POLITECNICO
MILANO 1863

DIPARTIMENTO DI ELETTRONICA
INFORMAZIONE E BIOINGEGNERIA

Z transform

DTFT definition

- The Discrete Time Fourier Transform (DTFT) of a sequence x(n) is defined as

$$X(f) = \sum_{-\infty}^{+\infty} x(n) \cdot e^{-j\omega n}$$

- ω is the normalized angular frequency in $[-\pi, \pi]$
- In the DTFT frequency is continuous, while time is discrete. We will skeep DTFT, directly going to DFT (discrete in time and frequency) in next classes.

Z transform definition

- The Z transform of a sequence x(n) is defined as

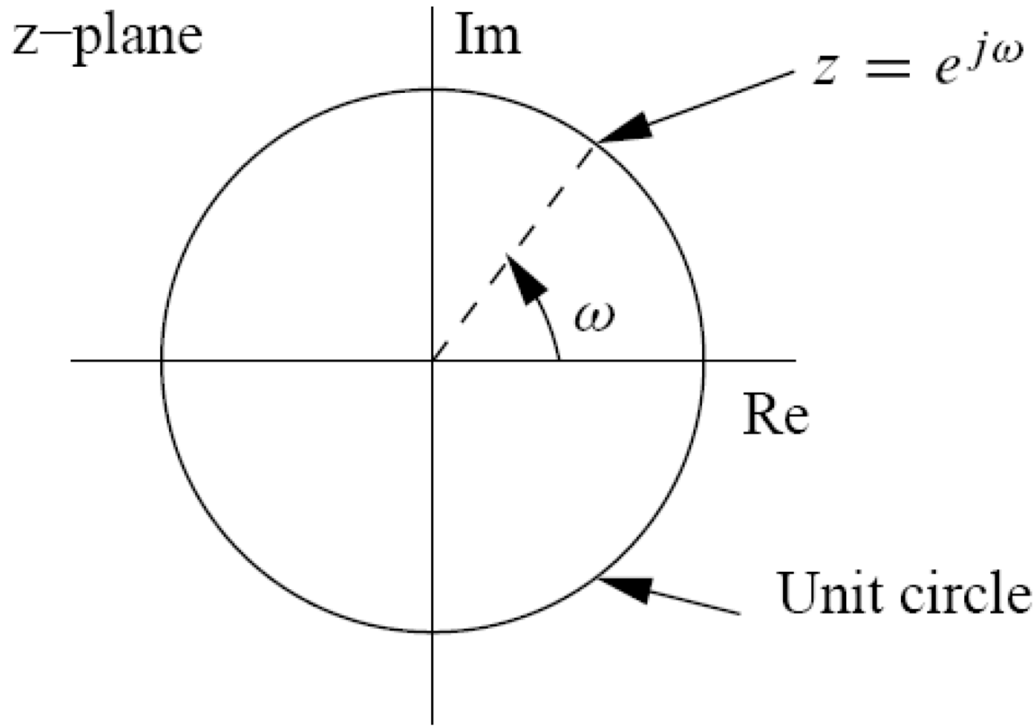
$$X(z) = \sum_{-\infty}^{+\infty} x(n) \cdot z^{-n}$$

- z is a complex variable, $z = \rho e^{j2\pi f}$
- Its relationship with the Discrete Time Fourier Transform is

$$X(f) = X(z)\Big|_{|z|=1}$$

Z transform definition

- Since z is a complex number, we can represent it in the complex plane.



$$X(f) = X(z)\Big|_{|z|=1}$$

Z transform example

- Given the sequence

$$x(n) = \delta(n) + \delta(n+1) + 2\delta(n-2)$$

$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{-\infty}^{\infty} \delta(n) z^{-n} + \sum_{-\infty}^{\infty} \delta(n+1) z^{-n} + \sum_{-\infty}^{\infty} 2\delta(n-2) z^{-n}$$

$$X(z) = 1 + z + 2z^{-2}$$

Z transform example

- Given the sequence $x(n) = a^n u(n)$

$$X(z) = \sum_{-\infty}^{\infty} a^n u(n) z^{-n}$$

$$X(z) = \sum_{0}^{\infty} a^n z^{-n} = \sum_{0}^{\infty} (az^{-1})^n$$

- Recalling geometric series:

$$\sum_{0}^{K} a^n = \frac{1 - a^{K+1}}{1 - a} \qquad \sum_{0}^{\infty} a^n = \frac{1}{1 - a}$$

$$\boxed{x(n) = a^n u(n) \Rightarrow X(z) = \frac{1}{1 - az^{-1}}}$$

Z transform ROC

- The Z transform converges only if

$$X(z) = \sum_{-\infty}^{\infty} |x(n) z^{-n}| < \infty$$

- The set of values z for which $X(z)$ converges is called Region of Convergence (ROC) $\Rightarrow$ $R_- \leq |z| \geq R_+$

- Example: given $x(n) = 2^n u(n)$

- Z-transform $X(z) = \sum_{0}^{\infty} (2z^{-1})^n$ converges only if

$$|2z^{-1}| < 1 \Rightarrow |z| > 2$$

Z transform properties

- $Z\{ax(n)+by(n)\} = aX(z)+bY(z)$, ROC = ROC$_x$ ∩ ROC$_y$
- $Z\{x(n-k)\} = X(z)z^{-k}$, ROC = ROC$_x$
- $Z\{x(n)a^n\} = X\left(\frac{z}{a}\right)$, ROC = ROC$_x$ /|a|
- $Z\{x(-n)\} = X(z^{-1})$, ROC = 1/ROC$_x$
- $Z\{nx(n)\} = -z\frac{dX(z)}{dz}$, ROC = ROC$_x$
- $\boxed{Z\{x(n)*y(n)\} = X(z)\cdot Y(z)}$, ROC = ROC$_x$ ∩ ROC$_y$

Convolution theorem

POLITECNICO
MILANO 1863

DIPARTIMENTO DI ELETTRONICA
INFORMAZIONE E BIOINGEGNERIA

Exercises

Es 10a: Z-transform convolution property

- Given a signal x(n) = [3, 2, 1, 0, 1], n in [-2, 2]
- Given a LTI system with h(n) = [1, 3, 2.5, 4, 2], n in [0, 4]
- Compute the output of the system using 'conv'. Which is the support of y(n)?
- Write the expression of H(z).
- Which is the order of polynomial H(z)?

Z-transform expressions

Z transform expression

There are several useful ways to represent Z-transform.

1. $X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + ... + a_D z^{-D}}$,

- N(z) and D(z) are polynomial expressed in z^-1
- Useful to compute the Inverse Z transform

2. $X(z) = z^{D-N} \frac{b_0}{a_0} \frac{\prod_{i=1}^{N}(z - z_i)}{\prod_{i=1}^{D}(z - p_i)}$

- Useful to for filter characterization
- z_i are called ‘zeros’, p_i are called ’poles’

Z transform relationship with LTI systems

Given x(n) and h(n) (impulse response of LTI system):

$$y(n) = x(n) * h(n)$$

The same system can also be described by a linear difference equation with constant coefficients

$$\sum_{k=0}^{D} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k)$$

$$\Rightarrow \quad y(n) = \boxed{\sum_{k=0}^{N} b_k x(n-k)} - \boxed{\sum_{k=1}^{D} a_k y(n-k)}$$

Moving Average (FIR) Autoregressive (IIR)

Z transform relationship with LTI systems

$$\sum_{k=0}^{D} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k)$$

Converting in Z domain

$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{k=0}^{N} b_k z^{-k}$$

Since $y(n) = x(n) * h(n)$,

Thanks to the convolution theorem: $Y(z) = X(z)H(z)$

$$H(z) = \cancel{\frac{X(z)}{Y(z)}} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\sum_{k=0}^{D} a_k z^{-k}}$$

inverso

Inverse Z transform

$$H(z) = \frac{X(z)}{Y(z)} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\sum_{k=0}^{D} a_k z^{-k}}$$

$$\Rightarrow \quad h(n) = Z^{-1}\{H(z)\}$$

$h(n)$ can be computed in different ways:

1. Long division
2. Partial fract expansion
3. Viewing H(z) as cascade of filters $h_1(n) * h_2(n) * h_3(n) \ldots$

Inversion of a polynomial Z transform

- Given the Z-transform of h(n), $H(z) = \sum_{n=0}^{k} h(n) z^{-n}$

- We can compute its root decomposition:

$$H(z) = h_0 \prod_{n=1}^{k} (1 - z_n z^{-1}), h_0 = H(n = 0)$$

- z_n are called roots of the polynomial $H(z)$, $H(z = z_n) = 0$

- Thanks to the convolution theorem,

$$H(z) = H_1(z) H_2(z) H_3(z) ... H_k(z)$$

$$\Downarrow$$

$$h(n) = h_1(n) * h_2(n) * h_3(n) * ... * h_k(n)$$

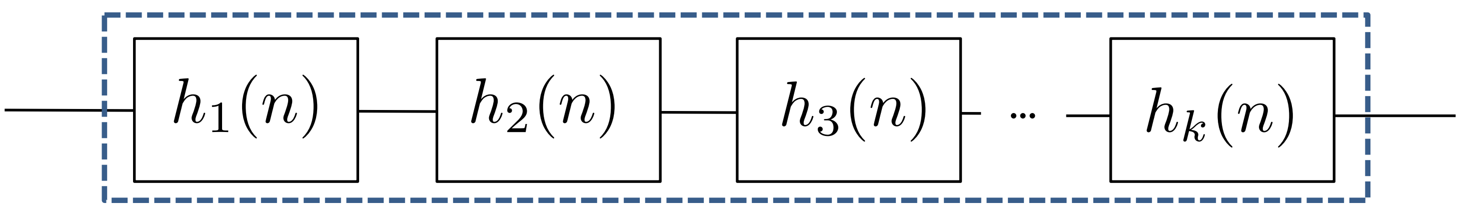
Inversion of a polynomial Z transform

- $H(z) = h_0 \prod_{n=1}^{k} (1 - z_n z^{-1})$

- Given the roots z_n, $H(z = z_n) = 0$

- Thanks to the convolution theorem we can derive the impulse response as the cascade of multiple filters written as elementary sequences . For example,

$$h_1(n) = Z^{-1}\{1 - z_1 z^{-1}\} = \delta(n) - z_1\delta(n-1)$$



$$h(n) = h_1(n) * h_2(n) * h_3(n) * \ldots * h_k(n)$$

Exercises

Es 10b: Z-transform convolution property

- Given a signal x(n) = [3, 2, 1, 0, 1], n in [-2, 2]
- Given a LTI system with h(n) = [1, 3, 2.5, 4, 2], n in [0, 4]
- Compute the output of the system using 'conv'. Which is the support of y(n)?
- Exploiting the convolution theorem, compute Y(z) = X(z) H(z)
- Write the expression of H(z).
- Which is the order of polynomial H(z)?
- Compute the roots of H(z).
- Write y1(n) as the convolution of x(n) with the filter cascade:

$$h(n) = h_0 \cdot h_1(n) * h_2(n) * h_3(n) * \ldots * h_k(n)$$

- Plot y(n) and y1(n) in the same figure and check if y1(n) = y(n)

Partial fract expansion for computing Z^{-1}

$$H(z) = \frac{X(z)}{Y(z)} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\sum_{k=0}^{D} a_k z^{-k}}$$

$$\Rightarrow \quad h(n) = Z^{-1}\{H(z)\}$$

$$H(z) = \sum_{k=1}^{D} \sum_{m=1}^{M} \frac{r_{k_m}}{(1 - p_k z^{-1})^m} + \boxed{\sum_{k=0}^{N-D} c_k z^{-k}}$$

$$N \geq D$$

k=diff gradi=N-D

M is the molteplicity of the root (or 'pole') p_k .

The Z transform inversion is the sum of simple inversions.

Partial fract expansion for computing Z^{-1}

$$H(z) = \sum_{k=1}^{D} \sum_{m=1}^{M} \frac{r_{k_m}}{(1 - p_k z^{-1})^m} + \boxed{\sum_{k=0}^{N-D} c_k z^{-k}}$$

$$N \geq D$$

The Z transform inversion is the sum of simple inversions (causal):

- $Z^{-1}\left\{\frac{r_{k_1}}{(1 - p_k z^{-1})}\right\} = r_{k_1} \cdot (p_k)^n u(n)$
- $Z^{-1}\left\{\frac{r_{k_2}}{(1 - p_k z^{-1})^2}\right\} = r_{k_2} \cdot (n+1)(p_k)^n u(n)$
- $Z^{-1}\left\{c_k z^{-k}\right\} = c_k \cdot \delta(n-k)$

Partial fract expansion for computing Z^{-1}

$$\boxed{H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\sum_{k=0}^{D} a_k z^{-k}} = \sum_{k=1}^{D} \sum_{m=1}^{M} \frac{r_{k_m}}{(1 - p_k z^{-1})^m} + \sum_{k=0}^{D-N} c_k z^{-k}}$$

- Residues r_{k_m}, poles p_k and c_k can be found using the MATLAB function '[residues, poles, c_k] = residuez(b, a)'
- 'b' is the vector of the numerator coefficients (ordered from b_0 to b_N)
- 'a' is the vector of the denominator coefficients (ordered from a_0 to a_D).

Another MATLAB solution for Z^{-1}

- Given $H(z) = \frac{X(z)}{Y(z)} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\sum_{k=0}^{D} a_k z^{-k}}$
- The inverse Z transform can be found using the MATLAB function 'h= filter(b, a, x)'
- 'b' is the vector of the numerator coefficients (ordered from b_0 to b_N)
- 'a' is the vector of the denominator coefficients (ordered from a_0 to a_D)
- 'x' is the input signal to the system. To find h(n), x must be…?

Exercises

Es 11.a: Partial fract expansion of Z transform

- Given a LTI system with this transfer function:

$$H(z) = \frac{z^{-5} + z^{-4} - 3z^{-3} - 8z^{-2} + 7z^{-1} + 9}{z^{-3} - 2z^{-2} - z^{-1} + 2}$$

- Find its partial fract expansion:
  - Save in a vector r the residues
  - Save in a vector p the poles
  - Save in a vector c the coefficients of the polynomial $\sum_{k=0}^{N-D} c_k z^{-k}$
- Find h(n) using 'filter', n = 0:100.
- Find h(n) as the sum of elementary filters founded with the partial fract expansion, n = 0:100.

Es 11.b: Partial fract expansion of Z transform

- Given a LTI system with this transfer function:

$$H(z) = \frac{z^{-5} + 4z^{-4} + 6z^{-3} + z^{-2} - 2z^{-1} - 3}{z^{-3} + z^{-2} - z^{-1} - 1}$$

- Find its partial fract expansion:
  - Save in a vector r the residues
  - Save in a vector p the poles
  - Save in a vector c the coefficients of the polynomial $\sum_{k=0}^{N-D} c_k z^{-k}$
- Find h(n) using 'filter', n = 0:100.
- Find h(n) as the sum of elementary filters founded with the partial fract expansion, n = 0:100.

Zeros-Poles factorization

Zeros-Poles factorization

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\sum_{k=0}^{D} a_k z^{-k}} = z^{D-N} \frac{b_0}{a_0} \frac{\prod_{i=1}^{N} (z - z_i)}{\prod_{i=1}^{D} (z - p_i)}$$

- Roots of numerator are called ‘zeros’
- Roots of denominator are called ‘poles’
- In MATLAB we can plot poles and zeros with the function
- ‘zplane(z, p)’ (zeros, poles, in column vectors)
- ‘zplane(b, a)’ (numerator, denominator, in row vectors)

System stability

- For a system to be stable,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

- If all the poles of H(z) are inside the unitary circle $(|p_i| < 1,\ \forall i)$, the system is stable.
- If one positive zero is inside the unitary circle, it is called 'minimum phase'
- If one positive zero is outside the unitary circle, it is called 'maximum phase'

POLITECNICO
MILANO 1863

DIPARTIMENTO DI ELETTRONICA
INFORMAZIONE E BIOINGEGNERIA

Exercises

Es 12: zeros-poles factorization

- Given $y(n) = x(n) - ax(n-1) + by(n-1)$
- Which is the expression of H(z)?
- Which is the value of h(0)? Derive it from H(z). è sempre il rapporto tra i coeff di grado 0 (aka z^0) del nume e den della tf
- Compute the zeros and poles.
- Plot h(n) for n in [0, 50] for a = 0.5 and b = 0.2.
- Plot h(n) for n in [0, 50] for a = 1.2 and b = 0.2
- Plot h(n) for n in [0, 50] for a = 1.2 and b = 1.1
- In which situations is the system stable?
- When the zeros are minimum phase?

Es 13: zeros-poles factorization

- Given

$$y(n) = -2\rho cos(\theta)y(n-1) - \rho^2 y(n-2) + x(n) + 2x(n-1) + x(n-2)$$

- $\rho = 0.9, \theta = \pi/8$
- Which is the expression of H(z)?
- Which is the value of h(0)? Derive it from H(z).
- Which is the expression of h(n)? Try using ‘filter’ and ‘residuez’.
- Compute its zeros and poles.
- Plot its zeros and poles.
- Is the system stable?