conjecture, we can prove the locl indeomposability. This generalizes the result in [Hi13b] to general abelian varieties.

Project IIIc: Local indecomposability for automorphic representations. This project will be carried out jointly with J. Tilouine. We will explore such generalization. By p-adic Siegel modularity of abelian surfaces by Tilouine and others (see §2.1 on such geometric modularity results) and potential modularity by Taylor and his followers, once an abelian variety fits into a p-adic analytic family of automorphic forms, it would prove local indecomposability for the Λ-adic Galois representations. Our hope is to find out what we can say about indecomposability and finer properties of the Galois representations.

1.2.4. *Project IV: Level of modular Galois image.* Under some mild assumptions, the PI has found in the elliptic modular case that if the Hecke eigen formal scheme $\mathrm{Spf}(\mathbb{I})$ does not have CM (i.e., not made of CM theta series), the associated $\mathbb{I}$-adic Galois representation $\rho_{\mathbb{I}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{I})$ has big image containing a principal congruence subgroup

$$\Gamma(\mathfrak{c}) = \{x \in SL_2(\Lambda) | x \equiv 1 \mod \mathfrak{c}M_2(\Lambda)\}.$$

Here $\Lambda = \mathbb{Z}_p[[T]]$ is the weight variable Iwasawa algebra and $0 \neq \mathfrak{c} \subset \Lambda$ is an ideal of Λ. Note that $\mathbb{I}$ canonically contains Λ and $\rho_{\mathbb{I}}$ is unramified outside Np for the prime-to-p level N of $\mathbb{I}$. The representation is characterized by the fact that $\mathrm{Tr}(\rho_{\mathbb{I}}(Frob_l)) \in \mathbb{I}$ is the image $a(l)$ of the Hecke operator $T(l)$ for almost all l. The maximal $\mathfrak{c}$ as above is called the Galois conductor of $\rho_{\mathbb{I}}$ which depends on the realization of $\rho_{\mathbb{I}}$ but its reflexive closure $(L) = \bigcap_{(\lambda)\supset\mathfrak{c}}(\lambda)$ is well defined invariant of the isomorphism class of $\rho_{\mathbb{I}}$ (called the *Galois level* of $\mathbb{I}$). Moreover the generator L can be made precise as a factor of either $(1+T)^{p^m}-1$ (for $m \gg 0$) or of the square of the Katz p-adic L-function or the Kubota–Leopold p-adic L–function (see [Hi15a]). We plan to make a more precise determination of the image and generalize the result to Shimura varieties than the elliptic Shimura curve. This project will be carried out jointly with Jacques Tilouine and the PI's student Jaclyn Lang.

Project IVa: Pinpointing the image of $\rho_{\mathbb{I}}$. By local indecomposbility of weight 2 classical Galois representation proven in [Hi13b] and [Z14], the PI has eliminated the assumptions made in [Hi15a] (see [HiT15]) and the existence of non-trivial Galois level is now unconditional. Moreover there should be a canonical choice of a Λ-subalgebra $\mathbb{I}_0 \subset \mathbb{I}$ such that $\mathrm{Im}(\rho_V)$ is very close to $SL_2(\mathbb{I}_0)$ up to an abelian error. Under the restrictive condition that the residual image of ρ_V containing $SL_2(\mathbb{F}_p)$ (the fullness assumption), such a question was studied by a former student A. Fischman [Fs02] of the PI to some extent. However, in the most interesting cases where the level L is related to a p-adic L–function, this fullness assumption fails. We want to determine $\mathbb{I}_0$ in general by a method different from [Fs02]. In addition, we expect that L is either a factor of $t^{p^m} - 1$ $(t = 1 + T)$ or of Katz p-adic L-function (not the square of it) or of the Kubota–Leopoldt p-adic L-function.

To give a conjectural description of $\mathbb{I}_0$ and the Galois image, we prepare some notation. A prime $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ is arithmetic if $P|(t^{p^n} - \gamma^k)$ for an integer $k \geq 2$ and $0 < n \in \mathbb{Z}$ for $\gamma = 1 + p$ and $t = 1 + T$. Write $\kappa(P)$ for the residue field of P. If there exists a finite order character $\eta : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{I}^\times$ and an automorphism σ_η of the quotient field $Q(\mathbb{I})$ of $\mathbb{I}$ such that $\sigma_\eta(a(l)) = \eta(l)a(l)$ for almost all primes l, we call σ_η a *self-conjugate twist* of $\mathbb{I}$. Then the collection of all self-conjugate twists $\Gamma_{\mathbb{I}}$ is a finite subgroup of $\mathrm{Aut}(\mathbb{I}_{/\Lambda})$

and is isomorphic to a subgroup of the character group of $(\mathbb{Z}/N\mathbb{Z})^\times$ by $\sigma_\eta \leftrightarrow \eta$. Thus identifying $\mathrm{Gal}(\mathbb{Q}[\mu_N]/\mathbb{Q})$ with $(\mathbb{Z}/N\mathbb{Z})^\times$ by class field theory, the Pontryagin dual $\Gamma_{\mathbb{I}}^*$ of $\Gamma_{\mathbb{I}}$ is canonically isomorphic to $\mathrm{Gal}(k_{\mathbb{I}}/\mathbb{Q})$ for a subfield $k_{\mathbb{I}} \subset \mathbb{Q}[\mu_N]$. For each arithmetic specialization $\rho_P = (\rho \mod P)$ of $\rho_{\mathbb{I}}$, write MT_P for the Mumford–Tate group of the compatible system containing ρ_P and $G_{P/\kappa(P)}$ for the Zariski closure in $GL_2(\kappa(P))$ of $\mathrm{Im}(\rho_P)$. Let G_P° (resp. G'_P) be the connected component of G_P (resp. the derived group of G_P°). The PI plan to solve the following conjecture with Jaclyn Lang.

Conjecture 1.1. *Suppose that $\mathbb{I}$ is a non CM component.*

1. *There exists a simple linear algebraic group G' defined over $\mathbb{Q}_p$ independent of the arithmetic point $P \in \mathrm{Spec}(\mathbb{I})$ such that $G'_P \cong G' \times_{\mathbb{Q}_p} \kappa(P)$, and $\mathrm{Res}_{\kappa(P)/\mathbb{Q}_p} G_P$ is the (ordinary) factor of $MT_P \times_{\mathbb{Q}} \mathbb{Q}_p$;*
2. *G_P/G_P° is canonically isomorphic to the Pontryagin dual Γ_P^* of $\Gamma_P := \{\sigma \in \Gamma_{\mathbb{I}} | \sigma(P) = P\}$;*
3. *For the field k_P° fixed by $\rho_P^{-1}(G_P^\circ(\kappa(P)))$, we have $\mathrm{Gal}(k_P^\circ/\mathbb{Q}) \cong \Gamma_P^*$, and k_P is the fixed subfield of $k_{\mathbb{I}}$ by $\mathrm{Ker}(\Gamma_{\mathbb{I}}^* \twoheadrightarrow \Gamma_P^*) \cong \Gamma_{\mathbb{I}}/\Gamma_P$;*
4. *Writing k° for the composite of k_P° for all arithmetic P, we have $k^\circ = k_{\mathbb{I}}$ and $\mathrm{Gal}(k^\circ/\mathbb{Q}) \cong \Gamma_{\mathbb{I}}^*$;*
5. *Let $\mathbb{I}_0$ be the fixed sbring of $\mathbb{I}$ by $\Gamma_{\mathbb{I}}$. Then $\mathbb{I}_0$ is generated by $\mathrm{Tr}(Ad(\rho_{\mathbb{I}})(\sigma))$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, where $Ad(\rho_{\mathbb{I}})$ is the adjoint representation of $\rho_{\mathbb{I}}$ acting on $\mathfrak{sl}_2(\mathbb{I})$;*
6. *$\mathrm{Im}(\rho_{\mathbb{I}}) \cong \mathrm{Im}(\mathrm{Ind}_{k^\circ}^{\mathbb{Q}} \rho_0)$, in particular, we have an exact sequence $1 \to \mathrm{Im}(\rho_0) \to \mathrm{Im}(\rho_{\mathbb{I}}) \to \Gamma_{\mathbb{I}}^* \to 1$;*
7. *There exists a non-zero ideal $\mathfrak{C}$ of $\mathbb{I}_0$ such that $\mathrm{Im}(\rho_0)$ contains $\Gamma_{\mathbb{I}_0}(\mathfrak{C}) \subset SL_2(\mathbb{I}_0)$.*

Once we are able to prove the above conjecture, we plan to relate the reflexive closure of $\mathfrak{C}$ above to p-adic L in a way neater than the result of [Hi15a] if the residual representation is dihedral.

Project IVb: Generalization to Hilbert modular cases. First we need to give a good definition of the Galois level of $\rho_{\mathbb{I}}$ in the Hilbert modular case, where $\mathbb{I}$ has many more variables (and hence the techniques invented by the PI do not apply immediately). We are fairly optimistic on this front. Assuming the level is well defined by the isomorphism class of $\rho_{\mathbb{I}}$, we try to prove the Hilbert modular version of the above conjecture. Once this is done, the PI suspects that in the Hilbert modular case over a totally real field $F \neq \mathbb{Q}$, the level is either close to trivial (like a factor of $t^{p^m} - 1$ in the elliptic modular case) or a factor of the anticyclotomic Katz p-adic L-function (in other words, the Deligne–Ribet p-adic L-function does not appear).

Project IVc: Generalization to automorphic forms on $\mathbb{Q}_p$-split groups. The PI has started jointly with Jacques Tiouine a project aiming at finding the level for Siegel modular $\mathbb{I}$-adic Galois representations Tilouine–Urban constructed (see Projects 3a, 3b of Tilouine). Indeed, for algebraic automorphic forms on $GSp(2n)$, we expect to have the associated Galois representation into its Langlands dual $GSpin(2n+1)$ (not this is known if $n \leq 2$ as $GSpin(3) = GL(2)$ and $GSpin_5 = GSp(4)$). Assuming the existence of such Galois representations, we have the big Galois representation $\rho_{\mathbb{I}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GSpin_{2n+1}(\mathbb{I})$. Under the following condition

(Adi) the adjoint $Ad(\rho_{\mathbb{I}})$ acting on $\mathfrak{spin}(2n+1)$ is absolutely irreducble, we have shown the existence of non-trivial Galois level (see [HiT15]). The condition (Adi) is an analogue of non CM condition (see (5) in the conjecture). Once the level is defined, we plan to relate the level to the characteristic power series of the (Siegel modular) congruence module of the Hecke eigen formal scheme $\mathrm{Spf}(\mathbb{I})$. More details of this project joint with Tilouine is described in Tilouin's projects.

Along with studying Siegel modular case, Tilouine and the PI plan to study the big automorphic Galois representations attached to automorphic forms on general $\mathbb{Q}_p$-split groups along the same line.

1.3. **Broader Impact.** The PI's research work in the past and the projects described in the proposal have already had and will continue to have a significant impact on education of graduate students (all over the world) and on research of young and senior mathematicians as well. His influence is well documented in the Bourbaki seminar talk by M. Emerton (see [E11]), and this point is also confirmed by many of the latest ICM talks (at Madrid, 2006 and Hyderabad in 2010).

1.3.1. *Educational activities.* The PI regularly gives graduate courses at UCLA on cutting-edge research materials, and out of his lecture notes he has produced six books ([LFE], [MFG], [GME], [PAF], [HMI] and [EAI]) which have been useful for educational purposes in graduate schools allover the world (as well as for professional researchers; see the introduction of [Hs12c] about Hsieh's opinion about these books). In the past five years, the PI wrote two books: the second expanded edition of [GME] and a new book [EAI]. The content of the two new books is described in Section 1.1. Once again, the PI plans to write two books giving expositions on the following two topics:

(1) p-adic L-functions and the proof of the main conjecture of CM field;
(2) arithmetic invariants in Hilbert modular and quaternionic cases,

(possibly in addition to a new edition of [PAF]). For the first topic, it seems that Hsieh [Hs12c] has made a good progress in proving the CM main conjecture in full after the work of the PI and J. Tilouine on the anti-cyclotomic main conjectue (see [HiT94], [Hi06b] and [Hi09e]). Thus the time would mature within 5 years to have a good expository book. As for the second topic, the PI has written a book [EAI] dealing with arithmetic invariant for elliptic modular forms. However there is no good *arithmetic* account dealing with

(a) integral model of quaternionic Shimura varieties (except for Carayol's paper for Shimura curves and the work of Deligne–Pappas on Hilbert modular varieties),
(b) exact computation of Waldspurger's formula of the central critical values for general quaternion algebra (except for [Hi10b] covering $M_2(\mathbb{Q})$ only; see Project Ia),
(c) the non-triviality problem of arithmetic invariant related to the above two topics,

in a way accessible by graduate students. If the PI writes the continuation of [EAI] taking case of these three topics, it would be a good service to the mathematical community, and also it would serve as a good introduction to the more difficult topics covered by [PAF].

The PI is now advising three graduate students: **Bin Zhao**, **Ashay Burungale** and **Jaclyn Lang**. The level of graduate students in the number theory group at UCLA goes up higher and higher every year. So it is now common that graduate students write fine research articles (at the level of top-notch research journals) before finishing Ph D thesis. For example, Zhao is a fourth year student and has now written a preprint proving local