1. VC-dimension

Suppose we have an infinite collection of sets $\mathcal{F}$. Take n many of those sets. They generate a boolean algebra. Count the number of atoms in it. There can be at most 2^n atoms, though depending on the collection there may be much less. For a given n, out of all choices of n sets, record the highest possible number of atoms generated. We define that to be a shatter function.

Definition 1.1.

$\pi_{\mathcal{F}}(n) = \max\{\#\text{ of atoms in boolean algebra generated by } S \mid S \subset \mathcal{F} \text{ and } |S| = n\}$

Example 1.2.

(1) Let $\mathcal{F}$ be a set of lines on a plane. Then
$$\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$$
(2) Let $\mathcal{F}$ be a set of disks on a plane. Then
$$\pi_{\mathcal{F}}(n) = n^2 - n + 2$$
(3) Let $\mathcal{F}$ be a set of balls in $\mathbb{R}^3$. Then
$$\pi_{\mathcal{F}}(n) = n(n^2 - 3n + 8)/3$$
(4) Let $\mathcal{F}$ be a set of intervals on a line. Then
$$\pi_{\mathcal{F}}(n) = 2n$$
(5) Let $\mathcal{F}$ be a set of half-planes. Then
$$\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$$
(6) Let $\mathcal{F}$ be a collection of finite subsets of $\mathbb{N}$. Then
$$\pi_{\mathcal{F}}(n) = 2^n$$
(7) Let $\mathcal{F}$ be a collection of polygons in a plane. Then
$$\pi_{\mathcal{F}}(n) = 2^n$$

Theorem 1.3 (Sauer-Shelah)**.** *Shatter function is either 2^n or bounded by a polynomial.*

Definition 1.4. Families of sets with polynomially bounded shatter functions are said to have a finite VC-dimension.

Definition 1.5. Suppose $\mathcal{F}$ has a finite VC-dimension. Let k be the smallest real such that
$$\pi_{\mathcal{F}}(n) = O(n^k)$$
We define such k to be the vc-density of $\mathcal{F}$.