QC-CNN Image Segmentation *

Abstract. This project joint-force the power of deforming templates by quasi-conformal geometry and the power of convolutional neural network (CNN) to achieve a high quality segmentation of medical images with shape/topological prior.

Key words. image segmentation, prior, neural network, medical imaging

AMS subject classifications.

1. Introduction.

(Introduction)

The paper is organized as follows.

2. Previous Works.

In this section, previous works closely related to the tools used in the proposed framework are briefly reviewed after a short survey on general image segmentation.

2.1. Image Segmentation and Registration.

The snake model, or active contour model, for segmentation was first introduced in [22] and has been improved since its inception in terms of capture range [11, 46] and removal of dependency on parametrization [24, 4]. A thorough survey on active contour models may be found in [2]. By representing the contour by a level set of a function (very often a *signed-distance function*, i.e. a function with unit gradient) rather than parametrizing the contour explicitly, Chan-Vese model allows for topological change in the contour [9]. Improvements to the model have been made in [36, 10]. A thorough survey on Chan-Vese model may be found in [7].

Many different methods have been developed for image registration, like feature-based methods [17, 15] and mutual information-based methods [45, 39]. A thorough survey may be found in [49]. In [41, 44], the non-parametric registration problem was solved by morphing one image to the other by a vector field, and the action is ensured to be diffeomorphic in the latter paper.

Segmentation and registration are interrelated. In particular, segmentation may be guided by registration by registering with a template [48, 27, 21].

2.2. Segmentation with Prior.

Prior knowledge is often useful for guiding segmentation. Topological-prior segmentation [19, 40, 26] ensures the segmented object has the prescribed topology. For stronger priors, the statistics of control points of snakes, and statistics of level-set functions or template functions are used in [14] and [29, 13, 8] respectively.

Less stringent than these geometrical priors is the prior of convexity. It was proposed in [38] to segment convex regions by prescribing a set of sufficiently dense set of orientations, as well as their orthogonal complements, and partition the image into regions with linear boundaries with the prescribed orientations, such that the central region will be convex. A graph theory-based method was proposed in [35] to enforce a graph-based discrete convexity constraint via imposing a linear inequality for each path in each segmented region. Gorelic

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et al. [18] proposed penalizing the number of triplets of collinear pixels (p, q, r) such that p and r lies inside the reigon, while q lies between p and r but outside the region. In [1], the prior is enforced by penalizing the L^1 norm of the curvature, which necessitates solving a high order PDE. In [47], the prior is enforced by restricting from the space of all level-set functiosn to that of subharmonic signed-distance functions, whose level sets are all convex. This is implemented in an alternating manner.

2.3. Discrete Conformal Geometry. Conformal maps be approximated in discrete settings [30, 16, 33, 20], say by approximating Cauchy-Riemann equation. Alternative to the equation-solving approach is the circle packing approach, which is based on the principle that conformal maps map infinitesimal circles to infinitesimal circles. Circle packing was first proposed in [42] for theoretical study of manifolds, and discretized in [43], and implemented in [12]. Circle Pattern, which allows trasversely intersecting circles, is a more relaxed setting for computation. Its theory, in the form of dihedral angles[1], was first proposed in [34] for the study of Euclidean simplicial surfaces with cone-like singularities and was extended in [28, 3]. It was applied in [23] for mesh flattening. A similar framework was used in [37] for texture mapping. Further details about dihedral angles may be found in Chapter **??**.

Allowing, and accounting for, conformal distortion in a discrete map gives rise to discrete quasiconformal geometry. It has been applied in diverse setting ranging from shape analysis [32, 5] and map compression [31] to registration [25] and segmentation with topological prior [6]. Quasiconformal geometry will be reviewed in subsection 3.1.

3. Mathematical Background.

In this section, the mathematical tools pertinent to the proposed framework are reviewed.

3.1. Quasi-Conformal Geometry. The foundation of the proposed framework is based on quasi-conformal geometry. Quasi-conformal maps generalize conformal maps. Given a domain $\Omega \subset \mathbb{C}$, a mapping $f : \Omega \to \mathbb{C}$ is said to be quasi-conformal if there exists a Lebesgue-measurable $\mu : \Omega \to \mathbb{C}$ such that

$$\frac{\partial f}{\partial \bar{z}}(z) = \mu(z)\frac{\partial f}{\partial z}(z) \tag{3.1}$$

and

$$||\mu||_\infty < 1. \tag{3.2}$$

(3.1) is called the Beltrami equation, and μ is called the Beltrami coefficient of f. Roughly speaking, quasi-conformal maps are orientation-preserving homeomorphisms between Riemann surfaces with a bounded conformality distortion, and the distortion can be effectively controlled using the Beltrami coefficient μ of f. The following theorem, whose proof can be found in [31], is a well-established explanation of the relationship between a mapping f and its Beltrami coefficient μ:

Theorem 3.1. *Given a domain $\Omega \subset \mathbb{C}$, let $f : \Omega \to \mathbb{C}$ be a mapping, by defining*

$$\mu(z) = \lim_{\hat{z} \to z} \left(\frac{\partial f}{\partial \bar{z}}(\hat{z}) \Big/ \frac{\partial f}{\partial z}(\hat{z}) \right), \tag{3.3}$$

[1] Circle pattern is phrased in terms of the supplement of dihedral angles.

then $||\mu||_\infty < 1$ *if and only if* f *is an orientation-preserving homeomorphism.*

Therefore, in particular, any diffeomorphic deformation on Ω must be a quasi-conformal mapping. This provides an alternative interpretation of diffeomorphic deformations, that requiring a deformation to be diffeomorphic is equivalent to requiring its Beltrami coefficient to have sup norm strictly less than 1.

In the infinitesimal scale, a quasi-conformal mapping f on Ω has its local parametric expression as

$$f(z) \approx f(0) + f_z(0)z + f_{\bar{z}}(0)\bar{z} = f(0) + f_z(0)(z + \mu(0)\bar{z}). \tag{3.4}$$

From the expression (3.4), it can be seen that the non-conformal part of f comes from the term $D(z) = z + \mu(0)\bar{z}$ which is essentially contributed by the Beltrami coefficient μ of f only. Indeed, the Beltrami coefficient μ has a one-to-one correspondence with the quasi-conformal mapping f. Given a quasi-conformal mapping f, its Beltrami coefficient can be uniquely determined by (3.1). The converse is guaranteed by the following famous theorem:

Theorem 3.2 (Measurable Riemann Mapping Theorem). *Suppose* $\mu : \mathbb{C} \to \mathbb{C}$ *is Lebesgue measurable satisfying* $||\mu||_\infty < 1$*, then there exists a quasi-conformal mapping* $f : \mathbb{C} \to \mathbb{C}$ *in the Sobolev space* $W^{1,2}$ *that satisfies the Beltrami equation in the distribution sense. Furthermore, assuming that the mapping is stationary at* $0, 1$ *and* ∞*, then the associated quasi-conformal mapping* f *is uniquely determined.*

The existence and uniqueness of the corresponding quasi-conformal mapping f from a given admissible Beltrami coefficient μ is not just guaranteed in theory. In practice, given a Beltrami coefficient $\mu : \Omega \to \mathbb{C}$ with sup-norm strictly less than 1, the corresponding quasi-conformal mapping f can be explicitly determined by the *Linear Beltrami Solver* (LBS), whose details may be found in [31]. With LBS, deformation can be directly controlled by perturbing the Beltrami coefficient and hence deformations can now be prescribed to be diffeomorphic.

Now, let us consider the composition of quasi-conformal maps. Given two quasi-conformal mappings $f, g : \Omega \to \Omega$, by using μ_f and μ_g to denote their Beltrami coefficients respectively, the Beltrami coefficient $\mu_{g\circ f}$ of the composite mapping $g \circ f$ is given by

$$\mu_{g\circ f} = \frac{\mu_f + (\mu_g \circ f)\tau}{1 + \bar{\mu}_f(\mu_g \circ f)\tau}, \quad \tau = \frac{\bar{f}_z}{f_z}. \tag{3.5}$$

This provides a more convenient way to compute the composition of diffeomorphisms. That is, using the above notations, the composite mapping $g \circ f$ on Ω can be obtained by applying the Linear Beltrami Solver on the its Beltrami coefficient $\mu_{g\circ f}$ which can be directly determined via (3.5).

4. Proposed Model.

In this section, the proposed model is to be explained in detail.

Let $\Omega \subset \mathbb{R}^2$ be the rectangular image domain, $I : \Omega \to [0, 1]^2$ be the target image. The target object lies in a region $D \subset \Omega$ and is revealed in the image I. Suppose prior knowledge (e.g. topology, geometry) of the target object is given, a template image $J : \Omega \to [0, 1]^2$ can be constructed to reveal the prior knowledge. The template image J usually contains a simple object (e.g. disk, ellipse, rectangle, etc.) with known boundary. In particular, there is a pre-defined region $\hat{D} \subset \Omega$ such that

$$J = \begin{cases} c_1 & \text{if } z \in \hat{D}, \\ c_2 & \text{if } z \notin \hat{D}. \end{cases}$$

The strategy is to deform the template image by a diffeomorphic mapping $f : \Omega \to \Omega$ such that $I \circ f = J$, and therefore D can be captured by $D = f(\hat{D})$.

To construct the function f to capture the target object D, the proposed model utilizes the power of the quasi-conformal geometry to gradually construct a diffeomorphism f such that the corresponding Beltrami coefficient $\mu : \mathbb{C} \to \mathbb{C}$ satisfies $||\mu||_\infty < 1$, assisted by a pre-trained CNN for segmentation to guide the construction of f in the process. In particular, our proposed model is formulated as a variational model:

$$f = \underset{g:\Omega\to\Omega}{\arg\min}\, E(\mu(g)),$$

where

$$E(\mu(g)) = \int_\Omega (|\mu|^2 + \alpha(I \circ g^\mu - J)^2 + \beta|\nabla\mu|^2)dz + \gamma||C \otimes D(g^\mu) - C||_2^2.$$

Here, $g = g^\mu$ is the diffeomorphic mapping corresponding to μ, α, β, γ are weighting parameters, C is a $n \times n$ matrix, $D : \{g : \Omega \to \Omega\} \to [0, 1]^{n\times n}$ maps g to a $n \times n$ matrix, and $\otimes$ is the Hadamard product such that

$$(A \otimes B)_{ij} = (A)_{ij}(B)_{ij}.$$

for any matrices A, B of the same size. The matrices C, D are corresponded to a pre-trained segmentation network $\hat{p} : I \to [0, 1]^2$.

Given a segmentation CNN $\hat{p}$, the technique of truncating network is utilized to extract a feature vector from each receptive field of an image. That is, a network p is constructed by truncating the last few layers of $\hat{p}$, leaving a network that outputs a hidden layer in the original network. Given an image I, it is firstly cut into overlapping patches with constant size $h \times w$. Each patch is then passed to the truncated network p, giving a feature vector of size d by vertical stacking of the output (hidden) layer. That is, p is a mapping from each image patch of size $h \times w$ to $\mathbb{R}^d$.

In the variational model, while the first term is designed to construct a diffeomorphism f matching J to I, the last term is a novel fidelity term giving a descent direction to help drive the whole segmentation process based on a pre-trained segmentation network. The variables involved in defined as follows.

Given the truncated network h, the matrix $\widetilde{C}$ is defined to correspond each patch:

$$(\widetilde{C})_{ij} = \left\langle \frac{p(x_i^1)}{|p(x_i^1)|}, \frac{p(x_j^2)}{|p(x_j^2)|} \right\rangle.$$

where x_i^1, x_j^2 are the centers of the image patches i on target image I and j on template image J respectively. Each row of $\widetilde{C}$ is normalized to form a matrix $\hat{C}$:

$$(\hat{C})_{ij} = \frac{\widetilde{C}_{ij} - \mu_i}{\sigma_i},$$

where μ_i and σ_i are the mean and standard deviation of row i of $\widetilde{C}$ respectively. For medical images, it is often to see that some rows of the given image have identical intensity values (black background for example). To deal with these rows, a $n \times n$ elimination matrix E is defined by

$$(E)_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 0, & \text{if } i = j \text{ and row } i \text{ is not unique in } \hat{C}, \\ 1, & \text{if } i = j \text{ and row } i \text{ is unique in } \hat{C} \end{cases}.$$

Therefore, the background rows in $\hat{C}$ can be removed by

$$\acute{C} = E\hat{C}.$$

Finally, the matrix C in the variational model is obtained by sparsifying $\acute{C}$, that is, for each row in $\acute{C}$, all but the largest entry are re-set as 0.

Then, the matrix D is defined to be a distance measuring function between the target image and the deformed template image:

$$(D(g^\mu))_{ij} = \exp\left(-\frac{||g(x_i^1) - x_j^2||_2^2}{\sigma^2}\right).$$

where σ is a parameter.

Now, by the Hadamard product, the last term in the variational model actually reads:

$$||C \otimes D(g^\mu) - C||_2^2 = \sum_{ij} C_{ij}((D(g^\mu))_{ij} - 1)^2.$$

Note that $(D(g^\mu))_{ij} = 1$ if and only if $g(x_i^1) = x_j^2$, i.e. g^μ maps the patch i on the template image to patch j on the target image. Also, in this formulation, if $(C)_{ij}$ is large, i.e. the correlation between the two patches is high as determined by the pre-trained network, then $(D(g^\mu))_{ij}$ should be close to 1. Otherwise if $(C)_{ij}$ is small, the requirement for $(D(g^\mu)) = 1$ can be relaxed.

In solving the variational model, the descent direction $df : \{x_i^1\}_{i=1}^{n^2}$ of the network fidelity term is

$$df(x_i^1) = \frac{4}{\sigma^2}\sum_j (C)_{ij}\left(\exp\left(-\frac{||g^\mu(x_i^1) - x_j^2||_2^2}{\sigma^2}\right) - 1\right)(g^\mu(x_i^1) - x_j^2).$$

Therefore, if $(C)_{ij} \approx 0$, i.e. the pre-trained network determines that patch i on the template image is not correlated to patch j on the target image, the corresponding descent direction for the (i, j) pair will be close to 0. In other words, the Hadamard product allows the elimination of unwanted descent directions based on prior estimation given by the pre-trained network, which is encoded in the matrix C.

5. Numerical Implementation.

By splitting variables, the variational model can be written as

$$E(\nu, f) = \int_\Omega (|\nu|^2 + \alpha(I \circ g^\mu - J)^2 + \beta|\nabla\nu|^2 + \lambda|\nu - \mu(g)|^2)dz + \gamma||C \otimes D(g) - C||_2^2.$$

That is, the mapping $g = g^\mu$ should have its Beltrami coefficient closing to ν, which is altered by the variational model to construct the desired deformation.

Fixing $g = g_n$, it suffices to minimize

$$E(\nu, g_n) = \int_\Omega (|\nu|^2 + \beta|\nabla\nu|^2 + \lambda|\nu - \mu(g_n)|^2)dz.$$

By the Euler-Lagrange equation, the minimizer of the above energy is given by solving the equation

$$(2I - \beta\Delta + 2\lambda I)\nu_{n+1} = 2\lambda\mu(g_n).$$

This is a sparse system and can be solved directly.

Fixing $\nu - \nu_n$, it suffices to minimize

$$E(\nu_n, g) = \int_\Omega (\alpha(I \circ g - J)^2 + \lambda|\nu_n - \mu(g)|^2)dz + \gamma||C \otimes D(g) - C||_2^2.$$

6. Experimental Results.

7. Conclusion.

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