Fix a formula $\phi(x, y)$ that is a minimal extension $M/\{x, y\}$.

- $\dim(M/\{x, y\}) = -\epsilon$
- there are no edges between x and y.
- there are no edges between x.

Let $Y = \dim(y)$

Let n be such that $n\epsilon < Y$ but $(n + 1)\epsilon > Y$. Fix a parameter set A, strongly embedded and disconnected (thus indiscernible).

1. Lower bound

Pick a finite $B \subset A^{|x|}$.

Consider the graph $x \cup y$. If y/x is not a proper extension, then ϕ has no realizations over B. If it is, abstractly make a realization of y, label it by b.

Fix arbitrary elements of B, label them a_i for $i = [0..n]$, with each $|a_i| = |x|$. Abstractly adjoin $M_i/\{a_i, b\} = M/\{x, y\}$ for each i. Let $\bar{M} = \bigcup M_i$.

Claim: $A \leq \bar{M}$. It's total dimension is $Y - n\epsilon > 0$ and all subextensions are positive as well.

Thus a copy of $\bar{M}$ can be embedded over A into our ambient model. Choice of elements of B was arbitrary, thus showing that any n elements can be traced out. Thus we have $O(|B|^n)$ many traces showing vc-density of n.

2. Upper bound

Pick a trace of $\phi(x, y)$ on $A^{|x|}$ by a parameter b.

$$B = \left\{a \in A^{|x|} \mid \phi(a, b)\right\}$$

Pick $B' \subset B$, ordered $B' = \{a_1, \ldots\}$ such that

$$a_i \cap \bigcup_{j \neq i} a_j \neq \emptyset$$

This is always possible by starting with B and taking away elements one by one. Call such a set a *generating set* of B.

Let $M_i/\{a_i, b\}$ be a witness of $\phi(a_i, b)$. Let $\bar{M} = \bigcup M_i$. Consider $\bar{M}/A$.

Claim: $\dim(\bar{M}/A)$ is maximized when all M_i are disjoint. Suppose not.

$\bar{M} \cap A \leq \bar{M}$ as A is strong. (Make sure M is not too big!) Let $\bar{A} = A - \{a_i\}_{i \in I}$. Suppose $\bar{A} \cap \bar{M} \neq \emptyset$. Then we can abstractly reembed $\mathcal{M}$ over A such that $\bar{A} \cap \bar{M} = \emptyset$. This would increase the dimension, contradicting maximality. Thus we can assume $A \cap \bar{M} = \{a_i\}_{i \in I}$

Suppose there is j such that

$$M_j \cap \bigcup_{i \neq j} M_i \neq \emptyset$$

Let $\bar{M}' = \bigcup_{i \neq j} M_i$. Apply lemma to $\bar{M}' \cup \{a_j\}$ and $M_j/\{a_j, b\}$. There are two cases

1. $M_j \subset \bar{M}' \cup \{a_j\}$. In this case there are edges between $\{a_j\}$ and M_j that contribute to dimension less than $-\epsilon$.
2. Otherwise M_j adds extra dimension less than $-\epsilon$

In either case replacing M_j by an isomorphic copy disjoint from $\bar{M}'$ would increase dimension, contradicting minimality.

Thus as A is strong we need $|B'|\epsilon < Y$. This gives us $|B'| \leq n$. Finally we need to relate $|B'|$ to $|B|$.

Suppose we have $C \subset A^{|x|}$, finite with $|C| = N$. A generating set for a trace has to have size $\leq n$. Thus there are $\binom{N}{n} \leq N^n$ choices for a generating set. A set generated from set of size n can have at most $(x|n|)^{|x|}$ elements. Thus a given set of size n can generate at most

$$2^{(x|n|)^{|x|}}$$

sets. Thus the number of possible traces on C is bounded above by

$$2^{(x|n|)^{|x|}} \cdot N^n = O(N^n)$$

This bounds the vc-density by n.

Lemma

Suppose we have a set B and a minimal pair (M, A) with $A \subset M$ and $\dim(M/A) = -\epsilon$. Then either $M \subseteq B$ or $\dim((M \cup B)/B) < -\epsilon$.

Proof

By diamond construction

$$\dim((M \cup B)/B) \leq \dim(M/(M \cap B))$$

and

$$\begin{aligned} \dim(M/(M \cap B)) &= \dim(M/A) - \dim(M/(M \cap B)) \\ \dim(M/A) &= -\epsilon \\ \dim(M/(M \cap B)) &> 0 \end{aligned}$$

Lemma

Suppose we have a set B and a minimal chain M_n with $M_0 \subset B$ and dimensions $-\epsilon_i$. Let ϵ be the minimal of ϵ_i. Then either $M_n \subseteq B$ or $\dim((M_n \cup B)/B) < -\epsilon$.

Proof

Let $\bar{M}_i = M_i \cup B$

$$\dim(\bar{M}_n/B) = \dim(\bar{M}_n/\bar{M}_{n-1}) + \ldots + \dim(\bar{M}_2/\bar{M}_1) + \dim(\bar{M}_1/B)$$

Either $M_n \subseteq B$ or one of the summands above is nonzero. Apply previous lemma.

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