

# VC-density in model theoretic structures

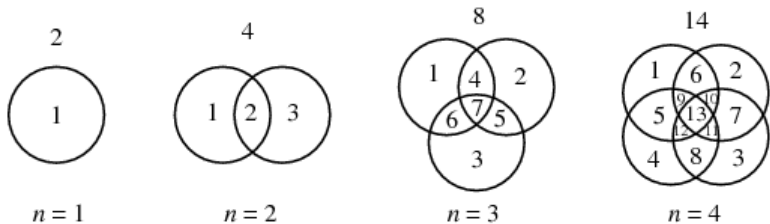
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Suppose we have an (infinite) collection of sets  $\mathcal{F}$ .  
We define a shatter function  $\pi_{\mathcal{F}}(n)$

$$\pi_{\mathcal{F}}(n) = \max\{\# \text{ of atoms in boolean algebra generated by } S \\ | S \subset \mathcal{F} \text{ with } |S| = n\}$$

Example: Let  $\mathcal{F}$  consist of all discs on a plane.



$$\pi_{\mathcal{F}}(1) = 2 \quad \pi_{\mathcal{F}}(2) = 4 \quad \pi_{\mathcal{F}}(3) = 8 \quad \pi_{\mathcal{F}}(4) = 14$$

$$\pi_{\mathcal{F}}(n) = n^2 - n + 2$$

More examples:

1. Lines on a plane  $\pi_{\mathcal{F}}(n) = n^2/2 + n/2 + 1$
2. Disks on a plane  $\pi_{\mathcal{F}}(n) = n^2 - n + 2$
3. Balls in  $\mathbb{R}^3$   $\pi_{\mathcal{F}}(n) = n^3/3 - n^2 + 8n/3$
4. Intervals on a line  $\pi_{\mathcal{F}}(n) = 2n$
5. Half-planes on a plane  $\pi_{\mathcal{F}}(n) = n(n+1)/2 + 1$
6. Finite subsets of  $\mathbb{N}$   $\pi_{\mathcal{F}}(n) = 2^n$
7. Polygons in a plane  $\pi_{\mathcal{F}}(n) = 2^n$

## Theorem (Sauer-Shelah '72)

*Shatter function is either  $2^n$  or bounded by a polynomial.*

### Definition

Suppose growth of shatter function for  $\mathcal{F}$  is polynomial. Let  $r$  be the smallest real such that

$$\pi_{\mathcal{F}}(n) = O(n^r)$$

We define such  $r$  to be the vc-density of  $\mathcal{F}$ ,  $\text{vc}(\mathcal{F})$ . If shatter function grows exponentially, we let the vc-density to be infinite.

# Applications

- ▶ NIP theories
- ▶ VC-Theorem in probability (VapnikChervonenkis 1971)
- ▶ Computational learning theory (PAC learning)
- ▶ Computational geometry
- ▶ Functional analysis (Bourgain-Fremlin-Talagrand theory)
- ▶ Abstract topological dynamics (tame dynamical systems)

# History

- ▶ VapnikChervonenkis 1971 - introduce VC-dimension
- ▶ NIP theories (Shelah 1971, 1990)
- ▶ vc-density (Aschenbrenner, Dolich, Haskell, Macpherson, Starchenko '13)

# Model Theory

Model Theory studies definable sets in first-order structures.

$$(\mathbb{Q}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) = \exists y \ y \cdot y = x$$

In the structure above  $\phi(x)$  defines a set of numbers that are a square.



$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\phi(x) = \exists y \ y \cdot y = x$$

In the structure above  $\phi(x)$  defines the set  $[0, \infty)$ .

$$(\mathbb{R}, 0, 1, +, \cdot, \leq)$$

$$\psi(x_1, x_2) = (x_1 \cdot x_1 + x_2 \cdot x_2 \leq 1.5) \wedge (x_1^2 \leq x_2)$$

This defines a set in  $\mathbb{R}^2$ .

We work with families of uniformly definable sets. Fix a formula  $\phi(x_1 \dots x_n, y_1, \dots y_m) = \phi(\vec{x}, \vec{y})$ . Plug in elements from the model for  $y$  variables to get a family of definable sets in  $M^n$ .

$$\mathcal{F}_\phi^M = \{\phi(x_1, \dots, x_n, a_1, \dots a_n) \mid a_1, \dots a_n \in M\}$$

Define  $\text{vc}^M(\phi)$  to be the vc-density of the family  $\mathcal{F}_\phi^M$

Open Question: it is possible for  $\text{vc}^M(\phi)$  to be irrational?

$$\phi(x_1, x_2, y_1, y_2, y_3) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq y_3^2$$

In structure  $(\mathbb{R}, +, \cdot, \leq)$  given  $a, b, r \in \mathbb{R}$  the formula  $\phi(x_1, x_2, a, b, r)$  defines a disk in  $\mathbb{R}^2$  with radius  $r$  with center  $(a, b)$ . Thus  $\mathcal{F}_\phi^\mathbb{R}$  is a collection of all disks in  $\mathbb{R}^2$ .

Shelah ('90) classified number of isomorphic classes for non-standard models. Important groups of structures included: stable, NIP, simple. A model  $M$  is said to be NIP if all uniformly definable families in it have finite vc-density.

- ▶  $(\mathbb{C}, 0, 1, +, \cdot)$  is stable (so both NIP and simple)
- ▶  $(\mathbb{R}, 0, 1, +, \cdot, \leq)$  is NIP and not stable
- ▶  $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$  is NIP and not stable
- ▶ Random graph  $(V, R)$  is simple and not stable.
- ▶ Pseudo-finite fields are simple and not stable.
- ▶  $(\mathbb{Q}, 0, 1, +, \cdot)$  is in neither of those categories.

Given an NIP structure  $M$  we define a  $\text{vc}$ -function of  $n$  to be the largest  $\text{vc}$ -density achieved by families of uniformly definable sets in  $M^n$ .

$$\text{vc}^M(n) = \max \{ \text{vc}(\phi) \mid \phi(\vec{x}, \vec{y}) \text{ with } |\vec{x}| = n \}$$

Easy to show  $\text{vc}_M(n) \geq n \text{vc}_M(1)$ ,  $\text{vc}_M(1) \geq 1$

Open question: Is  $\text{vc}_M(n) = n \text{vc}_M(1)$ ? If not, is there a linear relationship?

## Examples

- ▶  $(\mathbb{R}, 0, 1, +, \cdot, \leq)$  has  $\text{vc}(n) = n$  (true for all quasi o-minimal structures)
- ▶  $(\mathbb{C}, 0, 1, +, \cdot)$  has  $\text{vc}(n) = n$
- ▶  $(\mathbb{Q}_p, 0, 1, +, \cdot)$  has  $\text{vc}(n) \leq 2n - 1$
- ▶ ACVF has  $\text{vc}(n) \leq 2n$ .

## vc-density in trees

Consider structure  $(T, \leq)$  where elements of  $T$  are vertices of a rooted tree and we say that  $a \leq b$  if  $a$  is below  $b$  in the tree.

- ▶ Trees are NIP (Parigot '82)
- ▶ Trees are dp-minimal (Simon '11)
- ▶ Trees have  $vc(n) = n$  (B. '13)



$\text{tp}(a)$ , a type of an element  $a$  is a set of all the formulas that are true about  $a$ . Parigot's observation: there is a natural way to split a tree into parts  $A, B$  such that for  $a \in A$  and  $b \in B$  we have

$$\text{tp}(a), \text{tp}(b) \vdash \text{tp}(ab)$$

This allows us to decompose complex types into simple parts, which we can use to compute vc-density.

# Further applications

- ▶  $(\mathbb{Q}_p, 0, 1, +, \cdot, |)$
- ▶ other partial orderings, lattices

## vc-density in Shelah-Spencer graphs

Consider a random graph on  $n$  vertices where the probability of the given two vertices having an edge is  $n^{-\alpha}$ . Shelah-Spencer graph is a limit of such graphs for  $\alpha$  irrational in  $(0, 1)$ . We view it in a language with a single binary relation.

- ▶ Shelah-Spencer graphs can be axiomatized (Shelah-Spencer '88)
- ▶ Shelah-Spencer graphs are stable (Baldwin-Shi '96, Baldwin-Shelah '97 )

We show that  $\text{vc}^V(1) = \infty$ , so vc-function is infinite. However to any formula  $\phi(\vec{x}, \vec{y})$  we can prescribe a natural notion of dimension  $\epsilon$ , and we have

$$\text{vc}^V(\phi) < \frac{|x|}{\epsilon}$$

So even though vc-function is not well-behaved, there is still a linear structure on vc-density.

To a finite graph  $A$  assign a dimension  $\delta(A) = |V| - \alpha|E|$ .  $B/A$  is an extension.  $\delta(B/A)$  is  $\delta(A) = |V_B/V_A| - \alpha|E_B/E_A|$ .  $B/A$  is called minimal if its dimension is negative, but every subextension is positive.  $(A_0, \dots, A_n)$  is a  $n$ -minimal chain if  $A_{i+1}/A_i$  is minimal and adds no more than  $n$  new vertices. For any  $A_0$  and  $n$  there exists a maximal  $n$ -minimal chain, moreover the largest set in that chain is unique. Such sets are called  $n$ -strong as every extension with no more than  $n$  new vertices