## QUANTIFIER ELIMINATION IN SHELAH-SPENCER GRAPHS

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ABSTRACT. We simplify [?] proof of quantifier elimination in Shelah-Spencer graphs.

# 1. Introduction

Laskowski's paper [?] provides a combinatorial proof of quantifier elimination in Shelah-Spencer graphs. Here we provide a simplification of the proof using only maximal chains and avoiding the use of proposition 3.1 and technical lemmas of section 4.

We will use notation of [?], in particular things like  $K_{\alpha}$ ,  $\delta(\mathcal{A}/\mathcal{B})$ ,  $X_m(\mathcal{A})$ ,  $S_{\alpha}$ , maximal embedding, etc.

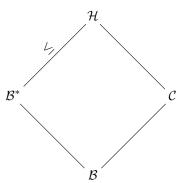
### 2. Omitting Lemma

**Definition 2.1.** Let  $\mathcal{M} \models S_{\alpha}$ ,  $\mathcal{B} \in \mathbf{K}_{\alpha}$ , embedding  $f : \mathcal{B} \to \mathcal{M}$ ,  $\Phi$  finite subset of  $\mathbf{K}_{\alpha}$ 

- (1) Say that f omits  $\Phi$  if there are no  $\mathcal{C} \in \Phi$  and  $g: \mathcal{C} \to \mathcal{M}$  extending f.
- (2) Say that f admits  $\Phi$  if for every  $\mathcal{C} \in \Phi$  there is  $g: \mathcal{C} \to \mathcal{M}$  extending f.

Note 2.2. Take notation as above and a structure  $C \in K_{\alpha}$  extending B. Then f doesn't omit  $\{C\}$  iff f admits  $\{C\}$ .

**Definition 2.3.** Fix  $\mathcal{B}, \mathcal{C} \in K_{\alpha}$ , and  $m \in \omega$  such that  $|C \setminus B| < m$ . Define  $Z(\mathcal{B}, \mathcal{C}, m)$  to be all  $\mathcal{B}^* \in X_m(\mathcal{B})$  such that there are no  $\mathcal{H}$  with  $|H \setminus B^*| < m$  satisfying



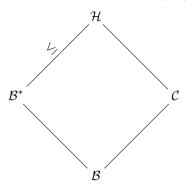
**Lemma 2.4.** Let  $\mathcal{B}, \mathcal{C} \in K_{\alpha}$ , and  $m \in \omega$  such that  $|C \setminus B| < m$ . Also let  $\mathcal{M} \models S_{\alpha}$  and  $f : \mathcal{B} \to \mathcal{M}$  an embedding. The following are equivalent:

- (1) f omits  $\{C\}$ .
- (2) There exists  $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$  maximally embeddable into  $\mathcal{M}$  over f.

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*Proof.* For the proof we identify  $\mathcal{B}$  with  $f(\mathcal{B})$ , i.e. for ease of notation assume that  $\mathcal{B} \subset \mathcal{M}$ .

 $(1) \Rightarrow (2)$  By remark 5.3 of [?] there is some  $B^* \in X_m(\mathcal{B})$  maximally embeddable in  $\mathcal{M}$  over f. Such embedding is unique by Lemma 3.8 of [?]. Again, we identify  $B^*$  with its maximal embedding into  $\mathcal{M}$ . To show (2) we need to verify that  $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$ . Suppose not. Then there is  $\mathcal{H}$  with  $|H \setminus B^*| < m$  satisfying



As  $\mathcal{B}^* \leq \mathcal{H}$  and  $\mathcal{B} \subset \mathcal{M}$  we can embed  $\mathcal{H}$  into  $\mathcal{M}$  (as  $\mathcal{M} \models S_{\alpha}$ ). But this would witness  $\mathcal{C}$  extending  $\mathcal{B}$  in  $\mathcal{M}$  which is impossible as we assumed that f omits  $\{\Phi\}$ .

 $(2) \Rightarrow (1)$  Suppose f doesn't omit  $\{C\}$ . Then by the note above f admits  $\{C\}$ , i.e. there is an embedding of  $\mathcal{C}$  into M over f. We identify  $\mathcal{C}$  with the image of that embedding. Similarly we identify  $\mathcal{B}^*$  with the image of its maximal embedding over f. That is we may assume  $\mathcal{C}, \mathcal{B}^* \subset \mathcal{M}$ . Let H be the substructure of M induced by vertices  $C \cup B^*$ . As  $|C \setminus B| < m$  we have  $|H \setminus B^*| < m$ .  $\mathcal{B}^*$  is m-strong by remark 5.3 of [?]. This forces  $\mathcal{B}^* \leq H$ . But this contradicts the fact that  $\mathcal{B}^* \in Z(\mathcal{B}, \mathcal{C}, m)$ .  $\square$ 

Corollary 2.5. With the setup of the previous lemma, the following are equivalent:

- (1) f admits  $\{C\}$ .
- (2) There exists  $\mathcal{B}^* \in X_m(\mathcal{B}) \backslash Z(\mathcal{B}, \mathcal{C}, m)$  maximally embeddable into  $\mathcal{M}$  over f.

For quantifier elimination we need to track multiple structures being admitted and omitted, hence the following definition.

**Definition 2.6.** Let  $\mathcal{B} \in K_{\alpha}$ ,  $\Phi, \Gamma$  finite subsets of  $K_{\alpha}$ , and  $m \in \omega$  such that for each  $\mathcal{C} \in \Phi$  or  $\mathcal{C} \in \Gamma$  we have  $\mathcal{B} \subseteq \mathcal{C}$  and  $|C \setminus B| < m$ . Define

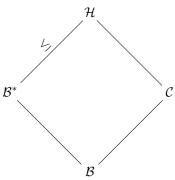
$$Y(\mathcal{B}, \Phi, \Gamma, m) = \{ B^* \in X_m(\mathcal{B}) \mid \forall \mathcal{C} \in \Phi \ B^* \in Z(\mathcal{B}, \mathcal{C}, m)^* \text{ and}^*$$
$$\forall \mathcal{D} \in \Gamma \ B^* \notin Z(\mathcal{B}, \mathcal{C}, m) \}$$

**Lemma 2.7.** Let  $\mathcal{B} \in K_{\alpha}$ ,  $\Phi, \Gamma$  finite subsets of  $K_{\alpha}$ , and  $m \in \omega$  such that for each  $\mathcal{C} \in \Phi$  or  $\mathcal{C} \in \Gamma$  we have  $\mathcal{B} \subseteq \mathcal{C}$  and  $|\mathcal{C} \setminus \mathcal{B}| < m$ . The following are equivalent:

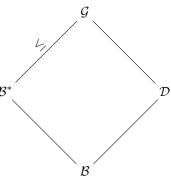
- (1) f omits  $\Phi$  and admits  $\Gamma$ .
- (2) There exists  $\mathcal{B}^* \in Z(\mathcal{B}, \Phi, \Gamma, m)$  maximally embeddable into  $\mathcal{M}$  over f.

*Proof.* (1)  $\Rightarrow$  (2) Identify  $\mathcal{B}$  with  $f(\mathcal{B})$ , i.e. for ease of notation assume that  $\mathcal{B} \subset \mathcal{M}$ . By remark 5.3 of [?] there is some  $B^* \in X_m(\mathcal{B})$  maximally embeddable in  $\mathcal{M}$  over f. Such embedding is unique by Lemma 3.8 of [?]. Again, we identify  $B^*$  with its maximal embedding into  $\mathcal{M}$ . To show (2) we need to verify that  $\mathcal{B}^* \in Z(\mathcal{B}, \Phi, \Gamma, m)$ .

Suppose not. Two things can go wrong. First, there can be  $\mathcal{H}$  with  $|H \setminus B^*| < m$  and  $\mathcal{C} \in \Phi$  satisfying



As  $\mathcal{B}^* \leq \mathcal{H}$  and  $\mathcal{B} \subset \mathcal{M}$  we can embed  $\mathcal{H}$  into  $\mathcal{M}$  (as  $\mathcal{M} \models S_{\alpha}$ ). But this would witness  $\mathcal{C}$  extending  $\mathcal{B}$  in  $\mathcal{M}$  which is impossible as we assumed that f omits  $\Phi$ . Another thing that could go wrong is that there could be  $\mathcal{D} \in \Gamma$  and no  $\mathcal{G}$  with  $|G \setminus B^*| < m$  satisfying



As f admits

### References

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- [3] P. Simon, On dp-minimal ordered structures, J. Symbolic Logic 76 (2011), no. 2, 448460. E-mail address: bobkov@math.ucla.edu