

# SOME VC-DENSITY COMPUTATIONS IN SHELAH-SPENCER GRAPHS

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ABSTRACT. We compute vc-densities of minimal extension formulas in Shelah-Spencer random graphs.

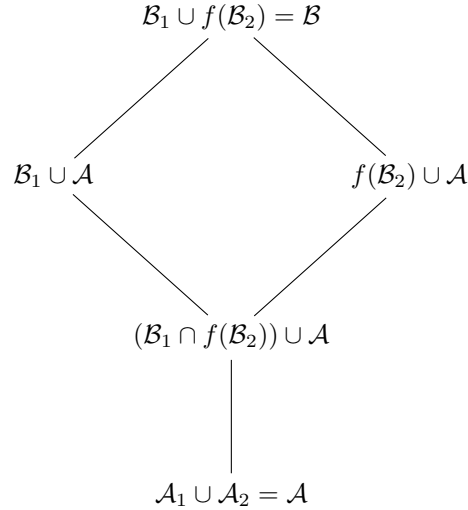
We fix the density of the graph  $\alpha$ .

**Lemma 0.1.** *For any  $\mathcal{A} \in K_\alpha$  and  $\epsilon > 0$  there exists an  $\mathcal{B}$  such that  $(\mathcal{A}, \mathcal{B})$  is minimal and  $\delta(\mathcal{B}/\mathcal{A}) < \epsilon$ .*

*Proof.* Let  $m$  be an integer such that  $m\alpha < 1 < (m+1)\alpha$ . Suppose  $\mathcal{A}$  has less than  $m+1$  vertices. Make a construction  $\mathcal{A}_0 = \mathcal{A}$  and  $\mathcal{A}_{i+1}$  is  $\mathcal{A}_i$  with one extra vertex connected to every single vertex of  $\mathcal{A}_i$ . Stop when the total number of vertices is  $m+1$ . Proceed as in [1] 4.1. Resulting construction is still minimal.  $\square$

**Lemma 0.2.** *Let  $\mathcal{A}_1 \subset \mathcal{B}_1$  and  $\mathcal{A}_2 \subset \mathcal{B}_2$  be  $K_\alpha$  structures with  $(\mathcal{A}_2, \mathcal{B}_2)$  a minimal pair. Let  $M$  be some ambient structure. Fix embeddings of  $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2$  into  $M$ . Assume that it is not the case that  $\mathcal{A}_2 \subset \mathcal{B}_2$ . Now consider all possible embeddings  $f: \mathcal{B}_2 \rightarrow M$  over  $\mathcal{A}_1$ . Let  $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$  and  $\mathcal{B}_f = \mathcal{B}_1 \cup f(\mathcal{B}_2)$  with  $\delta_f = \delta(\mathcal{B}_f/\mathcal{A})$ . Then  $\delta_f$  is at most  $\delta(\mathcal{B}_1 \cup \mathcal{A}/\mathcal{A}) + \delta(\mathcal{B}_2/\mathcal{A}_2)$*

Fix an embedding  $f$ . It induces the following substructure diagram in  $M$ .



From the diagram we see that

$$\delta(\mathcal{B}/\mathcal{A}) \leq \delta(\mathcal{B}_1 \cup \mathcal{A}/\mathcal{A}) + \delta((\mathcal{B}_2 \cup \mathcal{A})/((\mathcal{B}_1 \cap \mathcal{B}_2) \cup \mathcal{A}))$$

Thus all we need to do is to verify that

$$\delta((\mathcal{B}_2 \cup \mathcal{A})/((\mathcal{B}_1 \cap \mathcal{B}_2) \cup \mathcal{A})) \leq \delta(\mathcal{B}_2/\mathcal{A}_2)$$

It is easy to show that this construction induces a proper subpair in  $(\mathcal{A}_2, \mathcal{B}_2)$  which has to have smaller dimension.

Let  $\phi(x, y)$  be a formula in a random graph with  $|x| = |y| = 1$  saying that there exists a minimal extension  $M$  over  $\{x, y\}$  of relative dimension  $\epsilon$ . Let  $n$  be such that  $n\epsilon < 1 < (n+1)\epsilon$ . Then we argue that  $vc(\phi) = n$ .

Fix a  $m$ -strong (for any  $m > |M|$ ) set of non-connected vertices  $B$ . Fix some  $a$ . We investigate the trace of  $\phi(x, a)$  on  $B$ . Suppose we have  $b_1, \dots, b_k$  satisfying  $\phi(b_i, a)$  as witnessed by  $M_j$ . Relative dimension of  $M_1 \cup M_2 \cup \dots \cup M_j \cup a$  is minimized when all  $M_j$  are disjoint (by minimality). Thus for that dimension to be positive we can have at most  $n$  extensions.

#### REFERENCES

- [1] Michael C. Laskowski, *A simpler axiomatization of the Shelah-Spencer almost sure theories*, Israel J. Math. **161** (2007), 157-186. MR MR2350161  
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