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Computations of Vapnik-Chervonenkis Density in Various Model-Theoretic Structures

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Anton Bobkov

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ABSTRACT OF THE DISSERTATION

Computations of Vapnik-Chervonenkis Density
in Various Model-Theoretic Structures

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Doctor of Philosophy in Mathematics
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Professor Matthias J. Aschenbrenner, Chair

Aschenbrenner et al. have studied Vapnik-Chervonenkis density (VC-density) in the model-theoretic context. We investigate it further by computing it in some common structures: trees, Shelah-Spencer graphs, and an additive reduct of the field of p-adic numbers. In the theory of infinite trees we establish an optimal bound on the VC-density function. This generalizes a result of Simon showing that trees are dp-minimal. In Shelah-Spencer graphs we provide an upper bound on a formula-by-formula basis and show that there isn't a uniform lower bound, forcing the VC-density function to be infinite. In addition we show that Shelah-Spencer graphs do not have a finite dp-rank, so they are not dp-minimal. There is a linear bound for the VC-density function in the field of p-adic numbers, but it is not known to be optimal. We investigate a certain P-minimal additive reduct of the field of p-adic numbers and use a cell decomposition result of Leenknegt to compute an optimal bound for that structure. Finally, following the results of Podewski and Ziegler we show that superflat graphs are dp-minimal.

The dissertation of Anton Bobkov is approved.

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To my family and friends

who have been unerringly supportive

throughout my career path

TABLE OF CONTENTS

LIST OF FIGURES

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VITA

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CHAPTER 1

Introduction and Preliminaries

1.1 Introduction

My research concentrates on the concept of VC-density, a recently introduced notion of rank in NIP theories. The study of a structure in model theory usually starts with quantifier elimination in a natural language, followed by a finer analysis of definable functions and interpretability. The study of VC-density goes one step further, looking at the asymptotic growth of finite definable families. In the most geometric examples, VC-density is closely related to the natural notion of dimension. However, no geometric structure is required for the definition of VC-density, thus we can get some notion of geometric dimension for families of sets given without any geometric context.

In 2013, Aschenbrenner et al. investigated and developed a notion of VC-density for NIP structures, an analog of geometric dimension in an abstract setting [ADH16]. Their applications included a bound on the VC-density of definable families in the field of p-adic numbers, an object of great interest in many fields of mathematics. My research concentrates on expanding techniques of that paper to improve the known bounds as well as computing VC-density for other NIP structures of interest. I am able to obtain new bounds for a certain additive reduct of the p-adic numbers, trees, and some families of graphs. Recent research by Chernikov and Starchenko in 2015 [CS15] suggests that having good bounds on VC-density in the p-adic numbers opens a path for applications to incidence combinatorics (e.g. Szemeredi-Trotter theorem).

The concept of VC-dimension was first introduced in 1971 by Vapnik and Chervonenkis for set systems in a probabilistic setting [VC71]. The theory grew rapidly and found wide use in geometric combinatorics, computational learning theory, and machine learning. Around the same time Shelah was developing the notion of NIP ("not having the independence property"), a natural tameness property of (complete theories of) structures in model theory [She71]. In 1992, Laskowski noticed the connection between the two: theories where all uniformly definable families of sets have finite VC-dimension are exactly NIP theories [Las92]. This is a wide class of theories including algebraically closed fields, differentially closed fields, modules, free groups, o-minimal structures, and ordered abelian groups. A variety of valued fields fall into this category as well, including the p-adic numbers.

The p-adic numbers were first introduced by Hensel in 1897 in [Hen97], and over the following century a powerful theory was developed around them with numerous applications across a variety of disciplines, primarily in number theory, but also in physics and computer science. In 1965, Ax, Kochen [AK65] and Ershov [Ers65] axiomatized the theory of p-adic numbers. A key insight was that properties of the value group and residue field determine the properties of the valued field itself. In 1976, Macintyre proved a quantifier elimination result in a definitional expansion of the field language [Mac76]. In 1984, Denef proved a cell decomposition result which clarified the geometry of definable sets in $\mathbb{Q}_p$ [Den84]. This result was soon generalized to p-adic subanalytic and rigid analytic extensions, allowing for the later development of the powerful technique of motivic integration. The conjunction of those model theoretic results allowed to solve a number of outstanding open problems in number theory (e.g., Artin's Conjecture on p-adic homogeneous forms).

In 1997, Karpinski and Macintyre computed VC-density bounds for o-minimal structures and asked about similar bounds for the p-adic numbers [KM97]. VC-density is a concept closely related to VC-dimension. It comes up naturally in combinatorics with relation to packings, Hamming metric, entropic dimension and discrepancy. VC-density is also the decisive parameter in the Epsilon-Approximation Theorem, which is one of the crucial tools for applying VC theory in computational geometry. In a model theoretic setting it is computed for families of uniformly definable sets. In 2013, Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko computed a bound for VC-density in the field of p-adic numbers and a number of other NIP structures [ADH16]. They observed connections to dp-rank and dp-minimality, notions first introduced by Shelah. In well behaved NIP structures, families of uniformly definable sets in n-space tend to have VC-density bounded above by n, a simple linear behavior. In many cases (including the p-adic numbers) this bound is not known to be optimal. My research concentrates on improving those bounds and adapting those techniques to compute VC-density in other common NIP structures.

Some other well behaved NIP structures of a combinatorial flavor are Shelah-Spencer graphs and superflat graphs. Shelah-Spencer graphs are the limit structures of random graphs arising naturally in a combinatorial context. Their model theory was studied by Shelah and Spencer in 1988 in [SS88] and then refined by Baldwin, Shi, and Shelah in 1997 in [BS96], [BS97]. Later work of Laskowski in 2007 provided a quantifier simplification result [Las07]. Superflat graphs were introduced by Podewski-Ziegler in 1978, who showed their stablity [PZ78]. Later results gave a criterion for superstability [HMS83]. Superflat graphs also come up naturally in combinatorics (for example, see the work of Nesetril and Ossona de Mendez [NM11]).

The first chapter of my dissertation introduces some basics of model theory and defines VC-density and VC-dimension.

The second chapter concentrates on trees. I answer an open question from [ADH16], computing VC-density for trees viewed as a partial order. The main idea is to adapt a technique of Parigot [Par82] to partition trees into weakly interacting parts, with simple bounds of VC-density on each.

In the third chapter I work with Shelah-Spencer graphs. I show that they don't have finite dp-rank, so they are poorly behaved as NIP structures. I also show that one can obtain arbitrarily high VC-density when looking at uniformly definable families in a fixed ambient space. However, I'm able to bound VC-density of individual formulas in terms of edge density of the graphs they define.

The fourth chapter deals with p-adic numbers. I show that VC-density is linear for an additive reduct of p-adic numbers using a cell decomposition result from the work of Leenknegt in 2014 in [Lee14].

In chapter five I investigate superflat graphs using the techniques of Podewski-Ziegler [PZ78]. I am able to show that superflat graphs are dp-minimal, an important first step before establishing bounds on VC-density.

1.2 Basic Model Theory

This section goes through some of the basic model theoretic concepts that are used throughout this text. We assume the reader's familiarity with the fundamental notions of languages, formulas, structures, and theories. For an introduction to these topics, we refer to Chapters 1 and 2 of [TZ12].

We work with structures $\mathbb{M} = (M, \ldots)$ in finite or countable languages $\mathcal{L}$. Generally, in this thesis the structures have infinite universes M. For a tuple of variables x, let $|x|$ denote its arity. Similarly, for a tuple $a \in M^n$ let $|a| = n$. We often confuse a tuple $a = (a_1, \ldots, a_n) \in M^n$ with its underlying set $a = \{a_1, \ldots, a_n\} \subseteq M$. We study definable sets given by first-order formulas $\phi(x)$. Abusing notation, we denote $\mathcal{L}$ to be the set of all first-order formulas in the language $\mathcal{L}$. For a parameter set $A \subseteq M$ the expression $\mathcal{L}(A)$ denotes the collection of formulas with parameters from A (or simply A-formulas).

More generally, we work with complete theories and their models. Throughout this text we often confuse the two. This is justified for properties that can be described by a collection of first-order sentences. Then a theory has this property if and only any (all) models have this property. An example of that is the notion of stability.

Stability is a deep subject, with a lot of theory developed around it. We won't work with it directly, but it is a property of some of the structures we study. We present a definition for completeness and refer the reader to Chapter 8 of [TZ12] or to [Pil13] for a more complete introduction.

Definition 1.2.1.

- Suppose we have a structure $\mathbb{M}$. A formula $\phi(x, y)$ is called unstable in $\mathbb{M}$ if for all natural numbers n there exist $a_i \in M^{|x|}, b_i \in M^{|y|}$ for $0 \leq i \leq n$ such that

$$\mathbb{M} \models \phi(a_i, b_j) \iff i \leq j.$$

- A formula is stable if it is not unstable.
- A structure $\mathbb{M}$ is stable if all of its formulas are stable in $\mathbb{M}$.
- A complete theory T is stable if any (all) of its models are stable.

Definable sets are subsets of our structure given by formulas. More precisely:

Definition 1.2.2. Let $\mathbb{M}$ be a structure, $A \subseteq M$ a parameter set and $\phi(x)$ be an A-formula. Then

$$\phi(M^{|x|}) = \left\{ b \in M^{|x|} \mid \mathbb{M} \models \phi(b) \right\}$$

is referred to as the A-definable subset of $M^{|x|}$ defined by ϕ.

More generally, we will need a slightly more refined notion of trace:

Definition 1.2.3. Suppose we have a structure $\mathbb{M}$, a formula $\phi(x, y)$, tuples $a \in M^{|x|}, b \in M^{|y|}$, and sets $A \subseteq M^{|x|}, B \subseteq M^{|y|}$. Define

$$\phi(A, b) = \{a \in A \mid \mathbb{M} \models \phi(a, b)\},$$
$$\phi(a, B) = \{b \in B \mid \mathbb{M} \models \phi(a, b)\}.$$

These sets will be informally referred to as traces (on A, respectively B). Similarly, let

$$\phi(A, B) = \{\phi(A, b) \mid b \in B\} \subseteq \mathcal{P}(A)$$

denote the collection of traces on A parametrized by B.

Types are one of the main objects of study in model theory.

Definition 1.2.4. Let $\mathbb{M}$ be a structure, $B \subseteq M$. Also fix a tuple of variables x.

- A partial type over B is a collection of formulas in the variables x with parameters from B.

- A tuple $a \in M^{|x|}$ is called realization of a partial type $p(x)$ if $\mathbb{M} \models \phi(a)$ for all $\phi(x) \in p(x)$.

- A partial type $p(x)$ is consistent if every finite subset $p(x)$ has a realization in $\mathbb{M}$.

- Let $a \in M^{|x|}$ and $\Delta \subseteq \mathcal{L}(B)$ be a collection of formulas $\delta(x)$. Define the Δ-type of a over B to be the collection of formulas $\delta(x) \in \Delta$ such that $\mathbb{M} \models \delta(a)$. Denote it as $\mathrm{tp}_{\Delta}(a/B)$.

- Let $a \in M^{|x|}$. Define the type of a over B as the Δ-type of a over B for $\Delta = \mathcal{L}(B)$. Denote it as $\mathrm{tp}(a/B)$.

Many model theoretic arguments are simplified when done inside saturated structures. This is the next important construction that we turn our attention to:

Definition 1.2.5. Let κ be a cardinal. A structure $\mathbb{M}$ is called κ-saturated if for all $B \subseteq M$ with $|B| < \kappa$ all consistent partial types over B are realized in $\mathbb{M}$.

Indiscernible sequences will be useful to us to describe dp-rank and dp-minimality. They come up often in model theory as a way to leverage symmetry present in sequences and sets.

Definition 1.2.6.

- Suppose we have a sequence $(a_i)_{i \in \mathcal{I}}$ where $\mathcal{I}$ is an ordered index set and $a_i \in M^m$ for all i. For a subset $\mathcal{J} \subseteq \mathcal{I}$ let $a(\mathcal{J}) \in M^{m|\mathcal{J}|}$ denote the tuple obtained by the concatenation of the sequence $(a_j)_{j \in \mathcal{J}}$ (where $\mathcal{J}$ is ordered using the order induced by $\mathcal{I}$).

- Suppose $\mathbb{M}$ is a structure, $B \subseteq M$, and $\mathcal{I}$ is an ordered index set. A sequence $(a_i)_{i \in \mathcal{I}}$ with each $a_i \in M^m$ is called indiscernible over B (or B-indiscernible) if for any two finite subsets $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{I}$ of equal size we have

$$\mathrm{tp}(a(\mathcal{J}_1)/B) = \mathrm{tp}(a(\mathcal{J}_2)/B).$$

- If we use the same definition, but allow tuples $a(\mathcal{J}_1), a(\mathcal{J}_2)$ to be concatenated in arbitrary order, then we obtain the definition of a sequence that is totally indiscernible over B (alternatively called an indiscernible set).

Here is an important property of indiscernible sequences in stable theories:

Lemma 1.2.7 (see Lemma 9.1.1 in [TZ12])**.** *If $\mathbb{M}$ is stable then every indiscernible sequence (over a subset of M) is totally indiscernible.*

Instead of starting with an indiscernible sequence, we sometimes wish to construct one from a sequence with some degree of symmetry:

Lemma 1.2.8 (see Lemma 5.1.3 in [TZ12])**.** *Work in a κ-saturated structure $\mathbb{M}$. Let $B \subseteq M$ with $|B| < \kappa$. Fix a tuple of variables x and a collection of B-formulas $\Delta(x_1, \ldots, x_n)$ with $|x_i| = |x|$. Suppose we can find an arbitrarily long finite sequence $(a_i)_{i \in \mathcal{I}}$ with $a_i \in M^{|x|}$ such that for any subset $\mathcal{J} \subseteq \mathcal{I}$ of length n we have*

$$\mathbb{M} \models \Delta(a(\mathcal{J})).$$

Then there exists an infinite B-indiscernible sequence $(b_i)_{i \in \mathbb{N}}$ such that for any subset $\mathcal{J} \subseteq \mathbb{N}$ of length n we have

$$\mathbb{M} \models \Delta(b(\mathcal{J})).$$

Instead of working with types directly, it is often more convenient to work with automorphisms:

Definition 1.2.9. Suppose $\mathbb{M}$ is a structure and $A \subseteq M$. An automorphism of $\mathbb{M}$ over A is a bijection $f\colon M \longrightarrow M$ with $f(a) = a$ for all $a \in A$ which preserves the interpretations of all constant, relation, and function symbols in $\mathbb{M}$. We denote by $\mathrm{Aut}(\mathbb{M}/A)$ the group of all automorphisms of $\mathbb{M}$ over A. For a tuple $a = (a_1, \ldots, a_m) \in M^m$ let $f(a) = (f(a_1), \ldots, f(a_m))$.

The following lemma is obvious from the definition of an automorphism:

Lemma 1.2.10. *Let $\mathbb{M}$ be a structure, $A \subseteq M$, $f \in \mathrm{Aut}(\mathbb{M}/A)$, and $a \in M^m$. Then $tp(a/A) = tp(f(a)/A)$.*

The converse of this result holds in a special type of structures:

Definition 1.2.11. Let $\mathbb{M}$ be a structure and κ a cardinal. Then $\mathbb{M}$ is called strongly κ-homogeneous if for all $A \subseteq M$ with $|A| < \kappa$ and all $a, b \in M^n$, if $tp(a/A) = tp(b/A)$ then there exists $f \in \mathrm{Aut}(\mathbb{M}/A)$ such that $f(a) = b$.

Luckily, for a given theory one can always find a model sufficiently saturated and homogeneous:

Lemma 1.2.12 (see Theorem 6.1.7 in [TZ12])**.** *Let T be a complete theory and κ be a cardinal. There exists a model of T that is κ-saturated and strongly κ-homogeneous.*

1.3 VC-dimension and VC-density

Throughout this section we work with a collection $\mathcal{F}$ of subsets of an infinite set X. We call the pair $(X, \mathcal{F})$ a set system.

Definition 1.3.1.

- Given a subset A of X, we define the set system $(A, A \cap \mathcal{F})$ where

$$A \cap \mathcal{F} = \{A \cap F \mid F \in \mathcal{F}\}.$$

- For $A \subseteq X$ we say that $\mathcal{F}$ shatters A if $A \cap \mathcal{F} = \mathcal{P}(A)$ (the power set of A).

Definition 1.3.2. We say $(X, \mathcal{F})$ has VC-dimension n if the largest subset of X shattered by $\mathcal{F}$ is of size n. If $\mathcal{F}$ shatters arbitrarily large subsets of X, we say that $(X, \mathcal{F})$ has infinite VC-dimension. We denote the VC-dimension of $(X, \mathcal{F})$ by $\mathrm{VC}(X, \mathcal{F})$.

Note 1.3.3. We may drop X from the notation $\mathrm{VC}(X, \mathcal{F})$, as the VC-dimension doesn't depend on the base set and is determined by $(\bigcup \mathcal{F}, \mathcal{F})$.

Set systems of finite VC-dimension tend to have good combinatorial properties, and we consider set systems with infinite VC-dimension to be poorly behaved.

Another natural combinatorial notion is that of the dual system:

Definition 1.3.4. For $a \in X$ define $X_a = \{F \in \mathcal{F} \mid a \in F\}$. Let $\mathcal{F}^* = \{X_a \mid a \in X\}$. We call $(\mathcal{F}, \mathcal{F}^*)$ the dual system of $(X, \mathcal{F})$. The VC-dimension of the dual system of $(X, \mathcal{F})$ is referred to as the dual VC-dimension of $(X, \mathcal{F})$ and denoted by $\mathrm{VC}^*(\mathcal{F})$. (As before, this notion doesn't depend on X.)

Lemma 1.3.5 (see 2.13b in [Ass83])**.** *A set system $(X, \mathcal{F})$ has finite VC-dimension if and only if its dual system has finite VC-dimension. More precisely*

$$\mathrm{VC}^*(\mathcal{F}) \leq 2^{1+\mathrm{VC}(\mathcal{F})}.$$

For a more refined notion of complexity of $(X, \mathcal{F})$ we look at the traces of our family on finite sets:

Definition 1.3.6. Define the shatter function $\pi_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ and the dual shatter function $\pi^*_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ by

$$\pi_{\mathcal{F}}(n) = \max\{|A \cap \mathcal{F}| \mid A \subseteq X \text{ and } |A| = n\},$$
$$\pi^*_{\mathcal{F}}(n) = \max\{\mathrm{atoms}(B) \mid B \subseteq \mathcal{F}, |B| = n\}$$

where $\mathrm{atoms}(B)$ is the number of atoms in the boolean algebra of sets generated by B. Note that the dual shatter function is precisely the shatter function of the dual system: $\pi^*_{\mathcal{F}} = \pi_{\mathcal{F}^*}$.

A simple upper bound is $\pi_{\mathcal{F}}(n) \leq 2^n$ (same for the dual). If the VC-dimension of $\mathcal{F}$ is infinite then clearly $\pi_{\mathcal{F}}(n) = 2^n$ for all n. Conversely we have the following remarkable fact:

Theorem 1.3.7 (see [Sau72], [She72])**.** *If the set system $(X, \mathcal{F})$ has finite VC-dimension d then $\pi_{\mathcal{F}}(n) \leq \binom{n}{\leq d}$ for all n, where $\binom{n}{\leq d} = \binom{n}{d} + \binom{n}{d-1} + \ldots + \binom{n}{1}$.*

Thus the systems with a finite VC-dimension are precisely the systems where the shatter function grows polynomially. The VC-density of $\mathcal{F}$ quantifies the growth of the shatter function of $\mathcal{F}$:

Definition 1.3.8. Define the VC-density and dual VC-density of $\mathcal{F}$ as

$$\mathrm{vc}(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\},$$
$$\mathrm{vc}^*(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}.$$

Generally speaking, a shatter function that is bounded by a polynomial doesn't itself have to be a polynomial. Proposition 4.12 in [ADH16] gives an example of a shatter function that grows like $n \log n$ (so it has VC-density 1).

So far the notions that we have defined are purely combinatorial. We now adapt VC-dimension and VC-density to the model theoretic context.

Definition 1.3.9. Work in a first-order structure $\mathbb{M}$. Fix a finite collection of formulas $\Phi(x, y)$ in the language $\mathcal{L}(M)$ of $\mathbb{M}$.

- For $\phi(x, y) \in \mathcal{L}(M)$ and $b \in M^{|y|}$ let

$$\phi(M^{|x|}, b) = \{a \in M^{|x|} \mid \phi(a, b)\} \subseteq M^{|x|}.$$

- Let $\Phi(M^{|x|}, M^{|y|}) = \{\phi(M^{|x|}, b) \mid \phi \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|})$.
- Let $\mathcal{F}_\Phi = \Phi(M^{|x|}, M^{|y|})$, giving rise to a set system $(M^{|x|}, \mathcal{F}_\Phi)$.
- Define the VC-dimension $\mathrm{VC}(\Phi)$ of Φ to be the VC-dimension of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.
- Define the VC-density $\mathrm{vc}(\Phi)$ of Φ to be the VC-density of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.

We will also refer to the VC-density and VC-dimension of a single formula ϕ by viewing it as a one element collection $\Phi = \{\phi\}$.

Counting atoms of a boolean algebra in a model theoretic setting corresponds to counting types, so it is instructive to rewrite the shatter function in terms of types.

Definition 1.3.10.

$$\pi^*_\Phi(n) = \max \{\text{number of } \Phi\text{-types over } B \mid B \subseteq M, |B| = n\}.$$

Here a Φ-type over B is a maximal consistent collection of formulas of the form $\phi(x, b)$ or $\neg\phi(x, b)$ where $\phi \in \Phi$ and $b \in B$.

The functions π^*_Φ and $\pi^*_{\mathcal{F}_\Phi}$ do not have to agree, as one fixes the number of generators of a boolean algebra of sets and the other fixes the size of the parameter set. However, as the following lemma demonstrates, they both give the same asymptotic definition of dual VC-density.

Lemma 1.3.11.

$$\mathrm{vc}^*(\Phi) = \textit{degree of polynomial growth of } \pi^*_\Phi(n) = \limsup_{n\to\infty} \frac{\log \pi^*_\Phi(n)}{\log n}.$$

Proof. With a parameter set B of size n, we get at most $|\Phi|n$ sets $\phi(M^{|x|}, b)$ with $\phi \in \Phi, b \in B$. We check that asymptotically it doesn't matter whether we look at growth of boolean algebra of sets generated by n or by $|\Phi|n$ many sets. We have:

$$\pi^*_{\mathcal{F}_\Phi}(n) \leq \pi^*_\Phi(n) \leq \pi^*_{\mathcal{F}_\Phi}(|\Phi|n).$$

Hence:

$$\begin{aligned}
&\mathrm{vc}^*(\Phi) \leq \limsup_{n\to\infty} \frac{\log \pi^*_\Phi(n)}{\log n} \leq \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log n} = \\
&= \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log |\Phi|n} \frac{\log |\Phi|n}{\log n} = \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log |\Phi|n} \leq \\
&\leq \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(n)}{\log n} = \mathrm{vc}^*(\Phi).
\end{aligned}$$

□

One can check that the shatter function and hence VC-dimension and VC-density of a formula are elementary notions, so they only depend on the first-order theory of the structure $\mathbb{M}$.

NIP theories are a natural context for studying VC-density. In fact, we can take the following as the definition of NIP:

Definition 1.3.12. Define ϕ to be NIP if it has finite VC-dimension in a theory T. A theory T is NIP if all the formulas in T are NIP.

In a general combinatorial context (for arbitrary set systems), VC-density can be any real number in $0 \cup [1, \infty)$ (see [Ass85]). Less is known if we restrict our attention to NIP theories.

Proposition 4.6 in [ADH16] gives examples of formulas that have non-integer rational VC-density in an NIP theory, however it is open whether one can get an irrational VC-density in this model-theoretic setting.

Instead of working with a theory formula by formula, we can look for a uniform bound for all formulas:

Definition 1.3.13. For a given NIP structure $\mathbb{M}$, define the VC-density function

$$\begin{aligned}\mathrm{vc}^{\mathbb{M}}(n) &= \sup\{\mathrm{vc}^*(\phi(x,y)) \mid \phi \in \mathcal{L}(M), |x| = n\} \\ &= \sup\{\mathrm{vc}(\phi(x,y)) \mid \phi \in \mathcal{L}(M), |y| = n\} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}\,.\end{aligned}$$

As before this definition is elementary, so it only depends on the theory of $\mathbb{M}$. We omit the superscript $\mathbb{M}$ if it is understood from the context. One can easily check the following bounds:

Lemma 1.3.14 (Lemma 3.22 in [ADH16])**.** *We have* $\mathrm{vc}(1) \geq 1$ *and* $\mathrm{vc}(n) \geq n\,\mathrm{vc}(1)$.

However, it is not known whether the second inequality can be strict or even just whether $\mathrm{vc}(1) < \infty$ implies $\mathrm{vc}(n) < \infty$.

1.4 Dp-rank and dp-minimality

Dp-rank is a popular dimension notion used in the study of NIP theories, and is used to define dp-minimality. Those notions originated in [She14], and were further studied in [KOU13], where it was shown, for example, that dp-rank is additive. Here it is easiest for us to introduce dp-rank in terms of VC-density over indiscernible sequences.

Definition 1.4.1.

- Work in an $\aleph_1$-saturated structure $\mathbb{M}$. Fix a finite collection of formulas $\Phi(x, y)$ in the language of $\mathbb{M}$. Suppose $A = (a_i)_{i\in\mathbb{N}}$ is an $\emptyset$-indiscernible sequence with each $a_i \in M^{|x|}$. Let

$$\Phi(A, M^{|y|}) = \{\phi(A, b) \mid \phi \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|}).$$

This gives a rise to a set system $(M^{|x|}, \Phi(A, M^{|y|}))$.

- Define

$$\mathrm{vc}_{\mathrm{ind}}(\Phi) = \sup\left\{\mathrm{vc}(\Phi(A, M^{|y|})) \mid A = (a_i)_{i\in\mathbb{N}} \text{ is } \emptyset\text{-indiscernible}\right\}.$$

- The dp-rank of an $\aleph_1$-saturated structure $\mathbb{M}$ is $\leq n$ if $\mathrm{vc}_{\mathrm{ind}}(\phi) \leq n$ for all formulas ϕ.

- The dp-rank of a theory T is $\leq n$ if dp-rank is $\leq n$ for any (each) $\aleph_1$-saturated model of T.

- A theory T is said to have a finite dp-rank if its dp-rank is $\leq n$ for some n.

- A theory T is dp-minimal if its dp-rank ≤ 1.

Refer to [GH14] for the connection between the classical definition of dp-rank and the definition given here.

There is a characterization of dp-minimality in terms of indiscernible sequences that will be useful for what we do:

Lemma 1.4.2 (see Lemma 1.4 in [Sim11]). *Suppose $\mathbb{M}$ is an $\aleph_1$-saturated structure. Then the following are equivalent:*

- $\mathbb{M}$ *is dp-minimal.*

- *For any countable $\emptyset$-indiscernible sequence $(a_i)_{i\in\mathcal{I}}$ indexed by a dense linear order $\mathcal{I}$, and any $c \in M$, there is i_0 in the completion of $\mathcal{I}$ such that the two sequences $(\mathrm{tp}(a_i/c) \mid i < i_0)$ and $(\mathrm{tp}(a_i/c) \mid i > i_0)$ are constant.*

CHAPTER 2

Trees

In this chapter we establish an optimal bound on the VC-density function in the theory of infinite trees. This generalizes a result of Simon showing that trees are dp-minimal.

We work with trees viewed as posets. Parigot in [Par82] showed that such structures have NIP. This result was strengthened by Simon in [Sim11] showing that trees are dp-minimal. The paper [ADH16] poses the following problem:

Problem 2.0.3. (see section 6.3 in [ADH16]) Determine the VC-density function of each infinite tree.

Here we solve this problem by showing that any infinite model of the theory of trees has $\mathrm{vc}(n) = n$ for each n.

In Section 1 we introduce proper subdivisions – the main tool that we use to analyze trees. We also prove some basic properties of proper subdivisions. Section 2 introduces the key constructions of proper subdivisions which will be used in the proof. Section 3 presents the proof of $\mathrm{vc}(n) = n$ via subdivisions. In the concluding section we state open questions and outline future work.

The language of trees consists of a single binary predicate symbol $\leq$. The theory of trees states that $\leq$ defines a partial order and that for every element a the set $\{x \mid x \leq a\}$ is linearly ordered by $\leq$. For visualization purposes we assume that trees grow upwards, with the smaller elements on the bottom and the larger elements on the top. If $a \leq b$ we will say that a is below b and b is above a.

Definition 2.0.4. Work in a tree $\boldsymbol{T} = (T, \leq)$. For $x \in T$ let $T^{\leq x} = \{t \in T \mid t \leq x\}$ denote the set of all elements of T below x. Two elements a, b are said to be in same <u>connected</u>

component if $T^{\leq a} \cap T^{\leq b}$ is non-empty. The meet of two elements a, b of T is the greatest element of $T^{\leq a} \cap T^{\leq b}$ (if one exists) and is denoted by $a \wedge b$.

The theory of meet trees requires that any two elements in the same connected component have a meet. Colored trees are trees with a finite number of colors added via unary predicates.

From now on assume that all trees are colored. We allow our trees to be disconnected (so really, we work with forests) or finite unless otherwise stated.

2.1 Proper Subdivisions: Definition and Properties

We work with finite relational languages. Given a formula we define its complexity as the depth of quantifiers used to build up the formula. More precisely:

Definition 2.1.1. Define the complexity of a formula by induction:

$$\text{Complexity}(\text{q.f. formula}) = 0$$

$$\text{Complexity}(\exists x \phi(x)) = \text{Complexity}(\phi(x)) + 1$$

$$\text{Complexity}(\phi \wedge \psi) = \max(\text{Complexity}(\phi), \text{Complexity}(\psi))$$

$$\text{Complexity}(\neg \phi) = \text{Complexity}(\phi)$$

A simple inductive argument verifies that there are (up to logical equivalence) only finitely many formulas when the complexity and free variables are fixed. We will use the following notation for types:

Definition 2.1.2. Let $\boldsymbol{B}$ be a structure, $A \subseteq B$ be a finite parameter set, a, b be tuples in $\boldsymbol{B}$, and m, n be natural numbers.

- $\text{tp}^n_{\boldsymbol{B}}(a/A)$ will stand for the set of all A-formulas of complexity $\leq n$ that are true of a in $\boldsymbol{B}$. If $A = \emptyset$ we may also write this as $\text{tp}^n_{\boldsymbol{B}}(a)$. The subscript $\boldsymbol{B}$ will be omitted as well if it is clear from context. Note that if A is finite, then there are finitely many such formulas (up to equivalence and renaming free variables). The conjunction of finitely many formulas of complexity $\leq n$ still has complexity $\leq n$ and so we can just associate a single formula with complexity $\leq n$ to every type $\mathrm{tp}^n_{\boldsymbol{B}}(a/A)$ defining the set of realizations of $\mathrm{tp}^n_{\boldsymbol{B}}(a/A)$ in $\boldsymbol{B}$.

- $\boldsymbol{B} \models a \equiv^n_A b$ means that a, b have the same type with complexity $\leq n$ over A in $\boldsymbol{B}$, i.e., $\mathrm{tp}^n_{\boldsymbol{B}}(a/A) = \mathrm{tp}^n_{\boldsymbol{B}}(b/A)$.

- $S^n_{\boldsymbol{B},m}(A)$ will stand for the set of all m-types of complexity $\leq n$ over A:

$$S^n_{\boldsymbol{B},m}(A) = \{\mathrm{tp}^n_{\boldsymbol{B}}(a/A) \mid a \in B^m\}\,.$$

Definition 2.1.3.

- Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{T}$ be structures in some (possibly different) finite relational languages. If the underlying sets A, B of $\boldsymbol{A}, \boldsymbol{B}$ partition the underlying set T of $\boldsymbol{T}$ (i.e. $T = A \sqcup B$), then we say that $(\boldsymbol{A}, \boldsymbol{B})$ is a subdivision of $\boldsymbol{T}$.

- A subdivision $(\boldsymbol{A}, \boldsymbol{B})$ of $\boldsymbol{T}$ is called n-proper if given $p, q \in \mathbb{N}$, $a_1, a_2 \in A^p$ and $b_1, b_2 \in B^q$ with

$$\boldsymbol{A} \models a_1 \equiv_n a_2$$
$$\boldsymbol{B} \models b_1 \equiv_n b_2$$

we have

$$\boldsymbol{T} \models a_1 b_1 \equiv_n a_2 b_2.$$

- A subdivision $(\boldsymbol{A}, \boldsymbol{B})$ of $\boldsymbol{T}$ is called proper if it is n-proper for all $n \in \mathbb{N}$.

Lemma 2.1.4. *Consider a subdivision $(\boldsymbol{A}, \boldsymbol{B})$ of $\boldsymbol{T}$. If $(\boldsymbol{A}, \boldsymbol{B})$ is 0-proper then it is proper.*

Proof. We prove that the subdivision $(\boldsymbol{A}, \boldsymbol{B})$ is n-proper for all n by induction. The case $n = 0$ is given by the assumption. Suppose we have $\boldsymbol{T} \models \exists x\, \phi^n(x, a_1, b_1)$ where ϕ^n is some formula of complexity n. Let $a \in T$ witness the existential claim, i.e., $\boldsymbol{T} \models \phi^n(a, a_1, b_1)$. We can have $a \in A$ or $a \in B$. Without loss of generality assume $a \in A$. Let $\boldsymbol{p} = \mathrm{tp}^n_{\boldsymbol{A}}(a, a_1)$. Then we have

$$\boldsymbol{A} \models \exists x\ \mathrm{tp}^n_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$$

(with $\mathrm{tp}^n_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$ a shorthand for $\phi_{\boldsymbol{p}}(x)$ where $\phi_{\boldsymbol{p}}$ is a formula that determines the type $\boldsymbol{p}$). The formula $\mathrm{tp}^n_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$ is of complexity $\leq n$ so $\exists x \ \mathrm{tp}^n_{\boldsymbol{A}}(x, a_1) = \boldsymbol{p}$ is of complexity $\leq n + 1$. By the inductive hypothesis we have

$$\boldsymbol{A} \models \exists x \ \mathrm{tp}^n_{\boldsymbol{A}}(x, a_2) = \boldsymbol{p}\,.$$

Let a' witness this existential claim, so that $\mathrm{tp}^n_{\boldsymbol{A}}(a', a_2) = \boldsymbol{p}$, hence $\mathrm{tp}^n_{\boldsymbol{A}}(a', a_2) = \mathrm{tp}^n_{\boldsymbol{A}}(a, a_1)$, that is, $\boldsymbol{A} \models a'a_2 \equiv_n aa_1$. By the inductive hypothesis we therefore have $\boldsymbol{T} \models aa_1b_1 \equiv_n a'a_2b_2$; in particular $\boldsymbol{T} \models \phi^n(a', a_2, b_2)$ as $\boldsymbol{T} \models \phi^n(a, a_1, b_1)$, and $\boldsymbol{T} \models \exists x \phi^n(x, a_2, b_2)$. $\square$

This lemma is general, but we will use it specifically applied to (colored) trees. Suppose $\boldsymbol{T}$ is a (colored) tree in some language $\mathcal{L} = \{\leq, \ldots\}$ expanding the language of trees by finitely many predicate symbols. Suppose $\boldsymbol{A}, \boldsymbol{B}$ are some structures in languages $\mathcal{L}_A, \mathcal{L}_B$ which expand $\mathcal{L}$, with the $\mathcal{L}$-reducts of $\boldsymbol{A}, \boldsymbol{B}$ substructures of $\boldsymbol{T}$. Furthermore suppose that $(\boldsymbol{A}, \boldsymbol{B})$ is 0-proper. Then by the previous lemma $(\boldsymbol{A}, \boldsymbol{B})$ is a proper subdivision of $\boldsymbol{T}$. From now on all the subdivisions we work with will be of this form.

Example 2.1.5. Suppose a tree consists of two connected components C_1, C_2. Then those components $(C_1, \leq)$, $(C_2, \leq)$ viewed as substructures form a proper subdivision. To see this we only need to show that this subdivision is 0-proper. But that is immediate as any $c_1 \in C_1$ and $c_2 \in C_2$ are incomparable.

Example 2.1.6. Fix a tree $\boldsymbol{T}$ in the language $\{\leq\}$, and let $a \in T$. Let $B = \{t \in T \mid a < t\}$, $S = \{t \in T \mid t \leq a\}$, $A = T - B$. Then $(A, \leq, S)$ and $(B, \leq)$ form a proper subdivision, where $\mathcal{L}_A$ has a unary predicate interpreted by S. To see this, again, we show that the subdivision is 0-proper. The only time $a \in A$ and $b \in B$ are comparable is when $a \in S$, and this is captured by the language. (See proof of Lemma 2.2.7 for more details.)

Definition 2.1.7. For $\phi(x, y)$, $A \subseteq T^{|x|}$ and $B \subseteq T^{|y|}$

- let $\phi(A, b) = \{a \in A \mid \phi(a, b)\} \subseteq A$, and
- let $\phi(A, B) = \{\phi(A, b) \mid b \in B\} \subseteq \mathcal{P}(A)$.

Thus $\phi(A, B)$ is a collection of subsets of A definable by ϕ with parameters from B. We notice the following bound when A, B are parts of a proper subdivision.

Corollary 2.1.8. *Let $\boldsymbol{A}, \boldsymbol{B}$ be a proper subdivision of $\boldsymbol{T}$ and $\phi(x, y)$ be a formula of complexity n. Then $|\phi(A^{|x|}, B^{|y|})| \leq |S^n_{\boldsymbol{B},|y|}|$.*

Proof. Take some $a \in A^{|x|}$ and $b_1, b_2 \in B^{|y|}$ with $\mathrm{tp}^n_{\boldsymbol{B}}(b_1) = \mathrm{tp}^n_{\boldsymbol{B}}(b_2)$. We have $\boldsymbol{B} \models b_1 \equiv_n b_2$ and (trivially) $\boldsymbol{A} \models a \equiv_n a$. Thus we have $\boldsymbol{T} \models ab_1 \equiv_n ab_2$, so $\boldsymbol{T} \models \phi(a, b_1) \leftrightarrow \phi(a, b_2)$. Since a was arbitrary we have $\phi(A^{|x|}, b_1) = \phi(A^{|x|}, b_2)$ as different traces can only come from parameters of different types. Thus $|\phi(A^{|x|}, B^{|y|}) \leq |S^n_{\boldsymbol{B},|y|}|$. ☐

We note that the size of the type space $|S^n_{\boldsymbol{B},|y|}|$ can be bounded uniformly:

Definition 2.1.9. Fix a (finite relational) language $\mathcal{L}_B$. Let $N = N(n, m, \mathcal{L}_B)$ be smallest integer such that for any structure $\boldsymbol{B}$ in $\mathcal{L}_B$ we have $|S^n_{\boldsymbol{B},m}| \leq N$. This integer exists as there is a finite number (up to logical equivalence) of possible formulas of complexity $\leq n$ with free variables $x_1, x_2, \ldots, x_m$. Note that $N(n, m, \mathcal{L}_B)$ is increasing in all parameters:

$$n \leq n', m \leq m', \mathcal{L}_B \subseteq \mathcal{L}'_B \Rightarrow N(n, m, \mathcal{L}_B) \leq N(n', m', \mathcal{L}'_B).$$

2.2 Proper Subdivisions: Constructions

Throughout this section, $\boldsymbol{T} = (T, \leq, C_1, \ldots, C_n)$ denotes a colored meet tree. First, we describe several constructions of proper subdivisions that are needed for the proof of our main theorem.

Definition 2.2.1. We use $E(b, c)$ to express that b and c are in the same connected component of T:

$$E(b, c) \Leftrightarrow \exists x\, (b \geq x) \wedge (c \geq x).$$

Definition 2.2.2. Given an element a of the tree $\boldsymbol{T}$ we call the set of all elements strictly above a, i.e., the set $T^{>a} = \{x \mid x > a\}$, the <u>open cone</u> above a. Connected components of that cone can be thought of as closed cones above a. With that interpretation in mind, the notation $E_a(b,c)$ means that b and c are in the same closed cone above a. More formally:

$$E_a(b,c) \Leftrightarrow E(b,c) \text{ and } (b \wedge c) > a.$$

Fix a language for our colored tree $\mathcal{L} = \{\leq, C_1, \ldots C_n\}$. Put $\vec{C} = \{C_1, \ldots C_n\}$. Given $A \subseteq T$ we put $\vec{C} \cap A = \{C_1 \cap A, \ldots, C_n \cap A\}$. In that case $(A, \leq, \vec{C} \cap A)$ is a substructure of $\boldsymbol{T}$ (here as usual by abuse of notation $\leq$ denotes the restriction of the ordering $\leq$ of T to A.) For the following four definitions fix an expansion $\mathcal{L}^* = \mathcal{L} \cup \{U\}$ of the language $\mathcal{L}$ by a unary predicate symbol U. We refer to the figures below for illustrations of these definitions.

Definition 2.2.3. Fix $c_1 < c_2$ in T. Let

$$\begin{aligned} B &= \{b \in T \mid E_{c_1}(c_2, b) \wedge \neg(b \geq c_2)\}, \\ A &= T - B, \\ T^{<c_1} &= \{t \in T \mid t < c_1\}, \\ T^{<c_2} &= \{t \in T \mid t < c_2\}, \\ S_B &= T^{<c_2} - T^{<c_1}, \\ T^{\geq c_2} &= \{t \in T \mid c_2 \leq t\}. \end{aligned}$$

Define structures $\boldsymbol{A}^{c_1}_{c_2} = (A, \leq, \vec{C} \cap A, T^{<c_1}, T^{\geq c_2})$ in a language expanding $\mathcal{L}$ by two unary predicate symbols and $\boldsymbol{B}^{c_1}_{c_2} = (B, \leq, \vec{C} \cap B, S_B)$ in the language $\mathcal{L}^*$ (as defined above). Note that $c_1, c_2 \notin B$.

Definition 2.2.4. Fix c in T. Let

$$\begin{aligned} B &= \{b \in T \mid \neg(b \geq c) \wedge E(b,c)\}, \\ A &= T - B, \\ T^{<c_1} &= \{t \in T \mid t < c\}. \end{aligned}$$

Define structures $\boldsymbol{A}_c = (A, \leq, \vec{C} \cap A)$ in the language $\mathcal{L}$ and $\boldsymbol{B}_c = (B, \leq, \vec{C} \cap B, T^{<c_1})$ in the language $\mathcal{L}^*$. Note that $c \notin B$. (cf. example 2.1.6).

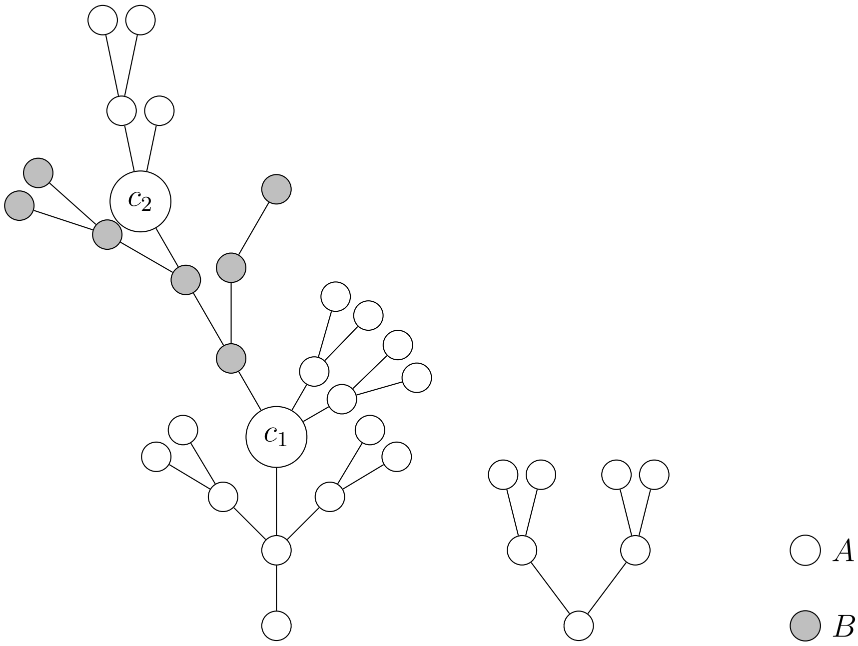

Figure 2.1: Proper subdivision $(\boldsymbol{A}, \boldsymbol{B}) = (\boldsymbol{A}_{c_2}^{c_1}, \boldsymbol{B}_{c_2}^{c_1})$

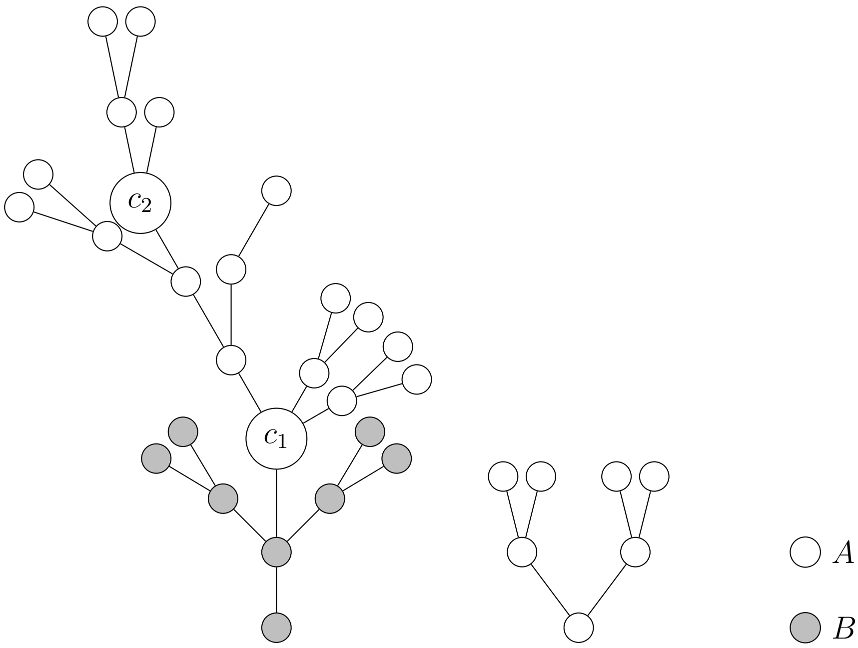

Figure 2.2: Proper subdivision $(\boldsymbol{A}, \boldsymbol{B}) = (\boldsymbol{A}_{c_1}, \boldsymbol{B}_{c_1})$

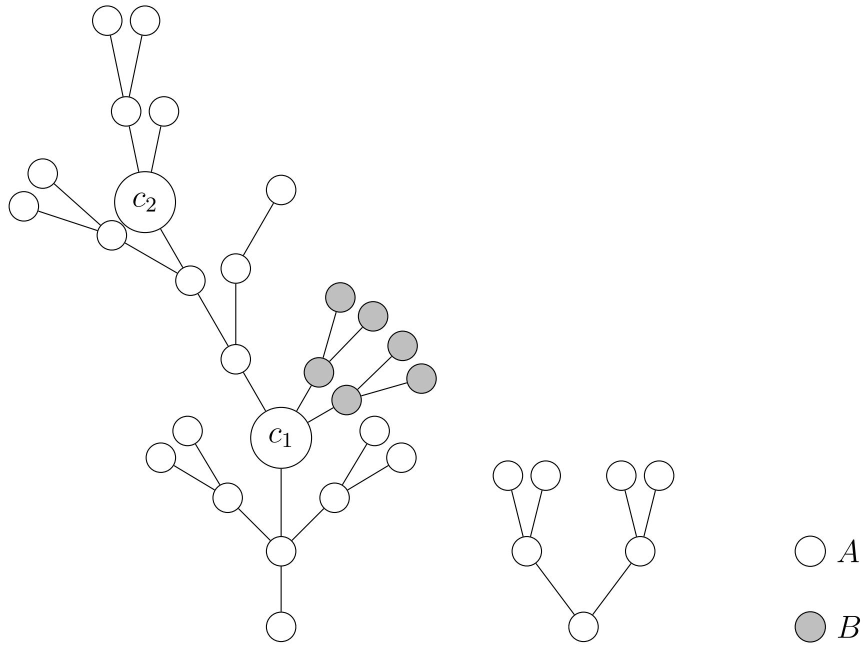

Figure 2.3: Proper subdivision $(\boldsymbol{A}, \boldsymbol{B}) = (\boldsymbol{A}_S^{c_1}, \boldsymbol{B}_S^{c_1})$ for $S = \{c_2\}$

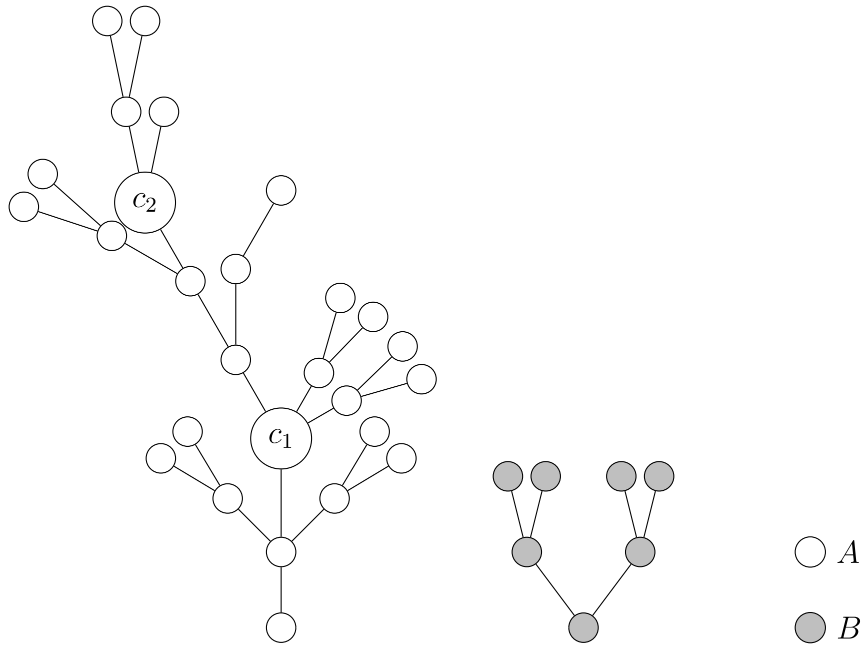

Figure 2.4: Proper subdivision $(\boldsymbol{A}, \boldsymbol{B}) = (\boldsymbol{A}_S, \boldsymbol{B}_S)$ for $S = \{c_1, c_2\}$

Definition 2.2.5. Fix c in T and a finite subset $S \subseteq T$. Let

$$\begin{aligned} B &= \{b \in T \mid (b > c) \text{ and for all } s \in S \text{ we have } \neg E_c(s,b)\}, \\ A &= T - B, \\ T^{\leq c_1} &= \{t \in T \mid t \leq c\}. \end{aligned}$$

Define structures $\boldsymbol{A}_S^c = (A, \leq, \vec{C} \cap A, T^{\leq c_1})$ and $\boldsymbol{B}_S^c = (B, \leq, \vec{C} \cap B, B)$ both in the language $\mathcal{L}^*$. Note that $c \notin B$ and $S \cap B = \emptyset$.

Definition 2.2.6. Fix a finite subset $S \subseteq T$. Let

$$\begin{aligned} B &= \{b \in T \mid \text{ for all } s \in S \text{ we have } \neg E(s,b)\}, \\ A &= T - B. \end{aligned}$$

Define structures $\boldsymbol{A}_S = (A, \leq, \vec{C} \cap A)$ in the language $\mathcal{L}$ and $\boldsymbol{B}_S = (B, \leq, \vec{C} \cap B, B)$ in the language $\mathcal{L}^*$. (cf. example 2.1.5)

Note that we forced the structures $\boldsymbol{B}_{c_2}^{c_1}, \boldsymbol{B}_c, \boldsymbol{B}_S^c, \boldsymbol{B}_S$ to have the same language $\mathcal{L}^*$. This is done for uniformity to simplify Lemma 2.3.1. By comparison, the corresponding structures denoted by $\boldsymbol{A}$ with decorations have different languages.

Lemma 2.2.7. *The pairs of structures defined above are all proper subdivisions of $\boldsymbol{T}$.*

Proof. We only show this holds for the pair $(\boldsymbol{A}, \boldsymbol{B}) = (\boldsymbol{A}_{c_2}^{c_1}, \boldsymbol{B}_{c_2}^{c_1})$. The other cases follow by a similar argument. The sets A, B partition T by definition, so (A, B) is a subdivision of $\boldsymbol{T}$. To show that it is proper, by Lemma 2.1.4 we only need to check that it is 0-proper. Suppose we have

$$\begin{aligned} a &= (a_1, a_2, \ldots, a_p) \in A^p, \\ a' &= (a'_1, a'_2, \ldots, a'_p) \in A^p, \\ b &= (b_1, b_2, \ldots, b_q) \in B^q, \\ b' &= (b'_1, b'_2, \ldots, b'_q) \in B^q, \end{aligned}$$

with $\boldsymbol{A} \models a \equiv_0 a'$ and $\boldsymbol{B} \models b \equiv_0 b'$. We need to show that ab has the same quantifier-free type in $\boldsymbol{T}$ as $a'b'$. Any two elements in T can be related in the four following ways:

$$\begin{aligned}
&x = y,\\
&x < y,\\
&x > y, \text{ or}\\
&x, y \text{ are incomparable.}
\end{aligned}$$

We need to check that for all i, j the same relations hold for (a_i, b_j) as do for (a'_i, b'_j).

- It is impossible that $a_i = b_j$ as they come from disjoint sets.
- Suppose $a_i < b_j$. This forces $a_i \in T^{<c_1}$ thus $a'_i \in T^{<c_1}$ and $a'_i < b'_j$.
- Suppose $a_i > b_j$. This forces $b_j \in S_B$ and $a \in T^{\geq c_2}$, thus $b'_j \in S_B$ and $a'_i \in T^{\geq c_2}$, so $a'_i > b'_j$.
- Suppose a_i and b_j are incomparable. Two cases are possible:
  - $b_j \notin S_B$ and $a_i \in T^{\geq c_2}$. Then $b'_j \notin S_B$ and $a'_i \in T^{\geq c_2}$ making a'_i, b'_j incomparable.
  - $b_j \in S_B$, $a_i \notin T^{\geq c_2}$, $a_i \notin T^{<c_1}$. Similarly this forces a'_i, b'_j to be incomparable.

Also we need to check that ab has the same colors as $a'b'$. But that is immediate as having the same color in a substructure means having the same color in the tree. $\square$

2.3 Main proof

The basic idea for the proof is as follows. Suppose we have a formula with q parameters over a parameter set of size n. We are able to split our parameter space into $O(n)$ many sets by designating subdivisions described in the previous section based on the parameter set (see figures in the previous section for an example of a partition of the tree into four sets using parameter set $\{c_1, c_2\}$). Each of the q parameters can come from any of those $O(n)$ components giving us $O(n)^q$ many choices for parameter configuration. When every parameter is coming from a fixed component, the number of definable sets is constant and in fact is uniformly bounded above by some N (independent of the parameter set). This gives us at most $N \cdot O(n)^q$ possibilities for different definable sets.

First, we generalize Corollary 2.1.8. (This is required for computing VC-density of formulas $\phi(x, y)$ with $|y| > 1$).

Lemma 2.3.1. *Consider a finite collection $(\boldsymbol{A}_i, \boldsymbol{B}_i)_{i \leq n}$ satisfying the following properties:*

- *$(\boldsymbol{A}_i, \boldsymbol{B}_i)$ is either a proper subdivision of $\boldsymbol{T}$ or $A_i = T$ and $B_i = \{b_i\}$,*
- *all $\boldsymbol{B}_i$ have the same language $\boldsymbol{\mathcal{L}}^*$, and*
- *the sets $\{B_i\}_{i \leq n}$ are pairwise disjoint.*

Let $A = \bigcap_{i \in I} A_i$. Fix a formula $\phi(x, y)$ of complexity m . Let $N = N(m, |y|, \boldsymbol{\mathcal{L}}^*)$ be as in Definition 2.1.9. Let $B = B_1^{i_1} \times B_2^{i_2} \times \ldots \times B_n^{i_n} \subseteq T^{|y|}$ where $i_1 + i_2 + \ldots + i_n = |y|$ (some of the indices can be zero). Then we have the following bound:

$$\phi(A^{|x|}, B) \leq N^{|y|}.$$

Proof. We show this result by counting types.

Claim 2.3.2. *Suppose we have*

$$\begin{aligned} &b_1, b_1' \in B_1^{i_1} \text{ with } b_1 \equiv_m b_1' \text{ in } \boldsymbol{B}_1, \\ &b_2, b_2' \in B_2^{i_2} \text{ with } b_2 \equiv_m b_2' \text{ in } \boldsymbol{B}_2, \\ &\quad \ldots \\ &b_n, b_n' \in B_n^{i_n} \text{ with } b_n \equiv_m b_n' \text{ in } \boldsymbol{B}_n. \end{aligned}$$

Then

$$\phi(A^{|x|}, b_1, b_2, \ldots b_n) \iff \phi(A^{|x|}, b_1', b_2', \ldots b_n').$$

Proof. Define $\bar{b}_i = (b_1, \ldots, b_i, b_{i+1}', \ldots, b_n') \in B$ for $i = 0, \ldots, n$. We have

$$\phi(A^{|x|}, \bar{b}_i) \iff \phi(A^{|x|}, \bar{b}_{i+1}),$$

as either $(\boldsymbol{A}_{i+1}, \boldsymbol{B}_{i+1})$ is m-proper or $\boldsymbol{B}_{i+1}$ is a singleton, and the implication is trivial. (Notice that $b_i \in \boldsymbol{A}_j$ for $j \neq i$ by disjointness assumption.) Thus, by induction we get $\phi(A^{|x|}, \bar{b}_0) \iff \phi(A^{|x|}, \bar{b}_n)$ as needed. □

Thus $\phi(A^{|x|}, B)$ only depends on the choice of the types for the tuples:

$$|\phi(A^{|x|}, B)| \leq |S^m_{\boldsymbol{B}_1, i_1}| \cdot |S^m_{\boldsymbol{B}_2, i_2}| \cdot \ldots \cdot |S^m_{\boldsymbol{B}_n, i_n}|$$

Now for each type space we have an inequality

$$|S^m_{\boldsymbol{B}_j, i_j}| \leq N(m, i_j, \mathcal{L}^*) \leq N(m, |y|, \mathcal{L}^*) \leq N.$$

(For singletons $|S^m_{\boldsymbol{B}_j, i_j}| = 1 \leq N$). Only non-zero indices contribute to the product and there are at most $|y|$ of those (by the equality $i_1 + i_2 + \ldots + i_n = |y|$). Thus we have

$$|\phi(A^{|x|}, B)| \leq N^{|y|}$$

as needed. □

For subdivisions to work out properly, we will need to pass to subsets closed under meets. We observe that the closure under meets doesn't add too many new elements:

Lemma 2.3.3. *Suppose $S \subseteq T$ is a finite subset of size $n \geq 1$ in a meet tree and S' is its closure under meets. Then $|S'| \leq 2n - 1$.*

Proof. We can partition S into connected components and prove the result separately for each component. Thus we may assume all elements of S lie in the same connected component. We prove the claim by induction on n. The base case $n = 1$ is clear. Suppose we have S of size k with closure of size at most $2k - 1$. Take a new point s, and look at its meets with all the elements of S. Those are linearly ordered, so we can pick the smallest one, s'. Then $S \cup \{s, s'\}$ is closed under meets. □

Putting all of those results together we are able to compute the VC-density of formulas in meet trees:

Theorem 2.3.4. *Let $\boldsymbol{T}$ be an infinite (colored) meet tree and $\phi(x, y)$ a formula with $|x| = p$ and $|y| = q$. Then* $\mathrm{vc}(\phi) \leq q$.

Proof. Pick a finite subset of $S_0 \subseteq T^p$ of size n. Let $S_1 \subseteq T$ consist of the components of the elements of S_0. Let $S \subseteq T$ be the closure of S_1 under meets. Using Lemma 2.3.3 we have $|S| \leq 2|S_1| \leq 2p|S_0| = 2pn = O(n)$. We have $S_0 \subseteq S^p$, so $|\phi(S_0, T^q)| \leq |\phi(S^p, T^q)|$. Thus it is enough to show $|\phi(S^p, T^q)| = O(n^q)$.

Label $S = \{c_i\}_{i \in I}$ with $|I| \leq 2pn$. For every c_i we construct two subdivisions in the following way. We have that c_i is either minimal in S or it has a predecessor in S (greatest element less than c). If it is minimal, construct $(\boldsymbol{A}_{c_i}, \boldsymbol{B}_{c_i})$. If there is a predecessor p, construct $(\boldsymbol{A}^p_{c_i}, \boldsymbol{B}^p_{c_i})$. For the second subdivision let G be all the elements in S greater than c_i and construct $(\boldsymbol{A}^c_G, \boldsymbol{B}^c_G)$. So far we have constructed two subdivisions for every $i \in I$. Additionally construct $(\boldsymbol{A}_S, \boldsymbol{B}_S)$. We end up with a finite collection of proper subdivisions $(\boldsymbol{A}_j, \boldsymbol{B}_j)_{j \in J}$ with $|J| = 2|I| + 1$. Before we proceed, we note the following two lemmas describing our subdivisions.

Lemma 2.3.5. *For all $j \in J$ we have $S \subseteq A_j$. Thus $S \subseteq \bigcap_{j \in J} A_j$ and $S^p \subseteq \bigcap_{j \in J} (A_j)^p$.*

Proof. Check this for each possible choice of subdivision. Cases for subdivisions of the type $\boldsymbol{A}_S, \boldsymbol{A}^c_G, \boldsymbol{A}_c$ are easy. Suppose we have a subdivision $(\boldsymbol{A}, \boldsymbol{B}) = (\boldsymbol{A}^{c_1}_{c_2}, \boldsymbol{B}^{c_1}_{c_2})$. We need to show that $B \cap S = \emptyset$. By construction we have $c_1, c_2 \notin B$. Suppose we have some other $c \in S$ with $c \in B$. We have $E_{c_1}(c_2, c)$, i.e., there is some b such that $b > c_1$, $b \leq c_2$ and $b \leq c$. Consider the meet $c \wedge c_2$. We have $c \wedge c_2 \geq b > c_1$. Also as $\neg(c \geq c_2)$ we have $c \wedge c_2 < c_2$. To summarize: $c_2 > c \wedge c_2 > c_1$. But this contradicts our construction as S is closed under meets, so $c \wedge c_2 \in S$ and c_1 is supposed to be a predecessor of c_2 in S. $\square$

Lemma 2.3.6. *$\{B_j\}_{j \in J}$ is a partition of $T - S$, i.e., $T = \bigsqcup_{j \in J} B_j \sqcup S$.*

Proof. This more or less follows from the choice of subdivisions. Pick any $b \in T - S$. Let a be the minimal element of S with $a > b$, and let c be the maximal element of S with $c < b$ (if such elements exist). Also let G be the set of elements of S incomparable to b. If both

a and c exist we have $b \in \boldsymbol{B}_c^a$. If only the upper bound exists we have $b \in \boldsymbol{B}_G^a$. If only the lower bound exists we have $b \in \boldsymbol{B}_c$. If neither exists we have $b \in \boldsymbol{B}_G$. □

Note that those two lemmas imply $S = \bigcap_{j \in J} A_j$.

For the one-dimensional case $q = 1$ we don't need to do any more work. We have partitioned the parameter space into $|J| = O(n)$ many pieces and over each piece the number of definable sets is uniformly bounded. By Corollary 2.1.8 we have that $|\phi((A_j)^p, B_j)| \leq N$ for any $j \in J$ (letting $N = N(n_\phi, q, \mathcal{L} \cup \{S\})$ where n_ϕ is the complexity of ϕ and S is a unary predicate). Compute

$$
\begin{aligned}
|\phi(S^p, T)| &= \left| \bigcup_{j \in J} \phi(S^p, B_j) \cup \phi(S^p, S) \right| \leq \\
&\leq \sum_{j \in J} |\phi(S^p, B_j)| + |\phi(S^p, S)| \leq \\
&\leq \sum_{j \in J} |\phi((A_j)^p, B_j)| + |S| \leq \\
&\leq \sum_{j \in J} N + |I| \leq \\
&\leq (4pn + 1)N + 2pn = (4pN + 2p)n + N = O(n).
\end{aligned}
$$

The basic idea for the general case $q \geq 1$ is that we have q parameters and $|J| = O(n)$ many components to pick each parameter from, giving us $|J|^q = O(n^q)$ choices for the parameter configuration, each giving a uniformly constant number of definable subsets of S. (If every parameter is picked from a fixed component, Lemma 2.3.1 provides a uniform bound). This yields $\mathrm{vc}(\phi) \leq q$ as needed. The rest of the proof is stating this idea formally.

First, we extend our collection of subdivisions $(\boldsymbol{A}_j, \boldsymbol{B}_j)_{j \in J}$ by the following singleton sets. For each $c_i \in S$ let $A_i = T, B_i = \{c_i\}$ and add $(\boldsymbol{A}_i, \boldsymbol{B}_i)$ to our collection with U in the language $\mathcal{L}^*$ of B_i interpreted arbitrarily. We end up with a new collection $(\boldsymbol{A}_k, \boldsymbol{B}_k)_{k \in K}$ indexed by some set K with $|K| = |I| + |J|$ (we added $|S|$ many new pairs). Now $\{B_k\}_{k \in K}$ partitions T, so $T = \bigsqcup_{k \in K} B_k$ and $S = \bigcap_{j \in J} A_j = \bigcap_{k \in K} A_k$. For $(k_1, k_2, \ldots k_q) = \vec{k} \in K^q$

denote

$$B_{\vec{k}} = B_{k_1} \times B_{k_2} \times \ldots \times B_{k_q}.$$

Then we have the following identity

$$T^q = (\bigsqcup_{k \in K} B_k)^q = \bigsqcup_{\vec{k} \in K^q} B_{\vec{k}}.$$

Thus we have that $\{B_{\vec{k}}\}_{\vec{k} \in K^q}$ partition T^q. Compute

$$\begin{aligned} |\phi(S^p, T^q)| &= \left| \bigcup_{\vec{k} \in K^q} \phi(S^p, B_{\vec{k}}) \right| \leq \\ &\leq \sum_{\vec{k} \in K^q} |\phi(S^p, B_{\vec{k}})|. \end{aligned}$$

We can bound $|\phi(S^p, B_{\vec{k}})|$ uniformly using Lemma 2.3.1. The family $(\boldsymbol{A}_k, \boldsymbol{B}_k)_{k \in K}$ satisfies the requirements of the lemma and $B_{\vec{k}}$ looks like B in the lemma after possibly permuting some variables in ϕ. (For example we would need to permute $B_{(2,1,2)} = B_2 \times B_1 \times B_2$ into $B_1 \times B_2^2$ so it has the appropriate form for Lemma 2.3.1.) Applying the lemma we get

$$|\phi(S^p, B_{\vec{k}})| \leq N^q$$

with N only depending on q and the complexity of ϕ. We complete our computation:

$$\begin{aligned} |\phi(S^p, T^q)| &\leq \sum_{\vec{k} \in K^q} |\phi(S^p, B_{\vec{k}})| \leq \sum_{\vec{k} \in K^q} N^q \leq \\ &\leq |K^q| N^q \leq (|J| + |I|)^q N^q \leq \\ &\leq (4pn + 1 + 2pn)^q N^q = N^q (6p + 1/n)^q n^q = O(n^q). \end{aligned}$$

□

Corollary 2.3.7. *In the theory of infinite (colored) meet trees we have $vc(n) = n$ for all n.*

We get the general result for trees that aren't necessarily meet trees via an easy application of interpretability.

Corollary 2.3.8. *In the theory of infinite (colored) trees we have $vc(n) = n$ for all n.*

Proof. Let $\boldsymbol{T}'$ be a tree. We can embed it in a larger tree $\boldsymbol{T}$ that is closed under meets. Expand $\boldsymbol{T}$ by an extra color and interpret it by coloring the subset $\boldsymbol{T}'$. Thus we can interpret $\boldsymbol{T}'$ in T. By Corollary 3.17 in [ADH16] we get that $\mathrm{vc}^{\boldsymbol{T}'}(n) \leq \mathrm{vc}^{T}(1 \cdot n) = n$ thus $\mathrm{vc}^{\boldsymbol{T}'}(n) = n$ as well. □

2.4 Conclusion

This settles the question of determining VC-density function for infinite trees. Lacking a quantifier elimination result in a natural language, a lot is still not known. One can try to adapt these techniques to compute the VC-density on the formula by formula basis:

Open Question 2.4.1. *What is the VC-density of individual formulas in infinite trees? Can it take non-integer values?*

It is also not known whether trees have the VC 1 property (see Definition 5.2 in [ADH16]; this is a quantified formulation of uniform definability of types over finite sets). It seems that our techniques can be used to show that the VC 2 property holds but this doesn't give the optimal VC-density function.

Open Question 2.4.2. *For which n do infinite trees have the VC n property?*

One can also try to apply similar techniques to more general classes of partially ordered sets. For example, the VC-density function is not known for lattices, and it is also not known whether lattices are dp-minimal. Similarly, relaxing the ordering condition, one can look at nicely behaved families of graphs, such as planar graphs or flat graphs (see [PZ78]). Those are known to be dp-minimal (see Theorem 5.3.4), so one would expect a simple VC-density function. It is this author's hope that the techniques developed in this chapter can be adapted to yield fruitful results for such more general classes of structures.

CHAPTER 3

Shelah-Spencer Graphs

In this chapter we investigate VC-density of definable sets in Shelah-Spencer graphs. We provide an upper bound on a formula-by-formula basis and show that there isn't a uniform lower bound, forcing the VC-density function to be infinite. In addition we show that Shelah-Spencer graphs do not have a finite dp-rank, thus, in particular, they are not dp-minimal.

A Shelah-Spencer graph is a limit of random structures $G(n, n^{-\alpha})$ for an irrational $\alpha \in (0, 1)$. Here $G(n, n^{-\alpha})$ is a random graph on n vertices with edge probability $n^{-\alpha}$. (The model theory of $G(n, n^{-\alpha})$ as $n \to \infty$ is much less pleasant if $\alpha \in (0, 1)$ is rational, see [BL12].) In [SS88] Shelah and Spencer showed that such structures have a 0-1 law, thus obtaining a complete (first-order) theory of Shelah-Spencer graphs. These structures are of a general combinatorial interest as well. For example, [ABC95] computes the VC-dimension of neighborhood sets in finite Shelah-Spencer graphs. Our treatment of Shelah-Spencer graphs closely follows the one in [Las07].

Our first result is that Shelah-Spencer graphs have $\mathrm{vc}(n) = \infty$ for each n. Our second result gives an upper bound on the VC-density of a given formula $\phi(x, y)$:

$$\mathrm{vc}(\phi) \leq \left\lfloor |y| \frac{K(\Phi)}{\epsilon(\Phi)} \right\rfloor$$

where $K(\phi), \epsilon(\phi)$ are explicitly computable expressions involving the number of vertices and edges defined by ϕ. For example, let $\phi(x, y)$ be a formula that says that there is an edge between x and y. Our bound gives $\mathrm{vc}(\phi) \leq \left\lfloor \frac{2}{\alpha} \right\rfloor$. With a more careful computation one can get the exact value $\mathrm{vc}(\phi) = \left\lfloor \frac{1}{\alpha} \right\rfloor$ (see 4.9 in [ADH16]).

Section 1 summarizes notation and basic facts concerning Shelah-Spencer graphs. We direct the reader to [Las07] for a more in-depth treatment. In Section 2 we introduce key lemmas that will be useful in our proofs. Section 3 computes a lower bound for VC-density to demonstrate that $\mathrm{vc}(n) = \infty$. We also do some computations involving dp-rank. Section 4 computes an upper bound for VC-density on a formula-by-formula basis. The concluding section discusses open questions and future work.

3.1 Graph Combinatorics

Throughout this chapter A, B, C, M (sometimes with decorations) denote finite graphs, and $\mathbb{D}$ is used to denote potentially infinite graphs. All graphs are undirected and asymmetric. For a graph A the set of its vertices is denoted by $v(A)$, and the set of its edges by $e(A)$. The number of vertices of A is denoted by $|A|$. Subgraph always means induced subgraph and $A \subseteq \mathbb{D}$ means that A is a subgraph of $\mathbb{D}$. For two subgraphs A, B of a larger graph, the union $A \cup B$ denotes the graph induced by $v(A) \cup v(B)$. Similarly, $A - B$ means a subgraph of A induced by the vertices of $v(A) - v(B)$. For $A \subseteq B \subseteq \mathbb{D}$ and $A \subseteq C \subseteq \mathbb{D}$, graphs B, C are said to be disjoint over A if $v(B) - v(A)$ is disjoint from $v(C) - v(A)$ and there are no edges from $v(B) - v(A)$ to $v(C) - v(A)$ in $\mathbb{D}$. We often confuse a tuple of vertices $a = (a_1, \ldots, a_n) \in \mathbb{D}^n$ with the subgraph $a = \{a_1, \ldots, a_n\} \subseteq \mathbb{D}$.

For the remainder of the chapter fix $\alpha \in (0, 1)$, irrational.

Definition 3.1.1.

- For a graph A let $\dim(A) = |A| - \alpha|e(A)|$. (Note that this may be negative.)
- For A, B with $A \subseteq B$ define $\dim(B/A) = \dim(B) - \dim(A)$.
- We say that $A \leq B$ if $A \subseteq B$ and $\dim(A'/A) > 0$ for all $A \subsetneq A' \subseteq B$.
- Define A to be positive if for all $A' \subseteq A$ we have $\dim(A') \geq 0$.
- We work in theory S_α in the language of graphs axiomatized by the following conditions:
  - Every finite substructure is positive.

– Given a model $\mathbb{G}$ and graphs $A \leq B$, every embedding $f : A \longrightarrow \mathbb{G}$ extends to an embedding $g : B \longrightarrow \mathbb{G}$.

(Here an embedding is taken in the model-theoretic sense, of structures in the language of graphs, so each embedding maps edges to edges and nonedges to nonedges.) This theory is complete and stable (see 5.7 and 7.1 in [Las07]). From now on, fix an ambient model $\mathbb{G} \models S_\alpha$. This will be the only infinite graph we work with.

- Given $S \in \mathbb{N}$, a graph $S \subseteq \mathbb{G}$ is called <u>S-strong</u> if for any $R \subseteq T \subseteq \mathbb{G}$ with $|T| - |R| \leq S$ we have $R \leq T$.

- For A, B positive, (A, B) is called a <u>minimal pair</u> if $A \subseteq B$, $\dim(B/A) < 0$ but $\dim(A'/A) \geq 0$ for all proper $A \subseteq A' \subsetneq B$. We call B a <u>minimal extension</u> of A. The dimension of a minimal pair is defined as $|\dim(B/A)|$.

- A sequence $\langle M_i \rangle_{0 \leq i \leq n}$ of finite graphs is called a <u>minimal chain</u> if (M_i, M_{i+1}) is a minimal pair for all $0 \leq i < n$.

- Suppose we have a graph A with vertices $v(A) = \{x_1, \ldots, x_n\}$ with pairwise disjoint x_i. For the variable tuple of vertices $x = (x_1, \ldots, x_n)$ let $\mathrm{diag}_A(x)$ be the atomic diagram of A, i.e., the first-order formula recording whether there is an edge or a nonedge between every pair of vertices. So for a graph $\mathbb{D}$ and a tuple $a = (a_1, \ldots, a_n)$ we have $\mathbb{D} \models \mathrm{diag}_A(a)$ if and only if there exists an embedding $f\colon A \longrightarrow \mathbb{D}$ such that $f(x_i) = a_i$.

- Given $A \subseteq B$ let

$$\phi_{A,B}(x) = \mathrm{diag}_A(x) \wedge \exists z \ \mathrm{diag}_B(x, z).$$

Any graph isomorphic to B is called a <u>witness of $\phi_{A,B}$</u>. Work in a graph $\mathbb{D}$. Suppose $\mathbb{D} \models \phi_{A,B}(a)$ for some tuple $a = (a_1, \ldots, a_m)$ and we have a finite subgraph $B' \subseteq \mathbb{D}$ with vertices $v(B') = \{b_1, \ldots b_n\}$ such that $b_i = a_i$ for $i = 1, \ldots, m$ and $\mathbb{D} \models \mathrm{diag}_B(b)$. In this case we call such a graph B' a <u>witness of $\phi_{A,B}(a)$</u>.

- A formula $\phi_{A,B}$ is called a <u>basic formula</u> if there is a minimal chain $\langle M_i \rangle_{0 \leq i \leq n}$ such that $A = M_0$ and $B = M_n$. We also denote such a formula by $\phi_{\langle M_i \rangle_{0 \leq i \leq n}}$.

Theorem 3.1.2 (Quantifier simplification, 5.6 in [Las07]). *In the theory S_α every formula is equivalent to a boolean combination of basic formulas.*

3.2 Basic Definitions and Lemmas

We require the following lemmas from [Las07]:

Lemma 3.2.1. *[See 2.3 in [Las07]] Let $A, B \subseteq \mathbb{D}$. Then*

$$\dim(A \cup B/A) \leq \dim(B/A \cap B).$$

Moreover,

$$\dim(A \cup B/A) = \dim(B/A \cap B) - \alpha E,$$

where E is the number of edges connecting the vertices of $B - A$ to the vertices of $A - B$.

Lemma 3.2.2. *[See 4.1 in [Las07]] Suppose A is a positive graph with $\lceil 1/\alpha \rceil + 2$ vertices. Then for any $\epsilon > 0$ there exists a graph B such that (A, B) is a minimal pair with dimension $\leq \epsilon$. Moreover, every vertex in A is connected to a vertex in $B - A$.*

Lemma 3.2.3. *[See 4.4 in [Las07]] Suppose we have $A \subseteq \mathbb{G}$. Then for any integer $S \geq 0$ there exists an embedding $f\colon A \longrightarrow \mathbb{G}$ such that $f(A)$ is S-strong in $\mathbb{G}$.*

Lemma 3.2.4. *[See 3.8 in [Las07]] For all $S > 0$ there exists $M = M(S, \alpha) \in \mathbb{N}$ with the following property. Suppose $A \subseteq \mathbb{G}$. Then there exists B with $A \subseteq B \subseteq \mathbb{G}$ such that B is S-strong in $\mathbb{G}$ and $|B| \leq M|A|$.*

We conclude this section by stating a couple of technical lemmas that will be useful in our proofs later. In these lemmas we work in some ambient graph $\mathbb{D}$; that is, all the finite graphs that come up are assumed to be subgraphs of $\mathbb{D}$.

Lemma 3.2.5. *Let B be a graph and (A, M) be a minimal pair with $A \subseteq B$ and $\dim(M/A) = -\epsilon$. Then either $M \subseteq B$ or $\dim(M \cup B/B) < -\epsilon$.*

Proof. By Lemma 3.2.1 we have

$$\dim(M \cup B/B) \leq \dim(M/M \cap B),$$

and as $A \subseteq M \cap B \subseteq M$ we get

$$\dim(M/A) = \dim(M/M \cap B) + \dim(M \cap B/A).$$

In addition we are given $\dim(M/A) = -\epsilon$. If $M \not\subseteq B$ then $A \subseteq M \cap B \subsetneq M$ and by minimality $\dim(M \cap B/A) > 0$. Combining the inequalities above we obtain the desired result:

$$\dim(M \cup B/B) \leq \dim(M/M \cap B) = \dim(M/A) - \dim(M \cap B/A) < -\epsilon.$$

□

Lemma 3.2.6. *Let B be a graph and $\langle M_i \rangle_{0 \leq i \leq n}$ be a minimal chain with dimensions*

$$\dim(M_{i+1}/M_i) = -\epsilon_i$$

and $M_0 \subseteq B$. Let $\epsilon = \min_{0 \leq i \leq n} \epsilon_i$. Then either $M_n \subseteq B$ or $\dim((M_n \cup B)/B) < -\epsilon$.

Proof. Let $\overline{M}_i = M_i \cup B$. Then:

$$\dim(\overline{M}_n/B) = \dim(\overline{M}_n/\overline{M}_{n-1}) + \ldots + \dim(\overline{M}_2/\overline{M}_1) + \dim(\overline{M}_1/B).$$

Either $M_n \subseteq B$ or at least one of the summands above is nonzero. Apply the previous lemma. □

Lemma 3.2.7. *Let (A, M) be a minimal pair with dimension ϵ and let $B \subseteq M$. Then*

$$\dim B/(A \cap B) \geq -\epsilon.$$

Moreover if $B \cup A \neq M$ then $\dim B/(A \cap B) \geq 0$.

Proof. We have $\dim(B \cup A/A) \leq \dim(B/A \cap B)$ by Lemma 3.2.1. Note that $A \subseteq B \cup A \subseteq M$. If $B \cup A \neq M$ then we have $\dim(B \cup A/A) \geq 0$ by minimality. If $B \cup A = M$ then we have $\dim(B \cup A/A) = -\epsilon$. □

Lemma 3.2.8. *Let* $\langle M_i \rangle_{0 \le i \le n}$ *be a minimal chain with dimensions*

$$\dim(M_i/M_{i-1}) = -\epsilon_i.$$

Let

$$\epsilon = \sum_{i=1}^{n} \epsilon_i,$$

and let $B \subseteq M_n$*. Then* $\dim(B/M_0 \cap B) \ge -\epsilon$*.*

Proof. Let $B_i = B \cap M_i$. We have $\dim(B_{i+1}/B_i) \ge \dim(M_{i+1}/M_i)$ by the previous lemma. Thus

$$\dim(B/M_0 \cap B) = \dim(B_n/B_0) = \sum_{i=1}^{n} \dim(B_{i+1}/B_i) \ge -\epsilon.$$

□

3.3 Lower bound

Definition 3.3.1. Suppose $\phi_{A,B}(x, y)$ is a basic formula. Define X to be the graph on the vertices x with edges defined by ϕ (equivalently it is a subgraph of A induced by the vertices x). Similarly define Y. Note that X, Y are positive as A is positive. Additionally, let Y' be a subgraph of Y induced by vertices of Y that are connected to $B - (X \cup Y)$.

In this section we restrict our attention to the following family of basic formulas $\phi(x, y)$:

- All formulas have $Y' = Y$.
- All formulas define no edges between X and Y.
- The minimal chain of $\phi(x, y)$ consists of one step, that is we only have one minimal extension as opposed to a chain of minimal extensions.
- The dimension of that minimal extension is smaller than α.

We obtain a lower bound for the formulas that are boolean combinations of basic formulas of this type written in disjunctive-normal form. First, define $\epsilon_L(\phi)$.

Definition 3.3.2.

- For a basic formula $\phi = \phi_{M_0,M_1}(x,y)$ let $\epsilon(\phi) = -\dim(M_1/M_0)$.
- (Negation) If ϕ is a basic formula, then define

$$\epsilon_L(\neg\phi) = \epsilon_L(\phi).$$

- (Conjunction) Take a finite collection of formulas $\phi_i(x,y)$ where each ϕ_i is a positive or a negative basic formula and $\phi = \bigwedge_i \phi_i$. If both positive and negative formulas are present then $\epsilon_L(\phi) = \infty$. We don't have a lower bound for that case. If different formulas define X or Y differently then let $\epsilon_L(\phi) = \infty$. In the case of conflicting definitions, the formula would have no realizations. Otherwise, let

$$\epsilon_L\left(\bigwedge_i \phi_i\right) = \sum_i \epsilon_L(\phi_i).$$

- (Disjunction) Take a collection of formulas ψ_i with each formula a conjunction as above. Also assume that all the basic formulas that appear agree on X and Y. Then let

$$\epsilon_L\left(\bigvee \psi_i\right) = \min \epsilon_L(\psi_i).$$

Theorem 3.3.3. *For a formula ψ as above we have*

$$\mathrm{vc}(\psi) \geq \left\lfloor \frac{Y(\psi)}{\epsilon_L(\psi)} \right\rfloor,$$

where $Y(\psi) = \dim(Y)$ (well-defined, as all basic components agree on Y).

Proof. First, work with a formula that is a conjunction of positive basic formulas $\psi = \bigwedge_{i\in I} \phi_i$. Then as we have defined above

$$\epsilon_L(\psi) = \sum_{i\in I} \epsilon_L(\phi_i).$$

If W_i is a witness of ϕ_i, let $S_i = |W_i|$. Let n_1 be the largest natural number such that

$$n_1 \epsilon_L(\psi) < Y(\psi).$$

Let ϵ' be the smallest value among $\epsilon_L(\phi_i)$ corresponding to the formula ϕ'. Let n_2 be the largest natural number such that

$$n_1 \epsilon_L(\psi) + n_2 \epsilon' < Y(\psi).$$

Fix some $N > n_1 + n_2$. Let J be the set of first N natural numbers. Let $\{a_j\}_{j \in J}$ be a pairwise disjoint collection of graphs, where each a_i is isomorphic to X. Let $A = \bigcup_{1 \leq j \leq N} a_j$. Let

$$S = |Y| + (n_1 + n_2 + 1) \sum_{i \in I} S_i.$$

By Lemma 3.2.3 the graph A can be embedded into $\mathbb{G}$ as an S-strong graph. Abusing notation, we identify A with this embedding. Thus we have $A \subseteq \mathbb{G}$, S-strong.

Let J_1, J_2 be disjoint subsets of J, of sizes n_1, n_2 respectively. Let b be a graph isomorphic to Y. For each $i \in I, j \in J_1$ let W_{ij} be a witness of $\phi_i(a_j, b)$. (Note that then $(a_j \cup b, W_{ij})$ is a minimal pair. Also note that we are not assuming yet that $W_{ij} \subseteq \mathbb{G}$.) For each $j \in J_1$ let W_j be a union of $\{W_{ij}\}_{i \in I}$ disjoint over $a_j \cup b$. For each $j \in J_2$ let W_j be a witness of $\phi'(a_j, b)$. Let W' be a union of $\{W_j\}_{j \in J_1 \cup J_2}$ disjoint over b. Let W be a union of W' and A disjoint over $\{a_j\}_{j \in J_1 \cup J_2}$.

Lemma 3.3.4. *We have $A \leq W$.*

Proof. Consider some $A \subsetneq B \subseteq W$. We need to show $\dim(B/A) > 0$. Let $\overline{A} = A \cup b$. We have

$$\dim(B/A) = \dim(B/B \cap \overline{A}) + \dim(B \cap \overline{A}/A).$$

Let $B_{ij} = B \cap W_{ij}$. Let $B_j = B \cap W_j$. To unify indices, relabel all the graphs above as $\{B_k\}_{k \in K}$ for some index set K. By the construction of W we have

$$\dim(B/B \cap \overline{A}) = \sum_{k \in K} \dim(B_k/B_k \cap \overline{A}).$$

Fix k. We have $B_k \subseteq W_k$, where W_k is a minimal extension of $M_0^k = a \cup b$ for some $a \in A$. Let ϵ_k be the dimension of this minimal extension. We have $\dim(B_k/B_k \cap \overline{A}) = \dim(B_k/a \cup (B \cap b))$.

Case 1: $B \cap b = b$. Then $M_0^k \subseteq B_k \subseteq W_k$ and

$$\dim(B_k/a \cup (B \cap b)) = \dim(B_k/M_0^k).$$

By minimality of (M_0^k, B_k) we have $\dim(B_k/M_0^k) \geq -\epsilon_k$. Thus

$$\dim(B/B \cap \overline{A}) \geq -\sum_{k \in K} \epsilon_k = -\left(n_1 \epsilon_L(\psi) + n_2 \epsilon'\right).$$

In addition

$$\dim(B \cap \overline{A}/A) = \dim(b) = Y(\psi).$$

Combining the two, we get

$$\dim(B/A) \geq Y(\psi) - \left(n_1 \epsilon_L(\psi) + n_2 \epsilon'\right),$$

which is positive by the construction of n_1, n_2 as needed.

Case 2: $B \cap b \subsetneq b$.

Claim 3.3.5. *We have* $\dim(B_k/B_k \cap \overline{A}) > 0$*.*

Proof. Recall that

$$\dim(B_k/B_k \cap \overline{A}) = \dim(B_k/a \cup (B \cap b)).$$

First, suppose that $B_k \cup M_0^k \neq W_k$. Then by Lemma 3.2.7 we get the required inequality. Thus we may assume that $B_k \cup M_0^k = W_k$. By Lemma 3.2.1 we have

$$\dim(B_k \cup M_0^k/M_0^k) = \dim(B_k/B_k \cap M_0^k) - \alpha E,$$

where E is the number of edges connecting the vertices of $B_k - M_0^k = B_k \cup M_0^k - M_0^k$ to the vertices of $M_0^k - B_k = M_0^k - B_k \cap M_0^k$. Noting that $B_k \cup M_0^k = W_k$, $\dim W_k/M_0^k = -\epsilon_k$, and $B_k \cap M_0^k = a \cup (B \cap b)$ we may rewrite the equality above as

$$\dim(B_k/a \cup (B \cap b)) = \alpha E - \epsilon,$$

and E is the number of edges connecting the vertices of $W_k - M_0^k$ to the vertices of $M_0^k - a \cup (B \cap b)$. As $Y = Y'$ and $B \cap b \subsetneq b$ we must have $E \geq 1$. But then as $\alpha > \epsilon$ we have $\dim(B_k/a \cup (B \cap b)) > 0$ as needed. $\square$

(Continuing the proof of Lemma 3.3.4) Now, recall that

$$\dim(B/A) = \dim(B \cap \overline{A}/A) + \sum_{k \in K} \dim(B_k/B_k \cap \overline{A}).$$

By the claim above, each of $\dim(B_k/B_k \cap \overline{A}) > 0$, thus

$$\dim(B/A) > \dim(B \cap \overline{A}/A).$$

In addition,

$$\dim(B \cap \overline{A}/A) = \dim(B \cap b) \geq 0,$$

as b is positive. Thus $\dim(B/A) > 0$ as needed. $\square$

As $A \leq W$ and $A \subseteq \mathbb{G}$, we can embed W into $\mathbb{G}$ over A. Abusing notation again, we identify W with its embedding $A \leq W \subseteq \mathbb{G}$. In particular, now we have $b \in \mathbb{G}$. Also note that

$$\begin{aligned} \dim(W/A) &= Y(\psi) - (n_1 \epsilon_L(\psi) + n_2 \epsilon'), \\ |W| - |A| &\leq |b| + (n_1 + n_2) \sum_{i \in I} S_i. \end{aligned}$$

Lemma 3.3.6. *We have*

$$\{a_j\}_{j \in J_1} \subseteq \psi(A, b) \subseteq \{a_j\}_{j \in J_1 \cup J_2}.$$

Proof. First inclusion $\{a_j\}_{j \in J_1} \subseteq \psi(A, b)$ is immediate from the construction of W, as W_{ij} witnesses that $\phi_i(a_j, b)$ holds. For the second inclusion, suppose that there is $a \in A - \{a_j\}_{j \in J_1 \cup J_2}$ such that $\psi(a, b)$ holds. Let $W' \subseteq \mathbb{G}$ be a witness of $\phi_1(a, b)$. First, note that the case $W' \subseteq W$ is impossible as there are no edges between a and $W - a$, but there are edges between a and $W' - a$. Thus assume $W' \nsubseteq W$. As $(a \cup b, W')$ is a minimal pair, by Lemma 3.2.5 we have $\dim(W' \cup W/W) < -\epsilon_1$. Therefore

$$\dim(W' \cup W/A) = \dim(W' \cup W/W) + \dim(W/A) < Y(\psi) - (n_1\epsilon_L(\psi) + n_2\epsilon') - \epsilon_1,$$

which is negative by the construction of n_1, n_2. Thus $A \nleq W \cup W'$, as then it would have a positive dimension. Additionally,

$$|W' \cup W| - |A| \leq |W' - W| + |W| - |A| \leq S_1 + |b| + (n_1 + n_2) \sum_{i \in I} S_i \leq S,$$

but then this is a contradiction as A is S-strong but $A \nleq W \cup W'$. $\square$

In the construction of W we could have chosen the index sets J_1, J_2 arbitrarily. In particular, suppose we let J_2 be the last n_2 indices of J and J_1 an arbitrary n_1-element subset of the first $N - n_2$ elements of J. Each of those choices would then yield a different trace $\psi(A, b)$ by the lemma above. Thus $\psi(A, M^{|y|}) \geq \binom{N-n_2}{n_1}$ and therefore $\mathrm{vc}(\psi) \geq n_1$. By the definition of n_1 we have $n_1 = \left\lfloor \frac{Y(\psi)}{\epsilon_L(\psi)} \right\rfloor$, so this proves the theorem for ψ.

Now consider a formula which is a conjunction of negative basic formulas $\psi = \bigwedge_{i \in I} \neg\phi_i$. Let $\bar{\psi} = \bigwedge_{i \in I} \phi_i$. Do the construction above for $\bar{\psi}$ and suppose its trace is $X \subseteq A$ for some b. Then $A - X$ is the trace $\psi(A, b)$. Therefore we get as many traces as above and thus the same bound.

Finally, consider a formula $\theta = \bigvee_{k \in K} \psi_k$ which is a disjunction of the formulas considered in the previous paragraph. Choose the one with the smallest ϵ_L, say ψ_k, and repeat the construction above for ψ_k. Any trace for ψ_k is automatically a trace for θ, so we get as many traces as above, and thus the same bound. $\square$

Corollary 3.3.7. *The VC-density function is infinite in the theory of Shelah-Spencer graphs:*

$$\mathrm{vc}^{S_\alpha}(n) = \infty.$$

Proof. Let A be a graph consisting of $\lceil 1/\alpha \rceil + 2$ disconnected vertices. Fix $\epsilon > 0$. By Lemma 3.2.2, there exists B such that (A, B) is a minimal pair with dimension $\leq \epsilon$ and every vertex in A is connected to a vertex in $B - A$. Consider a basic formula $\psi_{A,B}(x, y)$ where

$|x| = \lceil 1/\alpha \rceil + 1$ and $|y| = 1$. Then by the theorem above we have $\mathrm{vc}^{S_\alpha}(1) \geq \mathrm{vc}(\psi_{A,B}) \geq \left\lfloor \frac{1}{\epsilon} \right\rfloor$. As ϵ was arbitrary, we can make this number as large as we want, giving $\mathrm{vc}^{S_\alpha}(1) = \infty$. By Lemma 1.3.14 we then have $\mathrm{vc}^{S_\alpha}(n) = \infty$ as needed. □

Corollary 3.3.8. *The theory of Shelah-Spencer graphs doesn't have finite dp-rank. In particular it is not dp-minimal.*

Proof. Suppose that the ambient model $\mathbb{G}$ is $\aleph_1$-saturated. It suffices to show that $\mathbb{G}$ doesn't have finite dp-rank. We would like to modify the proof of Theorem 3.3.3 to make A an infinite $\emptyset$-indiscernible sequence. (Note that as S_α is stable, all indiscernible sequences are totally indiscernible.) In the proof we can construct S-strong sets $A = \{a_j\}_{j \in J}$ of arbitrary finite size. Moreover, for every $J' \subseteq J$, the set $A = \{a_j\}_{j \in J'}$ is still S-strong. Thus by Lemma 1.2.8 we can find an infinite $\emptyset$-indiscernible sequence $A = \{a_j\}_{j \in \mathbb{N}}$ in $\mathbb{G}$ that is S-strong. Repeating the construction of the corollary above, we can obtain a formula with an arbitrarily large VC-density over this indiscernible sequence A. By Definition 1.4.1 this means that $\mathbb{G}$ doesn't have finite dp-rank. □

3.4 Upper bound

Let $\phi(x, y)$ be a basic formula associated to a minimal chain $\langle M_i \rangle_{0 \leq i \leq n}$ with dimensions $\dim(M_{i+1}/M_i) = -\epsilon_i$. Define

$$\epsilon(\phi) = \min \{\epsilon_i\}_{0 \leq i \leq n_\phi}$$
$$K(\phi) = |M_n|.$$

Now consider a finite collection of basic formulas

$$\Phi = \Phi(x, y) = \{\phi_i(x, y)\}_{i \in I}.$$

Define

$$K(\Phi) = \max \{K(\phi_i)\}_{i \in I},$$
$$\epsilon(\Phi) = \min \{\epsilon(\phi_i)\}_{i \in I} \cup \{\alpha\}.$$

Theorem 3.4.1. *If ϕ is a boolean combination of formulas from Φ, then*

$$\mathrm{vc}(\phi) \leq \left\lfloor |y| \frac{K(\Phi)}{\epsilon(\Phi)} \right\rfloor .$$

We first reduce Theorem 3.4.1 to a combinatorial statement (Theorem 3.4.6 below), the proof of which takes up the rest of this section.

Let

$$S = \left\lceil \left(\frac{|y|}{\epsilon(\phi)} + 1 \right) K(\phi) \right\rceil .$$

Fix a finite parameter set $A_0 \subseteq \mathbb{G}^{|x|}$ with $|A_0| = N_0$. We would like to bound $\left|\phi(A_0, \mathbb{G}^{|y|})\right|$ in terms of Φ and N_0. Let $A_1 \subseteq \mathbb{G}$ consist of the components of the tuples of A_0 (so $A_0 \subseteq A_1^{|x|}$). Then $|A_1| \leq |x| N_0$. Using Lemma 3.2.4 let A be a graph $A_1 \subseteq A \subseteq \mathbb{G}$, S-strong in $\mathbb{G}$. Let $N = |A|$. Then we have $N \leq |x| N_0 M$ (where $M = M(S, \alpha)$ is the constant from Lemma 3.2.4). As $A_0 \subseteq A^{|x|}$ we have

$$\left|\phi(A_0, \mathbb{G}^{|y|})\right| \leq \left|\phi(A^{|x|}, \mathbb{G}^{|y|})\right| .$$

Therefore it suffices to bound $\left|\phi(A^{|x|}, \mathbb{G}^{|y|})\right|$ uniformly in $\Phi, |A|$.

Definition 3.4.2. For $A \subseteq \mathbb{G}^{|x|}, B \subseteq \mathbb{G}^{|y|}, b \in \mathbb{G}^{|y|}$ define

$$\Phi(A, b) = \{(a, i) \in A \times I \mid \mathbb{G} \models \phi_i(a, b)\} \subseteq A \times I,$$
$$\Phi(A, B) = \{\Phi(A, b) \mid b \in B\} \subseteq \mathcal{P}(A \times I).$$

Lemma 3.4.3. *For $A \subseteq \mathbb{G}^{|x|}, B \subseteq \mathbb{G}^{|y|}$ if ϕ is a boolean combination of formulas from Φ then $|\phi(A, B)| \leq |\Phi(A, B)|$.*

Proof. Clear, as for all $a \in A, b \in B$ the set

$$\Phi(a, b) = \{i \in I \mid \mathbb{G} \models \phi_i(a, b)\}$$

determines the truth value of $\phi(a, b)$. □

Thus it suffices to bound $\left|\Phi(A^{|x|}, \mathbb{G}^{|y|})\right|$ in terms of $\Phi, |A|$.

Definition 3.4.4.

- For all $i \in I, a \in A^{|x|}, b \in \mathbb{G}^{|y|}$ if $\phi_i(a,b)$ holds, fix $W^i_{a,b} \subseteq \mathbb{G}$, a witness of $\phi_i(a,b)$.
- For $b \in \mathbb{G}^{|y|}$ let
$$W_b = \bigcup \left\{ W^i_{a,b} \mid a \in A^{|x|}, i \in I, \mathbb{G} \models \phi_i(a,b) \right\}.$$
- Suppose A, B are subgraphs of $\mathbb{G}$ such that $v(A), v(B)$ are disjoint. Then define $\mathscr{E}(A,B)$ to be the number of edges between the vertices in $v(A)$ and the vertices in $v(B)$.
- For $C, B \subseteq \mathbb{G}$ define the <u>boundary</u> of C over B
$$\partial(C,B) = \{b \in B \mid \mathscr{E}(b, C-B) \neq 0\} \subseteq B.$$
- For $b \in \mathbb{G}^{|y|}$ let $\partial_b = \partial(W_b, A) \subseteq A$.
- For $b \in \mathbb{G}^{|y|}$ let $\overline{W}_b = (W_b - A) \cup \partial_b$.
- For $b_1, b_2 \in \mathbb{G}^{|y|}$ we say that $b_1 \sim b_2$ if $\partial_{b_1} = \partial_{b_2}$, $b_1 \cap A = b_2 \cap A$, and there exists a graph isomorphism from $\overline{W}_{b_1} \cup b_1$ to $\overline{W}_{b_2} \cup b_2$ that fixes ∂_{b_1} and maps b_1 to b_2. One easily checks that this defines an equivalence relation.

Lemma 3.4.5. *For $b_1, b_2 \in \mathbb{G}^{|y|}$ if $b_1 \sim b_2$ then $\Phi(A^{|x|}, b_1) = \Phi(A^{|x|}, b_2)$.*

Proof. Fix a graph isomorphism $\bar{f} \colon \overline{W}_{b_1} \cup b_1 \longrightarrow \overline{W}_{b_2} \cup b_2$. Extend it to an isomorphism $f \colon W_{b_1} \cup A \longrightarrow W_{b_2} \cup A$ by making it an identity map on the new vertices. Suppose $\mathbb{G} \models \phi_i(a, b_1)$ for some $a \in A^{|x|}$. Then $f(W^i_{a,b_1})$ is a witness of $\phi_i(a,b_2)$ (though not necessarily equal to W^i_{a,b_2}) and thus $\mathbb{G} \models \phi_i(a,b_2)$. As a was arbitrary, this proves the equality of the traces. $\square$

Thus to bound the number of traces it is sufficient to bound the number of $\sim$-equivalence classes.

Theorem 3.4.6. *Suppose we have $b \in \mathbb{G}^{|y|}$. Let $Y = |b - A|$. Then*

$$|\partial_b| \leq \left\lfloor Y \frac{K(\Phi)}{\epsilon(\Phi)} \right\rfloor,$$
$$|\overline{W}_b| \leq \left\lfloor 3Y \frac{K(\Phi)}{\epsilon(\Phi)} \right\rfloor.$$

From this theorem we get the desired result:

Corollary 3.4.7. *(Theorem 3.4.1) If ϕ is a boolean combination of formulas from Φ, then* $\mathrm{vc}(\phi) \leq \left\lfloor |y| \frac{K(\Phi)}{\epsilon(\Phi)} \right\rfloor$.

Proof of Theorem 3.4.1 (based on Theorem 3.4.6). We count possible equivalence classes of $\sim$. This amounts to bounding the possibilities for ∂_b, $b \cap A$, and the number of isomorphism classes of W_b. Fix $b \in \mathbb{G}^{|y|}$ and let $Y = |b - A|$. Let

$$D = \left\lfloor Y \frac{K(\Phi)}{\epsilon(\Phi)} \right\rfloor,$$
$$D' = \left\lfloor 3Y \frac{K(\Phi)}{\epsilon(\Phi)} \right\rfloor,$$
$$D'' = \left\lfloor |y| \frac{K(\Phi)}{\epsilon(\Phi)} \right\rfloor.$$

By the first part of Theorem 3.4.6 there are $\binom{N}{D}$ choices for the boundary ∂_b. By the second part of Theorem 3.4.6 there are at most D' vertices in $\overline{W}_b$. Thus to determine the graph $\overline{W}_b$ we need to choose how many vertices it has and then decide where edges go. This amounts to at most $D' 2^{(D')^2}$ choices. Additionally there are $\binom{N}{|y|-Y}$ choices for $b \cap A$. In total this gives us at most

$$\binom{N}{D} \cdot \binom{N}{|y| - Y} \cdot D' 2^{(D')^2} = O\left(N^{D+|y|-Y}\right)$$

choices. By Lemma 3.4.5 we have $\left|\Phi(A^{|x|}, \mathbb{G}^{|y|})\right| = O\left(N^{D+|y|-Y}\right)$. As $\frac{K(\Phi)}{\epsilon(\Phi)} \geq 1$ we have

$$D + |y| - Y = \left\lfloor Y \frac{K(\Phi)}{\epsilon(\Phi)} \right\rfloor + |y| - Y \leq \left\lfloor |y| \frac{K(\Phi)}{\epsilon(\Phi)} \right\rfloor = D''.$$

Thus

$$\left|\Phi(A^{|x|}, \mathbb{G}^{|y|})\right| = O\left(N^{D+|y|-Y}\right) = O\left(N^{D''}\right).$$

Recall that

$$\left|\phi(A_0, \mathbb{G}^{|y|})\right| \leq \left|\Phi(A^{|x|}, \mathbb{G}^{|y|})\right|.$$

Therefore we have

$$\begin{aligned}\left|\phi(A_0, \mathbb{G}^{|y|})\right| &= O\left(N^{D''}\right) = O\left((|x|N_0M)^{D''}\right) = \\ &= O\left((|x|M)^{D''} N_0^{D''}\right) = O\left(N_0^{D''}\right).\end{aligned}$$

As A_0 was arbitrary, this shows that $\mathrm{vc}(\phi) \leq D'' = \left\lfloor |y|\frac{K(\Phi)}{\epsilon(\Phi)}\right\rfloor$ as needed. (Note that throughout this proof we are using the fact that quantities $K(\Phi), \epsilon(\Phi), M$ are completely determined by Φ, thus independent from A_0.) $\square$

Proof of Theorem 3.4.6. The graph W_b is a union of witnesses of the form $W_{a,b}$ for some $a \in A^{|x|}, b \in \mathbb{G}^{|y|}$. Enumerate all of them as $\{W_j\}_{1\leq j\leq J}$. Define $M_j = \bigcup_{k=1}^{j} W_k$ for $1 \leq j \leq J$ and let $M_0 = b, M_{-1} = \emptyset$. Let $\overline{A} = A \cup b$.

Definition 3.4.8. For $0 \leq j \leq J$:

- Let $v_j = 1$ if new vertices are added to M_j outside of $\overline{A}$, that is if $M_j - \overline{A} \neq M_{j-1} - B$, and let it be 0 otherwise.
- Let $E_j = \partial(A - W_j, M_j - A)$.
- Let

$$m_j = \sum_{k=0}^{j}(v_j + |E_j|).$$

Lemma 3.4.9. *For $0 \leq j \leq J$ we have*

$$|\partial(M_j, A)| \leq |E_0| + m_j K(\Phi).$$

Proof. Proceed by induction on $j = 0, \ldots, J$. The base case $j = 0$ is clear. For the inductive step suppose that

$$|\partial(M_{j-1}, A)| \leq m_{j-1}K(\Phi)$$

holds. Let

$$\begin{aligned}\delta_1 &= \partial(M_j, A) - \partial(M_{j-1}, A) = \\ &= \{a \in A \mid \mathscr{E}(a, M_j - A) \neq 0 \text{ and } \mathscr{E}(a, M_{j-1} - A) = 0\}\,.\end{aligned}$$

If $M_j - A = M_{j-1} - A$ then $\delta_1 = \emptyset$ and we are done as m_j is increasing. Suppose not. We have $|\delta_1| = |\delta_1 \cap W_j| + |\delta_1 - W_j|$, and

$$\delta_1 - W_j = \{a \in A - W_j \mid \mathscr{E}(a, M_j - A) \neq 0 \text{ and } \mathscr{E}(a, M_{j-1} - A) = 0\}\,.$$

But then it's clear that $\delta_1 - W_j \subseteq E_j$ as

$$\begin{aligned}&W_j - M_{j-1} - A \subseteq M_j - A, \\ &(W_j - M_{j-1} - A) \cap (M_{j-1} - A) = \emptyset.\end{aligned}$$

As $b \in M_{j-1}$ and $M_j - A \neq M_{j-1} - A$, then $M_j - \overline{A} \neq M_{j-1} - \overline{A}$, and thus $v_j = 1$. Therefore we have

$$\begin{aligned}|\delta_1| &= |\delta_1 \cap W_j| + |\delta_1 - W_j| \leq |W_j| + |E_j| \leq \\ &\leq K(\Phi) + |E_j| \leq (v_j + |E_j|)K(\Phi) \leq (m_j - m_{j-1})K(\Phi),\end{aligned}$$

as needed. □

Lemma 3.4.10. *For $0 \leq j \leq J$ we have*

$$|M_j - \overline{A}| \leq \left(\sum_{k=0}^{j} v_k\right) K(\Phi).$$

Proof. Proceed by induction on $j = 0, \ldots, J$. The base case $j = 0$ is clear. For the inductive step suppose that

$$|M_{j-1} - \overline{A}| \leq \left(\sum_{k=0}^{j-1} v_k\right) K(\Phi)$$

holds. If $M_j - \overline{A} = M_{j-1} - \overline{A}$ then the inequality is immediate as $v_j \geq 0$. Therefore assume this is not the case. Then $v_j = 1$ and $|M_j - A| - |M_{j-1} - A| \leq |W_j| \leq v_j K(\Phi)$ so we get the required inequality. □

Lemma 3.4.11. *For $0 \leq j \leq J$ we have*

$$\dim(M_j \cup \overline{A}/\overline{A}) \leq -m_j\epsilon(\Phi),$$

Proof. Proceed by induction on $j = 0, \dots, J$. The base case $j = 0$ is clear. For the inductive step suppose that

$$\dim(M_{j-1} \cup \overline{A}/\overline{A}) \leq -m_{j-1}\epsilon(\Phi)$$

holds. We have

$$\begin{aligned}\dim(M_j \cup \overline{A}/\overline{A}) &= \dim(M_j \cup \overline{A}/M_{j-1} \cup \overline{A}) + \dim(M_{j-1} \cup \overline{A}/\overline{A}) \leq \\ &\leq \dim(M_j \cup \overline{A}/M_{j-1} \cup \overline{A}) - m_{j-1}\epsilon(\Phi).\end{aligned}$$

Let $\overline{M}_{j-1} = M_{j-1} \cup \overline{A}$. By Lemma 3.2.1 we have

$$\dim(M_j \cup \overline{A}/M_{j-1} \cup \overline{A}) = \dim(W_j \cup \overline{M}_{j-1}/\overline{M}_{j-1}) = \dim(W_j/W_j \cap \overline{M}_{j-1}) - E\alpha$$

where E is the number of edges connecting the vertices of $\overline{M}_{j-1} - W_j$ to the vertices of $W_j - \overline{M}_{j-1}$. Recall that $E_j = \partial(A - W_j, M_j - A)$. We have $A - W_j \subseteq \overline{M}_{j-1} - W_j$ (as $A \subseteq \overline{M}_{j-1}$) and $W_j - M_{j-1} - A = W_j - \overline{M}_{j-1}$ (as for $j > 1$, we have $b \subseteq M_{j-1}$). Thus $|E_j| \leq E$, and we get

$$\dim(M_j \cup \overline{A}/M_{j-1} \cup \overline{A}) \leq \dim(W_j/W_j \cap \overline{M}_{j-1}) - |E_j|\alpha.$$

If $W_j \subseteq \overline{M}_{j-1}$ then $\dim(W_j/W_j \cap \overline{M}_{j-1}) = 0$. If not, then by Lemma 3.2.6 we have $\dim(W_j/W_j \cap \overline{M}_{j-1}) \leq -\epsilon(\Phi)$. Either way, we have $\dim(W_j/W_j \cap \overline{M}_{j-1}) \leq -v_j\epsilon(\Phi)$. Using this and the fact that $\epsilon(\Phi) \leq \alpha$, we obtain

$$\dim(M_j \cup \overline{A}/M_{j-1} \cup \overline{A}) \leq -v_j\epsilon(\Phi) - |E_j|\epsilon(\Phi) = -(m_j - m_{j-1})\epsilon(\Phi).$$

Finally,

$$\begin{aligned}\dim(M_j \cup \overline{A}/\overline{A}) &\leq \dim(M_j \cup \overline{A}/M_{j-1} \cup \overline{A}) - m_{j-1}\epsilon(\Phi) \leq \\ &\leq -(m_j - m_{j-1})\epsilon(\Phi) - m_{j-1}\epsilon(\Phi) = -m_j\epsilon(\Phi),\end{aligned}$$

as needed. $\square$

(Proof of Theorem 3.4.6 continued) For any $0 \leq j \leq J$ we have

$$\begin{aligned}\dim(M_j \cup A/A) &= \dim(\overline{A}/A) + \dim(M_j \cup \overline{A}/\overline{A}) \\ &\leq Y - |E_0|\alpha + \dim(M_j \cup \overline{A}/\overline{A}).\end{aligned}$$

Lemma 3.4.11 gives us

$$\dim(M_j \cup \overline{A}/\overline{A}) \leq -m_j\epsilon(\Phi).$$

Thus

$$\dim(M_j \cup A/A) \leq Y - |E_0|\alpha - m_j\epsilon(\Phi).$$

Suppose j is an index such that

$$\begin{aligned}&Y - |E_0|\alpha - m_j\epsilon(\Phi) \geq 0,\\ &Y - |E_0|\alpha - m_{j+1}\epsilon(\Phi) < 0\end{aligned}$$

if one exists. Then

$$m_j \leq \frac{Y - |E_0|\alpha}{\epsilon(\Phi)}.$$

By Lemma 3.4.10 we have

$$\begin{aligned}|M_{j+1} - A| &\leq \left(\sum_{k=1}^{j+1} v_k\right) K(\Phi) \leq (m_j + 1)K(\Phi) \leq \\ &\leq \left(\frac{Y - |E_0|\alpha}{\epsilon(\Phi)} + 1\right) K(\Phi) \leq S.\end{aligned}$$

This is a contradiction, as A is S-strong and $\dim(M_{j+1} \cup A/A)$ is negative. Thus $Y - |E_0|\alpha - m_j\epsilon(\Phi) \geq 0$ for all $j \leq J$. In particular $Y - |E_0|\alpha - m_J\epsilon(\Phi) \geq 0$, so $m_J \leq \frac{Y - |E_0|\alpha}{\epsilon(\Phi)}$. Noting that $M_J = W_b$, Lemma 3.4.9 gives us

$$|\partial_b| = |\partial(W_b, A)| \leq |E_0| + m_J K(\Phi) \leq |E_0| + K(\Phi)\frac{Y - |E_0|\alpha}{\epsilon(\Phi)}.$$

As $K(\Phi) \geq 1$ and $\epsilon(\Phi) \geq \alpha$, we get

$$|\partial_b| \leq K(\Phi)\frac{Y}{\epsilon(\Phi)} = Y\frac{K(\Phi)}{\epsilon(\Phi)}.$$

As $|\partial_b|$ is an integer we have $|\partial_b| \leq \left\lfloor Y\frac{K(\Phi)}{\epsilon(\Phi)}\right\rfloor$. But this is precisely the first inequality we need to prove. For the second inequality, Lemma 3.4.10 gives us

$$\left|W_b - \overline{A}\right| \leq Y + \left(\sum_{k=0}^{J} v_k\right) K(\Phi) \leq Y + m_J K(\Phi) \leq$$
$$\leq Y + K(\Phi)\frac{Y}{\epsilon(\Phi)} \leq 2Y\frac{K(\Phi)}{\epsilon(\Phi)}.$$

As $\left|W_b - \overline{A}\right|$ is an integer we have $\left|W_b - \overline{A}\right| \leq \left\lfloor 2Y\frac{K(\Phi)}{\epsilon(\Phi)}\right\rfloor$. Thus we have

$$|\overline{W}_b| \leq |W_b - A| + |\partial_b| \leq \left\lfloor 3Y\frac{K(\Phi)}{\epsilon(\Phi)}\right\rfloor,$$

as needed. This ends the proof of Theorem 3.4.6. □

3.5 Conclusion

We have computed upper and lower bounds for certain types of formulas in Shelah-Spencer graphs. The bounds are not tight: in the best case scenario for a basic formula $\phi(x, y)$ defining a minimal extension of dimension ϵ we have

$$\frac{|y|}{\epsilon} \leq \mathrm{vc}(\phi) \leq K\frac{|y|}{\epsilon},$$

where K is the number of vertices in this minimal extension. Thus there is a multiple of K gap between lower and upper bounds. It is this author's hope that a refinement of the presented techniques can yield better estimates of the VC-density. One potential way to achieve this goal is to conduct a closer study on how multiple minimal extensions can intersect without increasing the overall dimension.

One direction for the future work is to ask what these bounds on VC-density can tell about the structure of large finite random graphs, along the lines of results in [ABC95].

Note that this chapter doesn't answer the question on whether there can be exotic values for the VC-density of individual formulas, such as non-integer or irrational values. A better bound can help address this.

Open Question 3.5.1. *In Shelah-Spencer graphs can a formula have non-integer or irrational VC-density?*

Another observation is that while the VC-density function is infinite there seems to be a good structural behavior of the VC-density for individual formulas. This suggests that perhaps the VC-density function is not the best tool to describe the behavior of definable sets in Shelah-Spencer graphs, and some more refined measure might be required. One potential way to do this is to separate the formulas based on the values of $K(\phi)$ and $\epsilon(\phi)$. Once those are bounded, VC-density seems to be well-behaved. The author hopes to explore this further in his future work.

CHAPTER 4

An Additive Reduct of the P-adic Numbers

For $\mathbb{Q}_p$ in the language $\mathcal{L}_{Mac}$ of Macintyre, the paper [ADH16] computes an upper bound for the VC-density function to be $2n-1$, and it is not known whether it is optimal for $n \geq 2$. This same bound holds in any reduct of the field of p-adic numbers, but one may expect that the simplified structure of suitable reducts would allow a better bound. In this chapter we investigate a reduct of the field of p-adic numbers and use a cell decomposition result of Leenknegt to compute an optimal bound for that structure.

In [Lee14], Leenknegt provides a cell decomposition result for a certain P-minimal additive reduct of the field of p-adic numbers. Using this result we improve the bound for the VC-density function, showing that in Leenknegt's structure $\mathrm{vc}(n) = n$ for each n. Using Definition 1.4.1 this also proves that this structure is dp-minimal which is more direct than using the fact that the field of p-adics is dp-minimal.

In Section 1 we recall some basic facts about the theory of p-adic numbers. Here we also introduce the reduct with which we will be working. Section 2 sets up basic definitions and lemmas that will be needed for the proof. We define trees and intervals and show how they help with VC-density calculations. Section 3 finishes the proof. In the concluding section we state open questions and discuss future work.

4.1 P-adic numbers

The field $\mathbb{Q}_p$ of p-adic numbers is often studied in the language of Macintyre

$$\mathcal{L}_{Mac} = \{0, 1, +, -, \cdot, |, \{P_n\}_{n\in\mathbb{N}}\}$$

which is a language $\{0, 1, +, -, \cdot\}$ of rings together with unary predicates P_n interpreted in $\mathbb{Q}_p$ so as to satisfy

$$P_n x \leftrightarrow \exists y \; y^n = x$$

and a divisibility relation where $a|b$ holds in $\mathbb{Q}_p$ when $\operatorname{val} a \leq \operatorname{val} b$.

Note that $P_n \setminus \{0\}$ is a multiplicative subgroup of $\mathbb{Q}_p$ with finitely many cosets.

Theorem 4.1.1 (see [Mac76])**.** *The $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$ has quantifier elimination.*

There is also a cell decomposition result for definable sets in this structure:

Definition 4.1.2. Define <u>k-cells</u> recursively as follows. A <u>0-cell</u> is the singleton $\mathbb{Q}_p^0$. A <u>$(k+1)$-cell</u> is a subset of $\mathbb{Q}_p^{k+1}$ of the following form:

$$\{(x, t) \in D \times \mathbb{Q}_p \mid \operatorname{val} a_1(x) \,\square_1\, \operatorname{val}(t - c(x)) \,\square_2\, \operatorname{val} a_2(x), t - c(x) \in \lambda P_n\}$$

where D is a k-cell, $a_1(x), a_2(x), c(x)$ are definable functions $D \longrightarrow \mathbb{Q}_p$, each of $\square_i$ is $<, \leq$ or no condition, $n \in \mathbb{N}$, and $\lambda \in \mathbb{Q}_p$.

Theorem 4.1.3 (see [Den84])**.** *Any subset of $\mathbb{Q}_p^k$ defined by an $\mathcal{L}_{Mac}$-formula decomposes into a finite disjoint union of k-cells.*

In [ADH16], Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko show that the $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$ satisfies $\operatorname{vc}(n) \leq 2n - 1$ for each $n \geq 1$, however, it is not known whether this bound is optimal.

In [Lee14], Leenknegt analyzes the reduct of $\mathbb{Q}_p$ to the language

$$\mathcal{L}_{aff} = \left\{0, 1, +, -, \{\bar{c}\}_{c \in \mathbb{Q}_p}, |, \{Q_{m,n}\}_{m,n \in \mathbb{N}}\right\}$$

where $\bar{c}$ denotes the scalar multiplication by c, $a|b$ as above stands for $\operatorname{val} a \leq \operatorname{val} b$, and $Q_{m,n}$ is a unary predicate interpreted as

$$Q_{m,n} = \bigcup_{k \in \mathbb{Z}} p^{km}(1 + p^n \mathbb{Z}_p).$$

Note that $Q_{m,n} \setminus \{0\}$ is a subgroup of the multiplicative group of $\mathbb{Q}_p$ with finitely many cosets. One can check that these extra relation symbols are definable in the $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$. The paper [Lee14] provides a cell decomposition result based on the following notion of a cell:

Definition 4.1.4. A 0-cell is the singleton $\mathbb{Q}_p^0$. A $(k+1)$-cell is a subset of $\mathbb{Q}_p^{k+1}$ of the following form:

$$\{(x,t) \in D \times \mathbb{Q}_p \mid \operatorname{val} a_1(x) \;\square_1\; \operatorname{val}(t - c(x)) \;\square_2\; \operatorname{val} a_2(x), t - c(x) \in \lambda Q_{m,n}\}$$

where D is a k-cell, called the base of the cell, $a_1(x), a_2(x), c(x)$ are polynomials of degree ≤ 1, called the defining polynomials, each of $\square_1, \square_2$ is $<$ or no condition, $m, n \in \mathbb{N}$, and $\lambda \in \mathbb{Q}_p$. We call $Q_{m,n}$ the defining predicate of our cell.

Theorem 4.1.5 (see [Lee14])**.** *Any definable subset of $\mathbb{Q}_p^k$ defined by an $\mathcal{L}_{aff}$-formula decomposes into a finite disjoint union of k-cells.*

Moreover, [Lee14] shows that $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ is a P-minimal reduct of the $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$, that is, the one-variable definable sets of the $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ coincide with the one-variable definable sets of the full structure $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$.

The main result of this chapter is the computation of the VC-density function for this structure:

Theorem 4.1.6. *The $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ satisfies $\operatorname{vc}(n) = n$ for all n.*

Unlike the bound on the VC-density function of the $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$ from [ADH16] which was obtained via a quantified version of uniform definability of types over finite sets, we will directly count the number of ϕ-types over a finite set of parameters.

4.2 Key Lemmas and Definitions

To show that $\operatorname{vc}(n) = n$ it suffices to bound $\operatorname{vc}^*(\phi) \leq |x|$ for every $\mathcal{L}_{aff}$-formula $\phi(x, y)$. Fix such a formula $\phi(x, y)$. Instead of working with this formula directly, we first simplify it using quantifier elimination. The required quantifier elimination result can be easily obtained from cell decomposition:

Lemma 4.2.1. *Any $\mathcal{L}_{aff}$-formula $\phi(x, y)$ is equivalent in the $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ to a boolean* *combination of formulas from a collection*

$$\Phi(x,y) = \{\mathrm{val}(p_i(x)-c_i(y)) < \mathrm{val}(p_j(x)-c_j(y))\}_{i,j\in I} \cup$$
$$\{p_i(x)-c_i(y) \in \lambda_k Q_{m,n}\}_{i\in I, k\in K}$$

of $\mathcal{L}_{aff}$-formulas where I, K are finite index sets, each p_i is a degree ≤ 1 polynomial in x without a constant term, each c_i is a degree ≤ 1 polynomial in y, $m, n \in \mathbb{N}$, and $\lambda_k \in \mathbb{Q}_p$.

Proof. Let $l = |x| + |y|$. Using Theorem 4.1.5 partition the subset of $\mathbb{Q}_p^l$ defined by ϕ to obtain $\mathscr{D}^l$, a collection of l-cells. Let $\mathscr{D}^{l-1}$ be the collection of the bases of the cells in $\mathscr{D}^l$. Similarly, construct by induction $\mathscr{D}^i$ for each $0 \leq j < l$, where $\mathscr{D}^j$ is the collection of j-cells which are the bases of cells in $\mathscr{D}^{j+1}$. Set

$$m = \prod \left\{m' \mid Q_{m',n'} \text{ is the defining predicate of a cell in } \mathscr{D}^j \text{ for } 0 \leq j \leq l\right\},$$
$$n = \max \left\{n' \mid Q_{m',n'} \text{ is the defining predicate of a cell in } \mathscr{D}^j \text{ for } 0 \leq j \leq l\right\}.$$

This way, if a, a' are in the same coset of the definable predicate $Q_{m',n'}$ of a cell in $\mathscr{D}^j$ ($0 \leq j \leq l$), then they are in the same coset of $Q_{m,n}$. Choose $\{\lambda_k\}_{k\in K}$ to range over all representations of cosets of $Q_{m,n}$. Let $q_i(x,y)$ enumerate all of the defining polynomials $a_1(x), a_2(x), t - c(x)$ that show up in the cells of $\mathscr{D}^j$ for any j. All of those are polynomials of degree ≤ 1 in the variables x, y. We can split each of them as $q_i(x,y) = p_i(x) - c_i(y)$ where the constant term of q_i is substituted by c_i. This gives us the appropriate finite collection Φ of formulas. From the cell decomposition it is easy to see that when a, a' have the same Φ-type, then they have the same ϕ-type. Thus ϕ can be written as a boolean combination of formulas from Φ. □

Lemma 4.2.2. *Let $\Phi(x,y)$ be a finite collection of formulas. If ϕ can be written as a boolean combination of formulas from Φ then* $\mathrm{vc}^*(\phi) \leq \mathrm{vc}^*(\Phi)$.

Proof. If a, a' have the same Φ-type over B, then they have the same ϕ-type over B, where B is some parameter set. Therefore the number of ϕ-types is bounded by the number of Φ-types. The bound follows from Lemma 1.3.11. □

For the remainder of the chapter fix $\Phi(x,y)$ to be a collection of formulas as in Lemma 4.2.1. By the previous lemma, to show that $\mathrm{vc}^*(\phi) \le |x|$, it suffices to bound $\mathrm{vc}^*(\Phi) \le |x|$. More precisely, it is sufficient to show that given a parameter set B of size N, the number of Φ-types over B is $O(N^{|x|})$. Fix such a parameter set B and work with it from now on. We will compute a bound for the number of Φ-types over B.

Consider the finite set $T = T(\Phi, B) = \{c_i(b) \mid b \in B, i \in I\} \subseteq \mathbb{Q}_p$. In this definition $c_i(b)$ come from the collection of formulas Φ (see Lemma 4.2.1). View T as a tree as follows:

Definition 4.2.3.

- For $c \in \mathbb{Q}_p, \alpha \in \mathbb{Z}$ define the (open) ball

$$B(c,\alpha) = \{c' \in \mathbb{Q}_p \mid \mathrm{val}\,(c'-c) > \alpha\}$$

  of radius α and center c. We also let $B(c,-\infty) = \mathbb{Q}_p$ and $B(c,+\infty) = \emptyset$.

- Define the collection of balls $\mathscr{B} = \{B(t_1, \mathrm{val}(t_1 - t_2))\}_{t_1,t_2 \in T}$. Note that $\mathscr{B}$ is a (directed) boolean algebra of sets in $\mathbb{Q}_p$. We refer to the atoms in that algebra as intervals. Note that the intervals partition $\mathbb{Q}_p$ so any element $a \in \mathbb{Q}_p$ belongs to a unique interval.

- Let's introduce some notation for the intervals. For $t \in T$ and $\alpha_L, \alpha_U \in \mathbb{Z} \cup \{-\infty, +\infty\}$ define

$$\mathrm{I}(t,\alpha_L,\alpha_U) = B(t,\alpha_L) \setminus \bigcup \{B(t',\alpha_U) \mid t' \in T, \mathrm{val}(t'-t) \ge \alpha_U\}$$

  (this is sometimes referred to as the swiss cheese construction). One can check that every interval is of the form $\mathrm{I}(t,\alpha_L,\alpha_U)$ for some values of t, α_L, α_U. The quantities α_L, α_U are uniquely determined by the interval $\mathrm{I}(t,\alpha_L,\alpha_U)$, while t might not be.

- Intervals are a natural construction for trees, however we will require a more refined notion to make Lemma 4.2.12 below work. Define a larger collection of balls

$$\mathscr{B}' = \mathscr{B} \cup \{B(c_i(b), \mathrm{val}(c_j(b) - c_k(b)))\}_{i,j,k \in I, b \in B}.$$

  Similarly to the previous definition, we define a subinterval to be an atom of the boolean algebra generated by $\mathscr{B}'$. Subintervals refine intervals. Moreover, as before, each subinterval can be written as $\mathrm{I}(t, \alpha_L, \alpha_U)$ for some values of t, α_L, α_U. As before, α_L, α_U are uniquely determined by the subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$, while t might not be.

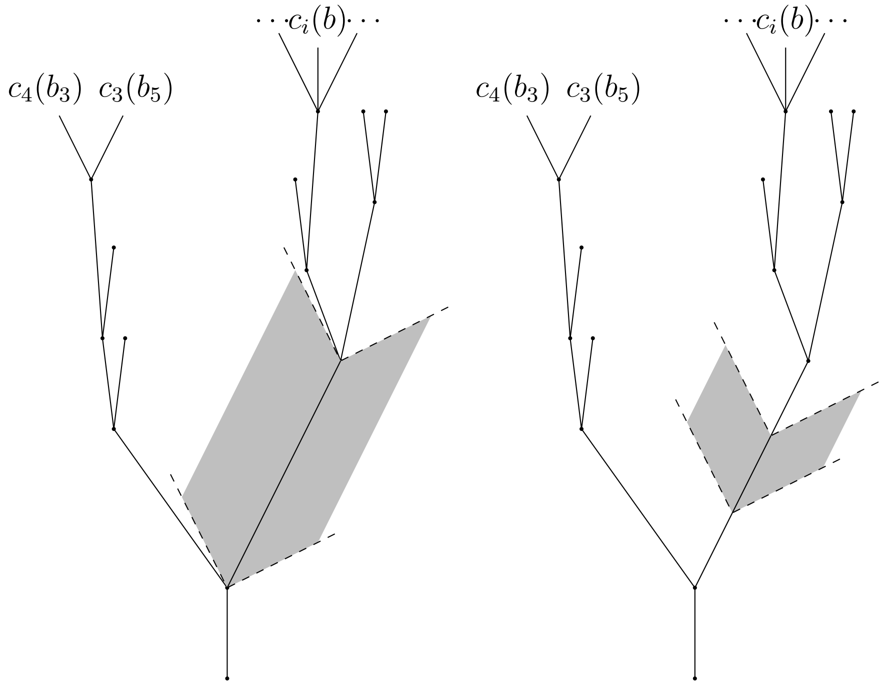

Figure 4.1: A typical interval (left) and subinterval (right) in a tree consisting of branches $c_i(b)$ with $i \in I$ and $b \in B$.

Subintervals are fine enough to make Lemma 4.2.12 below work while coarse enough to be $O(N)$ few:

Lemma 4.2.4.

- *There are at most* $2|T| = 2N|I| = O(N)$ *different intervals.*
- *There are at most* $2|T| + |B| \cdot |I|^3 = O(N)$ *different subintervals.*

Proof. Each new element in the tree T adds at most two intervals to the total count, so by induction there can be at most $2|T|$ many intervals. Each new ball in $\mathscr{B}' \backslash \mathscr{B}$ adds at most one subinterval to the total count, so by induction there are at most $|\mathscr{B}' \backslash \mathscr{B}|$ more subintervals than there are intervals. $\square$

Definition 4.2.5. Suppose $a \in \mathbb{Q}_p$ lies in the interval $\mathrm{I}(t, \alpha_L, \alpha_U)$. Define the T-valuation of a to be $T\text{-val}(a) = \mathrm{val}(a - t)$.

This is a natural notion having the following properties:

Lemma 4.2.6.

(a) $T\text{-val}(a)$ *is well-defined, independent of choice of* t *to represent the interval.*

(b) If $a \in \mathbb{Q}_p$ *lies in the subinterval* $\mathrm{I}(t, \alpha_L, \alpha_U)$, *then* $T\text{-val}(a) = \mathrm{val}(a - t)$.

(c) If $a \in \mathbb{Q}_p$ *lies in the (sub)interval* $\mathrm{I}(t, \alpha_L, \alpha_U)$ *then* $\alpha_L < T\text{-val}(a) \leq \alpha_U$.

(d) For any $a \in \mathbb{Q}_p$ *lying in the (sub)interval* $\mathrm{I}(t, \alpha_L, \alpha_U)$ *and* $t' \in T$:

- *If* $\mathrm{val}(t - t') \geq \alpha_U$, *then* $\mathrm{val}(a - t') = T\text{-val}(a)$.
- *If* $\mathrm{val}(t - t') \leq \alpha_L$, *then* $\mathrm{val}(a - t') = \mathrm{val}(t - t')\ (\leq \alpha_L < T\text{-val}(a))$.

Proof. (a)-(c) are clear. For (d) fix $t' \in T$ and suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\mathrm{I}(t, \alpha'_L, \alpha'_U)$. This subinterval lies inside of a unique interval $\mathrm{I}(t, \alpha_L, \alpha_U)$ for some choice of α_L, α_U and by the definition of intervals (or more specifically $\mathscr{B}$):

$$\mathrm{val}(t - t') \geq \alpha_U \iff \mathrm{val}(t - t') \geq \alpha'_U,$$
$$\mathrm{val}(t - t') \geq \alpha_L \iff \mathrm{val}(t - t') \geq \alpha'_L.$$

Therefore without loss of generality we may assume that $a \in \mathbb{Q}_p$ lies in an interval $\mathrm{I}(t, \alpha_L, \alpha_U)$. By (c) and the definition of intervals one of the three following cases has to hold.

Case 1: $\mathrm{val}(t - t') \geq \alpha_U$ and $T\text{-val}(a) < \alpha_U$. Then

$$\mathrm{val}(t - t') \geq \alpha_U > T\text{-val}(a) = \mathrm{val}(a - t),$$

thus $\mathrm{val}(a - t') = \mathrm{val}(a - t) = T\text{-val}(a)$ as needed.

Case 2: $\mathrm{val}(t - t') \geq \alpha_U$ and $T\text{-val}(a) = \alpha_U$. Then

$$T\text{-val}(a) = \mathrm{val}(a - t) = \mathrm{val}(t - t') \geq \alpha_U,$$

thus $\mathrm{val}(a-t') \geq \alpha_U$. The interval $\mathrm{I}(t,\alpha_L,\alpha_U)$ is disjoint from the ball $B(t',\alpha_U)$, so $a \notin B(t',\alpha_U)$, that is, $\mathrm{val}(a-t') \leq \alpha_U$. Combining this with the previous inequality we get that $\mathrm{val}(a-t') = \alpha_U = T\text{-}\mathrm{val}(a)$ as needed.

Case 3: $\mathrm{val}(t-t') \leq \alpha_L$. Then

$$\mathrm{val}(t-t') \leq \alpha_L < T\text{-}\mathrm{val}(a) = \mathrm{val}(a-t),$$

thus $\mathrm{val}(a-t') = \mathrm{val}(t-t')$ as needed. $\square$

Definition 4.2.7. Suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\mathrm{I}(t,\alpha_L,\alpha_U)$. We say that a is far from the boundary (tacitly: of $\mathrm{I}(t,\alpha_L,\alpha_U)$) if

$$\alpha_L + n \leq T\text{-}\mathrm{val}(a) \leq \alpha_U - n.$$

Here n is as in Lemma 4.2.1. Otherwise we say that it is close to the boundary (of $\mathrm{I}(t,\alpha_L,\alpha_U)$).

Definition 4.2.8. Suppose $a_1, a_2 \in \mathbb{Q}_p$ lie in the same subinterval $\mathrm{I}(t,\alpha_L,\alpha_U)$. We say a_1, a_2 have the same subinterval type if one of the following holds:

- Both a_1, a_2 are far from the boundary and $a_1 - t, a_2 - t$ are in the same $Q_{m,n}$-coset. (Here $Q_{m,n}$ is as in Lemma 4.2.1.)
- Both a_1, a_2 are close to the boundary and

$$T\text{-}\mathrm{val}(a_1) = T\text{-}\mathrm{val}(a_2) \leq \mathrm{val}(a_1 - a_2) - n.$$

Definition 4.2.9. For $c \in \mathbb{Q}_p$ and $\alpha, \beta \in \mathbb{Z}, \alpha < \beta$ define $c \restriction [\alpha,\beta)$ to be the record of the coefficients of c for the valuations between $[\alpha,\beta)$. More precisely write c in its power series form

$$c = \sum_{\gamma \in \mathbb{Z}} c_\gamma p^\gamma \text{ with } c_\gamma \in \{0,1,\ldots,p-1\}.$$

Then $c \restriction [\alpha,\beta)$ is just $(c_\alpha, c_{\alpha+1}, \ldots c_{\beta-1}) \in \{0,1,\ldots,p-1\}^{\beta-\alpha}$.

The following lemma is an adaptation of Lemma 7.4 in [ADH16].

Lemma 4.2.10. *Fix $m, n \in \mathbb{N}$. For any $x, y, c \in \mathbb{Q}_p$, if*

$$\operatorname{val}(x-c) = \operatorname{val}(y-c) \leq \operatorname{val}(x-y) - n,$$

then $x-c, y-c$ are in the same coset of $Q_{m,n}$.

Proof. Call $a, b \in \mathbb{Q}_p$ <u>similar</u> if $\operatorname{val} a = \operatorname{val} b$ and

$$a \restriction [\operatorname{val} a, \operatorname{val} a + n) = b \restriction [\operatorname{val} b, \operatorname{val} b + n).$$

If a, b are similar then

$$a \in Q_{m,n} \iff b \in Q_{m,n}.$$

Moreover for any $\lambda \in \mathbb{Q}_p^\times$, if a, b are similar then so are $\lambda a, \lambda b$. Thus if a, b are similar, then they belong to the same coset of $Q_{m,n}$. The hypothesis of the lemma force $x-c, y-c$ to be similar, thus belonging to the same coset. $\square$

Lemma 4.2.11. *For each subinterval there are at most $K = K(Q_{m,n})$ many subinterval types (with K not depending on B or on the subinterval).*

Proof. Let $a, a' \in \mathbb{Q}_p$ lie in the same subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$.

Suppose a, a' are far from the boundary. Then they have the same subinterval type if $a-t, a'-t$ are in the same $Q_{m,n}$-coset. So the number of such subinterval types is bounded by the number of $Q_{m,n}$-cosets.

Suppose a, a' are close to the boundary and

$$\begin{aligned} &T\text{-val}(a) - \alpha_L = T\text{-val}(a') - \alpha_L < n \text{ and} \\ &a \restriction [T\text{-val}(a), T\text{-val}(a) + n) = a' \restriction [T\text{-val}(a'), T\text{-val}(a') + n). \end{aligned}$$

Then a, a' have the same subinterval type. Such a subinterval type is thus determined by $T\text{-val}(a) - \alpha_L$ and the tuple $a \restriction [T\text{-val}(a), T\text{-val}(a) + n)$, therefore there are at most np^n many such types.

A similar argument works for a with $\alpha_U - T\text{-val}(a) \leq n$.

Adding all this up we get that there are at most

$$K = (\text{number of } Q_{m,n} \text{ cosets}) + 2np^n$$

many subinterval types. $\square$

The following critical lemma relates tree notions to Φ-types.

Lemma 4.2.12. *Suppose $d, d' \in \mathbb{Q}_p^{|x|}$ satisfy the following three conditions:*

- *For all $i \in I$ $p_i(d)$ and $p_i(d')$ are in the same subinterval.*
- *For all $i \in I$ $p_i(d)$ and $p_i(d')$ have the same subinterval type.*
- *For all $i, j \in I$, $T\text{-val}(p_i(d)) > T\text{-val}(p_j(d))$ iff $T\text{-val}(p_i(d')) > T\text{-val}(p_j(d'))$.*

Then d, d' have the same Φ-type over B.

Proof. There are two kinds of formulas in Φ (see Lemma 4.2.1). First we show that d, d' agree on formulas of the form $p_i(x) - c_i(y) \in \lambda_k Q_{m,n}$. It is enough to show that for every $i \in I, b \in B$, $p_i(d) - c_i(b), p_i(d') - c_i(b)$ are in the same $Q_{m,n}$-coset. Fix such i, b. For brevity let $a = p_i(d), a' = p_i(d')$ and $Q = Q_{m,n}$. We want to show that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

Suppose a, a' are close to the boundary. Then $T\text{-val}(a) = T\text{-val}(a') \leq \text{val}(a - a') - n$. Using Lemma 4.2.6d, we have

$$\text{val}(a - c_i(b)) = \text{val}(a' - c_i(b)) \leq T\text{-val}(a) \leq \text{val}(a - a') - n.$$

Lemma 4.2.10 shows that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

Now, suppose both a, a' are far from the boundary. Let $\mathrm{I}(t, \alpha_L, \alpha_U)$ be the interval containing a, a'. Then we have

$$\alpha_L + n \leq \text{val}(a - t) \leq \alpha_U - n,$$

$$\alpha_L + n \leq \text{val}(a' - t) \leq \alpha_U - n$$

(as being far from the subinterval's boundary also makes a, a' far from interval's boundary). We have either $\mathrm{val}(t - c_i(b)) \geq \alpha_U$ or $\mathrm{val}(t - c_i(b)) \leq \alpha_L$ (as otherwise it would contradict the definition of intervals, or more specifically $\mathscr{B}$).

Suppose it is the first case $\mathrm{val}(t - c_i(b)) \geq \alpha_U$. Then using Lemma 4.2.6d

$$\mathrm{val}(a - c_i(b)) = \mathrm{val}(a - t) \leq \alpha_U - n \leq \mathrm{val}(t - c_i(b)) - n.$$

So by Lemma 4.2.10 elements $a - c_i(b), a - t$ are in the same Q-coset. By an analogous argument, $a' - c_i(b), a' - t$ are in the same Q-coset. As a, a' have the same subinterval type, $a - t, a' - t$ are in the same Q-coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

For the second case, suppose $\mathrm{val}(t - c_i(b)) \leq \alpha_L$. Then using Lemma 4.2.6d

$$\mathrm{val}(a - c_i(b)) = \mathrm{val}(t - c_i(b)) \leq \alpha_L \leq \mathrm{val}(a - t) - n,$$

so by Lemma 4.2.10 elements $a - c_i(b), t - c_i(b)$ are in the same Q-coset. Similarly $a' - c_i(b), t - c_i(b)$ are in the same Q-coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

Next, we need to show that d, d' agree on formulas of the form $\mathrm{val}(p_i(x) - c_i(y)) < \mathrm{val}(p_j(x) - c_j(y))$ (again, referring to the presentation in Lemma 4.2.1). Fix $i, j \in I, b \in B$. We would like to show the following equivalence:

$$\begin{aligned}\mathrm{val}(p_i(d) - c_i(b)) < \mathrm{val}(p_j(d) - c_j(b)) \iff \\ \iff \mathrm{val}(p_i(d') - c_i(b)) < \mathrm{val}(p_j(d') - c_j(b)) \quad (4.2.1)\end{aligned}$$

Suppose $p_i(d), p_i(d')$ are in the subinterval $\mathrm{I}(t_i, \alpha_i, \beta_i)$ and $p_j(d), p_j(d')$ are in the subinterval $\mathrm{I}(t_j, \alpha_j, \beta_j)$. Lemma 4.2.6d yields the following four cases.

Case 1:

$$\mathrm{val}(p_i(d) - c_i(b)) = \mathrm{val}(p_i(d') - c_i(b)) = \mathrm{val}(t_i - c_i(b))$$

$$\mathrm{val}(p_j(d) - c_j(b)) = \mathrm{val}(p_j(d') - c_j(b)) = \mathrm{val}(t_j - c_j(b))$$

Then it is clear that the equivalence (4.2.1) holds.

Case 2:

$$\mathrm{val}(p_i(d) - c_i(b)) = T\text{-val}(p_i(d)) \text{ and } \mathrm{val}(p_i(d') - c_i(b)) = T\text{-val}(p_i(d'))$$
$$\mathrm{val}(p_j(d) - c_j(b)) = T\text{-val}(p_j(d)) \text{ and } \mathrm{val}(p_j(d') - c_j(b)) = T\text{-val}(p_j(d'))$$

Then the equivalence (4.2.1) holds by the third hypothesis of the lemma (that order of T-valuations is preserved).

Case 3:

$$\mathrm{val}(p_i(d) - c_i(b)) = \mathrm{val}(p_i(d') - c_i(b)) = \mathrm{val}(t_i - c_i(b))$$
$$\mathrm{val}(p_j(d) - c_j(b)) = T\text{-val}(p_j(d)) \text{ and } \mathrm{val}(p_j(d') - c_j(b)) = T\text{-val}(p_j(d'))$$

If $p_j(d), p_j(d')$ are close to the boundary, then $T\text{-val}(p_j(d)) = T\text{-val}(p_j(d'))$ and the equivalence (4.2.1) clearly holds. Suppose then that $p_j(d), p_j(d')$ are far from the boundary.

$$\alpha_j + n \leq T\text{-val}(p_j(d)), T\text{-val}(p_j(d')) \leq \beta_j - n$$
$$\alpha_j < T\text{-val}(p_j(d)), T\text{-val}(p_j(d')) < \beta_j$$

and $\mathrm{val}(t_i - c_i(b))$ lies outside of the (α_j, β_j) by the definition of subinterval (more specifically definition of $\mathscr{B}'$). Therefore (4.2.1) has to hold. (Note that we always have

$$T\text{-val}(p_j(d)), T\text{-val}(p_j(d')) \in (\alpha_j, \beta_j]$$

by Lemma 4.2.6c, so we only need the condition on being far from the boundary to avoid the edge case of equality to β_j.)

Case 4:

$$\mathrm{val}(p_i(d) - c_i(b)) = T\text{-val}(p_i(d)) \text{ and } \mathrm{val}(p_i(d') - c_i(b)) = T\text{-val}(p_i(d'))$$
$$\mathrm{val}(p_j(d) - c_j(b)) = \mathrm{val}(p_j(d') - c_j(b)) = \mathrm{val}(t_j - c_j(b)).$$

Similar to case 3 (switching i, j). □

The previous lemma gives us an upper bound on the number of types - there are at most $|2I|!$ many choices for the order of T-val, $O(N)$ many choices for the subinterval for each p_i, and K many choices for the subinterval type for each p_i (where K is as in Lemma 4.2.11), giving a total of $O(N^{|I|}) \cdot K^{|I|} \cdot |I|! = O(N^{|I|})$ many types. This implies $\mathrm{vc}^*(\Phi) \leq |I|$. The biggest contribution to this bound are the choices among the $O(N)$ many subintervals for each p_i with $i \in I$. Are all of those choices realized? Intuitively there are $|x|$ many variables and $|I|$ many equations, so once we choose a subinterval for $|x|$ many p_i's, the subintervals for the rest should be determined. This would give the required bound $\mathrm{vc}^*(\Phi) \leq |x|$. The next section outlines this idea formally.

4.3 Main Proof

Given a homogenenous linear polynomial $p(x)$ with coefficients in $\mathbb{Q}_p$ and $c \in \mathbb{Q}_p^{|x|}$, an alternative way to write $p(c)$ is as a scalar product $\vec{p} \cdot \vec{c}$, where $\vec{p}$ and $\vec{c}$ are vectors in $\mathbb{Q}_p^{|x|}$.

Lemma 4.3.1. *Suppose we have a finite collection of vectors $\{\vec{p}_j\}_{j \in J}$ with each $\vec{p}_j \in \mathbb{Q}_p^{|x|}$. Suppose $\vec{p} \in \mathbb{Q}_p^{|x|}$ satisfies $\vec{p} \in \operatorname{span}\{\vec{p}_j\}_{j \in J}$, and we have $\vec{c} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$ with $\mathrm{val}(\vec{p}_j \cdot \vec{c}) > \alpha$ for all $j \in J$. Then $\mathrm{val}(\vec{p} \cdot \vec{c}) > \alpha - \gamma$ for some $\gamma \in \mathbb{N}$. Moreover γ can be chosen independently from $\vec{c}, \alpha$ depending only on $\{\vec{p}_j\}_{j \in J}$.*

Proof. For some $c_j \in \mathbb{Q}_p$ for $j \in J$ we have $\vec{p} = \sum_{j \in J} c_j \vec{p}_j$, hence $\vec{p} \cdot \vec{c} = \sum_{j \in J} c_j \vec{p}_j \cdot \vec{c}$. Thus

$$\mathrm{val}\left(c_j \vec{p}_j \cdot \vec{c}\right) = \mathrm{val}\left(c_j\right) + \mathrm{val}\left(\vec{p}_j \cdot \vec{c}\right) > \mathrm{val}\left(c_j\right) + \alpha.$$

Let $\gamma = \max(0, -\max_{j \in J} \mathrm{val}\left(c_j\right))$. Then we have

$$\mathrm{val}(\vec{p} \cdot \vec{c}) = \mathrm{val}\left(\sum_{j \in J} c_j \vec{p}_j \cdot \vec{c}\right) \geq$$

$$\geq \min_{j \in J} \mathrm{val}\left(\sum_{j \in J} c_j \vec{p}_j \cdot \vec{c}\right) > \min_{j \in J} \mathrm{val}(c_j) + \alpha \geq \alpha - \gamma$$

as required. ☐

Corollary 4.3.2. *Suppose we have a finite collection of vectors $\{\vec{p}_i\}_{i \in I}$ with each $\vec{p}_i \in \mathbb{Q}_p^{|x|}$. Suppose $J \subseteq I$ and $i \in I$ satisfy $\vec{p}_i \in \operatorname{span}\{\vec{p}_j\}_{j \in J}$, and we have $\vec{c} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$ with*

$\mathrm{val}(\vec{p}_j \cdot \vec{c}) > \alpha$ *for all* $j \in J$. *Then* $\mathrm{val}(\vec{p}_i \cdot \vec{c}) > \alpha - \gamma$ *for some* $\gamma \in \mathbb{N}$. *Moreover* γ *can be chosen independently from* $J, j, \vec{c}, \alpha$ *depending only on* $\{\vec{p}_i\}_{i \in I}$.

Proof. The previous lemma shows that we can pick such γ for a given choice of i, J, but independent from $\alpha, \vec{c}$. To get a choice independent from i, J, go over all such eligible choices (i ranges over I and J ranges over subsets of I), pick γ for each, and then take the maximum of those values. $\square$

Recall that we have confined to work with collection $\Phi(x, y)$ of formulas from Lemma 4.2.1. Fix γ according to Corollary 4.3.2 corresponding to $\{\vec{p}_i\}_{i \in I}$ given by Φ. (The lemma above is a general result, but we only use it applied to the vectors given by Φ.)

Definition 4.3.3. Suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$. Define the <u>$T$-floor</u> of a to be $T\text{-fl}(a) = \alpha_L$.

Definition 4.3.4. Let $f : \mathbb{Q}_p^{|x|} \longrightarrow \mathbb{Q}_p^I$ with $f(c) = (p_i(c))_{i \in I}$. Define the segment space Sg to be the image of f. Equivalently:

$$\mathrm{Sg} = \left\{ (p_i(c))_{i \in I} \mid c \in \mathbb{Q}_p^{|x|} \right\} \subseteq \mathbb{Q}_p^I.$$

Without loss of generality, we may assume that $I = \{1, 2, \ldots, k\}$ (that is the formulas are labeled by consecutive natural numbers). Given a tuple $(a_i)_{i \in I}$ in the segment space, look at the corresponding T-floors $\{T\text{-fl}(a_i)\}_{i \in I}$ and T-valuations $\{T\text{-val}(a_i)\}_{i \in I}$. Partition the segment space by the order types of $\{T\text{-fl}(a_i)\}_{i \in I}$ and $\{T\text{-val}(a_i)\}_{i \in I}$ (as subsets of $\mathbb{Z}$).

Work in a fixed set Sg' of the partition. After relabeling the p_i we may assume that

$$T\text{-fl}(a_1) \geq T\text{-fl}(a_2) \geq \ldots \text{ for all } a_i \in \mathrm{Sg}'.$$

Consider the (relabeled) sequence of vectors $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_I$. There is a unique subset $J \subseteq I$ such that the set of all vectors with indices in J is linearly independent, and all vectors with indices outside of J are a linear combination of preceding vectors. (We can pick those using a greedy algorithm for finding a linearly independent subset of vectors.) We call indices in I <u>independent</u> and we call the indices in $I \setminus J$ <u>dependent</u>.

Definition 4.3.5.

- Denote $\{0, 1, \ldots, p-1\}$ as $\underline{\mathrm{Ct}}$.
- Let $\underline{\mathrm{Tp}}$ be the space of all subinterval types. By Lemma 4.2.11 we have $|\,\mathrm{Tp}\,| \leq K$.
- Let $\underline{\mathrm{Sub}}$ be the space of all subintervals. By Lemma 4.2.4 we have $|\,\mathrm{Sub}\,| \leq 3|I|^2 \cdot N = O(N)$.

Definition 4.3.6. Now, we define a function

$$g_{\mathrm{Sg}'} : \mathrm{Sg}' \longrightarrow \mathrm{Tp}^I \times \mathrm{Sub}^J \times \mathrm{Ct}^{I \setminus J}$$

as follows:

Let $a = (a_i)_{i \in I} \in \mathrm{Sg}'$. To define $g_{\mathrm{Sg}'}(a)$ we need to specify where it maps a in each individual component of the product.

For each a_i record its subinterval type, giving the first component in Tp^I.

For a_j with $j \in J$, record the subinterval of a_j, giving the second component in Sub^J.

For the third component (an element of $\mathrm{Ct}^{I \setminus J}$) do the following computation. Pick a_i with i dependent. Let j be the largest independent index with $j < i$. Record $a_i \restriction [T\text{-fl}(a_j) - \gamma, T\text{-fl}(a_j))$.

Combine $g_{\mathrm{Sg}'}$ for all sets Sg' in our partition of Sg to get a function

$$g : \mathrm{Sg} \longrightarrow \mathrm{Tp}^I \times \mathrm{Sub}^J \times \mathrm{Ct}^{I \setminus J}.$$

Lemma 4.3.7. *Suppose we have $c, c' \in \mathbb{Q}_p^{|x|}$ such that $f(c), f(c')$ are in the same set Sg' of the partition of Sg and $g(f(c)) = g(f(c'))$. Then c, c' have the same Φ-type over B.*

Proof. Let $a_i = \vec{p}_i \cdot \vec{c}$ and $a'_i = \vec{p}_i \cdot \vec{c}\,'$ so that

$$f(c) = (p_i(c))_{i \in I} = (\vec{p}_i \cdot \vec{c})_{i \in I} = (a_i)_{i \in I},$$
$$f(c') = (p_i(c'))_{i \in I} = (\vec{p}_i \cdot \vec{c}\,')_{i \in I} = (a'_i)_{i \in I}.$$

For each i we show that a_i, a'_i are in the same subinterval and have the same subinterval type, so the conclusion follows by Lemma 4.2.12 (the tuples $f(c), f(c')$ are in the same partition ensuring the proper order of T-valuations for the 3rd condition of the lemma). Tp records the subinterval type of each element, so if $g(\bar{a}) = g(\bar{a}')$ then a_i, a_i' have the same subinterval type for all $i \in I$. Thus it remains to show that a_i, a_i' lie in the same subinterval for all $i \in I$. Suppose i is an independent index. Then by construction, Sub records the subinterval for a_i, a_i', so those have to belong to the same subinterval. Now suppose i is dependent. Pick the largest $j < i$ such that j is independent. We have $T\text{-fl}(a_i) \leq T\text{-fl}(a_j)$ and $T\text{-fl}(a_i') \leq T\text{-fl}(a_j')$. Moreover $T\text{-fl}(a_j) = T\text{-fl}(a_j')$ as a_j, a_j' lie in the same subinterval (using the earlier part of the argument as j is independent).

Claim 4.3.8. *We have* $\mathrm{val}(a_i - a_i') > T\text{-fl}(a_j) - \gamma$.

Proof. Let K be the set of the independent indices less than i. Note that by the definition for dependent indices we have $\vec{p}_i \in \mathrm{span}\,\{\vec{p}_k\}_{k \in K}$. We also have

$$\mathrm{val}(a_k - a_k') > T\text{-fl}(a_k) \text{ for all } k \in K$$

as a_k, a_k' lie in the same subinterval (using the earlier part of the argument as k is independent). Now $\mathrm{val}(a_k - a_k') > T\text{-fl}(a_j)$ for all $k \in K$ by monotonicity of $T\text{-fl}(a_k)$. Moreover $a_k - a_k' = \vec{p}_k \cdot \vec{c} - \vec{p}_k \cdot \vec{c}' = \vec{p}_k \cdot (\vec{c} - \vec{c}')$. Combining the two, we get that $\mathrm{val}(\vec{p}_k \cdot (\vec{c} - \vec{c}')) > T\text{-fl}(a_j)$ for all $k \in K$. Now observe that $K \subseteq I, i \in I, \vec{c} - \vec{c}' \in \mathbb{Q}_p^{|x|}, T\text{-fl}(a_j) \in \mathbb{Z}$ satisfy the requirements of Lemma 4.3.2, so we apply it to obtain $\mathrm{val}(\vec{p}_i \cdot (\vec{c} - \vec{c}')) > T\text{-fl}(a_j) - \gamma$. Similar to before, we have $\vec{p}_i \cdot (\vec{c} - \vec{c}') = \vec{p}_i \cdot \vec{c} - \vec{p}_i \cdot \vec{c}' = a_i - a_i'$. Therefore we can conclude that $\mathrm{val}(a_i - a_i') > T\text{-fl}(a_j) - \gamma$ as needed, finishing the proof of the claim. □

Additionally a_i, a_i' have the same image in the Ct component, so we have $\mathrm{val}(a_i - a_i') > T\text{-fl}(a_j)$. We now would like to show that a_i, a_i' lie in the same subinterval. As $T\text{-fl}(a_i) \leq T\text{-fl}(a_j)$, $T\text{-fl}(a_i') \leq T\text{-fl}(a_j')$ and $T\text{-fl}(a_j) = T\text{-fl}(a_j')$ we have that $\mathrm{val}(a_i - a_i') > T\text{-fl}(a_i)$ and $\mathrm{val}(a_i - a_i') > T\text{-fl}(a_i')$. Suppose that a_i lies in the subinterval $\mathrm{I}(t, T\text{-fl}(a_i), \alpha_U)$ and that a_i' lies in the subinterval $\mathrm{I}(t', T\text{-fl}(a_i'), \alpha_U')$. Without loss of generality assume that $T\text{-fl}(a_i) \leq T\text{-fl}(a_i')$. As $\mathrm{val}(a_i - a_i') > T\text{-fl}(a_i')$, this implies that $a_i \in B(a_i', T\text{-fl}(a_i'))$. Then $a_i \in B(t', T\text{-fl}(a_i'))$ as $\mathrm{val}(a_i - t') > T\text{-fl}(a_i')$. This implies that $B(t, T\text{-fl}(a_i)) \cap$

$B(t', T\text{-fl}(a_i')) \neq \emptyset$ as they both contain a_i. As balls are directed, the non-zero intersection means that one ball has to be contained in another. Given our assumption that $T\text{-fl}(a_i) \leq T\text{-fl}(a_i')$, we have $B(t, T\text{-fl}(a_i)) \subseteq B(t', T\text{-fl}(a_i'))$. For the subintervals to be disjoint we need $\mathrm{I}(t, T\text{-fl}(a_i), \alpha_U) \cap B(t', T\text{-fl}(a_i')) = \emptyset$. But $\mathrm{val}(t' - a_i) > T\text{-fl}(a_i')$ implying that $a_i \in \mathrm{I}(t, T\text{-fl}(a_i), \alpha_U) \cap B(t', T\text{-fl}(a_i'))$ giving a contradiction. Therefore the subintervals coincide. $\square$

Corollary 4.3.9. *The dual VC-density of* $\Phi(x, y)$ *is* $\leq |x|$.

Proof. Suppose we have $c, c' \in \mathbb{Q}_p^{|x|}$ such that $f(c), f(c')$ are in the same partition and $g(f(c)) = g(f(c'))$. Then by the previous lemma c and c' have the same Φ-type. Thus the number of possible Φ-types is bounded by the size of the range of g times the number of possible partitions

$$(\text{number of partitions}) \cdot |\operatorname{Tp}|^{|I|} \cdot |\operatorname{Sub}|^{|J|} \cdot |\operatorname{Ct}|^{|I-J|}.$$

There are at most $(|2I|!)^2$ many partitions of Sg, so in the product above the only component dependent on B is

$$|\operatorname{Sub}|^{|J|} \leq (N \cdot 3|I|^2)^{|J|} = O(N^{|J|}).$$

Every p_i is an element of an $|x|$-dimensional vector space, so there can be at most $|x|$ many independent vectors. Thus we have $|J| \leq |x|$ and the bound follows. $\square$

Corollary 4.3.10 (Theorem 4.1.6)**.** *The* $\mathcal{L}_{aff}$*-structure* $\mathbb{Q}_p$ *satisfies* $\mathrm{vc}(n) = n$ *for each* n.

Proof. The previous lemma implies that $\mathrm{vc}^*(\phi) \leq \mathrm{vc}^*(\Phi) \leq |x|$. As our choice of ϕ was arbitrary, this implies that the VC-density of any formula is bounded by $|x|$. $\square$

4.4 Conclusion

This proof relies heavily on the linearity of the defining polynomials a_1, a_2, c in the cell decomposition result (see Definition 4.1.4). Linearity is used to separate the x and y variables as well as for Corollary 4.3.2 to reduce the number of independent factors from $|I|$ to $|x|$. The paper [Lee14] has cell decomposition results for more expressive reducts of $\mathbb{Q}_p$, including, for example, restricted multiplication. While our results don't apply to them directly, it is this author's hope that similar techniques can be used to also compute the VC-density function for those structures.

Open Question 4.4.1. *Compute the VC-density function for $\mathbb{Q}_p$-reducts studied in [Lee14].*

Another interesting question is whether the reduct studied in this chapter has the VC 1 property (see Definition 5.2 in [ADH16]). If so, this would imply the linear VC-density bound directly. The techniques used in the paper [ADH16] make it seem likely that the reduct has the VC 2 property (just as the $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$). While there are techniques for showing that a structure has a given VC n property, less is known about showing that a structure doesn't have a given VC n property. Perhaps the simple structure of the $\mathcal{L}_{aff}$-reduct can help understand this phenomenon better.

Open Question 4.4.2. *For which n does the $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ have the VC n property?*

CHAPTER 5

Dp-minimality in Superflat Graphs

Superflat graphs were introduced in [PZ78] as a natural class of stable graphs. This family of graphs is known in a combinatorial context as nowhere dense graphs, see [AA10] and [NM11]. In this chapter we prove that superflat graphs are dp-minimal. It may be possible to prove dp-minimality by combining a characterization of dp-minimality in stable theories studied in [OU11] with the results on forking in superflat graphs given in [Iva93]. Here, however, we present a direct proof using the characterization of dp-minimality by Lemma 1.4.2.

Section 1 gives all the necessary combinatorial and model-theoretic definitions. In addition, we list several basic results involving connectivity hulls and superflat graphs. In Section 2 we study how to expand parameter sets of indiscernible sequences to increase the distance between the elements of those sequences. Section 3 applies a special case of this result to show dp-minimality of superflat graphs via Lemma 1.4.2. In the concluding section we outline directions for future work.

5.1 Preliminaries

First, we introduce some basic graph-theoretic definitions.

Definition 5.1.1. Work in a possibly infinite graph $\mathbb{G}$. Let $A, B, S, V \subseteq G$ where G is the set of vertices of $\mathbb{G}$.

1. A <u>path</u> is a subgraph of $\mathbb{G}$ with distinct vertices $v_0, v_1, \ldots, v_n$ and an edge between v_{i-1}, v_i for all $i = 1, \ldots n$. It is called a path from A to B if $v_0 \in A$ and $v_n \in B$. The <u>length</u> of such a path is n.

2. Two paths are disjoint if, excluding endpoints, they have no vertices in common.

3. For $a, b \in G$ define the distance $d(a, b)$ between a and b to be the length of the shortest path from a to b in G. If no such path exists then the distance is infinite.

4. For $a, b \in G - A$ define $d_A(a, b)$ to be the distance between a and b in the subgraph of $\mathbb{G}$ induced on the set of vertices $G - A$. Equivalently it is the shortest path between a and b that avoids the vertices in A.

5. We say that S separates A from B if there exists $a \in A - S$ and $b \in B - S$ with $d_S(a, b) = \infty$.

6. We say that A separates V if it separates V from itself.

7. We say that V has connectivity n if there is a set of size n that separates V, but there are no sets of size $n - 1$ that separate V.

8. Suppose V has connectivity n. The connectivity hull of V is defined to be the union of all sets of size n separating V.

In [AB09] we find a generalization of Menger's Theorem for infinite graphs:

Theorem 5.1.2. *Let A and B be two sets of vertices in a possibly infinite graph. Then there exists a set P of disjoint paths from A to B, and a set S of vertices separating A from B, such that S consists of a choice of precisely one vertex from each path in P.*

We use the following easy consequences:

Corollary 5.1.3. *Let V be a subset of vertices of a graph $\mathbb{G}$ with connectivity n. Then there exists a set of n disjoint paths from V into itself.*

Corollary 5.1.4. *With assumptions as above, the connectivity hull of V is finite.*

Proof. All the separating sets have to have exactly one vertex in each of those paths. $\square$

Definition 5.1.5.

- K_n^m denotes the graph obtained from a complete graph on n vertices by adding m vertices to every edge. K_∞^m denotes the same construction on a complete graph with an infinite countable number of vertices.

- A graph is called flat if for every $m \in \mathbb{N}$ the graph avoids K_∞^m as a subgraph.

- A graph is called superflat if for every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that the graph avoids K_n^m as a subgraph.

It is easy to see by compactness that a graph is superflat if and only if there is an elementary extension which is flat. By the same line of reasoning, in uncountably saturated structures the notions of flatness and superflatness coincide.

Theorem 2 in [PZ78] gives a useful characterization of superflat graphs.

Theorem 5.1.6. *The following are equivalent:*

1. *$\mathbb{G}$ is superflat.*

2. *For every $n \in \mathbb{N}$ and an infinite set $A \subseteq G$, there exists a finite $B \subseteq G$ and an infinite $A' \subseteq A$ such that for all $a, b \in A'$ we have $d_B(a, b) > n$.*

Roughly, in superflat graphs every infinite set contains a sparse infinite subset (possibly after throwing away finitely many vertices).

We also note the stability result:

Theorem 5.1.7 (see Corollary 10 in [PZ78])**.** *Every superflat graph is stable.*

5.2 Indiscernible sequences

Fix an uncountable cardinal κ. Work in a superflat graph $\mathbb{S}$ that is κ-saturated and strongly κ-homogeneous. Fix a parameter set $A \subseteq S$ with $|A| < \kappa$. Let $I = (a_i)_{i \in \mathcal{I}}$ be a countable A-indiscernible sequence. Stability implies that I is totally indiscernible (see Lemma 1.2.7).

Definition 5.2.1.

- For a subsequence $\mathcal{J} \subseteq \mathcal{I}$ let $a(\mathcal{J})$ denote the tuple obtained by concatenating $(a_j)_{j \in \mathcal{J}}$.
- Let m be the arity of elements of I, that is, $a_i \in S^m$. We call a set $H \subseteq S$ uniformly definable from I if there is a formula $\phi(x, y_1, \ldots, y_k)$ with $|y_i| = m$ such that for every $\mathcal{J} \subseteq \mathcal{I}$ of size k we have $H = \phi(G, a(\mathcal{J}))$.

First suppose that I consists of singletons, that is $a_i \in S$.

Definition 5.2.2. Let $V \subseteq S$. Define $P_n(V)$, a subgraph of $\mathbb{S}$, to be the union of all paths of length $\leq n$ between the vertices of V.

Lemma 5.2.3. *Let $n \in \mathbb{N}$. There exists a finite set $B \subseteq S$ such that*

$$\forall i \neq j \ d_B(a_i, a_j) > n.$$

Proof. By Theorem 5.1.6 we can find an infinite $\mathcal{J} \subseteq \mathcal{I}$ and a finite set B' such that each pair from $J = (a_j)_{j \in \mathcal{J}}$ has distance $> n$ over B'. By total indiscernibility there exists an automorphism mapping J to I and fixing A. The image of B' under this automorphism is the required set B. □

In other words, B separates I when viewed inside the subgraph $P_n(I)$. This shows that I has finite connectivity in $P_n(I)$. Applying Corollary 5.1.4 we obtain that the connectivity hull of I in $P_n(I)$ is finite.

Definition 5.2.4. Given a graph $\mathbb{G}$ and $V \subseteq G$ define $H(\mathbb{G}, V) \subseteq G$ to be the connectivity hull of V in $\mathbb{G}$. Note that if V is finite, then $H(P_n(V), V)$ is V-definable.

Lemma 5.2.5. *Let H be the connectivity hull of I inside the graph $P_n(I)$, that is, $H = H(P_n(I), I)$. Then H is uniformly definable from I in $\mathbb{S}$.*

Proof. Using total indiscernibility we may assume without the loss of generality that I is indexed by $\mathbb{N}$. Let $I_i = (a_0, a_1, \ldots, a_{i-1})$ a finite subsequence of the sequence I. Let N be the connectivity of I inside of $P_n(I)$.

First note that any finite set $H \subseteq P_n(I)$ will be contained in $P_n(I_i)$ for large enough i. Every element of H is inside a path of length $\leq n$ and endpoints of that path are eventually going to be inside I_i. (Here the assumption that I is enumerated by $\mathbb{N}$ is important.)

Vertices a_0, a_1 cannot be separated by less than N elements inside of $P_n(I)$ (as this would contradict connectivity being N). Thus by Theorem 5.1.2 there are N disjoint paths inside of $P_n(I)$ connecting a_0 to a_1. For large enough i, say $i \geq M_1$, all those paths are contained inside of $P_n(I_i)$. Those paths also witness that vertices a_0, a_1 cannot be separated by less than N elements inside of $P_n(I_i)$. As the set $P_n(I_i)$ is I_i-definable and I is indiscernible, we have that no two vertices can be separated by less than N elements inside of $P_n(I_i)$ Thus I_i has connectivity $\geq N$ inside of $P_n(I_i)$ for $i \geq M_1$.

Consider a set S of size N that separates I inside of $P_n(I)$. This is witnessed by two elements of I that are separated. There are finitely many such sets S as connectivity hull is finite. Thus for large enough i, say $i \geq M_2$, for each such S the segment I_i contains a pair of vertices witnessing that S is a separating set.

Corollary 5.1.3 tells us that there are finitely many paths between elements of V such that $H(P_n(I), I)$ is inside the union of those paths. For large enough i, say $i \geq M_3$, $P_n(I_i)$ will contain all of those paths, and thus $H(P_n(I), I) \subseteq P_n(I_i)$.

Combine those three observations. Let $M = \max(M_1, M_2, M_3)$. Then for $i \geq M$ the set $P_n(I_i)$ contains all the N-element sets separating I in $P_n(I)$, those sets separate I_i in $P_n(I_i)$, and the connectivity of I_i in $P_n(I_i)$ is at most N. But this means that the connectivity of I_i in $P_n(I_i)$ has to be exactly N, and $H(P_n(I), I) \subseteq H(P_n(I_i), I_i)$.

For $i \geq M$ define

$$E_i = \bigcap_{j=M}^{i} H(P_n(I_j), I_j).$$

We have $H(P_n(I), I) \subseteq E_i$ and E_i is a decreasing chain. Suppose

$$H(P_n(I), I) \subsetneq H(P_n(I_M), I_M),$$

that is $H(P_n(I_M), I_M) - H(P_n(I), I) \neq \emptyset$. Then there exists a set S of size N that separates I_M in $P_n(I_M)$ but does not separate I in $P_n(I)$. Thus there has to be a finite subgraph of

$P_n(I)$ disjoint from S that connects all the elements of I_M (witness of failure of separation). For large enough i, say $i \geq M_S$, this subgraph is contained in $P_n(I_i)$. There are finitely many possibilities for S (as connectivity hull of I_M in $P_n(I_M)$ is finite). Let $M_4 = \max_S(M_S)$. Then for $i \geq \max(M_4, M)$ we have

$$H(P_n(I_i), I_i) \cap (H(P_n(I_M), I_M) - H(P_n(I), I)) = \emptyset,$$

and thus $E_i = H(P_n(I), I)$. As E_i is I_i-definable, this shows that $H(P_n(I), I)$ is I_i-definable. Now we need to show uniform definability. Suppose I' is a subsequence of I of length i. There is an automorphism mapping I_i to I' that fixes I setwise. But this automorphism has to fix $H(P_n(I), I)$ setwise, so it maps an I_i-definition of $H(P_n(I), I)$ to an I'-definition of $H(P_n(I), I)$. As I' was arbitrary this shows uniformity. □

Corollary 5.2.6. *Let $H_n = H(P_n(I), I)$. Then*

$$\forall i \neq j \ d_{H_n}(a_i, a_j) > n.$$

Proof. The set H_n separates I inside of $P_n(I)$. In particular there exist $i \neq j$ such that $d_{H_n}(a_i, a_j) = \infty$ inside $P_n(I)$. This means that $d_{H_n}(a_i, a_j) > n$ inside of $\mathbb{S}$. But then by total indiscernibility and using the fact that H_n is uniformly I-definable, this holds for all $i \neq j$. □

We would like to start working with tuples now instead of singletons. We need some notation to extract individual elements of a tuple:

Definition 5.2.7. Suppose $a = (a_1, \ldots, a_m)$ is a tuple of arity m. Let $a^{(j)}$ denote the j'th component, that is $a^{(j)} = a_j$.

More generally, now suppose that I consists of tuples of arity m, that is $a_i \in S^m$.

Definition 5.2.8.

- We would like extract j'th components out of elements of I. Let $I^{(j)} = ({a_i}^{(j)})_{i \in \mathcal{I}}$, an A-indiscernible sequence of singletons.

- Let ${H_n}^{(j)} = H(P_n(I^{(j)}), I^{(j)})$.
- Let

$$B_n = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} {H_n}^{(j)}.$$

Note that B_n is finite as each ${H_n}^{(j)}$ is finite by Corollary 5.1.4.

Lemma 5.2.9. *The sequence I is indiscernible over $A \cup B_n$.*

Proof. By Lemma 5.2.5 the set ${H_n}^{(j)}$ is uniformly $I^{(j)}$-definable. Thus it is uniformly I-definable. Then B_n is a finite union of uniformly I-definable sets, thus also uniformly I-definable.

By uniform definability there is a formula $\phi(z, w_1, \ldots, w_k)$ with $|z| = 1$ and $|w_i| = m$ such that for any subsequence $\mathcal{J} \subseteq \mathcal{I}$ of length k we have $\phi(G, a(\mathcal{J})) = B_n$. Fix such a subsequence $\mathcal{J}$.

Let $\psi(x_1, \ldots, x_l, y)$ be an arbitrary A-formula with $|x_i| = m$. Consider the collection of traces (i.e., a collection of subsets of $B_n^{|y|}$)

$$\left\{\psi(a(\mathcal{J}'), B_n^{|y|}) \mid \ \mathcal{J}' \text{ a subsequence of } \mathcal{I} \text{ of length } l \text{ disjoint from } \mathcal{J}\right\}.$$

If two of the traces are distinct, then by indiscernibility all of them are (using the fact that B_n is uniformly definable). But that is impossible as B_n is finite and thus has finitely many subsets. Thus all such traces are identical. As the choice of $\mathcal{J}$ was arbitrary, we can drop the condition that $\mathcal{J}'$ is disjoint from $\mathcal{J}$. This shows that for any $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{I}$ of length l and $h \in B_n^{|y|}$ we have

$$\mathbb{S} \models \psi(a(\mathcal{J}_1), h) \iff \mathbb{S} \models \psi(a(\mathcal{J}_2), h).$$

As the choice of ψ was arbitrary, this shows that I is indiscernible over $A \cup B_n$ as needed. $\square$

Definition 5.2.10. For tuples a, b of the same arity m and $B \subseteq S$ define

$$d_B(a, b) = \min_{1 \leq i,j \leq m} d_B(a^{(i)}, b^{(j)}).$$

Lemma 5.2.11.

$$\forall i \neq j \ d_{B_n}(a_i, a_j) > n/2.$$

Proof. Towards a contradiction suppose we have some $i \neq j$ and k, l such that

$$d_{B_n}({a_i}^{(k)}, {a_j}^{(l)}) \leq n/2.$$

As B_n is uniformly I-definable, by total indiscernibility we have that this inequality holds for all $i \neq j$. Assuming for convenience that I is enumerated by naturals, let $b_1 = {a_1}^{(k)}$, $b_2 = {a_2}^{(l)}$, $b_3 = {a_3}^{(k)}$ (note the superscripts). Then we have

$$d_{B_n}(b_1, b_2) \leq n/2,$$
$$d_{B_n}(b_3, b_2) \leq n/2.$$

By the triangle inequality

$$d_{B_n}(b_1, b_3) \leq n,$$
$$d_{B_n}({a_1}^{(k)}, {a_3}^{(k)}) \leq n.$$

But this is a contradiction, as Corollary 5.2.6 gives us

$$\forall i \neq j \ d_{{H_n}^{(k)}}({a_i}^{(k)}, {a_j}^{(k)}) > n$$

and we have ${H_n}^{(k)} \subseteq B_n$. □

Corollary 5.2.12. *There is a countable B such that I is indiscernible over $A \cup B$ and*

$$\forall i \neq j \ d_B(a_i, a_j) = \infty.$$

Proof. Let B_n as above. By Lemma 5.2.11 we have

$$\forall i \neq j \ d_{B_n}(a_i, a_j) > n/2,$$

and I is indiscernible over $A \cup B_n$ by Lemma 5.2.9. Let $B = \bigcup_{n \in \mathbb{N}} B_n$. Then

$$\forall i \neq j \ d_B(a_i, a_j) = \infty.$$

As $B_n \subseteq B_{n+1}$, the sequence I is indiscernible over $A \cup B$ as needed. □

Thus I can be upgraded to have infinite distance over its parameter set.

5.3 Superflat graphs are dp-minimal

Definition 5.3.1. For $B \subseteq S$ define an equivalence relation $\sim_B$ on $S - B$ as follows: two vertices b, c are $\sim_B$-equivalent if $d_B(b, c)$ is finite.

Lemma 5.3.2. *Fix tuples a, b, c in S, with a, b having the same arity. Also let $B \subseteq S$. Suppose* $\operatorname{tp}(a/B) = \operatorname{tp}(b/B)$ *and* $d_B(a, c) = d_B(b, c) = \infty$*. Then* $\operatorname{tp}(a/Bc) = \operatorname{tp}(b/Bc)$*.*

Proof. Suppose $a = (a_1, a_2, \ldots, a_m)$ and $b = (b_1, b_2, \ldots, b_m)$. Define X_j to be the $\sim_B$-equivalence class of a_j or $X_j = \emptyset$ if $a_j \in B$. Similarly define Y_j for b_j. There is an automorphism f of $\mathbb{S}$ fixing B with $f(a) = b$. It's easy to see that $f(X_j) = Y_j$ setwise. We would like to define a function $g \colon S \longrightarrow S$. For each j let $g = f$ on X_j. Additionally if $X_j \neq Y_j$ then also let $g = f^{-1}$ on Y_j. Define g to be identity on the rest of S. It is easy to check that g is a well-defined automorphism fixing Bc that maps a to b. This shows that $\operatorname{tp}(a/Bc) = \operatorname{tp}(b/Bc)$. □

Lemma 5.3.3. *Let $b \in G$. There exists $c \in \mathcal{I}$ such that all $(a_i)_{i \in \mathcal{I} - c}$ have the same type over Ab.*

Proof. Use Corollary 5.2.12 to find $B \supseteq A$ such that I is indiscernible over B and has infinite distance over B. All the tuples of the indiscernible sequence fall into distinct $\sim_B$-equivalence classes. If $b \in B$ we are done. Otherwise, there can be at most one element of the sequence that is in the same $\sim_B$-equivalence class as b. Exclude that element from the sequence. Remaining sequence elements are all infinitely far away from b over B. By the previous lemma we have that the elements of the indiscernible sequence all have the same type over Bb as needed. □

Theorem 5.3.4. *Superflat graphs are dp-minimal.*

Proof. It suffices to show that $\mathbb{S}$ is dp-minimal. Using Lemma 1.4.2, by total indiscernibility it is enough to show that if $b \in S$ and I is a countable $\emptyset$-indiscernible sequence then one element can be excluded from I, so that the remaining elements have the same type over b. But this is precisely Lemma 5.3.3. □

5.4 Conclusion

The determination of dp-minimality is the first step towards establishing bounds on VC-density. It is this author's hope that the simple structure of superflat graphs yields nicely behaved VC-density. We pose the following question for the future work:

Open Question 5.4.1. *What are the bounds on VC-density function in superflat graphs? In particular, do we have* $\mathrm{vc}(1) = 1$ *or* $\mathrm{vc}(n) = n\,\mathrm{vc}(1)$*? Are the bounds better in specific classes of superflat graphs, such as planar graphs, graphs with bounded tree-width, or graphs excluding certain classes of subgraphs?*

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