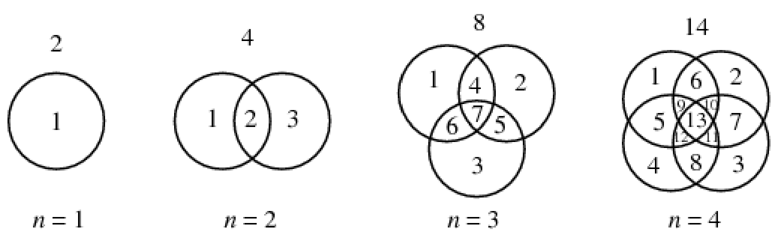
1. COMBINATORICS

Suppose we have an infinite collection of sets $\mathcal{F}$. Take n many of those sets. They generate a boolean algebra. Count the number of atoms in it. There can be at most 2^n atoms, though depending on the collection there may be much less. For a given n, out of all choices of n sets, record the highest possible number of atoms generated. We define that to be a shatter function.

Definition 1.1.

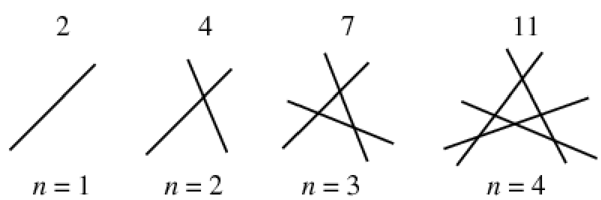
$$\pi_{\mathcal{F}}(n) = \max\{\# \text{ of atoms in boolean algebra generated by } S \mid S \subset \mathcal{F} \text{ and } |S| = n\}$$



Example 1.2. Let $\mathcal{F}$ consist of all discs in the plane.

$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 8 \quad \pi_{\mathcal{F}}(4) = 14$$

$$\pi_{\mathcal{F}}(n) = n^2 - n + 2$$



Example 1.3. Let $\mathcal{F}$ consist of all half-planes in the plane.

$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 7 \quad \pi_{\mathcal{F}}(4) = 11$$

$$\pi_{\mathcal{F}}(n) = n^2/2 + n/2 + 1$$

Example 1.4.

(1) Let $\mathcal{F}$ be a set of lines on a plane. Then
$$\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$$
(2) Let $\mathcal{F}$ be a set of disks on a plane. Then
$$\pi_{\mathcal{F}}(n) = n^2 - n + 2$$
(3) Let $\mathcal{F}$ be a set of balls in $\mathbb{R}^3$. Then
$$\pi_{\mathcal{F}}(n) = n(n^2 - 3n + 8)/3$$

(4) Let $\mathcal{F}$ be a set of intervals on a line. Then
$$\pi_{\mathcal{F}}(n) = 2n$$
(5) Let $\mathcal{F}$ be a set of half-planes. Then
$$\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$$
(6) Let $\mathcal{F}$ be a collection of finite subsets of $\mathbb{N}$. Then
$$\pi_{\mathcal{F}}(n) = 2^n$$
(7) Let $\mathcal{F}$ be a collection of polygons in a plane. Then
$$\pi_{\mathcal{F}}(n) = 2^n$$

Theorem 1.5 (Sauer-Shelah)**.** *Shatter function is either 2^n or bounded by a polynomial.*

Definition 1.6. Suppose growth of shatter function for $\mathcal{F}$ is polynomial. Let r be the smallest real such that
$$\pi_{\mathcal{F}}(n) = O(n^r)$$
We define such r to be the vc-density of $\mathcal{F}$. If shatter function grows exponentially, we let vc-density to be infinite.

2. Model Theory

Consider a structure with a language
$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

We work with subsets definable by first-order formulas. Those are called definable sets.

$\phi(x) := ((5 \leq x) \wedge (x \leq 7.7)) \vee (x \leq 0)$ (either x is between 5 and 7.7 or it is non-negative)

$\psi(x) := (\exists y\ y \cdot y = x)$ (there exists y such that y squared is x)

$\gamma(x) := (x \cdot x \cdot x \cdot x = 2)$ (x to the fourth power is 2)

- $\phi(\mathbb{R})$ defines the set $[5, 7.7] \cup (-\infty, 0]$
- $\psi(\mathbb{R})$ defines the set $[0, \infty)$

(1) in rationals $(\mathbb{Q}, \cdot)$, $\gamma(\mathbb{Q})$ defines an empty subset
(2) in reals $(\mathbb{R}, \cdot)$, $\gamma(\mathbb{R})$ defines a subset with two elements
(3) in complex numbers $(\mathbb{C}, \cdot)$, $\gamma(\mathbb{C})$ defines a subset with four elements
(4) in quaternions $(\mathbb{H}, \cdot)$, $\gamma(\mathbb{H})$ defines an infinite subset

$$\begin{aligned}\theta(x) = &\forall y \exists z\ x \leq z \leq y \\ &\text{for all } y \text{ there exists } z \text{ such that } x \leq z \leq y\end{aligned}$$

(1) in $(\mathbb{Q}, \leq)$, $\theta(\mathbb{Q})$ defines an empty subset
(2) in $(\mathbb{N}, \leq)$, $\theta(\mathbb{N})$ defines an empty subset
(3) in $(\mathbb{Q}^{\geq 0}, \leq)$, $\theta(\mathbb{Q}^{\geq 0})$ defines the set $\{0\}$

Definition 2.1. for a formula $\phi(x_1 \dots x_n, y_1, \dots y_m)$ we can plug in elements of our structure as parameters in places of y variables. This gives us a collection of definable sets.

Example 2.2.

$$\phi(x_1, x_2, y_1, y_2, y_3) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq y_3^2$$

In structure $(\mathbb{R}, +, \cdot, \leq)$ given $a, b, r \in \mathbb{R}$ the formula $\phi(x_1, x_2, a, b, r)$ defines a disk in $\mathbb{R}^2$ with radius r with center (a, b).

Thus all discs in $\mathbb{R}^2$ are defined uniformly by ϕ.

What are the collection of sets we can consider when working with a model?

We can look at all definable subsets. That's not interesting, always has an infinite vc-density. Uniformly definable families offer more interesting behavior.

A model is said to be NIP if all uniformly definable families have finite vc-density.

For a given model M, let vc function of n to be the largest vc-density achieved by n-dimensional families of uniformly definable sets.

$$\mathrm{vc}^M(n) = max\,\{\mathrm{vc}(\phi) \mid \phi(\vec{x}, \vec{y}) \text{ with } |\vec{x}| = n\}$$

It is easy to show that $\mathrm{vc}_M(n) \geq n$ for all models.
Open questions about vc functions.
Is $\mathrm{vc}_M(n) = n\,\mathrm{vc}_M(1)$? If not, is there a linear relationship?