Algorithms: Seminar 2

Solving recurrence relations

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Homogeneous linear recurrences with const. coefficients (1)

- **Homogeneous linear recurrences with constant coeficients:**
  Equations of the form:

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = 0 \tag{1}$$

  t_i are the *values of the recurrence*.
  In addition, k values of t_i: the *initial conditions*
  $\rightarrow$ allow to go from infinite solutions to a single solution.

- **Example: Fibonacci**

$$f_n - f_{n-1} - f_{n-2} = 0$$

$$\begin{cases} k = 2 \\ a_0 = 1 \\ a_1 = a_2 = -1 \end{cases} \qquad \begin{cases} f_0 = 0 \\ f_1 = 1 \end{cases}$$

Homogeneous linear recurrences with const. coefficients (2)

- **Rule:** *every linear combination of solutions is a solution*
  If $\sum_{i=0}^{k} a_i f_{n-i} = 0$ and $\sum_{i=0}^{k} b_i g_{n-i} = 0$
  i.e. if f_n satisfies (1), and g_n also satisfies (1),
  and $t_n = cf_n + dg_n$ (where c and d are arbitrary constants)
  then t_n is also a solution of (1).
  $\rightarrow$ This is generalised to any number of solutions.
- **Look for a solution of the form $t_n = x^n$?**
  (1): $a_0 x^n + a_1 x^{n-1} + \cdots + a_k x^{n-k} = 0$ (Trivial solution: x=0)
  $\Leftrightarrow \underbrace{a_0 x^k + a_1 x^{k-1} + \cdots + a_k}_{p(x)} = 0$
  is the *characteristic equation* of (1);
  $p(x)$ is the *characteristic polynomial* of (1).
  $\rightarrow$ Characteristic equation technique

- **Characteristic equation technique:**
  Every polynomial of degree k has k roots: $p(x) = \prod_{i=1}^{k}(x - r_i)$
  r_i: may be complex, unique solutions of $p(x) = 0$, i.e. $p(r_i) = 0$.
  $x = r_i$ is solution of the characteristic equation;
  $\rightarrow r_i^n$ is solution of the recurrence relation (1).

$$t_n = \sum_{i=1}^{k} c_i r_i^n \tag{2}$$

(2) *satisfies the recurrence* (1) with c_i: suitable constants;
it is the *most general solution*.
(1) only has solutions of this form, as long as all the r_i are different.
The k constants are determined from the k initial conditions, solving a system of k linear equations with k unknowns.

- **Example: Fibonacci (Cont.)**

$$f_n - f_{n-1} - f_{n-2} = 0$$
$$p(x) = x^2 - x - 1 \Rightarrow r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$$
$$\Rightarrow f_n = c_1 r_1^n + c_2 r_2^n$$

Initial conditions:

$$n = 0, f_0 = 0 \wedge f_0 = c_1 + c_2$$
$$n = 1, f_1 = 1 \wedge f_1 = c_1 r_1 + c_2 r_2$$

System of equations to solve: $\begin{cases} c_1 + c_2 = 0 \\ c_1 r_1 + c_2 r_2 = 1 \end{cases}$

$$\Rightarrow c_1 = \frac{1}{\sqrt{5}}, c_2 = -\frac{1}{\sqrt{5}}$$
$$\Rightarrow f_n = \frac{1}{\sqrt{5}}[(\underbrace{\frac{1+\sqrt{5}}{2}}_{\phi})^n - (\frac{1-\sqrt{5}}{2})^n]$$

More precise than just saying

$f_n = O(\phi^n)$

Homogeneous linear recurrences with const. coefficients (5)

- **Problem**: the k roots are not all different from each other?
  $\rightarrow$ (2) is still solution, but no longer the most general.
  Other solutions?
  $\rightarrow$ Let r be a multiple root (of multiplicity 2):
  by definition, there exists a polynomial q(x), of degree k-2, such that

$$p(x) = (x-r)^2 q(x)$$

For all $n \geq k$, let us consider:

$$u_n(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_k x^{n-k}$$

$$v_n(x) = a_0 n x^n + a_1 (n-1) x^{n-1} + \cdots + a_k (n-k) x^{n-k}$$

$$\rightarrow v_n(x) = x u_n'(x) \qquad (*)$$

$$u_n(x) = x^{n-k} p(x) = x^{n-k}(x-r)^2 q(x) = (x-r)^2 [x^{n-k} q(x)]$$

$$u_n'(x) = 2(x-r) x^{n-k} q(x) + (x-r)^2 [x^{n-k} q(x)]'$$

$$\Rightarrow u_n'(r) = 0$$

$$(*) \Rightarrow v_n(r) = x u_n'(r) = 0, \qquad \forall n \geq k$$

$$\Leftrightarrow a_0 n r^n + a_1 (n-1) r^{n-1} + \cdots + a_k (n-k) r^{n-k} = 0$$

$\Rightarrow t_n = n r^n$ **is also a (new) solution of (1)**

- More generally, if r has multiplicity m:

$$t_n = r^n, t_n = nr^n, t_n = n^2 r^n, \ldots, t_n = n^{m-1} r^n$$

are solutions of (1).

- General solution: linear combination
  $\rightarrow r_1, r_2, \ldots, r_l$: distinct roots of $p(x)$,
  with multiplicities $m_1, m_2, \ldots, m_l \Rightarrow$

$$t_n = \sum_{i=1}^{l} \sum_{j=0}^{m_i - 1} c_{ij} n^j r_i^n \qquad (3)$$

(3) is the general solution of (1).

- The c_{ij}'s are determined from the initial conditions:
  $\rightarrow$ they are k constants, $c_1, c_2, \ldots, c_k$.
  $\sum_{i=1}^{l} m_i = k$: sum of multiplicities, total number of roots.

- **Example:**

$$t_n = \begin{cases} n & n = 0, 1, 2 \\ 5t_{n-1} - 8t_{n-2} + 4t_{n-3} & \textit{otherwise} \end{cases}$$

With the form of equation (1):

$$t_n - 5t_{n-1} + 8t_{n-2} - 4t_{n-3} = 0$$

$$p(x) = x^3 - 5x^2 + 8x - 4 = (x-1)(x-2)^2$$

$$\begin{cases} r_1 = 1 & m_1 = 1 \\ r_2 = 2 & m_2 = 2 \end{cases}$$

$$(3) \Rightarrow t_n = c_1 1^n + c_2 2^n + c_3 n 2^n$$

$$\left.\begin{array}{lll} n = 0: & c_1 + c_2 & = 0 \\ n = 1: & c_1 + 2c_2 + 2c_3 & = 1 \\ n = 2: & c_1 + 4c_2 + 8c_3 & = 2 \end{array}\right\} \begin{array}{l} c_1 = -2 \\ c_2 = 2 \\ c_3 = -\frac{1}{2} \end{array}$$

$$\Rightarrow t_n = 2^{n+1} - n2^{n-1} - 2$$

- **Non-homogeneous recurrences:** The linear combination is no longer = 0
  $\rightarrow$ a linear combination of solutions is no longer a solution.
  We initially consider recurrences of the form:
  $$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = b^n p(n) \qquad (4)$$
  where b is a constant and p(n) a polynomial of degree d.
- **Example 1:** $t_n - 2t_{n-1} = 3^n$ $\quad$ $^*p(n) = 1$: pol. of degree 0
  Reduce to the homogeneous case:
  Multiply by 3: $\quad \rightarrow 3t_n - 6t_{n-1} = 3^{n+1}$
  Replace n with n-1: $\quad \rightarrow 3t_{n-1} - 6t_{n-2} = 3^n$
  Difference between 2 equations:
  $$\begin{array}{ccccc} & t_n & -2t_{n-1} & & = & 3^n \\ - & & 3t_{n-1} & -6t_{n-2} & = & 3^n \\ \hline & t_n & -5t_{n-1} & +6t_{n-2} & = & 0 \end{array}$$
  $$\Rightarrow \underbrace{x^2 - 5x + 6}_{p(x)} = (x-2)(x-3)$$

- **Example 1:** (Cont'd)
  $\Rightarrow$ The solutions are of the form: $t_n = c_1 2^n + c_2 3^n$
  And as $t_n \geq 0 \quad \forall n \geq 0$, we deduce that $t_n = O(3^n)$
  But c_1 and c_2 are no longer determined from initial conditions:
  $t_n = 2^n$ and $t_n = 3^n$ are **not** solutions of the original recurrence.
  $\rightarrow t_n - 2t_{n-1} = 3^n \Rightarrow t_1 = 2t_0 + 3 \Rightarrow$
  $$\begin{cases} c_1 + c_2 & = & t_0 & n = 0 \\ 2c_1 + 3c_2 & = & 2t_0 + 3 & n = 1 \end{cases}$$
  $$\Rightarrow \begin{cases} c_1 = t_0 - 3 \\ c_2 = 3 \end{cases} \Rightarrow t_n = (t_0 - 3)2^n + 3^{n+1} = \theta(3^n)$$
  (General solution, regardless of the initial conditions).
  Another reasoning:
  $$\begin{aligned} 3^n &= t_n - 2t_{n-1} \\ &= (c_1 2^n + c_2 3^n) - 2(c_1 2^{n-1} + c_2 3^{n-1}) \\ &= c_2 3^{n-1} \end{aligned}$$
  $\Rightarrow c_2 = 3$, indep. of t_0, i.e. we discard that $c_2 = 0$.
  $\Rightarrow t_n = \theta(3^n)$

- **Example 2:**

$$
\begin{array}{lcccccl}
 & t_n & -2t_{n-1} & & & = & (\overbrace{n+5}^{d=1})3^n \\
\star(-6), n-1: & & -6t_{n-1} & +12t_{n-2} & & = & -6(n+4)3^{n-1} \\
\star 9, n-2: & & & 9t_{n-2} & -18t_{n-3} & = & 9(n+3)3^{n-2} \\
\hline
 & t_n & -8t_{n-1} & +21t_{n-2} & -18t_{n-3} & = & 0
\end{array}
$$

$p(x): x^3 - 8x^2 + 21x - 18 = (x-2)(x-3)^2$

$\Rightarrow t_n = c_1 2^n + c_2 3^n + c_3 n 3^n$: General solution.

The original recurrence imposes the following constraints:

$t_1 = 2t_0 + 18$

$t_2 = 2t_1 + 63 = 4t_0 + 99$

$$
\Rightarrow \left\{
\begin{array}{llllcl}
n=0: & c_1 & +c_2 & & = & t_0 \\
n=1: & 2c_1 & +3c_2 & +3c_3 & = & 2t_0 + 18 \\
n=2: & 4c_1 & +9c_2 & +18c_3 & = & 4t_0 + 99
\end{array}
\right.
$$

Non-homogeneous recurrences (4)

- **Example 2: (Cont'd)**

$$\Rightarrow \begin{cases} c_1 = t_0 - 9 \\ c_2 = 9 \\ c_3 = 3 \end{cases}$$

General solution:

$$t_n = (t_0 - 9)2^n + (n+3)3^{n+1}$$

$t_n = \theta(n3^n)$ independently of t_0.

Alternatively, substituting the general solution in the original recurrence, we reach the same result (exercise).

- **Generalisation:**

$$
\begin{array}{llll}
Ej1: & t_n - 2t_{n-1} = 3^n & \rightarrow & p(x) = (x-2)(x-3)^{0+1} \\
Ej2: & \underbrace{t_n - 2t_{n-1}}_{(x-2)} = (n+5)3^n & \rightarrow & p(x) = (x-2)(x-3)^{1+1}
\end{array}
$$

$\rightarrow$ to solve (4), use:

$$(a_0x^k + a_1x^{k-1} + \cdots + a_k)(x-b)^{d+1} \tag{5}$$

We proceed as in the homogeneous case, except that some equations to determine constants are obtained from the recurrence.

- **Example 3: Towers of Hanoi**

  procedure Hanoi (m, i, j)

  si $m > 0$ **entonces** Hanoi $(m-1, i, 6-i-j)$;
  move_ring (i, j); $\{$*elementary instr.*$\}$
  Hanoi $(m-1, 6-i-j, j)$

  Monks'problem: Hanoi (64,1,2)?

  $\Rightarrow t(m)$: no. of exec. of the elementary instr. in Hanoi$(m, -, -)$.

  $$t(m) = \begin{cases} 0 & m = 0 \\ 2t(m-1)+1 & \textit{otherwise} \end{cases}$$

  $\rightarrow t(m) - 2t(m-1) = 1$ (4) with $b = 1$ and $p(n) = 1$, $d = 0$

  (5) $\Rightarrow p(x) = (x-2)(x-1)$

  All the solutions are of the form $t(m) = c_1 1^m + c_2 2^m$

  $$\begin{cases} t(0) = 0 \\ t(1) = 2t(0)+1 = 1 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 0 & m = 0 \\ c_1 + 2c_2 = 1 & m = 1 \end{cases} \Rightarrow$$

  $$\begin{cases} c_1 = -1 \\ c_2 = 1 \end{cases} \Rightarrow t(m) = 2^m - 1$$

Non-homogeneous recurrences (7)

- **Example 3: Towers of Hanoi** (Cont'd)

  Determine θ without calculating c_1 and c_2?

  $$\left.\begin{array}{l} t(m) = c_1 + c_2 2^m \\ t(m) \geq m \end{array}\right. \Rightarrow c_2 > 0 \Rightarrow t(m) = \theta(2^m)$$

- **Example 4:**

  $t_n = 2t_{n-1} + n \rightarrow t_n - 2t_{n-1} = n$ (4): $b = 1, p(n) = n, d = 1$

  (5) $\Rightarrow p(x) = (x-2)(x-1)^2$

  $\Rightarrow$ Solution of the form $t_n = c_1 2^n + c_2 1^n + c_3 n 1^n$

  $(t_0 \geq 0 \Rightarrow t_n \geq 0 \;\; \forall n) \Rightarrow t_n = O(2^n)$

  ¿$t_n = \theta(2^n)$? $\Leftrightarrow c_1 > 0$?

  $$\begin{aligned} n &= t_n - 2t_{n-1} \\ &= (c_1 2^n + c_2 + c_3 n) - 2(c_1 2^{n-1} + c_2 + c_3(n-1)) \\ &= \underbrace{2c_3 - c_2}_{=0} \underbrace{-c_3}_{=1} n \end{aligned} \Rightarrow$$

  $$\begin{cases} c_3 = -1 \\ c_2 = -2 \\ c_1 = ? \end{cases}$$

  $$\rightarrow \begin{array}{l} t_n = c_1 2^n - n - 2 \\ (t_0 \geq 0 \Rightarrow t_n \geq 0 \;\; \forall n) \end{array} \Rightarrow c_1 > 0 \Rightarrow t_n = \theta(2^n)$$

  (c_1 can also be calculated, but it is not necessary)

  Problem: this method cannot be used always...

Non-homogeneous recurrences (9)

- **Example 5:** $t_n = \begin{cases} 1 & n = 0 \\ 4t_{n-1} - 2^n & otherwise \end{cases}$

$t_n - 4t_{n-1} = -2^n \qquad (4): b = 2, p(n) = -1, d = 0$

$(5) \Rightarrow p(x) = (x-4)(x-2)^{0+1}$

$\Rightarrow$ Solution of the form $t_n = c_1 4^n + c_2 2^n$

$\rightarrow t_n = \theta(4^n)$?

$$\begin{aligned} -2^n &= t_n - 4t_{n-1} \\ &= (c_1 4^n + c_2 2^n) - 4(c_1 4^{n-1} + c_2 2^{n-1}) \\ &= -\underbrace{c_2}_{=1} 2^n \end{aligned}$$

Calculation of c_1?

i system of equations

ii $\begin{cases} t_n = c_1 4^n + 2^n \\ t_0 = 1 \end{cases} \Rightarrow 1 = c_1 + 1 \Rightarrow c_1 = 0 \quad !$

$\Rightarrow t_n = 2^n \neq \theta(4^n)$

But with $t_0 > 1, t_n = \theta(4^n)$

Conclusion: for some recurrences initial conditions are crucial, while in others it only matters that $t_0 \geq 0$.

- **Generalisation:**
  Recurrences of the form

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = b_1^n p_1(n) + b_2^n p_2(n) + \ldots \quad (6)$$

where b_i are constants and $p_i(n)$ are polynomials of degree d_i, can be solved with

$$(a_0 x^k + a_1 x^{k-1} + \cdots + a_k)(x - b_1)^{d_1+1}(x - b_2)^{d_2+1} \ldots \quad (7)$$

(One factor corresponds to the left side, one factor for each term at the right side).

Non-homogeneous recurrences (11)

- **Example 6:** $t_n = \begin{cases} 0 & n = 0 \\ 2t_{n-1} + n + 2^n & otherwise \end{cases}$

  $t_n - 2t_{n-1} = n + 2^n \qquad (6): \begin{cases} b_1 = 1, p_1(n) = n, d_1 = 1 \\ b_2 = 2, p_2(n) = 1, d_2 = 0 \end{cases}$

  $(7) \Rightarrow (x-2)(x-1)^2(x-2) = (x-1)^2(x-2)^2$

  $\Rightarrow t_n = c_1 1^n + c_2 n 1^n + c_3 2^n + c_4 n 2^n$

  $t_n = O(n2^n)$

  $¿t_n = \theta(n2^n)? \Leftrightarrow ¿c_4 > 0?$

  i (in the original recurrence) $\rightarrow n + 2^n = (2c_2 - c_1) - c_2 n + \underbrace{c_4}_{=1} 2^n \Rightarrow t_n = \theta(n2^n)$

  ii (system of equations) $\rightarrow t_n = n2^n + 2^{n+1} - n - 2 \Rightarrow t_n = \theta(n2^n)$

- **Example 1:**

$$T(n) = \begin{cases} 1 & n = 1 \\ 3T(n/2) + n & n = 2^i, n > 1 \end{cases}$$

$$\begin{array}{rcl} \text{Replace } n \text{ with } 2^i & \rightarrow & \text{new recurrence } t_i = T(2^i) \\ n/2 & \rightarrow & 2^i/2 = 2^{i-1} \\ T(n) \text{ function of } T(n/2) & \rightarrow & t_i \text{ function of } t_{i-1}\text{: what we know how to solve} \end{array}$$

$$\Rightarrow t_i = T(2^i) = 3T(2^{i-1}) + 2^i = 3t_{i-1} + 2^i$$

$$\Rightarrow t_i - 3t_{i-1} = 2^i \text{ (4)} \Rightarrow \text{(5) } (x-3)(x-2)$$

$$\Rightarrow \left. \begin{array}{l} t_i = c_1 3^i + c_2 2^i \\ T(2^i) = t_i \Leftrightarrow T(n) = t_{log_2 n}, n = 2^i \end{array} \right\} \Rightarrow T(n) = c_1 3^{log_2 n} + c_2 2^{log_2 n}$$

$$\Rightarrow T(n) = c_1 n^{log_2 3} + c_2 n \Rightarrow T(n) = O(n^{log 3})$$

$$T(n) = \theta(n^{log3}) \Leftrightarrow c_1 > 0?$$

$$\begin{array}{rl} n & = T(n) - 3T(n/2) \\ & = (c_1 n^{log_2 3} + c_2 n) - 3(c_1 (n/2)^{log_2 3} + c_2(n/2)) \quad \leftarrow (1/2)^{log_2 3} = 1/ \\ & = -c_2(n/2) \end{array}$$

$$\rightarrow \left. \begin{array}{l} c_2 = -2 \\ T(n) > 0 \end{array} \right\} \Rightarrow c_1 > 0 \Rightarrow T(n) = \theta(n^{log 3}) \text{ if n power of 2}$$

Change of variable (2)

- **Example 2: Divide and Conquer recurrences**

$$T(n) = \ell T(n/b) + cn^k, n > n_0 \qquad (8)$$

With $\ell \geq 1, b \geq 2, k \geq 0, n_0 \geq 1 \in \aleph$ and $c > 0 \in \Re$,
when n/n_0 is an exact power of b ($n \in \{bn_0, b^2 n_0, b^3 n_0 \dots\}$).
Change of variable: $n = b^i n_0$

$$\begin{aligned} \Rightarrow t_i = T(b^i n_0) &= \ell T(b^{i-1} n_0) + c(b^i n_0)^k \\ &= \ell t_{i-1} + cn_0^k b^{ik} \end{aligned}$$

$\Rightarrow t_i - \ell t_{i-1} = cn_0^k (b^k)^i \quad$ (4): $p(i) = cn_0^k, d = 0, b = b^k$

(5) $\Rightarrow (x - \ell)(x - b^k)$

$\Rightarrow t_i = c_1 \ell^i + c_2 (b^k)^i \qquad (\star)$

$i = log_b(n/n_0)$ when n/n_0 is an exact power of b

$\Rightarrow d^i = (n/n_0)^{log_b d}$ for $d > 0$

$$\begin{aligned} \Rightarrow T(n) &= (c_1/n_0^{log_b \ell}) n^{log_b \ell} + (c_2/n_0^k) n^k \\ &= c_3 n^{log_b \ell} + c_4 n^k \qquad (\star\star) \end{aligned}$$

(in original recurrence)

$\rightarrow cn^k \quad = T(n) - \ell T(n/b) = \dots$

- Asymptotic notation for $T(n)$?
  $\leftrightarrow$ dominant term in $(\star\star)$?
  1. $\ell < b^k \Rightarrow c_4 > 0 \wedge k > log_b \ell \Rightarrow c_4 n^k$ dominates
     $\Rightarrow T(n) = \Theta(n^k)$
  2. $\ell > b^k \Rightarrow c_4 < 0 \wedge log_b \ell > k \Rightarrow c_3 > 0, c_3 n^{log_b \ell}$ dominates
     $\Rightarrow T(n) = \Theta(n^{log_b \ell})$
  3. $\ell = b^k \Rightarrow c_4 = c/0$!
     Pb: $(\star)$ does not yield the general solution of the recurrence
     $$\begin{aligned} (x-\ell)(x-b^k) \quad & \rightarrow (x-b^k)^2 \\ & \Rightarrow t_i = c_5 (b^k)^i + c_6 i (b^k)^i \\ & \Leftrightarrow T(n) = c_7 n^k + c_8 n^k log_b(n/n_0) \end{aligned}$$
     (in original recurrence) $\rightarrow c_8 = c > 0 \Rightarrow cn^k log_b(n/n_0)$ dominates
     $\Rightarrow T(n) = \Theta(n^k logn)$

- **Divide and Conquer theorem**:
  If a recurrence is of the form (8), we apply

$$T(n) = \begin{cases} \theta(n^k) & if \quad \ell < b^k \\ \theta(n^k logn) & if \quad \ell = b^k \\ \theta(n^{log_b \ell}) & if \quad \ell > b^k \end{cases} \tag{9}$$

  In analysis of algorithms, inequalities are commonly used:
  $T(n) \leq \ell T(n/b) + cn^k, n > n_0$ with n/n_0 exact power of b

$$\Rightarrow T(n) = \begin{cases} O(n^k) & if \quad \ell < b^k \\ O(n^k logn) & if \quad \ell = b^k \\ O(n^{log_b \ell}) & if \quad \ell > b^k \end{cases}$$

- **Other techniques:**
  - interval transformations
  - asymptotic recurrences

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