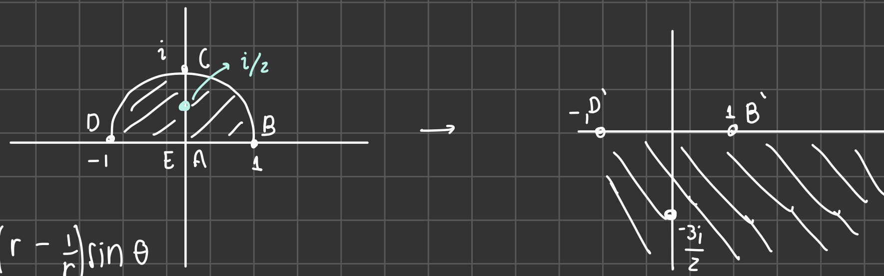
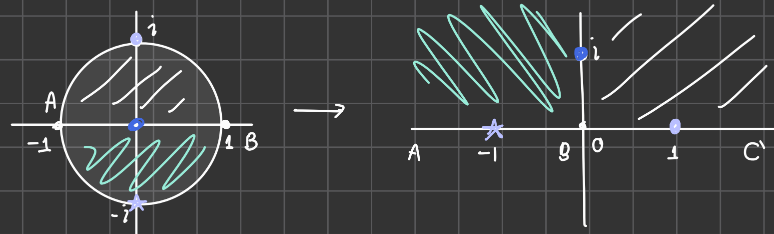
Mapa de Joukowski

- $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$



- $w = \frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta + i\,\frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta$

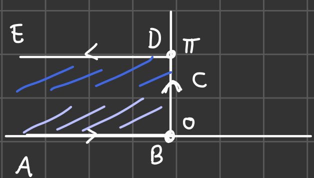
- Transf: $w = i\,\frac{(1-z)}{(1+z)}$

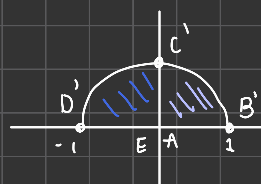


→ $w = \tan(\theta/2) = u+iv$

- $w = e^z$

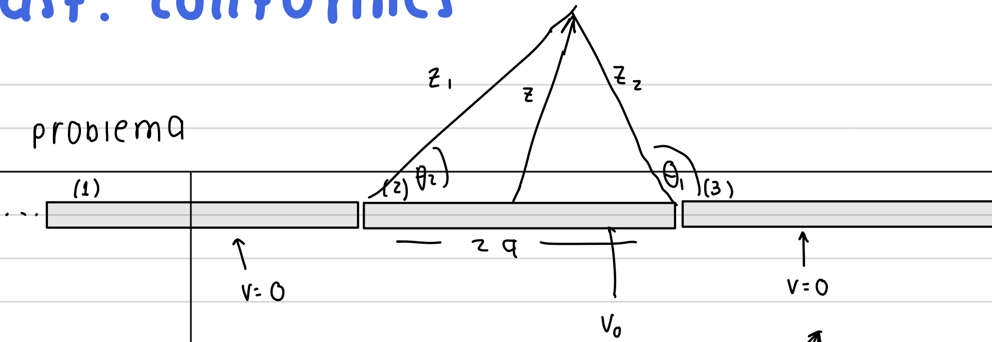
$w = (e^x \cos y) + i\,(e^x \sin y) = u+iv$





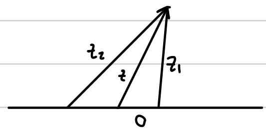
Trasf. conformes

otro problema



$\cdot\ a + z_1 = z$

$-a + z_2 = z$



$z_1 = z - a$

$z_2 = z + a$

$\text{arctg}\ \frac{y}{x} \Rightarrow$ satisfacen laplace

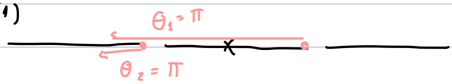
parte real e imag. satisfacen laplace $\leftarrow$ $\ln(z_1) = \ln(z-a) = \ln(|z-a|) + i\theta_1$

$\ln(z_2) = \ln(z+a) = \ln(|z+a|) + i\theta_2$

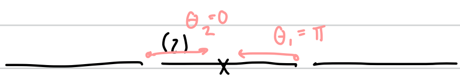
cumple $\nabla^2 V = 0$ ✓ $\leftarrow$ $V = c_0 + c_1\theta_1 + c_2\theta_2$

falta ver si podemos dar cuenta de C.B.

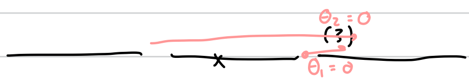
(1)



$V = 0 = \cancel{c_0}^{\,0} + c_1\pi + c_2\pi.$



$V_0 = c_1\pi + c_2 \cdot 0.$



$V = 0 = c_0 \longrightarrow \boxed{c_0 = 0}$

$\Rightarrow C_0 = 0$

$\Rightarrow C_1 = \frac{V_0}{\pi}$

$\Rightarrow C_2 = -\frac{V_0}{\pi}$

$$\boxed{V = \frac{V_0}{\pi} \cdot \theta_1 - \frac{V_0}{\pi}\theta_2}$$

$\longrightarrow$ θ_1 y θ_2 no se visualiza (aunque es soluØ)

expresar la solución en términos de las coord. físicas. $\nearrow$

$* \; z_1 = z + a = (x + a) + iy$

$$\theta_1 = \arctan\left(\frac{y}{x-a}\right) \qquad \left(\frac{\mathrm{im}(z_1)}{\mathrm{Re}(z_1)}\right)$$

$$\theta_2 = \arctan\left(\frac{y}{x+a}\right)$$

$$\boxed{V = \frac{V_0}{\pi}\left[\arctan\left(\frac{y}{x-a}\right) - \arctan\left(\frac{y}{x+a}\right)\right]}$$

simplificando $\longrightarrow$

$$* \; \overbrace{\arctan(A)}^{\alpha} + \overbrace{\mathrm{arctg}(B)}^{\beta} = ?$$

$$\mathrm{arctg}(\mathrm{tg}(\alpha + \beta))$$

$$\operatorname{sen}(\alpha+\beta) = \operatorname{sen}\alpha\cos\beta + \cos\alpha\operatorname{sen}\beta.$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta$$

$$\tan(\alpha+\beta) = \frac{tg\alpha + tg\beta}{1 - tg\alpha\, tg\beta}.$$

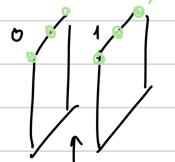
Tenemos :

$$V = \frac{V_0}{\pi}\operatorname{arctg}\left(\frac{A-B}{1+AB}\right)$$

$$= \frac{V_0}{\pi}\operatorname{arctg}\left(\frac{\frac{y}{x-a} - \frac{y}{x+a}}{1 + \frac{y^2}{(x-a)(x+a)}}\right)$$

$$= \frac{V_0}{\pi}\operatorname{arctg}\left(\frac{y(x+a) - y(x-a)}{(x^2-a^2)+y}\right)$$

$$\boxed{V = \frac{V_0}{\pi}\operatorname{arctg}\left(\frac{2ya}{x^2-a^2+y}\right)}$$



condiciones de borde de forma discreta.

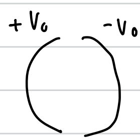
es analítica.

$$u + iv = b_0 + b_1 z + b_2 z^2 + \dots$$ → Identificar ctes.

$re^{i\dots}$

parte real. b_0 $\quad b_1(x)$ $\quad b_2(x^2 - y^2)$

ej]



→ ver insinuaciones del profe.

Problema de electrostática

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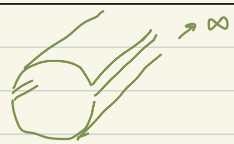
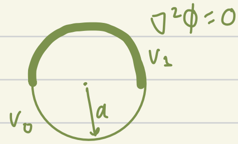
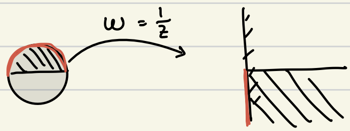
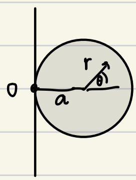






$$\nabla^2 \phi(u,v) = 0$$

$$\nabla^2 \phi(x,y) = 0$$



$$z = a + re^{i\theta}$$

$$r : 0 \to a$$

$$\theta : -\pi \to \pi$$

$$w = \frac{1}{z} = \frac{1}{a+re^{i\theta}} = \frac{(a+re^{-i\theta})}{a+re^{i\theta}\cdot(a+re^{-i\theta})}$$

$$= \frac{a+re^{-i\theta}}{a^2+r^2+are^{i\theta}+are^{-i\theta}}$$

$$= \frac{a+r\cos\theta - ir\,\mathrm{sen}\,\theta}{a^2+r^2+2ar\cos\theta}$$

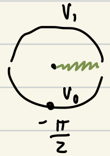
$$u+iv = \frac{a+r\cos\theta}{a^2+r^2+2ar\cos\theta} - \frac{i\,\mathrm{sen}\,\theta}{a^2+r^2+2ar\cos\theta}$$

Identificamos contorno

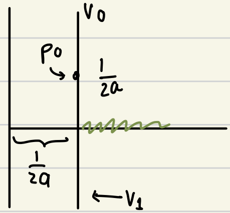
Hacemos $r = a \Rightarrow$

$$u + iv = \underbrace{\frac{a + a\cos\theta}{2a^2(1+\cos\theta)}}_{\frac{1}{2a}} - i\,\frac{a\,\text{sen}\,\theta}{2a^2(1+\cos\theta)}$$

$$\frac{1}{2a} - i\,\frac{\text{sen}\,\theta}{2a(1+\cos\theta)}$$



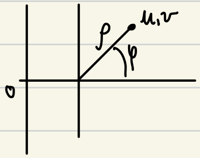
· $\theta = 0 \Rightarrow u + iv =$



$\nabla^2 \phi = 0$

conocemos la solución.

· solución $\phi = A + B\varphi$



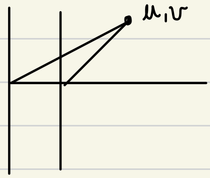
Para $\varphi = \frac{\pi}{2}$, $\phi = V_0$

$$V_0 = A + B\frac{\pi}{2}$$

$$V_1 = A - B\frac{\pi}{2}$$

$$A = \frac{V_0 + V_1}{2} \quad , \quad B = \frac{V_0 - V_1}{\pi}$$

$$\phi = \left(\frac{V_0 + V_1}{2}\right) + \left(\frac{V_0 - V_1}{\pi}\right)\varphi$$



De la geometría $\rightarrow w = \frac{1}{2a} + \rho\cos\varphi + i\rho\,\mathrm{sen}\,\varphi$

$$\hookrightarrow = \frac{1}{a + re^{i\theta}} = \left(\frac{a + r\cos\theta}{a^2 + r^2 + 2ar\cos\theta}\right) - i\,\frac{r\,\mathrm{sen}\,\theta}{\%}$$

$$\frac{1}{2a} + \rho\cos\varphi = \frac{a + r\cos\theta}{(\%)}$$

$$\boxed{\rho\,\mathrm{sen}\,\varphi = -\frac{r\,\mathrm{sen}\,\theta}{(\%)}}$$

$$\rho\cos\varphi = \frac{a + \cancel{r\cos\theta} - \frac{1}{2a}\left(a^2 + r^2 + \cancel{2ar\cos\theta}\right)}{\%}$$

$$= \frac{\frac{a}{2} - \frac{r^2}{2a}}{\%} = \frac{1}{2a}\,\frac{(a^2 - r^2)}{\%}$$

$$\boxed{\rho\cos\varphi = \frac{1}{2a}\,\frac{(a^2 - r^2)}{\%}}$$

$$\tan\varphi = -\frac{r\,sen\theta\cdot 2a}{a^2-r^2}$$

$$\varphi = -\arctan\left[\frac{2ar\,sen\theta}{a^2-r^2}\right]$$

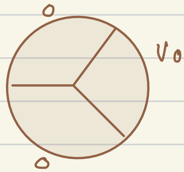
$$\phi(\overbrace{r,\theta}^{x,y}) = \frac{V_0+V_1}{2} - \left(\frac{V_0-V_1}{\pi}\right)\underbrace{\arctan\left(\frac{2ar\,sen\theta}{a^2-r^2}\right)}_{\frac{\pi}{2} = arctg(+\infty)}$$

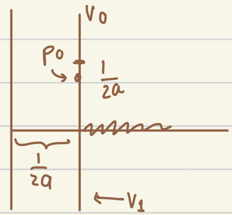
verificar que es → correcto

para $r \to a$

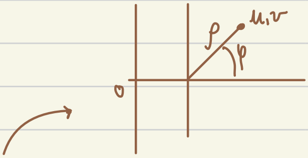
$$\frac{V_0+V_1}{2} - \frac{V_0-V_1}{2} = \boxed{V_1}$$ → es correcto

Variante





$B\varphi$



$\varphi = \frac{\pi}{2}$, $\phi = V_0$ y 0

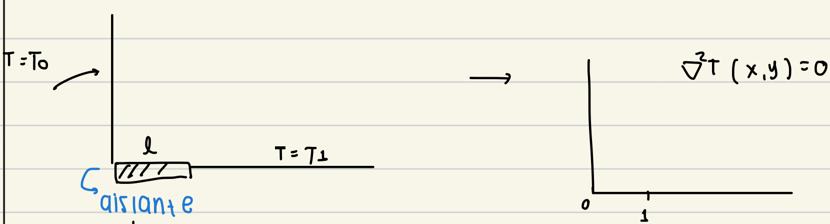
¿cómo se hace?

Problema inducción de calor

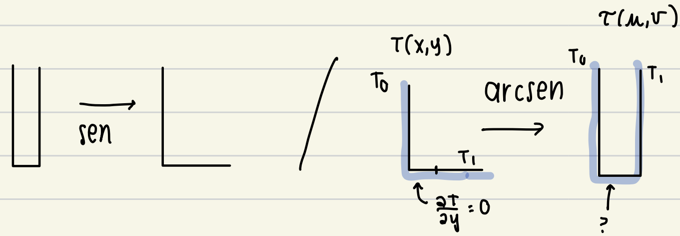
→ lo escaló a una variable adimensional: $x \to \frac{x}{\ell}$

Electrostática $\nabla^2 \phi = 0 \longrightarrow \vec{E} = -\vec{\nabla}\phi$

Termodinámica $\nabla^2 T = 0 \longrightarrow \vec{\nabla} T$



$\frac{\partial T}{\partial y} = 0 \sim$ cambio de T en y vale 0.



$w = \arcsen(z) = u + iv$

$$z = \operatorname{sen}(u+iv) = \frac{e^{u+iv} - e^{-(u+iv)}}{2i}$$

→ $x = \operatorname{sen} u \cosh v$ ①

$y = \cos u \operatorname{senh} v$. ②

$x^2 = \operatorname{sen}^2 u \cosh^2 v$

$y^2 = \cos^2 u \operatorname{senh}^2 v$

$\rightarrow x^2 + y^2 = \operatorname{sen}^2 u \cosh^2 v + \cos^2 u \operatorname{senh}^2 v$

$x^2 + y^2 = (1 - \cos^2 u)\cosh^2 v + \cos^2 u \operatorname{senh}^2 v$

$= \cosh^2 v + \cos^2 u(\operatorname{senh}^2 v - \cosh^2 v) = \cosh^2 v - \cos^2 u$

$\frac{x^2}{\operatorname{sen}^2 u} = \cosh^2 v$

$\cos^2 u = \cosh^2 v - x^2 - y^2$

$\cos^2 u = \frac{x^2}{\operatorname{sen}^2 u} - x^2 - y^2 \rightarrow \cos^2 u = \frac{x^2 - x^2 \operatorname{sen}^2 u - y^2 \operatorname{sen}^2 u}{\operatorname{sen}^2 u}$

$\rightarrow \cos^2 u \operatorname{sen}^2 u = x^2 \cos^2 u - y^2 \operatorname{sen}^2 u \quad \cdot \frac{1}{\cos^2 u}$

$\rightarrow \operatorname{sen}^2 u = x^2 - y^2 \operatorname{tg}^2 u$

$\rightarrow \operatorname{sen}^2 u \left(1 - \frac{1}{\cos^2 u}\right) = x^2$

Relación de T con μ e v $\rightarrow$

$$\frac{\partial T}{\partial y} = \frac{\partial \tau}{\partial \mu}\cdot \cancel{\frac{\partial \mu}{\partial y}}_{0} + \frac{\partial \tau}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial \tau}{\partial v}\cdot \underbrace{\frac{\partial v}{\partial y}}_{\neq 0} = 0$$

$$\downarrow$$

$$\frac{\partial \tau}{\partial v} = 0$$

$\phi \longrightarrow \Phi$

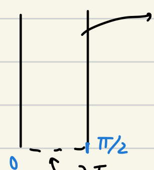
$T(x,y) \longrightarrow \tau(\mu, v)$

① $0 = \cos\mu \dfrac{\partial \mu}{\partial y} \cdot \underset{1}{\cosh v} + \operatorname{sen}\mu \underset{0}{\operatorname{senh} v} \dfrac{\partial v}{\partial y}$

en $v=0$ $\Longrightarrow$ $\dfrac{\partial \mu}{\partial y} = 0$

② $1 = \underbrace{\operatorname{sen}\mu \dfrac{\partial \mu}{\partial y}\cdot \operatorname{senh} v}_{0} + \cos\mu \underset{1}{\cosh v}\cdot\dfrac{\partial v}{\partial y}$. $\Rightarrow \dfrac{\partial v}{\partial y} \neq 0$

en $v=0$



$\dfrac{\partial \tau}{\partial r} =$

$$\frac{\partial^2 \tau}{\partial \mu^2} + \cancel{\frac{\partial^2 \tau}{\partial r^2}}^{0} = 0$$

$$\tau = A + B\mu$$

$$\left.\begin{aligned} T_0 &= A \\ T_1 &= A + B\frac{\pi}{2} \end{aligned}\right\} \boxed{\tau = T_0 + \left(\frac{T_1 - T_0}{\pi/2}\right)\mu}$$

$\rightarrow$ desacernos de μ con $\cosh^2 v - \operatorname{senh}^2 v = 1$.

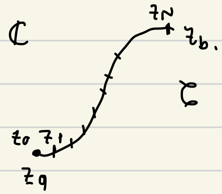
$\Longrightarrow \operatorname{arctg}\left(\sqrt{1 - x^2 + y^2}\right) = \mu$

$\rightsquigarrow \tau = T_0 + \left(\dfrac{T_1 + T_0}{\pi/2}\right)\operatorname{arctg}\left(\sqrt{1 - x^2 + y^2}\right)$

Integración

13 · abril

* en las $\vec{f}$ conservativas (W no depende de la trayectoria) → ptos extremos importan.

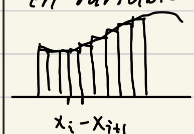


pensamos en

$$\lim_{N\to\infty}\sum_{i=1}^{N} f(\xi_i)(z_i - z_{i-1})$$

$$= \int_C f(z)\,dz$$

en variable real:



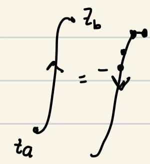
f(pto. medio)

$$I = \sum_{i}^{n} f(\bar{x}_i)(x_{i+1} - x_i)$$

$$\downarrow \lim_{N\to\infty}$$

$$\int$$

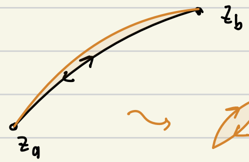
$$= \int_{z_a}^{z_b} f(z)\,dz \leftarrow$$



Obs → $$\int_{z_a,C}^{z_b} f(z)\,dz = -\int_{z_b,C}^{z_a} f(z)\,dz$$

Teorema de Cauchy-Goursat.

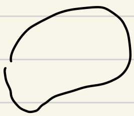
f(z) es analítica



$$\rightarrow \int_{z_a,C}^{z_b} f(z)\,dz + \int_{z_b}^{z_a} f(z)\,dz = 0$$

Es posible si $f(z)$ es analitica y <u>no</u> hay <u>singularidades</u>, ie, divisiones por 0.

sea $\mathcal{C}$ una trayectoria cerrada en $\mathbb{C}$ y $f(z)$ analítica dentro de $\mathcal{C}$ y sobre $\mathcal{C}$.



$$\oint_{\mathcal{C}} f(z)\,dz = 0$$

uso de la Analiticidad →

$$f(z) = f(x+iy) = u + iv$$

$$z = x + iy$$

$$dz = dx + i\,dy.$$

$$\rightarrow f(z)\,dz = (u+iv)(dx + i\,dy).$$

$$= \underbrace{(u\,dx - v\,dy)} + i\underbrace{(v\,dx + u\,dy)}.$$

$$\vec{A} \equiv (u,-v) \quad + \quad \vec{B} = (v,u)$$

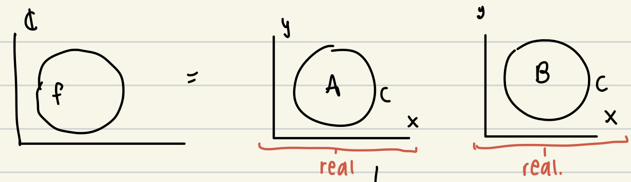
$$d\vec{l} = (dx,dy) \qquad d\vec{l} = (dx,dy)$$

prod. punto. →

$$= \vec{A}\cdot d\vec{l} + i\vec{B}\cdot d\vec{l}$$

→ rotor: circulación por unidad de área.
integral cerrada en un elem. infinitesimal de superficie.

$$\oint f(z)\,dz = \oint \vec{A}\cdot d\vec{l} + i\,\vec{B}\cdot d\vec{l}\,.$$



↓ stokes.

$$\oint_c \vec{F}\cdot d\vec{l} = \iint_{\Sigma(c)} (\vec{\nabla}\times\vec{F})\cdot d\vec{\Sigma}$$

superficie
$d\vec{\Sigma} = ds\cdot\hat{n}$

calculemos $(\vec{\nabla}\times\vec{A})_z$

$$(\vec{\nabla}\times\vec{A})_z = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \cancel{\partial_z} \\ A_x & A_y & \cancel{A_z} \end{vmatrix} = (\partial_x A_y - \partial_y A_x)\,\hat{z}$$

$$= \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} = -\left(\frac{\partial v}{\partial x} + \underbrace{\frac{\partial u}{\partial y}}_{-\partial v/\partial x}\right) = 0$$

C.R.

para $\vec{B} \longrightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix} \rightarrow \underset{u}{\partial_x B_y} - \underset{v}{\partial_y B_x}$

$$\rightarrow \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \oint \vec{A} \cdot d\vec{l} = 0 \quad \wedge \quad i \oint \vec{B} \cdot d\vec{l} = 0$$

$$\Rightarrow \oint f(z) = 0$$

* Importante que no haya singularidad en el camino.

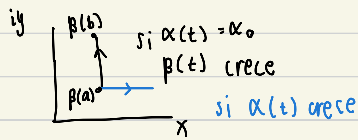
Definiciones

· a, b var. reales

① curva es un mapa

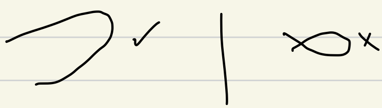
$$\gamma : [a, b] \text{ en } \mathbb{C}$$

$$\gamma(t) = \alpha(t) + i\beta(t).$$

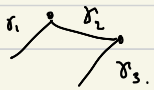


$\gamma(a)$ y $\gamma(b)$ son los puntos extremos.

② Arco simple : (arco de Jordan) : curva que no se cruza.



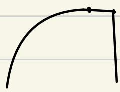
③ Trayectoria: colección de arcos simples unidos en sus extremos



④ Arco suave: $\frac{\partial \gamma}{\partial t} \neq 0$.

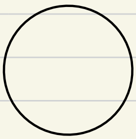
Que esté asociado a un desplazamiento

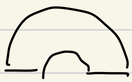
⑤ Contorno: colección (unida) de arcos suaves.

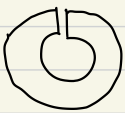


⑥ Contorno cerrado:

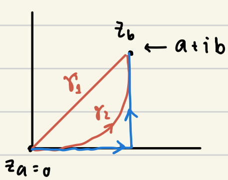
cuando los puntos extremos sí se unen.







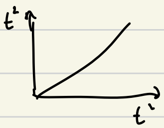
Ejemplos



$$\int_{z_a, \gamma}^{z_b} z\, dz$$

* $(a+ib)t^2$

queda lineal



ⓐ $\int_{\gamma_1} z\,dz$

$z = (a+ib)t$

$t : 0 \to 1.$

$dz = (a+ib)\,dt$

$z\,dz = (a+ib)^2 t \cdot dt$

$$\int_{\gamma_1} z\,dz = \int_0^1 (a+ib)^2 t\,dt$$

$$= \frac{1}{2}(a+ib)^2 = \frac{1}{2} z_b^2 \quad ✓$$

ⓑ $\int_{\gamma_2} z\,dz$

$z = at + ibt^2$

$t : 0 \to 1$

$dz = a\,dt + ib \cdot 2t\,dt$

$z\,dz = (at + ibt^2)(a\,dt + 2ibt\,dt).$

$= a^2 t\,dt - 2b^2 t^3 dt + i(abt^2 dt + 2abt^2 dt)$

$= a^2 t\,dt - 2b^2 t^3 dt + i\,3abt^2 dt.$

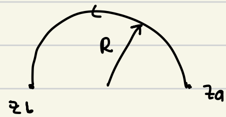
$$\int z\,dz = \int_0^1 (\quad)\,dt$$

$$= \left(\frac{a^2}{2} - 2b^2\frac{1}{4}\right) + iab\left(\underbrace{\frac{1}{2} + \frac{2}{3}}_{1}\right)$$

$$= \frac{1}{2}\left[a^2 - b^2 + 2iab\right] = \frac{1}{2} z_b^2$$

$$\int_C \frac{1}{z}\,dz$$

(a)



$z = Re^{i\theta}$

$\theta : 0 \to \pi$

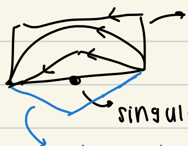
$dz = Re^{i\theta} \cdot i\,d\theta$

$$\frac{dz}{z} = \frac{\cancel{Re^{i\theta}}}{\cancel{Re^{i\theta}}}\, i\,d\theta \longrightarrow \int \frac{dz}{z} = \int_0^{\pi} i\,d\theta = i\pi$$

$\neq$

$$\int \frac{dz}{z} = \int_0^{-\pi} i\,d\theta = -i\pi$$

Propuesto

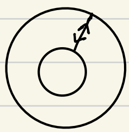


estos 3 dan igual resultado.

singularidad.

al pasar x acá, las cosas cambian.

Evitamos esto:

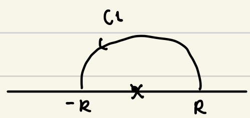


$$\oint + \int + \int + \int = 0$$

calcularemos:

$f(z) = \frac{1}{z}$, $\oint_C f(z)\,dz$

Examinaremos $\int_c \frac{dz}{z}$ para $\neq$ c.



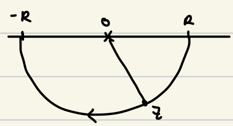
→ eje real.

parametrizamos:

$z = Re^{i\theta}$, $\theta : 0 \to \pi$

$dz = Re^{i\theta} i\,d\theta$

$$I_1 = \int_{C_1} \frac{dz}{z} = \int_{\theta=0}^{\pi} \frac{Re^{i\theta} i\,d\theta}{\underbrace{Re^{i\theta}}_{z}} = i\int_0^{\pi} d\theta = +i\pi \quad , \cancel{\oint} \frac{dz}{z}$$

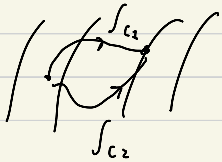


→ Igual parametrizaØ:

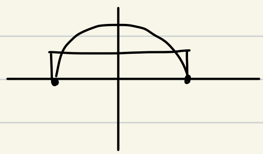
$I_2 = \int_{C_2} \frac{dz}{z}$ $\quad \theta : 0 \to -\pi$

$= i\int_0^{-\pi} d\theta = -i\pi$

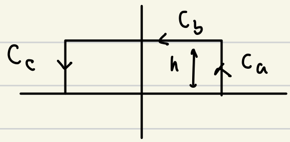
Obs si $f(z)$ analítica



$\rightarrow \int$ iguales (no hay singularidad).



propuesto



$$\int \frac{dz}{z} = \int_{C_a} + \int_{C_b} + \int_{C_c}.$$

para C_1 : $z = R + it$ $\quad t: 0 \rightarrow h.$

$dz = i\,dt$

$$\int_{C_1} = \int_0^h \frac{i\,dt}{R+it} = \ln(R+it)\Big|_0^h = \ln\left(\frac{R+ih}{R}\right)$$

$\int_{C_c}$ es análoga

→ *del apunte:* Si f analítica dentro y sobre un contorno cerrado C, con z_0 fuera de la trayectoria definida por C, entonces:

$$\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2}$$

ej. apunte

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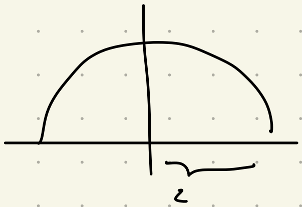
ⓐ

$$I = \int_C \frac{z+2}{z}\,dz \qquad \to z = 2e^{i\theta} \qquad \to dz = 2ie^{i\theta}\,d\theta$$



$$I = \int_0^{\pi} \frac{\cancel{2}(e^{i\theta}+1)}{\cancel{2}\cancel{e^{i\theta}}} \cdot 2i\cancel{e^{i\theta}}\,d\theta$$

$$= -2i\int_0^{\pi}(e^{i\theta}+1)d\theta = 2i\int_0^{\pi} e^{i\theta}d\theta + 2i\int_0^{\pi} d\theta.$$

$$= +2i\left[\frac{e^{i\theta}}{i}\right]_0^{\pi} + 2i\pi$$

$$= +2e^{i\pi} - 2e^{i0} + 2i\pi \qquad * \; e^{i\pi} = \cos(\pi) + i\,\mathrm{sen}\,\pi$$

$$= +2e^{i\pi} - 2 + 2i\pi \qquad = -1$$

$$[I = -4 + 2i\pi]$$

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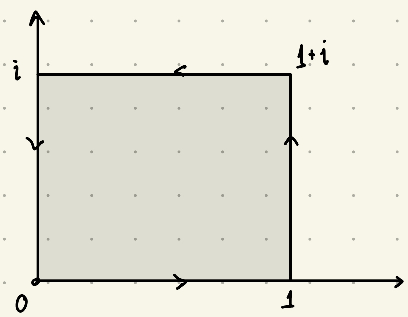

- $z_1 = t$
  $dz = dt$
  $t: 0 \to 1$
- $z_2 = 1 + ti$
  $dz = i\,dt$
  $t: 0 \to 1$
- $z_3 : t + i$
  $dz = dt$
  $t: 1 \to 0$
- $z_4 : ti$
  $dz = dt\,i$
  $t: 1 \to 0$

$$\Rightarrow = \int_0^1 e^{\pi t}dt + \int_0^1 e^{\pi(1-ti)}i\,dt + \int_1^0 e^{\pi(t-i)}dt + \int_1^0 e^{-\pi t i}i\,dt$$

$$= \int_0^1 e^{\pi t}dt + \int_0^1 e^{\pi}e^{-\pi t i}i\,dt + \int_1^0 e^{\pi t}\cdot e^{-\pi i}dt + \int_1^0 e^{-\pi t i}i\,dt$$

$$= \left[\frac{e^{\pi}}{\pi} - \frac{1}{\pi}\right] + e^{\pi}i\int_0^1 e^{-\pi t i}dt + e^{-\pi i}\int_1^0 e^{\pi t}dt + \left[\frac{\cancel{i}e^{-\pi t i}}{-\pi\cancel{i}}\right]_1^0$$

$$= -\frac{1}{\pi} + \frac{e^{\pi}}{\pi} + e^{\pi}\cancel{i}\left[\frac{e^{-\pi t i}}{-\pi\cancel{i}}\right]_0^1 + e^{-\pi i}\left[\frac{e^{\pi t}}{\pi}\right]_1^0 + \frac{e^{-\pi i}}{\pi} - \frac{1}{\pi}$$

$$= \frac{e^{\pi}}{\pi} - \frac{1}{\pi} - e^{\pi}\left(\frac{e^{-\pi i}}{\pi} - \frac{1}{\pi}\right) + e^{-\pi i}\left(\frac{1}{\pi} - \frac{e^{\pi}}{\pi}\right) + \frac{e^{-\pi i}}{\pi} - \frac{1}{\pi}$$

$$= \frac{2e^{\pi}}{\pi} - \frac{2e^{\pi}\overbrace{e^{-\pi i}}^{-1}}{\pi} + \frac{2\overbrace{e^{-\pi i}}^{-1}}{\pi} - \frac{2}{\pi}$$

$$= \frac{2e^{\pi}}{\pi} + \frac{2e^{\pi}}{\pi} + \frac{-2}{\pi} - \frac{2}{\pi}$$

$$= \frac{4e^{\pi}}{\pi} - \frac{4}{\pi}$$

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para C_b:

parametrizamos $z = t + ih$

$dz = dt$

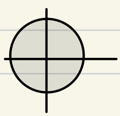
$t: R \to -R$

$$\int_{C_b} = \int_R^{-R} \frac{dt}{t+ih} = \ln\left(t+ih\right)\Big|_R^{-R} = \operatorname{Ln}\left(\frac{-R+ih}{R+ih}\right)$$

propuesto: verificar que

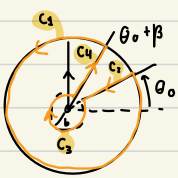
$$\int_{C_a} + \int_{C_b} + \int_{C_c} = i\pi$$

Sea esta vez C = circunferencia de radio R

 $\longrightarrow$ 

$$\oint \frac{dz}{dz} = 0$$

↳ en el interior de la trayectoria no hay singularidad



$$\oint = 0$$

↳ $= \int_1 + \int_2 + \int_3 + \int_4$

C_1 : $z = Re^{i\theta}$

$dz = Re^{i\theta}\, i\,d\theta$.

θ: $\theta_0+\beta \rightarrow \underbrace{\theta_0+\beta+2\pi}_{1\text{ vuelta}} - \beta. = \theta_0 + 2\pi.$

$$\int_1 = \int_{\theta_0+\beta}^{\theta_0+2\pi} i\,d\theta = i\,(\theta_0+2\pi-(\theta_0+\beta)) = i\,(2\pi-\beta).$$

$$\int_2 + \int_4 = 0.$$

C_3 : Parametrizaø =, pero θ cambia $\rightarrow$ cambia signo.

$$\int_3 = (\beta-2\pi)\,i$$

C_2 : $z = te^{i\theta_0}$

$dz = e^{i\theta_0}\,dt$

t : $R \longrightarrow b$.

$$\int_2 = \int_R^b \frac{dz}{z} = \int_R^b \frac{dt}{t} = \ln(t)\Big|_R^b = \ln\left(\frac{b}{R}\right).$$

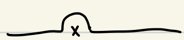
$$\int_4 = \int_b^R \frac{dz}{z} = \ln(t)\Big|_b^R = \ln\left(\frac{R}{b}\right). //$$

$$\int_{C_1} \frac{dz}{z} + \ln\left(\frac{b}{R}\right) + \int_{C_3} + \ln\left(\frac{R}{b}\right) = 0$$

$$\int_{C_1} \frac{dz}{z} = \lim_{\beta \to 0} (i\beta - 2\pi) = -2\pi i$$

Resumimos:

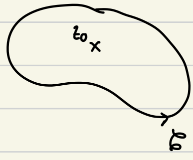
$$\oint_C \frac{dz}{z} = \begin{cases} 0 & \text{si } C \text{ excluye } z=0 \\ 2\pi i & \text{si } C \text{ incluye } z=0 \end{cases}$$



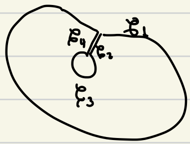
Fórmula integral de Cauchy:

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz}$$

- $f(z)$ analítica sobre y dentro de C.
- z_0 esté dentro de la trayectoria.



Dem



$$\to \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0$$

$*$ $*$

$$\int_{C_1} \frac{f(z)}{z-z_0}\,dz$$

Integramos $\int_{C_3}$ con C_3: circunferencia radio ε recorrida angularmente entre θ_0 y $\theta_0 - 2\pi - \delta$.

$z = z_0 + \varepsilon e^{i\theta}$

$dz = \varepsilon e^{i\theta} i\, d\theta$

$$\int_{C_3} \frac{f(z)}{z-z_0}\,dz = \int_{\theta_0}^{\theta_0 - 2\pi - \delta} \frac{f(z_0 + \varepsilon e^{i\theta})}{\cancel{\varepsilon e^{i\theta}}} \cancel{\varepsilon e^{i\theta}}\, i\, d\theta$$

$$= \int_{\theta_0}^{\theta_0 - 2\pi - \delta} f(z_0 + \varepsilon e^{i\theta})\, i\, d\theta$$

xq f analítica. $\leftarrow$ $\Big\downarrow \varepsilon \to 0$

$$= \int_{\theta_0}^{\theta_0 - 2\pi - \delta} f(z_0)\, i\, d\theta \longrightarrow -2\pi i\, f(z_0)$$

Entonces:

$$\oint_{\xi} \frac{f(z)}{z-z_0} = 2\pi i\, f(z_0)$$

$$\Rightarrow \boxed{f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-z_0}\, dz}$$

en sentido antihorario.

Expresión para las deriv. de f.

Ajuste de notación:

Tenemos $f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-z_0}\, dz$

$z \to z_0$, $\xi \to z$, $\xi \to$ "x_1"

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\xi)}{\xi - z}\, d\xi.$$

$$\frac{df}{dz} = f'(z) = \frac{1}{2\pi i} \frac{d}{dz} \oint \frac{f(\xi)\, d\xi}{\xi - z}$$

$$= \frac{1}{2\pi i} \oint f(\xi)\, d\xi\, \underbrace{\frac{d}{dz}\left(\frac{1}{\xi - z}\right)}_{\frac{1}{(\xi - z)^2}}$$

$$f'(z) = \frac{1}{2\pi i} \oint \frac{f(\xi)\,d\xi}{(\xi - z)^2}$$

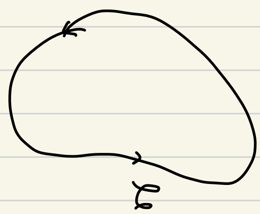
Prop: $\left[\frac{d^n}{dz^n}\left(\frac{1}{a-z}\right) = n!\,\frac{1}{(a-z)^{n+1}} \right]$

$$\frac{d^n f}{dz^n} = \frac{n!}{2\pi i} \oint \frac{f(\xi)\,d\xi}{(\xi - z)^{n+1}}$$

19·abr.

→ Mañana: Integrað

Recordar:



$$f(z) = \frac{1}{2\pi i} \oint_{\xi} \frac{f(\xi)\,d\xi}{\xi - z}$$

$$\boxed{f^{(n)}(z) = \frac{1}{2\pi i} \oint_{\xi} \frac{n!}{(\xi - z)^{n+1}} f(\xi)\,d\xi.}$$

Sucesiones y series

Sucesión: $\omega_1, \omega_2, \omega_3 \dots \in \mathbb{C}$

Suma parcial: $S_k = 1 + \omega_1 + \dots + \omega_k$

$$= \sum_{j=1}^{k} \omega_j$$

$S_\infty = \sum_{j=1}^{\infty} \omega_j$ es convergente

si S_∞ existe.

Criterios de convergencia.

ⓐ serie es absolutaⓜ convergente.

$\sum_{n=1}^{\infty} |\omega_n|$ es convergente

ⓑ serie condicionalmente convergente.

cuando $\sum_{n=1}^{\infty} \omega_n$ converge.

pero $\sum_{n=1}^{\infty} |\omega_n|$ diverge

diverge logarítmica

* $\sum \frac{1}{n} \to \int \frac{dx}{x}$ ⓜ.

* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$$\boxed{\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.}$$

$\approx \frac{1}{n^2}$

converge.

· serie geométrica:

$\omega_n = q^n \quad , q \in \mathbb{C}$

$$S_n = \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

dem. serie geométrica.

$S_n = 1 + q + q^2 + \dots + q^n$

$q S_{n+1} = \underbrace{1 + q + q^2 + \dots + q^n}_{S_n} + q^{n+1}$

$1 + q S_n = S_n + q^{n+1} \longrightarrow$ despejando S_n

· S_n converge?

$S_n = \dfrac{1-q^{n+1}}{1-q} \quad , q = \rho e^{i\alpha}$

$q^{n+1} = \rho^{n+1} e^{i(n+1)\alpha}$

tiende a 0 si $\rho < 1$

$* \quad \rho^2 < 1 \quad / \cdot \rho$

$\rho^2 < \rho < 1 \quad / \cdot \rho$

$\rho^3 < \rho^2 < \rho < 1$

$$\boxed{S_\infty = \frac{1}{1-q}}$$ → serie geométrica.

si $q < 1$.

* aritmética: $\dfrac{n(n+1)}{2} = \sum_{i}^{n} i$

$\int_0^L x\,dx = \dfrac{L^2}{2}$

· $\sum \dfrac{1}{n^2} =$

sum $\dfrac{1}{n^2}$

Test de convergencia

1.- De comparación

2.- Del cuociente

3.- De cauchy.

1° de convergencia

$S = \sum_{n=1}^{\infty} \omega_n = \overbrace{\sum u_n}^{Re} + i \overbrace{\sum v_n}^{Im}$ → Suponiendo u y v positivos.

$\sum a_n$ ← convergente.

Si $u_n < a_n$

$\sum b_n$ } conocida y convergente.

$v_n < b_n \Rightarrow \sum v$ converge.

2° del cuociente

$$\rho_n = \left|\frac{\omega_{n+1}}{\omega_n}\right| \xrightarrow{n\to\infty} \begin{cases} <1 & \text{converge} \\ =1 & \text{no se sabe (indeterm.)} \\ >1 & \text{diverge.} \end{cases}$$

· $\dfrac{|q^{n+1}|}{|q^n|} = q$

3° de cauchy.

$, \quad |\omega_n|^{1/n} = |q^n|^{1/n} = |q|.$

serie de función:

En este caso $\omega_k \to \omega_k(z)$

$$S_n(z) = \sum_{k=1}^{n} \omega_k(z)$$

$$\downarrow$$

$$S_\infty(z) = S(z) = \sum_{k=1}^{\infty} \omega_k(z)$$

$$\downarrow$$

$$f(z) = \sum_{k=1}^{\infty} \omega_k(z)$$

$$\cdot\ f(z) = \lim_{n\to\infty} \sum_{k=1}^{n} \omega_k(z)$$

para que esta suma tenga sentido exigimos que : $\forall\ \epsilon$
$\exists\ m$
tal que. $|f(z) - S_n(z)| < \epsilon$

serie geom: $\to \sum_{0}^{\infty} (\omega_k(z) = z^k)$

$$f(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

Suma $S_n = \sum_{0}^{n} z^k = \dfrac{1-z^{k+1}}{1-z}$

* suponemos que $z<1$.

$$\left| \frac{1}{1-z} - \frac{1-z^{n+1}}{1-z} \right| = \frac{|z|^{n+1}}{|1-z|} < \epsilon \quad /\ln.$$

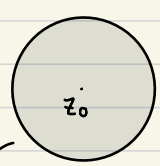
* $a < b$
$\ln a < \ln b \to$ creciente

$$\hookrightarrow (n+1)\ln|z| - \ln|1-z| < \ln(\epsilon)$$

$$(n+1) < \frac{\ln(\epsilon) + \ln|1-z|}{\ln|z|.} = \frac{\ln\left((\epsilon)|1-z|\right)}{\ln|z|} \to \text{negativo.}$$

Serie de taylor

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\xi)d\xi}{(\xi - z)}$$

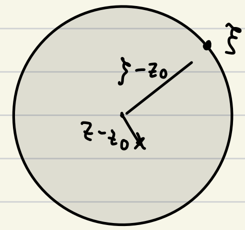


$$f(z) = f(z_0) + f'(z_0)(z-z_0) + f^{(2)}(z_0)(z-z_0)^2.$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) + (z_0 - z)}$$

$$= \overbrace{\frac{1}{1 - \underbrace{\frac{(z-z_0)}{\xi - z_0}}_{<1}}}^{\chi} \cdot \frac{1}{\xi - z_0}$$



$$\Rightarrow f(z) = \frac{1}{2\pi i} \oint d\xi \, f(\xi) \cdot \frac{1}{\underbrace{1 - \chi}} \cdot \frac{1}{\xi - z_0}$$

$* \quad \chi = \left| \frac{z - z_0}{\xi - z_0} \right| < 1$

podemos rep.
como serie geom.

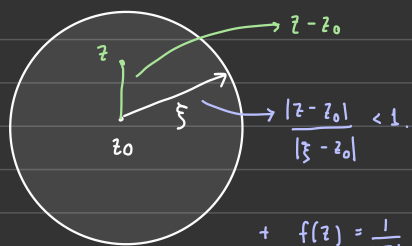
$$= \frac{1}{2\pi i} \oint d\xi \, f(z) \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n \frac{1}{\xi - z_0}.$$

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}} (z - z_0)^n \cdot \frac{n!}{n!}$$

$$\boxed{f(z) = \sum \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.}$$

Taylor: (en torno a z_0)

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$



$$+ \quad f(z) = \frac{1}{2\pi i} \oint \frac{f(\xi)\, d\xi}{(\xi - z)} \quad (*)$$

Usamos $\frac{1}{\xi - z} = \frac{1}{\xi - z_0 + z_0 - z} = \frac{1}{(\xi - z_0)\underbrace{\left(1 + \frac{z_0 - z}{\xi - z_0}\right)}_{\text{expresión cerrada suma geom.}}}$

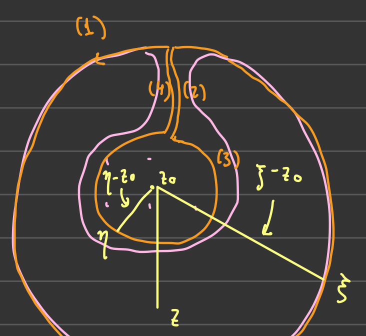
$\hookrightarrow \sum q^n = \frac{1}{1-q}$

$$= \frac{1}{\xi - z_0} \underbrace{\sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n}$$

Reemplazar en (*)

$$\Rightarrow f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint f(\xi)\, d\xi \, \frac{1}{\xi - z_0} \left(\frac{z - z_0}{\xi - z_0} \right)^n$$

$$= \sum \underbrace{\frac{1}{2\pi i} \left[\oint \frac{f(z)\, d\xi}{(z - z_0)^{n+1}} \right]}_{\frac{f^{(n)}(z_0)}{n!}} (z - z_0)^n.$$



$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\xi)\, d\xi}{(\xi - z)}.$$

$$\oint_1 + \underbrace{\int_2} + \oint_3 + \underbrace{\int_4} = 2\pi i\, f(z).$$

* si no hay un corte en las funøes.

$$\oint + \oint = 2\pi i\, f(z).$$

$\downarrow$

$$\oint_1 - \oint_2 = 2\pi i\, f(z).$$

$$2\pi i\, f(z) = \oint \frac{f(\xi)\,d\xi}{\xi - z} - \oint \frac{f(\eta)\,d\eta}{\eta - z}$$

$$\rightarrow \frac{1}{\xi - z} = \frac{1}{\xi - z_0 + z_0 - z} = \frac{1}{(\xi - z_0)\left(1 + \frac{z_0 - z}{\xi - z}\right)} = \frac{1}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n$$

$$\rightarrow \frac{1}{\eta - z} = \frac{1}{\underbrace{\eta - z_0}_{<} + \underbrace{z_0 - z}_{>}} = \frac{1}{\left(\left(\frac{\eta - z_0}{z_0 - z}\right) + 1\right)} \cdot \frac{1}{z_0 - z} = \frac{1}{z_0 - z}\sum_{n=0}^{\infty}\left(\frac{\eta - z_0}{z - z_0}\right)^n$$

Así $$\oint_z \frac{f(\eta)\,d\eta}{\eta - z} = \frac{-1}{z - z_0}\sum \oint_z \frac{(\eta - z_0)^n}{(z - z_0)^n} f(\eta)\,d\eta$$

$$= -\sum \oint \frac{(\eta - z_0)^n}{(z - z_0)^{n+1}} f(\eta)\,d\eta$$

$$= -\sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} \underbrace{\oint (\eta - z_0)^n f(\eta)\,d\eta}_{2\pi i\, b_n}.$$

, $n = n' - 1$

$n = 0 \Rightarrow n' = 1$

$$\hookrightarrow -\sum_{n'=1} \frac{1}{(z - z_0)^{n'}} \oint f(\eta)(\eta - z_0)^{n'-1}\,d\eta$$

como n' es muda:

$$-\sum_{n=1} \frac{1}{(z - z_0)^n} \oint f(\eta)(\eta - z_0)^{n-1}\,d\eta.$$

Así, luego de cancelar los "$2\pi i$"

$$f(z) = \sum_{n=0} a_n (z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$$

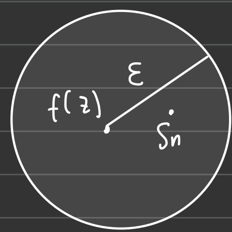
$$\boxed{a_n = \frac{1}{2\pi i}\oint \frac{f(\xi)\,d\xi}{\xi - z_0}}$$

$$\boxed{b_n = \frac{1}{2\pi i}\oint f(\eta)(\eta - z_0)^{n-1}\,d\eta.}$$

$$\rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \underbrace{\frac{b_1}{z - z_0}}_{\text{polo simple}} + \underbrace{\frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_N}{(z - z_0)^N}}_{\text{polos}} + \ldots$$

· $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2} \cdot \frac{1}{2!} + \ldots$ tiene ∞ términos.

Clase anterior:



$|z|^{n+1} < \varepsilon|1-z|$

$(n+1)\ln|z| < \ln(\varepsilon|1-z|)$. , $|z|<1$

$(n+1) > \dfrac{\ln(\varepsilon|1-z|)}{\ln|z|}$

* Taller jueves: series de Taylor y Laurent.

Taylor: a_n → derivada, $\oint$

Laurent: b_n → $\oint$, por inspección (funØes racionales).

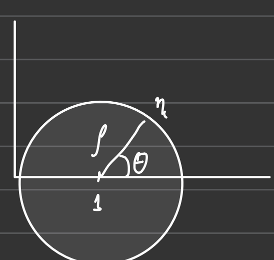
Ej

sea $f(z) = \dfrac{1}{1-z}$, $b_1 = 1$, $b_2 = 0 \ldots$

calcular b_n's para serie en torno a $z_0 = 1$.

$b_n = \dfrac{1}{2\pi i}\oint (\eta - z_0)^{n-1} f(\eta)\, d\eta$.

en torno a $z_0 = 1$



$\eta = \rho e^{i\theta} + 1$

$\theta = 0 \to 2\pi$

$d\eta = \rho e^{i\theta}\, i\, d\theta$

$b_n = \dfrac{1}{2\pi i}\displaystyle\int_0^{2\pi} (\eta - 1)^{n-1} \frac{1}{1-\eta}\rho e^{i\theta} i\, d\theta$.

$= \dfrac{1}{2\pi i} - \displaystyle\int_0^{2\pi} (\eta-1)^{n-2} \rho e^{i\theta} i\, d\theta = \frac{-1}{2\pi i}\int_0^{2\pi} \frac{(\rho e^{i\theta})^n}{(\rho e^{i\theta})^2} \cancel{\rho e^{i\theta}}\, i\, d\theta$

$$b_n = -\frac{1}{2\pi i}\, \rho^{n-1} \int_0^{2\pi} e^{i\theta(n-1)}\, i\, d\theta. \quad \begin{cases} 0 & , n \neq 1 \\ -1 & , n = 1 \end{cases}$$

$* \; b_{n=1} = -1$

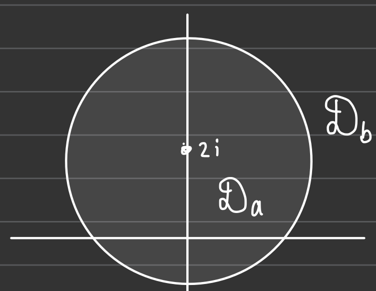
hacer c/ integral! $\longrightarrow$ $f(z) = \frac{z}{4+z^2}$ en torno a $z_0 = 2i$

$$= \frac{z}{(z+2i)(z-2i)} = \frac{1}{2}\left(\underbrace{\frac{1}{(z+2i)}} + \underbrace{\frac{1}{z-2i}} \right)$$

$- z_0 + z_0$

$$= \frac{1}{2}\left(\frac{1}{z-2i+4i} + \frac{1}{z-2i} \right)$$

$$* \; \frac{1}{(z-2i)+4i} = \begin{cases} \frac{1}{z-2i} \cdot \frac{1}{1+\left(\frac{4i}{z-2i}\right)} & \left|\frac{4}{z-2i}\right| < 1 \\ \frac{1}{4i} \cdot \frac{1}{1+\frac{(z-2i)}{4i}} \end{cases}$$

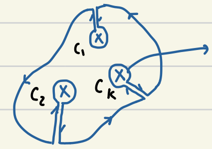


Teorema de los residuos

26·abr

se quiere $\oint_C f(z)\,dz$

· hay N singularidades 



singularidad en z_k

$$\cdot\ f(z) = \sum_{n=0} a_n (z - z_k)^n + \frac{b_1}{z - z_k} + \frac{b_n}{(z-z_k)^2} + \dots \frac{b_m}{(z-z_k)^m}$$

$$\Rightarrow \varphi^{(z)} = f(z)\,(z - z_k)^m \text{ es regular}$$

↳ asegurarse de que el integrando sea regular.

$$\cdot\ \oint_C + \oint_{C_1} + \dots + \oint_{C_N} = 0$$

$$\oint_C = \oint_{C_1} + \oint_{C_2} + \dots \oint_{C_k} + \dots + \oint_{C_N}$$

prestamos atención a $\oint_{C_k} f(z)\,dz$, C_k enlaza a z_k

$$\oint_{C_k} (z - z_k)^m \frac{f(z)}{(z - z_k)^m}\,dz = \underbrace{\oint \frac{\varphi(z)}{(z - z_k)^m}\,dz}$$

fórmula integral de cauchy. si $m=1$

1.63 Recordamos que si $\varphi(z)$ es analítica,

→ pág 41

$$\oint \frac{\varphi(z)\,dz}{(z - z_0)^m} = \frac{2\pi i}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left(f(z)\,(z - z_k)^m \right)$$

$$f^{(n)}(z) = \frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint \frac{f(\xi)\,d\xi}{(\xi - z)^{n+1}}$$

Así, $\displaystyle\oint_{C_k} f(z)\,dz = \frac{2\pi i}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[f(z)\,(z - z_k)^m \right]$

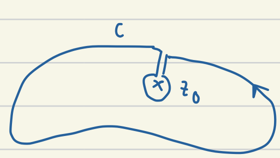
Entonces :

$$\oint_C f(z)\,dz = \sum_{\substack{\text{singularidades} \\ z_k}} \frac{2\pi i}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left(f(z)\,(z - z_k)^m \right)$$

$m \to m_k$ (ajustarlo al k ésimo polo).

Singularidades de primer orden.

$f(z) \sim \frac{1}{z - z_0}$ ← cercanía de z_0

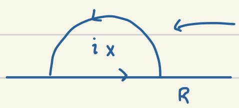


$$\oint_c f(z)\,dz = \frac{2\pi i}{1} \underbrace{\lim_{z\to z_0} (z-z_0) f(z)}_{\text{residuo}}$$

· Un ejemplo simple:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \arctan(x)\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - -\frac{\pi}{2} = \pi$$

En el plano $\mathbb{C}$.



$f(z) = \frac{1}{1+z^2}$

· $\oint = \int + \int$

$\hookrightarrow = 2\pi i \sum \text{residuos}$

$f(z)$ es singular para $z = +i$

$z = -i$

· $f(z) = \frac{1}{(z-i)(z+i)}$

Residuo en $z = i$

$$(z-i)\cdot f(z) = \cancel{(z-i)} \frac{1}{(z+i)\cancel{(z-i)}} = \frac{1}{z+i}\Big|_{\text{polo}} = \boxed{\frac{1}{2i}}$$

$\uparrow$ i

Resumiendo:

$$\Rightarrow \int + \int = 2\pi i \cdot \frac{1}{2i} = \pi$$

$$\int_{-R}^{R} \frac{dx}{1+x^2} + \int = \pi$$

$$\hookrightarrow \int \frac{dz}{z^2+1} \quad \begin{cases} R e^{i\theta} = z \\ R e^{i\theta} i\, d\theta = dz \\ \theta : 0 \to \pi \end{cases} \quad , \quad \frac{1}{1+z^2} = \frac{1}{1+R^2 e^{2i\theta}}$$

$$\hookrightarrow \int_0^{\pi} \frac{R e^{i\theta} i\, d\theta}{1+R^2 e^{2i\theta}} \underset{R \gg 1}{\sim} \int_0^{\pi} \frac{R e^{i\theta} i\, d\theta}{R^2 e^{2i\theta}} = \int_0^{\pi} \frac{i\, d\theta}{R e^{i\theta}} = \frac{i}{R} \int_0^{\pi} e^{-i\theta}\, d\theta$$

$\downarrow R \to \infty$

0.

Así, si $R \to \infty$

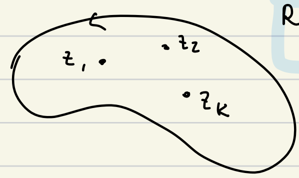
$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} + 0 = \pi \quad ✓$$

* Prop. $\int \frac{\sqrt{x}\, dx}{x^2+1}$

Teo. de los residuos.

Recordemos.

$$\oint_C f(z)\,dz = 2\pi i \sum_k \operatorname{Res} f(z_k)$$



$$\operatorname{Res} f(z_k) = \lim_{z\to z_k} \left[(z-z_k) f(z)\right] \quad \text{orden } 1.$$

$$= \lim_{z\to z_k} \frac{d}{dz}\left[(z-z_k)^2 f(z)\right] \quad \text{orden } 2.$$

$$\vdots$$

$$= \lim_{z\to z_k} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_k)^m f(z)\right]\frac{1}{(m-1)!} \quad \text{orden } m.$$

Abordaremos integrales del tipo:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$; $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \times \begin{Bmatrix} \operatorname{sen} kx \\ \cos kx \end{Bmatrix} dx$

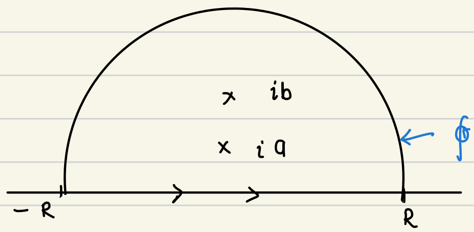
Integrales del tipo (a):

$$\frac{P(x)}{Q(x)} \overset{x\to+\infty}{\sim} \frac{1}{x^{1+\epsilon}}$$

$$\frac{x\cdot P(x)}{Q(x)} \to 0 \quad \text{para } x \to \pm\infty$$

→ exigimos que $Q(x)$ no tenga ceros en el eje real.

$\boxed{\text{Ej}}$ $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{(a+b)ab}$ → si $a=b \Rightarrow$ polo de segundo orden $(\pi/2a)$.



. polos : $z = \pm ia$
$z = \pm ib$.

$\oint = 2\pi i \sum \text{Res}.$

$\oint = -2\pi i \sum \text{Res}.$

$$\oint = \int_{-R}^{R} + \boxed{\oint}$$

$\downarrow$ sí $R\to\infty$

0

$$\oint \underset{\substack{\downarrow \\ z = Re^{i\theta} \\ dz = Re^{i\theta} i\, d\theta \\ \theta : 0 \to \pi}}{=} \int_0^{\pi} \frac{a_0 + a_1 z + \dots + a_n z^n}{b_0 + b_1 z + \dots + b_m z^n} Re^{i\theta} i\, d\theta .$$

debe ser par para evitar tener ceros en el eje real.

$$\sim \int \frac{a_n R^n e^{in\theta}}{b_n R^m e^{im\theta}} \cdot R\, e^{i\theta} d\theta = \frac{R^{n+1}}{R^m} \text{finito.}$$

$\downarrow R\to\infty$

0

Polos: $z_1 = ia$

$z_2 = ib$

para z_1: $(z - z_1) \cdot \dfrac{1}{(z^2+a^2)(z^2+b^2)}$

$$(z - ia) \cdot \frac{1}{(z-ia)(z+ia)} \cdot \frac{1}{(z^2+b^2)} \to \frac{1}{2ia} \cdot \frac{1}{(b^2-a^2)}$$

para z_2: $\dfrac{1}{2ib} \cdot \dfrac{1}{(a^2-b^2)}$

Así, $2i\pi \sum \text{Res}$

$$= 2\cancel{i}\pi \left[\frac{1}{\cancel{2i}a} \frac{1}{(b^2-a^2)} + \frac{1}{\cancel{2i}b} \frac{1}{a^2-b^2} \right]$$

$$= \pi \left[\left(\frac{1}{a} - \frac{1}{b}\right)\left(\frac{1}{b^2-a^2}\right) \right]$$

$$= \pi \left[\frac{\cancel{b-a}}{ab} \cdot \frac{1}{(\cancel{b-a})(b+a)} \right]$$

$$= \frac{\pi}{(b+a)ab}$$

Si $a = b$

$$\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{2a^3}$$

· Verifiquemos consistencia:

$$\int \frac{dz}{(z^2+a^2)^2} = 2\pi i \operatorname{Res} \frac{1}{(z^2+a^2)^2}\Big|_{z=ia}$$

$$\cdot\ (z-z_1)^2 \frac{1}{(z^2+a^2)^2} = \frac{(\cancel{z-ia})^2}{(z+ia)^2(\cancel{z-ia})^2}$$

$$= \frac{1}{(z+ia)^2}$$

$$\cdot\ \frac{d}{dz}\left[(\ )^2 \cdot \frac{1}{(\ )^2}\right] = -2 \cdot \frac{1}{(z+ia)^3} \xrightarrow{z=ia} \frac{-2}{(2ia)^3} = \frac{2}{8ia^3} = \frac{1}{4ia^3}.$$

evaluamos después de derivar.

$$\rightarrow \int \frac{dz}{(z^2+a^2)^2} = \cancel{2} i\pi \cdot \frac{1}{{}_2\cancel{4}ia^3} = \frac{\pi}{2a^3}$$

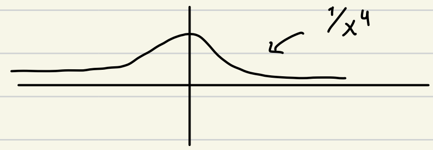
ⓑ Integral del tipo $\int_{-\infty}^{\infty} \frac{P}{Q} \begin{Bmatrix} \operatorname{sen} kx \\ \cos kx \end{Bmatrix}$ ↳ puede ser cualquiera.

· $\operatorname{sen} kx = \operatorname{Im} e^{ikx}$

· $\cos kx = \operatorname{Re} e^{ikx}$

$$\rightarrow \operatorname{Re} \int_{-\infty}^{\infty} \frac{P}{Q} e^{ikx}\, dx = \int_{-\infty}^{\infty} \frac{P}{Q} \cos(kx)\, dx$$

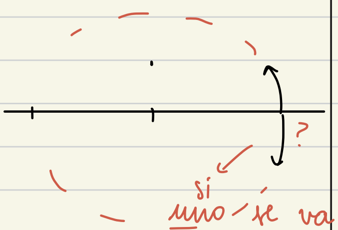
$\boxed{Ej}$ sea $f(x) = \dfrac{1}{(1+\mu^2x^2)^2}$



$$I(k) = \int_{-\infty}^{\infty} \frac{\cos(kx)}{(1+\mu^2x^2)^2}$$

si es sen(x) va a oscilar y $\int_{-\infty}^{\infty}$ será 0

$$\rightarrow \int_{-\infty}^{\infty} \frac{e^{ikz}}{(1+\mu^2z^2)^2}\,dz \qquad z = \pm\frac{i}{\mu}$$



Examinemos x donde cerrar:

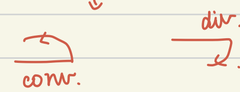
si uno se va por el camino equivocado diverge.

$$z = Re^{i\theta} = R\cos\theta + iR\,\text{sen}\theta$$

$$e^{ikz} = e^{ik[R\cos\theta + iR\,\text{sen}\theta]}$$

$$= \left(e^{ikR\cos\theta}\right)\left(e^{-ikR\,\text{sen}\theta}\right).$$

si encierro x arriba, sen$\theta>0$, $k>0$, $R>0$.



En este caso, para $k>0$ cerramos ↶

$$\rightarrow \text{Re} \int_{-\infty}^{\infty} \frac{e^{ikz}}{(1+\mu^2z^2)^2} = 2i\pi\ \text{Res}\ f(z_1).$$

Residuo: $\dfrac{d}{dz}\left[(z-z_1)^2 \dfrac{e^{ikz}}{\mu^4(z-z_1)^2(z+z_1)^2}\right]$

$$\frac{1}{(1+\mu^2z^2)^2} = \mu^4\left(\frac{1}{\mu^2}+z^2\right)^2 = \mu^4\left(\frac{i}{\mu}-z\right)\left(\frac{i}{\mu}+z\right).$$

$$= \frac{1}{\mu^4}\left[\frac{e^{ikz}ik}{(z+z_1)^2} - \frac{2e^{ikz}}{(z+z_1)^3}\right]$$

$$= \frac{e^{ikz}}{\mu^4}\ \frac{ik(z+z_1)-2}{(z+z_1)^3} \qquad \text{ev. } z=\frac{i}{\mu}.$$

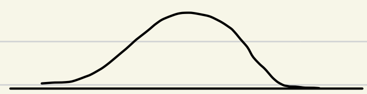
$$= \frac{e^{-\frac{k}{\mu}}}{\mu^4}\left[\frac{-\frac{2k}{\mu}-2}{\left(\frac{2i}{\mu}\right)^3}\right] = \frac{+2}{\mu^4}\ \frac{e^{-k/\mu}\left(1+\frac{k}{\mu}\right)\cdot\mu^3}{8i}$$

$$= \frac{1}{4i\mu}\cdot e^{-k/\mu}\left(1+\frac{k}{\mu}\right).$$

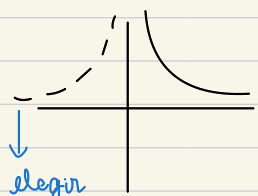
$$\int_{-\infty}^{\infty} \frac{e^{ikz}\,dz}{(1+\mu^2x^2)^2} = \frac{2i\pi}{4i\mu}\, e^{-k/\mu}\left(1+\frac{k}{\mu}\right) = \frac{\pi}{2\mu}\, e^{-k/\mu}\left(1+\frac{k}{\mu}\right)$$

Obs

Espacio 'real'

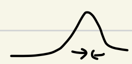


Espacio de "Fourier"



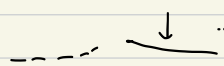
hay que elegir el otro camino $(k<0)$

$$\frac{1}{(1+\mu^2x^2)}$$ → si μ aumenta



→ espacio real

identificação espacial.



→ espacio de fourier.

↳ identificação. del momentum

Ppio. de Incertidumbre:

$(\Delta x)(\Delta p) \geq \hbar/2$.

10 · mayo

* Mañana: Aplicaciones teo. residuos.

Integrales del tipo: $\int_0^{2\pi} F(\operatorname{sen}\theta, \cos\theta)\, dx$

$x \to z = e^{i\theta}$ $\begin{cases} \operatorname{sen}\theta = \dfrac{z^2-1}{2iz} \\ \cos\theta = \dfrac{z^2+1}{2z} \end{cases}$

· $dz = e^{i\theta}\, i\, d\theta \rightsquigarrow d\theta = e^{-i\theta}(-i) = \frac{-i}{z} dz$

· $I = \oint -\frac{i\,dz}{z} \cdot F\left(\frac{z^2-1}{2iz}, \frac{z^2+1}{2z}\right)$

no tiene primitiva.

$$\left[\frac{1}{1-q} = \sum q^n\right]$$
↳ serie geométrica

$\boxed{\text{Ej}}$ $I(b) = \int_0^{2\pi} \frac{d\theta}{b+\cos\theta} \longrightarrow \frac{1}{b+\cos\theta} = \frac{1}{b(1+\frac{1}{b}\cos\theta)} = \frac{1}{b}\sum_{n=0}^{\infty}\left(-\frac{\cos\theta}{b}\right)^n$

serie geométrica.

b debe ser > 1 para que converga

con residuos.

$$= \oint \frac{-i\,dz}{z} \cdot \frac{1}{b + \frac{z^2+1}{2z}}$$

$$= -2i \oint \frac{dz}{2bz + z^2 + 1}$$

$$= -2i \oint \frac{dz}{z^2 + 2bz + 1}$$

Polos $z = \dfrac{-2b \pm \sqrt{4b^2-4}}{2}$

$= -b \pm \sqrt{b^2-1}$

$z_\pm = -b \pm \sqrt{(b-1)(b+1)}$

casos de interés (supondremos b positivo)

· $|b| < 1 \longrightarrow z_\pm = -b \pm i\sqrt{1-b^2} = Re^{i\varphi}$

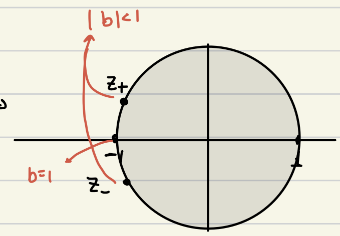
$\sqrt{b^2+1-b^2} = 1$

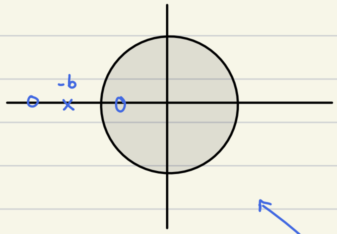
· $|b| = 1 \longrightarrow z_\pm = -1$

· $|b| > 1$

↳ $z_\pm = -b \pm \sqrt{b^2-1} \to -b + \sqrt{b^2-1}$ ← fuera del 0

$\to -b - \sqrt{b^2-1}$ ← nos permite def.





$1 > -b + \sqrt{b^2-1} > -1$?

$|-b+\sqrt{b^2-1}| < 1$?

$\iff (-b+\sqrt{b^2-1})^2 < 1$

$(b^2-1) - 2\sqrt{\ }\, b + b^2 < 1$

$2b^2 - 1 - 2\sqrt{\ }\, b < 1$

$b^2 - 1 < b\sqrt{b^2-1}$

$\sqrt{b^2-1} < b$

$b^2 - 1 < b^2$

$-1 < 0$ ✓ se cumple //

este polo se localiza dentro del círculo y podemos calcular el residuo.

Tenemos

$$\boxed{I(b) = \underbrace{-2i \oint \frac{dz}{z^2+2bz+1}}_{2i\pi \sum \text{Res.}}}$$

polos dentro de circ de radio unitario sólo para $b > 1$

$z_+ = -b + \sqrt{b^2-1}$ (dentro de O)

$z_- = -b - \sqrt{b^2-1}$ (fuera del O).

$$\text{Res}: \quad \cancel{(z-z_+)} \cdot \frac{1}{\cancel{(z-z_+)}(z-z_-)} = \frac{1}{z-z_-}\bigg|_{z_+} = \frac{1}{z_+ - z_-} = \frac{1}{2\sqrt{b^2-1}}$$

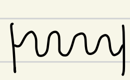
$$I(b) = \frac{4\pi}{2\sqrt{b^2-1}} = \frac{2\pi}{\sqrt{b^2-1}}.$$

$$\boxed{\int_{-\infty}^{\infty} e^{ikx - bx^2}\, dx = \sqrt{\frac{\pi}{b}}\; e^{-k^2/2b}.}$$

series de fourier. $\sum_{k_n} \to \int$

contexto:

$$f(x) = \sum C_n \operatorname{sen}\Big(\underbrace{\frac{n\pi x}{L}}_{k_n}\Big) + \sum b_n \cos\Big(\underbrace{\frac{n\pi x}{L}}_{k_n}\Big)$$



$f(x) = \int c_k e^{ikx} \, dk \quad \cdot e^{-ik'x}$

e^{-bx^2}

$\cdot \int_{-\infty}^{\infty} e^{-x^2} dx = I_G$ ↗ I gaussiana

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) = I^2$$

$$\int_{-\infty}^{\infty}\int dx\, dy\, e^{-r^2}$$

$$\overset{\shortparallel}{dA}\, e^{-r^2} = I^2$$

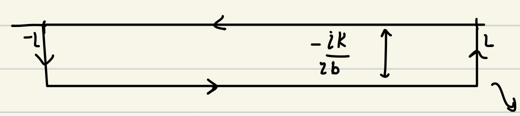
$$2\pi \int_0^{\infty} r\, dr\, e^{-r^2} = \pi = I^2 \rightarrow \boxed{I = \sqrt{\pi}}$$

$$\frac{e^{-r^2}}{2}$$

$$\cdot -bx^2 + ikx = -b\left(x - \frac{ik}{2b}\right)^2 - \frac{k^2}{4b}.$$

$$e^{-\frac{k^2}{4b}} \int_{-\infty}^{\infty} e^{-b\left(x - \frac{ik}{2b}\right)} dx$$

$$= \int_{-\infty}^{\infty} e^{ikx - bx^2} dx$$



$x \rightarrow z$

$z = L + iy.$

$$\oint = 0 = \int_{\rightarrow} + \int_{\uparrow} + \int_{\leftarrow} + \int_{\downarrow}$$

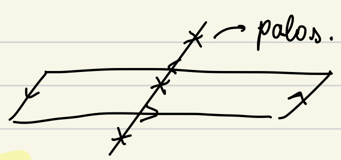
conocida (x eje real)

tiende a 0

* ver en pág 57 → integral Gaussiana.

↳ Rener? ⇝ truco: construye $G(z) = \dfrac{e^{-z^2}}{1 + e^{-2az}}$, a complejo $a = \sqrt{\frac{\pi}{2}}\,(1+i)$.

$a^2 = i\pi$



→ resultado: $\sqrt{\pi}$.

lo mismo pero del libro:

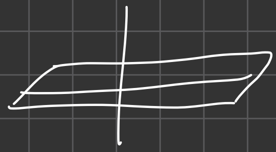
Integral Gaussiana

- $I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ $\rightarrow$ $G_R(x) = \frac{\exp(-z^2)}{1+\exp(-2az)}$

- a complejo de magnitud $\sqrt{2\pi}$ formando un ángulo polar de $\pi/4$. $a = \sqrt{\frac{\pi}{2}}(1+i)$

$G_R(z)$ singular para: $-2az = (2m+1)i\pi$

$G_R(z)$ se ubican en: $z = \left(\frac{1}{2}+n\right)a.$



$g_R(z) - g_R(z+a) = \exp(-z^2).$

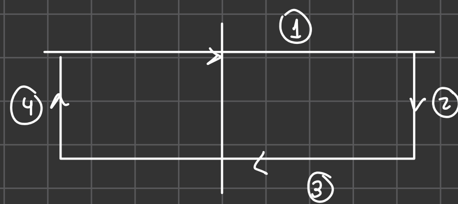
$\vdots$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2i\pi \, \mathrm{Res}\, g_R(z) = -2i\pi \left[\frac{\exp(-a^2/4)}{2a\exp(-a^2)}\right] = \sqrt{\pi}$$

Integral $\boxed{I = \int_{-\infty}^{\infty} e^{ikx-bx^2} dx}$

- $ikx - bx^2 = -b\left(x - \frac{ik}{2b}\right)^2 - \frac{k^2}{4b}$

$\rightarrow I = e^{-k^2/4b} \underbrace{\int_{-\infty}^{\infty} e^{-b(x-\frac{ik}{2b})^2} dx}_{I_z} = e^{-k^2/4b} I_z.$



$\rightarrow I = e^{-k^2/4b} I_z$

1 $z = x$; $x : [-R, R]$; $dz = dx$;
Obtenemos:

$$\int_{1} e^{-bz^2} dz = \int_{-R}^{R} e^{-bx^2} dx \rightarrow \int_{-\infty}^{\infty} e^{-bx^2} = \sqrt{\frac{\pi}{b}}$$ $\rightarrow$ Gaussiana en el eje real.

3 $-I_z$ en el lím. $R \rightarrow \infty$)

$z = x - ik/2b$; $x \in [R, -R]$, $dz = dx$

$$\int_{3} e^{-bz^2} dz = \int_{-R}^{R} e^{-b(x-ik/2b)^2} dx \rightarrow -I_z$$

2 y 4 $\quad R \to \pm\infty$ se anulan:

$\hookrightarrow z = R + it \quad , \quad t : [0, -k/2b] \quad , \quad dz = i\,dt$

$$\int_{2} e^{-bz^2} dz = \int_0^{-k/2b} e^{-b(R-it)^2}\, i\,dt = i\,e^{-bR^2}\left(\int_0^{-k/2b} e^{-2iRbt} e^{bt^2} dt\right) \to 0$$

$$\therefore \int_{-\infty}^{\infty} e^{ikx - bx^2} dx = \sqrt{\frac{\pi}{b}}\, e^{-k^2/4b}$$

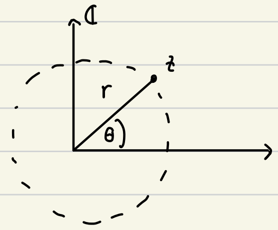
Hojas y superficies de Riemann.

15. mayo

→ Dibujos en apunte nuevo.

Ej $f(z) = \sqrt{z}$

$z = re^{i\theta}$



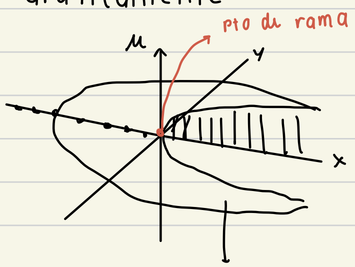
$f(z) = \sqrt{z} = u + iv$

$\hookrightarrow z^{1/2} = (re^{i\theta})^{1/2} = \sqrt{r}\, e^{i\theta/2}$

Hacemos $\theta \to \theta + 2\pi$

$f(t) = u + iv = \sqrt{r}\, e^{i\left(\frac{\theta+2\pi}{2}\right)} = \sqrt{r}\, e^{i\theta/2} \underbrace{e^{i\pi}}_{-1}$

Gráficamente:



⤳ si $\theta = \pi$, pasa x 0. → multievaluada.

⤳ si $\theta = 2\pi$, pasa por $-\sqrt{x}$

* Si la hoja se cierra ⇒ superficie de Riemann.

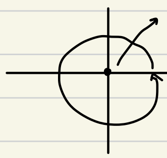
↓

si $z^{1/3}$ ⤳ la superficie debe dar 2 vueltas

func no coincide luego de dar una vuelta completa

z_0 es pto. de rama.

si $f(z_0 + \rho e^{i\theta}) \neq f(z_0 + \rho e^{i(\theta + 2\pi)})$



no necesaria# en el eje!

· Si $n = \frac{2}{3}$, $(e^{i\theta})^{1/n} = e^{\frac{i\theta}{n}} = e^{\frac{i3\theta}{2}} \to \boxed{\frac{k(2\pi)}{n} = 2\pi}$

k vueltas

$\Rightarrow k \cdot 2\pi \cdot \frac{2}{3} = 2\pi \Rightarrow \boxed{k = 3/2}$

· Si $n = \frac{1}{\sqrt{2}}$, pero k es racional

$k = \frac{1}{\sqrt{2}}$ ⤳ ∞ hojas de Riemann

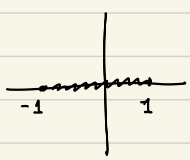
cada hoja es 1 ciclo

si z es irracional.

· Singularidad

↳ probable pto de corte.

→ $\frac{1}{(z-1)(z+1)^q}$

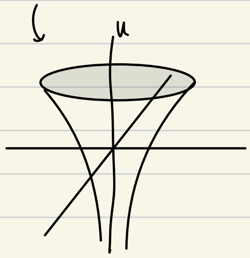


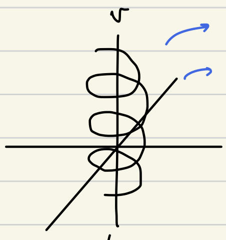
* aunque $\theta = \pm\pi$ representan = z en el plano complejo. → discontinuidad

Integrales con func. multievaluadas

* prob. prop' de integrales x rama!

· $\ln(z)$ también es una funø multievaluada.





→ ∞ hojas
→ no es cerrada.

$\sqrt{z} \to p - 1 = \frac{1}{2}$

$\Rightarrow \boxed{\frac{3}{2} = p}$

Calculemos:

llevaría al plano $\mathbb{C}$ ←

· Escoger trayect. cerrada que no transite x el corte.

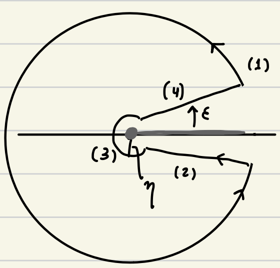
$$I(p) = \int_0^\infty \frac{x^{p-1}}{1+x^2}\,dx \longrightarrow \oint \underbrace{\frac{z^{p-1}}{1+z^2}}_{f(z)}\,dz$$

→ $1+z^2$ no es multieval.
→ $1/1+z^2$ " " "
↳ no considero polos.

con $0 < p < 2$

· $z = i + \rho e^{i\varphi}$
↳ revisar que no es multieval

$z = 0$ pto de rama



$$\oint f(z)\,dz = 2i\pi \sum_k \operatorname{Res} f(z_k)$$

· polos: $z = 0$ no cuenta: excluido

→ $z_1 = i$

$z_2 = -i$

con un ε y lo hacemos tender a 0 ←

(1) $z = Re^{i\theta}$

$\theta: \varepsilon \to 2\pi - \varepsilon$

$dz = Re^{i\theta}\, i\,d\theta$

$$\int_1 = \int_\varepsilon^{2\pi-\varepsilon} \frac{R^{p-1} e^{i\theta(p-1)} Re^{i\theta} i\,d\theta}{1 + R^2 e^{2i\theta}}$$

↓

si R es muy grande, 1 es despreciable ←

$$\sim \frac{R^{p-1}R}{R^2} \underbrace{\int_\varepsilon^{2\pi-\varepsilon} \frac{e^{ip\theta}}{e^{2i\theta}}\,d\theta}_{\text{finito}} \sim R^{p-2} \to \boxed{p-2<0}$$

↓

exige $p<2$.

→ tiende a 0 para $p<2$.

(3) $z = \eta e^{i\theta}$

$\theta : 2\pi - \varepsilon \rightarrow \varepsilon$

$dz = \eta e^{i\theta} i d\theta$

$$\int_3 = i \int_{2\pi-\varepsilon}^{\varepsilon} \frac{\eta^p e^{ip\theta}}{1+\eta^2 e^{2i\theta}} d\theta$$

$\downarrow$ tiende a 0 solo si $p>0$.

(4) $z = r e^{i\varepsilon}$

$dz = e^{i\varepsilon} dr$

$r : \eta \rightarrow \infty$

$$\int_4 = \int_\eta^\infty \frac{(r e^{i\varepsilon})^{p-1} e^{i\varepsilon} dr}{1+r^2 e^{i2\varepsilon}} = e^{i\varepsilon p} \int_\eta^\infty \frac{r^{p-1} dr}{1+r^2 e^{2i\varepsilon}}$$

$\downarrow \begin{matrix} \varepsilon \to 0 \\ \eta \to 0 \end{matrix}$

$= I(p)$

(2) $z = r e^{i(2\pi-\varepsilon)}$

$dz = dr\, e^{i(2\pi-\varepsilon)}$

$r : \infty \rightarrow \eta$

Toma de lím. $\leftarrow$

$\eta \to 0$ y $\varepsilon \to 0$

$$\int_2 = \int_\infty^\eta \frac{r^{p-1} dr\, e^{i(p-\cancel{1})(2\pi-\varepsilon)} \cancel{e^{i(2\pi-\varepsilon)}}}{1+r^2 e^{i2(2\pi-\varepsilon)}}.$$

$$= \left(\int_\infty^0 \frac{r^{p-1} dr}{1+r^2}\right) e^{ip2\pi} = -e^{2ip\pi} I(p).$$

$(z-z_1) \dfrac{z^{p-1}}{(z-z_1)(z-z_2)}$

$\rightarrow \dfrac{z_1^{p-1}}{(z_1-z_2)} = -\frac{1}{2} e^{ip\pi/2}$

$(z-z_2) \dfrac{z^{p-1}}{(z-z_1)(z-z_2)}$

$\rightarrow \dfrac{z_2^{p-1}}{(z_2-z_1)} = -\frac{1}{2} e^{3ip\pi/2}$

$\text{Res}(z_1) = -\frac{1}{2} e^{ip\pi/2}$ $\longrightarrow \oint = 2\pi i \sum \text{Res}.$

$\text{Res}(z_2) = -\frac{1}{2} e^{3ip\pi/2}.$

$\rightarrow \int_2 + \int_4 = 2\pi i \left(\text{Res}_1 + \text{Res}_2\right)$

$I(p) = \frac{\pi}{2} \operatorname{cosec}\left(\frac{p\pi}{2}\right).$

Método del descenso más pronunciado

Steepest descent

$$I(\lambda) = \int_c e^{\lambda f(z)} g(z)\, dz$$

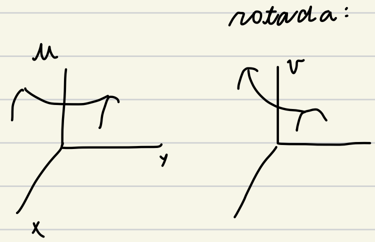
→ $f(z)$ y $g(z)$ analítica

→ Examinamos con λ muy grande.

$f(z) = u + iv$ → $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

→ $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

rotada:



$\lambda f = \lambda u + i\lambda v$

↓

$e^{\lambda f} = e^{\lambda u}\left(\underbrace{\cos(\lambda v) + i\,\mathrm{sen}(\lambda v)}_{\text{muy oscilatorias}}\right)$ → Trayect. donde las oscilaciones sean mínimas.

Buscamos z_0 para el cual $f'(z) = 0$ → $z = z_0$

$$f(z) = f(z_0) + \underbrace{f'(z_0)}_{0}(z - z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2$$

↓ montura

Hacemos: $f''(z_0) = 2Re^{i\varphi}$

$z - z_0 = re^{i\theta}$

$\Rightarrow f(z) - f(z_0) = r^2 R e^{i(\varphi + 2\theta)} = -|t|$ → describir el alejamiento de la función radial. Decrece parabólica.

$\mathrm{Re}\{f(z) - f(z_0)\} = r^2 R \cos(\varphi + 2\theta)$

$\mathrm{Im}\{f(z) - f(z_0)\} = r^2 R\, \mathrm{sen}(\underbrace{\varphi + 2\theta}_{n\pi})$ → controlamos las oscilaciones haciendo que son 0.

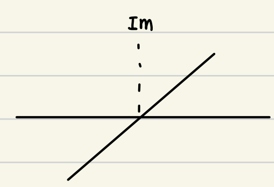
Tenemos $\varphi + 2\theta = n\pi$, $n = 0, 1, 2$ y 3 → los demás son equivalentes.

$\varphi + 2\theta_1 = 0 \Rightarrow \theta_1 = -\dfrac{\varphi}{2}$

$\varphi + 2\theta_2 = \pi \Rightarrow \theta_2 = \pi - \varphi/2$

$\varphi + 2\theta_3 = 2\pi \Rightarrow \theta_3 = 2\pi - \varphi/2$

$\varphi + 2\theta_4 = 3\pi \Rightarrow \theta_4 = 3\pi - \varphi/2$.

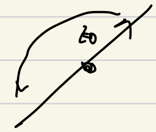


* n = 1 y n = 3 pasan por un máximo.

Para las trayectorias definidas por θ_1 y θ_3

$\rightarrow \operatorname{Re}\{f(z) - f(z_0)\} = -r^2 R \equiv -t^2$

$\operatorname{Im}\{f(z) - f(z_0)\} = 0$



$$e^{\lambda f(z)} \cong e^{\lambda f(z_0) - \lambda t^2}$$

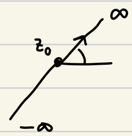
$$\Rightarrow \int e^{\lambda f(z)} g(z)\, dz \approx \int e^{\lambda f(z_0)} e^{-\lambda t} g(z)\, dz.$$

$$I(\lambda) = e^{\lambda f(z_0)} \int e^{-\lambda t^2} g(z)\, dz$$

$$z - z_0 = \frac{|t|}{\sqrt{R}} e^{i\theta} \qquad |t|^2 = r^2 R \Rightarrow r = \sqrt{\frac{|t|^2}{R}}$$

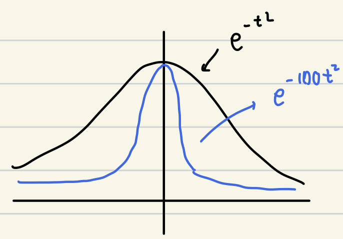
$$z - z_0 = re^{i\theta} = \frac{|t|}{\sqrt{R}} e^{i\theta}$$

$$z = z_0 + \frac{e^{i\theta_1} t}{\sqrt{R}}$$



$$dz = \frac{e^{i\theta_1} dt}{\sqrt{R}}$$

$$I(\lambda) = e^{\lambda f(z_0)} \int_{-\infty}^{\infty} e^{-\lambda t^2} \underbrace{g\left(z_0 + \frac{e^{i\theta_1} t}{\sqrt{R}}\right)}_{g(z_0)} \frac{e^{i\theta_1}}{\sqrt{R}}$$



Así

$$I(\lambda) \cong e^{\lambda f(z_0)} \frac{e^{i\theta_1}}{\sqrt{R}} \int_{-\infty}^{\infty} e^{-\lambda t^2} dt$$

$$\boxed{\int e^{\lambda f(z)} g(z)\, dz \cong e^{\lambda f(z_0)} g(z_0) \frac{e^{i\theta_1}}{\sqrt{R}} \sqrt{\frac{\pi}{\lambda}}}$$

$\varphi \rightarrow f''(z_0) = 2Re^{i\varphi}$

$\theta_1 = \frac{\pi - \varphi}{2}$

Fun∅ gamma

Aprox:

$\ln(n!) \approx n\ln n + \ldots$

La f_n gamma $\Gamma(z)$

$$\Gamma(t) = \int_0^\infty e^{-t}\, t^{z-1}\, dt$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$$

Prop. $z \to x$ real

$\Gamma(x+1) = x\,\Gamma(x)$

n entero

$$\Gamma(n+1) = n\,\Gamma(n) = n! \longrightarrow \Gamma(\underbrace{\text{argumento grande}}_{x+1}) = \int e^{-t}\, t^{x}\, dt$$

$$= \int_0^\infty e^{-z}\, z^{x}\, dz, \qquad z^x \to e^{x\ln(z)}$$

$$\int_0^\infty e^{-z}\, e^{x\ln z}\, dz$$

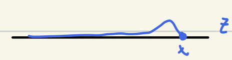
$$\int_0^\infty e^{x(\ln z - z/x)}\, dz$$

Entonces $f(z) = \ln(z) - z/x$

$g(z) = 1$

Buscamos z_0 y evaluamos $f''(z_0)$

$$f(z) = \ln(z) - \frac{z}{x} \Rightarrow f'(z) = \frac{1}{z} - \frac{1}{x} \Rightarrow \boxed{z_0 = x}$$



$$f''(z) = -\frac{1}{z^2} \qquad f''(z_0) = -\frac{1}{x^2} \Rightarrow \begin{cases} 2R = \frac{1}{x^2} \\ \varphi = \pi \end{cases} \Rightarrow \theta_1 = \frac{\pi - \varphi}{2} = 0$$

$|f''(z_0)| = |2R|\, e^{i\varphi}$

$$\underbrace{I(x)}_{\Gamma(x+1)} = \sqrt{\frac{\pi}{\lambda}}\, \frac{e^{i\theta_1}}{\sqrt{R}}\, g(z_0)\, e^{\lambda f(z_0)}$$

$\lambda \to x$, $\theta_1 \to 0$, $g(z_0) \to 1$, $\lambda \to x$, $\sqrt{R} \to \frac{1}{x\sqrt{2}}$, $f(z_0) \to \ln(x) - 1$

$$\Gamma(x+1) \cong \sqrt{2\pi x}\, e^{x(\ln(x) - 1)}$$

$$\boxed{\Gamma(n+1) \cong \sqrt{2\pi n}\, e^{n\ln n - n} = n!}$$

→ hasta acá el control!

Prolongación analítica:

22 - mayo

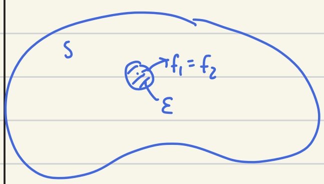
sean f_1 , f_2 definidas en S.

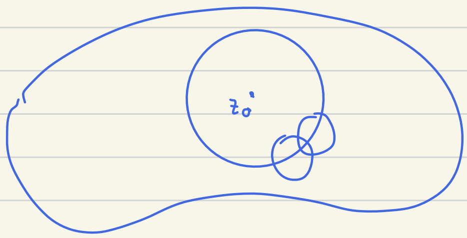
si $f_1 = f_2$ en ε

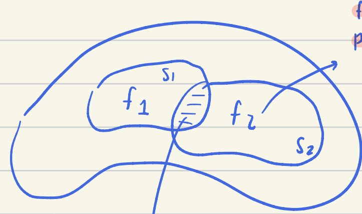
$\Rightarrow f_1 = f_2$ en S.

* contamos con la funø y sus ∞ deriv.

(contamos con la serie de Taylor en torno al pto)







f_2 es prolongación analítica de f_1 en S_2.

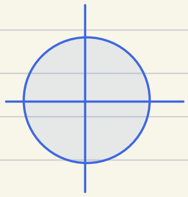
funøes iguales acá

Ej

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

en el interior. converge para $|z|<1$

→ analítica en todo el plano excepto $z=1$



$$\rightarrow g(z) = \frac{1}{1-z}$$

- $g(z)$ es la prolongación analítica de $f(z)$ a la zona $|z| \geq 1$, excluyendo $z=1$.

Extra

Ec. de schrodinger:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)\psi = E\psi$$

$$\underbrace{\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E\right)}\psi = -V\psi$$

$$(\hat{K} - E)\vec{\psi} = -\hat{V}\vec{\psi}$$

ec. de ondas planas.

sol. homogénea:

$$\rightarrow (\hat{K} - E)\vec{\psi}_0 = 0$$

ponemos una energía compleja

$$\vec{\psi} = \underset{\text{homo}}{\vec{\psi}_0} + \underset{\text{particular.}}{[\hat{K} - E]^{-1}(-\hat{V})\cdot\vec{\psi}}$$

extensión analítica

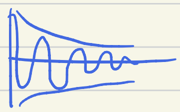
$$\Rightarrow \vec{\psi} = \vec{\psi}_0 + [\hat{K} - z]^{-1}(-\hat{V})\vec{\psi}$$

Ej 2

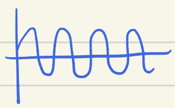
$$f_1(z) = \int_0^\infty e^{-zt}\, dt$$

$$\int_0^\infty e^{-(x+iy)t}\, dt = \int_0^\infty e^{-xt} \underbrace{(\cos yt - i\, \mathrm{sen}\, yt)}_{\pm 1}\, dt$$

· si $x > 0$:

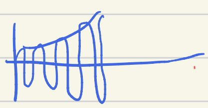


· si $x = 0$

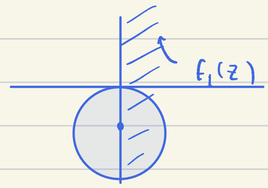


→ no converge.

· si $x < 0$



· $f(z)$ definida sólo para $\mathrm{Re}(z) > 0$



$$f_1(z) = \frac{1}{-z} e^{-zt}\Big|_0^\infty = \frac{1}{z}$$

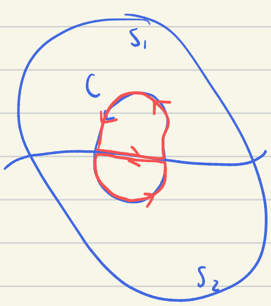
consideremos $\boxed{f_2(z) = i \sum_{n=0}^\infty \left(\frac{z+i}{i}\right)^n}$

↓ serie geométrica (converge para $\left|\frac{z+i}{i}\right| < 1 \Rightarrow |z+i| < 1$

$$f_2 = i \frac{1}{1 - \left(\frac{z+i}{i}\right)} = i \frac{i}{\cancel{i} - (z + \cancel{i})} = \frac{1}{z}$$

f_2 es la prolongaø analítica de $\frac{1}{z}$ en el círculo.

Otro teorema



frontera B

· S_1 definida en $S_1 \cup B$

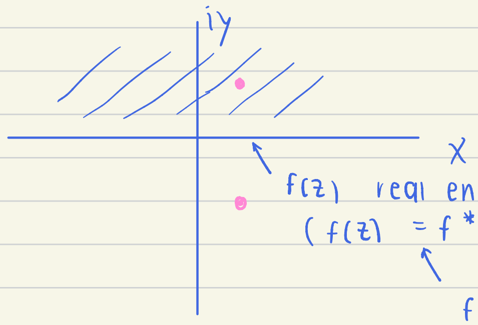
f_2 definida en $S_2 \cup B$.

· $f_1 = f_2$ en B.

$$\oint f(z)\, dz \quad , \quad f(z) = \begin{cases} f_1 & \text{en } S_1 \\ f_2 & \text{en } S_2 \end{cases}$$

$\rightarrow \underbrace{\oint_{C_1} f_1(z)}_{0} + \underbrace{\oint_{C_2} f_2(z)}_{0} = 0$ → analítica.

Principio de Reflexión de Schwartz.



f(z) real en el eje y conocida arriba

$(f(z) = f^*(z))$

$f(z) = z$

$f(z) = z^2$

$= \text{sen}\, z$

$= 1 + z^2$

funciones que satisfacen.

· $g(z) = f^*(z^*)$

$= a + ib.$

g satisface las cond. de Riemman dado que f las satisface.

* una analítica que no lo satisface: $i+z$ (no es real en el eje real).

$= i + (x + iy)$

$= i(1+i)y + x$

* $\frac{iz+1}{i+z}$ → en el eje real $(y=0)$

Relaciones dispersión

$\chi(w)$ ← c.r. son relaciones locales (van punto a punto) en torno a cada $w \in \mathbb{C}$

Relaciones de dispersión → globales.

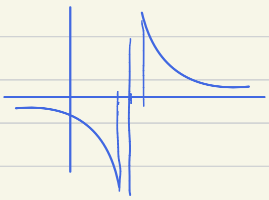
- representaciones espectrales
- Kröning y Kramers
- Transf. de Hilbert

$$\int \frac{\chi(\omega)\,d\omega}{\omega-\omega_0} = \underbrace{P\int \frac{\chi(\omega)\,d\omega}{\omega-\omega}}_{} - i\pi\,\chi(\omega_0) = 0.$$

$$\int_{-\infty}^{\omega_0-\varepsilon} \frac{\chi\,d\omega}{\omega-\omega_0} + \int_{\omega_0+\varepsilon}^{\infty} \frac{\chi\,d\omega}{\omega-\omega_0}$$

la singularidad se evita simétrica#

la int. sobre el eje real



$$P\int \frac{\chi(\omega)\,d\omega}{\omega-\omega_0} = i\pi\,\chi(\omega_0)$$

· sea $\chi(\omega) = Re\,\chi + i\,Im\,\chi = i\pi\,Re\,\chi(\omega_0) - \pi\,Im\,\chi(\omega_0)$.

$$\rightarrow\ P\int \frac{Re\,\chi(\omega)\,d\omega}{\omega-\omega_0} = -\pi\,Im\,\chi(\omega_0)$$

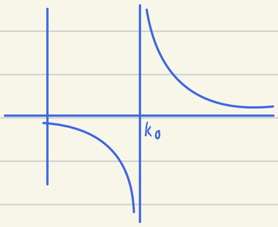
$$P\int \frac{Im\,\chi(\omega)\,d\omega}{\omega-\omega_0} = \pi\,Re\,\chi(\omega_0)$$

$\longrightarrow$ se puede obtener la parte real a partir de la parte imaginaria.

$$\frac{1}{(x-x_0)+i\eta} = \frac{(x-x_0)-i\eta}{(x-x_0)^2+\eta^2}$$

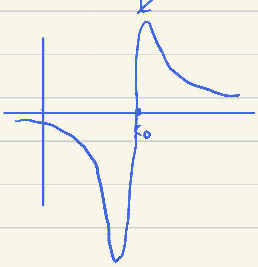
Obs considerar $\boxed{\dfrac{1}{(k-k_0)+i\eta} = G(k)}$ $\rightarrow$ aparece en física de part., nuclear, ...

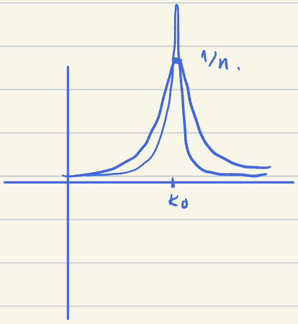
si $\eta = 0$



si $\eta \neq 0 \rightarrow \dfrac{1}{k-k_0+i\eta} = \dfrac{(k-k_0)-i\eta}{(k-k_0)^2+\eta}$

$$= \frac{k-k_0}{(k-k_0)^2+\eta^2} - i\,\frac{\eta}{(k-k_0)^2+\eta^2}$$





Cap 2: coord. curvilíneas

24 · mayo

$\nabla^2 \Phi = 0$

$(\nabla^2 + k^2)\phi = 0$

$(\nabla^2 - k^2)\Phi = 0$

$x \quad y \quad z \longrightarrow x^1 \quad x^2 \quad x^3$

$r \quad \theta \quad \varphi \longrightarrow u^1 \quad u^2 \quad u^3$

para que $(x^1 \; x^2 \; x^3) \overset{\text{invertible}}{\longleftrightarrow} (u^1 \; u^2 \; u^3)$

$u^1 = fn(x^1 \; x^2 \; x^3)$

$$\begin{cases} u^1 = F_1(x^1 \; x^2 \; x^3) \\ u^2 = F_2(x^1 \; x^2 \; x^3) \\ u^3 = F_3(x^1 \; x^2 \; x^3) \end{cases}$$

$$|J| = \begin{vmatrix} \frac{\partial u^1}{\partial x^1} & \frac{\partial u^2}{\partial x^2} & \frac{\partial u^1}{\partial x^3} \\ \vdots & & \vdots \\ & & \frac{\partial u^3}{\partial x^3} \end{vmatrix}$$

- $x^1 = f_1(u^1 \; u^2 \; u^3)$

$x^2 = f_2(u^1 \; u^2 \; u^3)$

$x^3 = f_3(u^1 \; u^2 \; u^3)$

Polares: $(x^1)\; x = r\cos\theta \qquad u_1 = r$

$(x^2)\; y = r\,\text{sen}\,\theta \qquad u_2 = \theta$

excepto para el origen. (orígenes son sigulares: $x=0$ e $y=0$)

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\,\text{sen}\,\theta \\ \text{sen}\,\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\,\text{sen}^2\theta = \boxed{r}$$

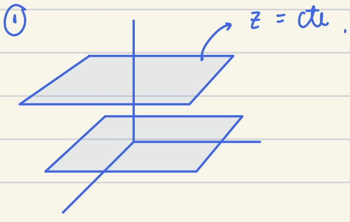
Elementos que surgen:

- superficies coordenadas
- líneas "
- ejes "
- coord. curvilíneas.
- coord. curvilíneas ortogonales. $\longrightarrow$ constatar que las líneas que pasan x ese punto son ortogonales.

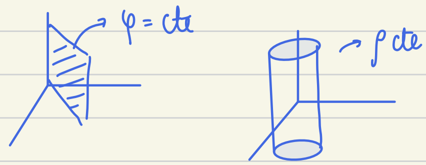
Superficies coordenadas

- Rectangular: $x \; y \; z$ ①
- cilíndricas: $\rho \; \varphi \; z$ ②
- Esféricas: $r \; \theta \; \varphi$ ③

①



(2)



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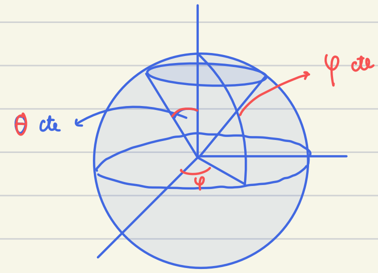
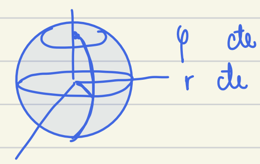


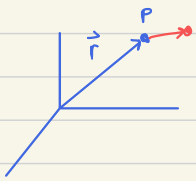
· líneas coord. : intersecø de sup. coordenadas.



. ejes coord. : ejes tangentes a c/e pto

coef. métricos (g_{ij})

desplazo p variando ← u



$$\vec{r} = \vec{r}(u^1\ u^2\ u^3).$$

$$\downarrow$$

$$d\vec{r} = \left[\frac{\partial \vec{r}}{\partial u^1}\right]\delta u^1 + \left[\frac{\partial \vec{r}}{\partial u^2}\right]\delta u^2 + \left[\frac{\partial \vec{r}}{\partial u^3}\right]\delta u^3.$$

para hablar de desplaza@. ←

$$\cdot\ (d\vec{s}\cdot d\vec{s}) = (ds)^2 = \Big(\sum_i \underbrace{\frac{\partial \vec{r}}{\partial u^i}}_{\vec{a}_i}\delta u^i\Big)\Big(\sum_j \underbrace{\frac{\partial \vec{r}}{\partial u^j}}_{\vec{a}_j}\delta u^j\Big) = \sum_i \vec{a}_i\cdot\delta u^i \sum_j \vec{a}_j\cdot\delta u^j$$

$$(ds)^2 = \sum_{ij} \underbrace{(\vec{a}_i\cdot\vec{a}_j)}_{g_{ij}}\delta u_i\,\delta u_j$$

$\vec{a}_i$: tangente a línea coordenada u_i

$$\downarrow$$

$$\hat{e}_i \equiv \frac{\vec{a}_i}{|\vec{a}_i|}$$

$$\Rightarrow (d\vec{s})^2 = \sum_{ij} |\vec{a}_i||\vec{a}_j|\ \underbrace{\hat{e}_i\cdot\hat{e}_j}_{\text{son ortogonales}}|\ \delta u_i\cdot\delta u_j$$

en el caso ortogonal

$$= |\vec{a}_1|^2(\delta u_1)^2 + |\vec{a}_2|^2(\delta u_2)^2 + |\vec{a}_2|^2(\delta u_2)^2 + |\vec{a}_3|(\delta u_3)^2.$$

Tenemos:

$$(d\vec{s})^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 + g_{33}(du^3)^2$$

$$\equiv (ds_1)^2 + (ds_2)^2 + (ds_3)^2$$

Así: $ds_1 = \boxed{\sqrt{g_{11}}}\, du^1 \quad \Rightarrow ds_1 = h_1\, du^1.$

h_1 = factor de escala

$$ds^2 = \sum_i dx^i dx^i = \sum_k \left[\sum_i \frac{\partial x^k}{\partial u^i} du^i\right]\left[\sum_j \frac{\partial x^k}{\partial u^j} du^j\right]$$

$$= \sum_{ij} \left[\underbrace{\sum_k \left(\frac{\partial x^k}{\partial u^i}\right)\left(\frac{\partial x^k}{\partial u^j}\right)}_{g_{ij}}\right] du^i du^j$$

$$\cdot\ g_{ij} = \sum_k \left(\frac{\partial x^k}{\partial u^i}\right)\left(\frac{\partial x^k}{\partial u^j}\right)$$

Ej Polares

x^1 $\quad x = \overset{u^1}{r}\cos\overset{u^2}{\theta}$

x^2 $\quad y = r\,\mathrm{sen}\,\theta$

$$\cdot\ g_{11} = \left(\overset{\cos\theta}{\frac{\partial x}{\partial u^1}} \cdot \frac{\partial x}{\partial u^1}\right) + \overset{\mathrm{sen}\,\theta}{\frac{\partial y}{\partial u^1}} \cdot \frac{\partial y}{\partial u^1}$$

$$g_{22} = \underset{= -r\,\mathrm{sen}\,\theta}{\frac{\partial x}{\partial u^2}} \cdot \frac{\partial x}{\partial u^2} + \underset{r\cos\theta}{\frac{\partial y}{\partial u^2}} \cdot \frac{\partial y}{\partial u^2} = r^2$$

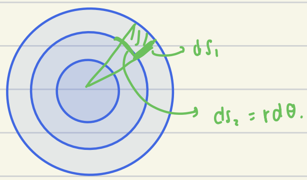
$ds_1 = h_1 \overset{dr}{(du^1)}$ $\quad , h_1 = 1 = \sqrt{g_{11}}$

$ds_2 = h_2 \underset{d\theta}{(du^2)}$ $\quad h_2 = r = \sqrt{g_{22}}$

* pág 90 → versión reciente.

$ds_1 = dr$

$ds_2 = r\, d\theta$



Def: Elementos de línea

$$ds_i = h_i du^i$$

De superficie $\quad d\sigma_{ij} = h_i h_j du^i du^j$

De volumen $\quad d\tau = h_1 h_2 h_3 du^1 du^2 du^3$

· Gradiente: $\vec{\nabla}\Phi(x\,y\,z)$

$$\delta\Phi = \frac{\partial\phi}{\partial x}\boxed{\delta x} + \frac{\partial\phi}{\partial y}\boxed{\delta y} + \frac{\partial\phi}{\partial z}\boxed{\delta z}$$

$$= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)\cdot(\delta x, \delta y, \delta z) = \vec{\nabla}\cdot\Phi\cdot\delta\vec{r} \rightarrow$$

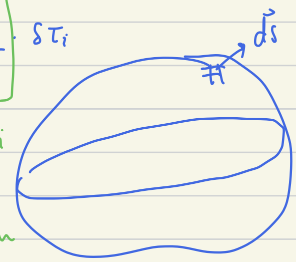
si δr apunta en la direcø del gradiente, para que $\delta\phi$ es máximo.

· Divergencia: $\rightarrow$

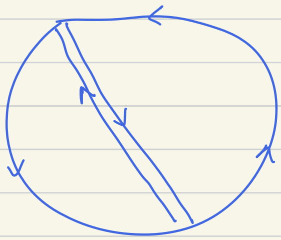
$$\oiint \vec{F}\cdot d\vec{s} = \sum_{sup.\,j}^{N} \boxed{\frac{\left(\oint_{\sigma_j} \vec{F}\cdot d\vec{s}\right)}{\delta\tau_i}}\cdot\delta\tau_i$$

$(\vec{\nabla}\cdot\vec{F})$ celda i

es un flujo neto x unidad de volumen



· Rotor:



$$\oint = \sum \boxed{\frac{\left(\oint \vec{F}\cdot d\vec{l}\right)}{\delta S}}\cdot\delta\sigma$$

$(\nabla\times\vec{F}) \longrightarrow$ circulación por unidad de área.

Divergencia

$$(d\vec{s})^2 = \underbrace{g_{11}(du^1)^2}_{(ds_1)^2} + g_{22}(du^2)^2 + g_{33}(du^3)^2$$

$$ds_1 = h_1\, du^1$$
$$ds_2 = h_2\, du^2$$
$$ds_3 = h_3\, du^3$$

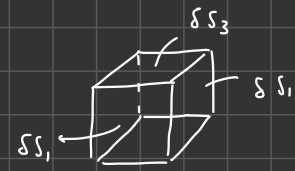
Gauss

$$\oiint \vec{F}\cdot d\vec{s} = \sum_{\text{celda } k} \left[\frac{\left(\oiint_{\Sigma_k} \vec{F}\cdot d\vec{s}\right)}{\delta \tau_k} \right] \delta \tau_k = \iiint_{V(\Sigma)} (\vec{\nabla}\cdot \vec{F})\, d\tau \qquad * \quad \vec{\nabla}\cdot\vec{F} = \lim_{\delta\tau\to 0} \frac{\oiint_{\delta\Sigma} \vec{F}\cdot d\vec{s}}{\delta \tau}$$

$\sum_{\text{celda } k} \to \iiint_{V(\Sigma)}$

$\delta\tau_k$: volumen de la celda.

$\equiv \vec{\nabla}\cdot\vec{F}$ en celda k.

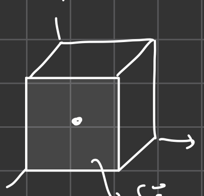


Así:

$$\vec{\nabla}\cdot\vec{F} = \lim_{\delta\tau\to 0} \frac{1}{\boxed{\delta\tau}} \left[\iint_{\delta\Sigma_1} + \iint_{\delta\Sigma_2} + \cdots \iint_{\delta\Sigma_6} \right]$$

$$\delta s_1\, \delta s_2\, \delta s_3 = h_1 h_2 h_3\, \delta u_1\, \delta u_2\, \delta u_3$$

· $\delta\Sigma_1$ en cara 1:



$$\delta\vec{s}_1 = (h_2 h_3)\Big|_{(u^1+\delta u^1)\bar{u}^2\bar{u}^3} \delta u^2\, \delta u^3\, \hat{e}_1$$

valor medio

$$\delta\vec{s}_6 = (h_2 h_3)\Big|_{(u^1)\bar{u}^2\bar{u}^3} \delta u^2\, \delta u^3 (-\hat{e}_1)$$

$$\vec{F}\Big|_{\text{cara}_1} \cdot \delta\vec{s}_1 + \vec{F}\Big|_{\text{cara } 6} \cdot \delta\vec{s}_6$$

$$\Big\{ F_1(u^1+\delta u^1, \bar{u}^2, \bar{u}^3)(h_2 h_3)\Big|_{u^1+\delta u^1, \bar{u}^2, \bar{u}^3} \delta u^2 \delta u^3$$
$$- F_1(u^1, \bar{u}^2, \bar{u}^3)(h_2 h_3)\Big|_{u^1, \bar{u}^2, u^3} \delta u^2 \delta u^3 \Big\}$$

Entonces

$$\frac{1}{\delta\tau}\left\{ \iint_1 + \iint_6 \right\} = \frac{1}{h_1 h_2 h_3\, \delta u^1\, \cancel{\delta u^2 \delta u^3}} \Big[F_1(u^1+\delta u^1, \bar{u}^2, \bar{u}^3)(h_2 h_3)\Big|_{u^1+\delta u^1}$$
$$- F_1(u^1, \bar{u}^2, \bar{u}^3)(h_2 h_3)\Big|_{u^1} \Big] \cancel{\delta u^2 \delta u^3}.$$

$$\downarrow$$

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{u^1}\left(h_2 h_3 F_1 \right) \right]$$

Extendiendo este resultado a las seis caras:

$$\boxed{\vec{\nabla}\cdot\vec{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u^1}(h_2h_3F_1) + \frac{\partial}{\partial u^2}(h_1h_3F_2) + \frac{\partial}{\partial u^3}(h_1h_2F_3)\right]}$$

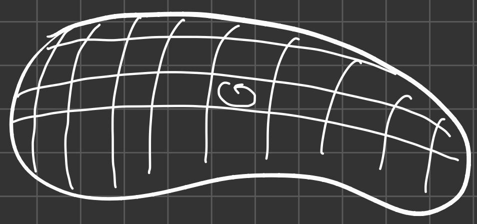
- $\vec{\nabla}\cdot\vec{F} \equiv$ Flujo neto por unidad de volumen.

El rotor

$$\rightarrow \quad W = \oint_{\mathcal{C}} \vec{F}\cdot d\vec{l} = \sum_{\text{celdas}} \boxed{\frac{\left(\oint \vec{F}\cdot d\vec{l}\right)\hat{n}}{(\delta\ \text{superf})}} \cdot \delta\,(\text{superf})\,\hat{n}$$

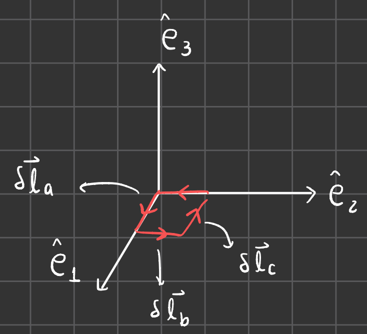
$$\downarrow$$

$$= \iint_{\Sigma(\mathcal{C})} \left[\vec{\nabla}\times\vec{F}\right]\tau\vec{s}$$



$$\rightarrow (\vec{\nabla}\times\vec{F}) = \lim_{\delta s\to 0} \frac{\left(\oint \vec{F}\cdot d\vec{l}\right)\hat{n}}{\delta s.}$$

· circulación por unidad de área.



$$\oint = \int_a + \int_b + \int_c + \int_d$$

$$= \left(\vec{F}_a\cdot\delta\vec{l}_a + \vec{F}_c\cdot\delta\vec{l}_c\right) + (\qquad\qquad)$$

$$\vec{F}_a\cdot\underset{\uparrow}{\delta\vec{l}_a} = F_1(\bar{u}^1, u^2, u^3)\cdot h_1(\bar{u}^1 u^2 u^3)\,\delta u^1$$

$$h_1\big|\delta u^1\,\hat{e}_1\Big|_{\bar{u}^1 u^2 u^3}$$

$$\vec{F}_c\cdot\delta\vec{l}_c = -\vec{F}_1(\bar{u}^1 \quad u^2+\delta u^2 \quad u^3)\,h_1(\bar{u}^1 \quad u^2+\delta u^2 \quad u^3).\ \delta u^1$$

sumando el par $\longrightarrow$ $-\left[F_1h_1\Big|_{\bar{u}^1(u^2+\delta u^2)\ u^3} - F_1h_1\Big|_{\bar{u}^1\ u_2\ u^3}\right]\cancel{\delta u^1}$

Dividiendo por $\delta s = h_1h_2\cancel{\delta u^1}\,\delta u^2$

$$\frac{\int_a + \int_c}{\delta\text{sup}}\Bigg|_{\text{par}} = \frac{1}{h_1h_2}\left[-\frac{\partial}{\partial u^2}(h_1F_1)\right]$$

$$\frac{\int_b + \int_d}{\delta\text{sup}}\Bigg|_{\text{par}} = \frac{1}{h_1h_2}\left[\frac{\partial}{u^1}(h_2F_2)\right]$$

- $(\vec{\nabla} \times \vec{F})_3 = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial u^1} (h_2 F_2) - \frac{\partial}{\partial u^2} (h_1 F_1) \right]$

rectangulares.

$= \frac{\partial}{\partial x}(A_y) - \frac{\partial A_x}{\partial y}$

- Laplaciano $\quad \vec{\nabla}^2 \Phi = \vec{\nabla} \cdot (\vec{\nabla} \Phi)$

$$\vec{\nabla} \Phi = \frac{1}{h_1} \frac{\partial \Phi}{\partial u^1} \hat{e}_1 + \dots$$

$$\boxed{\nabla^2 \Phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u^1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \Phi}{\partial u^1} \right) + \dots \right]}$$

cambia cíclica# este término

$*\ (-\nabla^2 \Phi) + \frac{1}{r} \Phi =$

↳ coordinadas parabólicas.

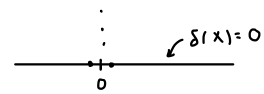
La función delta

31 · mayo.

- Fn delta de Dirac
- Funciones impropias.

Aplicación:
→ distribuc. de carga que está en un punto pero la densidad de carga es 0 en el espacio salvo en el pto.

Def $\delta(x) = 0 \quad \forall \neq 0$ → tiene argumentos reales



$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

Prop

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)} \xrightarrow{\text{dem}} \int_{-\infty}^{\infty} f(0)\,\delta(x)\,dx = f(0)\underbrace{\int_{-\infty}^{\infty}\delta(x)\,dx}_{1}$$

Otras propiedades

1) $\delta(-x) = \delta(x)$ (par)

2) $x\,\delta(x) = 0$

3) $\delta(ax) = \dfrac{1}{|a|}\,\delta(x)$

4) $\delta(x^2 - a^2) = \dfrac{1}{2|a|}\left(\delta(x+a) + \delta(x-a)\right)$ caso part.

$\uparrow \frac{p^2}{2m} \quad \uparrow \frac{p_0^2}{2m}$

5) $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$

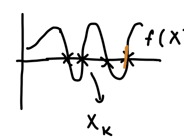
6) $\int \delta(x-a)\,\delta(x-b)\,dx = \delta(a-b).$

7) $\delta(f(x)) = \sum \dfrac{1}{|f'(x_k)|}\,\delta(x-x_k)$

↳ contribuye en los ceros de la funø.

$f'(x_0)(x-x_k)$

α



8) $\int \delta'(x)\,f(x)\,dx = -f'(0)$

9) $x\,\delta'(x) = -\delta(x)$

Def

$$\int_{-\infty}^{\infty} \delta^{(n)}(x)\,f(x)\,dx = (-)^n f^{(n)}(0)$$

→ se puede dem. integrando x partes.

$$\int f(x)\,D_1(x)\,dx = \int f(x)\,D_2(x)\,dx$$

Demostremos $\delta(x^2 - a^2) = \dfrac{1}{2|a|}\left(\delta(x+a) + \delta(x-a)\right)$

$$\int_{-\infty}^{\infty} f(x)\,\delta(\underbrace{x^2-a^2}_{(x-a)(x+a)})\,dx = \int_{-\infty}^{0} + \int_{0}^{\infty}$$

no se hace 0

$$= \int_{-\infty}^{0} f(x)\,\delta[(x-a)(x+a)]\,dx + \int_{0}^{\infty} f(x)\,\delta[\underline{(x-a)}(x+a)]\,dx$$

$\delta[-2a(x+a)]$ $\qquad$ $\delta[2a(x-a)]$

$\dfrac{1}{|2a|}(x+a)$ $\qquad$ $\dfrac{1}{2|a|}\,\delta(x-a)$

☺

Demostremos que $\delta(ax) = \dfrac{1}{|a|}\,\delta(x)$

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax)\,dx$$

→ sea $a > 0$

→ $u = ax$ $\qquad x \to -\infty \equiv u \to -\infty$

$du = a\,dx$ $\qquad x \to +\infty \equiv u \to +\infty$

$$I = \int_{-\infty}^{\infty} f\left(\frac{u}{a}\right)\delta(u)\,\frac{1}{a}\,du = \frac{1}{a}\int_{-\infty}^{\infty} f\left(\frac{u}{a}\right)\delta(u)\,du$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f(0)\,\delta(u)\,du = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx.$$

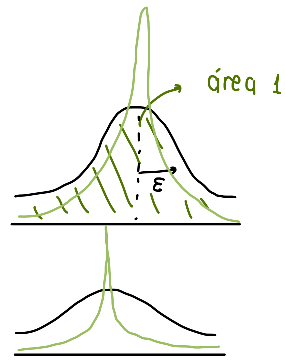
$$I = \int_{-\infty}^{\infty} f(x)\boxed{\delta(ax)}\,dx = \int_{-\infty}^{\infty} f(x)\boxed{\frac{1}{a}\,\delta(x)}\,dx$$

Representaciones:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

1) $\delta(x) = \lim_{\varepsilon \to 0} \dfrac{e^{-\frac{x^2}{2\varepsilon^2}}}{\sqrt{2\pi}\,\varepsilon}$

2) $\delta(x) = \lim_{\varepsilon \to 0} \dfrac{1}{\pi}\,\dfrac{\varepsilon}{\varepsilon^2 + x^2}$



matemáticamente

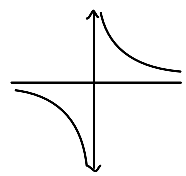
$$\delta(x) = \lim_{n\to\infty} \frac{n}{\sqrt{\pi}}\, e^{-n^2x^2}$$

$$\delta(x) = \lim_{n\to\infty} \frac{n}{\pi}\,\frac{1}{1+n^2x^2}$$

$$\delta(x - x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{k(x-x')}\,dk$$

Parte principal de una integral

$$I = \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\,dx$$



parte principal: $\boxed{P\int \frac{f(x)\,dx}{x - x_0} \equiv \lim_{\varepsilon\to 0} \int_{-\infty}^{x-\varepsilon} \% + \int_{x_0+\varepsilon}^{\infty} \%}$

$$P\int \frac{dx}{x - x_0} = -\int_{-\infty}^{x_0-\varepsilon} \frac{dx}{-x + x_0} + \int_{x_0+\varepsilon}^{\infty} \frac{dx}{x - x_0}$$

$$= -(-\ln(x_0 - x))\Big|_{-L}^{x_0-\varepsilon} + \underbrace{\ln(x - x_0)\Big|_{x_0+\varepsilon}^{L}}$$

$$= \ln\left[\frac{\varepsilon}{x_0 + L}\right] + \ln\left[\frac{L - x_0}{\varepsilon}\right]$$

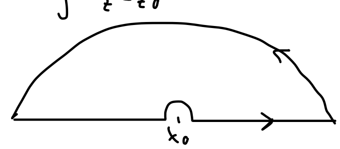
sumando:

$$I = \ln\left[\frac{\varepsilon}{L + x_0}\,\frac{L - x_0}{\varepsilon}\right] = \ln\left[\frac{L - x_0}{L + x_0}\right] \xrightarrow{L\to\infty} 0.$$

$$\frac{L - x_0}{L + x_0} \to 1$$

si no tiene polos arriba

$$\oint \frac{f(z)}{z - z_0}\,dz = 0$$



teníamos

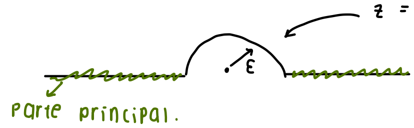
$$I = \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\,dx \longrightarrow \int_{-\infty}^{\infty} \frac{f(x)}{z - x_0}\,dz$$

$$\oint = 0$$

en el eje real

$$\hookrightarrow = P\int \frac{f(z)}{x - x_0}\,dx + \underbrace{\int \frac{f(z)}{z - x_0}\,dz}$$

$z = x_0 + \varepsilon e^{i\varphi}$



$\varphi : \pi \to 0$

$dz = \varepsilon\, e^{i\varphi}\, i\, d\varphi$

$$\int_{\smallfrown} = \int_{\pi}^{0} \frac{f(x_0+\varepsilon e^{i\varphi})}{\cancel{\varepsilon e^{i\varphi}}}\, \cancel{\varepsilon e^{i\varphi}}\, i\, d\varphi$$

$$\xrightarrow{\varepsilon\to 0} \quad f(x_0)\int_{\pi}^{0} i\, d\varphi = f(x_0)(-i\pi)$$

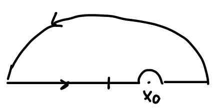
Entonces $\quad \oint = 0 = P\int \frac{f(x)}{x-x_0}dx - i\pi f(x_0)$

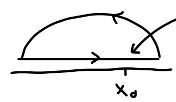
$\delta(x-x_0)f(x)$

$$\frac{1}{z-x_0} = P\,\frac{1}{x-x_0} - i\pi\,\delta(x-x_0)$$

Hemos encontrado:

$$\oint \frac{f(z)\,dz}{z-x_0} = P\int \frac{f(x)\,dx}{x-x_0} - i\pi f(x_0)$$

Obs 

= 

$z = x + i\varepsilon$
$dz = dx$

$$\oint = \int + \underbrace{\int_{\smallfrown}}_{0}$$

$$= \int_{-\infty}^{\infty} \frac{f(x+i\varepsilon)}{x+i\varepsilon-x_0}dx = \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x+i\varepsilon-x_0} = P\int \frac{f(x)\,dx}{x-x_0} - i\pi\int f(x)\,\delta(x-x_0)\,dx$$

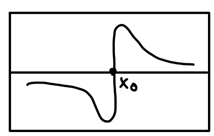
f continua

Así identificamos

$$\frac{1}{x+i\varepsilon-x_0} = P\,\frac{1}{x-x_0} - i\pi\,\delta(x-x_0)$$

$$\downarrow$$

$$\frac{1}{(x-x_0)+i\varepsilon} = \frac{(x-x_0)-i\varepsilon}{(x-x_0)^2+\varepsilon^2} = \boxed{\frac{x-x_0}{(x-x_0)^2+\varepsilon^2}} \underbrace{- \frac{i\varepsilon}{(x-x_0)^2+\varepsilon^2}\cdot\frac{\pi}{\pi}}_{-i\,\delta(x-x_0)\,\pi}$$



Series y transformada de fourier

5 junio

Teorema de Dirichlet.

Sea $S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \operatorname{sen} nx)$

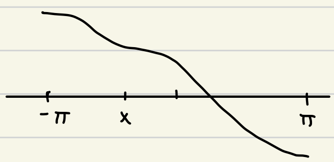
con $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

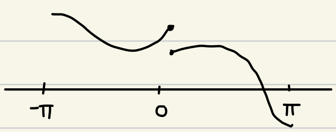
$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \operatorname{sen} nx\,dx$

$\cdot S_N(x) \xrightarrow[N\to\infty]{} \frac{1}{2}\left[f(x+\varepsilon) + f(x-\varepsilon)\right]$

Donde $f(x)$ es periódica $(f(x) = f(x+2\pi))$





$S_\infty = f(x)$

Así ·

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \operatorname{sen} nx)$

Cambio $-\pi \to \pi$ $\qquad -L \to L$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \operatorname{sen} \frac{n\pi x}{L}\right)$

Trabajemos con $e^{i(\%)}$

$e^{iq} = \cos q + i \operatorname{sen} q$

$\to \cos q = \frac{e^{iq} + e^{-iq}}{2}$

$\to \operatorname{sen} q = \frac{e^{iq} - e^{-iq}}{2i}$

$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(e^{in\pi x/L} + e^{\frac{-in\pi x}{L}}\right) \cdot \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{2i} b_n \left(e^{i\pi x/L} - e^{-i\pi x n/L}\right)$

$= \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/L}$

* $C_0 = \frac{a_0}{2}$

$n > 0$: $C_n = \frac{1}{2}(a_n - ib_n)$

$n < 0$: $C_n = \frac{1}{2}(a_{-n} + i b_{-n})$

Tenemos: $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{+in\pi x/L}$ $\cdot e^{-im\pi x/L}$

luego

$\int_{-L}^{L} dx \ldots$

¿ C_n?

$\int_{-L}^{L} f(x) e^{-\frac{im\pi x}{L}} dx = \sum_n C_n \int_{-L}^{L} e^{i\underbrace{(n-m)}_{n \neq m}\frac{x\pi}{L}} dx$ $\qquad \begin{cases} 0 & \text{si } m \neq n \\ 2L & \text{si } m = n. \end{cases}$

$= \sum_n C_n (2L) \delta_{nm} = (2L) C_m \Rightarrow \boxed{C_m = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-im\pi x/L} dx}$

Paralelo con vectores en $\mathbb{R}^N$

vector $\vec{F}$ base $\{\hat{e}_1, \hat{e}_2 \dots \hat{e}_N\}$.

$$\vec{F} = \sum_k C_k \, \hat{e}_k \qquad /\cdot \, \hat{e}_m$$

$$\vec{F} \cdot \underbrace{\hat{e}_m}_{k} = \sum_k C_k \underbrace{\hat{e}_k \cdot \hat{e}_m}_{\delta_{km}} = \underbrace{C_m}_{k}$$

$$\rightarrow \quad \vec{F} = \sum_k \underbrace{(\vec{F} \cdot \hat{e}_k)}_{C_k} \cdot \hat{e}_k$$

Acomodamos:

Definimos: Producto interno:

F prod. interno g

$$\langle f | g \rangle = \int_{-L}^{L} f^*(x)\, g(x)\, dx$$

Elementos de la base

$$\boxed{\phi_n(x) \equiv \frac{1}{\sqrt{2L}} e^{in\pi x/L}}$$

Así: $$\langle \phi_n | \phi_m \rangle = \int_{-L}^{L} \frac{1}{\sqrt{2L}} e^{-in\pi x/L} \cdot \frac{1}{\sqrt{2L}} e^{in\pi x/L} dx$$

$$= \frac{1}{2L} \int_{-L}^{L} e^{i(n-m)\pi x/L} dx$$

$$= \frac{2L\,\delta_{nm}}{2L} \quad \Rightarrow \text{ortogonales}$$

Obs. importante:

$$f(x) = \sum_n B_n \, \phi_n(x) \qquad /\cdot \, \phi_n^*(x) \quad , \quad \xrightarrow{\text{luego}} \int_{-L}^{L} dx$$

$$\boxed{\int_{-L}^{L} f(\underset{x'}{x})\, \phi_m^*(\underset{x'}{x})\, \underset{dx'}{dx}} = \sum_n B_n \underbrace{\int_{-L}^{L} \phi_m^*(x) \phi_n(x)\, dx}_{\delta_{nm}} = \boxed{B_m}$$

$$\Rightarrow f(x) = \sum_m \Big(\underbrace{\int_{-L}^{L} f(x')\, \phi_n^*(x')\, dx}_{B_n} \Big) \phi_n(x)$$

$$f(x) = \int_{-L}^{L} \underbrace{\Big(\sum_n \phi_n^*(x')\, \phi_n(x) \Big)}_{\delta(x-x')} f(x')\, dx' \quad \longrightarrow \quad \text{comparando: } f(x) = \int \delta(x-x')\, f(x')\, dx'$$

$\leftarrow$ completitud

$$\delta(x-x') = \sum_n \underbrace{\phi_n^*(x')}_{\langle \phi_n | x' \rangle} \underbrace{\phi_n(x)}_{\langle \phi_n || x \rangle} \qquad \text{completitud.}$$

$$\sum_n \langle x | \phi_n \rangle \langle \phi_n | x' \rangle \rightarrow \langle x | \Big(\overbrace{\sum_n |\phi_n\rangle \langle \phi_n|}^{I} \Big) | x' \rangle = \langle x | x' \rangle$$

Paso al continuo

$(-L, L) \longrightarrow (-\infty, \infty)$

$$\phi_n = \frac{1}{\sqrt{2L}} e^{in\pi x/L} \quad ; \quad \sum_n$$

$$f(x) = \sum_n c_n \phi_n(x)$$

Tenemos $e^{i\left(\frac{n\pi}{L}\right)x}$, $\frac{n\pi}{L} = k_n = n\left(\frac{\pi}{L}\right) \Rightarrow \delta k_n = \frac{\pi}{L}$

Además :

$$\delta(x-x') = \sum_n \frac{1}{\sqrt{2L}} e^{-i\frac{n\pi}{L}x'} \cdot \frac{1}{\sqrt{2L}} e^{i\frac{n\pi}{L}x}$$

k_n

$$= \frac{1}{\pi}\sum_n \frac{1}{2L}\pi\, e^{-ik_n x'} \cdot e^{ik_n x}$$

$$= \frac{1}{2\pi} \sum \delta k_n\, e^{-ik_n(x'-x)} \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\, dk$$

$$\rightarrow \int_{-\infty}^{\infty} e^{ik(x-x')}\, dk = 2\pi\,\delta(x-x').$$

$\downarrow$

Así, los elementos de nuestra base serán:

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

Teníamos $\int_{-L}^{L} \phi_n^* \phi_{n'} = \delta_{nn'}$

$$\int_{-\infty}^{\infty} \phi_k^*(x)\,\phi_{k'}(x)\, dx = \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} e^{-ikx}\cdot e^{ik'x}\, dx}_{2\pi\,\delta(k-k')}$$

$$f(x) = \int A(k)\,\phi_k(x)\, dk \qquad / \cdot \phi_{k'}^*(x) \quad \int_{-\infty}^{\infty} dx$$

$$\int_{-\infty}^{\infty} \phi_{k'}^*(x)\, dx = \int dk\, A(k)\int_{-\infty}^{\infty} \phi_{k'}^*(x)\,\phi_k(x) = A(k')$$

Recordar → discreto:

7· jun.

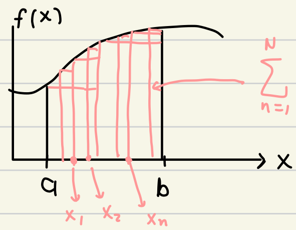
$$\delta(x'-x) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{2}\boxed{\frac{\pi}{L}}}_{\delta k} e^{in\underbrace{\boxed{\frac{\pi}{L}}}_{\delta k}(x-x')}$$

k

↓ continuo

$$\delta(x'-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk$$

Repaso



$$\sum_{n=1}^{N} f(x_n)(\delta x) \xrightarrow{N\to\infty} = \int_a^b f(x)\,dx$$

$a + \delta x = x_1$

$b = x_N$

$\underline{x_n = n\,\delta x}$

$\delta x = \frac{b-a}{N} \xrightarrow{N\to\infty} dx$

Transformada de Fourier

Observar lo siguiente

$f(x)$

↓

def
$$\boxed{\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{-ikx}\,dx}$$

puede ser ikx

$\int \cdot\, e^{ikx'}$ — x var. muda.

$\int dk$

* despejar $f(x)$!

$$\int_{-\infty}^{\infty} dk\,\tilde{f}(k)e^{ikx'} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} dk\,f(x)\,e^{ik(x'-x)}$$

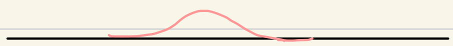
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\,f(x)\underbrace{\int_{-\infty}^{\infty} e^{ik(x'-x)}\,dk}_{2\pi\,\delta(x-x')}$$

$$= \sqrt{2\pi}\cdot f(x')$$

$$\Rightarrow \boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\,\tilde{f}(k)e^{ikx}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t}$$

Ej 66



$\downarrow$ t_0 e^{-ax^2}

¿Cuál es T en todo el espacio después de un tiempo t ?

$$T(x,t) = \int_{-\infty}^{\infty} a(k,t)\, \boxed{e^{ikx}}\, dx$$

$$\int (-k^2 e^{ikx})\, a(k,t) = \frac{1}{\sigma}\int \frac{da}{dt} e$$

$\longrightarrow$ multiplicamos x el conjugado e integramos.

Def Transformada de Fourier en 3D

$$f(x) \longrightarrow F(x,y,z) \equiv F(\vec{r})$$

$$\vec{F}(k_x, k_y, k_z) = \frac{1}{(2\pi)^{3/2}}\int dx \int dy \int dz\, F(x,y,z)\, e^{-i(k_x x + k_y y + k_z z)}$$

$$\boxed{\vec{F}(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\int d^3r\, F(\vec{r})\, e^{-i(\vec{k}\cdot\vec{r})}}$$

$\rightarrow$ transformada de Fourier.

Ejemplos:

$$\boxed{\tilde{f}(k) = \frac{\sigma}{\sqrt{2\pi}}\, e^{-k^2\sigma^2/2}}$$

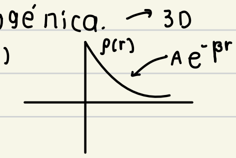
1D

ⓐ $\tilde{f}(k)$ para $f(x)$ una gaussiana. $\longrightarrow$ Transf. de Fourier de una gaussiana es una gaussiana.

* long. de onda de D'broglie : $p = \hbar h$.

ⓑ Distribuø hidrogénica. $\rightarrow$ 3D

$\rho(\vec{r}) \longrightarrow \rho(r)$



densidad de prob. esférica# simétrica.

$\downarrow$

$\tilde{\rho}(\vec{k})$?

* dual outputs

ⓐ $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\, e^{-ikx}\, dx$$

$$= \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2\sigma^2} + ikx\right)}\, dx$$

$$* \quad \frac{x^2}{2\sigma^2} + ikx = \frac{1}{2\sigma^2}\left(x + ik\sigma^2\right)^2 + \frac{(k\sigma^2)^2}{2\sigma^2}$$

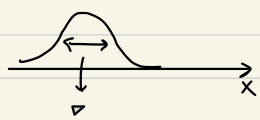
$$\frac{1}{2\sigma^2}(x + ik\sigma^2)^2 + \frac{k^2\sigma^2}{2}$$

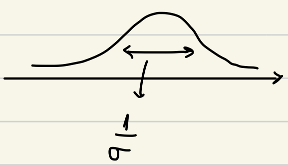
$$\rightarrow \frac{1}{2\pi\sigma}\left[\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2\sigma^2}(x+ik\sigma^2)^2}\right] e^{-k^2\pi^2/2}$$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{z^2}{2\sigma^2}} = \sqrt{2\pi}\,\sigma$$

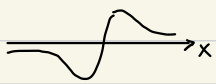
$$\rightarrow \frac{1}{2\pi\sigma}\sqrt{2\pi}\,\sigma\, e^{-k^2\sigma^2/2} = \frac{1}{\sqrt{2\pi}}\,\sigma\, e^{-\frac{k^2\sigma^2}{2}}$$

La dif:





* Prop: $\tilde{F}$ de $f(x) = x\, e^{-x^2}$



(b) $\tilde{\rho}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} A \iiint d^3r\, e^{-\beta r} e^{-i\vec{k}\cdot\vec{r}}$

$d^3r = r^2 dr\, d\Omega$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^{\infty} r^2 dr\, \rho(r) \cdot \underbrace{\left(\iint d\Omega\, e^{i\vec{k}\cdot\vec{r}}\right)}_{4\pi\, j_0(kr)}$$

$j_0(x) = \frac{\operatorname{sen} x}{x}$

$$\int d\Omega\, e^{i\vec{k}\cdot\vec{r}} = \iint \operatorname{sen}\theta\, d\theta\, d\varphi\; e^{ikr\cos\theta}$$

$$= 2\pi \underbrace{\int_0^{\pi} \operatorname{sen}\theta\, d\theta\, e^{ikr\cos\theta}}_{\underbrace{\int_{-1}^{1} du\, e^{ikru}}_{2\, j_0(kr)}}$$

$$= \frac{4\pi}{(2\pi)^{3/2}} \int_0^{\infty} r^2 dr\, j_0(kr)\, \rho(r)$$

$$= \frac{4\pi}{(2\pi)^{3/2}} \cdot \frac{1}{k} \int_0^{\infty} r\, dr\, \underbrace{\operatorname{sen}(kr)}_{\frac{e^{ikr}-e^{-ikr}}{2i}}\, \rho(r)$$

también $\left(-\frac{\partial}{\partial k}\right)\cos(kr)$

$\rho(r) \rightarrow e^{-\beta r}$

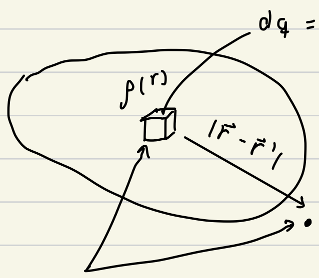
$$\int \left(-\frac{\partial}{\partial \beta}\right) \frac{e^{ikr-\beta r} - e^{-ikr-\beta r}}{2i}$$

→ result.: algo parecido a:

$$\rightarrow \frac{1}{(1+k^2/\beta^2)^2}$$

Teorema de Combolución

12 · junio.



$$\Phi(\vec r) = \iiint \frac{\rho(\vec r\,')}{|\vec r - \vec r\,'|} d\tau.$$

$$\Phi(\vec r) = \iiint \vec r(\vec r\,') \, G(\vec r - \vec r\,') \, d\tau.$$

convoluø

· Trasf. de Fourier de convolución es igual al producto de las transf. de Fourier de los términos (salvo cte).

Dem

· Sea $\qquad \varphi(x) = \int f(x') \, g(x-x') \, dx' \quad / \cdot e^{-ikx} \quad \int dx$

$$\int_{-\infty}^{\infty} \varphi(x) e^{-ikx} dx = \int_{-\infty}^{\infty} dx \, e^{-ikx} \int_{-\infty}^{\infty} dx' \, f(x') \, g(x-x')$$

$*$ $ikx = ik(x-x') + ikx'$

$$\rightarrow = \int_{-\infty}^{\infty} dx \, \boxed{e^{-ik(x-x')}} \, e^{ikx} \, f(x') \, \boxed{g(x-x')}$$

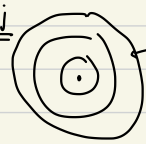
$x - x' \equiv y \qquad dx = dy$

$$= \iint dx' \, dy \, e^{-iky} \, e^{-ikx'} \, f(x') \, g(y).$$

Así,

$$\underbrace{\int_{-\infty}^{\infty} \varphi(x) \, e^{-ikx} dx}_{\sqrt{2\pi}\,\tilde\varphi(k)} = \underbrace{\left[\int_{-\infty}^{\infty} f(x') e^{-ikx'} dx'\right]}_{\sqrt{2\pi}\,\tilde f(k)} \underbrace{\left[\int_{-\infty}^{\infty} dy \, e^{-iky} g(y)\right]}_{\sqrt{2\pi}\,\tilde g(k)}$$

Ej



$\rho(\vec r) = \rho_0 e^{-r^2/2R^2}$

$\phi(r)$?

elemento de vol. 3 dimensional.

$d^3k \rightarrow d\vec k = k^2 dk \, d\hat k$

$$\phi(r) = \iiint \rho(\vec r\,') \underbrace{\frac{1}{|\vec r - \vec r\,'|}}_{f} d\tau \longrightarrow \tilde\phi(k) = (2\pi)^{3/2} \tilde\rho(k) \underbrace{\boxed{\tilde f(k)}}_{\sim \frac{1}{k^2}} \Longrightarrow \phi(\vec r) = \frac{1}{(2\pi)^{3/2}} \iiint d\vec k \, e^{i\vec k \cdot \vec r} \tilde\phi(\vec k)$$

$* \quad \tilde{f}(k) = \sqrt{\frac{2}{\pi}}\frac{1}{k^2} \longrightarrow$ si el potencial de una carga puntual unitaria es $f(x) = \frac{1}{r}$

$$\tilde{\Phi}(k) = \frac{4\pi}{k^2}\tilde{\rho}(k) \qquad * \; k\cdot r = kr\cos\theta$$

$$\rightarrow \tilde{\rho}(k) = \frac{2\pi}{(2\pi)^{3/2}} \int_0^\infty r^2 dr\, \rho(r) \int_0^\pi \sin\theta \, d\theta \, e^{-ikr\cos\theta}$$

$$\tilde{\rho}(k) = \sqrt{\frac{2}{\pi}}\frac{1}{k}\int_0^\infty r\, dr\, \rho(r) \sin kr.$$

$$\tilde{\rho}(k) = \sqrt{\frac{2}{\pi}}\frac{\rho_0}{k}\int_0^R r\, dr \sin kr = \sqrt{\frac{2}{\pi}}\frac{\rho_0}{k}\left(-\frac{\partial}{\partial k}\right)\underbrace{\int_0^R dr \sin kr}_{\sin(kR)/k}$$

$$\tilde{\rho}(k) = \sqrt{\frac{2}{\pi}}\frac{\rho_0 R^2}{k} j_1(kR) \quad , \; j_0(t) = \frac{\sin t}{t} \quad , \; j_1(t) = -\frac{\partial j_0}{\partial t}$$

pasos ...

$$\tilde{\Phi}(k) = \frac{4\pi}{k^2}\tilde{\rho}(k) \; //$$

$$\tilde{\rho}(k) \sim e^{-k^2/\#}$$

$* \int e^{i\vec{k}\cdot d\vec{r}}\, d\hat{r}$ $\quad\quad\quad$ $* \int \left(\frac{e^{-\mu r}}{r}\right)$

$\hookrightarrow 4\pi\, j_0(kr).$

calculemos $\tilde{f}(k)$ para $f(\vec{r}) = f_0 \dfrac{e^{-\mu r}}{r}$

Luego tomamos límite $\mu \to 0$.

$$\tilde{f}(k) = \frac{1}{(2\pi)^{3/2}} \cdot f_0 \cdot \iiint d\vec{r}\, \frac{e^{-\mu r}}{r}\, e^{-i\vec{k}\cdot\vec{r}}$$

Recordamos
$$\iint e^{i\vec{k}\cdot\vec{r}}\, \underset{\sin\theta\, d\theta d\varphi.}{d\Omega} = 4\pi\, j_0(kr) = 4\pi \frac{\operatorname{sen}(kr)}{kr}$$
$\longrightarrow$ ya que el resto no tiene dependencia angular.

$$\to \tilde{f}(k) = \frac{1}{(2\pi)^{3/2}} f_0\, 4\pi \int_0^\infty \cancel{r^2} dr\, \frac{e^{-\mu r}}{\cancel{r}} \frac{\operatorname{sen}(kr)}{k\cancel{r}}$$

$$= \sqrt{\frac{2}{\pi}} \frac{f_0}{k} \left[\underbrace{\int_0^\infty e^{-\mu r} e^{ikr} dr}_{\frac{1}{(-\mu+ik)} e^{*}\big|_0^\infty} - \int_0^\infty e^{-\mu r}\cdot e^{-ikr} dr \right] \cdot \frac{1}{2i}$$

acotado

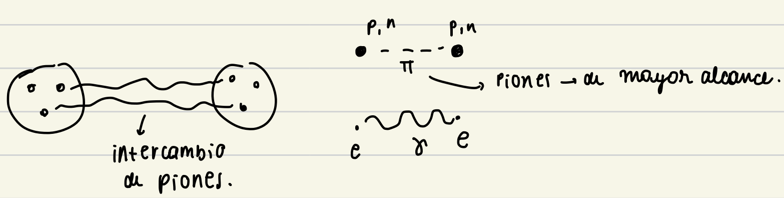
$$+ \frac{1}{\mu + ik}\, e^{*}\Big|_0^\infty$$

$$\sqrt{\frac{2}{\pi}} \frac{f_0}{k} (-) \left[\frac{1}{(-\mu+ik)} + \frac{1}{\mu+ik} \right] \frac{1}{2i}$$

$$= \sqrt{\frac{2}{\pi}} \frac{f_0}{\cancel{k}} (-) \frac{-\cancel{2ik}}{\mu^2+k^2} \cdot \frac{1}{\cancel{2i}}$$

$$= \underline{\sqrt{\frac{2}{\pi}} \frac{f_0}{(\mu^2+k^2)}} \xrightarrow{\mu\to 0} \sqrt{\frac{2}{\pi}} f_0 \frac{1}{k^2}$$

$\cdot\ \dfrac{e^{-\mu r}}{r}$ ← potencial de Yukawa.



Relaø de Parseval

$$\int_{-\infty}^{\infty} f^*(x)g(x)\,dx = \int_{-\infty}^{\infty} \tilde{f}^*(k)\tilde{g}(k)\,dk$$

$$f \begin{cases} f(x) = \langle x|f\rangle \\ \tilde{f}(x) = \langle k|f\rangle \end{cases}$$

Dem

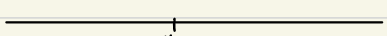
$$\int_{-\infty}^{\infty} \tilde{f}^*(k)\,\tilde{g}(k)\,dk = \int dk\,dx\,dx' \left[f(x)e^{-ikx}\right]^* \left[g(x')\,e^{-ikx'}\right] \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2\pi}}$$

$$= \int \boxed{dk}\, dx\, dx' \left[f^*(x)\,\boxed{e^{ik(x-x')}}\,g(x)\right]\frac{1}{2\pi}$$

Recordar $\displaystyle\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk = 2\pi\,\delta(x-x')$

$$\rightarrow = \int dx \int dx'\, f^*(x)\,g(x')\,\delta(x-x') = \int dx\, f^*(x)\,g(x).$$

Ej – 1D



$x=0$

se calienta el centro

$T = T_0\, e^{-x^2/2L^2}$

Sea $\displaystyle T(x,t) = \int a(k,t)\,e^{ikx}\,dk$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t}$$

$T = T(x,t) = T_0\, e^{-x^2/2L}$

$$\frac{\partial T}{\partial x} = \int_{-\infty}^{\infty} a(k,t)(ik)\,e^{ikx}\,dk$$

$$\frac{\partial^2 T}{\partial x^2} = \int_{-\infty}^{\infty} \underbrace{-k^2 a(k,t)}_{A(k)}\,e^{ikx}\,dk$$

$$\frac{\partial T}{\partial t} = \int_{-\infty}^{\infty} \underbrace{\frac{\partial a(k,t)}{\partial t}}_{B(k)}\,e^{ikx}\,dk$$

Multiplicando $\cdot\, e^{-ik'x}$, $\int dx$

$$\rightarrow \int A(k)\,e^{ikx}\,dk = \frac{1}{\sigma}\int B(k)\,e^{ikx}\,dk$$

$$\rightarrow A(k') = \frac{B(k')}{\sigma}$$

$$\rightarrow A(k') = \frac{1}{\sigma}B(k')$$

Luego de multiplicar e^x e integrar,

- $$-k^2 a(k,t) = \frac{1}{\sigma}\frac{\partial a(k,t)}{\partial t}$$

- $$\boxed{a(k,t) = w_k\, e^{-k^2\sigma t}}$$

$$T(x,t) = \int_{-\infty}^{\infty} w_k e^{-k^2\sigma t} e^{ikx}\, dk$$

$t=0$ ← condiø inicial: $T_0 e^{-x^2/2L^2} = \int_{-\infty}^{\infty} w_k e^{ikx}\, dk$ $\quad \int \cdot\, e^{-ik'x} \cdot \int_{-\infty}^{\infty} dx$

$2\pi\,\delta(k'-k)$

"Despejar" w_k

$$\int_{-\infty}^{\infty} dx\, T_0\, e^{-x^2/2L^2} \cdot e^{-ik'x} = 2\pi\, w_{k'}$$

$$w_k = \frac{T_0}{2\pi} \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2L^2} - ikx}$$

$$T(x,t) = \int_{-\infty}^{\infty} dk\, \frac{T_0 \sqrt{2\pi}\, L}{2\pi} e^{-k^2L^2/2} e^{-k^2\sigma t} e^{ikx}.$$

* ⇝ $k^2\left(\frac{L^2}{2} + \sigma t\right)$

$$= \frac{T_0\sqrt{2\pi}}{2\pi} L \int_{-\infty}^{\infty} e^{-k^2\left(\frac{L}{2}+\sigma t\right)} e^{ikx}\, dk$$

#

→ 3D con el Laplaciano.

* $\frac{x^2}{2\sigma^2} + ikx = \frac{1}{2\sigma^2}\left(x + ik\sigma^2\right)^2 + \frac{(k\sigma^2)^2}{2\sigma^2}$

$\frac{1}{2\sigma^2}\left(x + ik\sigma^2\right)^2 + \frac{k^2\sigma^2}{2}$

Cap 5. Ecuaciones diferenciales.

14 junio 22

Elípticas:
→ Laplace: $\nabla^2\phi = 0$
→ Helmholtz: $\nabla^2\phi + k^2\phi = 0$
→ Poisson: $\nabla^2\phi = f$

Parabólicas:
→ Difusión: $\nabla^2 u = \partial_t u$
→ Schrödinger: $-\nabla^2\psi + U\psi = -i\partial_t\psi$

Hiperbólicas:
→ ondas mecánicas: $\nabla^2\phi - \partial_t^2\phi = 0$
→ Klein-Gordon: $\nabla^2\phi - m^2\phi = \partial_t^2\phi$

$\mathcal{L}$ operador diferencial lineal.

$$\underset{\uparrow}{\mathcal{L}f = af} \longrightarrow \underset{\swarrow}{f_a} + \text{prod. interno}$$

soluc. propias asociadas a a.

$$\langle f_a | f_b \rangle \rightarrow \int w(x)\cdot f_a^*(x)\, f_b(x)\, dx \quad \sim \delta_{ab}.$$

$$f = \sum c_n f_n$$

Casos de interés:

a) $f \sim \text{sen}(\ )$, $\cos(\ )$ Fourier
$e^{i(\ )}$

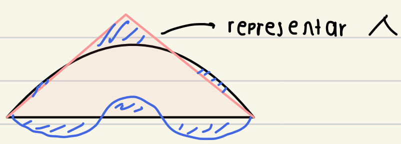
b) Funciones cilíndricas de Bessel

$J_\nu(\rho)$ $\quad Y_\nu(\rho)$ $\sim \nu$ entero

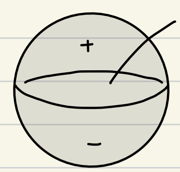
Funciones esféricas de
Bessel $\quad$ Newmann
$j_\ell(x)$ $\quad n_\ell(x)$ $\sim \nu$ semientero

c) Polinomios de Legendre → para desarrollos multipolares.

$P_\ell(u)$ $\quad Q_\ell$



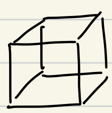
$$P_\ell(\cos\theta) \begin{cases} \ell = 0 \rightarrow 1 \\ \ell = 1 \rightarrow \cos\theta \\ \ell = 2 \\ \end{cases}$$



$$\sum C_n(r)\, P_\ell \cos\theta$$

d) Esféricos armónicos

$Y_\ell^m(\theta, \varphi)$ → en esféricas.



Separación de variables

Supongamos $D_{\vec{r}}$ operador diferencial que depende de $\vec{r}$ $\begin{cases} x\,y\,z \\ r\,\theta\,\varphi \\ \rho\,\varphi\,z \\ \vdots \end{cases}$
↳ Laplaciano, gradiente.

D_t operador que depende de la variable tiempo.

$$D_{\vec{r}}\, \underbrace{\Psi(\vec{r},t)} = D_t\, \Psi(\vec{r},t)$$

expresaría como el prod. de soluc. → la soluc. general es la c.l. de las soluc.

sea $\Psi(\vec{r},t) = R(\vec{r})T(t)$

$$D_{\vec{r}}\,\psi = \underbrace{D_{\vec{r}}}_{}\left(R(\vec{r})T(t)\right)$$

$D_{\vec{r}}$ no tiene op. en el t

$$D_t\psi = R(D_tT)$$

$$(D_{\vec{r}}R)\,T = R(D_tT) \;/\cdot\frac{1}{RT}$$

$$\frac{(D_{\vec{r}}R)}{R} = \frac{(D_tT(t))}{T(t)} \Rightarrow = \lambda \quad , \quad \lambda \text{ cte}$$

sólo depende de r

sólo depende de t

$$\boxed{D_tT = \lambda T}$$

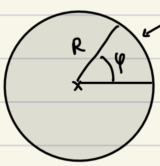
$$\boxed{D_{\vec{r}}R = \lambda R}$$

Ec. de Laplace en un disco:

→ otra manera de resolver este prob.

$\nabla^2\Phi = 0$

2D + C.B.



$f(\varphi)$ → dato.
↓
potencial

∇^2 en cilíndricas.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0.$$

$\Phi = \Phi(\rho,\varphi)$.

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

$\Phi(\rho,\varphi) = \Omega(\rho)\,\chi(\varphi)$ → separación de variables

$$\cdot\ \frac{\partial\Omega(\rho)}{\partial\rho} = \frac{d\Omega(\rho)}{d\rho} = \Omega'$$

$$\rightarrow \frac{1}{\Omega}\left[\rho\frac{d}{d\rho}(\rho\Omega')\right] + \boxed{\frac{1}{\chi}\chi''} = 0 \quad , \varphi \text{ indep. de } \rho.$$

$-n^2$

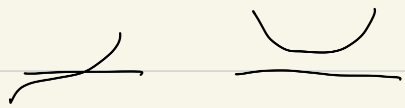
$\chi'' = \dfrac{d^2\chi}{\partial\varphi^2}$

2 casos

$\chi'' = \cancel{k^2}\chi \rightarrow \chi = A\cos(k\varphi) + B\,\text{sen}(k\varphi)$. → la descartamos xq por lo menos queremos que la función sea univaluada.

$\chi'' = -k^2\chi \rightarrow \chi = A\,\text{senh}(k\varphi) + B\cosh(k\varphi)$.

$= A\cos(k\varphi + c)$ → resolvimos la parte angular senh :



$\rightarrow \quad X(\varphi) = X = (\varphi + 2\pi) \rightarrow$ lo exigimos

$\Rightarrow k$ entero.

$\rightarrow \quad X''(\varphi) = -n^2 X$

$\Rightarrow \quad \boxed{X_n(\varphi) = A_n \cos(n\varphi) + B_n \operatorname{sen}(n\varphi).}$

Resolvemos

$$\frac{1}{\Omega}\left(\rho \frac{d}{d\rho}(\rho\Omega')\right) = n^2$$

$$\rho \frac{d}{d\rho}\left(\rho \frac{d\Omega}{d\rho}\right) = n^2\Omega$$

$* \quad \boxed{\Omega = c\rho^p} \longrightarrow$ sale de $x\frac{df}{dx} = c \cdot f \Rightarrow \frac{df}{dx} = \left(\frac{cf}{x}\right)$

$$\rho \frac{d\Omega}{d\rho} = c \cdot p \cdot \rho^p$$

$$\rho \frac{d}{d\rho}[\,//\,] = cp^2\rho^p = p^2\Omega$$

$* \quad p^2\Omega = n^2\Omega$

$p^2 = n^2$

$p = \pm n$

$\boxed{\Omega = c_n\rho^n + \cancel{d_n\rho^{-n}}} \longrightarrow$ ambas funcan.

$d_n = 0$

contiene el origen.

Caso n=0

$$\rho \frac{d}{d\rho}\left(\rho \frac{d}{d\rho}\Omega\right) = 0$$

$$\rho \frac{d}{d\rho}\Omega = cte_1 \Rightarrow \frac{d\Omega}{d\rho} = \frac{cte_1}{\rho} \Rightarrow \Omega = cte_2 + \cancel{cte_1 \ln(\rho)}.$$ $\rightarrow$ origen en el interior

$$\Phi(\rho,\varphi) = \underbrace{\Omega(\rho)}_{(c_n\rho^n \,;\, c_0)}\, X(\varphi).$$

$\uparrow$ encontramos : $A_n \cos(n\varphi) + B_n \operatorname{sen}(n\varphi)$

♥♥♥ solución general

$$\Phi(\rho,\varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos(n\varphi) + B_n \operatorname{sen}(n\varphi)\right]\rho^n.$$

condición de Borde

$\Phi(R,\varphi) = f(\varphi)$

$$\Rightarrow f(\varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos(n\varphi) + B_n \operatorname{sen}(n\varphi)\right]R^n$$

Para obtener A_0, A_n, B_n

$$\int_0^{2\pi} \text{sen}(n'x) \ldots \qquad \int_0^{2\pi} \cos(n'x) \ldots$$

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\varphi)\, d\varphi$$

$$A_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\varphi')\cos(n\varphi')\, d\varphi$$

$$B_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\varphi)\sin(n\varphi')\, d\varphi$$

$$\Phi(\rho,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi')\, d\varphi' + \sum_{n=1}^{\infty} \frac{\rho^n}{\pi R^n}\int_{-\pi}^{\pi} f(\varphi')\Big[\underbrace{\cos(n\varphi')\cos(n\varphi) + \text{sen}(n\varphi')\text{sen}(n\varphi)}_{\cos(n(\varphi-\varphi'))}\Big]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi')\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{\rho}{R}\right)^n \cos(n(\varphi-\varphi'))\right] d\varphi'$$

$$\frac{e^{in(\varphi-\varphi')} + e^{-i\theta}}{2}$$

$$* \left(\frac{\rho}{R} e^{i\theta}\right)^n \rightarrow \text{serie geométrica!}$$

$$\boxed{\Phi(\rho,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi+\varphi')\, \frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\rho')}\, d\varphi'}$$

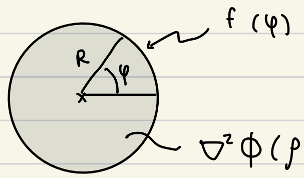
↳ La expansión no se factoriza.

↳ si $\rho = R$ debería dar el argumento ¿Cómo se demuestra?

* Taller: técnicas de Fourier para encontrar una funcø en el t.

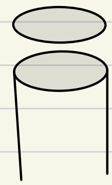
Resolvimos

Miérc. 28 · jun



$\nabla^2 \phi(\rho, \varphi) = 0$

$$\rightarrow \phi(\rho,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_0(\varphi' + \varphi)\,\frac{(R^2-\rho^2)\,d\varphi'}{R^2+\rho^2-2R\rho\cos\varphi'}$$



$$\nabla^2\phi = \pm\,\omega^2\phi$$

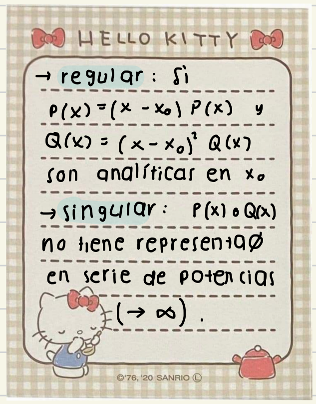
Funciones cilíndricas de Bessel.

$\underbrace{J_\nu(z)}_{\text{regular } \forall z}$, $\underbrace{Y_\nu(z)}_{\text{singular en } z=0.}$ } dan cuenta del comporta@ radial. (ρ)

Funciones modificadas de Bessel: $I_\nu(z)$, $K_\nu(z)$

Funøes esféricas de Bessel $\nabla^2 \rightarrow \nabla^2\big|_{\text{parte radial.}}$ } comporta@ radial (r)

$\underbrace{j_\ell(z)}_{\text{regular } \forall z}$, $\underbrace{n_\ell(z)}$



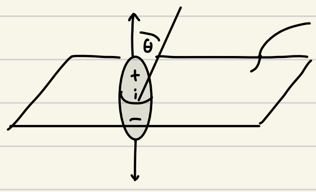
$$j_\ell(z) = \sqrt{\frac{\pi}{2z}}\; J_{\ell+1/2}(z)$$

$$n_\ell(z) = (-)^{\ell+1}\sqrt{\frac{\pi}{2z}}\; J_{-\ell-1/2}(z)$$

Sobre la parte angular: $\nabla^2 \longrightarrow (\nabla^2)_{radial} + (\nabla^2)_{ang.}$

[Geometría cilíndrica $(\nabla^2)_{ang}$
[Soluc. $\cos(m\varphi)$, $\operatorname{sen}(m\varphi)$.

Geometría esférica:
↳ simetría axial
↳ sin simetría axial



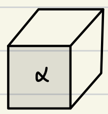
$\nabla^2 \rightarrow (\nabla^2)_{radial} + \underbrace{(\nabla^2)_\theta}$

$\cdot\,\underbrace{P_\ell(\cos\theta)}_{\text{reg}}$, $\underbrace{Q_\ell(\cos\theta)}_{\text{sing.}}$ → Polinomios de Legendre.

$\cdot\, j_\ell(\ \cdot r)\cdot P_\ell(\ )$

$\rightarrow J_\nu(\ )\left(A\cos(\ ) + B\,\text{sen}(\ )\right)$

sin simetría axial:



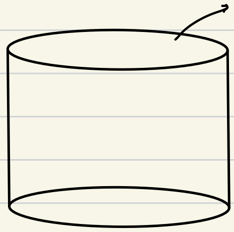
$$\nabla^2 \longrightarrow (\nabla^2{}_r) + \underbrace{(\nabla^2)_{\theta,\varphi}}.$$

$$Y_{\ell m}(\theta,\varphi) \rightarrow F(\theta)\cdot G(\varphi)$$

→ esféricas armónicas.

Problema

Ondas de presión en un tarro cilíndrico



$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

$$\Phi = \Phi(\rho, z, \varphi, t)$$
$$= \chi(\vec{r})\,T(t)$$

$$\rightarrow \frac{1}{\chi}(\nabla^2\chi) - \underbrace{\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2}}_{-\omega^2} = 0$$

$$\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\omega^2 \longrightarrow \frac{\partial^2 T}{\partial t^2} = -\omega^2 T$$

$$\longrightarrow T = T(t)$$

$$\longrightarrow \frac{\partial T}{\partial t} = \frac{dT}{dt} \rightarrow \ddot{T} = -\omega^2 T \longrightarrow T = Ae^{i\omega t}$$

parte espacial:

$$\nabla^2\chi + \frac{\omega^2}{c^2}\chi = 0$$

ρ, z, φ

Def: $\frac{\omega}{c} = k$, $c = \frac{\omega}{k}$

$$\boxed{\nabla^2\chi(\rho,\varphi,z) + k^2\chi = 0}$$

$$\nabla^2\chi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\chi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\chi}{\partial\varphi^2} + \frac{\partial^2\chi}{\partial z^2} + k\chi = 0$$

$$\chi(\rho,\varphi,z) = R(\rho)\,F(\varphi)\,\zeta(z).$$

z

$$\Rightarrow \underbrace{\frac{1}{\rho R}\frac{d}{d\rho}(\rho R') + \frac{1}{\rho^2 F}F''}\ + \underbrace{\frac{\zeta''}{\zeta}}_{p^2\ \text{cte}} + k^2 = 0.$$

$$\left[\begin{array}{l} \zeta'' + k^2\zeta = p^2\zeta \\ \zeta'' = -(k^2 - p^2)\zeta \Rightarrow \zeta(z). \end{array}\right]$$

Multiplicar por ρ^2

$$\frac{\rho}{R}\frac{d}{d\rho}(\rho R') + \boxed{\frac{F''}{F}} + p^2\rho^2 = 0.$$

$$\frac{F''(\varphi)}{F(\varphi)} = \text{cte} = -n^2$$

→ queremos que F sea cíclica (solución después de dar una vuelta sea la misma)

↳ (solución de F del tipo sin y cos).

$$\Rightarrow \frac{\rho}{R}\frac{d}{d\rho}(\rho R') - n^2 + p^2\rho^2 = 0$$

Entonces:

$$\boxed{\rho\frac{d}{d\rho}(\rho R') - n^2 R + p^2\rho^2 R = 0}$$

Definamos $x = p\rho$. , $R(\rho) = J(x)$

$$R'(\rho) = \frac{d}{d\rho}R = \frac{d}{d\rho}(J(x)) = \frac{d}{dx}J(x)\cdot\overbrace{\frac{dx}{d\rho}}^{p} = \boxed{J'\cdot p}$$

Ec. dif. de Bessel.

$$x\frac{d}{dx}\left[x\frac{dJ}{dx}\right] - n^2 J + x^2 J = 0$$

↓ forma gral:

$$x\frac{d}{dx}\left(x\frac{dJ_\nu}{dx}\right) - \nu^2 J_\nu + x^2 J_\nu = 0$$

Planteamos sol. en serie de Potencias:

→ Texto : $R \longrightarrow J_\nu$

$z \longleftarrow x$

∗ $J_\nu(z)$

$$\cdot\ z\frac{d}{dz}\left(z\frac{dJ_\nu}{dz}\right) - \nu^2 J_\nu + z^2 J_\nu = 0$$

sea $J_\nu(z) = z^r\sum_{k=0}^{\infty} a_k z^k \equiv z^r \underbrace{S(z)}_{\text{Texto}}$

$$\frac{dJ_\nu}{dz} = r z^{r-1}\cdot S + z^r S'$$

$$z\frac{dJ_\nu}{dz} = r z^r S + z^{r+1} S'$$

$$\frac{d}{dz}(\ ) = r^2 z^{r-1} S + r z^r S' + (r+1) z^r S' + z^{r+1} S''$$

$$\Rightarrow r^2 z^r S + r z^{r+1} S' + (r+1) z^{r+1} S' + z^{r+2} S'' - \nu^2 z^r S(z) + z^{2+r} S = 0$$

$$\Rightarrow (r^2 - \nu^2) S + \sum_{k=1}^{\infty} (2rk + k^2) a_k z^k - \sum_{k=0}^{\infty} a_k z^{k+2} = 0$$

$$\Rightarrow B_0 + B_1 z + \sum_{2} B_k z^k = 0, \quad \forall z$$

$\Rightarrow B_0 = 0 \quad \rightarrow (r^2 - \nu^2) a_0 = 0$

$B_1 = 0 \quad \rightarrow ((r+1)^2 - \nu^2) a_1$

$B_k = 0 \quad \forall k > 2 \quad \rightarrow (r^2 - \nu^2) a_k + (2rk + k^2) a_k + a_{k+2} = 0.$

de z en z.

cuál es la relaø de recurrencia que surge ?

→ posibles escenarios :

· Si $a_0 \neq 0 \longrightarrow r = \pm \nu \Rightarrow a_1 = 0$. Términos pares participan

↳ $r = +\nu$

$$a_k = -\frac{a_{k-2}}{(2\nu + k)k}$$ ⤳ permite det $a_2, a_4, a_6 \ldots$ a partir de a_0.

Podemos definir los nuevos coef. b_m $(m = 0, 1, 2 \ldots)$ mediante $b_m = a_{2m}$.

$$\Rightarrow b_m = -\frac{b_{m-1}}{4(\nu + m) m}$$

$$\Rightarrow b_m = \frac{(-)^m b_0}{4^m (\nu + m) \cdots (\nu + 1) m!}$$

· $b_0 = \dfrac{1}{2^\nu \Gamma(\nu + 1)} \quad \Longrightarrow \quad b_m = \dfrac{(-)^m}{2^\nu 2^{2m} \Gamma(\nu + m + 1)}$

Así, $R(z) = J_\nu(z) \longrightarrow J_\nu(z) \equiv \displaystyle\sum_{m=0}^{\infty} \frac{(-)^m}{m! \, \Gamma(\nu + m + 1)} \left(\frac{z}{2}\right)^{2m+\nu}$ = funø de Bessel del 1° tipo

↳ $r = -\nu$ se genera $J_{-\nu}(z)$ que es l.i. de $J_\nu(z)$. Si ν es entero positivo $(\nu \to n)$ $\Rightarrow J_{-n}(z) = (-)^n J_n(z)$

Funøes de Neumann def. por : $Y_n(z) \equiv \displaystyle\lim_{\nu \to n} \frac{J_\nu(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi}$

→ L'Hôpital: $Y_n(z) = \dfrac{1}{\pi} \left[\dfrac{dJ_\nu}{d\nu} - (-)^n \dfrac{dJ_{-\nu}}{d\nu} \right]$

Evaluando las deriv. y reordenando :

$$Y_n(z) = \frac{2}{\pi} \left\{ \left[\ln\left(\frac{z}{2}\right) + \gamma \right] J_n(z) \; - \text{serie regular en } z \right\}$$

Aquí se denota: $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots - \ln n\right) \approx 0{,}577$ la cte de Euler.

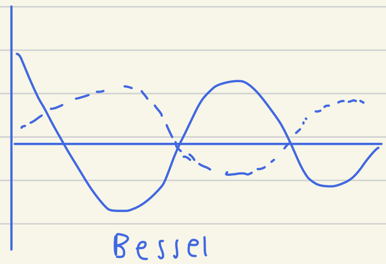
· Para $a_1 \neq 0$; $r+1 = \pm\nu$, con $a_0 = 0$. Sólo se dan términos impares de la serie: $a_1, a_3, a_5 \dots$

↳ $r = \nu - 1$ resolvemos a_k

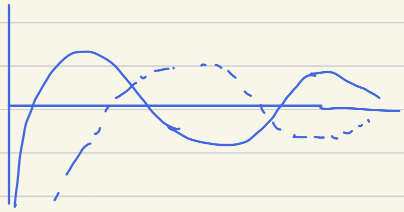
$$a_k = -\frac{a_{k-2}}{(k+r)^2 - \nu^2} \longrightarrow -\frac{a_{k-2}}{(k-1)(2\nu + k - 1)}$$

Haciendo $b_m = a_{2m+1}$, con $m = 0, 1 \dots$ se obtiene

$$b_m = -\frac{b_{m-1}}{4(\nu+m)m}$$



Bessel



* $J_0(0) = 1$
  $n \geq 1$, $J_n(0) = 0$

La singularidad de $Y_0(z)$ es del tipo logarítmica , $n \geq 1$ es del tipo $\sim 1/z^n$.

$$J_0(z) \approx 1 - \left(\frac{z}{2}\right)^2$$

$$J_n(z) \approx \frac{1}{\Gamma(n+1)}\left(\frac{z}{2}\right)^n , \quad n \geq 1$$

$$Y_0(z) \approx \frac{2}{\pi}\left[\ln\left(\frac{z}{2}\right) + \gamma\right]$$

$$Y_n(z) \approx -\frac{(n-1)!}{n}\left(\frac{2}{z}\right)^n \quad n \geq 1$$

3·Julio

Ecuaø dif. de Bessel (cilíndricas)

$$z\frac{d}{dz}\left(z\frac{dJ_\nu}{dz}\right)-\nu^2 J_\nu+z^2 J_\nu=0$$

$z=\beta x$
$J_\nu \to y$

Alternativamente:

$$x^2y''+xy'+\beta^2x^2y-\nu^2y=0 \leftarrow \begin{cases} J_\nu(\beta x) \\ Y_\nu(\beta x)\end{cases}$$

$$x^2y''+xy'-\beta^2x^2y-\nu^2y=0 \leftarrow \begin{cases} J_\nu(i\beta x) \longrightarrow i^{-\nu}J_\nu(i\beta x)=I_\nu(\beta x) \\ Y_\nu(i\beta x) \longrightarrow \frac{\pi}{2}\frac{I_{-\nu}-I_\nu}{\operatorname{sen}\pi\nu}=\underbrace{K_\nu(\beta x)}_{\text{real.}}\end{cases}$$

· se encontró serie de potencias para J_ν:

$$R(z)=z^{r}\ \text{Suma}$$

$z^r \to z^\nu$; Suma $=\sum_{k=0}^{\infty}a_kz^k$

Recurrencia para a_k de 2 en 2.
· coeficientes encontrados en taller 10

$$a_{2k}\to b_k \quad ; \quad b_0\equiv\frac{1}{2^\nu\Gamma(\nu+1)}$$

$$\to J_\nu(z)=\sum_{m=0}^{\infty}\frac{(-)^m}{m!\,\Gamma(\nu+m+1)}\cdot\left(\frac{z}{2}\right)^{2m+\nu}$$

* Γ en el caso de ν entero es el factorial.

Se necesita 2^{da} solución l.i.: → imponemos.

$Y_\nu(z)=J_{-\nu}(z) \longrightarrow$ sólo para ν no entero. * pero Y_n es l.d. de J_ν.

$$=J_{-n}(z)=(-)^nJ_n(z)$$

Para ν entero se define

$$Y_n(z)=\lim_{\nu\to n}\frac{J_\nu(z)\cos(\nu\pi)-J_{-\nu}(z)}{\sin(\nu\pi)}$$

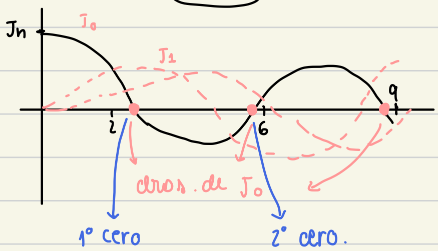
L'Hôpital

$$=\frac{2}{\pi}\left\{\left[\ln\left(\frac{z}{2}\right)+\gamma\right]J_n(z)+\text{serie regular en } t\right\}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln(n)\right)=\gamma=\text{cte de Euler: } 0{,}577216$$

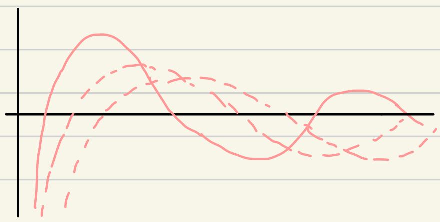
hace que no sea divergente.

b^2 $J_n \; Y_n$ $\qquad$ $I_n \; K_n$ β^2



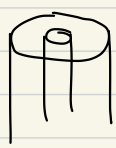
$J_0(\underbrace{z_{0,m}}_{\text{orden del 0}}) = 0$

$Y_n(z) \longrightarrow$ Newmann



I_n, K_n $\longrightarrow$ funções modificadas de Bessel.

→ cables coaxiales:



↓
no tiene ceros

Relações de recurrencia

$$z \cdot \frac{dJ_\nu}{dz} = \nu J_\nu - z J_{\nu+1}$$

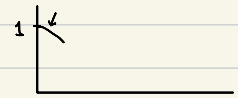
$$z \cdot \frac{dJ_\nu}{dz} = z J_{\nu-1} - \nu J_\nu .$$

$$\rightarrow J_{\nu+1} = \frac{2\nu}{z} J_\nu - J_{\nu-1}$$

$$\frac{d}{dz}\left(z^{-\nu} J_\nu\right) = -z^{-\nu} J_{\nu+1}$$

· Comportam. cerca del origen:

$$J = \begin{cases} J_0(z) \sim \left(1 - \left(\frac{z}{2}\right)^2\right) & ; n = 0 \\ J_n(z) \sim \frac{1}{\Gamma(n+1)} \left(\frac{z}{2}\right)^n & ; n \neq 0 \end{cases}$$



$$Y = \begin{cases} Y_0 \sim \frac{2}{\pi}\left[\underbrace{\ln\left(\frac{z}{2}\right)} + \gamma\right] & ; n = 0 \\ Y_n \approx -\frac{(n-1)!}{n} \cdot \left(\frac{2}{z}\right)^n & , n \neq 0 \end{cases}$$

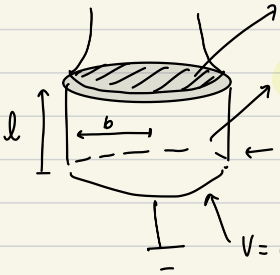
¡sólo las de Bessel (no newmann)!.

Examen: 1° Fourier + Laplace + series
2° Polinomios Legendre.

3hrs.

Laplace en cavidad cilíndrica

→ $V(\rho,\phi,L) = f(\rho,\phi)$
→ $V(\rho,\phi,0) = 0$
→ $V(b,\phi,z) = 0.$



$\nabla^2 V = 0$

$V = V(\rho,\varphi,z) = R(\rho)\,F(\varphi)\,\zeta(z)$

$$\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{F}\frac{1}{\rho^2}\frac{d^2F}{d\varphi^2} + \underbrace{\frac{1}{\zeta}\cdot\frac{d^2\zeta}{dz^2}}_{} = 0.$$

cte indep. ya que sólo depende de z

¡No entra! ← si es negativo → R ↔ modificada de Bessel → singular en $\rho = 0$ ($\to\infty$)
→ no se anula en el origen

$$\frac{d^2\zeta}{dz^2} = k^2\zeta \quad (\text{cte} > 0) \longrightarrow \begin{cases} A\,\text{senh}(kz) \\ B\cosh(kz). \end{cases}$$

→ soluø debe ser 0 en $z = 0$. ($V = 0$).

→ $\zeta(z) = A\sinh(kz) + B\cosh(kz)$

$$\frac{1}{R}\,\rho\,\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{F}\cdot\frac{d^2F}{d\varphi^2} + k^2\rho^2 = 0$$

depende de φ → cte. → entero negativo (si fueran >0 serían hiperbólicos y no es cíclico).

Exigimos

$$\frac{1}{F}\frac{d^2F}{d\varphi^2} = -m^2$$

Para que F sea cíclica. → $F = \begin{cases}\cos n\varphi \\ \text{sen}\, n\varphi.\end{cases}$

$\cdot\, F''/F$ negativo (F cíclica en φ).

$A\cos(m\varphi) + B\,\text{sen}(m\varphi).$

→ $V = R(\rho)\,F(\varphi)\,\zeta(z)$

$\text{senh}(kz)$

Tenemos $\quad \frac{1}{R}\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - m^2 + \rho^2k^2 = 0$

→ Es Bessel $\begin{cases} J_m(k\rho) \\ \cancel{Y_m(k\rho)} \end{cases}$ → spg. $R(\rho) = J_m(k\rho)$

singular en el origen. → zona incluida en región de soluø.

V es nulo para $\rho = b$.

⟹ $J_m(kb) = 0.$ ⟹ $\underbrace{kb = z_{m,n}}$ → raíces de $J_m(z)$

orden del 0.

muchas soluøes. → hay que sumarlos.

$k_{m,n} = \frac{z_{m,n}}{b}$ tabla de los ceros.

⇒ soluØes de la forma:

$$V = \sum_{mn} J_m(k_{m,n}\rho)\left[A_{mn}\cos(m\varphi) + B_{mn}\operatorname{sen}(m\varphi)\right]\sinh(k_{m,n}z).$$

En tapa superior: → Imponer $V(\rho,\varphi,L)=f(\rho,\varphi)$.

$V = f_0(\rho,\varphi)$.

$$f_0(\rho,\varphi) = \sum_{mn} J_m(k_{m,n}\rho)\left[A_{mn}\cos(m\varphi) + B_{mn}\operatorname{sen}(m\varphi)\right]\sinh(k_{m,n}L).$$

/ · $\operatorname{sen}(m'\varphi)$, $\int$

· $\cos(m'\varphi)$, $\int$.

Métodos Matemáticos de la Física II

Curso 2004-2005, grupo I, Pedro López Rodríguez

Primeros 15 ceros de las primeras 20 funciones de Bessel
de primera especie con 4 decimales exactos

	Cero 1	Cero 2	Cero 3	Cero 4	Cero 5	Cero 6	Cero 7	Cero 8	Cero 9	Cero 10	Cero 11	Cero 12	Cero 13	Cero 14	Cero 15
J_0	2,40483	5,52008	8,65373	11,7915	14,9309	18,0711	21,2116	24,3525	27,4935	30,6346	33,7758	36,9171	40,0584	43,1998	46,3412
J_1	3,83171	7,01559	10,1735	13,3237	16,4706	19,6159	22,7601	25,9037	29,0468	32,1897	35,3323	38,4748	41,6171	44,7593	47,9015
J_2	5,13562	8,41724	11,6198	14,796	17,9598	21,117	24,2701	27,4206	30,5692	33,7165	36,8629	40,0084	43,1535	46,298	49,4422
J_3	6,38016	9,76102	13,0152	16,2235	19,4094	22,5827	25,7482	28,9084	32,0649	35,2187	38,3705	41,5207	44,6697	47,8178	50,965
J_4	7,58834	11,0647	14,3725	17,616	20,8269	24,019	27,1991	30,371	33,5371	36,699	39,8576	43,0137	46,1679	49,3204	52,4716
J_5	8,77148	12,3386	15,7002	18,9801	22,2178	25,4303	28,6266	31,8117	34,9888	38,1599	41,3264	44,4893	47,6494	50,8072	53,963
J_6	9,93611	13,5893	17,0038	20,3208	23,5861	26,8202	30,0337	33,233	36,422	39,6032	42,7785	45,949	49,1158	52,2795	55,4406
J_7	11,0864	14,8213	18,2876	21,6415	24,9349	28,1912	31,4228	34,6371	37,8387	41,0308	44,2154	47,3942	50,5682	53,7383	56,9052
J_8	12,2251	16,0378	19,5545	22,9452	26,2668	29,5457	32,7958	36,0256	39,2404	42,4439	45,6384	48,8259	52,0077	55,1847	58,3579
J_9	13,3543	17,2412	20,807	24,2339	27,5837	30,8854	34,1544	37,4001	40,6286	43,8438	47,0487	50,2453	53,4352	56,6196	59,7993
J_{10}	14,4755	18,4335	22,047	25,5095	28,8874	32,2119	35,4999	38,7618	42,0042	45,2316	48,4472	51,6533	54,8516	58,0436	61,2302
J_{11}	15,5898	19,616	23,2759	26,7733	30,1791	33,5264	36,8336	40,1118	43,3684	46,6081	49,8347	53,0505	56,2576	59,4575	62,6512
J_{12}	16,6982	20,7899	24,4949	28,0267	31,46	34,83	38,1564	41,4511	44,7219	47,9743	51,212	54,4378	57,6538	60,8618	64,0629
J_{13}	17,8014	21,9562	25,7051	29,2706	32,7311	36,1237	39,4692	42,7804	46,0657	49,3308	52,5798	55,8157	59,0409	62,2572	65,4659
J_{14}	18,9	23,1158	26,9074	30,506	33,9932	37,4082	40,7728	44,1006	47,4003	50,6782	53,9387	57,1849	60,4194	63,6441	66,8605
J_{15}	19,9944	24,2692	28,1024	31,7334	35,2471	38,6843	42,0679	45,4122	48,7265	52,0172	55,2892	58,5458	61,7898	65,0231	68,2473
J_{16}	21,0851	25,417	29,2909	32,9537	36,4934	39,9526	43,3551	46,7158	50,0446	53,3483	56,6319	59,899	63,1524	66,3944	69,6267
J_{17}	22,1725	26,5598	30,4733	34,1673	37,7327	41,2136	44,6348	48,012	51,3553	54,6719	57,9671	61,2448	64,5078	67,7586	70,9989
J_{18}	23,2568	27,6979	31,6501	35,3747	38,9654	42,4678	45,9077	49,3011	52,6589	55,9885	59,2954	62,5836	65,8563	69,1159	72,3644
J_{19}	24,3382	28,8317	32,8218	36,5765	40,1921	43,7157	47,174	50,5837	53,9559	57,2984	60,617	63,9158	67,1982	70,4668	73,7234

RelaØ de ortogonalidad: Necesaria

$\boxed{\text{pp 128:}}$

$$(\alpha-\beta^2)\int_0^R \underbrace{x^2 J_\nu(\alpha x) J_\nu(\beta x)}_{0 \text{ si } \alpha\neq\beta} = 0 \quad , \alpha \neq 0.$$

↳ despeje coeficientes si $\alpha\neq\beta$.

$$\int_0^R x J_\nu^2(\alpha x) = \left[\frac{R^2}{2}\right]\left\{J_\nu'(z)^2 - \frac{1}{z}J_\nu(z)(zJ)'\right\}$$

Identidades para n y n' enteros:

$$\int_{-\pi}^{\pi}\cos(m\varphi)\cos(m'\varphi)\,d\varphi = \pi\delta_{mm'} \qquad \int_{-\pi}^{\pi}\sin(m\varphi)\sin(m'\varphi)\,d\varphi = \pi\delta_{mm'}$$

$$\int_{-\pi}^{\pi}\sin(m\varphi)\cos(m'\varphi)\,d\varphi = 0.$$

. Despejando A_{nm} y B_{nm}, mult. por $\cos(m'\varphi)$ e integramos φ:

$$\int_{-\pi}^{\pi}\cos(m\varphi)f(\rho,\varphi)\,d\varphi = \pi\sum_n A_{nm}J_m(k_{nm}\rho)\sinh(k_{nm}L).$$

. Haciendo uso la ec. de la ortogonalidad, multiplicando por $\rho J_m(k_{n'm}\rho)$ e integrando $\int_0^b d\rho$...

$$\int_0^b \rho\, d\rho\, J_m(k_{nm}\rho)\int_{-\pi}^{\pi} d\varphi\,\cos(m\varphi)\,f(\rho,\varphi) = \pi A_{nm}\sinh(k_{nm}L)\,\frac{b^2}{2}J_{m+1}^2(k_{nm}b).$$

Así,

$$A_{nm} = \frac{2\int_{-\pi}^{\pi}d\varphi\int_0^b \rho\, d\rho\, f(\rho,\varphi)\cos(m\varphi)J_m(k_{nm}\rho)}{\pi b^2\sinh(k_{nm}L)J_{m+1}^2(k_{nm}b)}$$

B_{nm} de forma análoga sustituyendo $\cos(m\varphi) \to \sin(m\varphi)$.

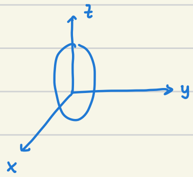
5.4 **Ecuación de Helmoltz** 5 julio

$$\nabla^2 \Psi + k^2 \Psi = 0$$

$$\downarrow$$

$$-\nabla^2 \Psi = k^2 \Psi$$

· simetría axial:



$\psi = \Psi(r,\theta) = R(r)\, P(\theta)$ $\longrightarrow$ $\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + (k^2 r^2 - \lambda) R = 0 \rightarrow$ esféricas de Bessel

$\frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \lambda P = 0 \rightarrow$ polinomios de Legendre.

$$\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2}\left[\frac{1}{\operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\frac{\partial\psi}{\partial\theta}\right)\right]\right] + k^2\psi = 0$$

$$\downarrow \qquad \psi = R\cdot P.$$

$$r\frac{\partial^2}{\partial r^2}(r\psi) + k^2 r^2 \psi + \underbrace{\frac{1}{\operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial\psi}{\partial\theta}\right)}_{cte} = 0$$

$$\frac{1}{\operatorname{sen}\theta}\frac{d}{d\theta}\left(\operatorname{sen}\theta\frac{dP}{d\theta}\right) = A$$

$\hookrightarrow \mu = \cos\theta$

$\hookrightarrow \frac{d}{d\theta} = \underbrace{\left(\frac{du}{d\theta}\right)}_{-\operatorname{sen}\theta}\frac{d}{du}$

$$\cancel{\frac{1}{\operatorname{sen}\theta}}(\cancel{-\operatorname{sen}\theta})\frac{d}{du}\left(\operatorname{sen}\theta \cancel{(-)}\operatorname{sen}\theta\frac{dP}{du}\right) = -\lambda P.$$

$$\rightarrow \boxed{\frac{d}{du}\left((1-u^2)\frac{dP}{du}\right) = -\lambda P.}$$

$$\frac{d}{du}\left[(1-u^2)\frac{dP}{du}\right] = (r-1)\, u^{r-2}\sum_{k=0} a_k u^k + r u^{r-1}\sum_{k=1} k a_k u^{k-1}$$

$$+ r u^{r-1}\sum_{k=1}^{\infty} k a_k \cdot u^{k-1} + u^r \sum_{k=1}^{\infty} k(k-1)\, a_k u^{k-2}$$

$$-r(r+1)u^r \sum_{k=0}^{\infty} a_k u^k - r u^{r+1}\sum_{k=1}^{\infty} k a_k u^k$$

$$- r u^{r-1}\sum_{k=1}^{\infty} k a_k u^{k+1} - u^r \sum_{k=}^{\infty}(k+1)\, k a_k u^k.$$

$$\cdot\ \frac{d}{du}\left[(1-u^2)\frac{dP}{du}\right] + \lambda P = 0.$$

ecuación de Legendre

$$\Rightarrow \boxed{(1-u^2)\frac{d^2P}{du^2} - 2u\frac{dP}{du} + \lambda P = 0.}$$

$\rightarrow a_0\, r(r-1) = 0$

$a_1\, r(r+1) = 0$

sea $P = u^r \sum_{k=0}^{\infty} a_k u^k$ $\longrightarrow$ Mismo que para Bessel (relaø de recurrencia para a_k).

$*\quad a_{k+2} = \left(\ \text{—}\ \right) a_k$ $\qquad \rightarrow a_{k+2} = \frac{(k+r)(k+r-1) + (k+r) - \lambda}{(k+r+1)(k+r+2)}\, a_k.$

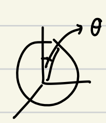
se observa que para k muy grande: $a_{k+2} \sim a_k \Rightarrow \frac{a_{k+2}}{a_k} \rightarrow 1$ $\rightarrow$ la serie de P diverge.

$\Rightarrow$ Soluø divergente para $u = 1$, o sea $\theta = 0$. No aceptable

sin embargo, la serie se trunca si es que

$\lambda = \underbrace{(\text{entero})}_{\ell}\underbrace{(\text{entero}+1)}_{\ell+1}.$ $\rightarrow \lambda = \ell(\ell+1)$

$\ell \downarrow$ momento angular orbital.



$\mu = \cos\theta$

$\rightarrow \mu \rightarrow 1 \Rightarrow \theta \rightarrow 0$

Hay una singularidad.

$$\rightarrow (1-u^2)\frac{d^2P_\ell}{du^2} - 2u\frac{dP_\ell}{du} + \ell(\ell+1)P_\ell = 0$$

soluciones son $P_\ell(u)$

$P_0(u) = 1 \rightarrow$ monopolo.

$P_1(u) = u \rightarrow$ dipolo

$P_2 = \frac{1}{2}(3u^2-1)$

$P_3 = \frac{1}{2}(5u^3-3u)$.

$P_\ell(-u) = (-)^\ell P_\ell(u)$

✓

discriminar transiØes prohibidas en niveles atómicos.

$$\boxed{P_\ell(u) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{du^\ell}\left[(u^2-1)^\ell\right]}$$

* fórmula:

$$P'_{\ell+1}(u) - uP_\ell = (\ell+1)P_\ell(u).$$

Propiedades:

$$\int_{-1}^{1} P_\ell(u)\,P_{\ell'}(u)\,du = \frac{2}{(2\ell+1)}\delta_{\ell\ell'} \rightarrow \text{relaØ de ortogonalidad}$$

FunØ generatriz:

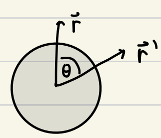
$$\frac{1}{\sqrt{1+t^2-2ut}} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(u).$$

$\longrightarrow$ en física:

$$\frac{1}{|\vec r - \vec r'|} = \frac{1}{\sqrt{(\vec r-\vec r')\cdot(\vec r-\vec r')}} = \frac{1}{\sqrt{r^2+r'^2-\underbrace{2\vec r\cdot\vec r'}_{rr'\cos\theta \;\; rr'u.}}}$$

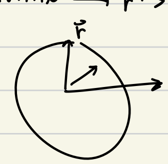
$\downarrow$

potencial electrostático.



$$\frac{1}{|\vec r-\vec r'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'u}}$$

$\rightarrow$ distinØ $r'<r$, $r'>r$



* $r'<r$ $\rightarrow$ para que $t<1$ y diverge la serie.

$$= \frac{1}{r\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}u}} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^\ell P_\ell(u)$$

$r'>r$

$$= \frac{1}{r}\frac{1}{\sqrt{1+\left(\frac{r}{r'}\right)^2-2\left(\frac{r}{r'}\right)u}} = \frac{1}{r'}\sum_{\ell=0}^{\infty}\left(\frac{r}{r'}\right)^\ell P_\ell(u).$$

$$= \boxed{\frac{1}{r_>}\sum\left(\frac{r_<}{r_>}\right)^\ell P_\ell(u)}$$

→ 4 escenarios que surgen de $r(r-1)a_0 = 0$

$r(r+1)a_1 = 0$.

- Si $a_0 \neq 0 \Rightarrow r = 0, 1$, sumatoria mult. por 1 o u, con coef. $\{a_0, a_2, a_4 \ldots\}$
- Si $a_1 \neq 0 \Rightarrow r = 0, -1$, " " " 1 o $1/u$, " " $\{a_1, a_3, a_5 \ldots\}$

Sólo 2 de ellas independientes → series de potencias impares en u.
Obtenidas exigiendo $r=0$. Así:

$$a_{k+2} = \frac{k(k+1) - \ell(\ell+1)}{(k+1)(k+2)} a_k.$$

$$P(u) \to P_\ell(u) = \sum_{k=0}^{K} \frac{(-)^k (2\ell-2k)!}{2^\ell k!(\ell-k)!(\ell-2k)!} u^{\ell-2k}.$$

donde $\ell = 2K$ para ℓ par, $\ell = 2K+1$ para ℓ impar

$$\frac{d^\ell (u^{2\ell-2k})}{du^\ell} = \frac{(2\ell-2k)!}{(\ell-2k)!} u^{\ell-2k}$$

Es posible dem, usando la fórmula binomial que:

$$P_\ell(u) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{du^\ell}\left[(u^2-1)^\ell\right]$$

fórmula de Rodrigues:

$P_0(u) = 1$

$P_1(u) = u$

$P_2 = \frac{1}{2}(3u^2-1)$

$P_3 = \frac{1}{2}(5u^3-3u)$.

$$\boxed{P_\ell(u) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{du^\ell}\left[(u^2-1)^\ell\right]}$$

Propiedades:

$$\int_{-1}^{1} P_\ell(u) P_{\ell'}(u)\, du = \frac{2}{(2\ell+1)} \delta_{\ell\ell'} \xrightarrow{\text{dem}} \frac{d}{du}\left[(1-u^2) P_m P_\ell' - (1-u^2) P_\ell P_m'\right]$$
$$+ \left[\ell(\ell+1) - m(m+1)\right] P_\ell P_m = 0$$

Integrando $\int_{-1}^{1} du \ldots$ se encuentra.

$$\left[\ell(\ell+1) - m(m+1)\right] \int_{-1}^{1} du\, P_\ell P_m = 0$$

Propiedades:

- $(1-u^2)P_l'' - 2uP_l' + l(l+1) = 0$
- $\frac{d}{du}\left[(1-u^2)\frac{dP_l}{du}\right] + l(l+1)P_l = 0$
- $(l+1)P_{l+1} - (2l+1)uP_l + lP_{l-1} = 0$
- $P'_{l+1} - uP'_l - (l+1)P_l = 0.$

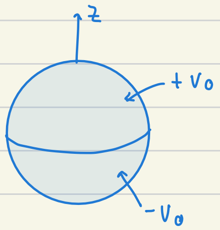
$\rightarrow P'_{l+1} - P'_{l-1} - (2l+1)P_l = 0$

$P_l(-u) = (-)^l P_l(u)$

$\rightarrow P_0 = 1$
$P_1 = u$

Ecuaø de Laplace en una cavidad esférica:

Ej



$$\nabla^2\Phi = 0$$

$$\Phi = \Phi(r,\theta)$$

$$= R(r)\,P_l(\cos\theta).$$

$$* \quad \underbrace{\frac{1}{\text{sen}\theta}\frac{d}{d\theta}\left(\text{sen}\,\frac{\partial P}{\partial\theta}\right)}_{-\lambda P}$$

$\cdot(O_p \cdot \text{dif})P = \lambda P. \longrightarrow$ autofunø

$$\frac{1}{R}\frac{1}{r}\frac{\partial^2}{\partial r^2}(rR) + \frac{1}{P}\frac{1}{r^2}\frac{1}{\text{sen}\theta}\frac{\partial}{\partial\theta}\left(\text{sen}\theta\,\frac{\partial P}{\partial\theta}\right) = 0$$

$$\frac{1}{R}r\frac{\partial^2}{\partial r^2}(rR) + \frac{1}{P}\underbrace{\frac{1}{\text{sen}\theta}\frac{\partial}{\partial\theta}\left(\text{sen}\theta\,\frac{\partial P}{\partial\theta}\right)}_{-\lambda P = -l(l+1)\cdot P} = 0.$$

$$* \quad \boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)} = \frac{1}{r^2}\left(2r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2}\right)$$

$$= \frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} \checkmark$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2} \sim \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}r\right)\right]$$

$$= \frac{\partial}{\partial r}\left(1 + r\frac{\partial}{\partial r}\right)$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r} + \frac{\partial}{\partial r} + r\frac{\partial^2}{\partial r^2}\right)$$

$$\frac{1}{R}r\frac{\partial^2}{\partial r^2}(rR) - l(l+1) = 0$$

Resolvemos R

$$r\frac{\partial^2}{\partial r^2}(rR) - l(l+1)R = 0$$

$R = r^p \longrightarrow$ probando $R = r^p$

$$\frac{\partial}{\partial r}(rR) = \frac{\partial}{\partial r}(r^{p+1}) = (p+1)\cdot r^p$$

$$\frac{\partial^2}{\partial r^2}(rR) = p(p+1)r^p.$$

$\rightarrow p(p+1) - l(l+1) = 0.$

Ec cuadrática para p: sol$_1 \longrightarrow p = l$

sol$_2 \longrightarrow p = -(l+1)$

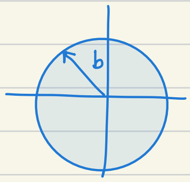
$$p(p+1) = -(l+1)(-l-1+1).$$

Hay 2 soluções para R:

$R \to r^{\ell}$

$R \to \frac{1}{r^{\ell+1}}$

$$\boxed{\Phi = \sum_{\ell}\left(A_\ell r^{\ell} + B_\ell \cdot \frac{1}{r^{\ell+1}}\right) \cdot P_\ell(u).}$$ //



En el interior : $B_\ell = 0$

" el exterior : $A_\ell = 0$ $\longrightarrow$ soluø que decaiga.

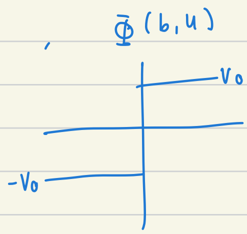
<u>En el exterior.</u>

$$\Phi = \sum_\ell B_\ell \frac{1}{r^{\ell+1}} P_\ell(\overset{\cos\theta}{u})$$

$$\Phi(b,\theta) = \begin{cases} +V_0 & \theta,\ 0 \to \pi/2 \equiv u\,[1,0] \\ -V_0 & \theta,\ \frac{\pi}{2} \to \pi \equiv u\,[0,-1]. \end{cases}$$

$$\Phi(b,u) = \sum_\ell \boxed{B_\ell} \frac{1}{b^{\ell+1}} P_\ell(u)$$

para B_ℓ : ortogonalidad



Ortogonalidad:

$$\int_{-1}^{1} P_\ell(u) \cdot P_m(u) = \frac{2}{2\ell+1} \delta_{\ell m}$$

$$\int_{-1}^{1} \Phi(\overset{\text{manto}}{b},u)\, P_m(u)\, du = \sum_\ell B_\ell \frac{1}{b^{\ell+1}} \underbrace{\int_{-1}^{1} P_\ell(u) P_m(u)\, du}_{\frac{2\delta_{\ell m}}{\ell+1}}.$$

$$B_m = \frac{2m+1}{2} b^{\ell+1} \underbrace{\int_{-1}^{1} \Phi(b,u)\, P_m(u)\, du}_{-\int_{-1}^{0} V_0 \cdot P_m + \int_{0}^{V} V_0 \cdot P_m}$$

$\rightarrow A_\ell = \frac{2\ell+2}{2b^\ell}\int_{-1}^{1} f_0(u)\,P_\ell(u)\,du$ $\qquad$ Origen: $\rightarrow f_0(u) = \sum_{\ell=0}^{\infty} A_\ell\, b^\ell\, P_\ell(u)$.

Caso $f_0(u) = \mathrm{sign}(u)\, V_0/2$, impar en u, los coef. de orden par se anulan. Porque $P_\ell(u)$ tiene paridad $(-)^\ell$.

$$A_{2k+2} = V_0\,\frac{2k+3/2}{b^{2k+1}}\int_0^1 P_{2k+1}(u)\,du.$$