RESEARCH PLAN: IMAGES OF p-ADIC FAMILIES OF GALOIS REPRESENTATIONS

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For a prime p, let $\Lambda = \mathbb{Z}_p[[T]]$ and $\mathbb{I}$ an integral domain that is finite flat over Λ. Recall that a Hida family F is a power series $F = \sum_{n=1}^{\infty} a_n q^n \in \mathbb{I}[[q]]$ that specializes to the q-expansion of a classical modular form for certain *arithmetic* primes of $\mathbb{I}$. We shall write $\rho_F : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{I})$ for the corresponding Galois representation and $\overline{\rho}_F$ for its residual representation. For an ideal $\mathfrak{a}$ of a ring A we write $\Gamma_A(\mathfrak{a}) = \ker(\mathrm{SL}_2(A) \to \mathrm{SL}_2(A/\mathfrak{a}))$.

Theorem 1 (Hida [3])**.** *Let $p > 3$ and let F be a non-CM Hida family that satisfies a minor regularity condition. Then there is a non-zero ideal $\mathfrak{a}$ of Λ such that, in an appropriate basis, $\Gamma_\Lambda(\mathfrak{a}) \subseteq \mathrm{Im}\,\rho_F$.*

The largest Λ-ideal $\mathfrak{a}$ such that $\Gamma_\Lambda(\mathfrak{a}) \subseteq \mathrm{Im}\,\rho_F$ is called the Λ-*level* $\mathfrak{c}_F$ of ρ_F. Hida determined the Λ-level of F in many cases, including those described in the following theorem.

Theorem 2 (Hida [3])**.** *Let F and p be as in Theorem 1.*

- *(a) If $\mathrm{Im}\,\bar{\rho}_F \supseteq \mathrm{SL}_2(\mathbb{F}_p)$, then $\mathfrak{c}_F \supseteq \mathfrak{m}_\Lambda^n$ for some non-negative integer n, where $\mathfrak{m}_\Lambda$ denotes the unique maximal ideal of Λ.*
- *(b) Suppose that $\bar{\rho}_F$ is absolutely irreducible and $\bar{\rho}_F \cong \mathrm{Ind}_M^{\mathbb{Q}}\,\bar{\psi}$ for an imaginary quadratic field M in which p splits and a character $\bar{\psi} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \overline{\mathbb{F}}_p^\times$. Assume M is the only such quadratic field. Under minor conditions on the tame level of F, there is a product $\mathcal{L}$ of anticyclotomic Katz p-adic L-functions such that $\mathfrak{c}_F | \mathcal{L}^2$. Furthermore, for every prime divisor P of $\mathcal{L}$ there exists a Hida family F such that $P | \mathfrak{c}_F$.*

My goal is to understand $\mathrm{Im}\,\rho_F$ as precisely as possible by using the following definition.

Definition 3. [Ribet [9]] Let F be as above and let $Q(\mathbb{I})$ the field of fractions of $\mathbb{I}$. An automoprhism σ of $Q(\mathbb{I})$ is a *conjugate self-twist* of F if there is a non-trivial Dirichlet character η_σ such that $a_\ell^\sigma = \eta_\sigma(\ell) a_\ell$ for almost all primes ℓ. Write Γ_F for the group generated by all conjugate self-twists and $\mathbb{I}_0$ for the integral closure of Λ in $Q(\mathbb{I})^\Gamma$.

My past work is a generalization of Hida's Theorem 1 and an analogue of Ribet [10] and Momose's [8] results in the classical setting. The main result of my paper [5] is the following theorem.

Theorem 4 (Lang [5])**.** *Let F be a non-CM Hida family such that $\bar{\rho}_F$ is absolutely irreducible and satisfies a minor $\mathbb{Z}_p$-regularity condition. Then there is a non-zero ideal $\mathfrak{a}_0$ of $\mathbb{I}_0$ and a basis for ρ_F such that $\mathrm{Im}\,\rho_F \supseteq \Gamma_{\mathbb{I}_0}(\mathfrak{a}_0)$.*

RESEARCH OBJECTIVES

Determining the $\mathbb{I}_0$-level and relation to p-adic L-functions. The first project is to strengthen Hida's Theorem 2 by replacing the Λ-level by the $\mathbb{I}_0$-level, enabling me to prove the following.

Conjecture 1. *Under the assumptions of Theorem 2:*

- *(a) If $\mathrm{Im}\,\rho_F \supseteq \mathrm{SL}_2(\mathbb{F}_p)$, then $\mathfrak{c}_{0,F} = \mathbb{I}_0$. That is, $\mathrm{Im}\,\rho_F \supseteq \mathrm{SL}_2(\mathbb{I}_0)$.*
- *(b) Under the hypotheses of Theorem 2(b) there is an $\mathbb{I}_0$-analogue $\mathcal{L}_0$ of $\mathcal{L}$ such that $\mathfrak{c}_{0,F}$ is a factor of $\mathcal{L}_0$. Furthermore, every prime factor of $\mathcal{L}_0$ divides $\mathfrak{c}_{0,F}$ for some F.*

Note that even when $\mathbb{I} = \Lambda$, Conjecture 1 is stronger than Theorem 2. Furthermore, Conjecture 1(a) is a natural extension of the work of Mazur-Wiles [7].

In proving case (a) of the conjecture, I will make use of Manoharmayum's recent work that shows $\operatorname{Im}\rho_F \supseteq \mathrm{SL}_2(W)$ for a finite unramified extension W of $\mathbb{Z}_p$ [6]. This will be combined with the Λ-module structure on the Pink Lie algebra associated to $\operatorname{Im}\rho_F$ that was used in the proof of Theorem 4 to get the desired result.

In proving case (b), I will relate the $\mathbb{I}_0$-level to the congruence ideal of F as in the proof of Theorem 2(b). The connection to Katz p-adic L-functions is then obtained by relating the congruence ideal to the p-adic L-function through known cases of the Main Conjecture of Iwasawa Theory. The idea is that replacing the Λ-level with the more precise $\mathbb{I}_0$-level will allow me to remove the ambiguity of the square factors that show up in Theorem 2(b).

Computing $\mathcal{O}_{0,\mathfrak{p}}$-levels of classical Galois representations. The goal of this project is to compute the level of Galois representations coming from classical modular forms and thus completely determine the image of such a representation. Let f be a non-CM classical Hecke eigenform, $\mathfrak{p}$ a prime of the ring of integers of the field generated by the Fourier coefficients of f, and $\rho_{f,\mathfrak{p}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ the associated p-adic representation. Let π be a uniformizer of the subring $\mathcal{O}_0$ of $\mathcal{O}$ fixed by all conjugate self-twists. By the work of Ribet [10] and Momose [8], there is a minimal non-negative integer $c(f,\mathfrak{p})$ such that $\operatorname{Im}\rho_f$ contains all matrices of determinant 1 that are congruent to the identity modulo $\pi^{c(f,\mathfrak{p})}$. Their work shows that $c(f,\mathfrak{p}) = 0$ for all but finitely many primes $\mathfrak{p}$. However, relatively little is known about the case when $c(f,\mathfrak{p})$ is positive and the weight of f is greater than 2. I plan to study how $c(f,\mathfrak{p})$ changes as f varies over the classical specializations of a non-CM Hida family that is congruent to a CM family.

This project will have both theoretical and computational components. First, I will establish a relationship between $c(f,\mathfrak{p})$ and the congruence number of f, which should also be related to values of the Katz p-adic L-function, as suggested by the proof of Theorem 2(b). Once this is established, I will create a method in the open-source software Sage [11] to compute $c(f,\mathfrak{p})$ by computing the congruence number of f. Using the new functionality, I will create a large data set of levels of classical Galois representations in Hida families, which will likely lead to new conjectures to be studied theoretically. I have experience working with Sage from my Women in Numbers 3 project [1] and from leading a project at Sage Days 69. Indeed, my project consisted of writing a method to test whether a modular form is CM, a first step in the eventual program to compute $c(f,\mathfrak{p})$.

Most of the computation that has been done surrounding levels of classical Galois representations has been for Galois representations arising from elliptic curves over $\mathbb{Q}$; relatively little is known for higher weight forms. Completing this project will begin to fill that gap in the literature. This project will bring number theorists close to being able to completely and computationally determine the image of a Galois representation associated to a classical modular form.

Analogue of the Mumford-Tate Conjecture in p-adic families. Ribet and Momose proved the Mumford-Tate Conjecture for compatible systems of Galois representations associated to classical modular forms [8, 10]. Hida has proposed an analogue of the Mumford-Tate Conjecture for p-adic families of Galois representations. For an arithmetic prime $\mathfrak{P}$ of $\mathbb{I}$, write $\mathrm{MT}_{\mathfrak{P}}$ for the Mumford-Tate group of the compatible system containing $\rho_{f_{\mathfrak{P}}}$, so $\mathrm{MT}_{\mathfrak{P}}$ is an algebraic group over $\mathbb{Q}$. Let $\kappa(\mathfrak{P}) = \mathbb{I}_{\mathfrak{P}}/\mathfrak{P}_{\mathfrak{P}}$, and write $G_{\mathfrak{P}}$ for the Zariski closure of $\operatorname{Im}\rho_{f_{\mathfrak{P}}}$ in $\mathrm{GL}_2(\kappa(\mathfrak{P}))$. Let $G_{\mathfrak{P}}^{\circ}$ be the connected component of the identity of $G_{\mathfrak{P}}$ and $G'_{\mathfrak{P}}$ the (closed) derived subgroup of $G_{\mathfrak{P}}$.

Conjecture 2 (Hida)**.** *Assume F is non-CM. There is a simple algebraic group G', defined over $\mathbb{Q}_p$, such that for all arithmetic primes $\mathfrak{P}$ of $\mathbb{I}$ one has $G'_{\mathfrak{P}} \cong G' \times_{\mathbb{Q}_p} \kappa(\mathfrak{P})$ and $\operatorname{Res}^{\kappa(\mathfrak{P})}_{\mathbb{Q}_p} G_{\mathfrak{P}}$ is (the ordinary factor of) $\mathrm{MT}_{\mathfrak{P}} \times_{\mathbb{Q}} \mathbb{Q}_p$. Furthermore, the component group $G_{\mathfrak{P}}/G^0_{\mathfrak{P}}$ is canonically isomorphic to the Pontryagin dual of the decomposition group of $\mathfrak{P}$ in Γ_F.*

By obtaining a sufficiently precise understanding of images of Galois representations attached to Hida families through the first project, I hopes to prove results along the lines of Conjecture 2. I have some preliminary results relating the Pontryagin dual of Γ_F to the quotient $(\mathrm{Im}\,\rho_F)/(\mathrm{Im}\,\rho_F|_H)$ for a certain finite index normal subgroup H of $G_{\mathbb{Q}}$.

Completing this research objective, or even any preliminary results in this direction, would reveal that the images of classical specializations of the Galois representation attached to a Hida family are even more related to one another than previously thought. Not only would they come as specializations of some group in $\mathrm{GL}_2(\mathbb{I})$, they could all be found simply by base change from a single group, at least up to abelian error.

Other settings. The above projects can be studied in more general settings than Hida families for GL_2. Hida and Tilouine proved an analogue of Theorem 1 for GSp_4-representations associated to Hida families of Seigel modular forms [4]. There are two main difficulties they overcome in their work: there are more cases than CM versus non-CM, and Pink's theory of Lie algebras is only valid for SL_2. The tools they developed to overcome these problems could be applied to study analogues of the above questions for groups other than GL_2.

Tilouine and his collaborators proved an analogue of Theorem 4 in the non-ordinary GL_2-setting [2] by building on my ideas in [5]. As their representation is not ordinary, they introduce the relative Sen operator to create a Λ-algebra structure on the Lie algebra of the image of ρ_F. This idea will be useful in studying the above questions in the non-ordinary setting.

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