SUPERFLAT GRAPHS ARE DP-MINIMAL

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Abstract. We show that the theory of superflat graphs is dp-minimal.

1. Preliminaries

We work with an infinite graph G and a subset of vertices $V \subset V(G)$. Say that V is n-connected if there aren't a set of $n-1$ vertices removing which disconnects every pair of vertices in V. Connectivity of V is the smallest n such that V are n-connected.

Definition 1.1. Suppose $V \subset V(G)$ has finite connectivity $n+1$. Let connectivity hull of V to be union of all n-point sets that disconnect it.

2. Connectivity hull is finite

Here we show our main technical lemma. This result is purely combinatorial, with no mention of model theory.

Lemma 2.1. *Suppose $\{a, b\}$ in G have finite connectivity $n+1$. Then there are finitely many n-point sets that disconnect a from b.*

Corollary 2.2. *Suppose a finite $V \subset V(G)$ has finite connectivity $n+1$. Then there are finitely many n-point sets that disconnect V.*

Proof. Fix set $P = p_1, \dots, p_m$ of all unordered pairs from V. Every pair p_i has connectivity $n_i \leq n+1$ and by previous lemma has finitely many sets of n_i points that disconnect it, denoted by S_i. Every minimal set that disconnects V can be written as (not necessarily unique) union

$$\bigcup_{i \leq m \ s_i \in S_i} s_i$$

There are finitely many ($\prod |S_i|$) ways to write that union giving finitely many minimal sets that can disconnect V. □

Corollary 2.3. *Suppose a countable $V \subset V(G)$ has finite connectivity $n+1$. Then there are finitely many n-point sets that disconnect V.*

Proof. Order $V = \{v_1, v_2, \dots\}$ and consider increasing finite parts $V_i = \{v_1, \dots, v_i\}$. By compactness connectivity becomes equal to $n+1$ for large enough i. Number of sets disconnecting V is bounded by number of sets disconnecting V_i for that large i, which has to be finite by previous lemma. □

3. Application to indiscernible sequences

In this section we work in a flat graph. It is stable so all the indiscernible sequences are totally indiscernible. Also note that by indiscernibility all pairwise distances between points are the same.

We need a refined notion of connectivity for the following argument to work. Suppose we have two points a, b distance n apart. Denote $P(a, b)$ union of all paths of length n going from a to b. If we have a collection of vertices V such that every two have distance n between them, denote

$$P(V) = \bigcup_{a \neq b \in V} P(a, b)$$

Lemma 3.1. *Let $(a_i)_{i\in I}$ be a countable indiscernible sequence over A. Let $n = d(a_i, a_j)$ for some (any) $i \neq j$. There exists a finite set B such that*

$$\forall i \neq j \ d_B(a_i, a_j) > n$$

Proof. By a flatness result we can find an infinite $J \subset I$ and a finite set B' such that each pair from $(a_j)_{j\in J}$ have infinite distance over B'. Using total indiscernibility we have an automorphism sending $(a_j)_{j\in J}$ to $(a_i)_{i\in I}$. Image of B' under this automorphism is the required set B. □

In other words, B disconnects $P(\{a_i\})$. This shows that $\{a_i\}$ has finite connectivity in $P(\{a_i\})$. Applying lemma from last section we obtain that connectivity hull of $\{a_i\}$ in $P(\{a_i\})$ is finite.

Lemma 3.2. *Connectivity hull described above is definable.*

Proof. Consider finite parts of the sequence $I_i = \{a_1, a_2, \ldots, a_i\}$. $P(I_i)$ is I_i-definable as union of all n-paths. Connectivity hull is I_i-definable as well. With increasing i it should stabilize. □

Lemma 3.3. *$\{a_i\}$ is indiscernible over the hull $\cup A$.*

Proof. Denote the hull by H. Fix an A-formula $\phi(x, y)$. Consider a collection of traces $\phi(\vec{a}_i, H^{\{}|y|\})$ for $i \in I$. Those are either all distinct or all the same. Finiteness of H forces latter. This shows indiscernability. □

Corollary 3.4. *Let $(a_i)_{i\in I}$ be a countable indiscernible sequence over A. Then there is a countable B such that (a_i) is indiscernible over $A \cup B$ and*

$$\forall i \neq j \ d_B(a_i, a_j) = \infty$$

Proof. Keep applying previous lemma to obtain larger B_i that provide higher separation while preserving indiscernibility. □

That is every indiscernible sequence can be upgraded to have infinite distance over its parameter set.

4. Superflat graphs are dp-minimal

Lemma 4.1. *Suppose $a \equiv_A b$ and $d_A(a, c) = d_A(b, c) = \infty$. Then $a \equiv_{Ac} b$*

Proof. partial automorphisms □

Theorem 4.2. *Let G be a flat graph with $(a_i)_{i\in\mathbb{Q}}$ indiscernible over A and $b \in G$. There exists $c \in \mathbb{Q}$ such that all $(a_i)_{i\in\{\mathbb{Q}-c\}}$ have the same type over Ab.*

Proof. Find $B \supseteq A$ such that (a_i) is indiscernible over B and has infinite distance over B. b can have finite distance over B to only one member of the sequence. Let C be that member. Remainder of elements have the same type over Bb by the lemma. □

Corollary 4.3. *Flat graphs are dp-minimal.*

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