

A Tiny Example

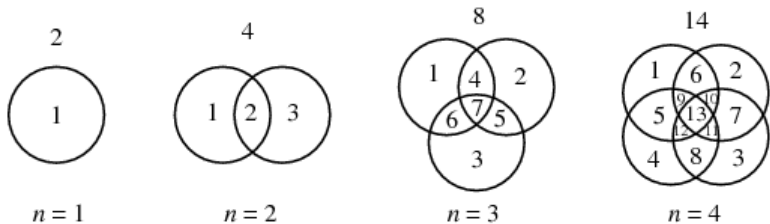
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Suppose we have an (infinite) collection of sets \mathcal{F} .
We define a shatter function $\pi_{\mathcal{F}}(n)$

$$\pi_{\mathcal{F}}(n) = \max\{\# \text{ of atoms in boolean algebra generated by } S \\ | S \subset \mathcal{F} \text{ with } |S| = n\}$$

Example: Let \mathcal{F} consist of all discs on a plane.



$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 8 \quad \pi_{\mathcal{F}}(4) = 14$$

$$\pi_{\mathcal{F}}(n) = n^2 - n + 2$$

More examples:

1. Lines on a plane $\pi_{\mathcal{F}}(n) = n^2/2 + n/2 + 1$
2. Disks on a plane $\pi_{\mathcal{F}}(n) = n^2 - n + 2$
3. Balls in \mathbb{R}^3 $\pi_{\mathcal{F}}(n) = n^3/3 - n^2 + 8n/3$
4. Intervals on a line $\pi_{\mathcal{F}}(n) = 2n$
5. Half-planes on a plane $\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$
6. Finite subsets of \mathbb{N} $\pi_{\mathcal{F}}(n) = 2^n$
7. Polygons in a plane $\pi_{\mathcal{F}}(n) = 2^n$

Theorem (Sauer-Shelah)

Shatter function is either 2^n or bounded by a polynomial.

Definition

Suppose growth of shatter function for \mathcal{F} is polynomial. Let r be the smallest real such that

$$\pi_{\mathcal{F}}(n) = O(n^r)$$

We define such r to be the vc-density of \mathcal{F} , $\text{vc}(\mathcal{F})$. If shatter function grows exponentially, we let the vc-density to be infinite.

Applications

- ▶ VapnikChervonenkis Theorem in probability
- ▶ Computability
- ▶ NIP theories

Model Theory

Model Theory studies definable sets in first-order structures.

$$(\mathbb{Q}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) = \exists y \ y \cdot y = x$$

In the structure above $\phi(x)$ defines a set of numbers that are a square.

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) = \exists y \ y \cdot y = x$$

In the structure above $\phi(x)$ defines the set $[0, \infty)$.

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\psi(x_1, x_2) = (x_1 \cdot x_1 + x_2 \cdot x_2 \leq 1.5) \wedge (x_1^2 \leq x_2)$$

This defines a set in \mathbb{R}^2 .

We work with families of uniformly definable sets. Fix a formula $\phi(x_1 \dots x_n, y_1, \dots y_m) = \phi(\vec{x}, \vec{y})$. Plug in elements from the model for y variables to get a family of definable sets in M^n .

$$\mathcal{F}_\phi^M = \{\phi(x_1, \dots, x_n, a_1, \dots a_n) \mid a_1, \dots a_n \in M\}$$

Define $\text{vc}^M(\phi)$ to be the vc-density of the family \mathcal{F}_ϕ^M

Open Question: it is possible for $\text{vc}^M(\phi)$ to be irrational?

$$\phi(x_1, x_2, y_1, y_2, y_3) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq y_3^2$$

In structure $(\mathbb{R}, +, \cdot, \leq)$ given $a, b, r \in \mathbb{R}$ the formula $\phi(x_1, x_2, a, b, r)$ defines a disk in \mathbb{R}^2 with radius r with center (a, b) . Thus $\mathcal{F}_\phi^\mathbb{R}$ is a collection of all disks in \mathbb{R}^2 .

A model M is said to have NIP property if all uniformly definable families in it have finite vc-density.

► Examples

- $(\mathbb{R}, 0, 1, +, \cdot, \leq)$
- $(\mathbb{C}, 0, 1, +, \cdot)$
- $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$

► Non-examples

- $(\mathbb{Q}, 0, 1, +, \cdot)$
- Random graph (V, R) .
- Pseudo-finite fields.

Given an NIP structure M we define a vc -function of n to be the largest vc -density achieved by families of uniformly definable sets in M^n .

$$\text{vc}^M(n) = \max \{ \text{vc}(\phi) \mid \phi(\vec{x}, \vec{y}) \text{ with } |\vec{x}| = n \}$$

Easy to show $\text{vc}_M(n) \geq n \text{vc}_M(1)$, $\text{vc}_M(1) \geq 1$

Open question: Is $\text{vc}_M(n) = n \text{vc}_M(1)$? If not, is there a linear relationship?

Examples

- ▶ $(\mathbb{R}, 0, 1, +, \cdot, \leq)$ has $\text{vc}(n) = n$ (true for all quasi o-minimal structures)
- ▶ $(\mathbb{C}, 0, 1, +, \cdot)$ has $\text{vc}(n) = n$
- ▶ $(\mathbb{Q}_p, 0, 1, +, \cdot)$ has $\text{vc}(n) \leq 2n - 1$
- ▶ ACVF has $\text{vc}(n) \leq 2n$.

vc-density in trees

Consider structure (T, \leq) where elements of T are vertices of a rooted tree and we say that $a \leq b$ if a is below b in the tree.

We were able to show that $\text{vc}^T(n) = n$.

p -adic numbers have a natural tree structure associated to them.

vc-density in Shelah-Spencer graphs

Shelah-Spencer random graph is a random graph which is a limit of finite random graphs with edge probability $n^{-\alpha}$ with α irrational in $(0, 1)$. We view it in the language (V, R) .

We show that $\text{vc}^V(1) = \infty$, so vc-function is infinite. However to any formula $\phi(\vec{x}, \vec{y})$ we can prescribe a natural notion of dimension ϵ , and we have

$$\text{vc}^V(\phi) < \frac{|x|}{\epsilon}$$

So even though vc-function is not well-behaved, there is still a linear structure on vc-density.