VC-DENSITY FOR TREES

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ABSTRACT. We show that for the theory of infinite trees we have $\mathrm{vc}(n) = n$ for all n.

VC density was introduced in [1] by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko as a natural notion of dimension for NIP theories. In a NIP theory we can define a VC function

$$\mathrm{vc} : \mathbb{N} \longrightarrow \mathbb{N}$$

Where $vc(n)$ measures complexity of definable sets n-dimensional space. Simplest possible behavior is $\mathrm{vc}(n) = n$ for all n. Theories with that property are known to be dp-minimal, i.e. having the smallest possible dp-rank. In general, it is not known whether there can be a dp-minimal theory which doesn't satisfy $\mathrm{vc}(n) = n$.

In this paper we work with infinite trees viewed as posets. Parigot in [3] showed that such theory is NIP. This result was strengthened by Simon in [2] showing that trees are dp-minimal. [1] has the following problem

Problem 0.1. ([1] p. 47) Determine the VC density function of each (infinite) tree.

Here we settle this question by showing that theory of trees has $\mathrm{vc}(n) = n$.

1. PRELIMINARIES

We use notation $a \in T^n$ for tuples of size n. For variable x or tuple a we denote their arity by $|x|$ and $|a|$ respectively.

We work with finite relational languages. Given a formula we can define its complexity n as the depth of quantifiers used to build up the formula. See for example [4] Definition 2D.4 pg.72. $S^n_{\boldsymbol{A}}(x)$ stands for all the types made up of formulas of complexity at most n in a structure $\boldsymbol{A}$. $\mathrm{tp}^n_{\boldsymbol{B}}(a)$ stands for such a type. For two structures $\boldsymbol{A}, \boldsymbol{B}$ we say $\boldsymbol{A} \equiv_n \boldsymbol{B}$ if two structures agree on all sentences of complexity at most n.

Note 1.1. Saying that $(\boldsymbol{A}, a_1) \equiv_n (\boldsymbol{A}, a_2)$ is the same as saying that a_1 and a_2 have the same n-complexity type in $\boldsymbol{A}$.

Language for the trees consists of a single binary predicate $\{\leq\}$. Theory of trees states that $\leq$ defines a partial order and for every element a we have $\{x \mid x < a\}$ a linear order. Theory of meet trees requires that in addition tree is closed under meet operation, i.e. for any a, b in the same connected component there exists the greatest upper bound for elements both $\leq$ than a and b. In this paper we will work with colored trees, where we simply add a finite number of unary predicates to our language. Note that we allow our trees to be disconnected or finite unless otherwise stated.

For visualization purposes we assume trees grow upwards, with smaller elements on the bottom and larger elements on the top. If $a < b$ we will say that a is below b and b is above a.

For completeness we also present definition of VC function. One should refer to [1] for more details. Suppose we have a collection $\mathcal{S}$ of subsets of X. We define a *shatter function* $\pi_{\mathcal{S}}(n)$

$$\pi_{\mathcal{S}}(n) = \max\{|A \cap \mathcal{S}| : A \subset X \text{ and } |A| = n\}$$

Sauer's Lemma asserts that asymptotically $\pi_{\mathcal{S}}$ is either 2^n or polynomial. In the polynomial case we define VC density of $\mathcal{S}$ to be power of polynomial that bounds $\pi_{\mathcal{S}}$. More formally

$$\mathrm{vc}(\mathcal{S}) = \limsup_{n\to\infty} \frac{\log \pi_{\mathcal{S}}}{\log n}$$

Given a model $M \models T$ and a formula $\phi(x, y)$ we define

$$\begin{aligned} \mathcal{S}_\phi &= \{\phi(M^{|n|}, b) : b \in M^{|y|}\} \\ \mathrm{vc}(\phi) &= \mathrm{vc}(\mathcal{S}_\phi) \end{aligned}$$

One has to check that this definition is independent of realization of T, see [1], Lemma 3.2. For a theory T we define the VC function

$$\mathrm{vc}(n) = \sup\{\mathrm{vc}(\phi(x, y)) : |x| = n\}$$

2. Proper Subdivisions

Definition 2.1. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{T}$ be models in (possibly different) finite relational languages. If A, B partition T (i.e. $T = A \sqcup B$) we say that $(\boldsymbol{A}, \boldsymbol{B})$ is a *subdivision* of $\boldsymbol{T}$.

Definition 2.2. $(\boldsymbol{A}, \boldsymbol{B})$ subdivision of $\boldsymbol{T}$ is called *n-proper* if for all $p, q \in \mathbb{N}$, for all $a_1, a_2 \in A^p$ and $b_1, b_2 \in B^q$ we have

$$\begin{aligned} (\boldsymbol{A}, a_1) &\equiv_n (\boldsymbol{A}, a_2) \\ (\boldsymbol{B}, b_1) &\equiv_n (\boldsymbol{B}, b_2) \end{aligned}$$

then

$$(\boldsymbol{T}, a_1, b_1) \equiv_n (\boldsymbol{T}, a_2, b_2)$$

Definition 2.3. $(\boldsymbol{A}, \boldsymbol{B})$ subdivision of $\boldsymbol{T}$ is called *proper* if it is n-proper for all $n \in \mathbb{N}$.

Lemma 2.4. *Consider a subdivision $(\boldsymbol{A}, \boldsymbol{B})$ of $\boldsymbol{T}$. If it is 0-proper then it is proper.*

Proof. Prove the subdivision is n-proper for all n by induction. Case $n = 0$ is given by the assumption. Suppose $n = k + 1$ and we have $\boldsymbol{T} \models \exists x\, \phi^k(x, a_1, b_1)$ where ϕ^k is some formula of complexity k. Let $a \in T$ witness the existential claim i.e.

$\boldsymbol{T} \models \phi^k(a, a_1, b_1)$. $a \in A$ or $a \in B$. Without loss of generality assume $a \in A$. Let $\boldsymbol{p} = \mathrm{tp}^k_{\boldsymbol{A}}(a, a_1)$. Then we have

$$\boldsymbol{A} \models \exists x \ \mathrm{tp}^k_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$$

Formula $\mathrm{tp}^k_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$ is of complexity k so $\exists x \ \mathrm{tp}^k_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$ is of complexity $k+1$ by inductive hypothesis we have

$$\boldsymbol{A} \models \exists x \ \mathrm{tp}^k_{\boldsymbol{A}}(x, a_2) = \boldsymbol{p}$$

Let a' witness this existential claim so that

$$\begin{aligned}
\mathrm{tp}^k_{\boldsymbol{A}}(a', a_2) &= \boldsymbol{p} \\
\mathrm{tp}^k_{\boldsymbol{A}}(a', a_2) &= \mathrm{tp}^k_{\boldsymbol{A}}(a, a_1) \\
(\boldsymbol{A}, a', a_2) &\equiv_k (\boldsymbol{A}, a, a_1)
\end{aligned}$$

by inductive assumption we have

$$\begin{aligned}
(\boldsymbol{T}, a, a_1, b_1) &\equiv_k (\boldsymbol{T}, a', a_2, b_2) \\
\boldsymbol{T} &\models \phi^k(a', a_2, b_2) && \text{as } \boldsymbol{T} \models \phi^k(a, a_1, b_1) \\
\boldsymbol{T} &\models \exists x \phi^k(x, a_2, b_2)
\end{aligned}$$

□

We use this lemma for colored trees. Suppose we have $\boldsymbol{T}$ to be a model of a colored tree in some language $\mathcal{L} = \{\leq, \ldots\}$ and $\boldsymbol{A}, \boldsymbol{B}$ be in some languages $\mathcal{L}_A, \mathcal{L}_B$ which will be expands of $\mathcal{L}$, with $\boldsymbol{A}, \boldsymbol{B}$ substructures of $\boldsymbol{T}$ as reducts to $\mathcal{L}$. In this case we'll refer to $(\boldsymbol{A}, \boldsymbol{B})$ as a *proper subdivision* (of $\boldsymbol{T}$).

Example 2.5. Suppose a tree consists of two connected components C_1, C_2. Then $(C_1, \leq)$ and $(C_2, \leq)$ form a proper subdivision.

Example 2.6. Fix $\boldsymbol{T}$ and $a \in T$. Let $B = \{t \in T \mid a < t\}$, $S = \{t \in T \mid t \leq a\}$, $A = T - B$. Then $(A, \leq, S)$ and $(B, \leq)$ form a proper subdivision, where $\mathcal{L}_A$ has a unary predicate interpreted by S.

Proof. By Lemma 2.4 we only need to check that it is 0-proper. Suppose we have

$$\begin{aligned}
a &= (a_1, a_2, \ldots, a_p) \in A^p \\
a' &= (a'_1, a'_2, \ldots, a'_p) \in A^p \\
b &= (b_1, b_2, \ldots, b_q) \in B^q \\
b' &= (b'_1, b'_2, \ldots, b'_q) \in B^q
\end{aligned}$$

with $(\boldsymbol{A}, a) \equiv_0 (\boldsymbol{A}, a')$ and $(\boldsymbol{B}, b) \equiv_0 (\boldsymbol{B}, b')$. We need to make sure that ab has the same quantifier free type as $a'b'$. Any two elements in T can be related in the four following ways

$$\begin{aligned}
&x = y \\
&x < y \\
&x > y \\
&x, y \text{ are incomparable}
\end{aligned}$$

We need to check that the same relations hold for pairs of $(a_i, b_j), (a'_i, b'_j)$ for all i, j.

- It is impossible that $a_i = b_j$ as they come from disjoint sets.
- It is impossible $a_i > b_j$
- Suppose $a_i < b_j$. This forces $a_i \in S$ thus $a'_i \in S$ and $a'_i < b'_j$
- Suppose a_i and b_j are incomparable. This forces $a_i \notin S$ so $a'_i \notin S$ making a'_i, b'_j incomparable.

$\square$

3. Intervals

Definition 3.1. For $c_1 > c_2$ in a tree $\boldsymbol{T}$ we define an (open) interval

$$(c_1, c_2) = \{t \in T \mid c_1 > t \text{ and } c_2 \not\geq t\}$$

We also allow endpoints to be infinite, $c_1 = \neg\infty$ or $c_2 = \infty$.

Lemma 3.2. *Suppose we have $I = (c_1, c_2)$, $a, b \in I$ and $a \equiv^n_{c_1 c_2} b$. Denote $D = T - I$. Then $a \equiv^n_D b$.*

To prove this, first we do the following two lemmas:

Lemma 3.3. *Suppose we have $I = (-\infty, c_2)$, $a, b \in I$ and $a \equiv^n_{c_2} b$. Denote $U = T - I$. Then $a \equiv^n_U b$.*

Proof. Consider subdivision as in the example 2.6 and fix a tuple $c \in U^{|c|}$. Universe and relations in $\boldsymbol{A}$ are c_2-definable. Then $a \equiv^n_{c_2} b$ implies that $(\boldsymbol{A}, a) \equiv_n (\boldsymbol{A}, b)$. We automatically have that $(\boldsymbol{B}, c) \equiv_n (\boldsymbol{B}, c)$. Thus by properness we get $T \models a \equiv^n_c b$. As choice of c was arbitrary we are done. $\square$

Lemma 3.4. *Suppose we have $I = (c_1, \infty)$, $a, b \in I$ and $a \equiv^n_{c_1} b$. Denote $L = T - I$. Then $a \equiv^n_L b$.*

Proof. Similar to the proof above. $\square$

Proof. (of lemma 3.2) Easy application of lemmas above. Pick an arbitrary tuple $d \in D^{|d|}$ and partition it as $d = (l, u)$ such that $l \in L^{|l|}$ and $u \in U^{|u|}$ (where L and U are as in the lemmas above). Then

$$\begin{aligned}
a &\equiv^n_{c_1} b && \\
a &\equiv^n_{c_2 l} b && \text{by lemma 3.4} \\
la &\equiv^n_{c_2} lb && \\
la &\equiv^n_{c_2 u} lb && \text{by lemma 3} \\
a &\equiv^n_{c_2 l u} b && \\
a &\equiv^n_{c_2 l u} b && \\
a &\equiv^n_{d} b &&
\end{aligned}$$

$\square$

Note 3.5. If we weren't restricting complexity of type, this lemma could be proven in an easier way by modifying automorphism sending a to b. However to have a uniform bound on complexity we need to use a finer analysis involving proper subdivisions.

Next we state a generalization of this result for a collection of disjoint intervals. For technical reasons we will allow some of the intervals to be single points $\{c\}$. This will only required for computing vc-density for formulas $\phi(x, y)$ with $|y| > 1$.

Corollary 3.6. *Let I_i with $i \in [1..m]$ be a collection of disjoint sets where each set is either interval (l_i, u_i) or a point $\{c_i\}$. We denote E_i denote endpoints of I_i. If $I = (l_i, u_i)$ then $E_i = \{l_i, u_i\}$ and if $I = \{c_i\}$ then $E_i = \{c_i\}$. Also we have tuples*

$$a = (a_1, a_2, \dots, a_m)$$
$$b = (b_1, b_2, \dots, b_m)$$

s.t. $a_i \in I_i^{|a_i|}$ and $b_i \in I_i^{|b_i|}$ for all $i \in [1..m]$ (i.e every a_i and b_i are also tuples). Moreover we require

$$a_i \equiv^n_{E_i} b_i \text{ for all } i \in [1..m]$$

Let $E = \bigcup_i E_i$ be a collection of all endpoints. Then we $a \equiv^n_E b$.

Proof. We apply lemma m times - one for each tuple. For example assume $I_1 = (l_1, u_1)$ and denote $C_1 = T - I_1$ (case for when I_i is singleton is trivial). Then

$$\begin{aligned} a_1 &\equiv^n_{l_1 u_1} b_1 \\ a_1 &\equiv^n_{C_1} b_1 \\ a_1 &\equiv^n_{E a_2 a_3 \dots a_m} b_1 \\ a_1 a_2 a_3 \dots a_m &\equiv^n_E b_1 a_2 a_3 \dots a_m \end{aligned}$$

Denote $C_2 = T - I_2$ and assume $I_2 = (l_2, u_2)$. Then

$$\begin{aligned} a_2 &\equiv^n_{l_2 u_2} b_2 \\ a_2 &\equiv^n_{C_2} b_2 \\ a_2 &\equiv^n_{E b_1 a_3 \dots a_m} b_2 \\ b_1 a_2 a_3 \dots a_m &\equiv^n_E b_1 b_2 a_3 \dots a_m \end{aligned}$$

Continuing in such fashion we get

$$\begin{aligned} a = a_1 a_2 a_3 \dots a_m &\equiv^n_E \\ b_1 a_2 a_3 \dots a_m &\equiv^n_E \\ b_1 b_2 b_3 \dots a_m &\equiv^n_E \\ &\dots \\ b_1 b_2 b_3 \dots b_m &= b \end{aligned}$$

i.e. $a \equiv^n_E b$ as needed. □

4. Partition into intervals

Here we show that every tree can be partitioned into intervals with prescribed endpoints. To do this we must restrict our attention to meet trees (closed under meets). Given a finite subset B of a meet tree T let $\mathrm{cl}(B)$ denote its closure under meets.

Lemma 4.1. *Suppose $S \subseteq T$ is a non-empty finite subset of size n in a meet tree and $S' = \mathrm{cl}(S)$ its closure under meets. Then $|S'| \leq 2n - 1$.*

Proof. We prove by induction on n. Base case $n = 1$ is clear. Suppose we have S of size k with closure of size at most $2k-1$. Take a new point and look at its meets with all the elements of S. Pick the largest one. That is the only element we need to add to S' to make sure the set is closed under meets. □

This way we make sure the set grows at most linearly when closed under meets.

Lemma 4.2. *Let C be a non-empty finite set closed under meets in a tree T. Then there is a collection of disjoint intervals $\{I_i = (a_i, b_i)\}_{1..n}$ such that*

1. *endpoints are either in C or infinite.*
2. *$n = |C| + 1$.*
3. *I_i partition $T - C$.*

Proof. For every $c \in C$ define c^+ to be supremum of all elements of C greater than c. If there aren't any, let $c^+ = \infty$. Also let c^- to be the supremum of elements in C. Our desired collection of intervals is $(c, c^+)_{c\in C}, (-\infty, c^-)$. □

5. Type counting

Definition 5.1. For $\phi(x, y)$, $A \subseteq T^{|x|}$ and $B \subseteq T^{|y|}$

- Let $\phi(A, b) = \{a \in A \mid \phi(a, b)\} \subseteq A$
- Let $\phi(A, B) = \{\phi(A, b) \mid b \in B\} \subseteq \mathcal{P}(A)$

That is $\phi(A, B)$ is a collection of subsets of A definable by ϕ with parameters from B.

Note 5.2. For $A_1 \subseteq A_2$ we have $|\phi(A_1, B)| \leq |\phi(A_2, B)|$.

Definition 5.3. Fix a (finite relational) language $\mathcal{L}_B$, and n, $|y|$. Let $N = N(n, |y|, \mathcal{L}_B)$ be smallest number such that for any structure $\boldsymbol{B}$ in $\mathcal{L}_B$ we have $|S^n_{\boldsymbol{B}}(y)| \leq N$. Note the following easy inequalities

$$n \leq m \Rightarrow N(n, |y|, \mathcal{L}_B) \leq N(m, |y|, \mathcal{L}_B)$$
$$|y| \leq |z| \Rightarrow N(n, |y|, \mathcal{L}_B) \leq N(n, |z|, \mathcal{L}_B)$$
$$\mathcal{L}_A \subseteq \mathcal{L}_B \Rightarrow N(n, |y|, \mathcal{L}_A) \leq N(n, |y|, \mathcal{L}_B)$$

$$N(n, |y|, \mathcal{L}_B) \cdot N(n, |z|, \mathcal{L}_B) \leq N(n, |y| + |z|, \mathcal{L}_B)$$

Proof. Take some $a \in A^{|x|}$. We have $(\boldsymbol{B}, b_1) \equiv_n (\boldsymbol{B}, b_2)$ and (trivially) $(\boldsymbol{A}, a) \equiv_n (\boldsymbol{A}, a)$. Thus by the Lemma 2.4 we have $(\boldsymbol{T}, a, b_1) \equiv_n (\boldsymbol{T}, a, b_2)$ so $\phi(a, b_1) \iff \phi(a, b_2)$. Since a was arbitrary we have $\phi(A^{|x|}, b_1) = \phi(A^{|x|}, b_2)$. □

Lemma 5.4. *Suppose $\phi(x, y)$ is a formula of complexity n in language $\mathcal{L}$ of trees. Let $I = (c_1, c_2)$ an interval and let its compliment be D. Then*

$$|\phi(D^{|x|}, I^{|y|})| < N(n, |y|, \mathcal{L}')$$

where $\mathcal{L}'$ is $\mathcal{L}$ with two extra constant symbols.

Note that the bound provided is uniform.

Proof. Fix $a, b \in I^{|y|}$. Then if $a \equiv^n_{c_1 c_2} b$ then $\phi(D^{|x|}, a) = \phi(D^{|x|}, b)$ by lemma **??** Thus $|\phi(D^{|x|}, I^{|y|})|$ is bounded by $|S^n(y/c_1c_2)|$. □

We extract a similar result from Lemma **??**.

Lemma 5.5. *Suppose $\phi(x, y)$ is a formula of complexity n. Let I_i with $i \in [1..m]$ be a disjoint collection of set that are either intervals or points. Denote collection of all endpoints by E. Moreover fix naturals $k_1, \ldots k_m$ such that $|y| = k_1 + k_2 + \ldots + k_n$. Then*

$$|\phi(E^{|x|}, I_1^{k_1} \times I_2^{k_2} \times \ldots \times I_m^{k_m})| \leq N(n, |y|, \mathcal{L}')$$

where $\mathcal{L}'$ is $\mathcal{L}$ with two extra constant symbols.

Proof. Fix $a_i, b_i \in I_i^{k_i}$ for each i such that $a_i \equiv_{E_i}^n b_i$ where E_i is endpoints of I_i. Then by lemma **??** $\phi(E^{|x|}, a_1, a_2, \dots, a_m) = \phi(E^{|x|}, b_1, b_2, \dots, b_m)$. This implies that

$$\begin{aligned}|\phi(E^{|x|}, I_1^{k_1} \times I_2^{k_2} \times \ldots \times I_m^{k_m})| &\leq |S_{k_1}^n(E_1)| \cdot |S_{k_2}^n(E_2)| \cdot \ldots \cdot |S_{k_m}^n(E_m)| \leq \\ &\leq N(n, k_1, \mathcal{L}') \cdot N(n, k_2, \mathcal{L}') \cdot \ldots \cdot N(n, k_m, \mathcal{L}') \leq \\ &\leq N(n, k_1 + k_2 + \ldots + k_m, \mathcal{L}') = N(n, |y|, \mathcal{L}')\end{aligned}$$

□

6. Proof for one-dimensional case

To demonstrate our method in a simple setting we present the computation of vc-density for formulas $\phi(x, y)$ with $|x| = |y| = 1$ in an infinite (colored) meet tree $\boldsymbol{T}$. Let B be an arbitrary finite set in T. We would like to show that $|\phi(B, T)| = O(|B|)$. This would show that vc-density of ϕ is 1. First let $C = \mathrm{cl}(B)$. By Lemma 4.1 we have $|C| \leq 2|B| + 1$. Using Lemma **??** we have a collection of intervals $\{I_i\}_{1..|C|}$ such that $T - C = \bigsqcup I_i$. Observe

$$\phi(C, T) = \phi(C, \bigsqcup I_i \sqcup C) = \bigcup \phi(C, I_i) \cup \phi(C, C)$$

Thus

$$|\phi(C, T)| = \left|\bigcup \phi(C, I_i) \cup \phi(C, C)\right| \leq \sum |\phi(C, I_i)| + |\phi(C, C)| = \sum_{1..|C|} |\phi(C, I_i)| + |C|$$

Let $N = N(n_\phi, 1, \mathcal{L}')$. Now observe that by Lemma **??** $|\phi(C, I_i)| \leq N$ for any i. Thus

$$|\phi(C, T)| \leq \sum_{1..|C|} |\phi(C, I_i)| + |C| \leq \sum_{1..|C|} N + |C| = |C|(N + 1)$$

Finally, as $B \subseteq C$ we have $|\phi(B, T)| \leq |\phi(C, T)|$ so

$$|\phi(B, T)| \leq |\phi(C, T)| \leq |C|(N + 1) \leq (2|B| + 1)(N + 1)$$

As N is independent from B, this finishes the proof.

7. Main proof

Basic idea for the proof is that we are able to divide our parameter space into $O(n)$ many intervals. Each of q parameters can come from any of those $O(n)$ intervals giving us $O(n)^q = O(n^q)$ many choices for parameter configuration. With every parameter coming from a fixed partition the number of definable sets is constant and in fact is uniformly bounded by some N. This gives us $NO(n^q) = O(n^q)$ possibilities for different definable sets.

Theorem 7.1. *Let $\boldsymbol{T}$ be an infinite (colored) meet tree and $\phi(x, y)$ a formula with $|x| = p$ and $|y| = q$. Then $\mathrm{vc}(\phi) \leq q$.*

Proof. Pick a finite subset of $S_0 \subset T^p$ of size n. Let $S_1 \subset T$ consist of coordinates of S_0. Let $S \subset T$ be a closure of S_1 under meets. Using Lemma 4.1 we have $|S| \leq 2|S_1| \leq 2p|S_0| = 2pn = O(n)$. We have $S_0 \subseteq S^p$, so $|\phi(S_0, T^q)| \leq |\phi(S^p, T^q)|$. Thus it is enough to show $|\phi(S^p, T^q)| = O(n^q)$. Using Lemma **??** construct collection of intervals $\{I_i\}_{i \in \mathcal{I}}$ for the set S. We have $|\mathcal{I}| = 2|S| + 1$. Extend this collection of intervals by adding singletons $\{s\}$ for each $s \in S$. Denote new collection $\{I_j\}_{j\in\mathcal{J}}$. We have $|\mathcal{J}| = |\mathcal{I}| + |S| = 3|S| + 1 \leq 6pn + 1 = O(n)$. We now have that $\{I_j\}_{j\in\mathcal{J}}$ partitions T i.e. $T = \bigsqcup_{j\in\mathcal{J}} I_j$.

Basic idea for the general case $q \geq 1$ is that we have q parameters and $|J| = O(n)$ partitions to pick each parameter from giving us $|J|^q = O(n^q)$ choices for parameter configuration, each giving uniformly constant number of definable subsets of S. (If every parameter is picked from a fixed partition, Lemma **??** provides a uniform bound). This yields $\mathrm{vc}(\phi) \leq q$ as needed. The rest of the proof is stating this idea formally.

For $(j_1, j_2, \ldots j_q) = \vec{j} \in \mathcal{J}^q$ denote

$$I_{\vec{j}} = I_{j_1} \times I_{j_2} \times \ldots \times I_{j_q}$$

Then we have the following identity

$$T^q = (\bigsqcup_{j\in\mathcal{J}} I_j)^q = \bigsqcup_{\vec{j}\in\mathcal{J}^q} I_{\vec{j}}$$

Thus we have that $\{I_{\vec{j}}\}_{\vec{j}\in\mathcal{J}^q}$ partition T^q. Compute

$$\begin{aligned}|\phi(S^p, T^q)| &= \left|\bigcup_{\vec{j}\in\mathcal{J}^q} \phi(S^p, I_{\vec{j}})\right| \leq \\ &\leq \sum_{\vec{j}\in\mathcal{J}^q} |\phi(S^p, I_{\vec{j}})|\end{aligned}$$

We can bound $|\phi(S^p, I_{\vec{j}})|$ uniformly using Lemma **??**. $\{I_j\}_{j\in\mathcal{J}}$ satisfies the requirements of the lemma and $I_{\vec{j}}$ looks like the argument in the lemma after possibly permuting some variables in ϕ. Applying the lemma we get

$$|\phi(S^p, I_{\vec{j}})| \leq N\,(= N(\text{complexity of } \phi, q, \mathcal{L}'))$$

with N only depending on q and complexity of ϕ. We complete our computation

$$\begin{aligned}|\phi(S^p, T^q)| &\leq \sum_{\vec{j}\in\mathcal{J}^q} |\phi(S^p, I_{\vec{j}})| \leq \\ &\leq \sum_{\vec{j}\in\mathcal{J}^q} N \leq \\ &\leq |\mathcal{J}^q| N \leq \\ &\leq (6pn+1)^q N = O(n^q)\end{aligned}$$

□

Corollary 7.2. *In the theory of infinite (colored) meet trees we have $vc(n) = n$ for all n.*

Corollary 7.3. *In the theory of infinite (colored) trees we have $vc(n) = n$ for all n.*

Proof. Let $\boldsymbol{T}$ be a tree and $\boldsymbol{T}'$ its closure under meets. $\boldsymbol{T}$ can be interpreted as a subset of $\boldsymbol{T}'$ using an extra color. □

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