) clearly holds. Suppose then that $p_j(d), p_j(d')$ are far from the boundary.

$$\alpha_j + n \leq \text{T-val}(p_j(d)), \text{T-val}(p_j(d')) \leq \beta_j - n$$

$$\alpha_j < \text{T-val}(p_j(d)), \text{T-val}(p_j(d')) < \beta_j$$

and $\mathrm{val}(t_i - c_i(b))$ lies outside of the (α_j, β_j) by the definition of subinterval (more specifically definition of $\mathscr{B}'$). Therefore (3.1) has to hold. (Note that we always have $\text{T-val}(p_j(d)), \text{T-val}(p_j(d')) \in (\alpha_j, \beta_j]$ by Lemma 3.6c, so we only need the condition on being far from the boundary to avoid the edge case of equality to β_j.)

Case 4:

$$\operatorname{val}(p_i(d)-c_i(b))=\text{T-val}(p_i(d)) \text{ and } \operatorname{val}(p_i(d')-c_i(b))=\text{T-val}(p_i(d'))$$

$$\operatorname{val}(p_j(d)-c_j(b))=\operatorname{val}(p_j(d')-c_j(b))=\operatorname{val}(t_j-c_j(b)).$$

Similar to case 3 (switching i,j). □

The previous lemma gives us an upper bound on the number of types - there are at most $|2I|!$ many choices for the order of T-val, $O(N)$ many choices for the subinterval for each p_i, and K many choices for the subinterval type for each p_i (where K is as in Lemma 3.11), giving a total of $O(N^{|I|})\cdot K^{|I|}\cdot|I|!=O(N^{|I|})$ many types. This implies $\mathrm{vc}^*(\Phi)\leq|I|$. The biggest contribution to this bound are the choices among the $O(N)$ many subintervals for each p_i with $i\in I$. Are all of those choices realized? Intuitively there are $|x|$ many variables and $|I|$ many equations, so once we choose a subinterval for $|x|$ many p_i's, the subintervals for the rest should be determined. This would give the required bound $\mathrm{vc}^*(\Phi)\leq|x|$. The next section outlines this idea formally.

4. Main Proof

An alternative way to write $p_i(c)$ is as a scalar product $\vec{p}_i\cdot\vec{c}$, where $\vec{p}_i$ and $\vec{c}$ are vectors in $\mathbb{Q}_p^{|x|}$ (as $p_i(x)$ is homogeneous linear).

Lemma 4.1. *Suppose we have a finite collection of vectors $\{\vec{p}_j\}_{j\in J}$ with each $\vec{p}_j\in\mathbb{Q}_p^{|x|}$. Suppose $\vec{p}\in\mathbb{Q}_p^{|x|}$ satisfies $\vec{p}\in\operatorname{span}\{\vec{p}_j\}_{j\in J}$, and we have $\vec{c}\in\mathbb{Q}_p^{|x|},\alpha\in\mathbb{Z}$ with $\operatorname{val}(\vec{p}_j\cdot\vec{c})>\alpha$ for all $j\in J$. Then $\operatorname{val}(\vec{p}\cdot\vec{c})>\alpha-\gamma$ for some $\gamma\in\mathbb{N}$. Moreover γ can be chosen independently from $\vec{c},\alpha$ depending only on $\{\vec{p}_j\}_{j\in J}$.*

Proof. For some $c_j\in\mathbb{Q}_p$ for $j\in J$ we have $\vec{p}=\sum_{j\in J}c_j\vec{p}_j$, hence $\vec{p}\cdot\vec{c}=\sum_{j\in J}c_j\vec{p}_j\cdot\vec{c}$. Thus

$$\operatorname{val}(c_j\vec{p}_j\cdot\vec{c})=\operatorname{val}(c_j)+\operatorname{val}(\vec{p}_j\cdot\vec{c})>\operatorname{val}(c_j)+\alpha.$$

Let $\gamma = \max(0, -\max_{j\in J} \operatorname{val}(c_j))$. Then we have

$$\operatorname{val}(\vec{p}\cdot\vec{c}) = \operatorname{val}\left(\sum_{j\in J} c_j\vec{p}_j\cdot\vec{c}\right) \geq$$

$$\geq \min_{j\in J} \operatorname{val}\left(\sum_{j\in J} c_j\vec{p}_j\cdot\vec{c}\right) > \min_{j\in J}\operatorname{val}(c_j) + \alpha \geq \alpha - \gamma$$

as required. □

Corollary 4.2. *Suppose we have a finite collection of vectors* $\{\vec{p}_i\}_{i\in I}$ *with each* $\vec{p}_i \in \mathbb{Q}_p^{|x|}$. *Suppose* $J \subseteq I$ *and* $i \in I$ *satisfy* $\vec{p}_i \in \operatorname{span}\{\vec{p}_j\}_{j\in J}$, *and we have* $\vec{c} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$ *with* $\operatorname{val}(\vec{p}_j\cdot\vec{c}) > \alpha$ *for all* $j \in J$. *Then* $\operatorname{val}(\vec{p}_i\cdot\vec{c}) > \alpha - \gamma$ *for some* $\gamma \in \mathbb{N}$. *Moreover* γ *can be chosen independently from* $J, j, \vec{c}, \alpha$ *depending only on* $\{\vec{p}_i\}_{i\in I}$.

Proof. The previous lemma shows that we can pick such γ for a given choice of i, J, but independent from $\alpha, \vec{c}$. To get a choice independent from i, J, go over all such eligible choices (i ranges over I and J ranges over subsets of I), pick γ for each, and then take the maximum of those values. □

Fix γ according to Corollary 4.2 corresponding to $\{\vec{p}_i\}_{i\in I}$ given by our collection of formulas Φ. (The lemma above is a general result, but we only use it applied to the vectors given by Φ.)

Definition 4.3. Suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$. Define the $\underline{T\text{-floor}}$ of a to be $\text{T-fl}(a) = \alpha_L$.

Definition 4.4. Let $f : \mathbb{Q}_p^{|x|} \longrightarrow \mathbb{Q}_p^I$ with $f(c) = (p_i(c))_{i\in I}$. Define the segment space Sg to be the image of f. Equivalently:

$$\mathrm{Sg} = \left\{(p_i(c))_{i\in I} \mid c \in \mathbb{Q}_p^{|x|}\right\} \subseteq \mathbb{Q}_p^I.$$

Without loss of generality, we may assume that $I = \{1, 2, \ldots, k\}$ (that is the formulas are labeled by consecutive natural numbers). Given a tuple $(a_i)_{i\in I}$ in the segment space, look at the corresponding T-floors $\{\text{T-fl}(a_i)\}_{i\in I}$ and T-valuations $\{\text{T-val}(a_i)\}_{i\in I}$. Partition the segment space by the order types of $\{\text{T-fl}(a_i)\}_{i\in I}$ and $\{\text{T-val}(a_i)\}_{i\in I}$ (as subsets of $\mathbb{Z}$).

Work in a fixed set Sg' of the partition. After relabeling the p_i we may assume that

$$\text{T-fl}(a_1) \geq \text{T-fl}(a_2) \geq \dots \text{ for all } a_i \in \text{Sg}'.$$

Consider the (relabeled) sequence of vectors $\vec{p}_1, \vec{p}_2, \dots, \vec{p}_I$. There is a unique subset $J \subseteq I$ such that the set of all vectors with indices in J is linearly independent, and all vectors with indices outside of J are a linear combination of preceding vectors. (We can pick those using a greedy algorithm for finding a linearly independent subset of vectors.) We call indices in I independent and we call the indices in $I \setminus J$ dependent.

Definition 4.5.

- Denote $\{0, 1, \dots, p-1\}$ as Ct.
- Let Tp be the space of all subinterval types. By Lemma 3.11 we have $|\,\text{Tp}\,| \leq K$.
- Let Sub be the space of all subintervals. By Lemma 3.4 we have $|\,\text{Sub}\,| \leq 3|I|^2 \cdot N = O(N)$.

Definition 4.6. Now, we define a function

$$g_{\text{Sg}'} : \text{Sg}' \longrightarrow \text{Tp}^I \times \text{Sub}^J \times \text{Ct}^{I\setminus J}$$

as follows:

Let $a = (a_i)_{i\in I} \in \text{Sg}'$. To define $g_{\text{Sg}'}(a)$ we need to specify where it maps a in each individual component of the product.

For each a_i record its subinterval type, giving the first component in Tp^I.

For a_j with $j \in J$, record the subinterval of a_j, giving the second component in Sub^J.

For the third component (an element of $\mathrm{Ct}^{I\setminus J}$) do the following computation. Pick a_i with i dependent. Let j be the largest independent index with $j < i$. Record $a_i \restriction [\text{T-fl}(a_j) - \gamma, \text{T-fl}(a_j))$.

Combine $g_{\mathrm{Sg}'}$ for all sets Sg' in our partition of Sg to get a function

$$g : \mathrm{Sg} \longrightarrow \mathrm{Tp}^I \times \mathrm{Sub}^J \times \mathrm{Ct}^{I\setminus J}.$$

Lemma 4.7. *Suppose we have $c, c' \in \mathbb{Q}_p^{|x|}$ such that $f(c), f(c')$ are in the same set Sg' of the partition of Sg and $g(f(c)) = g(f(c'))$. Then c, c' have the same Φ-type over B.*

Proof. Let $a_i = \vec{p}_i \cdot \vec{c}$ and $a'_i = \vec{p}_i \cdot \vec{c}'$ so that

$$f(c) = (p_i(c))_{i\in I} = (\vec{p}_i \cdot \vec{c})_{i\in I} = (a_i)_{i\in I}$$

$$f(c') = (p_i(c'))_{i\in I} = (\vec{p}_i \cdot \vec{c}')_{i\in I} = (a'_i)_{i\in I}$$

For each i we show that a_i, a'_i are in the same subinterval and have the same subinterval type, so the conclusion follows by Lemma 3.12 (the tuples $f(c), f(c')$ are in the same partition ensuring the proper order of T-valuations for the 3rd condition of the lemma). Tp records the subinterval type of each element, so if $g(\bar{a}) = g(\bar{a}')$ then a_i, a'_i have the same subinterval type for all $i \in I$. Thus it remains to show that a_i, a'_i lie in the same subinterval for all $i \in I$. Suppose i is an independent index. Then by construction, Sub records the subinterval for a_i, a'_i, so those have to belong to the same subinterval. Now suppose i is dependent. Pick the largest $j < i$ such that j is independent. We have $\text{T-fl}(a_i) \leq \text{T-fl}(a_j)$ and $\text{T-fl}(a'_i) \leq \text{T-fl}(a'_j)$. Moreover $\text{T-fl}(a_j) = \text{T-fl}(a'_j)$ as a_j, a'_j lie in the same subinterval (using the earlier part of the argument as j is independent).

Claim 4.8. $\mathrm{val}(a_i - a'_i) > \text{T-fl}(a_j) - \gamma$

Proof. Let K be the set of the independent indices less than i. Note that by the definition for dependent indices we have $\vec{p}_i \in \operatorname{span}\{\vec{p}_k\}_{k\in K}$. We also have

$$\mathrm{val}(a_k - a'_k) > \text{T-fl}(a_k) \text{ for all } k \in K$$

as a_k, a'_k lie in the same subinterval (using the earlier part of the argument as k is independent). Now $\mathrm{val}(a_k - a'_k) > \text{T-fl}(a_j)$ for all $k \in K$ by monotonicity of $\text{T-fl}(a_k)$. Moreover $a_k - a'_k = \vec{p}_k \cdot \vec{c} - \vec{p}_k \cdot \vec{c}' = \vec{p}_k \cdot (\vec{c} - \vec{c}')$. Combining the two, we get that $\mathrm{val}(\vec{p}_k \cdot (\vec{c} - \vec{c}')) > \text{T-fl}(a_j)$ for all $k \in K$. Now observe that $K \subseteq I, i \in I, \vec{c} - \vec{c}' \in \mathbb{Q}_p^{|x|}, \text{T-fl}(a_j) \in \mathbb{Z}$ satisfy the requirements of Lemma 4.2, so we apply it to obtain $\mathrm{val}(\vec{p}_i \cdot (\vec{c} - \vec{c}')) > \text{T-fl}(a_j) - \gamma$. Similarly to before, we have $\vec{p}_i \cdot (\vec{c} - \vec{c}') = \vec{p}_i \cdot \vec{c} - \vec{p}_i \cdot \vec{c}' = a_i - a'_i$. Therefore we can conclude that $\mathrm{val}(a_i - a'_i) > \text{T-fl}(a_j) - \gamma$ as needed, finishing the proof of the claim. □

Additionally a_i, a'_i have the same image in the Ct component, so we have $\mathrm{val}(a_i - a'_i) > \text{T-fl}(a_j)$. We now would like to show that a_i, a'_i lie in the same subinterval. As $\text{T-fl}(a_i) \leq \text{T-fl}(a_j)$, $\text{T-fl}(a'_i) \leq \text{T-fl}(a'_j)$ and $\text{T-fl}(a_j) = \text{T-fl}(a'_j)$ we have that $\mathrm{val}(a_i - a'_i) > \text{T-fl}(a_i)$ and $\mathrm{val}(a_i - a'_i) > \text{T-fl}(a'_i)$. Suppose that a_i lies in the subinterval $\mathrm{I}(t, \text{T-fl}(a_i), \alpha_U)$ and that a'_i lies in the subinterval $\mathrm{I}(t', \text{T-fl}(a'_i), \alpha'_U)$. Without loss of generality assume that $\text{T-fl}(a_i) \leq \text{T-fl}(a'_i)$. As $\mathrm{val}(a_i - a'_i) > \text{T-fl}(a'_i)$, this implies that $a_i \in B(a'_i, \text{T-fl}(a'_i))$. Then $a_i \in B(t', \text{T-fl}(a'_i))$ as $\mathrm{val}(a_i - t') > \text{T-fl}(a'_i)$. This implies that $B(t, \text{T-fl}(a_i)) \cap B(t', \text{T-fl}(a'_i)) \neq \emptyset$ as they both contain a_i. As balls are directed, the non-zero intersection means that one ball has to be contained in another. Given our assumption that $\text{T-fl}(a_i) \leq \text{T-fl}(a'_i)$, we have $B(t, \text{T-fl}(a_i)) \subseteq B(t', \text{T-fl}(a'_i))$. For the subintervals to be disjoint we need $\mathrm{I}(t, \text{T-fl}(a_i), \alpha_U) \cap B(t', \text{T-fl}(a'_i)) = \emptyset$. But $\mathrm{val}(t' - a_i) > \text{T-fl}(a'_i)$ implying that $a_i \in \mathrm{I}(t, \text{T-fl}(a_i), \alpha_U) \cap B(t', \text{T-fl}(a'_i))$ giving a contradiction. Therefore the subintervals coincide finishing the proof. □

Corollary 4.9. *The dual* vc*-density of* $\Phi(x, y)$ *is* $\leq |x|$.

Proof. Suppose we have $c, c' \in \mathbb{Q}_p^{|x|}$ such that $f(c), f(c')$ are in the same partition and $g(f(c)) = g(f(c'))$. Then by the previous lemma c, c' have the same Φ-type. Thus the number of possible Φ-types is bounded by the size of the range of g times the number of possible partitions

$$(\text{number of partitions}) \cdot |\operatorname{Tp}|^{|I|} \cdot |\operatorname{Sub}|^{|J|} \cdot |\operatorname{Ct}|^{|I-J|}.$$

There are at most $(|2I|!)^2$ many partitions of Sg, so in the product above, the only component dependent on B is

$$|\operatorname{Sub}|^{|J|} \leq (N \cdot 3|I|^2)^{|J|} = O(N^{|J|}).$$

Every p_i is an element of a $|x|$-dimensional vector space, so there can be at most $|x|$ many independent vectors. Thus we have $|J| \leq |x|$ and the bound follows. □

Corollary 4.10 (Theorem 2.6)**.** *The $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ satisfies* $\mathrm{vc}(n) = n$.

Proof. The previous lemma implies that $\mathrm{vc}^*(\phi) \leq \mathrm{vc}^*(\Phi) \leq |x|$. As choice of ϕ was arbitrary, this implies that the vc-density of any formula is bounded by the arity of x. □

This proof relies heavily on the linearity of the defining polynomials a_1, a_2, c in the cell decomposition result (see Definition 2.4). Linearity is used to separate the x and y variables as well as for Corollary 4.2 to reduce the number of independent factors from $|I|$ to $|x|$. The paper [5] has cell decomposition results for more expressive reducts of $\mathbb{Q}_p$, including, for example, restricted multiplication. While our results don't apply to them directly, it is this author's hope that similar techniques can be used to also compute the vc-function for those structures.

Another interesting question is whether the reduct studied in this paper has the VC 1 property (see [1], 5.2 for the definition). If so, this would imply the linear vc-density bound directly. The techniques used in paper [1] make it seem likely that the reduct has VC 2 property. While there are techniques for showing that a structure has a given VC property, less is known about showing that a structure doesn't have a given VC property. Perhaps the simple structure of the $\mathcal{L}_{aff}$-reduct can help understand this property better.

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