

QUANTIFIER ELIMINATION IN SHELAH-SPENCER GRAPHS

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ABSTRACT. We simplify [?] proof of quantifier elimination in Shelah-Spencer graphs.

1. INTRODUCTION

Laskowski's paper [?] provides a combinatorial proof of quantifier elimination in Shelah-Spencer graphs. Here we provide a simplification of the proof using only maximal chains and avoiding the use of proposition 3.1 and technical lemmas of section 4.

We will use notation of [?], in particular things like \mathbf{K}_α , $\delta(\mathcal{A}/\mathcal{B})$, $X_m(\mathcal{A})$, S_α , maximal embedding, etc. However we will give a different definition of Y .

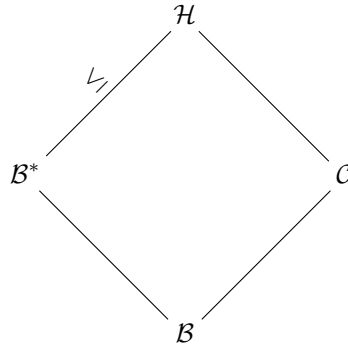
2. OMITTING LEMMA

Definition 2.1. Let $\mathcal{M} \models S_\alpha$, $\mathcal{B} \in \mathbf{K}_\alpha$, embedding $f: \mathcal{B} \rightarrow \mathcal{M}$, Φ finite subset of \mathbf{K}_α

- (1) Say that f *omits* Φ if there are no $\mathcal{C} \in \Phi$ and $g: \mathcal{C} \rightarrow \mathcal{M}$ extending f .
- (2) Say that f *admits* Φ if for every $\mathcal{C} \in \Phi$ there is $g: \mathcal{C} \rightarrow \mathcal{M}$ extending f .

Note 2.2. Take notation as above and a structure $\mathcal{C} \in \mathbf{K}_\alpha$ extending \mathcal{B} . Then f doesn't omit $\{\mathcal{C}\}$ iff f admits $\{\mathcal{C}\}$.

Definition 2.3. Fix $\mathcal{B}, \mathcal{C} \in \mathbf{K}_\alpha$, and $m \in \omega$ such that $|C \setminus B| < m$. Define $Z(\mathcal{B}, \mathcal{C}, m)$ to be all $\mathcal{B}^* \in X_m(\mathcal{B})$ such that there are no \mathcal{H} with $|H \setminus B^*| < m$ satisfying

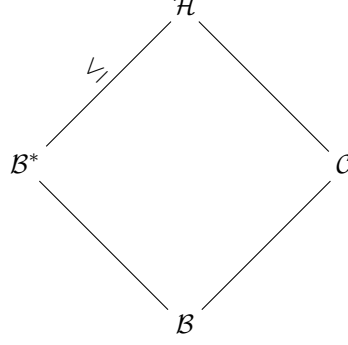


Lemma 2.4. Let $\mathcal{B}, \mathcal{C} \in \mathbf{K}_\alpha$, and $m \in \omega$ such that $|C \setminus B| < m$. Also let $\mathcal{M} \models S_\alpha$ and $f: \mathcal{B} \rightarrow \mathcal{M}$ an embedding. The following are equivalent:

- (1) f *omits* $\{\mathcal{C}\}$.
- (2) There exists $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into \mathcal{M} over f .

Proof. For the proof we identify \mathcal{B} with $f(\mathcal{B})$, i.e. for ease of notation assume that $\mathcal{B} \subset \mathcal{M}$.

(1) \Rightarrow (2) By remark 5.3 of [?] there is some $B^* \in X_m(\mathcal{B})$ maximally embeddable in \mathcal{M} over f . Such embedding is unique by Lemma 3.8 of [?]. Again, we identify B^* with its maximal embedding into \mathcal{M} . To show (2) we need to verify that $B^* \in Z(\mathcal{B}, \mathcal{C}, m)$. Suppose not. Then there is \mathcal{H} with $|H \setminus B^*| < m$ satisfying



As $B^* \leq \mathcal{H}$ and $\mathcal{B} \subset \mathcal{M}$ we can embed \mathcal{H} into \mathcal{M} (as $\mathcal{M} \models S_\alpha$). But this would witness \mathcal{C} extending \mathcal{B} in \mathcal{M} which is impossible as we assumed that f omits $\{\Phi\}$.

(2) \Rightarrow (1) Suppose f doesn't omit $\{C\}$. Then by the note above f admits $\{C\}$, i.e. there is an embedding of \mathcal{C} into \mathcal{M} over f . We identify \mathcal{C} with the image of that embedding. Similarly we identify B^* with the image of its maximal embedding over f . That is we may assume $\mathcal{C}, B^* \subset \mathcal{M}$. Let H be the substructure of \mathcal{M} induced by vertices $C \cup B^*$. As $|C \setminus B| < m$ we have $|H \setminus B^*| < m$. B^* is m -strong by remark 5.3 of [?]. This forces $B^* \leq H$. But this contradicts the fact that $B^* \in Z(\mathcal{B}, \mathcal{C}, m)$. \square

Corollary 2.5. *With the setup of the previous lemma, the following are equivalent:*

- (1) f admits $\{C\}$.
- (2) There exists $B^* \in X_m(\mathcal{B}) \setminus Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into \mathcal{M} over f .

For quantifier elimination we need to track multiple structures being admitted and omitted, hence the following definition.

Definition 2.6. Let $\mathcal{B} \in \mathbf{K}_\alpha$, Φ, Γ finite subsets of \mathbf{K}_α , and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|C \setminus B| < m$. Define

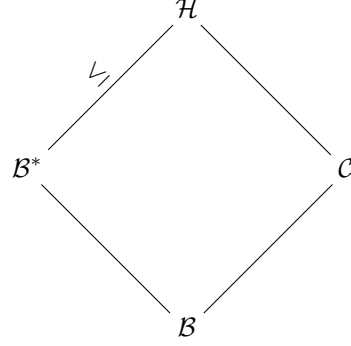
$$Y(\mathcal{B}, \Phi, \Gamma, m) = \{B^* \in X_m(\mathcal{B}) \mid \forall \mathcal{C} \in \Phi \ B^* \in Z(\mathcal{B}, \mathcal{C}, m) \text{ and } \forall \mathcal{D} \in \Gamma \ B^* \notin Z(\mathcal{B}, \mathcal{D}, m)\}$$

Lemma 2.7. *Let $\mathcal{B} \in \mathbf{K}_\alpha$, Φ, Γ finite subsets of \mathbf{K}_α , and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|C \setminus B| < m$. The following are equivalent:*

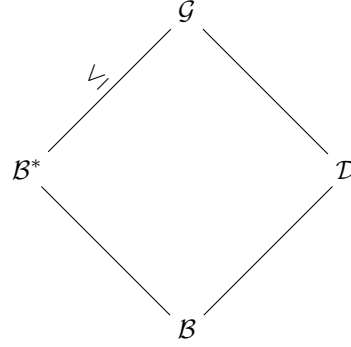
- (1) f omits Φ and admits Γ .
- (2) There exists $B^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$ maximally embeddable into \mathcal{M} over f .

Proof. (1) \Rightarrow (2) Identify \mathcal{B} with $f(\mathcal{B})$, i.e. for ease of notation assume that $\mathcal{B} \subset \mathcal{M}$. By remark 5.3 of [?] there is some $B^* \in X_m(\mathcal{B})$ maximally embeddable in \mathcal{M} over f . Such embedding is unique by Lemma 3.8 of [?]. Again, we identify B^* with its maximal embedding into \mathcal{M} . To show (2) we need to verify that $B^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$.

Suppose not. Two things can go wrong. First, there can be \mathcal{H} with $|H \setminus B^*| < m$ and $\mathcal{C} \in \Phi$ satisfying



As $\mathcal{B}^* \leq \mathcal{H}$ and $\mathcal{B} \subset \mathcal{M}$ we can embed \mathcal{H} into \mathcal{M} (as $\mathcal{M} \models S_\alpha$). But this would witness \mathcal{C} extending \mathcal{B} in \mathcal{M} which is impossible as we assumed that f omits Φ . Another thing that could go wrong is that there could be $\mathcal{D} \in \Gamma$ and no \mathcal{G} with $|G \setminus B^*| < m$ satisfying



As f admits

□

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