Université de Genève Autumn term 2021

Statistical Methods in Physics (14P058)

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Final project II – The muon polarisation

Submission due on: 24 January 2022

The study of the muon decay ($\mu^- \to e^- \nu_\mu \bar{\nu}_e$) provided significant information not only about the weak interaction, but also about the processes that generate muons. One possible quantity to measure is the muon polarisation, defined as the fraction of muons with positive helicity. The angular distribution of the electron produced in a single muon decay is described by the probability

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta} = \frac{1}{2}(1 - \frac{1}{3}P_\mu \cos\theta),\ \ P_\mu \in [-1, 1] \tag{1}$$

where P_μ is the muon polarisation that carries information about the muon production phenomena, and $\theta \in [0, \pi]$ is the angle between the electron and the muon polarisation vector in the muon rest frame.

Task 1: *Basic statistical properties*

Equation (1) fulfils the properties of a probability distribution function. The first task is to assess basic statistical properties of the distribution with the help of Monte Carlo (MC) processes.

a) Demonstrate that Eq. (1) is a probability density function.

b) Use two different kinds of MC processes to generate this probability density function for a large number of data points, $N \gg 1$. Fill your results in a histogram, plot them and verify their correctness by comparing them with the curve of the analytical form in Eq. (1).

c) The expectation value of $\cos\theta$ depends on the polarisation P_μ in a simple manner:

$$\mathbb{E}\left[\cos\theta\right] = -\frac{P_\mu}{9} \tag{2}$$

Compute the sample mean $\langle\cos\theta\rangle$ and compare with the expectation value.

d) What does the result in Eq. (2) imply for the muon production mechanisms?

e) Compute the variance, skewness and kurtosis of the distribution and comment.

Task 2: *Convergence*

If the experiment is repeated N_{exp} times, it can be seen that the sample means $\langle\cos\theta\rangle$ of the experiments are distributed according to a Gaussian distribution. In the following sub-tasks, we will look at the convergence behaviour of $\langle\cos\theta\rangle$. Each of the N_{exp} experiments still contains $N \gg 1$ data points generated with the MC process.

a) Take $N_{exp} = 1$ and show the law of large numbers based on the data points you generated.

b) Take $N_{exp} \gg 1$ and show the validity of the central limit theorem for $\langle \cos\theta \rangle$.

c) Compute the variance of $\langle \cos\theta \rangle$ and compare it with the value you expect.

Task 3: χ^2 *distribution*

By filling the numbers generated from the MC process into a histogram, we can study the χ^2 properties of the distribution. For now, we fix $N_{exp} = 1$ and $N \gg 1$, according to the probability density function in Eq. (1). Fill the obtained $\cos\theta$ values in a histogram.

a) What is the expected p.d.f. for the number of entries per bin?

b) Show that the χ^2 of the obtained entries per bin follows a χ^2 distribution. Instead of calculating the empirical mean value per bin from the experiments, use the nominal value from the p.d.f. formula per bin as the mean value per bin.

c) How does the χ^2 distribution change with the number of bins?

Task 4: *Parameter estimation*

We will now use three different methods to perform parameter estimation and compare the results from the different methods. Let's again start with one single experiment ($N_{exp} = 1$) with $N \gg 1$ data points according to the probability density function in Eq. (1).

a) Compute the log likelihood $ln\ L(x_1, x_2, ..., x_N \,|\, P_\mu)$ at a given P_μ. Then, use the maximum likelihood method to estimate P_μ and its variance.

b) Compute the goodness of fit at various P_μ and use the least squares method to estimate P_μ and its variance.

c) The parameter P_μ can also be estimated by computing the sample mean $\langle \cos\theta \rangle$ of Eq. (1) with the MC integral. Perform this estimation of P_μ and its corresponding variance.

d) Compare the results from the three questions above and discuss them.

Task 5: *Hypothesis testing*

Generate $N_{exp} = 200$ experiments and combine them into pairs, that is, forming a total of 100 pairs of experiments. Now let's assume we obtained these datasets in actual experiments and we would like to – for each pair – test the hypothesis that the two sets of experimental data originate from the same distribution. Set a threshold for the hypothesis testing and perform Kolmogorov tests for all dataset pairs. Then, comment on your outcome with respect to the threshold you set.

Task 6: *Bonus question: Measurement with systematic errors*

In an actual measurement, the obtained data may come with systematic errors. Let's simulate this effect by generating $N = 10^4$ data points which follow Eq. (1). We assume that there is an experimental error in determining the angle $\cos\theta$, which follows a Gaussian smearing with standard deviation $\sigma = 0.05$. Now, if we repeat the three estimation techniques in Task 4, how is the outcome influenced by this fluctuation? Which of the estimation techniques is more fragile and which one is more robust? Perform the simulation and discuss.

N.B. This task is not required for your report, but if you hand in a solution, it can be accredited for bonus points.