## 1. Combinatorics

Suppose we have an infinite collection of sets $\mathcal{F}$. Take $n$ many of those sets. They generate a boolean algebra. Count the number of atoms in it. There can be at most $2^n$ atoms, though depending on the collection there may be much less. For a given $n$, out of all choices of $n$ sets, record the highest possible number of atoms generated. We define that to be a shatter function.

**Definition 1.1.**

$\pi_{\mathcal{F}}(n) = \max\{\#$ of atoms in boolean algebra generated by $S \mid S \subset \mathcal{F}$ and $|S| = n\}$

Example: Let $\mathcal{F}$ consist of all discs in the plane.

$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 8 \quad \pi_{\mathcal{F}}(4) = 14$$

$$\pi_{\mathcal{F}}(n) = n^2 - n + 2$$

Example: Let $\mathcal{F}$ consist of all half-planes in the plane.

$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 7 \quad \pi_{\mathcal{F}}(4) = 11$$

$$\pi_{\mathcal{F}}(n) = n^2/2 + n/2 + 1$$

**Example 1.2.**

(1) Let $\mathcal{F}$ be a set of lines on a plane. Then
$$\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$$
(2) Let $\mathcal{F}$ be a set of disks on a plane. Then
$$\pi_{\mathcal{F}}(n) = n^2 - n + 2$$
(3) Let $\mathcal{F}$ be a set of balls in $\mathbb{R}^3$. Then
$$\pi_{\mathcal{F}}(n) = n(n^2 - 3n + 8)/3$$
(4) Let $\mathcal{F}$ be a set of intervals on a line. Then
$$\pi_{\mathcal{F}}(n) = 2n$$
(5) Let $\mathcal{F}$ be a set of half-planes. Then
$$\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$$
(6) Let $\mathcal{F}$ be a collection of finite subsets of $\mathbb{N}$. Then
$$\pi_{\mathcal{F}}(n) = 2^n$$
(7) Let $\mathcal{F}$ be a collection of polygons in a plane. Then
$$\pi_{\mathcal{F}}(n) = 2^n$$

**Theorem 1.3** (Sauer-Shelah)**.** *Shatter function is either $2^n$ or bounded by a polynomial.*

**Definition 1.4.** Suppose growth of shatter function for $\mathcal{F}$ is polynomial. Let $r$ be the smallest real such that

$$\pi_{\mathcal{F}}(n) = O(n^r)$$

We define such $r$ to be the vc-density of $\mathcal{F}$. If shatter function grows exponentially, we let vc-density to be infinite.

## 2. Model Theory

Consider a structure with a language

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

We work with subsets of the underlying set definable by first-order formulas. Those are called definable sets.

$$\begin{aligned}
\phi(x) &= 5 \leq x \leq 7.7 \vee x \leq 0 \\
\psi(x) &= \exists y \ y \cdot y = x \\
\gamma(x) &= x \cdot x \cdot x \cdot x = 2
\end{aligned}$$

$\phi(\mathbb{R})$ defines the set $[5, 7.7] \cup (-\infty, 0]$ in the structure above. $\psi(\mathbb{R})$ defines the set $[0, \infty)$ in the structure above.

(1) in rationals $(\mathbb{Q}, \cdot)$ $\gamma(x)$ defines an empty subset
(2) in reals $(\mathbb{R}, \cdot)$ $\gamma(x)$ defines a subset with two elements
(3) in complex numbers $(\mathbb{C}, \cdot)$ $\gamma(x)$ defines a subset with four elements
(4) in quaternions $(\mathbb{H}, \cdot)$ $\gamma(x)$ defines an infinite subset

$$\theta(x) = \forall y \exists z \ x \leq z \leq y$$

(1) in $(\mathbb{Q}, \leq)$ $\theta(x)$ defines an empty subset
(2) in $(\mathbb{N}, \leq)$ $\theta(x)$ defines an empty subset
(3) in $(\mathbb{Q}^{\geq 0}, \leq)$ $\theta(x)$ defines the set $\{0\}$

**Definition 2.1.** for a formula $\phi(x_1 \ldots x_n, y_1, \ldots y_m)$ we can plug in elements of our structure as parameters in places of $y$ variables. This gives us a collection of definable sets.

**Example 2.2.**

$$\phi(x_1, x_2, y_1, y_2, y_3) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq y_3^2$$

In structure $(\mathbb{R}, +, \cdot, \leq)$ given $a, b, r \in \mathbb{R}$ the formula $\phi(x_1, x_2, a, b, r)$ defines a disk in $\mathbb{R}^2$ with radius $r$ with center $(a, b)$.

Thus all discs in $\mathbb{R}^2$ are defined uniformly by $\phi$.

What are the collection of sets we can consider when working with a model?

We can look at all definable subsets. That's not interesting, always has an infinite vc-density. Uniformly definable families offer more interesting behavior.

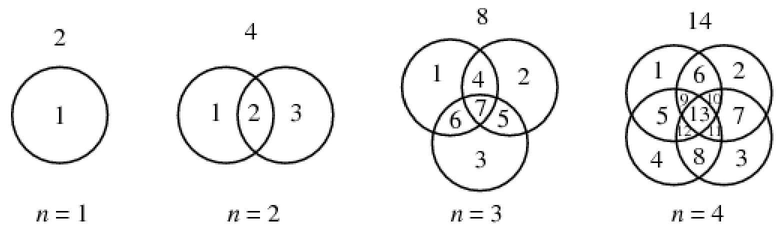
A model is said to be NIP if all uniformly definable families have finite vc-density.

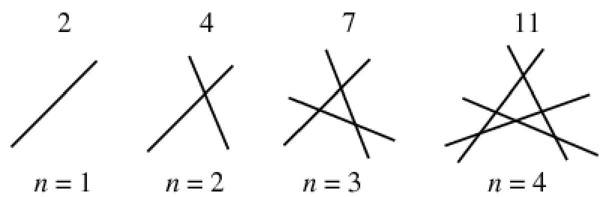
For a given model $M$, let vc function of $n$ to be the largest vc-density achieved by $n$-dimensional families of uniformly definable sets.

$$\mathrm{vc}^M(n) = max \{\mathrm{vc}(\phi) \mid \phi(\vec{x}, \vec{y}) \text{ with } |\vec{x}| = n\}$$

It is easy to show that $\mathrm{vc}_M(n) \geq n$ for all models.

Open questions about vc functions.
Is $\mathrm{vc}_M(n) = n\,\mathrm{vc}_M(1)$? If not, is there a linear relationship?


2
4
8
14
1
1 2 3
1 4 2 7 6 5 3
1 6 2 5 9 10 13 7 12 11 4 8 3
$n = 1$
$n = 2$
$n = 3$
$n = 4$


2
4
7
11
$n = 1$
$n = 2$
$n = 3$
$n = 4$