The following is the proof of the Theorem **7.1** in *Vapnik-Chervonenkis density in some theories without the independence property, I* without using a stong VC d property.

Theorem Assume that $\mathrm{vc}(m) \le r$ and the theory has the VC d property. Then $\mathrm{vc}(m+1) \le r+d$.

Proof Write $x = (x_0, x')$ with $x' = (x_1, \dots, x_m)$ so that $|x_0| = 1$ and $|x'| = m$. Let $\Delta(x;y)$ be given. Define

$$\Delta_0(x_0; x', y) = \{\phi(x_0; x', y) \mid \phi(x;y) \in \Delta\}$$

Applying VC d property to Δ_0 we have finitely many families

$$\mathcal{F}_i = (\phi_i(x', y; y_1, \dots, y_d))_{\phi \in \Delta} \qquad (i \in I)$$

of $\mathcal{L}$-formulas with the following property: for any $a_1 \in M^{|x_0|}$, $a_2 \in M^{|x'|}$, any finite $B \subset M^{|y|}$ there are $\vec{b} \in B^d$ and $i \in I$ such that $\mathcal{F}_i(a_2, y; \vec{b})$ defines $\mathrm{tp}_{\Delta_0}(a_1/a_2B)$, i.e. for all $\phi \in \Delta, b \in B$

$$\vDash \phi(a_1, a_2, b) \iff \vDash \phi_i(a_2, b, \vec{b})$$

For each $i \in I$ let

$$\Delta_i(x'; y, y_1 \dots y_d) = \{\phi_i(x'; y, y_1 \dots y_d) \mid \phi(x;y) \in \Delta\}$$

As $|x'| = m$ the assumption that $\mathrm{vc}(m) \le r$ applies to each Δ_i. Thus there is a constant K such that for any finite $C \subset (M^{|y|})^{(d+1)}$ there is a set of representatives for $S^{\Delta_i}(C)$ of size at most $K|C|^r$. (More precisely, for each Δ_i there is going to be such a constant K_i and we can take K to be the maximum of these). Now fix a finite set $B \subset M^{|y|}$. Let $N = K|B|^r$. For every element $\vec{b} \in B^d$ fix a set of representatives of $S^{\Delta_i}(B\vec{b})$ (Note that $|B| = |B\vec{b}|$)

$$\alpha_1^{i,\vec{b}}, \alpha_2^{i,\vec{b}}, \dots, \alpha_N^{i,\vec{b}}$$

(Some of the representatives may be repeated). Also fix a set of representatives of every type in $S^{\Delta}(B)$. That is we pick some functions

$$F_1 : S^{\Delta}(B) \longrightarrow M^{|x_0|}$$
$$F_2 : S^{\Delta}(B) \longrightarrow M^{|x'|}$$

such that for all $\boldsymbol{p}(x_0, x') \in S^{\Delta}(B)$ we have $\boldsymbol{p} = \mathrm{tp}^{\Delta}(F_1(\boldsymbol{p})F_2(\boldsymbol{p})/B)$, i.e. $(F_1(\boldsymbol{p}), F_2(\boldsymbol{p}))$ is a realization of $\boldsymbol{p}$. Now to every type in $S^{\Delta}(B)$ we assign a triple of elements $\langle i, \vec{b}, \alpha\rangle$ where $i \in I, \vec{b} \in B^d$ and α is one of the chosen representatives. This is done as follows. Given a type $\boldsymbol{p} \in S^{\Delta}(B)$ we pick its realization $(a_1, a_2) = (F_1(\boldsymbol{p}), F_2(\boldsymbol{p}))$. By definability of Δ_0 there is $j \in I$ and $\vec{b} \in B^d$ such that for all $\phi \in \Delta, b \in B$

$$\vDash \phi(a_1, a_2, b) \iff \vDash \phi_j(a_2, b, \vec{b})$$

Pick α, a representative of $S^{\Delta_j}(B\vec{b})$ that has the same Δ_j-type as a_2. ($\alpha = \alpha_k^{i,\vec{b}}$ for some $k \in [N]$). That is for all $b \in B, \phi \in \Delta$ we have

$$\vDash \phi_j(a_2, b, \vec{b}) \iff \vDash \phi_j(\alpha, b, \vec{b})$$

To the type $\boldsymbol{p}$ we associate triple $\langle j, \vec{b}, \alpha \rangle$. (In general there might be more than one choice for the triple. To ensure uniqueness pick the smallest triple after fixing some appropriate ordering) This defines a map

$$F : S^{\Delta}(B) \longrightarrow T$$

where T is the space of all possible resulting tuples. We have $|T| = I \cdot |B^d| \cdot N = |I||B|^d K|B|^r = K|I||B|^{d+r}$. Once we show that F is injective we will have $S^{\Delta}(B) \leq K|I||B|^{d+r}$. As choice of Δ and B was arbitrary we will be done. Thus, all that remains is to show injectivity of F. We claim that $\boldsymbol{p}$ is uniquely determined by its triple. Fix $\boldsymbol{p} \in S^{\Delta}(B)$ and let $F(\boldsymbol{p}) = \langle j, \vec{b}, \alpha \rangle$. Now for all $\phi \in \Delta, b \in B$ we have

$$\begin{aligned} \phi(x_0, x', b) \in \boldsymbol{p} &\iff \models \phi(F_1(\boldsymbol{p}), F_2(\boldsymbol{p}), b) \\ &\iff \models \phi_j(F_2(\boldsymbol{p}), b, \vec{b}) \iff \models \phi_j(\alpha, b, \vec{b}) \end{aligned}$$

This shows that two different types would have different tuples associated to them. Thus F is injective as needed. □

Corollary If $vc(1) = r$ and the theory has VC d property then $\mathrm{vc}(m) \leq r + d \cdot (m-1)$