A NOTE ON QUANTIFIER ELIMINATION IN SHELAH-SPENCER GRAPHS

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Abstract. We simplify [1]'s proof of quantifier elimination in Shelah-Spencer graphs.

1. Introduction

Laskowski's paper [1] provides a combinatorial proof of quantifier elimination in Shelah-Spencer graphs. Here we provide a simplification of the proof using only maximal chains and avoiding the use of proposition 3.1 and technical lemmas of section 4.

We will use notation of [1], in particular things like $\boldsymbol{K}_\alpha$, $\delta(\mathcal{A}/\mathcal{B})$, $X_m(\mathcal{A})$, S_α, $\mathcal{B}^* \sqsubseteq \mathcal{B}'$,maximal embedding, $\Delta_{\mathcal{A}}(x)$, $\Psi_{\mathcal{A},\mathcal{B}}(x)$ etc. However we will give a different definition of $Y(\ldots)$. When we write formulas $\theta(x, y)$ we may have x, y to be tuples.

2. Omitting lemma

Definition 2.1. Let $\mathcal{M} \models S_\alpha$, $\mathcal{B} \in \boldsymbol{K}_\alpha$, embedding $f\colon \mathcal{B} \to \mathcal{M}$, Φ finite subset of $\boldsymbol{K}_\alpha$

(1) Say that f *omits* Φ if there are no $\mathcal{C} \in \Phi$ and $g\colon \mathcal{C} \to \mathcal{M}$ extending f.
(2) Say that f *admits* Φ if for every $\mathcal{C} \in \Phi$ there is $g\colon \mathcal{C} \to \mathcal{M}$ extending f.

Note 2.2. Take notation as above and a structure $\mathcal{C} \in \boldsymbol{K}_\alpha$ extending $\mathcal{B}$. Then f doesn't omit $\{\mathcal{C}\}$ iff f admits $\{\mathcal{C}\}$.

Definition 2.3. Fix $\mathcal{B}$,$\mathcal{C} \in \boldsymbol{K}_\alpha$, and $m \in \omega$ such that $|C \backslash B| < m$. Define $Z(\mathcal{B}, \mathcal{C}, m)$ to be all $\mathcal{B}^* \in X_m(\mathcal{B})$ such that there are no $\mathcal{H}$ with $|H \backslash B^*| < m$ satisfying

$$\begin{tikzcd}
& \mathcal{H} & \\
\mathcal{B}^* \arrow[ur, dash, "\leq"] & & \mathcal{C} \arrow[ul, dash] \\
& \mathcal{B} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Lemma 2.4. *Let $\mathcal{B}$,$\mathcal{C} \in \boldsymbol{K}_\alpha$, and $m \in \omega$ such that $|C \backslash B| < m$. Also let $\mathcal{M} \models S_\alpha$ and $f\colon \mathcal{B} \to \mathcal{M}$ an embedding. The following are equivalent:*

(1) *f omits $\{\mathcal{C}\}$.*

(2) *There exists $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into $\mathcal{M}$ over f.*

Proof. For the proof we identify $\mathcal{B}$ with $f(\mathcal{B})$, i.e. for ease of notation assume that $\mathcal{B} \subset \mathcal{M}$.

(1) $\Rightarrow$ (2) By remark 5.3 of [1] there is some $B^* \in X_m(\mathcal{B})$ maximally embeddable in $\mathcal{M}$ over f. Such embedding is unique by Lemma 3.8 of [1]. Again, we identify B^* with its maximal embedding into $\mathcal{M}$. To show (2) we need to verify that $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$. Suppose not. Then there is $\mathcal{H}$ with $|H \backslash B^*| < m$ satisfying

$$\begin{tikzcd}
& \mathcal{H} & \\
\mathcal{B}^* \arrow[ur, dash, "\leq"] & & \mathcal{C} \arrow[ul, dash] \\
& \mathcal{B} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

As $\mathcal{B}^* \leq \mathcal{H}$ and $\mathcal{B} \subset \mathcal{M}$ we can embed $\mathcal{H}$ into $\mathcal{M}$ (as $\mathcal{M} \models S_\alpha$). But this would witness $\mathcal{C}$ extending $\mathcal{B}$ in $\mathcal{M}$ which is impossible as we assumed that f omits $\{\Phi\}$.

(2) $\Rightarrow$ (1) Suppose f doesn't omit $\{C\}$. Then by the note 2.2 f admits $\{C\}$, i.e. there is an embedding of $\mathcal{C}$ into M over f. We identify $\mathcal{C}$ with the image of that embedding. Similarly we identify $\mathcal{B}^*$ with the image of its maximal embedding over f. That is we may assume $\mathcal{C}, \mathcal{B}^* \subset \mathcal{M}$. Let H be the substructure of M induced by vertices $C \cup B^*$. As $|C \backslash B| < m$ we have $|H \backslash B^*| < m$. $\mathcal{B}^*$ is m-strong by remark 5.3 of [1]. This forces $\mathcal{B}^* \leq H$. But this contradicts the fact that $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$. □

Corollary 2.5. *With the setup of the previous lemma, the following are equivalent:*

(1) *f admits $\{\mathcal{C}\}$.*
(2) *There exists $\mathcal{B}^* \in X_m(\mathcal{B}) \backslash Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into $\mathcal{M}$ over f.*

For quantifier elimination we need to track multiple structures being admitted and omitted, hence the following definition.

Definition 2.6. Let $\mathcal{B} \in \boldsymbol{K}_\alpha$, Φ, Γ finite subsets of $\boldsymbol{K}_\alpha$, and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|C \backslash B| < m$. Define

$$\begin{aligned} Y(\mathcal{B}, \Phi, \Gamma, m) = \{B^* \in X_m(\mathcal{B}) \mid {} & \forall \mathcal{C} \in \Phi \ B^* \in Z(\mathcal{B}, \mathcal{C}, m) \text{ and} \\ & \forall \mathcal{D} \in \Gamma \ B^* \notin Z(\mathcal{B}, \mathcal{D}, m)\} \end{aligned}$$

Lemma 2.7. *Let $\mathcal{B} \in \boldsymbol{K}_\alpha$, Φ, Γ finite subsets of $\boldsymbol{K}_\alpha$, and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|C \backslash B| < m$. The following are equivalent:*

(1) *f omits Φ and admits Γ.*
(2) *There exists $\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$ maximally embeddable into $\mathcal{M}$ over f.*

Proof. Easy corollary of 2.4 and 2.5. □

3. Quantifier Elimination

Following proof of 5.6 in [1], we have a formula $\theta(x,y)$, some $\mathcal{A} \subseteq \mathcal{B} \in \boldsymbol{K}_\alpha$ with $\theta(x,y) \vdash \Delta_A(x) \wedge \Delta_{\mathcal{B}}(x,y)$. We may also assume that $\theta(x,y)$ is a conjunction of formulas of the type $\Psi_{\mathcal{B},\mathcal{C}}(x,y)$ and their negations. More precisely

$$\theta(x,y) \Leftrightarrow \bigwedge_{\mathcal{C}\in\Phi} \Psi_{\mathcal{B},\mathcal{C}}(x,y) \wedge$$
$$\bigwedge_{\mathcal{D}\in\Gamma} \neg\Psi_{\mathcal{B},\mathcal{D}}(x,y)$$

for some finite subsets Φ, Γ of $\boldsymbol{K}_\alpha$. Let $m = \max\{|C\backslash B| \mid \mathcal{C} \in \Phi \text{ or } \Gamma\}$. We claim that in $\mathcal{M} \models S_\alpha$

$$\exists y\theta(x,y) \Leftrightarrow \bigvee_{\mathcal{B}^*\in Y(\mathcal{B},\Phi,\Gamma,m)} (\mathcal{B}^* \text{ maximally embeds into } \mathcal{M} \text{ over } \mathcal{A})$$
$$\Leftrightarrow \bigvee_{\mathcal{B}^*\in Y(\mathcal{B},\Phi,\Gamma,m)} \left(\Psi_{\mathcal{A},\mathcal{B}^*}(x) \wedge \bigwedge_{\mathcal{B}*\sqsubseteq B', B'\in X_m(\mathcal{B})} \neg\Psi_{\mathcal{A},\mathcal{B}'}(x) \right)$$

Proof. ($\Rightarrow$) Fix $\mathcal{B} \subset \mathcal{M}$ witnessing existential statement. By remark 5.3 and lemma 3.8 There is a unique $\mathcal{B}^* \in X_m$ (uniquely embedded) maximally embeddable into $\mathcal{M}$ over $\mathcal{B}$. By lemma 2.7 $\mathcal{B}^* \in Y(\mathcal{B},\Phi,\Gamma,m)$.

($\Leftarrow$) Take the embedding $g\colon B^* \to \mathcal{M}$ and restrict it to $\mathcal{B} \subseteq \mathcal{B}^*$ i.e. $f = g \mid \mathcal{B}$. As $\mathcal{B}^* \in Y(\mathcal{B},\Phi,\Gamma,m)$ by lemma 2.7 f omits Φ and admits Γ. Thus is is a witness to $\exists y\theta(x,y)$. ☐

References

[1] Michael C. Laskowski, *A simpler axiomatization of the Shelah-Spencer almost sure theories,* Israel J. Math. **161** (2007), 157-186. MR MR2350161

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