Seminar 12

Aplicații

$\boxed{3.1.42}$ Care dintre următoarele aplicații sunt liniare? Iar pt aplicațiile care sunt liniare, determinați $Ker f$ și $Im f$

(1) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f[x_1, x_2, x_3] = [x_1 - x_2, x_2 - x_3, x_3 - x_1]$

f apl. liniară:

- $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ $\forall x, y \in \mathbb{R}^3$, $\beta, \alpha \in \mathbb{R}$
- $\begin{cases} f(x+y) = f(x) + f(y) & (I) \\ f(\alpha x) = \alpha f(x) & (II) \end{cases}$,

I $x, y \in \mathbb{R}^3$, $x = [x_1, x_2, x_3]$

$y = [y_1, y_2, y_3]$

$x + y = [x_1 + y_1, x_2 + y_2, x_3 + y_3]$

$f(x+y) = [x_1 + y_1 - x_2 - y_2, x_2 + y_2 - x_3 - y_3, x_3 + y_3 - x_1 - y_1]$

$$f(x)+f(y)=[x_1-x_2+y_1-y_2,\ x_2-x_3+y_2-y_3,\ x_3-x_1+y_3-y_1]$$

$$\Rightarrow f(x+y)=f(x)+f(y)$$

II $$f(\alpha x)=f[\alpha x_1,\alpha x_2,\alpha x_3]=$$

$$=[\alpha x_1-\alpha x_2,\ \alpha x_2-\alpha x_3,\ \alpha x_3-\alpha x_1]$$

$$=\alpha[x_1-x_2,\ x_2-x_3,\ x_3-x_1]=\alpha f(x)$$

I și II $\Rightarrow$ f este apl liniară

$$\operatorname{Ker} f=\{x\in\mathbb{R}^3 \mid f(x)=0\}$$

$$f(x)=0 \Leftrightarrow$$

$$\Leftrightarrow [x_1-x_2,\ x_2-x_3,\ x_3-x_1]=[0,0,0]$$

$$\Rightarrow \begin{cases} x_1-x_2=0 \\ x_2-x_3=0 \\ x_3-x_1=0 \end{cases} \Rightarrow \begin{cases} x_1=x_2 \\ x_2=x_3 \\ x_3=x_1 \end{cases} \Rightarrow x_1=x_2=x_3$$

$$\operatorname{Ker} f=\{[\beta,\beta,\beta]\in\mathbb{R}^3 \mid \beta\in\mathbb{R}\}$$

$$\operatorname{Im} f=\{y\in\mathbb{R}^3 \mid \exists\, x\in\mathbb{R}^3 \cdot f(x)=y\}$$

$$f(x)=y \Leftrightarrow [x_1-x_2,\ x_2-x_3,\ x_3-x_1]=[y_1,y_2,y_3]$$

$$\Rightarrow (S)\begin{cases} x_1 - x_2 = y_1 \\ x_2 - x_3 = y_2 \\ x_3 - x_1 = y_3 \end{cases}$$

$y \in \mathrm{Im} f \iff (S)$ este compatibil

$\iff$ rang A = rang $\bar{A}$ unde

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ -1 \end{matrix} \\ \begin{matrix} -1 & 0 \end{matrix} & 1 \end{pmatrix} \text{ și } \bar{A} = \left(\begin{array}{ccc|c} 1 & -1 & 0 & y_1 \\ 0 & 1 & -1 & y_2 \\ -1 & 0 & 1 & y_3 \end{array}\right)$$

$$\det A = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = \cancel{1} + 0 - \cancel{1} - 0 = 0$$

$\Rightarrow$ rang $A \leq 2$

$$d_2 = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \text{rang } A = 2$$

$$\text{rang } \bar{A} = 2 \iff \begin{vmatrix} 1 & -1 & y_1 \\ 0 & 1 & y_2 \\ -1 & 0 & y_3 \end{vmatrix} = 0$$

$\iff y_3 + y_2 + y_1 = 0$

$$\mathrm{Im} f = \{ y \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 = 0 \}$$

(2) $f\ \mathbb{R}^3 \to \mathbb{R}^3$

$$f[x_1, x_2, x_3] = [x_1 - 1, x_2 + 2, x_3 + 1]$$

I $\quad f(x+y) = f(x) + f(y)$, $\forall\ x, y \in \mathbb{R}^3$

II $\quad f(\alpha x) = \alpha f(x)$, $\alpha \in \mathbb{R}$

I. $\quad x = [x_1, x_2, x_3]$

$y = [y_1, y_2, y_3]$

$x + y = [x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3]$

$$f(x+y) = [x_1 + y_1 - 1,\ x_2 + y_2 + 2,\ x_3 + y_3 + 1]$$

$$f(x) + f(y) = f[x_1, x_2, x_3] + f[y_1, y_2, y_3] =$$

$$= [x_1 - 1,\ x_2 + 2,\ x_3 + 1] + [y_1 - 1,\ y_2 + 2,\ y_3 + 1]$$

$$= [x_1 + y_1 - 2,\ x_2 + y_2 + 4,\ x_3 + y_3 + 2]$$

$\Rightarrow f(x+y) \neq f(x) + f(y)$

$\Rightarrow f$ <u>nu</u> este apl lin

• Obs că $f(0) = f[0,0,0] = [-1, 2, 1] \neq [0, 0, 0]$

$\Rightarrow$ nu este liniară

(3) $\quad f: \mathbb{R}^3 \to \mathbb{R}^3$

$$f[x_1, x_2, x_3] = [2x_1 - 3x_2 + x_3,\ -x_1 + x_2 + 3x_3,\ x_1 + x_2 + x_3]$$

$$\mathrm{I} \quad f(x+y) = f(x) + f(y), \quad \forall\, x, y \in \mathbb{R}^3$$

$$\mathrm{II} \quad f(\alpha x) = \alpha f(x) \qquad \alpha \in \mathbb{R}$$

I

$$x = [x_1, x_2, x_3]$$

$$y = [y_1, y_2, y_3]$$

$$x+y = [\underbrace{x_1+y_1}_{\mathrm{I}},\ \underbrace{x_2+y_2}_{\mathrm{II}},\ \underbrace{x_3+y_3}_{\mathrm{III}}]$$

$$f(x+y) = [2(x_1+y_1) - 3(x_2+y_2) + x_3+y_3,\ -x_1-y_1+x_2+y_2+3x_3+3y_3,\ x_1+y_1+x_2+y_2+x_3+y_3]$$

$$f(x)+f(y) = [2x_1-3x_2+x_3,\ -x_1+x_2+3x_3,\ x_1+x_2+x_3] + [2y_1-3y_2+y_3,\ -y_1+y_2+3y_3,\ y_1+y_2+y_3]$$

$$= [2x_1-3x_2+x_3+2y_1-3y_2+y_3,\ -x_1+x_2+3x_3-y_1+y_2+3y_3,\ x_1+x_2+x_3+y_1+y_2+y_3]$$

$$\Rightarrow f(x+y) = f(x)+f(y), \quad \forall\, x, y \in \mathbb{R}^3$$

$\overline{II}$ $f(\alpha X) = f[\alpha X_1, \alpha X_2, \alpha X_3] =$

$= [2\alpha X_1 - 3\alpha X_2 + \alpha X_3, -\alpha X_1 + \alpha X_2 + 3\alpha X_3,$
$\alpha X_1 + \alpha X_2 + \alpha X_3]$

$= \alpha [2X_1 - 3X_2 + X_3, -X_1 + X_2 + 3X_3,$
$X_1 + X_2 + X_3] = \alpha f(X), \forall \alpha \in \mathbb{R}, X \in \mathbb{R}^3$

I, II $\Rightarrow$ f este apl lin

$\mathrm{Ker} f = \{ X \in \mathbb{R}^3 \mid f(X) = 0 \}$

$f(X) = 0 \iff$

$[2X_1 - 3X_2 + X_3, -X_1 + X_2 + 3X_3, X_1 + X_2 + X_3] = [0, 0, 0]$

$$\begin{cases} 2X_1 - 3X_2 + X_3 = 0 \\ -X_1 + X_2 + 3X_3 = 0 \\ X_1 + X_2 + X_3 = 0 \end{cases} \qquad A = \begin{pmatrix} 2 & -3 & 1 \\ -1 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 2 & -3 & 1 \\ -1 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \cancel{2} - \cancel{1} - 9 - \cancel{1} - 6 - 3 = -18 \neq 0$$

$\Rightarrow$ sist comp det

$\Rightarrow$ sol unică, sist omogen

$\Rightarrow X_1 = X_2 = X_3 = 0$

$\mathrm{Ker} f = \{ [0,0,0] \}$

$\mathrm{Im} f = \{ y \in \mathbb{R}^3 \mid \exists\, x \in \mathbb{R}^3 \cdot f(x) = y \}$

$f(x) = y \iff [2x_1 - 3x_2 + x_3, -x_1 + x_2 + 3x_3, x_1 + x_2 + x_3] = [y_1, y_2, y_3]$

$$(S) \begin{cases} 2x_1 - 3x_2 + x_3 = y_1 \\ -x_1 + x_2 + 3x_3 = y_2 \\ x_1 + x_2 + x_3 = y_3 \end{cases}$$

$$A = \begin{pmatrix} 2 & -3 & 1 \\ -1 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \qquad \bar{A} = \left(\begin{array}{ccc|c} 2 & -3 & 1 & y_1 \\ -1 & 1 & 3 & y_2 \\ 1 & 1 & 1 & y_3 \end{array}\right)$$

$y \in \mathrm{Im} f \iff (S)$ compatibil

$\iff \mathrm{rang}\, A = \mathrm{rang}\, \bar{A}$

cum $\det A \neq 0 \Rightarrow \mathrm{rang}\, A = 3$

$\Rightarrow \mathrm{rang}\, \bar{A} = 3$ „A" $(\bar{A} \in \mathcal{M}_{3,4}(\mathbb{R}))$

$\mathrm{Im} f = \mathbb{R}^3$

(4). $f : \mathbb{R}^2 \to \mathbb{R}^3$

$f[x_1, x_2] = [x_1 + x_2, x_1 - x_2, 2x_1 + x_2]$

I $\quad f(x+y) = f(x) + f(y) \quad \forall\, x, y \in \mathbb{R}^2$

$\overline{\text{II}}$ $f(\alpha x) = \alpha f(x)$ $\qquad \alpha \in \mathbb{R}$

$\overline{\text{I}}$ $x = [x_1, x_2]$

$y = [y_1, y_2]$

$x + y = [\underbrace{x_1 + y_1}, x_2 + y_2]$

$f(x+y) = [x_1 + y_1 + x_2 + y_2, x_1 + y_1 - x_2 - y_2,$
$2x_1 + 2y_1 + x_2 + y_2]$

$f(x) + f(y) = [x_1 + x_2, x_1 - x_2, 2x_1 + x_2] +$
$+ [y_1 + y_2, y_1 - y_2, 2y_1 + y_2]$

$= [x_1 + x_2 + y_1 + y_2, x_1 - x_2 + y_1 - y_2,$
$2x_1 + x_2 + 2y_1 + y_2]$
$= f(x+y), \forall x, y \in \mathbb{R}^2$

$\overline{\text{II}}$ $f(\alpha x) = [\alpha x_1 + \alpha x_2, \alpha x_1 - \alpha x_2,$
$2\alpha x_1 + \alpha x_2] =$

$= \alpha [x_1 + x_2, x_1 - x_2, 2x_1 + x_2]$
$= \alpha f(x), \quad \forall \alpha \in \mathbb{R}$
$x \in \mathbb{R}^2$

I, II $\Rightarrow$ f este apl liniară

$\ker f = \{x \in \mathbb{R}^2 \mid f(x) = 0\}$

$f(x)=0 \iff [x_1+x_2,\ x_1-x_2,\ 2x_1+x_2] =$
$= [0,0,0]$

$$\begin{cases} x_1+x_2=0 & (1) \\ x_1-x_2=0 & (2) \\ 2x_1+x_2=0 \end{cases}$$

"(1) + (2)" $\quad 2x_1=0$
$x_1=0 \implies x_2=0$

$\mathrm{Ker} f = \{[0,0]\}$

$\mathrm{Im} f = \{ y\in\mathbb{R}^3 \mid \exists\, x\in\mathbb{R}^2 \quad f(x)=y\}$

$f(x)=y \iff [x_1+x_2,\ x_1-x_2,\ 2x_1+x_2] =$
$= [y_1, y_2, y_3]$

$$(5)\begin{cases} x_1+x_2=y_1 \\ x_1-x_2=y_2 \\ 2x_1+x_2=y_3 \end{cases} \qquad A=\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$\bar{A}=\left(\begin{array}{cc:c} 1 & 1 & y_1 \\ 1 & -1 & y_2 \\ 2 & 1 & y_3 \end{array}\right)$$

$d_2 = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \neq 0$

$\implies \mathrm{rang}\, A = 2$

(5) comp $\iff$ rang $\bar{A} = 2$

$\iff \det \bar{A} = 0$

$$\begin{vmatrix} 1 & 1 & y_1 \\ 1 & -1 & y_2 \\ 2 & 1 & y_3 \end{vmatrix} = 0$$

$\iff \quad -y_3 + y_1 + 2y_2 + 2y_1 - y_2 - y_3 = 0$

$3y_1 + y_2 - 2y_3 = 0$

$$\mathrm{Im} f = \{ y = [y_1, y_2, y_3] \in \mathbb{R}^3 \mid 3y_1 + y_2 - 2y_3 = 0 \}$$

(5) $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f[x_1, x_2] = [a_{1,1} x_1 + a_{2,1} x_2,\ a_{1,2} x_1 + a_{2,2} x_2]$$

unde $\underbrace{a_{1,1}}_{a}, \overbrace{a_{1,2}}^{b}; \overbrace{a_{2,1}}^{c}; \underbrace{a_{2,2}}_{d} \in \mathbb{R}$ (fixate)

$$\begin{cases} \mathrm{I} & f(x+y) = f(x) + f(y) \quad \forall\ x, y \in \mathbb{R}^2 \\ \mathrm{II} & f(\alpha x) = \alpha f(x) \quad \alpha \in \mathbb{R} \end{cases}$$

I $\quad x = [x_1, x_2]$

$y = [y_1, y_2]$

$x + y = [x_1 + y_1, x_2 + y_2]$

$$f(x+y) = [a(x_1+y_1) + c(x_2+y_2),\ b(x_1+y_1) + d(x_2+y_2)]$$

$$f(x)+f(y) = [ax_1 + cx_2,\ bx_1 + dx_2] + [ay_1 + cy_2,\ by_1 + dy_2]$$

$$= [a(x_1+y_1) + c(x_2+y_2),\ b(x_1+y_1) + d(x_2+y_2)]$$

$$= f(x+y),\ \forall\ x, y \in \mathbb{R}^2$$

II $f(\alpha x) = f[\alpha x_1, \alpha x_2] =$

$$= [\alpha a x_1 + c\alpha x_2,\ \alpha b x_1 + d\alpha x_2]$$

$$= \alpha [ax_1 + cx_2,\ bx_1 + dx_2]$$

$$= \alpha f(x)$$

I, II $\Rightarrow$ f este apl lin

$$\operatorname{Ker} f = \{ x \in \mathbb{R}^2 \mid f(x) = 0 \}$$

$$f(x) = 0 \Leftrightarrow [ax_1 + cx_2,\ bx_1 + dx_2] = [0,0]$$

$$\begin{cases} ax_1 + cx_2 = 0 \\ bx_1 + dx_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_{1,1}x_1 + a_{2,1}x_2 = 0 \\ a_{1,2}x_1 + a_{2,2}x_2 = 0 \end{cases}$$

$$A = \begin{pmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{pmatrix}$$

Caz 1 $\det A \neq 0 \Rightarrow$ **$\text{rang } A = 2$**

$\Rightarrow$ sist comp det

$\Rightarrow$ sol unică $\Rightarrow \operatorname{Ker} f = \{[0,0]\}$

Caz 2 $\det A = 0$

- $\text{rang } A = 1$, adică unul dintre $a_{1,1}$, $a_{1,2}$, $a_{2,1}$ sau $a_{2,2}$ este nenul

$a_{1,1} \neq 0$

$$a_{1,1} x_1 + a_{2,1} x_2 = 0$$

$$x_1 = -\frac{a_{2,1} x_2}{a_{1,1}} ; \quad \beta = x_2$$

$$\operatorname{Ker} f = \left\{ \left[-\frac{a_{2,1}\beta}{a_{1,1}}, \beta \right] \middle| \beta \in \mathbb{R} \right\}$$

$$= \left\{ \beta \left[-\frac{a_{2,1}}{a_{1,1}}, 1 \right] \middle| \beta \in \mathbb{R} \right\}$$

- $\text{rang } A = 0$

$$\operatorname{Ker} f = \{ [x_1, x_2] \mid x_1, x_2 \in \mathbb{R} \} = \mathbb{R}^2$$

$$\mathrm{Im} f = \{ y \in \mathbb{R}^2 \mid \exists\, x \in \mathbb{R}^2,\ f(x) = y \}$$

$$\begin{cases} a_{1,1} x_1 + a_{2,1} x_2 = y_1 \\ a_{1,2} x_1 + a_{2,2} x_2 = y_2 \end{cases}$$

$$A = \begin{pmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{pmatrix}$$

<u>Cazul 1</u> $\det A \neq 0 \Rightarrow \operatorname{rang} A = 2$

$$\bar{A} = \left(\begin{array}{cc|c} a_{1,1} & a_{2,1} & y_1 \\ a_{1,2} & a_{2,2} & y_2 \end{array}\right) \in \mathcal{M}_{2,3}(\mathbb{R})$$

$\underset{\det A \neq 0}{\Rightarrow}$ $\operatorname{rang} \bar{A} = 2$, $\forall\ y_1, y_2 \in \mathbb{R}$

$\Rightarrow \mathrm{Im} f = \mathbb{R}^2$

<u>Cazul 2</u> $\det A = 0$

- $\operatorname{rang} A = 1$

p p $a_{1,1} \neq 0$

$$d_2 = \begin{vmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{vmatrix} = 0$$

$$d_3 = \begin{vmatrix} a_{1,1} & y_1 \\ a_{1,2} & y_2 \end{vmatrix} = 0$$

$$\Rightarrow a_{1,1} y_2 - a_{1,2} y_1 = 0$$

$$\mathrm{Im} f = \{ y = [y_1, y_2] \in \mathbb{R}^2 \mid a_{1,1} y_2 -$$

$a_{1,2} y_1 = 0 \}$

• $\text{rang } A = 0 \Rightarrow a_{1,1} = a_{1,2} = a_{2,1} = a_{2,2} = 0$

Ca $\text{rang } \bar{A} = 0 \Rightarrow y_1 = y_2 = 0$

$\text{Im} f = \{ [0, 0] \}$

(6) temă $f \ \mathbb{R}^2 \to \mathbb{R}$, $f[x_1, x_2] = x_1^2 - x_2^2$