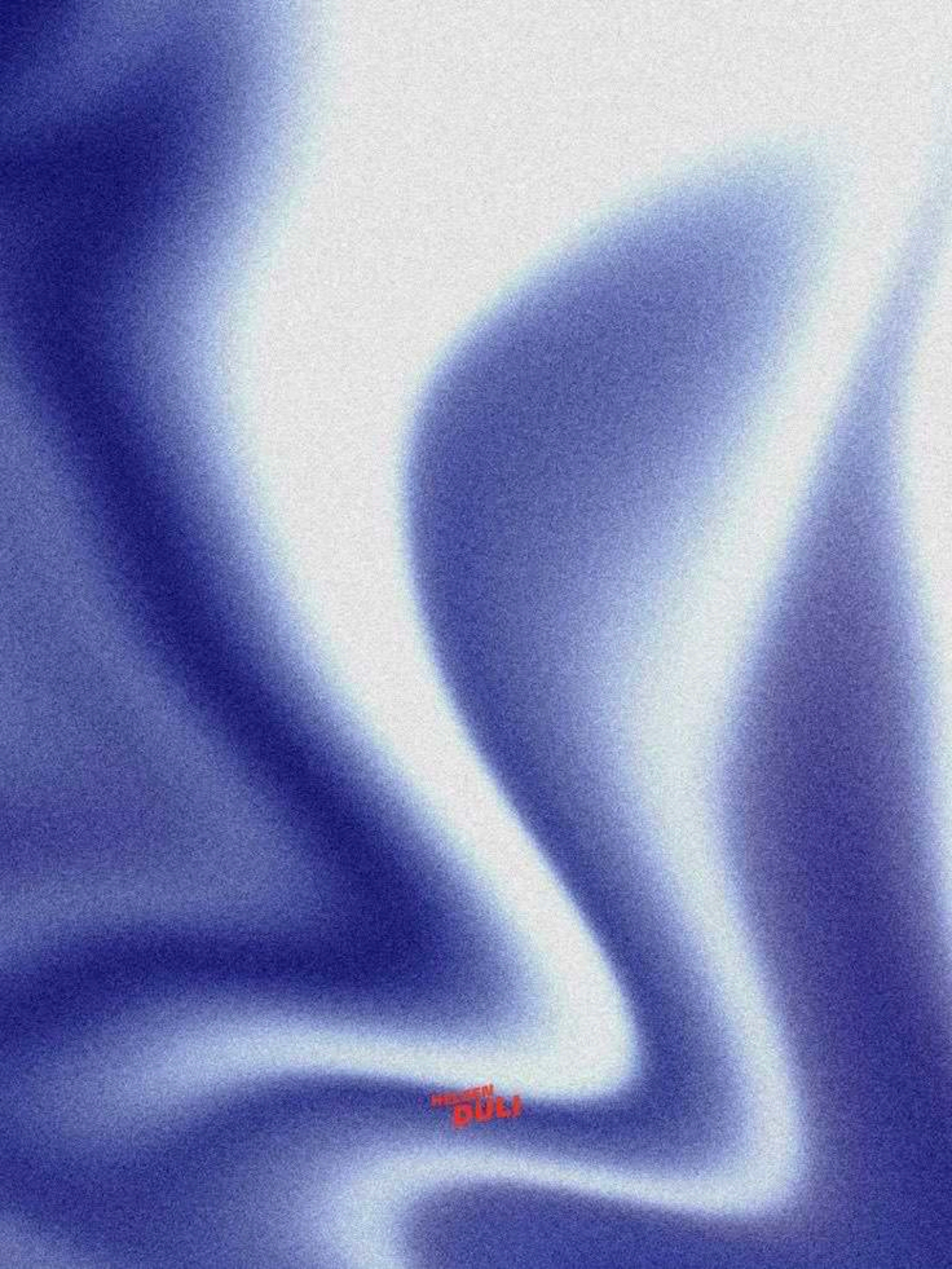


- Talleres → se pueden ver apuntes. } 60% de la nota
- Control (1 control)
- Examen 40% de la nota.

Propuesto

Evaluar numéricamente

π^{π} , i^{i} , π^{i} , i^{π}

36,462 159

Matrices de Pauli (introducción)

- Para representar el spin de las partículas.

Partícula cargada bajo un campo magnético uniforme $\vec{B} = B\hat{z}$

Newton $\vec{F} = m\vec{a} = m\dfrac{d\vec{v}}{dt}$

$$\vec{F} = q\,\underbrace{\vec{v}\times\vec{B}}_{E}$$

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\,\vec{v}\times\vec{B}$$

$$\left.\begin{array}{l}\vec{v} = v_x\hat{x} + v_y\hat{y}\\ \vec{B} = B\hat{z}\end{array}\right\} \vec{v}\times\vec{B} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ v_x & v_y & v_z\\ 0 & 0 & B\end{vmatrix} = v_y B\,\hat{x} - v_x B\,\hat{y}$$

$$\frac{d\vec{v}}{dt} = \frac{qB}{m}\left(v_y\hat{x} - v_x\hat{y}\right)$$

$$\frac{d}{dt}\begin{pmatrix}v_x\\ v_y\\ v_z\end{pmatrix} = \underbrace{\frac{qB}{m}}_{\omega}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_x\\ v_y\\ v_z\end{pmatrix} \Rightarrow \boxed{\frac{d}{dt}\begin{pmatrix}v_x\\ v_y\end{pmatrix} = \omega\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}v_x\\ v_y\end{pmatrix}} \Rightarrow \begin{array}{l}\dot{v}_x = -\omega v_y\\ \dot{v}_y = \omega v_x\end{array}$$

$$\frac{dv_z}{dt} = 0$$

$$\ddot{v}_x = -\omega^2 v_x \qquad \ddot{v}_y = -\omega^2 v_y$$

$$\rightsquigarrow \frac{df}{dx} = Mf$$

$$f = f_0\, e^{Mx}$$

Reescribimos

$$\frac{d}{dt}\begin{pmatrix}v_x\\ v_y\end{pmatrix} = -i\omega\underbrace{\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}_{\sigma_2\, =\, \sigma_y}\begin{pmatrix}v_x\\ v_y\end{pmatrix} \longrightarrow \frac{d\vec{v}}{dt} = -i\omega\sigma_2\vec{v} \xrightarrow{\text{sugiere}} \vec{v} = v_0\, e^{-i\omega\sigma_2 t}$$

note

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

· Obs: $\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Def

sea M matriz

$$e^M = \exp[M] = \sum_{k=0}^{\infty} \frac{1}{k!} M^k$$

$$= 1 + M + \frac{1}{2}M^2 + \frac{1}{3!}M^3 + \dots$$

· Prop.: sea t escalar

$$e^{t\cdot M} \rightarrow \frac{d}{dt}\left(e^{M\cdot t}\right) = \frac{d}{dt}\left(1 + Mt + \frac{1}{2}M^2t^2 + \frac{1}{3!}M^3t^3 + \dots\right)$$

$$= 0 + M + M^2t + \frac{1}{2}M^3t^2 + \frac{1}{3!}M^4t^3 + \dots$$

$$= M\left(1 + Mt + M\frac{t^2}{2!} + \dots\right)$$

$$= Me^{Mt} \Rightarrow \boxed{\frac{d}{dt}\left(e^{Mt}\right) = Me^{Mt}}$$

Con esto ⊛ $\Rightarrow$ $\boxed{\vec{v} = \vec{v}_0\, e^{-i\omega\sigma_2 t}}$

Expandimos $e^{-i\omega\sigma_2 t} \equiv A$

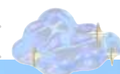
$$A = \sum_{k=0}^{\infty} \frac{1}{k!}\left(-i\,\omega t\,\sigma_2\right)^k$$

$$= \sum_{k\,par} \frac{1}{k!}\left(-i\omega t\right)^k \underbrace{(\sigma_2)^k}_{I} + \sum_{k\,impar} \frac{1}{k!}\left(-i\omega t\right)^k \underbrace{\sigma_2^k}_{\sigma_2}$$

$$= \sum_{k\,par} \frac{1}{k!}(i)^k(-\omega t)^k + \sum_{k\,impar} \frac{1}{k!}(i)^k(-\omega t)^k \cdot \sigma_2$$

prop. revisar

$$\boxed{A = \cos(\omega t)\cdot I - i\,\sigma_2 \operatorname{sen}(\omega t).}$$



Recuerdo:

$$\operatorname{sen}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$* \sum_{n=0}^{\infty}(-1)^n \cdot \sum_{n\,impar}^{\infty}$

$n=1 \longrightarrow (-1)^1 = -1$

$n=2 \longrightarrow (-1)^2 = 1$

$n=1 \rightarrow i^1 = i$

~~$n=2 \rightarrow i^2 = -1$~~

$n=3 \rightarrow i^3 = -i$

$n=5 \rightarrow i^5 = i$

Prop. clase pasada:

$$\pi^{\pi} = 36{,}46215$$
$$i^{i} = 0{,}20787957$$
$$\pi^{i} = 0{,}413292 + 0{,}91059\,i$$
$$i^{\pi} = 0{,}22058404 - 0{,}9753679\,i$$

15 · marzo

Recordamos:

$$m\vec{a} = q\vec{v}\times\vec{B}$$

$$\vec{B} = B\hat{z} \longrightarrow \dot{v}_z = 0 \Rightarrow v_z = cte$$

$$\frac{d}{dt}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = -i\underbrace{\frac{qB}{m}}_{\omega}\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_z}\begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

velocidad de la partícula cargada

$$\frac{d}{dt}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = -i\omega\sigma_z\begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = e^{-i\omega t\sigma_z}\begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix}$$

Obs

$$e^{-i\omega t\sigma_z} = (\cos\omega t)\,I - i\,\mathrm{sen}\,\omega t\,\sigma_z$$

$$= \begin{pmatrix} \cos\omega t & -\mathrm{sen}\,\omega t \\ \mathrm{sen}\,\omega t & \cos\omega t \end{pmatrix} \quad (*)$$

Notamos:

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\mathrm{sen}\,\omega t \\ \mathrm{sen}\,\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix}$$

sol. de la edo normal.

La posición:

$$\vec{v} = \frac{d\vec{r}}{dt} = e^{-i\omega t'\sigma_z}\begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix} \quad \Big/ \int_0^t \dots dt'$$

$$\vec{r}(t) - \vec{r}(0) = \int_0^t dt'\, e^{-i\omega t'\sigma_z}\begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix}. \quad (**)$$

Usando $(*)$ $\rightarrow$ $\vec{r}(t) - \vec{r}(0) = \frac{1}{\omega}\begin{pmatrix} \mathrm{sen}\,\omega t & \cos\omega t - 1 \\ 1-\cos\omega t & \mathrm{sen}\,\omega t \end{pmatrix}\begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix}$

$$= \frac{1}{\omega}\begin{pmatrix} v_{0x}\,\mathrm{sen}\,\omega t + (\cos\omega t - 1)v_{0y} \\ (1-\cos\omega t)v_{0x} + v_{0y}\sin\omega t \end{pmatrix}$$

Matrices de Pauli

Obs $\int e^{a\sigma_z x}\,dx \rightarrow \frac{1}{a\sigma_z} e^{a\sigma_z x} = \frac{1}{a}\sigma_z^{-1} e^{a\sigma_z x}$

· sale de la $\int$ xq es d_x

Retomamos (**):

$$\vec{r}(t) - \vec{r}_0 = \left[\int_0^t dt'\, e^{-i\omega t'\sigma_z}\right]\begin{pmatrix} v_{0x} \\ v_{0y}\end{pmatrix}$$

$$\left(-\frac{1}{i\omega}\sigma_z^{-1}\right)\underbrace{e^{-i\omega t'\sigma_z}\Big|_0^t}_{\left(e^{-i\omega t\sigma_z} - \mathbb{I}\right)}\left(\quad\right)$$

Propuesto: Obtener el radio de la trayectoria.

* Recordar coord. intrínsecas

$$\vec{a} = \dot{a}\hat{v} + \frac{\vec{a}\times\vec{v}}{\rho^2}$$

Las matrices de Pauli (canónicas)

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

· $\sigma_i^2 = 1$, $i = 1,2,3$

$\underbrace{\sigma_i^{+} = \sigma_i}_{\text{ie. hermíticas.}} \longrightarrow (\sigma_i^T)^* = \sigma_i$ (adjunto)

* ver conj. de matrices.

En coord. intrínsecas: $\vec{a}_t = \frac{dv}{dt}\vec{u}_t$, $\vec{a}_n = \frac{v^2}{\rho}\vec{u}_n \Rightarrow \vec{a} = \vec{a}_t + \vec{a}_n$.

Viene de:

$$\cdot\ \vec{v} = \frac{d\vec{r}}{dt} = \underbrace{\frac{d\vec{r}}{ds}}_{\hat{t}}\cdot\frac{ds}{dt} \rightsquigarrow \vec{a} = \frac{\partial\vec{v}}{\partial t} = \frac{d(v\hat{t})}{dt} = \frac{dv}{\partial t}\hat{t} + v\underbrace{\frac{d\hat{t}}{dt}}_{\frac{d\hat{t}}{ds}\cdot\frac{ds}{dt} = \frac{1}{\rho}\cdot\hat{n}v}$$

$$\rightsquigarrow \rho = \frac{v^3}{\|\vec{a}\times\vec{v}\|} = \text{radio de curvatura}$$

Propuestos

continuación

Así $\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\operatorname{sen}\omega t \\ \operatorname{sen}\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix} = \begin{pmatrix} \cos\omega t \cdot v_{0x} - v_{0y} \cdot \operatorname{sen}\omega t \\ v_{0x} \cdot \operatorname{sen}\omega t + v_{0y} \cdot \cos\omega t \end{pmatrix} \rightsquigarrow \|\vec{v}\| = \sqrt{v_x^2 + v_y^2}$

$$\to v = \|\vec{v}\| = \sqrt{\cos^2\omega t \cdot v_{0x}^2 - 2\cos\omega t \cdot \operatorname{sen}\cancel{\omega t} \cdot v_{0x} \cdot v_{0y} + v_{0y}^2 \operatorname{sen}^2\omega t + v_{0x}^2 \operatorname{sen}^2\omega t + 2 v_{0x} \cancel{v_{0y}} \operatorname{sen}\omega t \cdot \cos\omega t + v_{0y}^2 \cos^2\omega t}$$

$$= \sqrt{v_{0x}^2 \underbrace{(\cos^2\omega t + \operatorname{sen}^2\omega t)}_{1} + v_{0y}^2 \underbrace{(\ldots)}_{1}} = \sqrt{v_{0x}^2 + v_{0y}^2}$$

$$\to \rho = \frac{\sqrt{v_{0x}^2 + v_{0y}^2}^{\,3}}{\boxed{\|\vec{a} \times \vec{v}\|}}$$

$$\vec{a} = \begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix} = \begin{pmatrix} -v_y \\ \omega v_x \end{pmatrix} = \begin{pmatrix} \omega(-v_{0x}\operatorname{sen}\omega t - v_{0y}\cos\omega t) \\ \omega(\cos\omega t \cdot v_{0x} - v_{0y} \cdot \operatorname{sen}\omega t) \end{pmatrix}$$

$$\vec{a} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & 0 \\ v_x & v_y & 0 \end{vmatrix} = \hat{k}\,(a_x v_y - a_y v_x)$$

$$\|\vec{a} \times \vec{v}\| = \omega(-v_{0x}\operatorname{sen}\omega t - v_{0y}\cos\omega t) \cdot (v_{0x}\operatorname{sen}\omega t + v_{0y}\cos\omega t)$$
$$- \omega(\cos\omega t \cdot v_{0x} - v_{0y} \cdot \operatorname{sen}\omega t) \cdot (v_{0x}\cos\omega t - v_{0y}\operatorname{sen}\omega t)$$
$$= -v_{0x}^2 \operatorname{sen}^2\omega t - 2 v_{0y} \cos\cancel{\omega t} \cdot v_{0x}\operatorname{sen}\omega t - v_{0y}^2\cos^2\omega t$$
$$- (v_{0x}^2\cos^2\omega t - 2 v_{0y} \cancel{v_{0x}} \cos\omega t \operatorname{sen}\omega t + v_{0y}^2 \operatorname{sen}^2\omega t)$$
$$= \omega(-2 v_{0x}^2 (\operatorname{sen}^2 + \cos^2) - 2 v_{0y}^2 (\cos^2 + \operatorname{sen}^2))$$
$$= +2\omega(v_{0x}^2 + v_{0y}^2)$$

$$\rho = \frac{(v_{0x}^2 + v_{0y}^2)^{3/2}}{(v_{0x}^2 + v_{0y}^2) \cdot -2\omega} = + \frac{1}{(v_{0x}^2 + v_{0y}^2) \cdot 2\omega}$$

$$\hookrightarrow \rho = \frac{1}{2\omega v_0}$$

$\hookrightarrow$ * me queda un 2 :(

vectores prop. de la matriz hermítica asociados a los valores prop. ± 1.

$$\begin{pmatrix} a & c^* \\ c & b \end{pmatrix}\begin{pmatrix} \ \\ \ \end{pmatrix} = \pm 1 \begin{pmatrix} \ \\ \ \end{pmatrix}$$

↳ matriz hermítica si $M^+ = M$.

Calculemos $\sigma_1 \cdot \sigma_2$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\,\sigma_3$$

$$\sigma_2 \sigma_3 = i\,\sigma_1$$

Tensor o coef. de levi - civita:

ϵ_{ijk}

$\epsilon_{123} = 1$

$\epsilon_{213} = -1$ → si se permuta un coef. cambia el signo.

↳ antisimétrico

$\epsilon_{132} = -1$

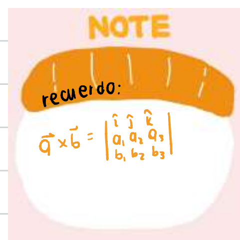
$\epsilon_{113} = 0$ → si se repiten vale 0.

$\epsilon_{212} = 0$

$$\boxed{\sigma_i \sigma_j = \delta_{ij} + i\,\underbrace{\epsilon_{ijk}\,\sigma_k}_{\text{suma } k = 1 \to 3.}}$$

↝ álgebra de las matrices de Pauli.

$$\rightarrow (\vec{a} \times \vec{b})_i = \sum_{jk} \epsilon_{ijk}\, a_j b_k$$



$$\begin{pmatrix} a & c^* \\ c & b \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \pm 1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Longrightarrow \begin{pmatrix} a v_1 + c^* v_2 \\ c v_1 + b v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \begin{matrix} a v_1 + c^* v_2 = v_1 \\ c v_1 + b v_2 = v_2 \end{matrix}$$

$$\to \frac{(1-a)}{c^*} v_1 = v_2 \longrightarrow c \cancel{v_1} + \frac{b(1-a)}{c^*} \cancel{v_1} = \frac{(1-a)}{c^*} \cancel{v_1}$$

$$\to c \cdot c^* + b(1-a) = 1-a$$

$$\to \frac{c \cdot c^* + b - 1}{b-1} = a \to \frac{c \cdot c^*}{b-1} + 1 = a$$

$$\Rightarrow \begin{pmatrix} a v_1 + c^* v_2 \\ c v_1 + b v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

δ_{ij} = delta de krönecker

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

variables complejas

$i^2 = -1$

Def $z = a + ib$

$\uparrow$ comp. real $\quad \uparrow$ com. imag.

Sean $z_1 = a_1 + i\, b_1$

$z_2 = a_2 + i\, b_2$

$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$.

Prop · Conmutatividad suma y prod.

· Asociatividad de la suma y prod.

$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$; $(z_1 z_2)\, z_3 = z_1 (z_2 z_3)$

· Distributividad c/r a suma.

$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$.

· Existencia del cero:

$0 = 0 + i0$, $z + 0 = (a + ib) + (0 + i0) = a + ib$

· Existencia de unidad:

$1 = 1 + i0$

$z \cdot 1 = 1 \cdot z = z$

· Existencia de inverso aditivo:

Dado $z = a + ib$. $\rightarrow (-z) = (-a) + i(-b)$. $\Big\}\; z + (-z) = a + (-a) + i(b + (-b)) = 0$.

· Inverso multiplicativo

z^{-1} , cumple $z \cdot z^{-1} = z^{-1} z = 1$

$$z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*} \quad , \quad z^* = a - ib.$$

$$\rightarrow z^{-1} = \frac{a-ib}{a^2+b^2} = \left(\frac{a}{a^2+b^2}\right) + i\left(\frac{-b}{a^2+b^2}\right)$$

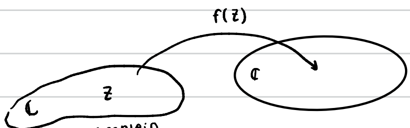
- Conjugado de z: $z = a+ib$

  $z^* = a - ib.$

- Desigualdades: $|z_1 + z_2| \leq |z_1| + |z_2|$

  $|z_1 + z_2| \geq |z_1| - |z_2|$

Funciones de variable compleja:



f(z) es complejo.

$z = x + iy.$ $\quad f(z) = u + iv$

$= u(x,y) + i\,v(x,y) = \omega$

Ej $f(z) = z^2$

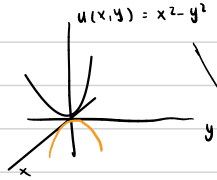
$= (x+iy)^2$

$= \underbrace{(x^2 - y^2)}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}. = \omega$

Analiticidad de funciones

20·mar

$u(x,y) = x^2 - y^2$ → gnuplot! (para graficar)



--→ plot x^2

splot $x^2 - y^2$ w l

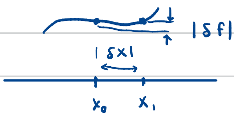
$v(x,y) = 2xy$ w pm3d

son las mismas pero rotadas en 45°.

una función $f(z)$ es analítica en z_0 si su derivada existe y es única.

previo: continuidad

En $\mathbb{R}$



$\forall \epsilon > 0, \exists \delta$ t.q $|x - x_0| < \delta \Rightarrow |f - f_0| < \epsilon$.

$f(z) = u + iv = w = w(x,y)$.

$z = x + iy$

En el caso de fn complejas

$\forall \epsilon > 0, \exists \delta$ t.q $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$

Verifiquemos continuidad de $f(z) = z^2$.

$|f(z) - f(z_0)|$

$$f(z) - f(z_0) = z^2 - z_0^2$$

$$= (z - z_0)(z + z_0)$$

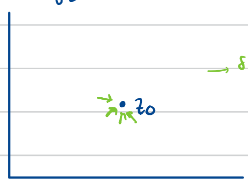
$$|f(z) - f(z_0)| = \underbrace{|(z - z_0)|}_{\delta}|z + z_0| < \varepsilon$$

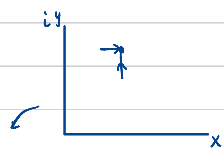
$$\Rightarrow \delta|z + z_0| < \varepsilon \rightarrow \delta = \frac{\varepsilon}{|z + z_0|}$$

Derivada

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$$

→ δz puede ir en ≠ direcciones





Imponemos derivada f' es la misma si $\begin{cases} \delta z = \delta x \\ \delta z = i\delta y \end{cases}$

Si $\delta z = \delta x$

$$\frac{u\overbrace{(x+\delta x,\ y)\ +iv(x+\delta x,y)}^{f(z+\delta z)}\ - u(x,y)\ - iv(x,y)}{\delta x}$$

$$= \frac{u(x,y+\delta y)+iv(x,y+\delta y)\ - u(x,y)\ - iv(x,y)}{i\,dy}.$$

En el limite $\delta x \to 0,\ dy \to 0 \ \equiv\ \delta z \to 0.$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

Cauchy - Riemann.

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{C.R.1.}$$

→ se deben cumplir ambas.

$$\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \quad \text{C.R.2.}$$

NO es analitica $f(z) = (z + z^*)\frac{1}{2}$ → en gral. una combinaø de z y z^* no es analitica.

$$u + iv = x$$

CR) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \longrightarrow 1 \neq 0$

Propiedad

Si $f(z) = u + iv$ es analítica $\Rightarrow$ u y v son armónicas: satisfacen la ecuación de Laplace en 2D:

CR1 $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \quad / \dfrac{\partial}{\partial x}$

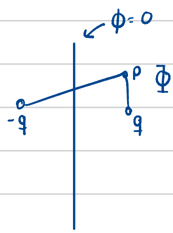
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\underbrace{\frac{\partial v}{\partial x}}_{\text{CR2}}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0}$$

Laplace en 2D.

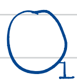
→ Revisar que Laplace c/ C.I. tiene sol. única.

↳ Revisar /estudiar: teo. de unicidad.



$\nabla^2 \Phi = 0$ → Método de imágenes (electro).

$\nabla^2 \phi = 0$



Prop: Notar que es lo mismo c/r a y.

Prop $f(z) = z^2 \rightarrow f'(z) = 2z$.

dem:

$f(z) = z^2 \longrightarrow w = u + iv$

$= (x^2 - y^2) + i(2xy)$

cauchy - riemann: $\left.\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right\}$ $2x = 2x$ ✓

$\left.\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.\right\}$ $-2y = -2y$. ✓

→ Dado que se cumple C-R, podemos derivar.

$f'(z) \rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i2y = 2(x+iy) = 2z$

Prop

sea $f(z) = z^3$ dem que es analítica y calc. $f'(z)$.

$z^3 = (x+iy)^3 = (\quad) + i(\quad)$

La exponencial

Denotamos la fn. exponencial de z como $\underbrace{\exp(z)}_{f(z)} \to e^z$

(a) $\exp(z)$ es analítica

(b) $f'(z) = f(z)$ $\longrightarrow$ ec. de valores propios.

$\hookrightarrow f' = \lambda f \longrightarrow$ sol: $e^{\lambda z}$

(c) $f(z_1 + z_2) = f(z_1) \cdot f(z_2)$.

De (c), si $z_1 = z_2 = 0$ $\to f(0) = f(0) \cdot f(0)$

tomamos $\underline{f(0) = 1}$

Exigimos $f'(z) = f(z)$

$$f(z) = u + iv \xrightarrow[\delta x]{} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u + iv \to \frac{\partial u}{\partial x} = u \quad (1)$$

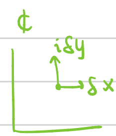
$$\to \frac{\partial v}{\partial x} = v \quad (2)$$

Tomando derivada c/r $\delta z = i\,\delta y$

$$\frac{1}{i}\frac{\partial u}{\partial y} + \frac{1}{\cancel{i}} \cdot \cancel{i}\,\frac{\partial v}{\partial y} = u + iv$$

$(1/i = -i)$

- $\dfrac{\partial u}{\partial y} = -v$ (3)
- $\dfrac{\partial v}{\partial y} = u$ (4)



$\delta z = \delta x + i\delta y$.

Buscamos $u(x,y)$ y $v(x,y)$

consideremos $h(x,y) = \dfrac{v(x,y)}{u(x,y)}$.

Examinamos $\dfrac{\partial h}{\partial x}$ o $\dfrac{\partial h}{\partial y}$.

⑤ $$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\left(\frac{v}{u}\right) = \frac{1}{u}\overset{v}{\left(\frac{\partial v}{\partial x}\right)} - v\,\frac{1}{u^2}\overset{u}{\frac{\partial u}{\partial x}} = \frac{v}{u} - \frac{v}{u} = 0$$

⑥ $$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(\frac{v}{u}\right) = \frac{1}{u}\underset{u}{\frac{\partial v}{\partial y}} - v\cdot\frac{1}{u^2}\underset{v}{\frac{\partial u}{\partial y}} = 1 + \frac{v^2}{u^2}$$

$\rightarrow \dfrac{\partial h}{\partial x} = 0 \Rightarrow \boxed{h = h(y)}$

$\rightarrow \boxed{\dfrac{\partial h}{\partial y} = \dfrac{dh}{dy}}$

usamos ⑤ $\dfrac{\partial h}{\partial y} = \dfrac{dh}{dy} = 1 + h^2$

$$\frac{dh}{1+h^2} = dy$$

$\rightarrow \arctg(h) = y + c$

$h(y) = \tan(y+c) = \dfrac{v}{u} \rightarrow \begin{array}{l} v = u\tan(y+c). \\ u = u \end{array}$

$$\Rightarrow \quad w = u + iv$$
$$= u + iu\tan(y+c)$$
$$\boxed{w = u(1 + i\tan(y+c))}$$

Imponemos $w(0,0) = 1 \Rightarrow u(0,0) = 1 \qquad u(0,0)\tan(0+c) = 0$

Escogemos $c = 0 \rightarrow w = u(1 + i\tan y)$.

$$= \frac{u}{\cos y}(\cos y + i\,\mathrm{sen}\,y).$$

debe ser e^x

prop!

* $e^z = e^{x+iy} = e^x e^{iy}$
$$= e^x(\cos y + i\,\mathrm{sen}\,y)$$

22-marzo

La exponencial : $f(z)$.

1) Analítica (C.R)

2) $f'(z) = f(z)$

3) $f(z_1 + z_2) = f(z_1) f(z_2)$.

· $f(z) = u(x,y) + i v(x,y) \equiv w$

Encontremos

$$\frac{\partial}{\partial x}\underbrace{\left(\frac{v}{u}\right)}_{h} = 0$$

$\Rightarrow h = h(y)$ → h no depende de x!

→ $h(y) = \tan y = \dfrac{v}{u}$ $\Rightarrow v = u \tan y$.

Entonces:

$$\boxed{w = u + i u \tan y}$$

· $f' = f \rightarrow \dfrac{\partial u}{\partial x} + i \dfrac{\partial}{\partial x}(u \tan y) = u + i u \tan y$.

$$\rightarrow \frac{\partial u}{\partial x} = u$$

$$\rightarrow \frac{\partial}{\partial x}(u \tan y) = u \tan y \Rightarrow \frac{\partial u}{\partial x} \cancel{\tan y} = u \cancel{\tan y}.$$

$$\frac{\partial u}{\partial x} = u$$

$$\downarrow$$

$$u(x,y) = g(y)e^{x} \quad (*)$$

$$* \quad \frac{\partial}{i\partial y} = \frac{1}{i} \cdot \frac{\partial}{\partial y} = -i\frac{\partial}{\partial y}.$$

$$\frac{1}{i}\frac{\partial}{\partial y}(u + i\,u\tan y) = u + i\,u\tan y.$$

$$\frac{\partial}{\partial y}(u\tan y) = u$$

$$-\frac{\partial u}{\partial y} = u\tan y.$$

$$\hookrightarrow -\frac{1}{u}\frac{\partial u}{\partial y} = \frac{\operatorname{sen} y}{\cos y} \xrightarrow[(*)]{\text{Usando}} -\frac{1}{g}\frac{\partial g}{\partial y} = \frac{\operatorname{sen} y}{\cos y}$$

$$-\frac{1}{g\cancel{e^{x}}} \cdot \cancel{e^{x}} \cdot \frac{\partial g}{\partial y}$$

$$\hookrightarrow \boxed{\frac{g'}{g} = -\frac{\operatorname{sen} y}{\cos y}.}$$

$$\hookrightarrow \frac{d}{dy}\ln g = \frac{d}{dy}\ln\cos y.$$

$$\frac{y'}{y} = \frac{d}{dx}\ln y.$$

$$\rightarrow \underline{g = C\cos y}$$

En $(*)$ $\longrightarrow$ $u(x,y) = Ce^{x}\cos y.$

$f(z) = u + iv$

$= u(1 + i\tan y)$.

$= c\, e^x \cos y$

$f(z) = c\, e^x(\cos y + i\,\operatorname{sen} y)$.

Puesto que $f(0) = 1 \Rightarrow c = 1$ $\to$ $f(0+1) = f(0)\cdot f(1)$

$$f(z) = w = \underbrace{(e^x \cos y)}_{u} + i\,\underbrace{(e^x \operatorname{sen} y)}_{v}$$

Denotamos

$$f(z) = \exp(z) = e^z$$

$$e^z = e^{x+iy} = e^x(\cos y + i\,\operatorname{sen} y)$$

$\to$ Exp. analítica en todo el plano complejo.

La exponencial es una **fn. entera.**

Definiciones:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$\cos(x + i\cancel{y}) = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right)$$

0

$$= \left(e^{-y+ix} + e^{y-ix}\right)\frac{1}{2}$$

$$= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\operatorname{sen} z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right).$$

$$\tan z = \frac{\operatorname{sen} z}{\cos z} = i\,\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

$$= \left(\cos x + \cancel{i\operatorname{sen} x} + \cos x - \cancel{i\operatorname{sen} x}\right)\frac{1}{2}$$

$$= \cos x \ ✓$$

* **Propuesto:**

$$e^{z_1} e^{z_2} = e^x(\cos y + i\,\operatorname{sen} y).$$

$$e^{x_1} e^{iy_1} e^{x_2} e^{iy_2}$$

$$= e^{(x_1+x_2)} e^{i(y_1+y_2)}$$

$$= e^{(x_1+x_2)+i(y_1+y_2)}$$

$$= e^{x_1+x_2}\left(\cos(y_1+y_2) + i\,\operatorname{sen}(y_1+y_2)\right)$$

- $\cosh z = \frac{1}{2}(e^{z} + e^{-z})$

- $\operatorname{senh} z = \frac{1}{2}(e^{z} - e^{-z})$ → lo mismo pero z no lleva i!

- $\tanh z = \frac{\operatorname{senh} z}{\cosh z}$

* hacer de los 10 ejercicios de la guía.

↳ 12, 13

* Traer calculadora!

Obs

$$e^{i\pi} = \cos \pi + i \operatorname{sen} \pi = -1.$$

$$e^{i\pi} + 1 = 0$$

Logaritmo

Fn inversa de la exponencial

$$e^z = w \quad / \ln$$
$$\ln(e^z) = \ln(w)$$
$$z = \ln(w)$$

· Nos preguntamos por $\operatorname{Ln}(z) \overset{?}{=} u + iv$

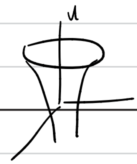
$$e^{\ln z} = e^{u+iv}$$
$$z = e^{u+iv}$$
$$\boxed{x + iy = e^{u}(\cos v + i\,\mathrm{sen}\, v)}$$

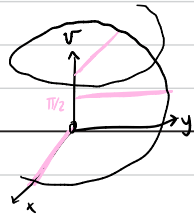
$$x = e^{u}\cos v$$
$$y = e^{u}\sin v$$

① $x^2 + y^2 = e^{2u}$

$$2u = \ln(x^2+y^2)$$
$$u = \ln(\underbrace{\sqrt{x^2+y^2}}_{|z|})$$
$$u = \ln|z|$$

② $\operatorname{tg} v = \frac{y}{x} \rightarrow v = \operatorname{arctg}\left(\frac{y}{x}\right) = v(x,y).$





* ln no es analítico en 0.

Def

$$z^a = e^{a \ln z}$$

* pág 18 (revisar). → la forma de z^z.

$z^z = ?$

$\frac{d}{dz}(\ln z) = \frac{\partial}{\partial x}[u + iv] = \underbrace{\frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}\frac{\cdot 2x}{\sqrt{x^2+y^2}}}_{\frac{\partial u}{\partial x}} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i \frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2}$

$\frac{\partial}{i\partial y}(\ )$

$= \left(\frac{x}{x^2+y^2} + - \frac{i\ y}{x^2+y^2}\right) \cdot \frac{(x+iy)}{(x+iy)}$

$= \frac{1}{x+iy} = \frac{1}{z}$

* $\frac{1}{1+\theta^2} \cdot \theta' = (\operatorname{arctg} \theta)'$

27 mar.

tenemos $f(z) = \ln z$

$f'(z) = 1/z$

· a puede ser complejo

<u>Def</u> $z^a \equiv e^{a\ln z} \rightarrow (e^{\ln z})^a$

$$\frac{d}{dz}(z^a) = \underbrace{\frac{de^{(\ )}}{d(\arg)}} \cdot \frac{d(\arg)}{dz}$$

$$e^{a\ln z} \cdot \frac{d(a\ln z)}{dz} = \frac{e^{a\ln z} \cdot a}{z} = \frac{z^a \cdot a}{z}$$

$$\Rightarrow \frac{d}{dz}(z^a) = a z^{a-1}$$

como demostramos que $z^a z^b = z^{a+b}$

$\rightarrow e^{a\ln z} e^{b\ln z} = e^{(a+b)\ln z} = z^{a+b}$

Propiedades geométricas : $z \xrightarrow{f} w$, f analítico.

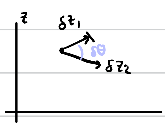
· ϕ ángulo. $\rightarrow$ \phi

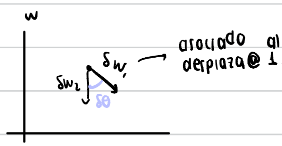
· φ campo $\rightarrow$ \var phi

$\hookrightarrow \underbrace{\nabla^2 \varphi = 0}_{\rho, \phi, z}$ //

$z = \rho\cos\theta + i\rho\,\text{sen}\,\theta$

$z = \rho e^{i\theta}$





· Si la deriv. no es nula, si f es analítica, los ángulos son iguales.

$\rightarrow \boxed{w = f(z)}$

$$\delta w = f'(z_0)\cdot \delta z$$

$$\delta w_1 = f'(z_0)\cdot \delta z_1$$

$$\delta w_2 = \text{''} \quad \delta z_2$$

$\downarrow$ deriv. no nula

Los ángulos son los mismos.

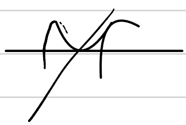
$$\frac{\delta w_2}{\delta w_1} = \frac{\cancel{f'(z_0)}}{\cancel{f'(z_0)}}\cdot\frac{\delta z_2}{\delta z_1} = \frac{\delta z_1}{\delta z_2} \longrightarrow e^{i(\theta_2-\theta_1)}.$$

· si $\delta z_2 = \delta s\, e^{i\theta_2}$

$\delta z_1 = \delta s\, e^{i\theta_1}$

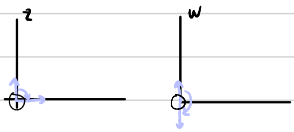
Qué pasa si $f'(z_0) = 0$

$$\delta w_2 = \cancel{f'(z_0)}^{\,0}\,\delta z_2 + \frac{1}{2} f''(z_0)\cdot(\delta z_2)^2$$



se duplicó el ángulo

$$\frac{\delta w_2}{\delta w_1} = \frac{(\delta z_2)^2}{(\delta z_1)^2} = e^{i2(\theta_2-\theta_1)}$$



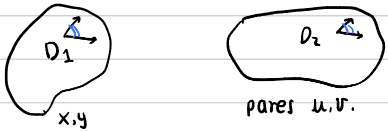
→ examinamos cómo se comporta localmente.

$$f'(z_0)\cdot\delta z_2 + \frac{1}{2} f''(z_0)\cdot(\delta z_2)^2 + \frac{1}{3} f^{(3)}(\delta z_2)^3 \dots$$

Transformaciones conformes:

· "Mapeo" de zona D_1 en $\mathbb{C}$ a zona D_2 en $\mathbb{C}$ vía una función analítica $f(z)$.

· se prod. una transformaø en el otro dominio con = ángulo.



· $w = f(z)$ (z = x+iy.)

$= u + iv$

· Algunas transformaciones de interés:

1) $w = z$ ident.

2) $w = z + z_0$ traslaø.

3) $w = iz$ rotaø.

4) $w = z^2$ (potenciación)

$w = z^{1/2}$ $z^{1/3}$ $z^{\text{pot. racional.}}$

5) $w = \frac{1}{z}$ (inversión)

6) $w = \operatorname{sen} z$; $\cos z$

7) $w = \operatorname{senh} z$, $\cosh z$

8) $w = \ln(z)$

9) $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ Joukowski

un ala de avión a una circunferencia.

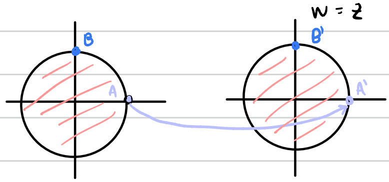
Ej

Nos damos D_1 y $f \to$ buscamos D_2.

1) círculo radio unitario en $\mathbb{C}$.

transformación $f(z) = z$

La región de puntos interiores del círculo será el interior de la transformac.



$u = x$

$v = y$

$u = \cos\theta$

$v = \operatorname{sen}\theta$

Parametrizando el contorno:

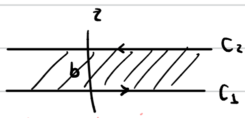
$y = \operatorname{sen}\theta$

$x = \cos\theta \quad \theta : 0 \to 2\pi$

1° ver dominio

2) $w = iz$

Franja



$e^{i\alpha} \to$ para otro ángulo.

Prop $w = e^{i\alpha} z$.

$w = i(x + iy) = -y + ix$

$u = -y$

$v = x$

C_1 | $x = t \quad , \quad t : -\infty \to \infty \qquad \to u = 0$

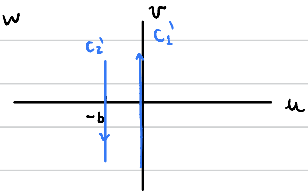
$y = 0 \qquad\qquad v = t \quad , \quad t : -\infty \to \infty$

Podría ser que las regiones de adentro se van a un punto que no sea intuitivo.

→ pág. 22 → fig 1.8

∼ tikz para hacer figuras.

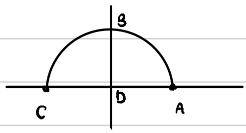
en w



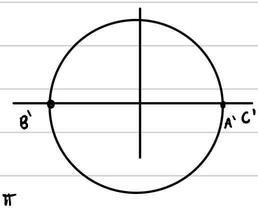
C_2 | $x = t$ $\quad t: +\infty \to -\infty \to u = -b$

$y = b$ $\quad v = t \quad , \; t: +\infty \to -\infty.$

3) $f(z) = z^2$



w



ABC $\quad z = e^{i\theta} \quad \theta: 0 \to \pi$

$z^2 = e^{2i\theta} \to \cos 2\theta + i\,\text{sen}\, 2\theta.$

$C: x, y = -1, 0$

$u, v = 1, 0.$

$\theta = 0 \Rightarrow u, v = 1, 0$

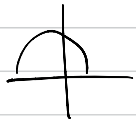
$\theta = \pi/2 \Rightarrow u, v = -1, 0$

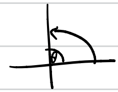
$D: 0, 0$

$D': 0, 0.$

· $f(z) = \sqrt{z}$

ángulo se reduce a la mitad.



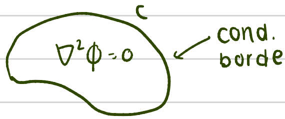


· $f(z) = z^{1/3} \rightarrow$ se reduce $1/3$.

Unicidad

29. mar.

· Por unicidad no hay + soluØes



ϕ_1 , ϕ_2 soluciones: $(\phi_1 \neq \phi_2)$

$\rightarrow \Phi = \phi_2 - \phi_1$

C.B Φ

C.B. $\underbrace{\Phi/_{contorno}}_{\Phi/_{\partial}} = 0$

$\Phi/_{\partial} = 0$

sea $\Phi \cdot \nabla^2 \Phi = 0$

$$\vec{\nabla} \cdot (\Phi \cdot \vec{\nabla}\Phi) = (\vec{\nabla}\cdot\Phi)\cdot(\vec{\nabla}\Phi) + \underbrace{\Phi\,\nabla^2\Phi}_{0} \quad / \iiint_{vol} d\tau$$

$$\iiint_{vol} \vec{\nabla}\cdot(\Phi \cdot \vec{\nabla}\Phi)\,d\tau = \iiint_{vol} |\vec{\nabla}\Phi|^2\,d\tau + 0$$

Teo. de la divergencia:

$$\oiint_{\Sigma vol} (\Phi \cdot \vec{\nabla}\Phi)\cdot d\vec{\Sigma} = \iiint |\vec{\nabla}\phi|^2\,d\tau$$

$\Phi/_{\partial} = 0$

$$0 = \iiint \underbrace{|\vec{\nabla}\phi|^2}_{\geq 0}\,d\tau$$

Transf

$\Rightarrow \quad \vec{\nabla}\Phi = 0$ en el interior

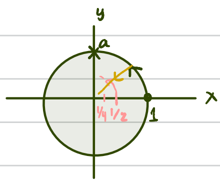
$\Phi = cte$ en el interior

$\Rightarrow \quad \phi_1(\vec{r}) - \phi_2(\vec{r}) = 0$

$\Phi /_{\partial} = 0 \quad \Longrightarrow \quad \Phi = 0$ en el interior.

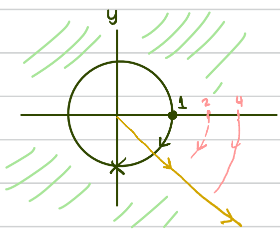
Ejemplos de transformaciones conformes:

<u>Inversión</u>: $f(z) = \frac{1}{z}$



$z = re^{i\theta}$ → puntos del círculo.

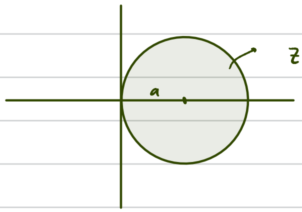
$$\omega = \frac{1}{re^{i\theta}} = \rho\, e^{i\phi}$$



→ $\rho = \frac{1}{r}$

→ $\phi = -\theta$

Otro dominio para $f(z) = 1/z$



$\rightarrow \frac{1}{z} = \frac{1}{a+re^{i\theta}} = u + iv$

multiplicar por el conjugado para que el denominador sea real. $\rightarrow$

$= \frac{a+re^{-i\theta}}{(a+re^{i\theta})(a+re^{-i\theta})}$

$= \frac{a+re^{-i\theta}}{a^2+r^2+\underbrace{are^{i\theta}+are^{-i\theta}}_{2ar\cos\theta}}$

$= \underbrace{\left(\frac{a+r\cos\theta}{a^2+r^2+2ar\cos\theta}\right)}_{u} + \underbrace{\left(i\frac{-r\,\text{sen}\,\theta}{a^2+r^2+2ar\cos\theta}\right)}_{v}$

Examinamos el contorno:

$r = a$

$u(r=a) = \frac{a(1+\cos\theta)}{a^2+a^2+2a^2\cos\theta} = \frac{a(1+\cos\theta)}{2a^2(1+\cos\theta)} = \frac{1}{2a}$ $\rightarrow$ una línea vertical (u cte).

→ Anotar identidades trigonométricas!!

↳ $\cos^2\frac{\theta}{2} - \text{sen}^2\frac{\theta}{2} = \cos\theta$

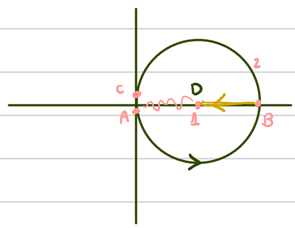
↳ $\cos^2\frac{\theta}{2} + \text{sen}^2\frac{\theta}{2} = 1$

$v(r=a) = i\frac{-a\,\text{sen}\,\theta}{2a^2+2a^2\cos\theta} = \frac{-\overbrace{\text{sen}\,\theta}^{\cancel{2}\text{sen}\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}}{2a\underbrace{(1+\cos\theta)}_{\cancel{2}\cos^{\cancel{2}}(\frac{\theta}{2})}} = \boxed{-\frac{\text{tg}(\theta/2)}{2a}}$

→ en $-\pi$ → $v = \frac{\text{tg}(\pi/2)}{2a} = \infty$

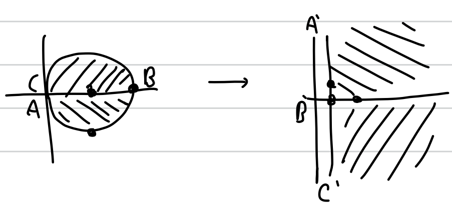
→ en 0 → $v = 0$

→ en π → $v = \frac{-\text{tg}(\pi/2)}{2a} = -\infty$



Apunte

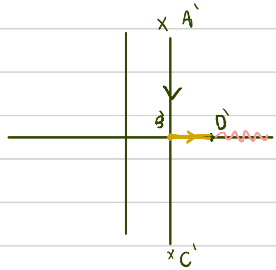
$\rightarrow \frac{1}{z}$:



→ google: Escher and conform Transformations

→ pág 25 → Mapa
↳ ej: del 18 al 21
↳ 1.5.4
(a)
(b)
(c) → examinaremos →

→ escribir sen( ) como
$\frac{e^{iz}-e^{-iz}}{2}$



$f(z) = \text{sen}(z)$

$$w = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right)$$

$$= \frac{1}{2i}\left(e^{-y}e^{ix} - e^{y}e^{-ix}\right)$$

e^{ix}: $\cos x + i\,\text{sen}\,x$; e^{-ix}: $\cos x - i\,\text{sen}\,x$

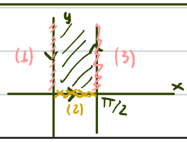
$$= \frac{1}{2i}\left[\cos x\left(e^{-y} - e^{y}\right) + i\,\text{sen}\,x\left(e^{-y}+e^{y}\right)\right]$$

$$= \frac{1}{i}\left[-\cos x\,\text{senh}\,y + i\,\text{sen}x\,\cosh y\right]$$

$$= \underbrace{\text{sen}\,y\,\cosh y}_{u} - i\underbrace{\cos x\,\text{senh}\,y}_{v}$$

$u = \text{sen}x\,\cosh y$
$v = \cos x\,\text{senh}y$

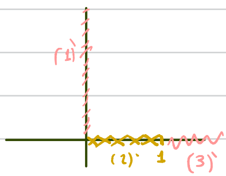
Dominio en x, y



· Tres tramos:

(1) $x = 0$
$y = t$, con $t\ (+\infty \to 0)$

$$\begin{cases} u = 0 \\ v = \operatorname{senh}(t) \end{cases} \longrightarrow \begin{array}{l} \operatorname{senh}(0) = 0 \\ \operatorname{senh}(\infty) = \infty \end{array}$$



(2) : $y = 0$

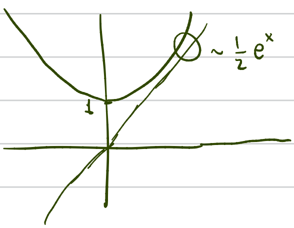
$x = t \quad , \underbrace{t : 0 \to \pi/2.}_{}$

$\rightarrow u = \operatorname{sen} t$ u crece de 0 a 1.

$v = 0$

* $\operatorname{senh} y = \dfrac{e^{y} - e^{-y}}{2} \xrightarrow{y \to \infty} \dfrac{e^{y}}{2}$

$\cosh y = \dfrac{e^{y} + e^{-y}}{2} \longrightarrow \dfrac{e^{y}}{2}$



plot 'senh(x), cosh(x)

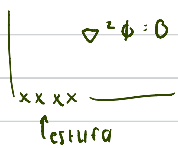
(3) : $x = \pi/2$

$y = t \quad , \underbrace{t : 0 \to \infty}_{}$ u de 1 a ∞

$\rightarrow u = \cosh t$

$v = 0$

Problema :



Identidades trigonométricas

- $\csc\theta = \frac{1}{\text{sen}\,\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\text{cotg}\,\theta = \frac{\cos\theta}{\text{sen}\,\theta}$

$\text{sen}(-\theta) = -\text{sen}\,\theta$
$\cos(-\theta) = \cos\theta$

- $1 + \text{tg}^2\theta = \sec^2\theta$
- $1 + \text{cotg}^2\theta = \csc^2\theta$.
- $\text{sen}(\alpha \pm \beta) = \text{sen}\,\alpha\cos\beta \pm \text{sen}\,\beta\cos\alpha$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \text{sen}\,\alpha\,\text{sen}\,\beta$.

$\text{tg}(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$.

→ $\text{sen}(2\theta) = 2\,\text{sen}\,\theta\cos\theta$

→ $\cos(2\theta) = 1 - 2\,\text{sen}^2\theta = \cos^2\theta - \text{sen}^2\theta = 2\cos^2\theta - 1$

→ $\text{tg}(2\theta) = \dfrac{2\,\text{tg}\,\theta}{1 - \text{tg}^2\theta}$

→ $\text{sen}\,\theta/2 = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$

→ $\cos\theta/2 = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$

→ $\text{tg}\,\theta/2 = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \dfrac{1-\cos\theta}{\text{sen}\,\theta} = \dfrac{\text{sen}\,\theta}{1+\cos\theta}$

→ $\text{sen}\,\alpha \pm \text{sen}\,\beta = 2\,\text{sen}\left(\dfrac{\alpha \pm \beta}{2}\right)\cos\left(\dfrac{\alpha \mp \beta}{2}\right)$

→ $\cos\alpha + \cos\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

→ $\cos\alpha - \cos\beta = -2\,\text{sen}\left(\dfrac{\alpha+\beta}{2}\right)\text{sen}\left(\dfrac{\alpha-\beta}{2}\right)$

→ $\text{sen}\,\alpha\,\text{sen}\,\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

→ $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$

→ $\text{sen}\,\alpha\cos\beta = \frac{1}{2}\left[\text{sen}(\alpha+\beta) + \text{sen}(\alpha-\beta)\right]$

ej)

[18] **Mapa de Joukowski** : $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Si $z = re^{i\theta}$ $\Longrightarrow$ $w = \frac{1}{2}\left(re^{i\theta} + \frac{1 \cdot re^{-i\theta}}{re^{-i\theta} \cdot re^{i\theta}}\right)$

$$= \frac{1}{2}\left(re^{i\theta} + \frac{e^{-i\theta}}{r}\right)$$

$$= \frac{1}{2r}\left(r^2 e^{i\theta} + e^{-i\theta}\right)$$

$$= \frac{1}{2r}\left(\underbrace{e^{\ln r^2 + i\theta}}_{e^{\ln r + \ln z}} + e^{-i\theta}\right)$$

$$\ln z = \ln(re^{i\theta})$$
$$= \ln(e^{\ln r} e^{i\theta})$$
$$= \ln(e^{\ln r + i\theta})$$
$$\ln z = \ln r + i\theta$$

$$\to \cos(\ln z) = \frac{1}{2}\left(e^{i\ln z} + e^{-i\ln z}\right)$$

$$= \frac{1}{2}\left(e^{i\ln z} + \frac{1}{e^{i\ln z}}\right)$$

$$= \frac{1}{2}\left(e^{\ln(z^i)} + \frac{1}{e^{\ln(z^i)}}\right) = \frac{1}{2}\left(z^i + \frac{1}{z^i}\right)$$

$e^{i\frac{\pi}{2}}$

$\cos(\ln z) + i\,\operatorname{sen}(\ln z)$

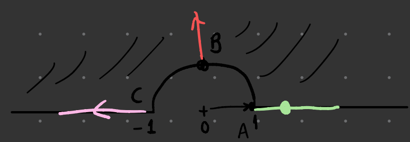
$$\to w = \frac{1}{2}\left(e^{\ln z} + e^{-\ln z}\right) = \frac{1}{2}\left(e^{-i(i\ln z)} + e^{i(i\ln z)}\right)$$

$$= \cos(i\ln z) - i\,\cancel{\operatorname{sen}(i\ln z)} + \cos(i\ln z) + i\,\cancel{\operatorname{sen}(i\ln z)}.$$

$$= \frac{1}{\cancel{2}}\,\cancel{2}\cos(i\ln z).$$

$$= \boxed{\cos(i\ln z)}\ ?$$

$\boxed{20}$ $w(z) = \frac{1}{z} - z \longrightarrow z = re^{i\theta}$



$$\Rightarrow w = \frac{1}{re^{i\theta}} - re^{i\theta} = \frac{e^{-i\theta} - r^2e^{i\theta}}{r} = \frac{\cos(\theta) - i\,sen\theta - r^2(\cos\theta + i\,sen\theta)}{r}$$

$$= \underbrace{\frac{\cos\theta(1-r^2)}{r}}_{u} + i\left(\underbrace{-\frac{sen\theta(1+r^2)}{r}}_{v}\right)$$

si $\theta = 0 \rightarrow v = 0$

si $\theta = \pi/2 \rightarrow v = -2$

si $\theta = \pi \rightarrow v = 0$

$\rightarrow$ si $r = 1$: $u = 0$, $v = -2\,sen\theta$

$\rightarrow$ si $r = -1$: $u = 0$, $v = \frac{-2\,sen\theta}{-1} = 2\,sen\theta$

$\rightarrow$ si $r = 0$: $u = v = \infty$

$\rightarrow$ si $r > 1$, $\theta = 0 \rightarrow u = \frac{(1-r^2)}{r} < 0$, $v = 0$.

$= \frac{1}{r} - r$

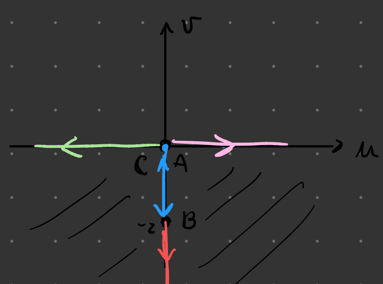
$r > 1$, $\theta = \pi \rightarrow u = \frac{-(1-r^2)}{r} > 0$, $v = 0$.

$= r - \frac{1}{r}$

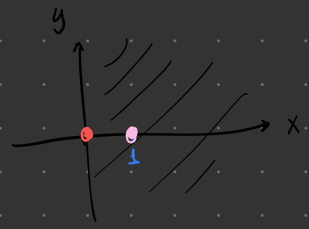
$\rightarrow$ si $r > 1$, $\theta = \frac{\pi}{2} \rightarrow$ $u = 0$

$v = -\frac{(1+r^2)}{r}$

$= -\left(\frac{1}{r} + r\right)$.



¿Está mal?



$w = \frac{z+1}{z-1} \longrightarrow z = re^{i\theta}$
$= x + iy$

$\hookrightarrow \frac{x+iy+1}{x+iy-1} = \frac{x+1+iy}{((x-1)+iy)}\frac{((x-1)-iy)}{((x-1)-iy)}$

$= \frac{((x+1)+iy)((x-1)-iy)}{(x-1)^2+y^2}$

$= (x^2-1) - (\cancel{x}+1)iy + iy(\cancel{x}-1) + y^2$

$= \frac{x^2+y^2-1-2iy}{x^2-2x+1+y^2}$

$= \underbrace{\frac{x^2+y^2-1}{x^2-2x+1+y^2}}_{u} + \underbrace{\frac{-2y}{x^2-2x+1+y^2}}_{v} i$

Si $y=0 \longrightarrow$

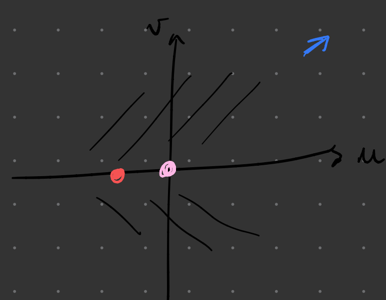
si $x^2-2x+1+\cancel{y^2}^{0} = 0 \Rightarrow x^2-2x+1 = 0$
$\Rightarrow u = v = \infty \qquad \Rightarrow (x-1) = 0$
$\Rightarrow x = 1.$

si $x^2 + \cancel{y}^{0} = 1 \Rightarrow u = 0 = v$

si $x = y = 0 \Rightarrow u = -1$
$v = 0$

Si $x = 0 \rightarrow \frac{y^2-1}{1+y^2} = u$

$v = \frac{-2y}{1+y^2}$



3 · abril

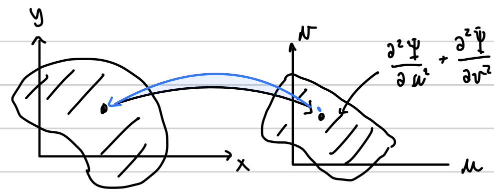
Si $\nabla^2 \Phi = 0$ + C.B

en un dominio dado $\rightarrow$ $\Phi = \Phi(x,y,z)$ es única

—//—

Transformaciones conformes aplicadas a $\nabla^2 \Psi = 0$ en dominio 2D.

i.- $w = u + iv = f(x+iy)$



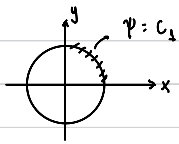
$\frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2}$ $\rightarrow$ sin ambiguedad

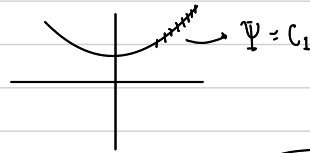
siendo $u = u(x,y)$, $\Psi(x,y) = \Psi(u,v)$.
$v = v(x,y)$

$$\underbrace{\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}}_{0} = |f'(z)|^2 \underbrace{\left(\frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2} \right)}_{0}$$

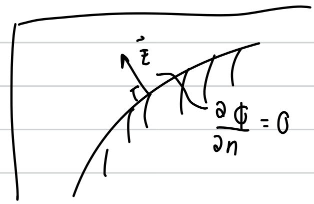
$\rightarrow$ siempre que $f'(z) \neq 0$.

iii.- Mapeo de las C.B.





$$\frac{\partial \psi}{\partial n} = 0 = \frac{\partial \Phi}{\partial n}$$



Demostramos ii.-

$$\cdot \ \frac{\partial \psi}{\partial x} = \left(\frac{\partial \Psi}{\partial u} \cdot \frac{\partial u}{\partial x}\right) + \left(\frac{\partial \Psi}{\partial v} \cdot \frac{\partial v}{\partial x}\right)$$

▽ importantes:

$$\cdot \ \frac{\partial^2 \psi}{\partial x^2} = \left(\boxed{\frac{\partial^2 \Psi}{\partial u^2} \cdot \left(\frac{\partial u}{\partial x}\right)^2} + \frac{\partial \Psi}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 \Psi}{\partial u \partial v} \cdot \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial x}\right)$$

$$+ \left(\boxed{\frac{\partial^2 \Psi}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2} + \frac{\partial \Psi}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 \Psi}{\partial v \partial u} \cdot \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial x}\right)$$

$$\cdot \ \frac{\partial \psi}{\partial y} = \frac{\partial \Psi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \Psi}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\cdot \ \frac{\partial^2 \psi}{\partial y^2} = \left(\frac{\partial^2 \Psi}{\partial u^2} \cdot \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial \Psi}{\partial u} \cdot \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 \Psi}{\partial u \partial v} \cdot \frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial y}\right)$$

$$+ \left(\frac{\partial^2 \Psi}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial \Psi}{\partial v} \cdot \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 \Psi}{\partial v \partial u} \cdot \frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial y}\right)$$

$\rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\rightarrow$ u y v son armónicas (C-R)

$\rightarrow$ sumando $\frac{\partial^2 \psi}{\partial x} + \frac{\partial^2 \psi}{\partial y}$

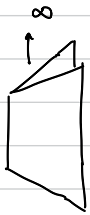
$$= \frac{\partial^2 \Psi}{\partial u^2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) + \frac{\partial^2 \Psi}{\partial v^2}\left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)$$

$$+ 2\frac{\partial^2 \Psi}{\partial u \partial v}\underbrace{\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right)}_{0 \text{ C-R}}$$

$$+ \frac{\partial \Psi}{\partial u}\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{0 \text{ (C-R)}} + \frac{\partial \Psi}{\partial v}\cdot\underbrace{\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}_{0 \text{ (C-R)}}$$

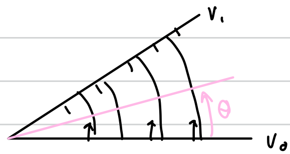
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \left[\frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2}\right]\underbrace{\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]}_{|f'(z)|^2}$$

Si $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \rightarrow \frac{\partial^2 \Psi}{\partial^2 u} + \frac{\partial^2 \Psi}{\partial v^2} = 0$



→ fórmula de esféricas, cilíndricas, etc, agregar al cuaderno.

problema de placas conductoras oblicuas a una diferencia de potencial.



Electromagnetismo:

coordenadas cilíndricas r, θ, z

$V = V(r, \theta, z) \longrightarrow V(r, \theta)$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 V = 0 \longrightarrow \frac{1}{r}\frac{\partial^2}{\partial r^2} rV + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + 0 = 0$$

solución:

$V = A_1 + A_2 \theta \longrightarrow \text{arctg}(\ )$

$f(z) = z^2 = (x^2 - y^2) + i2xy$

↳ si f es analítica ⇒ la parte real o imaginaria son sol. de Laplace.

Laplacianos:

Coord. cilíndricas: $$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

coord. esféricas: $$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sen(\theta)}\frac{\partial}{\partial \theta}\left(sen(\theta)\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 sen^2(\theta)}\left(\frac{\partial^2}{\partial \phi^2}\right)$$

$\ln|z| + i\theta = \ln z = u + iv$

arctg satisface laplace.

$= \text{sen}\,\theta$

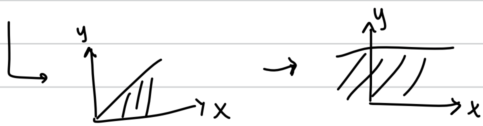
$\cos z$

senh

} son analíticas

$\Longrightarrow$ cada una de sus comp. satisfacen laplace.

$\rightarrow w = \ln z = \ln\sqrt{x^2+y^2} + i\theta$



hola Pichi :)

Aplicaciones.

5· Abril.

Obs

$f(z) = u + iv \rightarrow u, v$ armónicas

$z = x + iy$

$\nabla^2 u = 0$ (satisfacen

$\nabla^2 v = 0$ Laplace).

$$\ln(z) = \ln\left(\sqrt{x^2+y^2}\right) + i\arctg\left(\frac{y}{x}\right)$$

$$= \ln r + i\theta$$

Probemos $\nabla^2 f = 0$ con $f(x,y) = \arctan\left(\frac{y}{x}\right)$.

→ que sea armónica no es garantía que es la sol. del problema.
↳ faltan C.B!

$$\rightarrow \frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y}{(\quad)^2}\,2x = \frac{2xy}{(\,\cdot/\cdot\,)} \quad ①$$

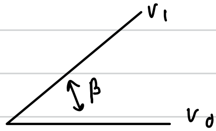
$$\rightarrow \frac{\partial f}{\partial y} = \frac{1}{1+\left(y/x\right)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

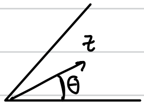
$$\frac{\partial^2 f}{\partial y^2} = \frac{-x}{(\quad)^2}\cdot 2y \quad ②$$

$$\rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

↓

armónica! ✨





$z = re^{i\theta}$

→ como es infinito, no depende de r.
↳ es igual que hacerlo en una vecindad cerca del centro

$$f(z) = \ln z = \underbrace{\ln r}_{u} + i\underbrace{\theta}_{v}$$

$$V(\theta) = A + B\theta$$

cond. de borde: $\theta = 0 \quad V(\theta = 0) = V_0 \Rightarrow A + B\cdot 0 = V_0$ ①

$\theta = \beta \quad V(\beta) = V_1 \Rightarrow A + B\cdot\beta = V_1$ ②

① → $A = V_0$

② → $B = \dfrac{V_1 - V_0}{\beta}$

$$V = V_0 + \frac{V_1 - V_0}{\beta}\underset{\operatorname{arctg}\left(\frac{y}{x}\right)}{\underset{\uparrow}{\theta}}$$

La solución.

$$\boxed{V = V_0 + \frac{V_1 - V_0}{\beta}\operatorname{arctg}\left(\frac{y}{x}\right)}$$

→ cumple $\nabla^2 V = 0$
→ cumple C.B.

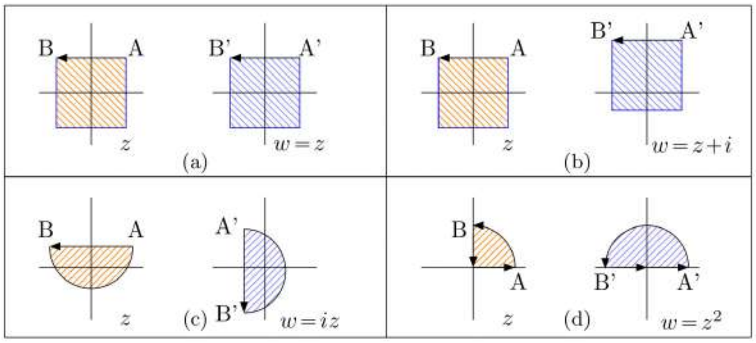


Fig. 1.8: Mapas conformes elementales.

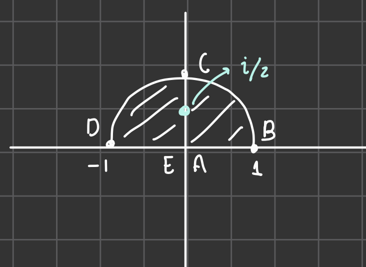
· Inversión: $\omega = 1/z$

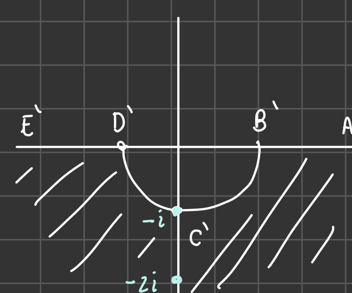
→ Disco centrado en el origen: $z = re^{i\theta}$ $\begin{cases} r: 0 \to a \\ \theta: -\pi \to \pi \end{cases}$

$$\cdot\ \omega = \frac{1}{re^{i\theta}} = \left(\frac{1}{r}\right) e^{-i\theta}$$

Introducimos: $\omega = \rho e^{i\phi}$

$$\cdot\ \left.\begin{array}{l} \rho = \frac{1}{r} \\ \phi = -\theta \end{array}\right\}$$





→ Centrado en $x = a$. $z = a + re^{i\theta}$ $\begin{cases} r: 0 \to a \\ \theta: -\pi \to \pi \end{cases}$

Construimos ω:

$$\omega = \frac{1}{z} = \frac{1}{a + re^{i\theta}}$$

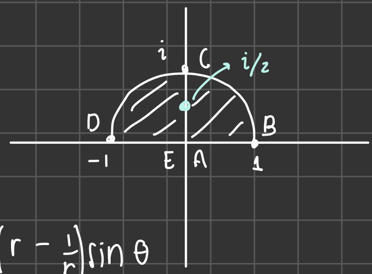
$$\Rightarrow \quad \omega = \left(\frac{a + r\cos\theta}{a^2 + r^2 + 2ar\cos\theta}\right) + i\left(\frac{-r\sin\theta}{a^2 + r^2 + 2ar\cos\theta}\right)$$

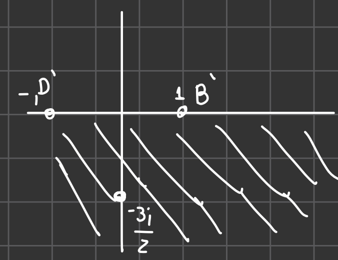
$$\omega\big|_a = \left(\frac{1}{2a}\right) + i\left(-\frac{1}{2a}\tan\theta/2\right) \longrightarrow \text{Recta con } u = 1/2a$$

$v: -\infty \to -\infty$

Mapa de Joukowski

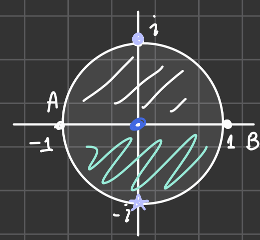
· $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$

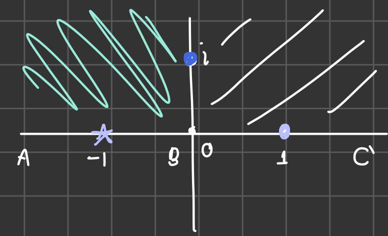




· $w = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta + i\,\frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta$

· Transf: $w = i\,\frac{(1-z)}{(1+z)}$





→ $w = \tan(\theta/2) = u + iv$

· $w = e^z$

$w = (e^x \cos y) + i\,(e^x \sin y) = u + iv$