# A Tiny Example

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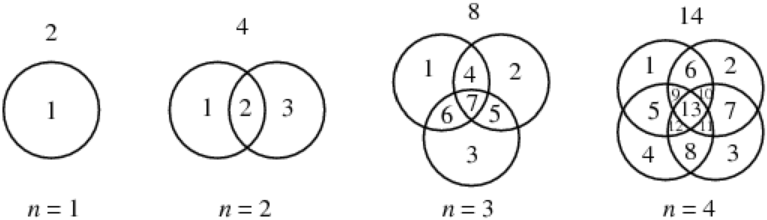
Suppose we have an (infinite) collection of sets $\mathcal{F}$.
We define a shatter function $\pi_{\mathcal{F}}(n)$

$$\pi_{\mathcal{F}}(n) = \max\{\# \text{ of atoms in boolean algebra generated by } S \\ \mid S \subset \mathcal{F} \text{ with } |S| = n\}$$

Example: Let $\mathcal{F}$ consist of all discs on a plane.




$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 8 \quad \pi_{\mathcal{F}}(4) = 14$$

$$\pi_{\mathcal{F}}(n) = n^2 - n + 2$$

More examples:

1. Lines on a plane $\pi_{\mathcal{F}}(n) = n^2/2 + n/2 + 1$
2. Disks on a plane $\pi_{\mathcal{F}}(n) = n^2 - n + 2$
3. Balls in $\mathbb{R}^3$ $\pi_{\mathcal{F}}(n) = n^3/3 - n^2 + 8n/3$
4. Intervals on a line $\pi_{\mathcal{F}}(n) = 2n$
5. Half-planes on a plane $\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$
6. Finite subsets of $\mathbb{N}$ $\pi_{\mathcal{F}}(n) = 2^n$
7. Polygons in a plane $\pi_{\mathcal{F}}(n) = 2^n$

### Theorem (Sauer-Shelah)

*Shatter function is either $2^n$ or bounded by a polynomial.*

### Definition

Suppose growth of shatter function for $\mathcal{F}$ is polynomial. Let $r$ be the smallest real such that

$$\pi_{\mathcal{F}}(n) = O(n^r)$$

We define such $r$ to be the vc-density of $\mathcal{F}$, $\mathrm{vc}(\mathcal{F})$. If shatter function grows exponentially, we let the vc-density to be infinite.

# Applications

- VapnikChervonenkis Theorem in probability
- Computability
- NIP theories

# Model Theory

Model Theory studies definable sets in first-order structures.

$$(\mathbb{Q}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) = \exists y \; y \cdot y = x$$

In the structure above $\phi(x)$ defines a set of numbers that are a square.

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) = \exists y \; y \cdot y = x$$

In the structure above $\phi(x)$ defines the set $[0, \infty)$.

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\psi(x_1, x_2) = (x_1 \cdot x_1 + x_2 \cdot x_2 \leq 1.5) \wedge (x_1^2 \leq x_2)$$

This defines a set in $\mathbb{R}^2$.

We work with families of uniformly definable sets. Fix a formula $\phi(x_1 \dots x_n, y_1, \dots y_m)$. Plug in elements from the model for $y$ variables to get a family of definable sets in $M^n$.

$$\mathcal{F}_\phi^M = \{\phi(x_1, \dots, x_n, a_1, \dots a_n) \mid a_1, \dots a_n \in M\}$$

Define $\mathrm{vc}^M(\phi)$ to be the vc-density of the family $\mathcal{F}_\phi^M$

$$\phi(x_1, x_2, y_1, y_2, y_3) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq y_3^2$$

In structure $(\mathbb{R}, +, \cdot, \leq)$ given $a, b, r \in \mathbb{R}$ the formula $\phi(x_1, x_2, a, b, r)$ defines a disk in $\mathbb{R}^2$ with radius $r$ with center $(a, b)$. Thus $\mathcal{F}_\phi^{\mathbb{R}}$ is a collection of all disks in $\mathbb{R}^2$.

A model $M$ is said to have NIP property if all uniformly definable families in it have finite vc-density.

- Examples
  - $(\mathbb{R}, 0, 1, +, \cdot, \leq)$
  - $(\mathbb{C}, 0, 1, +, \cdot)$
  - $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$
- Non-examples
  - $(\mathbb{Q}, 0, 1, +, \cdot)$
  - Random graph $(V, R)$.
  - Pseudo-finite fields.