SOME VC-DENSITY COMPUTATIONS IN SHELAH-SPENCER GRAPHS

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Abstract. We compute vc-densities of minimal extension formulas in Shelah-Spencer random graphs.

We fix the density of the graph α.

Lemma 0.1. *For any $\mathcal{A} \in \boldsymbol{K}_\alpha$ and $\epsilon > 0$ there exists an $\mathcal{B}$ such that $(\mathcal{A}, \mathcal{B})$ is minimal and $\delta(\mathcal{B}/\mathcal{A}) < \epsilon$.*

Proof. Let m be an integer such that $m\alpha < 1 < (m+1)\alpha$. Suppose $\mathcal{A}$ has less than $m+1$ vertices. Make a construction $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_{i+1}$ is $\mathcal{A}_i$ with one extra vertex connected to every single vertex of A_i. Stop when the total number of vertices is $m+1$. Proceed as in [**?**] 4.1. Resulting construction is still minimal. $\square$

Lemma 0.2. *Let $\mathcal{A}_1 \subset \mathcal{B}_1$ and $\mathcal{A}_2 \subset \mathcal{B}_2$ be $\boldsymbol{K}_\alpha$ structures with $(\mathcal{A}_2, \mathcal{B}_2)$ a minimal pair with $\epsilon = \delta(\mathcal{B}_2/\mathcal{A}_2)$. Let M be some ambient structure. Fix embeddings of $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2$ into M. Assume that it is not that case that $\mathcal{A}_2 \subset \mathcal{B}_2$ and $\mathcal{A}_1$ is disjoint from $\mathcal{A}_2$ (No!). Now consider all possible embeddings $f \colon \mathcal{B}_2 \to M$ over $\mathcal{A}_1$. Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{B}_f = \mathcal{B}_1 \cup f(\mathcal{B}_2)$ with $\delta_f = \delta(\mathcal{B}_f/\mathcal{A})$. Then δ_f is at most $\delta(\mathcal{B}_1 \cup \mathcal{A}/\mathcal{A}) + \epsilon$*

Fix an embedding f. It induces the following substructure diagram in M. Denote

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_1 \cup \mathcal{A}_2 \\
\mathcal{B}^*_f &= \mathcal{B}_1 \cup f(\mathcal{B}_2) \\
\mathcal{B}^*_1 &= \mathcal{B}_1 \cup \mathcal{A} \\
\mathcal{B}^*_2 &= f(\mathcal{B}_2) \cup \mathcal{A} \\
\mathcal{B}^* &= \mathcal{B}^*_1 \cap \mathcal{B}^*_2
\end{aligned}$$

$$\begin{tikzcd}
& \mathcal{B}_f \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{B}_1^* \arrow[dr, dash] & & \mathcal{B}_2^* \arrow[dl, dash] \\
& \mathcal{B}^* \arrow[d, dash] & \\
& \mathcal{A} &
\end{tikzcd}$$

From the diagram we see that

$$\delta(\mathcal{B}_f/\mathcal{A}) \leq \delta(\mathcal{B}_1^*/\mathcal{A}) + \delta(\mathcal{B}_2^*/\mathcal{B}^*)$$

Thus all we need to do is to verify that

$$\delta(\mathcal{B}_2^*/\mathcal{B}^*) \leq \epsilon$$

Let $\mathcal{B}'$ denote graph induced on all the vertices in $(f(B_2)/B_1) \cup A_2$. Then $\mathcal{B}'$ is a substructure of $\mathcal{B}_2$ over $\mathcal{A}_2$. By minimality we get that $\delta(\mathcal{B}'/\mathcal{A}_2) \leq \epsilon$. We need to show $\delta(\mathcal{B}_2^*/\mathcal{B}^*) \leq \delta(\mathcal{B}'/\mathcal{A}_2)$. Do the vertex computation

$$\begin{aligned} B_2^* - B^* &= \\ f(B_2) - (B_1 \cap f(B_2)) - A &= \\ f(B_2) - B_1 - A &= \\ f(B_2) - B_1 - A_2 & \end{aligned}$$

and

$$B' - A_2 = f(B_2) - B_1 - A_2$$

So the sets of the extra vertices in the extension are the same. The base $\mathcal{B}_2^*/\mathcal{B}^*$ is larger so we can introduce some extra edges but no new vertices. This means that $\delta(\mathcal{B}_2^*/\mathcal{B}^*) \leq \delta(\mathcal{B}'/\mathcal{A}_2)$ giving us the original statement.

Let $\phi(x, y)$ be a formula in a random graph with $|x| = |y| = 1$ saying that there exists $\mathcal{D}$ over $\mathcal{C} = \{x, y\}$ such that $(\mathcal{D}, \mathcal{C})$ is minimal with relative dimension ϵ. Let N be such that $N\epsilon < 1 < (N+1)\epsilon$. Then we argue that $vc(\phi) = N$.

Fix a m-strong (for any $m > |D|$) set of non-connected vertices A. Fix some $a*$. We invesitgate the trace of $\phi(x, a*)$ on A. Suppose we have $a_1, \ldots, a_k$ satisfying $\phi(a_i, a^*)$ as witnessed by $D_i/\{a_i, a*\}$. Let $\mathcal{D}^* = \bigcap \mathcal{D}_i$ and $\mathcal{C}^*$

Call $\mathcal{M}$ n-composite embedding if there are distinct vertices $a_1, \ldots a_n$ and $a*$ in M and there are an embeddings $\mathcal{D} \longrightarrow \mathcal{M}$ with $\mathcal{C}$ going to $\{a_i, a^*\}$. Image of i-th embedding is denoted $\mathcal{D}_i$. Note that images of embeddings can intersect each other or a_j's. Consider $\mathcal{D}^* = \bigcap \mathcal{D}_i$ and $\mathcal{C}^* = \{a_1, \ldots a_n, a^*\}$. Dimension of M is $\delta(\mathcal{D}^*/\mathcal{C}^*)$.

Lemma: Dimension of n-composite embedding is at most $-n\epsilon$.

Note: if $\mathcal{D}_i$ are disjoint over $\mathcal{C}^*$ then the dimension is exactly $-n\epsilon$.

Take n-composite embedding with maximal dimension. Suppose it is larger than $-n\epsilon$. Without loss of generality we may assume $\mathcal{D}_n$ intersects with $\mathcal{D}_1 \cup \ldots \cup \mathcal{D}_{n-1}$ over $\mathcal{C}^*$. Consider two cases. First, suppose that there is some element in $\mathcal{D}_n$ outside of $\mathcal{D}_1 \cup \ldots \cup \mathcal{D}_{n-1}$. Let $\mathcal{B}_1 = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_{n-1}$. Let $\mathcal{A}_1 = \{a_1, \ldots a_{n-1}\} \cup \{a*\}$. Let $\mathcal{B}_2 = \mathcal{D}_n$. Let $\mathcal{A}_2 = \{a_n, a*\}$.

Lemma applies to the above. Above dimension is minimized when $\mathcal{D}_n$ is disjoint. Contradiction.

Second, suppose that $\mathcal{D}_n \subseteq \mathcal{B}_1$. In particular $a_n \in \mathcal{B}_1$. Consider

Consider sets $\mathcal{B}_1 \ldots \mathcal{B}_n$ with

(1) $a_i \in \mathcal{B}_i$
(2) $a_i \in A$
(3) $a_i \neq a_j$
(4) $a* \in \bigcap \mathcal{B}_i$

and s.t. $\mathcal{B}_i/\{a*, a_i\}$ is isomorphic to $\mathcal{B}/\mathcal{A}$. We look at all the possible embeddings with those properties. We argue that a disjoint configuration minimizes total dimension of the whole construction.

We argue by induction on n. Fix an embedding $\mathcal{B}_1, \ldots \mathcal{B}_n$ and consider possible choices for $\mathcal{B}_{n+1}, a_{n+1}$. We can pick a_n to be an element of A not used so far and embed $\mathcal{B}_{n+1}$ over $\{a*, a_i\}$ disjoint from the entire construction. On the other hand suppose it is embedded such that there is an intersection. We set up to apply the previous lemma. Let

$$\mathcal{B}_1 = \bigcup_{1..n} \mathcal{B}_i$$
$$\mathcal{A}_1 = \{a_1, \ldots a_n\}$$
$$\mathcal{B}_2 = \mathcal{B}_{n+1}$$
$$\mathcal{A}_2 = \{a^*, a_{n+1}\}$$

Applying the lemma say that the extra dimension cannot be larger then ϵ.

References

[1] Michael C. Laskowski, *A simpler axiomatization of the Shelah-Spencer almost sure theories,* Israel J. Math. **161** (2007), 157-186. MR MR2350161

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