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VC-density Computations in Various Model-Theoritic Structures

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ABSTRACT OF THE DISSERTATION

VC-density Computations
in Various Model-Theoritic Structures

by

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Doctor of Philosophy in Mathematics
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Aschenbrenner et. al. studied vc-density in model-theoretic context. We investigate it further by computing in some common structures: trees, Shelah-Spencer graphs, and an additive reduct of p-adic numbers. We show that in the theory of infinite trees the vc-function is optimal. This generalizes a result of Simon showing that the trees are dp-minimal. In Shelah-Spencer graphs we provide an upper bound on a formula-by-formula basis and show that there isn't a uniform lower bound, forcing the vc-function to be infinite. In addition we show that Shelah-Spencer graphs do not have a finite dp-rank, in particular they are not dp-minimal. There is a linear bound for the vc-density function in the field of p-adic numbers, but this bound is not known to be optimal. We investigate a certain P-minimal additive reduct of the field of p-adic numbers and use a cell decomposition result of Leenknegt to compute an optimal bound for that structure.

The dissertation of Anton Bobkov is approved.

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To my family and friends

who have been unerringly supportive

throughout my career path

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CHAPTER 1

Inroduction and Preliminaries

1.1 Inroduction

My research concentrates on the concept of VC-density, a recent notion of rank in NIP theories. The study of a structure in model theory usually starts with quantifier elimination, followed by a finer analysis of definable functions and interpretability. The study of VC-density goes one step further, looking at a structure of the asymptotic growth of finite definable families. In the most geometric examples, VC-density coincides with the natural notion of dimension. However, no geometric structure is required for the definition of VC-density, thus we can get some notion of geometric dimension for families of sets given without any geometric context.

In 2013, Aschenbrenner et al. investigated and developed a notion of VC-density for NIP structures, an analog of geometric dimension in an abstract setting [ADH16]. Their applications included a bound for p-adic numbers, an object of great interest and a very active area of research in mathematics. My research concentrates on improving and expanding techniques of that paper to improve the known bounds as well as computing VC-density for other NIP structures of interest. I am able to obtain new bounds for the additive reduct of p-adic numbers, trees, and certain families of graphs. Recent research by Chernikov and Starchenko in 2015 [CS15] suggests that having good bounds on VC-density in p-adic numbers opens a path for applications to incidence combinatorics (e.g. Szemeredi-Trotter theorem).

The concept of VC-dimension was first introduced in 1971 by Vapnik and Chervonenkis for set systems in a probabilistic setting [VC71]. The theory grew rapidly and found wide use in geometric combinatorics, computational learning theory, and machine learning. Around the same time Shelah was developing the notion of NIP ("not having the independence property"), a natural tameness property of (complete theories of) structures in model theory [She71]. In 1992 Laskowski noticed the connection between the two: theories where all uniformly definable families of sets have finite VC-dimension are exactly NIP theories [Las92]. It is a wide class of theories including algebraically closed fields, differentially closed fields, modules, free groups, o-minimal structures, and ordered abelian groups. A variety of valued fields fall into this category as well, including the p-adic numbers.

The p-adic numbers were first introduced by Hensel in 1897 in [Hen97], and over the following century a powerful theory was developed around them with numerous applications across a variety of disciplines, primarily in number theory, but also in physics and computer science. In 1965 Ax, Kochen [AK65] and Ershov [Ers65] axiomatized the theory of p-adic numbers and proved a quantifier elimination result. A key insight was to connect properties of the value group and residue field to the properties of the valued field itself. In 1984 Denef proved a cell decomposition result for more general valued fields [Den84]. This result was soon generalized to p-adic subanalytic and rigid analytic extensions, allowing for the later development of a more powerful technique of motivic integration. The conjunction of those model theoretic results allowed to solve a number of outstanding open problems in number theory (e.g., Artin's Conjecture on p-adic homogeneous forms).

In 1997, Karpinski and Macintyre computed VC-density bounds for o-minimal structures and asked about similar bounds for p-adic numbers [KM97]. VC-density is a concept closely related to VC-dimension. It comes up naturally in combinatorics with relation to packings, Hamming metric, entropic dimension and discrepancy. VC-density is also the decisive parameter in the Epsilon-Approximation Theorem, which is one of the crucial tools for applying VC theory in computational geometry. In a model theoretic setting it is computed for families of uniformly definable sets. In 2013, Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko computed a bound for VC-density in p-adic numbers and a number of other NIP structures [ADH16]. They observed connections to dp-rank and dp-minimality, notions first introduced by Shelah. In well behaved NIP structures families of uniformly definable sets tend to have VC-density bounded by a multiple of their dimension, a simple linear behavior. In a lot of cases including p-adic numbers this bound is not known to be optimal. My research concentrates on improving those bounds and adapting those techniques to compute VC-density in other common NIP structures of interest to mathematicians.

Some of the other well behaved NIP structures are Shelah-Spencer graphs and flat graphs. Shelah-Spencer graphs are limit structures for random graphs arising naturally in a combinatorial context. Their model theory was studied by Baldwin, Shi, and Shelah in 1997 [BS96], [BS97]. Later work of Laskowski in 2006 [Las07] has provided a quantifier simplification result. Flat graphs were first studied by Podewski-Ziegler in 1978, showing that those are stable [PZ78], and later results gave a criterion for super stability. Flat graphs also come up naturally in combinatorics in work of Nesetril and Ossona de Mendez [NM11].

The first chapter of my dissertation introduces some basics of model theory and defines VC-density and VC-dimension.

The second chapter concentrates on trees. I answer an open question from [ADH16], computing VC-density for trees viewed as a partial order. The main idea is to adapt a technique of Parigot [Par82] to partition trees into weakly interacting parts, with simple bounds of VC-density on each.

In the third chapter of my dissertation I work with Shelah-Spencer graphs. I have shown that they have infinite dp-rank, so they are poorly behaved as NIP structures. I have also shown that one can obtain arbitrarily high VC-density when looking at uniformly definable families in a fixed dimension. However I'm able to bound VC-density of individual formulas in terms of edge density of the graphs they define.

The fourth chapter deals with p-adic numbers. I have shown that VC-density is linear for an additive reduct of p-adic numbers using a cell decomposition result from the work of Leenknegt in 2013 [Lee14].

In chapter five I investigate flat graphs using the work of Podewski-Ziegler [PZ78]. I am able to show that flat graphs are dp-minimal, an important first step before establishing bounds on VC-density.

1.2 Basic Model Theory

This section goes through the basics of the model theory used throughout this text. It is meant to be used mostly as a reference on the notation as opposed to a comprehensive summary. For a complete and more thorough introduction to the material, we refer the reader to Chapters 1 and 2 of [TZ12]. We begin with a short summary of languages, formulas, and structures:

Definition 1.2.1.

- A language is a collection of predicate, function, and constant symbols.
- Fix language $\mathcal{L}$ and a collection of variables. A term is an expression constructed out of constants, variables, and functions.
- An atomic formula is an expression constructed out of equality symbol or a predicate applied to terms.
- A (first-order) formula is an expression constructed out of atomic formulas using boolean connectives $\wedge, \vee, \neg$ and quantifiers $\exists, \forall$. We denote such a formula as $\phi(x)$ where x is a tuple of free variables, that is the variables used in ϕ that are not bound by quantifiers. Abusing notation, we denote $\mathcal{L}$ to be the set of all such formulas (so $\phi \in \mathcal{L}$).
- A formula without free variables is called a sentence.
- A quantifier-free formula is a formula that doesn't contain any quantifiers.
- A structure $\mathbb{M}$ consists of an infinite universe M and functions, predicates, and constants matching those of $\mathcal{L}$.
- For a variable tuple x, let $|x|$ be the arity of the tuple.
- Suppose we have a formula $\phi(x)$, structure $\mathbb{M}$, and $a \in M^{|x|}$. Then we say that $\mathbb{M}$ models $\phi(a)$, denoted as $\mathbb{M} \models \phi(a)$, if formula ϕ holds $\mathbb{M}$ when we plug in a into x.

- Suppose we have a structure $\mathbb{M}$ and $A \subseteq M$. Then $\mathcal{L}(A)$ denotes an expansion of $\mathcal{L}$ by constant symbols correspoding to elements in A. The structure $\mathbb{M}$ then can be viewed as a $\mathcal{L}(A)$-structure with the appropriate interpretations. Formulas $\phi \in \mathcal{L}(A)$ will be referred to as formulas with parameters from A or simply as A-formulas. In this context A is usually referred to as a parameter set.

- A theory is a collection of sentences.

- For a theory T and a structure $\mathbb{M}$, we say that $\mathbb{M}$ models T, or that $\mathbb{M}$ is a model of T, if $\mathbb{M}$ models every sentence in T.

- For a structure $\mathbb{M}$, a theory of $\mathbb{M}$ is a collection of all sentences that are modelled by $\mathbb{M}$.

- A theory is called complete if it is a theory of some structure $\mathbb{M}$.

Throughout this text we often confuse complete theories with their models. This is justfied for properties that can be described by a collection of first-order sentences. Then a theory has this property if and only any (all) models have this property. An example of that is a notion of stability.

Stability is a deep subject, with a lot of theory developed around it. We won't work with it directly, but it is a property of some of the structures we study. We present a definitnon for completeness and refer the reader to Chapter 8 of [TZ12] or to [Pil13] for a more complete introduction.

Definition 1.2.2.

- Suppose we have a structure $\mathbb{M}$. The formula $\phi(x, y)$ is called unstable if for all natural n there exist $a_i \in M^{|x|}, b_i \in M^{|y|}$ for $0 \leq i \leq n$ such that

$$\mathbb{M} \models \phi(a_i, b_j) \iff i \leq j.$$

- A formula is stable if it is not unstable.

- A structure $\mathbb{M}$ is stable if all of its formulas are stable.

- A complete theory T is stable if any (all) of its models are stable.

Definable sets are subsets of our structure given by formulas. More precisely:

Definition 1.2.3. Suppose we have a structure $\mathbb{M}$, a paramter set $A \subseteq M$ and an A-formula $\phi(x)$. Then

$$\phi(M^{|x|}) = \left\{m \in M^{|x|} \mid \mathbb{M} \models \phi(m)\right\}.$$

is referred as an <u>A-definable</u> subset of $M^{|x|}$ defined by ϕ.

More generally, we will need a slightly more refined notion of a trace:

Definition 1.2.4. Suppose we have a structure $\mathbb{M}$, a formula $\phi(x, y)$, tuples $a \in M^{|x|}, b \in M^{|y|}$, and sets $A \subseteq M^{|x|}, B \subseteq M^{|y|}$. Define

$$\phi(A, b) = \{a \in A \mid \mathbb{M} \models \phi(a, b)\},$$
$$\phi(a, B) = \{b \in B \mid \mathbb{M} \models \phi(a, b)\}.$$

These sets will be informally referred to as traces.

Types is one of the main tools of study in model theory:

Definition 1.2.5. Suppose $\mathbb{M}$ is a structure, $B \subseteq M$. Also fix a variable tuple x.

- <u>A partial type over B</u> is a collection of formulas in variable x with parameters from B.
- A partial type $p(x)$ has a realization in $\mathbb{M}$ if there exists $a \in M^{|x|}$ such that $\mathbb{M} \models \phi(a)$ for all $\phi(x) \in p(x)$.
- A partial type is <u>consistent</u> if its every finite subset of formulas has a realization.
- Suppose $a \in M^{|x|}$ and $\Delta \subseteq \mathcal{L}(B)$ a collection of formulas in x. Define the <u>Δ-type of a over B</u> to be a collection of formulas $\phi(x) \in \Delta$ such that $\mathbb{M} \models \phi(a)$. Denote it as $\mathrm{tp}_\Delta(a/B)$.
- Suppose $a \in M^{|x|}$. Define the <u>type of a over B</u> as the Δ-type of a over B for $\Delta = \mathcal{L}(B)$. Denote it as $\mathrm{tp}(a/B)$.

Saturated structures is another important construction that we will be using. Generally speaking, a lot of the model theory is done inside of saturated structures as it simplifies a lot of constructions. The definition is as follows:

Definition 1.2.6. Let κ be a cardinal. A structure $\mathbb{M}$ is called κ-saturated if for all $B \subset M$ with $|B| < \kappa$ we have that all consistent partial types over B are realized in $\mathbb{M}$.

Indiscernible sequences will be useful to us to describe dp-rank and dp-minimality. They come up often in model theory as a way to leverage symmetry present in sequences and sets.

Definition 1.2.7.

- Suppose we have a sequence $(a_i)_{i\in\mathcal{I}}$ where $\mathcal{I}$ is an ordered index set. For $\mathcal{J} \subset \mathcal{I}$ the expression $a_{\mathcal{J}}$ denotes a tuple obtained by concatenation of the sequence $(a_j)_{j\in\mathcal{J}}$ (the sequence is ordered using the order of $\mathcal{I}$).
- Suppose $\mathbb{M}$ is a structure, $B \subseteq M$, and $\mathcal{I}$ is an oredred index set. A sequence $(a_i)_{i\in\mathcal{I}}$ is called indiscernible over B if for any two subsets $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{I}$ of equal length we have
$$\mathrm{tp}(a_{J_1}/B) = \mathrm{tp}(a_{J_2}/B).$$
- If we use the same definition, but allow tuples a_{J_1}, a_{J_2} to be concatenated in arbitrary order, then we obtain the definition a sequence that is totally indiscernible over B (alternatively a totally indiscernible set).

Here is an important property of indiscernible sequences in stable thoeries:

Lemma 1.2.8 (see Lemma 9.1.1 in [TZ12])**.** *If a structure is stable then every indiscernible sequence is totally indiscernible.*

Sometimes instead of starting with an indiscernible sequence, we wish to construct one from a sequence with some degree of symmetry:

Lemma 1.2.9 (see Lemma 5.1.3 in [TZ12])**.** *Work in a $\aleph_1$-saturated structure $\mathbb{M}$. Suppose $B \subset M$. Fix a variable tuple x and a collection of formulas $\Delta(x_1, \ldots, x_n)$ with $|x_1| = |x|$.*

Suppose we can find an arbitrarily long sequence $(a_i)_{i \in \mathcal{I}}$ with $a_i \in M^{|x|}$ such that for any subset $\mathcal{J} \subseteq \mathcal{I}$ of length n we have

$$\mathbb{M} \models \Delta(a_{\mathcal{J}}).$$

Then there exists an infinite indiscernible sequence $(a'_i)_{i \in \omega}$ with

$$\mathbb{M} \models \Delta(a'_1, a'_2, \ldots, a'_n).$$

Instead of working with types directly, it is often more convenient to work with automorphisms:

Definition 1.2.10. Suppose $\mathbb{M}$ is a structure and $A \subset M$. An automorphism of $\mathbb{M}$ over A is a bijection of $f\colon M \longrightarrow M$ that fixes A and preserves constants, relations, and functions of $\mathbb{M}$. We use notation $f \in \mathrm{Aut}(\mathbb{M}/A)$.

The following result is easy to show directly from the definition:

Lemma 1.2.11. *Suppose $\mathbb{M}$ is a structure, $A \subset M$, and $f \in \mathrm{Aut}(\mathbb{M}/A)$. Suppose also that $a, b \in M^n$ such that $f(a) = b$. Then $tp(a/A) = tp(b/A)$.*

The converse of this result holds in a special type of structure:

Definition 1.2.12. Let $\mathbb{M}$ be a structure and κ a cardinal. Then $\mathbb{M}$ is called strongly κ-homogeneous if for all $A \subseteq M$ with $|A| < \kappa$ we have that if for $a, b \in M^n$ if $tp(a/A) = tp(b/A)$ then there exists $f \in \mathrm{Aut}(\mathbb{M}/A)$ such that $f(a) = b$.

Luckily, for a given theory one can always find a model sufficiently saturated and homogeneous:

Lemma 1.2.13 (see Theorem 6.1.7 in [TZ12])**.** *Let T be a complete theory and κ a cardinal. There exists a model of T that is κ-saturated and strongly κ-homogeneous.*

1.3 VC-dimension and vc-density

Throughout this section we work with a collection $\mathcal{F}$ of subsets of an infinite set X. We call the pair $(X, \mathcal{F})$ a set system.

Definition 1.3.1.

- Given a subset A of X, we define the set system $(A, A \cap \mathcal{F})$ where $A \cap \mathcal{F} = \{A \cap F \mid F \in \mathcal{F}\}$.
- For $A \subseteq X$ we say that $\mathcal{F}$ shatters A if $A \cap \mathcal{F} = \mathcal{P}(A)$ (the power set of A).

Definition 1.3.2. We say $(X, \mathcal{F})$ has VC-dimension n if the largest subset of X shattered by $\mathcal{F}$ is of size n. If $\mathcal{F}$ shatters arbitrarily large subsets of X, we say that $(X, \mathcal{F})$ has infinite VC-dimension. We denote the VC-dimension of $(X, \mathcal{F})$ by $\mathrm{VC}(X, \mathcal{F})$.

Note 1.3.3. We may drop X from the notation $\mathrm{VC}(X, \mathcal{F})$, as the VC-dimension doesn't depend on the base set and is determined by $(\bigcup \mathcal{F}, \mathcal{F})$.

Set systems of finite VC-dimension tend to have good combinatorial properties, and we consider set systems with infinite VC-dimension to be poorly behaved.

Another natural combinatorial notion is that of the dual system of a set system:

Definition 1.3.4. For $a \in X$ define $X_a = \{F \in \mathcal{F} \mid a \in F\}$. Let $\mathcal{F}^* = \{X_a \mid a \in X\}$. We call $(\mathcal{F}, \mathcal{F}^*)$ the dual system of $(X, \mathcal{F})$. The VC-dimension of the dual system of $(X, \mathcal{F})$ is referred to as the dual VC-dimension of $(X, \mathcal{F})$ and denoted by $\mathrm{VC}^*(\mathcal{F})$. (As before, this notion doesn't depend on X.)

Lemma 1.3.5 (see 2.13b in [Ass83])**.** *A set system $(X, \mathcal{F})$ has finite VC-dimension if and only if its dual system has finite VC-dimension. More precisely*

$$\mathrm{VC}^*(\mathcal{F}) \leq 2^{1+\mathrm{VC}(\mathcal{F})}.$$

For a more refined notion of complexity of $(X, \mathcal{F})$ we look at the traces of our family on finite sets:

Definition 1.3.6. Define the shatter function $\pi_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ and the dual shatter function $\pi^*_{\mathcal{F}} \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $\mathcal{F}$ by

$$\pi_{\mathcal{F}}(n) = \max \{ |A \cap \mathcal{F}| \mid A \subseteq X \text{ and } |A| = n \}$$
$$\pi^*_{\mathcal{F}}(n) = \max \{ \text{atoms}(B) \mid B \subseteq \mathcal{F}, |B| = n \}$$

where $\text{atoms}(B)$ = number of atoms in the boolean algebra of sets generated by B. Note that the dual shatter function is precisely the shatter function of the dual system: $\pi^*_{\mathcal{F}} = \pi_{\mathcal{F}^*}$.

A simple upper bound is $\pi_{\mathcal{F}}(n) \leq 2^n$ (same for the dual). If the VC-dimension of $\mathcal{F}$ is infinite then clearly $\pi_{\mathcal{F}}(n) = 2^n$ for all n. Conversely we have the following remarkable fact:

Theorem 1.3.7 (Sauer-Shelah '72, see [Sau72], [She72])**.** *If the set system $(X, \mathcal{F})$ has finite VC-dimension d then $\pi_{\mathcal{F}}(n) \leq \binom{n}{\leq d}$ for all n, where $\binom{n}{\leq d} = \binom{n}{d} + \binom{n}{d-1} + \ldots + \binom{n}{1}$.*

Thus the systems with a finite VC-dimension are precisely the systems where the shatter function grows polynomially. The vc-density of $\mathcal{F}$ quantifies the growth of the shatter function of $\mathcal{F}$:

Definition 1.3.8. Define the vc-density and dual vc-density of $\mathcal{F}$ as

$$\text{vc}(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\},$$
$$\text{vc}^*(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}}(n)}{\log n} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}.$$

Generally speaking a shatter function that is bounded by a polynomial doesn't itself have to be a polynomial. Proposition 4.12 in [ADH16] gives an example of a shatter function that grows like $n \log n$ (so it has vc-density 1).

So far the notions that we have defined are purely combinatorial. We now adapt VC-dimension and vc-density to the model theoretic context.

Definition 1.3.9. Work in a first-order structure M. Fix a finite collection of formulas $\Phi(x, y)$ in the language $\mathcal{L}(M)$ of M.

- For $\phi(x, y) \in \mathcal{L}(M)$ and $b \in M^{|y|}$ let

$$\phi(M^{|x|}, b) = \{a \in M^{|x|} \mid \phi(a, b)\} \subseteq M^{|x|}.$$

- Let $\Phi(M^{|x|}, M^{|y|}) = \{\phi(M^{|x|}, b) \mid \phi_i \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|})$.

- Let $\mathcal{F}_\Phi = \Phi(M^{|x|}, M^{|y|})$, giving rise to a set system $(M^{|x|}, \mathcal{F}_\Phi)$.

- Define the VC-dimension $\mathrm{VC}(\Phi)$ of Φ to be the VC-dimension of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.

- Define the vc-density $\mathrm{vc}(\Phi)$ of Φ to be the vc-density of $(M^{|x|}, \mathcal{F}_\Phi)$, similarly for the dual.

We will also refer to the vc-density and VC-dimension of a single formula ϕ viewing it as a one element collection $\Phi = \{\phi\}$.

Counting atoms of a boolean algebra in a model theoretic setting corresponds to counting types, so it is instructive to rewrite the shatter function in terms of types.

Definition 1.3.10.

$$\pi^*_\Phi(n) = \max \{\text{number of } \Phi\text{-types over } B \mid B \subseteq M, |B| = n\}.$$

Here a Φ-type over B is a maximal consistent collection of formulas of the form $\phi(x, b)$ or $\neg\phi(x, b)$ where $\phi \in \Phi$ and $b \in B$.

The functions π^*_Φ and $\pi^*_{\mathcal{F}_\Phi}$ do not have to agree, as one fixes the number of generators of a boolean algebra of sets and the other fixes the size of the parameter set. However, as the following lemma demonstrates, they both give the same asymptotic definition of dual vc-density.

Lemma 1.3.11.

$$\mathrm{vc}^*(\Phi) = \textit{degree of polynomial growth of } \pi^*_\Phi(n) = \limsup_{n\to\infty} \frac{\log \pi^*_\Phi(n)}{\log n}.$$

Proof. With a parameter set B of size n, we get at most $|\Phi|n$ sets $\phi(M^{|x|}, b)$ with $\phi \in \Phi, b \in B$. We check that asymptotically it doesn't matter whether we look at growth of boolean algebra of sets generated by n or by $|\Phi|n$ many sets. We have:

$$\pi^*_{\mathcal{F}_\Phi}(n) \leq \pi^*_\Phi(n) \leq \pi^*_{\mathcal{F}_\Phi}(|\Phi|n).$$

Hence:

$$\begin{aligned}
&\mathrm{vc}^*(\Phi) \leq \limsup_{n\to\infty} \frac{\log \pi^*_\Phi(n)}{\log n} \leq \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log n} = \\
&= \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log |\Phi|n} \frac{\log |\Phi|n}{\log n} = \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(|\Phi|n)}{\log |\Phi|n} \leq \\
&\leq \limsup_{n\to\infty} \frac{\log \pi^*_{\mathcal{F}_\Phi}(n)}{\log n} = \mathrm{vc}^*(\Phi).
\end{aligned}$$

□

One can check that the shatter function and hence VC-dimension and vc-density of a formula are elementary notions, so they only depend on the first-order theory of the structure M.

NIP theories are a natural context for studying vc-density. In fact we can take the following as the definition of NIP:

Definition 1.3.12. Define ϕ to be NIP if it has finite VC-dimension in a theory T. A theory T is NIP if all the formulas in T are NIP.

In a general combinatorial context (for arbitrary set systems), vc-density can be any real number in $0 \cup [1, \infty)$ (see [Ass85]). Less is known if we restrict our attention to NIP theories. Proposition 4.6 in [ADH16] gives examples of formulas that have non-integer rational vc-density in an NIP theory, however it is open whether one can get an irrational vc-density in this model-theoretic setting.

Instead of working with a theory formula by formula, we can look for a uniform bound for all formulas:

Definition 1.3.13. For a given NIP structure M, define the vc-function

$$\begin{aligned}\mathrm{vc}^M(n) &= \sup\{\mathrm{vc}^*(\phi(x,y)) \mid \phi \in \mathcal{L}(M), |x| = n\} \\ &= \sup\{\mathrm{vc}(\phi(x,y)) \mid \phi \in \mathcal{L}(M), |y| = n\} \in \mathbb{R}^{\geq 0} \cup \{+\infty\}\,.\end{aligned}$$

As before this definition is elementary, so it only depends on the theory of M. We omit the superscript M if it is understood from the context. One can easily check the following bounds:

Lemma 1.3.14 (Lemma 3.22 in [ADH16])**.** *We have* $\mathrm{vc}(1) \geq 1$ *and* $\mathrm{vc}(n) \geq n\,\mathrm{vc}(1)$.

However, it is not known whether the second inequality can be strict or even whether $\mathrm{vc}(1) < \infty$ implies $\mathrm{vc}(n) < \infty$.

Dp-rank is a common measure used in study of NIP theories, with dp-minimality being a special case. Those notions originated in [She14], and further studied in [KOU13], showing, for example, that dp-rank is additive. Here it is easiest for us to define dp-rank in terms of vc-density over indiscernible sequences.

Definition 1.3.15.

- Work in a $\aleph_1$-saturated first-order structure M. Fix a finite collection of formulas $\Phi(x,y)$ in the language of M. Suppose $A = (a_i)_{i\in\omega}$ is an indiscernible sequence with each $a_i \in M^{|x|}$. Let

$$\mathscr{I}(A,\Phi) = \{\phi(\bigcup_{i\in\mathbb{N}} a_i, b) \mid \phi \in \Phi, b \in M^{|y|}\} \subseteq \mathcal{P}(M^{|x|}).$$

  This gives rise to a set system $(M^{|x|}, \mathscr{I}(A,\Phi))$.

- Define

$$\mathrm{vc}_{\mathrm{ind}}(\Phi) = \sup\left\{\mathrm{vc}(\mathscr{I}(A,\Phi)) \mid A = (a_i)_{i\in\mathbb{N}} \text{ is indiscernible}\right\}.$$

- Dp-rank of an $\aleph_1$-saturated structure M is $\leq n$ if $\mathrm{vc}_{\mathrm{ind}}(\phi) \leq n$ for all formulas ϕ.

- Dp-rank of a theory T is $\leq n$ if dp-rank is $\leq n$ for any (all) $\aleph_1$-saturated model of T.

- A theory T is said to have finite dp-rank if its dp-rank is $\leq n$ for some n.

- A theory T is <u>dp-minimal</u> if its dp-rank ≤ 1.

Refer to [GH14] for the connection between to the classical definition of dp-rank and the definition given here.

There is a useful characterization of dp-minimality in terms of indiscernible sequences that will be useful for what we do:

Lemma 1.3.16 (see Lemma 1.4 in [Sim11])**.** *Suppose* $\mathbb{M}$ *is an* $\aleph_1$*-saturated structure. Then the following are equivalent:*

- $\mathbb{M}$ *is dp-minimal.*
- *For any countable indiscernible sequence* $(a_i)_{i \in \mathcal{I}}$ *indexed by a dense linear order* $\mathcal{I}$*, and any* $c \in M$*, there is* i_0 *in the completion of* I *such that the two sequences* $(\mathrm{tp}(a_i/c) \mid i < i_0)$ *and* $(\mathrm{tp}(a_i/c) \mid i > i_0)$ *are constant.*

CHAPTER 2

Trees

We show that for the theory of infinite trees we have $\mathrm{vc}(n) = n$ for all n. This generalizes a result of Simon in [Sim11] showing that the trees are dp-minimal.

VC-density was studied in [ADH16] by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko as a natural notion of dimension for NIP theories. In an NIP theory we can define a vc-function

$$\mathrm{vc} : \mathbb{N} \longrightarrow \mathbb{R} \cup \{\infty\},$$

where $\mathrm{vc}(n)$ measures the worst-case complexity of families of definable sets in an n-dimensional space. Simplest possible behavior is $\mathrm{vc}(n) = n$ for all n. Theories with the property that $\mathrm{vc}(1) = 1$ are known to be dp-minimal, i.e., having the smallest possible dp-rank. In general, it is not known whether there can be a dp-minimal theory which doesn't satisfy $\mathrm{vc}(n) = n$.

In this paper we work with trees viewed as posets. Parigot in [Par82] showed that such structures have NIP. This result was strengthened by Simon in [Sim11] showing that trees are dp-minimal. The paper [ADH16] poses the following problem:

Problem 2.0.17. ([ADH16] p. 47) Determine the VC density function of each (infinite) tree.

Here we settle this question by showing that any model of the theory of trees has $\mathrm{vc}(n) = n$.

Section 1 of the paper consists of a basic introduction to the concepts of VC-dimension and vc-density. In Section 2 we introduce proper subdivisions – the main tool that we use to analyze trees. We also prove some basic properties of proper subdivisions. Section 3 introduces the key constructions of proper subdivisions in tree which will be used in the proof. Section 4 presents the proof of $\mathrm{vc}(n) = n$ via the subdivisions.

We use notation $a \in T^n$ for the tuples of size n. For a variable x or a tuple a we denote their arity by $|x|$ and $|a|$ respectively.

The language for the trees consists of a single binary predicate $\{\leq\}$. The theory of trees states that $\leq$ defines a partial order and for every element a the set $\{x \mid x < a\}$ is linearly ordered by $<$. For visualization purposes we assume that trees grow upwards, with the smaller elements on the bottom and the larger elements on the top. If $a \leq b$ we will say that a is below b and b is above a.

Definition 2.0.18. Work in a tree $\mathcal{T} = (T, \leq)$. For $x \in T$ let $I(x) = \{t \in T \mid t \leq x\}$ denote all the elements below x. The *meet* of two tree elements a, b is the greatest element of $I(a) \cap I(b)$ (if one exists) and is denoted by $a \wedge b$.

The theory of meet trees requires that any two elements in the same connected component have a meet. Colored trees are trees with a finite number of colors added via unary predicates.

From now on assume that all trees are colored. We allow our trees to be disconnected (so really, we work with forests) or finite unless otherwise stated.

2.1 Proper Subdivisions: Definition and Properties

We work with finite relational languages. Given a formula we define its complexity as the depth of quantifiers used to build up the formula. More precisely:

Definition 2.1.1. Define *complexity* of a formula by induction:

$$\text{Complexity}(\text{q.f. formula}) = 0$$
$$\text{Complexity}(\exists x \phi(x)) = \text{Complexity}(\phi(x)) + 1$$
$$\text{Complexity}(\phi \wedge \psi) = \max(\text{Complexity}(\phi), \text{Complexity}(\psi))$$
$$\text{Complexity}(\neg \phi) = \text{Complexity}(\phi)$$

A simple inductive argument verifies that there are (up to equivalence) only finitely many formulas when the complexity and the number of free variables are fixed. We will use the following notation for types:

Definition 2.1.2. Let $\mathcal{B}$ be a structure, $A \subset B$ be a finite parameter set, and a, b be tuples in $\mathcal{B}$, and m, n be natural numbers.

- $\text{tp}^n_{\mathcal{B}}(a/A)$ will stand for the set of all A-formulas of complexity $\leq n$ that are true of a in $\mathcal{B}$. If $A = \emptyset$ we may also write this as $\text{tp}^n_{\mathcal{B}}(a)$. The subscript $\mathcal{B}$ will be omitted as well if it is clear from context. Note that if A is finite, there are finitely many such formulas (up to equivalence). The conjunction of those formulas still has complexity $\leq n$ and so we can just associate a single formula to every type $\text{tp}^n_{\mathcal{B}}(a/A)$.
- $\mathcal{B} \models a \equiv^n_A b$ means that a, b have the same type with complexity $\leq n$ over A in $\mathcal{B}$, i.e., $\text{tp}^n_{\mathcal{B}}(a/A) = \text{tp}^n_{\mathcal{B}}(b/A)$.
- $S^n_{\mathcal{B},m}(A)$ will stand for the set of all m-types of complexity $\leq n$ over A:

$$S^n_{\mathcal{B},m}(A) = \left\{\text{tp}^n_{\mathcal{B}}(a/A) \mid a \in B^m\right\}.$$

Definition 2.1.3.

- Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{T}$ be structures in some (possibly different) finite relational languages. If the underlying sets A, B of $\mathcal{A}, \mathcal{B}$ partition the underlying set T of $\mathcal{T}$ (i.e. $T = A \sqcup B$), then we say that $(\mathcal{A}, \mathcal{B})$ is a *subdivision* of $\mathcal{T}$.
- A subdivision $(\mathcal{A}, \mathcal{B})$ of $\mathcal{T}$ is called *n-proper* if given $p, q \in \mathbb{N}$, $a_1, a_2 \in A^p$ and $b_1, b_2 \in B^q$

with

$$\mathcal{A} \models a_1 \equiv_n a_2$$

$$\mathcal{B} \models b_1 \equiv_n b_2$$

we have

$$\mathcal{T} \models a_1 b_1 \equiv_n a_2 b_2.$$

- A subdivision $(\mathcal{A}, \mathcal{B})$ of $\mathcal{T}$ is called *proper* if it is n-proper for all $n \in \mathbb{N}$.

Lemma 2.1.4. *Consider a subdivision $(\mathcal{A}, \mathcal{B})$ of $\mathcal{T}$. If $(\mathcal{A}, \mathcal{B})$ is 0-proper then it is proper.*

Proof. We prove that the subdivision is n-proper for all k by induction. The case $n = 0$ is given by the assumption. Suppose we have $\mathcal{T} \models \exists x\, \phi^n(x, a_1, b_1)$ where ϕ^n is some formula of complexity n. Let $a \in T$ witness the existential claim, i.e., $\mathcal{T} \models \phi^n(a, a_1, b_1)$. We can have $a \in A$ or $a \in B$. Without loss of generality assume $a \in A$. Let $\boldsymbol{p} = \mathrm{tp}^n_{\mathcal{A}}(a, a_1)$. Then we have

$$\mathcal{A} \models \exists x \ \mathrm{tp}^n_{\mathcal{A}}(x, a_1) = \boldsymbol{p}$$

(with $\mathrm{tp}^n_{\mathcal{A}}(x, a_1) = \boldsymbol{p}$ a shorthand for $\phi_{\boldsymbol{p}}(x)$ where $\phi_{\boldsymbol{p}}$ is a formula that determines the type $\boldsymbol{p}$). The formula $\mathrm{tp}^n_{\mathcal{A}}(x, a_1) = \boldsymbol{p}$ is of complexity $\leq k$ so $\exists x \ \mathrm{tp}^n_{\mathcal{A}}(x, a_1) = \boldsymbol{p}$ is of complexity $\leq k + 1$. By the inductive hypothesis we have

$$\mathcal{A} \models \exists x \ \mathrm{tp}^n_{\mathcal{A}}(x, a_2) = \boldsymbol{p}\,.$$

Let a' witness this existential claim, so that $\mathrm{tp}^n_{\mathcal{A}}(a', a_2) = \boldsymbol{p}$, hence $\mathrm{tp}^n_{\mathcal{A}}(a', a_2) = \mathrm{tp}^n_{\mathcal{A}}(a, a_1)$, that is, $\mathcal{A} \models a' a_2 \equiv_n a a_1$. By the inductive hypothesis we therefore have $\mathcal{T} \models a a_1 b_1 \equiv_n a' a_2 b_2$; in particular $\mathcal{T} \models \phi^n(a', a_2, b_2)$as $\mathcal{T} \models \phi^n(a, a_1, b_1)$, and $\mathcal{T} \models \exists x \phi^n(x, a_2, b_2)$. □

This lemma is general, but we will use it specifically applied to (colored) trees. Suppose $\mathcal{T}$ is a (colored) tree in some language $\mathcal{L} = \{\leq, \ldots\}$. Suppose $\mathcal{A}, \mathcal{B}$ are some structures in languages $\mathcal{L}_A, \mathcal{L}_B$ which expand $\mathcal{L}$, with the $\mathcal{L}$-reducts of $\mathcal{A}, \mathcal{B}$ substructures of $\mathcal{T}$. Furthermore suppose that $(\mathcal{A}, \mathcal{B})$ is 0-proper. Then by the previous lemma $(\mathcal{A}, \mathcal{B})$ is a proper subdivision of $\mathcal{T}$. From now on all the subdivisions we work with will be of this form.

Example 2.1.5. Suppose a tree consists of two connected components C_1, C_2. Then those components $(C_1, \leq)$, $(C_2, \leq)$ viewed as substructures form a proper subdivision. To see that we only need to show that this subdivision is 0-proper. But that is immediate as any $c_1 \in C_1$ and $c_2 \in C_2$ are incomparable.

Example 2.1.6. Fix a tree $\mathcal{T}$ in the language $\{\leq\}$ and $a \in T$. Let $B = \{t \in T \mid a < t\}$, $S = \{t \in T \mid t \leq a\}$, $A = T - B$. Then $(A, \leq, S)$ and $(B, \leq)$ form a proper subdivision, where $\mathcal{L}_A$ has a unary predicate interpreted by S. To see this, again, we show that the subdivision is 0-proper. The only time $a \in A$ and $b \in B$ are comparable is when $a \in S$, and this is captured by the language. (See proof of Lemma 2.2.7 for more details.)

Definition 2.1.7. For $\phi(x, y)$, $A \subseteq T^{|x|}$ and $B \subseteq T^{|y|}$

- let $\phi(A, b) = \{a \in A \mid \phi(a, b)\} \subseteq A$, and
- let $\phi(A, B) = \{\phi(A, b) \mid b \in B\} \subseteq \mathcal{P}(A)$.

Thus $\phi(A, B)$ is a collection of subsets of A definable by ϕ with parameters from B. We notice the following bound when A, B are parts of a proper subdivision.

Corollary 2.1.8. *Let $\mathcal{A}, \mathcal{B}$ be a proper subdivision of $\mathcal{T}$ and $\phi(x, y)$ be a formula of complexity n. Then $|\phi(A^{|x|}, B^{|y|})|$ is bounded by $|S^n_{\mathcal{B},|y|}|$.*

Proof. Take some $a \in A^{|x|}$ and $b_1, b_2 \in B^{|y|}$ with $\mathrm{tp}^n_{\mathcal{B}}(b_1) = \mathrm{tp}^n_{\mathcal{B}}(b_2)$. We have $\mathcal{B} \models b_1 \equiv_n b_2$ and (trivially) $\mathcal{A} \models a \equiv_n a$. Thus we have $\mathcal{T} \models ab_1 \equiv_n ab_2$, so $T \models \phi(a, b_1) \leftrightarrow \phi(a, b_2)$. Since a was arbitrary we have $\phi(A^{|x|}, b_1) = \phi(A^{|x|}, b_2)$ as different traces can only come from parameters of different types. Thus $|\phi(A^{|x|}, B^{|y|}) \leq |S^n_{\mathcal{B},|y|}|$. □

We note that the size of the type space $|S^n_{\mathcal{B},|y|}|$ can be bounded uniformly:

Definition 2.1.9. Fix a (finite relational) language $\mathcal{L}_B$. Let $N = N(n, m, \mathcal{L}_B)$ be smallest integer such that for any structure $\mathcal{B}$ in $\mathcal{L}_B$ we have $|S^n_{\mathcal{B},m}| \leq N$. This integer exists as there is a finite number (up to equivalence) of possible formulas of complexity $\leq n$ with m free variables. Note that $N(n, m, \mathcal{L}_B)$ is increasing in all variables:

$$n \leq n', m \leq m', \mathcal{L}_B \subseteq \mathcal{L}'_B \Rightarrow N(n, m, \mathcal{L}_B) \leq N(n', m', \mathcal{L}'_B)$$

2.2 Proper Subdivisions: Constructions

Throughout this section, $\mathcal{T}$ denotes a colored meet tree. First, we describe several constructions of proper subdivisions that are needed for the proof.

Definition 2.2.1. We use $E(b,c)$ to express that b and c are in the same connected component:

$$E(b,c) \Leftrightarrow \exists x\,(b \geq x) \wedge (c \geq x).$$

Definition 2.2.2. Given an element a of the tree we call the sets fo all the elements above a, i.e. the set $\{x \mid x \geq a\}$, the *closed cone* above a. Connected components of that cone can be thought of as *open cones* above a. With that interpretation in mind, the notation $E_a(b,c)$ means that b and c are in the same open cone above a. More formally:

$$E_a(b,c) \Leftrightarrow E(b,c) \text{ and } (b \wedge c) > a.$$

Fix a language $\mathcal{L}$ for a colored tree $\mathcal{L} = \{\leq, C_1, \dots C_n\} = \{\leq, \vec{C}\}$. In the following four definitions structures denoted by $\mathcal{B}$ are going to be in the same language $\mathcal{L}_B = \mathcal{L} \cup \{U\}$ with U a unary predicate. It is not always necessary to have this predicate but we keep it for the sake of uniformity. Structures denoted by $\mathcal{A}$ will have different languages $\mathcal{L}_A$ (those are not as important in later applications).

Definition 2.2.3. Fix $c_1 < c_2$ in T. Let

$$\begin{aligned}
B &= \{b \in T \mid E_{c_1}(c_2, b) \wedge \neg(b \geq c_2)\},\\
A &= T - B,\\
S_1 &= \{t \in T \mid t < c_1\},\\
S_2 &= \{t \in T \mid t < c_2\},\\
S_B &= S_2 - S_1,\\
T_A &= \{t \in T \mid c_2 \leq t\}.
\end{aligned}$$

Define structures $\mathcal{A}^{c_1}_{c_2} = (A, \leq, \vec{C} \cap A, S_1, T_A)$ where $\vec{C} \cap A = \{C_1 \cap A, \dots, C_n \cap A\}$ and $\mathcal{B}^{c_1}_{c_2} = (B, \leq, \vec{C} \cap B, S_B)$ where $\mathcal{L}_A$ is an expansion of $\mathcal{L}$ by two unary predicate symbols (and $\mathcal{L}_B$ as defined above). Note that $c_1, c_2 \notin B$.

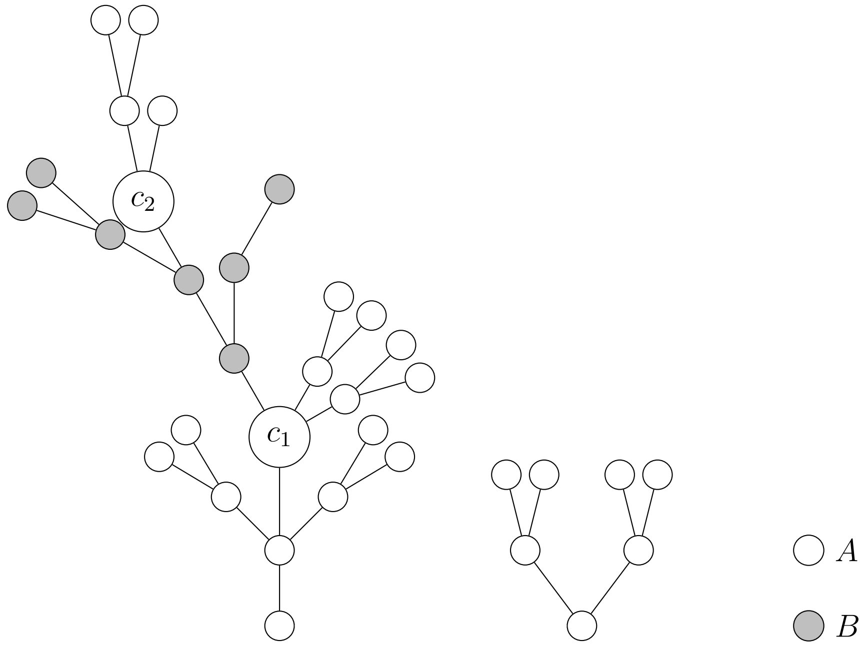

Figure 2.1: Proper subdivision for $(A, B) = (A^{c_1}_{c_2}, B^{c_1}_{c_2})$

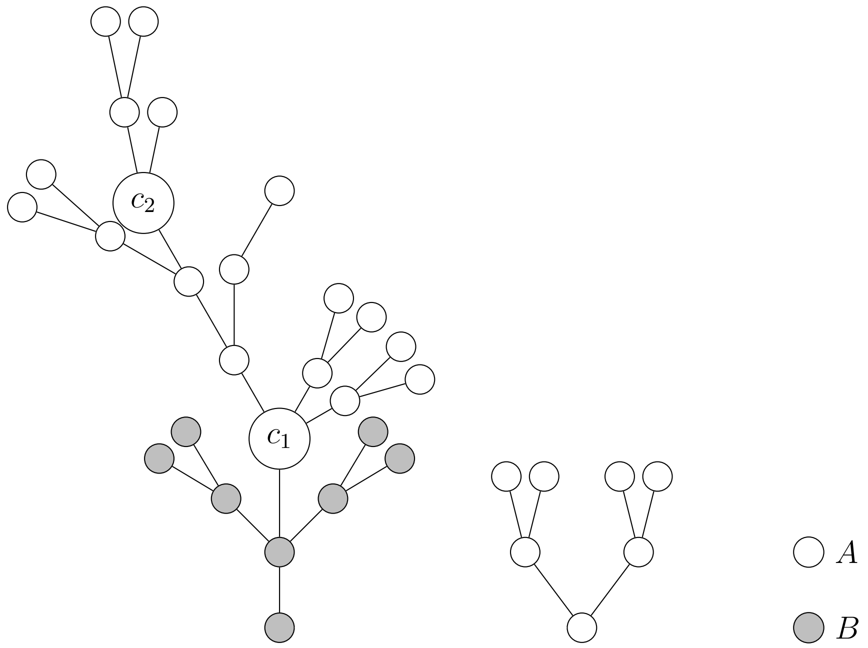

Figure 2.2: Proper subdivision for $(A, B) = (A_{c_1}, B_{c_1})$

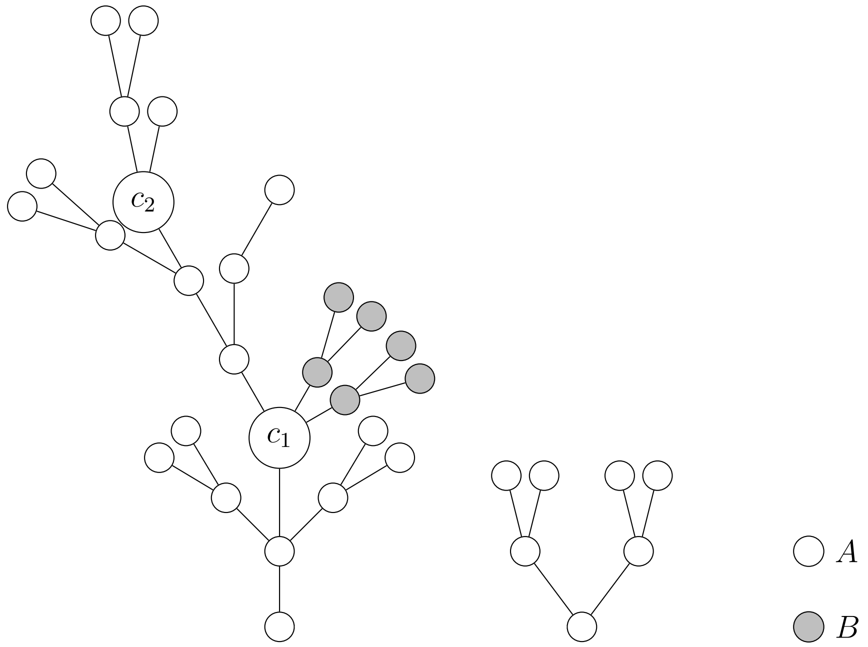

Figure 2.3: Proper subdivision for $(A, B) = (A_G^{c_1}, B_G^{c_1})$ for $S = \{c_2\}$

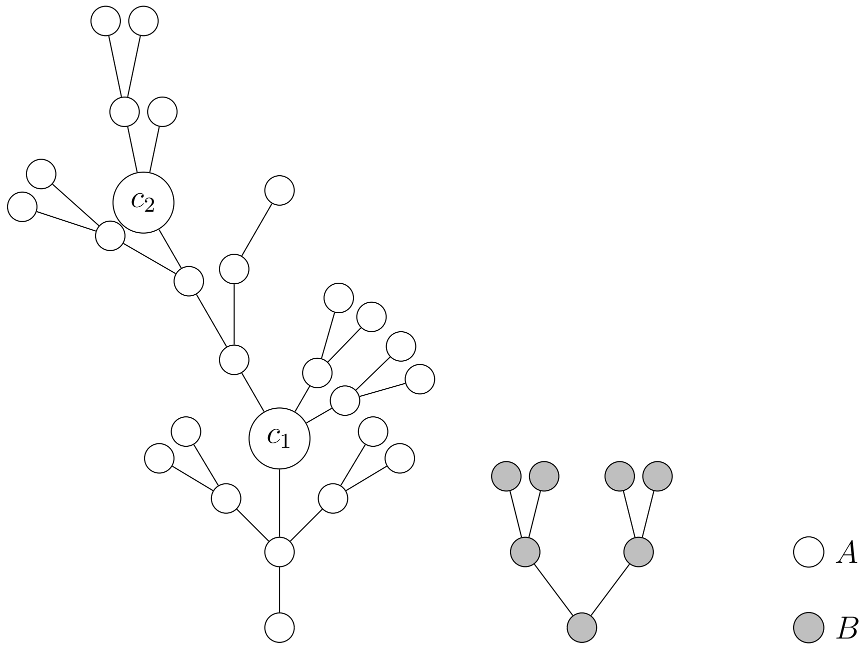

Figure 2.4: Proper subdivision for $(A, B) = (A_G, B_G)$ for $S = \{c_1, c_2\}$

Definition 2.2.4. Fix c in T. Let

$$B = \{b \in T \mid \neg(b \geq c) \wedge E(b, c)\},$$
$$A = T - B,$$
$$S_1 = \{t \in T \mid t < c\}.$$

Define structures $\mathcal{A}_c = (A, \leq, \vec{C} \cap A)$ and $\mathcal{B}_c = (B, \leq, \vec{C} \cap B, S_1)$ where $\mathcal{L}_A = \mathcal{L}$ (and $\mathcal{L}_B$ as defined above). Note that $c \notin B$. (cf example 2.1.6).

Definition 2.2.5. Fix c in T and a finite subset $S \subseteq T$. Let

$$B = \{b \in T \mid (b > c) \text{ and for all } s \in S \text{ we have } \neg E_c(s, b)\},$$
$$A = T - B,$$
$$S_1 = \{t \in T \mid t \leq c\}.$$

Define structures $\mathcal{A}^c_S = (A, \leq, \vec{C} \cap A, S_1)$ and $\mathcal{B}^c_S = (B, \leq, \vec{C} \cap B, B)$ where $\mathcal{L}_A$ is an expansion of $\mathcal{L}$ by a single unary predicate (and $U \in \mathcal{L}_B$ vacuously interpreted by B). Note that $c \notin B$ and $S \cap B = \emptyset$.

Definition 2.2.6. Fix a finite subset $S \subseteq T$. Let

$$B = \{b \in T \mid \text{ for all } s \in S \text{ we have } \neg E(s, b)\},$$
$$A = T - B.$$

Define structures $\mathcal{A}_S = (A, \leq)$ and $\mathcal{B}_S = (B, \leq, \vec{C} \cap B, B)$ where $\mathcal{L}_A = \mathcal{L}$ (and $U \in \mathcal{L}_B$ vacuously interpreted by B). Note that $S \cap B = \emptyset$. (cf. example 2.1.5)

Lemma 2.2.7. *The pairs of structures defined above are all proper subdivisions of $\mathcal{T}$.*

Proof. We only show this holds for the pair $(\mathcal{A}, \mathcal{B}) = (\mathcal{A}^{c_1}_{c_2}, \mathcal{B}^{c_1}_{c_2})$. The other cases follow by a similar argument. The sets A, B partition T by definition, so (A, B) is a subdivision of $\mathcal{T}$. To show that it is proper, by Lemma 2.1.4 we only need to check that it is 0-proper.

Suppose we have

$$a = (a_1, a_2, \ldots, a_p) \in A^p,$$
$$a' = (a'_1, a'_2, \ldots, a'_p) \in A^p,$$
$$b = (b_1, b_2, \ldots, b_q) \in B^q,$$
$$b' = (b'_1, b'_2, \ldots, b'_q) \in B^q.$$

with $\mathcal{A} \models a \equiv_0 a'$ and $\mathcal{B} \models b \equiv_0 b'$. We need to show that ab has the same quantifier-free type in $\mathcal{T}$ as $a'b'$. Any two elements in T can be related in the four following ways:

$$x = y,$$
$$x < y,$$
$$x > y, \text{ or}$$
$$x, y \text{ are incomparable.}$$

We need to check that for all i, j the same relations hold for (a_i, b_j) as do for (a'_i, b'_j).

- It is impossible that $a_i = b_j$ as they come from disjoint sets.
- Suppose $a_i < b_j$. This forces $a_i \in S_1$ thus $a'_i \in S_1$ and $a'_i < b'_j$.
- Suppose $a_i > b_j$. This forces $b_j \in S_B$ and $a \in T_A$, thus $b'_j \in S_B$ and $a'_i \in T_A$, so $a'_i > b'_j$.
- Suppose a_i and b_j are incomparable. Two cases are possible:
  - $b_j \notin S_B$ and $a_i \in T_A$. Then $b'_j \notin S_B$ and $a'_i \in T_A$ making a'_i, b'_j incomparable.
  - $b_j \in S_B$, $a_i \notin T_A$, $a_i \notin S_1$. Similarly this forces a'_i, b'_j to be incomparable.

Also we need to check that ab has the same colors as $a'b'$. But that is immediate as having the same color in a substructure means having the same color in the tree. $\square$

2.3 Main proof

The basic idea for the proof is as follows. Suppose we have a formula with q parameters over a parameter set of size n. We are able to split our parameter space into $O(n)$ many partitions. Each of q parameters can come from any of those $O(n)$ partitions giving us $O(n)^q$ many choices for parameter configuration. When every parameter is coming from a fixed partition the number of definable sets is constant and in fact is uniformly bounded above by some N. This gives us at most $N \cdot O(n)^q$ possibilities for different definable sets.

First, we generalize Corollary 2.1.8. (This is required for computing vc-density for formulas $\phi(x, y)$ with $|y| > 1$).

Lemma 2.3.1. *Consider a finite collection $(\mathcal{A}_i, \mathcal{B}_i)_{i\leq n}$ satisfying the following properties:*

- *$(\mathcal{A}_i, \mathcal{B}_i)$ is either a proper subdivision of $\mathcal{T}$ or $A_i = T$ and $B_i = \{b_i\}$,*
- *all $\mathcal{B}_i$ have the same language $\mathcal{L}_B$,*
- *sets $\{B_i\}_{i\leq n}$ are pairwise disjoint.*

Let $A = \bigcap_{i\in I} A_i$. Fix a formula $\phi(x, y)$ of complexity m . Let $N = N(m, |y|, \mathcal{L}_B)$ be as in Definition 2.1.9. Consider any $B \subseteq T^{|y|}$ of the form

$$B = B_1^{i_1} \times B_2^{i_2} \times \ldots \times B_n^{i_n} \text{ with } i_1 + i_2 + \ldots + i_n = |y|.$$

(some of the indeces can be zero). Then we have the following bound:

$$\phi(A^{|x|}, B) \leq N^{|y|}.$$

Proof. We show this result by counting types.

Claim 2.3.2. *Suppose we have*

$$b_1, b_1' \in B_1^{i_1} \text{ with } b_1 \equiv_m b_1' \text{ in } \mathcal{B}_1,$$
$$b_2, b_2' \in B_2^{i_2} \text{ with } b_2 \equiv_m b_2' \text{ in } \mathcal{B}_2,$$
$$\ldots$$
$$b_n, b_n' \in B_n^{i_n} \text{ with } b_n \equiv_m b_n' \text{ in } \mathcal{B}_n.$$

Then

$$\phi(A^{|x|}, b_1, b_2, \ldots b_n) \iff \phi(A^{|x|}, b_1', b_2', \ldots b_n').$$

Proof. Define $\bar{b}_i = (b_1, \ldots, b_i, b'_{i+1}, \ldots, b'_n) \in B$ for $i \in [0..n]$. (That is, a tuple where first i elements are without prime, and elements after that are with a prime.) We have $\phi(A^{|x|}, \bar{b}_i) \iff \phi(A^{|x|}, \bar{b}_{i+1})$ as either $(\mathcal{A}_{i+1}, \mathcal{B}_{i+1})$ is m-proper or $\mathcal{B}_{i+1}$ is a singleton, and the implication is trivial. (Notice that $b_i \in \mathcal{A}_j$ for $j \neq i$ by disjointness assumption.) Thus, by induction we get $\phi(A^{|x|}, \bar{b}_0) \iff \phi(A^{|x|}, \bar{b}_n)$ as needed. $\square$

Thus $\phi(A^{|x|}, B)$ only depends on the choice of the types for the tuples:

$$|\phi(A^{|x|}, B)| \leq |S^m_{\mathcal{B}_1, i_1}| \cdot |S^m_{\mathcal{B}_2, i_2}| \cdot \ldots \cdot |S^m_{\mathcal{B}_n, i_n}|$$

Now for each type space we have an inequality

$$|S^m_{\mathcal{B}_j, i_j}| \leq N(m, i_j, \mathcal{L}_B) \leq N(m, |y|, \mathcal{L}_B) \leq N$$

(For singletons $|S^m_{\mathcal{B}_j, i_j}| = 1 \leq N$). Only non-zero indices contribute to the product and there are at most $|y|$ of those (by the equality $i_1 + i_2 + \ldots + i_n = |y|$). Thus we have

$$|\phi(A^{|x|}, B)| \leq N^{|y|}$$

as needed. $\square$

For subdivisions to work out properly, we will need to work with subsets closed under meets. We observe that the closure under meets doesn't add too many new elements.

Lemma 2.3.3. *Suppose $S \subseteq T$ is a finite subset of size $n \geq 1$ in a meet tree and S' is its closure under meets. Then $|S'| \leq 2n - 1$.*

Proof. We can partition S into connected components and prove the result separately for each component. Thus we may assume all elements of S lie in the same connected component. We prove the claim by induction on n. The base case $n = 1$ is clear. Suppose we have S of size k with closure of size at most $2k - 1$. Take a new point s, and look at its meets with all the elements of S. Pick the smallest one, s'. Then $S \cup \{s, s'\}$ is closed under meets. $\square$

Putting all of those results together we are able to compute the vc-density of formulas in meet trees.

Theorem 2.3.4. *Let $\mathcal{T}$ be an infinite (colored) meet tree and $\phi(x, y)$ a formula with $|x| = p$ and $|y| = q$. Then* $\mathrm{vc}(\phi) \leq q$.

Proof. Pick a finite subset of $S_0 \subset T^p$ of size n. Let $S_1 \subset T$ consist of the components of the elements of S_0. Let $S \subset T$ be the closure of S_1 under meets. Using Lemma 2.3.3 we have $|S| \leq 2|S_1| \leq 2p|S_0| = 2pn = O(n)$. We have $S_0 \subseteq S^p$, so $|\phi(S_0, T^q)| \leq |\phi(S^p, T^q)|$. Thus it is enough to show $|\phi(S^p, T^q)| = O(n^q)$.

Label $S = \{c_i\}_{i \in I}$ with $|I| \leq 2pn$. For every c_i we construct two partitions in the following way. We have that c_i is either minimal in S or it has a predecessor in S (greatest element less than c). If it is minimal, construct $(\mathcal{A}_{c_i}, \mathcal{B}_{c_i})$. If there is a predecessor p, construct $(\mathcal{A}^p_{c_i}, \mathcal{B}^p_{c_i})$. For the second subdivision let G be all the elements in S greater than c_i and construct $(\mathcal{A}^c_G, \mathcal{B}^c_G)$. So far we have constructed two subdivisions for every $i \in I$. Additionally construct $(\mathcal{A}_S, \mathcal{B}_S)$. We end up with a finite collection of proper subdivisons $(\mathcal{A}_j, \mathcal{B}_j)_{j \in J}$ with $|J| = 2|I| + 1$. Before we proceed, we note the following two lemmas describing our partitions.

Lemma 2.3.5. *For all $j \in J$ we have $S \subseteq A_j$. Thus $S \subseteq \bigcap_{j \in J} A_j$ and $S^p \subseteq \bigcap_{j \in J} (A_j)^p$.*

Proof. Check this for each possible choice of partition. Cases for partitions of the type $\mathcal{A}_S, \mathcal{A}^c_G, \mathcal{A}_c$ are easy. Suppose we have a partition $(\mathcal{A}, \mathcal{B}) = (\mathcal{A}^{c_1}_{c_2}, \mathcal{B}^{c_1}_{c_2})$. We need to show that $B \cap S = \emptyset$. By construction we have $c_1, c_2 \notin B$. Suppose we have some other $c \in S$ with $c \in B$. We have $E_{c_1}(c_2, c)$ i.e. there is some b such that $(b > c_1)$, $(b \leq c_2)$ and $(b \leq c)$. Consider the meet $(c \wedge c_2)$. We have $(c \wedge c_2) \geq b > c_1$. Also as $\neg(c \geq c_2)$ we have $(c \wedge c_2) < c_2$. To summarize: $c_2 > (c \wedge c_2) > c_1$. But this contradicts our construction as S is closed under meets, so $(c \wedge c_2) \in S$ and c_1 is supposed to be a predecessor of c_2 in S. □

Lemma 2.3.6. *$\{B_j\}_{j \in J}$ is a disjoint partition of $T - S$ i.e. $T = \bigsqcup_{j \in J} B_j \sqcup S$*

Proof. This more or less follows from the choice of partitions. Pick any $b \in S - T$. Take all the elements in S greater than b and take the minimal one a. Take all the elements in S less than b and take the maximal one c (possible as S is closed under meets). Also take all the elements in S incomparable to b and denote them G. If both a and c exist we have

$b \in \mathcal{B}_c^a$. If only the upper bound exists we have $b \in \mathcal{B}_G^a$. If only the lower bound exists we have $b \in \mathcal{B}_c$. If neither exists we have $b \in \mathcal{B}_G$. $\square$

Note 2.3.7. Those two lemmas imply $S = \bigcap_{j \in J} A_j$.

Note 2.3.8. For one-dimensional case $q = 1$ we don't need to do any more work. We have partitioned the parameter space into $|J| = O(n)$ many pieces and over each piece the number of definable sets is uniformly bounded. By Corollary 2.1.8 we have that $|\phi((A_j)^p, B_j)| \leq N$ for any $j \in J$ (letting $N = N(n_\phi, q, \mathcal{L} \cup \{S\})$ where n_ϕ is the complexity of ϕ and S is a unary predicate). Compute

$$\begin{aligned}
|\phi(S^p, T)| = \left| \bigcup_{j \in J} \phi(S^p, B_j) \cup \phi(S^p, S) \right| &\leq \\
\leq \sum_{j \in J} |\phi(S^p, B_j)| + |\phi(S^p, S)| &\leq \\
\leq \sum_{j \in J} |\phi((A_j)^p, B_j)| + |S| &\leq \\
\leq \sum_{j \in J} N + |I| &\leq \\
\leq (4pn + 1)N + 2pn = (4pN + 2p)n + N = O(n)&
\end{aligned}$$

Basic idea for the general case $q \geq 1$ is that we have q parameters and $|J| = O(n)$ many partitions to pick each parameter from giving us $|J|^q = O(n^q)$ choices for the parameter configuration, each giving a uniformly constant number of definable subsets of S. (If every parameter is picked from a fixed partition, Lemma 2.3.1 provides a uniform bound). This yields $\mathrm{vc}(\phi) \leq q$ as needed. The rest of the proof is stating this idea formally.

First, we extend our collection of subdivisions $(\mathcal{A}_j, \mathcal{B}_j)_{j \in J}$ by the following singleton sets. For each $c_i \in S$ let $B_i = \{c_i\}$ and $A_i = T$ and add $(\mathcal{A}_i, \mathcal{B}_i)$ to our collection with $\mathcal{L}_B$ the language of B_i interpreted arbitrarily. We end up with a new collection $(\mathcal{A}_k, \mathcal{B}_k)_{k \in K}$ indexed by some K with $|K| = |J| + |I|$ (we added $|S|$ new pairs). Now $\{B_k\}_{k \in K}$ partitions T, so $T = \bigsqcup_{k \in K} B_k$ and $S = \bigcap_{j \in J} A_j = \bigcap_{k \in K} A_k$. For $(k_1, k_2, \ldots k_q) = \vec{k} \in K^q$ denote

$$B_{\vec{k}} = B_{k_1} \times B_{k_2} \times \ldots \times B_{k_q}$$

Then we have the following identity

$$T^q = (\bigsqcup_{k \in K} B_k)^q = \bigsqcup_{\vec{k} \in K^q} B_{\vec{k}}$$

Thus we have that $\{B_{\vec{k}}\}_{\vec{k} \in K^q}$ partition T^q. Compute

$$\begin{aligned} |\phi(S^p, T^q)| &= \left| \bigcup_{\vec{k} \in K^q} \phi(S^p, B_{\vec{k}}) \right| \leq \\ &\leq \sum_{\vec{k} \in K^q} |\phi(S^p, B_{\vec{k}})| \end{aligned}$$

We can bound $|\phi(S^p, B_{\vec{k}})|$ uniformly using Lemma 2.3.1. $(\mathcal{A}_k, \mathcal{B}_k)_{k \in K}$ satisfies the requirements of the lemma and $B_{\vec{k}}$ looks like B in the lemma after possibly permuting some variables in ϕ. Applying the lemma we get

$$|\phi(S^p, B_{\vec{k}})| \leq N^q$$

with N only depending on q and complexity of ϕ. We complete our computation

$$\begin{aligned} |\phi(S^p, T^q)| &\leq \sum_{\vec{k} \in K^q} |\phi(S^p, B_{\vec{k}})| \leq \\ &\leq \sum_{\vec{k} \in K^q} N^q \leq \\ &\leq |K^q| N^q \leq \\ &\leq (|J| + |I|)^q N^q \leq \\ &\leq (4pn + 1 + 2pn)^q N^q = N^q (6p + 1/n)^q n^q = O(n^q) \end{aligned}$$

□

Corollary 2.3.9. *In the theory of infinite (colored) meet trees we have $vc(n) = n$ for all n.*

We get the general result for the trees that aren't necessarily meet trees via an easy application of interpretability.

Corollary 2.3.10. *In the theory of infinite (colored) trees we have $vc(n) = n$ for all n.*

Proof. Let $\mathcal{T}'$ be a tree. We can embed it in a larger tree $\mathcal{T}$ that is closed under meets. Expand $\mathcal{T}$ by an extra color and interpret it by coloring the subset $\mathcal{T}'$. Thus we can interpret $\mathcal{T}'$ in T. By Corollary 3.17 in [ADH16] we get that $\mathrm{vc}^{\mathcal{T}'}(n) \leq \mathrm{vc}^{T}(1 \cdot n) = n$ thus $\mathrm{vc}^{\mathcal{T}'}(n) = n$ as well. ☐

This settles the question of *vc*-function for trees. Lacking a quantifier elimination result, a lot is still not known. One can try to adapt these techniques to compute the vc-density of a fixed formula, and see if it can take non-integer values. It is also not known whether trees have VC 1 property (see [ADH16] 5.2 for the definition). Our techniques can be used to show that VC 2 property holds but this doesn't give the optimal vc-function.

One can also try to apply similar techniques to more general classes of partially ordered sets. For example, vc-density values are not known for lattices. Similarly, dropping the order, one can look at nicely behaved families of graphs, such as planar graphs or flat graphs. Those are known to be dp-minimal, so one would expect a simple vc-function. It is this author's hope that the techniques developed in this paper can be adapted to yield fruitful results for a more general class of structures.

CHAPTER 3

Shelah-Spencer Graphs

We investigate vc-density in Shelah-Spencer graphs. We provide an upper bound on a formula-by-formula basis and show that there isn't a uniform lower bound, forcing the vc-function to be infinite. In addition we show that Shelah-Spencer graphs do not have a finite dp-rank, in particular they are not dp-minimal.

VC-density was studied in [ADH16] by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko as a natural notion of dimension for NIP theories. In a complete NIP theory T we can define the vc-function

$$\mathrm{vc}^T = \mathrm{vc} : \mathbb{N} \longrightarrow \mathbb{R} \cup \{\infty\}$$

where $\mathrm{vc}(n)$ measures the worst-case complexity of families of definable sets in an n-fold Cartesian power of the underlying set of a model of T (see 1.3.13 below for a precise definition of vc^T). We always have $\mathrm{vc}(n) \geq n$ for each n, and the simplest possible behavior is $\mathrm{vc}(n) = n$ for all n. Theories with the property that $\mathrm{vc}(1) = 1$ are known to be dp-minimal, i.e., having the smallest possible dp-rank (see Definition 1.3.15). It is not known whether there can be a dp-minimal theory which doesn't satisfy $\mathrm{vc}(n) = n$ (see [ADH16], diagram in section 5.3).

In this paper, we investigate vc-density of definable sets in Shelah-Spencer graphs. First major model-theoretic breakthrough for these structures was made in [SS88]. In our description of Shelah-Spencer graphs we follow closely the treatment in [Las07]. A Shelah-Spencer graph is a limit of random structures $G(n, n^{-\alpha})$ for an irrational $\alpha \in (0, 1)$. Here $G(n, n^{-\alpha})$ is a random graph on n vertices with edge probability $n^{-\alpha}$.

Our first result is that in Shelah-Spencer graphs

$$\mathrm{vc}(n) = \infty \text{ for each } n.$$

We also show that Shelah-Spencer graphs don't have a finite dp-rank, which in particular implies that they are not dp-minimal. Our second result provides an upper bound on the vc-density of a given formula $\phi(x, y)$:

$$\mathrm{vc}(\phi) \leq D(\phi)$$

where $D(\phi)$ is an expression involving $|y|$ and number of vertices and edges defined by ϕ.

Section 1 introduces basic facts about VC-dimension and vc-density. More can be found in [ADH16]. Section 2 summarizes notation and basic facts concerning Shelah-Spencer graphs. We direct the reader to [Las07] for a more in-depth treatment. In section 3 we introduce key lemmas that will be useful in our proofs. Section 4 computes a lower bound for vc-density to demonstrate that $\mathrm{vc}(n) = \infty$. Here we also do computations involving dp-rank. Section 5 computes an upper bound for vc-density on a formula-by-formula basis.

3.1 Graph Combinatorics

Throughout this paper A, B, C, M will denote finite graphs, and $\mathbb{D}$ will be used to denote potentially infinite graphs. For a graph $\mathcal{A}$ the set of its vertices is denoted by $v(\mathcal{A})$, and the set of its edges by $e(\mathcal{A})$. Number of vertices of $\mathcal{A}$ will be denoted as $|\mathcal{A}|$. Subgraph always means induced subgraph and $A \subseteq B$ means that A is a subgraph of B. For two subgraphs $\mathcal{A}, \mathcal{B}$ of a larger graph, the union $\mathcal{A} \cup \mathcal{B}$ denotes the graph induced by $v(\mathcal{A}) \cup v(\mathcal{B})$. Similarly, $A - B$ means a subgraph of A induced by the vertices of $v(A) - v(B)$. For $A \subseteq B \subseteq D$ and $A \subseteq C \subseteq D$, graphs B, C are said to be disjoint over A if $v(B) - v(A)$ is disjoint from $v(C) - v(A)$ and there are no edges from $v(B) - v(A)$ to $v(C) - v(A)$ in D.

For the remainder of the paper fix $\alpha \in (0, 1)$, irrational.

Definition 3.1.1.

- For a graph $\mathcal{A}$ let $\dim(\mathcal{A}) = |\mathcal{A}| - \alpha|e(\mathcal{A})|$.

- For $\mathcal{A}, \mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B}$ define $\dim(\mathcal{B}/\mathcal{A}) = \dim(\mathcal{B}) - \dim(\mathcal{A})$.
- We say that $\mathcal{A} \leq \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and $\dim(\mathcal{A}'/A) > 0$ for all $\mathcal{A} \subsetneq \mathcal{A}' \subseteq \mathcal{B}$.
- Define $\mathcal{A}$ to be positive if for all $\mathcal{A}' \subseteq \mathcal{A}$ we have $\dim(\mathcal{A}') \geq 0$.
- We work in theory S_α in the language of graphs axiomatized by:
  - Every finite substructure is positive.
  - Given a model $\mathbb{G}$ and graphs $\mathcal{A} \leq \mathcal{B}$, every embedding $f : \mathcal{A} \longrightarrow \mathbb{G}$ extends to an embedding $g : \mathcal{B} \longrightarrow \mathbb{G}$.

  (Here an embedding maps edges to edges and nonedges to nonedges.) This theory is complete and stable (see 5.7 and 7.1 in [Las07]). From now on fix an ambient model $\mathbb{G} \models S_\alpha$. This will be the only infinite graph we work with.
- For $\mathcal{A}, \mathcal{B}$ positive, $(\mathcal{A}, \mathcal{B})$ is called a minimal pair if $\mathcal{A} \subseteq \mathcal{B}$, $\dim(\mathcal{B}/\mathcal{A}) < 0$ but $\dim(\mathcal{A}'/\mathcal{A}) \geq 0$ for all proper $\mathcal{A} \subseteq \mathcal{A}' \subsetneq \mathcal{B}$. We call B a minimal extension of A. The dimension of a minimal pair is defined as $|\dim(B/A)|$.
- A sequence $\langle M_i \rangle_{0 \leq i \leq n}$ is called a minimal chain if (M_i, M_{i+1}) is a minimal pair for all $0 \leq i < n$.
- For a graph $\mathcal{A}$ with the tuple of vertices x let $\mathrm{diag}_{\mathcal{A}}(x)$ be the atomic diagram of $\mathcal{A}$, i.e. the first-order formula recording whether there is an edge between every pair of vertices.
- Given $\mathcal{A} \subseteq \mathcal{B}$ let

  $$\phi_{\mathcal{A},\mathcal{B}}(x) = \mathrm{diag}_{\mathcal{A}}(x) \land \exists z \ \mathrm{diag}_{\mathcal{B}}(x, z).$$

  Any graph isomorphic to $\mathcal{B}$ is called a witness of $\phi_{A,B}$.
- A formula $\phi_{A,B}$ is called a basic formula if there is a minimal chain $\langle M_i \rangle_{0 \leq i \leq n}$ such that $A = M_0$ and $B = M_n$.

Theorem 3.1.2 (Quantifier elimination, 5.6 in [Las07])**.** *In theory S_α every formula is equivalent to a boolean combination of basic formulas.*

Definition 3.1.3. A graph $S \subseteq \mathbb{D}$ is called N-strong if for any $S \subseteq T \subseteq D$ with $|T| - |S| \leq N$ we have $S \leq T$.

3.2 Basic Definitions and Lemmas

Definition 3.2.1. Suppose $\phi(x, y)$ is a basic formula. Define $\mathcal{X}$ to be the graph on vertices x with edges defined by ϕ. Similarly define $\mathcal{Y}$. Note that $\mathcal{X}$, $\mathcal{Y}$ are positive. Additionally, let $\mathcal{Y}'$ be a subgraph of $\mathcal{Y}$ induced by vertices of $\mathcal{Y}$ that are connected to $W - (X \cup Y)$, where W is a witness of ϕ.

Definition 3.2.2. Suppose A, B are subgraphs of $\mathcal{D}$ such that $v(A), v(B)$ are disjoint. Then define $\mathscr{E}(A, B)$ to be the number of edges between the vertices in $v(A)$ and the vertices in $v(B)$.

We will require the following lemmas from [Las07]:

Lemma 3.2.3. *[See 2.3 in [Las07]] Let $A, B \subseteq \mathbb{D}$. Then*

$$\dim(A \cup B/A) \leq \dim(\mathcal{B}/A \cap B).$$

Moreover,

$$\dim(A \cup B/A) = \dim(\mathcal{B}/A \cap B) - \alpha E,$$

where E is the number of edges connecting the vertices of $B - A$ to the vertices of $A - B$.

Lemma 3.2.4. *[See 4.1 in [Las07]] Suppose A is a positive graph, with at least $1/\alpha + 2$ vertices. Then for any $\epsilon > 0$ there exists a graph B such that (A, B) is a minimal pair with dimension $\leq \epsilon$. Moreover, every vertex in A is connected to a vertex in $B - A$.*

Lemma 3.2.5. *[See 4.4 in [Las07]] Suppose A is a positive graph, and $\mathcal{G}$ a model of S_α. Then for any integer S there exists an embedding $f\colon A \longrightarrow \mathcal{G}$ such that $f(A)$ is S-strong in $\mathcal{G}$.*

Lemma 3.2.6. *[See 3.8 in [Las07]] For all $S > 0$ there exists $M = M(S, \alpha) \in \mathbb{N}$ with the following property. Suppose $A \subseteq \mathcal{G}$ where $\mathcal{G}$ is a model of S_α. Then there exists B with $A \subseteq B \subseteq \mathcal{G}$ such that B is S-strong in $\mathbb{G}$ and $|B| \leq M|A|$.*

We conclude this section by stating a couple of technical lemmas that will be useful in our proofs later.

Lemma 3.2.7. *Work in an ambient graph $\mathbb{D}$. Suppose we have a set B and a minimal pair (A, M) with $A \subseteq B$ and $\dim(M/A) = -\epsilon$. Then either $M \subseteq B$ or $\dim(M \cup B/B) < -\epsilon$.*

Proof. By Lemma 3.2.3

$$\dim(M \cup B/B) \leq \dim(M/M \cap B),$$

and as $A \subseteq M \cap B \subseteq M$

$$\dim(M/A) = \dim(M/M \cap B) + \dim(M \cap B/A).$$

In addition we are given $\dim(M/A) = -\epsilon$. If $M \nsubseteq B$ then $A \subseteq M \cap B \subsetneq M$ and by minimality $\dim(M \cap B/A) > 0$. Combining the inequalities above we obtain the desired result:

$$\dim(M \cup B/B) \leq \dim(M/M \cap B) = \dim(M/A) - \dim(M \cap B/A) < -\epsilon.$$

□

Lemma 3.2.8. *Work in an ambient graph $\mathbb{D}$. Suppose we have a set B and a minimal chain $\langle M_i \rangle_{0 \leq i \leq n}$ with dimensions*

$$\dim(M_{i+1}/M_i) = -\epsilon_i$$

and $M_0 \subseteq B$. Let $\epsilon = \min_{0 \leq i \leq n} \epsilon_i$. Then either $M_n \subseteq B$ or $\dim((M_n \cup B)/B) < -\epsilon$.

Proof. Let $\bar{M}_i = M_i \cup B$. Then:

$$\dim(\bar{M}_n/B) = \dim(\bar{M}_n/\bar{M}_{n-1}) + \ldots + \dim(\bar{M}_2/\bar{M}_1) + \dim(\bar{M}_1/B).$$

Either $M_n \subseteq B$ or at least one of the summands above is nonzero. Apply previous lemma. □

Lemma 3.2.9. *Suppose we have a minimal pair (A, M) with dimension ϵ. Suppose we have some $B \subseteq M$. Then* $\dim B/(A \cap B) \geq -\epsilon$. *Moreover if $B \cup A \neq M$ then* $\dim B/(A \cap B) \geq 0$.

Proof. We have $\dim(B \cup A/A) \leq \dim B/(A \cap B)$ by Lemma 3.2.3. As $A \subseteq B \cup A \subseteq M$ we have $\dim(B \cup A/A) \geq -\epsilon$ by minimality. Moreover, minimality implies that it is positive if $B \cup A \neq M$. □

Lemma 3.2.10. *Suppose we have a minimal chain $\langle M_i \rangle_{0 \leq i \leq n}$ with dimensions*

$$\dim(M_{i+1}/M_i) = -\epsilon_i.$$

Let ϵ be the sum of all ϵ_i. Suppose we have a graph B with $B \subseteq M_n$. Then $\dim B/(M_0 \cap B) \geq -\epsilon$.

Proof. Let $B_i = B \cap M_i$. We have $\dim B_{i+1}/B_i \geq \dim M_{i+1}/M_i$ by the previous lemma. Thus

$$\dim B/(M_0 \cap B) = \dim B_n/B_0 = \sum \dim B_{i+1}/B_i \geq -\epsilon.$$

□

3.3 Lower bound

In this section we restrict our attention to the following family of basic formulas $\phi(x, y)$:

- All formulas have $\mathcal{Y}' = \mathcal{Y}$ (see Definition 3.2.1).
- All formulas define no edges between X and Y.
- Minimal chain of $\phi(x, y)$ consists of one step, that is we only have one minimal extension as opposed to a chain of minimal extensions.
- The dimension of that minimal extension is smaller than α.

We obtain a lower bound for the formulas that are boolean combinations of basic formulas written in the disjunctive-conjunctive form. First, define $\epsilon_L(\phi)$.

Definition 3.3.1. For a basic formula $\phi = \phi_{\langle M_i \rangle_{0 \le i \le n}}(x, y)$ let

- $\epsilon_i(\phi) = -\dim(M_i/M_{i-1})$.
- $\epsilon_L(\phi) = \sum_1^n \epsilon_i(\phi)$.

Definition 3.3.2 (Negation)**.** If ϕ is a basic formula, then define

$$\epsilon_L(\neg\phi) = \epsilon_L(\phi).$$

Definition 3.3.3 (Conjunction)**.** Take a collection of formulas $\phi_i(x, y)$ where each ϕ_i is a positive or a negative basic formula. If both positive and negative formulas are present then $\epsilon_L(\phi) = \infty$. We don't have a lower bound for that case. If different formulas define $\mathcal{X}$ or $\mathcal{Y}$ differently then $\epsilon_L(\phi) = \infty$. In the case of conflicting definitions the formula would have no realizations. Otherwise let

$$\epsilon_L\left(\bigwedge \phi_i\right) = \sum \epsilon_L(\phi_i).$$

Definition 3.3.4 (Disjunction)**.** Take a collection of formulas ψ_i where each instance is a conjunction as above all agreing on $\mathcal{X}$ and $\mathcal{Y}$. Then

$$\epsilon_L\left(\bigvee \psi_i\right) = \min \epsilon_L(\psi_i).$$

Theorem 3.3.5. *For a formula ψ as above we have*

$$\operatorname{vc}\psi \ge \left\lfloor \frac{Y(\psi)}{\epsilon_L(\psi)} \right\rfloor,$$

where $Y(\psi)$ is $\dim(Y)$ (as all basic componenets agree on $\mathcal{Y}$).

Proof. First, work with a formula that is a conjunction of positive basic formulas $\psi = \bigwedge_{i \in I} \phi_i$. Then as we have defined above

$$\epsilon_L(\psi) = \sum_{i \in I} \epsilon_L(\phi_i).$$

If W_i is a witness of ϕ_i, let $S_i = |W_i|$. Let n_1 be the largest natural number such that

$$n_1 \epsilon_L(\psi) < Y(\psi).$$

Let ϵ' be the smallest value among $\epsilon_L(\phi_i)$. Suppose it corresponds to the formula ϕ'. Let n_2 be the largest natural number such that

$$n_1\epsilon_L(\psi) + n_2\epsilon' < Y(\psi).$$

Fix some $N > n_1 + n_2$. Let

$$J = \{0 \leq j < N\} \subseteq \mathbb{N}.$$

Let a_j be a graph isomorphic to $\mathcal{X}$ for each $j \in J$, pairwise disjoint. Let $A = \bigcup_{1\leq j\leq N} a_j$. Let

$$S = |Y| + (n_1 + n_2 + 1)\sum_{i\in I} S_i.$$

By Lemma 3.2.5 the graph A can be embedded into $\mathbb{G}$ as an S-strong graph. Abusing notation, we identify A with this embedding. Thus we have $A \subseteq \mathbb{G}$, S-strong.

Let J_1, J_2 be disjoint subsets of J, of sizes n_1, n_2 respectively. Let b be a graph isomorphic to $\mathcal{Y}$. For each $i \in I, j \in J_1$ let W_{ij} be a witness of $\phi_i(a_j, b)$. (Note that then $(a_j \cup b, W_{ij})$ is a minimal pair.) For each $j \in J_1$ let W_j be a union of $\{W_{ij}\}_{i\in I}$ disjoint over $a_j \cup b$. For each $j \in J_2$ let W_j be a witness of $\phi'(a_j, b)$. Let W' be a union of $\{W_j\}_{j\in J_1\cup J_2}$ disjoint over b. Let W be a union of W' and A disjoint over $\{a_j\}_{j\in J_1\cup J_2}$.

Claim 3.3.6. *We have $A \leq W$.*

Proof. Consider some $A \subsetneq B \subseteq W$. We need to show $\dim(B/A) > 0$. Let $\bar{A} = A \cup b$. We have

$$\dim(B/A) = \dim(B/B\cap\bar{A}) + \dim(B\cap\bar{A}/A).$$

Let $B_{ij} = B \cap W_{ij}$. Let $B_j = B \cap W_j$. To unify indices, relabel all the graphs above as $\{B_k\}_{k\in K}$ for some index set K. By the construction of W we have

$$\dim(B/B\cap\bar{A}) = \sum_{k\in K} \dim(B_k/B_k\cap\bar{A}).$$

Fix k. We have $B_k \subseteq W_k$, where W_k is a minimal extension of $M_0^k = a \cup b$ for some $a \in A$. Let ϵ_k be the dimension of this minimal extension. We have $\dim(B_k/B_k \cap \bar{A}) = \dim(B_k/a \cup (B \cap b))$.

Case 1: $B \cap b = b$. Then $M_0^k \subseteq B_k \subseteq W_k$ and

$$\dim(B_k/a \cup (B \cap b)) = \dim(B_k/M_0^k).$$

By minimality of (M_0^k, B_k) we have $\dim(B_k/M_0^k) \geq -\epsilon_k$. Thus

$$\dim(B/B \cap \bar{A}) \geq -\sum_{k \in K} \epsilon_k = -\left(n_1 \epsilon_L(\psi) + n_2 \epsilon'\right).$$

In addition

$$\dim(B \cap \bar{A}/A) = \dim(b) = Y(\psi).$$

Combining the two, we get

$$\dim(B/A) \geq Y(\psi) - \left(n_1 \epsilon_L(\psi) + n_2 \epsilon'\right),$$

which is positive by the construction of n_1, n_2 as needed.

Case 2: $B \cap b \subsetneq b$.

Claim 3.3.7. *We have* $\dim(B_k/B_k \cap \bar{A}) > 0$*.*

Proof. Recall that $\dim(B_k/B_k \cap \bar{A}) = \dim(B_k/a \cup (B \cap b))$. First, suppose that $B_k \cup M_0^k \neq W_k$. Then by Lemma 3.2.9 we get the required inequality. Thus we may assume that $B_k \cup M_0^k = W_k$. By Lemma 3.2.3 we have

$$\dim(B_k \cup M_0^k/M_0^k) = \dim(B_k/B_k \cap M_0^k) - \alpha E,$$

where E is the number of edges connecting the vertices of $B_k - M_0^k = B_k \cup M_0^k - M_0^k$ to the vertices of $M_0^k - B_k = M_0^k - B_k \cap M_0^k$. Noting that $B_k \cup M_0^k = W_k$, $\dim W_k/M_0^k = -\epsilon_k$, and $B_k \cap M_0^k = a \cup (B \cap b)$ we may rewrite the equality above as

$$\dim(B_k/a \cup (B \cap b)) = \alpha E - \epsilon,$$

and E is the number of edges connecting the vertices of $W_k - M_0^k$ to the vertices of $M_0^k - a \cup (B \cap b)$. As $\mathcal{Y} = \mathcal{Y}'$ and $B \cap b \subsetneq b$ we must have $E \geq 1$. But then as $\alpha > \epsilon$ we have $\dim(B_k / a \cup (B \cap b)) > 0$ as needed. □

Now, recall that

$$\dim(B/A) = \dim(B \cap \bar{A}/A) + \sum_{k \in K} \dim(B_k / B_k \cap \bar{A}).$$

By the claim above each of $\dim(B_k / B_k \cap \bar{A}) > 0$, thus

$$\dim(B/A) > \dim(B \cap \bar{A}/A).$$

In addition

$$\dim(B \cap \bar{A}/A) = \dim(B \cap b) \geq 0,$$

as b is postive. Thus $\dim(B/A) > 0$ as needed. □

As $A \leq W$ and $A \subseteq \mathbb{G}$, we can embed W into $\mathbb{G}$ over A. Abusing notation again, we identify W with its embedding $A \leq W \subseteq \mathbb{G}$. In particular, now we have $b \in \mathbb{G}$. Also note that

$$\dim(W/A) = Y(\psi) - (n_1 \epsilon_L(\psi) + n_2 \epsilon'),$$
$$|W| - |A| \leq |b| + (n_1 + n_2) \sum_{i \in I} S_i.$$

Lemma 3.3.8. *We have*

$$\{a_j\}_{j \in J_1} \subseteq \psi(A, b) \subseteq \{a_j\}_{j \in J_1 \cup J_2}.$$

Proof. First inclusion $\{a_j\}_{j \in J_1} \subseteq \psi(A, b)$ is immediate from the construction of W, as W_{ij} witnesses that $\phi_i(a_j, b)$ holds. For the second inclusion, suppose that there is $a \in A - \{a_j\}_{j \in J_1 \cup J_2}$ such that $\psi(a, b)$ holds. Let $W' \subseteq \mathbb{G}$ be a witness of $\phi_1(a, b)$. First, note that the case $W' \subseteq W$ is impossible as there are no edges between a and $W - a$, but there are edges between a and $W' - a$. Thus assume $W' \not\subseteq W$. As $(a \cup b, W')$ is minimal, by Lemma 3.2.7 we have $\dim(W' \cup W/W) < -\epsilon_1$. Therefore

$$\dim(W' \cup W/A) = \dim(W' \cup W/W) + \dim(W/A) < Y(\psi) - (n_1 \epsilon_L(\psi) + n_2 \epsilon') - \epsilon_1,$$

which is negative by the construction of n_1, n_2. Thus $A \not\leq W \cup W'$, as then it would have a positive dimension. Additionally,

$$|W' \cup W| - |A| \leq |W' - W| + |W| - |A| \leq S_1 + |b| + (n_1 + n_2) \sum_{i \in I} S_i \leq S,$$

but then this contradicts that A is S-strong, as then we would have $A \leq W \cup W'$. $\square$

In the construction of W we have chosen indices J_1, J_2 arbitrarily. In particular, suppose we let J_2 to be the last n_2 indices of J and J_1 an arbitrary n_1-element subset of the first $N - n_2$ elements of J. Each of those choices would then yield a different trace $\psi(A, b)$ by the lemma above. Thus $\psi(A, M^{|y|}) \geq \binom{N-n_2}{n_1}$ and therefore $\mathrm{vc}(\psi) \geq n_1$. By the definition of n_1 we have $n_1 = \left\lfloor \frac{Y(\psi)}{\epsilon_L(\psi)} \right\rfloor$, so this proves the theorem for ψ.

Now consider a formula which is a conjunction consisting of negative basic formulas $\psi = \bigwedge_{i \in I} \neg\phi_i$. Let $\bar{\psi} = \bigwedge_{i \in I} \phi_i$. Do the construction above for $\bar{\psi}$ and suppose its trace is $X \subseteq A$ for some b. Then over b the same construction gives trace $(A - X)$ for ψ. Thus we get as many traces as above, and the same bound.

Finally consider a formula which is a disjunction of formulas considered above $\theta = \bigvee_{k \in K} \psi_k$. Choose the one with the smallest ϵ_L, say ψ_k, and repeat the construction above for ψ_k. Any trace we obtain is automatically a trace for θ, and thus we get as many traces as above, and the same bound. $\square$

Corollary 3.3.9. *VC-function is infinite in Shelah-Spencer random graphs:*

$$\mathrm{vc}(n) = \infty.$$

Proof. Let A be a graph consisting of $1/\alpha + 2 + n$ disconnected vertices. Fix $\epsilon > 0$. By Lemma 3.2.4, there exists B such that (A, B) is minimal with dimension $\leq \epsilon$. Consider a basic formula $\psi_{A,B}(x, y)$ where $|x| = 1/\alpha + 2$ and $|y| = n$. Then by the theorem above $\mathrm{vc}(n) \geq \mathrm{vc}(\psi_{A,B}) \geq \frac{n}{\epsilon}$. As ϵ was arbitrary, this number can be made arbitrarily large, giving $vc(n) = \infty$ as needed. $\square$

Corollary 3.3.10. *Shelah-Spencer random graphs don't have finite dp-rank. In particular they are not dp-minimal.*

Proof. We would like to modify the proof of Theorem 3.3.5 such that A is indiscernible. Note that in the proof we can construct sets $A = \{a_j\}_{j \in J}$ of arbitrary length. Moreover for every finite $J' \subseteq J$, the set $A = \{a_j\}_{j \in J'}$ is still S-strong. Thus we can find an infinite set $A = \{a_j\}_{j \in \mathbb{N}}$ indiscernible and S-strong. Repeating the construction of the corollary above, we can obtain a formula with an arbitrarily large vc-density over the indiscernible sequence A. $\square$

3.4 Upper bound

Consider a basic formula $\phi(x, y)$ with a minimal chain $\langle M_i \rangle_{0 \leq i \leq n_\phi}$ with dimensions $\dim(M_{i+1}/M_i) = -\epsilon_i$. Define

$$\begin{aligned} \epsilon(\phi) &= \min \{\epsilon_i\}_{0 \leq i \leq n_\phi} \\ K(\phi) &= |M_{n_\phi}|. \end{aligned}$$

Now consider a finite collection of basic formulas

$$\Phi = \Phi(\vec{x}, \vec{y}) = \{\phi_i(\vec{x}, \vec{y})\}_{i \in I}.$$

Define

$$\begin{aligned} \epsilon(\Phi) &= \min \{\epsilon(\phi_i)\}_{i \in I} \cup \{\alpha\}, \\ K(\Phi) &= \max \{K(\phi_i)\}_{i \in I}, \\ D_1(\Phi) &= \frac{K(\Phi)}{\epsilon(\Phi)}, \\ D(\Phi) &= |y| D_1(\Phi). \end{aligned}$$

We claim that

Theorem 3.4.1. *If ϕ is a boolean combination of formulas from Φ, then* $\mathrm{vc}(\phi) \leq D(\Phi)$.

Let

$$S = \left\lceil \left(\frac{|y|}{\epsilon(\phi)} + 1 \right) K(\phi) \right\rceil.$$

Suppose we have a finite parameter set $A_0 \subseteq \mathbb{G}^{|x|}$ with $|A_0| = N_0$. We would like to bound $\phi(A_0, \mathbb{G}^{|y|})$. Let $A_1 \subseteq \mathbb{G}$ consist of the components of the elements of A_0. Then $|A_1| \leq |x|N_0$. Using Lemma 3.2.6 let A be a graph $A_0 \subseteq A \subseteq \mathbb{G}$, S-strong in $\mathbb{G}$. Let $N = |A|$. We have $N \leq |x|N_0M$ (where M is the constant from the Lemma 3.2.6). As $A_0 \subseteq A^{|x|}$ we have

$$\left|\phi(A_0, \mathbb{G}^{|y|})\right| \leq \left|\phi(A^{|x|}, \mathbb{G}^{|y|})\right|.$$

Therefore it suffices to bound $\left|\phi(A^{|x|}, \mathbb{G}^{|y|})\right|$.

Definition 3.4.2. For $A \subseteq \mathbb{G}^{|x|}, B \subseteq \mathbb{G}^{|y|}$ define

$$\Phi(A, B) = \{(a, i) \in A \times I \mid \mathbb{G} \models \phi_i(a, b)\} \subseteq A \times I$$

Lemma 3.4.3. *For $A \subseteq \mathbb{G}^{|x|}, B \subseteq \mathbb{G}^{|y|}$ if ϕ is a boolean combination of formulas from Φ then*

$$|\phi(A, B)| \leq |\Phi(A, B)|$$

Proof. Clear, as for all $a \in A, b \in B$ the set $\Phi(a, b)$ determines the truth value of $\phi(a, b)$. $\square$

Thus it suffices to bound $\left|\Phi(A^{|x|}, \mathbb{G}^{|y|})\right|$.

Definition 3.4.4.

- For all $i \in I, a \in A^{|x|}, b \in \mathbb{G}^{|y|}$ if $\phi_i(a, b)$ holds fix $W^i_{a,b} \subseteq \mathbb{G}$, a witness of this formula.
- For $b \in \mathbb{G}^{|y|}$ let

$$W_b = \bigcup \left\{W^i_{a,b} \mid a \in A^{|x|}, i \in I, \mathbb{G} \models \phi_i(a, b)\right\}.$$

- For sets $C, B \subset \mathbb{G}$ define the <u>boundary</u> of C over B

$$\partial(C, B) = \{b \in B \mid \mathscr{E}(b, C - B) \neq \emptyset\}$$

  (see Definition 3.2.2 for $\mathscr{E}$).
- For $b \in \mathbb{G}^{|y|}$ let $\partial_b \subseteq A$ be the boundary $\partial(W_b, A)$.
- For $b \in \mathbb{G}^{|y|}$ let $\bar{W}_b = (W_b - A) \cup \partial_b$.

- For $b_1, b_2 \in \mathbb{G}^{|y|}$ we say that $b_1 \sim b_2$ if $\partial_{b_1} = \partial_{b_2}$, $b_1 \cap A = b_2 \cap A$, and there exists a graph isomorphism from $\bar{W}_{b_1} \cup b_1$ to $\bar{W}_{b_2} \cup b_2$ that fixes ∂_{b_1} and maps b_1 to b_2. One easily checks that this defines an equivalence relation.

- For $b \in \mathbb{G}^{|y|}$ define $\mathscr{I}_b$ to be the $\sim$-equivalence class of b.

Lemma 3.4.5. *For $b_1, b_2 \in \mathbb{G}^{|y|}$ if $b_1 \sim b_2$ then $\Phi(A^{|x|}, b_1) = \Phi(A^{|x|}, b_2)$.*

Proof. Fix a graph isomorphism $\bar{f}\colon \bar{W}_{b_1} \cup b_1 \longrightarrow \bar{W}_{b_2} \cup b_2$. Extend it to an isomorphism $f\colon W_{b_1} \cup A \longrightarrow W_{b_2} \cup A$ by being an identity map on the new vertices. Suppose $\mathbb{G} \models \phi_i(a, b_1)$ for some $a \in A^{|x|}$. Then $f(W^i_{a,b_1})$ is a witness for $\phi_i(a, b_2)$ (though not necessarily equal to W^i_{a,b_2}) and thus $\mathbb{G} \models \phi_i(a, b_2)$. As a was arbitrary, this proves the equality of the traces. □

Thus to bound the number of traces it is sufficient to bound the number of possibilities for $\mathscr{I}_b$.

Theorem 3.4.6. *Suppose we have $b \in \mathbb{G}^{|y|}$. Let $Y = |b - A|$. Then*

$$|\partial_b| \leq Y D_1(\phi)$$
$$|\bar{W}_b| \leq 3Y D_1(\phi)$$

From this theorem we get the desired result:

Corollary 3.4.7. *(Theorem 3.4.1) If ϕ is a boolean combination of formulas from Φ, then* $\mathrm{vc}(\phi) \leq D(\Phi)$.

Proof. We count possible equivalence classes of $\sim$. This essentially boils down to bounding possibilities for ∂_b, $b \cap A$, and number of isomorphism classes of W_b. Fix $b \in \mathbb{G}^{|y|}$ and let $Y = |b - A|$. Let $D = Y D_1(\Phi)$. By the first part of Theorem 3.4.6 there are $\binom{N}{D}$ choices for boundary ∂_b. By the second part of Theorem 3.4.6 there are at most $3D$ vertices in $\bar{W}_b$. Thus to determine the graph $\bar{W}_b$ we need to choose how many vertices it has and then decide where edges go. This amounts to at most $3D2^{(3D)^2}$ choices. Additionally there are $\binom{N}{|y|-Y}$

choices for $b \cap A$. In total this gives us at most

$$\binom{N}{D} \cdot \binom{N}{|y|-Y} \cdot 3D2^{(3D)^2} = O(N^{D+|y|-Y}) =$$
$$= O(N^{YD_1(\Phi)+|y|-Y}) = O(N^{|y|D_1(\Phi)}) = O(N^{D(\Phi)})$$

choices (second to last inequality uses $D_1(\Phi) \geq 1$). By Lemma 3.4.5 we have $\left|\Phi(A^{|x|}, \mathbb{G}^{|y|})\right| = O(N^{D(\Phi)})$. Recall that

$$\left|\phi(A_0, \mathbb{G}^{|y|})\right| \leq \left|\Phi(A^{|x|}, \mathbb{G}^{|y|})\right|.$$

Therefore we have

$$\left|\phi(A_0, \mathbb{G}^{|y|})\right| = O(N^{D(\Phi)}) = O((|x|N_0M)^{D(\Phi)}) =$$
$$= O((|x|M)^{D(\Phi)}\, N_0^{D(\Phi)}) = O(N_0^{D(\Phi)}).$$

As A_0 was arbitrary, this shows that $\mathrm{vc}(\phi) \leq D(\Phi)$ as needed. (Note that throughout this proof we are using the fact that quantities $D_1(\Phi), D(\Phi), M$ are completely determined by Φ, thus independent from A_0.) $\square$

Proof. (of Theorem 3.4.6)

The graph W_b is a union of witnesses of the from $W_{a,b}$ for some $a \in A^{|x|}, b \in \mathbb{G}^{|y|}$. Enumerate all of them as $\{W_j\}_{1 \leq j \leq J}$. Define $M_j = \bigcup_1^j W_{j'}$ for $1 \leq j \leq J$ and let $M_0 = b$. Let $\bar{A} = A \cup b$.

Definition 3.4.8. For $0 \leq j \leq J$ define:

- Let $v_j = 1$ if new vertices are added to M_j outside of $\bar{A}$, that is if $M_j - \bar{A} \neq M_{j-1} - \mathcal{B}$, and let it be 0 otherwise.
- Let $E_j = \partial(A - W_j, M_j - A)$.
- Let

$$m_j = \sum_{j'=0}^{j} (v_j + |E_j|).$$

(Here assume $M_{-1} = \emptyset$.)

Lemma 3.4.9. *For $0 \leq j \leq J$ we have*

$$|\partial(M_j, A)| \leq |E_0| + m_j K(\Phi)$$

Proof. Proceed by induction. The base case $j = 0$ is clear. For an induction step suppose that

$$|\partial(M_{j-1}, A)| \leq m_{j-1} K(\Phi)$$

holds. Let

$$\begin{aligned} \delta_1 &= \partial(M_j, A) - \partial(M_{j-1}, A) = \\ &= \{a \in A \mid \mathscr{E}(a, M_j - A) \neq \emptyset \text{ and } \mathscr{E}(a, M_{j-1} - A) = \emptyset\}\,. \end{aligned}$$

If $M_j - A = M_{j-1} - A$ then $\delta_1 = \emptyset$ and we are done as m_j is increasing. Suppose not. We have $|\delta_1| = |\delta_1 \cap W_j| + |\delta_1 - W_j|$, and

$$\delta_1 - W_j = \{a \in A - W_j \mid \mathscr{E}(a, M_j - A) \neq \emptyset \text{ and } \mathscr{E}(a, M_{j-1} - A) = \emptyset\}\,.$$

But then it's clear that $\delta_1 - W_j \subseteq E_j$ as

$$\begin{aligned} &W_j - M_{j-1} - A \subseteq M_j - A, \\ &(W_j - M_{j-1} - A) \cap (M_{j-1} - A) = \emptyset. \end{aligned}$$

As $b \in M_{j-1}$ and $M_j - A \neq M_{j-1} - A$, then $M_j - \bar{A} \neq M_{j-1} - \bar{A}$, and thus $v_j = 1$. Therefore we have

$$\begin{aligned} |\delta_1| &= |\delta_1 \cap W_j| + |\delta_1 - W_j| \leq |W_j| + |E_j| \leq \\ &\leq K(\Phi) + |E_j| \leq (v_j + |E_j|)K(\Phi) \leq (m_j - m_{j-1})K(\Phi), \end{aligned}$$

as needed. ∎

Lemma 3.4.10. *For $0 \leq j \leq J$ we have*

$$|M_j - \bar{A}| \leq \sum_{j'=0}^{j} v_{j'} K(\Phi)$$

Proof. Proceed by induction. The base case $j = 0$ is clear. For an induction step suppose that

$$|M_{j-1} - \bar{A}| \leq \sum_{j'=0}^{j-1} v_{j'} K(\Phi)$$

holds. If $M_j - \bar{A} = M_{j-1} - \bar{A}$ then the inequality is immediate as $v_j \geq 0$. Therefore assume this is not the case, so $v_j = 1$ and $|M_j - A| - |M_{j-1} - A| \leq |W_j| \leq v_j K(\Phi)$, and so we get the required inequality.

$\square$

Lemma 3.4.11. *For $0 \leq j \leq J$ we have*

$$\dim(M_j \cup \bar{A}/\bar{A}) \leq -m_j \epsilon(\Phi),$$

Proof. Proceed by induction. Base case $j = 0$ is clear. For an induction step suppose that

$$\dim(M_{j-1} \cup \bar{A}/\bar{A}) \leq -m_{j-1}\epsilon(\Phi)$$

holds. We have

$$\begin{aligned}\dim(M_j \cup \bar{A}/\bar{A}) &= \dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) + \dim(M_{j-1} \cup \bar{A}/\bar{A}) \leq \\ &\leq \dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) - m_{j-1}\epsilon(\Phi).\end{aligned}$$

Let $\bar{M}_{j-1} = M_{j-1} \cup \bar{A}$. By Lemma 3.2.3

$$\dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) = \dim(W_j \cup \bar{M}_{j-1}/\bar{M}_{j-1}) = \dim(W_j/W_j \cap \bar{M}_{j-1}) - e\alpha$$

where e is the number of edges connecting the vertices of $\bar{M}_{j-1} - W_j$ to the vertices of $W_j - \bar{M}_{j-1}$. Recall that $E_j = \partial(A - W_j, M_j - A)$. We have $A - W_j \subseteq \bar{M}_{j-1} - W_j$ (as $A \subseteq \bar{M}_{j-1}$) and $W_j - M_{j-1} - A = W_j - \bar{M}_{j-1}$ (as for $j > 1$, we have $b \subseteq M_{j-1}$). Thus $|E_j| \leq e$, and we get

$$\dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) \leq \dim(W_j/W_j \cap \bar{M}_{j-1}) - |E_j|\alpha.$$

If $W_j \subseteq \bar{M}_{j-1}$ then $\dim(W_j/W_j \cap \bar{M}_{j-1}) = 0$. If not, then by Lemma 3.2.8 we have $\dim(W_j/W_j \cap \bar{M}_{j-1}) \leq -\epsilon(\Phi)$. Either way, we have $\dim(W_j/W_j \cap \bar{M}_{j-1}) \leq -v_j\epsilon(\Phi)$. Using this and the fact that $\epsilon(\Phi) \leq \alpha$, we obtain

$$\dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) \leq -v_j\epsilon(\Phi) - |E_j|\epsilon(\Phi) = -(m_j - m_{j-1})\epsilon(\Phi).$$

Finally,

$$\begin{aligned}\dim(M_j \cup \bar{A}/\bar{A}) &\leq \dim(M_j \cup \bar{A}/M_{j-1} \cup \bar{A}) - m_{j-1}\epsilon(\Phi) \leq \\ &\leq -(m_j - m_{j-1})\epsilon(\Phi) - m_{j-1}\epsilon(\Phi) = -m_j\epsilon(\Phi),\end{aligned}$$

as needed. □

(Proof of Theorem 3.4.6 continued) For any $0 \leq j \leq J$ we have

$$\begin{aligned}\dim(M_j \cup A/A) &= \dim(\bar{A}/A) + \dim(M_j \cup \bar{A}/\bar{A}) \\ &\leq Y - |E_0|\alpha + \dim(M_j \cup \bar{A}/\bar{A}).\end{aligned}$$

Lemma 3.4.11 gives us

$$\dim(M_j \cup \bar{A}/\bar{A}) \leq -m_j\epsilon(\Phi).$$

Thus

$$\dim(M_j \cup A/A) \leq Y - |E_0|\alpha - m_j\epsilon(\Phi).$$

Suppose j is an index such that

$$Y - |E_0|\alpha - m_j\epsilon(\Phi) \geq 0,$$

$$Y - |E_0|\alpha - m_{j+1}\epsilon(\Phi) < 0$$

if one exists. Then

$$m_j \leq \frac{Y - |E_0|\alpha}{\epsilon(\Phi)}.$$

By Lemma 3.4.10 we have

$$\begin{aligned}|M_{j+1} - A| &\leq \left(\sum_{j'=1}^{j+1} v_{j'}\right) K(\Phi) \leq (m_j + 1) K(\Phi) \\ &\leq \left(\frac{Y - |E_0|\alpha}{\epsilon(\Phi)} + 1\right) K(\Phi) \leq S.\end{aligned}$$

This is a contradiction, as A is S-strong and $\dim(M_{j+1} \cup A/A)$ is negative. Thus $Y - |E_0|\alpha - m_j\epsilon(\Phi) \geq 0$ for all $j \leq J$. In particular $Y - |E_0|\alpha - m_J\epsilon(\Phi) \geq 0$, so $m_J \leq \frac{Y-|E_0|\alpha}{\epsilon(\Phi)}$. Noting that $M_J = W_b$, Lemma 3.4.9 gives us

$$|\partial_b| = |\partial(W_b, A)| \leq |E_0| + m_J K(\Phi) \leq |E_0| + K(\Phi)\frac{Y - |E_0|\alpha}{\epsilon(\Phi)}.$$

As $K(\Phi) \geq 1$ and $\epsilon(\Phi) \geq \alpha$, we get

$$|\partial_b| \leq K(\Phi)\frac{Y}{\epsilon(\Phi)} = Y D_1(\Phi).$$

But this is precisely the first inequality we need to prove. For the second inequality, Lemma 3.4.10 gives us

$$\begin{aligned}\left|W_b - \bar{A}\right| &\leq Y + \left(\sum_{j'=0}^{J} v_{j'}\right) K(\Phi) \leq Y + m_J K(\Phi) \leq \\ &\leq Y + K(\Phi)\frac{Y}{\epsilon(\Phi)} \leq 2Y D_1(\Phi).\end{aligned}$$

Thus we have

$$|\bar{W}_b| \leq |W_b - A| + |\partial_b| \leq 3Y D_1(\Phi),$$

as needed. This ends the proof for Theorem 3.4.6. □

3.5 Conclusion

This paper computes upper and lower bounds for certain types of formulas in Shelah-Spencer graphs. The bounds are not tight: in the best case scenario for a basic formula $\phi(x, y)$ defining a minimal extension of dimension ϵ we have

$$\frac{|y|}{\epsilon} \leq \mathrm{vc}(\phi) \leq K\frac{|y|}{\epsilon},$$

where K is the number of vertices in the minimal extension. Thus there is a multiple of K gap between lower and upper bounds. It is this author's hope that a refinement of presented techniques can yield better estimates of the vc-density. One potential direction towards this goal is to have a closer study on how multiple minimal extensions can intersect without increasing overall dimension.

Note that this paper doesn't answer the question whether there can be exotic values for vc-density of individual formulas, such as non-integer or irrational values. A better bound can help address this question.

Another observation is that while $\mathrm{vc}(n) = \infty$ there seems to be a good structural behavior of the vc-density for individual formulas. This perhaps suggests that the vc-function is not the best tool to describe behaviour of the definable sets in Shelah-Spencer graphs, and some more refined measure might be required. One potential way to do this is to separate the formulas based on values of $K(\phi), \epsilon(\phi)$. Once those are bounded, vc-density seems to be well-behaved. This author hopes to explore this further in his future work.

CHAPTER 4

An Additive Reduct of the P-adic Numbers

Aschenbrenner et. al. computed a linear bound for the vc-density function in the field of p-adic numbers, but it is not known to be optimal. In this paper we investigate a certain P-minimal additive reduct of the field of p-adic numbers and use a cell decomposition result of Leenknegt to compute an optimal bound for that structure.

P-adic numbers are a simple, yet a very deep construction. They were only discovered a hundred years ago, but could have been studied in classical mathematics when number theory was just forming. Their construction is simple enough to explain at the undergraduate level, yet has a very rich number theoretic structure. Normally the real numbers are constructed by first taking rational numbers in decimal form and allowing infinite decimal sequences after the decimal point.

Letting decimals be infinite before the decimal point yields a well behaved mathematical object as well, but with a drastically different behavior from real numbers, now depending on the base in which the decimals were written.

When the base is a prime number p, this constructs p-adic numbers. These were first studied exclusively within number theory, but later found applications in other areas of math, physics, and computer science. My research will allow for a finer understanding of the finite structure of polynomially definable sets in p-adic numbers. Model theory began with Gödel and Malcev in the 1930s, but first matured as a subject in the work of Abraham Robinson, Tarski, Vaught, and others in the 1950s. Model theory studies sets definable by first order formulas in a variety of mathematical objects. Restricting to subsets definable by simple formulas gives access to an array of powerful techniques such as indiscernible sequences and nonstandard extensions. These allow insights not otherwise accessible by classical methods.

Nonstandard real numbers, for example, formalize the notion of infinitesimals. Model theory is an extremely flexible field with applications in many areas of mathematics including algebra, analysis, geometry, number theory, and combinatorics as well as some applications to computer science and quantum mechanics.

VC-density was studied in model theory in [ADH16] by Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko as a natural notion of dimension for definable families of sets in NIP theories. In a complete NIP theory T we can define the vc-function

$$\mathrm{vc}^T = \mathrm{vc} : \mathbb{N} \longrightarrow \mathbb{R} \cup \{\infty\}$$

where $\mathrm{vc}(n)$ measures the worst-case complexity of families of definable sets in an n-fold Cartesian power of the underlying set of a model of T (see 1.3.13 below for a precise definition of vc^T). The simplest possible behavior is $\mathrm{vc}(n) = n$ for all n, satisfied, for example, if T is o-minimal. For $T = \mathrm{Th}(\mathbb{Q}_p)$, the paper [ADH16] computes an upper bound for this function to be $2n - 1$, and it is not known whether this is optimal. This same bound holds in any reduct of the field of p-adic numbers, but one may expect that the simplified structure of the reduct would allow a better bound. In [Lee14], Leenknegt provides a cell decomposition result for a certain P-minimal additive reduct of the field of p-adic numbers. Using this result, in this paper we improve the bound for the vc-function, showing that in Leenknegt's structure $\mathrm{vc}(n) = n$.

Section 1 defines vc-density and states some basic lemmas about it. A more in depth exposition of vc-density can be found in [ADH16]. Section 2 defines and states some basic facts about the theory of p-adic numbers. Here we also introduce the reduct which we will be working with. Section 3 sets up basic definitions and lemmas that will be needed for the proof. We define trees and intervals and show how they help with vc-density calculations. Section 4 concludes the proof.

Throughout the paper, variables and tuples of elements will be simply denoted as $x, y, a, b, \ldots$. We will occasionally write $\vec{a}$ instead of a for a tuple in $\mathbb{Q}_p^n$ to emphasize it as an element of the $\mathbb{Q}_p$-vector space $\mathbb{Q}_p^n$. We denote the arity of a tuple x of variables by $|x|$. Natural numbers are $\mathbb{N} = \{0, 1, \ldots\}$.

4.1 P-adic numbers

The field $\mathbb{Q}_p$ of p-adic numbers is often studied in the language of Macintyre

$$\mathcal{L}_{Mac} = \{0, 1, +, -, \cdot, |, \{P_n\}_{n\in\mathbb{N}}\}$$

which is a language $\{0, 1, +, -, \cdot\}$ of rings together with unary predicates P_n interpreted in $\mathbb{Q}_p$ so as to satisfy

$$P_n x \leftrightarrow \exists y\; y^n = x$$

and a divisibility relation where $a|b$ holds in $\mathbb{Q}_p$ when $\operatorname{val} a \leq \operatorname{val} b$.

Note that $P_n \setminus \{0\}$ is a multiplicative subgroup of $\mathbb{Q}_p$ with finitely many cosets.

Theorem 4.1.1 (Macintyre '76)**.** *The $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$ has quantifier elimination.*

There is also a cell decomposition result:

Definition 4.1.2. Define <u>k-cells</u> recursively as follows. A <u>0-cell</u> is a singleton subset of $\mathbb{Q}_p$. A <u>$(k+1)$-cell</u> is a subset of $\mathbb{Q}_p^{k+1}$ of the following form:

$$\{(x,t) \in D \times \mathbb{Q}_p \mid \operatorname{val} a_1(x) \,\square_1\, \operatorname{val}(t - c(x)) \,\square_2\, \operatorname{val} a_2(x), t - c(x) \in \lambda P_n\}$$

where D is a k-cell, $a_1(x), a_2(x), c(x)$ are definable functions $D \longrightarrow \mathbb{Q}_p$, each of $\square_i$ is $<, \leq$ or no condition, $n \in \mathbb{N}$, and $\lambda \in \mathbb{Q}_p$.

Theorem 4.1.3 (Denef '84)**.** *Any definable subset of $\mathbb{Q}_p^n$ defined by an $\mathcal{L}_{Mac}$-formula decomposes into a finite disjoint union of n-cells.*

In [ADH16], Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko show that $\mathbb{Q}_p$ as $\mathcal{L}_{Mac}$-structure satisfies $\operatorname{vc}(n) \leq 2n - 1$, however it is not known whether this bound is optimal.

In [Lee14], Leenknegt analyzes the reduct of $\mathbb{Q}_p$ to the language

$$\mathcal{L}_{aff} = \left\{0, 1, +, -, \{\bar{c}\}_{c \in \mathbb{Q}_p}, |, \{Q_{m,n}\}_{m,n \in \mathbb{N}}\right\}$$

where $\bar{c}$ denotes a scalar multiplication by c, $a|b$ as above stands for $\operatorname{val} a \leq \operatorname{val} b$, and $Q_{m,n}$ is a unary predicate interpreted as

$$Q_{m,n} = \bigcup_{k \in \mathbb{Z}} p^{km}(1 + p^n \mathbb{Z}_p).$$

Note that $Q_{m,n} \setminus \{0\}$ is a subgroup of the multiplicative group of $\mathbb{Q}_p$ with finitely many cosets. One can check that these extra relation symbols are definable in the $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$. The paper [Lee14] provides a cell decomposition result with the following cells:

Definition 4.1.4. A 0-cell is a singleton subset of $\mathbb{Q}_p$. A $(k+1)$-cell is a subset of $\mathbb{Q}_p^{k+1}$ of the following form:

$$\{(x, t) \in D \times \mathbb{Q}_p \mid \operatorname{val} a_1(x) \,\square_1\, \operatorname{val}(t - c(x)) \,\square_2\, \operatorname{val} a_2(x), t - c(x) \in \lambda Q_{m,n}\}$$

where D is a k-cell, called the base of the cell, $a_1(x), a_2(x), c(x)$ are polynomials of degree ≤ 1, called the defining polynomials each of $\square_1, \square_2$ is $<$ or no condition, $m, n \in \mathbb{N}$, and $\lambda \in \mathbb{Q}_p$. We call $\mathbb{Q}_{m,n}$ the defining predicate.

Theorem 4.1.5 (Leenknegt '12)**.** *Any definable subset of $\mathbb{Q}_p^n$ defined by an $\mathcal{L}_{aff}$-formula decomposes into a finite disjoint union of n-cells.*

Moreover, [Lee14] shows that $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ is a P-minimal reduct, that is, the one-dimensional definable sets of $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ coincide with the one-dimensional definable sets in the full structure $\mathcal{L}_{Mac}$-structure $\mathbb{Q}_p$.

The main result of this paper is the computation of the vc-function for this structure:

Theorem 4.1.6. *$\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$ has* $\operatorname{vc}(n) = n$.

4.2 Key Lemmas and Definitions

To show that $\operatorname{vc}(n) = n$ it suffices to bound $\operatorname{vc}^*(\phi) \leq |x|$ for every $\mathcal{L}_{aff}$-formula $\phi(x; y)$. Fix such a formula $\phi(x; y)$. Instead of working with it directly, we first simplify it using quantifier elimination. The required quantifier elimination result can be easily obtained from cell decomposition:

Lemma 4.2.1. *Any formula $\phi(x;y)$ in $\mathcal{L}_{aff}$-structure $\mathbb{Q}_p$. can be written as a boolean combination of formulas from a collection*

$$\Phi(x;y) = \{\mathrm{val}(p_i(x) - c_i(y)) < \mathrm{val}(p_j(x) - c_j(y))\}_{i,j \in I} \cup \\ \{p_i(x) - c_i(y) \in \lambda_k Q_{m,n}\}_{i \in I, k \in K}$$

of $\mathcal{L}_{aff}$-formulas where I, K are finite index sets, each p_i is a degree ≤ 1 polynomial in x without a constant term, each c_i is a degree ≤ 1 polynomial in y, $m, n \in \mathbb{N}$, and $\lambda_k \in \mathbb{Q}_p$.

Proof. Let $l = |x| + |y|$. Partition the subset of $\mathbb{Q}_p^l$ defined by ϕ to obtain $\mathscr{D}^l$, a collection of l-cells. Let $\mathscr{D}^{l-1}$ be the collection of the bases of the cells in $\mathscr{D}^l$. Similarly, construct by induction $\mathscr{D}^i$ for each $0 \leq j < l$, where $\mathscr{D}^j$ is the collection of j-cells which are the bases of cells in $\mathscr{D}^{j+1}$. Set

$$m = \prod \left\{m' \mid Q_{m',n'} \text{ is the defining predicate of a cell in } \mathscr{D}^j \text{ for } 0 \leq j \leq l\right\}$$
$$n = \max \left\{n' \mid Q_{m',n'} \text{ is the defining predicate of a cell in } \mathscr{D}^j \text{ for } 0 \leq j \leq l\right\}$$

This way if a, a' are in the same coset of $Q_{m',n'}$ then they are in the same coset of $Q_{m,n}$. Choose $\{\lambda_k\}_{k \in K}$ to range over all the cosets of $Q_{m,n}$. Let $q_i(x, y)$ enumerate all of the defining polynomials $a_1(x), a_2(x), t - c(x)$ that show up in the cells of $\mathscr{D}^j$ for any j. All if those are all polynomials of degree ≤ 1 in variables x, y. We can split each of them as $q_i(x, y) = p_i(x) - c_i(y)$ where the constant term of q_i goes into c_i. This gives us the appropriate finite collection of formulas Φ. From the cell decomposition it is easy to see that when a, a' have the same Φ-type, then they have the same ϕ-type. Thus ϕ can be written as a boolean combination of formulas from Φ. ☐

Lemma 4.2.2. *Let $\Phi(x;y)$ be a finite collection of formulas. If ϕ can be written as a boolean combination of formulas from Φ then*

$$\mathrm{vc}^*(\Phi) \leq r \implies \mathrm{vc}^*(\phi) \leq r \quad \textit{for all } r \in \mathbb{R}.$$

Proof. If a, a' have the same Φ-type over B, then they have the same ϕ-type over B, where B is some parameter set. Therefore the number of ϕ-types is bounded by the number of Φ-types. The bound follows from Lemma 1.3.11. $\square$

For the remainder of the paper fix $\Phi(x; y)$ to be the collection of formulas defined by Lemma 4.2.1. By the previous lemma, to show that $\mathrm{vc}^*(\phi) \leq |x|$, it suffices to bound $\mathrm{vc}^*(\Phi) \leq |x|$. More precisely, it is sufficient to show that if there is a parameter set B of size N then the number of Φ-types over B is $O(N^{|x|})$. Fix such a parameter set B and work with it from now on. We will compute a bound for the number of Φ-types over B.

Consider the set $T = T(\Phi, B) = \{c_i(b) \mid b \in B, i \in I\} \subset \mathbb{Q}_p$. In this definition B is the parameter set that we have fixed and $c_i(b)$ come from the collection of formulas Φ from the quantifier elimination above. View T as a tree as follows:

Definition 4.2.3.

- For $c \in \mathbb{Q}_p, \alpha \in \mathbb{Z}$ define a <u>ball</u>

$$B(c, \alpha) = \{c' \in \mathbb{Q}_p \mid \mathrm{val}\,(c' - c) > \alpha\}.$$

  We also let $B(c, -\infty) = \mathbb{Q}_p$ and $B(c, +\infty) = \emptyset$.

- Define a collection of balls $\mathscr{B} = \{B(t_1, \mathrm{val}(t_1 - t_2))\}_{t_1, t_2 \in T}$. Note that $\mathscr{B}$ is a (directed) boolean algebra of sets in $\mathbb{Q}_p$. We refer to the atoms in that algebra as <u>intervals</u>. Note that the intervals partition $\mathbb{Q}_p$ so any element $a \in \mathbb{Q}_p$ belongs to a unique interval.

- Let's introduce some notation for the intervals. For $t \in T$ and $\alpha_L, \alpha_U \in \mathbb{Z} \cup \{-\infty, +\infty\}$ define

$$\mathrm{I}(t, \alpha_L, \alpha_U) = B(t, \alpha_L) \backslash \bigcup \{B(t', \alpha_U) \mid t' \in T, \mathrm{val}(t' - t) \geq \alpha_U\}$$

  (this is sometimes referred to as the swiss cheese construction). One can check that every interval is of the form $\mathrm{I}(t, \alpha_L, \alpha_U)$ for some values of t, α_L, α_U. The quantities α_L, α_U are uniquely determined by the interval $\mathrm{I}(t, \alpha_L, \alpha_U)$, while t might not be.

- Intervals are a natural construction for trees, however we will require a more refined notion to make Lemma 4.2.12 below work. Define a larger collection of balls

$$\mathscr{B}' = \mathscr{B} \cup \{B(c_i(b), \mathrm{val}(c_j(b) - c_k(b)))\}_{i,j,k \in I, b \in B}\,.$$

  Similar to the previous definition, we define a subinterval to be an atom of the boolean algebra generated by $\mathscr{B}'$. Subintervals refine intervals. Moreover, as before, each subinterval can be written as $\mathrm{I}(t, \alpha_L, \alpha_U)$ for some values of t, α_L, α_U. As before, α_L, α_U are uniquely determined by the subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$, while t might not be.

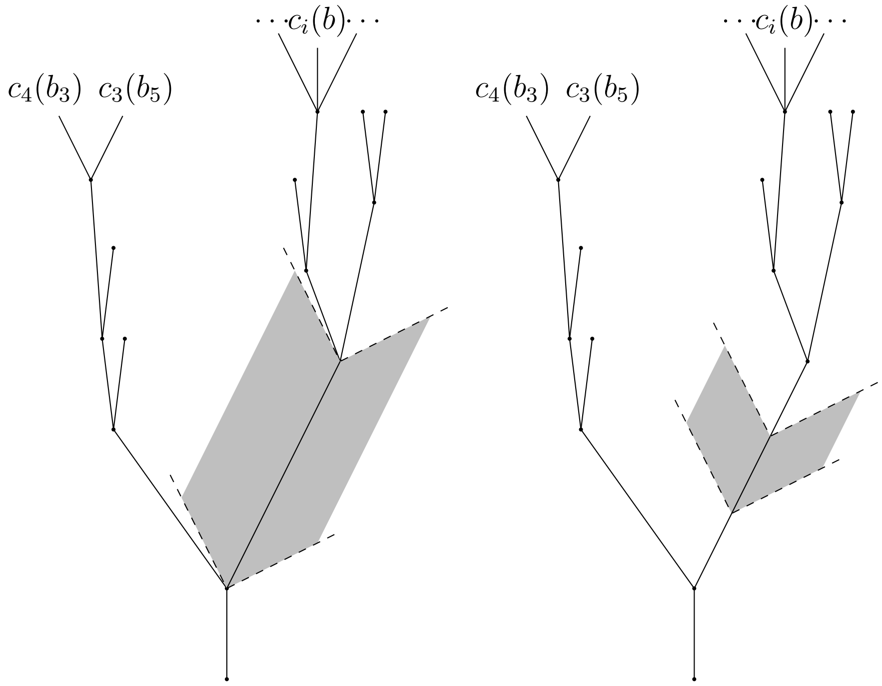

Figure 4.1: A typical interval (left) and subinterval (right) on a tree $\{c_i(b) \mid i \in I, b \in B\}$.

Subintervals are fine enough to make Lemma 4.2.12 below work while coarse enough to be $O(N)$ small:

Lemma 4.2.4.

- *There are at most $2|T| = 2N|I| = O(N)$ different intervals.*

- *There are at most $2|T| + |B| \cdot |I|^3 = O(N)$ different subintervals.*

Proof. Each new element in the tree T adds at most two intervals to the total count, so by induction there can be at most $2|T|$ many intervals. Each new ball in $\mathscr{B}'\backslash\mathscr{B}$ adds at most one subinterval to the total count, so by induction there are at most $|\mathscr{B}'\backslash\mathscr{B}|$ more subintervals than there are intervals. $\square$

Definition 4.2.5. Suppose $a \in \mathbb{Q}_p$ lies in the interval $\mathrm{I}(t, \alpha_L, \alpha_U)$. Define the T-valuation of a to be $\text{T-val}(a) = \mathrm{val}(a - t)$.

This is a natural notion having the following properties:

Lemma 4.2.6.

(a) $\text{T-val}(a)$ *is well-defined, independent of choice of* t *to represent the interval.*

(b) If $a \in \mathbb{Q}_p$ *lies in the subinterval* $\mathrm{I}(t, \alpha_L, \alpha_U)$*, then* $\text{T-val}(a) = \mathrm{val}(a - t)$*.*

(c) If $a \in \mathbb{Q}_p$ *lies in the (sub)interval* $\mathrm{I}(t, \alpha_L, \alpha_U)$ *then* $\alpha_L < \text{T-val}(a) \leq \alpha_U$*.*

(d) For any $a \in \mathbb{Q}_p$ *lying in the (sub)interval* $\mathrm{I}(t, \alpha_L, \alpha_U)$ *and* $t' \in T$*:*

- *If* $\mathrm{val}(t - t') \geq \alpha_U$*, then* $\mathrm{val}(a - t') = \text{T-val}(a)$*.*
- *If* $\mathrm{val}(t - t') \leq \alpha_L$*, then* $\mathrm{val}(a - t') = \mathrm{val}(t - t')\ (\leq \alpha_L < \text{T-val}(a))$*.*

Proof. (a)-(c) are clear. For (d) fix $t' \in T$ and suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\mathrm{I}(t, \alpha'_L, \alpha'_U)$. This subinterval lies inside of a unique interval $\mathrm{I}(t, \alpha_L, \alpha_U)$ for some choice of α_L, α_U and by the definition of intervals (or more specifically $\mathscr{B}$):

$$\mathrm{val}(t - t') \geq \alpha_U \iff \mathrm{val}(t - t') \geq \alpha'_U,$$
$$\mathrm{val}(t - t') \geq \alpha_L \iff \mathrm{val}(t - t') \geq \alpha'_L.$$

Therefore without loss of generality we may assume that $a \in \mathbb{Q}_p$ lies in an interval $\mathrm{I}(t, \alpha_L, \alpha_U)$. By (c) and the definition of intervals one of the three following cases has to hold.

Case 1: $\mathrm{val}(t - t') \geq \alpha_U$ and $\text{T-val}(a) < \alpha_U$. Then

$$\mathrm{val}(t - t') \geq \alpha_U > \text{T-val}(a) = \mathrm{val}(a - t),$$

thus $\text{val}(a-t') = \text{val}(a-t) = \text{T-val}(a)$ as needed.

Case 2: $\text{val}(t-t') \geq \alpha_U$ and $\text{T-val}(a) = \alpha_U$. Then

$$\text{T-val}(a) = \text{val}(a-t) = \text{val}(t-t') \geq \alpha_U,$$

thus $\text{val}(a-t') \geq \alpha_U$. The interval $\text{I}(t, \alpha_L, \alpha_U)$ is disjoint from the ball $B(t', \alpha_U)$, so $a \notin B(t', \alpha_U)$, that is, $\text{val}(a-t') \leq \alpha_U$. Combining this with the previous inequality we get that $\text{val}(a-t') = \alpha_U = \text{T-val}(a)$ as needed.

Case 3: $\text{val}(t-t') \leq \alpha_L$. Then

$$\text{val}(t-t') \leq \alpha_L < \text{T-val}(a) = \text{val}(a-t),$$

thus $\text{val}(a-t') = \text{val}(t-t')$ as needed. $\square$

Definition 4.2.7. Suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\text{I}(t, \alpha_L, \alpha_U)$. We say that a is far from the boundary tacitly (of $\text{I}(t, \alpha_L, \alpha_U)$) if

$$\alpha_L + n \leq \text{T-val}(a) \leq \alpha_U - n.$$

Here n is as in Lemma 4.2.1. Otherwise we say that it is close to the boundary (of $\text{I}(t, \alpha_L, \alpha_U)$).

Definition 4.2.8. Suppose $a_1, a_2 \in \mathbb{Q}_p$ lie in the same subinterval $\text{I}(t, \alpha_L, \alpha_U)$. We say a_1, a_2 have the same subinterval type if one of the following holds:

- Both a_1, a_2 are far from the boundary and $a_1 - t, a_2 - t$ are in the same $Q_{m,n}$-coset. (Here $Q_{m,n}$ is an in Lemma 4.2.1.)
- Both a_1, a_2 are close to the boundary and

$$\text{T-val}(a_1) = \text{T-val}(a_2) \leq \text{val}(a_1 - a_2) - n.$$

Definition 4.2.9. For $c \in \mathbb{Q}_p$ and $\alpha, \beta \in \mathbb{Z}, \alpha < \beta$ define $c \restriction [\alpha, \beta)$ to be the record of the coefficients of c for the valuations between $[\alpha, \beta)$. More precisely write c in its power series form

$$c = \sum_{\gamma \in \mathbb{Z}} c_\gamma p^\gamma \text{ with } c_\gamma \in \{0, 1, \ldots, p-1\}$$

Then $c \restriction [\alpha, \beta)$ is just $(c_\alpha, c_{\alpha+1}, \ldots c_{\beta-1}) \in \{0, 1, \ldots, p-1\}^{\beta - \alpha}$.

The following lemma is an adaptation of Lemma 7.4 in [ADH16].

Lemma 4.2.10. *Fix $m, n \in \mathbb{N}$. For any $x, y, c \in \mathbb{Q}_p$, if*

$$\mathrm{val}(x-c) = \mathrm{val}(y-c) \leq \mathrm{val}(x-y) - n,$$

then $x-c, y-c$ are in the same coset of $Q_{m,n}$.

Proof. Call $a, b \in \mathbb{Q}_p$ similar if $\mathrm{val}\, a = \mathrm{val}\, b$ and

$$a \restriction [\mathrm{val}\, a, \mathrm{val}\, a + n) = b \restriction [\mathrm{val}\, b, \mathrm{val}\, b + n).$$

If a, b are similar then

$$a \in Q_{m,n} \iff b \in Q_{m,n}.$$

Moreover for any $\lambda \in \mathbb{Q}_p^\times$, if a, b are similar then so are $\lambda a, \lambda b$. Thus if a, b are similar, then they belong to the same coset of $Q_{m,n}$. The hypothesis of the lemma force $x-c, y-c$ to be similar, thus belonging to the same coset. $\square$

Lemma 4.2.11. *For each subinterval there are at most $K = K(Q_{m,n})$ many subinterval types (with K not depending on B or on the subinterval).*

Proof. Let $a, a' \in \mathbb{Q}_p$ lie in the same subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$.

Suppose a, a' are far from the boundary. Then they have the same subinterval type if $a-t, a'-t$ are in the same $Q_{m,n}$-coset. So the number of such subinterval types is bounded by the number of $Q_{m,n}$-cosets.

Suppose a, a' are close to the boundary and

$$\begin{aligned} &\text{T-val}(a) - \alpha_L = \text{T-val}(a') - \alpha_L < n \text{ and} \\ &a \restriction [\text{T-val}(a), \text{T-val}(a) + n) = a' \restriction [\text{T-val}(a'), \text{T-val}(a') + n). \end{aligned}$$

Then a, a' have the same subinterval type. Such a subinterval type is thus determined by $\text{T-val}(a) - \alpha_L$ and the tuple $a \restriction [\text{T-val}(a), \text{T-val}(a) + n)$, therefore there are at most np^n many such types.

A similar argument works for a with $\alpha_U - \text{T-val}(a) \leq n$.

Adding all this up we get that there are at most

$$K = (\text{number of } Q_{m,n} \text{ cosets}) + 2np^n$$

many subinterval types. $\square$

The following critical lemma relates tree notions to Φ-types.

Lemma 4.2.12. *Suppose $d, d' \in \mathbb{Q}_p^{|x|}$ satisfy the follwing three conditions:*

- *For all $i \in I$ $p_i(d)$ and $p_i(d')$ are in the same subinterval.*
- *For all $i \in I$ $p_i(d)$ and $p_i(d')$ have the same subinterval type.*
- *For all $i, j \in I$, $\text{T-val}(p_i(d)) > \text{T-val}(p_j(d))$ iff $\text{T-val}(p_i(d')) > \text{T-val}(p_j(d'))$.*

Then d, d' have the same Φ-type over B.

Proof. There are two kinds of formulas in Φ (see Lemma 4.2.1). First we show that d, d' agree on formulas of the form $p_i(x) - c_i(y) \in \lambda_k Q_{m,n}$. It is enough to show that for every $i \in I, b \in B$, $p_i(d) - c_i(b), p_i(d') - c_i(b)$ are in the same $Q_{m,n}$-coset. Fix such i, b. For brevity let $a = p_i(d), a' = p_i(d')$ and $Q = Q_{m,n}$. We want to show that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

Suppose a, a' are close to the boundary. Then $\text{T-val}(a) = \text{T-val}(a') \leq \text{val}(a - a') - n$. Using Lemma 4.2.6d, we have

$$\text{val}(a - c_i(b)) = \text{val}(a' - c_i(b)) \leq \text{T-val}(a) \leq \text{val}(a - a') - n.$$

Lemma 4.2.10 shows that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

Now, suppose both a, a' are far from the boundary. Let $\mathrm{I}(t, \alpha_L, \alpha_U)$ be the interval containing a, a'. Then we have

$$\alpha_L + n \leq \text{val}(a - t) \leq \alpha_U - n,$$

$$\alpha_L + n \leq \text{val}(a' - t) \leq \alpha_U - n$$

(as being far from the subinterval's boundary also makes a, a' far from interval's boundary). We have either $\mathrm{val}(t - c_i(b)) \geq \alpha_U$ or $\mathrm{val}(t - c_i(b)) \leq \alpha_L$ (as otherwise it would contradict the definition of intervals, or more specifically $\mathscr{B}$).

Suppose it is the first case $\mathrm{val}(t - c_i(b)) \geq \alpha_U$. Then using Lemma 4.2.6d

$$\mathrm{val}(a - c_i(b)) = \mathrm{val}(a - t) \leq \alpha_U - n \leq \mathrm{val}(t - c_i(b)) - n.$$

So by Lemma 4.2.10 $a - c_i(b), a - t$ are in the same Q-coset. By an analogous argument, $a' - c_i(b), a' - t$ are in the same Q-coset. As a, a' have the same subinterval type, $a - t, a' - t$ are in the same Q-coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

For the second case, suppose $\mathrm{val}(t - c_i(b)) \leq \alpha_L$. Then using Lemma 4.2.6d

$$\mathrm{val}(a - c_i(b)) = \mathrm{val}(t - c_i(b)) \leq \alpha_L \leq \mathrm{val}(a - t) - n,$$

so by Lemma 4.2.10, $a - c_i(b), t - c_i(b)$ are in the same Q-coset. Similarly $a' - c_i(b), t - c_i(b)$ are in the same Q-coset. Thus by transitivity we get that $a - c_i(b), a' - c_i(b)$ are in the same Q-coset.

Next, we need to show that d, d' agree on formulas of the form $\mathrm{val}(p_i(x) - c_i(y)) < \mathrm{val}(p_j(x) - c_j(y))$ (again, referring to the presentation in Lemma 4.2.1). Fix $i, j \in I, b \in B$. We would like to show the following equivalence:

$$\begin{aligned} &\mathrm{val}(p_i(d) - c_i(b)) < \mathrm{val}(p_j(d) - c_j(b)) \iff \\ &\qquad \iff \mathrm{val}(p_i(d') - c_i(b)) < \mathrm{val}(p_j(d') - c_j(b)) \end{aligned} \tag{4.2.1}$$

Suppose $p_i(d), p_i(d')$ are in the subinterval $\mathrm{I}(t_i, \alpha_i, \beta_i)$ and $p_j(d), p_j(d')$ are in the subinterval $\mathrm{I}(t_j, \alpha_j, \beta_j)$. Lemma 4.2.6d yields the following four cases.

Case 1:

$$\mathrm{val}(p_i(d) - c_i(b)) = \mathrm{val}(p_i(d') - c_i(b)) = \mathrm{val}(t_i - c_i(b))$$
$$\mathrm{val}(p_j(d) - c_j(b)) = \mathrm{val}(p_j(d') - c_j(b)) = \mathrm{val}(t_j - c_j(b))$$

Then it is clear that the equivalence (4.2.1) holds.

Case 2:

$$\begin{aligned}
&\mathrm{val}(p_i(d) - c_i(b)) = \text{T-val}(p_i(d)) \text{ and } \mathrm{val}(p_i(d') - c_i(b)) = \text{T-val}(p_i(d')) \\
&\mathrm{val}(p_j(d) - c_j(b)) = \text{T-val}(p_j(d)) \text{ and } \mathrm{val}(p_j(d') - c_j(b)) = \text{T-val}(p_j(d'))
\end{aligned}$$

Then the equivalence (4.2.1) holds by the third hypothesis of the lemma (that order of T-valuations is preserved).

Case 3:

$$\begin{aligned}
&\mathrm{val}(p_i(d) - c_i(b)) = \mathrm{val}(p_i(d') - c_i(b)) = \mathrm{val}(t_i - c_i(b)) \\
&\mathrm{val}(p_j(d) - c_j(b)) = \text{T-val}(p_j(d)) \text{ and } \mathrm{val}(p_j(d') - c_j(b)) = \text{T-val}(p_j(d'))
\end{aligned}$$

If $p_j(d), p_j(d')$ are close to the boundary, then $\text{T-val}(p_j(d)) = \text{T-val}(p_j(d'))$ and the equivalence (4.2.1) clearly holds. Suppose then that $p_j(d), p_j(d')$ are far from the boundary.

$$\alpha_j + n \leq \text{T-val}(p_j(d)), \text{T-val}(p_j(d')) \leq \beta_j - n$$

$$\alpha_j < \text{T-val}(p_j(d)), \text{T-val}(p_j(d')) < \beta_j$$

and $\mathrm{val}(t_i - c_i(b))$ lies outside of the (α_j, β_j) by the definition of subinterval (more specifically definition of $\mathscr{B}'$). Therefore (4.2.1) has to hold. (Note that we always have $\text{T-val}(p_j(d)), \text{T-val}(p_j(d')) \in (\alpha_j, \beta_j]$ by Lemma 4.2.6c, so we only need the condition on being far from the boundary to avoid the edge case of equality to β_j.)

Case 4:

$$\begin{aligned}
&\mathrm{val}(p_i(d) - c_i(b)) = \text{T-val}(p_i(d)) \text{ and } \mathrm{val}(p_i(d') - c_i(b)) = \text{T-val}(p_i(d')) \\
&\mathrm{val}(p_j(d) - c_j(b)) = \mathrm{val}(p_j(d') - c_j(b)) = \mathrm{val}(t_j - c_j(b))
\end{aligned}$$

Similar to case 3 (switching i, j). □

The previous lemma gives us an upper bound on the number of types - there are at most $|2I|!$ many choices for the order of T-val, $O(N)$ many choices for the subinterval for each p_i, and K many choices for the subinterval type for each p_i (where K is as in Lemma 4.2.11), giving a total of $O(N^{|I|}) \cdot K^{|I|} \cdot |I|! = O(N^{|I|})$ many types. This implies $\mathrm{vc}^*(\Phi) \leq |I|$. The biggest contribution to this bound are the choices among the $O(N)$ many subintervals for each p_i with $i \in I$. Are all of those choices realized? Intuitively there are $|x|$ many variables and $|I|$ many equations, so once we choose a subinterval for $|x|$ many p_i's, the subintervals for the rest should be determined. This would give the required bound $\mathrm{vc}^*(\Phi) \leq |x|$. The next section outlines this idea formally.

4.3 Main Proof

An alternative way to write $p_i(c)$ is as a scalar product $\vec{p_i} \cdot \vec{c}$, where $\vec{p_i}$ and $\vec{c}$ are vectors in $\mathbb{Q}_p^{|x|}$ (as $p_i(x)$ is homogeneous linear).

Lemma 4.3.1. *Suppose we have a finite collection of vectors $\{\vec{p_j}\}_{j\in J}$ with each $\vec{p_j} \in \mathbb{Q}_p^{|x|}$. Suppose $\vec{p} \in \mathbb{Q}_p^{|x|}$ satisfies $\vec{p} \in \operatorname{span}\{\vec{p_j}\}_{j\in J}$, and we have $\vec{c} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$ with $\mathrm{val}(\vec{p_j} \cdot \vec{c}) > \alpha$ for all $j \in J$. Then $\mathrm{val}(\vec{p} \cdot \vec{c}) > \alpha - \gamma$ for some $\gamma \in \mathbb{N}$. Moreover γ can be chosen independently from $\vec{c}, \alpha$ depending only on $\{\vec{p_j}\}_{j\in J}$.*

Proof. For some $c_j \in \mathbb{Q}_p$ for $j \in J$ we have $\vec{p} = \sum_{j\in J} c_j \vec{p_j}$, hence $\vec{p} \cdot \vec{c} = \sum_{j\in J} c_j \vec{p_j} \cdot \vec{c}$. Thus

$$\mathrm{val}\left(c_j \vec{p_j} \cdot \vec{c}\right) = \mathrm{val}\left(c_j\right) + \mathrm{val}\left(\vec{p_j} \cdot \vec{c}\right) > \mathrm{val}\left(c_j\right) + \alpha.$$

Let $\gamma = \max(0, -\max_{j\in J} \mathrm{val}\left(c_j\right))$. Then we have

$$\mathrm{val}(\vec{p} \cdot \vec{c}) = \mathrm{val}\left(\sum_{j\in J} c_j \vec{p_j} \cdot \vec{c}\right) \geq$$
$$\geq \min_{j\in J} \mathrm{val}(\sum_{j\in J} c_j \vec{p_j} \cdot \vec{c}) > \min_{j\in J} \mathrm{val}(c_j) + \alpha \geq \alpha - \gamma$$

as required. $\square$

Corollary 4.3.2. *Suppose we have a finite collection of vectors $\{\vec{p_i}\}_{i\in I}$ with each $\vec{p_i} \in \mathbb{Q}_p^{|x|}$. Suppose $J \subseteq I$ and $i \in I$ satisfy $\vec{p_i} \in \operatorname{span}\{\vec{p_j}\}_{j\in J}$, and we have $\vec{c} \in \mathbb{Q}_p^{|x|}, \alpha \in \mathbb{Z}$ with $\mathrm{val}(\vec{p_j} \cdot \vec{c}) > \alpha$ for all $j \in J$. Then $\mathrm{val}(\vec{p_i} \cdot \vec{c}) > \alpha - \gamma$ for some $\gamma \in \mathbb{N}$. Moreover γ can be chosen independently from $J, j, \vec{c}, \alpha$ depending only on $\{\vec{p_i}\}_{i\in I}$.*

Proof. The previous lemma shows that we can pick such γ for a given choice of i, J, but independent from $\alpha, \vec{c}$. To get a choice independent from i, J, go over all such eligible choices (i ranges over I and J ranges over subsets of I), pick γ for each, and then take the maximum of those values. $\square$

Fix γ according to Corollary 4.3.2 corresponding to $\{\vec{p}_i\}_{i\in I}$ given by our collection of formulas Φ. (The lemma above is a general result, but we only use it applied to the vectors given by Φ.)

Definition 4.3.3. Suppose $a \in \mathbb{Q}_p$ lies in the subinterval $\mathrm{I}(t, \alpha_L, \alpha_U)$. Define the T-floor of a to be $\text{T-fl}(a) = \alpha_L$.

Definition 4.3.4. Let $f : \mathbb{Q}_p^{|x|} \longrightarrow \mathbb{Q}_p^I$ with $f(c) = (p_i(c))_{i\in I}$. Define the segment space Sg to be the image of f. Equvalently:

$$\mathrm{Sg} = \left\{(p_i(c))_{i\in I} \mid c \in \mathbb{Q}_p^{|x|}\right\} \subseteq \mathbb{Q}_p^I$$

Without loss of generality, we may assume that $I = \{1, 2, \ldots, k\}$ (that is the formulas are labeled by consecutive natural numbers). Given a tuple $(a_i)_{i\in I}$ in the segment space, look at the corresponding T-floors $\{\text{T-fl}(a_i)\}_{i\in I}$ and T-valuations $\{\text{T-val}(a_i)\}_{i\in I}$. Partition the segment space by the order types of $\{\text{T-fl}(a_i)\}_{i\in I}$ and $\{\text{T-val}(a_i)\}_{i\in I}$ (as subsets of $\mathbb{Z}$).

Work in a fixed set Sg' of the partition. After relabeling the p_i we may assume that

$$\text{T-fl}(a_1) \geq \text{T-fl}(a_2) \geq \ldots \text{ for all } a_i \in \mathrm{Sg}'$$

Consider the (relabeled) sequence of vectors $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_I$. There is a unique subset $J \subset I$ such that the set of all vectors with indices in J is linearly independent, and all vectors with indices outside of J are a linear combination of preceding vectors. (We can pick those using a greedy algorithm for finding a linearly independent subset of vectors.) We call indices in I independent and we call the indices in $I \setminus J$ dependent.

Definition 4.3.5.

- Denote $\{0, 1, \ldots, p-1\}$ as Ct.

- Let $\underline{\mathrm{Tp}}$ be the space of all subinterval types. By Lemma 4.2.11 we have $|\,\mathrm{Tp}\,| \leq K$.
- Let $\underline{\mathrm{Sub}}$ be the space of all subintervals. By Lemma 4.2.4 we have $|\,\mathrm{Sub}\,| \leq 3|I|^2 \cdot N = O(N)$.

Definition 4.3.6. Now, we define a function

$$g_{\mathrm{Sg}'} : \mathrm{Sg}' \longrightarrow \mathrm{Tp}^I \times \mathrm{Sub}^J \times \mathrm{Ct}^{I \setminus J}$$

as follows:

Let $a = (a_i)_{i \in I} \in \mathrm{Sg}'$. To define $g_{\mathrm{Sg}'}(a)$ we need to specify where it maps a in each individual component of the product.

For each a_i record its subinterval type, giving the first component in Tp^I.

For a_j with $j \in J$, record the subinterval of a_j, giving the second component in Sub^J.

For the third component (an element of $\mathrm{Ct}^{I \setminus J}$) do the following computation. Pick a_i with i dependent. Let j be the largest independent index with $j < i$. Record $a_i \restriction [\text{T-fl}(a_j) - \gamma, \text{T-fl}(a_j))$.

Combine $g_{\mathrm{Sg}'}$ for all the partitions to get a function

$$g : \mathrm{Sg} \longrightarrow \mathrm{Tp}^I \times \mathrm{Sub}^J \times \mathrm{Ct}^{I \setminus J}.$$

Lemma 4.3.7. *Suppose we have $c, c' \in \mathbb{Q}_p^{|x|}$ such that $f(c), f(c')$ are in the same set Sg' of the partition of Sg and $g(f(c)) = g(f(c'))$. Then c, c' have the same Φ-type over B.*

Proof. Let $a_i = \vec{p}_i \cdot \vec{c}$ and $a_i' = \vec{p}_i \cdot \vec{c}'$ so that

$$f(c) = (p_i(c))_{i \in I} = (\vec{p}_i \cdot \vec{c})_{i \in I} = (a_i)_{i \in I}$$
$$f(c') = (p_i(c'))_{i \in I} = (\vec{p}_i \cdot \vec{c}')_{i \in I} = (a_i')_{i \in I}$$

For each i we show that a_i, a_i' are in the same subinterval and have the same subinterval type, so the conclusion follows by Lemma 4.2.12 ($f(c), f(c')$ are in the same partition ensuring the proper order of T-valuations for the 3rd condition of the lemma). Tp records the subinterval type of each element, so if $g(\bar{a}) = g(\bar{a}')$ then a_i, a_i' have the same subinterval type for all $i \in I$. Thus it remains to show that a_i, a_i' lie in the same subinterval for all $i \in I$. Suppose i is an independent index. Then by construction, Sub records the subinterval for a_i, a_i', so those have to belong to the same subinterval. Now suppose i is dependent. Pick the largest $j < i$ such that j is independent. We have $\text{T-fl}(a_i) \leq \text{T-fl}(a_j)$ and $\text{T-fl}(a_i') \leq \text{T-fl}(a_j')$. Moreover $\text{T-fl}(a_j) = \text{T-fl}(a_j')$ as a_j, a_j' lie in the same subinterval (using the earlier part of the argument as j is independent).

Claim 4.3.8. $\text{val}(a_i - a_i') > \text{T-fl}(a_j) - \gamma$

Proof. Let K be the set of the independent indices less than i. Note that by the definition for dependent indices we have $\vec{p_i} \in \text{span}\,\{\vec{p}_k\}_{k \in K}$. We also have

$$\text{val}(a_k - a_k') > \text{T-fl}(a_k) \text{ for all } k \in K$$

as a_k, a_k' lie in the same subinterval (using the earlier part of the argument as k is independent). Now $\text{val}(a_k - a_k') > \text{T-fl}(a_j)$ for all $k \in K$ by monotonicity of $\text{T-fl}(a_k)$. Moreover $a_k - a_k' = \vec{p}_k \cdot \vec{c} - \vec{p}_k \cdot \vec{c}' = \vec{p}_k \cdot (\vec{c} - \vec{c}')$. Combining the two, we get that $\text{val}(\vec{p}_k \cdot (\vec{c} - \vec{c}')) > \text{T-fl}(a_j)$ for all $k \in K$. Now observe that $K \subset I, i \in I, \vec{c} - \vec{c}' \in \mathbb{Q}_p^{|x|}, \text{T-fl}(a_j) \in \mathbb{Z}$ satisfy the requirements of Lemma 4.3.2, so we apply it to obtain $\text{val}(\vec{p_i} \cdot (\vec{c} - \vec{c}')) > \text{T-fl}(a_j) - \gamma$. Similarly to before, we have $\vec{p_i} \cdot (\vec{c} - \vec{c}') = \vec{p_i} \cdot \vec{c} - \vec{p_i} \cdot \vec{c}' = a_i - a_i'$. Therefore we can conclude that $\text{val}(a_i - a_i') > \text{T-fl}(a_j) - \gamma$ as needed, finishing the proof of the claim. □

Additionally a_i, a_i' have the same image in the Ct component, so we have

$$\text{val}(a_i - a_i') > \text{T-fl}(a_j).$$

We now would like to show that a_i, a_i' lie in the same subinterval. As $\text{T-fl}(a_i) \leq \text{T-fl}(a_j)$, $\text{T-fl}(a_i') \leq \text{T-fl}(a_j')$ and $\text{T-fl}(a_j) = \text{T-fl}(a_j')$ we have that $\text{val}(a_i - a_i') > \text{T-fl}(a_i)$ and $\text{val}(a_i - a_i') > \text{T-fl}(a_i')$. Suppose that a_i lies in the subinterval $\text{I}(t, \text{T-fl}(a_i), \alpha_U)$ and that a_i' lies in the subinterval $\text{I}(t', \text{T-fl}(a_i'), \alpha_U')$. Without loss of generality assume that $\text{T-fl}(a_i) \leq \text{T-fl}(a_i')$. As $\text{val}(a_i - a_i') > \text{T-fl}(a_i')$, this implies that $a_i \in B(a_i', \text{T-fl}(a_i'))$. Then $a_i \in B(t', \text{T-fl}(a_i'))$ as $\text{val}(a_i - t') > \text{T-fl}(a_i')$. This implies that $B(t, \text{T-fl}(a_i)) \cap B(t', \text{T-fl}(a_i')) \neq \emptyset$ as they both contain a_i. As balls are directed, the non-zero intersection means that one ball has to be contained in another. Given our assumption that $\text{T-fl}(a_i) \leq \text{T-fl}(a_i')$, we have $B(t, \text{T-fl}(a_i)) \subset B(t', \text{T-fl}(a_i'))$. For the subintervals to be disjoint we need $\mathrm{I}(t, \text{T-fl}(a_i), \alpha_U) \cap B(t', \text{T-fl}(a_i')) = \emptyset$. But $\mathrm{val}(t' - a_i) > \text{T-fl}(a_i')$ implying that $a_i \in \mathrm{I}(t, \text{T-fl}(a_i), \alpha_U) \cap B(t', \text{T-fl}(a_i'))$ giving a contradiction. Therefore the subintervals coincide finishing the proof. □

Corollary 4.3.9. *The dual* vc-*density of* $\Phi(x, y)$ *is* $\leq |x|$.

Proof. Suppose we have $c, c' \in \mathbb{Q}_p^{|x|}$ such that $f(c), f(c')$ are in the same partition and $g(f(c)) = g(f(c'))$. Then by the previous lemma c, c' have the same Φ-type. Thus the number of possible Φ-types is bounded by the size of the range of g times the number of possible partitions

$$(\text{number of partitions}) \cdot |\operatorname{Tp}|^{|I|} \cdot |\operatorname{Sub}|^{|J|} \cdot |\operatorname{Ct}|^{|I-J|}.$$

There are at most $(|2I|!)^2$ many partitions of Sg, so in the product above, the only component dependent on B is

$$|\operatorname{Sub}|^{|J|} \leq (N \cdot 3|I|^2)^{|J|} = O(N^{|J|}).$$

Every p_i is an element of a $|x|$-dimensional vector space, so there can be at most $|x|$ many independent vectors. Thus we have $|J| \leq |x|$ and the bound follows. □

Corollary 4.3.10 (Theorem 4.1.6)**.** *The* $\mathcal{L}_{aff}$*-structure* $\mathbb{Q}_p$ *satisfies* $\mathrm{vc}(n) = n$.

Proof. The previous lemma implies that $\mathrm{vc}^*(\phi) \leq \mathrm{vc}^*(\Phi) \leq |x|$. As choice of ϕ was arbitrary, this implies that the vc-density of any formula is bounded by the arity of x. □

This proof relies heavily on the linearity of the defining polynomials a_1, a_2, c in the cell decomposition result (see Definition 4.1.4). Linearity is used to separate the x and y variables as well as for Corollary 4.3.2 to reduce the number of independent factors from $|I|$ to $|x|$. The paper [Lee14] has cell decomposition results for more expressive reducts of $\mathbb{Q}_p$, including, for example, restricted multiplication. While our results don't apply to it directly, it is this author's hope that similar techniques can be used to also compute the vc-function for those structures.

Another interesting question whether the reduct studied in this paper has VC 1 property (see [ADH16] 5.2 for the definition). If so, this would imply the linear vc-density bound directly. The paper [ADH16] implies that the reduct has VC 2 property. While there are techniques for showing that a structure has a given VC property, less is known about showing that a structure doesn't have a given VC property. Perhaps the simple structure of the $\mathcal{L}_{aff}$-reduct can help understand this property better.

CHAPTER 5

Dp-minimality in Flat Graphs

In this chapter we show that the theory of superflat graphs is dp-minimal.

5.1 Preliminaries

Superflat graphs were introduced in [PZ78] as a natural class of stable graphs. Here we present a direct proof showing dp-minimality.

First, we introduce some basic graph-theoretic definitions.

Definition 5.1.1. Work in a possibly infinite graph $\mathbb{G}$. Let $A, B, S, V \subseteq G$ where G is the set of vertices of G.

1. A path is a subgraph of $\mathbb{G}$ with distinct vertices $v_0, v_1, \ldots, v_n$ and an edge between v_{i-1}, v_i for all $i = 1, \ldots n$. It is called a path from A to B if there is an edge between v_0 and a vertex in A and there is an edge between v_n and a vertex in B. A length of such a path is $n + 2$.

2. For $a, b \in V(G)$ define the distance $d(a, b)$ to be the length of the shortest path between a and b in G.

3. For $a, b \in V(G)$ define $d_A(a, b)$ to be the distance between a and b in $G[V(G) - A]$. Equivalently it is the shortest path between a and b that avoids vertices in A.

4. We say that S separates A from B if there exist $a \in A - S$, $b \in B - S$, with $d_S(a, b) = \infty$.

5. We say that A separates V if it separates V from itself.

6. We say that V has connectivity n if there is a set of size n that separates V, but there are no sets of size $n-1$ that separate V.

7. Suppose V has connectivity n. A connectivity hull of V is defined to be the union of all sets separating V of size n.

In [AB09] we find a generalization of Megner's Theorem for infinite graphs:

Theorem 5.1.2. *Let A and B be two sets of vertices in a possibly infinite graph. Then there exists a set P of disjoint paths from A to B, and a set S of vertices separating A from B, such that S consists of a choice of precisely one vertex from each path in P.*

We use the following easy consequence:

Corollary 5.1.3. *Let V be a subset of vertices of a graph $\mathbb{G}$ with connectivity n. Then there exists a set of n disjoint paths from V into itself.*

Corollary 5.1.4. *With assumptions as above, the connectivity hull of V is finite.*

Proof. All the separating sets have to have exactly one vertex in each of those paths. $\square$

Definition 5.1.5.

- A graph K_n^m denotes a graph obtained from a complete graph on n vertices with m vertices added to every edge.

- A graph is called superflat if for every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that the graph avoids K_n^m as a subgraph.

Theorem 2 in [PZ78] gives a useful characterization of the superflat graphs.

Theorem 5.1.6. *The following are equivalent:*

1. *$\mathbb{G}$ is superflat.*

2. *For every $n \in \mathbb{N}$ and an infinite set $A \subseteq G$, there exists a finite $B \subseteq G$ and infinite $A' \subseteq A$ such that for all $x, y \in A'$ we have $d_B(x, y) > n$.*

Roughly, in superflat graphs every infinite set contains a sparse infinite subset (possibly after throwing away finitely many nodes).

5.2 Indiscernible sequences

Fix an uncoutable cardinal κ. Work in a superflat graph $\mathbb{S}$ that is κ-saturated and strongly κ-homogeneous. Fix a parameter set $A \subset S$ with $|A| < \kappa$. Let $I = (a_i)_{i \in \mathcal{I}}$ be a countable A-indiscernible sequence.Stability implies that I is totally indiscernible (see Lemma 1.2.8).

Definition 5.2.1. We call a set $H \subseteq S$ uniformly definable from I if there is a formula $\phi(x, y_1, \ldots, y_k)$ such that for every $J \subseteq I$ of size k we have

$$H = \{g \in G \mid \phi(g, J)\},$$

where J is considered a tuple.

First suppose that I consists of singletons, that is $a_i \in S$.

Definition 5.2.2. Let $V \subseteq S$. Define $P_n(V)$, a subgraph of $\mathbb{S}$, to be a union of all paths of length $\leq n$ between the vertices of V.

Lemma 5.2.3. *Fix $n \in \mathbb{N}$. Then there exists a finite set B such that*

$$\forall i \neq j \ d_B(a_i, a_j) > n.$$

Proof. By a 5.1.6 we can find an infinite $\mathcal{J} \subseteq \mathcal{I}$ and a finite set B' such that each pair from $J = (a_j : j \in \mathcal{J})$ has distance $> n$ over B'. Using total indiscernibility we have an automorphism sending J to I fixing A. Image of B' under this automorphism is the required set B. $\square$

In other words, B separates I when viewed inside the subgraph $P_n(I)$. This shows that I has finite connectivity in $P_n(I)$. Applying Corollary 5.1.4 we obtain that the connectivity hull of I in $P_n(I)$ is finite.

Definition 5.2.4. Given a graph $\mathbb{G}$ and $V \subseteq G$ define $H(\mathbb{G}, V) \subseteq G$ to be the connectivity hull of V in $\mathbb{G}$. Note that if V is finite, we have that $H(P_n(V), V)$ is V-definable.

Lemma 5.2.5. *Let H be the connectivity hull of I inside of graph $P_n(I)$, that is $H = H(\P_n(I), I)$. Then H is uniformly definable from I in $\mathbb{S}$.*

Proof. Using total indiscernability we may assume without the loss of generality that I is indexed by $\mathbb{N}$. Let $I_i = \{a_0, a_1, \ldots, a_{i-1}\}$ a finite segment of the sequence. Let N be the connectivity of I inside of $P_n(I)$.

First note that any finite set $H \subseteq P_n(I)$ will be contained in $P_n(I_i)$ for large enough i. Every element of H is inside a path of length $\leq n$ and endpoints of that path are eventually going to be inside I_i. Here the assumption that I is enumerated by $\mathbb{N}$ is important.

Vertices a_0, a_1 cannot be separated by less than N elements inside of $P_n(I)$. By Theorem 5.1.2 there are N disjoint paths inside of $P_n(I)$ connecting a_0 to a_1. For large enough i, say $i \geq M_1$, all these paths are contained inside of $P_n(I_i)$. Those paths witness that vertices a_0, a_1 cannot be separated by less than N elements inside of $P_n(I_i)$. Thus by indiscernibility, no two vertices can be separated by less than N elements inside of $P_n(I_i)$ (using the fact that $P_n(I_i)$ is I_i-definable). Thus I_i has connectivity $\geq N$ inside of $P_n(I_i)$ for $i \geq M_1$.

Consider a set S of size N that separates I inside of $P_n(I)$. This is witnessed by two elements of I that are separated. There are finitely many such sets S as connectivity hull is finite. Thus for large enough i, say $i \geq M_2$, for each such S the segment I_i contains a pair of vertices witnessing that S is a separating set.

The Corollary 5.1.3 tells us that there are finitely many paths between elements of V such that $H(P_n(I), I)$ is inside the union of those paths. For large enough i, say $i \geq M_3$, $P_n(I_i)$ will contain all of those paths, and thus $H(P_n(I), I) \subseteq P_n(I_i)$.

Combine those three observations. Let $M = \max(M_1, M_2, M_3)$. Then for $i \geq M$ the set $P_n(I_i)$ contains all the N-element sets separating I in $P_n(I)$, those sets separate I_i in $P_n(I_i)$, and the connectity of I_i in $P_n(I_i)$ is at most N. But this means that the connectity of I_i in $P_n(I_i)$ has to be exactly N, and $H(P_n(I), I) \subseteq H(P_n(I_i), I_i)$.

For $i \geq M$ define

$$E_i = \bigcap_{j=M}^{i} H(P_n(I_j), I_j).$$

We have $H(P_n(I), I) \subseteq E_i$ and E_i is a decreasing chain. Suppose $H(P_n(I), I) \subsetneq H(P_n(I_M), I_M)$, that is $H(P_n(I_M), I_M) - H(P_n(I), I) \neq \emptyset$. Then there exists a set S of size N that separates

I_M in $P_n(I_M)$ but does not separate I in $P_n(I)$. There is a finite subgraph of $P_n(I)$ disjoint from S that connects all the elements of I_M. For large enough i, say $i \geq M_S$, this subgraph is contained in $P_n(I_i)$. There are finite many possibilities for S (as connectivity hull of I_M in $P_n(I_M)$ is finite). Let $M_4 = \max_S(M_S)$. Then for $i \geq M_4$ we have

$$H(P_n(I_i), I_i) \cap (H(P_n(I_M), I_M) - H(P_n(I), I)) = \emptyset,$$

and thus $E_i = (P_n(I), I)$. As E_i is I_i-definable, we have that $(P_n(I), I)$ is I_i-definable. We need to show uniform definability. Suppose I' is a subsequence of I of length i. There is an automorphism mapping I_i to I' that is I-invariant. Thus it is $H(P_n(I), I)$-invariant and maps I_i-definition of $H(P_n(I), I)$ to a I'-definition of $H(P_n(I), I)$. As I' was arbitrary this shows uniformity. □

Corollary 5.2.6. *Let $H_n = H(P_n(I), I)$. Then*

$$\forall i \neq j \; d_{H_n}(a_i, a_j) > n.$$

Proof. The set H_n separates I inside of $P_n(I)$. In particular there exist $i \neq j$ such that $d_{H_n}(a_i, a_j) = \infty$ inside $P_n(I)$. This means that $d_{H_n}(a_i, a_j) > n$ inside of $\mathbb{S}$. But then by total indiscernibility and using the fact that H_n is uniformly I-definable, this holds for all $i \neq j$. □

We would like to start working with tuples now, instead of singletons. We need some notation to extract individual elements of a tuple:

Definition 5.2.7. Suppose $a = (a_1, \ldots, a_m)$ is a tuple of arity m. Let $a^{(j)}$ denote the i'th component, that is $a^{(j)} = a_j$.

More generally, now suppose that I consists of tuples of arity m, that is $a_i \in S^m$.

Definition 5.2.8.

- We would like extract j'th components out of elements of I. Let $I^{(j)} = ({a_i}^{(j)})_{i \in \mathcal{I}}$, an A-indiscernible sequence of singletons.
- Let ${H_n}^{(j)} = H(P_n(I^{(j)}), I^{(j)})$.

- Let

$$B_n = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} {H_n}^{(j)}.$$

Lemma 5.2.9. *The sequence I is indiscernible over the $A \cup B_n$.*

Proof. By Lemma 5.2.5 the set ${H_n}^{(j)}$ is uniformly $I^{(j)}$-definable. Thus it is uniformly I-definable. Then B_n is a finite union of uniformly I-definable sets, thus also uniformly I-definable.

For a subsequence $\mathcal{J} \subset \mathcal{I}$ let $a(\mathcal{J})$ denote the tuple obtained by concatenating $(a_j)_{j \in \mathcal{J}}$. By uniform definability there is a formula $\phi(z, w_1, \ldots, w_k)$ with $|z| = 1$ and $|w_i| = m$ such that for any subsequence $\mathcal{J} \subset \mathcal{I}$ of length k we have $\phi(G, a(\mathcal{J})) = B_n$. Fix such a subsequence $\mathcal{J}$.

Let $\psi(x_1, \ldots, x_l, y)$ be an arbitrary A-formula with $|x_i| = m$. Consider the collection of traces $\psi(a(\mathcal{J}'), B_n^{|y|})$ for subsequences $\mathcal{J}' \subset \mathcal{I}$ of length l and disjoint from $\mathcal{J}$. If two of the traces are distinct, then by indiscernibility all of them are. But that is impossible as B_n is finite and thus has finitely many subsets. Thus all such traces are identical. As the choice of $\mathcal{J}$ was arbitrary, we can drop the condition that $\mathcal{J}'$ is disjoint from $\mathcal{J}$. This shows that for any $\mathcal{J}_1, \mathcal{J}_2 \subset \mathcal{I}$ of length l and $h \in B_n^{|y|}$ we have $\psi(a(\mathcal{J}_1), h) \iff \psi(a(\mathcal{J}_2), h)$. As choice of ψ was arbitrary, this shows that I is indiscernible over $A \cup B_n$ as needed. □

Definition 5.2.10. For tuples a, b of the same arity m and $B \subset S$ define

$$d_B(a, b) = \min_{1 \leq i,j \leq m} d_B(a^{(i)}, b^{(j)}).$$

Lemma 5.2.11.

$$\forall i \neq j \ d_{B_n}(a_i, a_j) > n/2.$$

Proof. Towards a contradiction suppose we have some $i \neq j$ and k, l such that

$$d_{B_n}({a_i}^{(k)}, {a_j}^{(l)}) \geq n/2.$$

As B_n is uniformly I-definable, by total indiscernability we have that this inequality holds for all $i \neq j$. Let $b_1 = {a_1}^{(k)}$, $b_2 = {a_2}^{(l)}$, $b_3 = {a_3}^{(k)}$ (again, assuming for convenience that I is enumerated by naturals). Then we have

$$d_{B_n}(b_1, b_2) \geq n/2,$$
$$d_{B_n}(b_3, b_2) \geq n/2.$$

By triangle inequality

$$d_{B_n}(b_1, b_3) \geq n,$$
$$d_{B_n}({a_1}^{(k)}, {a_3}^{(k)}) \geq n.$$

But this is a contradiction, as Lemma 5.2.12 gives us

$$\forall i \neq j \; d_{{H_n}^{(k)}}({a_i}^{(k)}, {a_j}^{(k)}) > n$$

and we have ${H_n}^{(k)} \subseteq B_n$. □

Corollary 5.2.12. *There is a countable B such that I is indiscernible over $A \cup B$ and*

$$\forall i \neq j \; d_B(a_i, a_j) = \infty.$$

Proof. Let B_n as above. By Lemma 5.2.11 we have

$$\forall i \neq j \; d_{B_n}(a_i, a_j) > n,$$

and I is indiscernible over $A \cup B_n$ by Lemma 5.2.9. Let $B = \bigcup_{n \in \mathbb{N}} B_n$. Then

$$\forall i \neq j \; d_B(a_i, a_j) = \infty.$$

As $B_i \subseteq B_{i+1}$, the sequence I is indiscernible over $A \cup B$ as needed. □

That is I can be upgraded to have infinite distance over its parameter set.

5.3 Superflat graphs are dp-minimal

Definition 5.3.1. For $B \subseteq S$ define an equivalence relation $\sim_B$ on $S - B$. Two vertices b, c are $\sim_B$-equivalent if $d_B(b, c)$ is finite.

Lemma 5.3.2. *Fix tuples a, b, c in S, with a, b having the same arity. Also let $B \subseteq S$. Suppose* $\mathrm{tp}(a/B) = \mathrm{tp}(b/B)$ *and $d_B(a, c) = d_B(b, c) = \infty$. Then* $\mathrm{tp}(a/Bc) = \mathrm{tp}(b/Bc)$.

Proof. Suppose $a = (a_1, a_2, \dots, a_m)$ and $b = (b_1, b_2, \dots, b_m)$. Define X_j to be the $\sim_B$-equivalence class of a_j or $\emptyset$ if $a_j \in B$. Similarly define Y_j for b_j. There is an automorphism f of $\mathbb{S}$ fixing B mapping a to b. It's easy to see that $f(X_j) = Y_j$. We would like to define a function $g\colon S \longrightarrow S$. For each j let $g = f$ on X_j. Additionally if $X_j \neq Y_j$ then also let $g = f^{-1}$ on Y_j. Let g be identity on the rest of S. It is easy to check that g is a well-defined automorphism fixing Bc that maps a to b. This shows that $\mathrm{tp}(a/Bc) = \mathrm{tp}(b/Bc)$. ☐

Lemma 5.3.3. *Suppose $c \in G$. Then there exists $c \in \mathcal{I}$ such that all $(a_i)_{i \in \{\mathcal{I} - c\}}$ have the same type over Ab.*

Proof. Use Corollary 5.2.12 to find $B \supseteq A$ such that I is indiscernible over B and has infinite distance over B. All the tuples of the indiscernible sequence fall into distinct $\sim_B$-equivalence classes. If $b \in B$ we are done. Otherwise, there can be at most one element of the sequence that is in the same $\sim_B$-equivalence class as b. Exclude that element from the sequence. Remaining sequence elements are all infinitely far away from b over B. By the previous lemma we have that elements of indiscernible sequence all have the same type over Bb as needed. ☐

Corollary 5.3.4. *Flat graphs are dp-minimal.*

Proof. Using Lemma 1.3.16, by total indiscernibility it is enough to show that if I is a countable sequence indiscernible over $\emptyset$ and $b \in S$, then one element can be excluded from I, so that the remaining elements have the same type over b. But this is precisely Lemma 5.3.3. ☐

5.4 Conclusion

The determination of dp-minimality is the first step towards establishing bounds on vc-density. It is this author's hope that the simple structure of flat graphs yields nicely behaved vc-density. We pose the following question for the future work:

Open Question: What are bounds on vc-function $\mathrm{vc}(n)$ in flat graphs? In particular do we have $\mathrm{vc}(1) = 1$ or $\mathrm{vc}(n) = n\,\mathrm{vc}(1)$? Are the bounds better in specific classes of flat graphs, such as planar graphs, graphs with bounded tree-width, or graphs excluding certain types of subgraphs?

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