Fix a formula $\phi(x,y)$ that is a minimal extension $M/\{x,y\}$.

- $\dim(M/\{x,y\}) = -\epsilon$
- there are no edges between x and y.
- there are no edges between x.

Let $Y = \dim(y)$

Let n be such that $n\epsilon < Y$ but $(n+1)\epsilon > Y$. Fix a parameter set A, strongly embedded and disconnected (thus indiscernible).

1. Lower bound

Pick a finite $B \subset A^{|x|}$.

Consider the graph $x \cup y$. If y/x is not a proper extension, then ϕ has no realizations over B. If it is, abstractly make a realization of y, label it by b.

Fix arbitrary elements of B, label them a_i for $i = [0..n]$, with each $|a_i| = |x|$. Abstractly adjoin $M_i/\{a_i,b\} = M/\{x,y\}$ for each i. Let $\bar{M} = \bigcup M_i$.

Claim: $A \leq \bar{M}$. It's total dimension is $Y - n\epsilon > 0$ and all subextensions are positive as well.

Thus a copy of $\bar{M}$ can be embedded over A into our ambient model. Choice of elements of B was arbitrary, thus showing that any n elements can be traced out. Thus we have $O(|B|^n)$ many traces showing vc-density of n.

2. Upper bound

Pick a trace of $\phi(x,y)$ on $A^{|x|}$ by a parameter b.

$$B = \left\{a \in A^{|x|} \mid \phi(a,b)\right\}$$

Pick $B' \subset B$, ordered $B' = \{a_1, \ldots\}$ such that

$$a_i \cap \bigcup_{j<i} a_j \neq \emptyset$$

Let $M_i/\{a_i,b\}$ be a witness of $\phi(a_i,b)$. Let $\bar{M} = \bigcup M_i$. Consider $\bar{M}/A$.

Claim: $\dim(\bar{M}/A)$ is minimized when all M_i are disjoint. Suppose not. Suppose there is j such that

$$M_j \cap \bigcup_{i \neq j} M_i \neq \emptyset$$

Apply the key lemma to see that making it disjoint would reduce dimension contradicting minimality.

Thus as A is strong we need $|B'|\epsilon < Y$. This gives us $|B'| \leq n$. Finally we need to relate $|B'|$ to $|B|$. Take $A' = \bigcup B' =\subset A$. Note that $\bigcup B' = \bigcup B$, so all elements in B have to have components from some elements of B'. $|A'| \leq n|x|$, thus the size of B is at most $(n|x|)^|x|$.

Suppose we have $C \subset A^{|x|}$, finite with $|C| = N$. Then the underlying set $C' = \bigcup C$ has size at most $N|x|$. Suppose we pick a subset $B \subset C$ with underlying set $B' = \bigcup B$ with size at most $n|x|$. How many such choices are there for B? There are $\binom{N}{n}$ choices for underlying set (it has to be generated by n elements from C).

Given set $|A| = N$ how many subsets of $B \subset A^{|x|}$ can we pick such that with $A' = \bigcup B$ we have $|A'| \leq n|x|$? $\binom{N}{n}$

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