Notes on Surreal Numbers
Math 285: Fall 2014

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We define a map which will eventually be proven to be an ordered field isomorphism.

$$K = \mathbb{R}((t^{\mathbf{No}})) \xrightarrow{\sim} \mathbf{No}$$

We have an element written as

$$f = \sum_{\gamma \in \mathbf{No}} f_\gamma t^\gamma$$
$$\operatorname{supp}(f) = \{\gamma \colon f_\gamma \neq 0\}$$

where $\operatorname{supp}(f)$ is a well-ordered sub**set**. Now let $x = t^{-1}$ and write

$$f(x) = \sum_{i<\alpha} f_i x^{a_i}$$

where $(a_i)_{i<\alpha}$ is strictly decreasing in **No**, α ordinal and $f_i \in \mathbb{R}$ for $i < \alpha$. Also define $l(f(x))$ to be the order type of $\operatorname{supp}(f)$ (which may be smaller than α as we allow zero coefficients).

Question 1. *What is the relationship of what we are going to do with Kaplansky's results from valuation theory?*

For $f(x) = \sum_{i<\alpha} f_i x^{a_i}$ define $\sum_{i<\alpha} f_i \omega^{a_i} = f(\omega)$ recursively on α:
When $\alpha = \beta + 1$ is a successor:

$$\sum_{i<\alpha} f_i \omega^{a_i} = \left(\sum_{i<\beta} f_i \omega^{a_i} \right) + f_\beta \omega^{a_\beta}$$

When α is a limit ordinal:

$$\sum_{i<\alpha} f_i \omega^{a_i} = \{L \mid R\}$$
$$L = \left\{ \sum_{i<\beta} f_i \omega^{a_i} + (f_\beta - \epsilon)\omega^{a_\beta} : \beta < \alpha, \epsilon \in \mathbb{R}^{>0} \right\}$$
$$R = \left\{ \sum_{i<\beta} f_i \omega^{a_i} + (f_\beta + \epsilon)\omega^{a_\beta} : \beta < \alpha, \epsilon \in \mathbb{R}^{>0} \right\}$$

Simultaneously with this definition we prove the following statements by induction:

1. For

$$f(x) = \sum_{i<\alpha} f_i x^{a_i}$$
$$g(x) = \sum_{i<\alpha} g_i x^{a_i}$$

we have $f(x) > g(x) \Rightarrow f(\omega) > g(\omega)$

Tail property if $\gamma < \kappa < \alpha$

$$\left| \sum_{i<\alpha} f_i \omega^{a_i} - \sum_{i<\kappa} f_i \omega^{a_i} \right| << \omega^{a_\gamma}$$

Suppose we have

$$f(x) = \sum_{i<\alpha} f_i x^{a_i}$$
$$g(x) = \sum_{i<\alpha} g_i x^{a_i}$$

with $f(x) < g(x)$
Choose γ smallest such that $f_\gamma \neq g_\gamma$. It has to be that $f_\gamma > g_\gamma$. Also $f(x) \restriction_\gamma = g(x) \restriction_\gamma$
Case 1: $\alpha = \beta + 1$

$$f(x) = f(x) \restriction_\beta + f_\beta x^{a_\beta}$$
$$g(x) = g(x) \restriction_\beta + g_\beta x^{a_\beta}$$

Suppose $\gamma = \beta$. Then $\bar{f}(x) = \bar{g}(x)$, $\bar{f}(\omega) = \bar{g}(\omega)$, so compute

$$\begin{aligned} f(\omega) - g(\omega) = & \\ &= f(\omega) \restriction_\beta + f_\beta \omega^{a_\beta} - g(\omega) \restriction_\beta - g_\beta \omega^{a_\beta} \\ &= f_\beta \omega^{a_\beta} - g_\beta \omega^{a_\beta} \\ &= (f_\beta - g_\beta)\, \omega^{a_\beta} > 0 \end{aligned}$$

Now suppose $\gamma < \beta$.
Group the terms

$$f(\omega) = h(\omega) + f_\gamma \omega^{a_\gamma} + f^* + f_\beta \omega^{a_\beta}$$
$$g(\omega) = h(\omega) + g_\gamma \omega^{a_\gamma} + g^* + g_\beta \omega^{a_\beta}$$

where

$$h(\omega) = f(\omega) \restriction_\gamma = g(\omega) \restriction_\gamma$$
$$f^* = f(\omega) \restriction_\beta - f(\omega) \restriction_{\gamma+1}$$
$$g^* = g(\omega) \restriction_\beta - g(\omega) \restriction_{\gamma+1}$$

Then we have by tail property $f^* << x^{a_\gamma}$ and $g^* << x^{a_\gamma}$. Compute

$$f(\omega) - g(\omega) = (f_\gamma - g_\gamma) x^{a_\gamma} + (f * - g *) + (f_\beta - g_\beta) x^{a_\beta}$$

We have $f_\gamma > g_\gamma$. All $f*$, $g*$ and $(f_\beta - g_\beta)x^{a_\beta}$ are $<< x^{a_\gamma}$. Thus $f(\omega) - g(\omega) > 0$ as needed.

Case 2: α is a limit ordinal.

$f(\omega)$ and $g(\omega)$ are defined as

$$f(\omega) = \{L_f \mid R_f\}$$
$$g(\omega) = \{L_g \mid R_g\}$$

Recall that

$$L_f = \left\{ \sum_{i<\beta} f_i \omega^{a_i} + (f_\beta - \epsilon)\omega^{a_\beta} : \beta < \alpha, \epsilon \in \mathbb{R}^{>0} \right\}$$
$$R_g = \left\{ \sum_{i<\beta} g_i \omega^{a_i} + (g_\beta + \epsilon)\omega^{a_\beta} : \beta < \alpha, \epsilon \in \mathbb{R}^{>0} \right\}$$

Pick any β with $\gamma < \beta < \alpha$ and $\epsilon \in \mathbb{R}^{>0}$, and pick limit elements $\bar{f}(\omega) \in L_f$ and $\bar{g}(\omega) \in R_g$ corresponding to β, ϵ.

Then $\bar{f}(x) < \bar{g}(x)$ as first coefficient where they differ is x^{a_γ} and $f_\gamma > g_\gamma$. Thus by inductive hypothesis $\bar{f}(\omega) < \bar{g}(\omega)$. As choice of those was arbitrary we have $L_f < R_g$ so $f(\omega) > g(\omega)$.

Tail property

It is easy to see that statement holds for all $\gamma < \kappa < \alpha$ iff it holds for all $\gamma < \kappa \le \alpha$.

Case 1: $\alpha = \beta + 1$.

Suppose we have $\gamma < \kappa < \alpha$, then $\gamma < \kappa \le \beta$ and induction hypothesis applies.

$$\sum_{i<\alpha} f_i \omega^{a_i} - \sum_{i<\kappa} f_i \omega^{a_i} =$$
$$\left[\sum_{i<\beta} f_i \omega^{a_i} - \sum_{i<\kappa} f_i \omega^{a_i} \right] + f_\alpha \omega^{a_\alpha}$$

Expression $[\ldots]$ is $<< \omega^{a_\gamma}$ by induction hypothesis. $f_\alpha \omega^{a_\alpha} << \omega^{a_\gamma}$ as $a_\alpha < a_\gamma$. Thus the entire sum is $<< \omega^{a_\gamma}$ as needed.

Case 2: α is a limit ordinal.

Write definitions of $f(\omega)$ using κ

$$f(\omega) = \{L_f \mid R_f\}$$
$$F(\omega) = f(\omega) \restriction_\kappa = \sum_{i<\kappa} f_i \omega^{a_i}$$

$$L_f = \left\{ \sum_{i<\beta} f_i \omega^{a_i} + (f_\beta - \epsilon)\omega^{a_\beta} : \beta < \alpha, \epsilon \in \mathbb{R}^{>0} \right\}$$
$$R_f = \left\{ \sum_{i<\beta} f_i \omega^{a_i} + (f_\beta + \epsilon)\omega^{a_\beta} : \beta < \alpha, \epsilon \in \mathbb{R}^{>0} \right\}$$

Pick any β with $\kappa < \beta < \alpha$ and $\epsilon \in \mathbb{R}^{>0}$, and pick limit elements $\bar{l}(\omega) \in L_f$ and $\bar{r}(\omega) \in R_f$ corresponding to β, ϵ.

By induction hypothesis we have

$$\bar{l}(\omega) - F(\omega) = \bar{l}(\omega) - \bar{l}(\omega) \restriction_\kappa << \omega^{a_\kappa}$$
$$\bar{r}(\omega) - F(\omega) = \bar{r}(\omega) - \bar{r}(\omega) \restriction_\kappa << \omega^{a_\kappa}$$

$$l(\omega) \leq f(\omega) \leq r(\omega)$$

$$l(\omega) - F(\omega) \leq f(\omega) - F(\omega) \leq r(\omega) - F(\omega)$$

Thus $f(\omega) - F(\omega) << \omega^{a_\kappa}$ as it is between two elements that are $<< \omega^{a_\kappa}$.

We also need to check that the function is well-defined. For $f(x)$ define its reduced form, where we only keep non-zero coefficients.

$$f(x) = \sum_{i<\alpha} f_i \omega^{a_i}$$
$$\bar{f}(x) = \sum_{j<\alpha'} f'_j \omega^{a'_j}$$