VC-DENSITY FOR TREES

ANTON BOBKOV

ABSTRACT. We show that for the theory of infinite trees we have $\mathrm{vc}(n) = n$ for all n.

VC-density was introduced in [1] by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko as a natural notion of dimension for NIP theories. In an NIP theory we can define a vc-function

$$\mathrm{vc} : \mathbb{N} \longrightarrow \mathbb{N}$$

Where $\mathrm{vc}(n)$ measures complexity of definable sets in an n-dimensional space. Simplest possible behavior is $\mathrm{vc}(n) = n$ for all n. Theories with that property are known to be dp-minimal, i.e. having the smallest possible dp-rank. In general, it is not known whether there can be a dp-minimal theory which doesn't satisfy $\mathrm{vc}(n) = n$.

In this paper we work with trees viewed as posets. Parigot in [3] showed that such models have NIP. This result was strengthened by Simon in [2] showing that trees are dp-minimal. [1] has the following problem

Problem 0.1. ([1] p. 47) Determine the VC density function of each (infinite) tree.

Here we settle this question by showing that the theory of trees has $\mathrm{vc}(n) = n$.

1. PRELIMINARIES

We use notation $a \in T^n$ for the tuples of size n. For a variable x or a tuple a we denote their arity by $|x|$ and $|a|$ respectively.

We work with finite relational languages. Given a formula we can define its complexity as the depth of quantifiers used to build up the formula. More precisely

Definition 1.1. Define *complexity* of a formula by induction:

$$\text{Complexity}(\text{q.f. formula}) = 0$$

$$\text{Complexity}(\exists x \phi(x)) = \text{Complexity}(\phi(x)) + 1$$

$$\text{Complexity}(\phi \wedge \psi) = \max(\text{Complexity}(\phi), \text{Complexity}(\psi))$$

$$\text{Complexity}(\neg\phi) = \text{Complexity}(\phi)$$

A simple inductive argument verifies that there are (up to equivalence) only finitely many formulas when the complexity and the number of free variablesare fixed. We will use the following notation for types:

Definition 1.2. Let n, m be natural numbers, $\boldsymbol{B}$ a structure, A a parameter set and a, b tuples in $\boldsymbol{B}$.

- $\text{tp}^n_{\boldsymbol{B}}(a/A)$ will stand for all the A-formulas of complexity $\leq n$ that are true of a in $\boldsymbol{B}$. If $A = \emptyset$ we may write $\text{tp}^n_{\boldsymbol{B}}(a)$. $\boldsymbol{B}$ will be omitted as well if it is clear from context. Note that if A is finite, there are finitely many such formulas (up to an equivalence). Conjunction of those formulas would still have complexity $\leq n$ so we can just associate a single formula to every type.
- $\boldsymbol{B} \models a \equiv^n_A b$ means that a, b have the same type with complexity n over A in $\boldsymbol{B}$, i.e. $\text{tp}^n_{\boldsymbol{B}}(a/A) = \text{tp}^n_{\boldsymbol{B}}(b/A)$
- $S^n_{\boldsymbol{B},m}(A)$ will stand for all m-types of complexity n over A, for example $\text{tp}^n_{\boldsymbol{B}}(a/A) \in S^n_{\boldsymbol{B},m}(A)$ assuming $|a| = m$.

The language for the trees consists of a single binary predicate $\{\leq\}$. The theory of trees states that $\leq$ defines a partial order and for every element a we have that $\{x \mid x < a\}$ is a linear order. For visualization purposes we assume that trees grow upwards, with the smaller elements on the bottom and the larger elements on the top. If $a \leq b$ we will say that a is below b and b is above a.

Definition 1.3. Work in a tree $\boldsymbol{T}$. For $x \in T$ let $I(x) = \{t \in T \mid t \leq x\}$ denote all the elements below x. *Meet* of two tree elements a, b is the greatest element of $I(a) \cap I(b)$ (if one exists) and is denoted by $a \wedge b$.

The theory of meet trees requires that any two elements in the same connected component have a meet. Colored trees are trees with a finite number of colors added via unary predicates.

From now on assume that all trees are colored. We allow our trees to be disconnected or finite unless otherwise stated.

2. Proper Subdivisions: Definition and Properties

Definition 2.1. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{T}$ be models in some (possibly different) finite relational languages. If A, B partition T (i.e. $T = A \sqcup B$) we say that $(\boldsymbol{A}, \boldsymbol{B})$ is a *subdivision* of $\boldsymbol{T}$.

Definition 2.2. $(\boldsymbol{A}, \boldsymbol{B})$ subdivision of $\boldsymbol{T}$ is called *n-proper* if given $p, q \in \mathbb{N}$, $a_1, a_2 \in A^p$ and $b_1, b_2 \in B^q$ with

$$\boldsymbol{A} \models a_1 \equiv_n a_2$$

$$\boldsymbol{B} \models b_1 \equiv_n b_2$$

then we have

$$\boldsymbol{T} \models a_1 b_1 \equiv_n a_2 b_2$$

Definition 2.3. $(\boldsymbol{A}, \boldsymbol{B})$ subdivision of $\boldsymbol{T}$ is called *proper* if it is n-proper for all $n \in \mathbb{N}$.

Lemma 2.4. *Consider a subdivision $(\boldsymbol{A}, \boldsymbol{B})$ of $\boldsymbol{T}$. If it is 0-proper then it is proper.*

Proof. We prove that the subdivision is n-proper for all n by induction. Case $n = 0$ is given by the assumption. Suppose $n = k + 1$ and we have $\boldsymbol{T} \models \exists x\, \phi^k(x, a_1, b_1)$ where ϕ^k is some formula of complexity k. Let $a \in T$ witness the existential claim i.e. $\boldsymbol{T} \models \phi^k(a, a_1, b_1)$. We can have $a \in A$ or $a \in B$. Without loss of generality assume $a \in A$. Let $\boldsymbol{p} = \mathrm{tp}^k_{\boldsymbol{A}}(a, a_1)$. Then we have

$$\boldsymbol{A} \models \exists x \ \mathrm{tp}^k_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$$

Formula $\mathrm{tp}^k_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$ is of complexity k so $\exists x \ \mathrm{tp}^k_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$ is of complexity $k + 1$. By the inductive hypothesis we have

$$\boldsymbol{A} \models \exists x \ \mathrm{tp}^k_{\boldsymbol{A}}(x, a_2) = \boldsymbol{p}$$

Let a' witness this existential claim so that

$$\begin{aligned} &\mathrm{tp}^k_{\boldsymbol{A}}(a', a_2) = \boldsymbol{p} \\ &\mathrm{tp}^k_{\boldsymbol{A}}(a', a_2) = \mathrm{tp}^k_{\boldsymbol{A}}(a, a_1) \\ &\boldsymbol{A} \models a' a_2 \equiv_k a a_1 \end{aligned}$$

by the inductive hypothesis we have

$$\begin{aligned} \boldsymbol{T} \models a a_1 b_1 \equiv_k a' a_2 b_2 & \\ \boldsymbol{T} \models \phi^k(a', a_2, b_2) & \qquad \text{as } \boldsymbol{T} \models \phi^k(a, a_1, b_1) \\ \boldsymbol{T} \models \exists x \phi^k(x, a_2, b_2) & \end{aligned}$$

$\square$

We use this lemma for (colored) trees. Suppose $\boldsymbol{T}$ is a model of a (colored) tree in some language $\mathcal{L} = \{\leq, \ldots\}$. Suppose $\boldsymbol{A}, \boldsymbol{B}$ are some structures in languages $\mathcal{L}_A, \mathcal{L}_B$ which expand $\mathcal{L}$, with $\boldsymbol{A}, \boldsymbol{B}$ substructures of $\boldsymbol{T}$ as reducts to $\mathcal{L}$. Furthermore suppose that $(\boldsymbol{A}, \boldsymbol{B})$ is proper. In this case we'll refer to $(\boldsymbol{A}, \boldsymbol{B})$ as a *proper subdivision* (of $\boldsymbol{T}$).

Example 2.5. Suppose a tree consists of two connected components C_1, C_2. Then those components $(C_1, \leq, \ldots)$, $(C_2, \leq, \ldots)$ interpreted as substructures form a proper subdivision.

Example 2.6. Fix a tree $\boldsymbol{T}$ in the language $\{\leq\}$ and $a \in T$. Let $B = \{t \in T \mid a < t\}$, $S = \{t \in T \mid t \leq a\}$, $A = T - B$. Then $(A, \leq, S)$ and $(B, \leq)$ form a proper subdivision, where $\mathcal{L}_A$ has a unary predicate interpreted by S.

Definition 2.7. For $\phi(x, y)$, $A \subseteq T^{|x|}$ and $B \subseteq T^{|y|}$

- Let $\phi(A, b) = \{a \in A \mid \phi(a, b)\} \subseteq A$
- Let $\phi(A, B) = \{\phi(A, b) \mid b \in B\} \subseteq \mathcal{P}(A)$

$\phi(A, B)$ is a collection of subsets of A definable by ϕ with parameters from B. We notice the following bound when A, B are parts of a proper subdivision.

Corollary 2.8. *Let $\boldsymbol{A}, \boldsymbol{B}$ be a proper subdivision of $\boldsymbol{T}$ and $\phi(x, y)$ a formula of complexity n. Then $|\phi(A^{|x|}, B^{|y|})|$ is bounded by $|S^n_{\boldsymbol{B},|y|}|$.*

Proof. Take some $a \in A^{|x|}$ and $b_1, b_2 \in B^{|y|}$ with $\mathrm{tp}^n_{\boldsymbol{B}}(b_1) = \mathrm{tp}^n_{\boldsymbol{B}}(b_2)$. We have $\boldsymbol{B} \models b_1 \equiv_n b_2$ and (trivially) $\boldsymbol{A} \models a \equiv_n a$. Thus by the Lemma 2.4 we have $\boldsymbol{T} \models ab_1 \equiv_n ab_2$ so $\phi(a, b_1) \leftrightarrow \phi(a, b_2)$. Since a was arbitrary we have $\phi(A^{|x|}, b_1) = \phi(A^{|x|}, b_2)$ as different traces can only come from parameters of different types. Thus $|\phi(A^{|x|}, B^{|y|}) \leq |S^n_{\boldsymbol{B},|y|}|$. $\square$

We note that the size of type space $|S^n_{\boldsymbol{B},|y|}|$ can be bounded uniformly:

Definition 2.9. Fix a (finite relational) language $\mathcal{L}_B$, and n, $|y|$. Let $N = N(n, |y|, \mathcal{L}_B)$ be smallest number such that for any structure $\boldsymbol{B}$ in $\mathcal{L}_B$ we have $|S^n_{\boldsymbol{B},|y|}| \leq N$. This number is well-defined as there is a finite number (up to equivalence) of possible formulas of complexity $\leq n$ with $|y|$ free variables. Note the following easy inequalities

$$n \le m \Rightarrow N(n, |y|, \mathcal{L}_B) \le N(m, |y|, \mathcal{L}_B)$$

$$|y| \le |z| \Rightarrow N(n, |y|, \mathcal{L}_B) \le N(n, |z|, \mathcal{L}_B)$$

$$\mathcal{L}_A \subseteq \mathcal{L}_B \Rightarrow N(n, |y|, \mathcal{L}_A) \le N(n, |y|, \mathcal{L}_B)$$

3. Proper Subdivisions: Constructions

From now on work in meet trees unless mentioned otherwise. First, we describe several constructions of proper subdivisions that are needed for the proof.

Definition 3.1. If b and c are in the same connected component we denote it as $E(b, c)$

$$E(b, c) \Leftrightarrow \exists x\, (b \ge x) \wedge (c \ge x)$$

Definition 3.2. Given a tree element a we can look at all the elements above a, i.e. $\{x \mid x \ge a\}$. We can think about it as a *closed cone* above a. Connected components of that cone can be thought of as *open cones* over a. With that interpretation in mind, notation $E_a(b, c)$ means that b and c are in the same open cone over a. More formally:

$$E_a(b, c) \Leftrightarrow E(b, c) \text{ and } (b \wedge c) > a$$

Fix a language $\mathcal{L}$ for a colored tree $\mathcal{L} = \{\le, C_1, \dots C_n\} = \{\le, \vec{C}\}$. In the following four definitions $\boldsymbol{B}$-structures are going to be in the same language $\mathcal{L}_B = \mathcal{L} \cup \{U\}$ with U a unary predicate. It is not always necessary to have this predicate but we keep it for the sake of uniformity. $\boldsymbol{A}$-structures will have different $\mathcal{L}_A$ languages (those are not as important in later applications).

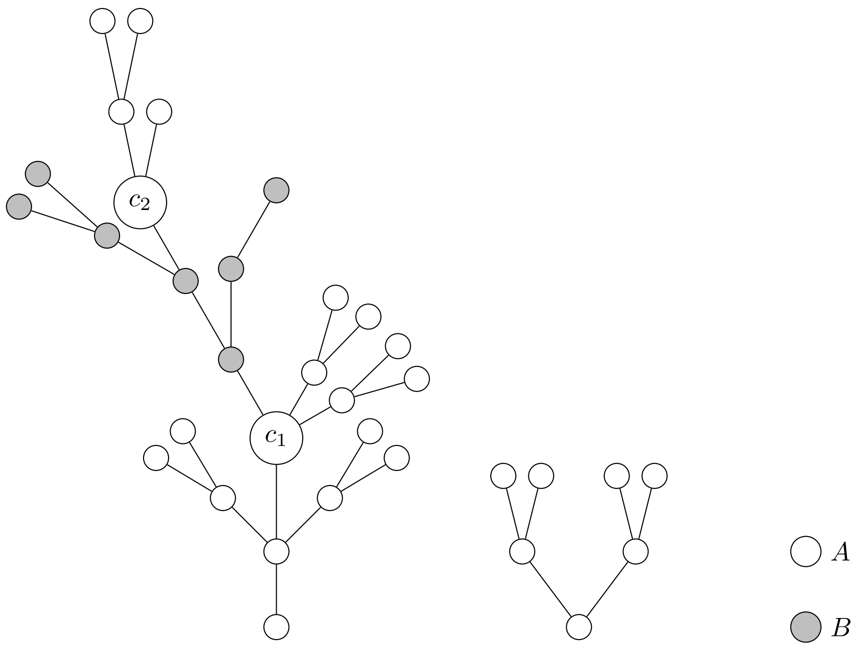

FIGURE 1. Proper subdivision for $(A, B) = (A^{c_1}_{c_2}, B^{c_1}_{c_2})$

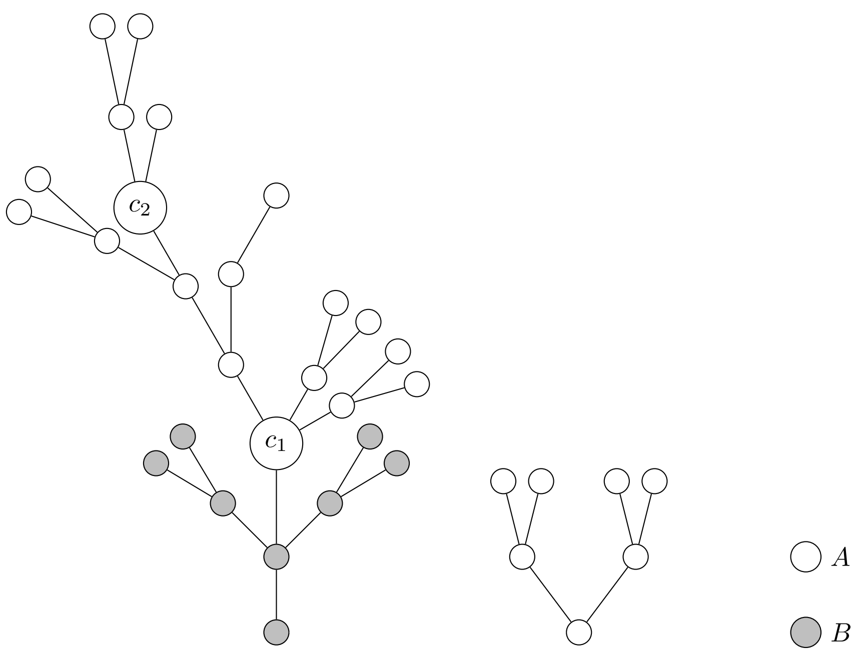

FIGURE 2. Proper subdivision for $(A, B) = (A_{c_1}, B_{c_1})$

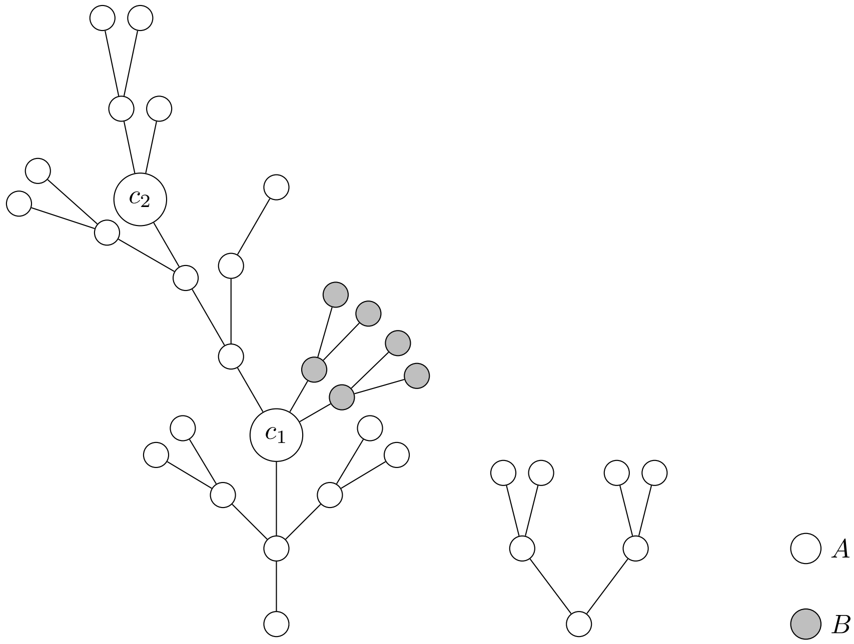

FIGURE 3. Proper subdivision for $(A, B) = (A_G^{c_1}, B_G^{c_1})$ for $G = \{c_2\}$

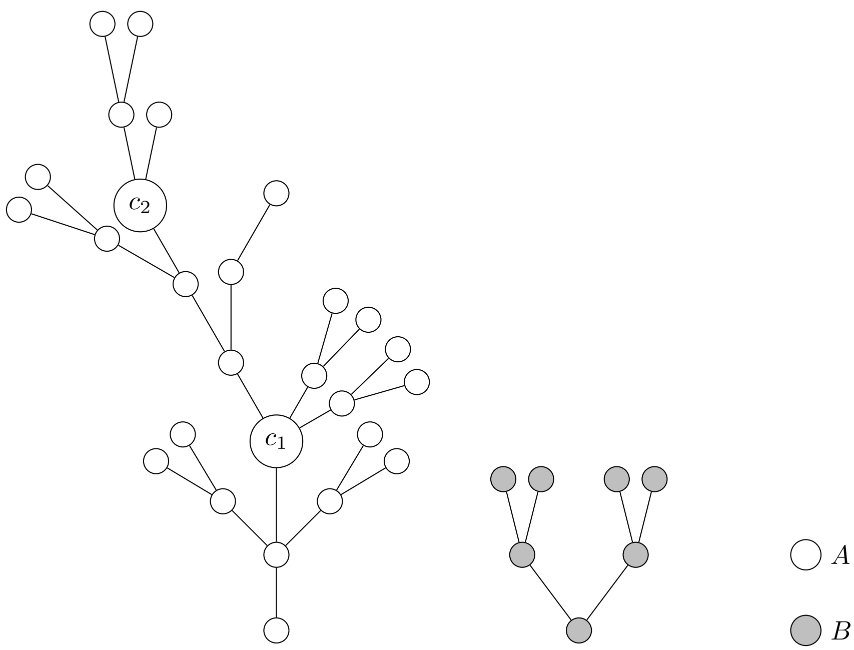

FIGURE 4. Proper subdivision for $(A, B) = (A_G, B_G)$ for $G = \{c_1, c_2\}$

Definition 3.3. Fix $c_1 < c_2$ in T. Let

$$B = \{b \in T \mid E_{c_1}(c_2, b) \wedge \neg(b \geq c_2)\}$$

$$A = T - B$$

$$S_1 = \{t \in T \mid t < c_1\}$$

$$S_2 = \{t \in T \mid t < c_2\}$$

$$S_B = S_2 - S_1$$

$$T_A = \{t \in T \mid c_2 \leq t\}$$

Define structures $\boldsymbol{A}_{c_2}^{c_1} = (A, \leq, \vec{C}, S_1, T_A)$ and $\boldsymbol{B}_{c_2}^{c_1} = (B, \leq, \vec{C}, S_B)$ where $\mathcal{L}_A$ is an expansion of $\mathcal{L}$ by two unary predicates (and $\mathcal{L}_B$ as defined above). Note that $c_1, c_2 \notin B$.

Definition 3.4. Fix c in T. Let

$$B = \{b \in T \mid \neg(b \geq c) \wedge E(b, c)\}$$

$$A = T - B$$

$$S_1 = \{t \in T \mid t < c\}$$

Define structures $\boldsymbol{A}_c = (A, \leq, \vec{C})$ and $\boldsymbol{B}_c = (B, \leq, \vec{C}, S_1)$ where $\mathcal{L}_A = \mathcal{L}$ (and $\mathcal{L}_B$ as defined above). Note that $c \notin B$. (cf example 2.6).

Definition 3.5. Fix c in T and $S \subseteq T$ a finite subset. Let

$$B = \{b \in T \mid (b > c) \text{ and for all } s \in S \text{ we have } \neg E_c(s, b)\}$$

$$A = T - B$$

$$S_1 = \{t \in T \mid t \leq c\}$$

Define structures $\boldsymbol{A}_S^c = (A, \leq, \vec{C}, S_1)$ and $\boldsymbol{B}_S^c = (B, \leq, \vec{C}, B)$ where L_A is an expansion of $\mathcal{L}$ by a single unary predicate (and $U \in \mathcal{L}_B$ vacuously interpreted by B). Note that $c \notin B$ and $S \cap B = \emptyset$.

Definition 3.6. Fix $S \subseteq T$ a finite subset. Let

$$B = \{b \in T \mid \text{ for all } s \in S \text{ we have } \neg E(s, b)\}$$
$$A = T - B$$

Define structures $\boldsymbol{A}_S = (A, \leq)$ and $\boldsymbol{B}_S = (B, \leq, \vec{C}, B)$ where $\mathcal{L}_A = \mathcal{L}$ (and $U \in \mathcal{L}_B$ vacuously interpreted by B). Note that $S \cap B = \emptyset$. (cf example 2.5)

Lemma 3.7. *The pairs of structures defined above are all proper subdivisions.*

Proof. We only show this holds for the first definition $\boldsymbol{A} = \boldsymbol{A}_{c_2}^{c_1}$ and $\boldsymbol{B} = \boldsymbol{B}_{c_2}^{c_1}$. Other cases follow by a similar argument. A, B partition T by definition, so it is a subdivision. To show that it is proper by Lemma 2.4 we only need to check that it is 0-proper. Suppose we have

$$a = (a_1, a_2, \dots, a_p) \in A^p$$
$$a' = (a'_1, a'_2, \dots, a'_p) \in A^p$$
$$b = (b_1, b_2, \dots, b_q) \in B^q$$
$$b' = (b'_1, b'_2, \dots, b'_q) \in B^q$$

with $(\boldsymbol{A}, a) \equiv_0 (\boldsymbol{A}, a')$ and $(\boldsymbol{B}, b) \equiv_0 (\boldsymbol{B}, b')$. We need to make sure that ab has the same quantifier free type as $a'b'$. Any two elements in T can be related in the four following ways

$$x = y$$
$$x < y$$
$$x > y$$
$$x, y \text{ are incomparable}$$

We need to check that the same relations hold for the pairs of $(a_i, b_j), (a'_i, b'_j)$ for all i, j.

- It is impossible that $a_i = b_j$ as they come from disjoint sets.
- Suppose $a_i < b_j$. This forces $a_i \in S_1$ thus $a'_i \in S_1$ and $a'_i < b'_j$.
- Suppose $a_i > b_j$ This forces $b_j \in S_B$ and $a \in T_A$, thus $b'_j \in S_B$ and $a'_i \in T_A$ so $a'_i > b'_j$.
- Suppose a_i and b_j are incomparable. Two cases are possible:
  - $b_j \notin S_B$ and $a_i \in T_A$. Then $b'_j \notin S_B$ and $a'_i \in T_A$ making a'_i, b'_j incomparable.
  - $b_j \in S_B$, $a_i \notin T_A$, $a_i \notin S_1$. Similarly this forces a'_i, b'_j incomparable.

Also we need to check that ab has the same colors as $a'b'$. But that is immediate as having the same color in the substructure means having the same color in the whole tree. □

4. Main proof

Basic idea for the proof is as follows. Suppose we have a formula with q parameters. We are able to split our parameter space into $O(n)$ many partitions. Each of q parameters can come from any of those $O(n)$ partitions giving us $O(n)^q$ many choices for parameter configuration. When every parameter is coming from a fixed partition the number of definable sets is constant and in fact is uniformly bounded above by some N. This gives us at most $N \cdot O(n)^q$ possibilities for different definable sets.

First, we generalize Corollary 2.8. (This is required for computing vc-density for formulas $\phi(x, y)$ with $|y| > 1$).

Lemma 4.1. *Consider a finite collection $(\boldsymbol{A}_i, \boldsymbol{B}_i)_{i \le n}$ where each $(\boldsymbol{A}_i, \boldsymbol{B}_i)$ is either a proper subdivision or a singleton: $B_i = \{b_i\}$ with $A_i = T$. Also assume that all $\boldsymbol{B}_i$ have the same language $\mathcal{L}_B$. Let $A = \bigcap_{i \in I} A_i$. Fix a formula $\phi(x, y)$ of complexity m . Let $N = N(m, |y|, \mathcal{L}_B)$ as in Definition 2.9. Consider any $B \subseteq T^{|y|}$ of the form*

$$B = B_1^{i_1} \times B_2^{i_2} \times \ldots \times B_n^{i_n} \text{ with } i_1 + i_2 + \ldots + i_n = |y|$$

(some of the indexes can be zero). Then we have the following bound

$$\phi(A^{|x|}, B) \leq N^{|y|}$$

Proof. We show this result by counting types. Suppose we have

$$b_1, b_1' \in B_1^{i_1} \text{ with } b_1 \equiv_m b_1' \text{ in } \boldsymbol{B}_1$$

$$b_2, b_2' \in B_2^{i_2} \text{ with } b_2 \equiv_m b_2' \text{ in } \boldsymbol{B}_2$$

$$\dots$$

$$b_n, b_n' \in B_n^{i_n} \text{ with } b_n \equiv_m b_n' \text{ in } \boldsymbol{B}_n$$

Then we have

$$\phi(A^{|x|}, b_1, b_2, \dots b_n) \Leftrightarrow \phi(A^{|x|}, b_1', b_2', \dots b_n')$$

This is easy to see by applying Corollary 2.8 one by one for each tuple. This works if $\boldsymbol{B}_i$ is a part of a proper subdivision; if it is a singleton then the implication is trivial as $b_i = b_i'$. Thus $\phi(A^{|x|}, B)$ only depends on the choice of the types for the tuples

$$|\phi(A^{|x|}, B)| \leq |S^m_{\boldsymbol{B}_1, i_1}| \cdot |S^m_{\boldsymbol{B}_2, i_2}| \cdot \ldots \cdot |S^m_{\boldsymbol{B}_n, i_n}|$$

Now for each type space we have an inequality

$$|S^m_{\boldsymbol{B}_j, i_j}| \leq N(m, i_j, \mathcal{L}_B) \leq N(m, |y|, \mathcal{L}_B) \leq N$$

(For singletons $|S^m_{\boldsymbol{B}_j, i_j}| = 1 \leq N$). Only non-zero indexes contribute to the product and there are at most $|y|$ of those (by equality $i_1 + i_2 + \ldots + i_n = |y|$). Thus we have

$$|\phi(A^{|x|}, B)| \leq N^{|y|}$$

as needed. ∎

For subdivisions to work out properly, we will need to work with subsets closed under meets. We observe that the closure under meets doesn't add too many new elements.

Lemma 4.2. *Suppose $S \subseteq T$ is a finite subset of size n in a meet tree and S' is its closure under meets. Then $|S'| \leq 2n$.*

Proof. We can partition S into connected components and prove the result separately for each component. Thus we may assume elements of S lie in the same connected component. We prove the claim by induction on n. Base case $n = 1$ is clear. Suppose we have S of size k with closure of size at most $2k - 1$. Take a new point s, and look at its meets with all the elements of S. Pick the smallest one, s'. Then $S \cup \{s, s'\}$ is closed under meets. □

Putting all of those results together we are able to compute the vc-density of formulas in meet trees.

Theorem 4.3. *Let $\boldsymbol{T}$ be an infinite (colored) meet tree and $\phi(x, y)$ a formula with $|x| = p$ and $|y| = q$. Then $\mathrm{vc}(\phi) \leq q$.*

Proof. Pick a finite subset of $S_0 \subset T^p$ of size n. Let $S_1 \subset T$ consist of coordinates of S_0. Let $S \subset T$ be a closure of S_1 under meets. Using Lemma 4.2 we have $|S| \leq 2|S_1| \leq 2p|S_0| = 2pn = O(n)$. We have $S_0 \subseteq S^p$, so $|\phi(S_0, T^q)| \leq |\phi(S^p, T^q)|$. Thus it is enough to show $|\phi(S^p, T^q)| = O(n^q)$.

Label $S = \{c_i\}_{i \in I}$ with $|I| \leq 2pn$. For every c_i we construct two partitions in the following way. We have that c_i is either minimal in S or it has a predecessor in S (greatest element less than c). If it is minimal, construct $(\boldsymbol{A}_{c_i}, \boldsymbol{B}_{c_i})$. If there is a predecessor p, construct $(\boldsymbol{A}^p_{c_i}, \boldsymbol{B}^p_{c_i})$. For the second subdivision let G be all the elements in S greater than c_i and construct $(\boldsymbol{A}^c_G, \boldsymbol{B}^c_G)$. So far we have constructed two subdivisions for every $i \in I$. Additionally construct $(\boldsymbol{A}_S, \boldsymbol{B}_S)$. We end up with a finite collection of proper subdivisons $(\boldsymbol{A}_j, \boldsymbol{B}_j)_{j \in J}$ with $|J| = 2|I| + 1$. Before we proceed, we note the following two lemmas describing our partitions.

Lemma 4.4. *For all $j \in J$ we have $S \subseteq A_j$. Thus $S \subseteq \bigcap_{j \in J} A_j$ and $S^p \subseteq \bigcap_{j \in J} (A_j)^p$.*

Proof. Check this for each possible choice of partition. Cases for partitions of the type $\boldsymbol{A}_S, \boldsymbol{A}_G^c, \boldsymbol{A}_c$ are easy. Suppose we have a partition $(\boldsymbol{A}, \boldsymbol{B}) = (\boldsymbol{A}_{c_2}^{c_1}, \boldsymbol{B}_{c_2}^{c_1})$. We need to show that $B \cap S = \emptyset$. By construction we have $c_1, c_2 \notin B$. Suppose we have some other $c \in S$ with $c \in B$. We have $E_{c_1}(c_2, c)$ i.e. there is some b such that $(b > c_1)$, $(b \leq c_2)$ and $(b \leq c)$. Consider the meet $(c \wedge c_2)$. We have $(c \wedge c_2) \geq b > c_1$. Also as $\neg(c \geq c_2)$ we have $(c \wedge c_2) < c_2$. To summarize: $c_2 > (c \wedge c_2) > c_1$. But this contradicts our construction as S is closed under meets, so $(c \wedge c_2) \in S$ and c_1 is supposed to be a predecessor of c_2 in S. □

Lemma 4.5. *$\{B_j\}_{j \in J}$ is a disjoint partition of $T - S$ i.e. $T = \bigsqcup_{j \in J} B_j \sqcup S$*

Proof. This more or less follows from the choice of partitions. Pick any $b \in S - T$. Take all the elements in S greater than b and take the minimal one a. Take all the elements in S less than b and take the maximal one c (possible as S is closed under meets). Also take all the elements in S incomparable to b and denote them G. If both a and c exist we have $b \in \boldsymbol{B}_c^a$. If only the upper bound exists we have $b \in \boldsymbol{B}_G^a$. If only the lower bound exists we have $b \in \boldsymbol{B}_c$. If neither exists we have $b \in \boldsymbol{B}_G$. □

Note 4.6. Those two lemmas imply $S = \bigcap_{j \in J} A_j$.

Note 4.7. For one-dimensional case $q = 1$ we don't need to do any more work. We have partitioned the parameter space into $|J| = O(n)$ many pieces and over each piece the number of definable sets is uniformly bounded. By Corollary 2.8 we have that $|\phi((A_j)^p, B_j)| \leq N$ for any $j \in J$ (letting $N = N(n_\phi, q, \mathcal{L} \cup \{S\})$ where n_ϕ is the complexity of ϕ and S is a unary predicate). Compute

$$
\begin{aligned}
|\phi(S^p, T)| = &\left|\bigcup_{j\in J} \phi(S^p, B_j) \cup \phi(S^p, S)\right| \leq \\
&\leq \sum_{j\in J} |\phi(S^p, B_j)| + |\phi(S^p, S)| \leq \\
&\leq \sum_{j\in J} |\phi((A_j)^p, B_j)| + |S| \leq \\
&\leq \sum_{j\in J} N + |I| \leq \\
&\leq (4pn+1)N + 2pn = (4pN + 2p)n + N = O(n)
\end{aligned}
$$

Basic idea for the general case $q \geq 1$ is that we have q parameters and $|J| = O(n)$ many partitions to pick each parameter from giving us $|J|^q = O(n^q)$ choices for the parameter configuration, each giving a uniformly constant number of definable subsets of S. (If every parameter is picked from a fixed partition, Lemma 4.1 provides a uniform bound). This yields $\mathrm{vc}(\phi) \leq q$ as needed. The rest of the proof is stating this idea formally.

First, we extend our collection of subdivisions $(\boldsymbol{A}_j, \boldsymbol{B}_j)_{j\in J}$ by the following singleton sets. For each $c_i \in S$ let $B_i = \{c_i\}$ and $A_i = T$ and add $(\boldsymbol{A}_i, \boldsymbol{B}_i)$ to our collection with $\mathcal{L}_B$ the language of B_i interpreted arbitrarily. We end up with a new collection $(\boldsymbol{A}_k, \boldsymbol{B}_k)_{k\in K}$ indexed by some K with $|K| = |J| + |I|$ (we added $|S|$ new pairs). Now $B_{k\,k\in K}$ partitions T, so $T = \bigsqcup_{k\in K} B_k$ and $S = \bigcap_{j\in J} A_j = \bigcap_{k\in K} A_k$. For $(k_1, k_2, \ldots k_q) = \vec{k} \in K^q$ denote

$$B_{\vec{k}} = B_{k_1} \times B_{k_2} \times \ldots \times B_{k_q}$$

Then we have the following identity

$$T^q = (\bigsqcup_{k\in K} B_k)^q = \bigsqcup_{\vec{k}\in K^q} B_{\vec{k}}$$

Thus we have that $\{B_{\vec{k}}\}_{\vec{k}\in K^q}$ partition T^q. Compute

$$\begin{aligned}|\phi(S^p,T^q)| &= \left|\bigcup_{\vec{k}\in K^q}\phi(S^p,B_{\vec{k}})\right| \leq \\ &\leq \sum_{\vec{k}\in K^q}|\phi(S^p,B_{\vec{k}})|\end{aligned}$$

We can bound $|\phi(S^p,B_{\vec{k}})|$ uniformly using Lemma 4.1. $(\boldsymbol{A}_k,\boldsymbol{B}_k)_{k\in K}$ satisfies the requirements of the lemma and $B_{\vec{k}}$ looks like B in the lemma after possibly permuting some variables in ϕ. Applying the lemma we get

$$|\phi(S^p,B_{\vec{k}})| \leq N^q$$

with N only depending on q and complexity of ϕ. We complete our computation

$$\begin{aligned}|\phi(S^p,T^q)| &\leq \sum_{\vec{k}\in K^q}|\phi(S^p,B_{\vec{k}})| \leq \\ &\leq \sum_{\vec{k}\in K^q} N^q \leq \\ &\leq |K^q|N^q \leq \\ &\leq (|J|+|I|)^q N^q \leq \\ &\leq (4pn+1+2pn)^q N^q = N^q(6p+1/n)^q n^q = O(n^q)\end{aligned}$$

□

Corollary 4.8. *In the theory of infinite (colored) meet trees we have $vc(n)=n$ for all n.*

We get the general result for the trees that aren't necessarily meet trees via an easy application of interpretability.

Corollary 4.9. *In the theory of infinite (colored) trees we have $vc(n)=n$ for all n.*

Proof. Let $\boldsymbol{T}'$ be a tree. We can embed it in a larger tree that is closed under meets $\boldsymbol{T}' \subset \boldsymbol{T}$. Expand $\boldsymbol{T}$ by an extra color and interpret it by coloring the subset $\boldsymbol{T}'$. Thus we can interpret $\boldsymbol{T}'$ in T^1. By Corollary 3.17 in [1] we get that $\mathrm{vc}^{\boldsymbol{T}'}(n) \leq \mathrm{vc}^{T}(1 \cdot n) = n$ thus $\mathrm{vc}^{\boldsymbol{T}'}(n) = n$ as well. □

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E-mail address: `bobkov@math.ucla.edu`