A NOTE ON QUANTIFIER ELIMINATION IN SHELAH-SPENCER GRAPHS

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ABSTRACT. We simplify [1]'s proof of quantifier elimination in Shelah-Spencer graphs.

1. INTRODUCTION

Laskowski's paper [1] provides a combinatorial proof of quantifier elimination in Shelah-Spencer graphs. Here we provide a simplification of the proof using only maximal chains and avoiding the use of proposition 3.1 and technical lemmas of section 4.

We will use notation of [1], in particular things like $\boldsymbol{K}_\alpha$, $\delta(\mathcal{A}/\mathcal{B})$, $X_m(\mathcal{A})$, S_α, maximal embedding, $\Delta_\mathcal{A}(x)$, $\Psi_{\mathcal{A},\mathcal{B}}(x)$ etc. However we will give a different definition of $Y(\ldots)$.

2. OMITTING LEMMA

Definition 2.1. Let $\mathcal{M} \models S_\alpha$, $\mathcal{B} \in \boldsymbol{K}_\alpha$, embedding $f\colon \mathcal{B} \to \mathcal{M}$, Φ finite subset of $\boldsymbol{K}_\alpha$

    (1) Say that f *omits* Φ if there are no $\mathcal{C} \in \Phi$ and $g\colon \mathcal{C} \to \mathcal{M}$ extending f.

    (2) Say that f *admits* Φ if for every $\mathcal{C} \in \Phi$ there is $g\colon \mathcal{C} \to \mathcal{M}$ extending f.

Note 2.2. Take notation as above and a structure $\mathcal{C} \in \boldsymbol{K}_\alpha$ extending $\mathcal{B}$. Then f doesn't omit $\{\mathcal{C}\}$ iff f admits $\{\mathcal{C}\}$.

Definition 2.3. Fix $\mathcal{B},\mathcal{C} \in \boldsymbol{K}_\alpha$, and $m \in \omega$ such that $|C \backslash B| < m$. Define $Z(\mathcal{B},\mathcal{C},m)$ to be all $\mathcal{B}^* \in X_m(\mathcal{B})$ such that there are no $\mathcal{H}$ with $|H \backslash B^*| < m$ satisfying
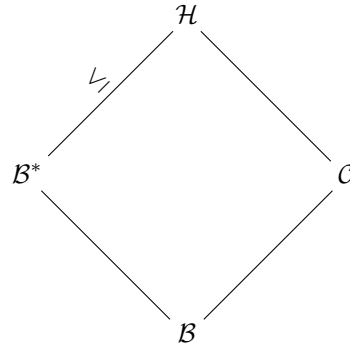

Lemma 2.4. *Let* $\mathcal{B},\mathcal{C} \in \boldsymbol{K}_\alpha$, *and* $m \in \omega$ *such that* $|C \backslash B| < m$. *Also let* $\mathcal{M} \models S_\alpha$ *and* $f\colon \mathcal{B} \to \mathcal{M}$ *an embedding. The following are equivalent:*

    (1) f *omits* $\{\mathcal{C}\}$.

(2) *There exists $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into $\mathcal{M}$ over f.*

Proof. For the proof we identify $\mathcal{B}$ with $f(\mathcal{B})$, i.e. for ease of notation assume that $\mathcal{B} \subset \mathcal{M}$.

$(1) \Rightarrow (2)$ By remark 5.3 of [1] there is some $B^* \in X_m(\mathcal{B})$ maximally embeddable in $\mathcal{M}$ over f. Such embedding is unique by Lemma 3.8 of [1]. Again, we identify B^* with its maximal embedding into $\mathcal{M}$. To show (2) we need to verify that $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$. Suppose not. Then there is $\mathcal{H}$ with $|H \backslash B^*| < m$ satisfying

$$
\begin{array}{ccc}
 & \mathcal{H} & \\
\mathcal{B}^* & & \mathcal{C} \\
 & \mathcal{B} &
\end{array}
$$
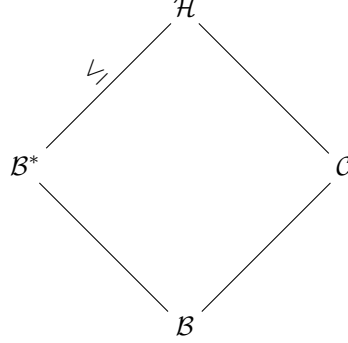
As $\mathcal{B}^* \leq \mathcal{H}$ and $\mathcal{B} \subset \mathcal{M}$ we can embed $\mathcal{H}$ into $\mathcal{M}$ (as $\mathcal{M} \models S_\alpha$). But this would witness $\mathcal{C}$ extending $\mathcal{B}$ in $\mathcal{M}$ which is impossible as we assumed that f omits $\{\Phi\}$.

$(2) \Rightarrow (1)$ Suppose f doesn't omit $\{C\}$. Then by the note **??** f admits $\{C\}$, i.e. there is an embedding of $\mathcal{C}$ into M over f. We identify $\mathcal{C}$ with the image of that embedding. Similarly we identify $\mathcal{B}^*$ with the image of its maximal embedding over f. That is we may assume $\mathcal{C}, \mathcal{B}^* \subset \mathcal{M}$. Let H be the substructure of M induced by vertices $C \cup B^*$. As $|C \backslash B| < m$ we have $|H \backslash B^*| < m$. $\mathcal{B}^*$ is m-strong by remark 5.3 of [1]. This forces $\mathcal{B}^* \leq H$. But this contradicts the fact that $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$. $\square$

Corollary 2.5. *With the setup of the previous lemma, the following are equivalent:*

   (1) *f admits $\{\mathcal{C}\}$.*
   (2) *There exists $\mathcal{B}^* \in X_m(\mathcal{B}) \backslash Z(\mathcal{B}, \mathcal{C}, m)$ maximally embeddable into $\mathcal{M}$ over f.*

For quantifier elimination we need to track multiple structures being admitted and omitted, hence the following definition.

Definition 2.6. Let $\mathcal{B} \in \boldsymbol{K}_\alpha$, Φ, Γ finite subsets of $\boldsymbol{K}_\alpha$, and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|C \backslash B| < m$. Define

$$Y(\mathcal{B}, \Phi, \Gamma, m) = \{B^* \in X_m(\mathcal{B}) \mid \forall \mathcal{C} \in \Phi \ B^* \in Z(\mathcal{B}, \mathcal{C}, m) \text{ and}$$
$$\forall \mathcal{D} \in \Gamma \ B^* \notin Z(\mathcal{B}, \mathcal{D}, m)\}$$

Lemma 2.7. *Let $\mathcal{B} \in \boldsymbol{K}_\alpha$, Φ, Γ finite subsets of $\boldsymbol{K}_\alpha$, and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|C \backslash B| < m$. The following are equivalent:*

   (1) *f omits Φ and admits Γ.*
   (2) *There exists $\mathcal{B}^* \in Y(\mathcal{B}, \Phi, \Gamma, m)$ maximally embeddable into $\mathcal{M}$ over f.*

Proof. Easy corollary of 2.4 and 2.5. $\square$

3. Quantifier Elimination

Following proof of 5.6 in [1], we have a formula $\theta(x,y)$, some $\mathcal{A} \subseteq \mathcal{B} \in \boldsymbol{K}_\alpha$ with $\theta(x,y) \vdash \Delta_A(x) \wedge \Delta_\mathcal{B}(x,y)$. We may also assume that $\theta(x,y)$ is a conjunction of formulas of the type $\Psi_{\mathcal{B},\mathcal{C}}(x,y)$ and their negations. More precisely

$$\theta(x,y) \Leftrightarrow \bigwedge_{\mathcal{C}\in\Phi} \Psi_{\mathcal{B},\mathcal{C}}(x,y) \wedge$$
$$\bigwedge_{\mathcal{D}\in\Gamma} \neg\Psi_{\mathcal{B},\mathcal{D}}(x,y)$$

for some finite subsets Φ, Γ of $\boldsymbol{K}_\alpha$.

References

[1] Michael C. Laskowski, *A simpler axiomatization of the Shelah-Spencer almost sure theories,* Israel J. Math. **161** (2007), 157-186. MR MR2350161

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