1. Graph Combinatorics

Throughout this paper A, B, C, M will denote finite graphs, and $\mathbb{D}$ will be used to denote potentially infinite graphs. For a graph A the set of its vertices is denoted by $v(A)$, and the set of its edges by $e(A)$. Number of vertices of A will be denoted as $|A|$. Subgraph always means induced subgraph and $A \subseteq B$ means that A is a subgraph of B. For two subgraphs A, B of a larger graph, the union $A \cup B$ denotes the graph induced by $v(A) \cup v(B)$. Similarly, $A - B$ means a subgraph of A induced by the vertices of $v(A) - v(B)$. For $A \subseteq B \subseteq D$ and $A \subseteq C \subseteq D$, graphs B, C are said to be <u>disjoint over A</u> if $v(B) - v(A)$ is disjoint from $v(C) - v(A)$ and there are no edges from $v(B) - v(A)$ to $v(C) - v(A)$ in D.

For the remainder of the paper fix $\alpha \in (0, 1)$, irrational.

Definition 1.1.

- For a graph A let $\dim(A) = |A| - \alpha|e(A)|$.
- For A, B with $A \subseteq B$ define $\dim(B/A) = \dim(B) - \dim(A)$.
- We say that $A \leq B$ if $A \subseteq B$ and $\dim(A'/A) > 0$ for all $A \subsetneq A' \subseteq B$.
- Define A to be <u>positive</u> if for all $A' \subseteq A$ we have $\dim(A') \geq 0$.
- We work in theory S_α in the language of graphs axiomatized by:
  - Every finite substructure is positive.
  - Given a model $\mathbb{G}$ and graphs $A \leq B$, every embedding $f : A \longrightarrow \mathbb{G}$ extends to an embedding $g : B \longrightarrow \mathbb{G}$.

  (Here an embedding maps edges to edges and nonedges to nonedges.) This theory is complete and stable (see 5.7 and 7.1 in [2]). From now on fix an ambient model $\mathbb{G} \models S_\alpha$. This will be the only infinite graph we work with.
- For A, B positive, (A, B) is called a <u>minimal pair</u> if $A \subseteq B$, $\dim(B/A) < 0$ but $\dim(A'/A) \geq 0$ for all proper $A \subseteq A' \subsetneq B$. We call B a <u>minimal extension</u> of A. The dimension of a minimal pair is defined as $|\dim(B/A)|$.
- A sequence $\langle M_i \rangle_{0 \leq i \leq n}$ is called a <u>minimal chain</u> if (M_i, M_{i+1}) is a minimal pair for all $0 \leq i < n$.

- For a graph A with the tuple of vertices x let $\mathrm{diag}_A(x)$ be the atomic diagram of A, i.e. the first-order formula recording whether there is an edge between every pair of vertices.
- Given $A \subseteq B$ let

$$\phi_{A,B}(x) = \mathrm{diag}_A(x) \wedge \exists z \ \mathrm{diag}_B(x, z).$$

  Any graph isomorphic to B is called a <u>witness</u> of $\phi_{A,B}$.
- A formula $\phi_{A,B}$ is called a <u>basic formula</u> if there is a minimal chain $\langle M_i \rangle_{0 \leq i \leq n}$ such that $A = M_0$ and $B = M_n$.

Theorem 1.2 (Quantifier elimination, 5.6 in [2])**.** *In theory S_α every formula is equivalent to a boolean combination of basic formulas.*

Definition 1.3. A graph $S \subseteq \mathbb{D}$ is called <u>N-strong</u> if for any $S \subseteq T \subseteq D$ with $|T| - |S| \leq N$ we have $S \leq T$.

2. Basic Definitions and Lemmas

Definition 2.1. Suppose $\phi(x, y)$ is a basic formula. Define X to be the graph on vertices x with edges defined by ϕ. Similarly define Y. Note that X, Y are positive. Additionally, let Y' be a subgraph of Y induced by vertices of Y that are connected to $W - (X \cup Y)$, where W is a witness of ϕ.

Definition 2.2. Suppose A, B are subgraphs of $\mathcal{D}$ such that $v(A), v(B)$ are disjoint. Then define $\mathscr{E}(A, B)$ to be the number of edges between the vertices in $v(A)$ and the vertices in $v(B)$.

We will require the following lemmas from [2]:

Lemma 2.3. *[See 2.3 in [2]] Let $A, B \subseteq \mathbb{D}$. Then*

$$\dim(A \cup B/A) \leq \dim(B/A \cap B).$$

Moreover,

$$\dim(A \cup B/A) = \dim(B/A \cap B) - \alpha E,$$

where E is the number of edges connecting the vertices of $B - A$ to the vertices of $A - B$.

Lemma 2.4. *[See 4.1 in [2]] Suppose A is a positive graph, with at least $1/\alpha + 2$ vertices. Then for any $\epsilon > 0$ there exists a graph B such that (A, B) is a minimal pair with dimension $\leq \epsilon$. Moreover, every vertex in A is connected to a vertex in $B - A$.*

Lemma 2.5. *[See 4.4 in [2]] Suppose A is a positive graph, and $\mathcal{G}$ a model of S_α. Then for any integer S there exists an embedding $f\colon A \longrightarrow \mathcal{G}$ such that $f(A)$ is S-strong in $\mathcal{G}$.*

Lemma 2.6. *[See 3.8 in [2]] For all $S > 0$ there exists $M = M(S, \alpha) \in \mathbb{N}$ with the following property. Suppose $A \subseteq \mathcal{G}$ where $\mathcal{G}$ is a model of S_α. Then there exists B with $A \subseteq B \subseteq \mathcal{G}$ such that B is S-strong in $\mathbb{G}$ and $|B| \leq M|A|$.*

We conclude this section by stating a couple of technical lemmas that will be useful in our proofs later.

Lemma 2.7. *Work in an ambient graph $\mathbb{D}$. Suppose we have a set B and a minimal pair (A, M) with $A \subseteq B$ and $\dim(M/A) = -\epsilon$. Then either $M \subseteq B$ or $\dim(M \cup B/B) < -\epsilon$.*

Proof. By Lemma 2.3

$$\dim(M \cup B/B) \leq \dim(M/M \cap B),$$

and as $A \subseteq M \cap B \subseteq M$

$$\dim(M/A) = \dim(M/M \cap B) + \dim(M \cap B/A).$$

In addition we are given $\dim(M/A) = -\epsilon$. If $M \not\subseteq B$ then $A \subseteq M \cap B \subsetneq M$ and by minimality $\dim(M \cap B/A) > 0$. Combining the inequalities above we obtain the desired result:

$$\dim(M \cup B/B) \leq \dim(M/M \cap B) = \dim(M/A) - \dim(M \cap B/A) < -\epsilon.$$

□

Lemma 2.8. *Work in an ambient graph $\mathbb{D}$. Suppose we have a set B and a minimal chain $\langle M_i \rangle_{0 \leq i \leq n}$ with dimensions*

$$\dim(M_{i+1}/M_i) = -\epsilon_i$$

and $M_0 \subseteq B$. Let $\epsilon = \min_{0 \leq i \leq n} \epsilon_i$. Then either $M_n \subseteq B$ or $\dim((M_n \cup B)/B) < -\epsilon$.

Proof. Let $\bar{M}_i = M_i \cup B$. Then:

$$\dim(\bar{M}_n/B) = \dim(\bar{M}_n/\bar{M}_{n-1}) + \ldots + \dim(\bar{M}_2/\bar{M}_1) + \dim(\bar{M}_1/B).$$

Either $M_n \subseteq B$ or at least one of the summands above is nonzero. Apply previous lemma. □

Lemma 2.9. *Suppose we have a minimal pair (A, M) with dimension ϵ. Suppose we have some $B \subseteq M$. Then $\dim B/(A \cap B) \geq -\epsilon$. Moreover if $B \cup A \neq M$ then $\dim B/(A \cap B) \geq 0$.*

Proof. We have $\dim(B \cup A/A) \leq \dim B/(A \cap B)$ by Lemma 2.3. As $A \subseteq B \cup A \subseteq M$ we have $\dim(B \cup A/A) \geq -\epsilon$ by minimality. Moreover, minimality implies that it is positive if $B \cup A \neq M$. □

Lemma 2.10. *Suppose we have a minimal chain $\langle M_i \rangle_{0 \leq i \leq n}$ with dimensions*

$$\dim(M_{i+1}/M_i) = -\epsilon_i.$$

Let ϵ be the sum of all ϵ_i. Suppose we have a graph B with $B \subseteq M_n$. Then $\dim B/(M_0 \cap B) \geq -\epsilon$.

Proof. Let $B_i = B \cap M_i$. We have $\dim B_{i+1}/B_i \geq \dim M_{i+1}/M_i$ by the previous lemma. Thus

$$\dim B/(M_0 \cap B) = \dim B_n/B_0 = \sum \dim B_{i+1}/B_i \geq -\epsilon.$$

□

3. Upper bound

Consider a basic formula $\phi(x, y)$ with a minimal chain $\langle M_i \rangle_{0 \leq i \leq n_\phi}$ with dimensions $\dim(M_{i+1}/M_i) = -\epsilon_i$. Define

$$\begin{aligned} \epsilon(\phi) &= \min \{\epsilon_i\}_{0 \leq i \leq n_\phi} \\ K(\phi) &= |M_{n_\phi}|. \end{aligned}$$

Now consider a finite collection of basic formulas

$$\Phi = \Phi(\vec{x}, \vec{y}) = \{\phi_i(\vec{x}, \vec{y})\}_{i \in I}.$$

Define

$$\begin{aligned} \epsilon(\Phi) &= \min \{\epsilon(\phi_i)\}_{i \in I} \cup \{\alpha\}, \\ K(\Phi) &= \max \{K(\phi_i)\}_{i \in I}, \\ D_1(\Phi) &= \frac{K(\Phi)}{\epsilon(\Phi)}, \\ D(\Phi) &= |y| D_1(\Phi). \end{aligned}$$

We claim that

Theorem 3.1. *If ϕ is a boolean combination of formulas from Φ, then* $\mathrm{vc}(\phi) \leq D(\Phi)$.

Let

$$S = \left\lceil \left(\frac{|y|}{\epsilon(\phi)} + 1 \right) K(\phi) \right\rceil .$$

Suppose we have a finite parameter set $A_0 \subseteq \mathbb{G}^{|x|}$ with $|A_0| = N_0$. We would like to bound $\phi(A_0, \mathbb{G}^{|y|})$. Let $A_1 \subseteq \mathbb{G}$ consist of the components of the elements of A_0. Then $|A_1| \leq |x| N_0$. Using Lemma 2.6 let A be a graph $A_0 \subseteq A \subseteq \mathbb{G}$, S-strong in $\mathbb{G}$. Let $N = |A|$. We have $N \leq |x| N_0 M$ (where M is the constant from the Lemma 2.6). As $A_0 \subseteq A^{|x|}$ we have

$$\left| \phi(A_0, \mathbb{G}^{|y|}) \right| \leq \left| \phi(A^{|x|}, \mathbb{G}^{|y|}) \right| .$$

Therefore it suffices to bound $\left| \phi(A^{|x|}, \mathbb{G}^{|y|}) \right|$.

Definition 3.2. For $A \subseteq \mathbb{G}^{|x|}, B \subseteq \mathbb{G}^{|y|}$ define

$$\Phi(A, B) = \{(a, i) \in A \times I \mid \mathbb{G} \models \phi_i(a, b)\} \subseteq A \times I$$

Lemma 3.3. *For $A \subseteq \mathbb{G}^{|x|}, B \subseteq \mathbb{G}^{|y|}$ if ϕ is a boolean combination of formulas from Φ then*

$$|\phi(A, B)| \leq |\Phi(A, B)|$$

Proof. Clear, as for all $a \in A, b \in B$ the set $\Phi(a, b)$ determines the truth value of $\phi(a, b)$. □

Thus it suffices to bound $\left| \Phi(A^{|x|}, \mathbb{G}^{|y|}) \right|$.

Definition 3.4.
- For all $i \in I, a \in A^{|x|}, b \in \mathbb{G}^{|y|}$ if $\phi_i(a, b)$ holds fix $W^i_{a,b} \subseteq \mathbb{G}$, a witness of this formula.
- For $b \in \mathbb{G}^{|y|}$ let

$$W_b = \bigcup \left\{ W^i_{a,b} \mid a \in A^{|x|}, i \in I, \mathbb{G} \models \phi_i(a, b) \right\} .$$

- For sets $C, B \subset \mathbb{G}$ define the boundary of C over B

$$\partial(C,B) = \{b \in B \mid \mathscr{E}(b, C-B) \neq \emptyset\}$$

(see Definition 2.2 for $\mathscr{E}$).

- For $b \in \mathbb{G}^{|y|}$ let $\partial_b \subseteq A$ be the boundary $\partial(W_b, A)$.
- For $b \in \mathbb{G}^{|y|}$ let $\bar{W}_b = (W_b - A) \cup \partial_b$.
- For $b_1, b_2 \in \mathbb{G}^{|y|}$ we say that $b_1 \sim b_2$ if $\partial_{b_1} = \partial_{b_2}$, $b_1 \cap A = b_2 \cap A$, and there exists a graph isomorphism from $\bar{W}_{b_1} \cup b_1$ to $\bar{W}_{b_2} \cup b_2$ that fixes ∂_{b_1} and maps b_1 to b_2. One easily checks that this defines an equivalence relation.
- For $b \in \mathbb{G}^{|y|}$ define $\mathscr{I}_b$ to be the $\sim$-equivalence class of b.

Lemma 3.5. *For $b_1, b_2 \in \mathbb{G}^{|y|}$ if $b_1 \sim b_2$ then $\Phi(A^{|x|}, b_1) = \Phi(A^{|x|}, b_2)$.*

Proof. Fix a graph isomorphism $\bar{f}\colon \bar{W}_{b_1} \cup b_1 \longrightarrow \bar{W}_{b_2} \cup b_2$. Extend it to an isomorphism $f\colon W_{b_1} \cup A \longrightarrow W_{b_2} \cup A$ by being an identity map on the new vertices. Suppose $\mathbb{G} \models \phi_i(a, b_1)$ for some $a \in A^{|x|}$. Then $f(W^i_{a,b_1})$ is a witness for $\phi_i(a, b_2)$ (though not necessarily equal to W^i_{a,b_2}) and thus $\mathbb{G} \models \phi_i(a, b_2)$. As a was arbitrary, this proves the equality of the traces. $\square$

Thus to bound the number of traces it is sufficient to bound the number of possibilities for $\mathscr{I}_b$.

Theorem 3.6. *Suppose we have $b \in \mathbb{G}^{|y|}$. Let $Y = |b - A|$. Then*

$$|\partial_b| \leq Y D_1(\phi)$$

$$|\bar{W}_b| \leq 3Y D_1(\phi)$$

From this theorem we get the desired result:

Corollary 3.7. *(Theorem 3.1) If ϕ is a boolean combination of formulas from Φ, then* $\mathrm{vc}(\phi) \leq D(\Phi)$.

Proof. We count possible equivalence classes of $\sim$. This essentially boils down to bounding possibilities for ∂_b, $b \cap A$, and number of isomorphism classes of W_b. Fix

$b \in \mathbb{G}^{|y|}$ and let $Y = |b - A|$. Let $D = YD_1(\Phi)$. By the first part of Theorem 3.6 there are $\binom{N}{D}$ choices for boundary ∂_b. By the second part of Theorem 3.6 there are at most $3D$ vertices in $\bar{W}_b$. Thus to determine the graph $\bar{W}_b$ we need to choose how many vertices it has and then decide where edges go. This amounts to at most $3D2^{(3D)^2}$ choices. Additionally there are $\binom{N}{|y|-Y}$ choices for $b \cap A$. In total this gives us at most

$$\binom{N}{D} \cdot \binom{N}{|y| - Y} \cdot 3D2^{(3D)^2} = O(N^{D+|y|-Y}) =$$
$$= O(N^{YD_1(\Phi)+|y|-Y}) = O(N^{|y|D_1(\Phi)}) = O(N^{D(\Phi)})$$

choices (second to last inequality uses $D_1(\Phi) \geq 1$). By Lemma 3.5 we have $\left|\Phi(A^{|x|}, \mathbb{G}^{|y|})\right| = O(N^{D(\Phi)})$. Recall that

$$\left|\phi(A_0, \mathbb{G}^{|y|})\right| \leq \left|\Phi(A^{|x|}, \mathbb{G}^{|y|})\right|.$$

Therefore we have

$$\left|\phi(A_0, \mathbb{G}^{|y|})\right| = O(N^{D(\Phi)}) = O((|x|N_0M)^{D(\Phi)}) =$$
$$= O((|x|M)^{D(\Phi)}\, N_0^{D(\Phi)}) = O(N_0^{D(\Phi)}).$$

As A_0 was arbitrary, this shows that $\mathrm{vc}(\phi) \leq D(\Phi)$ as needed. (Note that throughout this proof we are using the fact that quantities $D_1(\Phi), D(\Phi), M$ are completely determined by Φ, thus independent from A_0.) $\square$

Proof. (of Theorem 3.6)

The graph W_b is a union of witnesses of the from $W_{a,b}$ for some $a \in A^{|x|}, b \in \mathbb{G}^{|y|}$. Enumerate all of them as $\{W_j\}_{1 \leq j \leq J}$. Define $M_j = \bigcup_1^j W_{j'}$ for $1 \leq j \leq J$ and let $M_0 = b$. Let $\bar{A} = A \cup b$.

Definition 3.8. For $0 \leq j \leq J$ define:

- Let $v_j = 1$ if new vertices are added to M_j outside of $\bar{A}$, that is if $M_j - \bar{A} \neq M_{j-1} - B$, and let it be 0 otherwise.

- Let $E_j = \partial(A - W_j, M_j - A)$.
- Let

$$m_j = \sum_{j'=0}^{j} (v_j + |E_j|).$$

(Here assume $M_{-1} = \emptyset$.)

Lemma 3.9. *For* $0 \leq j \leq J$ *we have*

$$|\partial(M_j, A)| \leq |E_0| + m_j K(\Phi)$$

Proof. Proceed by induction. The base case $j = 0$ is clear. For an induction step suppose that

$$|\partial(M_{j-1}, A)| \leq m_{j-1} K(\Phi)$$

holds. Let

$$\begin{aligned} \delta_1 &= \partial(M_j, A) - \partial(M_{j-1}, A) = \\ &= \{a \in A \mid \mathscr{E}(a, M_j - A) \neq \emptyset \text{ and } \mathscr{E}(a, M_{j-1} - A) = \emptyset\}. \end{aligned}$$

If $M_j - A = M_{j-1} - A$ then $\delta_1 = \emptyset$ and we are done as m_j is increasing. Suppose not. We have $|\delta_1| = |\delta_1 \cap W_j| + |\delta_1 - W_j|$, and

$$\delta_1 - W_j = \{a \in A - W_j \mid \mathscr{E}(a, M_j - A) \neq \emptyset \text{ and } \mathscr{E}(a, M_{j-1} - A) = \emptyset\}.$$

But then it's clear that $\delta_1 - W_j \subseteq E_j$ as

$$\begin{aligned} &W_j - M_{j-1} - A \subseteq M_j - A, \\ &(W_j - M_{j-1} - A) \cap (M_{j-1} - A) = \emptyset. \end{aligned}$$

As $b \in M_{j-1}$ and $M_j - A \neq M_{j-1} - A$, then $M_j - \bar{A} \neq M_{j-1} - \bar{A}$, and thus $v_j = 1$. Therefore we have

$$|\delta_1| = |\delta_1 \cap W_j| + |\delta_1 - W_j| \leq |W_j| + |E_j| \leq$$
$$\leq K(\Phi) + |E_j| \leq (v_j + |E_j|)K(\Phi) \leq (m_j - m_{j-1})K(\Phi),$$

as needed. □

Lemma 3.10. *For $0 \leq j \leq J$ we have*

$$|M_j - \bar{A}| \leq \sum_{j'=0}^{j} v_{j'} K(\Phi)$$

Proof. Proceed by induction. The base case $j = 0$ is clear. For an induction step suppose that

$$|M_{j-1} - \bar{A}| \leq \sum_{j'=0}^{j-1} v_{j'} K(\Phi)$$

holds. If $M_j - \bar{A} = M_{j-1} - \bar{A}$ then the inequality is immediate as $v_j \geq 0$. Therefore assume this is not the case, so $v_j = 1$ and $|M_j - A| - |M_{j-1} - A| \leq |W_j| \leq v_j K(\Phi)$, and so we get the required inequality.

□

Lemma 3.11. *For $0 \leq j \leq J$ we have*

$$\dim(M_j \cup \bar{A}/\bar{A}) \leq -m_j \epsilon(\Phi),$$

Proof. Proceed by induction. Base case $j = 0$ is clear. For an induction step suppose that

$$\dim(M_{j-1} \cup \bar{A}/\bar{A}) \leq -m_{j-1} \epsilon(\Phi)$$

holds. We have

$$\dim(M_j \cup \bar{A}/\bar{A}) = \dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) + \dim(M_{j-1} \cup \bar{A}/\bar{A}) \leq$$

$$\leq \dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) - m_{j-1}\epsilon(\Phi).$$

Let $\bar{M}_{j-1} = M_{j-1} \cup \bar{A}$. By Lemma 2.3

$$\dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) = \dim(W_j \cup \bar{M}_{j-1}/\bar{M}_{j-1}) = \dim(W_j/W_j \cap \bar{M}_{j-1}) - e\alpha$$

where e is the number of edges connecting the vertices of $\bar{M}_{j-1} - W_j$ to the vertices of $W_j - \bar{M}_{j-1}$. Recall that $E_j = \partial(A - W_j, M_j - A)$. We have $A - W_j \subseteq \bar{M}_{j-1} - W_j$ (as $A \subseteq \bar{M}_{j-1}$) and $W_j - M_{j-1} - A = W_j - \bar{M}_{j-1}$ (as for $j > 1$, we have $b \subseteq M_{j-1}$). Thus $|E_j| \leq e$, and we get

$$\dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) \leq \dim(W_j/W_j \cap \bar{M}_{j-1}) - |E_j|\alpha.$$

If $W_j \subseteq \bar{M}_{j-1}$ then $\dim(W_j/W_j \cap \bar{M}_{j-1}) = 0$. If not, then by Lemma 2.8 we have $\dim(W_j/W_j \cap \bar{M}_{j-1}) \leq -\epsilon(\Phi)$. Either way, we have $\dim(W_j/W_j \cap \bar{M}_{j-1}) \leq -v_j\epsilon(\Phi)$. Using this and the fact that $\epsilon(\Phi) \leq \alpha$, we obtain

$$\dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) \leq -v_j\epsilon(\Phi) - |E_j|\epsilon(\Phi) = -(m_j - m_{j-1})\epsilon(\Phi).$$

Finally,

$$\dim(M_j \cup \bar{A}/\bar{A}) \leq \dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) - m_{j-1}\epsilon(\Phi) \leq$$

$$\leq -(m_j - m_{j-1})\epsilon(\Phi) - m_{j-1}\epsilon(\Phi) = -m_j\epsilon(\Phi),$$

as needed. □

(Proof of Theorem 3.6 continued) For any $0 \leq j \leq J$ we have

$$\dim(M_j \cup A/A) = \dim(\bar{A}/A) + \dim(M_j \cup \bar{A}/\bar{A})$$

$$\leq Y - |E_0|\alpha + \dim(M_j \cup \bar{A}/\bar{A}).$$

Lemma 3.11 gives us

$$\dim(M_j \cup \bar{A}/\bar{A}) \leq -m_j \epsilon(\Phi).$$

Thus

$$\dim(M_j \cup A/A) \leq Y - |E_0|\alpha - m_j \epsilon(\Phi).$$

Suppose j is an index such that

$$Y - |E_0|\alpha - m_j \epsilon(\Phi) \geq 0,$$

$$Y - |E_0|\alpha - m_{j+1} \epsilon(\Phi) < 0$$

if one exists. Then

$$m_j \leq \frac{Y - |E_0|\alpha}{\epsilon(\Phi)}.$$

By Lemma 3.10 we have

$$\begin{aligned} |M_{j+1} - A| &\leq \left(\sum_{j'=1}^{j+1} v_{j'} \right) K(\Phi) \leq (m_j + 1) K(\Phi) \\ &\leq \left(\frac{Y - |E_0|\alpha}{\epsilon(\Phi)} + 1 \right) K(\Phi) \leq S. \end{aligned}$$

This is a contradiction, as A is S-strong and $\dim(M_{j+1} \cup A/A)$ is negative. Thus $Y - |E_0|\alpha - m_j \epsilon(\Phi) \geq 0$ for all $j \leq J$. In particular $Y - |E_0|\alpha - m_J \epsilon(\Phi) \geq 0$, so $m_J \leq \frac{Y - |E_0|\alpha}{\epsilon(\Phi)}$. Noting that $M_J = W_b$, Lemma 3.9 gives us

$$|\partial_b| = |\partial(W_b, A)| \leq |E_0| + m_J K(\Phi) \leq |E_0| + K(\Phi) \frac{Y - |E_0|\alpha}{\epsilon(\Phi)}.$$

As $K(\Phi) \geq 1$ and $\epsilon(\Phi) \geq \alpha$, we get

$$|\partial_b| \leq K(\Phi) \frac{Y}{\epsilon(\Phi)} = Y D_1(\Phi).$$

But this is precisely the first inequality we need to prove. For the second inequality, Lemma 3.10 gives us

$$\left|W_b - \bar{A}\right| \le Y + \left(\sum_{j'=0}^{J} v_{j'}\right) K(\Phi) \le Y + m_J K(\Phi) \le$$
$$\le Y + K(\Phi)\frac{Y}{\epsilon(\Phi)} \le 2Y D_1(\Phi).$$

Thus we have

$$|\bar{W}_b| \le |W_b - A| + |\partial_b| \le 3Y D_1(\Phi),$$

as needed. This ends the proof for Theorem 3.6. $\square$

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