Statistical Methods in Physics (14P058)

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Final project II – The muon Michel Parameters

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The study of the muon decay ($\mu^- \to e^- \nu_\mu \bar{\nu}_e$) provided significant information about the weak interaction. Exploring the energy of the electron we can measure deviations of the vector-axial weak current Standard Model assumption. The cross-section can be parameterised as function of the potential scalar, pseudo-scalar, vector and axial currents. One possible quantity to measure is the electron energy divided by the muon mass in the muon rest-frame (x). The probability of finding an electron with fractional energy x per muon is given by the formula:

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}x\mathrm{d}\cos\theta} \propto x^2 \left(\frac{1}{2}(3-3x) + \frac{2}{3}\rho(4x-3) + 3\eta x_0 \frac{(1-x)}{x} + P_\mu \xi \cos\theta[(1-x) + \frac{2}{3}\delta(4x-3)]\right) \tag{1}$$

where P_μ is the muon polarisation that carries information about the muon production phenomena, $\theta \in [0,\pi]$ is the angle between the electron and the muon polarisation vector in the muon rest frame and x is the fraction of the muon mass carried by the electron. x_0 is a constant equal to 9.67×10^{-3}. The parameters $(\rho, \xi, \delta, \eta)$ are equal to $(\frac{3}{4}, 1, \frac{3}{4}, 0)$ in the Standard Model.

To simplify the problem, let's think that the muon is not polarised or that we do integrate over the polar angle θ. In this case the cross-section is given by:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}x} \propto x^2 \left(\frac{1}{2}(3-3x) + \frac{2}{3}\rho(4x-3) + 3\eta x_0 \frac{(1-x)}{x}\right) \tag{2}$$

In what follows, we will consider the Standard Model Michel parameters ($\rho = 3/4., \eta = 0$) for the generation of the event samples. We will use the full formula later to estimate the parameter δ and to explore a real experiment where we try to search for deviations from the model.

Task 1: *Basic statistical properties*

Equation (2) fulfils the properties of a probability distribution function. The first task is to assess basic statistical properties of the distribution with the help of Monte Carlo (MC) processes.

a) Demonstrate that Eq. (2) is a probability density function. Compute the normalization and the range of possible values of ρ and η that allows the cross-section to be a p.d.f.

   In both cases, use the Standard Model Michel parameters as inputs.

b) The expectation value of x depends on the polarisation Michel parameters in a simple manner:

$$\mathbb{E}[x] = \frac{\frac{9}{5} + 2x_0\eta + \frac{4}{5}\rho}{3 + 4x_0\eta} \tag{3}$$

and fixing to the SM parameters values:

$$\mathbb{E}[x] = \frac{9 + 4 \times 3/4}{15} = \frac{4}{5} \tag{4}$$

Compute the sample mean $\langle x \rangle$ and compare with the expectation value for the case of the Standard Model parameters.

c) Compute the variance, skewness and kurtosis of the Standard Model distribution and comment.

Task 2: *Convergence*

If the experiment is repeated N_{exp} times, it can be seen that the sample means $\langle x \rangle$ of the experiments are distributed according to a Gaussian distribution. In the following sub-tasks, we will look at the convergence behaviour of $\langle x \rangle$. Each of the N_{exp} experiments still contains $N \gg 1$ data points generated with the MC process.

a) Take $N_{exp} = 1$ and show the law of large numbers based on the data points you generated.

b) Take $N_{exp} \gg 1$ and show the validity of the central limit theorem for $\langle x \rangle$.

c) Compute the variance of $\langle x \rangle$ and compare it with the value you expect.

Task 3: χ^2 *distribution*

By filling the numbers generated from the MC process into a histogram, we can study the χ^2 properties of the distribution. For now, we fix $N_{exp} = 1$ and $N \gg 1$, according to the probability density function in Eq. (2). Fill the obtained x values in a histogram.

a) What is the expected p.d.f. for the number of entries per bin?

b) Show that the χ^2 of the obtained entries per bin follows a χ^2 distribution. Instead of calculating the empirical mean value per bin from the experiments, use the nominal value from the p.d.f. formula per bin as the mean value per bin.

c) How does the χ^2 distribution change with the number of bins?

Task 4: *Parameter estimation*

We will now use three different methods to perform parameter estimation and compare the results from the different methods. Let's again start with one single experiment ($N_{exp} = 1$) with $N \gg 1$ data points according to the probability density function in Eq. (2).

a) Compute the log likelihood $ln\ L(x_1, x_2, ..., x_N \,|\, \rho, \eta)$ at a given (ρ, η). Then, use the maximum likelihood method to estimate (ρ, η) and its covariance.

b) Compute the goodness of fit at various (ρ, η) and use the least squares method to estimate ρ, η and its covariance.

c) The parameter (ρ, η) cannot be estimated simultaneously by computing the sample mean $\langle x \rangle$ of Eq. (4) with the MC integral. Why? How would you do this using the method of the moments. Perform this estimation of ρ and its corresponding variance.

d) Compare the results from the three questions above and discuss them.

Task 5: *Hypothesis testing*

Generate $N_{exp} = 200$ experiments and combine them into pairs, that is, forming a total of 100 pairs of experiments. Now let's assume we obtained these datasets in actual experiments and we would like to – for each pair – test the hypothesis that the two sets of experimental data originate from the same distribution. Set a threshold for the hypothesis testing and perform Kolmogorov tests for all dataset pairs. Then, comment on your outcome with respect to the threshold you set.

Task 6: *Bonus question*

Generate N_{exp} number of experiments with N data in each. Compute the exact Fisher information with the equation

$$I_X(\rho) = \mathbb{E}\left[-\frac{\partial^2 \ln L(X|\rho)}{\partial^2 \rho}\bigg|_{\rho=\rho_0} \right] \tag{5}$$

Compare your results with the variance $V(\hat{\rho} - \rho)$ and discuss. (*)

N.B. This task is not required for your report, but if you hand in a solution, it can be accredited for bonus points.