VC-density in an additive reduct of p-adic numbers

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Abstract

Aschenbrenner et. al. computed a bound $\mathrm{vc}(n) \le 2n-1$ for the vc-density function in the field of p-adic numbers, but it is not known to be optimal. I investigate a certain P-minimal additive reduct of the field of p-adic numbers and compute an optimal bound $\mathrm{vc}(n) = n$ for it using a cell decomposition result of Leenknegt.

VC-density

Let M be a structure and $\Psi(x;y) = \{\phi_i(x;y)\}$ a finite collection of formulas in $L(M)$. Define the shatter function $\pi^M_\Psi \colon \mathbb{N} \longrightarrow \mathbb{N}$ of Ψ as

$$\pi^M_\Psi(n) = \max\{\text{number of } \Psi\text{-types over } B \mid B \subset M^{|y|} \text{ with } |B| = n\}.$$

Define the shatter function of a single formula ϕ as the shatter function of a one element collection $\{\phi\}$. The shatter function only depends on the theory of M. The following theorem is an important result concerning a dichotomy for the growth of the shatter function.

Theorem (Sauer-Shelah '72)

The shatter function either grows exponentially or is bounded by a polynomial.

In fact, a formula $\phi(x;y)$ is NIP precisely when its shatter function grows polynomially. From now on work with NIP theories, that is all formulas will have shatter functions that grow polynomially. The following definition captures the degree of polynomial growth.
For a collection of formulas $\Psi(x;y)$ in a model M let $\mathrm{vc}^M(\Psi)$ be the infimum of all positive reals r such that

$$\pi^M_\Psi(n) = O(n^r)$$

Call $\mathrm{vc}^M(\Psi)$ the vc-density of Ψ. As before for a single formula ϕ define $\mathrm{vc}(\phi)$ as the vc-density of a one element collection $\{\phi\}$. This allows formula by formula analysis of the growth rate for the shatter function. More generally, we look at the bounds of vc-density for all the formulas in a given structure.
Define the vc-function $\mathrm{vc}^M \colon \mathbb{N} \longrightarrow \mathbb{N}$ to be the largest vc-density achieved by formulas that define subsets of M^n.

$$\mathrm{vc}^M(n) = \sup\left\{\mathrm{vc}^M(\phi) \mid \phi(x,y) \text{ with } |x| = n\right\}$$

As before this only depends on the theory of M. There is a simple lower bound $\mathrm{vc}^M(n) \ge n$. More generally $\mathrm{vc}^M(n) \ge n\,\mathrm{vc}^M(1)$, and it is not known whether strict inequality can hold.

Application to p-adic numbers

A common example of a non-stable NIP structure is the field $\mathbb{Q}_p$ of p-adic numbers. In [1], Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko show that this structure has $\mathrm{vc}(n) \le 2n-1$. My work improves that bound in a reduct of the full structure. In [2], Leenknegt analyzes the reduct of p-adic numbers to the language

$$\mathcal{L}_{aff} = \left\{+, -, \{\bar{c}\}_{c\in\mathbb{Q}_p}, |, \{Q_{m,n}\}_{m,n\in\mathbb{N}}\right\}$$

where $\bar{c}$ is a scalar multiplication by c, $a|b$ stands for $\mathrm{val}\, a \le \mathrm{val}\, b$, and $Q_{m,n}$ is a unary predicate

$$Q_{m,n} = \bigcup_{k\in\mathbb{Z}} p^{km}(1 + p^n\mathbb{Z}_p).$$

Note that $Q_{m,n}$ is a subgroup of the multiplicative group of $\mathbb{Q}_p$ with finitely many cosets. One can check that the extra relation symbols are definable in the full structure. Moreover, [2] shows that $(\mathbb{Q}_p, \mathcal{L}_{aff})$ is a P-minimal reduct, that is one-dimensional definable sets coincide with one-dimensional definable sets in the full structure.

Theorem (B.)

$(\mathbb{Q}_p, \mathcal{L}_{aff})$ has $\mathrm{vc}(n) = n$.

Proof outline

In [2], Leenknegt provides the following cell decomposition result

Theorem (Leenknegt '12)

Any formula $\phi(t,x)$ in $(\mathbb{Q}_p, \mathcal{L}_{aff})$ with t singleton decomposes into the union of the following cells:

$$\{(t,x) \in K \times D \mid \mathrm{val}\, a_1(x) \,\square_1\, \mathrm{val}(t - c(x)) \,\square_2\, \mathrm{val}\, a_2(x), t - c(x) \in \lambda Q_{m,n}\}$$

where D is a cell of a smaller dimension, $a_1(x), a_2(x), c(x)$ are linear polynomials, $\square$ is $<$ or no condition, and $\lambda \in \mathbb{Q}_p$.

This can be used to eliminate quantifiers in the following way:

Corollary

Any formula $\phi(x;y)$ in $(\mathbb{Q}_p, \mathcal{L}_{aff})$ can be written as a boolean combination of formulas from the following collection

$$\Psi(x;y) = \{\mathrm{val}(p_i(x) - c_i(y)) < \mathrm{val}(p_j(x) - c_j(y))\}_{i,j\in I} \cup \{p_i(x) - c_i(y) \in \lambda_k Q_{m,n}\}_{i\in I, k\in K}$$

where I, K are finite index sets, each p_i is a linear polynomial in x without a constant term, each c_i is a linear polynomial in y, and $\lambda_k \in \mathbb{Q}_p$.

It is easy to show that $\mathrm{vc}(\phi) \le \mathrm{vc}(\Psi)$. Therefore to show that $\mathrm{vc}(n) = n$ it suffices to bound $\mathrm{vc}(\Psi) \le |x|$ for any such collection. More precisely, it is sufficient to show that if there is a parameter set B of size N then the number of Ψ-types over B is $O(N^{|x|})$. Fix a parameter set B of size N. Consider a set $T = \{c_i(b) \mid b \in B, i \in I\} \subset \mathbb{Q}_p$. View T as a tree as follows. Branches through the tree are elements of T. For $c \in \mathbb{Q}_p, \alpha \in \mathbb{Z}$ define a ball $B(c,\alpha) = \{c' \in \mathbb{Q}_p \mid \mathrm{val}\,(c' - c) \le \alpha\}$. With T we associate the balls $B(t_1, \mathrm{val}(t_1 - t_2))$ for all $t_1, t_2 \in T$. An interval is two balls $B(t_1, v_1) \supset B(t_2, v_2)$ with no balls in between. An element $a \in \mathbb{Q}_p$ belongs to this interval if $a \in B(t_1,v_1)\backslash B(t_2,v_2)$. There are at most $2|T| = 2N|I| = O(N)$ different intervals and they partition $\mathbb{Q}_p$. (See Figure 1).
Suppose $a \in \mathbb{Q}_p$ lies in an interval $B(t_L, \alpha_L)\backslash B(t_U, \alpha_U)$. Define T-valuation of a to be $\text{T-val}(a) = \mathrm{val}(a - t_U)$. Define floor of a to be $F(a) = \alpha_L$.
Suppose $a_1, a_2 \in \mathbb{Q}_p$ lie in our tree in the same interval $B(t_L, \alpha_L)\backslash B(t_U, \alpha_U)$. We say that a_i is close to boundary if $|\text{T-val}(a_i) - \alpha_L| \le m$ or $|\text{T-val}(a_i) - \alpha_U| \le m$. Otherwise we say that it is far from boundary. We say a_1, a_2 have the same interval type if one of the following holds (see Figure 2):

- Both a_1, a_2 are far from boundary and $a_1 - t_U, a_2 - t_U$ are in the same $Q_{m,n}$ coset.
- Both a_1, a_2 are close to boundary and $\mathrm{val}(a_1 - a_2) > \text{T-val}(a_1) + n = \text{T-val}(a_2) + n$.

One can check that for each interval there are at most $K = K(\Psi, Q_{m,n})$ many interval types (with K not dependent on B or the interval).

Lemma

Suppose $c_1, c_2 \in \mathbb{Q}_p^{|x|}$ satisfy the follwing three conditions

- For all $i \in I$ $p_i(c_1)$ and $p_i(c_2)$ are in the same interval.
- For all $i \in I$ $p_i(c_1)$ and $p_i(c_2)$ have the same interval type.
- For all $i, j \in I$, $\text{T-val}(p_i(c_1)) > \text{T-val}(p_j(c_1))$ iff $\text{T-val}(p_i(c_2)) > \text{T-val}(p_j(c_2))$.

Then c_1, c_2 have the same Ψ-type over B.

This gives us an upper bound on the number of types - there are at most $|I|!$ many choices for the order of T-val, $O(N)$ many choices for the interval for each p_i, and K many choices for the interval type for each p_i, giving a total of $O(N^{|I|}) \cdot K^{|I|} \cdot |I|! = O(N^{|I|})$ many types (see Figure 3). This implies $\mathrm{vc}(\Psi) \le |I|$. The biggest contribution to this bound are the choices among the $O(N)$ many intervals for each p_i with $i \in I$. Are all of those choices realized? Intuitively there are $|x|$ many variables and $|I|$ many equations, so once we choose an interval for $|x|$ many p_i's, the interval for the rest should be determined. This would give the required $\mathrm{vc}(\Psi) \le |x|$ bound. The remainder of the poster is a more formal outline of this idea.

Reduction from $|I|$ to $|x|$

For $c \in \mathbb{Q}_p$ and $\alpha, \beta \in \mathbb{Z}$ define $c \restriction [\alpha, \beta] \in (\mathbb{Z}/p\mathbb{Z})^{\beta-\alpha+1}$ to be the record of the coefficients of c for the valuations between α, β. More precisely write c in its power series form

$$c = \sum_{\gamma\in\mathbb{Z}} c_\gamma p^\gamma \text{ with } c_\gamma \in \mathbb{Z}/p\mathbb{Z}$$

Then $c \restriction [\alpha, \beta]$ is just $(c_\alpha, c_{\alpha+1}, \dots c_\beta)$.
Alternative way to write $p_i(x)$ is $\vec{p}_i \cdot \vec{x}$, where $\vec{p}_i$ and $\vec{x}$ are vectors in $\mathbb{Q}_p^{|x|}$.

Lemma

Suppose we have a finite collection of vectors $\{\vec{p}_i\}_{i\in I}$ with each $\vec{p}_i \in \mathbb{Q}_p^{|x|}$. Suppose $J \subset I$ and $j \in I$ satisfy $\vec{p}_j \in \mathrm{span}\,\{\vec{p}_i\}_{i\in J}$. and we have $\vec{x} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$ with $\mathrm{val}(\vec{p}_i \cdot \vec{x}) > \alpha$ for all $i \in J$. Then $\mathrm{val}(\vec{p}_j \cdot \vec{x}) > \alpha - \gamma$ for some $\gamma \in \mathbb{N}$. Moreover γ can be chosen independently from $J, j, \vec{x}, \alpha$ depending only on $\{\vec{p}_i\}_{i\in I}$.

Let $f : \mathbb{Q}_p^{|x|} \longrightarrow \mathbb{Q}_p^I$ with $f(c) = (p_i(c))_{i\in I}$. Define the segment space Sg to be the image of f. Given a tuple $(a_i)_{i\in I}$ in the segment space, look at the corresponding floors $\{F(a_i)\}_{i\in I}$ and T-valuations $\{\text{T-val}(a_i)\}_{i\in I}$. Partition the segment space by the order types of $\{F(a_i)\}$ and $\{\text{T-val}(a_i)\}$ (as subsets of $\mathbb{Z}$). Work in a fixed partition Sg'. After relabeling we may assume that $F(a_1) \ge F(a_2) \ge \dots$. Consider the (relabeled) sequence of vectors $\vec{p}_1, \vec{p}_2, \dots, \vec{p}_I$. There is a unique subset $J \subset I$ such that all vectors with indices in J are linearly independent, and all vectors with indices outside of J are a linear combination of the preceding vectors. For any index $i \in I$ we call it independent if $i \in J$ and we call it dependent otherwise.
Now, we define the following function $g_{\mathrm{Sg}'} : \mathrm{Sg}' \longrightarrow \mathrm{It}^I \times \mathrm{Pt}^J \times \mathrm{Ct}^{I-J}$. Let $a = (a_i)_{i\in I} \in \mathrm{Sg}'$. To define $g_{\mathrm{Sg}'}(a)$ we need to specify where it maps a in each individual component of the product.
For all a_i record its interval type, giving the first component.
For a_j with $j \in J$, record the interval of a_j, giving the second component.
For the third component do the following computation. Pick a_i with i dependent. Let j be the largest independent index with $j < i$. Record $a_i \restriction [F(a_j) - \gamma, F(a_j)]$.
Combine $g_{\mathrm{Sg}'}$ for all the partitions to get a function $g : \mathrm{Sg} \longrightarrow \mathrm{It}^I \times \mathrm{Pt}^J \times \mathrm{Ct}^{I-J}$.

Lemma

Suppose we have $c_1, c_2 \in \mathbb{Q}_p^{|x|}$ such that $f(c_1), f(c_2)$ are in the same partition and $g(f(c_1)) = g(f(c_2))$. Then c_1, c_2 have the same Ψ-type over B.

To get the desired bound one checks that the number of partitions multiplied by the size of the range of g is $O(N^{|x|})$.

References

[1] M. Aschenbrenner, A. Dolich, D. Haskell, D. Macpherson, S. Starchenko, *Vapnik-Chervonenkis density in some theories without the independence property*, I, Trans. Amer. Math. Soc. 368 (2016), 5889-5949

[2] E. Leenknegt. *Reducts of p-adically closed fields*, Archive for Mathematical logic, 53(3):285-306, 2014