Reinforcement Learning
TP1

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November 11, 2018

1 Dynamic Programming

1.1 Question 1

The MDP model is defined by the following :

In [2]:
```
# Set of states
X = [0, 1, 2]

# Set of actions
A = [0, 1, 2]

# Transition probabilities
p = np.zeros((3, 3, 3))
p[0, 0] = [0.55, 0.45, 0.]
p[1, 0] = [0, 1, 0]
p[2, 0] = [0, 1, 0]
p[0, 1] = [0.3, 0.7, 0]
p[1, 1] = [0, 0.4, 0.6]
p[2, 1] = [0, 0.6, 0.4]
p[0, 2] = [1, 0, 0]
p[1, 2] = [0, 1, 0]
p[2, 2] = [0, 0, 1]


# Reinforcement
r = np.zeros((3, 3))
r[0, 0] = 0
r[1, 0] = 0
r[2, 0] = 0
r[0, 1] = 0
r[1, 1] = 0
r[2, 1] = 1
r[0, 2] = 5/100
r[1, 2] = 0
r[2, 2] = 9/10
```

In this simple MDP the optimal policy π^* is :

- In s_0 : do a_1
- In s_1 : do a_1
- In s_2 : do a_2

```
In [3]: # Optimal policy :
        pi_star = np.zeros(len(X), dtype=int)
        pi_star[0] = 1
        pi_star[1] = 1
        pi_star[2] = 2
```

1.2 Question 2

```
In [4]: gamma = 0.95
```

1.2.1 Policy Evaluation to compute v^*

```
In [5]: # Optimal value function with policy pi*
        p_pi = np.array([p[i, pi_star[i]] for i in X])
        r_pi = np.array([r[i, pi_star[i]] for i in X])
        v_star = np.linalg.inv(np.eye(3) - gamma*p_pi).dot(r_pi)
```

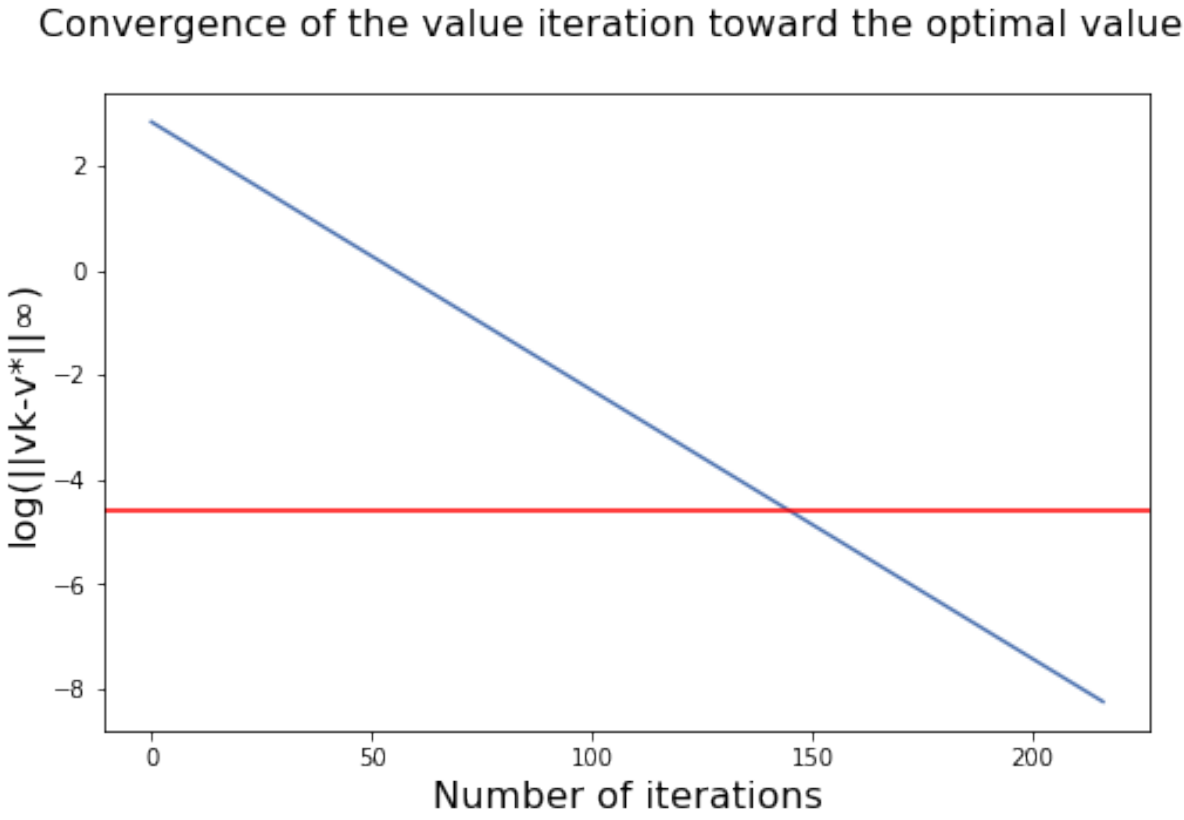
1.2.2 Value iteration

```
In [6]: # Bellman operator
        # Point-wise
        def bellman_op(v, x, X, a, p, r, gamma = 0.95):
            return r[x, a] + gamma * sum([p[x,a,y]*v[y] for y in X])
        def T_pw(v, x, X, A, p, r, gamma = 0.95):
            return max([bellman_op(v, x, X, a, p, r, gamma) for a in A])
        # Vector
        def T(v, X, A, p, r, gamma = 0.95):
            fun = lambda x: T_pw(v, x, X, A, p, r, gamma = 0.95)
            return np.array(list(map(fun, X)))
```

```
In [7]: # Value Iteration
        # Parameters
        n_iter = 0
        eta = 0.01
        gamma = 0.95
        eps = (1-gamma)*eta/(2*gamma)

        # Initialization
        v0 = [0, 0, 0]

        norm_inf_tab = []
        prev_v = v0
        v = T(v0, X, A, p, r)
```

```
        norm_inf_tab.append(np.linalg.norm(v_star-v, ord=inf))

        # Main loop
        while np.linalg.norm(v_star-v, ord=inf) > eps:
            prev_v = v
            v = T(v, X, A, p, r)

            norm_inf_tab.append(np.linalg.norm(v_star-v, ord=inf))
            n_iter += 1
```

In [8]:
```
        plt.figure(figsize=(8, 5))
        plt.plot(list(map(log,norm_inf_tab)), label = '||vk-v*||∞')
        plt.axhline(log(eta), c='r', label = 'Precisions criterion')
        plt.suptitle('Convergence of the value iteration toward the optimal value', size = 16)
        plt.xlabel('Number of iterations', size = 16)
        plt.ylabel('log(||vk-v*||∞)', size = 16)
        plt.show()
```



The graph above represent the evolution of Value Iteration algorithm compared to the value function of the optimal policy v^*. The redline represents a 0.01 difference. We see that we can achieve an arbitrary precision η by stopping when $||v^k - v^{k-1}||_\infty < \epsilon$ for $\epsilon = \frac{1-\gamma}{2\gamma}\eta$.

1.3 Question 3

1.4 Policy Iteration

```
In [9]: # Optimal value function with policy pi*
        def get_value_from_policy(pi, X, r, gamma = 0.95):
            '''Returns the value function for a given policy pi.'''
            p_pi = np.array([p[x, pi[x]] for x in X])
            r_pi = np.array([r[x, pi[x]] for x in X])
            return np.linalg.inv(np.eye(len(X)) - gamma*p_pi).dot(r_pi)

        def policy_iteration_state(v, x, X, A, p, r, gamma = 0.95):
            '''Compute a step of the policy interation algorithm for a given state x.'''
            return A[np.argmax([bellman_op(v, x, X, a, p, r, gamma) for a in A])]

        def policy_iteration(v, X, A, p, r, gamma = 0.95):
            '''Compute a step of the policy interation algorithm for the set of states X'''
            fun = lambda x: policy_iteration_state(v, x, X, A, p, r, gamma)
            return np.array(list(map(fun, X)))
```

```
In [10]: # Policy iteration algorithm
         # Initial policy
         pi = [0, 0, 0]
         iterations = 0
         while True:
             # Until convergence, repeat:
             prev_pi = pi
             # Policy evaluation
             v = get_value_from_policy(pi, X, r)
             # Policy improvement
             pi = policy_iteration(v, X, A, p, r)
             if (prev_pi == pi).all():
                 break
             iterations += 1
         print("Converged in {} iterations".format(iterations))
         print("Final policy : ", pi)
```

```
Converged in 3 iterations
Final policy :  [1 1 2]
```

1.5 Comments

- PI converges in only a few iterations, but requires the inversion of a N sized matrix at each step: complexity $O(n^3)$ with the Gaussian Elimination algorithm, $O(n^{2.373})$ with Optimized CW-like algorithms.
- In VI each iteration has a complexity $O(An^2)$ which is much smaller than for PI, however the algorithm will require more iterations and can only approach the optimal value function asymptoticaly.

2 Reinforcement Learning

```
In [11]: env = GridWorld1
```

2.1 Question 4

```
In [12]: # Question 4
         # Fixed policy : 'right when available, up otherwise'
         det_pol = [0 if (0 in s) else 3 for s in env.state_actions]
         print("Deterministic policy: ", det_pol)
```

```
Deterministic policy:  [0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 3]
```

```
In [13]: def trajectory_simulation(env, state, policy, Tmax = 200):
             '''Simulates a trajectory until an absorbing state is reached
             or Tmax steps have been taken. Return the list of steps and rewards'''
             s = state
             states = []
             rewards = []
             t = 0
             while t < Tmax:
                 a = policy[s]
                 next_state, r, absorb = env.step(s, a)
                 rewards.append(r)
                 states.append(next_state)
                 s = next_state
                 if absorb:
                     break
                 t += 1
             return states, rewards

         def get_discounted_rewards(rewards, gamma = 0.95, T_max = None):
             '''Computes the discouted rewards for a list of rewards and a discount rate gamma.'''
             if (T_max is not None):
                 rewards = rewards[:T_max]
             gammas = np.array([gamma**t for t in range(len(rewards))])
             return gammas.dot(np.array(rewards))
```

2.2 Estimation of μ

```
In [14]: def estimate_density(env, sample_size = 100):
             '''Estimates the starting state distribution of an environnement
             with a given sample size.'''
             i = 0
             mu = np.zeros(env.n_states)
             while i < sample_size:
                 s = env.reset()
                 mu[s] += 1
```

```
            i+=1
        mu = mu/np.sum(mu)
        return mu
```

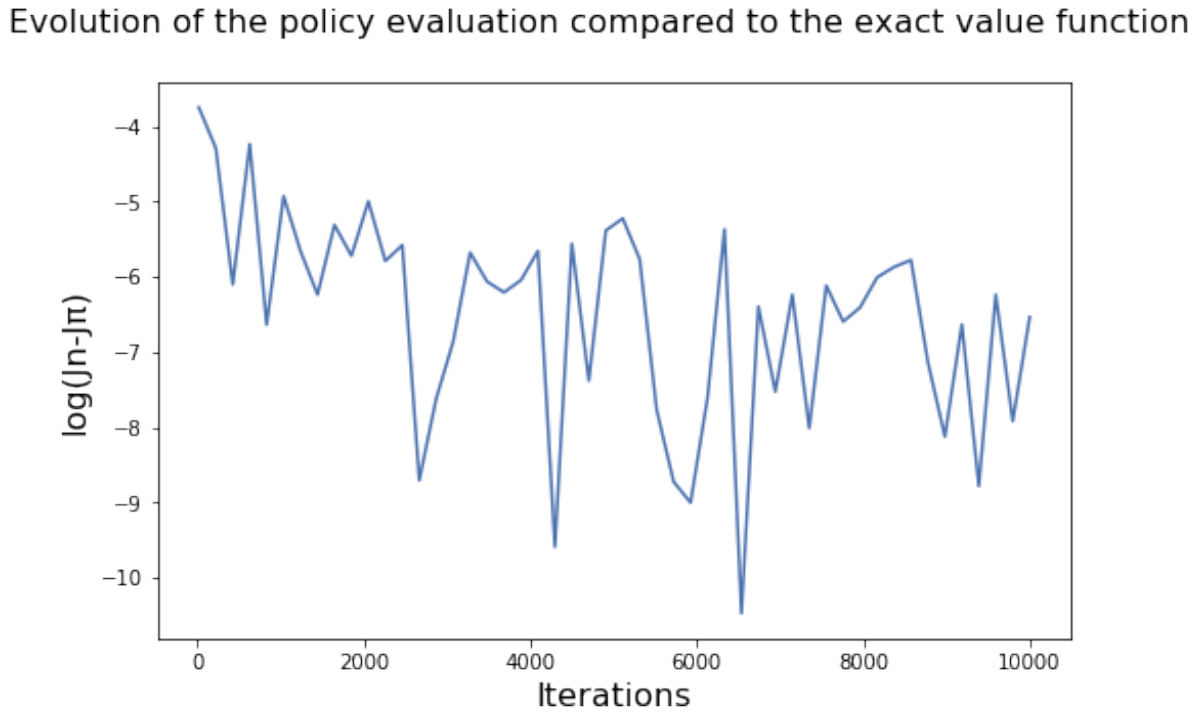
In [15]:
```
mu = estimate_density(env,sample_size=1e4)
print("Estimated starting state density with 10^4 iterations : \n", mu)
```

```
Estimated starting state density with 10^4 iterations :
 [0.1422 0.089  0.0899 0.0408 0.0901 0.0914 0.0901 0.0926 0.087  0.0979
 0.089 ]
```

In [24]:
```
def value_estimator(state, N, policy):
    '''Estimates the value of a state based on a given policy
    using a Monte-Carlo simulation of size N'''
    value = []
    for i in range(N):
        # Simulate a trajectory
        states, rewards = trajectory_simulation(env, state, policy)
        # Compute the discounted rewards
        disc_rewards = get_discounted_rewards(rewards)
        value.append(disc_rewards)
    return np.mean(np.array(value))

def value2vec(n, policy):
    '''Returns a Monte-Carlo estimation of size n of the value function given a policy'''
    n_s = n//env.n_states
    return np.array([value_estimator(s, n_s, policy) for s in range(env.n_states)])

def compute_J(mu, value_estimator):
    '''Computes the total value function weighted by the starting state distribution mu'''
    return np.array(mu).dot(np.array(value_estimator))
```

In [17]:
```
v_pi = [0.87691855, 0.92820033, 0.98817903, 0.00000000, 0.67106071, -0.99447514,
        0.00000000, -0.82847001, -0.87691855, -0.93358351, -0.99447514]
J_pi = compute_J(mu, v_pi)

n_tab=np.linspace(20, 1e4, 50)
delta_J = []
for n in n_tab:
    J_n = compute_J(mu, value2vec(n=int(n), policy=det_pol))
    delta_J.append(abs(J_n - J_pi))
```

In [18]:
```
plt.figure(figsize=(8,5))
plt.suptitle('Evolution of the policy evaluation compared to the exact value function',
             size = 16)
plt.plot(n_tab, list(map(log,delta_J)))
plt.xlabel('Iterations', size= 16)
plt.ylabel('log(Jn-Jπ)', size=16)
plt.show()
```



3 Question 5

• The following stepsize satisfies the Robbins-Monro conditions: $\forall x \in X, \forall a \in A, \alpha_i(x,a) = \frac{1}{i}$.

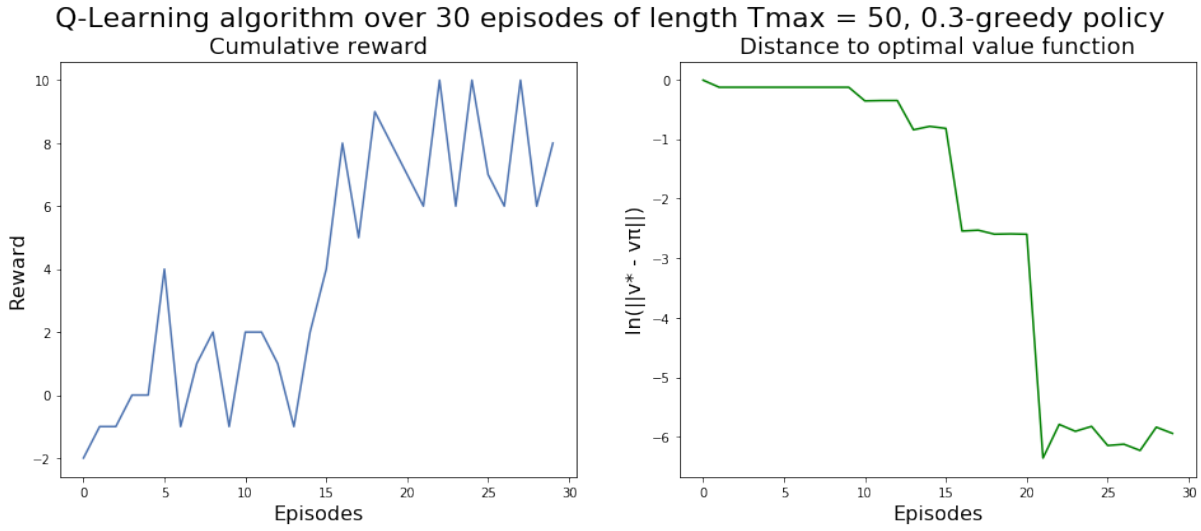
• Policy : exploitation with probability $1-\epsilon$, exploration with probability ϵ. For this relatively simple environnement we can set $\epsilon = 0.3$ for example.

```
In [19]: # Stepsize
         alpha = lambda i, x, a : 1/i

         # Random exploration policy
         exploration_policy = lambda env, s: np.random.choice(env.state_actions[s])

         # epsilon-greedy exploration
         def e_greedy_exploration(env, s, Q, eps):
             p = np.random.binomial(1, eps)
             if p == 1:
                 return exploration_policy(env, s)
             else:
                 return np.argmax(Q[s,:])

         def init_Q(env):
             '''Initialize randomly the matrix Q and set to -inf
             the value of non-admissible actions'''
```

```
    Q = np.random.uniform(size=(env.n_states, len(env.action_names)))
    Q = [[Q[s,a] if (a in env.state_actions[s]) else -np.Infinity
                 for a in range(len(env.action_names))]
                 for s in range(env.n_states)]
    return np.array(Q)

def Q_learning(env, episodes, Tmax, eps):
    '''Perform the Q-Learning with an epsilon-greedy exploration policy'''
    Q = init_Q(env)
    N = np.ones((env.n_states, len(env.action_names)))
    Q_epi = []
    rewards = []
    episode = 0
    while episode < episodes:
        Q, N, reward = Q_learning_episode(env, Q, N, Tmax, eps)
        Q_epi.append(Q)
        rewards.append(reward)
        episode += 1
    return Q_epi, rewards

def Q_learning_episode(env, Q, N, Tmax, eps):
    '''Perform one episode of the Q-Learning algorithm.
    Return matrices Q, N and the cumulative reward.'''
    t = 0
    s = env.reset()
    Q_p = Q.copy()
    cumulative_reward = 0
    while t < Tmax:
        # Pick an action according to the exploration policy
        a = e_greedy_exploration(env, s, Q_p, eps)
        # Take a step
        next_state, r, absorb = env.step(s, a)
        # Stepsize
        alp= alpha(N[s,a], s, a)
        # Update Q
        Q_p[s,a] = (1-alp) * Q_p[s,a] + alp * (r + gamma * max(Q_p[next_state,:]))
        N[s,a] += 1
        cumulative_reward += r
        # Reset if an absorbing state is reached
        if absorb:
            s = env.reset()
        else:
            s = next_state
        t+=1
    return Q_p, N, cumulative_reward

def Q2policy(Q):
    return np.argmax(Q, axis = 1)
```

In [20]:
```
# v*
v_opt = np.array([0.87691855, 0.92820033, 0.98817903, 0.00000000, 0.82369294, 0.92820033,
                  0.00000000, 0.77818504, 0.82369294, 0.87691855, 0.82847001])
# Q Learning algorithm
n_episodes = 30
Tmax = 50
N_value = int(1e4)
eps = 0.3
Q_epi, r = Q_learning(env, n_episodes, Tmax, eps)
delta_v = []
for Q in Q_epi:
    pol = Q2policy(Q)
    value = value2vec(N_value, pol)
    delta_v.append(np.linalg.norm(value-v_opt, ord = inf))
```

In [21]:
```
plt.figure(figsize=(16, 6))
plt.suptitle('Q-Learning algorithm over {} episodes of length Tmax = {}, {}-greedy policy'.
             format(n_episodes, Tmax, eps), size = 22)
plt.subplot(1, 2, 1)
plt.title('Cumulative reward', size = 18)
plt.plot(r)
plt.xlabel('Episodes', size= 16)
plt.ylabel('Reward', size = 16)
plt.subplot(1, 2, 2)
plt.plot(list(map(log, delta_v)), c = 'g')
plt.title('Distance to optimal value function', size = 18)
plt.xlabel('Episodes', size = 16)
plt.ylabel('ln(||v* - vπ||)', size= 16)
plt.show()
```



```
In [23]: print("Average performance over all states over last 5 episodes :", np.mean(delta_v[-5:]))

Average performance over all states over last 5 episodes : 0.0023749258180725706
```

- We see that the cumulative reward increases with the episodes, but stays erratic because of the exploration policy.
- The performance over all states converges in a few episodes, but then oscillates at around $0.002 - 0.003$.

4 Question 6

The initial distribution only impacts how often certain states will be visited compared to others. However **it will not affect the optimal policy**: for example the Q-Learning algorithm - which approaches the optimal policy asymptotically - makes the assumption that all state-action pairs are visited infinitely often; in this case having certain states visited more often than others do not have any impact on the convergence of the algorithm.