

VC-density in model theoretic structures

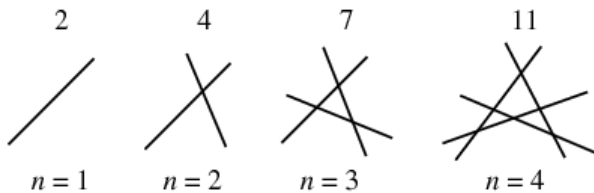
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Suppose we have an (infinite) collection of sets \mathcal{F} .
We define the shatter function $\pi_{\mathcal{F}}: \mathbb{N} \longrightarrow \mathbb{N}$ of \mathcal{F}

$$\pi_{\mathcal{F}}(n) = \max\{\# \text{ of atoms in the boolean algebra generated by } \mathcal{S} \\ | \mathcal{S} \subset \mathcal{F} \text{ with } |\mathcal{S}| = n\}$$

Example: Let \mathcal{F} consist of all half-planes in the plane.



$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 7 \quad \pi_{\mathcal{F}}(4) = 11$$

$$\pi_{\mathcal{F}}(n) = n^2/2 + n/2 + 1$$

More examples:

1. Disks in the plane: $\pi_{\mathcal{F}}(n) = n^2 - n + 2$
2. Balls in \mathbb{R}^3 : $\pi_{\mathcal{F}}(n) = n^3/3 - n^2 + 8n/3$
3. Intervals in the line: $\pi_{\mathcal{F}}(n) = 2n$
4. Finite subsets of \mathbb{N} : $\pi_{\mathcal{F}}(n) = 2^n$
5. Convex polygons in the plane: $\pi_{\mathcal{F}}(n) = 2^n$

Theorem (Sauer-Shelah '72)

The shatter function is either 2^n or bounded by a polynomial.

Definition

Suppose the growth of the shatter function of \mathcal{F} is polynomial. Let $\text{vc}(\mathcal{F})$ be the infimum of all positive reals r such that

$$\pi_{\mathcal{F}}(n) = O(n^r)$$

Call $\text{vc}(\mathcal{F})$ the vc -density of \mathcal{F} . If the shatter function grows exponentially, we let $\text{vc}(\mathcal{F}) := \infty$.

Applications

- ▶ Model Theory (NIP theories)
- ▶ VC-Theorem in probability (Vapnik-Chervonenkis '71)
- ▶ Computational learning theory (PAC learning, Warmuth conjecture)
- ▶ Computational geometry
- ▶ Functional analysis (Bourgain-Fremlin-Talagrand theory)
- ▶ Abstract topological dynamics (tame dynamical systems)

History

- ▶ VC-dimension defined by Vapnik-Chervonenkis '71
- ▶ NIP theories studied by Shelah '71
- ▶ vc-density in model theoretic context introduced by Aschenbrenner, Dolich, Haskell, Macpherson, Starchenko '13

Model Theory

Model Theory studies definable sets in first-order structures.

$$(\mathbb{Q}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) := \exists y \, y \cdot y = x$$

$\phi(\mathbb{Q})$ defines the set of rationals that are a square.

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) := \exists y \ y \cdot y = x$$

$\phi(\mathbb{R})$ defines the set $[0, \infty)$.

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\psi(x_1, x_2) := (x_1 \cdot x_1 + x_2 \cdot x_2 \leq 1.5) \wedge (x_1^2 \leq x_2)$$

$\phi(\mathbb{R}^2)$ defines the set in \mathbb{R}^2 that is an intersection of a disc with an inside of a parabola.

We work with families of uniformly definable sets. Fix a formula $\phi(x_1 \dots x_m, y_1, \dots y_n) = \phi(\vec{x}, \vec{y})$. Plug in elements from M for y variables to get a family of definable sets in M^m .

$$\mathcal{F}_\phi^M = \{\phi(M^m, a_1, \dots a_n) \mid a_1, \dots a_n \in M\}$$

Define $\text{vc}^M(\phi)$ to be the vc-density of the family \mathcal{F}_ϕ^M

$$\phi(x_1, x_2, y_1, y_2, y_3) := (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq y_3^2$$

In structure $(\mathbb{R}, 0, 1, +, \cdot, \leq)$ given $a, b, r \in \mathbb{R}$ the formula $\phi(x_1, x_2, a, b, r)$ defines a disk in \mathbb{R}^2 with radius r with center (a, b) . Thus $\mathcal{F}_\phi^\mathbb{R}$ is a collection of all disks in \mathbb{R}^2 .

Shelah ('78) classified number of isomorphism classes for structures elementarily equivalent to structure M . One of the important classes is NIP structures. Structure M is said to be NIP if all uniformly definable families in it have finite vc-density.

- ▶ $(\mathbb{C}, 0, 1, +, \cdot)$ is NIP
- ▶ $(\mathbb{R}, 0, 1, +, \cdot, \leq)$ is NIP
- ▶ $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$ is NIP
- ▶ Random graph (V, R) is not NIP
- ▶ $(\mathbb{Q}, 0, 1, +, \cdot)$ is not NIP.

Given an NIP structure M we define a vc-function of n to be the largest vc-density achieved by families of uniformly definable sets in M^n .

$$\text{vc}^M(n) = \max \left\{ \text{vc}^M(\phi) \mid \phi(\vec{x}, \vec{y}) \text{ with } |\vec{x}| = n \right\}$$

Easy to show $\text{vc}^M(n) \geq n \text{vc}^M(1) \geq n$

Open Question: If M is NIP, is it possible for $\text{vc}^M(\phi)$ to be irrational? Open Question: Is $\text{vc}^M(n) = n \text{vc}^M(1)$? If not, is there a linear relationship? If $\text{vc}(1) < \infty$ do we have $\text{vc}(2) < \infty$?

Examples

- ▶ $(\mathbb{R}, 0, 1, +, \cdot, \leq)$ has $\text{vc}(n) = n$ (true for o-minimal structures)
- ▶ $(\mathbb{C}, 0, 1, +, \cdot)$ has $\text{vc}(n) = n$
- ▶ $(\mathbb{Q}_p, 0, 1, +, \cdot)$ has $\text{vc}(n) \leq 2n - 1$

vc-density in trees

Consider structure (T, \leq) where elements of T are vertices of a rooted tree and we say that $a \leq b$ if a is below b in the tree.

- ▶ Trees are NIP (Parigot '82)
- ▶ Trees are dp-minimal (Simon '11)
- ▶ Trees have $vc(n) = n$ (B. '13)

proof background

$\text{tp}(a)$, a type of an element a is a set of all the formulas that are true about a .

Parigot's observation: there is a natural way to split a tree into parts A, B such that for $a \in A$ and $b \in B$ we have

$$\text{tp}(a), \text{tp}(b) \vdash \text{tp}(ab)$$

This allows us to decompose complex types into simple parts, which we can use to compute vc-density.

Shelah-Spencer graphs

Let α irrational $\in (0, 1)$. Consider a random graph on n vertices where the probability of any given two vertices having an edge is $n^{-\alpha}$. Shelah-Spencer ('88) showed that 0-1 law holds for first-order formulas. A structure satisfying those axioms is called a Shelh-Spencer graph.

- ▶ Shelah-Spencer graphs are stable (Baldwin-Shi '96, Baldwin-Shelah '97)

Background

Definition

- ▶ To a finite graph A assign a dimension $\delta(A) = |V| - \alpha|E|$.
- ▶ B/A is called a positive extension if quantity $\delta(B/A) = |V_B/V_A| - \alpha|E_B/E_A|$ is positive.
- ▶ B/A is called minimal if its dimension is negative, but every subextension is positive.
- ▶ (A_0, \dots, A_n) is a minimal chain if each A_{i+1}/A_i is minimal.

For B/A chain-minimal define

$$\phi_{A,B}(\vec{x}) = \exists \vec{x}^* \text{ such that } \vec{x}^* / \vec{x} \text{ is isomorphic to } B/A$$

Theorem (quantifier elimination, Laskowski '06)

In Shelah-Spencer graph every definable set can be defined by a boolean combination of formulas $\phi_{A_i, B_i}(\vec{x})$.

vc-density in Shelah-Spencer graphs

Theorem (B., '15)

For a formula $\phi(\vec{x}, \vec{y})$ we can define ϵ_L, ϵ_U explicitly computable from $\delta(B_i/A_i)$ such that

$$\epsilon_L |\vec{x}| \leq \text{vc}(\phi) \leq \epsilon_U |\vec{x}|$$

Corollary

$\text{vc}(1) = \infty$, so *vc-function is not well-behaved for this structure.*

Future work

- ▶ $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$
- ▶ Other partial orderings, lattices
- ▶ Other graph structures, in particular flat graphs