REDUCT

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Claim 0.1. *Suppose we have a collection of vectors* $\{\vec{p}_i\}_{i\in I}$ *with each* $\vec{p}_i \in \mathbb{Q}_p^m$*. Pick a subset* $J \subset I$ *and* $j \in I$ *such that*

$$\vec{p}_j \in \operatorname{span}\{\vec{p}_i\}_{i\in J}$$

Suppose we have $\vec{x} \in \mathbb{Q}_p^m, \alpha \in \mathbb{Z}$ *with*

$$\operatorname{val}(\vec{p}_i \cdot \vec{x}) > \alpha \text{ for all } i \in J$$

Then

$$\operatorname{val}(\vec{p}_j \cdot \vec{x}) > \alpha - \gamma$$

for some $\gamma \in \mathbb{Z}^{\geq 0}$*. Moreover* γ *can be chosen independent of choice of* $J, j, \vec{x}, \alpha$ *depending only on* $\{\vec{p}_i\}_{i\in I}$*.*

Suppose we have a finite $T \subset \mathbb{Q}_p, V \subset \mathbb{Z}$. We view it as a tree (T, V) as follows. Branches through the tree are elements of T. Branching points are defined by open balls as follows. Branching points is $(t_1, \operatorname{val}(t_1 - t_2))$ for all $t_1, t_2 \in T$. Branching point is also (t, v) for all $t \in T, v \in V$. An interval is two balls $(t_1, v_1) \supset (t_2, v_2)$ with no balls in between. There are at most $2|T| \cdot |V|$ different intervals.

We work with a collection of formulas $\Psi(\bar{x}, \vec{y})$ of the form

$$\vec{p}_i \cdot \bar{x} + c_i(\vec{y}) \in \lambda_i Q_i$$
$$\operatorname{val}(\vec{p}_i \cdot \bar{x} + c_i(\bar{y}))\ \square_i\ v_i$$

for $i \leq I$ with $|\bar{x}| = m$ with $Q_i = Q_{n_i, m_i}$ for some n_i, m_i. We work with a parameter set B of size N. Consider a tree (T, V)

$$T = \{c_i(b) \mid b \in B, i \leq I\}$$
$$V = \{v_i \mid i \leq I\}$$

This tree has at most $O(N) = N \cdot I \cdot I$ many intervals.

For some $x, x' \in \mathcal{M}$ we say they have the same Ψ-type if they have the same Ψ type over B.

For some $x, x' \in \mathcal{M}$ we say they have the same Q-type if

- $x + c_i(b)$ is in the same Q^i-coset as $x' + c_i(b)$ for all $i \leq I, b \in B$
- $\operatorname{val}(x + c_i(b)) \square_i\ v_i$ iff $\operatorname{val}(x' + c_i(b)) \square_i\ v_i$ for all $i \leq I, b \in B$

Lemma 0.2. $c, c' \in \mathcal{M}^m$ *have the same* Ψ*-type if all* $p_i(c), p_i(c')$ *have the same* Q^i*-type*

Lemma 0.3. *For any $Q = Q_{n,m}$ there exists θ_Q such that for all $\theta \geq \theta_Q$ the following holds. Suppose we have $x, y, c \in \mathcal{M}$ such that*

$$\operatorname{val}(x-y) - \theta > \operatorname{val}(x-c) = \operatorname{val}(y-c)$$

Then $x-c$ and $y-c$ lie in the same coset of Q.

Lemma 0.4. *Fix θ sufficiently large to satisfy previous lemma for all Q_i. Define an enumeration of near balls*

$$B_1(c,\alpha), B_2(c,\alpha), \ldots B_{N_\theta}(c,\alpha)$$

Definition 0.5. Let $c \in \mathcal{M}$. It lies in our tree between (c_L, α_L) and (c_U, α_U). Suppose c lies in one of the near balls in a branching point above or below it. Then define its interval type to be the index of that near ball. Otherwise define its interval type to be the coset of $c - c_U$ of Q_i for all $i \in I$. Denote the space of all the possible branch types Bt. We have

$$|\operatorname{Bt}| = N_\theta + \prod_{i \leq I} (\text{number of cosets of } Q_i)$$

depends only on Ψ, independent from B.

Lemma 0.6. *If c, c' are in the same interval and have the same interval type then they have the same Q-type.*

Definition 0.7. For $c \in \mathcal{M}$ and $\alpha, \beta \in \mathbb{Z}$ let $c \restriction [\alpha, \beta] \in \mathbb{Z}_p^{\beta-\alpha}$ be the record of coefficients of c for valuations between α, β. More precisely write c in its power series form

$$c = \sum_{\gamma \in Z} c_\gamma p^\gamma \text{ with } c_\gamma \in \mathbb{Z}_p$$

Then $c \restriction [\alpha, \beta]$ is just $(c_\alpha, c_{\alpha+1}, \ldots c_\beta)$.

For any c define $F(c)$, the floor of c to be the valuation of the largest branching point below c.

Let $f : \mathcal{M}^n \longrightarrow \mathcal{M}^I$ with $f(\bar{c}) = (p_i(\bar{c}))_{i \leq I}$. Define segment space Sg to be the image of f.

For some element (a_i) in segment space look at floors $F(a_i)$. Partition the segment space by order type of $\{F(a_i)\}$. Work in a fixed partition Sg'. After relabeling we may assume that

$$F(a_1) \geq F(a_2) \geq \ldots$$

Consider (relabeled) sequence of vectors $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_I$. Choose the unique subset of linearly independent vectors $J \subset I$. For any index $i \in I$ we call it independent if $i \in J$ and we call it dependent otherwise.

For all a_i record its interval type.

For a_i with i independent, record the interval of a_i.

Pick a_i with i dependent. Let j be the largest independent index with $j < i$. Record $a_i \restriction [F(a_j) - \gamma, F(a_j)]$.

Combining all the records defines a function

$$g : \operatorname{Sg}' \longrightarrow \operatorname{Bt}^I \times \operatorname{Pt}^m \times \operatorname{Ct}^I$$

We claim that for $\bar{a}, \bar{a}' \in \operatorname{Sg}'$ if we have $g(\bar{a}) = g(\bar{a}')$ then all a_i, a_i' have the same Q-type.

Proof. Suppose we have $\bar{a}, \bar{a}' \in \mathrm{Sg}'$ that map to the same image by g. Suppose i is independent. Then by construction, a_i, a_i' map to the same interval of the tree and have the same interval type. Thus they have the same Q-type. Otherwise, suppose i is dependent. Pick largest $j < i$ such that j is independent. We have $F(a_i) \leq F(a_j)$ and $F(a_i') \leq F(a_j')$. Moreover $F(a_j) = F(a_j')$ as they are mapped to the same interval (as j is independent).

Claim 0.8. $\mathrm{val}(a_i - a_i') > F(a_j) - \gamma$

Proof. Let $\bar{x}, \bar{x}' \in \mathbb{Q}_p^m$ be some elements with

$$\begin{aligned} \vec{p}_k \cdot \bar{x} &= a_k \\ \vec{p}_k \cdot \bar{x}' &= a_k' \text{ for all } k \leq I \end{aligned}$$

Let J be the set of independent indices less than i. We have

$$\mathrm{val}(a_k - a_k') > F(a_k) \text{ for all } k \leq J$$

as for independent indices a_k, a_k' lie in the same interval.

$$\begin{aligned} &\mathrm{val}(a_k - a_k') > F(a_j) \text{ for all } k \leq J \text{ by monotonicity of } F(a_k) \\ &\mathrm{val}(\vec{p}_k \cdot \bar{x} - \vec{p}_k \cdot \bar{x}') > F(a_j) \text{ for all } k \leq J \\ &\mathrm{val}(\vec{p}_k \cdot (\bar{x} - \bar{x}')) > F(a_j) \text{ for all } k \leq J \end{aligned}$$

J and i match the requirements of the claim above by independence so we conclude

$$\begin{aligned} &\mathrm{val}(\vec{p}_i \cdot (\bar{x} - \bar{x}')) > F(a_j) - \gamma \\ &\mathrm{val}(\vec{p}_i \cdot \bar{x} - \vec{p}_i \cdot \bar{x}') > F(a_j) - \gamma \\ &\mathrm{val}(a_i - a_i')) > F(a_j) - \gamma \end{aligned}$$

as needed. □

By record of continuations (which a_i, a_i' agree on) we have

$$a_i = a_i' \restriction F(a_j)$$

As $F(a_i) \leq F(a_j)$, a_i, a_i' have to lie in the same interval. They also agree on interval type. Thus they have the same Q-type.

□

Now suppose we have $c, c' \in \mathcal{M}^m$ such that $g(f(c)) = g(f(c'))$. Then $f(c)$ components have the same Q-type as $f(c')$ components. Then c, c' have the same Ψ-type. Thus the number of possible Ψ-types is bound by the size of the range of g.

$$\begin{aligned} &|\,\mathrm{Ct}\,| = p^\gamma \\ &|\,\mathrm{Pt}\,| \leq N \cdot I^2 \text{ (the only component dependent on } N) \end{aligned}$$

Moreover we need at most $I!$ many partitions of Sg. This gives us

$$I! \cdot |Bt|^I \cdot (N \cdot I^2)^m \cdot p^{\gamma I} = O(N^m)$$

upper bound for the possible number of Ψ-types.

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