Consider a finite family of formulas $\Psi(x;y) = \{\phi_i(x;y)\}$ in the language of a model M. We define the shatter function $\pi^M_\Psi : \mathbb{N} \longrightarrow \mathbb{N}$ of Ψ as

$$\pi^M_\Psi(n) = \max\{\text{number of } \Psi\text{-types over } B_0 \ \mid B_0 \subset M^{|y|} \text{ with } |B_0| = n\}.$$

For a single formula ϕ we define $\mathrm{vc}(\phi)$ as VC-density of a one element collection $\{\phi\}$. Shatter function only depends on the theory of M. The following theorem is an important result concerning dichotomy of shatter function growth.

Theorem (Sauer-Shelah '72)

The shatter function either grows exponentially or is bounded by a polynomial.

In fact, formula $\phi(x;y)$ is NIP precisely when its shatter function grows polynomially. From now on we restrict our attention to NIP theories, that is all formulas will have shatter functions that grow polynomially. The following definition captures the degree of polynomial growth.

Definition

For a formula $\phi(x;y)$ in model M let $\mathrm{vc}^M(\phi)$ be the infimum of all positive reals r such that

$$\pi^M_\phi(n) = O(n^r)$$

Call $\mathrm{vc}^M(\phi)$ the vc-density of ϕ.

This allows formula by formula analysis of the growth rate for the shatter function. More generally, we look at bounds of VC-density for all the formulas in a given structure.

Definition

Define vc-function $\mathrm{vc}^M : \mathbb{N} \longrightarrow \mathbb{N}$ to be the largest vc-density achieved by uniformly definable families in M^n.

$$\mathrm{vc}^M(n) = \sup\left\{\mathrm{vc}^M(\phi) \mid \phi(x,y) \text{ with } |x| = n\right\}$$

As before this only depends on the theory of M. There is a simple lower bound $\mathrm{vc}^M(n) \geq n$. More generally $\mathrm{vc}^M(n) \geq n\,\mathrm{vc}^M(1)$, and it is not known whether strict inequality can hold.
A common example of a non-stable NIP structure are p-adic numbers $\mathbb{Q}_p$ in the language of fields. Aschenbrenner et. al show that p-adic numbers have $\mathrm{vc}(n) \leq 2n - 1$. My work improves that bound in a reduct of the full structure.
In [**?**], Leenknegt analyzes the reduct of p-adic numbers to the language

$$\mathcal{L}_{aff} = \left\{\{Q_{n,m}\}_{n,m\in\mathbb{N}}, +, -, \{\bar{c}\}_{c\in\mathbb{Q}_p}, |\right\}$$

where $\bar{c}$ is a scalar multiplication by c, $a|b$ stands for $a0a \leq a0b$, and $Q_{n,m}$ is a unary predicate

$$Q_{n,m} = \bigcup_{k\in\mathbb{Z}} p^{kn}(1 + p^m\mathbb{Z}_p).$$

One can check that the extra relation symbols are definable in the full structure. Moreover [**?**] shows it is a P-minimal reduct, that is one-dimensional definable sets coincide with one-dimensional definable sets in the full structure.

Theorem (B.)

In $\mathcal{L}_{aff}$, $\mathbb{Q}_p$ has $\mathrm{vc}(n) = n$.

[**?**] provides the following cell decomposition result

Theorem

Any formula $\phi(t,x)$ with t singleton decomposes into the union of the following cells:

$$\{(t,x) \in K \times D \mid a0a_1(x)\square_1 a0(t - c(x))\square_2 a0a_2(x), t - c(x) \in \lambda Q_{n,m}\}$$

where D is a cell of a smaller dimension, a_1, a_2, c are linear polynomials in x, $\square$ is $<$ or no condition, $\lambda \in \mathbb{Q}_p$.

This can be adapted into a quantifier elimination result

Corollary

Any formula $\phi(x;y)$ can be written as a boolean combination of formulas from the following two collections of formulas

$$\Psi_1(x;y) = \{a0(p_i(x) - c_i(y)) < a0(p_j(x) - c_j(y))\}_{i,j\in I}$$
$$\Psi_2(x;y) = \{a0(p_i(x) - c_i(y)) \in \lambda_k Q_{n,m}\}_{i\in I, k\in K}$$

where I, K are finite index sets, p_i is a linear polynomial in x without a constant term, c_i is a linear polynomial in y, and $\lambda_k \in \mathbb{Q}_p$.

Letting $\Psi = \Psi_1 \cup \Psi_2$ it is easy to show that $\mathrm{vc}(\phi) \leq \mathrm{vc}(\Psi)$. Therefore to show that $\mathrm{vc}(n) = n$ it suffices to bound $\mathrm{vc}(\Psi) \leq |x|$ for any such collection. More precisely, we would like to show that if we have a parameter set B of size N then the number of Ψ-types over B is $O(N^{|x|})$.

Definition

For $c \in \mathbb{Q}_p, \alpha \in \mathbb{Z}$ we define a ball

$$B(c,\alpha) = \{c' \in \mathbb{Q}_p \mid a0\,(c' - c) \leq \alpha\}$$

Definition

Suppose we have a finite $T \subset \mathbb{Q}_p$. We view it as a tree as follows. Branches through the tree are elements of T. With this tree we associate balls $B(t_1, a0(t_1 - t_2))$ for all $t_1, t_2 \in T$. An interval is two balls $B(t_1, v_1) \supset B(t_2, v_2)$ with no balls in between. An element $a \in \mathbb{Q}_p$ belongs to this interval if $a \in B(t_1, v_1) \backslash B(t_2, v_2)$. There are at most $2|T|$ different intervals and they partition the entire space.
Fix a parameter set B of size N.
Consider a tree $T = \{c_i(b) \mid b \in B, i \in I\}$ It has at most $O(N) = N \cdot |I|$ many intervals.

Definition

Suppose $a \in \mathbb{Q}_p$ lies in an interval $B(t_L, \alpha_L)\backslash B(t_U, \alpha_U)$. Define T-valuation $\text{T-val}(a) = a0(a - t_U)$.

Definition

Suppose $a_1, a_2 \in \mathbb{Q}_p$ lie in our tree in the same interval $B(t_L, \alpha_L)\backslash B(t_U, \alpha_U)$. We say that a_i is close to boundary if $|a0(a_i - t_U) - \alpha_L| \leq m$ or $|a0(a_i - t_U) - \alpha_U| \leq m$. Otherwise we say that it is far from boundary. Say that a_1, a_2 have the same interval type if one of the following holds:

- Both a_1, a_2 are far from boundary and $a_1 - t_U, a_2 - t_U$ are in the same $Q_{n,m}$ coset.
- Both a_1, a_2 are close to boundary and $a0(a_1 - a_2) > \text{T-val}(a_1) + m = \text{T-val}(a_2) + m$.

Lemma

For each interval there are at most $M = M(\Psi, Q_{n,m})$ many interval types with M not dependent on B, or the interval.

Conclusion

Nunc tempus venenatis facilisis. **Curabitur suscipit** co
dolor, id ornare enim. Fusce quis massa dictum tortor **ti**
quis pretium at, laoreet scelerisque lacus. Nam quis odio
at *nulla elementum sollicitudin*.

Additional Information

Maecenas ultricies feugiat velit non mattis. Fusce tempu

- Curabitur pellentesque dignissim
- Eu facilisis est tempus quis
- Duis porta consequat lorem

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Acknowledgements

Nam mollis tristique neque eu luctus. Suspendisse rutrum congue
Pellentesque habitant morbi tristique senectus et netus et malesuad

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