# SUPERFLAT GRAPHS ARE DP-MINIMAL

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Abstract. We show that the theory of superflat graphs is dp-minimal.

## 1. Preliminaries

We work with an infinite graph $G$ and a subset of vertices $V \subset V(G)$. Say that $V$ is $n$-connected if there aren't a set of $n-1$ vertices removing which disconnects every pair of vertices in $V$. Connectivity of $V$ is the smallest $n$ such that $V$ are $n$-connected.

**Definition 1.1.** Suppose $V \subset V(G)$ has finite connectivity $n+1$. Let connectivity hull of $V$ to be union of all $n$-point sets that disconnect it.

## 2. Connectivity hull is finite

Here we show our main technical lemma. This result is purely combinatorial, with no mention of model theory.

**Lemma 2.1.** *Suppose $\{a, b\}$ in $G$ have finite connectivity $n+1$. Then there are finitely many $n$-point sets that disconnect $a$ from $b$.*

**Corollary 2.2.** *Suppose a finite $V \subset V(G)$ has finite connectivity $n+1$. Then there are finitely many $n$-point sets that disconnect $V$.*

**Corollary 2.3.** *Suppose a countable $V \subset V(G)$ has finite connectivity $n+1$. Then there are finitely many $n$-point sets that disconnect $V$.*

## 3. Application to indiscernible sequences

In this section we work in a flat graph. It is stable so all the indiscernible sequences are totally indiscernible.

We need a refined notion of connectivity for the following argument to work. Suppose we have two points $a, b$ distance $n$ apart. Denote $P(a, b)$ union of all paths of length $n$ going from $a$ to $b$. If we have a collection of vertices $V$ such that every two have distance $n$ between them, denote

$$P(V) = \bigcup_{a \neq b \in V} P(a, b)$$

**Lemma 3.1.** *Let $(a_i)_{i \in I}$ be a countable indiscernible sequence over $A$. Let $n = d(a_i, a_j)$ for some (any) $i \neq j$. There exists a finite set $B$ such that*

$$\forall i \neq j \ d_B(a_i, a_j) > n$$

*Proof.* By a flatness result we can find an infinite $J \subset I$ and a finite set $B'$ such that each pair from $(a_j)_{j \in J}$ have infinite distance over $B'$. Using total indiscernibility we have an automorphism sending $(a_j)_{j \in J}$ to $(a_i)_{i \in I}$. Image of $B'$ under this automorphism is the required set $B$. □

In other words, $B$ disconnects $P(\{a_i\})$. This shows that $\{a_i\}$ has finite connectivity in $P(\{a_i\})$. Applying lemma from last section we obtain that connectivity hull of $\{a_i\}$ in $P(\{a_i\})$ is finite.

**Lemma 3.2.** *Connectivity hull described above is definable.*

**Lemma 3.3.** $\{a_i\}$ *is indiscernible over the hull.*

**Corollary 3.4.** *Let $(a_i)_{i \in I}$ be a countable indiscernible sequence over $A$. Then there is a countable $B$ such that $(a_i)$ is indiscernible over $A \cup B$ and*

$$\forall i \neq j \ d_B(a_i, a_j) = \infty$$

That is every indiscernible sequence can be upgraded to have infinite distance over its parameter set.

## 4. Superflat graphs are dp-minimal

**Lemma 4.1.** *Suppose $a \equiv_A b$ and $d_A(a, c) = d_A(b, c) = \infty$. Then $a \equiv_{Ac} b$*

**Theorem 4.2.** *Let $G$ be a flat graph with $(a_i)_{i \in \mathbb{Q}}$ indiscernible over $A$ and $b \in G$. There exists $c \in \mathbb{Q}$ such that $(a_i)_{i \in \{\mathbb{Q} - c\}}$ is indiscernible over $Ab$.*

**Corollary 4.3.** *Flat graphs are dp-minimal.*

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