QUANTIFIER ELIMINATION IN SHELAH-SPENCER GRAPHS

ANTON BOBKOV

Abstract. We simplify [?] proof of quantifier elimination in Shelah-Spencer graphs.

1. Introduction

Laskowski's paper [?] provides a combinatorial proof of quantifier elimination in Shelah-Spencer graphs. Here we provide a simplification of the proof using only maximal chains and avoiding technical lemmas of sections 3 and 4.

We will use notation of [?], in particular things like $\boldsymbol{K}_\alpha$, $\delta(\mathcal{A}/\mathcal{B})$, $X_m(\mathcal{A})$, S_α, maximal embedding, etc.

2. Omitting lemma

Definition 2.1. Fix $\mathcal{B} \in \boldsymbol{K}_\alpha$, Φ, Γ finite subsets of $\boldsymbol{K}_\alpha$, and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|C \backslash B| < m$. Define $Z(\mathcal{B}, \Phi, \Gamma, m)$ to be all $\mathcal{B}^* \in X_m(\mathcal{B})$ such that

(1) For every $\mathcal{C} \in \Phi$ there *are no* $\mathcal{H}$ with $|H \backslash B^*| < m$ satisfying

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, dash, "\leq"'] \arrow[dr, dash] & \\ \mathcal{B}^* \arrow[dr, dash] & & \mathcal{C} \arrow[dl, dash] \\ & \mathcal{B} & \end{tikzcd}$$

(2) For every $\mathcal{D} \in \Gamma$ there *is some* $\mathcal{G}$ with $|G \backslash B^*| < m$ satisfying

$$\begin{tikzcd} & \mathcal{G} \arrow[dl, dash, "\leq"'] \arrow[dr, dash] & \\ \mathcal{B}^* \arrow[dr, dash] & & \mathcal{D} \arrow[dl, dash] \\ & \mathcal{B} & \end{tikzcd}$$

Definition 2.2. Let $\mathcal{M} \models S_\alpha$, $\mathcal{B} \in \boldsymbol{K}_\alpha$, embedding $f\colon \mathcal{B} \to \mathcal{M}$, Φ finite subset of $\boldsymbol{K}_\alpha$

(1) Say that f *omits* Φ if there are no $\mathcal{C} \in \Phi$ and $g\colon \mathcal{C} \to \mathcal{M}$ extending f.
(2) Say that f *admits* Φ if for every $\mathcal{C} \in \Phi$ there is $g\colon \mathcal{C} \to \mathcal{M}$ extending f.

Lemma 2.3. *Let $\mathcal{B} \in \boldsymbol{K}_\alpha$, Φ, Γ finite subsets of $\boldsymbol{K}_\alpha$, and $m \in \omega$ such that for each $\mathcal{C} \in \Phi$ or $\mathcal{C} \in \Gamma$ we have $\mathcal{B} \subseteq \mathcal{C}$ and $|C \backslash B| < m$. The following are equivalent:*

(1) *f omits Φ and admits Γ.*
(2) *There exists $\mathcal{B}^* \in Z(\mathcal{B}, \Phi, \Gamma, m)$ maximally embeddable into $\mathcal{M}$ over f.*

Proof. (1) $\Rightarrow$ (2) Identify $\mathcal{B}$ with $f(\mathcal{B})$, i.e. for ease of notation assume that $\mathcal{B} \subset \mathcal{M}$. By remark 5.3 of [**?**] there is some $B^* \in X_m(\mathcal{B})$ maximally embeddable in $\mathcal{M}$ over f. Such embedding is unique by Lemma 3.8 of [**?**]. Again, we identify B^* with its maximal embedding into $\mathcal{M}$. To show (2) we need to verify that $\mathcal{B}^* \in Z(\mathcal{B}, \Phi, \Gamma, m)$. Suppose not. Two things can go wrong. First, there can be $\mathcal{H}$ with $|H \backslash B^*| < m$ and $\mathcal{C} \in \Phi$ satisfying

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, dash, "\leq"'] \arrow[dr, dash] & \\ \mathcal{B}^* \arrow[dr, dash] & & \mathcal{C} \arrow[dl, dash] \\ & \mathcal{B} & \end{tikzcd}$$

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References

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E-mail address: bobkov@math.ucla.edu