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Gravitational multipolar expansion: gravito-electromagnetism analogy and circular orbits of spinning particles

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Within the context of the multipolar expansion in General Relativity we address the question of whether the gravito-electromagnetic analogy for the dynamics of spinning bodies can be extended, using tidal tensors, to the quadrupole order. Our analysis of the equations of motion shows that the analogy breaks down at the quadrupolar order since the gravitational interaction involves a gravitational tidal tensor with no electromagnetic analogy, namely the magnetic-magnetic tidal tensor. Additionally, we explore the limits of the pole-dipole approximation by studying particular solutions of the equations of motion for a spinning particle for which the 4-momentum does not necessarily is a timelike vector. In particular, we analyze exact solutions representing circular orbits of a gyroscope around a Kerr black hole and discuss the values of the parameters for which the solution involves a spacelike 4-momentum. We find two different values of the angular velocity of the circular orbit that determine three regions in angular velocity space with different signs of the square norm of the 4-momentum. We also analyze the values of the gyroscope spin necessary to sustain such solutions and we compare our solution with previous known results.

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1. Introduction

The equations that describe the movement of extended bodies in the presence of a gravitational field have been studied for a long time, since they allow us to model the dynamics of realistic bodies, such as satellites, stars or black holes. The so-called Mathisson-Papapetrou-Dixon (MPD) equations[1–5] model the dynamics of test particles with spin (gyroscopes), and can be obtained as the first order correction of the dynamical equations of motion when considering the multipolar expansion of the energy-momentum distribution of the body. For a recent discussion of the definition and properties of the multipolar moments and its role in the multipolar expansion of the equation of motion, see Ref. 6.

For this article, we study two different aspects of the multipolar expansion of the equation of motion. The first is to analyze how the analogy between gravitation and electromagnetism which holds for the first order of the multipolar expansion, namely a gyroscope and an electric charge with a magnetic dipole moment, still is valid at higher order. The analogy between a gyroscope and a magnetic dipole has been analyzed in Ref. 7 using tidal tensors. Within this analogy one is able to obtain the MPD equations from the corresponding dynamical equations in the electromagnetic case, by substituting the electromagnetic tidal tensors by the gravito-electromagnetic tidal tensors defined below, and also the magnetic moment by the spin vector. In section 3, we address the question if it is possible to use the same method to relate the gravitational dyanamical equations to the electromagnetic ones at the quadrupolar orders of the multipolar expansion.

The second goal of the present work is to analyze the cases where the pole-dipole approximation is no longer a good description of the gravitational dynamics of an extended body. In particular, we are interested in the properties of solutions of the MPD equations were the 4-momentum is not automatically timelike. This is due to the fact that the MPD equations do not guarantee the timelike character of the 4-momentum. As mentioned in Refs. 8 and 9, a solution with a spacelike 4-momentum provides hints about the limits of the pole-dipole approximation, and it might be necessary, in those cases, to consider higher orders in the multipolar expansion for a more complete description of the dynamics of the system. On the other hand, spacelike 4-momenta which seem to be physically reasonable for the description of a system under the influence of gravity have been discussed before. For instance, massless spinning particles with spacelike 4-momentum are discussed in Ref. 9.

As is well-known, the MPD need a suitable supplementary spin condition (SSC) in order to define a complete system of evolution equations. In our work, we consider the Mathisson-Pirani supplementary spin condition (SSC-MP). However, there are other possible choices, as for instance the Tulczyjew-Dixon supplementary spin condition (SSC-TD) and the Kyrian-Semeràk supplementary spin condition (SSC-KS), see Ref. 10.

In section 4 we solve the MPD equations for circular orbits in the equatorial plane. We analyze the range of values of the angular velocity ω for which the 4-momentum of the exact solution is a spacelike or timelike vector. Previous relevant studies of circular orbits of gyroscopes include, for instance, Ref. 11, where a numerically solution is presented for the Kerr spacetime, using the SSC-TD. Different properties of the orbit are analyzed, such as gravitational waves, angular momentum, radiated energy, among other. Furthermore, the coupling between the angular momentum of the gyroscope, its spin and the angular momentum of the black hole are also discussed. Solutions for circular orbits are also presented in Ref. 12, again using the SSC-TD. Additionally, the dynamics of gyroscopes in the equatorial plane for γ spacetimes, assuming again the SSC-TD, is discussed in Ref. 13, where circular orbits as well are also computed.

Our conventions are the following: we use letters of the Greek alphabet $\mu, \nu, \rho, \dots$ to refer to a 4-dimensional spatial-temporal index that varies from 0 to 3, and letters of the Latin alphabet $i, j, k, \dots$ to refer to at spatial indexes that vary from 1 to 3. We use $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ for the metric and we also set $c = 1$. The Levi-Civita fully antisymmetric tensor is defined as $\epsilon_{\mu\nu\rho\sigma} := \sqrt{-g}\hat{\epsilon}_{\mu\nu\rho\sigma}$ and such that $\hat{\epsilon}_{\mu\nu\rho\sigma}$ it is the totally antisymmetric symbol with $\hat{\epsilon}_{0123} = \hat{\epsilon}^{0123} = 1$. In addition, the contravariant tensor is defined as $\epsilon^{\mu\nu\rho\sigma} := g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta}\epsilon_{\alpha\beta\gamma\delta} = -\hat{\epsilon}^{\mu\nu\rho\sigma}/\sqrt{-g}$. The Riemann tensor is defined as $R_{\mu\nu\beta}{}^{\alpha} := \partial_\mu \Gamma^{\alpha}_{\beta\nu} + \Gamma^{\alpha}_{\delta\mu}\Gamma^{\delta}_{\beta\nu} - (\mu \leftrightarrow \nu)$.

2. Multipolar expansion

In this section we briefly review the main aspects of the multipolar expansion which will be relevant to our discussion. In Refs. 1 and 2 a method is proposed to decompose the energy-momentum tensor to an infinite set of multipolar moments. A different formulation is released in Ref. 6 where the equations of motion are also extended up to quadrupole order.

For a test mass modeled up to dipole order, the MPD equations are

$$\frac{\delta p^\mu}{\mathrm{d}s} = -\frac{1}{2}R_{\nu\gamma\rho}{}^{\mu}S^{\nu\gamma}u^\rho, \tag{1}$$

$$\frac{\delta S^{\mu\nu}}{\mathrm{d}s} = 2p^{[\mu}u^{\nu]}, \tag{2}$$

where $R_{\nu\gamma\rho}{}^{\mu}$ is the Riemann tensor, p^μ and $S^{\mu\nu}$ are the 4-momentum and the spin tensor of the gyroscope, respectively, $u^\mu := \mathrm{d}X^\mu/\mathrm{d}s$ is the 4-velocity of the gyroscope defined by a reference worldline X^μ and where $\delta/\mathrm{d}s$ denotes the covariant derivative along the reference worldline.

From (2), contracting with the 4-velocity u_ν, the following relationship between the 4-momentum and the spin tensor is obtained

$$p^\mu = Mu^\mu + u_\nu \frac{\delta S^{\mu\nu}}{\mathrm{d}s}, \tag{3}$$

where $M := p^\mu u_\mu$ is the rest mass of the body.

Up to quadrupole order, equations of motion are

$$\frac{\delta p^\mu}{\mathrm{d}s} = \frac{1}{2}R^{\mu}{}_{\nu\gamma\delta}u^\nu S^{\gamma\delta} - g^{\mu\xi}\frac{1}{6}\left(\nabla_\xi R_{\gamma\rho\delta\omega}\right)I^{\gamma\omega\delta\rho}, \tag{4}$$

$$\frac{\delta S^{\mu\nu}}{\mathrm{d}s} = 2p^{[\mu}u^{\nu]} + \frac{4}{3}R_{\gamma\rho\sigma}{}^{[\mu}I^{\nu]\gamma\rho\sigma}, \tag{5}$$

where the quantity $I^{\mu\nu\rho\sigma}$ is constructed from the components of the quadrupole moment, see Refs. 5 and 6 for more details.

For a complete description it is also necessary to choose a reference curve with which to describe the movement of the body. This choice is not unique, which is why (1)-(2) or (4)-(5) do not completely determine a solution. The choice of a reference worldline is equivalent to assume a supplementary condition for the spin,[14,15] this reduces the number of linearly independent elements, so that the number of equations equals the number of unknowns.

In the literature, there are several supplementary spin conditions of which we will highlight only two, the Mathisson-Pirani supplementary spin condition

$$S^{\mu\nu}u_\nu = 0, \tag{6}$$

also called the Frenkel supplementary spin condition,[6] and the Tulzcyjew-Dixon supplementary spin condition

$$S^{\mu\nu}p_\nu = 0. \tag{7}$$

The SSC-MP is associated with the center of mass/energy of the body, while the SSC-TD is associated with the center of the momentum frame of the body.[6,14] Besides, notice that under (6) we have that M is a conserved quantity along X^μ, see Ref. 6.

3. Gravito-electromagnetism

The term gravito-electromagnetism refers to an analogy between the theory of General Relativity and classical electromagnetic theory. However, even if both theories describe different phenomena, it is possible to construct quantities that can be treated similarly in both theories. In Refs. 7 and 16 a gravito-electromagnetic analogy is discussed in terms of tidal tensors.

For an arbitrary spacetime we define, following Ref. 7 but with a slightly different convention, the gravito-electric and gravito-magnetic tidal tensors, respectively, as

$$\mathbb{E}^\mu{}_\nu := R^\mu{}_{\alpha\nu\beta}u^\alpha u^\beta, \quad \mathbb{H}_{\mu\nu} := -\frac{1}{2}\epsilon_{\gamma\lambda\mu\sigma}R^{\gamma\lambda}{}_{\nu\delta}u^\sigma u^\delta. \tag{8}$$

Similarly, for a flat spacetime, the electric and magnetic tidal tensors are define, respectively, as

$$E^\mu{}_\nu := \left(\partial_\nu F^\mu{}_\alpha\right)u^\alpha, \quad B_{\mu\nu} := -\frac{1}{2}\epsilon_{\gamma\lambda\mu\sigma}\left(\partial_\nu F^{\gamma\lambda}\right)u^\sigma, \tag{9}$$

where $F_{\mu\nu}$ is the Faraday tensor of an external electromagnetic field.

We notice that using the tidal tensors we can write

$$\partial_\gamma F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}B^\rho{}_\gamma u^\sigma - 2u_{[\mu}E_{\nu]\gamma}, \tag{10}$$

$$R_{\mu\nu\gamma\xi}u^\xi = \epsilon_{\mu\nu\rho\sigma}\mathbb{H}^\rho{}_\gamma u^\sigma - 2u_{[\mu}\mathbb{E}_{\nu]\gamma}. \tag{11}$$

In this analogy, the electric/magnetic tidal tensor is analogous to the gravito-electric/gravito-magnetic tidal tensor. All those tensors encode information of the inhomogeneities of the electromagnetic and the gravitational field. Notice that the gravito-electric tidal tensor defined in (8), through the geodesic deviation equation, describes the evolution of the separation of a pair test particles, and in the newtonian limit $\mathbb{E}$ reduces to the newtonian tidal tensor, see for instance Ref. 17, page 452. Additionally, it is possible to describe the force acting on a gyroscope in a similar way to the force exerted by an electromagnetic field on a magnetic dipole, see Ref. 7.

From (8) and assuming the SSC-MP (6) we can rewrite (1)-(2) as

$$\frac{\delta p^\mu}{\mathrm{ds}} = -\mathbb{H}^{\mu\nu} S_\nu, \tag{12}$$

$$\frac{\delta S^\mu}{\mathrm{ds}} = -S_\nu a^\nu u^\mu \tag{13}$$

where $a^\nu := \delta u^\nu/\mathrm{d}s$ is the 4-acceleration and $S^\mu := -\epsilon^{\mu\gamma\delta\sigma} S_{\gamma\delta} u_\sigma/2$ is the spin vector of the dipole.

For the electromagnetic case, the force exerted by an arbitrary external field on a magnetic dipole with magnetic dipole moment[a] μ^μ, can be rewritten as

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}s} = \mathtt{B}^{\mu\nu} \mu_\nu, \tag{14}$$

see Ref. 18, page 318.

Equations (12) and (14) are quite similar, whereby exchanging $\mathtt{B} \leftrightarrow -\mathbb{H}$ and $\mu \leftrightarrow S$, the MPD equations for the evolution of the 4-momentum can be obtained from the familiar electromagnetic case, see Ref. 7 for more details. However, for the equations that model the spin dynamics, the same is not true. If we analyze the dynamics of the angular momentum, which is related to the magnetic moment by the gyromagnetic ratio, the spin dynamics of the magnetic dipole is determine by the following equations,

$$\frac{\mathrm{d}}{\mathrm{d}s} S^{\mu\nu} = 2p^{[\mu} u^{\nu]} + 2\eta^{\sigma[\mu} m^{\nu]\lambda} F_{\sigma\lambda}, \tag{15}$$

where $m^{\nu\lambda}$ is the electromagnetic dipole moment defined in Ref. 5.

From (15) we deduce that, in general, it is not possible to rewrite the spin dynamics under the influence of the electromagnetic field in terms of the electric/magnetic tidal tensors. This is because they are completely determined by the field values, not by the values of the inhomogeneities of the field. Besides, if we compare (2) with (15), it is clear that they are not related in the same way as the equation which determined the evolution of the 4-momentum.

The analysis above can be extended to the quadrupole order. Replacing (11) in (4) we can write, after some algebra, the 4-momentum equations of motion in

[a]The magnetic dipole moment is related to the spin of the magnetic dipole by $\mu^\mu = (q/2m)S^\mu$.

terms of the gravitational tidal tensors, obtaining

$$
\begin{aligned}
\frac{\delta}{\mathrm{d}s}p^{\mathrm{G}}_{\mu} &= \frac{1}{2}R_{\mu\nu\gamma\delta}u^{\nu}S^{\gamma\delta} - \frac{1}{6}\left(\nabla_{\mu}R_{\alpha\nu\lambda\beta}\right)I^{\alpha\beta\lambda\nu} \\
&= \frac{1}{2}R_{\gamma\delta\mu\nu}u^{\nu}S^{\gamma\delta} - \frac{1}{6}\left(\nabla_{\mu}\bar{R}_{\alpha\nu\lambda\beta}\right)I^{\alpha\beta\lambda\nu} - \frac{1}{6}\nabla_{\mu}\left(R_{\alpha\nu\lambda\sigma}u^{\sigma}u_{\beta}\right)I^{\alpha\beta\lambda\nu} \\
&= \frac{1}{2}\left(\epsilon_{\gamma\delta\xi\sigma}\mathbb{H}^{\xi}{}_{\mu}u^{\sigma} - 2u_{[\gamma}\mathbb{E}_{\delta]\mu}\right)S^{\gamma\delta} - \frac{1}{6}\left(\nabla_{\mu}\bar{R}_{\alpha\nu\lambda\beta}\right)I^{\alpha\beta\lambda\nu} \\
&\quad - \frac{1}{6}\nabla_{\mu}\left(\epsilon_{\alpha\nu\xi\sigma}\mathbb{H}^{\xi}{}_{\lambda}u^{\sigma}u_{\beta} - 2u_{[\alpha}\mathbb{E}_{\nu]\lambda}u_{\beta}\right)I^{\alpha\beta\lambda\nu} \\
&= \frac{1}{2}\epsilon_{\gamma\delta\xi\sigma}\mathbb{H}^{\xi}{}_{\mu}u^{\sigma}S^{\gamma\delta} - S^{\gamma\delta}u_{\gamma}\mathbb{E}_{\delta\mu} - \frac{1}{6}I^{\alpha\beta\lambda\nu}\epsilon_{\alpha\nu\xi\sigma}\nabla_{\mu}\left(\mathbb{H}^{\xi}{}_{\lambda}u_{\beta}u^{\sigma}\right) \\
&\quad + \frac{1}{3}I^{\alpha\beta\lambda\nu}\nabla_{\mu}\left(u_{[\alpha}\mathbb{E}_{\nu]\lambda}u_{\beta}\right) - \frac{1}{6}\left(\nabla_{\mu}\bar{R}_{\alpha\nu\lambda\beta}\right)I^{\alpha\beta\lambda\nu}, \qquad (16)
\end{aligned}
$$

where we defined

$$\bar{R}_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - R_{\mu\nu\rho\xi}u^{\xi}u_{\sigma}. \qquad (17)$$

In addition, the equations of motion for a charge distribution modeled including the dipole and quadrupole electromagnetic moments was derived in Ref. 5. They can be written, using the electromagnetic tidal tensors (9), as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}p^{\mathrm{EM}}_{\mu} &= m^{\nu\lambda}u_{[\mu}\mathtt{E}_{\lambda]\nu} + \frac{1}{2}m^{\nu\lambda}\epsilon_{\mu\lambda\rho\delta}\mathtt{B}^{\delta}{}_{\nu}u^{\rho} \\
&\quad + \frac{2}{3}m^{\alpha\beta\lambda}\left(\partial_{\beta}u_{[\mu}\mathtt{E}_{\lambda]\nu}\right) + \frac{1}{3}m^{\alpha\beta\lambda}\epsilon_{\mu\lambda\rho\delta}\left(\partial_{\beta}\mathtt{B}^{\delta}{}_{\nu}\right)u^{\rho}, \qquad (18)
\end{aligned}
$$

where $m^{\mu\nu\rho}$ is the quadrupole electromagnetic moment.

Equations (16) is obtained without assuming any supplementary condition, but even assuming one, the term proportional to $\bar{R}$ does not, in general, vanish. This shows that (16) and (18) are not analogous, in the sense that it is not possible to obtain the gravitational equations of motion for a particle modeled up to the quadrupole order from the electromagnetic equations of motion, using the same identification of the tidal tensors discussed above.

From (10) and (11), we see that the derivatives of the Faraday tensor and the contraction $R_{\mu\nu\gamma\xi}u^{\xi}$ of the curvature tensor behave similarly, in the sense that they can be written as a linear combination of the tidal tensors appearing in the equations of motion up to dipole order. However, if we extend the gravitational equations of motion to the quadrupole order, they involve all the curvature components present in the combination (17), some of which cannot be written in terms of the tidal tensors $\mathbb{E}$ and $\mathbb{H}$.

The same argument applies when we analyze the equations that determine the evolution of the spin tensor. For the gravitational case, the equations of motion were discussed in Ref. 5 which, written in terms of the tidal tensors, are given by

$$\begin{aligned}\frac{\delta}{\mathrm{d}s}S^{\mu\nu} &= 2p^{[\mu}u^{\nu]} + \frac{4}{3}R_{\gamma\rho\sigma}{}^{[\mu}I^{\nu]\gamma\rho\sigma}\\ &= 2p^{[\mu}u^{\nu]} + \frac{4}{3}g^{\delta[\mu}I^{\nu]\gamma\rho\sigma}\bar{R}_{\gamma\rho\sigma\delta} + \frac{4}{3}g^{\delta[\mu}I^{\nu]\gamma\rho\sigma}R_{\gamma\rho\sigma\xi}u^{\xi}u_{\delta}\\ &= 2p^{[\mu}u^{\nu]} + \frac{4}{3}g^{\delta[\mu}I^{\nu]\gamma\rho\sigma}\bar{R}_{\gamma\rho\sigma\delta} + \frac{4}{3}u^{[\mu}I^{\nu]\gamma\rho\sigma}\left(\epsilon_{\gamma\rho\alpha\beta}\mathbb{H}^{\alpha}{}_{\sigma}u^{\beta} - 2u_{[\gamma}\mathbb{E}_{\rho]\sigma}\right)\\ &= 2p^{[\mu}u^{\nu]} + \frac{4}{3}g^{\delta[\mu}I^{\nu]\gamma\rho\sigma}\bar{R}_{\gamma\rho\sigma\delta} + \frac{4}{3}\epsilon_{\gamma\rho\alpha\beta}u^{[\mu}I^{\nu]\gamma\rho\sigma}\mathbb{H}^{\alpha}{}_{\sigma}u^{\beta} - \frac{8}{3}u^{[\mu}I^{\nu]\gamma\rho\sigma}u_{[\gamma}\mathbb{E}_{\rho]\sigma}.\end{aligned} \tag{19}$$

Similarly, in the electromagnetic case and for an ideal quadrupole, that is, a charge distribution modeled using only the quadrupole moment, the spin equations of motion are given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}S^{\mu\nu} &= 2p^{[\mu}u^{\nu]} + \eta^{\mu\sigma}m^{\nu\rho\alpha}\partial_{\rho}F_{\sigma\alpha}\\ &= 2p^{[\mu}u^{\nu]} + \epsilon_{\sigma\alpha\delta\gamma}\eta^{\mu\sigma}m^{\nu\rho\alpha}B^{\delta}{}_{\rho}u^{\gamma} - 2\eta^{\mu\sigma}m^{\nu\rho\alpha}u_{[\sigma}E_{\alpha]\rho},\end{aligned} \tag{20}$$

see Ref. 5 for details.

From (19) and (20) we can draw the same conclusion as in the case for the evolution equations of the 4-momentum: the quadrupole contribution to the equations are not analogous, since the gravitational equations include terms proportional to $\bar{R}$ defined in (17). In general, the Riemann tensor has 20 independent components, while $\mathbb{E}$ and $\mathbb{H}$ together have $8+6$ independent components,[7] respectively. The information of the remaining 6 components are then encoded in a new gravitational tidal tensor, which we can define as

$$\mathbb{F}_{\mu\nu} := \frac{1}{4}\epsilon_{\gamma\lambda\mu\alpha}\epsilon_{\sigma\rho\nu\beta}R^{\gamma\lambda\sigma\rho}u^{\alpha}u^{\beta}, \tag{21}$$

which is referred to as the magnetic-magnetic part of the Riemann tensor, see for instance Refs. 15 and 19.

With the three tidal tensors $\{\mathbb{E},\mathbb{H},\mathbb{F}\}$ and the 4-velocity u^{μ}, it is possible to reconstruct the Riemann tensor using,

$$R^{\xi\delta\gamma\lambda} = \epsilon^{\gamma\lambda\nu\alpha}\epsilon^{\xi\delta\mu\eta}\mathbb{F}_{\mu\nu}u_{\alpha}u_{\eta} - 4u^{[\xi}\mathbb{E}^{\delta][\gamma}u^{\lambda]} - 2\left[\epsilon^{\gamma\lambda\mu\sigma}\mathbb{H}_{\mu}{}^{[\delta}u^{\xi]} + \epsilon^{\delta\xi\mu\sigma}\mathbb{H}_{\mu}{}^{[\gamma}u^{\lambda]}\right]u_{\sigma}. \tag{22}$$

We can also write $\bar{R}$ in terms of the tidal tensors as

$$\bar{R}^{\xi\delta\gamma\lambda} = \epsilon^{\gamma\lambda\nu\alpha}\epsilon^{\xi\delta\mu\eta}\mathbb{F}_{\mu\nu}u_{\alpha}u_{\eta} - 2u^{[\delta}\mathbb{E}^{\xi]\lambda}u^{\gamma} - 2\epsilon^{\gamma\lambda\mu\sigma}\mathbb{H}_{\mu}{}^{[\delta}u^{\xi]}u_{\sigma} + \epsilon^{\delta\xi\mu\sigma}\mathbb{H}_{\mu}{}^{\gamma}u^{\lambda}u_{\sigma}. \tag{23}$$

From (10) and (22) we see that (21) has no electromagnetic equivalent. In other words, one requires two tidal tensors for a complete description of the inhomogeneities of the electromagnetic field ($\mathtt{E},\mathtt{B}$), while we need three for the gravitational case ($\mathbb{E},\mathbb{H},\mathbb{F}$).

We conclude that kind of analogy presented in Ref. 7 cannot be extended to the quadrupole order for general spacetimes. Nevertheless, for vacuum spacetimes the

Riemann tensor has only 10 independent components, which are all encoded in $\mathbb{E}$ and $\mathbb{H}$. Therefore, particular configurations of the gravitational field where a closer analogy could be valid might still exist.

4. Circular orbits

Here we concentrate on the study of circular orbits in Kerr spacetime assuming the Mathisson-Pirani spin supplementary condition. The metric, in Boyer-Lindquist coordinates $x^\mu = [t, r, \theta, \phi]$, is given by

$$\mathrm{d}s^2 = \frac{\Delta}{\rho^2}\left(\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)\mathrm{d}\phi - a\mathrm{d}t\right]^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2, \quad (24)$$

where m and a are the mass and spin parameter of the black hole, respectively, and

$$\rho^2 := r^2 + a^2\cos^2\theta, \quad \Delta := r^2 - 2r + a^2. \quad (25)$$

In the equatorial plane, and under the SSC-MP, we solve (1) and (13) for a time-independent but otherwise arbitrary angular velocity ω. We assume a 4-velocity of the form

$$u^\mu = [\gamma,\ 0,\ 0,\ \gamma\omega], \quad (26)$$

with

$$\gamma := \sqrt{\frac{R}{R - 2m - 2a^2m\omega^2 - a^2\omega^2R + 4am\omega - \omega^2R^3}}, \quad (27)$$

so that $u^\mu u_\mu = 1$, and where R is the radius of the orbit. Also, notice that for a flat space γ is the Lorentz factor.
From (26) we compute the 4-acceleration of the gyroscope, obtaining

$$a^\mu = \left[0,\ -\frac{\gamma^2\Theta}{R^6}\left(a^2 - 2mR + R^2\right),\ 0,\ 0\right], \quad (28)$$

where

$$\Theta := R^2\left(a^2m\omega^2 - 2am\omega + m - \omega^2R^3\right). \quad (29)$$

Besides, we assume that the spin vector of the particle is parallel to the black hole rotation, i.e. $S^\mu = [0,\ 0,\ S^\theta,\ 0]$. From (3), under the SSC-MP (6), we can write

$$p^\mu = Mu^\mu - a_\nu S^{\mu\nu}, \quad (30)$$

where the spin tensor is given by

$$S^{\mu\nu} = \epsilon^{\mu\nu\gamma\sigma}S_\gamma u_\sigma. \quad (31)$$

Then, replacing (26), (28) and (31) in (30), we have that the 4-momentum is,

$$p^\mu = \left[\gamma M + \frac{\gamma^3 S\Theta}{R^6}\left(am - 2a^2\omega m + a^2\omega R + \omega R^3\right),\ 0,\ 0,\ \gamma M\omega - \frac{\gamma^3 S\Theta}{R^6}\left(2am\omega - 2m + R\right)\right]. \quad (32)$$

We can assume, as in more simple cases (as discussed for instance in Ref. 20), that the spin value is independent of the proper time, that is $\mathrm{d}S/\mathrm{d}s = 0$. Then, the 4-momentum (32) it is also time-independent. With this assumptions, Eq. (1) reduces to set of algebraic equations, namely

$$R_{\nu\gamma\rho}{}^{\mu} S^{\nu\gamma} u^{\rho} + 2\Gamma^{\mu}_{\rho\sigma} p^{\rho} u^{\sigma} = 0. \tag{33}$$

After replacing (26), (31) and (32) in (33), we solve S in terms of M, R and ω, obtaining

$$S = \frac{M}{\gamma^2}\frac{\Theta}{\Upsilon}, \tag{34}$$

where

$$\begin{aligned}\Upsilon :=\ & 5a^5m^2\omega^4 + 3a^5m\omega^4R - 20a^4m^2\omega^3 - 6a^4m\omega^3R + 3a^3m^2\omega^4R^2 + 30a^3m^2\omega^2 \\ & + 7a^3m\omega^4R^3 - 9a^2m^2\omega^3R^2 - 20a^2m^2\omega - 12a^2m\omega^3R^3 + 6a^2m\omega R + 9am^2\omega^2R^2 \\ & + 5am^2 + 6am\omega^4R^5 + 3am\omega^2R^3 - 3amR - 3m^2\omega R^2 - 6m\omega^3R^5 + 2m\omega R^3 + \omega^3R^6.\end{aligned} \tag{35}$$

Then, substituting (34) into (32), the values of 4-momentum and 4-spin vector are, respectively,

$$p^{\mu} = \left[\frac{M}{\gamma}\frac{\Lambda}{\Upsilon},\ 0,\ 0,\ \frac{M}{\gamma}\frac{\Omega}{\Upsilon}\right], \tag{36}$$

$$S^{\mu} = \left[0,\ 0,\ \frac{M}{\gamma^2}\frac{\Theta}{\Upsilon},\ 0\right], \tag{37}$$

where Λ and Ω are functions that depend only on the parameters of the orbit, given by

$$\begin{aligned}\Lambda :=\ & a^4m^2\omega^3 - 3a^3m^2\omega^2 - 3a^3m\omega^2R + 3a^2m^2\omega - 2a^2m\omega^3R^3 + 6a^2m\omega R - am^2 - 3amR \\ & + 2m\omega R^3 + \omega^3R^6,\end{aligned} \tag{38}$$

$$\Omega := m\,(1 - a\omega)\left(-a^2m\omega^2 + 3a^2\omega^2R + 2am\omega - 3a\omega R - m + 4\omega^2R^3\right). \tag{39}$$

The following invariants are computed too:

$$a^{\mu}a_{\mu} = -\frac{\gamma^4\Theta^2}{R^{10}}\left(a^2 - 2mR + R^2\right), \tag{40}$$

$$S^{\mu}S_{\mu} = -\frac{M^2\Theta^2R^2}{\Upsilon^2\gamma^4}, \tag{41}$$

$$p^{\mu}p_{\mu} = -\frac{M^2\left(2\Lambda^2m - \Lambda^2R - 4\Lambda\Omega am + 2\Omega^2a^2m + \Omega^2a^2R + \Omega^2R^3\right)}{\gamma^2\Upsilon^2R}. \tag{42}$$

A similar analysis is shown in Ref. 21, where the circular orbits are studied in order to analyze Aschenbach effect for test particles with spin. Our solution above is consistent with the one presented there. Notice however, that we have solved (32) for S, whereas the authors are interested in the solution for the angular frequency ω in terms of the spin S, which involves solving a quartic polynomial equation.

We are interested in the possible values of the norm of the 4-momentum in terms of the angular velocity, in order to find for which values of ω the 4-momentum can be a timelike vector. We first look for values of ω for which (42) vanishes, that is an eight-order polynomial equation for ω, including a cuadratic factor coming from the factor γ^{-2} in (42). This determines the first two analytic roots, namely

$$\omega_{\rm min} = \frac{2am - R\sqrt{a^2 - 2mR + R^2}}{2a^2m + a^2R + R^3}, \quad \omega_{\rm max} = \frac{2am + R\sqrt{a^2 - 2mR + R^2}}{2a^2m + a^2R + R^3}, \tag{43}$$

for which γ^{-2} vanishes and that represent the minimum and maximum values such that γ is real. Therefore, then the 4-velocity u^μ is a timelike vector provided $\omega_{\rm min} < \omega < \omega_{\rm max}$.

Furthermore, when $\omega = \omega_{\rm min}$ or $\omega = \omega_{\rm max}$, we can see that $p^\mu p_\mu \to 0$ and $S^\mu S_\mu \to 0$, while for the 4-acceleration we have $a^\mu a_\mu \to -\infty$. This behavior is due to the fact that for these angular velocity values the gyroscope reaches the speed of light.

Introducing the dimensionless variables

$$\bar{R} := \frac{R}{m}, \tag{44}$$

$$\bar{a} := \frac{a}{m}, \tag{45}$$

$$\bar{\omega} := \omega m, \tag{46}$$

we can recast (43) as

$$\bar{\omega}_{\rm min} = \frac{2\bar{a} - \bar{R}\sqrt{\bar{R}^2 - 2\bar{R} + \bar{a}^2}}{2\bar{a}^2 + \bar{a}^2\bar{R} + \bar{R}^3}, \quad \bar{\omega}_{\rm max} = \frac{2\bar{a} + \bar{R}\sqrt{\bar{R}^2 - 2\bar{R} + \bar{a}^2}}{2\bar{a}^2 + \bar{a}^2\bar{R} + \bar{R}^3}, \tag{47}$$

and the norm of 4-momentum may be rewritten as

$$\bar{p}^\mu \bar{p}_\mu := \frac{p^\mu p_\mu}{M^2} = -\frac{2\bar{\Lambda}^2 - \bar{\Lambda}^2\bar{R} - 4\bar{\Lambda}\bar{\Omega}\bar{a} + 2\bar{\Omega}^2\bar{a}^2 + \bar{\Omega}^2\bar{a}^2\bar{R} + \bar{\Omega}^2\bar{R}^3}{\gamma^2\bar{\Upsilon}^2\bar{R}}, \tag{48}$$

where

$$\bar{\Lambda} := \frac{\Lambda}{m^3} = \bar{R}^6\bar{\omega}^3 - 2\bar{R}^3\bar{\omega}^3\bar{a}^2 + 2\bar{R}^3\bar{\omega} - 3\bar{R}\bar{\omega}^2\bar{a}^3 + 6\bar{R}\bar{\omega}\bar{a}^2 - 3\bar{R}\bar{a} + \bar{\omega}^3\bar{a}^4 - 3\bar{\omega}^2\bar{a}^3 + 3\bar{\omega}\bar{a}^2 - \bar{a}, \tag{49}$$

$$\bar{\Omega} := \frac{\Omega}{m^2} = (1 - \bar{\omega}\bar{a})\left(4\bar{R}^3\bar{\omega}^2 + 3\bar{R}\bar{\omega}^2\bar{a}^2 - 3\bar{R}\bar{\omega}\bar{a} - \bar{\omega}^2\bar{a}^2 + 2\bar{\omega}\bar{a} - 1\right), \tag{50}$$

$$\begin{aligned}\bar{\Upsilon} := \frac{\Upsilon}{m^3} = {} & \bar{R}^6\bar{\omega}^3 + 6\bar{R}^5\bar{\omega}^4\bar{a} - 6\bar{R}^5\bar{\omega}^3 + 7\bar{R}^3\bar{\omega}^4\bar{a}^3 - 12\bar{R}^3\bar{\omega}^3\bar{a}^2 + 3\bar{R}^3\bar{\omega}^2\bar{a} + 2\bar{R}^3\bar{\omega} \\ & + 3\bar{R}^2\bar{\omega}^4\bar{a}^3 - 9\bar{R}^2\bar{\omega}^3\bar{a}^2 + 9\bar{R}^2\bar{\omega}^2\bar{a} - 3\bar{R}^2\bar{\omega} + 3\bar{R}\bar{\omega}^4\bar{a}^5 - 6\bar{R}\bar{\omega}^3\bar{a}^4 + 6\bar{R}\bar{\omega}\bar{a}^2 \\ & - 3\bar{R}\bar{a} + 5\bar{\omega}^4\bar{a}^5 - 20\bar{\omega}^3\bar{a}^4 + 30\bar{\omega}^2\bar{a}^3 - 20\bar{\omega}\bar{a}^2 + 5\bar{a},\end{aligned} \tag{51}$$

$$\gamma = \sqrt{\frac{\bar{R}}{\bar{R} - 2 - 2\bar{a}^2\bar{\omega}^2 - \bar{a}^2\bar{\omega}^2\bar{R} + 4\bar{a}\bar{\omega} - \bar{\omega}^2\bar{R}^3}}, \tag{52}$$

The general behavior of $p^\mu p_\mu$ is shown in figures 1(a) and 1(b). We see that there exist two additional roots for which (48) vanishes within the interval $\bar{\omega}_{\rm min} < \bar{\omega} < \bar{\omega}_{\rm max}$. The analytics expressions for those roots are rather involved, and we will not display them here, but we will refer to them as $\bar{\omega}_-$ and $\bar{\omega}_+$. The 4-momentum is always a spacelike vector for values of ω within the interval $\bar{\omega}_- < \bar{\omega} < \bar{\omega}_+$. The fact that the 4-momentum is assumed as a timelike vector is likely to mean that for those values of angular velocities the pole-dipole approximation is not longer valid.

Notice that in Minkowski spacetime the 4-momentum is always a timelike vector. For the Schwarzschild metric $\bar{p}^\mu \bar{p}_\mu$ has a divergence for $\bar{\omega} = 0$, see figure 1(a). This is because the 4-momentum, according to (30), is constructed from the usual kinetic term and one proportional to the spin of the body. While the 4-momentum remains finite in this case, the spin diverges, as can be seen from (34) or (30) since $\bar{\Upsilon}$ vanishes. For instance, if $\bar{R} = 5$ we have that the maximum/minimum values for angular velocity are $\bar{\omega}_{\rm min} \approx -0.155$, $\bar{\omega}_{\rm max} \approx 0.155$ and $\bar{\omega}_\pm \approx \pm 0.02$. For orbits farther away from the black hole, for instance $\bar{R} = 10$, we have that $\bar{\omega}_{\rm min} \approx -0.089$, $\bar{\omega}_{\rm max} \approx 0.089$ and $\bar{\omega}_\pm \approx \pm 0.005$.

Similarly in Kerr spacetime, we can see from figure 1(b) that the behavior $\bar{p}^\mu \bar{p}_\mu$ is quite similar to the case in Schwarzschild spacetime, but now the angular frequencies are shifted due to the black hole rotation. Now the value of $\bar{\omega}$ for which S and $\bar{p}^\mu \bar{p}_\mu$ diverges no longer vanishes. For instance $\bar{R} = 5$ and $\bar{a} = 1/2$, the minimum/maximum angular velocity of gyroscope are $\bar{\omega}_{\rm min} \approx -0.146$ and $\bar{\omega}_{\rm max} \approx 0.162$, respectively. Furthermore, the value of the spin is undetermined for $\bar{\omega} \approx 0.027$ because, for this set of parameters, it is a root of $\bar{\Upsilon}$. We will refer to this value as *critical* ω. It is interesting that even for $\bar{\omega} = 0$ there are circular orbits where the 4-momentum is a timelike vector, not so for Schwarzschild spacetime. Besides, for those parameters we have that $\bar{\omega}_- \approx 0.007$ and $\bar{\omega}_+ \approx 0.04$. In this way, we conclude that angular velocities around the *critical* ω given by $\bar{\omega} \approx 0.027$, that is, inside the range $\bar{\omega}_- < \bar{\omega} < \bar{\omega}_+$, the pole-dipole approximation is no longer valid.

For $\omega = 0$, that is, static trajectories outside the black hole, the values for 4-momentum, 4-spin and 4-acceleration are, respectively,

$$p^\mu = \left[\frac{M\left(3R+m\right)}{\gamma\left(3R-5m\right)},\ 0,\ 0,\ \frac{Mm}{\gamma a\left(3R-5m\right)}\right], \tag{53}$$

$$S^\mu = \left[0,\ 0,\ \frac{MR^2}{\gamma^2 a\left(3R-5m\right)},\ 0\right], \tag{54}$$

$$a^\mu = \left[0, -\frac{m\left(a^2-2mR+R^2\right)}{R^3\left(2m-R\right)},\ 0,\ 0\right], \tag{55}$$

where $\gamma = \sqrt{1/(1-2m/R)}$.

The follow invariants are computed too, and we see that

$$p^{\mu}p_{\mu} = \frac{M^2\left(-2m+R\right)\left(-12a^2m+9a^2R-m^2R\right)}{a^2\left(-5m+3R\right)^2}, \tag{56}$$

$$S^{\mu}S_{\mu} = -\frac{M^2R^4\left(2m-R\right)^2}{a^2\left(5m-3R\right)^2}, \tag{57}$$

$$a^{\mu}a_{\mu} = \frac{m^2\left(-a^2+2mR-R^2\right)}{R^4\left(2m-R\right)^2}. \tag{58}$$

As we expected, the 4-momentum and 4-spin diverge when $a \to 0$ because in Schwarzschild spacetime static trajectories would require an infinitely large spin to counteract the gravitational pull from the black hole.
In Kerr spacetime, the angular velocity for circular geodesics is given by $\omega_{g\pm} = 1/(a \pm \sqrt{r^3/m})$, see Ref. 14. We can also check that in this case our solution implies a vanishing spin, see Eq. (29), which is the correct limit for a monopole test particle, the lowest order in the multipolar expansion.
The solution in Minkowski spacetime is obtained from (36) and (37) in the limit $m \to 0$. For such case the 4-momentum, 4-spin and 4-acceleration are, respectively,

$$p^{\mu} = \left[\frac{M}{\gamma},\ 0,\ 0,\ 0\right], \tag{59}$$

$$S^{\mu} = \left[\ 0,\ 0,\ -\frac{M}{\gamma^2 v},\ 0\right], \tag{60}$$

$$a^{\mu} = \left[\ 0,\ -\gamma^2\omega^2 R,\ 0,\ 0\right], \tag{61}$$

where $\gamma = 1/\sqrt{1-v^2}$ and $v = \omega R$ is the tangential velocity of the gyroscope.
We notice that the 4-momentum is always a timelike vector, and the norm of the spin vector obeys

$$S := \sqrt{-S^{\mu}S_{\mu}} = \frac{MR}{\gamma^2 v}. \tag{62}$$

The 4-momentum (59) and 4-spin (60) agree with the solution found in Ref. 20, where circular orbits in Minkowski spacetime were studied under SSC-MP and a discussion of the physical interpretation of the solution is presented. On this way, the 4-momentum (36) and the 4-spin (37) represent a generalization of the solution presented in Ref. 20 to the case in Kerr spacetime.
Additionally, another interesting particular case of our general solution presented here is given by the following value of the angular velocity,

$$\omega = \frac{a}{a^2+R^2}, \tag{63}$$

which has the propriety that all the components of the gravito-magnetic tidal tensor vanishes, and then

$$\frac{\delta p^{\mu}}{\mathrm{d}s} = 0. \tag{64}$$

This is the case analogous to a magnetic dipole following a circular orbit with an angular velocity $\omega = \mu_{\rm s}/QR^2$, around a central electric charge Q with magnetic moment $\mu_{\rm s}$. For such case, the magnetic dipole does not experiment a magnetic field and the symmetric part of the magnetic tidal tensor vanishes. Here it is noticed that induction effects in electromagnetism establish a difference between General Relativity and the classical electromagnetic theory. For more details, see Ref. 14.
The 4-velocity of the gyroscope for this case is

$$u^\mu = \frac{1}{\sqrt{R^2+a^2-2mR}}\left[\frac{a^2+R^2}{R},\ 0,\ 0,\ \frac{a}{R}\right], \tag{65}$$

and the values of the 4-momentum, 4-spin and 4-acceleration are, respectively,

$$p^\mu = \frac{1}{\sqrt{R^2+a^2-2mR}}\left[M(m+R),\ 0,\ 0,\ \frac{Mm}{a}\right], \tag{66}$$

$$S^\mu = \left[0,\ 0,\ -\frac{MR}{a},\ 0\right], \tag{67}$$

$$a^\mu = \left[0,\ \frac{mR-a^2}{R^3},\ 0,\ 0\right]. \tag{68}$$

In this case, the 4-momentum is always spacelike outside the black hole, since

$$p^\mu p_\mu = -\frac{M^2R^2\left(m^2-a^2\right)}{a^2\left(R^2+a^2-2Rm\right)}. \tag{69}$$

Notice also that for $a \to 0$ we have $|S^\theta| \to \infty$. This is because $\omega \propto a$ and, as $a \to 0$, the gyroscope tends to remain static respect to the black hole.
The fact that the 4-momentum is usually assumed as a timelike vector means that in this case it should be necessary to consider higher orders in the multipolar expansion for a successful description of the gyroscope's dynamics.

5. Concluding remarks

We have considered the question of whether the gravito-electromagnetic analogy discussed by Costa and Herdeiro in Ref. 7 for the dynamics of a spinning particle is still valid at higher order in the multipolar expansion, and in particular when we include quadrupolar terms. The analogy is based on the fact that the gravitational equations of motion are of the same form as the electromagnetic ones, when one writes them in terms of the gravito-electric and gravito-magnetic tidal tensors $\mathbb{E}$ and $\mathbb{H}$ and in terms of the electromagnetic tidal tensors E and B, respectively. However, as we have discussed in section 3, when quadrupolar terms are included all the components of the curvature tensor are in general present in the equations of motion that determine the dynamics of both 4-momentum and 4-spin, and the analogy breaks down. One can certainly write the equations of motion at quadrupole order in terms of the gravitational tidal tensors $\mathbb{E}$ (6 independent components) and $\mathbb{H}$ (8 independent components), but necessarily also in terms of the magnetic-magnetic part $\mathbb{F}$ of the curvature tensor[19] (6 independent components, c.f. (21)). This tidal tensor, however, has no electromagnetic analog for general non-vacuum spacetimes.

Additionally, we have studied solutions of the gravitational equations of motion within the pole-dipole approximation, and found particular solutions for the case of circular orbits in a Kerr spacetime, assuming the Mathison-Pirani supplementary spin conditions. These solutions are summarized in the expressions for the 4-momentum and 4-spin (36) and (37), respectively. It is interesting to note that although the expressions (36) and (37), together with the original assumption (26), are indeed solutions of the equations of motion and the supplementary condition, the timelike character of the 4-momentum is not automatically guaranteed. This has been recognized before, see for instance Ref. 8. It is usually believed that those cases in which the solution for the momentum is spacelike would correspond to situations where the pole-dipole approximation is no longer valid. In any case, it is then instructive to analyze the conditions under which the 4-momentum is indeed timelike and, on the other hand, the parameter space for which the description is no longer valid since the 4-momentum becomes spacelike.

As shown in figures 1(a) and 1(b), for a Schwarzschild and Kerr spacetime respectively, there exist values for the angular velocity of the circular motion for which the 4-momentum is timelike, and other values for which it is spacelike. The change in sign of the norm of the 4-momentum occurs at particular values ω_- and ω_+ and are a consequence of the existence of a "critical angular velocity", $\omega_{\rm critical}$. The also noticed that for $\omega = a/(a^2 + R^2)$, which is the particular case discussed in Ref. 20 the 4-momentum turns out to be spacelike.

The solutions discussed here should be useful to understand the limits of validity of the pole-dipole approximation. It would be interesting to explore the question of how the inclusion of higher order terms, i.e at the quadrupole level and beyond, would alleviate the pathologies of the solutions of the equations of motion of a gravitational dipole.

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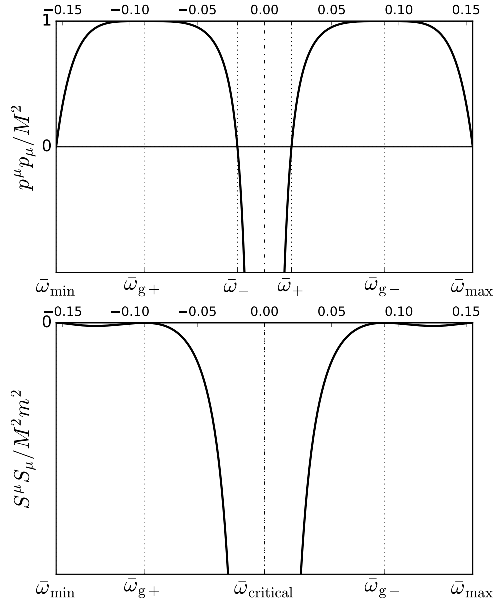
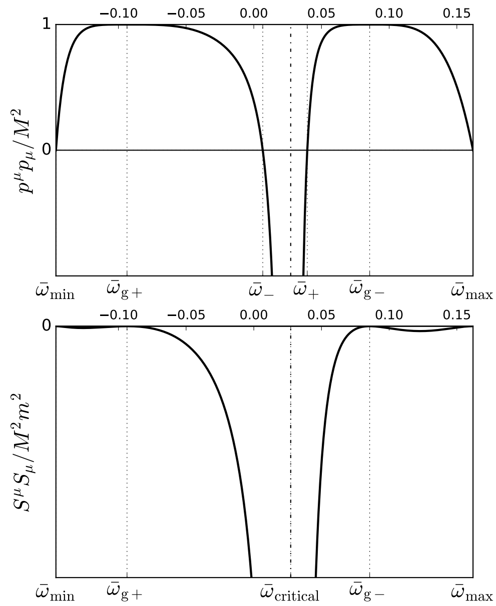


(a) $\bar{a} = 0$



(b) $\bar{a} = 1/2$

Figure 1. Norm of the 4-momentum and 4-spin for $\bar{R} = 5$.

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