Optimal Trajectory Planning for an Omni-Directional Mobile Robot with Static Obstacles: A Polynomial Based Approach

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***Abstract*— This paper presents a polynomial based optimal trajectory planning for an omni-directional mobile robot in presence of static obstacles with considering a limitation on velocity and acceleration of the robot. First, optimal trajectory planning problem is formulated as an optimal control problem which minimize a cost function of states and control efforts respect to constraints of the problem. To solve this optimal control problem, a state parameterization method is used. In fact, state variables of system are approximated by polynomial functions of time with unknown coefficients. Thus optimal control problem converts to a constraint optimization problem which is too easier than original optimal control problem. Then the polynomials coefficients are computed such that satisfy all the problem requirements and constraints. Simulation results show effectiveness of the proposed method under different situations.**

Keywords*— *omni-directional robot; optimal trajectory planning; state parameterization; static obstacles;

I. Introduction

Omni-directional robot is a kind of mobile robots which has the ability of moving along any direction, irrespective of the robot orientation. This maneuverability, make it an attractive option in static and dynamic environments. The small size league of the annual Robocop competition is an example where Omni-directional mobile robots have been used [1].

There are some papers which have presented kinematic and dynamic model of Omni-directional mobile robots [2-4]. In these models, dynamic behavior of drivers are considered but the nonlinear coupling between the wheels has been ignored. It causes robot dynamic model is simplified as a linear system. The model of the nonlinear coupled dynamics for three-wheeled Omni-directional robot was modelled in [5].

There are many works on control of Omni-directional mobile robots. In the motion control problems of omni-directional mobile robot many research have been done. In [6] a back-stepping controller based on the Newtonian mechanism and with proven global stability has been proposed. Ref [7] related to a back-stepping control method which synthesize a nonlinear controller for a three-wheel omni-directional mobile robot by using of squares technique. Authors of [1] have designed a nonlinear controller which consists of an outer-loop and an inner-loop controller by using the trajectory linearization control method and based on a nonlinear dynamic model.

In the research on omni-directional mobile robot, trajectory planning is an important issue. Generally, trajectory planning for mobile robots is finding the desired trajectory with minimization of a given objective function [8]. The trajectory planning problem can be formulated in terms of an optimal control problem [9]. It is clear that obtaining an exact analytical solution of nonlinear optimal control problem is out of reach. However, there are many numerical and approximate analytical methods which help us to find a solution. One of the methods for solving the optimal control problem which is applied in many works is using Hamiltonian approach [10]. Inequality states constraints exist in our problem formulation which the constraints lead to appear Heaviside functions. Existence of this functions cause complexity for solving the optimal control problem. Also, dynamic states constraints may not be entered in Hamiltonian approach. Thus in this work we use the parameterization method. This method transforms the optimal control problem into a parametric optimization problem which from computational point of view is too easier than the main optimal control problem. The parameterization method related to parameterization variables divided to parameterization of control variables which is applied in [11], parameterization of state variables which is presented in [12], and parameterization of both the state and control variables which proposed in [13].

In this paper we present a state parameterization method to find the states and energy consumption optimal solution for trajectory planning of an Omni-directional mobile robot. The objective of this paper is formulation of optimal trajectory planning problem as a nonlinear optimal control problem and finding an approximated solution by state parameterization for it. The goal of the trajectory planning is moving the robot to a desired location in a fixed time $t_0 - t_f$ without collisions

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with static obstacles respect to minimization of states and energy consumption and with considering all the velocity and acceleration constraints.

The organization of the paper is as follows. In section II, we introduce the requirements and assumption of this trajectory planning problem. In the section III, the dynamic model of an omni-directional robot is presented. Section IV, is related to formulation of the optimal trajectory planning as a optimal control problem. A solution to the optimal control problem is proposed in Section V. simulation results are given in section VI. To illustrate the effectiveness of the proposed method. Finally, the paper is concluded is Section VII.

II. Problem Statement

The goal of the optimal trajectory planning is finding a trajectory which based on it, robot will move from a known initial point to another known point as a final point.

- Avoiding collisions with static obstacles
- Minimization of states and energy consumption
- Limitation on maximum velocity and acceleration of robot
- Zero velocity of robot in initial and final point

To avoid discontinuities in the solution when getting close to the desired final state, final velocity always is chosen zero [2] and [14].

III. The Omni-Directional Mobile Robot Model

In this section the model of omni-directional mobile robot is presented. The simplified equations which were expressed in [2] to describe the decoupled motion equations are as following:

$$\ddot{x}(\mathrm{t}) + \dot{x}(\mathrm{t}) = q_x(\mathrm{t}), \tag{1}$$

$$\ddot{y}(\mathrm{t}) + \dot{y}(\mathrm{t}) = q_y(\mathrm{t}), \tag{2}$$

$$\ddot{\theta}(\mathrm{t}) + \frac{2mL^2}{J}\dot{\theta}(\mathrm{t}) = q_\theta(\mathrm{t}) \tag{3}$$

While these equations are linear and decoupled, the control efforts are remain coupled with the constraints

$$q_x^2(\mathrm{t}) + q_y^2(\mathrm{t}) \le \left(\frac{3 - |q_\theta(\mathrm{t})|}{2}\right)^2, \tag{4}$$

$$|q_\theta(\mathrm{t})| \le 3 \tag{5}$$

Where as shown in Fig.1, vector $p_0 = [x \quad y]^T$ is the position of the center of mass (CM) of the robot in a Newtonian frames, θ is the angle of counterclockwise rotation, L is the distance of the drive units from the CM, m is the mass of the robot, J is its moment of inertia, $q_x(\mathrm{t})$, $q_y(\mathrm{t})$, and $q_\theta(\mathrm{t})$ are the control efforts.

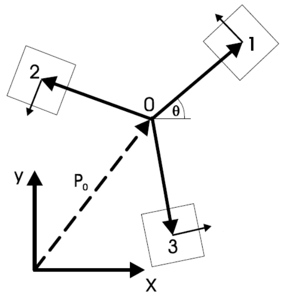

Fig. 1. Geometry of the omni-directional mobile robot [2]

In the remainder to decouple the θ equation from those of the translational ones, they set

$$|q_\theta(\mathrm{t})| \le 1 \tag{6}$$

Then the constraint on the control inputs becomes

$$q_x^2(\mathrm{t}) + q_y^2(\mathrm{t}) \le 1 \tag{7}$$

Finally, the linear equations of system are proposed as follows:

$$\ddot{x}(\mathrm{t}) + \dot{x}(\mathrm{t}) = q_x(\mathrm{t}), \tag{8}$$

$$\ddot{y}(\mathrm{t}) + \dot{y}(\mathrm{t}) = q_y(\mathrm{t}) \tag{9}$$

With constraint on the control inputs which are given by

$$q_x^2(\mathrm{t}) + q_y^2(\mathrm{t}) \le 1 \tag{10}$$

Now, to obtain state space equations of system, we consider four state variables as $x_1(\mathrm{t})$, $x_2(\mathrm{t})$, $x_3(\mathrm{t})$, and $x_4(\mathrm{t})$ Where $x_1(\mathrm{t})$ and $x_2(\mathrm{t})$ are the locations of robot in x and y direction respectively; $x_3(\mathrm{t})$ and $x_4(\mathrm{t})$ are the velocities of robot in x and y direction respectively. Also denote $u_1(\mathrm{t}) \triangleq q_x(\mathrm{t})$ and $u_2(\mathrm{t}) \triangleq q_y(\mathrm{t})$; thus state variables vector and inputs vector of system are given by follows:

$$X(\mathrm{t}) = \begin{bmatrix} x_1(\mathrm{t}) = x(\mathrm{t}) \\ x_2(\mathrm{t}) = y(\mathrm{t}) \\ x_3(\mathrm{t}) = \dot{x}(\mathrm{t}) = v_x(\mathrm{t}) \\ x_4(\mathrm{t}) = \dot{y}(\mathrm{t}) = v_y(\mathrm{t}) \end{bmatrix}, \; U(\mathrm{t}) = \begin{bmatrix} u_1(\mathrm{t}) \\ u_2(\mathrm{t}) \end{bmatrix}, \tag{11}$$

And the state space equations of system are as follows:

$$\begin{cases} \dot{x}_1(\mathrm{t}) = x_3(\mathrm{t}) = v_x(\mathrm{t}) \\ \dot{x}_2(\mathrm{t}) = x_4(\mathrm{t}) = v_y(\mathrm{t}) \\ \dot{x}_3(\mathrm{t}) = -x_3(\mathrm{t}) + u_1(\mathrm{t}) = a_x(\mathrm{t}) \\ \dot{x}_4(\mathrm{t}) = -x_4(\mathrm{t}) + u_2(\mathrm{t}) = a_y(\mathrm{t}) \end{cases} \tag{12}$$

With following constraint on the control inputs

$$u_1^2(\mathrm{t}) + u_2^2(\mathrm{t}) \le 1 \tag{13}$$

Which $a_x(\text{t})$ and $a_y(\text{t})$ are the accelerations of robot in x and y direction respectively; $u_1(\text{t})$ and $u_2(\text{t})$ are the control efforts of system in x and y direction respectively.

IV. Problem Formulation

The optimal trajectory planning can be formulated as an optimal control problem as follows:

Find the control $U^*(\text{t})$ in $t_0 \le t \le t_f$ which minimizes the cost function J :

$$J = \frac{1}{2}\int_{t_0}^{t_f} (\text{x}(t)^T Qx(t) + u(t)^T Ru(t))\,\text{dt}, \qquad (14)$$

Subject to robot dynamics (12), Initial and final conditions:

$$\begin{aligned} &x_1(t_0) = x_{1_0},\ x_2(t_0) = x_{2_0}, \\ &x_3(t_0) = x_{3_0},\ x_4(t_0) = x_{4_0}, \end{aligned} \qquad (15)$$

$$\begin{aligned} &x_1(t_f) = x_{1_f},\ x_2(t_f) = x_{2_f}, \\ &x_3(t_f) = 0,\ x_4(t_f) = 0, \end{aligned} \qquad (16)$$

State inequality constraints due to the i circular obstacles

$$\bigcup_i (\text{x}_1(\text{t}) - \text{x}_{c_1})^2 + (\text{x}_2(\text{t}) - \text{x}_{c_2})^2 \ge r^2, \qquad (17)$$

Control efforts constraint (13) and maximum limitation velocity and acceleration of the robot

$$\sqrt{v_x^2(\text{t}) + v_y^2(\text{t})} \le v_{\max}, \qquad (18)$$

$$\sqrt{a_x^2(\text{t}) + a_y^2(\text{t})} \le a_{\max} \qquad (19)$$

Which x_{1_0} and x_{2_0} are the initial positions in x and y direction, x_{1_f} and x_{2_f} are the final positions, x_{3_0} and x_{4_0} are the initial velocities, $X_c = \begin{bmatrix} x_{c_1} \\ x_{c_2} \end{bmatrix}$ is the center and r is the radius of circular obstacles, $v_{\max}$ and $a_{\max}$ are the maximum velocity and acceleration of robot respectively.

V. Solving Optimal Control Problem

In this section a solution method of the optimal control problem is proposed which is based on the parameterization method.

First, two states of system are considered as polynomial functions of time. Then from (12), the expression of control efforts and other two states are obtained as a function of unknown parameters of the approximated state variables.

Let consider $x_1(\text{t})$ and $x_2(\text{t})$ as polynomial functions of time as follows:

$$x_1(\text{t}) = \sum_{m=0}^{n} a_m t^m,\ n = 1,2,3,... \qquad (20)$$

$$x_2(\text{t}) = \sum_{p=0}^{n} b_p t^p,\ n = 1,2,3,... \qquad (21)$$

Thus the expression of $x_3(\text{t})$, $x_4(\text{t})$, $u_1(\text{t})$, and $u_2(\text{t})$ can be determined as follows:

$$x_3(\text{t}) = \sum_{m=1}^{n} m a_m t^{m-1}, \qquad (22)$$

$$x_4(\text{t}) = \sum_{p=1}^{n} p b_p t^{p-1}, \qquad (23)$$

$$u_1(\text{t}) = \sum_{m=1}^{n} m a_m t^{m-1} + \sum_{m=2}^{n} m(\text{m}-1) a_m t^{m-2}, \qquad (24)$$

$$u_2(\text{t}) = \sum_{p=1}^{n} p b_p t^{p-1} + \sum_{p=2}^{n} p(p-1) b_p t^{p-2} \qquad (25)$$

Now by substituting (20)-(25) into the cost function (14) and with considering Q and R as elementary matrices, J can be expressed as follows:

$$\begin{aligned} J(a_n, b_n) = \frac{1}{2}\int_{t_0}^{t_f} \Bigg\{ &\left(\sum_{m=0}^{n} a_m t^m\right)^2 + \left(\sum_{p=0}^{n} b_p t^p\right)^2 \\ &+ \left(\sum_{m=1}^{n} m a_m t^{m-1}\right)^2 + \left(\sum_{p=1}^{n} p b_p t^{p-1}\right)^2 \\ &+ \left(\sum_{m=1}^{n} m a_m t^{m-1} + \sum_{m=2}^{n} m(\text{m}-1) a_m t^{m-2}\right)^2 \\ &+ \left(\sum_{p=1}^{n} p b_p t^{p-1} + \sum_{p=2}^{n} p(p-1) b_p t^{p-2}\right)^2 \Bigg\} dt \end{aligned} \qquad (26)$$

From initial and final conditions (15) and (16) we get:

$$x_1(t_0) = \sum_{m=0}^{n} a_m t^m \Bigg|_{t=t_0} = x_{1_0}, \qquad (27)$$

$$x_2(t_0) = \sum_{p=0}^{n} b_p t^p \Bigg|_{t=t_0} = x_{2_0}, \qquad (28)$$

$$x_3(t_0) = \sum_{m=1}^{n} m a_m t^{m-1} \Bigg|_{t=t_0} = x_{3_0}, \qquad (29)$$

$$x_4(t_0)=\sum_{p=1}^{n} pb_p t^{p-1}\Big|_{t=t_0}=x_{4_0}, \qquad (30)$$

$$x_1(t_f)=\sum_{m=0}^{n} a_m t^{m}\Big|_{t=t_f}=x_{1_f}, \qquad (31)$$

$$x_2(t_f)=\sum_{p=0}^{n} b_p t^{p}\Big|_{t=t_f}=x_{2_f}, \qquad (32)$$

$$x_3(t_f)=\sum_{m=1}^{n} ma_m t^{m-1}\Big|_{t=t_f}=0, \qquad (33)$$

$$x_4(t_f)=\sum_{p=1}^{n} pb_p t^{p-1}\Big|_{t=t_f}=0 \qquad (34)$$

Also constraints of problem can be rewritten as follows:

$$\bigcup_i\left(\sum_{m=0}^{n} a_m t^m - x_{c_1}\right)^2+\left(\sum_{p=0}^{n} b_p t^p - x_{c_2}\right)^2 \ge r^2, \qquad (35)$$

$$\left(\sum_{m=1}^{n} ma_m t^{m-1}+\sum_{m=2}^{n} m(\mathrm{m}-1)a_m t^{m-2}\right)^2 +\left(\sum_{p=1}^{n} pb_p t^{p-1}+\sum_{p=2}^{n} p(p-1)b_p t^{p-2}\right)^2 \le 1, \qquad (36)$$

$$\sqrt{\left(\sum_{m=1}^{n} ma_m t^{m-1}\right)^2+\left(\sum_{p=1}^{n} pb_p t^{p-1}\right)^2} \le v_{\max}, \qquad (37)$$

$$\sqrt{a_x^2(\mathrm{t})+a_y^2(\mathrm{t})} \le a_{\max} \qquad (38)$$

Which

$$a_x(t)=-\sum_{m=1}^{n} ma_m t^{m-1}+\sum_{m=1}^{n} ma_m t^{m-1} +\sum_{m=2}^{n} m(\mathrm{m}-1)a_m t^{m-2}, \qquad (39)$$

$$a_y(t)=-\sum_{p=1}^{n} pb_p t^{p-1}+\sum_{p=1}^{n} pb_p t^{p-1} +\sum_{p=2}^{n} p(p-1)b_p t^{p-2} \qquad (40)$$

In order to solve this constraint optimization problem, we use Matlab's *fmincon* function which is a part of the optimization toolbox [15]. This function implements a sequential quadratic programming to solve nonlinearly constraint optimization problems.

VI. SIMULATION RESULTS

In this section we utilize the proposed approach in different conditions to illustrate the effectiveness of the method. In all simulations we consider $n=4$. Indeed, both of $x_1(\mathrm{t})$ and $x_2(\mathrm{t})$ are approximated by fourth-order polynomials function as follows:

$$x_1(\mathrm{t})=a_0+a_1t+a_2t^2+a_3t^3+a_4t^4, \qquad (41)$$

$$x_2(\mathrm{t})=b_0+b_1t+b_2t^2+b_3t^3+b_4t^4 \qquad (42)$$

Then $x_3(\mathrm{t})$, $x_4(\mathrm{t})$, $u_1(\mathrm{t})$, and $u_2(\mathrm{t})$ can be determined as follows:

$$x_3(\mathrm{t})=a_1+2a_2t+3a_3t^2+4a_4t^3, \qquad (43)$$

$$x_4(\mathrm{t})=b_1+2b_2t+3b_3t^2+4b_4t^3, \qquad (44)$$

$$u_1(\mathrm{t})=a_1+2a_2+(2a_2+6a_3)t+(3a_3+12a_4)t^2+4a_4t^3, \qquad (45)$$

$$u_2(\mathrm{t})=b_1+2b_2+(2b_2+6b_3)t+(3b_3+12b_4)t^2+4b_4t^3 \qquad (46)$$

Also since the soccer robots for the small-size League of RoboCup must fit within $180mm$ diameter circle, we consider radius of the all circular obstacles equal to $0.09m$.

A. Scenario1: in presence a static obstacle

The objective of this scenario is finding the states and control optimal trajectory which based on it, the omni-directional robot will move in the fixed time $t_f=4s$ from initial location $\begin{pmatrix}x_{1_0}\\ x_{2_0}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ to final location $\begin{pmatrix}x_{1_f}\\ x_{2_f}\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$, with avoiding collisions with a static obstacle represented by a circle of radius $r=0.09$ and centered at $X_c=(0.3\,\mathrm{m}, 0.4\,\mathrm{m})$. Also we consider initial velocities of robot equal to zero.

First, we formulate the optimal trajectory planning problem as follows:

Find the optimal control which minimizes the following cost function

$$J=\frac{1}{2}\int_0^4(\mathrm{x}_1^2+\mathrm{x}_2^2+\mathrm{x}_3^2+\mathrm{x}_4^2+u_1^2+u_2^2)dt, \qquad (47)$$

Subject to dynamic system (12) with initial and final conditions

$$x_1(0)=0,\ x_2(0)=0,\ x_3(0)=0,\ x_4(0)=0, \qquad (48)$$

$$x_1(4)=1,\ x_2(4)=1,\ x_3(4)=0,\ x_4(4)=0, \qquad (49)$$

The state inequality constraint due to the obstacle

$$(x_1-0.3)^2+(x_2-0.4)^2 \ge 0.1^2, \qquad (50)$$

And control inequality constraint due to the modeling

$$u_1^2(t) + u_2^2(t) \le 1 \tag{51}$$

Now, to solve this optimal control problem, we use the proposed method.

Fig. 2 and Fig. 3 depict the location, velocity, and acceleration of robot, control effort, the value of cost function and the optimal state and control trajectory in 2D coordinate with a static obstacle respectively.

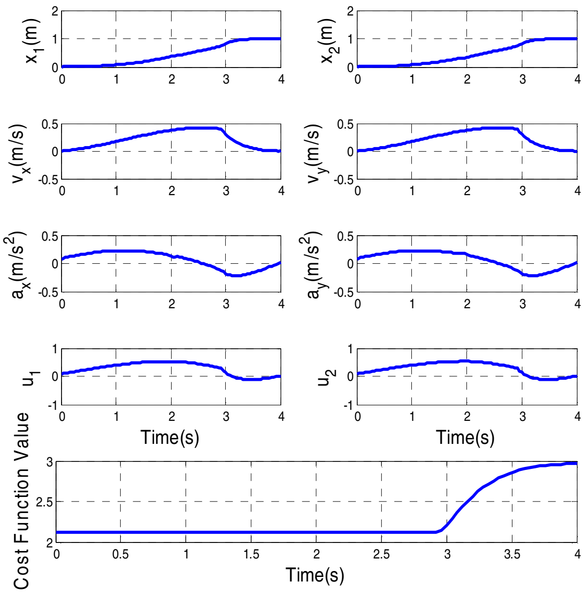

Fig. 2. Location, velocity, acceleration, control effort and The value of cost function with a static obtscle

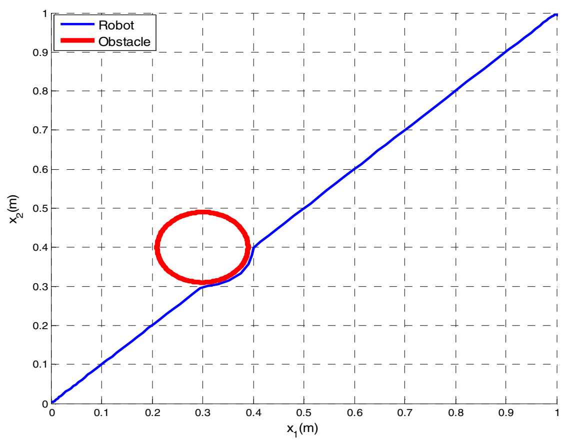

Fig. 3. Optimal trajectory in 2D coordinate with a static obtscle

B. *Scenario2 : in presence two static obstacles*

This scenario is totally similar with previous scenario only with considering two circular obstacles which are centered at

$$X_{c_1} = (0.2\,\mathrm{m}, 0.3\,\mathrm{m}), \quad X_{c_2} = (0.5\,\mathrm{m}, 0.4\,\mathrm{m}) \tag{52}$$

The state inequality constraints due to these obstacles are:

$$(x_1 - 0.2)^2 + (x_2 - 0.3)^2 \ge 0.1^2, \tag{53}$$

$$(x_1 - 0.5)^2 + (x_2 - 0.4)^2 \ge 0.1^2 \tag{54}$$

By using the proposed method, the position, velocity, acceleration of robot, control effort, the value of cost function and the optimal state and control trajectory in 2D coordinate are illustrated in Fig. 4 and Fig. 5.

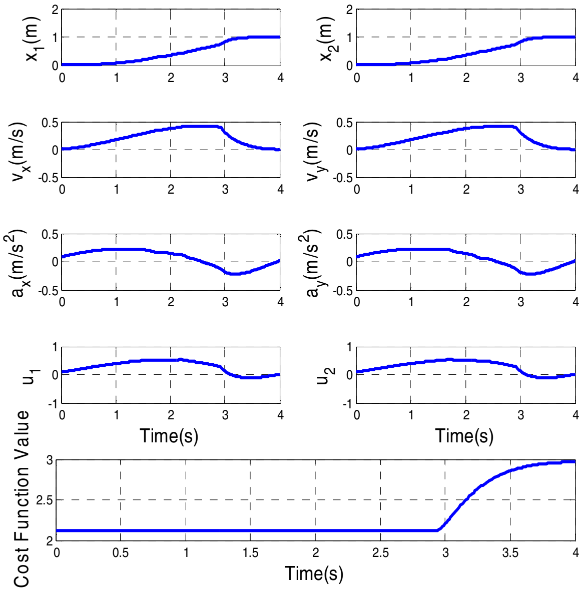

Fig. 4. Location, velocity, acceleration, control effort and The value of cost function with two static obtscles

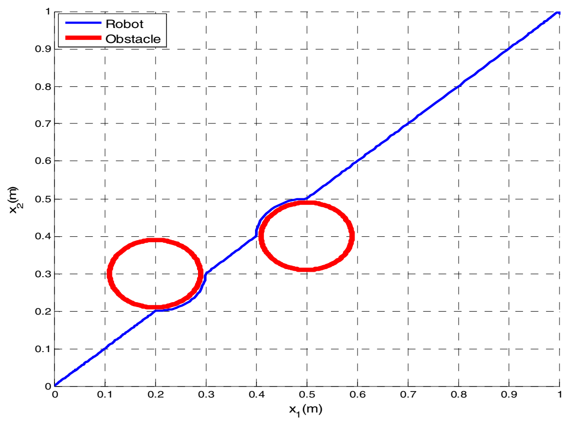

Fig. 5. Optimal states and control trajectory in 2D coordinate with two static obtscles

C. *Scenario3 : obstacle avoidance in presence two obstacles with considering limitation on velocity and acceleration*

In this scenario we consider a maximum limitation on velocity and acceleration of robot. The maximum velocity and acceleration are limited to $v_{\max} = 0.55\frac{m}{s}$ and $a_{\max} = 0.35\frac{m}{s^2}$ respectively. The states and control efforts inequality constraints due to these limitations are:

$$\sqrt{(x_3^2 + x_4^2)} \le 0.55, \tag{55}$$

$$\sqrt{(-x_3 + u_1)^2 + (-x_4 + u_2)^2} \le 0.35 \tag{56}$$

The simulation results for scenario3 are shown in Fig. 6 and Fig. 7.

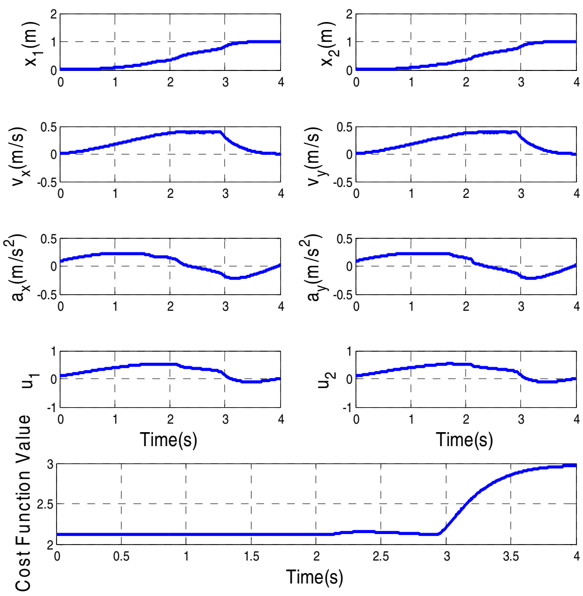

Fig. 6. Location, velocity, acceleration, control effort and the value of cost function with considering maximum velocity and acceleration constraint

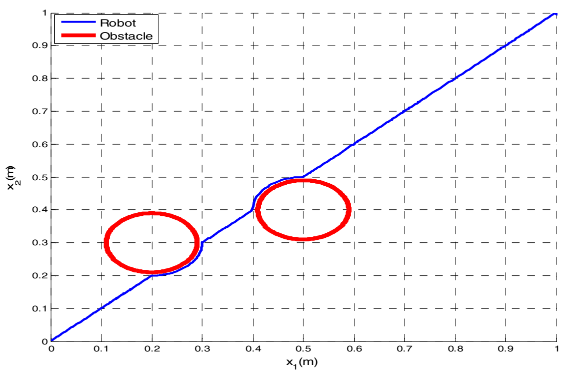

Fig. 7. Optimal states and control trajectory in 2D coordinate with considering maximum velocity and acceleration constraints

VII. Concolution

In this paper an optimal trajectory planning algorithm for omni-directional mobile robot in static environments was presented which takes the maximum limitation velocity and acceleration of robot into account. By formulating the problem as an optimal control problem, a state parameterization technique for solving this nonlinear optimal control problem was used. It converts the difficult nonlinear optimal control problem into an optimization problem with a few unknown parameters which can be solved easily. The proposed algorithm provides an efficient and low computational cost. The effectiveness of the proposed method was shown by different scenario simulations.

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