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MILANO 1863

DIPARTIMENTO DI ELETTRONICA
INFORMAZIONE E BIOINGEGNERIA

1D Digital filters

Filter definition in MATLAB

1. Given H(z) = B(z) / A(z):
   - [H(omega), omega] = freqz(B(z), A(z), N, ‘whole’): for both FIR and IIR.
   - h(n) = filter(B(z), A(z), delta(n)): precise with FIR, only an approximation for IIR.
2. Given h(n):
   - H(f) or H(omega) evaluated on N samples = fft(h, N)

Filter definition in MATLAB

- H(f) (or H(omega)) returned by MATLAB are defined over N samples.
- The DTFT is PERIODIC:
  - In frequency domain, period = Fs, f = [0, Fs] or f = [-Fs/2, Fs/2) [Hz]
  - In angular frequency domain, period = 2πFs, ω= [0, 2πFs) or ω =[-πFs, πFs) [rad/s]
  - In normalized frequency, period = 1, $\tilde{f}$ = [0, 1) or $\tilde{f}$ = [-0.5, 0.5)
  - In normalized angular frequency, period = 2π, $\tilde{\omega}$= [0, 2π) or $\tilde{\omega}$= [-π,π)

How to relate the MATLAB result with the actual Fourier spectrum?
→ How to express MATLAB samples as real frequencies in Hz or normalized frequencies?

MATLAB metrics conversion

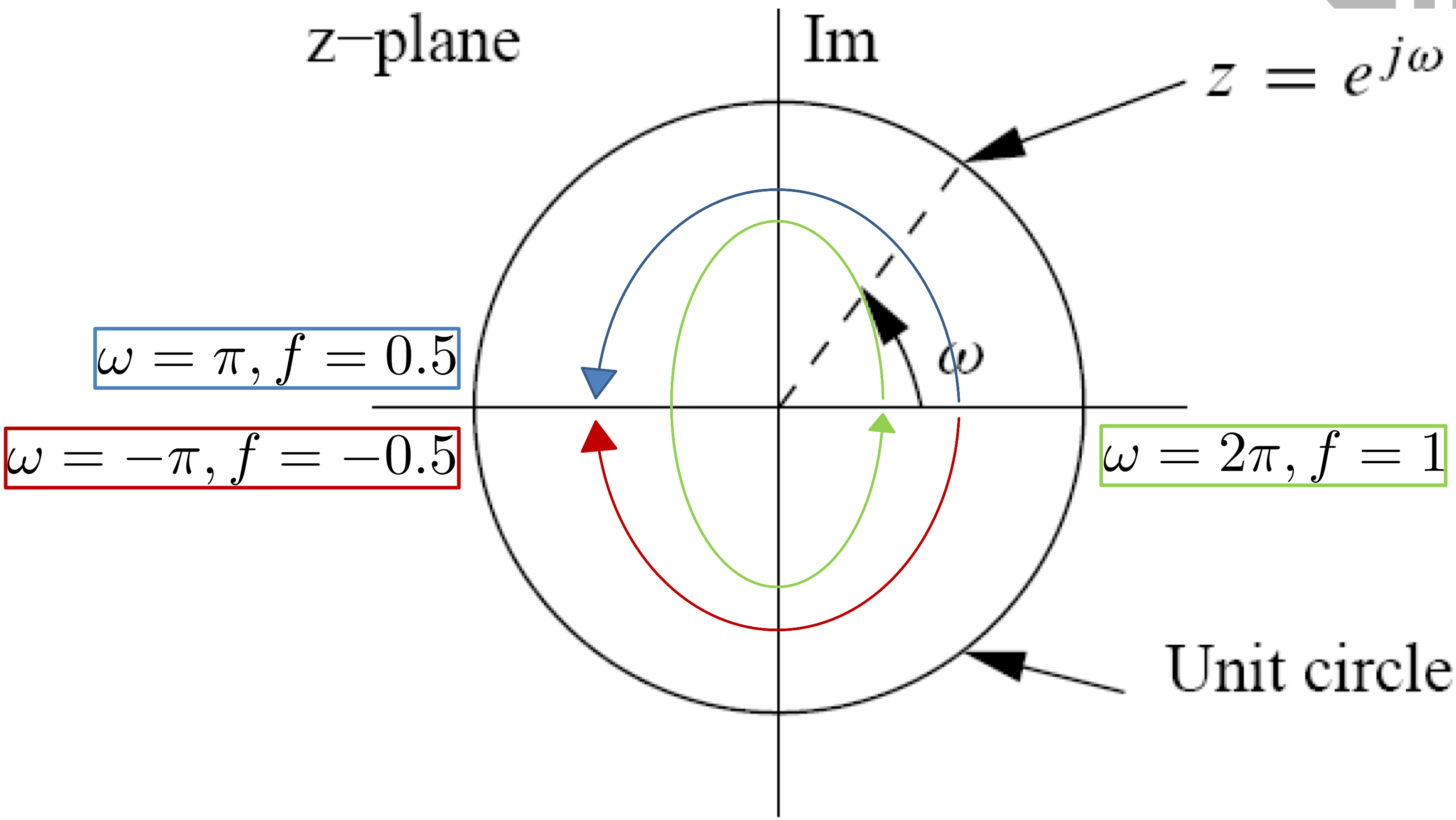
The N-th sample in the MATLAB result corresponds to the maximum frequency over 1 period of the periodic spectrum.

Given the array of MATLAB samples: $\mathbf{n} = [0, 1, 2, ..., N-1]$

- The frequency axis [Hz] = [0, Fs) is obtained as $\mathbf{n} \cdot \frac{\mathrm{F}_s}{N}$
- The angular frequency axis [rad/s] = $[0, 2\pi\mathrm{F}_s)$ is obtained as $\mathbf{n} \cdot \frac{2\pi\mathrm{F}_s}{N}$
- The normalized frequency axis = [0, 1) is obtained as $\mathbf{n} \cdot \frac{1}{N}$
- The normalized angular frequency axis = $[0, 2\pi)$ is obtained as $\mathbf{n} \cdot \frac{2\pi}{N}$

LEARNING BY HEART IS NOT NEEDED! *Just think at the units of measure*

From Z domain to normalized frequency domain



To pass from normalized domain to real frequency domain, multiply by Fs

FIR vs IRR filters in MATLAB

1. Given a FIR filter and the input signal x(n):
   - Use ‘conv’ if filter is expressed in time domain
   - Use ‘filter’ if you have H(z) or the time domain filter
   - Use the product of ‘fft’s in frequency domain
   - Use the product of signal ‘fft’ and filter ‘freqz’ in f domain
2. Given an IIR filter and the input signal x(n):
   - You cannot use ‘conv’! The result will be just an approximation because the filter has infinite duration
   - You can use all the other functions

Zeros and poles recall

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\sum_{k=0}^{D} a_k z^{-k}} = z^{D-N} \frac{b_0}{a_0} \frac{\prod_{i=1}^{N}(z - z_i)}{\prod_{i=1}^{D}(z - p_i)} = \frac{b_0}{a_0} \frac{\prod_{i=1}^{N}(1 - z_i z^{-1})}{\prod_{i=1}^{D}(1 - p_i z^{-1})}$$

- z_i = roots of numerator, called 'zeros'
- p_i = roots of denominator, called 'poles'

Zeros and poles recall: *the poles*

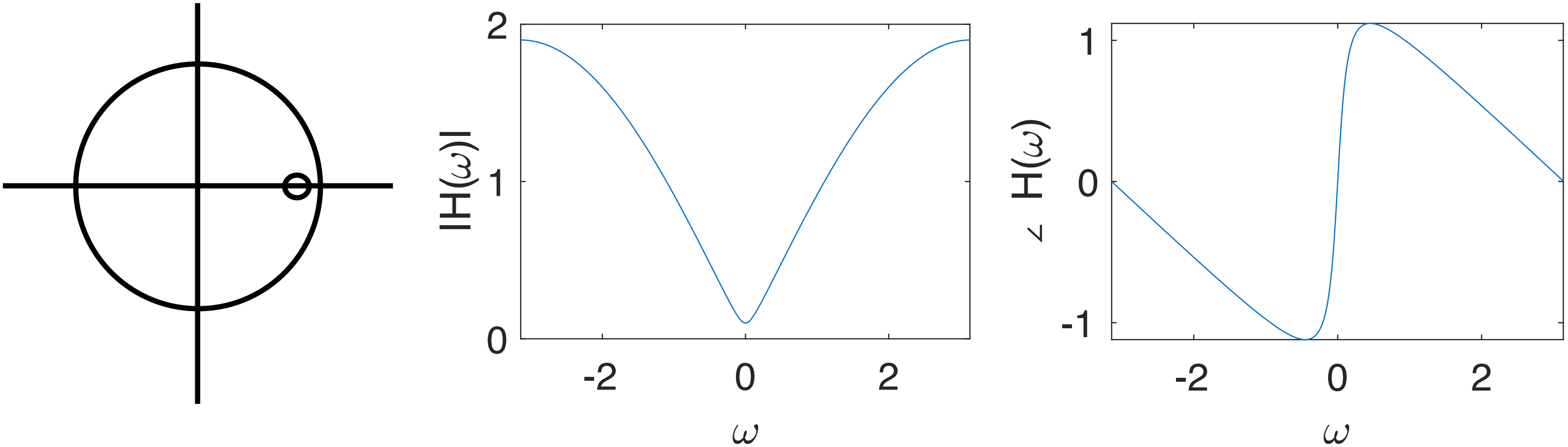
- The poles are associated with the autoregressive part of the filter → they generate IIR filters.
- The filter amplitude response enhances frequencies which are near the poles.
- If poles are outside the unit circle and the filter is causal, the system is unstable.

Zeros and poles recall: *the zeros*

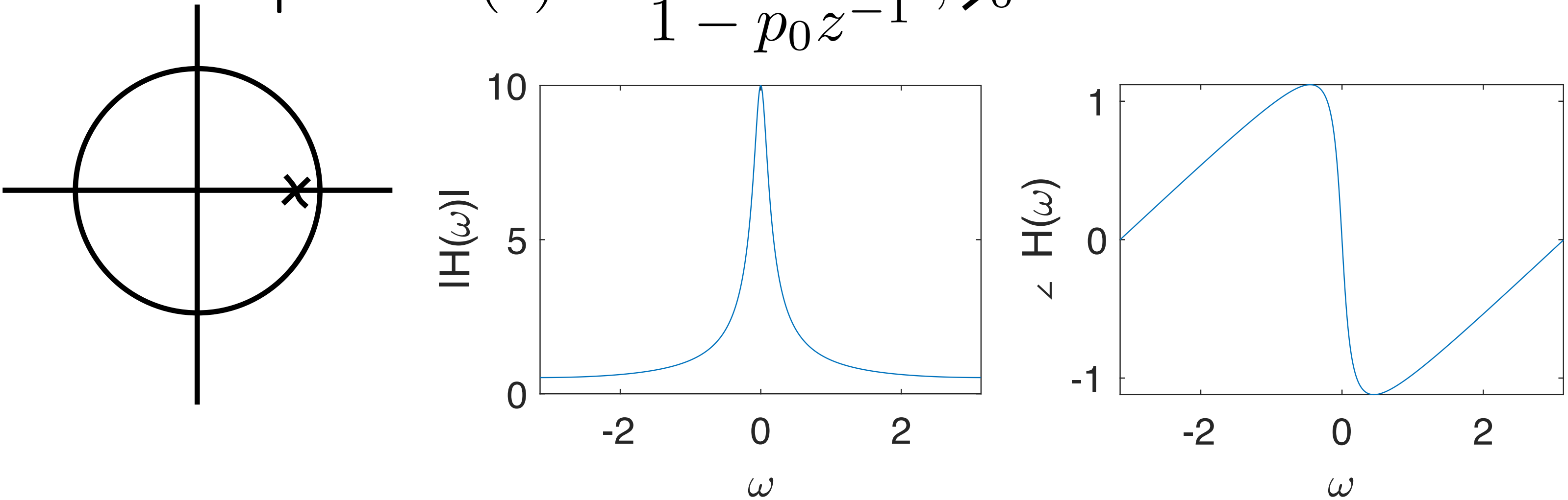
- The zeros are associated with the moving average part of the filter → they generate FIR filters
- The filter amplitude response attenuates frequencies which are near the zeros
- Zeros influence also the phase of the filter:
  - Minimum phase zeros if $z < 1$
  - Maximum phase zeros if $z \geq 1$

Filter design using zeros&poles

- 1 zero: $H(z) = 1 - z_0 z^{-1}$, $z_0 = 0.9$



- 1 pole: $H(z) = \dfrac{1}{1 - p_0 z^{-1}}$, $\overset{\text{p\_0}}{\cancel{z_0}} = 0.9$



Filter design using zeros&poles

- Place poles close to the unit circle in frequencies that must be emphasized
- Place zeros according to the desired phase response.

  The closer they are to the unit circle, the higher the frequency attenuation

Filter design using zeros&poles

Open ‘zpgui.m’

Remarkable LTI filters

Magnitude square function

- The magnitude response of a LTI system is:

$$M(f) = |H(f)|^2 = H(f) \cdot H^*(f) = H(z) \cdot H^*(z^{-1})\Big|_{|z|=1}$$

- Given a generic rational transfer function

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{i=1}^{N}(1 - z_i z^{-1})}{\prod_{i=1}^{D}(1 - p_i z^{-1})}$$

$$\Downarrow$$

$$M(z) = H(z)H^*(z^{-1}) = \frac{|b_0|^2}{|a_0|^2} \frac{\prod_{i=1}^{N}(1 - z_i z^{-1})(1 - z_i^* z)}{\prod_{i=1}^{D}(1 - p_i z^{-1})(1 - p_i^* z)}$$

Magnitude square function

$$M(z) = H(z)H^*(z^{-1}) = \frac{|b_0|^2}{|a_0|^2} \frac{\prod_{i=1}^{N}(1 - z_i z^{-1})(1 - z_i^* z)}{\prod_{i=1}^{D}(1 - p_i z^{-1})(1 - p_i^* z)}$$

quando calcolo il modulo ho nuovi poli e zeri: (per ottenere coeff a valore Reale)

- For each zero z_i of H(z), there is another zero at $\dfrac{1}{z_i^*}$
- For each pole p_i of H(z), there is another pole at $\dfrac{1}{p_i^*}$
- M(z) presents poles and zeros in conjugate reciprocal pairs

Magnitude square function

- Given a magnitude response requirement M(z) for the filter H(z)
- Given stability and causality requirements for the filter H(z)

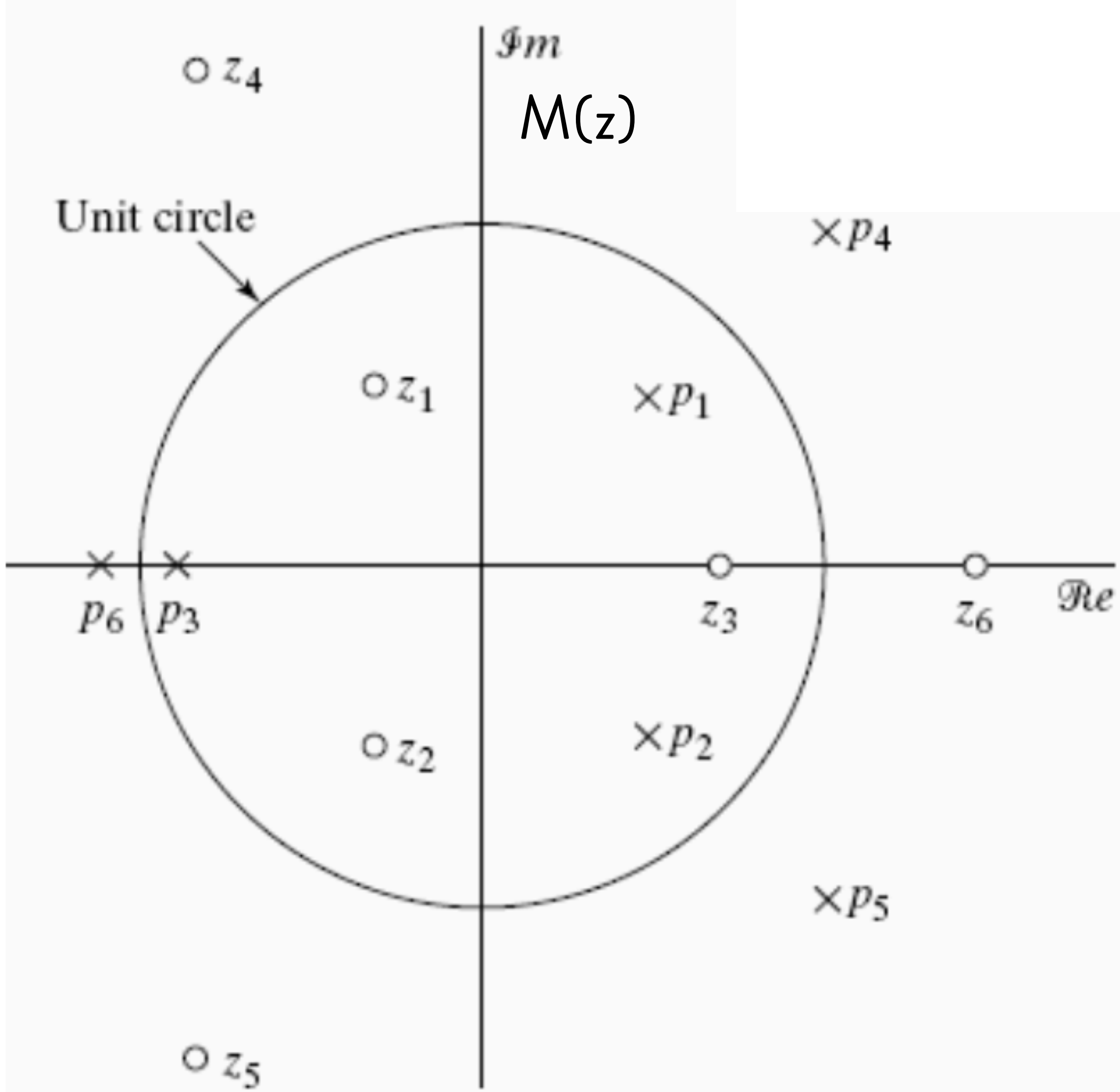
  only poles inside the unit circle, zero no constraint

⬇

- The poles of H(z) are those of M(z) inside the unit circle and are uniquely identified
- The zeros of H(z) are **not** uniquely identified
- Given a causal filter $H(z)$ of order N, it has the same magnitude response M(z) of the causal filter $G(z) = z^{-N}H^*(z^{-1})$

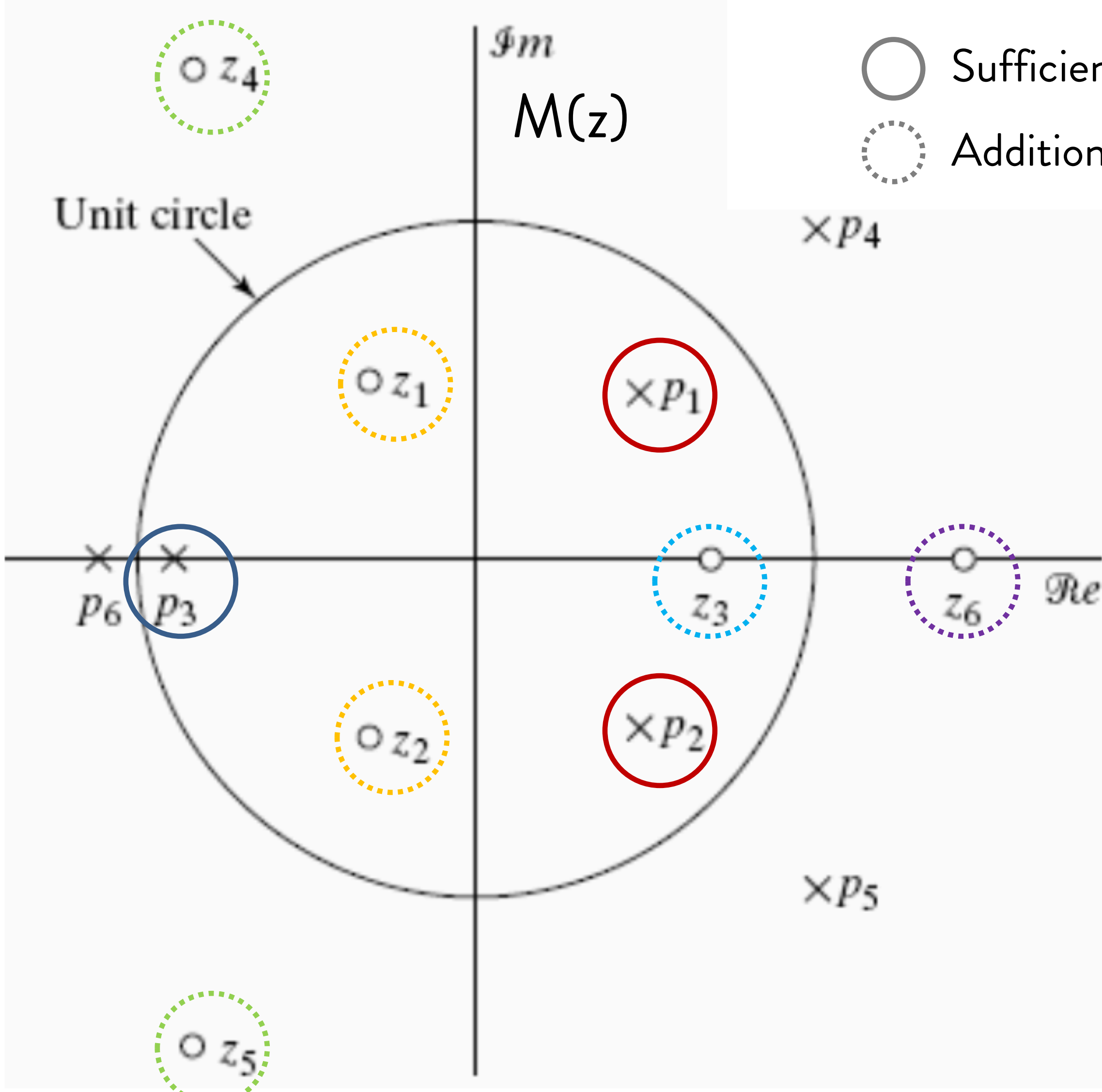
From M(z) to H(z)

*How to get a **causal stable** system with **real** coefficients?*



From M(z) to H(z)

*How to get a **causal stable** system with **real** coefficients?*



Sufficient configuration

Additional configuration

Es 22: magnitude response

- Given the filters:

$$H_1(z) = \frac{2(1 - z^{-1})(1 + 0.5z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}$$

$$H_2(z) = \frac{(1 - \boxed{z}^{-1})(1 + \boxed{2}z^{-1})}{(1 - \boxed{0.8e^{j\pi/4}}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}$$

per ricavare il polinomio (cioè la moltiplicazione di questi elementi fondamentali) posso fare la convoluzione tra i coeff, quindi la penso nel tempo; ottengo i coeff del polinomio

- Plot the zeros and the poles in the Z-plane using ‘zplane’
- Plot in the same figure the magnitude responses as a function of normalized omega, using N = 1024 samples
- How are the magnitudes related? Why?

se ho uno zero su H1 e lo stesso ma reciproco e coniugato in H2, il modulo NON CAMBIA

Allpass filters

Allpass filters are designed to have constant gain and any phase response:

$$|H_{ap}(f)| = |H_{ap}(z)|\Big|_{|z|=1} = 1$$

- Given the previous considerations, a generic causal allpass filter is:

$$H_{ap}(z) = z^{-K} e^{j\phi} \frac{A(z)}{\tilde{A}(z)}, \quad K \geq 0$$

where

ritardo il den poiché voglio un sist causale.
Non influisce nel modulo

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}$$

$$\tilde{A}(z) = z^{-N} A^*(z^{-1}) = a_N^* + a_{N-1}^* z^{-1} + \ldots + a_2^* z^{2-N} + a_1^* z^{1-N} + z^{-N}$$

Allpass filters

- Given an allpass filter: inverto l'ordine dei coeff, ne faccio il coniug, e aggiungo la pot di z

$$H_{ap}(z) = \frac{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}{a_N^* + a_{N-1}^* z^{-1} + \ldots + a_2^* z^{2-N} + a_1^* z^{1-N} + z^{-N}}$$

a general form to represent an **allpass real** valued impulse response is:

$$H_{ap}(z) = c_0 \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k)(z^{-1} - e_k^*)}{(1 - e_k^* z^{-1})(1 - e_k z^{-1})}$$

Zeros and poles occur in conjugate reciprocal pairs

Allpass filter properties

- The cascade of two allpass filters is again an allpass filter
- Each pole of an allpass system is paired with a conjugate reciprocal zero
- The magnitude of many cascaded allpass filters is always the same

un AP serve a cambiare la fase, in base agli zeri !

quando faccio la versione all pass, il polo (che era stabile) diventa uno zero max phase (poichè il suo valore è reciproco del polo). Quindi avrò sempre un max phase dovuto a questa inversione del polo

Es 23: allpass systems [~exam 19/02/2018, 7 pts]

- Write a MATLAB function ‘allpass.m’ which has the form:

  ‘[z, p, b, a]=all_pass(b,a)’
- Input: b, a = numerator and denominator of H(z)
- Output: z, p, b, a = zeros, poles, numerator, denominator of the allpass transfer function related to H(z)
- Use the function ‘allpass’ to compute the allpass transfer function related to the causal filter $H(z) = \dfrac{1+3z^{-1}}{1+0.5z^{-1}}$
- Plot the magnitude response of the filter vs normalized frequencies using N = 512 samples
- How do you expect the phase to behave?

Minimum phase filters

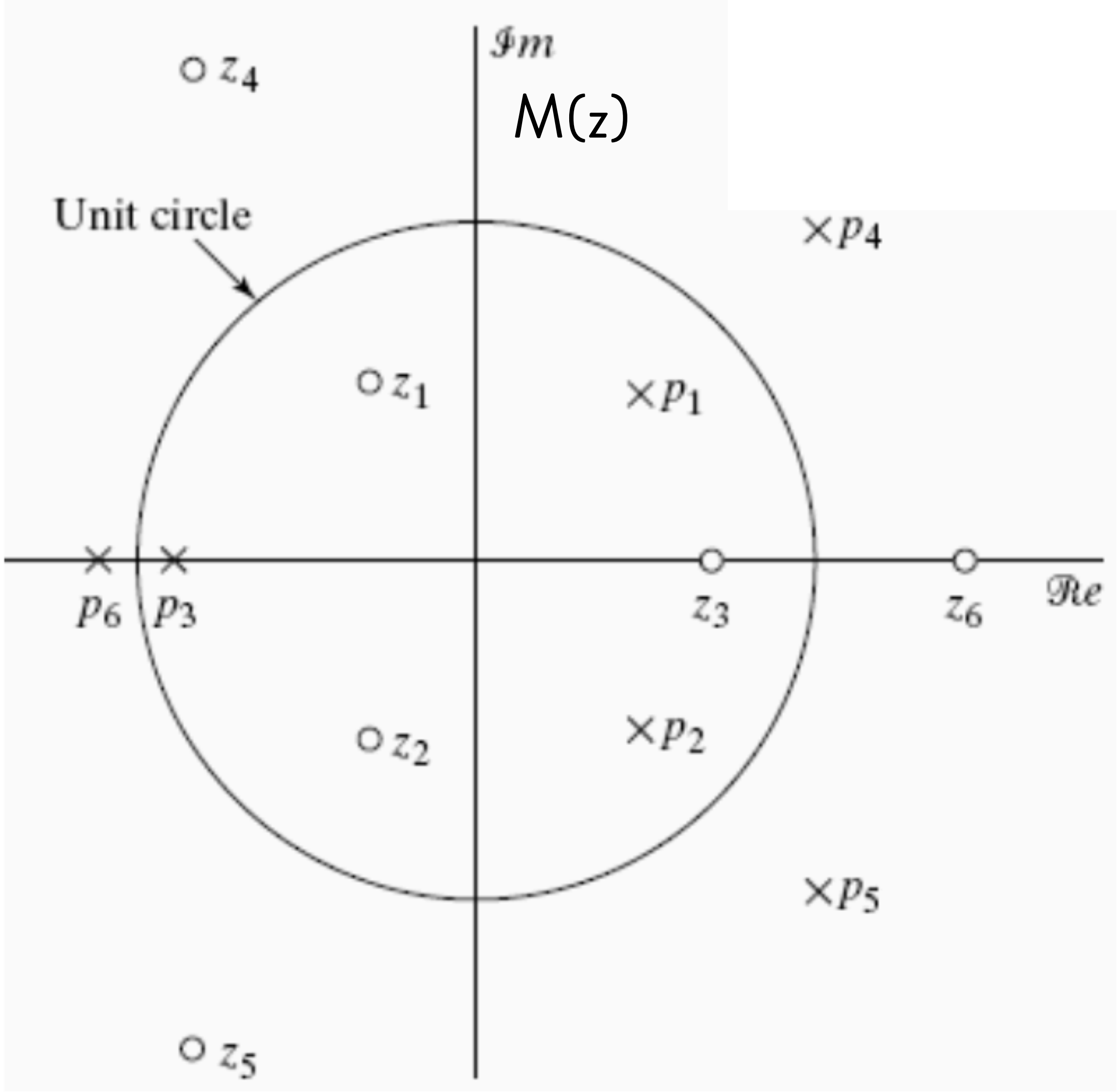
- Minimum phase filters are such that both H(z) and 1/H(z) are stable and causal

⇓

- The poles must be inside the unit circle
- The zeros must be inside the unit circle **perchè l’inversa avrà poli = inverso degli zeri**
- Given a square magnitude response M(z), there is a **unique** system whose zeros and poles are inside the unit circle and it is called minimum phase system

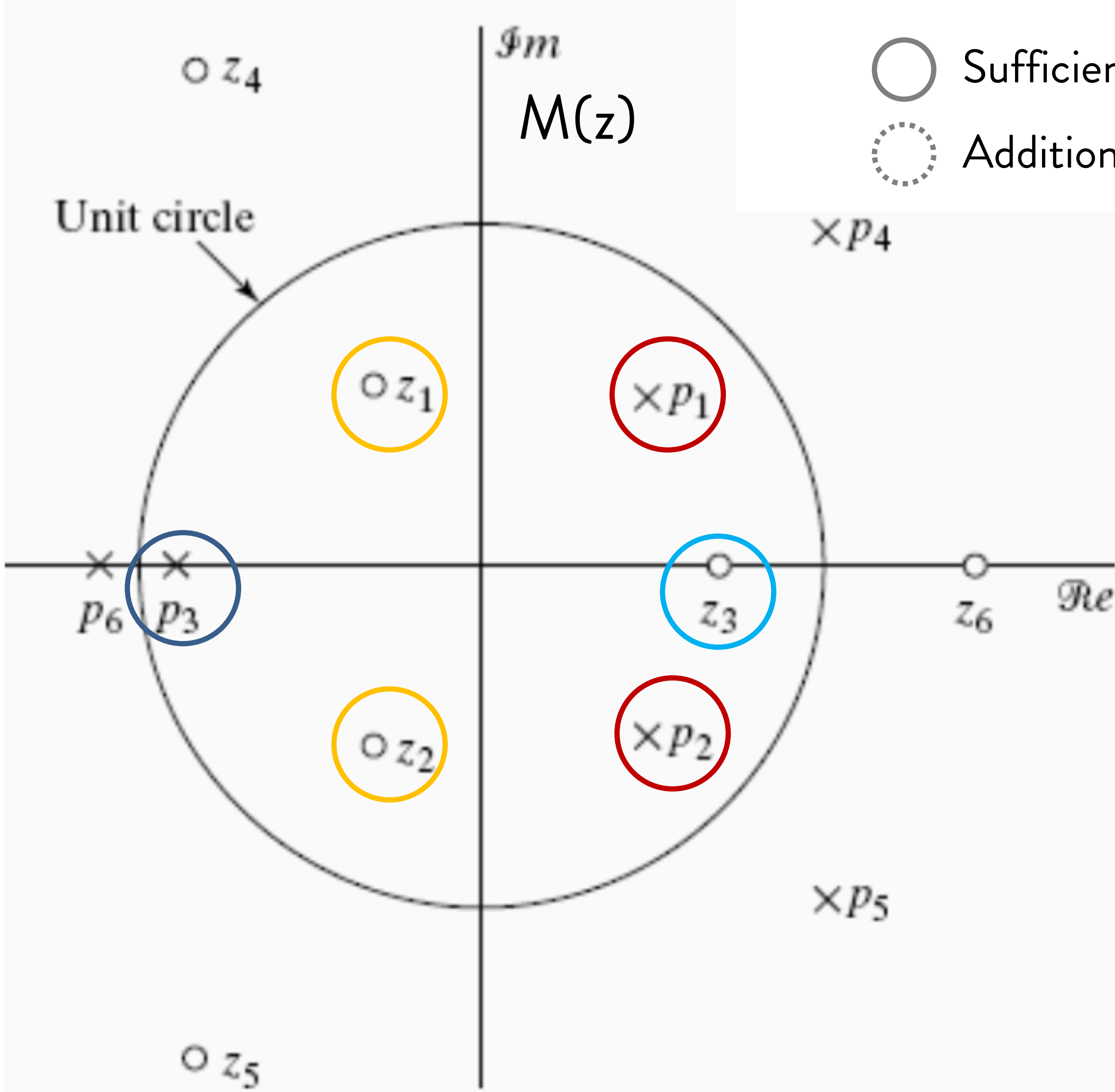
From M(z) to H(z)

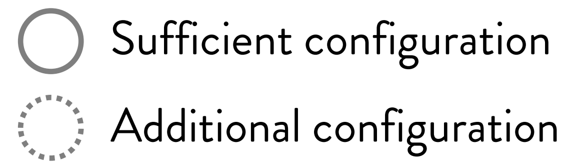
How to get a ***minimum phase*** *system?*



From M(z) to H(z)

*How to get a **minimum phase** system?*

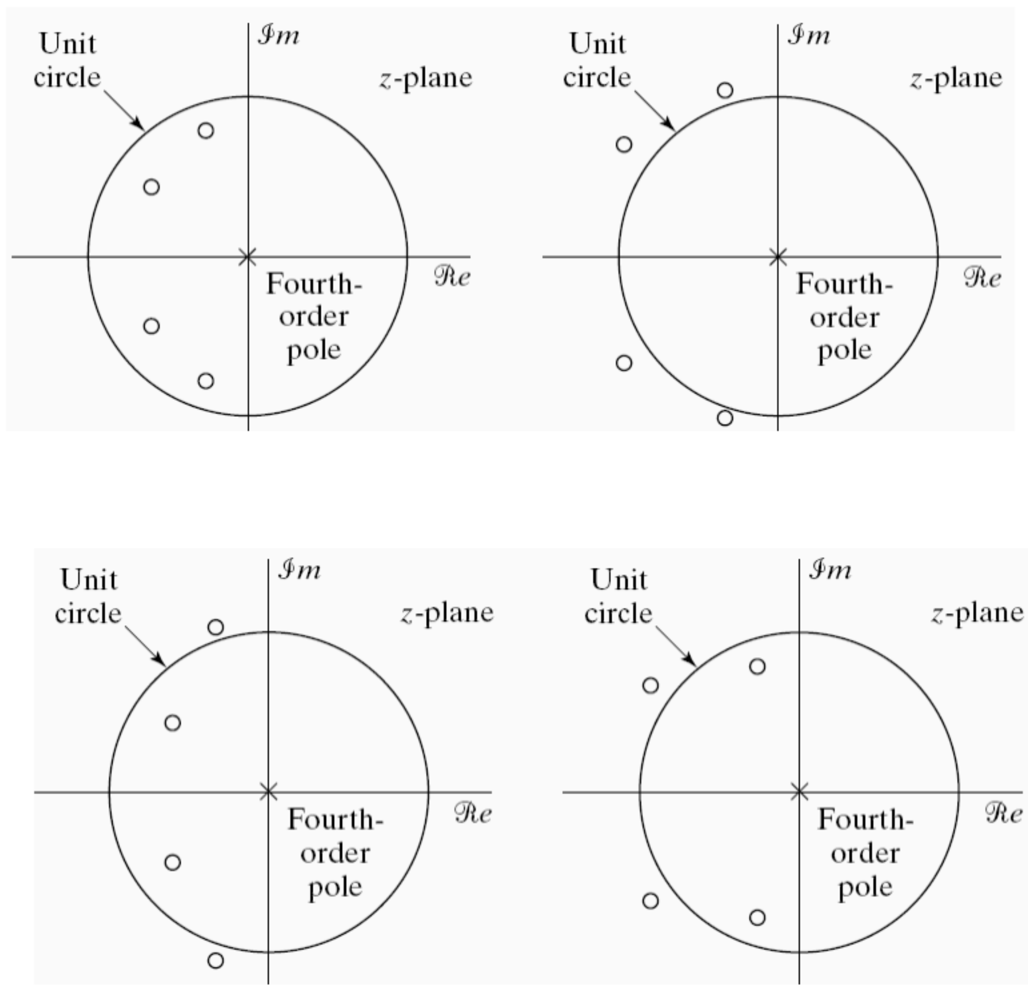




From M(z) to H(z)

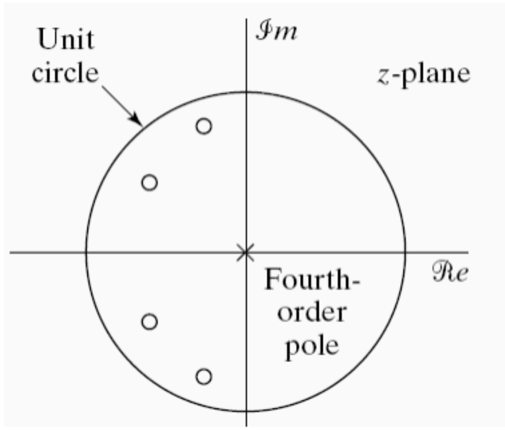
*Which is the **minimum phase** system?*

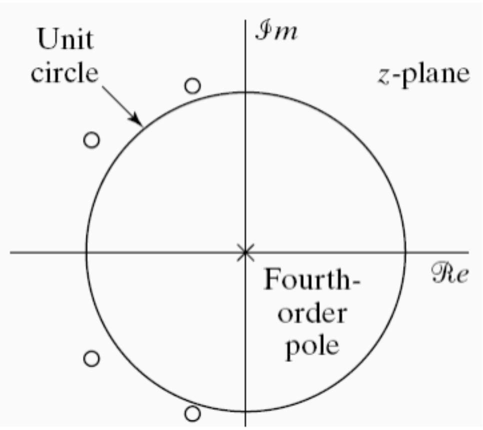
NB 4 filtri differenti, MA STESSA MAGNITUDE RESPONSE
ho diverse config, ma gli zeri sono in più comb ma conj e reciprocal

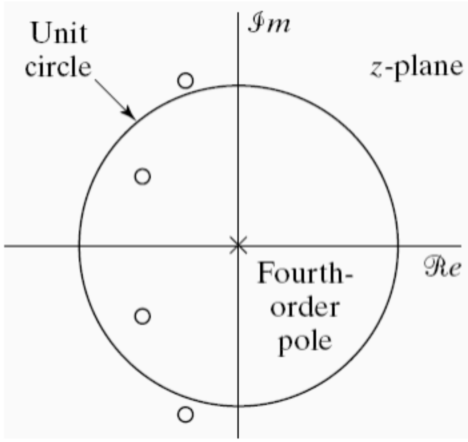


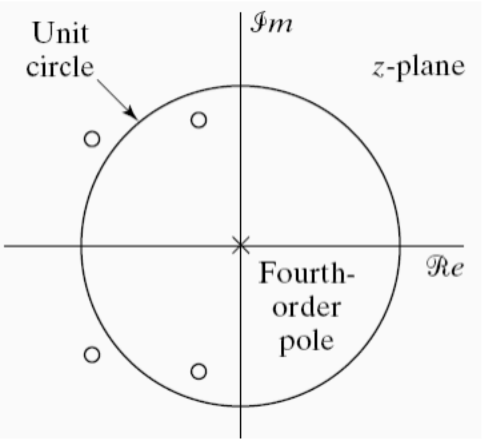
From M(z) to H(z)

*Which is the **minimum phase** system?*









Es 24: minimum phase systems [~exam 12/02/2019, 8pts]

- Write a MATLAB function 'typeOfFilter(b, a)' that receives as input the numerator and denominator coefficients of a causal filter H(z)=B(z)/A(z) and it returns:
  - -1 if the filter is not stable
  - 1 if the filter is stable and it is minimum phase
  - 0 if the filter is stable but it is not minimum phase
- If you test this function on a FIR filter, which is the output?
- Test the function on

$$H(z) = \frac{1 - 2z^{-1} - 0.5z^{-3} + 0.2z^{-4}}{1 + 0.08z^{-1} + 2z^{-3}}$$

Properties of Allpass – Minimum phase filters

Any rational system can be decomposed into the multiplication (the cascade) of a minimum phase system and an allpass system

i min phase filter hanno l'energia (nel tempo, quindi IR) conservata intorno allo zero, cioè decreasing veloce

$$H(z) = \underbrace{H_{min}(z)}_{\text{modifico il modulo}}\underbrace{H_{ap}(z)}_{\text{modifico la fase}}$$

- $H_{min}(z)$ contains:
  - the poles and zeros of H(z) that lie inside the unit circle
  - zeros that are conjugate reciprocals of the zeros of H(z) lying outside the unit circle. **, cioè quelli dell AllPass**
- $H_{ap}(z)$ contains:
  - all the zeros of H(z) that lie outside the unit circle
  - poles to cancel the conjugate reciprocal zeros inserted in $H_{min}(z)$

Es 25: Allpass-minimum phase conversion [~exam 22/07/2019, 4pts]

- Given the filter with B(z)=[1, -1.98, 1.77, -0.17, 0.21, 0.34], A(z)=[1, 0.08, 0.40 ,0.27]
- Compute the allpass-minimum phase decomposition of H(z)
- Check the results using 'zplane'
- Plot the magnitude of Hap(f) using N = 1024 samples
- Plot the first N samples of h_min(n)

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Digital filters' design

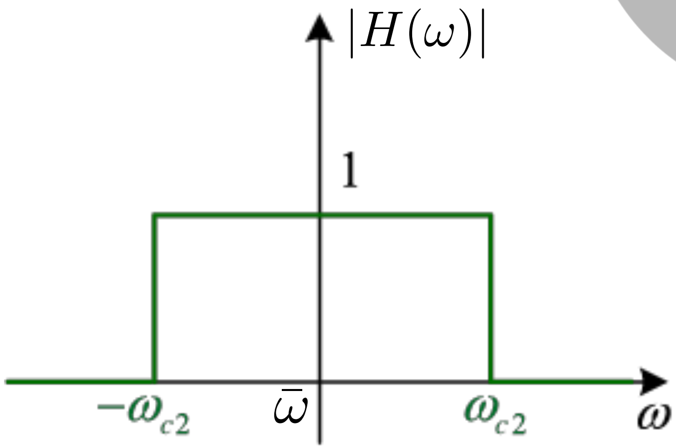
How to design filters

1. Specify always the characteristics of the filter in frequency domain, not in time domain (e.g., lowpass, highpass, bandpass..)
2. Approximate these properties using a discrete-time system → find the filter coefficients
3. Realize the system using finite precision arithmetic

Ideal filter

Ideal filter:

- low-pass if $\bar{\omega} = 0$
- band-pass if $0 < \bar{\omega} < \pi$
- high-pass if $\bar{\omega} = \pi$

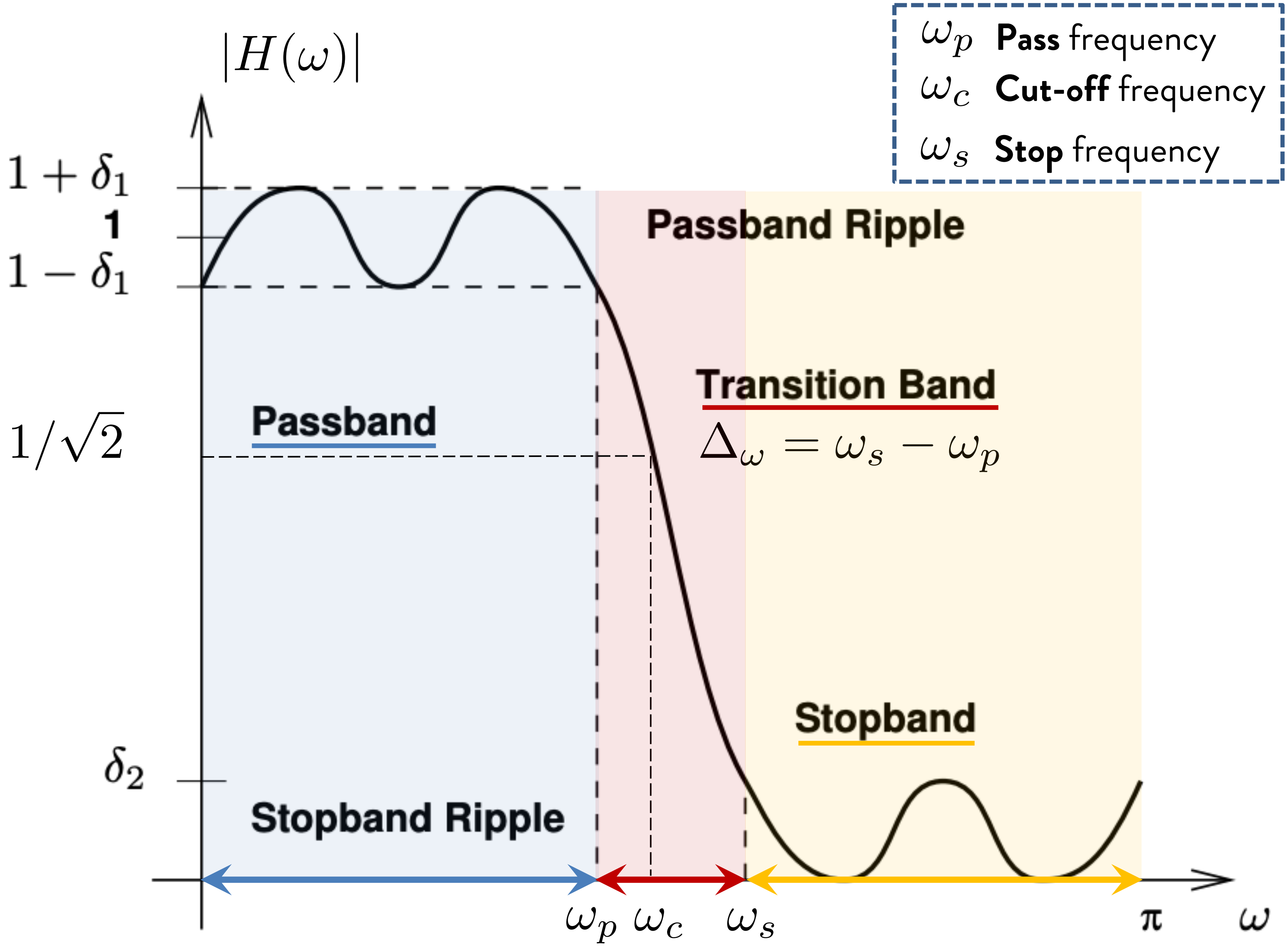


The impulse response of this filter is ≈ the sinc function.

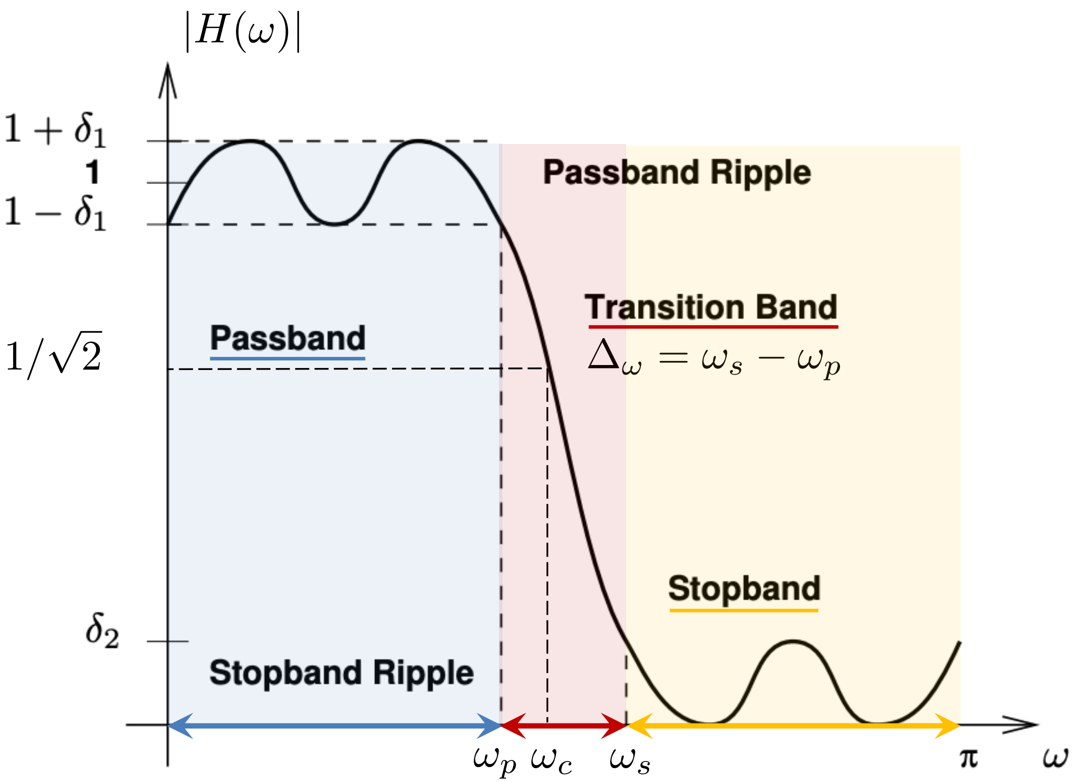
It is non-causal with an infinite delay →

Real systems can only approximate it

Real filters



Real filters



δ_1 Peak passband ripple

$1-\delta_1$ Minimum passband gain

$$1-\delta_1 = \frac{1}{\sqrt{1+\varepsilon^2}}$$

δ_2 Peak stobband ripple

$\frac{1}{\delta_2}$ Minimum stopband attenuation

ω_c 3dB or cut-off frequency

Towards ideal filters:

- Peak ripple → 0
- Transition band → 0

IIR vs FIR filters

FIR:

- Only zeros
- Always stable
- Can be linear phase
- It should be high order for best performances

IIR:

- Poles and zeros
- May be unstable
- Difficult to control phase
- Lower order (1/10-th of FIR) for high performances

IIR vs FIR

FIR:

- Only zeros
- Always stable
- Can be linear phase
- It should be high order for best performances

IIR:

- Poles and zeros
- May be unstable
- Difficult to control phase
- Lower order (1/10 of FIR)

We saw IIR design with poles&zeros

How to design FIR filters?

FIR filter design: windowing method

The ideal filter has an infinite time duration and infinite delay →

Idea: obtain a FIR filter by truncating an infinite duration impulse response

- Given an ideal $h_i(n)$, build $h(n) = h_i(n)w(n)$
- $w(n)$ is a finite duration window

  → <u>in frequency domain, product becomes convolution</u>
- <u>$H(f)$ is a blurred version of the ideal filter $H_i(f)$</u>

FIR filter design: windowing method

How to choose the window?

- As short as possible to minimize the cost of the FIR filter
- As narrow as possible in frequency to approach the ideal filter

FIR filter design: windowing method

How to choose the window?

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FIR filter design: windowing method

How to choose the window?

- As short as possible to minimize the cost of the FIR filter
- As narrow as possible in frequency to approach the ideal filter

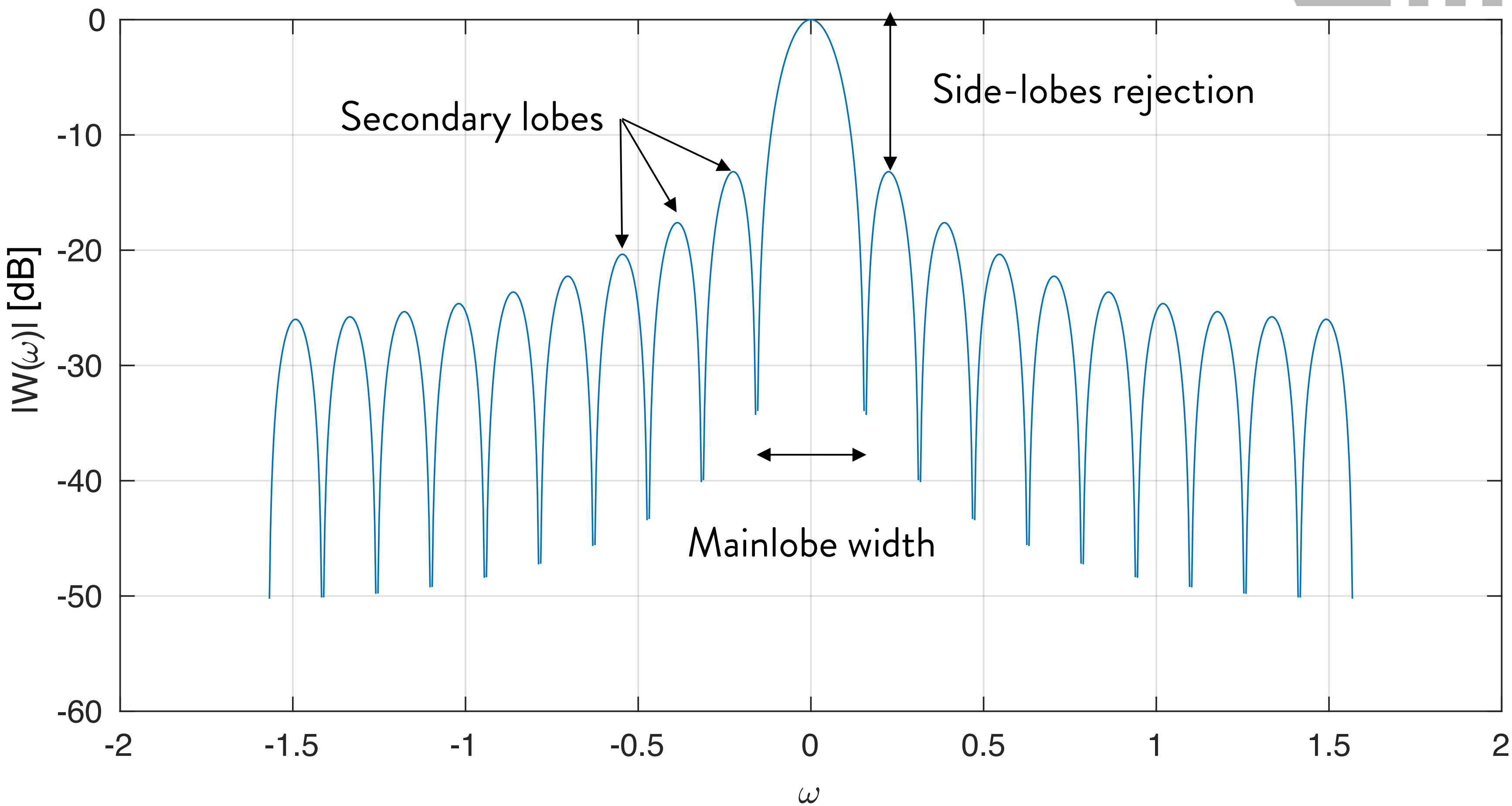
↓

Even though these requirements conflict each other, a good window is defined as the one introducing the minimum distortion.

→ $W(f)$ should look like a $\delta(f)$:

- its energy must be concentrated around $f=0$
- $W(f)$ should decay fast as frequency increases

FIR filter design: windowing method



FIR filter design: windowing method

Every window is characterized by:

- Main-lobe width: it decreases as the window length increases
- Side-lobes rejection: ratio between the main-lobe peak and 1° secondary lobe peak [dB]
- Side-lobes roll off: asymptotic decay of the side-lobe peaks vs frequency octave [dB/octave] or frequency decade [dB/decade]

Examples of windows:

- Rectangular → 'rectwin' in MATLAB
- Hanning → 'hann' in MATLAB
- Hamming → 'hamming' in MATLAB
- Blackman → 'blackman' in MATLAB and many others…

Window design in MATLAB

- You can use the function ‘window’ to design windows:

  ‘w = window(@window_name, Nsamples)’

- Otherwise, you can call specific functions named as the window, for instance:
  - ‘rect_w = rectwin(Nsamples)’
  - ‘hamming_w = hamming(Nsamples)’
  - ‘hann_w = hann(Nsamples)’
  - …

FIR design in MATLAB

ORDINE DEL FILTRO = NUMERO SAMPLES -1

SOLO IN QUESTO COMANDO ho range di 'freq' da 0 a 1

Use the function 'fir1' to implement window-based FIR filter design

h = fir1(filter_order,cut-off,filter_type,window_type(filter_order+1))

NB:

only on matlab (aka decay 1/2 of power instead of 1/sqrt(2))

- cut-off parameter sets the 6dB point (when H(f) is 6dB lower than the maximum peak)
- cut-off for MATLAB is between 0 and 1, but 1 corresponds to half the sampling frequency! e cioè 0,5, aka PI

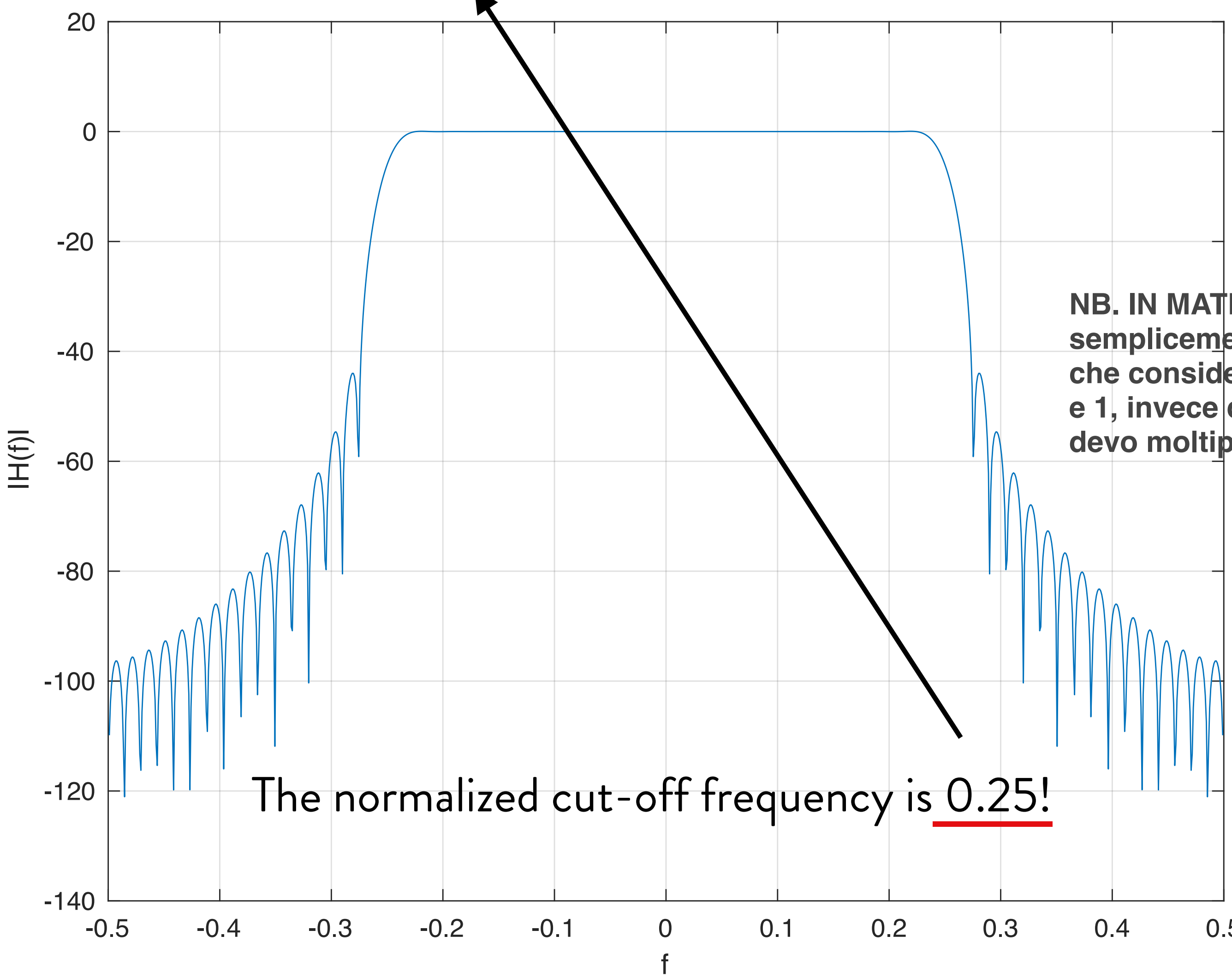
$$\boxed{\text{Cut-off} = 1 \leftrightarrow \text{normalized frequency} = 0.5}$$

- The filter order corresponds to the number of samples - 1

BIN sono in freq, TAP nel tempo

FIR design in MATLAB:fir1

Low-pass: ‘h = fir1(66, 0.5, hann(67))’

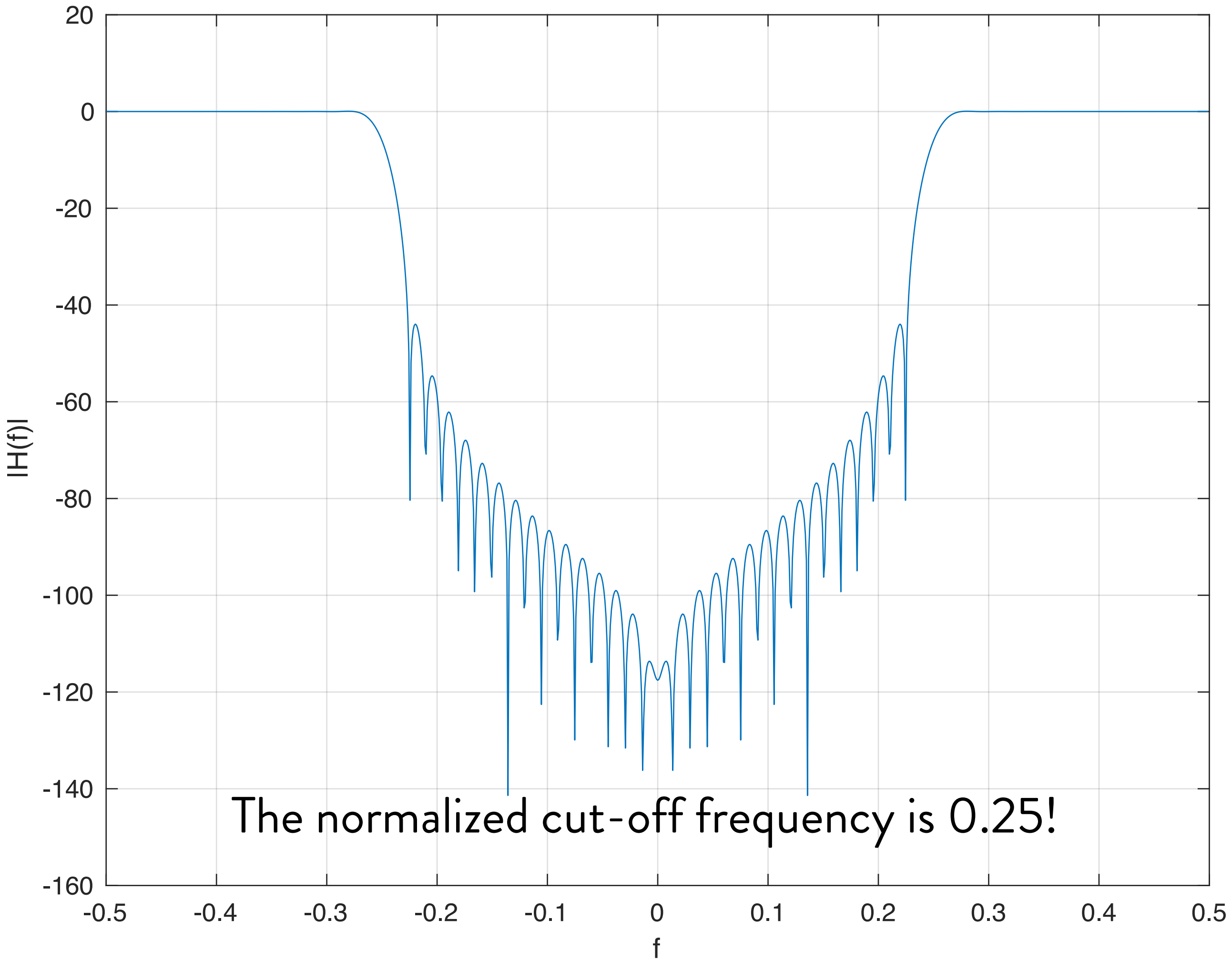


The normalized cut-off frequency is 0.25!

NB. IN MATLAB è semplicemente il doppio dato che considera intervallo tra 0 e 1, invece di 0 a 0,5, quindi devo moltiplicare per 2

FIR design in MATLAB:fir1

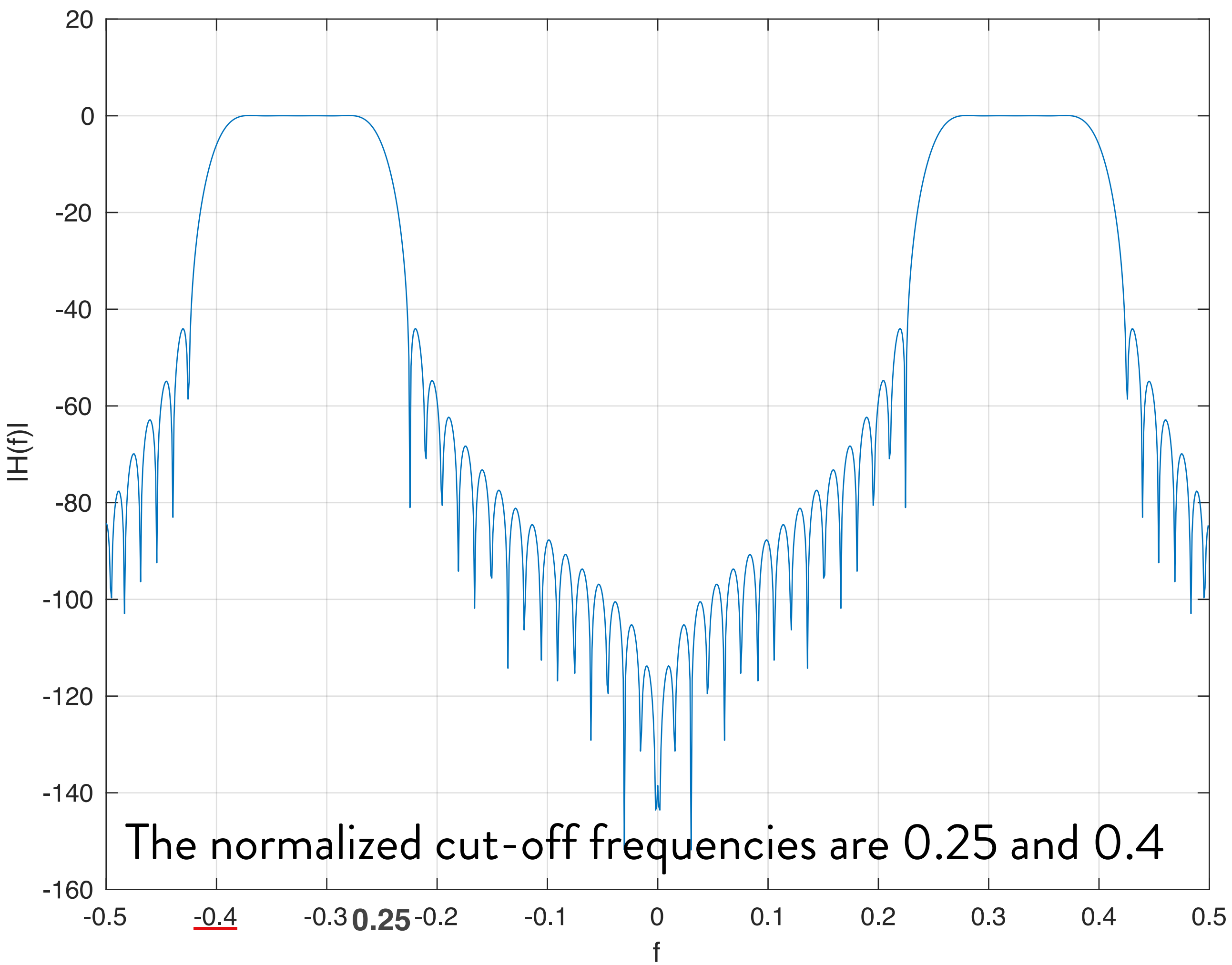
High-pass: ‘h = fir1(66, 0.5, ‘high’, hann(67))



FIR design in MATLAB:fir1

double cut off

Band-pass: ‘h = fir1(66, [0.5, 0.8], hann(67))



Es 26: windowing [~exam 10/09/2019, 4pts]

- Given x as a cosine wave sampled at Fs = 8KHz, duration 1 second, amplitude 1.5, frequency 1.1KHz, phase 45 deg.
- Plot the sinusoid vs time
- Compute y as x filtered with a low-pass filter with normalized cut-off frequency of 0.4 and 64 weights
- Apply a Hanning window to select the first 512 samples of y
- Plot the magnitude of the DFT of the windowed y versus frequency in Hz.
- If you change the cut-off frequency to 0.05, what do you expect to see in the spectrum of y?