

VC-DENSITY IN AN ADDITIVE REDUCT OF p -ADIC NUMBERS

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ABSTRACT. Aschenbrenner et. al. computed a bound $\text{vc}(n) \leq 2n - 1$ for the VC density function in the field of p -adic numbers, but it is not known to be optimal. I investigate a certain P -minimal additive reduct of the field of p -adic numbers and use a cell decomposition result of Leenknegt to compute an optimal bound $\text{vc}(n) = n$ for that structure.

VC density was introduced into model theory in [1] by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko as a natural notion of dimension for definable families of sets in NIP theories. In a NIP theory T we can define the vc -function

$$\text{vc}_T = \text{vc} : \mathbb{N} \longrightarrow \mathbb{N}$$

where $\text{vc}(n)$ measures the worst-case complexity of families of definable sets in an n -dimensional space. The simplest possible behavior is $\text{vc}(n) = n$ for all n . For $T = \text{Th}(\mathbb{Q}_p)$, the paper [1] computes an upper bound for this function to be $2n - 1$, and it is not known whether it is optimal. This same bound would hold in any reduct of the field of p -adic numbers, so one may expect that the simplified structure of the reduct would allow a better bound. In [2], Leenknegt provides a cell decomposition result for a certain P -minimal additive reduct of the field p -adic numbers. Using this result, in this paper we improve the bound for the VC function, showing that in Leenknegt's structure $\text{vc}(n) = n$.

Section 1 defines VC-density and states some basic lemmas about it. More in depth exposition of VC-density can be found in [1]. Section 2 defines and states some basic facts about theory of p -adic numbers. Here we also introduce the reduct we will be working with. Section 3 sets up basic definition and lemmas that will be needed for the proof. We define trees and intervals and show how it helps with VC-density calculations. Section 4 concludes the proof.

Throughout the paper, variables and tuples of elements will be simply denoted as x, y, a, b, \dots . We will occasionally write \vec{a} instead of a for a tuple in \mathbb{Q}_p^n to emphasize it as an element of \mathbb{Q}_p -vector space \mathbb{Q}_p^n . $|x|$ refers to the arity of the variable. First-order formulas will have parameter variables separated $\phi(x; y)$.

1. VC-DIMENSION AND VC-DENSITY

Definition 1.1. Throughout this section we work with a collection \mathcal{F} of subsets of a set X . We call the pair (X, \mathcal{F}) a set system.

- Given a subset A of X , we define the set system $(A, A \cap \mathcal{F})$ where $A \cap \mathcal{F} = \{A \cap F\}_{F \in \mathcal{F}}$.
- For $A \subset X$ we say that \mathcal{F} shatters A if $A \cap \mathcal{F} = \mathcal{P}(A)$.

Definition 1.2. We say (X, \mathcal{F}) has VC-dimension n if the largest subset of X shattered by \mathcal{F} is of size n . If \mathcal{F} shatters arbitrarily large subsets of X , we say that (X, \mathcal{F}) has infinite VC-dimension. We denote the VC-dimension of (X, \mathcal{F}) by $\text{VC}(\mathcal{F})$.

Note 1.3. We may drop X from the previous definition, as it VC-dimension doesn't depend on the base set and is determined by $(\bigcup \mathcal{F}, \mathcal{F})$.

This allows us to distinguish between well behaved set systems of finite VC-dimension which tend to have good combinatorial properties and poorly behaved set systems with infinite VC dimension.

Another natural combinatorial notion is that of a dual system:

Definition 1.4. For $a \in X$ define $X_a = \{F \in \mathcal{F} \mid a \in F\}$. Let $\mathcal{F}^* = \{X_a\}_{a \in X}$. We define $(\mathcal{F}, \mathcal{F}^*)$ as the dual system of (X, \mathcal{F}) . The VC-dimension of the dual system of (X, \mathcal{F}) is referred to as the dual VC-dimension of (X, \mathcal{F}) and denoted by $\text{VC}^*(\mathcal{F})$. (As before, this notion doesn't depend on X .)

Lemma 1.5. *A set system has finite VC-dimension if and only if its dual system has finite VC-dimension. More precisely*

$$\text{VC}^*(\mathcal{F}) \leq 2^{1+\text{VC}(\mathcal{F})}.$$

For a more refined notion we look at the traces of our family on finite sets:

Definition 1.6. Define the shatter function $\pi_{\mathcal{F}}: \mathbb{N} \rightarrow \mathbb{N}$ and the dual shatter function $\pi_{\mathcal{F}}^*: \mathbb{N} \rightarrow \mathbb{N}$ of \mathcal{F} by

$$\pi_{\mathcal{F}}(n) = \max \{|A \cap \mathcal{F}| \mid A \subset X \text{ and } |A| = n\}$$

$$\pi_{\mathcal{F}}^*(n) = \max \{\text{number of atoms in Boolean algebra generated by } B \mid B \subset \mathcal{F}, |B| = n\}$$

Note that the dual shatter function is precisely the shatter function of the dual system: $\pi_{\mathcal{F}}^* = \pi_{\mathcal{F}^*}$.

A simple upper bound is $\pi_{\mathcal{F}}(n) \leq 2^n$ (same for the dual). If VC-dimension is infinite then clearly $\pi_{\mathcal{F}}(n) = 2^n$ for all n . Conversely we have the following remarkable fact:

Theorem 1.7 (Sauer-Shelah '72). *If the set system (X, \mathcal{F}) has finite VC-dimension d then $\pi_{\mathcal{F}}(n) \leq \binom{n}{\leq d}$ where $\binom{n}{\leq d} = \binom{n}{d} + \binom{n}{d-1} + \dots + \binom{n}{1}$.*

Thus the systems with a finite VC-dimension are precisely the systems where the shatter function grows polynomially. Define VC-density to be the degree of that polynomial:

Definition 1.8. Define vc-density and dual vc-density of \mathcal{F} as

$$\text{vc}(\mathcal{F}) = \limsup_{n \rightarrow \infty} \frac{\log \pi_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}$$

$$\text{vc}^*(\mathcal{F}) = \limsup_{n \rightarrow \infty} \frac{\log \pi_{\mathcal{F}}^*(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}$$

Generally speaking a shatter function that is bounded by a polynomial doesn't itself have to be a polynomial. Proposition 4.12 in [1] gives an example of a shatter function that grows like $n \log n$ (so it has VC-density 1).

So far the notions that we have defined are purely combinatorial. We now adapt VC-dimension and VC-density to the model theoretic context.

Definition 1.9. Work in a structure M . Fix a finite collection of formulas $\Phi(x, y) = \{\phi_i(x, y)\}$.

- For $\phi(x, y) \in \mathcal{L}(M)$ and $b \in M^{|y|}$ let $\phi(M^{|x|}, b) = \{a \in M^{|x|} \mid \phi(a, b)\} \subseteq M^{|x|}$.
- Let $\Phi(M^{|x|}, M^{|y|}) = \{\phi_i(M^{|x|}, b) \mid \phi_i \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|})$.
- Let $\mathcal{F}_{\Phi} = \Phi(M^{|x|}, M^{|y|})$ giving a set system $(M^{|x|}, \mathcal{F}_{\Phi})$.
- Define VC-dimension of Φ , $\text{VC}(\Phi)$ to be the dual VC-dimension of $(M^{|x|}, \mathcal{F}_{\Phi})$.
- Define VC-density of Φ , $\text{vc}(\Phi)$ to be the dual VC-density of $(M^{|x|}, \mathcal{F}_{\Phi})$.

We will also refer to the VC-density and VC-dimension of a single formula ϕ viewing it as a one element collection $\{\phi\}$.

Counting atoms of a Boolean algebra in a model theoretic setting corresponds to counting types, so it is instructive to rewrite the shatter function in terms of types.

Definition 1.10.

$$\pi_{\Phi}(n) = \max \{\text{number of } \Phi\text{-types over } B \mid B \subset M, |B| = n\}$$

Lemma 1.11.

$$\text{vc}(\Phi) = \text{degree of polynomial growth of } \pi_{\Phi}(n) = \limsup_{n \rightarrow \infty} \frac{\log \pi_{\Phi}(n)}{\log n}$$

One can check that the shatter function and hence VC-dimension and VC-density of a formula are elementary notions, so they only depend on the first-order theory of the structure.

NIP theories are a natural context for studying VC-density. In fact we can take the following as the definition of NIP:

Definition 1.12. Define ϕ to be NIP if it has finite VC-dimension.

[?] shows that in a general combinatorial context, VC-density can be any real number in $0 \cup [1, \infty)$. Less is known if we restrict our attention to NIP theories. Proposition 4.6 in [1] gives examples of formulas that have non-integer rational VC-density in an NIP theory, however it is open whether one can get an irrational VC-density in this context.

In general, instead of working with a theory formula by formula, we can look for a uniform bound for all formulas:

Definition 1.13. For a given NIP structure M , define the vc-function

$$\text{vc}^M(n) = \sup \{\text{vc}(\phi(x, y)) \mid \phi \in \mathcal{L}(M), |x| = n\}$$

As before this definition is elementary, so it only depends on the theory of M . We omit the superscript M if it is understood from the context. One can easily check the following bounds:

Lemma 1.14 (Lemma 3.22 in [1]).

$$\begin{aligned} \text{vc}(1) &\geq 1 \\ \text{vc}(n) &\geq n \text{vc}(1) \end{aligned}$$

However, it is not known whether the second inequality can be strict or even whether $\text{vc}(1) < \infty$ implies $\text{vc}(n) < \infty$.

2. P -ADIC NUMBERS

The field of p -adic numbers is often studied in the language of Macintyre $\mathcal{L}_{Mac} = \{0, 1, +, -, \cdot, |, \{P_n\}_{n \in \mathbb{N}}\}$ which is a language of fields together with unary predicates P_n interpreted in \mathbb{Q}_p by

$$P_n x \leftrightarrow \exists y \ y^n = x$$

and a divisibility relation where $a|b$ holds when $\text{val } a \leq \text{val } b$.

Note that $P_n \setminus \{0\}$ is a multiplicative subgroup of \mathbb{Q}_p with finitely many cosets.

Theorem 2.1 (Macintyre '76). *The \mathcal{L}_{Mac} -structure \mathbb{Q}_p has quantifier elimination.*

There is also a cell decomposition result:

Definition 2.2. Define k -cell recursively. 0-cells are points in \mathbb{Q}_p . An $(k+1)$ -cell is a subset of \mathbb{Q}_p^{k+1} of the following form:

$$\{(x, t) \in D \times \mathbb{Q}_p \mid \text{val } a_1(x) \square_1 \text{val}(t - c(x)) \square_2 \text{val } a_2(x), t - c(x) \in \lambda P_n\}$$

where D is an k -cell, $a_1(x), a_2(x), c(x)$ are \emptyset -definable, \square is $<, \leq$ or no condition, and $\lambda \in \mathbb{Q}_p$.

Theorem 2.3 (Denef '84). *Any subset of \mathbb{Q}_p defined by a \mathcal{L}_{Mac} -formula $\phi(x, t)$ with $|x| = n$ and $|t| = 1$ decomposes into a finite union of $(k+1)$ -cells.*

In [1], Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko show that this structure has $\text{vc}(n) \leq 2n - 1$, however it is not known whether this bound is optimal.

In [2], Leenknegt analyzes the reduct of p -adic numbers to the language

$$\mathcal{L}_{aff} = \{0, 1, +, -, \{\bar{c}\}_{c \in \mathbb{Q}_p}, |, \{Q_{m,n}\}_{m,n \in \mathbb{N}}\}$$

where \bar{c} is a scalar multiplication by c , $a|b$ stands for $\text{val } a \leq \text{val } b$, and $Q_{m,n}$ is a unary predicate

$$Q_{m,n} = \bigcup_{k \in \mathbb{Z}} p^{km} (1 + p^n \mathbb{Z}_p).$$

Note that $Q_{m,n} \setminus \{0\}$ is a subgroup of the multiplicative group of \mathbb{Q}_p with finitely many cosets. One can check that the extra relation symbols are definable in the \mathcal{L}_{Mac} -structure \mathbb{Q}_p . The paper [2] provides a cell decomposition result with the following cells:

Definition 2.4. A 0-cell is a point in \mathbb{Q}_p . An $(k+1)$ -cell is a subset of \mathbb{Q}_p^{k+1} of the following form:

$$\{(x, t) \in D \times \mathbb{Q}_p \mid \text{val } a_1(x) \square_1 \text{val}(t - c(x)) \square_2 \text{val } a_2(x), t - c(x) \in \lambda Q_{m,n}\}$$

where D is an k -cell called the base of the cell, $a_1(x), a_2(x), c(x)$ are degree ≤ 1 polynomials, \square is $<$ or no condition, and $\lambda \in \mathbb{Q}_p$.

Theorem 2.5 (Leenknegt '12). *Any formula $\phi(x, t)$ in $(\mathbb{Q}_p, \mathcal{L}_{aff})$ with $|x| = n$ and $|t| = 1$ decomposes into a union of $(k+1)$ -cells.*

Moreover, [2] shows that $(\mathbb{Q}_p, \mathcal{L}_{aff})$ is a P -minimal reduct, that is the one-dimensional definable sets of $(\mathbb{Q}_p, \mathcal{L}_{aff})$ coincide with the one-dimensional definable sets in the full structure $(\mathbb{Q}_p, \mathcal{L}_{Mac})$.

I am able to compute the vc-function for this structure:

Theorem 2.6. *Theorem (B.) $(\mathbb{Q}_p, \mathcal{L}_{aff})$ has $\text{vc}(n) = n$.*

3. KEY LEMMAS AND DEFINITIONS

To show that $\text{vc}(n) = n$ it suffices to bound $\text{vc}(\phi) \leq |x|$ for every formula $\phi(x; y)$. Fix such a formula $\phi(x; y)$. Instead of working with it directly, we simplify it using quantifier elimination. The quantifier elimination result can be easily obtained from cell decomposition:

Lemma 3.1. *Any formula $\phi(x; y)$ in $(\mathbb{Q}_p, \mathcal{L}_{aff})$ can be written as a boolean combination of formulas from the following collection*

$$\Psi(x; y) = \{\text{val}(p_i(x) - c_i(y)) < \text{val}(p_j(x) - c_j(y))\}_{i,j \in I} \cup \{p_i(x) - c_i(y) \in \lambda_k Q_{m,n}\}_{i \in I, k \in K}$$

where I, K are finite index sets, each p_i is a degree ≤ 1 polynomial in x without a constant term, each c_i is a degree ≤ 1 polynomial in y , and $\lambda_k \in \mathbb{Q}_p$.

Proof. Let $l = |x| + |y|$. Apply the cell decomposition theorem to $\phi(x; y)$ to obtain \mathcal{D}^l , a collection of l -cells. Let \mathcal{D}^{l-1} be a collection $l-1$ of bases of cells in \mathcal{D}^l . Similarly, construct by induction \mathcal{D}^j for each $0 \leq j < l$, where \mathcal{D}^j is a collection of j -cells which are the bases of cells in \mathcal{D}_{j+1} . Let $\mathcal{D} = \bigcup \mathcal{D}^j$. Choose m, n large enough to cover all n', m' for $Q_{n', m'}$ that show up in the cells of \mathcal{D} . Choose λ_k to go over all the cosets of $Q_{m,n}$. Let $q_i(x, y)$ enumerate all of the polynomials $a_1(x), a_2(x), t - c(x)$ that show up in the cells of \mathcal{D} . Those are all polynomials of degree ≤ 1 in variables x, y . We can split each of them as $q_i(x, y) = p_i(x) - c_j(y)$ where the constant term goes into c_j . This gives us the appropriate finite collection of formulas Ψ . From the cell decomposition it is easy to see that when a, a' have the same Ψ -type, then they have the same ϕ -type. Thus ϕ can be written as a boolean combination of formulas from Ψ . \square

Lemma 3.2. *If ϕ can be written as a boolean combination of formulas from Ψ then*

$$\text{vc}(\Psi) \leq n \implies \text{vc}(\phi) \leq n$$

Proof. If a, a' have the same Ψ -type over B , then they have the same ϕ -type over B , where B is some parameter set. Therefore the number of ϕ -types is bounded by the number of Ψ -types. The bound follows from Lemma 1.11. \square

Therefore to show that $\text{vc}(\phi) \leq |x|$, it suffices to bound $\text{vc}(\Psi) \leq |x|$. More precisely, it is sufficient to show that if there is a parameter set B of size N then the number of Ψ -types over B is $O(N^{|x|})$. Fix such a parameter set B and work with it from now on. We will compute a bound for the number of Ψ -types over B .

Consider a set $T = \{c_i(b) \mid b \in B, i \in I\} \subset \mathbb{Q}_p$. In this definition B is the parameter set that we fixed and $c_i(b)$ come from the collection of formulas Ψ from the quantifier elimination above. View T as a tree as follows:

Definition 3.3.

- For $c \in \mathbb{Q}_p, \alpha \in \mathbb{Z}$ define a ball

$$B(c, \alpha) = \{c' \in \mathbb{Q}_p \mid \text{val}(c' - c) \geq \alpha\}.$$

- Define a collection of balls $\mathcal{B} = \{B(t_1, \text{val}(t_1 - t_2))\}_{t_1, t_2 \in T}$. An interval (B_1, B_2) is a set $B_1 \setminus B_2$ for $B_1, B_2 \in \mathcal{B}$ with $B_1 \supset B_2$ and no balls from \mathcal{B} in between. Note that intervals partition \mathbb{Q}_p .
- Define a collection of balls

$$\mathcal{B}' = \mathcal{B} \cup \{B(c_{i_1}(b), \text{val}(c_{i_2}(b) - c_{i_3}(b)))\}_{i_1, i_2, i_3 \in I, b \in B}.$$

A sub-interval is defined the same as an interval except using collection \mathcal{B}' instead of \mathcal{B} . Sub-intervals refine intervals.

Lemma 3.4.

- There are at most $2|T| = 2N|I| = O(N)$ different intervals.
- There are at most $2|T| + |B| \cdot |I|^3 = O(N)$ different sub-intervals.

Proof. Each new element in the tree T adds at most two intervals to the total count, so by induction there can be at most $2|T|$ many intervals. Each new ball in $\mathcal{B}' \setminus \mathcal{B}$ adds at most one interval to the total count, so by induction there are at most $|\mathcal{B}' \setminus \mathcal{B}|$ more sub-intervals than there are intervals. \square

Definition 3.5. Suppose $a \in \mathbb{Q}_p$ lies in an interval $(B(t_L, \alpha_L), B(t_U, \alpha_U))$.

- Define T-valuation of a to be $\text{T-val}(a) = \text{val}(a - t_U)$.
- Define the floor of a to be $F(a) = \alpha_L$.
- We say that a is close to boundary if $|\text{T-val}(a) - \alpha_L| \leq n$ or $|\text{T-val}(a) - \alpha_U| \leq n$. Otherwise we say that it is far from boundary.

Definition 3.6. Suppose $a_1, a_2 \in \mathbb{Q}_p$ lie in our tree in the same interval $(B(t_L, \alpha_L), B(t_U, \alpha_U))$. We say a_1, a_2 have the same interval type if one of the following holds:

- Both a_1, a_2 are far from boundary and $a_1 - t_U, a_2 - t_U$ are in the same $Q_{m,n}$ coset.
- Both a_1, a_2 are close to boundary and $\text{val}(a_1 - a_2) > \text{T-val}(a_1) + n = \text{T-val}(a_2) + n$.

Definition 3.7. For $c \in \mathbb{Q}_p$ and $\alpha, \beta \in \mathbb{Z}$ define $c \upharpoonright [\alpha, \beta] \in (\mathbb{Z}/p\mathbb{Z})^{\beta-\alpha}$ to be the record of the coefficients of c for the valuations between $[\alpha, \beta]$. More precisely write c in its power series form

$$c = \sum_{\gamma \in \mathbb{Z}} c_\gamma p^\gamma \text{ with } c_\gamma \in \mathbb{Z}/p\mathbb{Z}$$

Then $c \upharpoonright [\alpha, \beta]$ is just $(c_\alpha, c_{\alpha+1}, \dots, c_{\beta-1})$.

The following lemma is an adaptation of lemma 7.4 in [1].

Lemma 3.8. Fix $m, n \in \mathbb{N}$. For any $x, y, c \in \mathbb{Q}_p$, if

$$\text{val}(x - c) = \text{val}(y - c) < \text{val}(x - y) - n,$$

then $x - c, y - c$ are in the same coset of $Q_{m,n}$.

Proof. Call $a, b \in \mathbb{Q}_p$ similar if $\text{val } a = \text{val } b$ and

$$a \upharpoonright [\text{val } a, \text{val } a + n] = b \upharpoonright [\text{val } b, \text{val } b + n]$$

If a, b are similar then

$$a \in Q_{m,n} \leftrightarrow b \in Q_{m,n}$$

Moreover for any $\lambda \in \mathbb{Q}_p^\times$, if a, b are similar then so are $\lambda a, \lambda b$. Thus if a, b are similar, then they belong to the same coset of $Q_{m,n}$. Conditions of the lemma force $x - c, y - c$ to be similar, thus belonging to the same coset. \square

Lemma 3.9. For each interval there are at most $K = K(Q_{m,n})$ many interval types (with K not dependent on B or the interval).

Proof. Let $a, a' \in \mathbb{Q}_p$ lie in the same interval $(B(t_L, \alpha_L), B(t_U, \alpha_U))$.

Suppose a, a' are far from boundary. Then they have the same interval type if $a - t_U, a' - t_U$ are in the same $Q_{m,n}$ -coset. Number of such interval types is bounded by the number of $Q_{m,n}$ -cosets.

Suppose a, a' are close to boundary and

$$\begin{aligned} |\text{T-val}(a) - \alpha_L| &= |\text{T-val}(a') - \alpha_L| \leq n \\ a \upharpoonright [\text{T-val}(a), \text{T-val}(a) + n] &= a' \upharpoonright [\text{T-val}(a'), \text{T-val}(a') + n] \end{aligned}$$

Then a, a' have the same interval type. Such interval type is thus determined by $|\text{T-val}(a) - \alpha_L|$ and $a \upharpoonright [\text{T-val}(a), \text{T-val}(a) + n]$, therefore there are at most np^n many such types.

A similar argument works for a with $|\text{T-val}(a) - \alpha_U| \leq n$.

Adding those up we get that there are at most

$$K = (\text{number of } Q_{m,n} \text{ cosets}) + 2np^n$$

many interval types. \square

Lemma 3.10. Suppose $d_1, d_2 \in \mathbb{Q}_p^{|x|}$ satisfy the following three conditions

- For all $i \in I$ $p_i(d_1)$ and $p_i(d_2)$ are in the same sub-interval.
- For all $i \in I$ $p_i(d_1)$ and $p_i(d_2)$ have the same interval type.
- For all $i, j \in I$, $\text{T-val}(p_i(d_1)) > \text{T-val}(p_j(d_1))$ iff $\text{T-val}(p_i(d_2)) > \text{T-val}(p_j(d_2))$.

Then d_1, d_2 have the same Ψ -type over B .

Proof. There are two kinds of formulas in Ψ (see Lemma 3.1). First we show that d_1, d_2 agree on formulas of the form $p_i(x) - c_i(y) \in \lambda_k Q_{m,n}$. It is enough to show that for every $i \in I, b \in B$ we have $p_i(d_1) - c_i(b), p_i(d_2) - c_i(b)$ are in the same $Q_{m,n}$ -coset. Fix such i, b . For brevity let $a = p_i(d_1), a' = p_i(d_2)$ and $Q = Q_{m,n}$. We want to show that $a - c_i(b), a' - c_i(b)$ are in the same Q -coset.

Suppose a, a' are close to boundary. Then we have $\text{val}(a - c_i(b)) = \text{val}(a' - c_i(b)) < \text{val}(a - a') - n$. By Lemma 3.8 we have $a - c_i(b), a' - c_i(b)$ in the same Q -coset.

Now, suppose both a, a' are far from boundary. Label their interval as $B(t_L, \alpha_L) \setminus B(t_U, \alpha_U)$. Then we have

$$\begin{aligned} \alpha_L + n &< \text{val}(a - t_U) < \alpha_U - n \\ \alpha_L + n &< \text{val}(a' - t_U) < \alpha_U - n \end{aligned}$$

We have either $\text{val}(t_U - c_i(b)) \geq \alpha_U$ or $\text{val}(t_U - c_i(b)) \leq \alpha_L$ as otherwise it would contradict the definition of an interval.

Suppose it is the first case $\text{val}(t_U - c_i(b)) \geq \alpha_U$. Then

$$\text{val}(a - c_i(b)) = \text{val}(a - t_U) < \alpha_U - n \leq \text{val}(t_U - c_i(b)) - n$$

so by Lemma 3.8 we have $a - c_i(b), a - t_U$ are in the same Q -coset. By a parallel argument we have $a' - c_i(b), a' - t_U$ are in the same Q -coset. As a, a' have the same interval type, $a - t_U, a' - t_U$ are in the same Q -coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same Q -coset.

For the second case, suppose $\text{val}(t_U - c_i(b)) \leq \alpha_L$. Then

$$\text{val}(a - c_i(b)) = \text{val}(t_U - c_i(b)) \leq \alpha_L < \text{val}(a - t_U) - n$$

so by Lemma 3.8 we have $a - c_i(b), t_U - c_i(b)$ are in the same Q -coset. By a parallel argument we have $a' - c_i(b), t_U - c_i(b)$ are in the same Q -coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same Q -coset.

Next, we need to show that d_1, d_2 agree on formulas of the form $\text{val}(p_i(x) - c_i(y)) < \text{val}(p_j(x) - c_j(y))$ (see Lemma 3.1). TO DO THE REST OF THE PROOF \square

Note 3.11. This gives us an upper bound on the number of types - there are at most $|2I|!$ many choices for the order of T-val, $O(N)$ many choices for the sub-interval for each p_i , and K many choices for the interval type for each p_i , giving a total of $O(N^{|I|}) \cdot K^{|I|} \cdot |I|! = O(N^{|I|})$ many types. This implies $\text{vc}(\Psi) \leq |I|$. The biggest contribution to this bound are the choices among the $O(N)$ many sub-intervals for each p_i with $i \in I$. Are all of those choices realized? Intuitively there are $|x|$ many variables and $|I|$ many equations, so once we choose an sub-interval for $|x|$ many p_i 's, the sub-interval for the rest should be determined. This would give the required $\text{vc}(\Psi) \leq |x|$ bound. The next section outlines this idea formally.

4. MAIN PROOF

Alternative way to write $p_i(c)$ is $\vec{p}_i \cdot \vec{c}$, where \vec{p}_i and \vec{c} are vectors in $\mathbb{Q}_p^{|x|}$ (as $p_i(x)$ is linear).

Lemma 4.1. Suppose we have a finite collection of vectors $\{\vec{p}_i\}_{i \in I}$ with each $\vec{p}_i \in \mathbb{Q}_p^{|x|}$. Suppose $J \subset I$ and $i \in I$ satisfy

$$\vec{p}_i \in \text{span}\{\vec{p}_j\}_{j \in J},$$

and we have $\vec{c} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$ with

$$\text{val}(\vec{p}_j \cdot \vec{c}) > \alpha \text{ for all } j \in J$$

Then

$$\text{val}(\vec{p}_i \cdot \vec{c}) > \alpha - \gamma$$

for some $\gamma \in \mathbb{N}$. Moreover γ can be chosen independently from J, j, \vec{c}, α depending only on $\{\vec{p}_i\}_{i \in I}$.

Proof. Fix i, J satisfying the conditions of the lemma. For some $c_j \in \mathbb{Q}_p$ for $j \in J$ we have

$$\vec{p}_i = \sum_{j \in J} c_j \vec{p}_j,$$

hence

$$\vec{p}_i \cdot \vec{c} = \sum_{j \in J} c_j \vec{p}_j \cdot \vec{c}.$$

We have

$$\text{val}(c_j \vec{p}_j \cdot \vec{c}) = \text{val}(c_j) + \text{val}(\vec{p}_j \cdot \vec{c}) > \text{val}(c_j) + \alpha.$$

Let $\gamma = \max(0, -\max_{j \in J} \text{val}(c_j))$. Then we have

$$\text{val}(c_j \vec{p}_j \cdot \vec{c}) > \alpha - \gamma \text{ for all } j \in J$$

$$\text{val}\left(\sum_{j \in J} c_j \vec{p}_j \cdot \vec{c}\right) > \alpha - \gamma$$

$$\text{val}(\vec{p}_i \cdot \vec{c}) > \alpha - \gamma$$

This shows that we can pick such γ for a given choice of i, J , but independent from α, \vec{c} . To get a choice independent from i, J , go over all such eligible choices (i ranges over I and J ranges over subsets of I), pick γ for each, and then take the maximum of those values. \square

Fix γ according to Lemma 4.1 corresponding to $\{\vec{p}_i\}_{i \in I}$ given by our collection of formulas Ψ . (The lemma above is a general result, but we only use it applied to the vectors given by Ψ .)

Definition 4.2. Suppose $a \in \mathbb{Q}_p$ lies in a sub-interval $(B(t_L, \alpha_L), B(t_U, \alpha_U))$. Define floor of a to be $F(a) = \alpha_L$.

Definition 4.3. Let $f : \mathbb{Q}_p^{|x|} \rightarrow \mathbb{Q}_p^I$ with $f(c) = (p_i(c))_{i \in I}$. Define the segment space Sg to be the image of f .

Given a tuple $(a_i)_{i \in I}$ in the segment space, look at the corresponding floors $\{F(a_i)\}_{i \in I}$ and T-valuations $\{\text{T-val}(a_i)\}_{i \in I}$. Partition the segment space by the order types of $\{F(a_i)\}_{i \in I}$ and $\{\text{T-val}(a_i)\}_{i \in I}$ (as subsets of \mathbb{Z}).

Work in a fixed partition Sg' . After relabeling we may assume that

$$F(a_1) \geq F(a_2) \geq \dots$$

Consider the (relabelled) sequence of vectors $\vec{p}_1, \vec{p}_2, \dots, \vec{p}_I$. There is a unique subset $J \subset I$ such that all vectors with indices in J are linearly independent, and all vectors with indices outside of J are a linear combination of preceding vectors. For any index $i \in I$ we call it independent if $i \in J$ and we call it dependent otherwise.

Definition 4.4.

- Denote $\mathbb{Z}/p\mathbb{Z}^\gamma$ as Ct . Note that $|\text{Ct}| = p^\gamma$.
- Let It be the space of all interval types. By Lemma 3.9 $|\text{It}| \leq K$.
- Let Sub be the space of all sub-intervals. By Lemma 3.4 $|\text{Sub}| \leq 3|I|^2 \cdot N = O(N)$.

Definition 4.5. Now, we define the following function

$$g_{\text{Sg}'} : \text{Sg}' \rightarrow \text{It}^I \times \text{Sub}^J \times \text{Ct}^{I \setminus J}$$

Let $a = (a_i)_{i \in I} \in \text{Sg}'$. To define $g_{\text{Sg}'}(a)$ we need to specify where it maps a in each individual component of the product.

For each a_i record its interval type, giving the first component It^I .

For a_j with $j \in J$, record the sub-interval of a_j , giving the second component Sub^J .

For the third component $\text{Ct}^{I \setminus J}$ do the following computation. Pick a_i with i dependent. Let j be the largest independent index with $j < i$. Record $a_i \upharpoonright [F(a_j) - \gamma, F(a_j)]$.

Combine $g_{\text{Sg}'}$ for all the partitions to get a function

$$g : \text{Sg} \rightarrow \text{It}^I \times \text{Sub}^J \times \text{Ct}^{I \setminus J}.$$

Lemma 4.6. Suppose we have $c, c' \in \mathbb{Q}_p^{|x|}$ such that $f(c), f(c')$ are in the same partition and $g(f(c)) = g(f(c'))$. Then c, c' have the same Ψ -type over B .

Proof. Let $a_i = \vec{p}_i \cdot \vec{c}$ and $a'_i = \vec{p}_i \cdot \vec{c}'$ so that

$$\begin{aligned} f(c) &= (p_i(c))_{i \in I} = (\vec{p}_i \cdot \vec{c})_{i \in I} = (a_i)_{i \in I} \\ f(c') &= (p_i(c'))_{i \in I} = (\vec{p}_i \cdot \vec{c}')_{i \in I} = (a'_i)_{i \in I} \end{aligned}$$

For each i we show that a_i, a'_i are in the same sub-interval and have the same interval type, so the conclusion follows by Lemma 3.10 ($f(c_1), f(c_2)$ are in the same partition ensuring the proper order of T-valuations for the 3rd condition of the lemma). It records the interval type of each element, so if $g(\vec{a}) = g(\vec{a}')$ then a_i, a'_i have the same interval type for all $i \in I$. Thus it remains to show that a_i, a'_i lie in the same sub-interval for all $i \in I$. Suppose i is an independent index. Then by construction, Sub records the sub-interval for a_i, a'_i , so those have to belong to the same sub-interval. Now suppose i is dependent. Pick the largest $j < i$ such that j is independent. We have $F(a_i) \leq F(a_j)$ and $F(a'_i) \leq F(a'_j)$. Moreover $F(a_j) = F(a'_j)$ as a_j, a'_j lie in the same sub-interval (using the earlier part of the argument as j is independent).

Claim 4.7. $\text{val}(a_i - a'_i) \geq F(a_j) - \gamma$

Proof. Let K be the set of the independent indices less than i . Note that by the definition for dependent indices we have $\vec{p}_i \in \text{span}\{\vec{p}_k\}_{k \in K}$. We also have

$$\text{val}(a_k - a'_k) \geq F(a_k) \text{ for all } k \in K$$

as a_k, a'_k lie in the same sub-interval (using the earlier part of the argument as k is independent).

$$\text{val}(a_k - a'_k) \geq F(a_j) \text{ for all } k \in K \text{ by monotonicity of } F(a_k)$$

$$\text{val}(\vec{p}_k \cdot \vec{c} - \vec{p}_k \cdot \vec{c}') \geq F(a_j) \text{ for all } k \in K$$

$$\text{val}(\vec{p}_k \cdot (\vec{c} - \vec{c}')) \geq F(a_j) \text{ for all } k \in K$$

$K \subset I, i \in I, \vec{c} - \vec{c}' \in \mathbb{Q}_p^{|x|}, F(a_j) \in \mathbb{Z}$ satisfy the requirements of Lemma 4.1, so we apply it to conclude

$$\begin{aligned} \text{val}(\vec{p}_i \cdot (\vec{c} - \vec{c}')) &\geq F(a_j) - \gamma \\ \text{val}(\vec{p}_i \cdot \vec{c} - \vec{p}_i \cdot \vec{c}') &\geq F(a_j) - \gamma \\ \text{val}(a_i - a'_i) &\geq F(a_j) - \gamma \end{aligned}$$

as needed, finishing the proof of the claim. \square

Additionally a_i, a'_i have the same image in Ct component, so we have

$$\text{val}(a_i - a'_i) \geq F(a_j)$$

As $F(a_i) \leq F(a_j)$, a_i, a'_i have to lie in the same sub-interval. TO DO: PROVE THE PREVIOUS SENTENCE FORMALLY \square

Corollary 4.8. $\Psi(x, y)$ has VC-density $\leq |x|$.

Proof. Suppose we have $c, c' \in \mathbb{Q}_p^{|x|}$ such that $f(c), f(c')$ are in the same partition and $g(f(c)) = g(f(c'))$. Then by the previous lemma c, c' have the same Ψ -type. Thus the number of possible Ψ -types is bounded by the size of the range of g times the number of possible partitions

$$(\text{number of partitions}) \cdot |\text{It}|^{|I|} \cdot |\text{Sub}|^{|J|} \cdot |\text{Ct}|^{|I-J|}$$

There are at most $(|2I|!)^2$ many partitions of Sg, so in the product above, the only component dependent on B is

$$|\text{Sub}|^{|J|} \leq (N \cdot 3|I|^2)^{|J|} = O(N^{|J|})$$

Every p_i is an element of a $|x|$ -dimensional vector space, so there can be at most $|x|$ many independent vectors. Thus we have $|J| \leq |x|$ and the bound follows. \square

Corollary 4.9. In the language \mathcal{L}_{aff} we have $\text{vc}(n) = n$.

Proof. Previous lemma implies that $\text{vc}(\phi) \leq \text{vc}(\Psi) \leq |x|$. As choice of ϕ was arbitrary, this implies that VC-density of any formula is bounded by the arity of x . \square

This proof relies heavily on the linearity of functions a_1, a_2, c in the cell decomposition result (see Definition 2.4). Linearity is used to separate x and y variables as well as for Lemma 4.1 to reduce the number of independent factors from $|I|$ to $|x|$. The paper [2] has cell decomposition results for more expressive reducts of \mathbb{Q}_p , including, for example, restricted multiplication. While our results don't apply to it directly, it is this author's hope that similar techniques can be used to compute $\text{vc}(n)$ function for those structures.

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